UNIVERSITAT DE BARCELONA

On Quasiperiodic Perturbations of Ordinary Differential Equations

Àngel Jorba i Monte

On Quasiperiodic Perturbations of Ordinary Differential Equations

Àngel Jorba i Monte

Memòria presentada per a aspirar al grau de Doctor en Ciències Matemàtiques

Departament de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona

Barcelona, Agost del 1991

Appendix B

The Model Equations Near the Equilateral Points in the Earth-Moon System

B.1 Introduction

In this Appendix we have included the obtention of the model equations used to study the neighbourhood of L_4. To do that, the full solar system and radiation pressure are taken into account to write the Lagrangian and Hamiltonian of the problem. In that Lagrangian, the terms which contain Legendre polynomials (except those coming from the RTBP) are expanded as power series in x, y, z. Its coefficients are known functions of the positions of the bodies of the solar system. For a short time interval it can be assumed that these positions, and therefore the coefficients, are quasiperiodic functions of time.

A computation of these coefficients using a Fourier analysis shows that the relevant frequencies are the ones related to the following four angles:

1. The mean longitude of the Moon (equal to 1, because of the choice of the units).
2. The mean longitude of the lunar perigee.
3. The mean longitude of the ascending node of the Moon.
4. The mean elongation of the Sun.

All the contributions with amplitude less than 5×10^{-5} are dropped in order to keep a manageable number of terms. This leads to the fact that the perturbation coming from the planets, the radiation pressure and the aspherical terms coming from the

Earth and Moon can be neglected. Using this, it is possible to write the equations of motion in a simplified way.

The technical details can be found inside next sections.

B.2 Systems of Reference

We consider an inertial frame of reference with the origin at the centre of masses of the solar system and the axes parallel to the ecliptic ones. The equations of motion of a spacecraft in the solar system can be written as

$$\ddot{\vec{R}} = G \sum_{A \in \{S,E,P_1,\ldots,P_k\}} \frac{A(\vec{R}_A - \vec{R})}{\mid \vec{R}_A - \vec{R} \mid^3}.$$

However, the above system of reference is not convenient to study the motion of a spacecraft in the vicinity of the libration points L_4 or L_5 corresponding to the Earth-Moon system. As it is usual, the libration points are defined as the ones that form an equilateral triangle with the Earth and the Moon. These points are placed at the instantaneous plane of motion of the Moon around the Earth (see Figure B.1)

We define a normalized reference system centered at the libration points and given by the unitary vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$:

$$\begin{aligned}
\vec{e}_1 &= \frac{\vec{r}_{EM}}{\mid \vec{r}_{EM} \mid}, \\
\vec{e}_3 &= \frac{\dot{\vec{r}}_{EM} \wedge \vec{r}_{EM}}{\mid \dot{\vec{r}}_{EM} \wedge \vec{r}_{EM} \mid}, \\
\vec{e}_3 &= \vec{e}_3 \wedge \vec{e}_1.
\end{aligned}$$

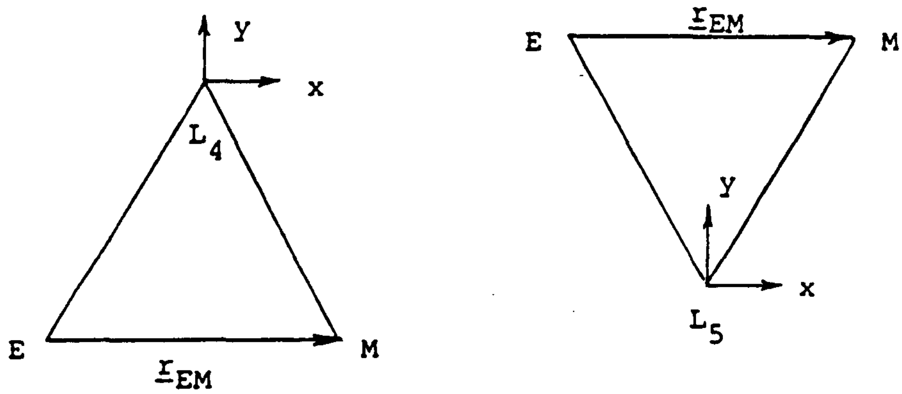

Figure B.1

We introduce a modified mass of the Earth, $\overline{E}$, to satisfy the third Kepler's law

$$K = G(\overline{E} + M) = n_M^2 a_M^3$$

where n_M and a_M are the mean motion and the semimajor axis of the Moon around the Earth. The rest of mass $\tilde{E} = E - \overline{E}$ will be considered as a perturbation. Then, in this normalized system (x, y, z), we adopt the following units:

- The unit of mass is chosen such that $G(\overline{E} + M) = 1$,
- The unit of time is defined to have $n_M = 1$,
- The unit of distance is taken as r_{EM}.

Therefore, in the L_4 case, the coordinates of the Earth and the Moon are:

- $(x_E, y_E, z_E) = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$,
- $(x_M, y_M, z_M) = (\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$.

In a similar way, for L_5 we have:

- $(x_E, y_E, z_E) = (-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$,
- $(x_M, y_M, z_M) = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$.

For a precise definition of the adopted system see [13].

B.3 The Lagrangian

In the normalized system of reference centered at the libration point L_i, $i = 4, 5$, the Lagrangian (see [13]) can be expanded in Legendre polynomials as:

$$
\begin{aligned}
L = & \frac{1}{2}\{k^2(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + 2k\dot{k}(x\dot{x} + y\dot{y} + z\dot{z}) + \\
& +2k^2\left(E(x\dot{y} - y\dot{x}) + F(y\dot{z} - z\dot{y})\right) + \dot{k}^2(x^2 + y^2 + z^2) + \\
& +k^2(Ax^2 + By^2 + Cz^2 + 2Dxz)\} - \\
& -\dot{k}^2(x_E x + y_E y) - k\dot{k}(x_E\dot{x} + y_e\dot{y}) - k^2(Ax_E x + By_E y + Dz_E z) - \\
& -k^2(-Ey_E\dot{x} + Ex_E\dot{y} + Fy_E\dot{z}) + \\
& +Kk^{-1}(1 - \mu_M + \mu_{\tilde{E}})\sum_{n\geq 1} a^n P_n(\cos E_1) + \\
& +Kk^{-1}\frac{\mu_{\tilde{S}}}{\tilde{r}_S}\sum_{n\geq 1}\left(\frac{a}{\tilde{r}_S}\right)^n P_n(\cos S_1) + \\
& +Kk^{-1}\sum_{A\in\{S,M,P_1,\dots,P_k\}} \mu_A\left(-\frac{a\cos A_2}{\tilde{r}_{EA}^2} + \frac{1}{\tilde{r}_A}\sum_{n\geq 1}\left(\frac{a}{\tilde{r}_A}\right)^n P_n(\cos A_1)\right),
\end{aligned}
$$

