Applications of Stochastic calculus in economy and statistics: Extensions of the Kyle-Back model. Ambit processes and power variation.

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Introduction

This thesis deals with three possible applications of stochastic calculus: modelling prices by supply and demand in a financial market where there is an informed trader, turbulence and financial models using ambit processes and the asymptotic analysis of certain power variation processes.

In Part I, basic facts and techniques of mathematics used in the latter chapters are presented, such as Lévy processes, enlargement of filtrations, filtering techniques and dynamic programming approach of stochastic optimal control.

In Part II, markets with the presence of the insider are studied. Such markets with asymmetric information have a great literature. We will take, as a base, Kyle's model, introduced in [Kyl85], an order-driven market of a risk-free bond and a risky asset. We can distinguish between two different approaches of pricing: endogenously and exogenously given prices. When prices are given exogenously, the price process of the assets are given and the participants try to maximize their profit. In real markets the prices are given endogenously, t.i. they are determined by supply and demand. In this case, all buyers and sellers display the price at which they are willing to buy or sell a security, and also the amount that they are willing to buy or sell. They are called bid and ask prices. When those requirements meet, trading is done. In the models studied in Part II, only the amount of bids and asks are set by the participants and designated specialists, the *market makers* set the prices of the assets. In this case, the equilibrium sought is one maximizing the profit of the informed trader in a way that the market makers set a rational pricing rule satisfying market efficiency conditions. A detailed description of these markets can be found in Chapter 2. Briefly, the following is studied in this Part.

We shall assume that there are three kinds of traders on the market: *noise traders* (or *liquidity traders*) who trade for hedging reasons, an *informed trader* (or *insider*), who is aware of the privilege information about the risky asset, such as the underlying value or the price to be announced later, and the *market makers*, who clear the market setting the prices according to the total demand of the noise traders and the informed trader. The demand of the insider is a function of the price and the information possessed by her, and the price of the risky asset is a function of the total demand, so the presence of the insider does have an impact on the market, the stock price also depends on her strategy. Thus, an equilibrium is sought, and generally, sufficient and necessary conditions are found, and the informed trader's strategy in equilibrium is described, as well. An important property of the model is that, in equilibrium, the insider is inconspicuous, t.i. the total demand being the demand of the noise traders' and insider's together, is of the same distribution on the market makers' information, as the demand of the noise trader on its own filtration, as it is without the presence of the insider. Kyle's model is constructed in three steps: first, a single-auction equilibrium model is described where at time 0 the insider learns some privileged information: the price of the risky asset at time 1, that is to be announced right after the trading. In this case, the noise traders' demand is given by a Gaussian variable. Then, an N-period model is described, in which the insider learns the same information as before, but there are N auctions at discrete times before the announcement, with the noise traders' cumulative demand following a discretization of a Brownian motion. Finally, a continuous model is described and also obtained as the limit of the N-period model, as N tends to infinity. This continuous model is studied in [Bac92] and sufficient and necessary conditions were found using a perturbation method, and also the originally used dynamic programming approach is presented. As in the model before, in the continuous one, the noise traders' cumulative demand is given by a Brownian motion. There come several possible extensions of the model regarding the kind of information possessed, the time horizon and the noise traders' demand, as well, as the participants on the market. The original models are presented in Chapter 2, and their extensions and related models in Chapter 3. In particular, allowing the noise traders' demand to be a Lévy process is studied in Section 3.1, a general model with possibly random deadline and a more complex information structure, with applications of enlargements of filtrations and filtering techniques, is studied in Section 3.2, with general results as well, as its applications to find the insider's strategy in special cases: models already introduced and studied. A summary of other related model can be found in Section 3.3. Section 3.1 and Section 3.2 summarize the results of [Cor14b] and [Cor14a], respectively, which can be found in the appendices. Extensions covered by these two generalizations and other related continuous models deal with different types of dynamic information [Dan10, CS10, CcD11, CcD13b], a weaker sense of equilibrium, [Wu99, KHOL10, Dan10], risk-averse insiders [Cho03] and different techniques to find optimal strategies [CcD11, CcD13b]. Possible extensions of the discrete model considering more than one insiders [NT06] and more than one signals [Jai99] are also presented.

Part III is dedicated to the recent research about ambit processes. The notion of ambit processes was introduced in [BNS07]. Since than, many properties and applications have been studied. Ambit fields are stochastic fields $\{Y(t,x)\}$ in space-time, where $t \in \mathbb{R}, x \in \mathbb{R}^n$, with the values of (t,x) depending on what happened prior to time t in a certain subset of $\mathbb{R}^n$ (meaning that in the model the future cannot influence the past). Then, an ambit process is $Y_t := Y(t, x(t))$, where $x(t)$ is a curve in $\mathbb{R}^n$. One particular case, used in the short-rate model, for example, is

$$X_t = \int_{-\infty}^{t} g(t-s)W(ds) \quad t \geq 0,$$

where W is a Gaussian white noise in $\mathbb{R}^n$, and $g \in L^2(\mathbb{R}_+)$. It is important to note the dependence of the path of the ambit process in the behavior of the weight function g near 0. Applications of ambit processes are presented: stochastic modelling in turbulence, models in energy markets are studied [CFV14], and a short rate model describing bond prices [CFSV13]. In the latter model, option prices and the completeness of the market are also studied, in particular a fractional version of the Cox-Ingersoll-Ross model together with a numerical method that can be applied, in case there is no exact formula for the price.

In Part IV, the power variations of processes of the form $\mathrm{d}Z_t = u_{s-}\mathrm{d}S_s^\alpha$ are considered, where $\alpha \in (0, 2]$, $(S_t^\alpha)_{t\geq 0}$ is an α-stable Lévy process, and where, roughly speaking, the power variation is defined as limit of

$$V_t^{(p)}(\Pi) = \sum_{k=1}^{m} \left| X_{t_k} - X_{t_{k-1}} \right|^p,$$

with Π being a partition on the period $[0, t]$ and with X being a stochastic process. Note, that with $p = 2$ it coincides with the well known quadratic variation. After reviewing the existing results for α-stable processes in Section 5.2, Section 5.3 summarizes the new results that is contained in [CF10], relaxing the conditions of the trajectories of u having a finite q-variation on any finite interval for some $q < \alpha / \max\{0, \alpha - 1\}$ to having

$$\int_0^t |u_s|^\alpha \, \mathrm{d}s < \infty,$$

showing that the same theorems describing its asymptotic behavior and the same Central Limit Theorem hold (Theorems 5.2.1, 5.2.2 and 5.2.3).

The thesis is organized as follows. Part I contains the basic facts and techniques of mathematics used in the latter parts. Part II deals with the markets with asymmetric information, Chapter 2 presents the basic models by Kyle and Back, and Chapter 3 presents the new results of Kyle's model with Lévy noise: [Cor14b] and a General Model: [Cor14a], and also a short summary of other related models. Part III is dedicated to ambit processes. Chapter 4 introduces ambit fields and processes and bond markets, summarizes the new results of some applications of ambit processes on energy markets and turbulence: [CFV14], and on a short rate model: [CFSV13]. In Part IV, power variation processes are introduced and new results of [CF10] are summarized in Section 5.3. Finally, the above mentioned articles are included in the appendices.

Part I

Basic facts and techniques

Chapter 1

Theory

In this Chapter, we do a short review of theories used in the thesis: the Lévy processes, initial and progressive enlargements of filtrations, filtering techniques and a dynamic programming approach of stochastic optimal control.

1.1 Lévy Processes

Assume that in all the definitions and results, the stochastic process $(X_t)_{t\geq 0}$ is defined on $\mathbb{R}$, even though most of the definitions and results are well-defined and hold for processes on $\mathbb{R}^n$, as well.

Definition 1.1.1 (Definition 1.6 in [Sat99]) *A real valued stochastic process $(X_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is called a time-homogeneous Lévy process, if the following conditions are satisfied:*

(a) it has independent increments, that is, for any $n \geq 1$ and $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent,

(b) $X_0 = 0$ almost surely,

(c) the distribution of $(X_{s+t} - X_s)_{t\geq 0}$ does not depend on s,

(d) it is stochastically continuous, that is for any $\varepsilon > 0$: $lim_{t\to 0} P\left(|X_{s+t} - X_s| > \varepsilon\right) = 0$,

(e) as a function of t, X_t is right continuous with left limits almost surely (càdlàg).

We refer to 4.2 in Chapter II in [JS00] to remark that the stochastic continuity condition follows from the others if all of them are satisfied, t.i. if (a), (b), (c) and (e) hold, then it implies that (d) holds, as well. Various generalizations of the previous type are used, such as *Lévy process in law*: if it satisfies (a)–(d), *time in-homogeneous Lévy process* or *additive process*: if it satisfies (a), (b), (d) and (e) and *additive process in law*: if it satisfies (a), (b) and (d).

Denote the convolution of two distributions μ_1 and μ_2 by

$$\mu_1 * \mu_2(B) = \int\int_{\mathbb{R}\times\mathbb{R}} 1_B(x+y)\mu_1(\mathrm{d}x)\mu_2(\mathrm{d}x),$$

and the n-fold convolution of $\mu_1, \dots \mu_n$ by μ_n^{n*}. A distribution μ is infinitely divisible if, for any n positive integer, there exists a distribution μ_n, so that $\mu = \mu_n^{n*}$. Denote the law (distribution) of a random variable X by $\mathcal{L}(X)$ and define the characteristic function of a distribution μ by $\Phi_\mu(\cdot) : \mathbb{R} \to \mathbb{C}$ as

$$\Phi_\mu(z) := \int_{\mathbb{R}} e^{izx}\mu(\mathrm{d}x), \qquad z \in \mathbb{R}.$$

Theorem 1.1.1 (Theorem 9.1 and Corollary 11.6 in [Sat99]) *If $(X_t)_{t\geq 0}$ is an additive process in law, then for any $t \geq 0$, $\mathcal{L}(X_t)$ is infinitely divisible. If μ is an infinitely divisible distribution, then there exists, unique in law, a Lévy process in law $(X_t)_{t\geq 0}$, such that $\mathcal{L}(X_1) = \mu$.*

Theorem 1.1.2 (Lévy-Khintchine representation, Theorem 8.1 in [Sat99]) *If μ is infinitely divisible, then*

$$\Phi_\mu(z) = \exp\left\{-\frac{1}{2}Az^2 + \int_{\mathbb{R}} \left(e^{izx} - 1 - izx1_{\{|x|\leq 1\}}(x)\right)\nu(\mathrm{d}x) + i\gamma z\right\},$$

where $A \geq 0$, ν is a measure on $\mathbb{R}$, satisfying

$$\nu(\{0\}) = 0 \quad \textit{and} \quad \int_{\mathbb{R}} \left(|x|^2 \wedge 1\right)\nu(\mathrm{d}x) < \infty,$$

and $\gamma \in \mathbb{R}$. This representation by (A, ν, γ) is unique. Conversely, for any choice of (A, ν, γ) satisfying the conditions above, there exists an infinitely divisible distribution μ having this characteristic function.

Let $(X_t)_{t\geq 0}$ be a Lévy process corresponding to an infinitely divisible distribution μ, as in Theorem 1.1.1, then it has the following characteristic function

$$\begin{aligned}\Phi_{X_t}(z) &= E\left[e^{izX_t}\right] = \left(\Phi_\mu(z)\right)^t \\ &= \exp\left\{t\left(-\frac{1}{2}Az^2 + \int_{\mathbb{R}} \left(e^{izx} - 1 - izx1_{\{|x|\leq 1\}}(x)\right)\nu(\mathrm{d}x) + i\gamma z\right)\right\}.\end{aligned}$$

The triple (A, ν, γ) is called the *generating triplet*. In particular, A is called the *Gaussian variance* (matrix, in case of d-dimensional processes), ν the *Lévy measure* of μ (or of the corresponding Lévy process). The value of γ depends on the choice of the term $izx1_{\{|x|<1\}}(x)$ in the integrand as it is a term to make it ν-integrable and does not have such a meaning as A or ν. Note, that if $\nu = 0$, then μ is Gaussian, and in case of having $A = 0$, we say that μ is purely non-Gaussian.

Let $c(\cdot)$ be a measurable, bounded function of $O(1/|x|)$ as $|x| \to \infty$ and $1 + o(|x|)$ as $x \to 0$, and define γ_c and rewrite the characteristic function as

$$\begin{aligned}
\gamma_c &= \gamma + \int_{\mathbb{R}} x(c(x) - 1_{\{|x|\leq 1\}}(x))\nu(\mathrm{d}x), \\
\Phi_{X_t}(z) &= \exp\left\{t\left(-\frac{1}{2}Az^2 + \int_{\mathbb{R}}\left(e^{izx} - 1 - izxc(x)\right)\nu(\mathrm{d}x) + i\gamma_c z\right)\right\}
\end{aligned}$$

Then, the triplet $(A, \nu, \gamma_c)_c$ is called a generating triplet, as well, with the chosen c. If we omit writing c, then we refer to $c(x) = 1_{\{|x|\leq 1\}}$, if ν satisfies $\int_{|x|\leq 1}|x|\,\nu(\mathrm{d}x) < \infty$, then we can use $c(\cdot) \equiv 0$ and we call γ_0 the drift, and if ν satisfies $\int_{|x|>1}|x|\,\nu(\mathrm{d}x) < \infty$ (or equivalently $\int_{\mathbb{R}}|x|\,\mu(\mathrm{d}x) < \infty$, see Theorem 6.1 in [Sat99]), then we can use $c(\cdot) \equiv 1$ and call γ_1 the center of μ, which, for such ν, equals γ_0 and the mean of μ.

A Lévy process is called non-trivial, if μ is non trivial (not concentrated to a point). It is a

- *Brownian motion* if $(A, \nu, \gamma_0)_0 = (1, 0, 0)_0$, with $\Phi_\mu(z) = \exp\left\{-\frac{1}{2}z^2\right\}$
- *Poisson processes* if $(A, \nu, \gamma_0)_0 = (0, c\delta_1, 0)_0$ where δ_a is the distribution on $\mathbb{R}$ concentrated on $a \in \mathbb{R}$, with $\Phi_\mu(z) = e^{c\left(e^{iz}-1\right)}$,
- *Compound Poisson process* if $(A, \nu, \gamma_0)_0 = (0, c\sigma, 0)$ with $c > 0$ and σ being a distribution on $\mathbb{R}$ with $\sigma(\{0\}) = 0$,
- Γ-*process* with the parameters c and α if $(A, \nu, \gamma_0)_0 = (0, \nu, 0)$, where $\nu(dx) = c1_{(0,\infty)}(x)x^{-1}e^{-\alpha x}\mathrm{d}x$, and in this case, we have

$$\Phi_\mu(z) = \exp\left\{c\int_0^\infty \left(e^{izx} - 1\right)\frac{e^{-\alpha x}}{x}\mathrm{d}x\right\}.$$

For details of the above mentioned examples and calculations, see Chapter 2 in [Sat99].

If a Lévy process is not time-homogeneous, then it determines a system of triplets $\{(A_t, \nu_t, \gamma_t) : t > 0\}$, where each (A_t, ν_t, γ_t) is the generating triplet of $\mathcal{L}(X_t)$. Consider an additive process $X.$ on the probability space $(\Omega, \mathcal{F}, P)$. Let $\Omega_0 \in \mathcal{F}$ such that $P(\Omega_0) = 1$ and for every $\omega \in \Omega_0$, the function $X_t(\omega)$ is right-continuous in $t \geq 0$

and has left limits in $t > 0$. Denote the Borel σ-algebra of a set A by $\mathcal{B}(A)$. For the definition of the Poisson random measure, see Definition 19.1 in [Sat99]. Then, we have the Lévy-Itô decomposition as follows:

Theorem 1.1.3 (Theorem 19.2 in [Sat99]) *Let $(X_t)_{t\geq 0}$ be an additive process defined on the probability space $(\Omega, \mathcal{F}, P)$ with he generating triplet $(A_t, \nu_t, \gamma(t))_{t\geq 0}$ and define the measure $\tilde{\nu}$ on H by $\tilde{\nu}((0,t] \times D) = \nu_t(D)$ for $D \in \mathcal{B}(\mathbb{R})$. Define, for $B \in \mathcal{B}(H)$ and Ω_0 as above,*

$$J(B,\omega) = \begin{cases} \#\{t : (t, X_t(\omega) - X_{t-}(\omega)) \in B\} & \text{for } \omega \in \Omega_0, \\ 0 & \text{for } \omega \notin \Omega_0. \end{cases}$$

Then, the following hold.

1. *$\{J(B) : B \in \mathcal{B}(H)\}$ is a Poisson random measure on H with intensity $\tilde{\nu}$*

2. *There is a $\Omega_1 \in \mathcal{F}$ with $P(\Omega_1) = 1$ such that, for any $\omega \in \Omega_1$,*

$$\begin{aligned} X_t^1 &= \lim_{\varepsilon \downarrow 0} \int_{(0,t]\times(\varepsilon,1]} [xJ(\mathrm{d}(s,x),\omega) - x\tilde{\nu}(\mathrm{d}(s,x))] \\ &\quad + \int_{(0,t]\times(1,\infty)} xJ(\mathrm{d}(s,x),\omega) \end{aligned}$$

   is defined for all $t \in [0,\infty)$ and the convergence is uniform in t on any bounded interval. The process $(X_t^1)_{t\geq 0}$ is an additive process on $\mathbb{R}$ with the generating triplet $(0, \nu_t, 0)_{t\geq 0}$.

3. *Define*

$$X_t^2(\omega) = X_t(\omega) - X_t^1(\omega) \quad \text{for } \omega \in \Omega_1.$$

   There is $\Omega_2 \in \mathcal{F}$ with $P(\Omega_2) = 1$ such that, for any $\omega \in \Omega_2$, $X_t^2(\omega)$ is continuous in t. The process $(X_t^2)_{t\geq 0}$ is an additive process on $\mathbb{R}$ with the generating triplet $(A_t, 0, \gamma(t))_{t\geq 0}$.

4. *The processes $(X_t^1)_{t\geq 0}$ and $(X_t^2)_{t\geq 0}$ are independent.*

Theorem 1.1.4 (Theorem 19.3 in [Sat99]) *Suppose that the additive process in Theorem 1.1.3 satisfies*

$$\int_{|x|\leq 1} |x|\, \nu_t(\mathrm{d}x) < \infty \quad \text{for all } t > 0.$$

Let $\gamma_0(t)$ be the drift of X_t. Then, there is a $\Omega_3 \in \mathcal{F}$ with $P(\Omega_3) = 1$ such that, for any $\omega \in \Omega_3$,

$$X_t^3(\omega) = \int_{(0,t]\times(0,\infty)} xJ(\mathrm{d}(s,x),\omega)$$

is defined for all $t \geq 0$. The process $(X_t^3)_{t\geq 0}$ is an additive process on $\mathbb{R}$ such that

$$E\left[e^{izX_t^3}\right] = \exp\left\{\int_{\mathbb{R}} \left(e^{izx} - 1\right) \nu_t\left(\mathrm{d}x\right)\right\}.$$

Define

$$X_t^4\left(\omega\right) = X_t\left(\omega\right) - X_t^3\left(\omega\right), \quad \text{for } \omega \in \Omega_3.$$

Then, for any $\omega \in \Omega_2 \cap \Omega_3$, $X_t^4(\omega)$ is continuous in t and $(X_t^4)_{t\geq 0}$ is an additive process on $\mathbb{R}$ such that

$$E\left[e^{izX_t^4}\right] = \exp\left\{-\frac{1}{2}z^2 A_t + i\gamma_0\left(t\right)z\right\}.$$

The two processes $(X_t^3)_{t\geq 0}$ and $(X_t^4)_{t\geq 0}$ are independent.

In the context of Theorem 1.1.4, $(X_t^3)_{t\geq 0}$ and $(X_t^4)_{t\geq 0}$ are called the jump part and the continuous part of $(X_t)_{t\geq 0}$, respectively. The processes $(X_t^1)_{t\geq 0}$ and $(X_t^2)_{t\geq 0}$ are called so, as well, but they do depend on the choice of the representation (for more details, see Remark 8.4 in [Sat99]).

Definition 1.1.2 (Definition 13.1 in [Sat99]) *An infinitely divisible probability measure μ on $\mathbb{R}$ is called (strictly) stable, if, for any $a > 0$, there are $b > 0$ and $c \in \mathbb{R}$ ($c = 0$ in case of strictly stable) such that*

$$\left[\Phi_\mu\left(z\right)\right]^a = \Phi_\mu\left(bz\right)e^{icz}.$$

It is called semi-stable (strictly semi-stable), if for some $a > 0$ with $a \neq 0$, there are $b > 0$ and $c \in \mathbb{R}$ ($c = 0$ in case of strictly stable) satisfying the previous equation.

Definition 1.1.3 (Definition 13.2 in [Sat99]) *Let X_t, $t \geq 0$ be a Lévy process. It is called a stable, strictly stable, semi stable or strictly semi-stable process if the distribution of X_1 is, respectively, stable, strictly stable, semi stable or strictly semi-stable.*

If μ is stable, then it is infinitely divisible, and the corresponding Lévy process is such that, for any $a > 0$, there are $b > 0$ and $c \in \mathbb{R}$ such that X_{at} and $bX_t + ct$ are of the same distribution (with $c = 0$ if μ is strictly stable). If μ is nontrivial, then b is uniquely determined by a, and there is an $\alpha = (1/\log_a b) \in (0, 2]$, equivalently $b = a^{1/\alpha}$. The corresponding nontrivial Lévy process is called (strictly) α-stable process. The characteristic exponent of such a process is

$$\Psi\left(iz\right) = -c\left|z\right|^\alpha\left(1 - i\beta\tan\frac{\pi\alpha}{2}\mathrm{sgn}\ z\right) + i\tau z$$

if $\alpha \neq 1$ and

$$\Psi(iz) = -c\left(|z| + i\beta\frac{2}{\pi}z\log|z|\right) + i\tau z,$$

for $\alpha = 1$, where $c > 0, \beta \in [-1,1]$ and $\tau \in \mathbb{R}$. (α, β, τ, c) are called the parameters of the non-Gaussian stable distribution or Lévy process. For instance, with parameter $(1/2, 1, 0, c)$, the density of μ can be written as

$$(2\pi)^{-1/2}\, ce^{-c^2/(2x)}x^{-3/2}1_{(0,\infty)}(x).$$

1.2 Enlargement of filtrations

Consider two filtrations $\mathbb{F} = (\mathcal{F}_t)_t$ and $\mathbb{H} = (\mathcal{H}_t)_t$. Let the first one represent the already known information and the latter one some new information. Then, we can define the enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_t$ with $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. For several research, we are interested in the Doob-Meyer decomposition with respect to the enlarged filtration in function of $\mathbb{F}$ and $\mathbb{H}$, and also to know, when an $\mathbb{F}$-semimartingale remains so with respect to $\mathbb{G}$. We distinguish between two cases: if $\mathcal{H}_t = \sigma(R)$ for some random variable, it is called initial enlargement, as (all) the new information did arrive at time 0. When it does not hold, it is called progressive enlargement. In the following, some important results are summarized. For a more detailed discussion, see [Cor14a], [Jeu80], [Jeu85], [Man06] and [CV11].

Initial enlargement of filtrations

Consider a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a $\mathcal{F}$-measurable random variable L with values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $\mathcal{G}_t := \cap_{s>t}(\mathcal{F}_t \vee \sigma(L))$ and $\mathbb{G} = (\mathcal{G}_t)$. Then, we have the following results.

Proposition 1.2.1 *Let η be the law of L. Then, $Q_t(\omega, \mathrm{d}x) \ll \eta(\mathrm{d}x)$ if and only if for all t, there exists a σ-finite measure η_t in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $Q_t(\omega, \cdot) \ll \eta_t$ where $Q_t(\omega, \mathrm{d}x)$ is a regular version of the law of $L|\mathcal{F}_t$.*

Proposition 1.2.2 *If $Q_t(\omega, \mathrm{d}x) \ll \eta(\mathrm{d}x)$, then there exists a $\mathcal{B}(\mathbb{R})\otimes\mathcal{F}_t$-measurable process $q_t^x(\omega)$ such that $Q_t(\omega, \mathrm{d}x) = q_t^x(\omega)\eta(\mathrm{d}x)$ and, for fixed x, q_t^x is an $\mathbb{F}$-martingale.*

Theorem 1.2.1 *Let M be a continuous local $\mathbb{F}$-martingale and consider $k_t^x(\omega)$ such that*

$$\langle q^x, M\rangle_t = \int_0^t k_s^x q_{s-}^x \mathrm{d}\langle M, M\rangle_s,$$

then

$$M - \int_0^{\cdot} k_s^L \mathrm{d}\langle M, M\rangle_s$$

is a $\mathbb{G}$*-martingale.*

Example 1.2.1 *With* M_t *being a Brownian motion and* $L = M_1$*, we can get*

$$q_t^x(\omega) \sim \frac{1}{(1-t)^{1/2}} \exp\left\{-\frac{1}{2(1-t)}(M_t(\omega)-x)^2 + \frac{x^2}{2}\right\},$$

by Itô's formula, we get

$$\begin{aligned} \mathrm{d}_t q_t^x &= q_t^x \frac{x - M_t}{1-t} \mathrm{d}M_t, \text{ so} \\ k_s^x &= \frac{x - M_t}{1-t}, \end{aligned}$$

and $M - \int_0^{\cdot} \frac{M_1 - M_s}{1-s} \mathrm{d}s$ *is an* $\mathbb{F}^M \vee \sigma(M_1)$ *martingale, and by the Lévy theorem, it is a standard* $\mathbb{G} := \mathbb{F}^M \vee \sigma(M_1)$*-Brownian motion and since* B_1 *is* $\mathcal{G}_0$*-measurable, it is independent of* W*.*

Example 1.2.2 *If the filtration* $\mathbb{F}$ *is generated by a Brownian motion* B*, then for any* $\mathbb{F}$*-martingale* $\mathrm{d}M_t = \sigma_t \mathrm{d}B_t$ *and* $\mathrm{d}\langle M, M\rangle_t = \sigma_t^2 \mathrm{d}t$*. Assuming that* $q_t^x(\omega) = h_t^x(B_t)$ *and that* $h \in C^{1,2}$*, we have* $\mathrm{d}_t q_t^x = \partial h_t^x(B_t)\mathrm{d}B_t$*, and*

$$k_t^x = \frac{\partial \log h_t^x(B_t)}{\sigma_t}.$$

Example 1.2.3 *Let* Y *be the Brownian semimartingale*

$$Y_t = Y_0 + \int_0^t \sigma(Y_s)\mathrm{d}B_s + \int_0^t b(Y_s)\mathrm{d}s,$$

and assume that $Y_1|\mathcal{F}_t \sim \pi(1-t, Y_t, x)\mathrm{d}x$*, with* π *smooth. Then, we can get*

$$Y_t = Y_0 + \int_0^t \sigma(Y_s)\mathrm{d}\tilde{B}_s + \int_0^t b(Y_s)\mathrm{d}s + \int_0^t \frac{\partial \log \pi}{\partial y}(1-s, Y_s, Y_1)\sigma^2(Y_s)\mathrm{d}s,$$

where $\tilde{B}$ *is an* $\mathbb{F} \vee \sigma(Y_1)$*-Brownian motion.*

Example 1.2.4 *Let* B *a Brownian motion and* $\tau = \inf\{t > 0, B_t = -1\}$*, it is known that*

$$P[\tau \le s|\mathcal{F}_t] = 2\Phi\left(-\frac{1+B_t}{\sqrt{s-t}}\right)\mathbf{1}_{\{\tau\wedge s > t\}} + \mathbf{1}_{\{s < \tau\wedge t\}},$$

where Φ is the cumulative distribution function of a standard normal distribution. Then, it can be shown (see [Cor14a] for details) that

$$B_t - \int_0^{t\wedge\tau}\left(\frac{1}{1+B_s} - \frac{1+B_s}{\tau - s}\right)\mathrm{d}s, \quad t \geq 0,$$

is a $\mathbb{G}$-martingale.

Progressive enlargement of filtrations

In the progressive enlargement of filtrations, we have $\mathbb{G} = (\mathcal{G}_t)$ with $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, where $\mathbb{H} = (\mathcal{H}_t)$ is another filtration. For the case where $\mathcal{H}_t = \sigma(\mathbf{1}_{\{\tau \leq t\}})$ with τ, see for instance [Jeu80], [Jeu85], [Man06].

Let V_0 be a zero mean normal random variable, (W^1, W^2) is a 2-dimensional Brownian motion independent of V_0, σ_s a deterministic function and $\mathcal{H}_t = \sigma(V_t)$ with

$$V_t = V_0 + \int_0^t \sigma_s \mathrm{d}W_s^1,$$

Proposition 1.2.3 *Assume that $Var(V_1) = 1$ and that*

$$\int_0^t \frac{\mathrm{d}s}{Var(V_s) - s} < \infty \textit{ for all } 0 \leq t < 1,$$

then

$$B_t = W_t^2 + \int_0^t \frac{V_s - B_s}{Var(V_s) - s}\mathrm{d}s, 0 \leq t \leq 1$$

is a Brownian motion with $B_1 = V_1$.

1.3 Filtering techniques

In this Section, some important results of filtering techniques are presented. Let $\left(\Omega, (\mathcal{F}_t)_{t\geq 0}, P\right)$ be a filtered probability space. Consider the two-dimensional Gaussian process $(\theta_t, \xi_t)_{0\leq t\leq T}$ satisfying

$$\begin{aligned} d\theta_t &= [a_0(t,\xi) + a_1(t,\xi)\theta_t]\,\mathrm{d}t + b_1(t,\xi)\,\mathrm{d}W_1(t) + b_2(t,\xi)\,\mathrm{d}W_2(t), \\ d\xi_t &= [A_0(t,\xi) + A_1(t,\xi)\theta_t]\,\mathrm{d}t + B(t,\xi)\,\mathrm{d}W_2(t), \end{aligned} \tag{1.1}$$

where $W_1(\cdot)$ and $W_2(\cdot)$ are two independent Brownian motions on $(\mathcal{F}_t)_{t\geq 0}$. $(\theta_t)_{t\in[0,T]}$ is a process inaccessible for observation. The observed values are $(\xi_t)_{t\in[0,T]}$. Assume that the measurable functionals $a_i(t,x), A_i(t,x), b_j(t,x), B(t,x)$, where $i = 0, 1$ and $j = 1, 2$, are non-anticipative, meaning that they are measurable with respect to the σ-algebra generated by the functions continuous on $[0,T]$. Denote the conditional expectation and variance of θ by $m_t = E[\theta_t|\mathcal{F}_t^\xi]$ and $\gamma_t = E[(\theta_t - m_t)^2|\mathcal{F}_t^\xi]$.

Theorem 1.3.1 (Theorem 12.1 in [Lip01]) *Assume that for any x continuous function on $[0,T]$ and for $i=0,1$, the functions $|a_i(\cdot,x)|$, $|A_i(\cdot,x)|$, $b_i^2(\cdot,x)$, $A_i^2(\cdot,x)$, and $B^2(\cdot,x)$ have finite integrals on $[0,T]$, $|a_1(\cdot,x)|<L$ and $|A_1(\cdot,x)|<L$ (for some L), $B^2(\cdot,x)\geq C>0$ for some C, for any x,y functions continuous on $[0,T]$, there exist $L_1,L_2\in\mathbb{R}$ and a $K(s)$ nondecreasing, right-continuous function with values in $[0,1]$ such that*

$$\begin{aligned}|B(t,x)-B(t,y)|^2 &\leq L_1\int_0^t |x_s-y_s|^2\,\mathrm{d}K(s)+L_2|x_t-y_t|^2,\\ B^2(t,x) &\leq L_1\int_0^t \left(1+x_s^2\right)\mathrm{d}K(s)+L_2\left(1+x_t^2\right),\end{aligned}$$

and

$$\begin{aligned}\int_0^T E\left[a_0^4(t,\xi)+b_1^4(t,\xi)+b_2^4(t,\xi)\right]\mathrm{d}t &< \infty,\\ E\left[\theta_0^4\right] &< \infty.\end{aligned}$$

If the conditional distribution of $\theta_0|\xi_0$ is Gaussian $N(m_0,\gamma_0)$, then m_t and γ_t satisfy

$$\begin{aligned}\mathrm{d}m_t &= [a_0(t,\xi)+a_1(t,\xi)m_t]\,\mathrm{d}t\\ &\quad+\frac{b_2(t,\xi)B(t,\xi)+\gamma_t A_1(t,\xi)}{B^2(t,\xi)}\left[\mathrm{d}\xi_t-(A_0(t,\xi)+A_1(t,\xi)m_t)\,\mathrm{d}t\right]\end{aligned}$$

and

$$\gamma_t'=2a_1(t,\xi)\gamma_t+b_1^2(t,\xi)-\left(\frac{b_2(t,\xi)B(t,\xi)+\gamma_t A_1(t,\xi)}{B(t,\xi)}\right)^2$$

subject to the conditions $m_0=E(\theta_0|\xi_0)$ and $\gamma_0=E\left[\left|(\gamma_0-m_0)^2\right|\xi_0\right]$.

In a particular case, we have:

Theorem 1.3.2 (Theorem 12.2 in [Lip01]) *Let $\theta=\theta(\omega)$ be a random variable with $E\theta^2<\infty$. Assume that ξ has the dynamics*

$$d\xi_t=[A_0(t,\xi)+A_1(t,\xi)\theta]\,\mathrm{d}t+B(t,\xi)\,\mathrm{d}W_2(t),$$

where the coefficients A_0,A_1,B satisfy the conditions of Theorem 1.3.1, and the conditional distribution of $\theta|\xi_0$ is Gaussian. Then, m_t and γ_t are given by

$$\begin{aligned}m_t &= \frac{m_0+\gamma_0+\int_0^t\frac{A_1(s,\xi)}{B^2(s,\xi)}\left[\mathrm{d}\xi_s-A_0(s,\xi)\,\mathrm{d}s\right]}{1+\gamma_0\int_0^t\left(\frac{A_1(s,\xi)}{B(s,\xi)}\right)^2\mathrm{d}s},\\ \gamma_t &= \frac{\gamma_0}{1+\gamma_0\int_0^t\left(\frac{A_1(s,\xi)}{B(s,\xi)}\right)^2\mathrm{d}s.}\end{aligned}$$

1.4 Stochastic Optimal Control

In this section, the problems and solutions are defined on $\mathbb{R}$, however, they can be easily extended to $\mathbb{R}^n$. We refer to [Bjö98] for a more detailed discussion. Assume, we have the following optimization problem on the finite time horizon $[0, T]$, where $T \in \mathbb{R}$. Let $\mu(t, y, x)$ and $\sigma(t, y, x)$ real deterministic functions defined for any $t \geq 0$ and $y, x \in \mathbb{R}$, and assume that the dynamics of a process Y_t is given by

$$\mathrm{d}Y_t = \mu(t, Y_t, X_t)\,\mathrm{d}t + \sigma(t, Y_t, X_t)\,\mathrm{d}W_t \quad \text{with} \quad Y_0 = 0, \tag{1.2}$$

also called controlled SDE, where $W.$ is a Brownian motion. Y and X are called the state process and control process (or law), respectively. Suppose that X_t is of the form $X_t = g(t, Y_t)$ for some deterministic function g. Then, in fact we can use the notation $X(t, Y_t)$ for the control process. We will call $X(t, y)_{t \geq 0, y \in \mathbb{R}}$ admissible, if for any $t \geq 0$ and $y \in \mathbb{R}$, there is a unique solution of the SDE

$$\mathrm{d}Y_s = \mu(s, Y_s, X(s, Y_s))\,\mathrm{d}s + \sigma(s, Y_s, X(s, Y_s))\,\mathrm{d}W_t \quad \text{with} \quad Y_t = y. \tag{1.3}$$

Denote the set of admissible control processes by $\mathcal{X}$.

Consider the real valued functions $F(t, y, x)$ and $\Phi(y)$ well defined for any $t \geq 0$, $y, x \in \mathbb{R}$, and define the value function for a control process by

$$J_0(X) = E\left[\int_0^T F(t, Y_t, X_t)\,\mathrm{d}t + \Phi(Y_T)\right], \tag{1.4}$$

where $Y.$ is the solution of (1.2) with $Y_0 = y_0$. Its optimal value is given by $\tilde{J}_0 = \sup_{X \in \mathcal{X}} J_0(X)$. If there is a control process $\tilde{X} \in \mathcal{X}$, such that $J_0\left(\tilde{X}\right) = \tilde{J}_0$, then we call it an optimal control process (or law).

A control problem $\mathcal{P}(t, y)$ is defined for fixed $t \geq 0, y \in \mathbb{R}$, as the problem to maximize

$$E_{t,x}\left[\int_t^T F(t, Y_t, X_t)\,\mathrm{d}t + \Phi(Y_T)\right]$$

given the dynamics by (1.3). Note that the original optimization problem is, then, $\mathcal{P}(0, y_0)$. Then, given this dynamics, the value function is defined as

$$J(t, y, X) = E\left[\int_t^T F(t, Y_t, X_t)\,\mathrm{d}t + \Phi(Y_T)\right],$$

and the optimal value function, the expected utility over the interval $[t, T]$, is given by $J(t, y) = \sup_{X \in \mathcal{X}} J(t, y, X)$.

Assume that there exists an optimal control process and that $J(\cdot, \cdot)$ is continuously differentiable with respect to the the first and twice continuously differentiable with respect to the second variable. Let $t \geq 0$, $y \in \mathbb{R}$ and $h \geq 0$ be fixed such that $t + h < T$, $\tilde{X}$ be an optimal control process, and X be a fixed, arbitrary control process, and define

$$X^*(s,x) = \begin{cases} X(s,x) & \text{if} \quad (s,x) \in [t, t+h] \times \mathbb{R} \\ \tilde{X}(s,x) & \text{if} \quad (s,x) \in [t+h, T] \times \mathbb{R} \end{cases}$$

Then, we will observe the difference between the optimal $\tilde{X}$ and the just defined X^* to derive a PDE for the dynamics of the value function. Using the optimal low, the expected utility coincides with the optimal one: $J\left(t, y, \tilde{X}\right) = V(t,x)$. Using X^*, over the interval $[t, t+h]$, it is given by

$$E_{t,y}\left[\int_t^{t+h} F(s, Y_s, X_s)\, \mathrm{d}s\right],$$

and over the interval $(t+h, T]$, as we start from the state $Y^X(t+h)$, where the superscript X of Y refers to the fact that until time $t+h$, the control process X has been used, is given by

$$E_{t,y}\left[J\left(t+h, Y_{t+h}^X\right)\right],$$

so the total expected utility is given by

$$J^*(t,y) = E_{t,y}\left[\int_t^{t+h} F(s, Y_s, X_s)\, \mathrm{d}s + J\left(t+h, Y_{t+h}^X\right)\right]$$

for which, because of the optimality of $\tilde{X}$, we have

$$J(t,y) \geq J^*(t,y). \tag{1.5}$$

Then, by Itô's formula, we get

$$\begin{aligned} J\left(t+h, Y_{t+h}^X\right) &= V(t,y) + \int_t^{t+h} \left(\partial_t J\left(s, Y_{t+h}^X\right) + \partial_{yy} J\left(s, Y_{t+h}^X\right)\right) \mathrm{d}s \\ &\quad + \int_t^{t+h} \partial_y J\left(s, Y^X(s)\right) \sigma(s, Y_s, X_s)\, \mathrm{d}W_s. \end{aligned}$$

Then, taking the expectation, and using (1.5) and assuming sufficient integrability, we get

$$E_{t,y}\left[\int_t^{t+h} \left(F\left(s, Y_s^X, X_s\right) + \partial_t J\left(s, Y_{t+h}^X\right) + \partial_{yy} J\left(s, Y_{t+h}^X\right)\right) \mathrm{d}s\right] \leq 0,$$

where, dividing by h and letting it tend to 0, assuming enough regularity so that we can change the order of the expectation and differentiation and using $Y_t = y$, we get

$$F\left(t, y, X\left(t, y\right)\right) + \partial_t J\left(t, y\right) + \partial_{yy} J\left(t, y\right) \leq 0,$$

which hold for any X, and equality holds if and only if $X = \tilde{X}$, so we have got the following equation:

$$\begin{aligned} \partial_t J\left(t, y\right) + \sup_{X \in \mathcal{X}} F\left(t, y, X\left(t, y\right) + \partial_{yy} J\left(t, y\right)\right) &= 0, \\ \text{with the boundary condition } J\left(T, y\right) &= \Phi\left(y\right). \end{aligned}$$

As (t, y) was fixed but arbitrary, the above PDE must be satisfied for $(t, y) \in (0, T) \times \mathbb{R}$. It is called the Hamilton-Jacobi-Bellman equation. So, under the assumptions made earlier, the above PDE is satisfied and the supremum is reached by $X = \tilde{X}$. This is a necessary condition. Also, a so called *Verification Theorem* can be proved saying that if some J' is sufficiently integrable and solves the above Hamilton-Jacobi-Bellman equation with the boundary condition and the supremum is reached by an admissible strategy X', then the optimal value function J to the control problem coincides with J' and the strategy is optimal, making it be a sufficient condition.

A possible generalization is the following controlled SDE:

$$\begin{aligned} Y_t &= \mu\left(t, Y_t, X_t\right) \mathrm{d}t + \sigma\left(t, Y_t, X_t\right) \mathrm{d}W_t \\ Y_0 &= y_0. \end{aligned}$$

on a fixed interval $[0, T]$, considering a stopping time $\tau = \inf\{t \geq 0 : Y_t = c\} \wedge T$, for some $c \in \mathbb{R}$, meaning the first time when Y hits the level c, with the control problem of maximizing

$$E\left[\int_0^\tau F\left(s, Y_s^X, X_s\right) \mathrm{d}s + \Phi\left(\tau, Y_\tau^X\right)\right].$$

It can be shown that the same equations and the Verification Theorem hold, in this case, as well, with the boundary condition $J\left(c, y\right) = \Phi\left(c, y\right)$.

Part II

Equilibrium models with asymmetric information

Chapter 2

Introduction

In Part II, order-driven market models are studied with the presence of insiders. Consider a market of a risk-less bond and a risky asset. The price of the risky asset will depend on the incoming market orders in the following way. First, market participants place their orders, displaying the amount of risky asset they want to buy or sell. Then, some market specialist, the so called market makers set the prices, at which trading will be done. Such models with endogenously given prices are the order driven markets. Another approach is having the prices given either exogenously described by their dynamics depending on their trajectory or their recent values, but not on the market orders. The markets studied throughout this Part are order driven markets, based on two models presented in [Kyl85] and [Bac92]. These original models and their extensions presented in this Part. A more realistic (limit order) model is studied in [BB04], in which buyers and sellers set not only the amount of assets they are willing to buy, but also the price at which they are willing to trade, called bid and ask prices. In this case, trading is done when those requirements meet.

In the models studied in these Chapters, there are three types of participants:

- *Noise traders* or *Liquidity traders*, who trade for liquidity or hedging reasons,
- *Informed traders* or *Insiders*, who are aware of some privilege information about the risky asset and try to maximize their profit, and
- *Market makers*, who set the price and clear the market.

Denote the noise traders' cumulative demand by Z, the informed trader's cumulative demand by X, the total demand by $Y = X + Z$ and the prices set by the market makers by P in all the models presented in this Part. The market is order driven, so the presence of the insider does have a impact on the market, the stock price also depends on her strategy. We will start, in this Chapter, from the discrete model introduced in [Kyl85], and see how to get from the single auction model to a continuous one (described and studied in [Bac92], as well) through a discrete, sequential auction model. In Chapter 3, its extensions and related models are presented. In all these models the private information is the price V right after an announcement to be made, or some equivalent quantity, and the market consist of a bank account with interest rate 0 and a risky asset. In the single auction model, the demand of the noise traders is a normal random variable, in the sequential one it is a discrete version of a Brownian motion and in the continuous model it is a Brownian motion. Its extension to Lévy processes are studied, as well. Let W denote the wealth of the insider (introduced and calculated later). Then, the insider tries to maximize her expected profit conditioned on V. The market efficiency condition says that the prices, set by the market makers, have to coincide with the expectation of V, conditioned on the market makers' information: Y.

We will refer to the model presented in [Kyl85] as Kyle's model and to the one presented in [Bac92] as Back's model. In the following, Kyle's and Back's original models can be found in Section 2.1, an extension allowing the noise traders' demand to be a Lévy process is studied in Section 3.1 summarizing the results of [Cor14b], a general framework including random announcement time and different structures of the private information is presented in Section 3.2 with the results of [Cor14a], and finally some related models are presented in Section 3.3, including more insiders on the market [NT06], more than one signal [Jai99], a risk-averse insider [Cho03,Cor14b] and a concept of a weaker equilibrium [Wu99, KHOL10, Dan10].

2.1 Models

In this Section, the previously mentioned models are presented. First, the discrete models can be found: the one-period and the N-period equilibrium model, then a bridge to the continuous case, and finally, the continuous model is solved. Afterwards, the original approaches of Kyle and Back are mentioned and referred.

2.1.1 Kyle's and Back's Models

Consider a market with two assets: we have a risk asset S and a bank account with interest rate r equal to zero. We consider N trading periods: $0 \le t_1 \le t_2 \le \cdots \le t_N$ and a liquidation value of the asset, V, which is announced just after time t_N. We repeat the behavior and the information of the three kinds of agents:

1. Let the noise traders' (aggregate) demand process be denoted by $(Z_k)_{0\le k\le N}$, suppose that $\Delta Z_k = Z_k - Z_{k-1}$ are independent, identically distributed random variables of law $N(0, \sigma_u^2 \Delta t_k)$, $\Delta t_k = t_k - t_{k-1}$, Z is independent of V and $Z_0 = 0$. We also assume that $V \sim N(p_0, \sigma_l^2)$.

2. Let the informed trader's (aggregate) demand process be denoted by $(X_k)_{0\leq k\leq N}$ with $X_0 = 0$, at time t_k it is supposed he knows the value of V and $(P_j)_{0\leq j\leq k-1}$, he tries to maximize his wealth.

3. The market makers *clear* the market fixing a *rational* price

$$P_k = E(S_1|Y_j, 0 \leq j \leq k), k = 1, .., N$$

where $Y = X + Z$. The process Y. is the information that the market makers have. Note that (P_k) is an $(\mathcal{F}_k)$- martingale, where $\mathcal{F}_k = \sigma(Y_j, 0 \leq j \leq k)$.

The optimality and rationality conditions are the following:

Definition 2.1.1 *Given a demand process Y, a pricing rule*

$$P_k = H(k, Y_j, 0 \leq j \leq k), k = 1, .., N$$

is rational if

$$H(k, Y_j, 0 \leq j \leq k) = E(V|Y_j, 0 \leq j \leq k), k = 1, .., N$$

Definition 2.1.2 *Given a pricing rule H, a trading strategy X is optimal if it maximizes the value of insider's portfolio.*

Definition 2.1.3 *An equilibrium is a pair (H, X) such that X is optimal given H and H is rational given X.*

We may have several equilibriums, then it is convenient the following definition

Definition 2.1.4 *If (H, X) is an equilibrium for any X then H is an equilibrium pricing rule.*

The value of insider's portfolio at time t_k is given by

$$W_k = \sum_{i=1}^{k} G_i$$

where G_i is the gain in the period $(i-1, i]$, that is the new value of the portfolio minus the initial value and minus what she spends in getting the new position:

$$\begin{aligned} G_i &= X_i P_i - X_{i-1} P_{i-1} - (X_i - X_{i-1}) P_i \\ &= X_{i-1}(P_i - P_{i-1}), \end{aligned}$$

so

$$W_N = \sum_{i=1}^{N} X_{i-1}(P_i - P_{i-1}).$$

Once the announcement is made, there is a new gain, say G_{N+}, given by

$$G_{N+} = (V - P_N)X_N,$$

so the total gain W_{N+} is given by

$$\begin{aligned}
W_{N+} &= (V - P_N)X_N + \sum_{i=1}^{N} X_{i-1}(P_i - P_{i-1}) \\
&= VX_N - P_N X_N + \sum_{i=1}^{N} X_{i-1}P_i - \sum_{i=1}^{N} X_{i-1}P_{i-1} \\
&= VX_N + \sum_{i=1}^{N} X_{i-1}P_i - \sum_{i=1}^{N} X_i P_i \\
&= VX_N - \sum_{i=1}^{N} P_i(X_i - X_{i-1}) = \sum_{i=1}^{N}(V - P_i)(X_i - X_{i-1}). \qquad (2.1)
\end{aligned}$$

Note that X_k is measurable with respect to the σ-field $\mathcal{G}_{k-1} = \sigma(V, P_1, ..., P_{k-1})$. If we consider the total portfolio of insider plus noise traders we have that its value, say Λ, is given by

$$\Lambda = \sum_{i=1}^{N}(V - P_i)(Y_i - Y_{i-1}),$$

and

$$\begin{aligned}
E(\Lambda) &= E\left[\sum_{i=1}^{N}(V - P_i)(Y_i - Y_{i-1})\right] \\
&= \sum_{i=1}^{N} E[(E(V|\mathcal{F}_i) - P_i)(Y_i - Y_{i-1})] \\
&= 0,
\end{aligned}$$

if the price is a rational price. So, in these conditions the gain of the market makers is zero in average and the insider's gain is due to the losses of the noise traders.

Equilibrium in one period

Consider now that $N = 1$, then

$$W_{1+} = (V - P_1)X_1$$

with X_1 a measurable function of V. First we consider linear strategies

$$X_1 = \alpha + \beta V$$

and we look for rational price rules

$$\begin{aligned}
P_1 &= E(V|Y_1) = E(V|\alpha + \beta V + Z_1) \\
&= p_0 + \frac{\text{Cov}(V, \beta V + Z_1)}{\text{Var}(\beta V + Z_1)}(\beta(V - p_0) + Z_1) \\
&= p_0 + \frac{\beta \sigma_l^2}{\beta^2 \sigma_l^2 + \sigma_u^2}(\beta(V - p_0) + Z_1) \\
&= \mu + \lambda Y_1,
\end{aligned}$$

so, in this situation, prices are also linear with

$$\mu = p_0 \frac{\sigma_u^2}{\beta^2 \sigma_l^2 + \sigma_u^2} - \lambda \alpha \tag{2.2}$$

$$\lambda = \frac{\beta \sigma_l^2}{\beta^2 \sigma_l^2 + \sigma_u^2} \tag{2.3}$$

The insider wants to maximize

$$\begin{aligned}
E(W_{1+}|V) &= E((V - P_1)X_1|V) \\
&= (V - \mu - \lambda X_1)\, X_1,
\end{aligned}$$

$$\begin{aligned}
\mu + \lambda X_1 &= \mu + \lambda \alpha + \lambda \beta V \\
&= p_0 \frac{\sigma_u^2}{\beta^2 \sigma_l^2 + \sigma_u^2} + \lambda \beta V,
\end{aligned}$$

$$V - \mu - \lambda X_1 = \frac{(V - p_0)\, \sigma_u^2}{\beta^2 \sigma_l^2 + \sigma_u^2},$$

so

$$E(W_{1+}|V) = \frac{(V - p_0)\, \sigma_u^2}{\beta^2 \sigma_l^2 + \sigma_u^2}(\alpha + \beta V),$$

and there is not equilibrium, since $E(W_{1+}|V)$ is not bounded, but note that we are trying to maximize the portfolio's wealth for different pricing rules.

Assume however that μ and λ are really constants which do not depend on the particular value of insider's demand. Then, we have to maximize

$$\begin{aligned} E(W_{1+}|V) &= E((V-P_1)X_1|V) \\ &= (V-\mu-\lambda X_1)\,X_1, \end{aligned}$$

with respect to X_1, so the optimal value is

$$X_1 = \frac{V-\mu}{2\lambda}.$$

We obtain that the optimal strategy is linear, that is, if we start with a linear pricing rule the optimal strategy in the set of all strategies is linear and we have

$$\alpha = -\frac{\mu}{2\lambda} \text{ and } \beta = \frac{1}{2\lambda}.$$

Now the coherent values of μ and λ with rational pricing rules should satisfy (2.2) and (2.3), so

$$\mu = p_0 \text{ and } \lambda = \frac{\sigma_l}{2\sigma_u}.$$

Then

$$X_1 = \frac{\sigma_u}{\sigma_l}(V-p_0).$$

We also have that the optimal wealth is given by

$$E(W_{1+}|V) = \frac{\sigma_u\,(V-p_0)^2}{2\sigma_l},$$

and

$$E(W_{1+}) = \frac{\sigma_u\sigma_l}{2}.$$

It is also worth to point that

$$\begin{aligned} \text{Var}(V-P_1) &= \text{Var}(V)+\text{Var}(P_1)-2\text{Cov}(V,P_1) \\ &= \sigma_l^2+\lambda^2(\beta^2\sigma_l^2+\sigma_u^2)-2\lambda\beta\sigma_l^2 \\ &= \sigma_l^2+\lambda\beta\sigma_l^2-2\lambda\beta\sigma_l^2 = \frac{\sigma_l^2}{2} \end{aligned}$$

and

$$\text{Cov}(V-P_1,P_1)=0,$$

so $V-P_1$ and P_1 are independent.

Equilibrium with N periods

Here, we present a way of finding equilibriums that reflects the dynamic programming method used in the continuous model in [Bac92]. We assume that pricing rules are linear in the demand process, more precisely

$$\begin{aligned} P_n &= \sum_{i=1}^{n} \lambda_i^{(n)} \Delta Y_i = \lambda_n^{(n)} \Delta Y_n + \sum_{i=1}^{n-1} \left(\lambda_i^{(n)} - \lambda_i^{(n-1)} \right) \Delta Y_i \\ &= \lambda_n \Delta Y_n + P_{n-1} + r_n(\Delta Y_1, \Delta Y_2, ..., \Delta Y_{n-1}), \end{aligned} \tag{2.4}$$

where $\lambda_n := \lambda_n^{(n)}$. Define

$$W_n := \sup_{\mathcal{X}} E \left[\sum_{i=n}^{N} (V - P_i)(X_i - X_{i-1}) \middle| \mathcal{G}_{n-1} \right], n = 1, ..., N,$$

where $\mathcal{X}$ is the set of (admissible, because we need the wealth process to be well defined) $(\mathcal{G}_n)$-previsible strategies (that is X_n is $\mathcal{G}_{n-1}$-measurable). Then

$$\begin{aligned} W_n &= \sup_{\mathcal{X}} E \left[\sum_{i=n+1}^{N} (V - P_i)(X_i - X_{i-1}) + (V - P_n)(X_n - X_{n-1}) \middle| \mathcal{G}_{n-1} \right] \\ &= \sup_{\mathcal{X}} E \left[W_{n+1} + (V - P_n)(X_n - X_{n-1}) | \mathcal{G}_{n-1} \right], \;\; n = 1, ..., N \end{aligned}$$

Then we can solve this backwards,

$$\begin{aligned} W_N &= \sup_{\mathcal{X}} E((V - P_N)(X_N - X_{N-1}) | \mathcal{G}_{N-1}) \\ &= \sup_{\mathcal{X}} E((V - P_{N-1} - \lambda_N \Delta Y_N - r_N) \Delta X_N) | \mathcal{G}_{N-1}) \\ &= \sup_{\mathcal{X}} (V - P_{N-1} - \lambda_N \Delta X_N - r_N) \Delta X_N), \end{aligned}$$

and the optimal strategy is

$$\Delta X_N = \frac{V - P_{N-1} - r_N}{2\lambda_N},$$

and the optimal wealth value

$$W_N = \frac{(V - P_{N-1} - r_N)^2}{4\lambda_N}.$$

If the pricing rule (2.4) is rational it must satisfy

$$\begin{aligned} 0 &= E(V - P_N | \mathcal{F}_{N-1}) = E(V - P_{N-1} - \lambda_N \Delta Y_N - r_N | \mathcal{F}_{N-1}) \\ &= E(V - P_{N-1} - \lambda_N \Delta X_N - r_N | \mathcal{F}_{N-1}) = E \left(\frac{V - P_{N-1} - r_N}{2} \middle| \mathcal{F}_{N-1} \right) \\ &= -\frac{r_N}{2}. \end{aligned}$$

Then, assume that $r_n = 0$ and that

$$W_n = \alpha_{n-1}(V - P_{n-1})^2 + \delta_{n-1},$$

Note that $r_N = 0$ and that $\alpha_{N-1} = \frac{1}{4\lambda_N}$. Now,

$$W_{n-1} = \sup_{\mathcal{X}} \left\{ E\left[W_n + (V - P_{n-1})(X_{n-1} - X_{n-2}) | \mathcal{G}_{n-2} \right] \right\},$$

$$\begin{aligned}
E(W_n|\mathcal{G}_{n-2}) &= \alpha_{n-1}E((V - P_{n-1})^2 \,|\mathcal{G}_{n-2}) + \delta_{n-1} \\
&= \alpha_{n-1}E((V - P_{n-2} - \lambda_{n-1}\Delta Y_{n-1} - r_{n-1})^2|\mathcal{G}_{n-2}) + \delta_{n-1} \\
&= \alpha_{n-1}(V - P_{n-2} - \lambda_{n-1}\Delta X_{n-1} - r_{n-1})^2 \\
&\quad +\alpha_{n-1}\lambda_{n-1}^2\sigma_u^2\Delta t_{n-1} + \delta_{n-1},
\end{aligned}$$

so

$$\begin{aligned}
W_{n-1} &= \sup_{\mathcal{X}}(\alpha_{n-1}(V - P_{n-2} - \lambda_{n-1}\Delta X_{n-1} - r_{n-1})^2 \\
&\quad +\alpha_{n-1}\lambda_{n-1}^2\sigma_u^2\Delta t_{n-1} + \delta_{n-1} \\
&\quad +(V - P_{n-2} - \lambda_{n-1}\Delta X_{n-1} - r_{n-1})\Delta X_{n-1}),
\end{aligned}$$

and we have that the optimal strategy is given by

$$\Delta X_{n-1} = \frac{1 - 2\lambda_{n-1}\alpha_{n-1}}{2\lambda_{n-1}(1 - \lambda_{n-1}\alpha_{n-1})}(V - P_{n-2} - r_{n-1}),$$

again, by the rationality pricing condition, $r_{n-1} = 0$ and

$$W_{n-1} = \frac{1}{4\lambda_{n-1}(1 - \lambda_{n-1}\alpha_{n-1})}(V - P_{n-2})^2 + \alpha_{n-1}\lambda_{n-1}^2\sigma_u^2\Delta t_{n-1} + \delta_{n-1},$$

so

$$\begin{aligned}
\alpha_{n-2} &= \frac{1}{4\lambda_{n-1}(1 - \lambda_{n-1}\alpha_{n-1})}, \\
\delta_{n-2} &= \alpha_{n-1}\lambda_{n-1}^2\sigma_u^2\Delta t_{n-1} + \delta_{n-1}.
\end{aligned}$$

Note that the second order condition is

$$\lambda_{n-1}(1 - \lambda_{n-1}\alpha_{n-1}) > 0.$$

Since the pricing rule is rational

$$E(V - P_{n-1}|\mathcal{F}_n) = P_n - P_{n-1} = \lambda_n\Delta Y_n,$$

we have

$$\begin{aligned} E(V - P_{n-1}|\mathcal{F}_n) &= E(V - P_{n-1}|\Delta Y_n) \\ &= \frac{\text{Cov}(V - P_{n-1}, \Delta X_n)}{\text{Var}(\Delta Y_n)} \Delta Y_n \\ &= \frac{\beta_n \sigma_{n-1}^2}{\beta_n^2 \Delta t_n \sigma_{n-1}^2 + \sigma_u^2} \Delta Y_n, \end{aligned}$$

where σ_n^2 :=Var$(V - P_n)$ and $\beta_n \Delta t_n := \frac{1-2\lambda_n\alpha_n}{2\lambda_n(1-\lambda_n\alpha_n)}, n = 1, ..., N$, where we take by definition $\alpha_N = 0$. Then

$$\lambda_n = \frac{\beta_n \sigma_{n-1}^2}{\beta_n^2 \Delta t_n \sigma_{n-1}^2 + \sigma_u^2}.$$

So summarizing we have the following equations for the parameters

$$\begin{aligned} \alpha_{n-1} &= \frac{1}{4\lambda_n(1 - \lambda_n\alpha_n)}, && (2.5) \\ \delta_{n-1} &= \alpha_n \lambda_n^2 \sigma_u^2 \Delta t_n + \delta_n, \\ \lambda_n &= \frac{\beta_n \sigma_{n-1}^2}{\beta_n^2 \Delta t_n \sigma_{n-1}^2 + \sigma_u^2}, && (2.6) \\ \beta_n \Delta t_n &= \frac{1 - 2\lambda_n\alpha_n}{2\lambda_n(1 - \lambda_n\alpha_n)}, && (2.7) \end{aligned}$$

$n = 1, ..., N$, where we take by definition $\delta_N = 0$. It is easy to show that we also have that

$$\sigma_n^2 = \sigma_{n-1}^2(1 - \lambda_n\beta_n\Delta t_n), n = 1, ..., N. \tag{2.8}$$

Bridge to the continuous model

From (2.7), by multiplying by λ_n, we get

$$\beta_n \Delta t_n \lambda_n = \frac{1 - 2\alpha_n\lambda_n}{1 + (1 - 2\alpha_n\lambda_n)}. \tag{2.9}$$

From (2.6), by multiplying by $\Delta t_n \beta_n$, we get

$$\beta_n \Delta t_n \lambda_n = \frac{\beta_n^2 \sigma_{n-1}^2 \Delta t_n}{\beta_n^2 \Delta t_n \sigma_{n-1}^2 + \sigma_u^2}, \tag{2.10}$$

which implies

$$1 - 2\alpha_n\lambda_n = \frac{\beta_n^2 \sigma_{n-1}^2 \Delta t_n}{\sigma_u^2}, \tag{2.11}$$

because both (2.9) and (2.10) can be written in the form of $z/(1+z)$ with z being equal to either side of (2.11). By substituting the equation for α (2.5) in (2.7), we get

$$\beta_n \Delta t_n = 2\alpha_{n-1}(1 - 2\alpha_n \lambda_n). \tag{2.12}$$

Using the fact that by (2.5), we know

$$\frac{\alpha_n}{\alpha_{n-1}} = 4\alpha_n \lambda_n (1 - \alpha_n \lambda_n),$$

we get

$$\frac{\alpha_n - \alpha_{n-1}}{\alpha_{n-1}} = -(1 - 4\alpha_n \lambda_n + 4\alpha_n^2 \lambda_n^2) = -(1 - 2\alpha_n \lambda_n)^2. \tag{2.13}$$

Now, define

$$\phi_n := \frac{4\alpha_n^2 \sigma_n^2}{\sigma_u^2}.$$

In (2.11), by substituting $\beta_n \Delta t_n$ as in (2.12), we get

$$1 = 2\alpha_{n-1}(1 - 2\alpha_n \lambda_n) 2\alpha_{n-1} \frac{\sigma_{n-1}^2}{\Delta t_n \sigma_u^2},$$

so

$$1 - 2\alpha_n \lambda_n = \frac{\Delta t_n}{\phi_{n-1}}. \tag{2.14}$$

Also, by (2.11), it is easy to check that

$$\left(1 + \frac{\Delta t_n}{\phi_{n-1}}\right)^{-1} = \frac{\sigma_u^2}{\beta_n \Delta t_n \sigma_{n-1}^2 + \sigma_u^2} = \frac{\sigma_n^2}{\sigma_{n-1}^2}. \tag{2.15}$$

Then, (2.13) and the definition of ϕ_n imply

$$\frac{\alpha_n}{\alpha_{n-1}} = \frac{\alpha_n - \alpha_{n-1}}{\alpha_{n-1}} + 1 = 1 - \frac{\Delta t_n^2}{\phi_n^2}. \tag{2.16}$$

By multiplying (2.15) and (2.16), we get an equation for ϕ_n that can be simplified to

$$\phi_n - \phi_{n-1} = -\Delta t_n - \frac{\Delta t_n^2}{\phi_{n-1}} + \frac{\Delta t_n^3}{\phi_{n-1}^2}. \tag{2.17}$$

These cubic equations have to be solved subject to a boundary condition $\phi_N = 0$. Equivalently,

$$0 = \phi_{n-1}^3 - (\Delta t_n + \phi_n)\phi_{n-1}^2 - \Delta t_n^2 \phi_{n-1} + \Delta t_n^3.$$

Because of the positivity of ϕ, we obtain $\phi_{N-1} = \Delta t_N$. Also, we know that

$$\frac{\phi_n - \phi_{n-1}}{\Delta t_n} = -1 - \frac{\Delta t_n}{\phi_{n-1}} + \frac{\Delta t_n^2}{\phi_{n-1}^2}.$$

Therefore, if we show that we have a solution with $\frac{\Delta t_n}{\phi_{n-1}}$ tending to zero, then, in every step only one solution makes economic sense and this satisfies

$$-\frac{5}{4} < \frac{\phi_n - \phi_{n-1}}{\Delta t_n} < -1 \tag{2.18}$$

and

$$\frac{\phi_n - \phi_{n-1}}{\Delta t_n} \to -1 \quad \text{as} \quad \frac{\phi_n}{\Delta t_n} \to \infty, \tag{2.19}$$

which imply that, for the continuous version of ϕ, we have

$$\phi(t) = 1 - t, \tag{2.20}$$

and the convergence is uniform on $[0, 1]$. Since we can write

$$\frac{\sigma_n^2 - \sigma_{n-1}^2}{\sigma_{n-1}^2} = -\frac{\Delta t_n}{1 - t_n} + o(|\Delta t|),$$

for the continuous version we have

$$\frac{(\sigma^2(t))'}{\sigma^2(t)} = -\frac{1}{1-t},$$

with uniform convergence on intervals not containing $t = 1$. Its solution is

$$\sigma^2(t) = (1 - t)\sigma_0^2.$$

The continuous model

In the following, the continuous version of Kyle's model is solved. Note that in [Bac92], the price at time t depends only on Y_t, while in Kyle's model, and also in the one presented here, it depends on the history $(Y_s)_{0 \le s \le t}$ through the price pressure λ. We consider the same market of a risky asset S and a bank account with interest rate r equal to zero with the trading continuous in time. The trading period is $[0, 1]$. There is to be a public release of information at time 1, revealing the value of the risky asset V (assumed to be a random variable with finite expectation and with distribution function F), at which price it will trade afterwards, t.i. at time $1+$. The price of the stock at time t is denoted by P_t and the filtration generated by it by $\mathbb{F}^P \left(\mathcal{F}_t^P\right)_{0 \le t \le 1}$ where

$\mathcal{F}_t^P = \sigma(P_s, 0 \leq s \leq t)$. Let Z be the aggregate demand process of the noise traders, a Brownian motion with a fixed volatility $\sigma : dZ_t = \sigma dB_t$, where $(B_t)_{t \geq 0}$ is a standard Brownian motion independent of V. Let X be the demand process of the informed trader, as she knows the value of V from the beginning, as well as $\{P_s : 0 \leq s \leq t\}$, X has to be adapted to the augmented filtration (completed with $\mathbb{P}$-null sets)

$$\mathbb{F}^{V,P} := \left(\mathcal{F}_t^{V,P}\right)_{0 \leq t \leq 1},$$

where

$$\mathcal{F}_t^{V,P} := \sigma(V, P_s, 0 \leq s \leq t),$$

generated by the random variable V and the process P. Because of the independency of $Z_\cdot$ and V, Z is an $\mathbb{F}^{V,Z}$-Brownian motion, as well. The informed trader tries to maximize her final wealth and the market makers set the rational price, given by

$$P_t = \mathbb{E}(V | Y_s, 0 \leq s \leq t), t \in [0,1]$$

where $Y = X + Z$ is the total demand market makers observe. Note that (P_t) is an $\mathbb{F}^Y$-martingale, where $\mathbb{F}^Y = \left(\mathcal{F}_t^Y\right)_{0 \leq t \leq 1}$ and $\mathcal{F}_t^Y = \sigma(Y_s, 0 \leq s \leq t)$. Here and in the sequel we always consider $\mathbb{P}$-augmented filtrations. Note that $\mathbb{F}^Y = \mathbb{F}^P$ and that $\mathbb{F}^{V,P} = \mathbb{F}^{V,Y} = \mathbb{F}^{V,X+Z}$.