where $\vec{a} = (x, y, z)$ is the position vector of the spacecraft, $\vec{\tilde{v}}$ stands for the vector $\vec{v}$ in the normalized system, $k = r_{EM}$ is a scaling factor, $\cos A_1 = (\vec{\tilde{r}}_A, \vec{a})/(\tilde{r}_A a)$ and $\cos A_2 = (\vec{\tilde{r}}_{EA}, \vec{a})/(\tilde{r}_{EA} a)$. We denote by V the modulus of the vector $\vec{V}$. The term with $\mu_{\tilde{S}}$ takes into account the radiation pressure of the Sun. The coefficients A, B, C, D, E, and F are defined by

$$\begin{aligned}
A &= \dot{\theta}^2 \cos^2\delta + \dot{\delta}^2, \\
B &= \frac{(\cos^2\delta + \sin^2\delta R^2)\dot{\theta}^2 + (2 + R^2)\dot{\delta}^2 + (\dot{R} + \sin\delta\dot{\theta})^2}{(1 + R^2)} - \frac{R^2\dot{R}^2}{(1 + R^2)^2}, \\
C &= \frac{R^2\dot{\theta}^2 - \dot{\delta}^2 + (\dot{R} + \sin\delta\dot{\theta})^2}{(1 + R^2)} - \frac{R^2\dot{R}^2}{(1 + R^2)^2}, \\
D &= -\sin\delta\cos\delta\dot{\theta}^2(1 + R^2)^{1/2} - \frac{\cos\delta\dot{\theta}\dot{R}}{(1 + R^2)^{1/2}}, \\
E &= \frac{\dot{\theta}\cos\delta + \dot{\delta}R}{(1 + R^2)^{1/2}}, \\
F &= \dot{\theta}\sin\delta + \frac{\dot{R}}{(1 + R^2)},
\end{aligned}$$

where

$$R = \frac{\dot{\delta}}{\dot{\theta}\cos\delta},$$

and θ, δ denote the geocentric longitude and latitude of the Moon, respectively.

From the Lagrangian, the equations of motion are easily written as:

$$\begin{aligned}
\ddot{x} - 2\dot{y} - x &= C(1)x + C(2)y + C(3)z + \\
&+ C(4)\dot{x} + C(5)\dot{y} + C(7) + \\
&+ Kk^{-3}(1 - \mu_M + \mu_{\tilde{E}})(x - x_E)\sum_{n\geq 2} a^{n-2}\overline{P}_n(\cos E_1) + \\
&+ Kk^{-3}\mu_{\tilde{S}}(x - x_S)\sum_{n\geq 2}\frac{a^{n-2}}{\tilde{r}_S^{n+1}}\overline{P}_n(\cos S_1) + \\
&+ Kk^{-3}\sum_{A\in\mathcal{P}}\mu_A\left(-\frac{x_{EA}}{\tilde{r}_{EA}^3} + (x - x_A)\sum_{n\geq 2}\frac{a^{n-2}}{\tilde{r}_A^{n+1}}\overline{P}_n(\cos A_1)\right), \\
\ddot{y} + 2\dot{x} - y &= C(11)x + C(12)y + C(13)z + \\
&+ C(14)\dot{x} + C(15)\dot{y} + C(16)\dot{z} + C(17) + \\
&+ Kk^{-3}(1 - \mu_M + \mu_{\tilde{E}})(y - y_E)\sum_{n\geq 2} a^{n-2}\overline{P}_n(\cos E_1) + \\
&+ Kk^{-3}\mu_{\tilde{S}}(y - y_S)\sum_{n\geq 2}\frac{a^{n-2}}{\tilde{r}_S^{n+1}}\overline{P}_n(\cos S_1) +
\end{aligned}$$

$$
\begin{aligned}
&+Kk^{-3}\sum_{A\in\mathcal{P}}\mu_A\left(-\frac{y_{EA}}{\bar{r}_{EA}^3}+(y-y_A)\sum_{n\geq 2}\frac{a^{n-2}}{\bar{r}_A^{n+1}}\overline{P}_n(\cos A_1)\right),\\
\ddot{z} \;=\; &C(21)x+C(22)y+C(23)z+\\
&+C(25)\dot{y}+C(26)\dot{z}+C(27)+\\
&+Kk^{-3}(1-\mu_M+\mu_{\bar{E}})z\sum_{n\geq 2}a^{n-2}\overline{P}_n(\cos E_1)+\\
&+Kk^{-3}\mu_{\bar{S}}(z-z_S)\sum_{n\geq 2}\frac{a^{n-2}}{\bar{r}_S^{n+1}}\overline{P}_n(\cos S_1)+\\
&+Kk^{-3}\sum_{A\in\mathcal{P}}\mu_A\left(-\frac{z_{EA}}{\bar{r}_{EA}^3}+(z-z_A)\sum_{n\geq 2}\frac{a^{n-2}}{\bar{r}_A^{n+1}}\overline{P}_n(\cos A_1)\right),
\end{aligned}
$$

where $\mathcal{P}=\{S,M,P_1,...,P_k\}$, $\overline{P}_n(\alpha)=-d(P_{n-1}(\alpha))/d\alpha$,

$$
\begin{aligned}
C(0) &= Kk^{-3}-1,\\
C(1) &= A-1-\ddot{k}k^{-1},\\
C(2) &= 2\dot{k}k^{-1}E+\dot{E},\\
C(3) &= D,\\
C(4) &= -2\dot{k}k^{-1},\\
C(5) &= 2(\dot{\theta}-\dot{\theta}^2+E-1),\\
C(7) &= (\ddot{k}k^{-1}-A)x_E-(2\dot{k}k^{-1}E+\dot{E})y_E,\\
C(12) &= B-1-\ddot{k}k^{-1},\\
C(13) &= 2\dot{k}k^{-1}F+\dot{F},\\
C(16) &= 2F,\\
C(17) &= (\ddot{k}k^{-1}-B)y_E+(2\dot{k}k^{-1}E+\dot{E})x_E,\\
C(23) &= C-\ddot{k}k^{-1},\\
C(27) &= -Dx_E+(2\dot{k}k^{-1}F+\dot{F})y_E,
\end{aligned}
$$

and $C(11)=-C(2)$, $C(14)=-C(5)$, $C(15)=C(4)$, $C(21)=C(3)$, $C(22)=-C(13)$, $C(25)=-C(16)$, $C(26)=C(4)$.

B.4 The Hamiltonian and the Related Expansions

The Hamiltonian will not be required in what follows. However we give it because of its possible use in future work.

We start with the expression of the Lagrangian (given in Section B.3) of the problem when normalized coordinates centered at the instantaneous $L_{4,5}$ point of the Earth-Moon system are used.