Definition 2.1.5 *Assume that λ is a positive smooth function, $H \in C^{1,2}$ and $H(t,\cdot)$ is strictly increasing for every $t \in [0,1]$. Denote the class of pairs (H, λ) above by $\mathcal{H}$. An element of $\mathcal{H}$ is called a pricing rule.*

Suppose that market makers fix prices through a pricing rule

$$P_t = H(t, \xi_t), t \in [0,1]$$

with

$$\xi_t := \int_0^t \lambda(s) \mathrm{d}Y_s$$

where λ is called price pressure. We also write $\xi(t, Y_t)$ for ξ_t. Assume that X is adapted to the filtration $\mathbb{F}^{V,Z}$, and that consequently $\mathbb{F}^Y \subseteq \mathbb{F}^{V,Z}$, in such a way that if $X_t = f(Y_s, 0 \leq s \leq t, V)$ for certain measurable function f we can write $X_t = g(Z_s, 0 \leq s \leq t, V)$ for another measurable function g.

Definition 2.1.6 *Denote, by $\mathcal{X}$, the set of $\mathcal{F}^{V,Z}$-adapted processes X satisfying*

$$\mathrm{d}X_t = \theta_t \mathrm{d}t \tag{2.21}$$

for some measurable θ. and such that $\forall\,(H,\lambda)\in\mathcal{H}$

$$E\left(\int_0^1 U\left(t,\int_0^t \lambda_s \mathrm{d}\left(X_s+Z_s\right)\right)^2\right)\mathrm{d}t<\infty \tag{2.22}$$

for both cases $U=H$ and $U=\frac{\partial}{\partial y}H$. The elements of $\mathcal{X}$ are called the strategies. We assume that $X\equiv 0$ is a strategy in $\mathcal{X}$.

The final wealth W of the insider, just after the announcement, can be written in the following way, analogously to the discrete version (2.1),

$$W_{1+}=\int_0^1 (V-P_{t-})\mathrm{d}X_t-[P,X]_1, \tag{2.23}$$

with X_{t-} denoting the limit $\lim_{s\uparrow t}X_s$. Assume that X is an $\mathbb{F}^{V,P}$-semimartingale (so that the integral can be seen as an Itô integral) and that P is an $\mathbb{F}^{V,P}$-semimartingale (to ensure the quadratic covariation $[P,X]$ is finite). The definitions of the rationality, optimality and equilibrium are as follows.

Definition 2.1.7 *Given a trading strategy X (and total demand $Y=X+Z$), the price process P is* rational*, if*

$$P_t=\mathbb{E}(V|Y_s,0\leq s\leq t),\;\; t\in[0,1]$$

Definition 2.1.8 *A strategy X is called* optimal *with respect to a price process P if it maximizes $E(W_{1+})$.*

Definition 2.1.9 *Let $(H,\lambda)\in\mathcal{H}$ and $X\in\mathcal{X}$. The triple (H,λ,X) is an equilibrium, if the price process $P_\cdot:=H(\cdot,\xi(\cdot,Y))$ is rational, given X, and the strategy X is optimal, given P.*

In Back's original model, a dynamic programming approach as introduced in Section 1.4 is used to find and describe the equilibria. It is presented later in Subsection 2.1.2. In the following, a perturbation method is used to find the equation corresponding to our problem. We have the following necessary condition for optimal strategies:

Proposition 2.1.1 *An admissible triple (H,λ,X) such that X is locally optimal for the insider, satisfies*

$$V-E(H(t,\xi_t)|\mathcal{H}_t)-\lambda(t)E\left[\int_t^1 \partial_2 H(s,\xi_s)\mathrm{d}X_s\middle|\,\mathcal{H}_t\right]=0,a.s, \tag{2.24}$$

for a.a. $0\leq t\leq 1$, and the strategy does not jump at 1, leading the price to V.

Proof. Denote the filtration generated by V and by $P_s : 0 \leq s \leq t$ by $\mathcal{H}_t$ and assume that the total wealth of the insider is given by (2.23). Consider

$$J(X) := E\left(W_{1+}\right) = E\left(\int_0^1 (V - H(t,\xi_t))\mathrm{d}X_t - [P,X]_1\right).$$

Suppose that X is (locally) optimal. Then, for all β such that $X_\cdot + \varepsilon \int_0^\cdot \beta_s \mathrm{d}s$ is admissible, with $\varepsilon > 0$ small enough, we have

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} J(X_\cdot + \varepsilon \int_0^\cdot \beta_s \mathrm{d}s)\Big|_{\varepsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} E\left(\int_0^1 \left[V - H\left(t, \int_0^t \lambda(s)\left(\mathrm{d}X_s + \varepsilon\beta_s \mathrm{d}s + \mathrm{d}Z_s\right)\right)\right]\left(\mathrm{d}X_t + \varepsilon\beta_t \mathrm{d}t\right)\right)\Big|_{\varepsilon=0} \\
&= E\left(\int_0^1 \left[V - H(t,\xi_t)\right]\beta_t \mathrm{d}t\right) + E\left(\int_0^1 -\partial_2 H(t,\xi_t)\left(\int_0^t \lambda(s)\beta_s \mathrm{d}s\right)\mathrm{d}X_t\right) \\
&= E\left(\int_0^1 \left((V - H(t,\xi_t)) - \lambda(t)\int_t^1 \partial_2 H(s,\xi_s)\mathrm{d}X_s\right)\beta_t \mathrm{d}t\right).
\end{aligned}$$

Since we can take $\beta_t = \mathbf{1}_{[u,u+h]}(t)\ \alpha_u$, with α_u $\mathcal{H}_u$-measurable and bounded, we have

$$E\left(\int_u^{u+h}\left(E((V - H(t,\xi_t))|\,\mathcal{H}_t) - \lambda(t)E\left(\int_t^1 \partial_2 H(s,\xi_s)\mathrm{d}X_s\middle|\,\mathcal{H}_t\right)\right)\mathrm{d}t\middle|\,\mathcal{H}_u\right) = 0 \tag{2.25}$$

and this means that the process:

$$M_t := \int_0^t \left(E(V|\mathcal{H}_u) - E(H(u,\xi_u)|\mathcal{H}_u) - \lambda(u)E\left[\int_u^1 \partial_2 H(s,\xi_s)\mathrm{d}X_s\middle|\,\mathcal{H}_u\right]\right)\mathrm{d}u$$

is an $\mathbb{H}$-martingale. Hence, knowing that $E(V|\mathcal{H}_u) = V$, this implies (2.24) for a.a. $0 \leq t \leq 1$, in particular $H\left(1,\xi_1\right) = V$. And since by the definition of $\mathcal{X}$ and $\mathcal{H}$

$$\begin{aligned}
& V - H(t,\xi_t) - \lambda(t)E\left(\int_t^1 \partial_2 H(s,\xi_s)\mathrm{d}X_s\middle|\,\mathcal{H}_t\right) \\
=\ & V - H(t,\xi_t) - \lambda(t)\int_t^1 E\left(\partial_2 H(s,\xi_s)\theta_s|\,\mathcal{H}_t\right)\mathrm{d}s \\
& -\lambda(t)\sum_{t\leq s\leq 1} E\left(\partial_2 H(s-,\xi_{s-})\Delta X_s|\,\mathcal{H}_t\right) \\
=\ & 0
\end{aligned}$$

And also, we have

$$\int_t^1 E\left(\partial_2 H(s,\xi_s)\left|\theta_s\right|\middle|\,\mathcal{H}_0\right)\mathrm{d}s < \infty,$$

then

$$\lim_{t\to 1} E\left(E\left(\int_t^1 \partial_2 H(s,\xi_s)|\theta_s|\mathrm{d}s \middle| \mathcal{H}_t \right) \middle| \mathcal{H}_0 \right) = 0,$$

and $E\left(\int_t^1 \partial_2 H(s,\xi_s)|\theta_s|\mathrm{d}s \middle| \mathcal{H}_t \right)$ converges in L^1 to zero, and since it is a positive super-martingale it converges almost surely to zero. The same reasoning holds for the term

$$\lambda(t) \sum_{t\le s\le 1} \mathbb{E}(\partial_2 H(s-,\xi_{s-})\Delta X_s|\mathcal{H}_t).$$

so, since $\lambda(t)$ is continuous, we get $V = H(1-,\xi_{1-})$, a.s. Now if we consider a locally optimal strategy with a jump at the end with respect to another without jump we have

$$\begin{aligned}
\Delta J(X) &= \mathbb{E}\left[(V - H(1-,\xi_{1-}))\Delta X_1 - \Delta H_1 \Delta X_1\right] \\
&= -\mathbb{E}\left(\Delta H_1 \Delta X_1\right) < 0,
\end{aligned}$$

since $H(1,\cdot)$ is strictly increasing. Therefore, an optimal strategy does not jump at the end and $V = H(1,\xi_1)$. ■

Then, apart from equation (2.24), we have the following characteristics in equilibrium:

Proposition 2.1.2 *Consider an admissible triple (H,λ,X). If (H,λ,X) is a local equilibrium, then Y_t is a local martingale and $\lambda_\cdot \equiv \lambda$ is constant, and the following equation holds*

$$0 = \partial_1 H(t,\xi_t) + \frac{1}{2}\partial_{22} H(t,\xi_t)\lambda^2\sigma^2 \quad \textit{a.s. on } [0,1]. \tag{2.26}$$

Remark 2.1.1 *Note, that $Y_t = Z_t + \int_0^t \theta_s \mathrm{d}s$ being a local martingale implies by Lévy's characterization that it is a Brownian motion on its filtration, as $Y_0 = 0$ and $[Y]_t = [Z]_t = \sigma^2 t$.*

Proof of Proposition 2.1.2. By using Itô's formula for $\frac{H(t,\xi_t)}{\lambda(t)}$, we have

$$\begin{aligned}
& E\left(\int_t^1 \frac{1}{\lambda(s)} \partial_2 H(s,\xi_s)\mathrm{d}\xi_s \,|\mathcal{H}_t \right) \\
= \; & E\left(\frac{H(1,\xi_1)}{\lambda(1)} \middle| \mathcal{H}_t \right) - \frac{H(t,\xi_t)}{\lambda(t)} \\
& -E\left(\int_t^1 \left(-\frac{\lambda'(s)}{\lambda^2(s)} H(s,\xi_s) + \frac{\partial_1 H(s,\xi_s)}{\lambda(s)} + +\frac{1}{2}\partial_{22} H(s,\xi_s)\lambda(s)\sigma^2 \right) \mathrm{d}s \middle| \mathcal{H}_t \right),
\end{aligned}$$

since X is of the form (2.21) and $\frac{\mathrm{d}[Y,Y]_s}{\mathrm{d}s} = \frac{\mathrm{d}[Z,Z]_s}{\mathrm{d}s} = \sigma^2$. Since X is locally optimal, given (H, λ), by the equation (2.24) and knowing that an optimal strategy leads to price to the final value: $H(1, \xi_1) = V$ from Proposition 2.1.1, we can write:

$$\begin{aligned} 0 &= V - \lambda(t) E\left(\frac{V}{\lambda(1)} \bigg| \mathcal{H}_t \right) \\ &+ \lambda(t) \int_t^1 E\left(-\frac{\lambda'(s)}{\lambda^2(s)} H(s,\xi_s) + \frac{\partial_1 H(s,\xi_s)}{\lambda(s)} + \frac{1}{2}\partial_{22} H(s,\xi_s)\lambda(s)\sigma^2 \bigg| \mathcal{H}_t \right) \mathrm{d}s. \end{aligned}$$

Hence, we have

$$\begin{aligned} 0 &= V\left(\frac{1}{\lambda(t)} - \frac{1}{\lambda(1)} \right) \\ &+ \int_t^1 E\left(-\frac{\lambda'(s)}{\lambda^2(s)} H(s,\xi_s) + \frac{\partial_1 H(s,\xi_s)}{\lambda(s)} + \frac{1}{2}\partial_{22} H(s,\xi_s)\lambda(s)\sigma^2 \bigg| \mathcal{H}_t \right) \mathrm{d}s. \end{aligned}$$

By identifying the predictive and martingale parts, we have that

$$0 = \frac{\lambda'(t)}{\lambda^2(t)} V - \frac{\lambda'(t)}{\lambda^2(t)} H(t,\xi_t) + \frac{\partial_1 H(t,\xi_t)}{\lambda(t)} + \frac{1}{2}\partial_{22} H(t,\xi_t)\lambda(t)\sigma^2 \tag{2.27}$$

Now, since we are in a local equilibrium, prices are rational given X, so by taking conditional expectations with respect to $\mathcal{F}_t$ and using $\mathbb{E}(V|\mathcal{F}_t) - \mathbb{E}(H(t,\xi_t)|\mathcal{F}_t) = 0$, we have

$$\frac{\partial_1 H(t,\xi_t)}{\lambda(t)} + \frac{1}{2}\partial_{22} H(t,\xi_t)\lambda(t)\sigma^2 = 0,$$

consequently

$$P_t = H(t,\xi_t) = H(0,\xi_0) + \int_0^t \lambda_s \partial_2 H(s,\xi_{s-}) \mathrm{d}Y_s \tag{2.28}$$

so,

$$\mathrm{d}Y_t = \frac{\mathrm{d}P_t}{\lambda_t \partial_2 H(t,\xi_{t-})}$$

and, since P_t is a martingale and $\lambda_t \partial_2 H(t,y) > 0$, we have that $Y_.$ is a local martingale. Finally, from (2.27) we have that

$$\frac{\lambda'(t)}{\lambda^2(t)} V - \frac{\lambda'(t)}{\lambda^2(t)} H(t,\xi_t) = 0,$$

then $V \neq H(t,\xi_t)$ implies that $\lambda'(t) = 0$, which together with (2.27) imply (2.26). ■

In equilibrium, the pricing rule (H, λ) satisfies (2.26). Consider, now, only pricing rules satisfying

$$0 = \partial_1 H(t, y) + \frac{1}{2}\partial_{22} H(t, y)\,\lambda_t^2 \sigma^2 = 0. \tag{2.29}$$

We have the following necessary and sufficient conditions for equilibria. Considering a wider set of admissible trading strategies, relaxing the condition (2.21), we find that even in that set, the optimal ones are, indeed, of form (2.21).

Theorem 2.1.1 *Consider an admissible triple (H, λ, X) with (H, λ) satisfying (2.29). Then, it is an equilibrium, if and only if*

(i) $\lambda(t) \equiv \lambda_0$,

(ii) $H(1, \xi_1) = V$ *a.s.,*

(ii) $[X^c, X^c] \equiv 0$,

(iv) X *has not jumps,*

(v) Y *is a local martingale.*

Proof. Denote throughout this proof the derivative with respect to the variables v, t and y by ∂_0, ∂_1 and ∂_2 respectively. Set

$$\begin{aligned} i(v, y) &= \int_y^{H^{-1}(1,\cdot)(v)} \frac{v - H(1, x)}{\lambda_0} \\ I(v, t, y) &= E\left[i\left(V, y + \lambda_0 (Z_1 - Z_t)\right) | V = v\right] = E\left[i\left(v, y + \lambda_0 (Z_1 - Z_t)\right)\right], \end{aligned}$$

and note, that since $(H(t, \lambda_0 Z_t))_t$ and $I(v, t, Z_t)_t$ are martingales, and Z is a Brownian motion, so it has independent increments, we have

$$\begin{aligned} H(t, y) &= E\left[H(1, \lambda_0 Z_1) | \lambda_0 Z_t = y\right] = E\left[H\left(1, y + \lambda_0 (Z_1 - Z_t)\right)\right], \\ I(v, t, y)_t &= E\left[i(v, \lambda_0 Z_1) | \lambda_0 Z_t = y\right] = E\left[i\left(v, y + \lambda_0 (Z_1 - Z_t)\right)\right], \quad \text{and} \\ \partial_2 I(v, t, y) &= E\left[\partial_2 i\left(v, y + \lambda_0 (Z_1 - Z_t)\right)\right] \\ &= E\left[-\frac{v - H\left(1, y + \lambda_0 (Z_1 - Z_t)\right)}{\lambda_0}\right] = -\frac{v - H(t, y)}{\lambda_0} \end{aligned} \tag{2.30}$$

where derivative can be taken under the integral sign, since $E\left[H(1, \lambda_0 Z_1)\right] < \infty$ and $H(1, \cdot)$ is monotone. Then,

$$\begin{aligned} 0 &= \partial_{12} I(v, t, y) + \frac{1}{2}\partial_{222} I(v, t, y)\,\lambda_0 \sigma^2, \quad \text{so} \\ C(v, t) &= \partial_1 I(v, t, y) + \frac{1}{2}\partial_{22} I(v, t, y)\,\lambda_0 \sigma^2, \end{aligned} \tag{2.31}$$

with $C(v,t)$ being a constant with respect to y, in fact being zero for a.a. $t \in [0,1]$, since $I(v,t,Z_t)_t$ is a martingale. Then, by Itô's formula, we get

$$\begin{aligned} I(v,1,\xi_1) &= I(v,0,0) + \int_0^1 \partial_1 I(v,t,\xi_t)\,\mathrm{d}t \\ &\quad + \int_0^1 \partial_2 I(v,t,\xi_{t-})\,\mathrm{d}\xi_t + \frac{1}{2}\int_0^1 \partial_{22} I(v,t,\xi_t)\,\mathrm{d}\left[\xi^c,\xi^c\right]_t \\ &\quad + \sum_{0\le t\le 1} \left[\Delta I(v,t,\xi_t) - \partial_2(v,t,\xi_{t-})\,\Delta\xi_t\right], \end{aligned}$$

where

$$\mathrm{d}\left[\xi^c,\xi^c\right]_t = \lambda_0^2 \mathrm{d}\left[X^c,X^c\right]_t + 2\lambda_0^2 \mathrm{d}\left[X^c,Z^c\right]_t + \lambda_0^2 \sigma \mathrm{d}t,$$

so by (2.31), we have

$$\begin{aligned} I(v,1,\xi_1) &= I(v,0,0) + \int_0^1 (P_{t-}-v)(\mathrm{d}X_t + \mathrm{d}Z_t) \\ &\quad + \frac{1}{2}\int_0^1 \partial_{22} I(v,t,\xi_t)\,\lambda_0^2 \mathrm{d}\left[X^c,X^c\right]_t \\ &\quad + \int_0^1 \partial_{22} I(v,t,\xi_t)\,\lambda_0^2 \mathrm{d}\left[X^c,Z^c\right]_t \\ &\quad + \sum_{0\le t\le 1} \left[\Delta I(v,t,\xi_t) - \partial_2(v,t,\xi_{t-})\,\lambda_0 \Delta Y_t\right]. \end{aligned}$$

Then, subtracting $[P,X]_1$ from both sides, we get

$$\begin{aligned} &\int_0^1 (v-P_{t-})\,\mathrm{d}X_t - [P,X]_1 - I(v,0,0) \\ = &\; -I(v,1,\xi_1) + \int_0^1 (P_{t-}-v)\,\mathrm{d}Z_t \\ &+ \frac{1}{2}\int_0^1 \partial_{22} I(v,t,\xi_t)\,\lambda_0^2 \mathrm{d}\left[X^c,X^c\right]_t + \int_0^1 \partial_{22} I(v,t,\xi_t)\,\lambda_0^2 \mathrm{d}\left[X^c,Z^c\right]_t \\ &+ \sum_{0\le t\le 1} \left[\Delta I(v,t,\xi_t) - \partial_2(v,t,\xi_{t-})\,\lambda_0 \Delta X_t\right] - [P,X]_1. \end{aligned}$$

Note that $I(v,0,0)$ is a lower bound for all strategies. We will show that taking the conditional expectation for $V=v$ the right hand side (so the left hand side, as well), is non-positive.

Note, that

$$[P,X]_1 = [P^c,X^c]_1 + \sum_{0\le t\le 1} \Delta P_t \Delta X_t,$$

where Itô's formula implies that the continuous local martingale part of P is

$$\int \partial_2 H(t, \xi_t)\, \mathrm{d}\xi_t^c,$$

so by (2.30),

$$\begin{aligned}
[P^c, X^c]_1 &= \left[\int \partial_2 H(t, \xi_t)\, \mathrm{d}\xi_t^c, X^c\right]_1 = \int_0^1 \partial_2 H(t, \xi_t)\, \mathrm{d}[\xi^c, X^c]_t \\
&= \int_0^1 \partial_{22} I(v, t, \xi_t)\, \lambda_0^2 \mathrm{d}[X^c, X^c]_t + \int_0^1 \partial_{22} I(v, t, \xi_t)\, \lambda_0^2 \mathrm{d}[X^c, Z]_t
\end{aligned}$$

and

$$\begin{aligned}
\lambda_0 \partial_2 I(v, t, \xi_{t-})\, \Delta X_t + \Delta P_t \Delta X_t &= (P_{t-} - v)\, \Delta X_t + \Delta P_t \Delta X_t \\
&= (P_t - v)\, \Delta X_t = \lambda_0 \partial_2 I(v, t, \xi_t)\, \Delta X_{t.}
\end{aligned}$$

Then, substituting them for $[P, X]_1$ on the right hand side, it simplifies to

$$\begin{aligned}
= \; & -I(v, 1, \xi_1) + \int_0^1 (P_{t-} - v)\, \mathrm{d}Z_t \\
& + \frac{1}{2} \int_0^1 \partial_{22} I(v, t, \xi_t)\, \lambda_0^2 \mathrm{d}[X^c, X^c]_t + \\
& + \sum_{0 \le t \le 1} [I(v, t, \xi_t) - I(v, t, \xi_{t-}) - \partial_2 (v, t, \xi_{t-})\, \lambda_0 \Delta X_t].
\end{aligned}$$

We have the following results:

1. Since

$$\begin{aligned}
\lambda_0 \partial_{22} I(V, 1, \xi_1) &= \partial_2 H(V, 1, \xi_1) > 0 \qquad \text{and} \\
\lambda_0 \partial_2 I(V, 1, \xi_1) &= -V + H(1, \xi_1),
\end{aligned}$$

   so by (ii), we have the maximum value of $-I(V, 1, \xi_1)$ for the strategy, and by its definition and (ii), we have $I(V, 1, \xi_1) = 0$.

2. The process $\int_0^1 (P_{t-} - v)\, \mathrm{d}Z_t$ is a $\mathbb{F}^{P,V}$-martingale and becomes zero when taking the expectation.

3. Because of H being increasing monotone, and (2.30), $\partial_{22} I > 0$ and the measure $\mathrm{d}[X^c, X^c] \ge 0$, so

$$-\frac{1}{2} \int_0^1 \partial_{22} I(v, t, \xi_t)\, \lambda_0^2 \mathrm{d}[X^c, X^c]_t \le 0,$$

   and it reaches its maximum if and only if $[X^c, X^c] = 0$.

4. The convexity of I, $\partial_{22}I > 0$ implies that

$$(v,t,x+h) - I\left(v,t,x\right) - \partial_2\left(v,t,x+h\right)h \leq 0,$$

so is the last term, and it reaches its maximum if and only if X does not have jumps (iv).

5. (2.22) and (v) imply that the prices are rational.

■

We will need the following Lemma:

Lemma 2.1.1 *Assume that a process G is $\mathbb{F}^Y$-adapted and*

$$G_t = M_t + \int_0^t \alpha_s ds,$$

where M is an $\mathbb{F}^{Z,V}$-martingale and α is $\mathbb{F}^{Z,V}$-adapted. Let $\mathbb{H}$ be a filtration such that $\mathbb{F}^Y \subseteq \mathbb{H} \subseteq \mathbb{F}^{Z,V}$. Then

$$G_t = N_t + \int_0^t \mathbb{E}\left[\alpha_s \left|\mathcal{H}_s\right.\right] ds,$$

where N is an $\mathbb{H}$-martingale.

Proof. First, I show that $\mathbb{E}\left[M_t \left|\mathcal{H}_t\right.\right]$ is an $\mathbb{H}$-martingale. Let $s \leq t \leq 1$, then since $\mathcal{H}_s \subseteq \mathcal{F}_s^{Z,V}$

$$\mathbb{E}\left[\mathbb{E}\left[M_t \left|\mathcal{H}_t\right.\right]\right|\mathcal{H}_s] = \mathbb{E}\left[M_t \left|\mathcal{H}_s\right.\right] = \mathbb{E}\left[\mathbb{E}\left[M_t \left|\mathcal{F}_s^{Z,V}\right.\right]\right|\mathcal{H}_s\right] = \mathbb{E}\left[M_s \left|\mathcal{H}_s\right.\right],$$

since M is an $\mathbb{F}^{P,V}$-martingale. Then, consider

$$G_t - G_s = M_t - M_s + \int_s^t \alpha_u \mathrm{d}u.$$

We have

$$\begin{aligned}
\mathbb{E}\left[G_t - G_s\right|\mathcal{H}_s] &= \mathbb{E}\left[M_t - M_s\right|\mathcal{H}_s] + \int_s^t \mathbb{E}\left[\alpha_u\right|\mathcal{H}_s]\,\mathrm{d}u \\
&= \mathbb{E}\left[\int_s^t \mathbb{E}\left[\alpha_u\right|\mathcal{H}_u]\,\mathrm{d}u\right|\mathcal{H}_s\right],
\end{aligned}$$

so

$$\mathbb{E}\left[G_t - G_s - \int_s^t \mathbb{E}\left[\alpha_u\right|\mathcal{H}_u]\,\mathrm{d}u\right|\mathcal{H}_s\right] = 0,$$

hence, $N_t := G_t - \int_0^t \mathbb{E}\left[\alpha_u\right|\mathcal{H}_u]\,\mathrm{d}u$ is an $\mathbb{H}$-martingale. ■

Then, the following conditions characterizes optimal strategies:

Proposition 2.1.3 *Let (X, H, λ) be a triplet with the pricing rule of class $\mathcal{H}$ that satisfies (2.29) and the strategy $X \in \mathcal{X}$ satisfying in (2.21). Then the following conditions are equivalent:*

$i)$ *The process* $(H(t, \xi_t))$ *is an* $\mathbb{F}^Y$*-martingale.*
$ii)$ $\mathbb{E}\left[\theta_t | \mathcal{F}_t^Y\right] = 0,$ *and*
$iii)$ *The process* Y_t *is an* $\mathbb{F}^Y$*-martingale (Brownian motion).*

Proof. By Itô's formula, we have

$$\begin{aligned}
H(t,\xi_t) &= H(0,0) + \int_0^t \lambda_s \theta_s \partial_2 H(s,\xi_s)\,\mathrm{d}s \\
&\quad + \int_0^t \left[\partial_1 H(s,\xi_s) + \frac{1}{2}\lambda_s^2 \sigma^2 \partial_{22} H(s,\xi_s)\right] \mathrm{d}s \\
&\quad + \int_0^t \partial_2 H(s,\xi_{s-}) \lambda_s \mathrm{d}B_s \\
&= M_t + \int_0^t \left[\partial_1 H(s,\xi_s) + \frac{1}{2}\lambda_s^2 \sigma^2 \partial_{22} H(s,\xi_s)\right] \mathrm{d}s \\
&\quad + \int_0^t \lambda_s \theta_s \partial_2 H(s,\xi_s)\,\mathrm{d}s.
\end{aligned}$$

where M is an $\mathcal{F}^{Z,V}$-martingale. Then, using Lemma 2.1.1, for some $\mathbb{F}^Y$-martingale N, we can write H as

$$\begin{aligned}
H(t,\xi_t) &= N_t + \int_0^t \left[\partial_1 H(s,\xi_s) + \frac{1}{2}\lambda_s^2 \sigma^2 \partial_{22} H(s,\xi_s)\right] \mathrm{d}s \\
&\quad + \int_0^t \lambda_s \mathbb{E}(\theta_s|\mathcal{F}_s^Y) \partial_2 H(s,\xi_s)\,\mathrm{d}s \\
&= N_t + \int_0^t \lambda_s \mathbb{E}(\theta_s|\mathcal{F}_s^Y) \partial_2 H(s,\xi_s)\,\mathrm{d}s,
\end{aligned}$$

Then, the equivalency of i) and ii) holds since, $(H(t,\xi_t))$ is an $\mathbb{F}^Y$-martingale if and only if $\mathbb{E}(\theta_s|\mathcal{F}_s^Y) = 0$. Then, we also know that $Y_t = Z_t + \int_0^t \theta_s \mathrm{d}s$. and that we can get Y, by Lemma 2.1.1, as

$$Y_t = U_t + \int_0^t \mathbb{E}(\theta_s|\mathcal{F}_s^Y)\mathrm{d}s$$

where U is an $\mathbb{F}^Y$-martingale, so ii) and iii) are equivalent. ■

Then, we have proved that having a pricing rule satisfying (2.29), the necessary and sufficient conditions for an equilibrium are that the insiders' a strategy is of form (2.21) leading the price to its final price (equivalently, leading the total demand to $H^{-1}(1, \lambda_0 \cdot)(V)$ and that $(Y_t)_t$) is an F^Y-Brownian motion (or equivalently, $\mathbb{E}\left[\theta_t \middle| \mathcal{F}_t^Y\right] = 0$).

Then, we have that (supposing that $\lambda \equiv 1$)

$$\begin{aligned} H(t, y) &= E\left[h\left(y + Z_1 - Z_t\right)\right], \\ X_t &= (1-t)\int_0^1 \frac{h^{-1}(V) - Z_s}{(1-s)^2}\mathrm{d}s = \int_0^t \frac{V - Y_s}{1-s}\mathrm{d}s \end{aligned}$$

is an equilibrium, where $h = F^{-1} \circ N$, with N being the normal distribution function of zero mean and variance σ^2.

2.1.2 Original approach

In [Kyl85], the already introduced discrete models and a continuous model are studied. In the single auction model, equilibrium is found with the linear regression formulas, which is extended to the N-period model recursively by backward induction. Also, it is shown that the continuous model is the limit of the N-period model as N tends to infinity. In [Bac92], the continuous model is studied and a dynamic programming method is used to find the equilibrium. Supposing that the insider's strategy is of the form $\mathrm{d}X_t = \theta_t \mathrm{d}t$, J is of the form

$$J(V, t, y) = \sup_{\theta, Y_t = y} E\left[\left.\int_t^1 (V - P_u)\theta_u du \right| \mathcal{F}_t^{Z,V}\right],$$

which, by splitting the integral into two parts: from t to $t+h$ and $t+h$ to 1, implies

$$0 = \sup_{\theta, Y_t = y} E\left[\left.\int_t^{t+h} (V - P_u)\theta_u du + J(V, t+h, Y_{t+h}) - J(V, t, Y_t) \right| \mathcal{F}_t^{Z,V}\right].$$

Then, by Itô's formula and taking the limit as $h \to 0$ and denoting by ∂_1 and ∂_2 the differentiation with respect to the variables t and y, respectively, we get

$$0 = \sup_{\theta}\left\{(V - P_t)\theta_t + \partial_1 J + \partial_2 J \theta_t + + \frac{1}{2}\partial_{22} J \sigma_t^2\right\},$$

which, being linear in θ, implies

$$\partial_2 J = H - V \quad \text{and} \quad \partial_1 J + \frac{1}{2}\sigma_t^2 \partial_{22} J = 0.$$

This can be extended to the case of having jumps in the noise traders' demand, as considered in [Cor14b], with Z given by

$$\mathrm{d}Z_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t + \mathrm{d}L_t, \quad t \in [0,1] \quad \text{with } Z_0 = 0,$$

where B is a Brownian motion, independent of V, and $\mu, \sigma : [0,1] \to \mathbb{R}$ are deterministic, càdlàg functions, and L is a pure jump Lévy process independent of V, which can be expressed by

$$L_t = \int_0^t \int_{\mathbb{R}} x \tilde{M}(\mathrm{d}t, \mathrm{d}x),$$

where $\tilde{M}(\mathrm{d}t, \mathrm{d}x) = M(\mathrm{d}t, \mathrm{d}x) - v_t(\mathrm{d}x)\mathrm{d}t$ is the compensated Poisson random measure associated with L, and with intensity $v_t(\mathrm{d}x)$. The conditional value function is defined as earlier, and when splitting the integral into two parts, using the fact that

$$\mathrm{d}\xi(t,\theta) = \lambda_t \theta_t \mathrm{d}t + \lambda_t \mu_t \mathrm{d}t + \lambda_t \sigma_t \mathrm{d}B_t + \lambda_t \mathrm{d}L_t,$$

by Itô's formula we get

$$\begin{aligned} J(t+h, \xi(t+h,\theta)) &= J(t, \xi(t,\theta)) \\ &\quad + \int_t^{t+h} \left[\partial_1 J + \lambda_s(\mu_s + \theta_s)\partial_2 J + \frac{1}{2}\lambda_s^2 \sigma_s^2 \partial_{22} J \right] \mathrm{d}s \\ &\quad + \int_t^{t+h} \partial_2 J \lambda_s \sigma_s \mathrm{d}B_s + \int_t^{t+h} \partial_2 J \lambda_s \mathrm{d}L_s \\ &\quad + \sum_{t \le s \le t+h} \left[\Delta J(s, \xi(s,\theta)) - \partial_2 J \Delta\xi(s,\theta) \right]. \end{aligned}$$

Since $\Delta\xi(t,\theta) = \lambda_s \Delta Y_s = \lambda_s \Delta Z_s$, we have

$$\begin{aligned} & E\left[\sum_{t \le s \le t+h} \Delta J(s, \xi(s,\theta)) - \partial_2 \Delta\xi(s,\theta) \,\middle|\, \mathcal{F}_t^{P,V} \right] \\ &= E\left[\sum_{t \le s \le t+h} J\left(s, \xi(s_-,\theta) + \lambda_s \Delta Z_s\right) - J\left(s, \xi(s-,\theta)\right) - \lambda_s \partial_2 J \Delta Z_s \,\middle|\, \mathcal{F}_t^{P,V} \right] \\ &= \int_t^{t+h} \int_{\mathbb{R}} E\left[J(s, \xi_{s-} + \lambda_s u) - J(s, \xi_{s-}) - u\lambda_s \partial_2 J \,\middle|\, \mathcal{F}_t^{P,V} \right] \nu_s(\mathrm{d}u)\mathrm{d}s. \end{aligned}$$

Therefore, we obtain the Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} 0 &= \sup_\theta \Big\{ (V-H)\theta_t + \partial_1 J + \partial_2 J \lambda_t \theta_t + \partial_2 J \lambda_t \mu_t + \frac{1}{2}\partial_{22} J \lambda_t^2 \sigma_t^2 \\ &\quad + \int_{\mathbb{R}} (J(t, y + \lambda_t u) - J(t,y) - u\lambda_t \partial_2 J(t,y))\nu_t(\mathrm{d}u) \Big\}, \end{aligned}$$

which is linear in θ, so we obtain

$$\lambda_t \partial_2 J(t,y) = H(t,y) - V$$

for every $(t,y) \in (0,1] \times \mathbb{R}$ and

$$\partial_1 J + \lambda_t \mu_t \partial_2 J + \frac{1}{2}\lambda_t^2 \sigma_t^2 \partial_{22} J + \int_{\mathbb{R}} \left(J\left(t, y + \lambda_t u\right) - J\left(t,y\right) - u \lambda_t \partial_2 J\left(t,y\right)\right) \nu_t\left(\mathrm{d}u\right) = 0$$

for $\forall (t,y) \in (0,1) \times \mathbb{R}$, where the $t = 1$ case follows from the continuity of $\partial_2 J$ and H.

Chapter 3

Extensions and related models

In this Chapter, two extensions of Kyle's and Back's model are presented, as well, as some related models. First, a model allowing jumps in the noise traders' demand and considering also risk-averse insiders, following [Cor14b], then a general model with examples as particular cases (already studied ones) can be found, following [Cor14a]. Finally other related models are summarized.

3.1 Kyle's model with the presence of Jumps

In this Section, the model studied in [Cor14b] is presented, in which the noise traders' demand is allowed to have jumps, modeled by a Lévy-process, and the risk-aversion of the insider is considered, as well. It is shown that with the informed trader being risk neutral, the price pressure is constant over time, and there is no equilibrium in the presence of jumps. Also, an approximation is studied. Finally, it is shown that the insider being risk-averse, equilibrium may exist only if the jump part as well, as the drift part of the noise traders' process Z, equal 0, in which case we have the model already studied in [Cho03], presented in Subsection 3.3.2.

3.1.1 The model

Consider the same market with the two assets and the same participant over the period $[0, 1]$ as they are considered in [Bac92]. Suppose, also, that the bank account has an interest rate of r equal to zero. The public announcement is made at time 1, and it reveals the value of the risky asset, at which price it will trade afterwards (that is to say, at time $1+$), denoted by V and assumed to be a random variable with finite expectation. The market is continuous in time and order driven. The informed trader is assumed to be aware of the V at time 0. All random variables are defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

As before, price of the stock at time t is denoted by P_t and $\mathbb{F}^P = \left(\mathcal{F}_t^P\right)_{0\leq t\leq 1}$ where $\mathcal{F}_t^P = \sigma(P_s, 0 \leq s \leq t)$. Let Z be the noise traders' aggregate demand process with possible drift and jumps given by

$$\mathrm{d}Z_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t + \mathrm{d}L_t, t \in [0,1], Z_0 = 0, \tag{3.1}$$

where B is a Brownian motion, independent of V, and $\mu, \sigma : [0,1] \to \mathbb{R}$ are deterministic, càdlàg functions, and L is an pure jump Lévy process independent of V and B. Assume also that the process L can be expressed by

$$L_t = \int_0^t \int_{\mathbb{R}} x\tilde{M}(\mathrm{d}t, \mathrm{d}x),$$

where $\tilde{M}(\mathrm{d}t, \mathrm{d}x) = M(\mathrm{d}t, \mathrm{d}x) - v_t(\mathrm{d}x)\mathrm{d}t$ is the compensated Poisson random measure associated with L, and with intensity $v_t(\mathrm{d}x)$.

Then, denote

$$\mathbb{F}^{V,P} := \left(\mathcal{F}_t^{V,P}\right)_{0\leq t\leq 1},$$

where

$$\mathbb{F}_t^{V,P} := \sigma(V, P_s, 0 \leq s \leq t)$$

and suppose that the market makers "clear" the market by fixing a competitive or rational price, given by

$$P_t = \mathbb{E}(V|Y_s, 0 \leq s \leq t), t \in [0,1]$$

where $Y = X + Z$ is the total demand that market makers observe. In this case, the definitions of optimality, rationality and equilibrium are as follows (the set of admissible strategies $\mathcal{X}$ and pricing rules $\mathcal{H}$ are defined later)

Definition 3.1.1 *Given a trading strategy X (and total demand $Y = X + Z$), the price process P is* rational, *if*

$$P_t = \mathbb{E}(V|Y_s, 0 \leq s \leq t), t \in [0,1]$$

Definition 3.1.2 *A strategy X is called* optimal *with respect to a price process P if it maximizes $E(W_{1+})$.*

Definition 3.1.3 *Let $(H, \lambda) \in \mathcal{H}$ and $X \in \mathcal{X}$. The triple (H, λ, X) is an equilibrium, if the price process $P_\cdot := H(\cdot, \xi(\cdot, Y))$ is rational, given X, and the strategy X is optimal, given P.*

3.1.2 The equilibrium

The Perturbation method is used to characterize the equilibria. We suppose that market makers fix prices through a pricing rule

$$P_t = H(t, \xi_t) \text{ where } \xi_t := \xi(t, Y_t) = \int_0^t \lambda(s)\mathrm{d}Y_s,$$

with $t \in [0, 1]$, where, the pressure $\lambda.$ is a positive smooth function, $H \in C^{1,2}$ and $H(t, \cdot)$ is strictly increasing for every $t \in [0, 1]$. Note, that $\mathbb{F}^Y = \mathbb{F}^P$, $\mathbb{F}^{V,P} = \mathbb{F}^{V,Y} = \mathbb{F}^{V,X+Z}$ and that we can assume that X is $\mathbb{F}^{V,Z}$-adapted, and that consequently $\mathbb{F}^Y \subseteq \mathbb{F}^{V,Z}$, in such a way that if $X_t = f(Y_s, 0 \le s \le t, V)$ for certain measurable function f we can write $X_t = g(Z_s, 0 \le s \le t, V)$ for another measurable function g.

Definition 3.1.4 *Denote the class of such pairs (H, λ) above by $\mathcal{H}$. An element of $\mathcal{H}$ is called a pricing rule.*

As shown in [Bac92] and [Cho03], in equilibrium, the optimal strategies are of the form

$$\mathrm{d}X_t = \theta_t \mathrm{d}t. \tag{3.2}$$

Definition 3.1.5 *Denote, by $\mathcal{X}$, the set of càdlàg $\mathbb{F}^{V,P}$-predictable processes with*

(A1) $X \in \mathcal{X}$ *satisfying* $X_t = M_t + A_t + \int_0^t \theta_s \mathrm{d}s$, *where M is a continuous $\mathbb{F}^{V,P}$-martingale, A is a càdlàg, finite variation predictable process with*

$$A_t = \sum_{0 \le s \le t} (X_s - X_{s-})$$

and θ is a càdlàg, $\mathbb{F}^{V,P}$-adapted process. And for all $X \in \mathcal{X}$ and $(H, \lambda) \in \mathcal{H}$, $\mathbb{P}$-a.s, a.a. $0 \le t \le 1$ we have:

(A2) $\mathbb{E}\left(\int_0^1 \left(\partial_2 H\left(t-, \xi_{t-}\right)\right)^2 \left(\mathrm{d}\left[Z, Z\right]_t + \mathrm{d}\left[M, M\right]_t\right)\right) < \infty,$

(A3) $\mathbb{E}\left(\int_0^1 \partial_2 H\left(t, \xi_t\right) \left|\theta_t\right| \mathrm{d}t\right) < \infty,$

(A4) $\mathbb{E}\left(\sum_0^1 \partial_2 H\left(t-, \xi_{t-}\right) \left|\Delta X_t\right|\right) < \infty$ *with* $\Delta X_t = X_t - X_{t-},$

(A5) $\int_{\mathbb{R}} \left(H\left(t, \xi_{t-} + \lambda_t u\right) - H\left(t, \xi_{t-}\right) - u\lambda_t \frac{\partial H}{\partial y}\left(t, \xi_{t-}\right)\right) \nu_t\left(\mathrm{d}u\right) < \infty,$

(A6) $0 \in \mathcal{X}.$

Then, by the Perturbation method, considering the total wealth being

$$J(X) := E\left(W_{1+}\right) = E\left(\int_0^1 (V - H(t,\xi_t))\mathrm{d}X_t - [P,X]_1\right)$$

and by Itô's formula, the following necessary conditions have been found.

Proposition 3.1.1 *Consider an admissible triple* (H,λ,X). *If it is a local equilibrium, then we have:*

(i) $V - H(t,\xi_t) - \lambda(t)\mathbb{E}\left[\int_t^1 \partial_2 H(s,\xi_{s-})\mathrm{d}X_s \middle| \mathcal{F}_t^{V,Z}\right] = 0,$ *a.s, a.a.* $0 \le t \le 1.$

(ii) $V = P_1 = H(1,\xi_1) = H(1-,\xi_{1-}) = P_{1-}$ *a.s.,* ,

(iii) $0 = \partial_1 H(t,\xi_t) + \lambda_t\mu_t\partial_2 H(t,\xi_t) + \frac{1}{2}\lambda_t^2\sigma_{Y,t}^2\partial_{22}H(t,\xi_t)$

$+\int_{\mathbb{R}}\left(H\left(t,\xi_{t-}+\lambda_t u\right) - H\left(t,\xi_{t-}\right) - u\lambda_t\partial_2 H\left(t,\xi_{t-}\right)\right)\nu_t\left(\mathrm{d}u\right),$ *a.s, a.a.* $0 \le t \le 1.$

(iv) $Y - \int_0^{\cdot}\mu_t\mathrm{d}t$ *is a local martingale*

(v) *If* $V \neq P_t$ $a.s.on$ $[0,1),$ *then* $\lambda(t) = \lambda_0$,

where $\sigma_{Y,t}^2 := \frac{\mathrm{d}[Y^c,Y^c]_s}{\mathrm{d}s}$.

Note, that while in [Bac92], it is assumed a priori that in equilibrium, the prices tend to the price at time $1+$, in this case, as it was shown also in [ABØ07], it follows from the optimality of the insider's strategy. In equilibrium, the pricing satisfies (iii) from Proposition 3.1.1. Then, restricting the set of pricing rules, we have the following necessary and sufficient conditions for equilibria.

Theorem 3.1.1 *Consider an admissible triple* (H,λ,X) *with* (H,λ) *satisfying for a.a.* $0 \le t \le 1$ *and* $y \in \mathbb{R}$

$$\begin{aligned} 0 &= \partial_1 H(t,y) + \partial_2 H(t,y)\lambda(t)\mu_t + \frac{1}{2}\partial_{22}H(t,y)\lambda(t)^2\sigma_t^2 \\ &\quad + \int_{\mathbb{R}}\left(H\left(t,y+\lambda(t)u\right) - H\left(t,y\right) - u\lambda(t)\frac{\partial H}{\partial y}\left(t,y\right)\right)\nu_t\left(\mathrm{d}u\right), \end{aligned} \quad (3.3)$$

then (H,λ,X) *is an equilibrium, if and only if :*

(i) $\lambda(t) = \lambda_0,$

(ii) $H(1,\xi_1) = V$ *a.s.*

(iii) $[X^c,X^c] \equiv 0,$

(iv) X *has not jumps*

$$(v)\ X + Z - \int_0^{\cdot}\mu_s\mathrm{d}s \text{ is a local martingale.} \quad (3.4)$$

Then, by Itô's formula and Lemma 2.1.1, it can be shown that Proposition 2.1.3 holds in this model, as well:

Proposition 3.1.2 *Let X be an admissible strategy in $\mathcal{X}$ and (H, λ) be a pricing rule of class $\mathcal{H}$ that satisfies (3.3). Then the following conditions are equivalent:*

$i)$ *The process* $(H(t, \xi_t))$ *is an* $\mathbb{F}^Y$*-martingale.*
$ii)$ $\mathbb{E}\left[\theta_t | \mathcal{F}_t^Y\right] = 0,$ *and*
$iii)$ *The process* $\left(Y_t - \int_0^t \mu_s \mathrm{d}s\right)$ *is an* $\mathbb{F}^Y$*-martingale.*

In this case, as well, as in the model in [Bac92], in equilibrium, Itô's formula applies that $H(1, \cdot)$ defines $H(\cdot, \cdot)$ by

$$H(t, y) = E[H(1, y + \lambda Z_1 - \lambda Z_y)].$$

Restricting the set of pricing rule to the ones satisfying (3.3) and considering only strategies of the form $X_\cdot = \int_0^\cdot \theta_s \mathrm{d}s$, Proposition 3.1.2 implies that if $\lambda \equiv \lambda_0 > 0$ and if the strategy leads the price to V and $E\left[\theta_t \left| \mathcal{F}_t^Y\right.\right] = 0$, then the pricing rule is rational and (H, λ, X) is an equilibrium. Then, for such X and H satisfying 3.3 with some constant $\lambda > 0$, the necessary and sufficient conditions are the following: the total demand minus the noise traders' drift is a martingale, and the strategy drives the total demand at the announcement, t.i:

1. $\left(Y_t - \int_0^t \mu_s \mathrm{d}s\right)$ is an $\mathbb{F}^y$-martingale, and
2. $Y_1 = H^{-1}(1, \lambda \cdot)(V)$

3.1.3 Examples

Four different cases are considered: Back's original model, drift or jumps in the noise traders demand, and the risk averse informed trader.

Back's original model

With $\sigma_\cdot \equiv \sigma$, $\mu_\cdot \equiv 0$ and without jumps, we have the results of the continuous model presented in 2.1, with the optimal strategy given by

$$\theta_t = \frac{Y_1 - Y_t}{1 - t}.$$

With a drift in the noise traders' demand

First, suppose that the noise traders' demand Z does not have a jump component. Then the equilibrium strategy is given by

$$\theta_t = \frac{Y_1 - Y_t - \int_t^1 \mu_s \mathrm{d}s}{\int_t^1 \sigma_s^2 \mathrm{d}s} \sigma_t^2$$

With a the presence of jumps

This is the main result of [Cor14b]:

Theorem 3.1.2 *If the demand of the liquidity traders Z has a jump component (i.e. $L \neq 0$), then there is not equilibrium.*

It is shown by reaching a contradiction when supposing rational prices: the jump part of the noise traders' process cannot be independent of the information to be released at the end of the trading period. What can be done in this case is an approximation in the following way. Although a jump in $X_{\cdot}$ makes it suboptimal, if there were a jump just at the same moment when there is a jump in the noise traders' demand, mathematically speaking:

$$X_t' = -L_{t-} + \int_0^t \frac{V - Y_s}{1 - s} \mathrm{d}s,$$

then it would lead us to the continuous version of the model. This strategy is not admissible, but $-L_{t-}$ can be approximated by

$$L_t^{j,\varepsilon} = \frac{1}{\varepsilon} \int_{t-\varepsilon}^t L_s^j \mathrm{d}s,$$

where L_s^j is the pure jump part of L. Then, the approximated strategy converges with probability 1 and also in L^1 to the X_t'. In case of L being a process that may have infinite activity, a moving average process can be used to approximate it, that has the same properties needed in this context.

Risk-averse insider

Finally, markets with risk-averse insiders are studied, as well, using the Hamilton-Jacobi-Bellman Equations as mentioned in Subsection 2.1.2. If the insider wants to maximize

$$E(u\left(W_{1+}\right)) = E(\gamma e^{\gamma W_{1+}}), \quad \text{where } \gamma < 0,$$

then the value function is given by

$$J(t,y) := \sup_{\tilde{\theta}:\xi(t,\tilde{\theta})=y} \mathbb{E}\left[\gamma \exp\left\{\gamma \int_t^1 (V - P_l)\tilde{\theta}_l \mathrm{d}l\right\}\middle| \mathcal{F}_t^{Z,V}\right],$$

and we can get the corresponding HJB Equations

$$\begin{aligned} 0 \quad = \quad & \sup_{\theta}\left\{ J\gamma(V-H)\theta_t + \frac{\partial J}{\partial t} + \lambda_t\theta_t\frac{\partial J}{\partial y} + \frac{\partial J}{\partial y}\lambda_t\mu_t + \frac{1}{2}\lambda_t^2\sigma_t^2\frac{\partial^2 J}{\partial y^2} \right. \\ & \left. + \int_{\mathbb{R}} (J(t, y+\lambda_t u) - J(t,y) - u\lambda_t\frac{\partial J}{\partial y}(t,y))\nu_t(\mathrm{d}u)\right\}, \end{aligned}$$

which is linear in θ, so we can get two equations as in the risk-free case, and by differentiating them, it is shown that there can not exist an equilibrium if either the drift part, or the jump part differs from zero in the noise traders' demand process. In case of both being zero, we are in the same situation as in [Cho03].

3.2 A general model

In this Section, a general model is presented allowing the pricing function to depend on the trajectory of the total demand, the announcement time to be random, and a more general set-up of the framework is studied. The private information owned only by the insider is the fundamental value of the stock at the time of the transactions. Two cases are distinguished: when the informed trader knows the (random) announcement time, and when she does not. It is shown that in the first case, the market is efficient, t.i. the market prices converge to the fundamental prices. In the case of her not knowing the exact announcement time, the prices become more stable as the announcement time is approaching, its sensitivity is decreasing as the probability of the announcement time is increasing. Explicit insider's strategies are calculated with the tools if initial and progressive enlargements of filtrations and filtering techniques. This model covers various extensions of Back's original model, which are included as Examples.

3.2.1 The model

The (order driven) market consists of the same three types of traders, as before. Trading is continuous in time over $[0,\infty)$. There is to be a release of information at a possibly random time τ. The information released at τ is the *fundamental* value of the stock, denoted by the process $V_\cdot$. Denote the price process by P_t, and assume that they do not coincide until the announcement, and that just afterwards they do: $P_t \neq V_t$ if $t \leq \tau$

and $P_t = V_t$ if $t > \tau$. The insider's cumulative demand is denoted by the process $X_\cdot$, and her information by $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ with

$$\mathcal{H}_t = \sigma(P_s, \eta_s, \tau : 0 \leq s \leq t)$$

in case she has knowledge of the time of release of information, and

$$\mathcal{H}_t = \sigma(P_s, \eta_s, \tau \wedge s : 0 \leq s \leq t)$$

in case she does not, but she will know it when it happens. Either way, she observes the market prices P and, in addition, she has access to some signal process η related to the firm value. The fundamental value is assumed to be a martingale with respect to $\mathbb{H}$:

$$V_t = \mathbb{E}(f(\eta_\tau)|\mathcal{H}_t), \quad t \geq 0,$$

where f is a non-negative deterministic function. Assume that the process V is continuous and that $\sigma_V^2(t) := \frac{d[V,V]_t}{dt}$ is well defined. Assume that the noise traders' cumulative demand process, denoted by Z, is a continuous $\mathbb{H}$-martingale, independent of V and η and that $\sigma_t^2 = \frac{\mathrm{d}[Z,Z]_t}{\mathrm{d}t}$ is also well defined. Denote the cumulative demand of the informed trader by X and total demand by $Y = Z + X$. Then, the definition of the optimality is as follows.

Definition 3.2.1 *A strategy X is called* optimal *with respect to a price process P if it maximizes $E(W_{\tau+})$.*

Assume the market makers' information flow is given by the total demand and by knowing if the announcement time has been reached

$$\mathcal{F}_t = \sigma(Y_s, \tau \wedge s, 0 \leq s \leq t).$$

Definition 3.2.2 *The market prices are rational if*

$$P_t = \mathbb{E}(V_t|Y_s, \tau \wedge s, 0 \leq s \leq t), \quad t \geq 0.$$

Let us suppose that P_t is given by $P_t = H(t, \xi_t), t \geq 0$ with $\xi_t := \xi(t, Y_t) = \int_0^t \lambda(s)\mathrm{d}Y_s$, where $\lambda \in C^1$ is a strictly positive deterministic function, $H \in C^{1,2}$, and $H(t, \cdot)$ is strictly increasing for every $t \geq 0$. Denote the class of such pairs (H, λ) above by $\mathfrak{H}$. An element of $\mathfrak{H}$ is called a pricing rule.

Definition 3.2.3 *Let $(H, \lambda) \in \mathfrak{H}$ and consider a strategy X. The triple (H, λ, X) is an (a local) equilibrium, if the price process $P_\cdot := H(\cdot, \xi_\cdot)$ is rational, given X, and the strategy X is (locally) optimal, given (H, λ).*

Note that τ is a stopping time with respect to the filtration generated by

$$(\sigma(\tau \wedge s, 0 \le s \le t))_t\,,$$

so it is a stopping time for the insider and for the market makers, as well. τ will be assumed to be bounded if known by the insider, and independent of (V, P, Z) if unknown.

3.2.2 The equilibrium

In the following, necessary and sufficient conditions are presented in the model just introduced. If we write the value function as (for detailes see the [Cor14b] or [Cor14a])

$$\begin{aligned} W_{\tau+} &= X_\tau V_\tau - \int_0^\tau P_{t-} \mathrm{d}X_t - [P, X]_\tau \\ &= \int_0^\tau (V_{t-} - P_{t-})\, \mathrm{d}X_t + \int_0^\tau X_{t-} \mathrm{d}V_t + [V, X]_\tau - [P, X]_\tau, \end{aligned}$$

where $P_{t-} = \lim_{s \uparrow t} P_s$ a.s., then the insider tries to maximize

$$\begin{aligned} J(X) &:= \mathbb{E}\,(W_{\tau+}) \\ &= \mathbb{E}\left(\int_0^\tau (V_t - H(t-, \xi_{t-}))\mathrm{d}X_t + \int_0^\tau X_{t-} dV_t + [V, X]_\tau - [P, X]_\tau \right), \end{aligned}$$

over all *admissible* (H, λ, X) with $(H, \lambda) \in \mathfrak{H}$ satisfying

(A1) $X_t = M_t + A_t + \int_0^t \theta_s \mathrm{d}s$, where M is a continuous $\mathbb{H}$-martingale, A a finite variation predictable process with $A_t = \sum_{0<s\le t} (X_s - X_{s-})$, and θ a càdlàg, $\mathbb{H}$-adapted, process.

(A2) $\mathbb{E}\left(\int_0^\tau (\partial_2 H(s, \xi_s))^2 \left(\sigma_s^2 \mathrm{d}s + \mathrm{d}[M, M]_s\right)\right) < \infty$.

(A3) $\mathbb{E}\left(\int_0^\tau \partial_2 H(s, \xi_s) |\theta_s| \mathrm{d}s \right) < \infty$.

(A4) $\mathbb{E}\left(\sum_0^\tau \partial_2 H(s-, \xi_{s-}) |\Delta X_s| \right) < \infty$,

(A5) $\mathbb{E}\left(\int_0^\tau |X_s|^2 \sigma_V^2(s) \mathrm{d}s \right) < \infty$.

where ∂_i indicates the derivative w.r.t. the i-th argument. Note that the martingale part of $X.$ cannot have jumps, as it has to be $\mathbb{H}$-predictable.

By (A5) and considering only two already mentioned kinds of stopping times τ: either bounded, or independent of (V, P, Z), $E\left(\int_0^\tau X_t \mathrm{d}V_t\right) = 0$ implies

$$J(X) := E\,(W_{\tau+}) = E\left(\int_0^\tau (V_t - H(t, \xi_t))\mathrm{d}X_t + [V, X]_\tau - [P, X]_\tau \right).$$

Then, by applying the Perturbation method, it can be shown, that in equilibrium, for a.a. $t \geq 0$, we have

$$\begin{aligned} 0 &= E(\mathbf{1}_{[0,\tau]}(t)V_t|\mathcal{H}_t) - E(\mathbf{1}_{[0,\tau]}(t)H(t,\xi_t)|\mathcal{H}_t) \\ &\quad -\lambda(t)E\left(\int_t^{\infty} \mathbf{1}_{[0,\tau]}(s)\partial_2 H(s,\xi_s)\mathrm{d}X_s \middle| \mathcal{H}_t\right), \text{ a.s.}, \end{aligned} \tag{3.5}$$

which implies

$$V_t - H(t,\xi_t) - \lambda(t)E\left(\int_t^{\tau} \partial_2 H(s,\xi_s)\mathrm{d}X_s \middle| \mathcal{H}_t\right) = 0, \text{ a.s. } t \in [0,\tau], \tag{3.6}$$

or equivalently for a.a. $\omega \in \{\tau \geq t\}$, since τ is an $\mathcal{H}_{\cdot}$-stopping time.

Now, suppose that τ is known to the insider. Then, it can be shown that optimal strategies lead the market price to the fundamental price making the market efficient, as it was first observed in [ABØ07], and found in case of Z having jumps in [Cor14b], and also in the model of [Cc07].

Proposition 3.2.1 *If τ is known to the insider and (H,λ,X) is admissible with X locally optimal, then the market is efficient, i.e.*

$$V_\tau = P_\tau = H(\tau,\xi_\tau) = H(\tau-,\xi_{\tau-}) = P_{\tau-} \quad \textit{a.s.}.$$

The following necessary conditions have been found:

Proposition 3.2.2 *Consider an admissible triple (H,λ,X). If (H,λ,X) is a local equilibrium, then we have:*

(i) $H(\tau,\xi_\tau) = V_\tau$ *a.s.,*

(ii) $\dfrac{\partial_1 H(t,\xi_t)}{\lambda(t)} + \dfrac{1}{2}\partial_{22}H(t,\xi_t)\lambda(t)\sigma_{Y,t}^2 = 0$ *a.s.on* $[0,\tau)$,

(iii) Y *is a local martingale,*

(iv) *If* $V_t \neq P_t$ *a.s.on* $[0,\tau)$, *then* $\lambda(t) = \lambda_0$,

where $\sigma_{Y,s}^2 := \frac{\mathrm{d}[Y,Y]_s}{\mathrm{d}s}$.

Following (ii) from Proposition 3.2.2, restricting the set of pricing rule to the ones satisfying

$$0 = \partial_1 H(t,y) + \frac{1}{2}\partial_{22}H(t,y)\lambda(t)^2\sigma_t^2 \ , \text{ a.a. } 0 \leq t \leq 1, \ y \in \mathbb{R}, \tag{3.7}$$

and assuming that the process σ_t^2 is deterministic (so that Z is of independent increments, and since it does not have jumps, it is Gaussian, as well), we have the following sufficient conditions:

Theorem 3.2.1 *Consider an admissible triple (H, λ, X) with (H, λ) satisfying (3.7). Then (H, λ, X) is an equilibrium, if and only if:*

$$(i)\ \lambda(t) = \lambda_0,$$
$$(ii)\ H(\tau, \xi_\tau) = V_\tau,$$
$$(iii)\ [X^c, X^c]_t \equiv 0,$$
$$(iv)\ X + Z \text{ is a local martingale without jumps.}$$

When τ is unknown to the insider, assume that τ is independent of (V, P, Z) and that $P(\tau > t) > 0$ for all $t \geq 0$. Then, in equilibrium, we have following necessary conditions:

Proposition 3.2.3 *Consider an admissible triple (H, λ, X). If (H, λ, X) is a local equilibrium, then we have:*

$$(i)\ Y \text{ is a local martingale,}$$
$$(ii)\ \text{If } V_t \neq P_t\ a.s. \text{on } [0, \tau),\ \text{then } \lambda(t) = cP(\tau > t), \quad \text{a.a. } t \geq 0 \quad (c > 0)\ .$$

Here, we can observe that when the (risk-neutral) insider does not know the release time of information, she would trade early in order to use her piece of information before the announcement time comes. This behavior would continue unless the price pressure decreases over time providing more favorable trading also at a later time, similarly to risk-averse case in [Cho03] (with deterministic release time), where in equilibrium, a risk-adverse insider would do most of his trading early to avoid the risk that the prices get closer to the asset value, unless the trading conditions become more favorable over time.

3.2.3 Examples

In this subsection, various already known extensions of the Kyle-Back model are studied as special cases of the just presented model, using techniques of enlargements of filtration and also filtering theory to explicitly compute the insider's optimal strategy.

The application of enlargements of filtration

Since optimal strategies are of the form $\mathrm{d}X_t = \theta_t \mathrm{d}t$, the total demand observed by the market makers is given by

$$Y_t = Z_t + \int_0^t \theta(Y_T; Y_u, 0 \leq u \leq s)\mathrm{d}s, \quad 0 \leq t \leq T. \tag{3.8}$$

We know that Z has to be adapted to $\mathbb{F}^{Y,\eta}$ and it is also a $\mathbb{F}^{Y,\eta}$-martingale, and also that in equilibrium, Y is a local martingale. Thus, (3.8) is the Doob-Meyer decomposition of Y when we enlarge the filtration $\mathbb{F}^Y$ with the process η. As in our case, Z is fixed, and we look for Y, we need a strong solution of (3.8). In the following, one can find how the initial and progressive enlargements of filtration techniques can be used to find optimal strategies.

Example 3.2.1 *We are in the situation of Back's original model, introduced in [Bac92], if we choose*

- *Z to be Brownian motion with variance σ^2,*
- *$\tau = 1$, and*
- *$V_{\cdot} \equiv V_1$ having a continuous cumulative distribution function and being independent of Z.*

Then, we can use the results of Example 1.2.1. We need $V_1 = H(1, Y_1)$, and that Y_1 is of standard normal distribution with zero mean and variance σ^2. It is possible, because we can choose freely $H(1, \cdot)$ without loss of generality, as the boundary condition of (3.7). Then, we have that

$$Y_t = Z_t + \int_0^t \frac{Y_1 - Y_s}{1 - s} \mathrm{d}s$$

is a Brownian motion with variance σ^2, so the prices are rational and the equilibrium strategy is

$$X_t = \int_0^t \frac{Y_1 - Y_s}{1 - s} \mathrm{d}s, \quad 0 \leq t < 1.$$

Example 3.2.2 *We get the model of [ABØ07] with*

- *$Z_t = \int_0^t \sigma_s dW_s$ where $W_{\cdot}$ is a Brownian motion, σ is a deterministic function,*
- *$\tau = 1$, and*
- *$V_{\cdot} \equiv Y_1$ being a Gaussian random variable with mean 0 and variance $\int_0^1 \sigma_s^2 \mathrm{d}s$ and independent of Z.*

Then, using the results of Example 1.2.2, we have that with

$$X_t = \int_0^t \frac{Y_s - Y_1}{\int_t^1 \sigma_u^2 \mathrm{d}u} \sigma_s^2 \mathrm{d}s,$$

being the strategy, $Y = X + Z$ is of the same law as Z.

Example 3.2.3 *We have the model of [CcD11], if we take*

- $\mathrm{d}Z_t = \sigma(Y_t)\mathrm{d}W_t$, *with* $W.$ *being a Brownian motion,*
- $\tau = 1$, *and*
- $V. \equiv \xi_1$, *where* $\xi_t = \int_0^t \sigma(\xi_s)\mathrm{d}B_s$, *and independent of* Z.

Then, by the results in the Example 1.2.3, denoting the transition density of $\xi.$ *by* $G(t, y, z)$, *we have that*

$$\mathrm{d}Y_t = \sigma(Y_t)\mathrm{d}W_t + \sigma^2(Y_t)\frac{\partial_y G(1-t, Y_t, \xi_1)}{G(1-t, Y_t, \xi_1)}\mathrm{d}t,$$

is a martingale.

Example 3.2.4 *To have the model of [Cc07], denote the first time* Y *hits* -1 *by* $\bar{\tau}$, *t. i.* $\bar{\tau} = \inf\{t \geq 0 : Y_t = -1\}$ *and set*

- Z *to be a Brownian motion,*
- $\tau = \bar{\tau} \wedge 1$, *and*
- $\eta_t \equiv \bar{\tau}$, $V_t = 1_{\bar{\tau}>1}$.

Then, we can use the results of Example 1.2.4 and get that the optimal strategy is

$$X_t = \int_0^t \left(\frac{1}{1+Y_s} - \frac{1+Y_s}{\bar{\tau}-s}\right)\mathbf{1}_{[0,\bar{\tau}]}(s)\mathrm{d}s.$$

Example 3.2.5 *Consider a model with the insider receiving a continuous signal, as in [BP98], [Wu99] and [Dan10]. Set*

- Z *to be a Brownian motion,*
- $\tau = 1$, *and*
- $\eta_t = \eta_0 + \int_0^t \sigma_s \mathrm{d}W_s$, *where* σ_s *is deterministic,* η_0 *is a zero mean normal random variable,* W *is a Brownian motion, both independent of* Z

Assuming $\mathit{Var}(\eta_1) = \mathit{Var}(\eta_0) + \int_0^1 \sigma_s^2 \mathrm{d}s = 1$, *Proposition 1.2.3 implies that the optimal strategy is given by*

$$X_t = \int_0^t \frac{\eta_s - Y_s}{\mathit{Var}(\eta_s) - s}\mathrm{d}s, \ 0 \leq t \leq 1.$$

The application of filtering techniques

Suppose that $V_\cdot \equiv V_0 = V$, so in equilibrium, we have rational prices: $P_t = E\left[V \,\middle|\mathcal{F}_t^Y\right]$ with dynamics, by (2.28),

$$\begin{aligned} \mathrm{d}P_t &= \lambda_t \partial_H(t,\xi_t)\mathrm{d}Y_t \\ &= \lambda_t a(t,P_\cdot)(\mathrm{d}Z_t + \theta(t,V,P_\cdot)\mathrm{d}t), \end{aligned}$$

for some function a. For V being Gaussian, optimal strategies can by calculated using the results of Theorem 1.3.1. This method can be generalized to having a signal being a (measurable) function of a Gaussian random variable, or to letting it depend on the time: V_t, and also to having a random announcement time τ being a stopping time on market makers' filtration.

Example 3.2.6 *The market with random announcement time considered in [CS10] is as follows. Let B^v and B^z be independent Brownian motions and $\sigma_v(\cdot)$ and $\sigma_z(\cdot)$ be deterministic functions, and set*

- $\mathrm{d}Z_t = \sigma_z(t)\,\mathrm{d}B_t^z$ *with* $Z_0 = 0$,
- τ *of exponential distribution with scale parameter* μ, *and*
- $\mathrm{d}V_t = \sigma_v(t)\,\mathrm{d}B_t^v$ *with* V_0 *of normal distribution.*

Consider strategies of the form $\mathrm{d}X_t = \beta_t(V_t - P_t)\,\mathrm{d}t$, where $\beta_\cdot$ is a deterministic function. Then, we have

$$\mathrm{d}P_t = \lambda_t\beta_t(V_t - P_t)\,\mathrm{d}t + \lambda_t\sigma_z(t)\,\mathrm{d}B_t^z.$$

Denoting $E\left(V_t|\mathcal{F}_t^Y\right)$ by m_t and the filtering error by Σ_t following the notation of the filtering techniques, we have

$$\begin{aligned} \mathrm{d}m_t &= \frac{\Sigma_t\beta_t}{\lambda_t\sigma_z^2(t)}\left(\mathrm{d}P_t - \lambda_t\beta_t(m_t - P_t)\,\mathrm{d}t\right) \quad \textit{and} \\ \Sigma_t' &= \sigma_v^2(t) - \frac{(\Sigma_t\beta_t)^2}{\sigma_z^2(t)}. \end{aligned}$$

Then, rationality of prices is just $P_t = m_t$, so we need $\Sigma_t\beta_t = \lambda_t\sigma_z^2(t)$ or equivalently $\beta_t = \lambda_t\sigma_z^2(t)/\Sigma_t$ to satisfy the first equation, which implies that the second one is of the form $\Sigma_t' = \sigma_v^2(t) - \sigma_z^2(t)\lambda_t^2$, so

$$\Sigma_t = \Sigma_0 + \int_0^t \sigma_v^2(s)\,\mathrm{d}s - \int_0^t \sigma_z^2(s)\lambda_s^2\mathrm{d}s.$$

Then, we get

$$Y_t = Z_t + \int_0^t \frac{V_s - \int_0^s \lambda_u \mathrm{d}Y_u}{\Sigma_s} \mathrm{d}s,$$

which is the Doob-Meyer decomposition of the martingale Y in the filtration generated by (Z, V), so the optimal strategy is given by

$$\theta_t = \frac{V_t - \int_0^t \lambda_u \mathrm{d}Y_u}{\Sigma_t}$$

Then, if τ is an exponential random variable with parameter μ, in equilibrium, (3.6) can be written as

$$0 = V_t - H(t, \xi_t) - \lambda_t \left(\int_t^\infty e^{-\mu(s-t)} \partial_2 H(s, \xi_s) \, \mathrm{d}X_s \middle| \mathcal{H}_t \right),$$

which, together with $V_t \neq H(t, \xi_t)$ implies $\lambda_t = \lambda_0 \mathrm{e}^{-\mu t}$.

Assume $\sigma_z^2(t) \equiv \sigma_z^2$. In this case, we obtain $\beta_t = \sigma_z^2 \lambda_0 \mathrm{e}^{-\mu t} / \Sigma_t$ with

$$\Sigma_t = \Sigma_0 + \int_0^t \sigma_v^2(s) \, \mathrm{d}s - \sigma_z^2(s) \frac{\lambda_0^2}{2\mu} \left(1 - e^{2\mu t}\right).$$

To fix λ_0, we can impose, for instance, $\lim_{t \to \infty} \Sigma_t = 0$ or to take T such that $\Sigma_t = 0$ for all $t \geq T$. In the first case, we get

$$\begin{aligned} 0 &= \Sigma_0 + \int_0^\infty \sigma_v^2(s) \, \mathrm{d}s - \sigma_z^2 \frac{\lambda_0^2}{2\mu} \quad \textit{and} \\ \lambda_0 &= \sqrt{\frac{2\mu \left(\Sigma_0 + \int_0^\infty \sigma_v^2(s) \, \mathrm{d}s\right)}{\sigma_z^2}}. \end{aligned}$$

In this case, if $\sigma_v^2(\cdot)$ is constant, there is no solution. In the second one, we get $P_t = V_t$ for all $t \geq T$ and, for $\sigma_v^2(\cdot)$ being a constant σ_v^2, we get

$$0 = \Sigma_0 + \sigma_v^2 T - \sigma_z^2 \frac{\lambda_0^2}{2\mu} \left(1 - e^{-2\mu T}\right) = \Sigma_0 + \sigma_v^2 T - \sigma_z^2 \frac{\lambda_T^2}{2\mu} \left(e^{2\mu T} - 1\right).$$

Then, assuming a smooth transition from the strategy, we have $\sigma_v^2 - \sigma_z^2 \lambda_t^2 = 0$, equivalently $\lambda_t = \lambda_T = \frac{\sigma_v}{\sigma_z}$ for all $t \geq T$, and

$$\mathrm{d}P_t = \lambda_t \mathrm{d}Y_t = \lambda_t \mathrm{d}X_t + \lambda_t \mathrm{d}Z_t = \mathrm{d}V_t \quad \textit{for all } t \geq T,$$

so the insider's strategy is given by

$$\mathrm{d}X_t = \frac{\sigma_z}{\sigma_v} \mathrm{d}V_t - \mathrm{d}Z_t$$

and T is the solution of

$$\Sigma_0 + \sigma_v^2 T = \frac{\sigma_v^2}{2\mu} \left(e^{2\mu T} - 1\right).$$

Example 3.2.7 *The market of a defaultable stock with the insider allowed to have information of the "future" is studied in [CcD13a] as follows. Let the time horizon be* 1 *and consider a defaultable stock with the default time* δ *being the first time a Brownian motion* $B.$ *hits the barrier* -1*, t.i.:* $\delta = \inf\{t \geq 0 : B_t = -1\}$*. It is not known by the insider, but is a stopping time for every trader. Assume that she observes* $B_{r(t)}$ *at time* t*, where* $r(\cdot)$ *is an increasing function with* $r(t) > t$ *for* $r \in (0,1)$ *and* $r(0) = 0, r(1) = 1$*. So, she observes the default time in advance, at time* $r^{-1}(\delta) < \delta$*. Let* $\bar{\tau} = r^{-1}(\delta) = \inf\left\{t \geq 0 : B_{r(t)} = -1\right\}$*. So, we have that*

- Z *is a Brownian motion,*
- $\tau = \delta \wedge 1$*, and*
- $\eta_t = B_{r(t)}$ *and* $V_t = 1_{\{\bar{\tau} > t\}} E\left[f(B_1) | B_{r(t)}\right]$*, where* $f(B_1)$ *is the payoff of the insider in case of no default.*

Then, the release time is $r(\bar{\tau})$*. Since* $\bar{\tau}$ *and* δ *are predictable stopping times, it can be shown that* $\lambda.$ *is constant in equilibrium and that the optimal strategy moves prices to the fundamental price:* $\lim_{\delta_n \uparrow \delta} P_{\delta_n} = V_\delta$*, where* (δ_n) *is any increasing sequence of stopping time that tends to* δ*. As from time* $\bar{\tau}$ *to* $V(\bar{\tau})$*, the insider already knows the default time, her strategy can be calculated as in the other models and we have*

$$Y_s = W_s + \int_0^s \left(\frac{1}{1+Y_u} - \frac{1+Y_u}{V(\bar{\tau}) - u}\right) \mathrm{d}s.$$

To get the insider's strategy until time τ*, enlargement of filtration and filtering techniques are used in [CcD13b].*

3.3 Other related models

In this Section, various models related to Kyle's discrete model and to Back's continuous one are presented.

3.3.1 Discrete models

We consider two possible extension of this model. First, a model with more insiders on the market, then a model in which the market makers also observe a signal are presented. We refer to [CS10] for a discussion of a discrete model of infinite horizon with random deadline.

More than one insider

In [NT06], a single auction equilibrium is considered as it is in [Kyl85] with noise traders, market makers and $N \geq 1$ risk-neutral insiders on the market. Let Z and V be independent and have finite second moment. After observing V, the informed traders simultaneously decide what to trade: their strategies are measurable functions $V \mapsto X_n(V)$, and the total demand is given by $Y = \sum_{n=1}^{N} X_n + Z$.

As earlier, the pricing rule P depends on the order flow and the expected profit of the insiders is $W_n = (V - P)X_n$. To make sure that this is well-defined, we consider only strategies and prices with finite second moment:

$$\begin{aligned}
\mathcal{X} &= \{X_n | E\left[X_n^2(V)\right] < \infty)\}, \\
\mathcal{P} &= \{P | \forall (X_1, \ldots, X_N) \in \mathcal{X}^N : E(P^2) < \infty\},
\end{aligned}$$

where $\mathcal{X}$ and $\mathcal{P}$ are the sets of admissible strategies and prices, respectively. Then, $(X_1, \ldots, X_N) \in \mathcal{X}^n$ is *optimal*, if for any $\tilde{X} \in \mathcal{X}$ strategy,

$$\begin{aligned}
E(W_n) &= E((V - P)X_n) \geq E\left[\left(V - \tilde{P}\right)\tilde{X}(V)\right] \quad n = 1, \ldots, N, \quad \text{where} \\
\tilde{P} &= P\left(\sum_{m \neq n} X_m(V) + \tilde{X}(V) + Z\right),
\end{aligned}$$

and P is *rational*, if $E[V - P|Y] = 0$.

Assume, the pricing rule is linear, as well. Then, we have that if a $(P, X_1, \ldots, X_N)$ is an equilibrium in the model (Z, V, N), then in the model $(a + bZ, c + dV, N)$ (where a, b, c, d are constants), $(c + dP, bX_1, \ldots, bX_N)$ is an equilibrium. Hence, we can restrict our study to distributions with zero mean and unit variance. Sufficient and necessary conditions are found for existence of a linear equilibrium:

Lemma 3.3.1 (Lemma 2 in [NT06]) *Suppose that we have*

$$\begin{aligned}
0 &= E[Z] = E[V] \text{ and} \\
1 &= \mathit{Var}[Z] = \mathit{Var}[V].
\end{aligned}$$

Then, a linear equilibrium in the model (Z, V, N) exists if and only if

$$0 = E[Z\sqrt{N} - V | Z + V\sqrt{N}] = 0.$$

The condition for the equilibrium is equivalent to

$$E[(Z\sqrt{N} - V)e^{it(Z+V\sqrt{N})}] = 0, \ \ \forall \in \mathbb{R},$$

as we know that, for two random variables ξ_1 and ξ_2, $E(\xi_1|\xi_2) = 0$ holds if and only if $E(\xi_1 f(\xi_2)) = 0$ holds for any bounded and measurable function f, which is equivalent to having $E(\xi_1 e^{it\xi_2}) = 0$ for all real t, as the set of these exponential functions form a generator on the space of the measurable functions. It has been proved by many that it can be true for two distinct values of N only if V and Z are normally distributed.

More than one signal

In [Jai99], a single auction model is considered like Kyle's one, but we assume that the market makers, as well, have a signal about the final price. While the informed trader knows the final value of the stock V, the market makers observe $V+\varepsilon$, where $\varepsilon \sim N(0,\sigma^2)$, independent of Z and V. Then, X is an *optimal strategy*, if it maximizes the expected profit $E(W)=E((V-P)X)$ and P is *rational*, if $P=E[V|Y,V+\varepsilon]$.

Since the insider's strategy and the price function are linear, it can be shown that $(V, X+Z, V+\varepsilon)$ are jointly normally distributed and by the linear regression formulas, one can find the following equilibrium (denote the mean and the variance of the random variables by $\mu_\cdot$ and $\sigma_\cdot^2$).

Proposition 3.3.1 (Proposition 1 in [Jai99]) *Let*

$$\begin{aligned}\mu_1 &= \frac{\sigma_V^2}{\sigma_V^2+2\sigma_\varepsilon^2} \text{ and}\\ \mu_2 &= \frac{\sigma_V\sigma_\varepsilon^2}{\sigma_Z(\sigma_V^2+2\sigma_\varepsilon^2)},\end{aligned}$$

Then, (X,P) form a linear equilibrium, where

$$\begin{aligned}X &= \frac{(1-\mu_1)(V-\mu_V)}{2\mu_2} \text{ and}\\ P &= (1-\mu_1)\mu_V+\mu_1(V+\varepsilon)+\mu_2(X+Z).\end{aligned}$$

In this model, the stock price reveals more than half of the information possessed by the insider, and this amount varies with the variance of V and ε. As a consequence of it, we find that ε has the effect of reducing the insider's profit, since the expected profits can be calculated in this case, as

$$W_1 = \frac{((1-\mu_1)(V-\mu_V))^2}{4\mu_2} = \frac{((V-\mu_V)\sigma_\varepsilon^2)^2\sigma_Z}{\sigma_V(\sigma_V^2+2\sigma_\varepsilon^2)^2}.$$

while the expected profit in Kyle's model can be written as

$$W_2 = \frac{(V-\mu_V)^2\sigma_Z}{2\sigma_V} > W_1.$$

3.3.2 Continuous models

In this section, I will review some models related to Back's continuous model. In the following, one can find models for a risk-averse informed trader [Cho03], relaxing the conditions of the independence of insider's signal of the noise traders' demand [ABØ07], and for a weaker sense of equilibrium [Wu99, KHOL10, Dan10].

We refer to the following related models: more risky assets including exogenously given prices are studied in [Las04b, Las04a], an option on the market is considered in [Bac93], Kyle's model is obtained as the limit of a limit order market in [BB04], a non-Gaussian generalization is studied and solved with filtering techniques in [CcD11] and the use of filtering techniques and enlargements of filtrations are developed in [CcD13b], with applications to insider trading.

Risk-averse insider

Assume, we have the model studied in Section 3.1 without jumps and with zero drift and that the insider is risk-averse, using a negative exponential utility function of the form $u(W) = \gamma e^{\gamma W}$ with $\gamma < 0$. Then, in [Cho03], the corresponding Hamilton-Jacobi-Bellman equation is calculated:

$$\sup_{\alpha} \left\{ \left[\lambda(t)\partial_2 J_{+}\gamma(V-H)J\right]\alpha + \partial_1 J + \frac{1}{2}\sigma^2\lambda^2(t)\partial_{22}J \right\} = 0.$$

Since it is linear in α, it is equivalent to the following two equations:

$$\begin{aligned} 0 &= \lambda(t)\partial_2 J + (V-H)\gamma J \\ 0 &= \partial_1 J + \frac{1}{2}\sigma^2\lambda^2(t)\partial_{22}J. \end{aligned}$$

It is shown that if there is a solution to these equations, then the pricing rule has to be linear: $H(t,\xi) = p_0 + q\xi$, and that $[\lambda(t)^{-1}]' = \gamma\sigma^2\partial_2 H(t,\xi)$ is a necessary condition. It is shown (see Proposition 4 in [Cho03]) that there exists an equilibrium only if V is normally distributed. Let $V \sim N(m,\Sigma)$, $\Gamma = -\gamma\Sigma/2$, $\nu = \sqrt{\Gamma/\sigma^2}$ and define

$$\begin{aligned} H(t,y) &= m + y \\ X_t &= \int_0^t \frac{V-P_s}{\lambda(1)(1-s)}\mathrm{d}s \\ \lambda(1) &= \sqrt{\nu^2+\Gamma^2} - \Gamma^2, \\ \lambda(t) &= \frac{\lambda(1)}{\gamma\sigma^2\lambda(1)(1-t)+1}. \end{aligned}$$

Then, (H,λ,X) is an equilibrium. Important properties of this equilibrium are that this price pressure is decreasing over time, and as γ tends to 0, the price pressure tends to the price pressure of the risk-neutral case, and it follows that the risk averse equilibrium converges to the risk neutral equilibrium.

Independence of the additional information

The independence of the private information of the demand of noise traders are released in [ABØ07]. Consider the model studied in Section 3.1 without jumps and with zero drift. Perturbation method is used to find equilibria and necessary and sufficient conditions. Forward integral (first defined in [GRV03]), an extension of the Itô integral, is used to calculate integrals to anticipating (non-adapted) functions.

It is found, as in [Bac92] and [Cor14b], that in equilibrium the law of Y coincides with the law of Z, and Y becomes a Brownian bridge starting at 0 and ending in V. Important results of this model compared to Back's model were:

- without assuming the independence of V and Z., the problem could be solved
- without assuming a priori (as it is in [Bac92]), the price at the end of the trading period is V
- without assuming so, the strategy turns out to be inconspicuous
- existence of a pricing rule was not assumed a priori

Weak equilibria

In [KHOL10], a weak formulation of equilibrium is considered. A general model is studied and applied to some examples with the privileged information being the maximum of the total demand Y, the time Y reaches its maximums or Y_{1-}. Note that while in the Kyle-Back model the insider knows P_{1+}, equivalently Y_{1+}, in this case she knows Y_{1-}. In [Dan10] and in [Wu99], models with increasing amount of information are considered. A weaker sense of equilibrium is considered defining rational prices as $P_t = E\left[P_{1-}|\mathcal{F}_t^Y\right]$ with P_{1-} and P_1 being of the same law and defining the expected combined profit of the informed and the uninformed traders as

$$E\left[(h(S_1,1) - h((Y_s)_{s<1}, 1-))Y_1 + \int_0^{1-} Y_t dh((Y_s)_{s<1}, s \leq t)\right]$$

is minimal.

Three related models are studied: noisy information, t.i. instead of V, the insider observes $V+\varepsilon$, where ε is a zero mean Gaussian random variable independent of V; delayed information, meaning that until some time $t_0 \in (0,1)$, the informed trader does not have any extra information; and two insiders on the market with different degrees of information.

Part III

Ambit Processes

Chapter 4

Ambit Processes and their applications

In Part III, ambit processes and their applications are studied. I present the recent articles: [CFV14] summarizing the research done so far about ambit processes and [CFSV13] about a short rate model using ambit processes.

4.1 Introduction

Ambit processes are used to model spheres of influence, especially in turbulence, electricity prices, risk management and derivative pricing. It was first introduced in [BNS07], and applied in [BNBV10a, BNBV10b, BNCP11, CFV14, CFSV13]. First, consider a partial differential operator

$$Lf = \frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2}, \text{ with } Lu = \varphi, u(0,x) = 0,$$

then there is a function G on $(\mathbb{R}_+, \mathbb{R})$ such that the solution can be written as

$$u(t,x) = \int_{\mathbb{R}_+ \times \mathbb{R}} G(t-s, x-y)\varphi(s,y)\mathrm{d}s\mathrm{d}y.$$

Now, consider that $\phi = W$, an L^2-noise in $\mathbb{R}_+ \times \mathbb{R}$, t.i.

$$\begin{aligned} W : \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}) &\longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}) \\ A &\longmapsto W(A), \end{aligned}$$

such that $W(\emptyset) = 0$ a.s and for all disjoint and bounded sets $A_1, A_{2,}...$ in $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$, $W(A_i)$ are independent and

$$W(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} W(A_i), a.s.$$

and where the convergence of the series is in $L^2(\mathbb{P})$. Then it is natural to consider that the solution of the proposed differential equation is given by

$$u(t,x) = \int_{\mathbb{R}_+ \times \mathbb{R}} G(t-s, x-y) W(\mathrm{d}s, \mathrm{d}y).$$

That is how the relation of one point to others of the space-time set is described, and this is the motivation for its definition:

Definition 4.1.1 *A tempo-spatial ambit field is defined as*

$$Y(t,x) = \int_{A(t,x)} g_{(t,x)}(s,\xi)\sigma(s,\xi)W(ds,d\xi) + \int_{B(t,x)} q_{(t,x)}(s,\xi)a(s,\xi)dsd\xi,$$

where $t \geq 0$, $x \in \mathbb{R}^n$, and $\mu \in \mathbb{R}, \xi \in \mathbb{R}^n$, W is a σ-finite, L^2-valued measure, $g_{(t,x)}(\cdot)$ and $q_{(t,x)}(\cdot)$ are deterministic kernels, $\sigma(\cdot,\cdot) \geq 0$ and $a(\cdot,\cdot)$ are predictable random fields and $A(t,x) \subseteq \mathbb{R}^{n+1}$ and $B(t,x) \subseteq \mathbb{R}^{n+1}$ are ambit sets. Then, $X_t = Y_t(x(t))$, for a curve $x(t)$ is an ambit process.

As mentioned before, ambit sets can be seen as areas of influence, with the only condition being that future cannot influence the past: as the ambit field $Y(t,x)$ depends on an ambit set containing points prior to time t. If W is a Gaussian noise, then $Y(t,y)$ is called a Brownian semi-stationary field (BSS), and if it is a Lévy noise, then Lévy semi-stationary field (LSS). A detailed discussion and applications are included in the next Section, summarizing the results of [CFV14].

The ambit fields used in practice are of the form

$$\begin{aligned} Y(t,x) &= \mu + \int_{A(t,x)} g_x(t-s,\xi)\sigma(s,\xi)W(\mathrm{d}s,\mathrm{d}\xi) \\ &\quad + \int_{B(t,x)} q_x(t-s,\xi)a(s,\xi)\mathrm{d}s\mathrm{d}\xi, \quad t \geq 0, x \in \mathbb{R}^n, \end{aligned}$$

where $A(t,x) = A + (t,x)$ and $B(t,x) = B + (t,x)$ with A and B containing only negative time coordinates (because of the causality principle). The part

$$X_t := \int_{A(t,x)} g_x(t-s,\xi)\sigma(s,\xi)W(\mathrm{d}s,\mathrm{d}\xi)$$

is called the core of Y, and σ the intermittency, volatility or modulating field or process. Consider the following specific case.

$$X_t = \int_{-\infty}^{t} g(t-s)W(ds), \tag{4.1}$$

where W is a Gaussian white noise in $\mathbb{R}$, σ an adapted càdlàg process and $g \in L^2(\mathbb{R}_+)$. An important fact is that X is not necessarily a semi-martingale, because g' may not be square integrable in the neighborhood of 0. We can get, by formal differentiation

$$\mathrm{d}X_t = g(0+)\mathrm{d}W(t) + \Big(\int_{-\infty}^{t} g'(t-s)W(ds)\Big)\mathrm{d}t,$$

and also that the necessary and sufficient condition for X_t to be a semi-martingale, are $g(0+) < \infty$ and $g' \in L(\mathbb{R}_+)$.

4.2 Applications of Ambit processes

In this section, we present the results of [CFV14] summarizing the research done so far about ambit processes. We start with models of turbulence. We refer to [Hed12, BNS07, BNS09] for a description of the approach studied in the article. The main component of velocity is described by

$$Y_t = \mu + \int_{-\infty}^{t} g(t-s)\sigma_s W(\mathrm{d}s) + \int_{-\infty}^{t} q(t-s)a_s \mathrm{d}s,$$

where μ is a constant, W is a Gaussian white noise on $\mathbb{R}$, g and q are nonnegative deterministic functions on $\mathbb{R}$, with $g\,(t) = q\,(t) = 0$ for $t \leq 0$, and σ and a are adapted càdlàg processes. Now, consider that $q(\cdot) \equiv 0$, and that

$$\int_{-\infty}^{t} g^2(t-s)\sigma_s^2 \mathrm{d}s < \infty, \quad \text{a.s.}$$

and also that the function g is continuously differentiable on $(0, \infty)$, $|g'|$ is non-increasing on (b, ∞) for some $b > 0$ and $g' \in L^2((\varepsilon, \infty))$ for any $\varepsilon > 0$. Moreover, we assume that for any $t > 0$

$$F_t = \int_{1}^{\infty} (g'(s))^2 \sigma_{t-s}^2 ds < \infty, \quad \text{a.s..}$$

See [BNCP11], for a discussion of this conditions. Realized multipower variation plays an important role when estimating σ, therefore, define it by

$$\sum_{i=1}^{[nt]-k+1} \prod_{j=1}^{k} |\Delta_{i+j-1}^n Y|^{p_j} \ , \ \text{where}\ \Delta_i^n Y = Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}} \ , \ \text{and}\, p_1, \ldots, p_k \geq 0 \ ,$$

for some fixed number $k \geq 1$.

Suppose that Y is observed at time points $t_i = i/n, i = 1, \dots, [nt]$ and that G is given by (4.1), as well. We are interested in the asymptotic behavior of the functionals

$$V(Y, p_1, \dots, p_k)^n_t = \frac{1}{n\tau_n^{p+}} \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^{k} |\Delta^n_{i+j-1} Y|^{p_j} , \qquad p_1, \dots, p_k \geq 0 ,$$

where $\Delta_i^n Y = Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}}$ and $\tau_n^2 = \bar{R}(1/n)$ with $\bar{R}(t) = \mathbb{E}[|G_{s+t} - G_s|^2]$, $t \geq 0$ and when n goes to infinity. Its asymptotic behavior is described by a Central Limit Theorem (see Theorem 1 in [CFV14]).

A bond market with a bond (as introduced later in Section 4.3.1) is modeled with

$$r_t = \int_{-\infty}^{t} g(t-s)\sigma_s W(\mathrm{d}s) + \mu_t \tag{4.2}$$

where W is an $(\mathcal{F}_t)$-Gaussian noise in $\mathbb{R}$ under the risk neutral probability, $\mathbb{P}^* \sim \mathbb{P}$, g is a deterministic function on $\mathbb{R}_+$, $g \in L^2((0,\infty))$, and $\sigma \geq 0$ and μ are also deterministic, under the assumption

$$\int_{-\infty}^{t} g^2(t-s)\sigma_s^2 \mathrm{d}s < \infty \quad \text{a.s.}$$

which ensures that r_t is well defined. For the summary of the results, see Section 4.3.

Ambit processes can be used to model Energy markets in the following way. Due to the properties of such markets (the fact that electricity spots cannot be stored in most of the cases, the possible presence of arbitrage and other empirical experience, discussed in more details in [CFV14]), we model spot and forward prices in the following way.

In the log-spot price $Y_\cdot$ is modeled by means of the *Lévy Semi-stationary Processes* (LSS), *i.e.*,

$$Y_t := \mu + \int_{-\infty}^{t} g(t-s)\sigma_s \mathrm{d}L_s + \int_{-\infty}^{t} q(t-s)a_s \mathrm{d}s,$$

where μ is a constant, $(L_t)_{t\in\mathbb{R}}$ is a two-sided Lévy process, g and q are non-negative deterministic functions on $\mathbb{R}$, with $g(t) = 0 = q(t)$ for $t \leq 0$, and $\sigma_\cdot$ and $a_\cdot$ are two càdlàg processes.

Consider a forward contract of delivering electricity at time T, for a predetermined price $F_t(T)$, the forward price, fixed today but payable at T with no other cash flow at $t < T$. It is fixed in such a way that the price of the contract, at the issue time t, is zero. Then by definition

$$0 = \mathbb{E}_{\mathbb{P}^*}\left[\exp\left\{ -\int_t^T r_u \mathrm{d}u \right\} (\exp\{Y_T\} - F_t(T)) \middle| \mathcal{F}_t \right] .$$

Then, it is shown that the price is of the form

$$F_t(T) = C(T) \exp\left\{ \int_{-\infty}^{t} g(T-s)\mathrm{d}W_s - \frac{1}{2}\int_{-\infty}^{t} g^2(T-s)\mathrm{d}s \right\}. \tag{4.3}$$

When modeling forward prices, instead of deducing it from the spot price, the forward price is modeled directly, supposing

$$\log F_t(x) := \int_{A(t,x)} g(\xi, t-s, x)\sigma_s(\xi) L(\mathrm{d}\xi, \mathrm{d}s),$$

where the spatial component in this formula models the time to maturity, *i.e.*, $x := T-t$, the ambit set is given by $A(t,x) := A_t := \{(\xi, s) \ : \ \xi > 0, s \leq t\}$, and the kernel g may be chosen in order to capture the so-called *Samuelson effect* (see [Sam65]). Traditionally, the forward price is modeled as a semi-martingale such that there is an equivalent martingale measure under which the price dynamics becomes a (local) martingale. According to Corollary 1 in [BNBV10a], $(F_t(T))_{t\in\mathbb{R}}$ is an $\mathbb{F}^L$-martingale if and only if the kernel g is deterministic and does not depend on t. For instance, one can consider

$$\log F_t(T-t) = \int_{A_t} \exp\{-\alpha(\xi + T - s)\}\sigma_s(\xi) W(\mathrm{d}\xi, \mathrm{d}s), \tag{4.4}$$

where $\alpha > 0$ and W a homogeneous Gaussian Lévy basis. Such rather strong condition rules out many interesting more general ambit fields, however, it still includes some *CARMA* and standard models.

4.3 A short rate model using ambit processes

In this section, I summarize the results of the paper [CFSV13]. First, find an introduction to the short rate models in general, then the results of the model of Bonds using ambit processes are presented.

4.3.1 Interest rate models

For a detailed discussion about interest rate models, see [Bjö98], whose line is followed in this introduction. A bond is a financial security that promises to pay a fixed, known income stream in the future. They can be characterized by their maturity date, face, par or principal value, coupon rate and number of coupon payments/year. We differentiate between zero coupon bonds, that pay no interest and coupon bonds, that pay fixed coupon at known times. Denote the underlying short term interest rate process by r_t.

Definition 4.3.1 *A zero coupon bond with maturity date T, is a contract which guarantees the holder 1 dollar to be paid on the date T. The price at time t of a bond with maturity date T is denoted by $P(t,T)$.*

Assume that there exists a market for zero coupon bonds for every $T > 0$, $P(t,t) =$ holds for any t and that for each fixed t, the bond price $P(t,T)$ is differentiable with respect to T. Let us fix S and T with $t \leq S \leq T$. Then the rate of return over the interval $[S,T]$ can be obtained as follows: at time t, we sell a zero coupon bond with maturity S, which will give us $P(t,S)$ dollars, what we use to buy $P(t,S)/P(t,T)$ bonds with maturity T, so that the net investment at t equals zero. Then time S we are obliged to put 1 dollar, and at time T we receive $P(t,S)/P(t,T)$ dollars, meaning that the investment of 1 dollar at time S has yielded $P(t,S)/P(t,T)$ dollars at time T. The simple forward rate (or LIBOR rate) is the solution of the equation

$$1 + (T-S)L = \frac{P(t,S)}{P(t,T)},$$

and the continuously compounded forward rate R is the solution of the equation

$$e^{R(T-S)} = \frac{P(t,S)}{P(t,T)}.$$

Then, we can define the following rates:

- the simple forward rate for $[S,T]$

$$L(t,S,T) = \frac{P(t,T) - P(t,S)}{(T-S)P(t,T)}$$

- the simple spot rate for $[S,T]$

$$L(S,T) = L(S,S,T) = \frac{P(S,T) - 1}{(T-S)P(S,T)}$$

- the continuously compounded forward rate for $[S,T]$

$$R(t,S,T) = -\frac{\log P(t,T) - \log P(t,S)}{T-S}$$

- the continuously compounded forward spot rate for $[S,T]$

$$R(t,S,T) = R(S,S,T) = -\frac{\log P(t,T)}{T-S}$$

- the instantaneous forward rate with maturity T, the limit of the continuously compounded forward rate as S tends to T

$$f(t,T) = -\frac{\partial \log P(t,T)}{\partial}$$

- the instantaneous short rate at time t

$$r(t) = f(t,t)$$

Then, we can define the money account as

$$B_t = \exp\left\{\int_0^t r(s)ds\right\},$$

and also can observe from the definitions that we have

$$P(t,T) = P(t,s)\exp\left\{-\int_s^T f(t,u)du\right\} = \exp\left\{-\int_t^T f(t,u)du\right\},$$

Consider short rate dynamics like

$$dr_t = a(t)dt + b(t)dW(t),$$

bond price dynamics like

$$dP(t,T) = P(t,T)m(t,T)dt + P(t,T)V(t,T)dW(t)$$

and forward rate dynamics like

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t),$$

where W is a Brownian motion. In case of allowing W to be vector valued, so are $v(t,T)$ and $\sigma(t,T)$. Assume that $m(t,T)$, $v(t,T)$, $\alpha(t,T)$ and $\sigma(t,T)$ are continuously differentiable in T, and that all processes are smooth enough to allow us differentiate under the integral and to change the order of integration. Denote the derivative of any function $f(\cdot)$ with respect to the variable T by $f_T(\cdot) = \frac{\partial f}{\partial T}(\cdot)$. Then we have the following relations between the dynamics just introduced:

- given the dynamics of $P(t,T)$ as above, for the forward rate dynamics, we have α and σ given by

$$\begin{cases} \alpha(t,T) = v_T(t,T)/v(t,T) - m_T(t,T), \\ \sigma(t,T) = -v_T(t,T), \end{cases}$$

- given the dynamics of $f(t,T)$ as above, for the short rate dynamics, we have a and b given by

$$\begin{cases} a(t) = f_T(t,t) + \alpha(t,t), \\ b(t) = \sigma(t,t), \end{cases}$$

- given the dynamics of $f(t,T)$ as above, $P(t,T)$ satisfies

$$dP(t,T) = P(t,T)\left\{r(t) + A(t,T) + \frac{1}{2}\|S(t,T)\|^2 dt + P(t,T)S(t,T)dW(t)\right\}$$

where $\|\cdot\|$ stands for the Euclidian norm and

$$\begin{cases} A(t,T) = -\int_t^T \alpha(t,s)ds, f_T(t,t) + \alpha(t,t), \\ B(t,T) = -\int_t^T \sigma(t,s)(t,s)ds. \end{cases}$$

Modeling the prices' dependence on the short rate of interest, consider the following SDE form:

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t), \tag{4.5}$$

with the necessary regularity assumptions on μ and σ to have a strong solution. Assume that there exists a market of zero-coupon bonds for any T, there is no arbitrage on the market, r follows the dynamics above and the price process B of the money account is given by

$$dB(t) = r(t)B(t)dt,$$

and that the price of a bond has the form

$$P(t,T) = F(t, r(t), T). \tag{4.6}$$

Considering a zero-coupon bond that is priced $P(t,T)$ at time $t < T$ to deliver $P_0(T,T) = 1$ at time T, in case of r_t being deterministic, we have

$$P(t,T) = e^{-\int_t^T r_s ds} \quad 0 \le t \le T,$$

and in case of r_t being an $\mathcal{F}_t$-adapted random process, it is given by

$$P(t,T) = \mathbb{E}^*\left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right] \quad 0 \le t \le T$$

under a risk-neutral measure $\mathbb{P}^*$. Then, it can be shown that the process

$$\tilde{P}(t,T) = e^{-\int_t^T r_s ds} P(t,T)$$

is a martingale under $\mathbb{P}^*$. Assume that

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dB_t,$$

where (B_t) is a standard Brownian motion under $\mathbb{P}^*$. Then, knowing that r_t is of the form (4.5) and F is given by (4.6), by Itô's formula, omitting writing the third variable of F, we have

$$xF(t,x) = \frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x}(t,x) + \sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x), \quad t, x \in \mathbb{R}, t \geq 0$$

with the boundary condition $F(T,x) = 1$, $x \in \mathbb{R}$, which implies

$$\frac{dP(t,T)}{P(t,T)} = r_t dt + \sigma(t, r_t)\frac{\partial \log F}{\partial x}(t, r_t)dB_t.$$

In case of a Vasicek model, t.i. r_t given as

$$dr_t = (a - br_t)dt + \sigma dB_t,$$

we get

$$\frac{dP(t,T)}{P(t,T)} = r_t dt - \frac{\sigma}{b}\left(1 - e^{-b(T-t)}\right) dB_t,$$

and generally, the pricing formula is often of the form

$$P(t,T) = e^{A(T-t)+C(T-t)r_t}.$$

The most important models of short rate are:

- Vasicek
$$dr_t = (b - ar_t)dt + \sigma dB_t$$

- Cox-Ingersoll-Ross (CIR)
$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dB_t$$

- Dothan
$$dr_t = ar_t dt + \sigma r_t dB_t$$

- Black-Derman-Toy
$$dr_t = \theta_t r_t dt + \sigma_t dB_t$$

- Ho-Lee
$$dr_t = \theta_t dt + \sigma dB_t$$

- Hull-White (extended Vasicek)

$$dr_t = (\theta_t - a_t r_t)dt + \sigma_t dB_t$$

- Hull-White (extended CIR)

$$dr_t = (\theta_t - a_t r_t)dt + \sigma_t \sqrt{r_t} dB_t.$$

The model possesses an Affine Term Structure, if F is given (4.6) and of the form

$$F(t, r_t, T) = e^{A(t,T) - B(t,T) r_t},$$

where A and B are deterministic functions. Then, it can be shown that the following equation must hold

$$A_t(t, T) - (1 + B_t(t, T)) r_t - \mu(t, r_t) B(r, T) + \frac{1}{2} \sigma^2(t, r) B^2(t, T) = 0.$$

Also, the boundary condition $F(T, r, T) = 1$ implies $A(T, T) = B(T, T) = 0$. Then, it can be shown that in case of μ and σ having the form

$$\begin{cases} \mu(t, r) = \alpha(t) r + \beta(t), \\ \sigma(t, r) = \sqrt{\gamma(t) r + \delta(t)}, \end{cases}$$

then, the following equations satisfy:

$$\begin{cases} B_t(t, T) + \alpha(t) B(t, T) - \frac{1}{2}\gamma(t) B^2(t, T) = 1, \\ B(T, T) = 0 \end{cases}$$

and

$$\begin{cases} A_t(t, T) = \beta(t) B(t, T) - \frac{1}{2}\gamma(t) B^2(t, T), \\ A(T, T) = 0 \end{cases}$$

Another model or method proposed by Heath-Jarrow-Morton (HJM) chooses the entire forward rate curve as state variable. It is described as follows. Assume that, for a fixed $T > 0$, the (instantaneous) forward rate $f(\cdot, T)$ has the following dynamics:

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW(t), \text{ and } f(0, T) = f^*(0, T),$$

where W is a d-dimensional Brownian motion and $\alpha(\cdot, T)$ and $\sigma(\cdot, T)$ are adapted processes. Then, we have the following results:

Theorem 4.3.1 (HJM drift condition) *Assume that the family of forward rates is given as above and that the induced bond market is arbitrage free. Then, there exist a d-dimensional column-vector process*

$$\lambda'(t) = [\lambda_1(t), \dots, \lambda_d(t)]^T$$

with, for all $T \geq 0$ and $t \leq T$

$$\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,s)^T ds - \sigma(t,T)\lambda(t),$$

where A^T denotes the transpose of A.

If the dynamics of f is given under a martingale, then, the bond prices are given by

$$p(0,T) = \exp\left\{-\int_0^T f(0,s)ds\right\} = E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\}\right],$$

where $r(t) = f(t,t)$. Then, we have:

Theorem 4.3.2 (HJM drift condition) *Under the martingale measure Q, the processes α and σ must satisfy the following relation for every t and $T \geq t$:*

$$\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,s)^T ds.$$

4.3.2 Results

In the following, bond markets are studied with short rates evolving as

$$r_t = \int_{-\infty}^t g(t-s)\sigma_s W(\mathrm{d}s),$$

where g is a real-valued, deterministic function, so is $\sigma \geq 0$ and W is a stochastic Wiener measure. The aim was to extend popular market models like the Vasicek model to these markets. Forward rates and bond prices are calculated with the result of having

$$\begin{aligned} P(t,T) &= \exp\left(A(t,T) - \int_{-\infty}^t \sigma_u c(u;t,T) W(\mathrm{d}u)\right), \text{ with} \\ A(t,T) &= \frac{1}{2}\int_t^T \sigma_u^2 c^2(u;u,T)\mathrm{d}u - \int_t^T \mu_s \mathrm{d}s, \end{aligned}$$

and it satisfies the HJM conditions and α and σ are given by

$$\begin{aligned}\sigma(t,T) &= \sigma_t g(T-t),\\ \alpha(t,T) &= \sigma_t^2 g(T-t)c(t;t,T) \text{ with}\\ c(u;t,T) &= \int_t^T g(s-u)\mathrm{d}s,\ t\geq u.\end{aligned}$$

Then, it is shown that

$$\begin{aligned}\tilde{P}(t,T) &:= \frac{P(t,T)}{\exp\left\{\int_0^t r_s \mathrm{d}s\right\}}\\ &= P(0,T)\exp\left(-\int_0^t \sigma_u c(u;u,T)W(\mathrm{d}u) - \frac{1}{2}\int_0^t \sigma_u^2 c^2(u;u,T)\mathrm{d}u\right).\end{aligned}$$

Therefore,

$$\mathrm{d}\tilde{P}(t,T) = -\tilde{P}(t,T)\sigma_t c(t;t,T)W(\mathrm{d}t),\ t\geq 0.$$

Let X be a P^*-square integrable, $\mathcal{F}_T$-measurable payoff. Consider the $(\mathcal{F}_t)$-martingale

$$M_t := E_{P^*}\left(X|\mathcal{F}_t\right), t\geq 0,$$

then by an extension of Brownian martingale representation theorem we can write

$$\mathrm{d}M_t = H_t W(\mathrm{d}t),$$

where H is an adapted square integrable process. Having (ϕ_t^0, ϕ_t^1) being a self-financing portfolio built with a bank account and a T-bond, its value process is given by

$$V_t = \phi_t^0 e^{\int_0^t r_s \mathrm{d}s} + \phi_t^1 P(t,T),$$

and, by the self-financing condition, the discounted value process $\tilde{V}_\cdot$, satisfies

$$\mathrm{d}\tilde{V}_t = \phi_t^1 \mathrm{d}\tilde{P}(t,T).$$

So, if we take

$$\phi_t^1 = -\frac{H_t}{\tilde{P}(t,T)\sigma_t c(t;t,T)}$$

we can replicate X. In particular the bond with maturity T^* can be replicated by taking

$$\frac{P(t,T^*)c(t;t,T^*)}{P(t,T)c(t;t,T)}$$

bonds with maturity time $T\geq T^*$.

Then, consider a bond with maturity $\bar{T} > T,$ where T is the maturity time of a call option for this bond with strike K. Its price is given by (see [Bjö98], Chapter 19)

$$\Pi(t;T) = P(t,\bar{T})P^{\bar{T}}(P(T,\bar{T}) \geq K|\mathcal{F}_t) - KP(t,T)P^{T}(P(T,\bar{T}) \geq K|\mathcal{F}_t)$$

and is shown to equal

$$\begin{aligned}
\Pi(t;T) &= P(t,\bar{T})\Phi(d_+) - KP(t,T)\Phi(d_-), \text{ where} \\
d\pm &= \frac{\log\frac{P(t,\bar{T})}{KP(t,T)} \pm \frac{1}{2}\Sigma^2_{t,T,\bar{T}}}{\Sigma_{t,T,\bar{T}}}, \text{ and} \\
\Sigma^2_{t,T,\bar{T}} &:= \int_t^T \sigma_u^2 c(u;T,\bar{T})^2 \mathrm{d}u.
\end{aligned}$$

It can be straightforwardly applied to the case $g(t) = e^{-bt}$, $\sigma_u = \sigma$, $\mu = a$ (Vasicek model) getting

$$\begin{aligned}
P(t,T) &= \exp\left(A(t,T) + aB(t,T) - r_t B(t,T)\right), \text{ with} \\
A(t,T) &= \frac{\sigma^2}{2}\int_t^T B(u,T)^2 \mathrm{d}u - a(T-t) \\
B(t,T) &= \frac{1}{b}(1 - e^{-b(T-t)})
\end{aligned}$$

Here

$$c(u;t,T) = \frac{1}{b}\left(e^{-b(t-u)} - e^{-b(T-u)}\right), \; u \leq t \leq T,$$

and

$$\operatorname{var}\left(-\frac{1}{T-t}\log P(t,T)\right) = \frac{\sigma^2}{2b^3}\frac{(1-e^{-b(T-t)})^2}{(T-t)^2} \sim T^{-2},$$

when $T \to \infty$. The corresponding instantaneous forward rates are given by

$$f(t,T) = -\frac{\sigma^2}{2b^2}\left(1 - e^{-b(T-t)}\right)^2 + \sigma e^{-b(T-t)}(r_t - a) + a.$$

$$\operatorname{var}\left(f(t,T)\right) = \sigma^2 \int_{-\infty}^t e^{-2b(T-u)}\mathrm{d}u = \frac{\sigma^2}{2b}e^{-2b(T-t)} \sim e^{-2bT},$$

when $T \to \infty$. Moreover the volatility of the forward rates is given by $\sigma(t,T) = \sigma e^{-b(T-t)}$ and this is not too realistic.

Also, consider the case of

$$g(t-u) = e^{-b(t-u)} \int_0^{t-u} e^{bs} \beta s^{\beta-1} \mathrm{d}s.$$

Then, we have that

$$\begin{aligned} c(u;t,T) &= c(0;0,T-u) - c(0;0,t-u), \quad \text{with} \\ c(0;0,x) &= e^{-bx} \int_0^x e^{bs} s^{\beta} \mathrm{d}s. \end{aligned}$$

Then

$$\operatorname{var}\left(-\frac{1}{T-t}\log P(t,T)\right) \sim \frac{1}{T^2}\int_0^t c(0;0,T-u)^2 \mathrm{d}u \sim T^{2\beta-2},$$

when $T \to \infty$. In fact

$$c(0;0,x) = e^{-bx}\int_0^x e^{bs} s^{\beta} \mathrm{d}s = x^{\beta}\int_0^x e^{-bs}(1-\frac{s}{x})^{\beta}\mathrm{d}s,$$

and by the monotone convergence theorem

$$\lim_{x\to\infty}\int_0^x e^{-bs}(1-\frac{s}{x})^{\beta}\mathrm{d}s = \int_0^{\infty} e^{-bs}\mathrm{d}s = \frac{1}{b}.$$

Moreover

$$\operatorname{var}\left(f(t,T)\right) = \int_{-\infty}^t \sigma_u^2 g^2(T-u)\mathrm{d}u \sim T^{2\beta-2}.$$

Since for $x \geq 0$

$$\begin{aligned} g(x) &= e^{-bx}\int_0^x e^{bs}\beta s^{\beta-1}\mathrm{d}s = \beta x^{\beta-1}\int_0^x e^{-bs}(1-\frac{s}{x})^{\beta-1}\mathrm{d}s \\ &= \beta x^{\beta-1}\left(\int_0^{x/2} e^{-bs}(1-\frac{s}{x})^{\beta-1}\mathrm{d}s + \int_{x/2}^x e^{-bs}(1-\frac{s}{x})^{\beta-1}\mathrm{d}s\right), \end{aligned}$$

and

$$\lim_{x\to\infty}\int_0^{x/2} e^{-bs}(1-\frac{s}{x})^{\beta-1}\mathrm{d}s = \int_0^{\infty} e^{-bs}\mathrm{d}s = \frac{1}{b},$$

$$\begin{aligned} \int_{x/2}^x e^{-bs}(1-\frac{s}{x})^{\beta-1}\mathrm{d}s &\leq e^{-bx/2}\int_{x/2}^x (1-\frac{s}{x})^{\beta-1}\mathrm{d}s \\ &= xe^{-bx/2}\int_0^{1/2} v^{\beta-1}\mathrm{d}v = \frac{xe^{-bx/2}}{\beta 2^{\beta}} \to 0, \end{aligned}$$

when $x \to \infty$. Also observe that the volatility of the forward rates $\sigma(t,T) = \sigma^2 g(T - t) \sim T^{\beta-1}$, when $T \to \infty$, that is more realistic than the exponential decay in the Vasicek model. For $\beta \in (-1/2, 0)$ consider the memory function

$$g(x) = e^{-bx}x^\beta + \beta \int_0^x (e^{-b(x-u)} - e^{-bx})u^{\beta-1}\mathrm{d}u,$$

and then

$$g(x) \sim x^{\beta-1}$$

when $x \to \infty$. In such a way that we obtain analogous asymptotic results to the previous case.

Ambit processes as noises of SDE are considered, as well, supposing we have processes like

$$W_t^g = \int_{-\infty}^t g(s,t)W(ds),$$

where $g(s,t)$ is a real-valued, deterministic function, continuously differentiable with respect to the second variable, equals 0 for $s > t$ and satisfies

$$\int_{-\infty}^t g^2(s,t)ds < \infty.$$

Stochastic calculus with respect to these processes is developed, as well, using the kernel $K_t^g(\cdot)(s,t)$ given by

$$K_t^g(f)(s,t) := \int_s^t (f(u,t) - f(s,t))\, \partial_u g(s,u)\mathrm{d}u + f(s,t)g(s,t).$$

Prices of defaultable zero coupon bonds are studied, as well. Knowing that the arbitrage-free price is given by

$$D(t,T) = 1_{\{\tau>t\}} E\left(1_{\{\tau>T\}} e^{-\int_t^T r_s \mathrm{d}s} \middle| \mathcal{G}_t \right),\ 0 \le t \le T,$$

where the expectation is taken with respect to a risk-neutral probability and the filtration $(\mathcal{G}_t)_{t\ge0}$ is the information available on the market. Then, an extension of the Vasicek model is

$$\begin{aligned} \mathrm{d}r_t &= b(a - r_t)\mathrm{d}t + \sigma \mathrm{d}W(t), \\ \mathrm{d}\lambda_t &= \breve{b}(\breve{a} - \lambda_t)\mathrm{d}t + \breve{\sigma}\mathrm{d}\breve{W}(t), \end{aligned}$$

where W and $\breve{W}$ are correlated Brownian motions, and the price of a zero coupon bond is

$$D(t,T) = 1_{\{\tau>t\}} \exp\left(A(t,T) - \int_{-\infty}^t (\sigma_u c(u;t,T)W(\mathrm{d}u) + \breve{\sigma}_u \breve{c}(u;t,T))\breve{W}(\mathrm{d}u) \right),$$

where

$$\begin{aligned}A(t,T) &= \frac{1}{2}\int_t^T \left(\sigma_u^2 c^2(u;t,T) + \breve{\sigma}_u^2 \breve{c}^2(u;t,T) + 2\rho\sigma_u\breve{\sigma}_u c(u;t,T)\breve{c}(u;t,T)\right)\mathrm{d}u \\ &\quad - \int_t^T (\mu_u + \breve{\mu}_u)\,\mathrm{d}u.\end{aligned}$$

and ρ is the correlation coefficient between W and $\breve{W}$.

An analogous of a CIR model is considered, supposing, to avoid negative short rates, that

$$r_t = \sum_{i=1}^{d}\left(\int_0^t g(t-s)\sigma_s \mathrm{d}W_i(s)\right)^2 + r_0, \quad t \geq 0, r_0 > 0.$$

where $(W_i))_{1\leq i\leq d}$ is a Brownian motion in $\mathbb{R}^d$. Then $P(0,T)$ can be rewritten as

$$\left(1+\sum_{n=1}^{\infty}\frac{(2T)^n}{n!}\int_0^1\cdots\int_0^1\begin{vmatrix} R(s_1,s_1) & \cdots & R(s_1,s_n) \\ \vdots & & \vdots \\ R(s_n,s_1) & \cdots & R(s_n,s_n)\end{vmatrix}\mathrm{d}s_1\cdots\mathrm{d}s_n\right)^{-d/2},$$

where the integrand is called Fredholm determinant and

$$\begin{aligned}R(u,v) &= \sigma_{Tu}\sigma_{Tv}c_2(Tu,Tv;T(u\vee v),T) \text{ with} \\ c_2(u,v;t,T) &= \int_t^T g(s-u)g(s-v)\mathrm{d}s.\end{aligned}$$

Then, for r_t being a Bessel process, as well, as for the classical CIR model, price can be given explicitly and for cases, when a closed formula has not been found, a numerical method is presented using Nyström-type approximation for the Fredholm determinant. The computation cost of the approximation is of order $O(m^3)$ and a simple Matlab code is given, as well. Finally, the characterization of the dynamics of such an r_t is developed.

Part IV

Power Variation of stable processes

Chapter 5

Power Variation for α-stable processes

In this Part, I present an introduction to Power Variation processes Stable processes, and then the details of [CF10].

5.1 Introduction

Originally, quadratic variation and power variation were introduced in the context of studying the path behavior of stochastic processes, but recently it has been introduced for statistical inference for integrals based on Brownian motion, as done in [BNS03], [BNGS04] and [Woe05], for integrated processes and Itô integrals (see [CNW07] and [BNS02] respectively) and more general Lévy processes in [Woe03].

Let $(X_t)_{0\leq t\leq T}$ be a stochastic process and p any natural number. Then, the p-th power variation is defined as

$$\sum_{i=1}^{[nt]} \left| X_{\frac{t}{n}} - X_{\frac{t-1}{n}} \right|^p .$$

The realized p-th variation process of (X_t) is defined as follows. Let

$$\Pi = \{t_0, t_1, \ldots, t_m\} \text{ where } 0 = t_0 \leq t_1 \leq \cdots \leq t_m = t$$

be a partition of $[0, t]$. Then, the p-th variation of X over Π is

$$V_t^{(p)}(\Pi) = \sum_{k=1}^{m} \left| X_{t_k} - X_{t_{k-1}} \right|^p .$$

If $V_t^{(p)}(\Pi)$ converges as $\max_{1\leq k\leq m} |t_k - t_{k-1}|$ tends to 0, then it is the realized p-th variation process. For $p = 2$ and any X square-integrable martingale, t.i. for such X

right continuous martingale that satisfies $EX_t^2 < \infty$ for every $t \geq 0$, it coincides with the original definition of quadratic variation: $V_t^{(2)} = A_t$, where $X_t^2 = M_t + A_t$ is the Doob-Meyer decomposition of X_t^2 with M_t being a right-continuous martingale and A is predictable and increasing (see Theorem 5.8. from [KS91]).

5.2 Stable processes

Let $(S_t^\alpha)_{t\geq 0}$ be an α-stable Lévy process with $\alpha \in (0,2)$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}_t^\alpha$ denotes the σ-field generated by $\{S_s^\alpha : 0 \leq s \leq t\}$ and the null sets. As defined and seen in Section 1.1, S^α is a process with stationary, independent increments and càdlàg, which can be characterized by

$$E\left[e^{iuS_t^\alpha}\right] = \exp\left\{t \int \left[e^{iux} - 1 - iuh\left(x\right)\right] \nu\left(\mathrm{d}x\right)\right\},$$

where $h = 1_{[1,2)}\left(\alpha\right) 1_{|\cdot|<1}\left(x\right)$, and the Lévy measure $\nu\left(\mathrm{d}x\right)$ is of the form

$$\nu\left(\mathrm{d}x\right) = rx^{-1-\alpha} 1_{x>0}\left(x\right) + q\left(-x\right)^{-1-\alpha} 1_{x<0}\left(x\right),$$

with $r, q \geq 0$, $r + q > 0$ and where $r = q = 1$ if $\alpha = 1$. It follows that S^α is self-similar: S^α is of the same law as $t^{-1/\alpha} S_t^\alpha$, and that it has all the moments of the order less than α, and for $\alpha < 1$ all the sample paths are of bounded variation, while for $\alpha > 1$, they are of unbounded variation [Sat99]. Define the p-variation (or strong variation) of a real valued function on the interval $[a, b]$ as

$$\mathrm{Var}_p\left(f, [a,b]\right) := \sup_{\pi \in P} \left\{\sum_{i=1}^n \left|f\left(t_i\right) - f\left(t_{i-1}\right)\right|^p\right\}^{1/p},$$

where P denotes the partitions of the interval:

$$P = \left\{\{a = t_0 < t_1 < \cdots < t_n = b\} : n \in \mathbb{N}\right\}.$$

It is known that for a pure jump Lévy process, the p-variation is finite for $p > \beta$, where β is the Blumenthal-Getoor index defined as

$$\beta = \inf_{\gamma \geq 0} \int 1 \wedge |x|^\gamma \, \nu\left(\mathrm{d}x\right) < \infty,$$

so in this case, it is finite for $p > \alpha$. For continuous processes, if f is α-Hölder continuous, then it has a finite $1/\alpha$-variation on any finite interval. Also, from [You36], we know that the Riemann-Stieltjes integral $\int_a^b f \mathrm{d}g$ exists if $f, g \in C$ and have finite p-variation and q-variation (respectively) on $[a, b]$ and $\frac{1}{p} + \frac{1}{q} > 1$. Moreover,

$$\left|\int_a^b f \mathrm{d}g - f(a)(g(b) - g(a))\right| \leq c_{p,q} \mathrm{Var}_p(f, [a,b]) \mathrm{Var}_q(f, [a,b]),$$

where

$$c_{p,q} = \zeta\left(\frac{1}{q} + \frac{1}{p}\right) \text{ with } \zeta(x) = \sum_{n \geq 1} \frac{1}{n^x}.$$

Consider stochastic processes of the form

$$\int_0^t u_{s-} \mathrm{d}S_s^\alpha,$$

where the stochastic integral is a pathwise refinement-Riemann-Stieltjes integral if $\alpha \geq 1$, and is a Lebesgue-Stieltjes integral if $\alpha < 1$. From [You36], the refinement-Riemann-Stieltjes integral exists, if the trajectories of $(u_t)_{t\geq 0}$ have a finite q-variation on any finite interval for some $q < \alpha / \max\{0, \alpha - 1\}$. Denote the uniform convergence in probability on $[0,T]$ by u.c.p., and the supremum norm on $[0,T]$ by $\|\cdot\|_\infty$. Write

$$V_p^n(Z)_t = \sum_{i=1}^{[nt]} \left| Z_{\frac{i}{n}} - Z_{\frac{i-1}{n}} \right|^p,$$

for any $p > 0$ real, $n \in \mathbb{N}$, and for any stochastic process $(Z_t)_{t\geq 0}$. For $p > \alpha$, it has been proved [Lep76, HM76] that the non-normed power variation tends to the p-th power of the absolute values of the jumps of Z, so we are only interested in the case of $p < \alpha$, where the non-normed power variation leads to an infinite limit. The following theorem is proved in [CNW07] about the convergence of such a process, normalized in an appropriate way:

Theorem 5.2.1 (Theorem 1 in [CNW07]) *Suppose that $(u_t)_{t\geq 0}$ is a stochastic process with càdlàg trajectories and, if $\alpha \geq 1$, with bounded q-variation on any finite interval, where $q < \frac{\alpha}{\alpha - 1}$. Set*

$$Z_t = \int_0^t u_{s-} \mathrm{d}S_s^\alpha$$

and $(Y_t)_{t\geq 0}$ is a stochastic process which satisfies

$$m^{-1+p/\alpha} V_p^m(Y)_t \xrightarrow{u.c.p.} 0$$

as m tends to infinity. Then, for any $p < \alpha$,

$$m^{-1+p/\alpha} V_p^m(Z+Y)_t \xrightarrow{u.c.p} c_p \int_0^t |u_s|^p \mathrm{d}s,$$

as m tends to infinity.

The condition for Y is satisfied if it is Hölder continuous of order $\gamma \in (1/\alpha, 1]$, and also for some semimartingales with jumps: assume that Y has a Blumenthal-Getoor index β and that it has a canonical representation

$$Y = Y_0 + B(h) + Y^c + h * (\mu - \nu) + (x - h(x)) * \mu,$$

where Y^c denotes the continuous local martingale, μ the jump measure and ν its compensator. Assume that $\langle Y^c \rangle = 0$, and in addition, if $\beta < 1$, then $B(h) + (x - h) * \nu$, as well. Then, it can be shown that the condition is satisfied for $\alpha > \max\{\beta, p\}$.

The following theorem shows that the properly normalized fluctuations of the power variation, for $p \in (0, \alpha/2)$ have Gaussian asymptotic distribution. Denote

$$v_p^2 = \mathrm{Var}\left(|S_1^\alpha|^p\right),$$

for any $p \in (0, \alpha/2)$. Then:

Theorem 5.2.2 (Theorem 2 in [CNW07]) *Fix $0 < p < \frac{1}{\alpha}$ and assume $0 < \alpha < 2$, then*

$$\left(S_t^\alpha, n^{-1/2+p/\alpha} V_p^n (S^\alpha)_t - c_p t n^{1/2}\right) \xrightarrow{\mathcal{L}} \left(S_t^\alpha, v_p W_t\right),$$

as n tends to infinity, where $(W_t)_{t\in[0,T]}$ is a Brownian motion independent of the process S^α, and the convergence is in the space $D([0,T])^2$ equipped with the Skohorod topology.

Condition 5.2.1 *Assume that, for $\gamma > 0$, u satisfies*

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left| |u|^\gamma (\eta_{n,j}) - |u|^\gamma (\chi_{n,j}) \right| \xrightarrow{a.s.} 0$$

as n tends to infinity, for any $(\eta_{n,j})$ and $(\chi_{n,j})$ such that

$$0 \le \chi_{n,1} \le \eta_{n,1} \le \frac{1}{n} \le \chi_{n,2} \le \eta_{n,2} \le \frac{2}{n} \le \cdots \le \chi_{n,n} \le \eta_{n,n} \le T.$$

Under this condition, the following central limit theorem can be proved:

Theorem 5.2.3 (Theorem 3 in [CNW07]) *Let S^α be an α-stable Lévy process with $\alpha \in (0,2)$. Fix $0 < p < \alpha/2$ and suppose that $(u_t)_{t\in[0,T]}$ is a càdlàg stochastic process, measurable with respect to $\mathcal{F}_T^\alpha$, satisfying Condition 5.2.1 with $\gamma = p$ and if $\alpha \ge 1$, with bounded q-variation with $q < 2p$. Furthermore, we assume that the stochastic process Y satisfies*

$$m^{-1/2+p/\alpha} V_p^m (Y)_t \xrightarrow{u.c.p} 0$$

as m tends to infinity. Setting $Z_t = \int_0^t u_{s-} \mathrm{d}S_s^\alpha$, we obtain

$$\left(S_t^\alpha, n^{-1/2+p/\alpha} V_p^n \left(Z+Y\right)_t - c_p\sqrt{n}\int_0^t \left|u_{s-}\right|^p \mathrm{d}s\right) \xrightarrow{\mathcal{L}} \left(S_t^\alpha, v_p \int_0^t \left|u_{s-}\right|^p \mathrm{d}W_s\right),$$

as n tends to infinity, where $(W_t)_{t\in[0,T]}$ is a Brownian motion independent of $\mathcal{F}_T^\alpha$, and the convergence is in $D\left([0,T]\right)^2$.

If u is independent of S^α, then it leads to

$$\frac{n^{-1/2+p/\alpha} V_p^n \left(Z+Y\right)_t - c_p\sqrt{n}\int_0^t \left|u_{s-}\right|^p \mathrm{d}s}{\int_0^t \left|u_{s-}\right| 2^p \mathrm{d}s} \xrightarrow{\mathcal{L}} N\left(0,1\right)$$

The condition on Y is satisfied if its Hölder continuous of the order b with $p\left(b-1/\alpha\right) > 1/2$, and also if it is a jump semimartingale with Blumethal-Getoor index β and $\frac{\alpha}{2} > p > \frac{\alpha\beta}{2(\alpha-\beta)}$.