The momenta p_x, p_y, p_z are introduced through

$$\begin{aligned} p_x &= \partial L/\partial \dot{x} = k^2\dot{x} + k\dot{k}x - k^2Ey - k\dot{k}x_E + k^2Ey_E, \\ p_y &= \partial L/\partial \dot{y} = k^2\dot{y} + k\dot{k}y + k^2(Ex - Fz) - k\dot{k}y_E - k^2Ex_E, \\ p_z &= \partial L/\partial \dot{z} = k^2\dot{z} + k\dot{k}z + k^2Fy - k^2Fy_E. \end{aligned}$$

From these relations it is easy to express $\dot{x}, \dot{y}, \dot{z}$ as functions of the positions and momenta, the coefficients being functions of time.

The Hamiltonian is obtained as

$$H = \sum p_i\dot{q}_i - L,$$

with

$$q = (x, y, z)^T, \;\; p = (p_x, p_y, p_z)^T,$$

where in $\dot{q}_i$, as well as in $\dot{x}^2 + \dot{y}^2 + \dot{z}^2$, $x\dot{x} + y\dot{y} + z\dot{z}$, $E(x\dot{y} - y\dot{x}) + F(y\dot{z} - z\dot{y})$, $x_E\dot{x} + y_E\dot{y}$ and $-Ey_E\dot{x} + Ex_E\dot{y} + Fy_E\dot{z}$, $\dot{x}, \dot{y}, \dot{z}$ are substituted by their expressions in terms of positions and momenta.

After a somewhat lengthly computations the following Hamiltonian is obtained

$$\begin{aligned} H = & \frac{1}{2}k^{-2}(p_x^2 + p_y^2 + p_z^2) - \dot{k}k^{-1}(xp_x + yp_y + zp_z) + \\ & +E(yp_x - xp_y) + F(zp_y - yp_z) + \\ & +(\dot{k}k^{-1}x_E - Ey_E)p_x + (\dot{k}k^{-1}y_E + Ex_E)p_y + Fy_Sp_z + \\ & +\text{Terms purely depending on time} - \\ & -Kk^{-1}(1 - \mu_M + \mu_{\tilde{E}})\sum_{n\geq 1} a^nP_n(\cos E_1) - \\ & -Kk^{-1}\frac{\mu_{\tilde{S}}}{\tilde{r}_S}\sum_{n\geq 1}\left(\frac{a}{\tilde{r}_S}\right)^n P_n(\cos S_1) + \\ & +Kk^{-1}\sum_{A\in\mathcal{P}}\mu_A\left(-\frac{a\cos A_2}{\tilde{r}_{EA}^2} + \frac{1}{\tilde{r}_A}\sum_{n\geq 1}\left(\frac{a}{\tilde{r}_A}\right)^n P_n(\cos A_1)\right), \end{aligned}$$

The time depending functions which appear as coefficients have been partly computed in the development of the Lagrangian. The only additional function is k^{-2}.

B.5 Some Useful Expansions

In order to get a quasiperiodic solution of the problem it is convenient to develop the right hand side of the equations as series of the following types

$$\sum a_{ijkr} x^i y^j z^k F(\nu_r t + \varphi_r),$$

or

$$\dot{v} \sum a_r F(\nu_r t + \varphi_r),$$

where $\dot{v}$ denotes $\dot{x}$, $\dot{y}$ or $\dot{z}$ and F stands for one of the trigonometric functions sinus or cosinus.

A Fourier analysis of the functions that appear in the equations will give the coefficients, a_{ijkr}, a_r, the frequencies, ν_r, and the phases, φ_r, involved in the dominant terms.

We deal separately with the terms of the equations which involve the Legendre polynomials. These terms come through derivation with respect to x, y or z, of the corresponding terms of the Lagrangian. So it is more convenient to develop the summations of the Lagrangian and then to compute the derivatives.

Using the normalized system of coordinates we have

$$\cos E_1 = \frac{xx_E + yy_E + zz_E}{a\tilde{r}_E} = -\frac{1}{2a}(x + s\sqrt{3}y),$$

$$\cos M_1 = \frac{xx_M + yy_M + zz_M}{a\tilde{r}_M} = \frac{1}{2a}(x - s\sqrt{3}y),$$

with $s = 1$ for L_4 and -1 for L_5.

Therefore, it is not difficult to see that

$$\sum_{n\geq 1} a^n P_n(\cos A_1) = \sum_{n\geq 1} \frac{1}{2^{2n}} \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^{k+nJ}}{(n-k)!} 2^{2k}(2(n-k))! \times$$

$$\times \sum_{m=0}^{n-2k} \frac{(r_s\sqrt{3})^m}{m!(n-2k-m)!} \sum_{n_1+n_2+n_3=k} \frac{1}{n_1!n_2!n_3!} x^{2n_1+n-2k-m} y^{2n_2+m} z^{2n_3}$$

where

$$J = \begin{cases} 1 & \text{if A = E,} \\ 0 & \text{if A = M,} \end{cases}$$

and

$$r = \begin{cases} 1 & \text{if A = E,} \\ -1 & \text{if A = M.} \end{cases}$$

A routine, EXPEAT, gives the coefficients of each monomial $x^i y^j z^k$ up to a given order $n = i + j + k$. This routine has been checked comparing the development with the generating function

$$\frac{1}{\sqrt{1 - 2a\cos A_1 + a^2}} - 1,$$

where $A = E$ or M. It has been seen that to get a difference of the order of 10^{-5}, terms up to order 20 are required if the magnitude of a is 0.5, but only order 8 is required if $a = 0.15$.

We note that the coefficients in the development of

$$\sum_{n\geq 1} a^n P_n(\cos A_1),$$

for $A = E$ or M are constants and so no Fourier analysis is needed for those terms.

For the planets and the Sun, similar expansions are obtained where the coefficients of the monomials are functions of time. We have:

- For the planets
$$Kk^{-1}\mu_A \frac{1}{\tilde{r}_A} \sum_{n\geq 1} \left(\frac{a}{\tilde{r}_A}\right)^n P_n(\cos A_1).$$
- For the Sun
$$Kk^{-1}(\mu_{\tilde{S}} + \mu_S) \frac{1}{\tilde{r}_S} \sum_{n\geq 1} \left(\frac{a}{\tilde{r}_S}\right)^n P_n(\cos S_1).$$

In general, we have the following expansion

$$\frac{1}{\tilde{r}_A} \sum_{n\geq 1} \left(\frac{a}{\tilde{r}_A}\right)^n P_n(\cos A_1) = \sum_{n\geq 1} \frac{1}{2^n} \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{(2(n-k))!}{(n-k)!} \times$$

$$\times \sum_{n_1=0}^{k} \sum_{n_2=0}^{k-n_1} \frac{1}{n_1! n_2! (k - n_1 - n_2)!} \sum_{l_1=0}^{n-2k} \sum_{l_2=0}^{n-2k-l_1} \frac{1}{l_1! l_2! (n - 2k - l_1 - l_2)!} \times$$

$$\times \left(\frac{x_A}{\tilde{r}_A}\right)^{l_1} \left(\frac{y_A}{\tilde{r}_A}\right)^{l_2} \left(\frac{z_A}{\tilde{r}_A}\right)^{l_3} \left(\frac{1}{\tilde{r}_A}\right)^{n+1} x^{2n_1+l_1} y^{2n_2+l_2} z^{2n_3+l_3},$$

where $l_3 = n - 2k - l_1 - l_2$.