5.3 Extensions

Consider stochastic processes of the same form as before: $\mathrm{d}Z_t = u_{s-}\mathrm{d}S_s^\alpha$, where $\alpha \in (0,2]$, $(S_t^\alpha)_{t\geq 0}$ is an α-stable Lévy process defined on $(\Omega, \mathcal{F}, P)$, where $(\mathcal{F}_t)_{t\geq 0}$ is a right continuous increasing family of P-complete sub-σ-fields of $\mathcal{F}$, and the integral is an Itô integral. Instead of assuming that the trajectories of u have a finite q-variation on any finite interval for some $q < \alpha/\max\{0, \alpha-1\}$, now it is generalized to having

$$\int_0^t \left|u_s\right|^\alpha \mathrm{d}s < \infty.$$

Then, we have $S_0^\alpha = 0$ almost surely and for every $0 \leq s \leq t, \lambda \in \mathbb{R}$

$$E\left[e^{i\lambda(S_t^\alpha - S_s^\alpha)} \left|\mathcal{F}_s\right.\right] = e^{-(t-s)|\lambda|^\alpha}.$$

Note, that it is of independent increments, α-self-similar, i. e., $(S_{at}^\alpha) \sim \left(a^{1/\alpha} S_t^\alpha\right)$ for $a > 0$, and for $\alpha = 2$, S^α equals $\sqrt{2}$ times a Brownian motion. Suppose, that u is an $(\mathcal{F}_t)$-adapted càdlàg process such that

$$\int_0^t E\left[\left|u_s\right|^\alpha\right] \mathrm{d}s < \infty,$$

then, the integral $\int_0^t u_{s-}\mathrm{d}S_s^\alpha$ is well defined. Also, by [GM83], we have, for all $\lambda > 0$,

$$P\left(\left|\int_0^t u_{s-}\mathrm{d}S_s^\alpha\right|^p > \lambda\right) \leq \frac{C}{\lambda^{\alpha/p}} \int_0^t E\left[\left|u_s\right|^\alpha\right] \mathrm{d}s,$$

(where C stands for a generic constant) which implies ([CF10]), that for $p < \alpha$,

$$E\left[\left|\int_0^t u_{s-} \mathrm{d}S_s^\alpha\right|^p\right] \leq C_p \left(\int_0^t E\left[|u_s|^\alpha\right] \mathrm{d}s\right)^{p/\alpha}.$$

For the case $p > \alpha$, it is known (see [Lep76] and [HM76]) that the non-normed power variation tends to the p-th power of the absolute values of the jumps of Z. For the case $p < \alpha$, the following can be shown:

Theorem 5.3.1 (Theorem 1 in [CF10]) *Under these assumptions, Theorem 5.2.1 holds.*

As before, for the case of $0 < p < \alpha/2$, we have the following result.

Theorem 5.3.2 (Theorem 2 in [CF10]) *Under these assumptions, Theorem 5.2.2 holds.*

Condition 5.3.1 *Assume that for some $\gamma \in (0,1)$, u satisfies*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[nT]} E\left[\sup_{t,s \in [(i-1)/n, i/n]} |u_t - u_s|^\gamma\right] \to 0$$

as n tends to infinity.

Then, the following Central Limit Theorem can be proved:

Theorem 5.3.3 (Theorem 3 in [CF10]) *Under these assumptions, supposing that Condition 5.3.1 holds (omitting Condition 5.2.1), Theorem 5.2.3 holds writing $(Z_t)_{t\geq 0}$ instead of $(Z_t + Y_t)_{t\geq 0}$.*

Appendices

Appendix A

Kyle-Back's model with Lévy noise

Kyle-Back's model with Lévy noise

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Abstract

The continuous-time version of Kyle's model [7], known as the Back's model [2], of asset pricing with asymmetric information, is studied. A larger class of price processes and a larger classes of noise traders' processes are studied. The price process, as in Kyle's model, is allowed to depend on the path of the market order. The process of the noise traders' is considered to be an inhomogeneous Lévy process. The solutions are found with the use of a perturbation method. With the informed agent being risk-neutral, the price pressure is constant over time, and there is no equilibirium in the presence of jumps. If the informed agent is risk-averse, there is no equilibirium in the presence of either jumps or drift in the process of the noise traders'.

Keywords: Market microstructure; insider trading; stochastic control; Lévy processes; semimartingales.

1 Introduction

Models of markets with the presence of an insider, that is to say, a trader who has some kind of additional information, have a great literature. In the approaches, we can distinguish two fundamentally different ones. One

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approach is considering the market with a bond and some stocks with prices given exogenously by their dynamics. The other one follows the idea of [7] where the price of the risky asset is led by the demand of the informed trader through some pricing rule. In the second case, the aim is to find or characterize an equilibrium where the informed agent maximizes her profits and the prices are set in a competitive way. In between one can find the model described in [9], where a bond and two risky assets are considered, one risky asset with prices given exogenously and one priced as it is in [7] (and [2]). A more general model is studied in [10], where more risky assets are involved. Following the Kyle-Back approach, [5] find equilibrium in the market of zero coupon bonds with default, and so does [3] in a market with options. Also the present paper follows the Kyle-Back approach but considers a time continuous trading where the noise traders' dynamics are allowed to have jumps. We study the existence of equilibria in this market model in presence of an insider taking advantage of asymmetric information, and we also consider different types of insider attitude to risk: risk neutral and risk-adverse.

The paper is organized as follows. In the next Section, the model is described and we formulate the wealth process. In the third Section, one can find an analysis of equilibrium and of its existence.

2 The Model and Equilibrium

We consider a market with two assets: we have a risky asset S and a bank account with interest rate r equal to zero for the sake of simplicity. The period in which the participants trade is $[0,1]$. There is to be a public release of information at time 1. The announcement reveals the value of the risky asset, at which price it will trade afterwards (that is to say, at time $1+$). This value is denoted by V and it is assumed to be a random variable with finite expectation. The market is continuous in time and order driven. There are three kinds of traders. Noise or liquidity traders, who trade for liquidity or hedging reasons, the informed trader or insider, who is aware of the privilege information at time 0, and market makers, who set the price and clear the market.

All random variables are defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote the price of the stock at time t by P_t and $\mathbb{F}^P = \left(\mathcal{F}^P_t\right)_{0\leq t\leq 1}$ where $\mathcal{F}^P_t = \sigma(P_s, 0 \leq s \leq t)$ augmented with the $\mathbb{P}$-null sets, here and in the sequel we always consider $\mathbb{P}$-augmented filtrations. With Z we indicate the aggregate demand process of the noise traders. The model we consider is an extension of the one in [2], where Z is a Brownian motion with a fixed volatility, to a more general set of processes. In [1] the authors consider a noise trader's demand with time-varying volatility. In this paper we consider processes that may have a drift and jumps, as well. More precisely we assume that

$$\mathrm{d}Z_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t + \mathrm{d}L_t, t \in [0,1], Z_0 = 0. \tag{1}$$

where B is a Brownian motion, independent of V, and $\mu, \sigma : [0,1] \to \mathbb{R}$ are deterministic, càdlàg functions, and L is a pure jump Lévy process independent of V and B. We also assume that the process L can be expressed by

$$L_t = \int_0^t \int_{\mathbb{R}} x \tilde{M}(\mathrm{d}t, \mathrm{d}x),$$

where $\tilde{M}(\mathrm{d}t, \mathrm{d}x) = M(\mathrm{d}t, \mathrm{d}x) - v_t(\mathrm{d}x)\mathrm{d}t$ is the compensated Poisson random measure associated with L, and with intensity $v_t(\mathrm{d}x)$.

Let X be the demand process of the informed trader and let $\mathbb{F}^{V,P}$ denote her flow of information:

$$\mathbb{F}^{V,P} = \left(\mathcal{F}_t^{V,P}\right)_{0 \le t \le 1},$$

as, at time t, she knows V, as well as $\{P_s : 0 \le s \le t\}$, thus, X has to be adapted to the filtration $\mathbb{F}^{V,P}$, with

$$\mathcal{F}_t^{V,P} := \sigma(V, P_s, 0 \le s \le t),$$

generated by the random variable V and the process P. Because of the independency assumed before, B is an $\mathbb{F}^{V,Z}$-Brownian motion and L is an $\mathbb{F}^{V,Z}$-pure jump Lévy process as well. The informed trader tries to maximize her final wealth, that is, she is risk-neutral (one may find a model with risk averse informed traders in [6] and we also study them in Subsection 2.6). Denoting by $W_.$ the wealth process corresponding to the insider's portfolio, we have the following definition for optimality:

Definition 1 *A strategy X is called* optimal *with respect to a price process P if it maximizes $E(W_{1+})$.*

Moreover, the market makers "clear" the market by fixing a competitive or rational price, given by

$$P_t = \mathbb{E}(V|Y_s, 0 \le s \le t), t \in [0,1] \tag{2}$$

where $Y = X + Z$ is the total demand that market makers observe. Note that (P_t) is an $\mathbb{F}^Y$-martingale, where $\mathbb{F}^Y = \left(\mathcal{F}_t^Y\right)_{0 \le t \le 1}$ and $\mathcal{F}_t^Y = \sigma(Y_s, 0 \le s \le t)$. Formally:

Definition 2 *Given a trading strategy X (and total demand $Y = X + Z$), the price process P is* rational*, if it satisfies (2).*

In the original model of Kyle, the current price depends on the past demand, while in Back's one it is supposed to be Markovian, depending only on the current total demand. [6] shows that Back's results hold in the original settings with the current price depending on the whole path. We also consider this case. Suppose that market makers fix prices through a pricing rule, in terms of formulas,

$$P_t = H(t, \xi_t), t \in [0, 1]$$

with

$$\xi_t := \int_0^t \lambda(s) \mathrm{d}Y_s$$

where λ, the so-called price pressure, is a positive smooth function, $H \in C^{1,2}$ and $H(t, \cdot)$ is strictly increasing for every $t \in [0, 1]$. We also write $\xi(t, Y_t)$ for ξ_t. Note that $\mathbb{F}^P = \mathbb{F}^Y$ and consequently $\mathbb{F}^{V,P} = \mathbb{F}^{V,Y} = \mathbb{F}^{V,X+Z} \subseteq \mathbb{F}^{V,Z}$, where for the last inclusion we assume that any strategy that is a measurable function of V and Y can be rewritten in terms of V and Z. Also we have that $\mathbb{F}^Z \subseteq \mathbb{F}^{V,Y}$, so $\mathbb{F}^{V,Z} = \mathbb{F}^{V,P}$.

Remark 3 *It is important to remark that the effect of the total demand in prices is due not only to the function λ but also to the function H. In fact, as we shall see later, in the equilibrium*

$$\mathrm{d}P_t = \frac{\partial H(t, \xi_t)}{\partial y} \lambda(t) \mathrm{d}Y_t,$$

and some authors call market depth to *the quantity*

$$\frac{1}{\frac{\partial H(t,Z_t)}{\partial y} \lambda(t)}.$$

So, to say that market depth is constant is not *equivalent to say that price pressure is constant. Only if the equilibrium pricing rule is linear, both results are equivalent. See [4].*

Definition 4 *Denote the class of such pairs (H, λ) above by $\mathcal{H}$. An element of $\mathcal{H}$ is called a pricing rule.*

Then, we can define the equilibrium in the class of the above pricing rules, and over a set of admissible strategies $\mathcal{X}$ introduced in the next Section in Definition 6:

Definition 5 *Let* $(H, \lambda) \in \mathcal{H}$ *and* $X \in \mathcal{X}$. *The triple* (H, λ, X) *is an equilibrium, if the price process* $P_\cdot := H(\cdot, \xi(\cdot, Y))$ *is rational, given* X, *and the strategy* X *is optimal, given* P.

2.1 Optimal strategies

The final wealth W_{1+} of the insider, just after the announcement, is computed as follows. Consider first a discrete model where trades are made at times $i = 1, 2, \ldots N$. If at time $i-1$, there is an order of buying $X_i - X_{i-1}$ shares, its cost will be $P_i(X_i - X_{i-1})$, so, there is a change in the bank account given by

$$-P_i(X_i - X_{i-1}).$$

Then the total change is

$$-\sum_{i=1}^{N} P_i(X_i - X_{i-1}),$$

and due to the announcement, just after the final time N, by the liquidation of the assets, there is the extra income: $X_N V$. So, the total wealth generated is

$$\begin{aligned} W_{N^+} &= -\sum_{i=1}^{N} P_i(X_i - X_{i-1}) + X_N V \\ &= -\sum_{i=1}^{N} P_{i-1}(X_i - X_{i-1}) - \sum_{i=1}^{N}(P_i - P_{i-1})(X_i - X_{i-1}) + X_N V \\ &= \sum_{i=1}^{N}(V - P_{i-1})(X_i - X_{i-1}) - \sum_{i=1}^{N}(P_i - P_{i-1})(X_i - X_{i-1}), \end{aligned}$$

where, without loss of generality, we assume $X_0 = 0$. Analogously, in the continuous model,

$$W_{1+} = \int_0^1 (V - P_{t-})\mathrm{d}X_t - [P, X]_1, \tag{3}$$

where (and throughout the whole article) X_{t-} denotes the left limit $\lim_{s\uparrow t} X_s$. We require that X is an $\mathbb{F}^{V,P}$-semimartingale, so that the integral can be seen as an Itô integral, and to ensure the quadratic covariation $[P, X]$ is finite we also assume that P is an $\mathbb{F}^{V,P}$-semimartingale.

Then we look for the optimal mean wealth of the insider, given by

$$J(X) := \mathbb{E}\left(W_{1+}\right) = \mathbb{E}\left(\int_0^1 (V - H(t,\xi_t))\mathrm{d}X_t - [P,X]_1\right), \tag{4}$$

over all admissible (H, λ, X), meaning that $(H, \lambda) \in \mathcal{H}$ and $X \in \mathcal{X}$ defined as:

Definition 6 *Denote, by $\mathcal{X}$, the set of càdlàg $\mathbb{F}^{V,P}$-predictable processes with*

(A1) $X \in \mathcal{X}$ satisfying $X_t = M_t + A_t + \int_0^t \theta_s \mathrm{d}s$, where M is a continuous $\mathbb{F}^{V,P}$-martingale, A is a càdlàg, finite variation predictable process with

$$A_t = \sum_{0 \leq s \leq t} (X_s - X_{s-})$$

and θ is a càdlàg, $\mathbb{F}^{V,P}$-adapted process. And for all $X \in \mathcal{X}$ and $(H, \lambda) \in \mathcal{H}$, $\mathbb{P}$-a.s, a.a. $0 \leq t \leq 1$ we have:

(A2) $\mathbb{E}\left(\int_0^1 \left(\partial_2 H\left(t-, \xi_{t-}\right)\right)^2 \left(\mathrm{d}\left[Z,Z\right]_t + \mathrm{d}\left[M,M\right]_t\right)\right) < \infty$,

(A3) $\mathbb{E}\left(\int_0^1 \partial_2 H\left(t, \xi_t\right) \left|\theta_t\right| \mathrm{d}t\right) < \infty$,

(A4) $\mathbb{E}\left(\sum_0^1 \partial_2 H\left(t-, \xi_{t-}\right) \left|\Delta X_t\right|\right) < \infty$ with $\Delta X_t = X_t - X_{t-}$,

(A5) $\int_{\mathbb{R}} \left(H\left(t, \xi_{t-} + \lambda_t u\right) - H\left(t, \xi_{t-}\right) - u\lambda_t \frac{\partial H}{\partial y}\left(t, \xi_{t-}\right)\right) \nu_t\left(\mathrm{d}u\right) < \infty$,

(A6) $0 \in \mathcal{X}$.

Where we write ∂_i to indicate the derivative w.r.t the i^{th} argument.

Remark 7 *Note that, since $(X_t)_{0 \leq t \leq 1}$ has to be a càdlàg $\mathbb{F}^{V,P}$-predictable process, its martingale part cannot have jumps.*

Remark 8 *Remember that $\mathbb{F}^{V,P} = \mathbb{F}^{V,Z}$.*

2.2 The optimality condition

Proposition 9 *Suppose that X is (locally) optimal and that the insider's wealth J is defined by (4). Then*

$$V - H(t,\xi_t) - \lambda(t)\mathbb{E}\left[\int_t^1 \partial_2 H(s,\xi_{s-})\mathrm{d}X_s \middle| \mathcal{F}_t^{V,Z}\right] = 0, \ \ a.s, \ a.a. \ 0 \le t \le 1. \tag{5}$$

and

$$V = P_1 = H(1,\xi_1) = H(1-,\xi_{1-}) = P_{1-} \quad a.s. \ . \tag{6}$$

Proof. For all β such that $X. + \varepsilon\int_0^{\cdot}\beta_s\mathrm{d}s$ is admissible, with $\varepsilon > 0$ small enough, by the local optimality of $X.$, we have

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} J(X. + \varepsilon\int_0^{\cdot}\beta_s\mathrm{d}s)\Big|_{\varepsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbb{E}\left(\int_0^1 \left[V - H\left(t, \int_0^{t-}\lambda(s)\left(\mathrm{d}X_s + \varepsilon\beta_s\mathrm{d}s + \mathrm{d}Z_s\right)\right)\right]\left(\mathrm{d}X_t + \varepsilon\beta_t\mathrm{d}t\right)\right)\Big|_{\varepsilon=0} \\
&= \mathbb{E}\left(\int_0^1 [V - H(t,\xi_t)]\,\beta_t\mathrm{d}t\right) + \mathbb{E}\left(\int_0^1 -\partial_2 H(t,\xi_{t-})\left(\int_0^t \lambda(s)\beta_s\mathrm{d}s\right)\mathrm{d}X_t\right) \\
&= \mathbb{E}\left(\int_0^1\left((V - H(t,\xi_t)) - \lambda(t)\int_t^1 \partial_2 H(s,\xi_{s-})\mathrm{d}X_s\right)\beta_t\mathrm{d}t\right).
\end{aligned}$$

Since we can take $\beta_t = \mathbf{1}_{[u,u+h]}(t)\ \alpha_u$, with α_u being $\mathcal{F}_u^{V,Z}$-measurable and bounded, we have

$$\mathbb{E}\left(\int_u^{u+h}\left(\mathbb{E}((V - H(t,\xi_t))|\,\mathcal{F}_t^{V,Z}) - \lambda(t)\mathbb{E}\left(\int_t^1 \partial_2 H(s,\xi_{s-})\mathrm{d}X_s\middle|\,\mathcal{F}_t^{V,Z}\right)\right)\mathrm{d}t\middle|\,\mathcal{F}_u^{V,Z}\right) = 0 \tag{7}$$

and this means that the process:

$$M_t := \int_0^t\left(\mathbb{E}(V|\mathcal{F}_u^{V,Z}) - \mathbb{E}(H(u,\xi_u)|\mathcal{F}_u^{V,Z}) - \lambda(u)\mathbb{E}\left[\int_u^1 \partial_2 H(s,\xi_{s-})\mathrm{d}X_s\middle|\,\mathcal{F}_u^{V,Z}\right]\right)\mathrm{d}u$$

is an $\mathbb{F}^{V,Z}$-martingale and this implies that, for a.a. $0 \le t \le 1$,

$$V - H(t,\xi_t) - \lambda(t)\mathbb{E}\left[\int_t^1 \partial_2 H(s,\xi_{s-})\mathrm{d}X_s\middle|\,\mathcal{F}_t^{V,Z}\right] = 0, a.s, \tag{8}$$

Then, (A1) and (A3) imply

$$\begin{aligned}
& V - H(t,\xi_t) - \lambda(t)\mathbb{E}\left(\int_t^1 \partial_2 H(s,\xi_{s-})\mathrm{d}X_s \middle| \mathcal{F}_t^{V,Z}\right) \\
= \; & V - H(t,\xi_t) - \lambda(t)\int_t^1 \mathbb{E}\left(\partial_2 H(s,\xi_s)\theta_s | \, \mathcal{F}_t^{V,Z}\right)\mathrm{d}s \\
& -\lambda(t)\sum_t^1 \mathbb{E}\left(\partial_2 H(s,\xi_{s-})\Delta X_s | \, \mathcal{F}_t^{V,Z}\right) \\
= \; & 0
\end{aligned}$$

And also by (A3), we have

$$\int_t^1 \mathbb{E}\left(\partial_2 H(s,\xi_s)\,|\theta_s|\right)\mathrm{d}s < \infty,$$

then

$$\lim_{t\to 1}\mathbb{E}\left(\mathbb{E}\left(\int_t^1 \partial_2 H(s,\xi_s)|\theta_s|\mathrm{d}s \middle| \mathcal{F}_t^{V,Z}\right)\right) = 0,$$

and $\mathbb{E}\left(\int_t^1 \partial_2 H(s,\xi_s)|\theta_s| \middle| \mathcal{F}_t^{V,Z}\right)$ converges in L^1 to zero, and since it is a positive supermartingale it converges almost surely to zero. Analogously for the term

$$\lambda(t)\sum_t^1 \mathbb{E}(\partial_2 H(s,\xi_{s-})\Delta X_s | \mathcal{F}_t^{V,Z}).$$

So, since $\lambda(t)$ is continuous $V = H(1-,\xi_{1-})$, a.s.. Moreover, if we consider a locally optimal strategy with a jump at the end with respect to another without jump we have

$$\begin{aligned}
\Delta J(X) &= \mathbb{E}\left[(V - H(1-,\xi_{1-}))\Delta X_1 - \Delta H_1 \Delta X_1\right] \\
&= -\mathbb{E}\left(\Delta H_1 \Delta X_1\right) < 0,
\end{aligned}$$

since $H(1,\cdot)$ is strictly increasing. So an optimal strategy does not jump at the end and $V_1 = H(1,\xi_1)$. ■

Remark 10 *Note that the property (6) was observed in [1].*

Now we can prove the following Proposition of necessary conditions for an equilibrium:

Proposition 11 *Consider an admissible triple* (H,λ,X). *If it is a local equilibrium, then we have:*

$$(i)\ V-H(t,\xi_t)-\lambda(t)\mathbb{E}\left[\int_t^1 \partial_2 H(s,\xi_{s-})\mathrm{d}X_s\right|\left.\mathcal{F}_t^{V,Z}\right]=0,\ \ a.s,\ a.a.\ 0\le t\le 1.$$

$$(ii)\ H(1,\xi_1)=V\ \ a.s.,\ \ ,$$

$$(iii)\ 0=\partial_1 H(t,\xi_t)+\lambda_t\mu_t\partial_2 H(t,\xi_t)+\frac{1}{2}\lambda_t^2\sigma_{Y,t}^2\partial_{22}H(t,\xi_t)$$

$$+\int_{\mathbb{R}}\left(H\left(t,\xi_{t-}+\lambda_t u\right)-H\left(t,\xi_{t-}\right)-u\lambda_t\partial_2 H\left(t,\xi_{t-}\right)\right)\nu_t\left(\mathrm{d}u\right),\ \ a.s,\ a.a.\ 0\le t\le 1.$$

$$(iv)\ Y-\int_0^{\cdot}\mu_t\mathrm{d}t\ \ is\ \ a\ local\ martingale$$

$$(v)\ If\ V\neq P_t\ a.s. on\ [0,1),\ \ then\ \lambda(t)=\lambda_0\ ,$$

where $\sigma_{Y,t}^2:=\frac{\mathrm{d}[Y^c,Y^c]_s}{\mathrm{d}s}$.

Proof. (i) and (ii) are just the Proposition 9. (iii) By using Itô's formula on $\frac{H(t,\xi_t)}{\lambda(t)}$, we have

$$\begin{aligned}
&\mathbb{E}\left(\int_t^1\frac{1}{\lambda(s)}\partial_2 H(s,\xi_{s-})\mathrm{d}\xi_s\left|\mathcal{F}_t^{V,Z}\right.\right)\\
=\ &\mathbb{E}\left(\frac{H(1,\xi_1)}{\lambda(1)}\right|\left.\mathcal{F}_t^{V,Z}\right)-\frac{H(t,\xi_t)}{\lambda(t)}\\
&-\mathbb{E}\left(\int_t^1\left(-\frac{\lambda'(s)}{\lambda^2(s)}H(s,\xi_s)+\frac{\partial_1 H(s,\xi_s)}{\lambda(s)}+\frac{1}{2}\partial_{22}H(s,\xi_s)\lambda(s)\sigma_{Y,s}^2\right)\mathrm{d}s\right|\left.\mathcal{F}_t^{V,Z}\right)\\
&-\mathbb{E}\left(\sum_{t\le s\le 1}\left(\frac{\Delta H(s,\xi_s)}{\lambda(s)}-\frac{\partial_2 H(s,\xi_{s-})\Delta\xi_s}{\lambda\left(s\right)}\right)\right|\left.\mathcal{F}_t^{V,Z}\right),
\end{aligned}$$

where $\sigma_{Y,s}^2:=\frac{\mathrm{d}[Y^c,Y^c]_s}{\mathrm{d}s}$. Since X is locally optimal, given (H,λ), by (i) and since $Z-\int_0^{\cdot}\mu_s\mathrm{d}s$ is an $\mathbb{F}_t^{V,Z}$-martingale, we can write:

$$\begin{aligned}
0\ =\ &V-\lambda(t)\mathbb{E}\left(\frac{V}{\lambda(1)}\right|\left.\mathcal{F}_t^{V,Z}\right)\\
&+\lambda(t)\int_t^1\mathbb{E}\left(-\frac{\lambda'(s)}{\lambda^2(s)}H(s,\xi_s)+\frac{\partial_1 H(s,\xi_s)}{\lambda(s)}+\frac{1}{2}\partial_{22}H(s,\xi_s)\lambda(s)\sigma_{Y,s}^2+\frac{\partial_2 H(s,\xi_s)\mu_s}{\lambda(s)}\right|\left.\mathcal{F}_t^{V,Z}\right)\mathrm{d}\\
&+\lambda(t)\sum_{t\le s\le 1}\mathbb{E}\left(\left(\frac{\Delta H(s,\xi_s)}{\lambda(s)}-\frac{\partial_2 H(s,\xi_{s-})\Delta\xi_s}{\lambda\left(s\right)}\right)\right|\left.\mathcal{F}_t^{V,Z}\right),
\end{aligned}$$

where, denoting $\{s \in A : \Delta X_s \neq 0\}$ and $\{s \in A : \Delta Z_s \neq 0\}$ by D_A^X and D_A^Z , respectively, for any $A \subseteq \mathbb{R}$, we get

$$\begin{aligned}
&\sum_{s\in D^Z_{[t,1]}} \mathbb{E}\left(\left(\frac{\Delta H(s,\xi_s)}{\lambda(s)} - \frac{\partial_2 H(s,\xi_{s-})\Delta\xi_s}{\lambda\,(s)}\right)\middle|\, \mathcal{F}_t^{V,Z}\right) \\
= &\sum_{s\in D^Z_{[t,1]}} \mathbb{E}\left(\left(\frac{H(s,\xi_{s-}+\lambda\,(s)\,\Delta Y_s)}{\lambda(s)} - \frac{H\,(s,\xi_{s-})}{\lambda\,(s)} - \Delta Y_s\partial_2 H\,(s,\xi_s)\right)\middle|\, \mathcal{F}_t^{V,Z}\right) \\
= &\int_t^1\int_{\mathbb{R}} \mathbb{E}\left[\frac{H\,(s,\xi_s+\lambda\,(s)\,u) - H\,(s,\xi_{s-})}{\lambda\,(s)} - u\partial_2 H\,(s,\xi_s)\middle|\, \mathcal{F}_t^{V,Z}\right]\nu_s\,(\mathrm{d}u)\,\mathrm{d}s
\end{aligned}$$

Hence, we have

$$\begin{aligned}
0 \;=\; & V\left(\frac{1}{\lambda(t)} - \frac{1}{\lambda(1)}\right) \\
&+\mathbb{E}\left(\int_t^1\left(-\frac{\lambda'(s)}{\lambda^2(s)}H(s,\xi_s) + \frac{\partial_1 H(s,\xi_s)}{\lambda(s)} + \frac{1}{2}\partial_{22}H(s,\xi_s)\lambda(s)\sigma^2_{Y,s} + \frac{\partial_2 H(s,\xi_s)\mu_s}{\lambda(s)}\right)\mathrm{d}s\middle|\, \mathcal{F}_t^{V,Z}\right) \\
&+\mathbb{E}\left(\sum_{s\in D^X_{[t,1]}} \frac{\Delta H(s,\xi_s)}{\lambda(s)} - \frac{\partial_2 H(s,\xi_{s-})\Delta\xi_s}{\lambda\,(s)}\right) \\
&+\int_t^1\int_{\mathbb{R}} \mathbb{E}\left[\frac{H\,(s,\xi_s+\lambda\,(s)\,u) - H\,(s,\xi_{s-})}{\lambda\,(s)} - u\partial_2 H\,(s,\xi_s)\middle|\, \mathcal{F}_t^{V,Z}\right]\nu_s\,(\mathrm{d}u)\,\mathrm{d}s.
\end{aligned}$$

By identifying the predictive and martingale parts we have that

$$\begin{aligned}
0 \;=\; & \frac{\lambda'(t)}{\lambda^2(t)}V - \frac{\lambda'(t)}{\lambda^2(t)}H(t,\xi_t) + \frac{\partial_1 H(t,\xi_t)}{\lambda(t)} \\
&+\frac{1}{2}\partial_{22}H(t,\xi_t)\lambda(t)\sigma^2_{Y,t} + \partial_2 H(t,\xi_t)\mu_t \\
&+\frac{\Delta H(t,\xi_t)}{\lambda(t)} - \frac{\partial_2 H(t,\xi_{t-})\Delta\xi_t}{\lambda\,(t)} \\
&+\int_{\mathbb{R}}\left[\frac{H\,(t,\xi_{t-}+\lambda\,(t)\,u) - H\,(t,\xi_{t-})}{\lambda\,(t)} - u\partial_2 H\,(t,\xi_t)\right]\nu_t\,(\mathrm{d}u)\,.
\end{aligned}$$

Then a.a $t \in [0,1]$ and $\mathbb{P}$-a.s., the continuous and jump parts of the r.h.s of the previous equation will be equal to zero.

$$\begin{aligned}
0 \;=\; & \frac{\lambda'(t)}{\lambda^2(t)}V - \frac{\lambda'(t)}{\lambda^2(t)}H(t,\xi_t) + \frac{\partial_1 H(t,\xi_t)}{\lambda(t)} \\
&+\frac{1}{2}\partial_{22}H(t,\xi_t)\lambda(t)\sigma^2_{Y,t} + \partial_2 H(t,\xi_t)\mu_t \\
&+\int_{\mathbb{R}}\left[\frac{H\,(t,\xi_{t-}+\lambda\,(t)\,u) - H\,(t,\xi_{t-})}{\lambda\,(t)} - u\partial_2 H\,(t,\xi_t)\right]\nu_t\,(\mathrm{d}u)\,, \quad (9)
\end{aligned}$$

and

$$\frac{\Delta H(t,\xi_t)}{\lambda(t)} - \frac{\partial_2 H(t,\xi_{t-})\Delta\xi_t}{\lambda\left(t\right)} = 0. \tag{10}$$

Now, since we are in an equilibrium, prices are rational given X, so by taking conditional expectations w.r.t. $\mathcal{F}_t^Y$ and using $\mathbb{E}(V|\mathcal{F}_t^Y) - \mathbb{E}(H(t,\xi_t)|\mathcal{F}_t^Y) = 0$, we have

$$\begin{aligned} 0 &= \frac{\partial_1 H(t,\xi_t)}{\lambda(t)} + \frac{1}{2}\partial_{22} H(t,\xi_t)\lambda(t)\sigma_{Y,t}^2 + \partial_2 H(t,\xi_t)\mu_t \\ &\quad + \int_{\mathbb{R}} \left[\frac{H\left(t,\xi_{t-}+\lambda\left(t\right)u\right) - H\left(t,\xi_{t-}\right)}{\lambda\left(t\right)} - u\partial_2 H\left(t,\xi_t\right)\right]\nu_t\left(\mathrm{d}u\right). \end{aligned}$$

(iv) Consequently

$$\begin{aligned} P_t &= H\left(t,\xi_t\right) = H\left(0,\xi_0\right) + \int_0^t \lambda_s \partial_2 H(s,\xi_{s-})\left(\mathrm{d}Y_s - \mu_s \mathrm{d}s\right) \\ &\quad + \left\{\sum_{t\in D^Z_{[0,t]}} \left(\frac{\Delta H(s,\xi_s)}{\lambda(s)} - \frac{\partial_2 H(s,\xi_{s-})\Delta\xi_s}{\lambda\left(s\right)}\right)\right. \\ &\quad \left. - \int_0^t \int_{\mathbb{R}} \left[\frac{H\left(s,\xi_{s-}+\lambda\left(s\right)u\right) - H\left(s,\xi_{s-}\right)}{\lambda\left(s\right)} - u\partial_2 H\left(s,\xi_s\right)\right]\nu_s\left(\mathrm{d}u\right)\right\}, \end{aligned}$$

so, denoting the second term by N_t

$$\mathrm{d}Y_t - \mu_t \mathrm{d}s = \frac{\mathrm{d}P_t - \mathrm{d}N_t}{\lambda_t \partial_2 H(t,\xi_{t-})}$$

and, since P_t and N_t are martingales and $\lambda_t \partial_2 H(t,y) > 0$, we have that $Y - \int_0^\cdot \mu_t \mathrm{d}t$ is a local martingale.

(v) Finally, from (9) we have that

$$\frac{\lambda'(t)}{\lambda^2(t)} V - \frac{\lambda'(t)}{\lambda^2(t)} H(t,\xi_t) = 0,$$

then $V \neq H(t,\xi_t)$ implies that $\lambda'(t) = 0$. ■

2.3 Characterization of the equilibrium

In this section we will study sufficient conditions for an equilibrium. We shall assume that the pricing rules satisfy

$$\begin{aligned} 0 &= \partial_1 H(t,y) + \partial_2 H(t,y)\mu_t + \frac{1}{2}\partial_{22} H(t,y)\lambda(t)^2\sigma_t^2 \\ &+ \int_{\mathbb{R}} \left(H\left(t,y+\lambda(t)u\right) - H\left(t,y\right) - u\lambda(t)\frac{\partial H}{\partial y}\left(t,y\right)\right)\nu_t\left(\mathrm{d}u\right), \text{ a.a. } 0 \le t \le 1,\ y \in \mathbb{R}, \end{aligned} \tag{11}$$

where σ_t is defined in (1). Note that this condition is close to the condition (iii) in Proposition 11, that is a necessary condition for the equilibrium. Then we have the following Theorem:

Theorem 12 *Consider an admissible triple (H, λ, X) with (H, λ) satisfying (11) then (H, λ, X) is an equilibrium, if and only if :*

(i) $\lambda(t) = \lambda_0$,
(ii) $H(1, \xi_1) = V$ *a.s.*
(iii) $[X^c, X^c] \equiv 0$,
(iv) X *has not jumps*
(v) $X + Z - \int_0^{\cdot} \mu_s \mathrm{d}s$ *is a local martingale.*

Proof. Set

$$i(v, y) := \int_y^{H^{-1}(1,\cdot)(v)} \frac{v - H(1, x)}{\lambda_0} \mathrm{d}x,$$

and

$$\begin{aligned} I(v, t, y) \quad &: = \mathbb{E}(i(V, y + \lambda_0(Z_1 - Z_t))|V = v) \\ &= \mathbb{E}(i(v, y + \lambda_0(Z_1 - Z_t))). \end{aligned}$$

Here, we write ∂_i to indicate the derivative w.r.t the $i^{th} + 1$ argument.

First note that

$$\mathbb{E}(H(1, y + \lambda_0(Z_1 - Z_t)) = H(t, y).$$

In fact, by (11) and (A2), (A5) and (A6) $(H(t, \lambda_0 Z_t))_{0 \leq t \leq 1}$ is a martingale, so, since Z has independent increments, we have that.

$$H(t, y) = \mathbb{E}(H(1, \lambda_0 Z_1)|\lambda_0 Z_t = y) = \mathbb{E}(H(1, y + \lambda_0(Z_1 - Z_t)).$$

$(I(v, t, Z_t))_{0 \leq t \leq 1}$ is also an $\mathbb{F}^Z$- martingale:

$$\begin{aligned} I(v, t, y) &= \mathbb{E}(i(v, y + \lambda_0(Z_1 - Z_t))) \\ &= \mathbb{E}(\, i(v, \lambda_0 Z_1)|\, \lambda_0 Z_t = y), \end{aligned}$$

and we have that

$$\begin{aligned} \partial_2 I(v, t, y) &= \mathbb{E}\left(\partial_1 i(v, y + \lambda_0(Z_1 - Z_t))\right) \\ &= \mathbb{E}\left(-\frac{v - H(1, y + \lambda_0(Z_1 - Z_t)}{\lambda_0}\right) = -\frac{v - H(t, y)}{\lambda_0}. \end{aligned} \tag{12}$$

We can take the derivative under the integral sign because $H(1,\cdot)$ is monotone and $\mathbb{E}(H(1,\lambda_0 Z_1))<\infty$. Now,

$$\begin{aligned}0 &= \partial_{12}I(v,t,y)+\partial_{22}I(v,t,y)\mu_t+\frac{1}{2}\partial_{222}I(v,t,y)\lambda_0^2\sigma_t^2\\ &\quad+\int_{\mathbb{R}}\left(I\left(v,t,y+\lambda_0 u\right)-I\left(v,t,y\right)-u\lambda_0\partial_2 I\left(v,t,y\right)\right)\nu_t\left(\mathrm{d}u\right),\end{aligned}$$

consequently

$$\begin{aligned}C(v,t) &= \partial_{1}I(v,t,y)+\partial_{2}I(v,t,y)\mu_t+\frac{1}{2}\partial_{22}I(v,t,y)\lambda_0^2\sigma_t^2 \\ &\quad+\int_0^t\int_{\mathbb{R}}\left(I\left(v,s,y+\lambda_0 u\right)-I\left(v,s,y\right)-u\lambda_0\partial_2 I\left(v,s,y\right)\right)\nu_s\left(\mathrm{d}u\right)\mathrm{d}s,\end{aligned} \tag{13}$$

where $C(v,t)$ is a constant that can depends on v and t but not on y. Now since $(I(v,t,Z_t))_{0\leq t\leq 1}$ is a martingale it turns out that $C(v,t)=0$ a.a. $t\in[0,1]$. Consider now any admissible strategy X, then, by using Itô's formula we have

$$\begin{aligned}I(v,1,\xi_1) &= I(v,0,0)+\int_0^1\partial_1 I(v,t,\xi_t)\mathrm{d}t\\ &\quad+\int_0^1\partial_2 I(v,t,\xi_{t-})\mathrm{d}\xi_t+\frac{1}{2}\int_0^1\partial_{22}I(v,t,\xi_t)\mathrm{d}[\xi^c,\xi^c]_t\\ &\quad+\sum_{0\leq t\leq 1}\left(\Delta I(v,t,\xi_t)-\partial_2 I(v,t,\xi_{t-})\Delta\xi_t\right),\end{aligned}$$

since, by construction, $\xi_0=0$ and since $\mathrm{d}\xi_t=\lambda_0\mathrm{d}Y_t$ by (i). Now we have that

$$\mathrm{d}[\xi^c,\xi^c]_t=\lambda_0^2\mathrm{d}[X^c,X^c]_t+2\lambda_0^2\mathrm{d}[X^c,Z^c]_t+\lambda_0^2\sigma_t^2\mathrm{d}t.$$

Then using (13), we get :

$$\begin{aligned}I(v,1,\xi_1) &= I(v,0,0)+\int_0^1(P_{t-}-v)(\mathrm{d}X_t+\mathrm{d}Z_t-\mu_t\mathrm{d}t)\\ &\quad+\frac{1}{2}\int_0^1\partial_{22}I(v,t,\xi_t)\lambda_0^2\mathrm{d}[X^c,X^c]_t\\ &\quad+\int_0^1\partial_{22}I(v,t,\xi_t)\lambda_0^2\mathrm{d}[X^c,Z^c]+\sum_{0\leq t\leq 1}\left(\Delta I(v,t,\xi_t)-\partial_2 I(v,t,\xi_{t-})\lambda_0\Delta Y_t\right)\end{aligned}$$

Subtracting $[P,X]_1$ from both sides and substituting, we obtain

$$
\begin{aligned}
& \int_0^1 (v-P_{t-})\mathrm{d}X_t - [P,X]_1 - I(v,0,0) \\
= \quad & -I(v,1,\xi_1) + \int_0^1 (P_{t-}-v)\left(\mathrm{d}Z_t - \mu_t \mathrm{d}t\right) \\
& +\frac{1}{2}\int_0^1 \partial_{22} I(v,t,\xi_{t-})\lambda_0^2 \mathrm{d}[X^c,X^c]_t + \int_0^1 \partial_{22} I(v,t,\xi_{t-})\lambda_0^2 \mathrm{d}[X^c,Z^c] \\
& + \sum_{t\in D^Z_{[0,1]}} \left((\Delta I(v,t,\xi_s) - \partial_2 I(v,t,\xi_{s-})\lambda_0 \Delta Z_t)\right. \\
& \left. - \int_0^1 \int_{\mathbb{R}} \left(I\left(v,t,y+\lambda_0 u\right) - I\left(v,t,y\right) - u\lambda_0 \partial_2 I\left(v,t,y\right)\right)\nu_t\left(\mathrm{d}u\right)\mathrm{d}t\right) \\
& + \sum_{t\in D^X_{[0,1]}} (\Delta I(v,t,\xi_t) - \partial_2 I(v,t-,\xi_{t-})\lambda_0 \Delta X_t) - [P,X]_1.
\end{aligned}
$$

Now it is important to note that $I(v,0,0)$ is, fixing $V=v$, a lower bound for *any* strategy. Then, we will show that by taking the conditional expectation of the left hand side for $V=v$ and seeing that it is non-positive by evaluating the right hand side.

First we have that

$$
[P,X]_1 \equiv [P^c,X^c]_1 + \sum_{0\le t\le 1} \Delta P_t \Delta X_t,
$$

then Itô's formula for H shows that the continuous local martingale part of P is $\int \partial_2 H(t,\xi_t)\mathrm{d}\xi_t^c$, so by (12) we obtain

$$
\begin{aligned}
[P^c,X^c]_1 &= \left[\int \partial_2 H(t,\xi_t)\mathrm{d}\xi_t^c, X^c\right]_1 = \int_0^1 \partial_2 H(t,\xi_t)\mathrm{d}\left[\xi^c,X^c\right]_t \\
&= \int_0^1 \partial_{22} I(v,t,\xi_t)\lambda_0^2 \mathrm{d}\left[X^c,X^c\right]_t + \int_0^1 \partial_{22} I(v,t,\xi_t)\lambda_0^2 \mathrm{d}\left[X^c,Z\right]_t,
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_0 \partial_2 I(v,t,\xi_{t-})\Delta X_t + \Delta P_t \Delta X_t &= (P_{t-}-v)\Delta X_t + \Delta P_t \Delta X_t \\
&= (P_t - v)\Delta X_t = \lambda_0 \partial_2 I(v,t,\xi_t)\Delta X_t.
\end{aligned}
$$

Substituting them for $[P,X]_1$ in the right hand side of the equation, it simplifies to

$$
\begin{aligned}
&-I(v,1,\xi_1)+\int_0^1 (P_t-v)\,(\mathrm{d}Z_t-\mu_t\mathrm{d}t)-\frac{1}{2}\int_0^1 \partial_{22}I(v,t,\xi_t)\lambda_0^2\mathrm{d}[X^c,X^c]_t\\
&+\sum_{t\in D^Z_{[0,1]}}\Big((\Delta I(v,t,\xi_s)-\partial_2 I(v,t,\xi_{s-})\lambda_0\Delta Z_t)\\
&-\int_0^1\int_{\mathbb{R}}\left(I\left(v,t,y+\lambda_0 u\right)-I\left(v,t,y\right)-u\lambda_0\partial_2 I\left(v,t,y\right)\right)\nu_t\left(\mathrm{d}u\right)\mathrm{d}t\Big)\\
&+\sum_{t\in D^X_{[0,1]}}\left(I(v,t,\xi_t)-I(v,t,\xi_{t-})-\lambda_0\partial_2 I(v,t,\xi_t)\Delta X_t\right).
\end{aligned}
$$

Now the result follows from the following points.

1. We know that $\lambda_0\partial_{22}I(V,1,\xi_1)=\partial_2 H(V,1,\xi_1)>0$ and that $\lambda_0\partial_2 I(V,1,\xi_1)=-V+H(1,\xi_1)$ so by hypothesis (ii) we have a maximum value of $-I(V,1,\xi_1)$ for our strategy and, according to the definition of I and condition (ii), $I(V,1,\xi_1)=0$.

2. The processes $\int_0^\cdot (P_t-V)\,(\mathrm{d}Z_t-\mu_t\mathrm{d}t)$ and

$$
\begin{aligned}
&\sum_{t\in D^Z_{[0,\cdot]}}\Big((\Delta I(V,t,\xi_t)-\partial_2 I(V,t,\xi_{t-})\lambda_0\Delta Z_t)\\
&-\int_0^\cdot\int_{\mathbb{R}}\left(I\left(V,t,y+\lambda_0 u\right)-I\left(V,t,y\right)-u\lambda_0\partial_2 I\left(V,t,y\right)\right)\nu_t\left(\mathrm{d}u\right)\mathrm{d}t\Big)
\end{aligned}
$$

   are $\mathbb{F}^{P,V}$-martingale, so they vanish when we take expectations.

3. By (12) and H being increasing monotone, we have that $\partial_{22}I>0$, and the measure $\mathrm{d}[X^c,X^c]\geq 0$, so

$$
-\frac{1}{2}\int_0^1 \partial_{22}I(v,t,\xi_t)\lambda_0^2\mathrm{d}[X^c,X^c]_t\leq 0,
$$

   and we obtain the maximum value for our strategy if and only if $[X^c,X^c]=0$.

4. $\partial_{22}I>0$ (convexity) implies that

$$
I(v,t,x+h)-I(v,t,x)-\partial_2 I(v,t,x+h)h\leq 0.
$$

So,

$$\sum_{t\in D^Z_{[0,1]}} \left(I(v,t,\xi_{t-}+\lambda_0\Delta X_t)-I(v,t,\xi_{t-})-\partial_2 I(v,t,\xi_t)\lambda_0\Delta X_t\right)\leq 0,$$

and it reaches its maximum if and only if $\Delta X_t = 0$, that is what we assume at (iv).

5. Assumption (v) together with condition (A2) and (A5) guarantee the rationality of prices.

■

Remark 13 *In [2], it is proved that, in equilibrium, the pricing rule is of the form*

$$H(t,y)=\mathbb{E}\left[H\left(1,y+\xi_1-\xi_t\right)\right]. \tag{14}$$

In [6], and in our case, as well, we find that in equilibrium, the price pressure λ is constant and the pricing rule is of the form (14), as (H,λ) is a solution of (9) and Itô's formula applied to $H(t,\lambda Z_t)$ implies

$$\begin{aligned} H(t,y) &= \mathbb{E}\left[H\left(1,\lambda Z_1\right)\middle|\,\lambda Z_t=y\right] \\ &= \mathbb{E}\left[H\left(1,\lambda Z_1-\lambda Z_t+\lambda Z_t\right)\middle|\,\lambda Z_t=y\right] \\ &= \mathbb{E}\left[H\left(1,\lambda Z_1-\lambda Z_t+y\right)\right]. \end{aligned}$$

We have seen that provided that (11) is satisfied, the equilibrium strategies are of the form

$$X=\int_0^{\cdot}\theta_s\mathrm{d}s.$$

Then, the following propositions give conditions on θ to be an equilibrium strategy.

Proposition 14 *Let (H,λ) be a pricing rule of class $\mathcal{H}$ that satisfies (11) and $X=\int_0^{\cdot}\theta_s\mathrm{d}s$ a strategy in $\mathcal{X}$. Then the following conditions are equivalent:*

i) The process $(H(t,\xi_t))$ is an $\mathbb{F}^Y$-martingale.

ii) $\mathbb{E}\left[\theta_t\middle|\,\mathcal{F}^Y_t\right]=0.$

iii) The process $\left(Y_t-\int_0^t\mu_s\mathrm{d}s\right)$ is an $\mathbb{F}^Y$-martingale.

For its proof, we will need the following Lemma:

Lemma 15 *Assume that a process G is $\mathbb{F}^Y$-adapted and*

$$G_t = M_t + \int_0^t \alpha_s ds,\ \ t \geq 0$$

where M is an $\mathbb{F}^{Z,V}$-martingale and α is $\mathbb{F}^{Z,V}$-adapted with mathbb$E(|\alpha_s|) < \infty$ for all $s \geq 0$. Let $\mathbb{H}$ be a filtration such that $\mathbb{F}^Y \subseteq \mathbb{H} \subseteq \mathbb{F}^{Z,V}$. Then

$$G_t = N_t + \int_0^t \mathbb{E}\left[\alpha_s \left|\mathcal{H}_s\right.\right] ds,\ \ t \geq 0$$

where N is an $\mathbb{H}$-martingale.

Proof. First, we show that $\mathbb{E}\left[M_t \left|\mathcal{H}_t\right.\right]$ is an $\mathbb{H}$-martingale. Let $s \leq t$, then since $\mathcal{H}_s \subseteq \mathcal{F}_s^{Z,V}$

$$\mathbb{E}\left[\mathbb{E}\left[M_t \left|\mathcal{H}_t\right.\right]\right| \mathcal{H}_s] = \mathbb{E}\left[M_t \left|\mathcal{H}_s\right.\right] = \mathbb{E}\left[\mathbb{E}\left[M_t \left|\mathcal{F}_s^{Z,V}\right.\right]\right| \mathcal{H}_s\right] = \mathbb{E}\left[M_s \left|\mathcal{H}_s\right.\right],$$

since M is an $\mathbb{F}^{Z,V}$-martingale. Then, consider

$$G_t - G_s = M_t - M_s + \int_s^t \alpha_u \mathrm{d}u.$$

We have

$$\begin{aligned}
\mathbb{E}\left[G_t - G_s \right| \mathcal{H}_s] &= \mathbb{E}\left[M_t - M_s \right| \mathcal{H}_s] + \int_s^t \mathbb{E}\left[\alpha_u \right| \mathcal{H}_s] \mathrm{d}u \\
&= \mathbb{E}\left[\int_s^t \mathbb{E}\left[\alpha_u \right| \mathcal{H}_u] \mathrm{d}u \right| \mathcal{H}_s\right],
\end{aligned}$$

so

$$\mathbb{E}\left[G_t - G_s - \int_s^t \mathbb{E}\left[\alpha_u \right| \mathcal{H}_u] \mathrm{d}u \right| \mathcal{H}_s\right] = 0,$$

hence, $N_t := G_t - \int_0^t \mathbb{E}\left[\alpha_u \right| \mathcal{H}_u] \mathrm{d}u$ is an $\mathbb{H}$-martingale. ■

Proof of Proposition 14. Let (H, λ) be a pricing rule, then Itô's formula says

$$
\begin{aligned}
H(t,\xi_t) &= H(0,0) + \int_0^t \lambda_s \theta_s \frac{\partial H}{\partial y}(s,\xi_s)\,\mathrm{d}s \\
&\quad + \int_0^t \left[\frac{\partial H}{\partial t}(s,\xi_s) + \frac{\partial H}{\partial y}(s,\xi_s)\,\lambda_s \mu_s + \frac{1}{2}\lambda_s^2 \sigma_s^2 \frac{\partial^2 H}{\partial y^2}(s,\xi_s) \right] \mathrm{d}s \\
&\quad + \int_0^t \frac{\partial H}{\partial y}(s,\xi_{s-})\left(\lambda_s \sigma_s \mathrm{d}B_s + \lambda_s \mathrm{d}L_s\right) \\
&\quad + \sum_{0 \le s \le t} \left[\Delta H(s,\xi_s) - \frac{\partial H}{\partial y}(s,\xi_{s-})\Delta \xi_s \right] \\
&= M_t + \int_0^t \left[\frac{\partial H}{\partial t}(s,\xi_s) + \lambda_s \mu_s \frac{\partial H}{\partial y}(s,\xi_s) + \frac{1}{2}\lambda_s^2 \sigma_s^2 \frac{\partial^2 H}{\partial y^2}(s,\xi_s) \right] \mathrm{d}s \\
&\quad + \int_0^t (H(s,\xi_{s-} + \lambda_s u) - H(s,\xi_{s-}) - u\lambda_s \frac{\partial H}{\partial y}(s,\xi_{s-}))\nu_s(\mathrm{d}u)\mathrm{d}s \\
&\quad + \int_0^t \lambda_s \theta_s \frac{\partial H}{\partial y}(s,\xi_s)\,\mathrm{d}s.
\end{aligned}
$$

where M is an $\mathbb{F}^{Z,V}$-martingale. Then, by Lemma 15 we know that H can be rewritten as

$$
\begin{aligned}
H(t,\xi_t) &= N_t + \int_0^t \left[\frac{\partial H}{\partial t}(s,\xi_s) + \frac{\partial H}{\partial y}(s,\xi_s)\,\lambda_s \mu_s + \frac{1}{2}\lambda_s^2 \sigma_s^2 \frac{\partial^2 H}{\partial y^2}(s,\xi_s) \right] \mathrm{d}s \\
&\quad + \int_0^t (H(s,\xi_{s-} + \lambda_s u) - H(s,\xi_{s-}) - u\lambda_s \frac{\partial H}{\partial y}(s,\xi_{s-}))\nu_s(\mathrm{d}u)\mathrm{d}s \\
&\quad + \int_0^t \lambda_s \mathbb{E}(\theta_s|\mathcal{F}_s^Y) \frac{\partial H}{\partial y}(s,\xi_s)\, s \\
&= N_t + \int_0^t \lambda_s \mathbb{E}(\theta_s|\mathcal{F}_s^Y) \frac{\partial H}{\partial y}(s,\xi_s)\,\mathrm{d}s,
\end{aligned}
$$

where N is an $\mathbb{F}^Y$-martingale. Then, $(H(t,\xi_t))$ is an $\mathbb{F}^Y$-martingale if and only if

$$\mathbb{E}(\theta_s|\mathcal{F}_s^Y) = 0,$$

which proves that i) and ii) are equivalent. Also, we know that

$$Y_t = Z_t + \int_0^t \theta_s \mathrm{d}s,$$

so

$$Y_t - \int_0^t \mu_s \mathrm{d}s = R_t + \int_0^t \theta_s \mathrm{d}s,$$

where R is an $\mathbb{F}^{Z,V}$-martingale. Then we can write, by Lemma 15,

$$Y_t - \int_0^t \mu_s \mathrm{d}s = U_t + \int_0^t \mathbb{E}(\theta_s|\mathcal{F}_s^Y)\mathrm{d}s$$

where U is an $\mathbb{F}^Y$-martingale which proves that ii) and iii) are equivalent. ■

Then, we have the following proposition.

Proposition 16 *Suppose,* $(H,\lambda) \in \mathcal{H}$ *is a solution of (11) with* $\lambda = \lambda_0 > 0$, $X = \int_0^\cdot \theta_s \mathrm{d}s$, $H(1,\xi_1) = V$ *and such that* $\mathbb{E}\left[\theta_t | \mathcal{F}_t^Y\right] = 0$, *then the pricing rule is rational, that is*

$$H(t,\xi_t) = \mathbb{E}[V|\mathcal{F}_t^Y], 0 \leq t \leq 1,$$

and (H,λ,X) *is an equilibrium.*

Proof. By the previous proposition $H(t,\xi_t)$ is an $\mathbb{F}^Y$-martingale. Then

$$H(t,\xi_t) = \mathbb{E}(H(1,\xi_1)|\mathcal{F}_t^Y) = \mathbb{E}(V|\mathcal{F}_t^Y),$$

therefore prices are rational. That (H,λ,X) is an equilibrium follows from the previous proposition and Theorem 12. ■

2.4 Existence of equilibrium

From Theorem 12 we have seen that, assuming (11) with $\lambda_t = \lambda_0 > 0$, necessary and sufficient conditions to have an equilibrium are to have a strategy $\int_0^\cdot \theta_s \mathrm{d}s \in \mathcal{X}$ satisfying:

1. the process $\left(Y_t - \int_0^t \mu_s \mathrm{d}s\right)$ is an $\mathbb{F}^Y$-martingale, where $Y_t = \int_0^t \theta_s \mathrm{d}s + Z_t$ is the total demand.
2. it drives the total demand to the value $R := H^{-1}(1,\lambda_0\cdot)(V)$, that is $Y_1 = R$.

First we have a simple case:

Proposition 17 *If the demand of the liquidity traders, Z, has not a jump component, then the equilibrium strategy is such that*

$$\theta_t = \frac{Y_1 - Y_t - \int_t^1 \mu_s \mathrm{d}s}{\int_t^1 \sigma_s^2 \mathrm{d}s} \sigma_t^2$$

Proof. *If $\bar{Y}_t := Y_t - \int_0^t \mu_s \mathrm{d}s = \int_0^t \sigma_s \mathrm{d}\tilde{B}_s$, where $\tilde{B}$ is a Brownian motion, then*

$$\bar{Y}_t - \int_0^t \frac{\bar{Y}_1 - \bar{Y}_t}{\int_s^1 \sigma_u^2 \mathrm{d}u} \sigma_s^2 \mathrm{d}s, 0 \leq t \leq 1,$$

is a process identical in law to $\int_0^{\cdot} \sigma_s \mathrm{d}\tilde{B}_s$ and independent of Y_1. ■

Theorem 18 *If the demand of the liquidity traders Z has a jump component (i.e. $L \neq 0$), then there is not equilibrium.*

Proof. Let Y be the total demand in an equilibrium, then we have

$$M_t := Y_t - \int_0^t \mu_s \mathrm{d}s = \int_0^t \sigma_s \mathrm{d}B_s + L_t + \int_0^t \theta_s(Y_1; Y_u, 0 \leq u \leq s)\mathrm{d}s, 0 \leq t \leq 1$$

so the r.h.s. is the Doob-Meyer decomposition of the $\mathbb{F}^Y$-martingale M in the filtration $\mathbb{F}^{Y,Y_1}$, since $\int_0^{\cdot} \sigma_s \mathrm{d}B_s + L.$ is an $\mathbb{F}^{Y,Y_1}$-martingale. Now, we can decompose the martingale M in its continuous and jump components,

$$\begin{aligned} M_t^c &= \int_0^t \sigma_s \mathrm{d}B_s + \Gamma_t, \\ M_t^d &= L_t + \Lambda_t. \end{aligned}$$

These two equalities give us the $\mathbb{F}^{Y,Y_1}$-Doob-Meyer decompositions of M^c and M^d respectively, with $\Gamma_t + \Lambda_t = \int_0^t \theta_s(Y_1; Y_u, 0 \leq u \leq s)\mathrm{d}s$. Note that we have

$$M_t^d - L_t = \int_0^t \int_{\mathbb{R}} x \left(\delta(\mathrm{d}s, \mathrm{d}x) - v_t(\mathrm{d}x)\mathrm{d}s\right) = \Lambda_t,$$

where $\left(\int_0^t \int_{\mathbb{R}} x\delta(\mathrm{d}s, \mathrm{d}x)\right)$ is the $\mathbb{F}^Y$-predictable compensator of the integer random measure in the process M^d. So Λ is $\mathbb{F}^Y$-predictable and does not depend on Y_1. Moreover $M_t^d - L_t$ is an $\mathbb{F}^Y$-martingale and consequently $\Lambda \equiv 0$, a.s..

So, if there is only jump part in the demand of liquidity traders, i.e. $Z \equiv L$ $M_t = Y_t = L_t$ and $R = L_1$ contradicting the hypothesis of independence between L and V. Therefore there is not equilibrium.

If, on the contrary, we have a continuous part in Z then the argument above yields

$$M_t^c = \int_0^t \sigma_s \mathrm{d}B_s + \int_0^t \theta_s(Y_1; Y_u, 0 \leq u \leq s)\mathrm{d}s, \tag{15}$$

and

$$M_t^d = L_t.$$

Note that, since B is independent of L, (15) is the Doob-Meyer decomposition of M^c in the filtration $(\sigma(Y_1; Y_u, 0 \leq u \leq s; L_u, 0 \leq u \leq 1))$.

To have optimality we need $M_1^c = R - L_1 - \int_0^1 \mu_s \mathrm{d}s$. Now, by the Dambis-Dubins-Schwarz theorem (see [11], Thm. V.1.6. and Prop.V.1.11), $M_t^c \sim \int_0^t \sigma_s \mathrm{d}\tilde{B}_s$ for certain Brownian motion $\tilde{B}$, then M_1^c is Gaussian and by hypothesis V and L are independent (so R and L as well), then, since L is not Gaussian, this is not possible (see Thm 2.3 in [8]) .

Therefore, in any case of Z with and without continuous component we obtain that L cannot be independent of R if we want to have rational prices. Hence there is not equilibrium. ■

2.5 Equilibrium limit strategy in case of jumps and diffusion term in the noise traders' process

As we have seen, in case of the noise process having jumps, there is not equilibrium. Here, for the sake of simplicity, we take $\sigma_t \equiv 1$ and $\mu_t \equiv 0$. In order to move prices to the final value V, or equivalently to move Y_t to $Y_1 = H^{-1}(1, \lambda_0 \cdot)(V)$ a strategy would be the one having jumps just after the same moment when the noise traders' demand does:

$$X_t' = -L_{t-} + \int_0^t \frac{Y_1 - Y_s}{1 - s} \mathrm{d}s.$$

It would kill the jumps of Z. The problem is that the time "just after" does not exist and this strategy has to be seen as a limit of càdlàg strategies that are now feasible but not optimal:

$$X_t' = -L_{t-\varepsilon} + \int_0^t \frac{Y_1 - Y_s}{1 - s} \mathrm{d}s.$$

The drawback of this strategy would be the fact that it has jumps and consequently, as we have seen in Theorem 12, it is suboptimal.

However, we could assume pricing rules satisfying

$$0 = \partial_1 H(t, y) + \frac{1}{2}\partial_{22} H(t, y)\lambda_0,$$

and to compensate the jumps of noise traders, that is that of L, by jumping in the oppposite way, but this can only be done in an approximate or limit way.

Another approximate equilibrium could be obtained by assuming pricing rules of the form

$$\begin{aligned} 0 &= \partial_1 H(t, y) + \frac{1}{2}\partial_{22} H(t, y)\lambda_0 \\ &+ \int_{\mathbb{R}} \left(H(t, y + \lambda_0 u) - H(t, y) - u\lambda_0 \frac{\partial H}{\partial y}(t, y) \right) \nu_t(\mathrm{d}u), \text{ a.a. } 0 \leq t \leq 1, \; y \in \mathbb{R}, \end{aligned}$$

that compensate the jumps of L, and to avoid jumps in X and at the same time moving the prices to V. To get this we can approximate X_t' by something smoother as follows. Suppose the following integrals exist and are finite and denote the pure jump L_t part and its compensator by

$$\begin{aligned} L_t^j &: = \int_0^t \int_{\mathbb{R}} x M(\mathrm{d}t, \mathrm{d}x) \quad \text{ and} \\ L_t^c &: = \int_0^t \int_{\mathbb{R}} x v_t(\mathrm{d}x)\mathrm{d}t, \end{aligned}$$

respectively and also denote

$$L_t^{j,\varepsilon} := \frac{1}{\varepsilon}\int_{t-\varepsilon}^t L_s^j \mathrm{d}s,$$

an absolutely continuous function that "absorves" the jump in ε time, and set

$$L_t^{\varepsilon} = L_t^{j,\varepsilon} - L_t^c.$$

Note that if there is no jump in $[t-\varepsilon, t]$, then $L_t^{j,\varepsilon} = L_t^j$. So, we can introduce the following suboptimal solution:

$$X_t^{\varepsilon} = -L_t^{\varepsilon} + \int_0^t \frac{Y_1 - Y_s}{1-s}\mathrm{d}s,$$

using which we have that for fixed t,

$$|X'_t - X^\varepsilon_t| = |L^\varepsilon_t - L'_t| = \left|\frac{1}{\varepsilon}\int_{t-\varepsilon}^t L^j_s \mathrm{d}s - L^j_t\right| \to 0$$

a.s. as $\varepsilon \to 0$, since the fact that $L.$ does not have a jump at t is of probability one, implies that $\frac{1}{\varepsilon}\int_{t-\varepsilon}^t L^j_s\mathrm{d}s$ tends to L^j_t with probability 1, and also in L^1, since

$$\begin{aligned}
\mathbb{E}\left[\left|\frac{1}{\varepsilon}\int_{t-\varepsilon}^t L^j_s\mathrm{d}s - L^j_t\right|\right] &\leq \mathbb{E}\left[\frac{1}{\varepsilon}\int_{t-\varepsilon}^t \left|\left(L^j_s - L^j_t\right)\right| \mathrm{d}s\right] \\
&= \frac{1}{\varepsilon}\int_{t-\varepsilon}^t \mathbb{E}\left[\left|\left(L^j_s - L^j_t\right)\right|\right] \mathrm{d}s \\
&\leq \max_{t-\varepsilon\leq s\leq t} \mathbb{E}\left[\left|L^j_s - L^j_t\right|\right] \to 0
\end{aligned}$$

In case of $L.$ being a process that may have infinite activity, introduce the moving average process of $L.$,

$$L^\varepsilon_t = \frac{1}{\varepsilon}\int_{t-\varepsilon}^t L_s \mathrm{d}s,$$

which has the same convergence properties as the one before.

2.6 When the insider is risk averse

In this section we study the case of a risk-averse insider. We restrict ourselves to the case of exponential utility. We are going to follow the dynamic programming approach and to obtain the Hamilton-Jacobi-Bellman (HJB) Equations as done in [6], not the Perturbation method presented in Subsection 2.2.

Assume that the insider wants to maximize $\mathbb{E}(u\left(W_{1+}\right)) = \mathbb{E}(\gamma \mathbb{E}^{\gamma W_{1+}})$, where $\gamma < 0$. We define the value function as

$$J(t,y) := \sup_{\tilde\theta:\xi(t,\tilde\theta)=y} \mathbb{E}\left[\gamma\exp\left\{\gamma\int_t^1 (V-P_l)\tilde\theta_l \mathrm{d}l\right\}\middle| \mathcal{F}^{Z,V}_t\right],$$

where we assume that $\mathbb{E}\left[\gamma\exp\left\{\gamma\int_t^1 (V-P_l)\tilde\theta_l\mathrm{d}l\right\}\middle|\mathcal{F}^{Z,V}_t\right]$ is a measurable function of $\xi(t,\tilde\theta) := \int_0^t \lambda_s \mathrm{d}Y^{\tilde\theta}_s = \int_0^t \lambda_s\left(\mathrm{d}Z_s + \tilde\theta\mathrm{d}s\right)$. Then, adding and subtracting $\gamma\exp\gamma\int_{t+h}^1 (V-P_l)\tilde\theta_l\mathrm{d}l$ under the expectation, we have

$$
\begin{aligned}
J(t,y) &= \sup_{\tilde{\theta}:\xi(t,\tilde{\theta})=y} \mathbb{E}\left[\gamma \exp\left\{\gamma \int_t^1 (V-P_l)\tilde{\theta}_l \mathrm{d}l\right\}\left(1-\exp\left\{-\gamma\int_t^{t+h}(V-P_l)\tilde{\theta}_l\mathrm{d}l\right\}\right)\right. \\
&\quad \left. +\gamma \exp\left\{\gamma\int_{t+h}^1 (V-P_l)\tilde{\theta}_l \mathrm{d}l\right\}\right| \mathcal{F}_t^{Z,V}\Bigg], \\
&= \sup_{\tilde{\theta}:\xi(t,\tilde{\theta})=y} \mathbb{E}\left[\gamma \exp\left\{\gamma \int_t^1 (V-P_l)\tilde{\theta}_l \mathrm{d}l\right\}\left(1-\exp\left\{-\gamma\int_t^{t+h}(V-P_l)\tilde{\theta}_l\mathrm{d}l\right\}\right)\right. \\
&\quad \left. +J(t+h,\xi(t+h,\tilde{\theta}))\right| \mathcal{F}_t^{Z,V}\Bigg].
\end{aligned}
$$

So, subtracting $J(t,y)$, we can apply Itô's formula to the difference $J(t+h,\xi(t+h,\tilde{\theta}))-J(t,\xi(t,\tilde{\theta}))$. Moreover note that

$$
\lim_{h\to 0}\frac{\left(1-\exp\left\{-\gamma\int_t^{t+h}(V-P_l)\tilde{\theta}_l dl\right\}\right)}{h} = \gamma(V-P_t)\tilde{\theta}_t.
$$

Hence, we get the following HJB equations, where of course $P_t = H(t,\xi_t)$.

$$
\begin{aligned}
0 &= \sup_{\theta}\left\{J\gamma(V-H)\theta_t + \frac{\partial J}{\partial t} + \lambda_t\theta_t\frac{\partial J}{\partial y} + \frac{\partial J}{\partial y}\lambda_t\mu_t + \frac{1}{2}\lambda_t^2\sigma_t^2\frac{\partial^2 J}{\partial y^2}\right. \\
&\quad \left. + \int_{\mathbb{R}}(J(t,y+\lambda_t u) - J(t,y) - u\lambda_t\frac{\partial J}{\partial y}(t,y))\nu_t(\mathrm{d}u)\right\}.
\end{aligned}
$$

Since the equation is linear in θ, we get the following two equations:

$$
\lambda_t\frac{\partial J}{\partial y}(t,y) = J\left(t,y\right)\gamma\left(H(t,y)-V\right) \quad \forall (t,y)\in(0,1]\times\mathbb{R}, \tag{16}
$$

and for all $(t,y)\in(0,1)\times\mathbb{R}$

$$
\begin{aligned}
0 &= \frac{\partial J}{\partial t} + \lambda_t\mu_t\frac{\partial J}{\partial y} + \frac{1}{2}\lambda_t^2\sigma_t^2\frac{\partial^2 J}{\partial y^2} \\
&\quad + \int_{\mathbb{R}}\left(J\left(t,y+\lambda_t u\right) - J\left(t,y\right) - u\lambda_t\frac{\partial J}{\partial y}\left(t,y\right)\right)\nu_t\left(\mathrm{d}u\right).
\end{aligned} \tag{17}
$$

Differentiating (16) by y we have

$$
\frac{\partial^2 J}{\partial y^2} = \frac{1}{\lambda_t^2}J\gamma\left[\lambda_t\frac{\partial H}{\partial y} + \left(H-V\right)^2\gamma\right],
$$

which plugged in to (17) implies

$$
\begin{aligned}
0 &= \frac{\partial J}{\partial t} + (H-V)\,\gamma J \mu_t + \frac{1}{2} J \gamma \sigma_t^2 \left[\lambda_t \frac{\partial J}{\partial y} + (H-V)^2 \gamma\right] \\
&\quad + \int_{\mathbb{R}} \left(J(t, y+\lambda_t u) - J(t,y) - u\lambda_t \frac{\partial J}{\partial y}(t,y) \right) \nu_t(\mathrm{d}u). \qquad (18)
\end{aligned}
$$

Denote $\int_{\mathbb{R}} \left(J(t, y+\lambda_t u) - J(t,y) - u\lambda_t \frac{\partial J}{\partial y}(t,y) \right) \nu_t(\mathrm{d}u)$ by $I(t,y)$. By differentiating the previous equation by y, we get

$$
\begin{aligned}
0 &= \frac{\partial J}{\partial t \partial y} + \frac{\partial H}{\partial y} \gamma J \mu_t + \frac{(H-V)^2 \gamma^2 J \mu_t}{\lambda_t} \\
&\quad + \frac{1}{2} \gamma \sigma_t^2 \left\{ \frac{(H-V)\,\gamma J}{\lambda_t} \left[\lambda_t \frac{\partial H}{\partial y} + (H-V)^2 \gamma\right] + J \left[\lambda_t \frac{\partial^2 H}{\partial y^2} + 2(H-V) \frac{\partial H}{\partial y} \gamma\right] \right\} \\
&\quad + I_y(t,y), \qquad (19)
\end{aligned}
$$

so

$$
\begin{aligned}
\frac{\partial J}{\partial t \partial y} &= -J\gamma\mu_t \left(\frac{\partial H}{\partial y} + \frac{(V-H)^2 \gamma}{\lambda_t} \right) \\
&\quad + J \frac{\gamma \sigma_t^2}{2} \left(3\gamma (V-H) \frac{\partial H}{\partial y} + \frac{\gamma^2}{\lambda_t} (V-H)^3 - \lambda_t \frac{\partial^2 H}{\partial y^2} \right) - I_y(t,y).
\end{aligned}
$$

While differentiating (16) by t, we get

$$
\lambda_t' \frac{\partial J}{\partial y} + \frac{\partial J}{\partial t \partial y} \lambda_t = \frac{\partial H}{\partial t} \gamma J + (H-V)\,\gamma \frac{\partial J}{\partial t}.
$$

Inserting this expression together with (16) into (17), we get

$$
\begin{aligned}
\frac{\partial J}{\partial t \partial y} &= J \left[(V-H)^2 \frac{\gamma^2}{\lambda_t} \mu_t + \frac{\gamma^3 \sigma_t^2}{2\lambda_t} (V-H)^3 + \frac{\gamma^2 \sigma_t^2}{2} (V-H) \frac{\partial H}{\partial y} \right. \\
&\quad \left. + \frac{\gamma}{\lambda_t} \frac{\partial H}{\partial t} + \gamma \frac{\lambda_t'}{\lambda_t^2} (V-H) \right] + \frac{\gamma (H-V)}{\lambda_t} I(t,y). \qquad (20)
\end{aligned}
$$

Subtracting (20) from (19), we obtain

$$
\begin{aligned}
0 &= -J\gamma\mu_t \frac{\partial H}{\partial y} + \frac{\partial H}{\partial t} + \frac{1}{2} \sigma_t^2 \lambda_t^2 \frac{\partial^2 H}{\partial y^2} - \lambda_t (V-H) \left[\left(\frac{1}{\lambda_t} \right)' + \gamma \sigma_t^2 \frac{\partial H}{\partial y} \right] \\
&\quad + \frac{\gamma (H-V)}{\lambda_t} I(t,y) - I_y(t,y).
\end{aligned}
$$

Also, (16) implies

$$\frac{\frac{\partial J}{\partial y}}{J} = \frac{(H - V)\,\gamma}{\lambda_t}.$$

Hence we have that

$$\begin{aligned}
J &= \exp\left\{\frac{\gamma}{\lambda_t}\int_0^y (H - V)\,du\right\} c_2(t) =: H^e(t,y)\,c_2(t) \\
J_y &= \frac{\partial H^e}{\partial y} = H^e\frac{\gamma}{\lambda_t}(H - V)\,c_2(t).
\end{aligned}$$

and

$$I(t,y) = c_2(t)\int_{\mathbb{R}} \left(H^e(t, y + \lambda_t u) - H^e(t,y) - u\gamma H^e(H(t,y) - V)\right)\nu_t(\mathrm{d}u).$$

So,

$$\begin{aligned}
\frac{\gamma(H(t,y) - V)}{\lambda_t} I(t,y) &= c_2(t)\frac{\gamma}{\lambda_t}\int_{\mathbb{R}} \left[(H(t,y) - V)\,H^e(t, y + \lambda_t u)\right. \\
&\quad - (H(t,y) - V)\,H^e(t,y) \\
&\quad \left. -uH^e\gamma(H(t,y) - V)^2\right]\nu_t(\mathrm{d}u)
\end{aligned}$$

and

$$\begin{aligned}
I_y(t,y) &= c_2(t)\frac{\gamma}{\lambda_t}\int_{\mathbb{R}} \left[H^e(t, y + \lambda_t u)(H(t, y + \lambda_t u) - V)\right. \\
&\quad - H^e(t,y)(H(t,y) - V) \\
&\quad \left. -u\gamma H^e(t,y)(H(t,y) - V)^2 + u\lambda_t H^e(t,y)\,H_y(t,y)\right]\nu_t(\mathrm{d}u).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\gamma(H - V)}{\lambda_t} I(t,y) - I_y(t,y) &= -c_2(t)\frac{\gamma}{\lambda_t}\int_{\mathbb{R}} \left[H^e(t, y + \lambda_t u)(H(t, y + \lambda_t u) - H(t,y))\right. \\
&\quad \left. -u\lambda_t H^e(t,y)\frac{\partial H}{\partial y}(t,y)\right]\nu_t(\mathrm{d}u).
\end{aligned}$$

Hence, we get the following equation for H. If there is solution (J, H, λ) satisfying the HJB Equations, (H, λ) has to satisfy

$$\begin{aligned}
0 &= -H^e(t,y)\,c_2(t)\,\gamma\mu_t\frac{\partial H}{\partial y} + \frac{\partial H}{\partial t} + \frac{1}{2}\sigma_t^2\lambda_t^2\frac{\partial^2 H}{\partial y^2} \\
&\quad -\lambda_t(V - H)\left[\left(\frac{1}{\lambda_t}\right)' + \gamma\sigma_t^2\frac{\partial H}{\partial y}\right] \\
&\quad -c_2(t)\frac{\gamma}{\lambda_t}\int_{\mathbb{R}} \left[H^e(t, y + \lambda_t u)\,H(t, y + \lambda_t u) - H(t,y)\,H^e(t, y + \lambda_t u)\right. \\
&\quad \left. -u\lambda_t H^e(t,y)(t,y)\right]\nu_t(\mathrm{d}u). \qquad (21)
\end{aligned}$$

We remark that the equation differs in two terms from the one in [6]: the first term is given by the presence of the drift μ and the last term which is given because of the jumps. If there are no jumps and drift, a solution can be found as done in [6].

Suppose that we have drift and diffusion part but that there are no jumps in the noise traders' process. The last equation reduces to

$$\begin{aligned} 0 &= -H^e(t,y)\,c_2(t)\,\gamma\mu_t\frac{\partial H}{\partial y}+\frac{\partial H}{\partial t}+\frac{1}{2}\sigma_t^2\lambda_t^2\frac{\partial^2 H}{\partial y^2} \\ &\quad -\lambda_t(V-H)\left[\left(\frac{1}{\lambda_t}\right)'+\gamma\sigma_t^2\frac{\partial H}{\partial y}\right]. \end{aligned}$$

Then

$$H^e(t,y)\,c_2(t)\,\gamma\mu_t\frac{\partial H}{\partial y}+\lambda_t(V-H)\left[\left(\frac{1}{\lambda_t}\right)'+\gamma\sigma_t^2\frac{\partial H}{\partial y}\right]$$

cannot depend on V, equivalently, by differentiating with respect to V, we have

$$H^e(t,y)\,\frac{\gamma}{\lambda_t}yc_2(t)\,\gamma\mu_t\frac{\partial H}{\partial y}=\lambda_t\left[\left(\frac{1}{\lambda_t}\right)'+\gamma\sigma_t^2\frac{\partial H}{\partial y}\right] \tag{22}$$

where, for $\mu_t\neq 0$, the right hand side is strictly increasing in V, while the the left hand side does not depend on it, which is a contradiction. Hence, we can have a solution only if $\mu_t\equiv 0$ which implies

$$\left(\frac{1}{\lambda_t}\right)'+\gamma\sigma_t^2\frac{\partial H}{\partial y}=0.$$

Note that this is the same situation as in [6]. With analogous reasoning, one can show that, allowing jumps and drift only we arrive to a contradiction.

In fact the equation (21) has the form

$$\begin{aligned} 0 &= -H^e(t,y)\,c_2(t)\,\gamma\mu_t\frac{\partial H}{\partial y}+\frac{\partial H}{\partial t}+ \\ &\quad -\lambda_t(V-H)\left(\frac{1}{\lambda_t}\right)' \\ &\quad -c_2(t)\,\frac{\gamma}{\lambda_t}\int_{\mathbb{R}}\Big[H^e(t,y+\lambda_t u)\,(H(t,y+\lambda_t u)-H(t,y)) \\ &\quad -u\lambda_t H^e(t,y)\,\frac{\partial H}{\partial y}(t,y)\Big]\,\nu_t(\mathrm{d}u), \end{aligned}$$

therefore,

$$
\begin{array}{l}
-H^{e}\left(t,y\right)c_{2}\left(t\right)\gamma\mu_{t}\dfrac{\partial H}{\partial y}-\lambda_{t}\left(V-H\right)\left(\dfrac{1}{\lambda_{t}}\right)^{\prime}\\
-c_{2}\left(t\right)\dfrac{\gamma}{\lambda_{t}}\displaystyle\int_{\mathbb{R}}\left[H^{e}\left(t,y+\lambda_{t}u\right)\left(H\left(t,y+\lambda_{t}u\right)-H\left(t,y\right)\right)\right.\\
\left.-u\lambda_{t}H^{e}\left(t,y\right)\dfrac{\partial H}{\partial y}\left(t,y\right)\right]\nu_{t}\left(\mathrm{d}u\right),
\end{array}
$$

does not depends on V. Then, by differentiation with respect to V, we obtain

$$
\begin{array}{rcl}
0 & = & H^{e}\left(t,y\right)c_{2}\left(t\right)\dfrac{\gamma^{2}}{\lambda_{t}}y\mu_{t}\dfrac{\partial H}{\partial y}-\lambda_{t}\left(\dfrac{1}{\lambda_{t}}\right)^{\prime}\\
 & & +c_{2}\left(t\right)\dfrac{\gamma^{2}}{\lambda_{t}^{2}}\displaystyle\int_{\mathbb{R}}\left[\left(y+\lambda_{t}u\right)H^{e}\left(t,y+\lambda_{t}u\right)\left(H\left(t,y+\lambda_{t}u\right)-H\left(t,y\right)\right)\right.\\
 & & \left.-u\lambda_{t}yH^{e}\left(t,y\right)\dfrac{\partial H}{\partial y}\left(t,y\right)\right]\nu_{t}\left(\mathrm{d}u\right).
\end{array}
$$

or equivalently,

$$
\begin{array}{rcl}
\dfrac{\lambda_{t}^{2}}{c_{2}\left(t\right)\gamma^{2}}\left(\dfrac{1}{\lambda_{t}}\right)^{\prime} & = & yH^{e}\left(t,y\right)\mu_{t}\dfrac{\partial H}{\partial y}\\
 & & +\dfrac{1}{\lambda_{t}}\displaystyle\int_{\mathbb{R}}\left[\left(y+\lambda_{t}u\right)H^{e}\left(t,y+\lambda_{t}u\right)\left[H\left(t,y+\lambda_{t}u\right)-H\left(t,y\right)\right]\right.\\
 & & \left.-u\lambda_{t}yH^{e}\left(t,y\right)\dfrac{\partial H}{\partial y}\left(t,y\right)\right]\nu_{t}\left(\mathrm{d}u\right).
\end{array}
$$

By differentiating again with respect to V, we obtain

$$
\begin{array}{rcl}
0 & = & y^{2}H^{e}\left(t,y\right)\mu_{t}\dfrac{\partial H}{\partial y}\\
 & & +\dfrac{1}{\lambda_{t}}\displaystyle\int_{\mathbb{R}}\left[\left(y+\lambda_{t}u\right)^{2}H^{e}\left(t,y+\lambda_{t}u\right)\left[H\left(t,y+\lambda_{t}u\right)-H\left(t,y\right)\right]\right.\\
 & & \left.-u\lambda_{t}y^{2}H^{e}\left(t,y\right)\dfrac{\partial H}{\partial y}\left(t,y\right)\right]\nu_{t}\left(\mathrm{d}u\right)\\
0 & = & y^{2}\mu_{t}\dfrac{\partial H}{\partial y}\\
 & & +\dfrac{1}{\lambda_{t}}\displaystyle\int_{\mathbb{R}}\left[\left(y+\lambda_{t}u\right)^{2}H^{E}\exp\left\{-\gamma Vu\right\}\left[H\left(t,y+\lambda_{t}u\right)-H\left(t,y\right)\right]\right.\\
 & & \left.-u\lambda_{t}y^{2}\dfrac{\partial H}{\partial y}\left(t,y\right)\right]\nu_{t}\left(\mathrm{d}u\right),
\end{array}
$$

where H^E denotes $\exp\left\{\frac{\gamma}{\lambda_t}\int_y^{y+\lambda_t u} H \mathrm{d}w\right\} > 0$. So again, we have an equation with the left hand side is independent of V, but the right hand side is strictly decreasing in V.

Note that we obtain the same results having only jumps, with the drift part being zero. So in the risk-averse case we can expect to find a solution to the existence of an equilibrium only in the case in which the noise trader's demand process presents only a diffusion part.

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Appendix B

A continuous auction model with insiders and random time of information release

A continuous auction model with insiders and random time of information release

José Manuel Corcuera[*], Giulia Di Nunno[†], Gergely Farkas[‡], Bernt Øksendal[§]

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Abstract

In a unified framework we study equilibrium in the presence of an insider having information on the signal of the firm value, which is naturally connected to the fundamental price of the firm related asset. The fundamental value itself is announced at a future random (stopping) time. We consider the two cases in which this release time of information is known and not known, respectively, to the insider. Allowing for very general dynamics, we study the structure of the insider's optimal strategies in equilibrium and we discuss market efficiency. With respect to market efficiency, we show that in the case the insider knows the release time of information, the market is fully efficient. In the case the insider does not know this random time, we see that there is no full efficiency, but there is nevertheless an equilibrium where the sensitivity of prices is decreasing in time according with the probability that the announcement time is greater than the current time. In other words, the prices become more and more stable as the announcement approaches. Finally we couple our results to the tools of initial and progressive enlargement of filtrations to compute explicit insider's strategies. New and extended results on the theory of enlargement of filtrations are also presented.

Key words: Market microstructure, equilibrium, insider trading, stochastic control, semimartingales, enlargement of filtrations.

JEL-Classification C61· D43· D44· D53· G11· G12· G14

MS-Classification 2010: 60G35, 62M20, 93E10, 94Axx

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1 Introduction

Models of financial markets with the presence of an insider or informational asymmetries have a large literature, see e.g Karatzas and Pikovsky (1996), Amendiger *et. al.* (1998), Imkeller *et. al.* (2001), Corcuera *et. al.* (2004), Biagini and Øksendal (2005), (2006), Kohatsu-Higa (2007), Di Nunno *et. al.* (2006, 2008), Biagini *et. al.* (2012) and the references therein. In most of these models prices are fixed exogenously, i.e. the insider does not affect the stock price dynamics, and the privileged information is a functional of the stock price process: the maximum, the final value, etc. As pointed by Danilova (2010), in an equilibrium situation market prices are determined by the demand of market participants, so in such a situation the privileged information cannot be a functional of the stock price process because this implies the knowledge of future demand and it is unrealistic. Then the privileged information is exogenous like the value of the fundamental price, or some signal of it, or the announcement time of the release of the fundamental price, which evolves independently of the demand. The questions considered in this paper deal with the existence of an equilibrium and the properties of the insider's optimal strategies. Moreover another question studied is the efficiency of the market, namely the conditions in which market prices converge to the fundamental one. These problems have been addressed in different works, with different degrees of generality, and with very different types of insider's privileged information. Starting from the seminal papers of Kyle (1985) and Back (1992), we can now refer to more recent publications such as Back and Pedersen (1998), Cho (2003), Lasserre (2004a, 2004b), Aase *et. al.* (2012a), (2012b), Campi and Çetin (2007), Danilova (2010), Caldentey and Stacchetti (2010) and Campi *et. al.* (2012).

The present paper extends the previous contributions in different ways. Indeed we consider prices determined by the demand of the market participants and their knowledge about the fundamental value of the asset. Specifically we consider the very general case in which an insider has access to some signal related to the firm value, which is in fact released at some stopping time. We first consider the case where the insider knows the random time of release of information and then the case where this is also unknown to her. We study these two situations in the same framework with the purpose of analyzing equilibrium and efficiency of the market. In this study we show that the presence of the insider can be beneficial to the market. In fact, if the insider knows the random release time, then the market is efficient. However, if this time of release is unknown also to the insider, then the market is *not* fully efficient, nevertheless there exists an equilibrium where the sensitivity of prices is decreasing in time according with the probability that the announcement time is greater than the current time. In other words, prices are becoming more and more stable as the announcement time is approaching.

As far as we know this generality of the insider's information together with the presence of a random time of release has never been studied before. Moreover, our contribution includes also very general dynamics for the demand process. In fact the insider's demand is allowed to be a general semimartingale. In this setting we also prove that, in the case when the insider knows the release time, market efficiency is reached if and only if the insider's demand is a finite variation process with continuous trajectories.

The present paper includes also various examples in which we give explicit insider's optimal strategies. Here we show how our results, coupled with the mathematical tools of enlargement of filtrations (both initial and progressive) allow to finding the insider's optimal strategy in various cases presented in the literature, but here treated in a unified framework. We remark that, to allow for applications, we have improved various results in the theory of progressive enlargement of filtrations, these results have also independent mathematical interest.

The paper is structured as follows. In the next section we describe the model that gives rise the stock prices. In the third section we discuss the insider's optimal strategies. In section four and five we discuss what happens when the release time is known to the insider or not, respectively. In section six we review the results about the enlargement of filtration problem and provide new ones. Finally we apply these results to find explicit equilibrium strategies.

2 The model and equilibrium

We consider a market with two assets, a stock of a firm and a bank account with interest rate r equal to zero for the sake of simplicity. With abuse of terminology we will just write prices even though they are sometimes "discounted" prices. The trading is continuous in time over the period $[0,\infty)$ and it is order driven. There is a (possibly random) release time τ where the *fundamental* value of the stock is revealed. The fundamental value process, that we shall define in a precise way later, is denoted by V. We shall denote the market price of the stock at time t by P_t. Just after the revelation time the market price and the fundamental value will coincide. So, in principle, it is possible that $P_t \neq V_t$ if $t \leq \tau$ and $P_t = V_t$ if $t > \tau$. Note that if V is continuous $P_{\tau+} = V_\tau$.

We take for granted that all the processes mentioned below are defined in the same, *complete*, probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and that the filtrations are *complete and right-continuous.*

There are three kinds of traders. A *large* number of liquidity traders, who trade for liquidity or hedging reasons, an informed trader or insider, who has privileged information about the firm and can deduce the fundamental price, and the market makers, who set the price and clear the market.

Let X be the demand process of the informed trader. At time t, her information is given by $\mathcal{H}_t$ and her flow of information is given by the filtration $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$. Hence X is an $\mathbb{H}$-predictable process. The informed trader, like any other trader, observes the market prices P and, in addition, she has access to some *signal process* η related to the firm value. Moreover, she will have *some* knowledge about the random time τ. In the sequel we will consider two cases:

- $\mathcal{H}_t = \sigma(P_s, \eta_s, \tau, 0 \leq s \leq t)$, i.e. the informed trader has knowledge of the time of release of information
- $\mathcal{H}_t = \sigma(P_s, \eta_s, \tau \wedge s, 0 \leq s \leq t)$, i.e. the informed trader has no knowledge of this release time, but she will instantly know when it happens.

In both cases, the insider has access to the fundamental value and, in terms of the insider's information flow, this is assumed to be a martingale of form:

$$V_t = \mathbb{E}(f(\eta_\tau)|\mathcal{H}_t), \quad t \geq 0.$$

where f is a non-negative deterministic function. The explicit presence of f gives more flexibility in the relationship between the type of signal and the fundamental price, see Example 28 and Remark 10. Moreover we assume that the process V is *continuous* and that $\sigma_V^2(t) := \frac{d[V,V]_t}{dt}$ is well defined.

The informed trader is assumed risk-neutral and she aims at maximizing her expected final wealth. Let W be the wealth process corresponding to insider's portfolio X.

Definition 1 *A strategy X is called* optimal *with respect to a price process P if it maximizes $E(W_{\tau+})$.*

Let Z be the *aggregate* demand process of the liquidity traders. We recall that these are a large number of traders motivated by liquidity or hedging reasons. They are perceived as constituting noise in the market, thus also called *noise* traders. From the insider's perspective we assume that Z is an $\mathbb{H}$-martingale, independent of η and V. Moreover, we are going to assume that Z is a *continuous* $\mathbb{H}$-martingale, even though some of the following calculations can be carried through in the case of jumps. For later use we also assume that $\sigma_t^2 := \frac{\mathrm{d}[Z,Z]_t}{\mathrm{d}t}$ is well defined.

Market makers clear the market giving the market prices. They rely on the information given by the total aggregate demand $Y := X + Z$ which they observe and, just like the noise traders, they instantly know about the time of release of information when that occurs. Hence their information flow is: $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, where $\mathcal{F}_t = \sigma(Y_s, \tau \wedge s, 0 \leq s \leq t)$. Due to the competition among market makers, the market prices are *rational*, or *competitive*, in the sense that

$$P_t = \mathbb{E}(V_t|\mathcal{F}_t), \quad t \geq 0.$$

Finally we suppose that market makers give market prices through a pricing rule, which consists of a formula, here assumed of the form:

$$P_t = H(t, \xi_t), t \geq 0$$

involving

$$\xi_t := \int_0^t \lambda(s) \mathrm{d}Y_s,$$

where $\lambda \in C^1$ is a strictly positive deterministic function, $H \in C^{1,2}$, and $H(t, \cdot)$ is strictly increasing for every $t \geq 0$. Note that $\mathcal{F}_t = \sigma(P_s, \tau \wedge s, 0 \leq s \leq t)$, for all t. We have the following definition.

Definition 2 *Denote the class of such pairs (H, λ) above by $\mathfrak{H}$. An element of $\mathfrak{H}$ is called a pricing rule.*

Remark 3 *It is important to remark that the effect of the total demand in prices is due not only to the function λ but also to the function H. In fact, as we shall see later, in the equilibrium*

$$\mathrm{d}P_t = \frac{\partial H(t, \xi_t)}{\partial y} \lambda(t) \mathrm{d}Y_t,$$

and some authors give the name market depth *to the quantity*

$$\frac{1}{\frac{\partial H(t, Z_t)}{\partial y} \lambda(t)}.$$

So, to say that market depth is constant is not *equivalent to say that $\lambda(t)$ is constant. Only if the equilibrium pricing rule is linear, the two statements are equivalent. See Back and Pedersen (1998).*

Remark 4 *We remark that the random release time τ is actually a* stopping time *with respect to the filtration $\mathbb{S} = (\mathcal{S}_t)_{t \geq 0}$, where $\mathcal{S}_t = \sigma\{\tau \wedge s, 0 \leq s \leq t\}$. Indeed, for all t, $\{\tau \leq t\} \in \mathcal{S}_t$:*

$$\{\tau \leq t\} = \bigcap_{n=1}^{\infty} \left\{ \tau < t + \frac{1}{n} \right\} = \bigcap_{n=1}^{\infty} \left\{ \tau \wedge (t + \frac{1}{n}) < t + \frac{1}{n} \right\} \in \bigcap_{n=1}^{\infty} \mathcal{S}_{t + \frac{1}{n}} = \mathcal{S}_t.$$

For the last equality we only need $\mathbb{S}$ to be complete since the process $(\tau \wedge t)_{t \geq 0}$ is continuous. Hence τ is actually a stopping time *for the insider and the market makers in the market.*

In the sequel we are going to consider two cases corresponding to the above different insider's information flows. In the first case, we will assume that τ is bounded, in the second case, we will assume that τ is independent of (V, P, Z).

We introduce the following definition.

Definition 5 *Let $(H, \lambda) \in \mathfrak{H}$ and consider a strategy X. The triple (H, λ, X) is an (a local) equilibrium, if the price process $P_{\cdot} := H(\cdot, \xi_{\cdot})$ is rational, given X, that is*

$$P_t = \mathbb{E}(V_t|\mathcal{F}_t),$$

and the strategy X is (locally) optimal, given (H, λ).

3 Insider's optimal strategies

To illustrate the relationship among the processes V, P, X, and W we first consider a multi-period model where trades are made at times $i = 1, 2, \dots N$, and where $\tau = N$ is random. If at time $i-1$, there is an order of buying $X_i - X_{i-1}$ shares, its *cost* will be $P_i(X_i - X_{i-1})$, so, there is a change in the bank account given by

$$-P_i(X_i - X_{i-1}).$$

Then the total (cumulated) change at $\tau = N$ is

$$-\sum_{i=1}^{N} P_i(X_i - X_{i-1}),$$

and due to the convergence of the market and the fundamental prices *just after* time $\tau = N$, there is the extra income: $X_N V_N$. So, the total wealth $W_{\tau+}$ (i.e. just after τ) is

$$\begin{aligned} W_{\tau+} &= -\sum_{i=1}^{N} P_i(X_i - X_{i-1}) + X_N V_N \\ &= -\sum_{i=1}^{N} P_{i-1}(X_i - X_{i-1}) - \sum_{i=1}^{N} (P_i - P_{i-1})(X_i - X_{i-1}) + X_N V_N \end{aligned}$$

Consider now the continuous time setting where we have the processes X, P, and V, and we take N trading periods, where N is random and the trading times are: $0 \le t_1 \le t_2 \le ... \le t_N = \tau$, then we have

$$W_{\tau+} = -\sum_{i=1}^{N} P_{t_{i-1}}(X_{t_i} - X_{t_{i-1}}) - \sum_{i=1}^{N} (P_{t_i} - P_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}) + X_{t_N} V_{t_N},$$

so if the time between trades goes to zero we will have

$$\begin{aligned} W_{\tau+} &= X_\tau V_\tau - \int_0^\tau P_{t-}\mathrm{d}X_t - [P,X]_\tau \\ &= \int_0^\tau X_{t-}\mathrm{d}V_t + \int_0^\tau V_{t-}\mathrm{d}X_t + [V,X]_\tau - \int_0^\tau P_{t-}\mathrm{d}X_t - [P,X]_\tau \\ &= \int_0^\tau (V_{t-} - P_{t-})\,\mathrm{d}X_t + \int_0^\tau X_{t-}\mathrm{d}V_t + [V,X]_\tau - [P,X]_\tau \end{aligned} \tag{1}$$

where (and throughout the whole article) $P_{t-} = \lim_{s\uparrow t} P_s$ a.s.. We recall that V is continuous, hence $V_\tau = V_{\tau+}$, and that X is an $\mathbb{H}$-adapted (in fact *predictable*) càdlàg process. In addition we require that X is an $\mathbb{H}$-semimartingale, so that the stochastic integrals above can be seen as Itô's integrals. Moreover, note that, because of the pricing rule, P is an $\mathbb{H}$-semimartingale.

In this section we discuss the characterization of an insider's optimal strategy in equilibrium in terms of fundamental value and insider information. Namely, we consider a process X that is optimal in the sense that it maximizes

$$J(X) := \mathbb{E}\left(W_{\tau+}\right) = \mathbb{E}\left(\int_0^\tau (V_t - H(t-,\xi_{t-}))\mathrm{d}X_t + \int_0^\tau X_{t-}dV_t + [V,X]_\tau - [P,X]_\tau\right),$$

for some pricing rule $(H,\lambda) \in \mathfrak{H}$. We characterize the *admissible* triplets (H,λ,X) as those processes X (that include, by hypothesis, the process $X \equiv 0$) and price functions $(H,\lambda) \in \mathfrak{H}$ satisfying:

(A1) $X_t = M_t + A_t + \int_0^t \theta_s \mathrm{d}s$, where M is a continuous $\mathbb{H}$-martingale, A a finite variation $\mathbb{H}$-predictable process with $A_t = \sum_{0<s\le t}(X_s - X_{s-})$, and θ a càdlàg, $\mathbb{H}$-adapted, process.

(A2) $\mathbb{E}\left(\int_0^\tau (\partial_2 H(s,\xi_s))^2 \left(\sigma_s^2 \mathrm{d}s + \mathrm{d}[M,M]_s\right)\right) < \infty$.

(A3) $\mathbb{E}\left(\int_0^\tau \partial_2 H(s,\xi_s)|\theta_s|\mathrm{d}s\right) < \infty$.

(A4) $\mathbb{E}\left(\sum_0^\tau \partial_2 H(s-,\xi_{s-})|\Delta X_s|\right) < \infty$, $\Delta X_s := X_s - X_{s-}$.

(A5) $\mathbb{E}\left(\int_0^\tau |X_s|^2 \sigma_V^2(s)\mathrm{d}s\right) < \infty$, where $\sigma_V^2(s) := \frac{\mathrm{d}[V,V]_s}{\mathrm{d}s}$.

∂_i indicates the derivative w.r.t. the i argument.

Remark 6 *Note that, since $(X_t)_{t\ge 0}$ is a càdlàg* $\mathbb{H}$-predictable *process, its martingale part cannot have jumps, see Corollary 2.31 in Jacod and Shiryaev (1987).*

3.1 The optimality condition

In the sequel we will consider two kinds of stopping times: τ bounded, or τ independent of (V, P, Z). In both cases, by the assumption (A5), we have that $\mathbb{E}(\int_0^\tau X_t \mathrm{d}V_t) = 0$. Hence,

$$J(X) := \mathbb{E}\left(W_{\tau+}\right) = \mathbb{E}\left(\int_0^\tau (V_t - H(t, \xi_{t-}))\mathrm{d}X_t + [V, X]_\tau - [P, X]_\tau\right).$$

Suppose that X is (locally) optimal. Then, for all β such that $X_\cdot + \varepsilon \int_0^\cdot \beta_s \mathrm{d}s$ is admissible, with $\varepsilon > 0$ small enough, we have

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} J(X_\cdot + \varepsilon \int_0^\cdot \beta_s \mathrm{d}s)\Big|_{\varepsilon=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbb{E}\left(\int_0^\tau (V_t - H(t, \int_0^{t-} \lambda(s)(\mathrm{d}X_s + \varepsilon\beta_s \mathrm{d}s + \mathrm{d}Z_s)))(\mathrm{d}X_t + \varepsilon\beta_t \mathrm{d}t)\right)\Big|_{\varepsilon=0} \\ &= \mathbb{E}\left(\int_0^\tau (V_t - H(t, \xi_t))\beta_t \mathrm{d}t\right) + \mathbb{E}\left(\int_0^\tau -\partial_2 H(t, \xi_{t-})\left(\int_0^t \lambda(s)\beta(s)\mathrm{d}s\right)\mathrm{d}X_t\right) \\ &= \mathbb{E}\left(\int_0^\tau \left((V_t - H(t, \xi_t)) - \lambda(t)\int_t^\tau \partial_2 H(s, \xi_{s-})\mathrm{d}X_s\right)\beta_t \mathrm{d}t\right). \end{aligned}$$

Since we can take $\beta_t = \mathbf{1}_{(u,u+h]}(t)\ \alpha_u$, with α_u $\mathcal{H}_u$-measurable and bounded, we have

$$\mathbb{E}\left(\int_u^{u+h}\left(\mathbb{E}(\mathbf{1}_{[0,\tau]}(t)\,(V_t - H(t, \xi_t))\big|\,\mathcal{H}_t) - \lambda(t)\mathbb{E}\left(\int_t^\infty \mathbf{1}_{[0,\tau]}(s)\partial_2 H(s, \xi_{s-})\mathrm{d}X_s\Big|\,\mathcal{H}_t\right)\right)\mathrm{d}t\Big|\,\mathcal{H}_u\right) = 0 \quad (2)$$

and this means that the process M_t, $t \geq 0$:

$$M_t := \int_0^t \left(\mathbb{E}(\mathbf{1}_{[0,\tau]}(u)V_u|\mathcal{H}_u) - \mathbb{E}(\mathbf{1}_{[0,\tau]}(u)H(u, \xi_u)|\mathcal{H}_u) - \lambda(u)\mathbb{E}(\int_u^\infty \mathbf{1}_{[0,\tau]}(s)\partial_2 H(s, \xi_{s-})\mathrm{d}X_s|\mathcal{H}_u\right)\mathrm{d}u$$

is an $\mathbb{H}$-martingale. In particular this implies that, for a.a. $t \geq 0$,

$$\mathbb{E}(\mathbf{1}_{[0,\tau]}(t)V_t|\mathcal{H}_t) - \mathbb{E}(\mathbf{1}_{[0,\tau]}(t)H(t, \xi_t)|\mathcal{H}_t) - \lambda(t)\mathbb{E}(\int_t^\infty \mathbf{1}_{[0,\tau]}(s)\partial_2 H(s, \xi_{s-})\mathrm{d}X_s|\mathcal{H}_t) = 0, a.s.. \quad (3)$$

Since τ is an $\mathbb{H}$-stopping time, then for a.a. t and for a.a. $\omega \in \{\tau \geq t\}$, or equivalently a.s. on the stochastic interval $[0, \tau]$, we can write

$$V_t - H(t, \xi_t) - \lambda(t)\mathbb{E}\left(\int_t^\tau \partial_2 H(s, \xi_{s-})\mathrm{d}X_s\Big|\,\mathcal{H}_t\right) = 0. \quad (4)$$

As a summary we have the following necessary condition to help identifying good candidates as insider's optimal strategies.

Proposition 7 *An admissible triple (H, λ, X) such that X is locally optimal for the insider, satisfies equation* (3) *or, equivalently, it satisfies equation* (4) *a.s. in* $[0, \tau]$.

In the sequel we study two different cases of knowledge of τ from the insider's perspective. First the case in which the insider knows τ, the exact time of release of information about the firm value, then we study the case when the insider does not know τ.

4 Case when τ is known to the insider

Let $\sigma(\tau)$ be the σ-algebra generated by τ. Then we consider the case in which $\sigma(\tau) \subseteq \mathcal{H}_0$. At any time t, the insider relies on the information given by:

$$\mathcal{H}_t = \sigma(P_s, \eta_s, \tau, 0 \leq s \leq t).$$

Moreover, we assume that τ is bounded, so the analysis here below is consistent with the one of the previous section.

Recall that $V_{\tau-} = V_\tau = V_{\tau+} = P_{\tau+}$. However, the relationship between V and P up to τ is a matter of study. Our first observation is that optimal strategies lead the market price to the fundamental one, making the market be efficient. In fact we have the following

Proposition 8 *If τ is known to the insider and (H, λ, X) is admissible with X locally optimal then the market is efficient, i.e.*

$$V_\tau = P_\tau = H(\tau, \xi_\tau) = H(\tau-, \xi_{\tau-}) = P_{\tau-} \quad a.s..$$

Proof. By the assumptions (A1) and (A2), equation (4) can be rewritten:

$$\begin{aligned}
& V_t - H(t, \xi_t) - \lambda(t)\mathbb{E}(\int_t^\tau \partial_2 H(s-, \xi_{s-}) \mathrm{d}X_s | \mathcal{H}_t) \\
= \; & V_t - H(t, \xi_t) - \lambda(t)\mathbb{E}(\int_t^\tau \partial_2 H(s, \xi_s) \theta_s \mathrm{d}s | \mathcal{H}_t) \\
& -\lambda(t)\mathbb{E}(\sum_t^\tau \partial_2 H(s-, \xi_{s-}) \Delta X_s | \mathcal{H}_t) \\
= \; & V_t - H(t, \xi_t) - \lambda(t) \int_t^\tau \mathbb{E}(\partial_2 H(s, \xi_s) \theta_s | \mathcal{H}_t) \mathrm{d}s \\
& -\lambda(t) \sum_t^\tau \mathbb{E}(\partial_2 H(s-, \xi_{s-}) \Delta X_s | \mathcal{H}_t) \\
= \; & 0, \text{ a.s. on } [0, \tau].
\end{aligned}$$

Now by the assumption (A3) we have that

$$\int_t^\tau E\left(\partial_2 H(s,\xi_s)|\theta_s| : |\mathcal{H}_0\right) \mathrm{d}s < \infty, \text{ a.s. on } [0,\tau]$$

then

$$\lim_{t\uparrow\tau} \mathbb{E}\left(\mathbb{E}\left(\int_t^\tau \partial_2 H(s,\xi_s)|\theta_s| \mathrm{d}s \middle| \mathcal{H}_t \right) \middle| \mathcal{H}_0 \right) = 0, \quad \text{a.s..}$$

and $\mathbb{E}\left(\int_t^\tau \partial_2 H(s,\xi_s)|\theta_s|\middle| \mathcal{H}_t\right) \mathrm{d}s$ converges in L^1 to zero (where the expectation is taken with respect to the conditional probability, fixed τ) and since it is a positive supermartingale it converges almost surely to zero. Analogously for the term

$$\lambda(t) \sum_t^\tau \mathbb{E}(\partial_2 H(s-,\xi_{s-})\Delta X_s|\mathcal{H}_t).$$

So, since $\lambda(t)$ is continuous, then $V_\tau = V_{\tau-} = H(\tau-,\xi_{\tau-}) = P_{\tau-}$, a.s.. On the other side, we recall that $V_\tau = V_{\tau+} = P_{\tau+} = H(\tau+,\xi_{\tau+})$, a.s.. ■

Remark 9 *In Aase, Bjuland and Øksendal (2012a) it was already observed that market efficiency, that is the convergence of market prices to the fundamental ones, is a consequence of the optimality of the insider's strategy. Here we obtain an extension of this result for a more general framework.*

Remark 10 *This efficiency situation is also the case in Campi and Çetin (2007). In our notation they have the signal $\eta = \bar{\tau}$, with $\bar{\tau}$ an $\mathbb{H}$-stopping time, $V_t = \mathbf{1}_{\{\bar{\tau}>1\}}$ and the release time is $\tau = \bar{\tau} \wedge 1$. So, $\tau \in \mathcal{H}_0$ and it is bounded. Then, they obtain*

$$\mathbf{1}_{\{\bar{\tau}>1\}} - H(\bar{\tau}\wedge 1, \xi_{\bar{\tau}\wedge 1}) = 0, a.s..$$

They also assume that $\bar{\tau}$ is the first passage time of a standard Brownian motion that is independent of Z.

Remark 11 *If we take $V_t \equiv V$ and $\tau \equiv 1$ then we are in Back's framework (1992). There it is shown that market prices converge to V when $t \to 1$.*

Proposition 12 *Consider an admissible triple (H,λ,X) then if (H,λ,X) is a local equilibrium, we have:*

(i) $H(\tau,\xi_\tau) = V_\tau$ *a.s.,* ,

(ii) $\dfrac{\partial_1 H(t,\xi_t)}{\lambda(t)} + \dfrac{1}{2}\partial_{22} H(t,\xi_t)\lambda(t)\sigma_{Y,t}^2 = 0$ *a.s.on* $[0,\tau)$

(iii) Y *is a local martingale*

(iv) *If* $V_t \neq P_t$ *a.s.on* $[0,\tau)$, *then* $\lambda(t) = \lambda_0$, *where* $\sigma^2_{Y,s} := \frac{\mathrm{d}[Y,Y]_s}{\mathrm{d}s}$

Proof. (i) It is just Proposition 8. (ii) By using Itô's formula on $\frac{H(t,\xi_t)}{\lambda(t)}$, we have

$$\begin{aligned}
& \mathbb{E}\left(\int_t^\tau \frac{1}{\lambda(s)}\partial_2 H(s,\xi_{s-})\mathrm{d}\xi_s \,|\mathcal{H}_t\right) \\
= \; & \mathbb{E}\left(\frac{H(\tau,\xi_\tau)}{\lambda(\tau)}\bigg|\,\mathcal{H}_t\right) - \frac{H(t,\xi_t)}{\lambda(t)} \\
& -\mathbb{E}\left(\int_t^\tau \left(-\frac{\lambda'(s)}{\lambda^2(s)}H(s,\xi_s) + \frac{\partial_1 H(s,\xi_s)}{\lambda(s)} + \frac{1}{2}\partial_{22}H(s,\xi_s)\lambda(s)\sigma^2_{Y,s}\right)\mathrm{d}s\bigg|\,\mathcal{H}_t\right) \\
& -\mathbb{E}\left(\sum_{t\le s\le \tau}\left(\frac{\Delta H(s,\xi_s)}{\lambda(s)} - \partial_2 H(s,\xi_{s-})\Delta X_s\right)\bigg|\,\mathcal{H}_t\right),
\end{aligned}$$

where $\sigma^2_{Y,s} := \frac{\mathrm{d}[Y,Y]_s}{\mathrm{d}s}$. Now X is locally optimal, given (H,λ) , by the equation (4) and the Proposition 8 we can write:

$$\begin{aligned}
0 \; = \; & V_t - \lambda(t)\mathbb{E}\left(\frac{V_\tau}{\lambda(\tau)}\bigg|\,\mathcal{H}_t\right) \\
& +\lambda(t)\int_t^\tau \mathbb{E}\left(-\frac{\lambda'(s)}{\lambda^2(s)}H(s,\xi_s) + \frac{\partial_1 H(s,\xi_s)}{\lambda(s)} + \frac{1}{2}\partial_{22}H(s,\xi_s)\lambda(s)\sigma^2_{Y,s}\bigg|\,\mathcal{H}_t\right)\mathrm{d}s \\
& +\lambda(t)\sum_{t\le s\le\tau}\mathbb{E}\left(\left(\frac{\Delta H(s,\xi_s)}{\lambda(s)} - \partial_2 H(s,\xi_{s-})\Delta X_s\right)\bigg|\,\mathcal{H}_t\right)
\end{aligned}$$

Hence, we have

$$\begin{aligned}
0 \; = \; & \frac{V_t}{\lambda(t)} - \frac{V_t}{\lambda(\tau)} + \int_t^\tau \mathbb{E}\left(-\frac{\lambda'(s)}{\lambda^2(s)}H(s,\xi_s) + \frac{\partial_1 H(s,\xi_s)}{\lambda(s)} + \frac{1}{2}\partial_{22}H(s,\xi_s)\lambda(s)\sigma^2_s\bigg|\,\mathcal{H}_t\right)\mathrm{d}s \\
& +\sum_{t\le s\le\tau}\mathbb{E}\left(\left(\frac{\Delta H(s,\xi_s)}{\lambda(s)} - \partial_2 H(s,\xi_{s-})\Delta X_s\right)\bigg|\,\mathcal{H}_t\right).
\end{aligned}$$

By identifying the predictive and martingale parts we have that

$$\begin{aligned}
0 \; = \; & \frac{\lambda'(t)}{\lambda^2(t)}V_t - \frac{\lambda'(t)}{\lambda^2(t)}H(t,\xi_t) + \frac{\partial_1 H(t,\xi_t)}{\lambda(t)} + \frac{1}{2}\partial_{22}H(t,\xi_t)\lambda(t)\sigma^2_{Y,t} \\
& +\frac{\Delta H(t,\xi_t) - \partial_2 H(t,\xi_{t-})\Delta\xi_t}{\lambda(t)}, \text{ a.s. on } [0,\tau] \; . \qquad (5)
\end{aligned}$$

Then a.s on $[0,\tau]$, the continuous and jump parts of the r.h.s of the previous equation will be equal to zero. So

$$\frac{\Delta H(t,\xi_t) - \partial_2 H(t,\xi_{t-})\Delta\xi_t}{\lambda(t)} = 0, \text{ a.s. on } [0,\tau]$$

and

$$0 = \frac{\lambda'(t)}{\lambda^2(t)} V_t - \frac{\lambda'(t)}{\lambda^2(t)} H(t,\xi_t) + \frac{\partial_1 H(t,\xi_t)}{\lambda(t)} + \frac{1}{2}\partial_{22} H(t,\xi_t)\lambda(t)\sigma^2_{Y,t}.$$

(iii) Now, since we are in a local equilibrium, prices are rational, given X, so by taking conditional expectations w.r.t $\mathcal{F}_t$ we have

$$\begin{aligned} 0 &= \frac{\lambda'(t)}{\lambda^2(t)} \left(\mathbb{E}(V_t|\mathcal{F}_t) - \mathbb{E}(H(t,\xi_t)|\mathcal{F}_t)\right) + \frac{\partial_1 H(t,\xi_t)}{\lambda(t)} + \frac{1}{2}\partial_{22} H(t,\xi_t)\lambda(t)\sigma^2_{Y,t} \\ &= \frac{\partial_1 H(t,\xi_t)}{\lambda(t)} + \frac{1}{2}\partial_{22} H(t,\xi_t)\lambda(t)\sigma^2_{Y,t}, \end{aligned} \tag{6}$$

consequently

$$\mathrm{d}P_t = dH(t,\xi_t) = \lambda_t \partial_2 H(t,\xi_{t-})\mathrm{d}Y_t,$$

and, since $P.$ is a martingale and $\lambda_t \partial_2 H(t,y) > 0$, we have that Y is a local martingale. (iv) Finally, from (5) we have that

$$\frac{\lambda'(t)}{\lambda^2(t)} V_t - \frac{\lambda'(t)}{\lambda^2(t)} H(t,\xi_t) = 0$$

then $V_t \neq H(t,\xi_t)$ implies that $\lambda'(t) = 0$. ■

4.1 Characterization of the equilibrium

In this subsection we shall give sufficient conditions to guarantee that (H,λ,X) is an equilibrium. We shall assume that the pricing rules satisfy

$$0 = \partial_1 H(t,y) + \frac{1}{2}\partial_{22} H(t,y)\lambda(t)^2\sigma_t^2 \ , \ \text{a.a. } 0 \le t \le 1, \ y \in \mathbb{R}, \tag{7}$$

and note that this condition is close to condition (ii) in Proposition 12. that is a necessary condition for the equilibrium. We shall also assume that $\sigma_t^2 = \sigma^2(t)$, deterministic, in such a way that Z is a process with independent increments (since it has not jumps it is in fact a Gaussian process). Then we have the following sufficient condition for the equilibrium:

Theorem 13 *Consider an admissible triple (H,λ,X) with (H,λ) satisfying (7), then (H,λ,X) is an equi-*

librium, if and only if:

$$(i)\ \lambda(t) = \lambda_0,$$

$$(ii)\ H(\tau, \xi_\tau) = V_\tau$$

$$(iii)\ [X^c, X^c]_t \equiv 0,$$

(iv) $X + Z$ *is a local martingale without jumps .*

Proof. Assume (i)-(iv). The proof follows the same steps as in Corcuera et. al. (2014). Set

$$i(v,y) := \int_y^{H^{-1}(1,\cdot)(v)} \frac{v - H(1,x)}{\lambda_0} \mathrm{d}x,$$

and

$$\begin{aligned} I(v,t,y) &:= \mathbb{E}(i(V_t, y + \lambda_0(Z_1 - Z_t))|V_t = v) \\ &= \mathbb{E}(i(v, y + \lambda_0(Z_1 - Z_t))), \quad t \in [0,1]. \end{aligned}$$

Note that in this proof, we write ∂_i to indicate the derivative w.r.t the $i^{th} + 1$ argument.

First note that

$$\mathbb{E}(H(1, y + \lambda_0(Z_1 - Z_t)) = H(t,y).$$

In fact, by (7) and (A2) (also for $X \equiv 0$) , $(H(t, \lambda_0 Z_t))_{0 \le t \le 1}$ is a martingale, so, since Z has independent increments, we have that

$$H(t,y) = \mathbb{E}(H(1, \lambda_0 Z_1)|\lambda_0 Z_t = y) = \mathbb{E}(H(1, y + \lambda_0(Z_1 - Z_t)).$$

$(I(v,t,Z_t))_{0 \le t \le 1}$ is also an $\mathbb{F}^Z$- martingale (where $\mathbb{F}^Z$ is the filtration generated by Z):

$$\begin{aligned} I(v,t,y) &= \mathbb{E}(i(v, y + \lambda_0(Z_1 - Z_t))) \\ &= \mathbb{E}(\ i(v, \lambda_0 Z_1)|\ \lambda_0 Z_t = y), \end{aligned}$$

and we have that

$$
\begin{aligned}
\partial_2 I(v,t,y) &= \mathbb{E}\left(\partial_1 i(v, y+\lambda_0(Z_1-Z_t))\right) \\
&= \mathbb{E}\left(-\frac{v-H(1,y+\lambda_0(Z_1-Z_t))}{\lambda_0}\right) = -\frac{v-H(t,y)}{\lambda_0}.
\end{aligned} \tag{8}
$$

We can take the derivative under the integral sign because $H(1,\cdot)$ is monotone and $\mathbb{E}(H(1,\lambda_0 Z_1)) < \infty$ and, from (7) we obtain

$$
\partial_{12} I + \frac{1}{2}\partial_{222} I \lambda_0^2 \sigma_t^2 = 0
$$

so

$$
\partial_1 I + \frac{1}{2}\partial_{22} I \lambda_0^2 \sigma_t^2 = C(t,v).
$$

Now since $(I(v,t,Z_t))_{0\le t\le 1}$ is a martingale, it turns out that $C(v,t)=0$ a.a. $t\in[0,1]$. Then we obtain that

$$
\partial_1 I + \frac{1}{2}\partial_{22} I \lambda_0^2 \sigma_t^2 = 0. \tag{9}
$$

Now, consider any admissible strategy X, by using Itô's formula, we have

$$
\begin{aligned}
I(V_\tau,\tau,\xi_\tau) &= I(V_0,0,0) + \int_0^\tau \partial_0 I(V_t,t,\xi_t)\mathrm{d}V_t + \int_0^\tau \partial_1 I(V_t,t,\xi_t)\mathrm{d}t \\
&\quad + \int_0^\tau \partial_2 I(V_t,t,\xi_{t-})\mathrm{d}\xi_t + \frac{1}{2}\int_0^\tau \partial_{22} I(V_t,t,\xi_t)\mathrm{d}[\xi^c,\xi^c]_t \\
&\quad + \int_0^\tau \partial_{02} I(V_t,t,\xi_t)\mathrm{d}[\xi^c,V]_t + \frac{1}{2}\int_0^\tau \partial_{00} I(V_t,t,\xi_t)\sigma_V^2\mathrm{d}t \\
&\quad + \sum_{0\le t\le\tau}\left(\Delta I(V_t,t,\xi_t) - \partial_2 I(V_t,t,\xi_{t-})\Delta\xi_t\right).
\end{aligned}
$$

By construction, $\xi_0=0$, by (i) $\mathrm{d}\xi_t = \lambda_0 \mathrm{d}Y_t$. Now we have that

$$
\mathrm{d}[\xi^c,\xi^c]_t = \lambda_0^2\mathrm{d}[X^c,X^c]_t + 2\lambda_0^2\mathrm{d}[X^c,Z]_t + \lambda_0^2\sigma_t^2\mathrm{d}t.
$$

Also by (8) and the fact that V and Z are independent,

$$
\partial_{02} I(V_t,t,\xi_t)\mathrm{d}[\xi^c,V]_t = -\frac{1}{\lambda_0}\mathrm{d}[\xi^c,V]_t = -\mathrm{d}[X,V]_t,
$$

then using (8) and (9), and the fact that Z has not jumps, we get

$$\begin{aligned} I(V_\tau,\tau,\xi_\tau) &= I(V_0,0,0)+\int_0^\tau \partial_0 I(V_t,t,\xi_{t-})\mathrm{d}V_t+\int_0^\tau (P_{t-}-V_t)(\mathrm{d}X_t+\mathrm{d}Z_t)\\ &+\frac{1}{2}\int_0^\tau \partial_{22}I(V_t,t,\xi_{t-})\lambda_0^2\mathrm{d}[X^c,X^c]_t-[X,V]_\tau+\frac{1}{2}\int_0^\tau \partial_{00}I(V_t,t,\xi_t)\sigma_V^2\mathrm{d}t\\ &+\int_0^\tau \partial_{22}I(V_t,t,\xi_t)\lambda_0^2\mathrm{d}[X^c,Z^c]+\sum_{0\le t\le\tau}\big(\Delta I(V_t,t,\xi_t)-\partial_2 I(V_t,t,\xi_{t-})\lambda_0\Delta X_t\big)\end{aligned}$$

Subtracting $[P,X]_\tau$ from both sides and rearranging the terms, we obtain

$$\begin{aligned} &\int_0^\tau (V_t-P_{t-})\mathrm{d}X_t-[P,X]_\tau+[X,V]_\tau-\left(I(V_0,0,0)+\frac{1}{2}\int_0^\tau \partial_{00}I(V_t,t,\xi_t)\sigma_V^2\mathrm{d}t\right)\\ =\;& -I(V_\tau,\tau,\xi_\tau)+\int_0^\tau \partial_0 I(V_t,t,\xi_{t-})\mathrm{d}V_t+\int_0^\tau (P_{t-}-V_t)\mathrm{d}Z_t\\ &+\frac{1}{2}\int_0^\tau \partial_{22}I(V_t,t,\xi_{t-})\lambda_0^2\mathrm{d}[X^c,X^c]_t+\int_0^\tau \partial_{22}I(V_t,t,\xi_{t-})\lambda_0^2\mathrm{d}[X^c,Z^c]\\ &+\sum_{0\le t\le\tau}\big(\Delta I(V_t,t,\xi_t)-\partial_2 I(V_t,t,\xi_{t-}),\lambda_0\Delta X_t\big)-[P,X]_\tau. \end{aligned} \tag{10}$$

We have that

$$[P,X]_\tau=[P^c,X^c]_\tau+\sum_{0\le t\le\tau}\Delta P_t\Delta X_t.$$

Then Itô's formula for H shows that the continuous local martingale part of P is $\int \frac{\partial H}{\partial y}(t,\xi_t)\mathrm{d}\xi_t^c$, so by using (8), we obtain

$$\begin{aligned}[P^c,X^c]_\tau &= \left[\int \partial_1 H(t,\xi_t)\mathrm{d}\xi_t^c,X^c\right]_\tau=\int_0^\tau \partial_1 H(t,\xi_t)\mathrm{d}\left[\xi^c,X^c\right]_t\\ &= \int_0^\tau \partial_{22}I(V_t,t,\xi_t)\lambda_0^2\mathrm{d}\left[X^c,X^c\right]_t+\int_0^\tau \partial_{22}I(V_t,t,\xi_t)\lambda_0^2\mathrm{d}\left[X^c,Z\right]_t,\end{aligned}$$

and

$$\begin{aligned}\lambda_0\partial_2 I(V_t,t,\xi_{t-})\Delta X_t+\Delta P_t\Delta X_t &= (P_{t-}-V_t)\Delta X_t+\Delta P_t\Delta X_t\\ &= (P_t-V_t)\Delta X_t=\lambda_0\partial_2 I(V_t,t,\xi_t)\Delta X_t.\end{aligned}$$

Substituting the above relationships in the right-hand side of the equation (10), we obtain that

$$\begin{aligned}&-I(V_\tau,\tau,\xi_\tau)+\int_0^\tau \partial_0 I(V_t,t,\xi_t)\mathrm{d}V_t+\int_0^\tau (P_t-V_t)\mathrm{d}Z_t-\frac{1}{2}\int_0^\tau \partial_{22}I(V_t,t,\xi_t)\lambda_0^2\mathrm{d}[X^c,X^c]_t\\ &+\sum_{0\le t\le\tau}\big(I(V_t,t,\xi_t)-I(V_t,t,\xi_{t-})-\lambda_0\partial_2 I(V_t,t,\xi_t)\Delta X_t\big).\end{aligned}$$

Now it is important to note that $\partial_{00}I(v,t,y)$ does not depend on y and so $\partial_{00}I(V_t,t,\xi_t)$ does not depend of ξ. Then $I(V_0,0,0)+\frac{1}{2}\int_0^\tau \partial_{00}I(V_t,t,\xi_{t-})\sigma_V^2 \mathrm{d}t$ is actually fixed ω, a lower bound for *any* strategy. Then we will show that, taken the expectation, the right-hand side of (10) is non-positive. The result follows from the following points.

1. We know that $\lambda_0\partial_{22}I(V_\tau,\tau,\xi_\tau)=\partial_2 H(\tau,\xi_\tau)>0$ and that $\lambda_0\partial_2 I(V_\tau,\tau,\xi_\tau)=-V_\tau+H(\tau,\xi_\tau)$ so by hypothesis (ii) we have a maximum value of $-I(V_\tau,\tau,\xi_\tau)$ for our strategy.

2. The processes $\int_0^\cdot \partial_0 I(V_t,t,\xi_t)\mathrm{d}V_t$ and $\int_0^\cdot (P_t-V_t)\mathrm{d}Z_t$ are $\mathbb{F}^{P,V}$-martingale, so they vanish when we take expectations.

3. By (8) and H being increasing monotone, we have that $\partial_{22}I>0$, and the measure $\mathrm{d}[X^c,X^c]\geq 0$, so

$$-\frac{1}{2}\int_0^\tau \partial_{22}I(V_t,t,\xi_t)\lambda_0^2\mathrm{d}[X^c,X^c]_t\leq 0,$$

   and by hypothesis (iv) we obtain the maximum value for our strategy.

4. $\partial_{22}I>0$ (convexity) implies that

$$I(v,t,x+h)-I(v,t,x)-\partial_2 I(v,t,x+h)h\leq 0.$$

   So,

$$\sum_{0\leq t\leq\tau}\big(I(V_t,t,\xi_{t-}+\lambda_0\Delta X_t)-I(V_t,t,\xi_{t-})-\partial_2 I(V_t,t,\xi_t)\lambda_0\Delta X_t\big)\leq 0,$$

   and has its maximum if and only if $\Delta X_t=0$, which is assumed at (iv).

5. Assumption (iv) together with condition (A2) guarantee the rationality of prices.

Conversely, if (H,λ,X) is an equilibrium, (i) is obtained in Proposition 12 and (ii) in Proposition 8. The points 3. and 4. above together with Proposition 12 give (iii) and (iv). ■

5 Case when τ is unknown to the insider

In this section we consider the case when the insider does not know the precise time τ of release of information. Namely, the insider's information flow is given by:

$$\mathcal{H}_t=\sigma(P_s,\eta_s,\tau\wedge s,0\leq s\leq t).$$

Moreover we assume that τ is independent of (V, P, Z), so the analysis here below is consistent with the one in Section 3, and that $\mathbb{P}(\tau > t) > 0$ for all $t \geq 0$. In this context we have the following result.

Proposition 14 *Consider an admissible triple (H, λ, X). If (H, λ, X) is a local equilibrium, we have:*

(i) Y is a local martingale

(ii) If $V_t \neq P_t$ a.s.on $[0, \tau)$, then $\lambda(t) = c\mathbb{P}(\tau > t)$, a.a.$t \geq 0$ $(c > 0)$.