The terms of the expansion can be collected in the form

$$\sum_{q_1,q_2,q_3} f(q_1,q_2,q_3) x^{q_1} y^{q_2} z^{q_3},$$

where

$$f(q_1,q_2,q_3) = \sum_{l_1,l_2,l_3} c_{q_1 q_2 q_3 l_1 l_2 l_3} \left(\frac{x_A}{\tilde{r}_A}\right)^{l_1} \left(\frac{y_A}{\tilde{r}_A}\right)^{l_2} \left(\frac{z_A}{\tilde{r}_A}\right)^{l_3} \left(\frac{1}{\tilde{r}_A}\right)^{q_1+q_2+q_3+1}$$

The coefficients, $c_{q_1q_2q_3l_1l_2l_3}$ and the exponents, $q_1, q_2, q_3, l_1, l_2, l_3$, are computed up to a given order $n = q_1 + q_2 + q_3$ in the routine EXPEAT. The routine has been checked using the generating function

$$\frac{1}{\sqrt{\tilde{r}_A^2 - 2a\tilde{r}_A \cos A_1 + a^2}} - \frac{1}{\tilde{r}_A}.$$

Some computations for the Sun, Venus and Jupiter show that order 3 is needed for the Sun, and order 2 is sufficient for the planets in order to have a difference of order 10^{-5}.

A routine, FUN4, has been done to compute the time dependent functions at a given day. This routine will be used by the program that performs the Fourier analysis.

B.6 Fourier Analysis: Relevant Frequencies and Related Coefficients

First of all the equations of motion as given before, are rewritten in the more convenient form displayed below:

$$\begin{aligned}
&\ddot{x} - 2\dot{y} - 3x_E^2 x - 3(1-2\mu_M)x_E y_E y = \\
&\quad \overline{C(1)}x + \overline{C(2)}y + C(3)z + C(4)\dot{x} + C(5)\dot{y} + \overline{C(7)} + \\
&\quad + Kk^{-3}[-3(1-2m_M)x_E x^2 - 3y_E xy] + \\
&\quad + Kk^{-3}(1-\mu_M)(x - x_E)\sum_{n\geq 4} a^{n-2}\overline{P}_n(\cos E_1) + \\
&\quad + Kk^{-3}\mu_{\tilde{E}}(x - x_E)\sum_{n\geq 2} a^{n-2}\overline{P}_n(\cos E_1) + \\
&\quad + Kk^{-3}\mu_{\tilde{S}}(x - x_S)\sum_{n\geq 2}\frac{a^{n-2}}{\tilde{r}_S^{n+1}}\overline{P}_n(\cos S_1) + \\
&\quad + Kk^{-3}\mu_M(x - x_M)\sum_{n\geq 4}\frac{a^{n-2}}{\tilde{r}_M^{n+1}}\overline{P}_n(\cos M_1) + \\
&\quad + Kk^{-3}\sum_{A\in\mathcal{P}'}\mu_A\left(-\frac{x_{EA}}{\tilde{r}_{EA}^3} + (x - x_A)\sum_{n\geq 2}\frac{a^{n-2}}{\tilde{r}_A^{n+1}}\overline{P}_n(\cos A_1)\right),
\end{aligned}$$

$$\begin{aligned}
&\ddot{y} + 2\dot{x} - 3y_E^2 y - 3(1-2\mu_M)x_E y_E x = \\
&\quad \overline{C(11)}x + \overline{C(12)}y + C(13)z + C(14)\dot{x} + C(15)\dot{y} + C(16)\dot{z} + \overline{C(17)} + \\
&\quad + Kk^{-3}[-3(1-2m_M)x_E xy - 3y_E y^2] +
\end{aligned}$$

$$+Kk^{-3}(1-\mu_M)(y-y_E)\sum_{n\geq 4} a^{n-2}\overline{P}_n(\cos E_1)+$$
$$+Kk^{-3}\mu_{\bar{E}}(y-y_E)\sum_{n\geq 2} a^{n-2}\overline{P}_n(\cos E_1)+$$
$$+Kk^{-3}\mu_{\bar{S}}(y-y_S)\sum_{n\geq 2} \frac{a^{n-2}}{\tilde{r}_S^{n+1}}\overline{P}_n(\cos S_1)+$$
$$+Kk^{-3}\mu_M(y-y_M)\sum_{n\geq 4} \frac{a^{n-2}}{\tilde{r}_M^{n+1}}\overline{P}_n(\cos M_1)+$$
$$+Kk^{-3}\sum_{A\in\mathcal{P}'}\mu_A\left(-\frac{y_{EA}}{\tilde{r}_{EA}^3}+(y-y_A)\sum_{n\geq 2}\frac{a^{n-2}}{\tilde{r}_A^{n+1}}\overline{P}_n(\cos A_1)\right),$$

$$\ddot{z}+z=$$
$$C(21)x+C(22)y+\overline{C(23)}z+C(25)\dot{y}+C(26)\dot{z}+C(27)+$$
$$+Kk^{-3}(1-\mu_M)z\sum_{n\geq 3} a^{n-2}\overline{P}_n(\cos E_1)+$$
$$+Kk^{-3}\mu_{\bar{E}}z\sum_{n\geq 2} a^{n-2}\overline{P}_n(\cos E_1)+$$
$$+Kk^{-3}\mu_{\bar{S}}(z-z_S)\sum_{n\geq 2} \frac{a^{n-2}}{\tilde{r}_S^{n+1}}\overline{P}_n(\cos S_1)+$$
$$+Kk^{-3}\mu_M z\sum_{n\geq 3} \frac{a^{n-2}}{\tilde{r}_M^{n+1}}\overline{P}_n(\cos M_1)+$$
$$+Kk^{-3}\sum_{A\in\mathcal{P}'}\mu_A\left(-\frac{z_{EA}}{\tilde{r}_{EA}^3}+(z-z_A)\sum_{n\geq 2}\frac{a^{n-2}}{\tilde{r}_A^{n+1}}\overline{P}_n(\cos A_1)\right),$$

where

$$\begin{aligned}
\mathcal{P}' &= \{S, P_1, ..., P_k\},\\
\overline{C(1)} &= C(1)+(Kk^{-3}-1)(-1+3x_E^2),\\
\overline{C(2)} &= C(2)+(Kk^{-3}-1)3(1-2\mu_M)x_E y_E,\\
\overline{C(7)} &= C(7)+Kk^{-3}x_E,\\
\overline{C(11)} &= C(11)+(Kk^{-3}-1)3(1-2\mu_M)x_E y_E,\\
\overline{C(12)} &= C(12)+(Kk^{-3}-1)(-1+3y_E^2),\\
\overline{C(17)} &= C(17)+Kk^{-3}y_E,\\
\overline{C(23)} &= C(23)+(Kk^{-3}-1),
\end{aligned}$$

and, in fact, $\tilde{r}_M$ is equal to 1 in the normalized system.