Proof. Going back to Proposition 7, we can see that, on $[0, \tau]$, equation (4) can be written as:

$$V_t - H(t, \xi_t) - \lambda(t)\mathbb{E}(\int_t^\infty \mathbf{1}_{[0,\tau]}(s)\partial_2 H(s, \xi_s)\mathrm{d}X_s|\mathcal{H}_t) = 0.$$

Here we recall that the optimal total demand X for the insider satisfies (A1), (A2), (A3). Then, provided that, for all t, $P(\tau > t) > 0$, we have, on $[0, \tau]$,

$$\begin{aligned} & V_t - H(t, \xi_t) - \lambda(t)\mathbb{E}(\int_t^\infty \mathbb{P}\left(\tau > s|\mathcal{H}_t\right)\partial_2 H(s, \xi_s)\mathrm{d}X_s|\mathcal{H}_t) \\ = \; & V_t - H(t, \xi_t) - \frac{\lambda(t)}{P(\tau > t)}\mathbb{E}(\int_t^\infty \mathbb{P}\left(\tau > s\right)\partial_2 H(s, \xi_s)\mathrm{d}X_s|\mathcal{H}_t) = 0 \end{aligned} \tag{11}$$

and

$$\begin{aligned} 0 \; = \; & V_t - H(t, \xi_t) - \frac{\lambda(t)}{\mathbb{P}(\tau > t)}\mathbb{E}(\int_t^\infty \mathbb{P}\left(\tau > s\right)\partial_2 H(s, \xi_s)\mathrm{d}X_s|\mathcal{H}_t) \\ = \; & V_t - H(t, \xi_t) - \frac{\lambda(t)}{\mathbb{P}(\tau > t)}\mathbb{E}(\int_t^\infty \mathbb{P}\left(\tau > s\right)\partial_2 H(s, \xi_s)\theta_s\mathrm{d}s|\mathcal{H}_t) \\ & - \frac{\lambda(t)}{\mathbb{P}(\tau > t)}\mathbb{E}(\sum_t^\infty \mathbb{P}\left(\tau > s\right)\partial_2 H(s, \xi_{s-})\Delta X_s|\mathcal{H}_t\big) \, . \end{aligned} \tag{12}$$

First of all we note that

$$\lim_{t\to\infty} \mathbb{E}\left(\left|\int_t^\infty \mathbb{P}(\tau > s)\partial_2 H(s, \xi_s)|\theta_s|\mathrm{d}s\right|\right) = 0,$$

by assumption (A3) and applying the dominated convergence theorem. Hence

$$\lim_{t\to\infty} \mathbb{E}\left(\left|\int_t^\infty \mathbb{P}(\tau > s)\partial_2 H(s, \xi_s)|\theta_s|\mathrm{d}s\right|\Big|\mathcal{H}_t\right) = 0, \quad in\ L^1$$

and, since the process $\big(\mathbb{E}\left(\left|\int_t^\infty \mathbb{P}(\tau > s)\partial_2 H(s, \xi_s)|\theta_s|\mathrm{d}s\right|\big|\mathcal{H}_t\right)\big)_{t\geq 0}$ is a positive supermartingale, the convergence holds also a.s.. Analogously for $\mathbb{E}(\sum_t^\infty \mathbb{P}\left(\tau > s\right)\partial_2 H(s, \xi_{s-})|\Delta X_s||\mathcal{H}_t\big)$. Then, from (12), we have that

$$\lim_{t\to\infty} \frac{(V_t - H(t,\xi_t))\,\mathbb{P}(\tau > t)}{\lambda(t)} = 0 \tag{13}$$

in L_1 and a.s.. Applying the Itô's formula to $\frac{H(t,\xi_t)\mathbb{P}(\tau>t)}{\lambda(t)}$, $t \le T$, and studying the limit for $T \to \infty$, we have

$$\begin{aligned}
&\mathbb{E}\left(\int_t^\infty \mathbb{P}(\tau > s)\,\partial_2 H(s,\xi_s)\mathrm{d}X_s \middle| \mathcal{H}_t\right) \\
=\;& \lim_{T\to\infty} \mathbb{E}\left(\frac{H(T,\xi_T)\mathbb{P}(\tau > T)}{\lambda(T)} \middle| \mathcal{H}_t\right) - \frac{H(t,\xi_t)\mathbb{P}(\tau > t)}{\lambda(t)} \\
& -\mathbb{E}\left(\int_t^\infty (\partial_s \left(\frac{\mathbb{P}(\tau > s)}{\lambda(s)}\right) H(s,\xi_s) + \frac{\mathbb{P}(\tau > s)}{\lambda(s)}\partial_1 H(s,\xi_s)\right. \\
& \left. +\frac{1}{2}\partial_{22} H(s,\xi_s)\mathbb{P}(\tau > s)\,\lambda(s)\sigma_s^2)\mathrm{d}s \middle| \mathcal{H}_t\right) \\
& -\mathbb{E}\left(\sum_t^\infty \frac{\mathbb{P}(\tau > s)\Delta H}{\lambda(s)} - \mathbb{P}(\tau > s)\partial_2 H(s,\xi_{s-})\Delta X_s \middle| \mathcal{H}_t\right),
\end{aligned} \tag{14}$$

where $\sigma_s^2 := \frac{\mathrm{d}[Y,Y]}{\mathrm{d}s}$. Moreover, by (13), we have

$$\begin{aligned}
\lim_{T\to\infty} \mathbb{E}\left(\frac{H(T,\xi_T)\mathbb{P}(\tau > T)}{\lambda(T)} \middle| \mathcal{H}_t\right) &= \lim_{T\to\infty} \mathbb{E}\left(\frac{V_T\mathbb{P}(\tau > T)}{\lambda(T)} \middle| \mathcal{H}_t\right) \\
&= V_t \lim_{T\to\infty} \frac{\mathbb{P}(\tau > T)}{\lambda(T)} := V_t c.
\end{aligned} \tag{15}$$

With $\lim_{T\to\infty} \frac{\mathbb{P}(\tau>T)}{\lambda(T)} = c$. By substituting (14) and (15) into (11), we obtain the equation

$$\begin{aligned}
0 =\;& V_t\left(c - \frac{\mathbb{P}(\tau > t)}{\lambda(t)}\right) - \mathbb{E}\left(\int_t^\infty \left(\partial_s\left(\frac{\mathbb{P}(\tau > s)}{\lambda(s)}\right) H(s,\xi_s)\right.\right. \\
& \left.\left. +\frac{\mathbb{P}(\tau > s)}{\lambda(s)}\partial_1 H(s,\xi_s) + \frac{1}{2}\partial_{22} H(s,\xi_s)\mathbb{P}(\tau > s)\,\lambda(s)\sigma_s^2\right)\mathrm{d}s \middle| \mathcal{H}_t\right) \\
& -\mathbb{E}\left(\sum_t^\infty \frac{\mathbb{P}(\tau > s)\Delta H}{\lambda(s)} - \mathbb{P}(\tau > s)\partial_2 H(s,\xi_{s-})\Delta X_s \middle| \mathcal{H}_t\right).
\end{aligned} \tag{16}$$

By identifying the predictive and martingale parts we have

$$\begin{aligned}
0 =\;& \partial_t\left(\frac{\mathbb{P}(\tau > t)}{\lambda(t)}\right)(V_t - H(t,\xi_t)) + \\
& +\frac{\mathbb{P}(\tau > t)}{\lambda(t)}\partial_1 H(t,\xi_t) + \frac{1}{2}\partial_{22} H(t,\xi_t)\mathbb{P}(\tau > t)\,\lambda(t)\sigma_t^2 \\
& +\left(\frac{\mathbb{P}(\tau > t)\Delta H}{\lambda(t)} - \frac{\mathbb{P}(\tau > t)}{\lambda(t)}\partial_2 H(t,\xi_{t-})\Delta\xi_t\right).
\end{aligned} \tag{17}$$

Now since we are in a local equilibrium prices are rational and by taking conditional expectations w.r.t $\mathcal{F}_t$, we obtain

$$\begin{aligned} 0 &= \frac{\mathbb{P}(\tau > t)}{\lambda(t)} \partial_1 H(t, \xi_t) + \frac{1}{2} \partial_{22} H(t, \xi_t) \mathbb{P}(\tau > t) \lambda(t) \sigma_t^2 \\ &\quad + \left(\frac{\mathbb{P}(\tau > t) \Delta H}{\lambda(t)} - \frac{\mathbb{P}(\tau > t)}{\lambda(t)} \partial_2 H(t, \xi_{t-}) \Delta \xi_t \right). \end{aligned} \tag{18}$$

Consequently

$$\mathrm{d}P_t = \mathrm{d}H_t = \lambda_t \partial_2 H(t, \xi_{t-}) \mathrm{d}Y_t,$$

and, since $P.$ is a martingale and $\lambda_t \partial_2 H(t, y) > 0$, we have that Y is a local martingale and (i) is proved.

(*ii*) From (17) and (18) we have that

$$\partial_t \left(\frac{\mathbb{P}(\tau > t)}{\lambda(t)} \right) (V_t - H(t, \xi_t)) = 0.$$

Then $V_t \neq H(t, \xi_t)$ implies that $\partial_t \left(\frac{\mathbb{P}(\tau > t)}{\lambda(t)} \right) = 0$ and $\lambda(t) = c\mathbb{P}(\tau > t)$ a.a. $t \geq 0$. ■

Remark 15 *Here we can draw conclusions similar to the one in Cho (2003) where he considers a risk-averse insider (and a deterministic release time). Cho concludes that, in equilibrium, a risk-adverse insider would do most of her trading early to avoid the risk that the prices get closer to the asset value, unless the trading conditions become more favourable over time. Similarly in our case, when the (risk-neutral) insider does not know the release time of information, she would trade early in order to use her piece of information before the announcement time comes. This behaviour would continue unless the price pressure decreases over time providing more favourable trading also at a later time. A similar conclusion is obtained by Baruch (2002), who studies exactly the same problem about the effect of risk-aversion for the insider, by assuming that the noise trading is Brownian motion with time varying instantantenous variance.*

Example 16 *We can consider the context of Caldentey and Stacchetti (2010) where the authors assume that V and Z are arithmetic Brownian motion with variances σ_V and σ_Z respectively, and τ follows an exponential distribution with scale parameter μ, independent of $(V, P, Z)_{0 \leq t}$. Then, applying the arguments above, we have that, for a.a. t and a.a. $\omega \in \{t < \tau\}$,*

$$V_t - H(t, \xi_t) - \lambda(t) \mathbb{E} \left(\int_t^\infty e^{-\mu(s-t)} \partial_2 H(s, \xi_s) \mathrm{d}X_s \middle| \mathcal{H}_t \right) = 0.$$

And to have a local equlibrium, provided that $V_t - H(t, \xi_t) \neq 0$, we need $\lambda(t) = \lambda_0 e^{-\mu t}$.

6 Explicit insider's optimal strategies and enlargement of filtrations

In this section we shall apply our results to explicitly find the insider's optimal strategy in equilibrium. We will show how our general framework serves different models known in the literature presented as extensions of the Kyle-Back model. In order to perform the explicit computations we will use techniques of enlargements of filtrations. Hereafter we present two subsections dedicated to this mathematical techniques in the case of initial enlargement and in the case of progressive enlargement of filtrations. Here we include new results and extensions of known facts. These subsections have mathematical value also independent of the present application.

To explain how enlargement of filtration enters the topic we consider a total demand $Y = Z+X$ in equilibrium given by:

$$Y_t = Z_t + \int_0^t \theta(\eta_t; Y_u, 0 \leq u \leq s)\mathrm{d}s, \quad 0 \leq t \leq T. \tag{19}$$

Here X is absolutely continuous process with respect to the Lebesgue measure. We recall that Z is perceived by the insider as an $\mathbb{H}$-martingale independent of $V_\cdot = E(f(\eta_\tau)|\mathcal{H}_\cdot)$ and η. So since $\mathbb{F}^{Y,\eta} \subseteq \mathbb{H}$ and Z is adapted to $\mathbb{F}^{Y,\eta}$, it is also an $\mathbb{F}^{Y,\eta}$-martingale. On the other hand, as we have shown in Proposition 12 and in Proposition 14 Y is a local martingale when in equilibrium. Consequently (19) becomes the Doob-Meyer decomposition of Y when we enlarge the filtration $\mathbb{F}^Y$ with the process η. We are then into a problem of *enlargement of filtrations*. However, in our problem Z is fixed in advance and we want to obtain Y as a function of Z, fixed η, so we look in fact for *strong* solutions of (19), whereas the results on enlargement of filtrations provide weak solutions. In this sense the celebrated Yamada-Watanabe's theorem is the result, when Z is Gaussian, that can be used to obtain strong solutions from week solutions. See, for instance, Theorem 1.5.4.4. in Jeanblanc *et. al.* (2009). In the following two sections we remind the reader some useful results on enlargement of filtrations.

6.1 Initial enlargement of filtrations

Consider a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a $\mathcal{F}$-measurable random variable L with values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $\mathcal{G}_t := \cap_{s>t} (\mathcal{F}_t \vee \sigma(L))$ and $\mathbb{G} = (\mathcal{G}_t)$.

Condition A. For all t, there exists a σ-finite measure η_t in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $Q_t(\omega, \cdot) \ll \eta_t$ where $Q_t(\omega, \mathrm{d}x)$ is a regular version of the law of $L|\mathcal{F}_t$.

Proposition 17 *Condition A is equivalent to $Q_t(\omega, \mathrm{d}x) \ll \eta(\mathrm{d}x)$ where η is the law of L.*

Proof. By Condition A we have that $Q_t(\omega, \mathrm{d}x) = q_t^x(\omega)\eta_t(\mathrm{d}x)$, where $q_t^x(\omega)$ is $\mathcal{B}(\mathbb{R})\otimes\mathcal{F}_t$ measurable then we can write $Q_t(\omega, \mathrm{d}x) = \hat{q}_t^x(\omega)\eta(\mathrm{d}x)$ with $\hat{q}_t^x(\omega) = \frac{q_t^x(\omega)}{E(q_t^x(\omega))}$. ■

Proposition 18 *Under Condition A there exists a $\mathcal{B}(\mathbb{R})\otimes\mathcal{F}_t$-measurable process $q_t^x(\omega)$ such that $Q_t(\omega, \mathrm{d}x) = q_t^x(\omega)\eta(\mathrm{d}x)$ and, for fixed x, q_t^x is an $\mathbb{F}$-martingale.*

Proof. See Jacod (1985) Lemma 1.8. ■

Theorem 19 *Let M be a continuous local $\mathbb{F}$-martingale and consider $k_t^x(\omega)$ such that*

$$\langle q^x, M\rangle_t = \int_0^t k_s^x q_{s-}^x \mathrm{d}\langle M, M\rangle_s,$$

then

$$M - \int_0^{\cdot} k_s^L \mathrm{d}\langle M, M\rangle_s$$

is a $\mathbb{G}$-martingale.

Proof. Except for a localization procedure (see details in Jacod (1985) Theorem 2.1) the proof is the following: let $Z \in \mathcal{F}_s$ bounded and g be Borelian and bounded. Then, for $s \leq t$,

$$\begin{aligned}
E(Zg(L)(M_t - M_s)) &= E(E(Zg(L)(M_t - M_s)|\mathcal{F}_t)) \\
&= E(Z(M_t - M_s)E(g(L)|\mathcal{F}_t)) \\
&= \int_{\mathbb{R}} g(x)\eta(\mathrm{d}x)E(Z(M_t - M_s)q_t^x) \\
&= \int_{\mathbb{R}} g(x)\eta(\mathrm{d}x)E(Z(M_t q_t^x - M_s q_s^x)) \\
&= \int_{\mathbb{R}} g(x)\eta(\mathrm{d}x)E(Z(\langle M, q^x\rangle_t - \langle M, q^x\rangle_s)) \\
&= \int_{\mathbb{R}} g(x)\eta(\mathrm{d}x)E(Z(\int_s^t k_u^x q_{u-}^x \mathrm{d}\langle M, M\rangle_u)) \\
&= E(Zg(L)(\int_s^t k_u^L \mathrm{d}\langle M, M\rangle_u)),
\end{aligned}$$

where we have used Proposition 18. ■

Example 20 *Take $M_t = B_t$ where B is a standard Brownian motion and $L = B_1$. Then*

$$q_t^x(\omega) \sim \frac{1}{(1-t)^{1/2}} \exp\left\{-\frac{1}{2(1-t)}(B_t(\omega)-x)^2 + \frac{x^2}{2}\right\},$$

by Ito's formula

$$\mathrm{d}_t q_t^x = q_t^x \frac{x - B_t}{1-t} \mathrm{d}B_t,$$

then $k_s^x = \frac{x-B_t}{1-t}$ and

$$B - \int_0^{\cdot} \frac{B_1 - B_s}{1-s} \mathrm{d}s$$

is an $\mathbb{F}^B \vee \sigma(B_1)$ martingale. Note that, by the Lévy theorem, $W = B - \int_0^{\cdot} \frac{B_1 - B_s}{1-s} \mathrm{d}s$ is a (standard) $\mathbb{G} := \mathbb{F}^B \vee \sigma(B_1)$ -Brownian motion and since B_1 is $\mathcal{G}_0$-measurable, it is independent of W.

Example 21 *Note that if the filtration $\mathbb{F}$ is the one generated by a Brownian motion, B, then for any $\mathbb{F}$-martingale*

$$\mathrm{d}M_t = \sigma_t \mathrm{d}B_t$$

and

$$\mathrm{d}\langle M, M\rangle_t = \sigma_t^2 \mathrm{d}t.$$

Also, assuming that

$$q_t^x(\omega) = h_t^x(B_t)$$

and $h \in C^{1,2}$ we will have that

$$\mathrm{d}_t q_t^x = \partial h_t^x(B_t) \mathrm{d}B_t,$$

and

$$k_t^x = \frac{\partial \log h_t^x(B_t)}{\sigma_t}.$$

Example 22 *In fact the previous example is a particular case of the following one: let Y be the Brownian semimartingale*

$$Y_t = Y_0 + \int_0^t \sigma(Y_s)\mathrm{d}B_s + \int_0^t b(Y_s)\mathrm{d}s,$$

and assume that

$$Y_1 | \mathcal{F}_t \sim \pi(1-t, Y_t, x)\mathrm{d}x.$$

with π smooth. We know that $(\pi(1-t,Y_t,x))_t$ is an $\mathbb{F}$-martingale, then

$$\mathrm{d}\pi(1-t,Y_t,x)=\frac{\partial\pi}{\partial y}(1-t,Y_t,x)\sigma(Y_s)\mathrm{d}B_s$$

and by the Jacod theorem

$$\int_0^t\sigma(Y_s)\mathrm{d}B_s-\int_0^t\frac{\partial\log\pi}{\partial y}(1-s,Y_s,Y_1)\sigma^2(Y_s)\mathrm{d}s$$

is an $\mathbb{F}\vee\sigma(Y_1)$-martingale, and we can write

$$Y_t=Y_0+\int_0^t\sigma(Y_s)\mathrm{d}\tilde{B}_s+\int_0^t b(Y_s)\mathrm{d}s+\int_0^t\frac{\partial\log\pi}{\partial y}(1-s,Y_s,Y_1)\sigma^2(Y_s)\mathrm{d}s,$$

where $\tilde{B}$ is an $\mathbb{F}\vee\sigma(Y_1)$-Brownian motion.

Example 23 *Let B a Brownian motion and $\tau=\inf\{t>0,B_t=-1\}$ it is well known that*

$$P[\tau\le s|\mathcal{F}_t]=2\Phi(-\frac{1+B_t}{\sqrt{s-t}})\mathbf{1}_{\{\tau\wedge s>t\}}+\mathbf{1}_{\{s<\tau\wedge t\}},$$

where Φ is the cumulative distribution function of a standard normal distribution. Then in $t<s\wedge\tau$ we have, by Ito's formula,

$$P[\tau\le s|\mathcal{F}_t]=2\Phi(-\frac{1}{\sqrt{s}})+\sqrt{\frac{2}{\pi}}\int_0^t\frac{1}{\sqrt{s-u}}e^{-\frac{(1+B_u)^2}{2(s-u)}}\mathrm{d}B_u,$$

so

$$\mathrm{d}\langle P[\tau\le s|\mathcal{F}_\cdot],B\rangle_t=-\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{s-t}}e^{-\frac{(1+B_t)^2}{2(s-t)}}\mathrm{d}t,$$

and

$$\begin{aligned}&\alpha_t^sQ_t(\cdot,\mathrm{d}s)\\=\;&\frac{\partial}{\partial s}\left(\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{s-t}}e^{-\frac{(1+B_t)^2}{2(s-t)}}\right)=\frac{1}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{(s-t)^3}}-\frac{(1+B_t)^2}{\sqrt{(s-t)^5}}\right)e^{-\frac{(1+B_t)^2}{2(s-t)}},\end{aligned}$$

finally

$$Q_t(\cdot,\mathrm{d}s)=\frac{\partial}{\partial s}P[\tau>s|\mathcal{F}_t]=\frac{e^{-\frac{(1+B_t)^2}{2(s-t)}}}{\sqrt{2\pi}\sqrt{(s-t)^3}}\left(1+B_t\right),$$

and

$$\alpha_t^s = \frac{\frac{\partial}{\partial s}\left(\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{s-t}}e^{-\frac{(1+B_t)^2}{2(s-t)}}\right)}{\frac{\partial}{\partial s}P[\tau > s|\mathcal{F}_t]} = \frac{1}{1+B_t} - \frac{1+B_t}{s-t}.$$

Consequently

$$B_t - \int_0^{t\wedge\tau}\left(\frac{1}{1+B_s} - \frac{1+B_s}{\tau - s}\right)\mathrm{d}s, \quad t \geq 0,$$

is a $\mathbb{G}$*-martingale.*

6.2 Progressive enlargement of filtrations

In the progressive enlargement of filtrations $\mathbb{G} = (\mathcal{G}_t)$ is $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, where $\mathbb{H} = (\mathcal{H}_t)$ is another filtration. The case where $\mathcal{H}_t = \sigma(\mathbf{1}_{\{\tau\leq t\}})$ with τ a random time has been extensively studied, see for instance Jeulin (1980), Jeulin and Yor (1985) or Mansuy and Yor (2006), among others, however few studies has been developed in the general setting. One exception is when $\mathcal{H}_t = \sigma(J_t)$, for $J_t = \inf_{s\geq t} X_s$ and when X is a 3-dimensional Bessel process, see section 1.2.2 in Mansuy and Yor (2006), but this case can be reduced in fact to a case with random times taking into account that

$$\{J_t < a\} = \{t < \Lambda_a\},$$

where $\Lambda_a = \sup\{t, X_t = a\}$. Another exception is the case when $\mathcal{H}_t = \sigma(L_t)$, for $L_t = G(X, Y_t)$, with X and $\mathcal{F}_T$-measurable random variable, Y a process independent of $\mathcal{F}_T$, and G a Borel function, see Corcuera *et al. (2004).* However all these mentioned results do not apply to our context since we require the independency of η and Z.

Hereafter we suggest the following new result. Let $\mathcal{H}_t = \sigma(V_t)$ for

$$V_t = V_0 + \int_0^t \sigma_s \mathrm{d}W_s^1,$$

where σ_s is a deterministic function, V_0 is a zero mean normal random variable, and $\left(W^1, W^2\right)$ is a 2-dimensional Brownian motion independent of V_0. We have the following proposition:

Proposition 24 *Assume that* $Var(V_1) = 1$ *and that*

$$\int_0^t \frac{\mathrm{d}s}{Var(V_s) - s} < \infty \text{ for all } 0 \leq t < 1,$$

then

$$B_t = W_t^2 + \int_0^t \frac{V_s - B_s}{Var(V_s) - s}\mathrm{d}s, 0 \le t \le 1$$

is a Brownian motion with $B_1 = V_1$.

Proof. Denote $v_r := Var(V_r)$

$$B_t = \int_0^t \exp\left(-\int_u^t \frac{1}{v_r - r}\mathrm{d}r\right)\mathrm{d}W_u^2 + \int_0^t \exp\left(-\int_u^t \frac{1}{v_r - r}\mathrm{d}r\right)\frac{V_u}{v_u - u}\mathrm{d}u,$$

so B is a centered Gaussian process, and for $s \le t < 1$,

$$\begin{aligned}
E\left(B_t B_s\right) &= \exp\left(-\int_s^t \frac{1}{v_r - r}\mathrm{d}r\right) \\
&\quad + E\left(\int_0^t\int_0^s \exp\left(-\int_u^t \frac{1}{v_r - r}\mathrm{d}r\right)\exp\left(-\int_v^s \frac{1}{v_r - r}\mathrm{d}r\right)\frac{V_u V_v}{\left(v_u - u\right)\left(v_v - v\right)}\mathrm{d}u\mathrm{d}v\right) \\
&= \exp\left(-\int_s^t \frac{1}{v_r - r}\mathrm{d}r\right)\int_0^s \exp\left(-2\int_u^s \frac{1}{v_r - r}\mathrm{d}r\right)\mathrm{d}u \\
&\quad + \int_s^t\int_0^s \exp\left(-\int_u^t \frac{1}{v_r - r}\mathrm{d}r\right)\exp\left(-\int_v^s \frac{1}{v_r - r}\mathrm{d}r\right)\frac{v_v}{\left(v_u - u\right)\left(v_v - v\right)}\mathrm{d}u\mathrm{d}v \\
&\quad + 2\int_0^s\int_0^u \exp\left(-\int_u^t \frac{1}{v_r - r}\mathrm{d}r\right)\exp\left(-\int_v^s \frac{1}{v_r - r}\mathrm{d}r\right)\frac{v_v}{\left(v_u - u\right)\left(v_v - v\right)}\mathrm{d}u.
\end{aligned}$$

Then , since

$$\int_0^s \exp\left(-\int_v^s \frac{1}{v_r - r}\mathrm{d}r\right)\frac{v_v}{v_v - v}\mathrm{d}v = s,$$

and

$$2\int_0^s \exp\left(-2\int_v^s \frac{1}{v_r - r}\mathrm{d}r\right)\frac{v_v}{v_v - v}\mathrm{d}v = 2s + \int_0^s \exp\left(-2\int_u^s \frac{1}{v_r - r}\mathrm{d}r\right)\mathrm{d}u$$

we obtain that $E\left(B_t B_s\right) = s$. So for $0 \le t < 1$ we have that (B_t) is a standard Brownian motion. On the other hand

$$\begin{aligned}
E(B_t V_t) &= E\left(\int_0^t \exp\left(-\int_u^t \frac{1}{v_r - r}\mathrm{d}r\right)\frac{V_u V_t}{v_u - u}\mathrm{d}u\right) \\
&= \int_0^t \exp\left(-\int_u^t \frac{1}{v_r - r}\mathrm{d}r\right)\frac{v_u}{v_u - u}\mathrm{d}u \\
&= t,
\end{aligned}$$

therefore

$$\begin{aligned} E((B_t - V_t)^2) &= E(B_t^2) + E(V_t{}^2) - 2E(B_t V_t) \\ &= t + v_t - 2t = v_t - t, \end{aligned}$$

and, since by hypothesis $v_1 = 1$, this means that

$$\lim_{t \to 1} B_t \overset{L^2}{=} V_1,$$

then for all $0 \leq t < 1$

$$E\left(\int_0^t \frac{|V_s - B_s|}{v_s - s} \mathrm{d}s\right) < \int_0^t \frac{E\left((V_s - B_s)^2\right)^{\frac{1}{2}}}{v_s - s} \mathrm{d}s = \int_0^t \sqrt{v_s - s} \mathrm{d}s < \sqrt{2},$$

and this implies, by the monotone convergence theorem, that

$$\lim_{t \to 1} \int_0^t \frac{|V_s - B_s|}{v_s - s} \mathrm{d}s = \int_0^1 \frac{|V_s - B_s|}{v_s - s} \mathrm{d}s < \infty$$

and that $B_1 = \lim_{t \to 1} B_t$ is well defined. Now, we have, by the uniqueness of the limit in probability, that $V_1 = B_1$ a.s. ■

6.3 Application to find the equilibrium strategy

In this section we shall apply the results of the previous section to find the equilibrium strategy of the insider. We will see trough different examples how this can be done. These various examples correspond to different models that are extensions of the Kyle-Back model.

Example 25 *(Back (1992)) Assume that Z is a Brownian motion with variance σ^2 , $V_{\cdot} \equiv V_1$ and, the release time, $\tau = 1$. In equilibrium, if the strategy of the insider is optimal $V_1 = H(1, Y_1)$. Since $H(1, \cdot)$ can be chosen freely because it is the boundary condition of equation (7) and if V_1 has a continuous cumulative distribution function, we can assume w.l.o.g that $Y_1 \equiv N(0, \sigma^2)$. It is assumed that V_1 (and consequently Y_1) is independent of Z. Then by the calculations in the Example 20 we have that*

$$Y_t = Z_t + \int_0^t \frac{Y_1 - Y_s}{1 - s} \mathrm{d}s,$$

is a Brownian motion with variance σ^2. Hence, prices are rational and we recognize the equilibrium strategy *to be*

$$X_t = \int_0^t \frac{Y_1 - Y_s}{1-s}\mathrm{d}s, 0 \le t < 1.$$

Example 26 *(Aase, Bjuland, Øksendal (2012a)) Assume that $\tau = 1$ and suppose that Z is given by*

$$Z_t = \int_0^t \sigma_s dW_s$$

where σ is deterministic and $V_. \equiv Y_1$ is a $N(0, \int_0^1 \sigma_s^2 \mathrm{d}s)$ independent of Z. Then $V_.|\mathcal{F}_t^Y \sim N(Y_t, \int_t^1 \sigma_s^2 \mathrm{d}s)$ and by the results in the Example 21

$$Y_t = Z_t + \int_0^t \frac{Y_s - Y_1}{\int_t^1 \sigma_u^2 \mathrm{d}u}\sigma_s^2 \mathrm{d}s,$$

has the same law as Z. Then

$$X_t = \int_0^t \frac{Y_s - Y_1}{\int_t^1 \sigma_u^2 \mathrm{d}u}\sigma_s^2 \mathrm{d}s$$

is the optimal strategy.

We have a similar result if σ is random, in fact we have the following example:

Example 27 *(Campi, Çetin, Danilova (2009)) If $\mathrm{d}Z_t = \sigma(Y_t)\mathrm{d}W_t$, $\tau = 1$ and $V_. \equiv \xi_1$. Where $\xi_t = \int_0^t \sigma(\xi_s)\mathrm{d}B_s$, and independent of Z, then by the results in the Example 22*

$$\mathrm{d}Y_t = \sigma(Y_t)\mathrm{d}W_t + \sigma^2(Y_t)\frac{\partial_y G(1-t, Y_t, \xi_1)}{G(1-t, Y_t, \xi_1)}\mathrm{d}t,$$

where $G(t, y, z)$ is the transition density of $\xi_.$, is a martingale.

Example 28 *(Campi and Çetin (2007)) If we want the aggregate process Y to be a Brownian motion that reaches the value -1 for the first time at time $\bar{\tau}$, and Z is also a Brownian motion then, by the results in the Example 23:*

$$Y_t = Z_t + \int_0^t \left(\frac{1}{1+Y_s} - \frac{1+Y_s}{\bar{\tau} - s}\right)\mathbf{1}_{[0,\bar{\tau}]}(s)\mathrm{d}s,$$

so, in this case $\eta_t \equiv \bar{\tau}$, $V_t \equiv \mathbf{1}_{\{\bar{\tau}>1\}}$ and the release time is $\bar{\tau} \wedge 1$.

Example 29 *(Back and Pedersen (1998), Wu (1999), Danilova (2010)) The insider receives a continuous signal*

$$\eta_t = \eta_0 + \int_0^t \sigma_s \mathrm{d}W_s,$$

where σ_s is deterministic, η_0 is a zero mean normal random variable, W is a Brownian motion, both independent of the Brownian motion Z, $\tau = 1$. It is assumed that $var(\eta_1) = var(\eta_0) + \int_0^1 \sigma_s^2 \mathrm{d}s = 1$, then, by Proposition 24,

$$Y_t = Z_t + \int_0^t \frac{\eta_s - Y_s}{var(\eta_s) - s} \mathrm{d}s, \;\; 0 \leq t \leq 1.$$

is a Brownian motion.

Another view of the problem of finding the equilibrium strategy is the following. Market makers observe Y with dynamics

$$\mathrm{d}Y_t = \mathrm{d}Z_t + \theta(V_t, Y_s, 0 \leq s \leq t)\mathrm{d}t,$$

V is not observed. Then, the dynamics of $m_t := \mathbb{E}(V_t | \mathcal{F}_t^Y)$ can be obtained in certain cases, basically when Z and V are Gaussian diffussions, from the filtering theory, see for instance Theorem 12.1 in Liptser and Shiryayev (1978). Now we can try to deduce $\theta(V_t, Y_s, 0 \leq s \leq t)$ from the equilibrium condition: $P_t = m_t$. Even, if (V_t) is not a Gaussian diffusion but can be written in the form $V_t = h(D_t)$ where h is a strictly increasing function and D is a Gaussian diffusion, we can apply the filtering results for the couple (Y, D).

In the following example we use the filtering approach to find the equilibrium strategy.

Example 30 *(Caldentey and Stacchetti (2010)) τ is unknown (so we cannot apply Proposition 24),*

$$\mathrm{d}V_t = \sigma_v(t)\mathrm{d}B_t^v, V_0 \sim N(P_0, \Sigma_0), \qquad \mathrm{d}Z_t = \sigma_z(t)\mathrm{d}B_t^z, Z_0 = 0.$$

B^v and B^z being independent Brownian motions, $\sigma_v(t)$ and $\sigma_z(t)$ deterministic functions. Then, if we look for pricing rules such that

$$\mathrm{d}P_t = \lambda_t \mathrm{d}Y_t$$

and strategies

$$\mathrm{d}X_t = \beta_t(V_t - P_t)\mathrm{d}t$$

with β_t deterministic, we have

$$\mathrm{d}P_t = \lambda_t\beta_t(V_t - P_t)\mathrm{d}t + \lambda_t\sigma_z(t)\mathrm{d}B_t^z.$$

Let denote $m_t = E(V_t|\mathcal{F}_t^Y)$, by standard filtering results (see for instance Lipster and Shiryayev (2001)) we have

$$\mathrm{d}m_t = \frac{\Sigma_t\beta_t}{\lambda_t\sigma_z^2(t)}(\mathrm{d}P_t - \lambda_t\beta_t(m_t - P_t)\mathrm{d}t), \quad \frac{\mathrm{d}}{\mathrm{d}t}\Sigma_t = \sigma_v^2(t) - \frac{(\Sigma_t\beta_t)^2}{\sigma_z^2(t)},$$

where Σ_t is the filtering error. Now, we can recover the identity $P_t = m_t$, if and only if we impose $\Sigma_t\beta_t = \lambda_t\sigma_z^2(t)$ (remember that by construction $P_0 = m_0 = E(V_0)$) . Then

$$\Sigma_t = \Sigma_0 + \int_0^t \sigma_v^2(s)\mathrm{d}s - \int_0^t \sigma_z^2(s)\lambda_s^2\mathrm{d}s,\ \beta_t = \frac{\lambda_t\sigma_z^2(t)}{\Sigma_t}.$$

Note that in particular we obtain that

$$Y_t = Z_t + \int_0^t \frac{\lambda_s\sigma_z^2(s)\left(V_s - \int_0^s \lambda_u \mathrm{d}Y_u\right)}{\Sigma_s}\mathrm{d}s,$$

is the Doob-Meyer decomposition of the martingale Y in the filtration generated by (Z, V). Now if we assume $\sigma_z^2(t) = \sigma_z^2$, independent of t, and we take into account that in the equilibrium $\lambda_t = \lambda_0 e^{-\mu t}$, we have that

$$\Sigma_t = \Sigma_0 + \int_0^t \sigma_v^2(s)\mathrm{d}s - \sigma_z^2\frac{\lambda_0^2}{2\mu}(1 - e^{-2\mu t}), \beta_t = \frac{\sigma_z^2\lambda_0 e^{-\mu t}}{\Sigma_t}.$$

However λ_0 is not determined. We need an additional condition to fix λ_0. One possibility is to impose that

$$\lim_{t\to\infty} \Sigma_t = 0.$$

In such a case

$$0 = \Sigma_0 + \int_0^\infty \sigma_v^2(s)\mathrm{d}s - \sigma_z^2\frac{\lambda_0^2}{2\mu},$$

and

$$\lambda_0 = \sqrt{\frac{2\mu(\Sigma_0 + \int_0^\infty \sigma_v^2(s)\mathrm{d}s)}{\sigma_z^2}}.$$

Note that if $\sigma_v^2(t) = \sigma_v^2$ there is no solution! Another possibility, according with Proposition 14, is to take T such that

$$\Sigma_t = 0, \text{ for all } t \geq T$$

and then $P_t = V_t$ for $t \geq T$. But this implies, for $\sigma_v^2(t) = \sigma_v^2$,

$$\begin{aligned} 0 &= \Sigma_0 + \sigma_v^2 T - \sigma_z^2\frac{\lambda_0^2}{2\mu}(1 - e^{-2\mu T}) \\ &= \Sigma_0 + \sigma_v^2 T - \sigma_z^2\frac{\lambda_T^2}{2\mu}(e^{2\mu T} - 1). \end{aligned}$$

Now if we assume a smooth transition from the absolutely continuous strategy then $\sigma_v^2 - \sigma_z^2\lambda_t^2 = 0$ for all

$t \geq T$ and $\lambda_t = \lambda_T = \frac{\sigma_v}{\sigma_z}$, for all $t \geq T$. Finally

$$\mathrm{d}P_t = \lambda_t \mathrm{d}Y_t = \lambda_t \mathrm{d}X_t + \lambda_t \mathrm{d}Z_t = \mathrm{d}V_t,\ t \geq T$$

so

$$\mathrm{d}X_t = \frac{\sigma_z}{\sigma_v}\mathrm{d}V_t - \mathrm{d}Z_t,$$

and T is the solution of

$$\Sigma_0 + \sigma_v^2 T = \frac{\sigma_v^2}{2\mu}(e^{2\mu T} - 1).$$

This is exactly what Caldentey and Stacchetti (2010) obtain. It is important to remark that the authors obtain a limit of optimal strategies when passing from the discrete version of the model to the continuous one. This limit strategy is such that there is an endogenously determined time T such that, if $t \leq T$, then the limit strategy is absolutely continuous with respect to the Lebesgue measure and, if $t > T$, the strategy is not of bounded variation. In this case an insider's optimal strategy, between times T and τ, would yield to giving out the full information to the market by making the market prices match the fundamental value. They claim that this limit strategy is not optimal for the continuous time model and that we need to consider the discrete time model to realize about its existence. However this limit strategy can be obtained has a limit of strategies for the continuous model when we restrict the class of strategies to set of absolutely continuous strategies and we try to maximize the wealth. In fact if we have a sequence of strategies $\left(X^{(n)}\right)_{n\geq 1}$, their corresponding wealth is given by

$$W_\tau^{(n)} = X_\tau^{(n)} V_\tau^{(n)} - \int_0^\tau P_{t-}^{(n)} \mathrm{d}X_t^{(n)} - [P^{(n)}, X^{(n)}]_\tau.$$

Then, if we assume that $(X^{(n)}, P^{(n)}, V^{(n)}) \underset{n\to\infty}{\overset{u.c.p}{\longrightarrow}} (X, P, V)$ we obtain that

$$X_\tau^{(n)} V_\tau^{(n)} - \int_0^\tau P_{t-}^{(n)} \mathrm{d}X_t^{(n)} \underset{n\to\infty}{\overset{u.c.p}{\longrightarrow}} X_\tau V_\tau - \int_0^\tau P_{t-} \mathrm{d}X_t$$

but in general

$$[P^{(n)}, X^{(n)}]_\tau \nrightarrow [P, X]_\tau,$$

For instance if $X^{(n)}$ is a bounded variation process X is not necessarily a bounded variation one. Then the *gain limit for this limit of strategies after* T*, on the set* $\{\tau > T\}$*, is given by*

$$\begin{aligned} V_\tau X_\tau - V_T X_T - \int_T^\tau P_{t-}\mathrm{d}X_t &= \int_T^\tau X_{t-}\mathrm{d}V_t + \int_T^\tau V_{t-}\mathrm{d}X_t + \int_T^\tau \mathrm{d}[V,X]_t - \int_T^\tau P_{t-}\mathrm{d}X_t \\ &= \int_T^\tau (V_{t-} - P_{t-})\,\mathrm{d}X_t + \int_T^\tau \mathrm{d}[V,X]_t + \int_T^\tau X_{t-}\mathrm{d}V_t. \end{aligned}$$

Now if we take the (conditional) expectation, last term of the right-hand side cancels and we obtain that the gain from time T *onward is given by*

$$\mathbb{E}\left(\int_T^\tau (V_{t-} - P_{t-})\,\mathrm{d}X_t + \int_T^\tau \mathrm{d}[V,X]_t \middle| \mathcal{H}_T \right).$$

Finally, since for the limit strategy $V_{t-} = P_{t-}$, $t > T$*, in the conditions of Example 16, we obtain that there is a profit after* T *given by*

$$\mathbb{E}\left(\int_T^\infty e^{-\mu(t-T)}\mathrm{d}[V,X]_t \middle| \mathcal{H}_T \right) = \sigma_z\sigma_v \int_T^\infty e^{-\mu(t-T)}\mathrm{d}t = \frac{\sigma_z\sigma_v}{\mu} > 0.$$

Now we can justify the condition $\dot{\Sigma}_T = 0$*. The expected wealth for the insider with this kind of strategies is given by*

$$\begin{aligned} J(X) &= \mathbb{E}\left(\int_0^{T\wedge\tau} (V_t - P_t)\theta_t\mathrm{d}t \right) + \mathbb{E}\left(\int_{T\wedge\tau}^\tau \mathrm{d}[V,X]_t \right) = \mathbb{E}\left(\int_0^{T\wedge\tau} \beta_t(V_t - P_t)^2\mathrm{d}t \right) + \mathbb{E}\left(\int_{T\wedge\tau}^\tau \mathrm{d}[V,X]_t \right) \\ &= \mathbb{E}\left(\int_0^T \mathbf{1}_{[0,\tau]}(t)\beta_t(V_t - P_t)^2\mathrm{d}t \right) + \mathbb{E}\left(\int_T^\infty \mathbf{1}_{[0,\tau]}(t)\mathrm{d}[V,X]_t \right) = \int_0^T \mathbb{P}(\tau > t)\beta_t\Sigma_t\mathrm{d}t + \int_T^\infty \mathbb{P}(\tau > t)\frac{\sigma_v^2}{\lambda_t}\mathrm{d}t \\ &= \int_0^T e^{-\mu t}\beta_t\Sigma_t\mathrm{d}t + \sigma_v^2 \int_T^\infty \frac{e^{-\mu t}}{\lambda_t}\mathrm{d}t = \sigma_z^2 \int_0^T e^{-\mu t}\lambda_t\mathrm{d}t + \sigma_v^2 \int_T^\infty \frac{e^{-\mu t}}{\lambda_t}\mathrm{d}t. \end{aligned}$$

Then if we impose that T *is optimal, we have the condition*

$$\sigma_z^2 e^{-\mu T}\lambda_T - \sigma_v^2 \frac{e^{-\mu T}}{\lambda_T} = 0,$$

that is

$$\lambda_T = \frac{\sigma_v}{\sigma_z},$$

and this is equivalent to $\dot{\Sigma}_T = 0$*. Note that other equilibria are possible by taking* $\lambda_t \neq \lambda_T$ *when* $t > T$*.*

Remark 31 *It can we proved that the linearity of the strategies assumed in the previous example implies that the equilibrium pricing rules have to be linear as well. This interesting result can be seen in Aase et al. (2012a).*

Example 32 *Another interesting example is that of Campi et al. (2013). There, authors consider a defaultable stock. The default time is modeled as the first time that a Brownian motion, say B, hits the barrier -1, as in the above Example 28 . However in this case the default time, $\delta = \inf\{t \geq 0, B_t = -1\}$, is not known by the insider, but it is a stopping time for every trader. Instead, she observes the process $\left(B_{r(t)}\right)$ where $r(t)$ is a deterministic, increasing function with $r(t) > t$ for $t \in (0,1)$, $r(0) = 0$, and $r(1) = 1$. This circumstance allows the insider to know in advance the default time. The horizon of the market is $t = 1$.They also consider a payoff of the kind $f(B_1)$ in case of no default. Note that $\delta = r(\tau)$, where $\tau = \inf\{t \geq 0, B_{r(t)} = -1\}$. Then, in this example the release time $r(\tau)$, the signal is $\eta_t = B_{r(t)}$ and the fundamental value is*

$$V_t = \mathbf{1}_{\{\tau > t\}} \mathbb{E}(f(B_1)|B_{r(t)}).$$

Moreover the aggregate demand of noise traders follows a Brownian motion, say W, so $Z = W$. Even though τ, and consequently, δ is not known for the insider, they are predictable stopping times, and, by an extension of the case considered in section 4, we will have that, the price pressure is constant and that the optimal strategy moves prices to the fundamental one:

$$\lim_{\delta_n \uparrow \delta} P_{\delta_n} = V_\delta,$$

where (δ_n) is any increasing sequence of stopping times that grows to δ. To find the explicit form of an equilibrium strategy is not straightforward. However, if $\tau \leq s \leq V(\tau)$ an equilibrium strategy is obtained from a strong solution of

$$Y_s = W_s + \int_0^s \left(\frac{1}{1 + Y_u} - \frac{1 + Y_u}{V(\tau) - u} \right)(u)\mathrm{d}u,$$

as we deduce from Example 28 above, the difficult part is to see what happens until time τ. It requires a quite involved use of enlargement of filtrations and filtering techniques. See Campi et al. (2013b) for the details.

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Appendix C

Ambit processes, their volatility determination and their applications

Ambit processes, their volatility determination and their applications

José M. Corcuera, Gergely Farkas and Arturo Valdivia

Abstract In this paper we try to review the research done so far about ambit processes, and their applications. The notion of ambit process was introduced by Barndoff-Nielsen and Schmiegel in 2007. Since then, many papers have been written studying their properties and applying them to model in different natural or economic phenomena. As, it is shown in the paper, these processes share their mathematical structure with the solutions of random evolution equations allowing them great flexibility for modelling. The goal of this paper is fourth-fold: to show the main characteristics of these processes; how to determine their main structural component: their volatility; how they can be used for modelling different random phenomena like turbulence or financial prices; and last but not least the mathematics behind.

1 Introduction

The notion of ambit process was introduced by Barndoff-Nielsen and Schmiegel in 2007, see [12]. Since then, many papers have been written studying their properties and applying them to model in different natural or economic phenomena, see [7], [5], [8], [12], [24], among others. In the present paper we try to review all this work and to enlighten the notion of ambit process and its flexibility for modelling.

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Before giving the definition of ambit processes let us justify the generality and, consequently, the flexibility of such processes. Here we follow [6].

Let L be a partial differential operator, for instance the wave operator in dimension one

$$Lf = \frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2},$$

then, it is well known that there is a function G in $(\mathbf{R}_+, \mathbf{R})$ such that the solution of the PDE

$$Lu = \varphi, u(0,x) = 0,$$

where φ is a *test* function, can be written

$$u(t,x) = \int_{\mathbf{R}_+ \times \mathbf{R}} G(t-s, x-y)\varphi(s,y)\mathrm{d}s\mathrm{d}y.$$

Imagine now we have the SPDE

$$Lu = W, u(0,x) = 0 \tag{1}$$

where W is an L^2-noise in $\mathbf{R}_+ \times \mathbf{R}$, that is a map

$$\begin{aligned} \mathscr{B}(\mathbf{R}_+ \times \mathbf{R}) &\longrightarrow L^2(\Omega, \mathscr{F}, \mathbf{P}) \\ A &\longmapsto W(A), \end{aligned}$$

such that

1. $W(\emptyset) = 0$ a.s
2. For all disjoint and bounded sets $A_1, A_2, \ldots$ in $\mathscr{B}(\mathbf{R}_+ \times \mathbf{R})$, $W(A_i)$ are independent and

$$W(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} W(A_i), a.s.$$

and where the convergence of the series is in $L^2(\mathbf{P})$. Then it is natural to consider that the solution of (1) is given by

$$u(t,x) = \int_{\mathbf{R}_+ \times \mathbf{R}} G(t-s, x-y) W(\mathrm{d}s, \mathrm{d}y). \tag{2}$$

This kind of solution is named a *mild* solution. In general, if we have a random phenomenon with a certain dynamics, the tempo-spatial derivatives of the magnitude in a point will be connected with the *driving* noise at that point and this will imply that the value of the magnitude is related with the value of the driving noise in other points of the space-time set, as it can be appreciated in (2). Then, when modelling random phenomena, we can opt for proposing a kind of global dependency directly instead of a point-wise dynamical dependency. This is the motivation for the following definition,

Definition 1. A tempo-spatial *ambit field* is defined as

$$Y(t,x) = \mu + \int_{A(t,x)} g_{(t,x)}(s,\xi)\sigma(s,\xi)W(\mathrm{d}s,\mathrm{d}\xi) \\ + \int_{B(t,x)} q_{(t,x)}(s,\xi)a(s,\xi)\mathrm{d}s\mathrm{d}\xi, \quad t \geq 0, x \in \mathbf{R}^n$$

where $\mu \in \mathbf{R}$, $\xi \in \mathbf{R}^n$, W is a σ-finite, L^2-valued measure, $g_{(t,x)}(\cdot)$ and $q_{(t,x)}(\cdot)$ are deterministic kernels, $\sigma(\cdot,\cdot) \geq 0$, and $a(\cdot,\cdot)$ are predictable random fields and $A(t,x) \subseteq \mathbf{R}^{n+1}$ and $B(t,x) \subseteq \mathbf{R}^{n+1}$ are *ambit sets*. Then, $X_t := Y_t(x(t))$, for a curve $x(t)$, is called an *ambit process*.

In this definition the stochastic integral is assumed in the sense of Walsh, see for instance [42] and the more recent reference [32]. However a slight extension of this integral is considered here, in fact, in the integral, time coordinate moves in $\mathbf{R}$ more than in $\mathbf{R}_+$. This extension has been studied recently in [20]. Another extension, now for the case when ξ is infinite-dimensional and $W_s(\mathrm{d}\xi) := W([0,s],\mathrm{d}\xi)$, $s \geq 0$ is a cylindrical Brownian motion, can be found in [17].

The paper is organized as follows. Section 2 contains some properties and particularities of the ambit processes. Section 3 is devoted to see the application of ambit processes to modelling in Turbulence and to study their statistical properties in the context on infill asymptotics. Section 4 is devoted to study their applications in quantitative finance to modelling term structures and energy markets.

2 Ambit processes

The general concept of ambit field consists of a stochastic field $(Y(t,x))$ in space-time, $t \in \mathbf{R}$, $x \in \mathbf{R}^n$, where the values of $Y(t,x)$ depend on *innovations* prior to or a time t and that happened in a certain subset of $\mathbf{R}^n$. In other words, $Y(t,x)$ depends on what happened in a time-space subset (the so-called ambit set), $A(t,x) = \{(s,y) \in \mathbf{R}^{n+1}, s \in \Upsilon \subseteq (-\infty,t], y \in \Lambda_s \subseteq \mathbf{R}^n\}$. Then, if we take a curve $x(t)$ in $\mathbf{R}^n$ we have an ambit *process* $Y_t := Y(t,x(t))$. Evidently we can substitute a more abstract space, like a Hilbert space, for $\mathbf{R}^n$ to get a more general object. Another natural extension is to assume that Y takes values in $\mathbf{R}^n$, or even a Banach space. In any case we need further mathematical structure if we want to say something concrete about Y. The structure considered is that given in the Definition 1,

$$Y(t,x) = \mu + \int_{A(t,x)} g_{(t,x)}(s,\xi)\sigma(s,\xi)W(\mathrm{d}s,\mathrm{d}\xi) \\ + \int_{B(t,x)} q_{(t,x)}(s,\xi)a(s,\xi)\mathrm{d}s\mathrm{d}\xi, \quad t \geq 0, x \in \mathbf{R}^n, \tag{3}$$

where $\mu \in \mathbf{R}$, $\xi \in \mathbf{R}^n$, W is a σ-finite, L^2-noise, $g_{(t,x)}(\cdot)$ and $q_{(t,x)}(\cdot)$ are deterministic kernels, $\sigma(\cdot,\cdot) \geq 0$, and $a(\cdot,\cdot)$ are predictable random fields and $A(t,x) \subseteq \mathbf{R}^{n+1}$

and $B(t,x) \subseteq \mathbf{R}^{n+1}$ are *ambit sets*. Ambit sets can be seen as *areas of influence* or *causality* and this part of the structure could be seen as the only dynamic condition in these kind of processes or fields. The condition is that future cannot influence the past. Nevertheless the ambit fields used in practice are of the form

$$\begin{aligned} Y(t,x) = \mu &+ \int_{A(t,x)} g_x(t-s,\xi)\sigma(s,\xi)W(\mathrm{d}s,\mathrm{d}\xi) \\ &+ \int_{B(t,x)} q_x(t-s,\xi)a(s,\xi)\mathrm{d}s\mathrm{d}\xi, \quad t \geq 0, x \in \mathbf{R}^n, \end{aligned}$$

where $A(t,x) = A+(t,x)$, with A involving only negative time coordinates, in agreement with the *causality principle,* and analogously for $B(t,x)$. In such a situation this class of fields include the class of stationary fields in time and, by this reason, they are called *semistationary*. If W is a Lévy noise the field (or process) is called Lévy semistationary field (or process) ($\mathscr{LSS}$) and for the particular case where W is a Gaussian noise is called Brownian semistationary ($\mathscr{BSS}$). It is also said that

$$X_t := \int_{A(t,x)} g_x(t-s,\xi)\sigma(s,\xi)W(\mathrm{d}s,\mathrm{d}\xi)$$

is the *core* of Y. Moreover σ is referred to as the *intermittency, volatility* or *modulating* field or process.

It is difficult to say interesting statements for such general objects. To obtain something remarkable about, for instance, how the trajectories are or if the ambit process is a semimartingale or not, we need specific kernels, volatilities and noises.

Consider just the particular case $(X_t)_{t\in\mathbf{R}}$ of the form

$$X_t = \int_{-\infty}^{t} g(t-s)W(ds),$$

where W is a Gaussian white noise in $\mathbf{R}$, σ an adapted càdlag process and $g \in L^2(\mathbf{R}_+)$.

The path properties of the process $(X_t)_{t\in\mathbf{R}}$ crucially depend on the behaviour of the weight function g near 0. When $g(x) = x^{\alpha}L_g(x)$ (where $L_g(x)$ is a slowly varying function at 0) with $\alpha \in (-\frac{1}{2},0)\cup(0,\frac{1}{2})$, X has r-Hölder continuous paths for any $r < \alpha+\frac{1}{2}$. The analysis of the regularity of the sample paths follows the same routes that in the case of Volterra processes, see [37]. In fact X is a Volterra process though starting at $-\infty$.

Another important fact is that X is not a semimartingale, because g' is not square integrable in the neighbourhood of 0. In fact, observing the decomposition

$$X_{t+\Delta} - X_t = \int_t^{t+\Delta} g(t+\Delta-s)W(ds) + \int_{-\infty}^{t} \{g(t+\Delta-s) - g(t-s)\}W(ds),$$

we obtain by formal differentiation that

$$\mathrm{d}X_t = g(0+)\mathrm{d}W(t) + \Big(\int_{-\infty}^{t} g'(t-s)W(ds)\Big)\mathrm{d}t,$$

Then, the Gaussian process X is an Itô semimartingale when $g(0+) < \infty$ and $g' \in L^2(\mathbf{R}_+)$ and this property also transfers to the $\mathscr{BSS}$ process Y under mild assumptions. It can be shown, see [14], that the conditions $g(0+) < \infty$ and $g' \in L^2(\mathbf{R}^+)$ are also necessary conditions for X to be a semimartingale. So, if we assume that $g(x) = x^{\alpha} L_g(x)$, with $\alpha \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$, we have that $g' \notin L^2(\mathbf{R}^+)$ and the process X, and so the process Y (unless $\sigma = 0$), is not a semimartingale.

A similar analysis can be done to see if a $\mathscr{LSS}$ is a semimartingale. See for instance [9].

Moreover ambit processes can be used as leading noises of stochastic differential equations and we can construct a stochastic calculus with respect to this processes, see section 4.1 in [24].

3 Models in turbulence

In the framework of *stochastic modelling in turbulence*, see [28] for a description of this approach, Barndoff-Nielsen and Schmiegel [12] and [13] propose to model the main component of the velocity by a process of the form

$$Y_t = \mu + \int_{-\infty}^{t} g(t-s)\sigma_s W(\mathrm{d}s) + \int_{-\infty}^{t} q(t-s)a_s \mathrm{d}s, \tag{4}$$

where μ is a constant, W is a Gaussian white noise on $\mathbf{R}$, g and q are nonnegative deterministic functions on $\mathbf{R}$, with $g(t) = q(t) = 0$ for $t \leq 0$, and σ and a are adapted càdlàg processes.

Other approaches, out of the scope of this paper, combine the classical Navier-Stokes equation for a fluid, and randomness. The results in this framework are however quite implicit, see for instance [15], [35] or the more oriented toward applications [19].

3.1 Volatility determination

One crucial *quantity* in the model (4) is the volatility and some effort has been done to estimate σ. It is apparent, from [23], [22], [3], [4], [8] and [10], that a key tool to estimate σ is the *realized multipower variation (*RMV*)* of the process Y. It is an object of the type

$$\sum_{i=1}^{[nt]-k+1} \prod_{j=1}^{k} |\Delta_{i+j-1}^n Y|^{p_j} , \qquad \Delta_i^n Y = Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}} , \qquad p_1, \ldots, p_k \geq 0 ,$$

for some fixed number $k \geq 1$.

For simplicity of the exposition we shall consider the *core* of (4)

$$Y_t = \int_{-\infty}^{t} g(t-s)\sigma_s W(\mathrm{d}s), \tag{5}$$

where we assume that

$$\int_{-\infty}^{t} g^2(t-s)\sigma_s^2 \mathrm{d}s < \infty, \quad \text{a.s..}$$

and also that the function g is continuously differentiable on $(0,\infty)$, $|g'|$ is non-increasing on (b,∞) for some $b>0$ and $g' \in L^2((\varepsilon,\infty))$ for any $\varepsilon > 0$. Moreover, we assume that for any $t>0$

$$F_t = \int_1^{\infty} (g'(s))^2 \sigma_{t-s}^2 ds < \infty, \quad \text{a.s..}$$

See [8] for a discussion of this latter conditions.

The process Y is supposed to be observed at time points $t_i = i/n$, $i = 1,\ldots,[nt]$. Now, let G be the stationary Gaussian process defined as

$$G_t = \int_{-\infty}^{t} g(t-s)W(\mathrm{d}s).$$

We are interested in the asymptotic behaviour of the functionals

$$V(Y,p_1,\ldots,p_k)_t^n = \frac{1}{n\tau_n^{p_+}} \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^{k} |\Delta_{i+j-1}^n Y|^{p_j}, \qquad p_1,\ldots,p_k \geq 0,$$

where $\Delta_i^n Y = Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}}$ and $\tau_n^2 = \bar{R}(1/n)$ with $\bar{R}(t) = \mathbf{E}[|G_{s+t} - G_s|^2]$, $t \geq 0$ and when n goes to infinity. In such a way that we are in the context of infill asymptotics.

We define the correlation function of the increments of G:

$$r_n(j) = \mathrm{cov}\Big(\frac{\Delta_1^n G}{\tau_n}, \frac{\Delta_{1+j}^n G}{\tau_n}\Big) = \frac{\bar{R}(\frac{j+1}{n}) + \bar{R}(\frac{j-1}{n}) - 2\bar{R}(\frac{j}{n})}{2\tau_n^2}, \qquad j \geq 0.$$

Next, we introduce a class of measures:

$$\pi^n(A) = \frac{\int_A (g(x-\frac{1}{n}) - g(x))^2 \mathrm{d}x}{\int_0^{\infty} (g(x-\frac{1}{n}) - g(x))^2 \mathrm{d}x}, \qquad A \in \mathscr{B}(\mathbf{R}).$$

Finally, we define

$$\rho_{p_1,\ldots,p_k}^{(n)} = \mathbf{E}\Big[\Big|\frac{\Delta_1^n G}{\tau_n}\Big|^{p_1} \cdots \Big|\frac{\Delta_k^n G}{\tau_n}\Big|^{p_k}\Big].$$

To have a weak law of large numbers we require the following assumptions: **(LLN)**: There exists a sequence $r(j)$ with

$$r_n^2(j) \le r(j)\ , \qquad \frac{1}{n}\sum_{j=1}^{n-1} r(j) \to 0.$$

Moreover, it holds that

$$\lim_{n\to\infty} \pi^n((\varepsilon,\infty)) = 0$$

for any $\varepsilon > 0$.

For the **CLT** we need to introduce another Gaussian process. Let $(Q_i)_{i\ge1}$ be a non-degenerate stationary centered (discrete time) Gaussian process with variance 1 and correlation function

$$\rho(j) = \mathrm{cor}\left(Q_1, Q_{1+j}\right)\ , \qquad j \ge 1.$$

Define

$$V_Q(p_1,\dots,p_k)_t^n = \frac{1}{n}\sum_{i=1}^{[nt]-k+1}\prod_{j=1}^{k} |Q_{i+j-1}|^{p_j}$$

and let $\rho_{p_1,\dots,p_k} = \mathbf{E}(|Q_1|^{p_1}\cdots|Q_k|^{p_k})$

Now we can specify the condition (**CLT**): Assume **(LLN)** holds, and

$$r_n(j) \to \rho(j)\ , \qquad j \ge 0,$$

where $\rho(j)$ is the correlation function of (Q_i). Furthermore, there exists a sequence $r(j)$ such that, for any $j,n \ge 1$,

$$r_n^2(j) \le r(j)\ , \qquad \sum_{j=1}^{\infty} r(j) < \infty,$$

and we have

$$\mathbf{E}[|\sigma_t - \sigma_s|^A] \le C|t-s|^{A\gamma},$$

for any $A > 0$, with $\gamma(p\wedge 1) > \frac{1}{2}$, and $p = \min_{1\le i\le k, 1\le j\le d}(p_i^j)$. Finally we assume that there exists a constant $\lambda < -\frac{1}{p\wedge 1}$ such that for any $\varepsilon_n = O(n^{-\kappa})$, $\kappa \in (0,1)$, we have

$$\pi^n((\varepsilon_n,\infty)) = O(n^{\lambda(1-\kappa)}).$$

Set $p^+ = \sum_{l=1}^k p_l$. We have the following main theorem, see [8].

Theorem 1. *Consider the process Y given by (5). Assume that the condition **(CLT)** holds, then we obtain the stable convergence*

$$\sqrt{n}\Big(V(Y,p_1^j,\dots,p_k^j)_t^n - \rho^{(n)}_{p_1^j,\dots,p_k^j}\int_0^t |\sigma_s|^{p_+^j}\mathrm{d}s\Big)_{1\le j\le d} \xrightarrow{st} \int_0^t A_s^{1/2}\mathrm{d}B_s,$$

where B is a d-dimensional Brownian motion independent of $\mathscr{F}$, and A is a $d\times d$-dimensional process given by

$$A_s^{ij} = \beta_{ij}|\sigma_s|^{p_+^i + p_+^j}\ , \qquad 1 \le i,j \le d\ ,$$

with β the $d \times d$ matrix given by

$$\beta_{ij} = \lim_{n\to\infty} n\, cov\Big(V_Q(p_1^i,\ldots,p_k^i)_1^n, V_Q(p_1^j,\ldots,p_k^j)_1^n\Big)\ , \qquad 1 \le i,j \le d\ .$$

In [8] we worked with the function g

$$g(t) = t^{\nu-1} e^{-\lambda t} 1_{(0,\infty)}(t)$$

for $\lambda > 0$ and with $\nu > \frac{1}{2}$. For t near 0, $g(t)$ behaves as t^{δ} with $\delta = \nu - 1$. If we check the conditions for the CTL we have the restriction $1/2 < \nu < 1$. This forced us to consider higher order differences:

$$\diamondsuit_i^n X = X_{i\Delta_n} - 2X_{(i-1)\Delta_n} + X_{(i-2)\Delta_n}.$$

and to study the multipower variation of the second order differences of the $\mathscr{BSS}$ process X, i.e.

$$MPV^{\diamondsuit}(X,p_1,\ldots,p_k)_t^n = \Delta_n (\tau_n^{\diamondsuit})^{-p^+} \sum_{i=2}^{[t/\Delta_n]-2k+2} \prod_{l=0}^{k-1} |\diamondsuit_{i+2l}^n X|^{p_l},$$

where $(\tau_n^{\diamondsuit})^2 = \mathbf{E}(|\diamondsuit_i^n G|^2)$ and $p^+ = \sum_{l=1}^k p_l$.

See [10] and [25] for the development and application to real turbulence data of the high-order multipower variation.

It is worthwhile to comment that the limit theory for multipower variation of Lévy semistationary processes does not yet exist.

3.1.1 Volatility determination in an ambit field setting

Now we try to show a relation between the realized quadratic variation (RQV) along a curve and the volatility of the underlying random field. We refer to [11] for more details.

Consider a random field

$$Y(x) = \int_{A(x)} g(x-\xi)\sigma(\xi) W(\mathrm{d}\xi),$$

where $x \in \mathbf{R}^n$, W is the Gaussian white noise in $\mathbf{R}^n$, $g : \mathbf{R}^n \to \mathbf{R}$, with $g(x_1,..,x_n) = 0$ if $x_1 < 0$ (the first coordinate indicates time) and σ is either deterministic or independent of W. Then, assume that $A(x) = A + x$,

$$Y(x) = \int_{A(x)} g(x-\xi)\sigma(\xi) W(\mathrm{d}\xi) = \int_{\mathbf{R}^n} g\mathbf{1}_{-A}(v)\sigma(x-v) W(x-\mathrm{d}v),$$

In such a way that

$$Y(x+\Delta x)-Y(x)=\int_{\mathbf{R}^n}(g\mathbf{1}_{-A}(v+\Delta x)-g\mathbf{1}_{-A}(v))\,\sigma(x-v)W(x-\mathrm{d}v),$$

and

$$\mathbf{E}\left[\left.(Y(x+\Delta x)-Y(x))^2\right|\sigma\right]=\int_{\mathbf{R}^n}(g\mathbf{1}_{-A}(v+\Delta x)-g\mathbf{1}_{-A}(v))^2\,\sigma^2(x-v)\mathrm{d}v.$$

Then

$$\begin{aligned}&\sum_{i=1}^{n}\mathbf{E}\left[\left.(Y(x_{i-1}+\Delta x_i)-Y(x_i))^2\right|\sigma\right]\\&=\int_{\mathbf{R}^n}\sum_{i=1}^{n}(g\mathbf{1}_{-A}(v+\Delta x_i)-g\mathbf{1}_{-A}(v))^2\,\sigma^2(x_i-v)\mathrm{d}v.\end{aligned}$$

Assume now that $\Delta x_i=\Delta x(\delta)=(\tau_1(\delta),\tau_2(\delta),...,\tau_n(\delta))$ for all $i=1,...,n$, with $\tau_1(\delta)=\delta$, (in particular this happens if we are moving along a straight line). We take $n=[t/\delta]$. Then if we define

$$\pi_\delta(\mathrm{d}v):=\frac{(g\mathbf{1}_{-A}(v+\Delta x(\delta))-g\mathbf{1}_{-A}(v))^2}{c(\delta)}\mathrm{d}v,$$

where $c(\delta)=\int_{\mathbf{R}^n}(g\mathbf{1}_{-A}(u+\Delta x_i)-g\mathbf{1}_{-A}(u))^2\,\mathrm{d}u$ we have that

$$\begin{aligned}\frac{\delta}{c(\delta)}\sum_{i=1}^{[t/\delta]}\mathbf{E}\left[\left.(Y(x_{i-1}+\Delta x_i)-Y(x_i))^2\right|\sigma\right]&=\int_{\mathbf{R}^n}\delta\sum_{i=1}^{[t/\delta]}\sigma^2(x_i(\delta)-v)\pi_\delta(\mathrm{d}v)\\&\overset{\delta\to 0}{\longrightarrow}\int_{\mathbf{R}^n}\left(\int_0^t\sigma^2(x(s)-v)\mathrm{d}s\right)\pi_0(\mathrm{d}v),\end{aligned}$$

provided that

$$\pi_\delta\overset{\delta\to 0}{\longrightarrow}\pi_0$$

and σ is continuous. We have also the following result, see [11].

Proposition 1. *If π_0 is concentrated on $-\partial A$ then*

$$\mathrm{var}\left(\left.\frac{\delta}{c(\delta)}\sum_{i=1}^{[t/\delta]}(Y(x_{i-1}+\Delta x_i)-Y(x_i))^2\right|\sigma\right)\overset{\delta\to 0}{\to}0.$$

As a corollary, we have the convergence in probability

$$\frac{\delta}{c(\delta)}\sum_{i=1}^{[t/\delta]}(Y(x_{i-1}+\Delta x_i)-Y(x_i))^2\overset{\delta\to 0}{\longrightarrow}\int_{\mathbf{R}^n}\left(\int_0^t\sigma^2(x(s)-v)\mathrm{d}s\right)\pi_0(\mathrm{d}v).$$

But when is π_0 concentrated on $-\partial A$? In [11] authors give some sufficient conditions for A (bounded, closed, convex with non empty interior and piecewise smooth boundary) and g, but they are quite restrictive.

The behaviour of the RQV along smooth curves and for some particular shapes of A, for instance $A = (\mathbf{R}_+)^n$, and memory functions of the kind $g(x) = ||x||^{\alpha} L_g(||x||)$ is a topic of present research. The purpose is to relate σ or some integral of it, with the limit of the RQV along lines, or surfaces.

To remark that the asymptotic behaviour of the multipower variation of general tempo-spatial ambit fields is an open problem.

4 Models in Finance

4.1 A short rate model

4.1.1 The model

Let $(\Omega, \mathscr{F}, \mathbf{F}, \mathbf{P})$ be a filtered, complete probability space with $\mathbf{F} = (\mathscr{F}_t)_{t\in\mathbf{R}_+}$. Assume that, in this probability space

$$r_t = \int_{-\infty}^{t} g(t-s)\sigma_s W(\mathrm{d}s) + \mu_t \tag{6}$$

where W is an $(\mathscr{F}_t)$-Gaussian noise in $\mathbf{R}$ under the risk neutral probability, $\mathbf{P}^* \sim \mathbf{P}$, g is a deterministic function on $\mathbf{R}_+$, $g \in L^2((0,\infty))$, and $\sigma \geq 0$ and μ are also deterministic. Notice that the process r is not a semimartingale if $g' \notin L^2((0,\infty))$. Furthermore, we also assume that

$$\int_{-\infty}^{t} g^2(t-s)\sigma_s^2 \mathrm{d}s < \infty \quad \text{a.s.}$$

which ensures that r_t is well defined. Then, we consider a financial bond market with short rate r. Here we follow [24].

4.1.2 Bond prices

Assume that $\exp\left\{-\int_0^T r_s \mathrm{d}s\right\} \in L^1(\mathbf{P}^*)$ and denote $P(t,T)$ and $\tilde{P}(t,T)$ the price and the discounted price at t of the zero coupon bond with maturity time T:

$$\begin{aligned} P(t,T) &= \mathbf{E}_{\mathbf{P}^*}\left[\exp\left\{-\int_t^T r_s \mathrm{d}s\right\} \middle| \mathscr{F}_t\right] \\ \tilde{P}(t,T) &= P(t,T)\exp\left\{-\int_0^t r_s \mathrm{d}s\right\}, \end{aligned}$$

where $\tilde{P}(t,T)$ is a $\mathbf{P}^*$-martingale. Then, writing $c(u;t,T) := \int_t^T g(s-u)\mathrm{d}s$ for $t \geq u$, and by using Fubini's theorem, we have,

$$\begin{aligned}\int_t^T r_s \mathrm{d}s = &\int_{-\infty}^t \sigma_u c(u;t,T)W(\mathrm{d}u)\\ &+\int_t^T \sigma_u c(u;u,T)W(\mathrm{d}u)+\int_t^T \mu_s \mathrm{d}s.\end{aligned}$$

Then

$$P(t,T) = \exp\left\{A(t,T) - \int_{-\infty}^t \sigma_u c(u;t,T)W(\mathrm{d}u)\right\},$$

where

$$\begin{aligned}A(t,T) &= \log \mathbf{E}_{\mathbf{P}^*}\left[\exp\left\{-\int_t^T \sigma_u c(u;u,T)W(\mathrm{d}u) - \int_t^T \mu_s \mathrm{d}s\right\}\middle|\mathscr{F}_t\right]\\ &= \frac{1}{2}\int_t^T \sigma_u^2 c^2(u;u,T)\mathrm{d}u - \int_t^T \mu_s \mathrm{d}s.\end{aligned}$$

and the variance of the yield $-\frac{1}{T-t}\log P(t,T)$ is given by

$$\mathrm{var}\left(-\frac{1}{T-t}\log P(t,T)\right) = \frac{1}{(T-t)^2}\int_{-\infty}^t \sigma_u^2 c^2(u;t,T)\mathrm{d}u.$$

The corresponding forward rates are given by

$$\begin{aligned}f(t,T) &= -\partial_T \log P(t,T)\\ &= -\int_t^T \sigma_u^2 g(T-u)c(u;u,T)\mathrm{d}u + \int_{-\infty}^t \sigma_u g(T-u)W(\mathrm{d}u) + \mu_T\end{aligned}$$

and

$$\mathrm{var}\left(f(t,T)\right) = \int_{-\infty}^t \sigma_u^2 g^2(T-u)\mathrm{d}u.$$

Note that

$$\mathrm{d}_t f(t,T) = \alpha(t,T)\mathrm{d}t + \sigma(t,T)W(\mathrm{d}t),$$

with

$$\begin{aligned}\sigma(t,T) &= \sigma_t g(T-t),\\ \alpha(t,T) &= \sigma_t^2 g(T-t)c(t;t,T).\end{aligned}$$

4.1.3 Completeness of the market

It is easy to see that

$$\tilde{P}(t,T) := \frac{P(t,T)}{\exp\left\{\int_0^t r_s \mathrm{d}s\right\}} = P(0,T)\exp\left\{-\int_0^t \sigma_u c(u;u,T)W(\mathrm{d}u) - \frac{1}{2}\int_0^t \sigma_u^2 c(u;u,T)^2 \mathrm{d}u\right\},$$

so we have

$$\begin{aligned} P(t,T) &= P(0,T)\exp\left\{-\frac{1}{2}\int_0^t \sigma_u^2 c^2(u;u,T)\mathrm{d}u + \int_0^t \mu_s \mathrm{d}s\right\} \\ &\quad \times \exp\left\{\int_{-\infty}^0 \sigma_u c(u;0,t)W(\mathrm{d}u) - \int_0^t \sigma_u c(u;t,T)W(\mathrm{d}u)\right\} \end{aligned}$$

and

$$\tilde{P}(t,T) = P(0,T)\exp\left\{-\int_0^t \sigma_u c(u;u,T)W(\mathrm{d}u) - \frac{1}{2}\int_0^t \sigma_u^2 c^2(u;u,T)\mathrm{d}u\right\}.$$

Therefore,

$$\mathrm{d}\tilde{P}(t,T) = -\tilde{P}(t,T)\sigma_t c(t;t,T)W(\mathrm{d}t), t \geq 0,$$

Let X be a $\mathbf{P}^*$-square integrable, $\mathscr{F}_T$-measurable payoff. Consider the $(\mathscr{F}_t)$-martingale

$$M_t := \mathbf{E}_{\mathbf{P}^*}\left[X|\mathscr{F}_t\right], t \geq 0,$$

then, by an extension of Brownian martingale representation theorem, we can write

$$\mathrm{d}M_t = H_t W(\mathrm{d}t),$$

where H is an adapted square integrable process.

Let $\left(\phi_t^0, \phi_t^1\right)$ be a self-financing portfolio built with a bank account and a bond with maturity T, its value process is given by

$$V_t = \phi_t^0 e^{\int_0^t r_s \mathrm{d}s} + \phi_t^1 P(t,T),$$

and, by the self-financing condition, the discounted value process $\tilde{V}$, satisfies

$$\mathrm{d}\tilde{V}_t = \phi_t^1 \mathrm{d}\tilde{P}(t,T).$$

So, if we take

$$\phi_t^1 = -\frac{H_t}{\tilde{P}(t,T)\sigma_t c(t;t,T)}$$

we can replicate X. In particular the bond with maturity T^* can be replicated by taking

$$\frac{P(t,T^*)c(t;t,T^*)}{P(t,T)c(t;t,T)}$$

bonds with maturity time $T \geq T^*$.

4.1.4 Examples

Example 1. With $g(t) = e^{-bt}, \quad \sigma_u = \sigma$ and $\mu = a$, we have

$$r_t = r_0 e^{-bt} + a(1 - e^{-bt}) + e^{-bt} \int_0^t e^{bs} \sigma W(\mathrm{d}s),$$

$$P(t,T) = \exp\left(A(t,T) + aB(t,T) - r_t B(t,T)\right),$$

with

$$B(t,T) = \frac{1}{b}(1 - e^{-b(T-t)})$$

and

$$A(t,T) = \frac{\sigma^2}{2} \int_t^T B(u,T)^2 \mathrm{d}u - a(T-t).$$

Then,

$$\operatorname{var}\left(-\frac{1}{T-t} \log P(t,T)\right) = \frac{\sigma^2}{2b^3} \frac{(1 - e^{-b(T-t)})^2}{(T-t)^2} \sim T^{-2},$$

when $T \to \infty$, and the corresponding instantaneous forward rates and their variance are given by

$$f(t,T) = -\frac{\sigma^2}{2b^2}\left(1 - e^{-b(T-t)}\right)^2 + \sigma e^{-b(T-t)}(r_t - a) + a.$$

$$\operatorname{var}\left(f(t,T)\right) = \frac{\sigma^2}{2b} e^{-2b(T-t)} \sim e^{-2bT},$$

when $T \to \infty$. Moreover the volatility of the forward rates is given by $\sigma(t,T) = \sigma e^{-b(T-t)}$ and this is not too realistic.

Example 2. Assume that $\sigma_t = \sigma \mathbf{1}_{\{t \geq 0\}}$ and

$$g(t) = e^{-b(t)} \int_0^t e^{bs} \beta s^{\beta-1} \mathrm{d}s,$$

for $\beta \in (0, 1/2)$. Then

$$\operatorname{var}\left(f(t,T)\right) = \int_{-\infty}^t \sigma_u^2 g^2(T-u)\mathrm{d}u \sim T^{2\beta-2}.$$

And that the volatility of the forward rates are given by

$$\sigma(t,T) = \sigma^2 g(T-t) \sim T^{\beta-1},$$

when $T \to \infty$, that is more realistic (see [21, Section 4.1] and also [2]) than the exponential decay in the Vasicek model. For $\beta \in (-1/2, 0)$ consider the memory function

$$g(t) = e^{-bt}t^{\beta} + \beta \int_0^t (e^{-b(t-u)} - e^{-bt})u^{\beta-1}\mathrm{d}u,$$

and then

$$g(t) \sim t^{\beta-1}$$

when $x \to \infty$. In such a way that we obtain analogous asymptotic results to the previous case.

4.1.5 The analoge of a CIR model

One of the drawbacks of the previous model is that it allows for negative short rates. An obvious way of avoiding this is to take

$$r_t = \sum_{i=1}^{d} \left(\int_0^t g(t-s)\sigma_s \mathrm{d}W_i(s) \right)^2 + r_0, \quad t \geq 0, r_0 > 0.$$

where $((W_i)_{1\leq i\leq d})$ is a Brownian motion in $\mathbf{R}^d$.

Bond prices

Given

$$r_t = \sum_{i=1}^{d} \int_0^t \int_0^t g(t-u)g(t-v)\sigma_s\sigma_u \mathrm{d}W_i(u)\mathrm{d}W_i(v),$$

(where by simplicity we take $r_0 = 0$), we have

$$\begin{aligned}
\int_t^T r_s \mathrm{d}s = & \sum_{i=1}^{d} \int_0^t \int_0^t \sigma_u\sigma_v c_2(u,v;t,T)\mathrm{d}W_i(u)\mathrm{d}W_i(v) \\
& + 2\sum_{i=1}^{d} \int_0^t \int_t^T \sigma_u\sigma_v c_2(u,v;u,T)\mathrm{d}W_i(u)\mathrm{d}W_i(v) \\
& + \sum_{i=1}^{d} \int_t^T \int_t^T \sigma_u\sigma_v c_2(u,v;u\vee v,T)\mathrm{d}W_i(u)\mathrm{d}W_i(v),
\end{aligned}$$

with $c_2(u,v;t,T) := \int_t^T g(s-u)g(s-v)\mathrm{d}s$. Then, using this, we have

$$\begin{aligned}
P(0,T) &= \mathbf{E}\left[\exp\left\{-\int_0^T r_s \mathrm{d}s\right\}\right] \\
&= \prod_{i=1}^{d} \mathbf{E}\left[\exp\left\{-T\int_0^1 \int_0^1 \sigma_{Tu}\sigma_{Tv}c_2(Tu,Tv;T(u\vee v),T)\mathrm{d}W_i(u)\mathrm{d}W_i(v)\right\}\right] \\
&= d\,(2T)^{-d/2},
\end{aligned}$$

where $d(\lambda)$is the Fredholm determinant

$$d(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \int_0^1 \cdots \int_0^1 \begin{vmatrix} R(s_1,s_1) & \cdots & R(s_1,s_n) \\ \vdots & & \vdots \\ R(s_n,s_1) & \cdots & R(s_n,s_n) \end{vmatrix} \mathrm{d}s_1 \cdots \mathrm{d}s_n$$

where

$$R(u,v) = \sigma_{Tu}\sigma_{Tv}c_2(Tu,Tv;T(u\vee v),T).$$

Example 3. Assume that $g(t) = \mathbf{1}_{\{t\geq 0\}}$ and $\sigma_t = \sigma$. Then r_t is a squared Bessel process of dimension d , see for instance [30], and

$$R(u,v) = \sigma^2 T(1-(u\vee v)),$$

consequently

$$P(0,T) = \left(\cosh(\sqrt{2}\sigma T)\right)^{-\frac{d}{2}} = \frac{2^{\frac{d}{2}}}{\left(e^{\sqrt{2}\sigma T} + e^{-\sqrt{2}\sigma T}\right)^{\frac{d}{2}}},$$

see [40] for the calculations of the Fredholm determinant. Another procedure to calculate the Fredholm determinants is given in [31], where it is shown that provided the kernel $R(u,v)$ is of the form

$$R(u,v) = M(u\vee v)N(u\wedge v)$$

$$d(\lambda) = B_\lambda(1),$$

and therefore

$$P(0,T) = (B_{2T}(1))^{-\frac{d}{2}},$$

where, in our case of having $M(t) = \sigma^2 T(1-t)$ and $N(t) = 1$ and we obtain

$$B_\lambda(t) = \sigma^2 T^2 \left((1-t)\frac{e^{\sigma\sqrt{\lambda T}t} - e^{-\sigma\sqrt{\lambda T}t}}{\sigma\sqrt{\lambda T}} + \frac{e^{\sigma\sqrt{\lambda T}t} + e^{-\sigma\sqrt{\lambda T}t}}{\left(\sigma\sqrt{\lambda T}\right)^2} \right).$$

Note that we can consider squared Bessel processes of dimension $d \geq 0$, where d is not necessarily integer, see [30] and Corollary 6.2.5.5 therein. Due to the fact that discount values are in close form under the model, a calibration performs very fast.

Example 4. Another interesting example is the classical CIR model. In such a case

$$\begin{aligned} R(u,v) &= \sigma^2 \int_{T(u\vee v)}^{T} e^{-b(s-u)} e^{-b(s-v)} \mathrm{d}s = \frac{\sigma^2}{2b} e^{bT((u\wedge v)-1)} \left(e^{-bT((u\vee v)-1)} - e^{bT((u\vee v)-1)}\right) \\ &= M(u\vee v)N(u\wedge v), \end{aligned}$$

where

$$M(t) = \frac{\sigma}{\sqrt{2b}}\left(e^{-bT(t-1)} - e^{bT(t-1)}\right), \qquad \text{and} \qquad N(t) = \frac{\sigma}{\sqrt{2b}} e^{bT(t-1)}.$$

We obtain

$$B_{2T}(1) = \frac{1}{2\sqrt{b^2+2\sigma^2}}\Big((b+\sqrt{b^2+2\sigma^2})e^{T(-b+\sqrt{b^2+2\sigma^2})} + (-b+\sqrt{b^2+2\sigma^2})e^{-T(b+\sqrt{b^2+2\sigma^2})}\Big).$$

4.2 Models in energy markets

Like in other traditional commodities or stock markets, in the electricity market one finds trade in spot, forward/futures contracts as well as European options written on these (see [36, Capter 1] for the definition and terminology of these contracts). Despite this parallelism, the distinctive features of the electricity market lead to specific problems of pricing and hedging. Let us mention two examples of such features. On the one hand, power market trades in contracts which deliver power over a delivery period. This adds an extra dimension to the models for forward dynamics which generally depend only on the current time and the maturity of the contract. On the other hand, the electricity spot cannot be stored directly except via reservoirs for hydro–generated power, or large and expensive batteries. This implies that prices may vary significatively when demand increases, for instance, due to a temperature drop. Moreover, due to the non-storability issue, the electricity spot cannot be held in a portfolio. Hence, the usual *buy–and–hold* hedging arguments break down, and the requirement of being a martingale under an *equivalent martingale measure* (EMM) is not necessary. Similarly, from a liquidity point of view, it would be possible to use non–martingales for modelling forward prices since in many emerging electricity markets, one may not be able to find any buyer to get rid of a forward contract, nor a seller when one wants to enter into one. Thus the illiquidity prevents possible arbitrage opportunities from being exercised.

These features, along with empirical evidence (see [16, 39, 29]) and statistical studies (see [33]), point to random field models in time and space which, in addition, allow for stochastic volatility. We present below two examples of modelling spot and forward prices via ambit processes; these models grant rich flexibility and account for some of the stylized features in the context of energy markets. We note here that since spot prices are determined by supply and demand, strong mean-reversion can be observed; the spot prices have clear deterministic patterns over the year, week and intra-day.

4.2.1 Modelling spot prices

In [5] the log-spot price $Y_\cdot$ is modelled by means of the *Lévy Semistationary Processes* ($\mathscr{LSS}$) presented in Section 2, *i.e.*, processes of the form

$$Y_t := \mu + \int_{-\infty}^{t} g(t-s)\sigma_s \mathrm{d}L_s + \int_{-\infty}^{t} q(t-s)a_s \mathrm{d}s, \tag{7}$$

where μ is a constant, $(L_t)_{t\in\mathbf{R}}$ is a two-sided Lévy process, g and q are non-negative deterministic functions on $\mathbf{R}$, with $g(t) = 0 = q(t)$ for $t \leq 0$, and $\sigma_\cdot$ and $a_\cdot$ are two càdlàg processes. The $\mathscr{LSS}$ are analytically tractable and encompasses some classical models, as that of Schwartz [39], along with a wider class of continuous-time autoregressive moving-average (*CARMA*) processes. Note that in (7) the log-spot price is modelled directly, as opposed to traditional approaches that focus on modelling the dynamics of the spot price.

Consider a forward contract stating the agreement to deliver electricity at time T, for a predetermined price $F_t(T)$, fixed today but payable at T with no other cash flow at $t < T$. This price is referred to as *forward price*, and it is fixed in such a way that the price of the contract, at the issue time t, is zero. Then by definition

$$0 = \mathbf{E}_{\mathbf{P}^*}\left[\exp\left\{-\int_t^T r_u \mathrm{d}u\right\}(\exp\{Y_T\} - F_t(T))\middle|\mathscr{F}_t\right].$$

From this equation and the abstract Bayes' rule (see [36, Lemma A.1.4]), which links the risk-neutral measure $\mathbf{P}^*$ with the T-forward measure

$\mathbf{P}^T$, we get, provided integrability conditions on $\exp\{Y_T\}$,

$$F_t(T) = \mathbf{E}_{\mathbf{P}^T}[\exp\{Y_T\}|\,\mathscr{F}_t]. \tag{8}$$

As mentioned before, due to the lack of an underlying, any measure $\mathbf{P}^T$ equivalent to $\mathbf{P}$ maybe chosen as pricing measure. If we assume that under $\mathbf{P}^T$ the dynamics of the log-spot price is given by 7 with $(L_t)_{t\in\mathbf{R}} = (W_t)_{t\in\mathbf{R}}$ being a two sided Brownian motion, then for a constant volatility $\sigma_s \equiv 1$ we have the simple expression for the forward price

$$F_t(T) = C(T)\exp\left\{\int_{-\infty}^{t} g(T-s)\mathrm{d}W_s - \frac{1}{2}\int_{-\infty}^{t} g^2(T-s)\mathrm{d}s\right\}. \tag{9}$$

We refer to [41] for a multivariate version of (7), and a detailed empirical study using data from the European Energy Exchange.

4.2.2 Modelling forward prices

In [7] forward prices are modelled directly, rather than modelling the spot price and deducing the forward price from the conditional expectation of the spot at delivery (*cf.* 8). Moreover, as opposed to existing literature, the dynamics of the forward price are not specified; instead, the authors specify an ambit field which explicitly describes the forward price. More precisely, for each maturity T, the deseasonalized log-forward price at time t is modelled by

$$\log F_t(x) := \int_{A(t,x)} g(\xi, t-s, x)\sigma_s(\xi)L(\mathrm{d}\xi, \mathrm{d}s), \tag{10}$$

where the spatial component in (10) models the time to maturity, *i.e.*, $x := T-t$, the ambit set is given by $A(t,x) := A_t := \{(\xi,s) : \xi > 0, s \leq t\}$, and the kernel g may be chosen in order to capture the so-called *Samuelson effect* (see [38]). In addition, the fact that forward contracts close in maturity dates are strongly correlated may be captured by assuming that the volatility is another ambit field, independent of L, and with a kernel warranting that $\mathrm{Cor}(\sigma_t^2(x), \sigma_t^2(\bar{x}))$ is high for values of x and $\bar{x}$ close to 0.

Traditionally, the forward price is modelled as a semimartingale such that there is an E(L)MM under which the price dynamics becomes a (local) martingale. According to [7, Corollary 1], $(F_t(T))_{t\in\mathbf{R}}$ is an $\mathbf{F}^L$-martingale if and only if the kernel g in (10) is deterministic and does not depend on t. For instance, one can consider

$$\log F_t(T-t) = \int_{A_t} \exp\{-\alpha(\xi + T - s)\}\sigma_s(\xi)W(\mathrm{d}\xi, \mathrm{d}s), \tag{11}$$

where $\alpha > 0$ and W a homogeneous Gaussian Lévy basis. Such rather strong condition rules out many interesting more general ambit fields, however, it still includes some *CARMA* and standard models as those of Heath *et al.* and Audet *et al.* (see [27] and [1], respectively). Nevertheless, it would be possible to use non–martingales for modelling forward prices without given place to arbitrage opportunities, due to the specific features of electricity markets mentioned above.

Finally, let us mention that (10) induces a model for the log-spot price $Y.$ which is consistent with that in (7). In particular (see [7, Example 2]) the example in (11) leads to

$$Y_t = \int_{-\infty}^{t} \exp\{-\alpha(t-s)\}\mathrm{d}W_s.$$

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Appendix D

A short rate model using ambit processes

Chapter 24
A Short Rate Model Using Ambit Processes

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Abstract In this article, we study a bond market where short rates evolve as

$$r_t = \int_{-\infty}^{t} g(t-s)\sigma_s W(\mathrm{d}s)$$

where $g : (0, \infty) \to \mathbb{R}$ is deterministic, $\sigma \geq 0$ is also deterministic, and W is the stochastic Wiener measure. Processes of this type are also called Brownian semistationary processes and they are particular cases of ambit processes. These processes are, in general, not of the semimartingale kind. We also study a fractional version of the Cox–Ingersoll–Ross model. Some calibration and simulations are also done.

Keywords Bond market • Gaussian processes • Nonsemimartingales • Short rates • Volatility • Cox–Ingersoll–Ross model

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1 Introduction

In this paper we study a bond market where short rates evolve as

$$r_t = \int_{-\infty}^{t} g(t-s)\sigma_s W(\mathrm{d}s),$$

where $g : (0,\infty) \to \mathbb{R}$ is deterministic, $\sigma \geq 0$ is also deterministic, and W is the stochastic Wiener measure. Processes of this type are particular cases of ambit processes. These processes are, in general, not of the semimartingale kind. Our purpose is to see if these new models can capture the features of the bond market by extending popular models like the Vasicek model. Affine models are quite popular as short rate models (see for instance [5]) but they imply a perfect correlation between bond prices and short rates, something unobservable in real markets. Moreover, the long-range dependence in the short interest rates (see [7]) and also in the intensity of default in credit risk models (see [3, 8]) is not captured by these affine models.

We model the short rates under the risk neutral probability and we obtain formulas for bond prices and options on bonds. We also consider defaultable bonds where the short and intensity rates show long-range dependence. We also try to establish the dynamics corresponding to this ad hoc or statistical modelling. This leads us to study the stochastic calculus associated with certain ambit processes. The paper is structured as follows: in the next section we introduce the short rate model. In the second section we calculate the bond and option prices as well as the hedging strategies. In the third section we look for a dynamic version of the model that lead us to a stochastic calculus in a nonsemimartingale setting. In the fourth we discuss a credit risk model with long-range dependence and finally, in the fifth section, we discuss the analogous of the Cox–Ingersoll–Ross (CIR) model in this context and we do some calibration and simulations to see, as a first step, how these models can work in practice.

2 The Model of Short Rates

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered, complete probability space with $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Assume that, in this probability space

$$r_t = \int_{-\infty}^{t} g(t-s)\sigma_s W(\mathrm{d}s) + \mu_t, \tag{24.1}$$

where W is the stochastic Wiener measure under the risk neutral probability, $P^* \sim P$, g is a deterministic function on $\mathbb{R}_+$, $g \in L^2((0,\infty))$, and $\sigma \geq 0$ and μ are also deterministic. Notice that the process r is not a semimartingale if $g' \notin L^2((0,\infty))$. Furthermore, we also assume that

$$\int_{-\infty}^{t} g^2(t-s)\sigma_s^2 \mathrm{d}s < \infty \tag{24.2}$$

which ensures that $r_t < \infty$ almost surely. By an $(\mathcal{F}_t)$-stochastic Wiener measure we understand an L^2-valued measure such that, for any Borelian set A with $E(W(A)^2) < \infty$

$$W(A) \sim N(0, m(A)),$$

where m is the Lebesgue measure and if $A \subseteq [t, +\infty)$ then $W(A)$ is independent of $\mathcal{F}_t$. Note that for $a \in \mathbb{R}$ the process $\left\{B_t := \int_a^{t+a} W(\mathrm{d}s), t \geq 0\right\}$ is a standard Brownian motion.

3 Pricing and Hedging

3.1 Bond Prices

Set

$$P(t,T) = E_{P^*}\left(\exp\left(-\int_t^T r_s \mathrm{d}s\right)\bigg| \mathcal{F}_t\right)$$

for the price at t of the zero-coupon bond with maturity time T. We assume that $\exp\left(-\int_0^T r_s \mathrm{d}s\right) \in L^1(P^*)$ in such a way that the discounted prices $\tilde{P}(t,T) := P(t,T)\exp\left\{-\int_0^t r_s \mathrm{d}s\right\}$ are P^*-martingales. Then we have

$$\begin{aligned}
\int_t^T r_s \mathrm{d}s &= \int_t^T \left(\int_{-\infty}^s g(s-u)\sigma_u W(\mathrm{d}u)\right)\mathrm{d}s + \int_t^T \mu_s \mathrm{d}s \\
&= \int_{-\infty}^t \sigma_u \left(\int_t^T g(s-u)\mathrm{d}s\right) W(\mathrm{d}u) \\
&\quad + \int_t^T \sigma_u \left(\int_u^T g(s-u)\mathrm{d}s\right) W(\mathrm{d}u) + \int_t^T \mu_s \mathrm{d}s \\
&= \int_{-\infty}^t \sigma_u c(u;t,T) W(\mathrm{d}u) \\
&\quad + \int_t^T \sigma_u c(u;u,T) W(\mathrm{d}u) + \int_t^T \mu_s \mathrm{d}s,
\end{aligned}$$

where

$$c(u;t,T) := \int_t^T g(s-u)\mathrm{d}s, t \geq u$$

and where we use the stochastic Fubini theorem. Its use is guaranteed by (24.2). Then

$$P(t,T) = \exp\left(A(t,T) - \int_{-\infty}^{t} \sigma_u c(u;t,T) W(\mathrm{d}u)\right),$$

where

$$\begin{aligned} A(t,T) &= \log E_{P*}\left(\exp\left(-\int_t^T \sigma_u c(u;u,T)W(\mathrm{d}u) - \int_t^T \mu_s \mathrm{d}s\right)\middle| \mathcal{F}_t\right) \\ &= \frac{1}{2}\int_t^T \sigma_u^2 c^2(u;u,T)\mathrm{d}u - \int_t^T \mu_s \mathrm{d}s \end{aligned}$$

and the variance of the yield $-\frac{1}{T-t}\log P(t,T)$ is given by

$$\operatorname{var}\left(-\frac{1}{T-t}\log P(t,T)\right) = \frac{1}{(T-t)^2}\int_{-\infty}^{t} \sigma_u^2 c^2(u;t,T)\mathrm{d}u.$$

The corresponding forward rates are given by

$$\begin{aligned} f(t,T) &= -\partial_T \log P(t,T) \\ &= -\partial_T\left(\frac{1}{2}\int_t^T \sigma_u^2 c^2(u;u,T)\mathrm{d}u\right) + \partial_T\left(\int_{-\infty}^{t} \sigma_u c(u;t,T)W(\mathrm{d}u)\right) \\ &\quad + \partial_T\left(\int_t^T \mu_s \mathrm{d}s\right) \\ &= -\int_t^T \sigma_u^2 g(T-u)c(u;u,T)\mathrm{d}u + \int_{-\infty}^{t} \sigma_u g(T-u)W(\mathrm{d}u) + \mu_T \end{aligned}$$

and

$$\operatorname{var}\left(f(t,T)\right) = \int_{-\infty}^{t} \sigma_u^2 g^2(T-u)\mathrm{d}u.$$

Note that

$$\begin{aligned} \mathrm{d}_t f(t,T) &= \sigma_t^2 g(T-t)c(t;t,T)\mathrm{d}t + \sigma_t g(T-t)W(\mathrm{d}t) \\ &= \alpha(t,T)\mathrm{d}t + \sigma(t,T)W(\mathrm{d}t), \end{aligned}$$

with

$$\begin{aligned} \sigma(t,T) &= \sigma_t g(T-t), \\ \alpha(t,T) &= \sigma_t^2 g(T-t)c(t;t,T). \end{aligned}$$

Obviously it satisfies the HJM condition (see Chap. 18 in [5]) of absence of arbitrage:

$$\begin{aligned}\alpha(t,T) &= \sigma(t,T)\int_t^T \sigma(t,s)\mathrm{d}s \\ &= \sigma_t g(T-t)\int_t^T \sigma_t g(t-s)\mathrm{d}s \\ &= \sigma_t^2 g(T-t)c(t;t,T).\end{aligned}$$

3.2 Completeness of the Market

It is easy to see that

$$\begin{aligned}\tilde{P}(t,T) &:= \frac{P(t,T)}{\exp\left\{\int_0^t r_s \mathrm{d}s\right\}} \\ &= P(0,T)\exp\left(-\int_0^t \sigma_u c(u;u,T)W(\mathrm{d}u) - \frac{1}{2}\int_0^t \sigma_u^2 c(u;u,T)^2 \mathrm{d}u\right).\end{aligned}$$

In fact

$$\begin{aligned}A(0,T) &= \frac{1}{2}\int_0^T \sigma_u^2 c(u;u,T)^2\mathrm{d}u - \int_0^T \mu_s \mathrm{d}s \\ &= A(t,T) - \frac{1}{2}\int_0^t \sigma_u^2 c(u;u,T)^2 \mathrm{d}u - \int_0^t \mu_s \mathrm{d}s,\end{aligned}$$

so

$$\begin{aligned}P(t,T) &= \exp\left(A(t,T) - \int_{-\infty}^t \sigma_u c(u;t,T)W(\mathrm{d}u)\right) \\ &= \exp\left(A(0,T) - \int_{-\infty}^0 \sigma_u c(u;0,T)W(\mathrm{d}u)\right) \\ &\quad \times \exp\left(-\frac{1}{2}\int_0^t \sigma_u^2 c^2(u;u,T)\mathrm{d}u + \int_0^t \mu_s \mathrm{d}s\right) \\ &\quad \times \exp\left(\int_{-\infty}^0 \sigma_u\left(c(u;0,T) - c(u;t,T)\right)W(\mathrm{d}u)\right) \\ &\quad \times \exp\left(-\int_0^t \sigma_u c(u;t,T)W(\mathrm{d}u)\right),\end{aligned}$$

consequently

$$P(t,T) = P(0,T)\exp\left(-\frac{1}{2}\int_0^t \sigma_u^2 c^2(u;u,T)\mathrm{d}u + \int_0^t \mu_s \mathrm{d}s\right)$$
$$\times \exp\left(\int_{-\infty}^0 \sigma_u c(u;0,t)W(\mathrm{d}u) - \int_0^t \sigma_u c(u;t,T)W(\mathrm{d}u)\right),$$

$$\exp\left\{\int_0^t r_s \mathrm{d}s\right\} = \exp\left\{\int_0^t \left(\int_{-\infty}^s \sigma_u g(s-u)W(\mathrm{d}u)\right)\mathrm{d}s + \int_0^t \mu_s \mathrm{d}s\right\}$$
$$= \exp\left\{\int_{-\infty}^0 \sigma_u c(u;0,t)W(\mathrm{d}u) + \int_0^t \sigma_u c(u;u,t)W(\mathrm{d}u) + \int_0^t \mu_s \mathrm{d}s\right\},$$

$$\tilde{P}(t,T) = P(0,T)\exp\left(-\frac{1}{2}\int_0^t \sigma_u^2 c^2(u;u,T)\mathrm{d}u\right)$$
$$\times \exp\left(-\int_0^t \sigma_u(c(u;t,T) + c(u;u,t))W(\mathrm{d}u)\right)$$
$$= P(0,T)\exp\left(-\int_0^t \sigma_u c(u;u,T)W(\mathrm{d}u) - \frac{1}{2}\int_0^t \sigma_u^2 c^2(u;u,T)\mathrm{d}u\right).$$

Therefore,

$$\mathrm{d}\tilde{P}(t,T) = -\tilde{P}(t,T)\sigma_t c(t;t,T)W(\mathrm{d}t), t \geq 0.$$

Let X be a P^*-square integrable, $\mathcal{F}_T$-measurable payoff. Consider the $(\mathcal{F}_t)$-martingale

$$M_t := E_{P^*}(X|\mathcal{F}_t), t \geq 0,$$

then by an extension of Brownian martingale representation theorem we can write

$$\mathrm{d}M_t = H_t W(\mathrm{d}t),$$

where H is an adapted square integrable process. The proof of this extension follows the same steps as the proof of the classical result (for more details, see [14], pp. 198–200). But we need a wider set of functions $\mathcal{E} = \left\{\exp\left(\int_{-\infty}^T f(s)\, W(\mathrm{d}s)\right) : f \in \mathcal{S}\right\}$ as total set in $L^2(\mathcal{F}_T, P^*)$, where $\mathcal{S}$ is the set of step functions with compact support on $(-\infty, T]$.

Let $\left(\phi_t^0, \phi_t^1\right)$ be a self-financing portfolio built with a bank account and a T-bond; its value process is given by

$$V_t = \phi_t^0 \mathrm{e}^{\int_0^t r_s \mathrm{d}s} + \phi_t^1 P(t,T),$$

and, by the self-financing condition, the discounted value process $\tilde{V}_{\cdot}$ satisfies

$$d\tilde{V}_t = \phi_t^1 d\tilde{P}(t,T).$$

So, if we take

$$\phi_t^1 = -\frac{H_t}{\tilde{P}(t,T)\sigma_t c(t;t,T)}$$

we can replicate X. In particular the bond with maturity T^* can be replicated by taking

$$\frac{P(t,T^*)c(t;t,T^*)}{P(t,T)c(t;t,T)}$$

bonds with maturity time $T \geq T^*$.

3.3 Option Prices

Consider a bond with maturity $\bar{T} > T$, where T is the maturity time of a call option for this bond with strike K. Its price is given by (see [5], Chap. 19)

$$\begin{aligned}\Pi(t;T) &= P(t,\bar{T})P^{\bar{T}}(P(T,\bar{T}) \geq K|\mathcal{F}_t) - KP(t,T)P^T(P(T,\bar{T}) \geq K|\mathcal{F}_t) \\ &= P(t,\bar{T})P^{\bar{T}}\left(\frac{P(T,T)}{P(T,\bar{T})} \leq \frac{1}{K}\middle|\mathcal{F}_t\right) - KP(t,T)P^T\left(\frac{P(T,\bar{T})}{P(T,T)} \geq K\middle|\mathcal{F}_t\right),\end{aligned}$$

where P^T is the T-forward measure and analogously for $P^{\bar{T}}$. Define

$$U(t,T,\bar{T}) := \frac{P(t,T)}{P(t,\bar{T})}.$$

Then

$$U(t;T,\bar{T}) = \exp\left\{-A(t,\bar{T}) + A(t,T) - \int_{-\infty}^{t} \sigma_u \left(c(u;t,T) - c(u;t,\bar{T})\right) W(du)\right\}.$$

If we take the $\bar{T}$-forward measure $P^{\bar{T}}$, we will have that

$$W(du) = W^{\bar{T}}(du) - a(u)du,$$

where $W^{\bar{T}}(du)$ is a random Wiener measure in $\mathbb{R}$ again. Then, since $U(t,T,\bar{T})$ has to be a martingale with respect to $P^{\bar{T}}$, $a(u)$ is deterministic and we also have that

$$U(t;T,\bar{T}) = \exp\left\{-\int_{-\infty}^{t} \sigma_u \left(c(u;t,T) - c(u;t,\bar{T})\right) W^{\bar{T}}(\mathrm{d}u) \right.$$
$$\left. -\frac{1}{2}\int_{-\infty}^{t} \sigma_u^2 \left(c(u;t,T) - c(u;t,\bar{T})\right)^2 \mathrm{d}u\right\},$$

so

$$U(T) := U(T;T,\bar{T}) = U(t;T,\bar{T})\exp\left\{\int_t^T \sigma_u c(u;T,\bar{T}) W^{\bar{T}}(\mathrm{d}u) \right.$$
$$\left. -\frac{1}{2}\int_t^T \sigma_u^2 c(u;T,\bar{T})^2 \mathrm{d}u\right\}$$

and analogously

$$U(T)^{-1} = U(T;\bar{T},T) = U^{-1}(t;T,\bar{T})\exp\left\{-\int_t^T \sigma_u c(u;T,\bar{T}) W^{T}(\mathrm{d}u) \right.$$
$$\left. -\frac{1}{2}\int_t^T \sigma_u^2 c(u;T,\bar{T})^2 \mathrm{d}u\right\}.$$

Therefore

$$\begin{aligned}
\Pi(t;T) &= P(t,\bar{T})P^{\bar{T}}(U(T) \leq \frac{1}{K}|\mathcal{F}_t) - KP(t,T)P^{T}(U^{-1}(T) \geq K|\mathcal{F}_t) \\
&= P(t,\bar{T})P^{\bar{T}}(\log U(T) \leq -\log K|\mathcal{F}_t) - KP(t,T)P^{T}(\log U^{-1}(T) \\
&\geq \log K|\mathcal{F}_t) \\
&= P(t,\bar{T})\Phi(d_+) - KP(t,T)\Phi(d_-),
\end{aligned}$$

where

$$d\pm = \frac{\log \frac{P(t,\bar{T})}{KP(t,T)} \pm \frac{1}{2}\Sigma_{t,T,\bar{T}}^2}{\Sigma_{t,T,\bar{T}}}$$

and

$$\Sigma_{t,T,\bar{T}}^2 := \int_t^T \sigma_u^2 c(u;T,\bar{T})^2 \mathrm{d}u.$$

3.4 Examples

Example 3.1. If

$$g(t) = \mathrm{e}^{-bt}, \quad \sigma_u = \sigma, \text{ and } \mu = a,$$

we have

$$\begin{aligned}
r_t &= a + \mathrm{e}^{-bt} \int_{-\infty}^{0} \mathrm{e}^{bs} \sigma W(\mathrm{d}s) + \mathrm{e}^{-bt} \int_{0}^{t} \mathrm{e}^{bs} \sigma W(\mathrm{d}s) \\
&= r_0 \mathrm{e}^{-bt} + a(1 - \mathrm{e}^{-bt}) + \mathrm{e}^{-bt} \int_{0}^{t} \mathrm{e}^{bs} \sigma W(\mathrm{d}s),
\end{aligned}$$

that is the Vasicek model, and

$$\begin{aligned}
P(t,T) &= \exp\left(A(t,T) - \int_t^T \left(\int_{-\infty}^{t} \sigma g(s-u) W(\mathrm{d}u) \right) \mathrm{d}s \right) \\
&= \exp\left(A(t,T) - \int_t^T \left(\int_{-\infty}^{t} \frac{g(s-u)}{g(t-u)} \sigma g(t-u) W(\mathrm{d}u) \right) \mathrm{d}s \right) \\
&= \exp\left(A(t,T) - \int_t^T \mathrm{e}^{-b(s-t)} \left(\int_{-\infty}^{t} \sigma \mathrm{e}^{-b(t-u)} W(\mathrm{d}u) \right) \mathrm{d}s \right) \\
&= \exp\left(A(t,T) - (r_t - a) \int_t^T \mathrm{e}^{-b(s-t)} \mathrm{d}s \right) \\
&= \exp\left(A(t,T) + aB(t,T) - r_t B(t,T) \right),
\end{aligned}$$

with

$$B(t,T) = \frac{1}{b}(1 - \mathrm{e}^{-b(T-t)})$$

and

$$\begin{aligned}
A(t,T) &= \frac{\sigma^2}{2} \int_t^T \left(\int_u^T g(s-u) \mathrm{d}s \right)^2 \mathrm{d}u - a(T-t) \\
&= \frac{\sigma^2}{2} \int_t^T B(u,T)^2 \mathrm{d}u - a(T-t).
\end{aligned}$$

Here

$$c(u;t,T) = \frac{1}{b} \left(\mathrm{e}^{-b(t-u)} - \mathrm{e}^{-b(T-u)} \right), u \leq t \leq T,$$

so

$$\begin{aligned}
\operatorname{var}\left(-\frac{1}{T-t} \log P(t,T) \right) &= \frac{1}{(T-t)^2} \int_{-\infty}^{t} \sigma_u^2 c^2(u;t,T) \mathrm{d}u \\
&= \frac{\sigma^2}{2b^3} \frac{(1 - \mathrm{e}^{-b(T-t)})^2}{(T-t)^2} \sim T^{-2},
\end{aligned}$$

when $T \to \infty$. The corresponding instantaneous forward rates are given by

$$f(t,T) = -\frac{\sigma^2}{2b^2}\left(1 - \mathrm{e}^{-b(T-t)}\right)^2 + \sigma \mathrm{e}^{-b(T-t)}(r_t - a) + a,$$

$$\begin{aligned}\operatorname{var}(f(t,T)) &= \int_{-\infty}^{t} \sigma_u^2 g^2(T-u)\mathrm{d}u \\ &= \sigma^2 \int_{-\infty}^{t} \mathrm{e}^{-2b(T-u)}\mathrm{d}u = \frac{\sigma^2}{2b}\mathrm{e}^{-2b(T-t)} \sim \mathrm{e}^{-2bT},\end{aligned}$$

when $T \to \infty$. Moreover the volatility of the forward rates is given by $\sigma(t,T) = \sigma \mathrm{e}^{-b(T-t)}$ and this is not too realistic.

Example 3.2. Assume that $\sigma_t = \sigma \mathbf{1}_{\{t \geq 0\}}$ and

$$g(t-u) = \mathrm{e}^{-b(t-u)} \int_0^{t-u} \mathrm{e}^{bs} \beta s^{\beta-1}\mathrm{d}s,$$

for $\beta \in (0, 1/2)$. We have that

$$c(u;t,T) := \int_t^T g(s-u)\mathrm{d}s = c(0;0,T-u) - c(0;0,t-u),$$

with

$$c(0;0,x) = \mathrm{e}^{-bx} \int_0^x \mathrm{e}^{bs} s^{\beta}\mathrm{d}s.$$

Then

$$\begin{aligned}\operatorname{var}\left(-\frac{1}{T-t}\log P(t,T)\right) &= \frac{1}{(T-t)^2}\int_{-\infty}^{t} \sigma_u^2 c^2(u;t,T)\mathrm{d}u \\ &= \frac{\sigma^2}{2}\frac{1}{(T-t)^2}\int_0^t (c(0;0,T-u) - c(0;0,t-u))^2\mathrm{d}u \\ &\sim \frac{1}{T^2}\int_0^t c(0;0,T-u)^2\mathrm{d}u \sim T^{2\beta-2},\end{aligned}$$

when $T \to \infty$. In fact

$$c(0;0,x) = \mathrm{e}^{-bx}\int_0^x \mathrm{e}^{bs}s^{\beta}\mathrm{d}s = x^{\beta}\int_0^x \mathrm{e}^{-bs}\left(1 - \frac{s}{x}\right)^{\beta}\mathrm{d}s,$$

and by the monotone convergence theorem

$$\lim_{x\to\infty}\int_0^x \mathrm{e}^{-bs}(1 - \frac{s}{x})^{\beta}\mathrm{d}s = \int_0^{\infty}\mathrm{e}^{-bs}\mathrm{d}s = \frac{1}{b}.$$

Moreover

$$\operatorname{var}(f(t,T)) = \int_{-\infty}^{t} \sigma_u^2 g^2(T-u)\mathrm{d}u \sim T^{2\beta-2}.$$

Since for $x \geq 0$

$$\begin{aligned} g(x) &= \mathrm{e}^{-bx}\int_0^x \mathrm{e}^{bs}\beta s^{\beta-1}\mathrm{d}s = \beta x^{\beta-1}\int_0^x \mathrm{e}^{-bs}(1-\frac{s}{x})^{\beta-1}\mathrm{d}s \\ &= \beta x^{\beta-1}\left(\int_0^{x/2} \mathrm{e}^{-bs}(1-\frac{s}{x})^{\beta-1}\mathrm{d}s + \int_{x/2}^{x} \mathrm{e}^{-bs}(1-\frac{s}{x})^{\beta-1}\mathrm{d}s\right), \end{aligned}$$

and

$$\lim_{x\to\infty}\int_0^{x/2} \mathrm{e}^{-bs}(1-\frac{s}{x})^{\beta-1}\mathrm{d}s = \int_0^{\infty} \mathrm{e}^{-bs}\mathrm{d}s = \frac{1}{b},$$

$$\begin{aligned} \int_{x/2}^{x} \mathrm{e}^{-bs}(1-\frac{s}{x})^{\beta-1}\mathrm{d}s &\leq \mathrm{e}^{-bx/2}\int_{x/2}^{x}(1-\frac{s}{x})^{\beta-1}\mathrm{d}s \\ &= x\mathrm{e}^{-bx/2}\int_0^{1/2} v^{\beta-1}\mathrm{d}v = \frac{x\mathrm{e}^{-bx/2}}{\beta 2^{\beta}} \to 0, \end{aligned}$$

when $x \to \infty$. Also observe that the volatility of the forward rates $\sigma(t,T) = \sigma^2 g(T-t) \sim T^{\beta-1}$, when $T \to \infty$, that is more realistic (see Sect. 4.1 in [7] and also [2]) than the exponential decay in the Vasicek model. For $\beta \in (-1/2, 0)$ consider the memory function

$$g(x) = \mathrm{e}^{-bx}x^{\beta} + \beta\int_0^x (\mathrm{e}^{-b(x-u)} - \mathrm{e}^{-bx})u^{\beta-1}\mathrm{d}u,$$

and then

$$g(x) \sim x^{\beta-1}$$

when $x \to \infty$. In such a way that we obtain analogous asymptotic results to the previous case.

4 An SDE Approach

We have postulated that

$$r_t = \int_{-\infty}^{t} g(t-s)\sigma_s W(\mathrm{d}s) + \mu_t,$$

and the question is if this process $(r_t)_{t\in\mathbb{R}}$ can be seen as the solution of such a stochastic differential equation. For instance, assume that

$$\mathrm{d}r_t = b(a - r_t)\mathrm{d}t + \sigma W(\mathrm{d}t),$$

then we have

$$r_t = r_0 \mathrm{e}^{-bt} + a(1 - \mathrm{e}^{-bt}) + \mathrm{e}^{-bt} \int_0^t \mathrm{e}^{bs} \sigma W(\mathrm{d}s),$$

and if we take

$$r_0 = \int_{-\infty}^0 \mathrm{e}^{bs} \sigma W(\mathrm{d}s) + a,$$

we obtain that

$$r_t = a + \int_{-\infty}^t \mathrm{e}^{-b(t-s)} \sigma W(\mathrm{d}s).$$

So, it corresponds to $g(t) = \mathrm{e}^{-bt}$, $\sigma_s = \sigma$, and $\mu_t = a$.

4.1 Ambit Processes as Noises of SDE

Consider the processes W^g given by

$$W_t^g := \int_{-\infty}^t g(s,t) W(\mathrm{d}s),$$

where $g : \mathbb{R}^2 \to \mathbb{R}$ deterministic, continuously differentiable with respect to the second variable, $g(s,t) = 0$ if $s > t$ and $\int_{-\infty}^t g^2(s,t)\mathrm{d}s < \infty$. In this section we explain how a stochastic calculus can be developed with respect to these processes. Here we follow [1, 7, 13]. First, formally,

$$W_t^g(\mathrm{d}t) = g(t,t)W(\mathrm{d}t) + \left(\int_{-\infty}^t \partial_t g(s,t) W(\mathrm{d}s) \right) \mathrm{d}t,$$

and for a deterministic function $f(\cdot,\cdot)$, we can define

$$\begin{aligned}
&\int_{-\infty}^t f(u,t) W_t^g(\mathrm{d}u) \\
&\quad = \int_{-\infty}^t f(u,t) \left(g(u,u)W(\mathrm{d}u) + \left(\int_{-\infty}^u \partial_u g(s,u) W(\mathrm{d}s) \right) \mathrm{d}u \right) \\
&\quad = \int_{-\infty}^t \left(\int_{-\infty}^u \left(f(u,t) - f(s,t) \right) \partial_u g(s,u) W(\mathrm{d}s) \right) \mathrm{d}u
\end{aligned}$$

$$
\begin{aligned}
&+\int_{-\infty}^{t}\left(\int_{s}^{t} f(s,t)\partial_u g(s,u)\mathrm{d}u\right) W(\mathrm{d}s)\\
&+\int_{-\infty}^{t} f(u,t)g(u,u)W(\mathrm{d}u)\\
=&\int_{-\infty}^{t}\left(\int_{s}^{t} \left(f(u,t)-f(s,t)\right)\partial_u g(s,u)\mathrm{d}u\right) W(\mathrm{d}s)\\
&+\int_{-\infty}^{t} f(s,t)g(s,t)W(\mathrm{d}s)\\
=&\int_{-\infty}^{t}\left(\int_{s}^{t} \left(f(u,t)-f(s,t)\right)\partial_u g(s,u)\mathrm{d}u + f(s,t)g(s,t)\right) W(\mathrm{d}s).
\end{aligned}
$$

Then, the latest integral is well defined in an L^2 sense, provided that

$$\int_{-\infty}^{t}\left(\int_{s}^{t} \left(f(u,t)-f(s,t)\right)\partial_u g(s,u)\mathrm{d}u + f(s,t)g(s,t)\right)^2 \mathrm{d}s<\infty.$$

Now, if we construct the operator

$$K_t^g(f)(s,t):=\int_{s}^{t} \left(f(u,t)-f(s,t)\right)\partial_u g(s,u)\mathrm{d}u + f(s,t)g(s,t),$$

it is natural to define

$$\int_{-\infty}^{t} f(s,t)W_t^g(\mathrm{d}s):=\int_{-\infty}^{t} K_t^g(f)(s,t)W(\mathrm{d}s),$$

provided that $f(\cdot,t)\in\left(K_t^g\right)^{-1}(L^2(-\infty,t])$.

Note that if $g(s,s)=0$, then we can write

$$K_t^g(f)(s,t):=\int_{s}^{t} f(u,t)\partial_u g(s,u)\mathrm{d}u, \tag{24.3}$$

and in the particular case that $\Delta f=0$, we have

$$
\begin{aligned}
K_t^g(f)(s,t)&=\partial_t\int_{s}^{t} f(u,t)g(s,u)\mathrm{d}u\\
&=\partial_t\left(f*g\right)(s,t),
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{-\infty}^{t} f(s,t)W_t^g(\mathrm{d}s)&=\int_{-\infty}^{t}\left(\partial_t\int_{s}^{t} f(u,t)g(s,u)\mathrm{d}u\right)W(\mathrm{d}s)\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{t}\int_{s}^{t} f(u,t)g(s,u)\mathrm{d}uW(\mathrm{d}s)
\end{aligned}
$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{t} f(u,t) \left(\int_{-\infty}^{u} g(s,u) W(\mathrm{d}s) \right) \mathrm{d}u$$
$$= \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{t} f(u,t) W_u^g \mathrm{d}u.$$

Consider now

$$r_t = b \int_0^t (a - r_s) \mathrm{d}s + \sigma \int_0^t (t-s)^\beta W(\mathrm{d}s),$$

with $\beta \in (-1/2, 0) \cup (0, 1/2)$, then if we define

$$W_t^\beta := \int_0^t (t-s)^\beta W(\mathrm{d}s),$$

$$r_t = b \int_0^t (a - r_s) \mathrm{d}s + \sigma W^\beta(t).$$

In such a way that (r_t) is an Ornstein–Uhlenbeck process driven by W^β.

We obtain

$$r_t = r_0 \mathrm{e}^{-bt} + a(1 - \mathrm{e}^{-bt}) + \mathrm{e}^{-bt} \int_0^t \mathrm{e}^{bs} \sigma W^\beta(\mathrm{d}s)$$
$$= r_0 \mathrm{e}^{-bt} + a(1 - \mathrm{e}^{-bt}) + \int_0^t \sigma g(t-s) W(\mathrm{d}u).$$

Then, if $\beta \in (0, 1/2)$, by (24.3) we have

$$\int_0^t \mathrm{e}^{-b(t-s)} W^\beta(\mathrm{d}s) = \int_0^t \left(\int_u^t \mathrm{e}^{-b(t-s)} \beta (s-u)^{\beta-1} \mathrm{d}s \right) W(\mathrm{d}u)$$
$$= \int_0^t \left(\int_0^{t-u} \mathrm{e}^{-b(t-s-u)} \beta s^{\beta-1} \mathrm{d}s \right) W(\mathrm{d}u).$$
$$= \int_0^t \mathrm{e}^{-b(t-u)} \left(\int_0^{t-u} \mathrm{e}^{bs} \beta s^{\beta-1} \mathrm{d}s \right) W(\mathrm{d}u).$$

In such a way that

$$g(t-s) = \mathrm{e}^{-b(t-s)} \left(\int_0^{t-s} \mathrm{e}^{bu} \beta u^{\beta-1} \mathrm{d}u \right),$$

and if $\beta \in (-1/2, 0)$

$$g(t-s) = \mathrm{e}^{-b(t-s)} (t-s)^\beta + \beta \mathrm{e}^{-b(t-s)} \int_0^{t-s} (\mathrm{e}^{bu} - 1) v^{\beta-1} \mathrm{d}u.$$

5 A Defaultable Zero-Coupon Bond

The purpose in this section is to price a zero-coupon bond with possibility of default. The payoff of this contract at the maturity time is $1_{\{\tau>T\}}$, where τ is the default time. Then, an arbitrage free price at time t is given by

$$D(t,T) = 1_{\{\tau>t\}} E\left(1_{\{\tau>T\}} \mathrm{e}^{-\int_t^T r_s \mathrm{d}s} \middle| \mathcal{G}_t\right), 0 \leq t \leq T,$$

where the expectation is taken with respect to a risk neutral probability, P^*, and where the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ represents the information available to the market. Here we follow the hazard process approach (for more details, see Sect. 8.2 in [4]). In this approach we consider two filtrations, one is the default-free filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ that typically incorporates the history of the short rates. The default time is modelled by a random variable τ that is not necessarily an $\mathbb{F}$-stopping time, then the other filtration is $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, where

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau \wedge t),$$

in such a way that τ is a $\mathbb{G}$-stopping time. Now, if we assume that there exists an $\mathbb{F}$-adapted process $(\lambda_t)_{t\geq 0}$, the so-called hazard process, such that

$$P^*(\tau > t|\mathcal{F}_t) = \mathrm{e}^{-\int_0^t \lambda_s \mathrm{d}s},$$

it can be shown (see [12], Chap. 8) that

$$D(t,T) = 1_{\{\tau>t\}} E\left(1_{\{\tau>T\}} \mathrm{e}^{-\int_t^T r_s \mathrm{d}s} \middle| \mathcal{G}_t\right) = 1_{\{\tau>t\}} E\left(\mathrm{e}^{-\int_t^T (r_s+\lambda_s)\mathrm{d}s} \middle| \mathcal{F}_t\right).$$

Then we need a model for $(r_t)_{t\geq 0}$ and $(\lambda_t)_{t\geq 0}$. A classical model is a Vasicek model for both processes

$$\begin{aligned}
\mathrm{d}r_t &= b(a - r_t)\mathrm{d}t + \sigma \mathrm{d}W(t),\\
\mathrm{d}\lambda_t &= \breve{b}(\breve{a} - \lambda_t)\mathrm{d}t + \breve{\sigma}\mathrm{d}\breve{W}(t),
\end{aligned}$$

where W and $\breve{W}$ are correlated Brownian motions and here $\mathcal{F}_t = \sigma(W_s, \breve{W}_s, 0 \leq s \leq t)$. The idea is to extend this model by considering ambit processes as noises in the stochastic differential equations. For instance we can have

$$\begin{aligned}
r_t &= \int_{-\infty}^t \sigma_s g(t-s) W(\mathrm{d}s) + \mu_t,\\
\lambda_t &= \int_{-\infty}^t \breve{\sigma}_s \breve{g}(t-s) \breve{W}(\mathrm{d}s) + \breve{\mu}_t.
\end{aligned}$$

See [3] for a similar modelling. Then, the price of a defaultable zero-coupon bond at time t will be given by

$$D(t,T) = 1_{\{\tau>t\}} \exp\left(A(t,T) - \int_{-\infty}^{t} (\sigma_u c(u;t,T) W(\mathrm{d}u) + \breve{\sigma}_u \breve{c}(u;t,T)) \breve{W}(\mathrm{d}u)\right),$$

where

$$\begin{aligned} A(t,T) = \frac{1}{2} \int_t^T & \left(\sigma_u^2 c^2(u;t,T) + \breve{\sigma}_u^2 \breve{c}^2(u;t,T) + 2\rho\sigma_u \breve{\sigma}_u c(u;t,T)\breve{c}(u;t,T)\right) \mathrm{d}u \\ & - \int_t^T (\mu_u + \breve{\mu}_u)\,\mathrm{d}u \end{aligned}$$

and ρ is the correlation coefficient between W and $\breve{W}$. Interesting cases are $\sigma_u = \sigma 1_{\{u\geq 0\}}$, $\sigma_u = \breve{\sigma} 1_{\{u\geq 0\}}$, $\mu_u = \mu$, $\breve{\mu}_u = \breve{\mu}$,

$$\begin{aligned} g(t-s) &= \mathrm{e}^{-b(t-s)} \int_0^{t-s} \mathrm{e}^{bu} \beta u^{\beta-1} \mathrm{d}u, \\ \breve{g}(t-s) &= \mathrm{e}^{-\breve{b}(t-s)} \int_0^{t-s} \mathrm{e}^{\breve{b}u} \breve{\beta} u^{\breve{\beta}-1} \mathrm{d}u, \end{aligned}$$

$\beta, \breve{\beta} \in (-1/2, 0) \cup (0, 1/2)$. Note that

$$\operatorname{var}\left(-\frac{1}{T-t} \log D(t,T)\right) \sim T^{2(\beta\vee\breve{\beta})-2}.$$

6 The Analogue of a CIR Model

One of the drawbacks of the previous model is that it allows for negative short rates. An obvious way of avoiding this is to take

$$r_t = \sum_{i=1}^{d} \left(\int_0^t g(t-s)\sigma_s \mathrm{d}W_i(s)\right)^2 + r_0, \quad t \geq 0, r_0 > 0,$$

where $(W_i))_{1\leq i\leq d}$ is a Brownian motion in $\mathbb{R}^d$.

6.1 Bond Prices

$$r_t = \sum_{i=1}^{d} \int_0^t \int_0^t g(t-u)g(t-v)\sigma_s\sigma_u \mathrm{d}W_i(u)\mathrm{d}W_i(v),$$

where by simplicity we take $r_0 = 0$, then

$$
\begin{aligned}
\int_t^T r_s \mathrm{d}s &= \sum_{i=1}^d \int_t^T \left(\int_0^s g(s-u)g(s-v)\sigma_u\sigma_v \mathrm{d}W_i(u)\mathrm{d}W_i(v) \right) \mathrm{d}s \\
&= \sum_{i=1}^d \int_0^t \int_0^t \sigma_u\sigma_v \left(\int_t^T g(s-u)g(s-v)\mathrm{d}s \right) \mathrm{d}W_i(u)\mathrm{d}W_i(v) \\
&\quad +2 \sum_{i=1}^d \int_0^t \int_t^T \sigma_u\sigma_v \left(\int_u^T g(s-u)g(s-v)\mathrm{d}s \right) W_i(\mathrm{d}u)W_i(\mathrm{d}v) \\
&\quad + \sum_{i=1}^d \int_t^T \int_t^T \sigma_u\sigma_v \left(\int_{u\vee v}^T g(s-u)g(s-v)\mathrm{d}s \right) W_i(\mathrm{d}u)W_i(\mathrm{d}v) \\
&= \sum_{i=1}^d \int_0^t \int_0^t \sigma_u\sigma_v c_2(u,v;t,T)\mathrm{d}W_i(u)\mathrm{d}W_i(v) \\
&\quad +2 \sum_{i=1}^d \int_0^t \int_t^T \sigma_u\sigma_v c_2(u,v;u,T)\mathrm{d}W_i(u)\mathrm{d}W_i(v) \\
&\quad + \sum_{i=1}^d \int_t^T \int_t^T \sigma_u\sigma_v c_2(u,v;u\vee v,T)\mathrm{d}W_i(u)\mathrm{d}W_i(v),
\end{aligned}
$$

with $c_2(u,v;t,T) := \int_t^T g(s-u)g(s-v)\mathrm{d}s$.

$$
\begin{aligned}
P(0,T) &= E\left(\exp\left\{ -\int_0^T r_s \mathrm{d}s \right\} \right) \\
&= E\left(\exp\left\{ -\sum_{i=1}^d \int_0^T \int_0^T \sigma_u\sigma_v c_2(u,v;u\vee v,T)\mathrm{d}W_i(u)\mathrm{d}W_i(v) \right\} \right) \\
&= \prod_{i=1}^d E\left(\exp\left\{ -T \int_0^1 \int_0^1 \sigma_{Tu}\sigma_{Tv} c_2(Tu,Tv;T(u\vee v),T)\mathrm{d}W_i(u)\mathrm{d}W_i(v) \right\} \right) \\
&= \left(1 + \sum_{n=1}^\infty \frac{(2T)^n}{n!} \int_0^1 \cdots \int_0^1 \begin{vmatrix} R(s_1,s_1) & \cdots & R(s_1,s_n) \\ \vdots & & \vdots \\ R(s_n,s_1) & \cdots & R(s_n,s_n) \end{vmatrix} \mathrm{d}s_1 \cdots \mathrm{d}s_n \right)^{-d/2},
\end{aligned}
$$

where

$$R(u,v) = \sigma_{Tu}\sigma_{Tv} c_2(Tu,Tv;T(u\vee v),T).$$

In the second equality we use the scaling property of the Brownian motion and in the third Corollary 4 in [15].

Example 6.1. Assume that $g(t) = \mathbf{1}_{\{t\geq 0\}}$ and $\sigma_t = \sigma$. Then r_t is a squared Bessel process of dimension d (see for instance [10]) and

$$R(u,v) = \sigma^2 T(1-(u\vee v)),$$

consequently

$$P(0,T) = (\cosh(\sqrt{2}\sigma T)^{-\frac{d}{2}} = \frac{2^{\frac{d}{2}}}{\left(\mathrm{e}^{\sqrt{2}\sigma T}+\mathrm{e}^{-\sqrt{2}\sigma T}\right)^{\frac{d}{2}}}$$

(see [15] for the calculations of the Fredholm determinant),

$$d(\lambda) := \left(1+\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\int_0^1\cdots\int_0^1\begin{vmatrix} R(s_1,s_1) & \cdots & R(s_1,s_n) \\ \vdots & & \vdots \\ R(s_n,s_1) & \cdots & R(s_n,s_n)\end{vmatrix}\mathrm{d}s_1\cdots\mathrm{d}s_n\right).$$

Another procedure to calculate the Fredholm determinants is given in [11], where it is shown that provided the kernel $R(u,v)$ is of the form

$$R(u,v) = M(u\vee v)N(u\wedge v)$$

we have that

$$d(\lambda) = B_\lambda(1),$$

and therefore

$$P(0,T) = (B_{2T}(1))^{-\frac{d}{2}},$$

where $B_\lambda(t)$is defined by the linear differential equation system

$$\begin{pmatrix}\dot{A}_\lambda(t)\\ \dot{B}_\lambda(t)\end{pmatrix} = \lambda\begin{pmatrix}-N(t)M(t) & N^2(t)\\ -M^2(t) & N(t)M(t)\end{pmatrix}\begin{pmatrix}A_\lambda(t)\\ B_\lambda(t)\end{pmatrix},$$

$$\begin{pmatrix}A_\lambda(0)\\ B_\lambda(0)\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}.$$

In our case $M(t) = \sigma^2 T(1-t)$ and $N(t) = 1$ and by straightforward calculations we obtain

$$B_\lambda(t) = \sigma^2 T^2\left((1-t)\frac{\mathrm{e}^{\sigma\sqrt{\lambda T}t}-\mathrm{e}^{-\sigma\sqrt{\lambda T}t}}{\sigma\sqrt{\lambda T}}+\frac{\mathrm{e}^{\sigma\sqrt{\lambda T}t}+\mathrm{e}^{-\sigma\sqrt{\lambda T}t}}{\left(\sigma\sqrt{\lambda T}\right)^2}\right).$$

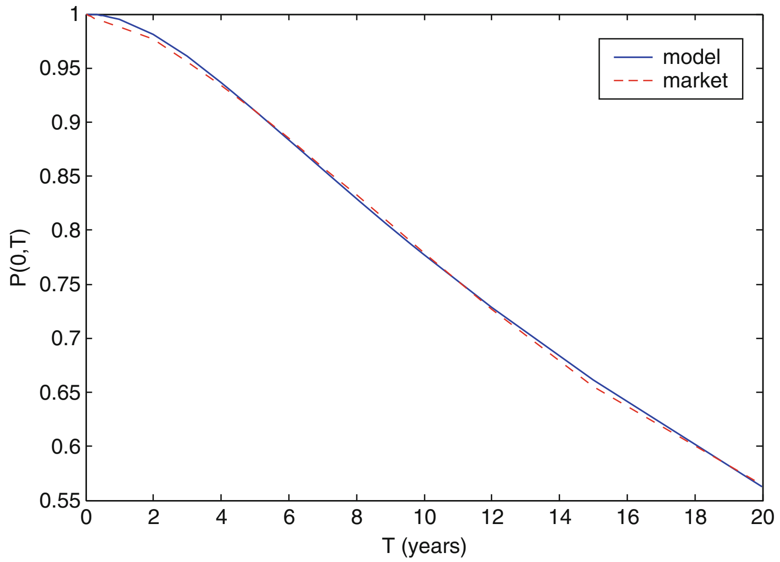

Fig. 24.1 EUR – Discount curve 04/11/2011 : $\sigma = 21.90\%$ and d $= 0.2093$

Note that we can consider squared Bessel processes of dimension $d \geq 0$, where d is not necessarily integer (see [10] and Corollary 6.2.5.5 therein). A calibration of this model is given in Fig. 24.1. We have performed a calibration of the model on the market discount curve of the 4th of November 2011. More precisely, we have on that date calibrated the d and σ parameters on the EUR market implied discount curve up to 20 years of maturity. The optimal parameters were obtained using a least-squared-error minimization employing a Nelder–Mead search algorithm. The calibrating is performed very fast and the optimal parameters are obtained in less than a second, due to the fact that discount values under the model are available in close form. Even though this model is not mean reverting the fit to real data is quite good.