The coefficients $C(j)$ and $\overline{C(j)}$ and the ones coming from the effect of the Sun and planets are computed by the routine FUN4. We shall use the following notation for further use:

$$
\begin{array}{lll}
SO(1) = Kk^{-3} - 1, & SO(6) = C(5), & SO(11) = C(16), \\
SO(2) = \overline{C(1)}, & SO(7) = \overline{C(7)}, & SO(12) = \overline{C(17)}, \\
SO(3) = \overline{C(2)}, & SO(8) = \overline{C(11)}, & SO(13) = \overline{C(23)}, \\
SO(4) = C(3), & SO(9) = \overline{C(12)}, & \\
SO(5) = C(4), & SO(10) = C(13) &
\end{array}
$$

for terms and coefficients coming from the noncircular motion of the Moon, and we expand the term

$$Kk^{-1}\mu_S \frac{1}{r_S} \sum_{n\geq 1} \left(\frac{a}{\tilde{r}_S}\right)^n P_n(\cos S_1),$$

accounting for part of the contribution of the Sun to the Hamiltonian as:

$$\text{ctant} + \overline{SOS(1)}x + \overline{SOS(2)}y + \overline{SOS(3)}z + SOS(4)x^2 + SOS(5)xy +$$

$$+ SOS(6)xz + SOS(7)y^2 + SOS(8)yz + SOS(9)z^2 + SOS(10)x^3 + \cdots$$

The contributions of the planets can be expanded in a similar way.

Then the contribution of the Sun to the equations is of the following form:

- 1st equation: $SOS(1) + 2SOS(4)x + SOS(5)y + SOS(6)z + \cdots$,
- 2nd equation: $SOS(2) + SOS(5)x + 2SOS(7)y + SOS(8)z + \cdots$,
- 3rd equation: $SOS(3) + SOS(6)x + SOS(8)y + 2SOS(9)z + \cdots$,

where:

$$
\begin{aligned}
SOS(1) &= \overline{SOS(1)} - Kk^{-3}\mu_S \frac{x_{ES}}{\tilde{r}_{ES}^3}, \\
SOS(2) &= \overline{SOS(2)} - Kk^{-3}\mu_S \frac{y_{ES}}{\tilde{r}_{ES}^3}, \\
SOS(3) &= \overline{SOS(3)} - Kk^{-3}\mu_S \frac{z_{ES}}{\tilde{r}_{ES}^3},
\end{aligned}
$$

which is quite useful because of the cancellations (i.e. $\overline{SOS(j)}$, $j = 1, 2, 3$ is quite large but $SOS(j)$, $j = 1, 2, 3$ is rather small).

A Fourier analysis of the functions $SO(i)$, $i = 1,\ldots,13$ and $SOS(i)$ $i = 1,\ldots,9$ has been performed. Other functions $SOS(i)$, $i \geq 10$ have also been computed but the fact that in the denominator appears at least $\tilde{r}_S^{-3}$ makes them negligible.

The Fourier analysis has been done in several steps. First we have performed several FFT in order to identify the relevant frequencies. Some of them appear both in the $SO(i)$ functions and in the $SOS(i)$ ones.

The frequencies retained are in principle, the ones such that the peak of the modulus of the discrete Fourier Transform is greater than 10^{-4}. The unit of frequency has been taken equal to the frequency of the mean longitude of the Moon. Then the frequencies have been checked against linear combinations of the four more importatnt terms appearing in the motion of the Moon (the motion of the Earth-Moon barycentre perhelion has been neglected). Let ν be one of the obtained frequencies. We look for expressions of the type:

$$\nu = n_1\nu_1 + n_2\nu_2 + n_3\nu_3 + n_4\nu_4,$$

where

ν_1 = frequency of the mean longitude of the Moon = 1,
ν_2 = frequency of the mean longitude of the lunar perigee,
ν_3 = frequency of the mean longitude of the ascending node of the Moon,
ν_4 = frequency of the mean elongation of the Sun.

The FFT have been carried out using 2^{16} points in a time interval equal to 8192 sidereal revolutions of the Moon centered at the epoch $\tau = 1 - 0.5/36525$ julian centuries since 1900.0. The results obtained for the frequencies are given in the Table B.1 where there appear also the phases at the epoch τ. The values of FFT are related to the approximate frequencies in the following way:

$$\text{frequency} = \frac{\text{FFT - 1}}{\text{no. revolutions of the secondary}}$$

The normalized time is taken with origin at the epoch τ and such that the sidereal period of the Moon is equal to 2π. Hence if JD is the julian date counted from 1950.0 (and then τ = 18262 JD) we have the relation

$$t_n = (JD - 18262)\overline{n}_M$$

where t_n stands for the normalized time and $\overline{n}_M$ for the mean sidereal motion of the Moon in rad/day (note that $\overline{n}_M = \nu_1$ letting aside the units).

The results of the FFT analysis for the functions $SO(i)$, i = 1,...,13 and $SOS(i)$ i=1,...,9 have been computed. We list a sample containing $SO(1)$ and $SO(13)$ and all the $SOS(j)$ larger than some tolerance in the Table B.2 and B.3 for the L_4 equilibrium point.

N	FFT	Frequency	Phase	n_1	n_2	n_3	n_4
1	1088	$.1326984719029E+00$	$.5820858844883E+00$	2	−2	0	−2
2	6424	$.7840427350868E+00$	$.3056181124683E+01$	−2	1	0	3
3	6934	$.8463731326263E+00$	$.2388511629532E+01$	−1	0	1	2
4	7037	$.8588467495676E+00$	$.1659447913254E+01$	−1	1	0	2
5	7511	$.9167412069898E+00$	$.3638267009167E+01$	0	−1	0	1
6	7580	$.9251959855191E+00$	$.5092083509088E+01$	0	0	0	1
7	8124	$.9915452214706E+00$	$.2241533797741E+01$	1	−1	0	0
8	8226	$.1004018838412E+01$	$.1512470081464E+01$	1	0	−1	0
9	8328	$.1016492455353E+01$	$.7834063651862E+00$	1	1	−2	0
10	8736	$.1066349235951E+01$	$.8448005863156E+00$	2	−1	0	−1
11	14072	$.1717693499135E+01$	$.3318895826509E+01$	−2	2	0	4
12	14547	$.1775587956557E+01$	$.5297714922424E+01$	−1	0	0	3
13	15159	$.1850391971038E+01$	$.3900981710996E+01$	0	0	0	2
14	16246	$.1983090442941E+01$	$.4483067595483E+01$	2	−2	0	0
15	22195	$.2709238720606E+01$	$.5560429624249E+01$	−1	1	0	4
16	23282	$.2841937192509E+01$	$.6142515508738E+01$	1	−1	0	2
17	24369	$.2974635664412E+01$	$.4414160860469E+00$	3	−3	0	0
18	30318	$.3700783942076E+01$	$.1518778114812E+01$	0	0	0	4
19	31405	$.3833482413979E+01$	$.2100863999301E+01$	2	−2	0	2
20	31507	$.3845956030921E+01$	$.1371800283021E+01$	2	−1	−1	2
21	32594	$.3978654502824E+01$	$.1953886167510E+01$	4	−3	−1	0

Table B.1. Values of the frequencies and phases.