Example 6.2. Another interesting example is the classical CIR model. In such a case

$$\begin{aligned} R(u,v) &= \sigma^2 \int_{T(u\vee v)}^{T} \mathrm{e}^{-b(s-u)}\mathrm{e}^{-b(s-v)}\mathrm{d}s = \frac{\sigma^2}{2b}\mathrm{e}^{bT((u\wedge v)-1)}(\mathrm{e}^{-bT((u\vee v)-1)} \\ &\quad -\mathrm{e}^{bT((u\vee v)-1)}) \\ &= M(u\vee v)N(u\wedge v), \end{aligned}$$

where

$$M(t) = \frac{\sigma}{\sqrt{2b}} \left(\mathrm{e}^{-bT(t-1)} - \mathrm{e}^{bT(t-1)} \right),$$
$$N(t) = \frac{\sigma}{\sqrt{2b}} \mathrm{e}^{bT(t-1)}.$$

Then we have the system

$$\begin{pmatrix} \dot{A}_\lambda(t) \\ \dot{B}_\lambda(t) \end{pmatrix} = \frac{\lambda\sigma^2}{2b} \begin{pmatrix} \mathrm{e}^{2bT(t-1)} \left(1 - \mathrm{e}^{-2bT(t-1)}\right) & \mathrm{e}^{2bT(t-1)} \\ -\left(\mathrm{e}^{2bT(t-1)} - 1\right)\left(1 - \mathrm{e}^{-2bT(t-1)}\right) & -\left(\mathrm{e}^{2bT(t-1)} - 1\right) \end{pmatrix}$$
$$\times \begin{pmatrix} A_\lambda(t) \\ B_\lambda(t) \end{pmatrix},$$
$$\begin{pmatrix} A_\lambda(0) \\ B_\lambda(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So,

$$\dot{B}_\lambda(t) = \left(\mathrm{e}^{-2bT(t-1)} - 1\right) \dot{A}_\lambda(t)$$

and

$$\ddot{A}_\lambda(t) = 2bT\dot{A}_\lambda(t) + \lambda\sigma^2 T A_\lambda(t);$$

from here we obtain that

$$A_{2T}(t) = C\left(\mathrm{e}^{T(b+\sqrt{b^2+2\sigma^2})t} - \mathrm{e}^{T(b-\sqrt{b^2+2\sigma^2})t}\right)$$

and that

$$B_{2T}(t) = C\left(\mathrm{e}^{-2bT(t-1)} - 1\right)\left(\mathrm{e}^{T\left(b+\sqrt{b^2+2\sigma^2}\right)t} - \mathrm{e}^{T(b-\sqrt{b^2+2\sigma^2})t}\right)$$
$$+C(-2b)T\mathrm{e}^{2bT}\left(\frac{\mathrm{e}^{T\left(-b+\sqrt{b^2+2\sigma^2}\right)t}}{-b+\sqrt{b^2+2\sigma^2}} - \frac{\mathrm{e}^{T\left(-b-\sqrt{b^2+2\sigma^2}\right)t}}{-b-\sqrt{b^2+2\sigma^2}}\right),$$

where $C = -\frac{\sigma^2 \mathrm{e}^{-2bT}}{2bT\sqrt{b^2+2\sigma^2}}$. Therefore

$$B_{2T}(1) = \frac{1}{2\sqrt{b^2+2\sigma^2}}\left((b+\sqrt{b^2+2\sigma^2})\mathrm{e}^{T(-b+\sqrt{b^2+2\sigma^2})}\right.$$
$$\left.+(-b+\sqrt{b^2+2\sigma^2})\mathrm{e}^{-T(b+\sqrt{b^2+2\sigma^2})}\right).$$

6.2 Numerical Methods for Pricing

In case that the Fredholm determinant appearing in the price formula cannot be calculated analytically, efficient numerical methods are known [6]. The idea of the approximation is the following: first let denote

$$d^R(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_0^1 \cdots \int_0^1 \begin{vmatrix} R(s_1, s_1) & \cdots & R(s_1, s_n) \\ \vdots & & \vdots \\ R(s_n, s_1) & \cdots & R(s_n, s_n) \end{vmatrix} \mathrm{d}s_1 \cdots \mathrm{d}s_n,$$

the price we are looking for equals $\left[d^R(2T)\right]^{-d/2}$; then, for a given quadrature formula

$$Q_m(f) = \sum_{j=1}^{m} w_j f\left(x_j\right) \approx \int_0^1 f(x)\,\mathrm{d}x,$$

we consider the *Nyström-type* approximation of $d(\lambda)$:

$$d^R_{Q_m}(\lambda) = \det\left[\delta_{ij} + \lambda w_i R\left(x_i, x_j\right)\right]_{i,j=1}^{m}. \tag{24.4}$$

By the von Koch formula (see [6]), we can write

$$d^R_{Q_m}(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} Q^n_m(R_n)$$

where, for functions f on $\mathbb{R}^n$,

$$Q^n_m(f) := \sum_{j_1,\ldots,j_n=1}^{m} w_{j_1} \ldots w_{j_n} f\left(x_{j_1}, \ldots x_{j_n}\right)$$

and $R_n(s_1, \ldots, s_n) := \det\left[R\left(s_i, s_j\right)\right]_{i,j=1}^{n}$. Note that the previous series terminates in fact at $n = m$. Nevertheless, the error is given by the exponentially generating function of the quadrature errors for the functions R_n

$$d^R_Q(\lambda) - d(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \left[Q^n_m(R_n) - \int_{[0,1]^n} R_n(t_1, \ldots, t_n)\,\mathrm{d}t_1 \cdots \mathrm{d}t_n\right].$$

So, this method approximates the Fredholm determinant by the determinant of an $m \times m$ matrix applied in (24.4). If the weights are positive (which is always a better choice), its equivalent symmetric variant is

$$d^R_{Q_m}(\lambda) = \det\left[\delta_{ij} + \lambda w_i^{1/2} R\left(x_i, x_j\right) w_j^{1/2}\right]_{i,j=1}^{m}.$$

Using Gauss–Legendre quadrature rule, the computation cost is of order $O\left(m^3\right)$ and simple codes for Matlab and Mathematica can be found on page in [6]. Also, Theorem 6.1 in [6] shows that if a family Q_m of quadrature rules converges for continuous functions, when m goes to infinity, then the corresponding Nyström-type approximation of the Fredholm determinant converges to $d\,(\lambda)$, uniformly for bounded λ. Moreover Theorem 6.2 in [6] shows that if $R \in C^{k-1,1}\left([0,1]^2\right)$, then for each quadrature rule Q of order $v \geq k$ with positive weights there holds the error estimate

$$\left| d^R_{Q_m}(\lambda) - d^R(\lambda) \right| \leq c_k 2^k (b-a)\, v^{-k} \Phi\left(|z|\,(b-a)\,\|R\|_k\right),$$

where c_k is a constant depending only on k:

$$\|R\|_k = \max_{i+j\leq k} \left\| \partial_1^i \partial_2^j R \right\|_{L^\infty}$$

and

$$\Phi(z) = \sum_{n=1}^{\infty} \frac{n^{(n+2)/2}}{n!} z^n$$

is an entire function on $\mathbb{C}$.

Figure 24.2 shows the relative error

$$R(T) = \left| \frac{P(0,T) - d^R_{Q_{100}}(2T)}{P(0,T)} \right|$$

in the classical CIR model as presented in Example 6.2 (with $m = 100$).

Now, we can apply this method to evaluate numerically Fredholm determinants and consequently prices for bonds in the CIR models. With the notation used above, we have the following proposition:

Proposition 6.1. *Assume $\sigma_t = \mathbf{1}_{\{t\geq 0\}}$, $g(s) = s^\alpha$, for $\alpha \in (-1/2, 1/2)$, let*

$$\tilde{R}(u,v) = \left[\frac{2(1-u)(1-v)}{2-u-v}\right]^{2\alpha+1} - \frac{1}{2}\left(\frac{|u-v|}{2}\right)^{2\alpha+1} \times \left[B\left(\frac{1}{2}-\alpha, \alpha+1\right) - B^\gamma\left(\frac{1}{2}-\alpha,\alpha+1\right)\right]$$

for $\gamma = \left(\frac{u-v}{2-(u+v)}\right)^2$, and where B and B^γ are the beta and the incomplete beta functions, respectively. Then, the price of a zero-coupon bond, for the corresponding CIR model, is given by

$$P(0,T) = \left[d^R(2T)\right]^{-d/2} = \left[d^{\tilde{R}}\left(\frac{2T^{2\alpha+2}}{1+2\alpha}\right)\right]^{-d/2} \approx \left[d^{\tilde{R}}_{Q_m}\left(\frac{2T^{2\alpha+2}}{1+2\alpha}\right)\right]^{-d/2}.$$

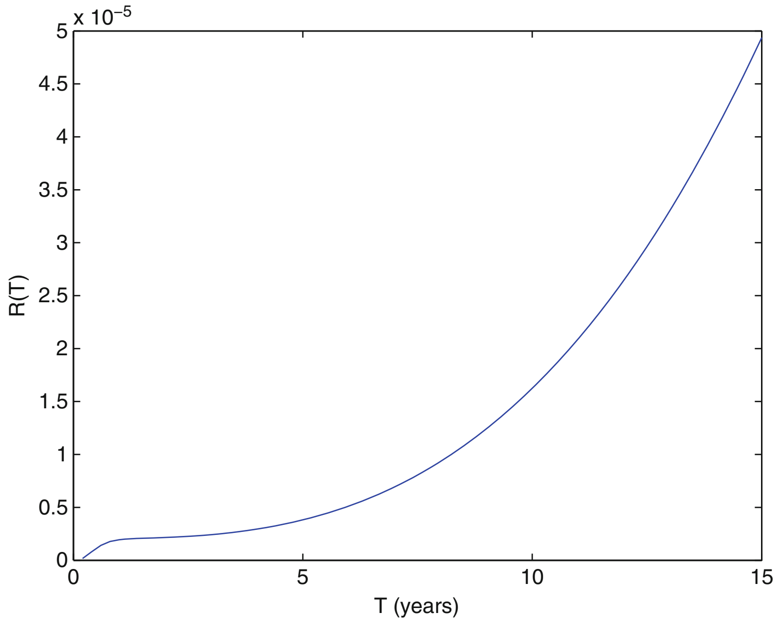

Fig. 24.2 R(T), Relative error, classical CIR model, d = 2, sigma = 0.2, m = 100

Proof. Assume that $0 \leq v \leq u \leq 1$, then

$$\begin{aligned} c_2(Tu, Tv; Tu, T) &= \int_{Tu}^{T} g(s - Tu) g(s - Tv) \mathrm{d}s = \int_{Tu}^{T} (s - Tu)^{\alpha} (s - Tv)^{\alpha} \, \mathrm{d}s \\ &= T^{2\alpha+1} \int_{u}^{1} (s - u)^{\alpha} (s - v)^{\alpha} \, \mathrm{d}s = T^{2\alpha+1} c_2(u, v; u, 1). \end{aligned} \tag{24.5}$$

Now, for $u \neq v$, we have

$$\int_{u}^{1} (s - u)^{\alpha} (s - v)^{\alpha} \, \mathrm{d}s = \left(\frac{u - v}{2}\right)^{2\alpha} \int_{u}^{1} \left[\left(s \frac{2}{u - v} - \frac{u + v}{u - v} \right)^2 - 1 \right]^{\alpha} \mathrm{d}s,$$

and we obtain

$$c_2(u, v; u, 1) = \left(\frac{u - v}{2}\right)^{2\alpha+1} \int_{1}^{b} \left(x^2 - 1\right)^{\alpha} \mathrm{d}x,$$

where $b = \frac{2-(u+v)}{u-v}$. Now, by writing $1/b^2 = \gamma$, we have

$$\begin{aligned}
&\int_1^b \left(x^2-1\right)^\alpha \mathrm{d}x \\
&= \frac{1}{2}\int_\gamma^1 (1-x)^\alpha\, x^{-\frac{3}{2}-\alpha}\mathrm{d}x = \int_\gamma^1 (1-x)^{\alpha+1}\, x^{-\frac{3}{2}-\alpha}\mathrm{d}x \\
&\quad + \int_\gamma^1 (1-x)^\alpha\, x^{-\frac{1}{2}-\alpha}\mathrm{d}x \\
&= \frac{1}{2}\left\{\left[(1-x)^{\alpha+1}\,\frac{x^{-\frac{1}{2}-\alpha}}{-\frac{1}{2}-\alpha}\right]_\gamma^1 - \frac{1}{1+2\alpha}\int_\gamma^1 (1-x)^\alpha\, x^{-\frac{1}{2}-\alpha}\mathrm{d}x\right\} \\
&= \frac{1}{1+2\alpha}\left\{-\left[(1-x)^{\alpha+1}\, x^{-\frac{1}{2}-\alpha}\right]_\gamma^1 - \int_\gamma^1 (1-x)^\alpha\, x^{-\frac{1}{2}-\alpha}\mathrm{d}x\right\} \\
&= \frac{1}{1+2\alpha}\left\{2\gamma^{-\frac{1}{2}-\alpha}\,(1-\gamma)^{\alpha+1} - \int_\gamma^1 (1-x)^{\alpha+1-1}\, x^{\left(\frac{1}{2}-\alpha\right)-1}\mathrm{d}x\right\}.
\end{aligned}$$

Then, since $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, $\frac{1}{2}-\alpha > 0$, and $\alpha+1 > 0$, and we can write

$$B\left(\frac{1}{2}-\alpha, \alpha+1\right) = \int_0^1 (1-x)^{\alpha+1-1}\, x^{\left(\frac{1}{2}-\alpha\right)-1}\mathrm{d}x,$$

where $B(\cdot,\cdot)$ is the beta function. If we denote the incomplete beta function by $B^z(\cdot,\cdot)$

$$B^z(\alpha,\beta) = \int_0^z x^{\alpha-1}\,(1-x)^{\beta-1}\,\mathrm{d}x, \qquad \alpha,\beta > 0,$$

we can also write, for $v < u \le 1$,

$$\begin{aligned}
&c_2(u,v;u,1) \\
&= \frac{1}{1+2\alpha}\left(\frac{u-v}{2}\right)^{2\alpha+1} \\
&\quad \times\left\{\gamma^{-\frac{1}{2}-\alpha}\,(1-\gamma)^{\alpha+1} - \frac{1}{2}\left(B\left(\frac{1}{2}-\alpha,\alpha+1\right) - B^\gamma\left(\frac{1}{2}-\alpha,\alpha+1\right)\right)\right\} \\
&= \frac{1}{1+2\alpha}\left\{\left(\frac{2(1-u)(1-v)}{2-u-v}\right)^{2\alpha+1}\right. \\
&\quad \left. - \frac{1}{2}\left(\frac{u-v}{2}\right)^{2\alpha+1}\left[\left(B\left(\frac{1}{2}-\alpha,\alpha+1\right) - B^\gamma\left(\frac{1}{2}-\alpha,\alpha+1\right)\right)\right]\right\}.
\end{aligned} \tag{24.6}$$

In case of $v = u \leq 1$,

$$c_2(u, u; u, 1) = \int_u^1 (s-u)^{2\alpha}\, \mathrm{d}s = \left[\frac{(s-u)^{2\alpha+1}}{2\alpha+1}\right]_u^1 = \frac{(1-u)^{2\alpha+1}}{2\alpha+1}$$

Then, by (24.5) and (24.6), we have

$$R(u, v) = \frac{T^{2\alpha+1}}{1+2\alpha}\left\{\left[\frac{2(1-u)(1-v)}{2-u-v}\right]^{2\alpha+1} - \frac{1}{2}\left(\frac{u-v}{2}\right)^{2\alpha+1} \times\left[\left(B\left(\frac{1}{2}-\alpha, \alpha+1\right) - B^{\gamma}\left(\frac{1}{2}-\alpha, \alpha+1\right)\right)\right]\right\}.$$

Therefore

$$\begin{aligned} d^R(\lambda) &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\int_0^1 \cdots \int_0^1 \begin{vmatrix} R(s_1, s_1) & \cdots & R(s_1, s_n) \\ \vdots & & \vdots \\ R(s_n, s_1) & \cdots & R(s_n, s_n) \end{vmatrix} \mathrm{d}s_1 \cdots \mathrm{d}s_n \\ &= \sum_{n=0}^{\infty} \frac{\left(\frac{\lambda T^{2\alpha+1}}{1+2\alpha}\right)^n}{n!}\int_0^1 \cdots \int_0^1 \begin{vmatrix} \tilde{R}(s_1, s_1) & \cdots & \tilde{R}(s_1, s_n) \\ \vdots & & \vdots \\ \tilde{R}(s_n, s_1) & \cdots & \tilde{R}(s_n, s_n) \end{vmatrix} \mathrm{d}s_1 \cdots \mathrm{d}s_n \\ &= d^{\tilde{R}}\left(\frac{\lambda T^{2\alpha+1}}{1+2\alpha}\right), \end{aligned}$$

and the price is given by

$$P(0, T) = \left[d^R(2T)\right]^{-d/2} = \left[d^{\tilde{R}}\left(\frac{2T^{2\alpha+2}}{1+2\alpha}\right)\right]^{-d/2} \approx \left[d^{\tilde{R}}_{Q_m}\left(\frac{2T^{2\alpha+2}}{1+2\alpha}\right)\right]^{-d/2}.$$

Remark 6.1. In order to include the case of the volatility not being constant, one only has to substitute $\sigma_{Tu}\sigma_{Tv}R(u, v)$ for $R(u, v)$ or $\sigma_{Tu}\sigma_{Tv}\tilde{R}(u, v)$ for $\tilde{R}(u, v)$.

Remark 6.2. The incomplete beta function ratio defined by

$$I_x(\alpha, \beta) = \frac{1}{B(\alpha, \beta)}\int_0^x t^{\alpha-1}(1-t)^{\beta-1}$$

can be obtained by using the function `betainc`$(\mathrm{x}, \alpha, \beta)$ in matlab, so we can compute $B^b(\alpha, \beta)$ easily.

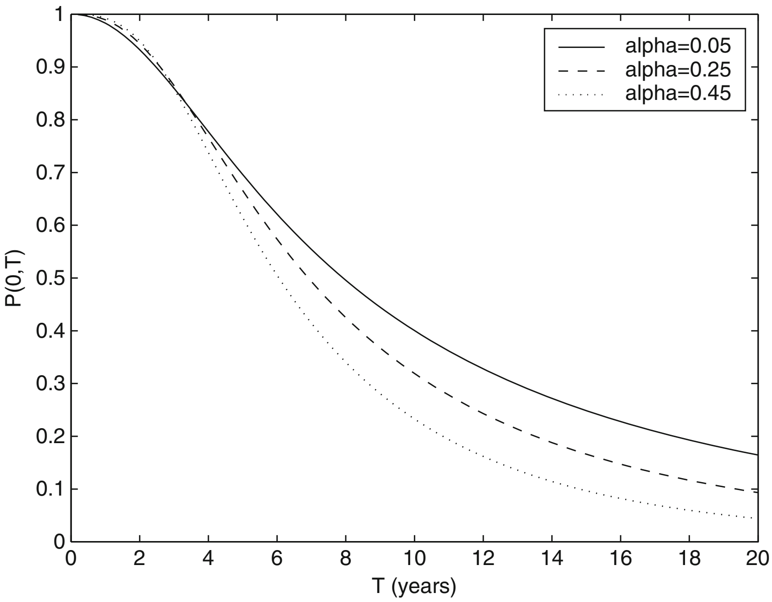

Fig. 24.3 Approximation of prices, d = 2, sigma = 0.2, alpha > 0

Figures 24.3 and 24.4 show the approximated price $P(0,T)$ under the circumstances of Proposition 6.1, for $T \in (0,20)$ in years, $d = 2$, $\sigma = 0.2$, and $\alpha \in \{-0.45, -0.25, -0.05, 0.05, 0.25, 0.45\}$.

6.3 The Dynamics of the CIR Model

A natural question, as we did in Sect. 4, is if the process

$$r_t = \sum_{i=1}^{d} \left(\int_0^t g(t-s)\sigma_s \mathrm{d}W_i(s) \right)^2$$

can be seen as the solution of certain SDE. Write

$$Y_i(t) := \int_0^t g(t-s)\sigma_s \mathrm{d}W_i(s),$$

then

$$r_t = \sum_{i=1}^{d} Y_i^2(t).$$

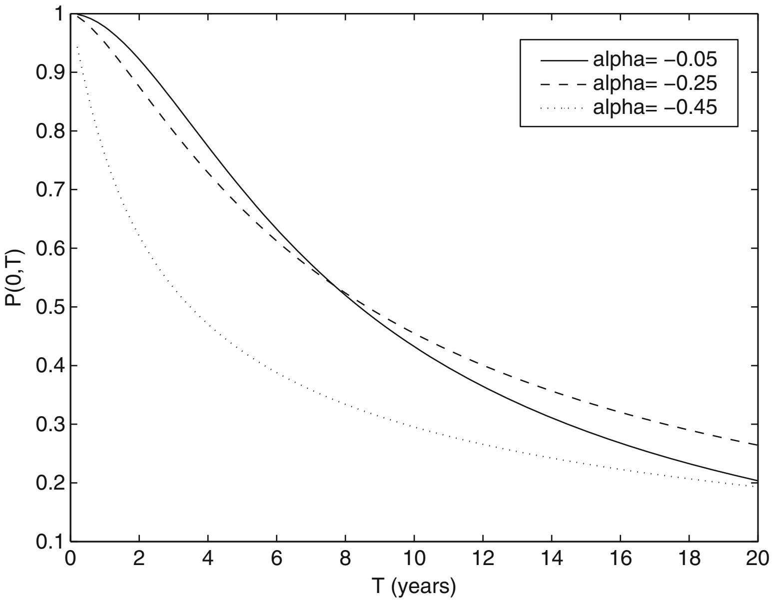

Fig. 24.4 Approximation of prices, d = 2, sigma = 0.2, alpha < 0

Assume that $g \in C^1$and it is square integrable, then Y is a semimartingale with

$$\mathrm{d}Y_i(t) = g(0)\sigma_t \mathrm{d}W_i(t) + \left(\int_0^t g'(t-s)\sigma_s \mathrm{d}W_i(s)\right)\mathrm{d}t,$$

suppose $g(0) \neq 0$ as well. If we apply the Itô formula for continuous semimartingales we have

$$\begin{aligned}
\mathrm{d}r_t &= \sum_{i=1}^d 2Y_i(t)\mathrm{d}Y_i(t) + \sum_{i=1}^d \mathrm{d}[Y_i, Y_i]_t \\
&= \sum_{i=1}^d 2g(0)\sigma_t Y_i(t)\mathrm{d}W_i(t) + \sum_{i=1}^d 2Y_i(t)\left(\int_0^t g'(t-s)\sigma_s \mathrm{d}W_i(s)\right)\mathrm{d}t \\
&\quad + \sum_{i=1}^d g^2(0)\sigma_t^2 \mathrm{d}t \\
&= 2g(0)\sigma_t\sqrt{r_t}\sum_{i=1}^d \frac{Y_i(t)}{\sqrt{r_t}}\mathrm{d}W_i(t) \\
&\quad + \left(dg^2(0)\sigma_t^2 + \sum_{i=1}^d 2Y_i(t)\left(\int_0^t g'(t-s)\sigma_s \mathrm{d}W_i(s)\right)\right)\mathrm{d}t.
\end{aligned}$$

Then it is easy to see, by using the Lévy characterization of the Brownian motion, that

$$\sum_{i=1}^{d} \frac{Y_i(t)}{\sqrt{r_t}} \mathrm{d}W_i(t) = \mathrm{d}B(t),$$

where B is a Brownian motion. Finally if $g'(t) = -bg(t)$, $g(0) = 1$, $\sigma_t = \sigma$, we have

$$\mathrm{d}r_t = (d\sigma^2 - 2br_t)\mathrm{d}t + 2\sigma\sqrt{r_t}\mathrm{d}B(t)$$

that is the dynamics of a CIR process. If g' is not square integrable then the process

$$Y_i(t) := \int_0^t g(t-s)\sigma_s \mathrm{d}W_i(s)$$

is not a semimartingale and we cannot apply the usual Itô formula. In the particular case that

$$g(t-s) = \mathrm{e}^{-b(t-s)} \int_0^{t-s} \mathrm{e}^{bu} \beta u^{\beta-1} \mathrm{d}u, \beta \in (-1/2, 0) \cup (0, 1/2),$$

and $\sigma_u = \sigma$

$$Y_i(t) = \int_0^t \sigma \mathrm{e}^{-b(t-s)} W_i^{\beta}(\mathrm{d}s)$$

$$W_i^{\beta}(t) := \int_0^t (t-s)^{\beta} W(\mathrm{d}s),$$

so

$$Y_i(t) = -b \int_0^t Y_i(s)\mathrm{d}s + \sigma W_i^{\beta}(t)$$

and, by the Itô formula for these processes, we have [1]

$$\begin{aligned} \mathrm{d}r_t &= \sum_{i=1}^{d} 2\sigma Y_i(t)\partial W_i^{\beta}(t) - 2br(t)\mathrm{d}t + \sum_{i=1}^{d} \sigma^2 \left(\int_0^t (t-u)^{\beta} \mathrm{d}u \right) \mathrm{d}t \\ &= \left(d\sigma^2 t^{2\beta} - 2br(t) \right) \mathrm{d}t + 2\sigma\sqrt{r_t} \sum_{i=1}^{d} \frac{Y_i(t)}{\sqrt{r_t}} \partial W_i^{\beta}(t). \end{aligned}$$

But we do not have a characterization of the process

$$Z_t := \sum_{i=1}^{d} \int_0^t \frac{Y_i(s)}{\sqrt{r_s}} \partial W_i^{\beta}(s), t \geq 0.$$

In the case that $b = 0$,

$$Z_t := \sum_{i=1}^{d} \int_0^t \frac{W_i^{\beta}(s)}{\sqrt{r_s}} \partial W_i^{\beta}(s), t \geq 0,$$

and it can be shown that Z is 2β-self-similar [9].

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Appendix E

Power variation for Itô integrals with respect to α-stable processes

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Power variation for Itô integrals with respect to α-stable processes

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In this article we consider the asymptotic behavior of the power variation of processes of the form $\int_0^t u_{s-}\,\mathrm{d}S_s^\alpha$, where S^α is an α-stable process with index of stability $0<\alpha<2$ and the integral is an Itô integral. We establish stable convergence of corresponding fluctuations. These results provide statistical tools to infer the process u from discrete observations.

Keywords and Phrases: stable processes, central limit theorem, power variation.

1 Introduction

We study the power variation of a process of the form

$$Z_t = \int_0^t u_{s-}\mathrm{d}S_s^\alpha,$$

where S^α is a symmetric α-stable Lévy process and the integral is an Itô integral. Instead of requiring that trajectories of u have finite q-variation on any finite interval for some

$$q < \frac{\alpha}{\max(0, \alpha-1)},$$

as in Corcuera, Nualart and Woerner (2007), we assume that

$$\int_0^t |u_s|^\alpha \mathrm{d}s < \infty$$

by allowing more general integrands. This is an interesting extension from a modeling perspective.

Originally, the concept of power variation was introduced in the context of studying the path behavior of stochastic processes, but recently has been introduced for statistical inference for integrals based on Brownian motion (Barndorff-Nielsen and Shephard, 2003; Barndorff-Nielsen *et al.*, 2006; Woerner, 2005), for fractionally integrated processes (Corcuera, Nualart and Woerner, 2006); for Itô integrals

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with respect to symmetric stable processes (Barndorff-Nielsen and Shephard, 2006) and more general Lévy processes (Woerner, 2003); in these latter cases, integrands and integrators are assumed to be independent. The power variation for pathwise integrals α-stable Lévy process has been considered in Corcuera *et al.* (2007), the power and bipower variation for pathwise integrals with respect to Gaussian process with stationary increments in Barndorff-Nielsen, Corcuera and Podolskij (2009a) and Barndorff-Nielsen *et al.* (2009b) and for Brownian semi-stationary processes in Barndorff-Nielsen, Corcuera and Podolskij (2009c). It is also interesting to mention the work by Jacod (2004) where closely related quantities to the quadratic variation of certain integrated stable processes are treated.

The paper is organized as follows: in section 2, we establish our asymptotic result in uniform convergence in probability (u.c.p.), and in section 3, we provide a central limit theorem.

2 Power variation for integrated stable processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_t)$ a right continuous increasing family of P-complete sub-σ-fields of $\mathcal{F}$.

An $(\mathcal{F}_t)$-adapted càdlàg process $S^\alpha = \{S^\alpha_t, t \geq 0\}$ is a (symmetric) α-stable Lévy process with index of stability $\alpha \in (0, 2]$ if for every $0 \leq s < t$

$$E(\exp(i\lambda(S^\alpha_t - S^\alpha_s)) \mid \mathcal{F}_s) = \exp(-(t-s) \mid \lambda \mid^\alpha), \quad \lambda \in \mathbb{R}, \quad S^\alpha_0 = 0 \text{ a.s.}$$

In particular, for $\alpha = 2$, S^α equals $\sqrt{2}$ times a standard Brownian motion. Also, note that S^α is a process with independent and homogeneous increments and α-self-similar:

$$\left(S^\alpha_{at}\right) \sim \left(a^{1/\alpha} S^\alpha_t\right), \quad a > 0.$$

See Sato (1999) for more details.

If u is an $(\mathcal{F}_t)$-adapted càdlàg process such that

$$\int_0^t E(\mid u_s \mid^\alpha) \,\mathrm{d}S < \infty,$$

we can define the Itô integral

$$\int_0^t u_{s-} \,\mathrm{d}S^\alpha_s$$

and we have the following inequality:

$$P\left\{ \left| \int_0^t u_{s-} \,\mathrm{d}S^\alpha_s \right|^p > \lambda \right\} \leq \frac{C}{\lambda^{\alpha/p}} \int_0^t E(\mid u_s \mid^\alpha) \,\mathrm{d}S, \qquad \text{for all } \lambda > 0.$$

In the following, we shall write C for any generic constant. This bound was obtained by Giné and Marcus (1983); see also Rosinski and Woyczynski (1986). Note that as a consequence, we obtain the following lemma:

LEMMA 1. *Fix $p<\alpha$. Then*

$$E\left(\left|\int_0^t u_{s-}\,\mathrm{d}S_s^\alpha\right|^p\right)\le C_p\left(\int_0^t E(|u_s|^\alpha)\,\mathrm{d}s\right)^{p/\alpha},$$

where C_p is a constant that depends on p.

PROOF.

$$\begin{aligned}E\left(\left|\int_0^t u_{s-}\,\mathrm{d}S_s^\alpha\right|^p\right)&=\int_0^\infty P\left\{\left|\int_0^t u_{s-}\mathrm{d}S_s^\alpha\right|^p>x\right\}\mathrm{d}x\\&=\int_0^K P\left\{\left|\int_0^t u_{s-}\mathrm{d}S_s^\alpha\right|^p>x\right\}\mathrm{d}x\\&\quad+\int_K^\infty P\left\{\left|\int_0^t u_{s-}\mathrm{d}S_s^\alpha\right|^p>x\right\}\mathrm{d}x\\&\le K+CK^{-\alpha/p+1}\int_0^t E(|u_s|^\alpha)\mathrm{d}s,\end{aligned}$$

for all $K>0$. The minimum of this bound is:

$$\left(C\left(\frac{\alpha}{p}-1\right)\right)^{p/\alpha}\left(1+\frac{1}{1-\alpha/p}\right)\left(\int_0^t E(|u_s|^\alpha)\,\mathrm{d}s\right)^{p/\alpha}. \qquad \square$$

For any $p>0$, a natural number n and for any stochastic process $Z=\{Z_t, t\in[0,T]\}$ the pth power variation is defined as:

$$V_p^n(Z)_t:=\sum_{i=1}^{[nt]}\left|Z_{\frac{i}{n}}-Z_{\frac{(i-1)}{n}}\right|^p.$$

The following theorem provides a result for the convergence of the appropriately normalized power variation of integrated stable processes where we denote u.c.p. in the time interval $[0,T]$ and $\|\cdot\|_\infty$ for the supremum norm on $[0,T]$.

In the following, we are only interested in the case $p<\alpha$, where the non-normed power variation leads to an infinite limit and, hence we need a norming sequence that converges to zero in an appropriate way. For the case $p>\alpha$ and Z an α-stable Lévy process, it is well-established (LEPINGLE, 1976; HUDSON and MASON, 1976) that the non-normed power variation tends to the pth power of the absolute values of the jumps of Z. First of all, we are concerned with limits in probability or in law, by a standard localization procedure, we can assume without loss of generality that u_- is bounded.

THEOREM 1. *Suppose that $u=\{u_t,t\ge 0\}$ is an $(\mathcal{F}_t)$-adapted stochastic process with càdlàg trajectories. Set*

$$Z_t=\int_0^t u_{s_-}\,\mathrm{d}S_s^\alpha.$$

Then, for any $p<\alpha$

$$m^{-1+p/\alpha}V_p^m(Z)_t \xrightarrow{u.c.p.} c_p\int_0^t |u_s|^p\,\mathrm{d}s,$$

as m *tends to infinity, where* $c_p=E(|S_1^\alpha|^p)$.

PROOF. For the case $\alpha<1$, the integral is a Lebesgue–Stieltjes integral and we can write

$$Z_t=\int_0^t u_{s_-}\,\mathrm{d}S_s^\alpha=\sum_{0\le s\le t} u_{s_-}\Delta S_s^\alpha,$$

where

$$\Delta S_s^\alpha=S_s^\alpha-S_{s-}^\alpha$$

and where

$$\textstyle\sum_{0\le s\le t}|\Delta S_s^\alpha|<\infty.$$

We obtain, for any $m\ge n$,

$$\begin{aligned}
&\left|m^{-1+p/\alpha}V_p^m(Z)_t-c_p\int_0^t |u_s|^p\mathrm{d}s\right|\\
&\le\left|\sum_{j=1}^{[mt]} m^{-1+p/\alpha}\left(\left|\int_{(j-1)/m}^{j/m}u_{s_-}\,\mathrm{d}S_s^\alpha\right|^p\right.\right.\\
&\quad\left.\left.-\sum_{i=1}^{[nt]}\left|u_{(i-1)/n-}\right|^p\sum_{j\in I_n(i)} m^{-1+p/\alpha}\left|S_{j/m}^\alpha-S_{(j-1)/m}^\alpha\right|^p\right)\right|\\
&\quad+\left|\sum_{i=1}^{[nt]}|u_{(i-1)/n-}|^p\sum_{j\in I_n(i)} m^{-1+p/\alpha}|S_{j/m}^\alpha-S_{(j-1)/m}^\alpha|^p-c_p\frac{1}{n}\sum_{i=1}^{[nt]}|u_{(i-1)/n-}|^p\right|\\
&\quad+c_p\left|\frac{1}{n}\sum_{i=1}^{[nt]}\left|u_{(i-1)/n-}\right|^p-\int_0^t|u_s|^p\,\mathrm{d}s\right|\\
&=\tilde{A}_t^{(m)}+C_t^{(n,m)}+D_t^{(n,m)},
\end{aligned}$$

where for each $i=1,\dots,n$, $I_n(i)=\left\{j:\frac{j}{m}\in\left(\frac{(i-1)}{n},\frac{i}{n}\right]\right\}$, $1\le i\le[nt]$.

For the terms $C_t^{(n,m)}$ and $D_t^{(n,m)}$ the convergence to zero may be shown analogously as in CORCUERA *et al.* (2006), noting that we have to use the scaling relation for stable processes and the law of large numbers instead of the ergodic theorem.

For the term $\tilde{A}_t^{(m)}$, since $p<\alpha<1$, we have,

$$\left|\sum_{j=1}^{[mt]} m^{-1+p/\alpha}\left|\int_{(j-1)/m}^{j/m} u_{s_-}\,\mathrm{d}S_s^{\alpha}\right|^p - \sum_{i=1}^{[nt]}\left|u_{(i-1)/n-}\right|^p \sum_{j\in I_n(i)} m^{-1+p/\alpha}\left|S_{j/m}^{\alpha}-S_{(j-1)/m}^{\alpha}\right|^p\right|$$
$$\leq \sum_{i=1}^{[nT]}\sum_{j\in I_n(i)} m^{-1+p/\alpha}\left|\sum_{s\in(\frac{j-1}{m},\frac{j}{m}]} u_{s_-}\Delta S_s^{\alpha} - u_{(i-1)/n-}(S_{j/m}^{\alpha}-S_{(j-1)/m}^{\alpha})\right|^p$$
$$+\|u\|_{\infty} m^{-1+p/\alpha}\sup_{0\leq t\leq T}\sum_{m/n[nt]\leq j\leq m/n([nt]+1)} |S_{j/m}^{\alpha}-S_{(j-1)/m}^{\alpha}|^p$$
$$\leq \sum_{i=1}^{[nT]}\sup_{s\in(i-2/n,i/n]}|u_{s_-}-u_{(i-1)/n-}|\sum_{j\in I_n(i)} m^{-1+p/\alpha}|S_{j/m}^{\alpha}-S_{(j-1)/m}^{\alpha}|^p$$
$$+\|u\|_{\infty} m^{-1+p/\alpha}\sup_{0\leq t\leq T}\sum_{m/n[nt]\leq j\leq m/n([nt]+1)} |S_{j/m}^{\alpha}-S_{(j-1)/m}^{\alpha}|^p.$$

As m tends to infinity, by the law of large numbers, this converges in probability to

$$\frac{c_p}{n}\left(\sum_{i=1}^{[nT]}\sup_{s\in((i-2)/n,i/n]}|u_{s_-}-u_{(i-1)/n-}|^p+\|u\|_{\infty}\right)$$

and, since u is càdlàg, this tends alost surely to zero as n tends to infinity. Now, suppose that $\alpha\geq 1$ and $p\leq 1$. Again, for any $m\geq n$,

$$|m^{-1+p/\alpha}V_p^m(Z)_t - c_p\int_0^t |u_s|^p\,\mathrm{d}s|$$
$$\leq \left|\sum_{j=1}^{[mt]} m^{-1+p/\alpha}\left(\left|\int_{(j-1)/m}^{j/m} u_{s_-}\,\mathrm{d}S_s^{\alpha}\right|^p - \left|u_{(j-1)/m-}(S_{j/m}^{\alpha}-S_{(j-1)/m}^{\alpha})\right|^p\right)\right|$$
$$+\left|\sum_{j=1}^{[mt]} m^{-1+p/\alpha}\left|u_{(j-1)/m-}(S_{j/m}^{\alpha}-S_{(j-1)/m}^{\alpha})\right|^p - \sum_{i=1}^{[nt]}\left|u_{(i-1)/n-}\right|^p\right.$$
$$\left.\times\sum_{j\in I_n(i)} m^{-1+p/\alpha}|S_{j/m}^{\alpha}-S_{(j-1)m}^{\alpha}|^p\right|$$
$$+\left|\sum_{i=1}^{[nt]}\left|u_{(i-1)/n-}\right|^p\sum_{j\in I_n(i)} m^{-1+p/\alpha}\left|S_{j/m}^{\alpha}-S_{(j-1)/m}^{\alpha}\right|^p\right.$$
$$\left.-c_p 1/n\sum_{i=1}^{[nt]}|u_{(i-1)/n-}|^p\right| + c_p\left|1/n\sum_{i=1}^{[nt]}\left|u_{(i-1)/n-}\right|^p - \int_0^t |u_s|^p\,\mathrm{d}s\right|$$
$$=A_t^{(m)}+B_t^{(n,m)}+C_t^{(n,m)}+D_t^{(n)}.$$

For the terms $B_t^{(n,m)}, C_t^{(n,m)}$and $D_t^{(n)}$, as before, the convergence to zero may be shown analogously as in Corcuera *et al.* (2006). The term $A_t^{(m)}$ can be bounded in the following way:

$$|A_t^{(m)}| \le m^{-1+p/\alpha} \sum_{j=1}^{[mt]} \left| \int_{(j-1)/m}^{j/m} u_{s_-}\, \mathrm{d}S_s^\alpha - u_{(j-1)/m-} \left(S_{j/m}^\alpha - S_{(j-1)/m}^\alpha\right) \right|^p$$

then,

$$E\left(\|A^{(m)}\|_\infty\right) \le m^{-1+p/\alpha} \sum_{j=1}^{[mT]} E\left(\left| \int_{(j-1)/m}^{j/m} \left(u_{s_-} - u_{(j-1)/m-}\right) \mathrm{d}S_s^\alpha \right|^p \right).$$

By lemma 1,

$$E\left(\left| \int_{(j-1)/m}^{j/m} \left(u_{s_-} - u_{(j-1)/m-}\right)\, \mathrm{d}S_s^\alpha \right|^p \right) \le C \left(\int_{(j-1)/m}^{j/m} E\left(|u_{s_-} - u_{(j-1)/m-}|^\alpha\right)\, \mathrm{d}s \right)^{p/\alpha},$$

$$\begin{aligned} E\left(\|A^{(m)}\|_\infty\right) &\le m^{-1+p/\alpha} \sum_{j=1}^{[mT]} \left| \int_{(j-1)/m}^{j/m} E\left(|u_{s_-} - u_{(j-1)/m-}|^\alpha\right) \mathrm{d}s \right|^{p/\alpha} \\ &\le \left(m^{-1} \sum_{j=1}^{[mT]} E\left(\sup_{s\in((j-1)/m, j/m]} |u_{s_-} - u_{(j-1)/m-}|^\alpha \right) \right)^{p/\alpha}, \end{aligned}$$

since u is càdlàg, for any $\kappa > 0$ and fixed ω, there exists m large enough such that

$$\sup_{s\in((j-1)/m)j/m]} |u_{s_-} - u_{(j-1)/m-}|^\alpha \le C(\kappa + |\Delta u_{(j-1)/m}|^\alpha 1_{\{|\Delta u_{(j-1)/m}|^\alpha > \kappa\}}), 1 \le j \le [mT],$$

so

$$m^{-1} \sum_{j=1}^{[mT]} \sup_{s\in((j-1)/m, j/m]} |u_{s_-} - u_{(j-1)/m-}|^\alpha \overset{\text{a.s.}}{\to} 0.$$

Then by the boundness of u_{s_-} and the dominated convergence theorem, this convergence is also in L^1.

In the case $p > 1$, we make use of Minkowski's inequality:

$$\begin{aligned} &\left| \left(m^{-1+p/\alpha} V_p^m(Z)_t\right)^{1/p} - \left(c_p \int_0^t |u_s|^p\, \mathrm{d}s\right)^{1/p} \right| \\ &\le \left(\sum_{j=1}^{[mT]} m^{-1+p/\alpha} \left| \int_{(j-1)/m}^{j/m} u_{s-}\, \mathrm{d}S_s^\alpha - u_{(j-1)/m-}(S_{j/m}^\alpha - S_{(j-1)/m}^\alpha) \right|^p \right)^{1/p} \\ &\quad + \left(\sum_{i=1}^{[nt]} \sum_{j\in I_n(i)} m^{-1+p/\alpha} \left| \left(u_{(j-1)/m-} - u_{(i-1)/n-}\right) (S_{j/m}^\alpha - S_{(j-1)/m}^\alpha) \right|^p \right)^{1/p} \end{aligned}$$

$$
\begin{aligned}
&+\left|\left(\sum_{i=1}^{[nt]} |\, u_{(i-1)/n-}\,|^p \sum_{j\in I_n(i)} m^{-1+p/\alpha}\,|\, S^{\alpha}_{j/m}-S^{\alpha}_{(j-1)/m}\,|^p\right)^{1/p}\right.\\
&-\left.\left(\frac{1}{n}c_p\sum_{i=1}^{[nt]}|u_{(i-1)/n-}\,|^p\right)^{1/p}\right|+c_p^{1/p}\left|\left(1/n\sum_{i=1}^{[nt]} |\, u_{(i-1)/n-}\,|^p\right)^{1/p}\right.\\
&-\left.\left(\int_0^t |u_s|^p\,\mathrm{d}s\right)^{1/p}\right| = A_t^{(m)}+B_t^{(n,m)}+C_t^{(n,m)}+D_t^{(n)},
\end{aligned}
$$

and we make use of the same arguments as before using Minkowski's inequality instead of the triangle inequality. □

REMARK 1.

$$E(\,|\,S_1^{\alpha}\,|^p)=2^p\Gamma\left(\frac{1+p}{2}\right)\Gamma\left(\frac{\alpha-p}{\alpha}\right)/\left(\sqrt{\pi}\Gamma\left(\frac{2-p}{2}\right)\right);$$

see SATO (1999, p. 163).

3 Central limit theorem for the power variation

Fluctuations of the power variation, for $0<p<\frac{\alpha}{2}$ properly normalized, have asymptotically mixed Gaussian distributions. To establish this result we first introduce some notation.

For any $0<p<\frac{\alpha}{2}$, we put

$$v_p^2=\mathrm{var}(\,|\,S_1^{\alpha}\,|^p).$$

We will first show a functional limit theorem for the realized power variation of a stable process.

THEOREM 2. *Fix* $0<p<\alpha/2$ *and assume* $0<\alpha<2$. *Then*

$$(S_t^{\alpha}, n^{-1/2+p/\alpha}V_p^n(S^{\alpha})_t - c_p t n^{1/2}) \xrightarrow{\mathcal{L}} (S_t^{\alpha}, v_p W_t), \tag{1}$$

as n tends to infinity, where $W=\{W_t, t\in[0,T]\}$ *is a Brownian motion independent of the process* S^{α}*-, and the convergence is in the space* $\mathcal{D}([0,T])^2$ *equipped with the Skorohod topology.*

PROOF. The proof will be done in two steps. Set

$$Z_t^{(n)}=n^{-1/2+p/\alpha}V_p^n(S^{\alpha})_t-c_p t n^{1/2}.$$

Step 1. *We will first show the convergence of finite-dimensional distributions. Let* $J_k = (a_k, b_k]$, $k = 1, \ldots, N$ *be pairwise disjoint intervals contained in* $[0, T]$. *Define the random vectors* $S = (S^\alpha_{b_1} - S^\alpha_{a_1}, \ldots, S^\alpha_{b_N} - S^\alpha_{a_N})$ *and* $X^{(n)} = (X^{(n)}_1, \ldots, X^{(n)}_N)$, *where*

$$\begin{aligned} X^{(n)}_k &= n^{-1/2+p/\alpha} \sum_{[na_k]<j\leq[nb_k]} |\, S^\alpha_{j/n} - S^\alpha_{(j-1)/n} \,|^p - n^{1/2} c_p \,|\, J_k \,| \\ &= Z^{(n)}_{b_k} - Z^{(n)}_{a_k} + o(1), \end{aligned}$$

$k = 1, \ldots, N$ *and* $|\, J_k \,| = b_k - a_k$. *We claim that*

$$(S, X^{(n)}) \xrightarrow{\mathcal{L}} (S, V), \tag{2}$$

where S and V are independent and V is a Gaussian random vector with zero mean, and independent components of variances $v^2_p \,|\, J_k \,|$.

By the self-similarity of the stable process, the sequence

$$\left(n^{p/\alpha} \left| S^\alpha_{j/n} - S^\alpha_{(j-1)/n} \right|^p - c_p \right)_{1\leq j\leq n}$$

has the same law as

$$\left(|\, S^\alpha_j - S^\alpha_{j-1} \,|^p - c_p \right)_{1\leq j\leq n}.$$

Set

$$X_j = |\, S^\alpha_j - S^\alpha_{j-1} \,|^p - c_p.$$

Then, $\{X_j, j \geq 1\}$ is a stationary sequence with zero mean, independent increments and variance v^2_p.

Thus, the convergence (Equation 2) is equivalent to the convergence in the distribution of $(S^{(n)}, Y^{(n)})$ to (S, V), where

$$S^{(n)}_k = n^{-1/\alpha} \sum_{[na_k]<j\leq[nb_k]} (S^\alpha_j - S^\alpha_{j-1}), \quad 1 \leq k \leq N \tag{3}$$

and

$$Y^{(n)}_k = 1/\sqrt{n} \sum_{[na_k]<j\leq[nb_k]} \left(|\, S^\alpha_j - S^\alpha_{j-1} \,|^p - c_p \right), \quad 1 \leq k \leq N. \tag{4}$$

But for any $1 \leq k \leq N$, we have stable convergence of $Y^{(n)}_k$, by theorem 2 in Aldous and Eagleson (1978), so we have joint convergence of $(S^{(n)}_k, Y^{(n)}_k)$ to $(S_k, v_p(W_{b_k} - W_{a_k}))$ and since $(S_k, v_p(W_{b_k} - W_{a_k}))$ has an infinitely divisible law by being a limit of infinitely divisible laws; see theorem 8.7 in Sato (1999), and one component is Gaussian and the other has no Gaussian component that are independent. Finally, for different values of k, the components are independent, so we have that $(S^{(n)}, Y^{(n)}) \xrightarrow{\mathcal{L}} (S, V)$.

Step 2. Tightness condition of the sequence of processes $Z^{(n)}$ follows from the fact that

$$\sum_{1\le j\le N} (|S_j^\alpha - S_{j-1}^\alpha|^p - c_p), \quad N \ge 1$$

has independent and stationary increments with second-order moments; see Billingsley (1968, theorem 16.1). □

The convergence established in theorem 2 can be also expressed in terms of the *stable convergence* (Aldous and Eagleson, 1978). In fact, for any bounded random variable X measurable with respect to the σ-field $\mathcal{F}_T^\alpha$ generated by $\{S_t^\alpha, 0 \le t \le T\}$, and for any continuous and bounded function ϕ on the Skorohod space $\mathcal{D}([0,T])$, we have

$$\lim_{n\to\infty} E(X\phi(Z^{(n)})) = E(X)E(\phi(W)).$$

If X is a continuous functional of $\{S_t^\alpha, 0 \le t \le T\}$, this convergence is an immediate consequence of theorem 2, and the general case follows by an easy approximation argument. Then by proposition 1, condition (C), in Aldous and Eagleson (1978) we have stable convergence if we take $\mathcal{F} = \mathcal{F}_T^\alpha$, but by condition (D) the same is true if we take $\mathcal{F} \supseteq \mathcal{F}_T^\alpha$; so, by condition (B) we have the joint convergence $(X, Z^{(n)})$, X being $\mathcal{F}$-measurable.

Then, as a consequence of theorem 2 we can derive the following central limit theorem for the realized power variation of the stochastic integrals studied above. Unfortunately, we also need an additional condition on the process u.

Condition 1. *Assume that, for some $\gamma \in (0,1)$, u satisfies:*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[nT]} E\left(\sup_{t,s\in[(i-1)/n,\, i/n]} |u_t - u_s|^\gamma \right) \underset{n\to\infty}{\to} 0.$$

Theorem 3. *Let S^α be an $(\mathcal{F}_t)$-adapted α-stable Lévy process with $\alpha \in (0,2)$. Fix $0 < p < \alpha/2$ and suppose that $u = \{u_t, t \in [0,T]\}$ is an $(\mathcal{F}_t)$-adapted càdlàg stochastic process and satisfies condition 1 with $\gamma = p$. Setting*

$$Z_t = \int_0^t u_{s-} \mathrm{d}S_s^\alpha,$$

we obtain

$$\left(S_t^\alpha, n^{-1/2+p/\alpha} V_p^n(Z)_t - c_p\sqrt{n} \int_0^t |u_s|^p \mathrm{d}s \right) \xrightarrow{\mathcal{L}} \left(S_t^\alpha, v_p \int_0^t |u_s|^p \mathrm{d}W_s \right),$$

as n tends to infinity, where $W = \{W_t, t \in [0,T]\}$ is a Brownian motion independent of $\mathcal{F}$, defined on an extension of $(\Omega, \mathcal{F}, P)$, and the convergence is in $\mathcal{D}([0,T])^2$.

PROOF. Consider first $\alpha \geq 1$. The proof will be based on theorem 2. For any $m \geq n$ and with the same notation as in theorem 1, we can write,

$$m^{-1/2+p/\alpha} V_p^m(Z)_t - \sqrt{m} c_p \int_0^t |u_s|^p \,\mathrm{d}s = A_t^{(m)} + B_t^{(n,m)} + C_t^{(n,m)} + D_t^{(m)},$$

where

$$A_t^{(m)} = m^{-1/2+p/\alpha} \sum_{j=1}^{[mt]} \left(\left| \int_{(j-1)/m}^{j/m} u_{s-} \,\mathrm{d}S_s^\alpha \right|^p - | u_{(j-1)/m-} (S_{j/m}^\alpha - S_{(j-1)/m}^\alpha) |^p \right),$$

$$\begin{aligned} B_t^{(n,m)} = {} & m^{-1/2+p/\alpha} \sum_{j=1}^{[mt]} | u_{(j-1)/m-} (S_{j/m}^\alpha - S_{(j-1)/m}^\alpha) |^p \\ & - m^{-1/2} c_p \sum_{j=1}^{[mt]} | u_{(j-1)m-} |^p - \sum_{i=1}^{[nt]} | u_{(i-1)/n-} |^p \sum_{j \in I_n(i)} m^{-1/2+p/\alpha} | S_{j/m}^\alpha - S_{\frac{(j-1)}{m}}^\alpha |^p \\ & + \frac{\sqrt{m}}{n} c_p \sum_{i=1}^{[nt]} | u_{(i-1)/n-} |^p, \end{aligned}$$

$$C_t^{(n,m)} = \sum_{i=1}^{[nt]} |u_{(i-1)/n-}|^p \sum_{j \in I_n(i)} m^{-1/2+p/\alpha} |S_{j/m}^\alpha - S_{(j-1)/m}^\alpha|^p - \frac{\sqrt{m}}{n} c_p \sum_{i=1}^{[nt]} | u_{(i-1)/n-} |^p$$

and

$$D_t^{(m)} = m^{-1/2} c_p \sum_{j=1}^{[mt]} | u_{(j-1)/m-} |^p - \sqrt{m} c_p \int_0^t | u_s |^p \mathrm{d}s.$$

First, we show that $\| D^{(m)} \|_\infty \to 0$ as $m \to \infty$. We have that

$$\| D_t^{(m)} \|_\infty \leq c_p m^{-1/2} \sum_{j=1}^{[mT]} \sup_{s \in \mathcal{I}_m(j)} | u_{(j-1)/m-} - u_s |^p + \frac{c_p}{\sqrt{m}} \| \, | u |^p \|_\infty.$$

Hence, $\| D^{(m)} \|_\infty \to 0$ by the conditions on u.

Let us now study the term $C_t^{(n,m)}$. Set

$$Y_{n,m}^i := \sum_{j \in I_n(i)} m^{-1/2+p/\alpha} | S_{j/m}^\alpha - S_{(j-1)/m}^\alpha |^p - \frac{\sqrt{m}}{n} c_p.$$

By theorem 2 and taking into account that it implies the stable convergence of $\{ Y_{n,m}^1, Y_{n,m}^2, \ldots, Y_{n,m}^n \}_{m \geq 1}$ for any n (see the comment after theorem 2 and ALDOUS and EAGLESON 1978, proposition 1), we have that for any $\mathcal{F}$-measurable random variable

$$| u_{(i-1)/n-} |^p,$$

as $m \to \infty$

$$\left(| u_{(i-1)/n-} |^p, Y_{n,m}^i \right)_{1 \leq i \leq [nt]} \xrightarrow{\mathcal{L}} \left(| u_{(i-1)/n-} |^p, v_p \left(W_{i/n} - W_{(i-1)/n} \right) \right)_{1 \leq i \leq [nt]},$$

where W is a Brownian motion independent of $\mathcal{F}_T^{\alpha}$. Hence,

$$C_t^{(n,m)} \xrightarrow{\mathcal{L}} v_p \sum_{i=1}^{[nt]} |\, u_{(i-1)/n-} \,|^p \left(W_{i/n} - W_{(i-1)/n} \right)$$

as m tends to infinity, and this convergence is also stable (see Aldous and Eagleson, (1978, Theorem 1). However,

$$\sum_{i=1}^{[nt]} |\, u_{(i-1)/n-} \,|^p \left(W_{i/n} - W_{(i-1)/n} \right)$$

converges u.c.p. to

$$\int_0^t |\, u_s \,|^p \,\mathrm{d} W_s,$$

as n tends to infinity. This implies, by first letting m tend to infinity and then letting n tend to infinity that $C_t^{(n,m)}$ converges in distribution and stably to

$$v_p \int_0^t |\, u_s \,|^p \,\mathrm{d} W_s$$

in $\mathcal{D}([0,T])$.

We want to show that

$$\lim_n \limsup_m P\left(\left\| B^{(n,\,m)} \right\|_{\infty} > \varepsilon \right) = 0.$$

Using the mean value theorem, we can rewrite $B_t^{(n,\,m)}$ as follows:

$$\begin{aligned}
|B_t^{(n,\,m)}| = \Bigg| & \sum_{i=1}^{[nt]} \sum_{j \in I_n(i)} |u_{(j-1)/m-}|^p \left(m^{-1/2+p/\alpha} |S^{\alpha}_{j/m} - S^{\alpha}_{(j-1)/m}|^p - m^{-1/2} c_p \right) \\
& - \sum_{i=1}^{[nt]} |u_{(i-1)/n-}|^p \left(\sum_{j \in I_n(i)} m^{-1/2+p/\alpha} |S^{\alpha}_{j/m} - S^{\alpha}_{(j-1)/m}|^p - \frac{\sqrt{m}}{n} c_p \right) \\
& + \sum_{j \geq \frac{m}{n}[nt]}^{[mt]} |u_{(j-1)/m-}|^p \left(m^{-1/2+p/\alpha} |S^{\alpha}_{j/m} - S^{\alpha}_{(j-1)/m}|^p - m^{-1/2} c_p \right) \Bigg| \\
\leq \Bigg| & \sum_{i=1}^{[nt]} |\tilde{u}|^p \sum_{j \in I_n(i)} \left(m^{-1/2+p/\alpha} |S^{\alpha}_{j/m} - S^{\alpha}_{(j-1)/m}|^p - m^{-1/2} c_p \right) \\
& - \sum_{i=1}^{[nt]} |u_{(i-1)/n-}|^p \left(\sum_{j \in I_n(i)} m^{-1/2+p/\alpha} |S^{\alpha}_{j/m} - S^{\alpha}_{(j-1)/m}|^p - \frac{\sqrt{m}}{n} c_p \right) \Bigg| \\
& + \frac{c_p}{\sqrt{m}} \sum_{i=1}^{[nt]} \sup_{s,\, t \in \mathcal{I}_n(i)} |u_s - u_t|^p
\end{aligned}$$

$$
\begin{aligned}
&+\sup_{0\le t\le T}\left|\sum_{m/n[nt]\le j\le[mt]}|u_{(j-1)/m}|^p\left(m^{-1/2+p/\alpha}|S^\alpha_{j/m}-S^\alpha_{(j-1)/m}|^p-m^{-1/2}c_p\right)\right|\\
&\le\sum_{i=1}^{[nT]}\sup_{s\in\mathcal{I}_n(i)\cup\mathcal{I}_n(i-1)}|u_s-u_{(i-1)/n}|^p\left|Y^i_{n,m}\right|+\frac{c_p}{\sqrt{m}}\,\||u|^p\|_\infty\\
&+\frac{c_p}{\sqrt{m}}\sum_{i=1}^{[nT]}\sup_{s,\,t\in\mathcal{I}_n(i)}|u_s-u_t|^p\\
&+\sup_{0\le t\le T}\left|\sum_{m/n[nt]\le j\le[mt]}|u_{(j-1)/m}|^p\left(m^{-1/2+p/\alpha}|S^\alpha_{j/m}-S^\alpha_{(j-1)/m}|^p-m^{-1/2}c_p\right)\right|,
\end{aligned}
$$

where

$$
\min_{s\in\mathcal{I}_n(i)}\cup\mathcal{I}_n(i-1)|u_s|\le|\tilde{u}|\le\max_{s\in\mathcal{I}_n(i)}\cup\mathcal{I}_n(i-1)|u_s|.
$$

Then, by theorem 2, and the condition on u, for any $\varepsilon>0$ we obtain

$$
\begin{aligned}
\limsup_{m\to\infty}P\left(\left\|B^{(n,m)}\right\|_\infty>\varepsilon\right)\le P\Bigg(\Bigg(&v_p\sum_{i=1}^{[nT]}\sup_{s\in\mathcal{I}_n(i)\cup\mathcal{I}_n(i-1)}\\
&|u_s-u_{(i-1)/n-}|^p\left|W_{i/n}-W_{(i-1)/n}\right|+v_p\,\||u|^p\|_\infty\,1/n\sup_{0\le t\le T}\left|W_t-W_{[nt]/n}\right|\Bigg)>\varepsilon\Bigg).
\end{aligned}
$$

Then, since u and W are independent, we can apply the condition on u and obtain that

$$
\sum_{i=1}^{[nT]}n^{-1/2}\sup_{s\in\mathcal{I}_n(i)\cup\mathcal{I}_n(i-1)}|u_s-u_{(i-1)/n-}|^p\left|W_{i/n}-W_{(i-1)/n}\right|
$$

converges to zero in L^1 as n tends to infinity. Additionally,

$$
\frac{1}{n}\sup_{0\le t\le T}\left|W_t-W_{[nt]/n}\right|\xrightarrow[n\to\infty]{\text{a.s.}}0,
$$

and we deduce the desired result.

Finally, we have to show that

$$
\|A^{(m)}\|_\infty\xrightarrow{P}0
$$

as $m\to\infty$. Then,

$$
\|A^{(m)}\|_\infty\le m^{-1/2+p/\alpha}\sum_{j=1}^{[mT]}\left|\int_{\frac{(j-1)}{m}}^{\frac{j}{m}}u_{s-}\mathrm{d}S^\alpha_s-u_{\frac{(j-1)}{m}-}(S^\alpha_{\frac{j}{m}}-S^\alpha_{\frac{(j-1)}{m}})\right|^p
$$

and, as in theorem 1, by lemma 1, we have

$$
E(\|A^{(m)}\|_\infty)\le m^{-1/2+p/\alpha}\sum_{j=1}^{[mT]}E\left(\left|\int_{(j-1)/m}^{j/m}\left(u_{s_-}-u_{(j-1)/m-}\right)\mathrm{d}S^\alpha_s\right|^p\right)
$$

$$\leq m^{-1/2+p/\alpha}\sum_{j=1}^{[mT]}\left(\left|\int_{(j-1)/m}^{j/m}E\left(|u_{s_-}-u_{(j-1)/m-}|^{\alpha}\right)\mathrm{d}s\right|^{p/\alpha}\right)$$
$$\leq m^{-1/2}\sum_{j=1}^{[mT]}E\left(\sup_{s,t\in[(j-1)/m,j/m]}|u_{s-}-u_{t-}|^{p}\right).$$

Then by the condition on u, we conclude that $E(\|A^{(m)}\|_{\infty})$ goes to zero as m goes to infinity. Finally, for $\alpha<1,$ a similar proof can be given by substituting $B_t^{(n,m)}$ for the term

$$\tilde{A}_t^{(n,m)}=m^{-1/2+p/\alpha}\sum_{j=1}^{[mt]}\left|\sum_{s\in((j-1)/m,j/m]}u_{s-}\Delta S_s^{\alpha}\right|^{p}-m^{-1/2}c_p\sum_{j=1}^{[mt]}|u_{(j-1)/m-}|^{p}$$
$$-\sum_{i=1}^{[nt]}|u_{(i-1)/n-}|^{p}\sum_{j\in I_n(i)}m^{-1/2+p/\alpha}|S_{j/m}^{\alpha}-S_{(j-1)/m}^{\alpha}|^{p}$$
$$+\frac{\sqrt{m}}{n}c_p\sum_{i=1}^{[nt]}|u_{(i-1)/n-}|^{p},$$

and eliminating the term $A_t^{(n,m)}$. □

REMARK 2. Note that condition 1 implies that if u has a continuous part, then $\gamma>1/2$ and, consequently if we take $\gamma=p$ this means that $\alpha>1$ in the previous theorem, limiting seriously its scope. Then, it would be good to have an alternative to the condition 1. Nevertheless inequality (3.16) of JACOD (2004) can help to elucidate which processes fulfill condition 1.

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