FFT	Cosinus coeffficient	Sinus coeffficient
1	$4.736558433505679E-03$	$0.0E+00$
1088	$-1.442878806787767E-03$	$9.486955994339119E-04$
6424	$1.732884404452373E-04$	$1.319772662125154E-03$
7037	$-2.328750090605973E-03$	$-2.614600756885180E-02$
7511	$-7.188469510604870E-04$	$-8.334045780647701E-04$
7580	$2.527291376289149E-04$	$6.772719887250550E-04$
8124	$9.112384873999765E-02$	$.114730314343880E+00$
8328	$5.519636493992337E-04$	$-2.929283508050680E-04$
8737	$3.427934502822663E-04$	$4.720421965804989E-04$
14072	$1.505348196583156E-04$	$-1.990498598218668E-05$
14547	$-9.838830990845423E-04$	$1.041806911006114E-03$
15159	$-1.434663778272931E-02$	$1.362093368173893E-02$
16246	$2.038370016127341E-03$	$-8.718756403069540E-03$
22195	$5.837752807074912E-04$	$5.138938756818489E-04$
23282	$-3.977169383514819E-03$	$-5.612676063410181E-04$
24369	$9.015594811462270E-04$	$-4.245134436628341E-04$
30318	$-4.837067869558339E-06$	$1.571617391046836E-04$
31405	$-1.194870945858466E-04$	$-2.090091434778585E-04$

Table B.2. FFT of function $SO(1)$. Total number of terms = 18

FFT	Cosinus coeffficient	Sinus coeffficient
1	$-6.346799886093270E-03$	$0.0E+00$
1088	$1.871401381441822E-03$	$-1.229933345372316E-03$
6424	$-2.093906569989454E-04$	$-1.610294549322988E-03$
7037	$2.926146024377030E-03$	$3.282871340455004E-02$
7511	$9.223549824103249E-04$	$1.067890991980766E-03$
7580	$-3.222400272087621E-04$	$-8.689878798638993E-04$
7649	$7.062830239169652E-05$	$1.117174376136566E-04$
$e124$	$-.120996727832802E+00$	$-.152343904826198E+00$
8328	$-7.390184267466767E-04$	$3.950008491334309E-04$
8737	$-4.699097976667718E-04$	$-6.493835058819712E-04$
9211	$1.078825676320622E-04$	$1.036011623173255E-05$
14547	$1.939818528408744E-03$	$-2.048900191259839E-03$
15159	$2.887777795874210E-02$	$-2.741286236759826E-02$
16246	$-3.150038221555223E-03$	$1.347080813131899E-02$
22195	$-1.235076277776033E-03$	$-1.088438002013584E-03$
23282	$8.348499106275216E-03$	$1.178953359826934E-03$
24369	$-1.551432331495733E-03$	$7.315205923789502E-04$
30318	$-1.991186202809775E-05$	$2.910381907009060E-04$
31405	$-1.924523895413760E-04$	$-3.200122329176353E-04$

Table B.2 (Continuation). FFT of function $SO(13)$. Total number of terms = 19

FFT	Cosinus coeffficient	Sinus coeffficient
1	$1.076041971379874E-03$	$0.0E+00$
7037	$5.540968232159205E-04$	$2.576994925933965E-04$
8124	$-1.371567874626327E-04$	$2.223408884711102E-05$
14547	$9.452870104655420E-05$	$3.175589405929647E-04$
15159	$1.146265957232343E-03$	$4.775571350912270E-03$
22195	$1.297644728743069E-04$	$-4.380310979254232E-05$
23282	$-4.299773470473283E-04$	$5.490768205089372E-04$

Table B.3. FFT of function $SOS(1)$. Total number of terms = 7. Factor = 0.8

FFT	Cosinus coeffficient	Sinus coeffficient
1	$9.849081716565741E-04$	$0.0E+00$
7037	$1.291139808833131E-04$	$-2.767737044324544E-04$
14547	$1.587833276134036E-04$	$-4.736270317474194E-05$
15159	$2.387847913536772E-03$	$-5.735836085890310E-04$
23282	$2.744346740034203E-04$	$2.148819384748705E-04$

Table B.3 (Continuation). FFT of function $SOS(2)$. Total number of terms = 5. Factor = 0.4

FFT	Cosinus coeffficient	Sinus coeffficient
1	$8.608527472909735E-04$	$0.0E+00$
7037	$2.137106439286387E-05$	$2.433446750792411E-04$
14547	$-9.112096775198372E-05$	$9.635116100675502E-05$
15159	$-1.425152968611055E-03$	$1.352586742233675E-03$
23282	$-2.761985649308407E-04$	$-3.906650547346975E-05$

Table B.3 (Continuation). FFT of function $SOS(4)$. Total number of terms = 5. Factor = 0.64

FFT	Cosinus coeffficient	Sinus coeffficient
7037	$2.435908829381741E-04$	$-2.147050054569744E-05$
14547	$9.627416669766357E-05$	$9.104857039991896E-05$
15159	$1.352265753433075E-03$	$1.424840670512157E-03$
23282	$-3.913691359113510E-05$	$2.761690038753613E-04$

Table B.3 (Continuation). FFT of function $SOS(5)$. Total number of terms = 4. Factor = 0.32

FFT	Cosinus coeffficient	Sinus coeffficient
1	$2.274585745896600E-04$	$0.0E+00$
15159	$3.562730984701729E-04$	$-3.381233036219169E-04$

Table B.3 (End). FFT of function $SOS(7)$. Total number of terms = 2. Factor = 0.16

The output of the FFT as it is given has been filtered using the following criteria: Only the modulus of an harmonic is considered. If this modulus times a certain factor is greater than a certain tolerance (10^{-4}) then the term is retained. The factor has been taken equal to one for all the $SO(i)$ functions and it is explicitely given for the $SOS(i)$ functions in the tables. Its value depends on the power of x, y, z which multiplies the function in the equations.

When the frequencies are known it is possible to obtain the related coefficients by means of the Fourier integral

$$\frac{1}{T}\int_{t_0}^{t_0+T} e^{-it} f(t)dt,$$

where the time interval T is a multiple of the period associated to the frequency. The integral has been discretized using the trapezoidal rule. Several time intervals and steps have been used in the numerical integration to allow for interval checks. This task is performed by routine FURLE4 of program FURI4. The routine analyzes, for a given frequency all the functions. In the Tables B.4, B.5 and B.6 we present a sample of the results obtained by this routine for the first three frequencies.

SO(1)	.17382394166E-02	-.59997788432E-05	.47395438954E-02
SO(2)	.10254938500E-02	-.44121424690E-05	.36027986003E-02
SO(3)	.22031065393E-02	.38134680368E-04	.60072135347E-02
SO(4)	-.17639764051E-06	.19321594920E-06	-.17353689874E-07
SO(5)	-.17307793543E-05	.72200869768E-04	-.90353795223E-06
SO(6)	-.26882174545E-03	-.53853805348E-05	.32977405368E-04
SO(7)	-.13914423698E-03	.39655171023E-04	.24057388853E-04
SO(8)	.22032251981E-02	-.53343756680E-04	.60072434427E-02
SO(9)	.36333639998E-02	-.13413292051E-04	.10713307661E-01
SO(10)	.10485703027E-05	-.11465982572E-06	.41263815965E-06
SO(11)	-.92204252159E-07	-.12856628128E-06	-.90175501289E-07
SO(12)	-.24044366876E-03	-.22794948912E-04	.42731885096E-04
SO(13)	-.22537364397E-02	.78832163123E-05	-.63511715714E-02

Table B.4. Fourier coefficients of the $SO(i)$ and $SOS(i)$ functions, related to the frequency number 1. The first column corresponds to the cosinus coefficients, the second one to the sinus ones, and the last one to the independent term. The integration has been done for 8192 revolutions of the secondary and taking 8 points in each revolution.

SO(1)	.30632775366E-03	.12346476348E-02	.47341287409E-02
SO(2)	.24249374638E-03	.96888738711E-03	.36012017019E-02
SO(3)	.33831721635E-03	.15780928718E-02	.60005784389E-02
SO(4)	-.13503954382E-05	.14476202672E-05	.42641555370E-06
SO(5)	.57330463257E-03	-.14472472257E-03	-.49781903353E-05
SO(6)	.35050838445E-03	.14174950728E-02	.29956955699E-04
SO(7)	-.36878758031E-04	.32894793201E-04	.25487464934E-04
SO(8)	.43820510280E-03	.15516641720E-02	.60001514495E-02
SO(9)	.70196530135E-03	.28207723232E-02	.10703587254E-01
SO(10)	-.13869493463E-05	-.10175523278E-05	-.55271606980E-06
SO(11)	.28201432971E-05	-.27481969897E-05	-.17430992628E-05
SO(12)	.35994617897E-04	.30471828167E-04	.44751279129E-04
SO(13)	-.36918032250E-03	-.15037606804E-02	-.63420936877E-02

Table B.5. Same as Table B.4 but for the frequency number 2. Number of revolutions: 4096. Points per revolution: 16.

SO(1)	-.12594557653E-02	-.10753826679E-02	.46883313243E-02
SO(2)	-.97557845230E-03	-.83490712227E-03	.35655706288E-02
SO(3)	-.15752246597E-02	-.13900301966E-02	.59405518526E-02
SO(4)	.20009741744E-05	-.72477983103E-03	-.11795161652E-05
SO(5)	-.63634153151E-03	.67359537336E-03	-.32478296958E-04
SO(6)	-.15605501736E-02	-.13249168303E-02	-.38796341658E-04
SO(7)	.27775849432E-05	-.37580911560E-04	.23131449008E-04
SO(8)	-.16174194937E-02	-.13359997934E-02	.59440843559E-02
SO(9)	-.28647541603E-02	-.24479837772E-02	.10599262504E-01
SO(10)	.60705362417E-03	.17137791843E-05	.43077937962E-06
SO(11)	-.40866601041E-05	.14431431247E-02	.51020775047E-05
SO(12)	-.37377039494E-04	-.11063942554E-04	.44632152297E-04
SO(13)	.16058308817E-02	.13682229965E-02	-.62825187859E-02
SOS(1)	.34000091963E-04	-.10493905037E-04	.13450566027E-02
SOS(2)	-.10881522867E-04	-.34846744803E-04	.24626744913E-02
SOS(3)	.63677501510E-03	.36331197235E-03	-.33152319641E-06
SOS(4)	.13120464456E-04	.12482105204E-04	.13455711899E-02
SOS(5)	.24023399950E-04	-.26527334802E-04	-.57890491722E-06
SOS(6)	-.91503256635E-06	.72517896787E-03	.11565824739E-05
SOS(7)	-.13263767627E-04	-.12414720462E-04	.14220109638E-02
SOS(8)	.73582861048E-03	.84265032711E-06	-.10549880246E-05
SOS(9)	.14330317099E-06	-.67384742087E-07	-.27675821537E-02

Table B.6. Same as Table B.4 but for the frequency number 3. Number of revolutions: 2048. Points per revolution: 32.

With these values in hand we have performed a second Fourier analysis of the real functions minus the values obtained by the Fourier computations (routine FUN4A computes these new modified functions). This second step can be seen as a check of the results obtained and also allows the detection of other frequencies, very close to the previously detected and having a not too small amplitude. Though there appear some new frequencies it has been seen that their contribution is not significant. A sample of the results obtained by routine FURLEN4 for these modified functions is given in the Tables B.7, B.8 and B.9 for the first three frequencies. It can be seen that the coefficients obtained for the cosinus and sinus terms are rather small.

$SO(1)$	$.35568417572E-06$	$-.40322419498E-06$	$.28559174786E-06$
$SO(2)$	$.70115792837E-06$	$.38139633818E-06$	$-.29681465409E-06$
$SO(3)$	$.12146877298E-07$	$.48569426731E-06$	$.64051007480E-06$
$SO(4)$	$-.15731356753E-06$	$.23708440564E-06$	$.16329791151E-08$
$SO(5)$	$-.21103029284E-06$	$.75922810896E-06$	$.23284808734E-06$
$SO(6)$	$-.38749253552E-06$	$-.97659341736E-07$	$.14632035060E-06$
$SO(7)$	$.32141020623E-07$	$.48594035631E-06$	$.39486426620E-09$
$SO(8)$	$-.20011280866E-06$	$-.49103131773E-06$	$.62693609842E-06$
$SO(9)$	$.24369851555E-06$	$-.25774766661E-06$	$-.89674748628E-04$
$SO(10)$	$.13989979809E-06$	$.11957476295E-06$	$-.18440157474E-07$
$SO(11)$	$.41108634080E-06$	$-.60856366639E-06$	$.17516324602E-08$
$SO(12)$	$-.12746328656E-05$	$-.26368164664E-07$	$-.41002471794E-06$
$SO(13)$	$-.73948979187E-07$	$.35524554515E-06$	$-.38449423876E-06$

Table B.7. Fourier coefficients of the modified $SO(i)$ and $SOS(i)$ functions, related to the frequency number 1. The integration has been done as in Table B.4 .

$SO(1)$	$-.31467465776E-06$	$.70569124026E-07$	$.69848479382E-06$
$SO(2)$	$.23324209725E-06$	$.41624917891E-06$	$-.29517876782E-06$
$SO(3)$	$.33006088003E-06$	$-.56178918144E-07$	$.11569840516E-05$
$SO(4)$	$-.52035238471E-07$	$.67768182828E-07$	$-.71518814952E-07$
$SO(5)$	$-.40097876687E-06$	$-.32471236087E-06$	$.32022021226E-08$
$SO(6)$	$-.22078701162E-06$	$.45483402084E-06$	$.31237004204E-06$
$SO(7)$	$-.19852744985E-06$	$.30875099175E-08$	$-.16030402325E-06$
$SO(8)$	$-.39933817738E-07$	$-.18197969459E-06$	$.11573071351E-05$
$SO(9)$	$.24188191871E-06$	$-.59286057368E-07$	$-.89053298322E-04$
$SO(10)$	$-.61594687680E-07$	$-.41949297687E-07$	$.63634463143E-07$
$SO(11)$	$.80007061269E-07$	$-.15246825854E-06$	$.27512674965E-06$
$SO(12)$	$.19118334596E-06$	$.41542217047E-06$	$-.65647123209E-06$
$SO(13)$	$.46797557741E-06$	$-.44283782648E-06$	$-.94246025229E-06$

Table B.8. Same as Table B.7 but for the frequency number 2. The integration as in Table B.5.

$SO(1)$	$.83862202057E-07$	$.12053975395E-06$	$.75000519757E-06$
$SO(2)$	$.15084988671E-06$	$.21198080749E-06$	$.41922643094E-07$
$SO(3)$	$.10131300913E-06$	$.78747132178E-07$	$.12087952354E-05$
$SO(4)$	$-.14977378829E-07$	$-.45068790493E-06$	$.17041202246E-06$
$SO(5)$	$.45668273592E-07$	$-.42364167443E-07$	$.77235136276E-08$
$SO(6)$	$.21389373857E-06$	$.16964795833E-06$	$.61346136009E-06$
$SO(7)$	$.47234336885E-07$	$-.11750549388E-07$	$-.19035975416E-07$
$SO(8)$	$.11483834669E-06$	$.23676579824E-06$	$.12361078470E-05$
$SO(9)$	$.28735451872E-06$	$.14052713876E-05$	$-.88639938983E-04$
$SO(10)$	$.74150799164E-06$	$.25949499804E-08$	$-.11900690689E-06$
$SO(11)$	$.86156201155E-07$	$.36440373236E-06$	$-.67459315737E-06$
$SO(12)$	$.58220691088E-05$	$.16316368674E-05$	$-.32477746550E-06$
$SO(13)$	$-.98281403861E-07$	$-.16817065629E-06$	$-.10120677298E-05$

Table B.9. Same as Table B.7 but for the frequency number 3. The integration as in Table B.6.

B.7 Simplified Normalized Equations

We shall use a simplified version of the equations of motion that retains the terms called $SO(i)$ and $SOS(i)$ and the ones coming from the RTBP.

Let us introduce some auxiliary functions $P(i)$, i=1,...,20 which account for the terms appearing in $SO(i)$ and $SOS(i)$ according to the following definition

$$\begin{aligned}
P(1) &= SOS(1) + SO(7), \\
P(2) &= 1 + 2SOS(4) + 0.25SO(1) + \alpha SO(2), \\
P(3) &= SOS(5) + SO(3) + \alpha SO(1), \\
P(4) &= SOS(6) + SO(4), \\
P(5) &= SO(5), \\
P(6) &= 2 + SO(6), \\
P(7) &= 1 + SO(1), \\
P(8) &= SOS(2) + SO(12), \\
P(9) &= SOS(5) + SO(8) + \alpha SO(1), \\
P(10) &= 1 + 2SOS(7) + SO(9) - 1.25SO(1), \\
P(11) &= SOS(8) + SO(10), \\
P(12) &= -2 - SO(6), \\
P(13) &= SO(5), \\
P(14) &= SO(11),
\end{aligned}$$

$$
\begin{aligned}
P(15) &= SOS(3) + y_E SO(10) + 0.5SO(4), \\
P(16) &= SOS(6) + SO(4), \\
P(17) &= SOS(8) - SO(10), \\
P(18) &= 2SOS(9) + SO(13) + SO(1), \\
P(19) &= -SO(11), \\
P(20) &= SO(5),
\end{aligned}
$$

where α stands for $1.5y_E(1-2\mu_M)$. With these definitions the functions $P(i)$ can be expressed as

$$P(i) = A_{i,0} + \sum_{j=1}^{m} A_{i,j}\cos\theta_j + \sum_{j=1}^{m} B_{i,j}\sin\theta_j,$$

where $\theta_j = \nu_j t + \varphi_j$, with ν_j, φ_j being the frequency and phase as given in the preceeding section, and t denotes the normalized time. The values of the coefficients, given in a suitable way to be used by the program that computes quasiperiodic orbits, are given in Table B.10 .

1						.00137	2	0	0	0	4	0	.00102
1	2	−2	0	−2	0	−.00014	2	2	−2	0	2	0	.00109
1	−1	1	0	2	1	.00158	2	4	−3	−1	0	0	.00006
1	1	−1	0	0	0	.00031	3						.00000
1	1	−1	0	0	1	.00017	3	−2	1	0	3	0	−.00005
1	−2	2	0	4	0	.00033	3	−1	1	0	2	1	.00181
1	−2	2	0	4	1	−.00031	3	1	−1	0	0	1	.00020
1	−1	0	0	3	0	.00086	3	−2	2	0	4	1	−.00036
1	−1	0	0	3	1	−.00020	3	−1	0	0	3	0	.00100
1	0	0	0	2	1	−.01428	3	−1	0	0	3	1	−.00021
1	−1	1	0	4	0	−.00011	3	0	0	0	2	1	−.01649
1	−1	1	0	4	1	−.00015	3	−1	1	0	4	1	−.00017
1	1	−1	0	2	1	−.00154	3	1	−1	0	2	1	−.00179
1	0	0	0	4	0	.00042	3	0	0	0	4	1	.00014
1	0	0	0	4	1	.00011	3	2	−2	0	2	0	−.00006
1	2	−2	0	2	0	.00037	3	2	−2	0	2	1	.00032
1	2	−2	0	2	1	.00026	3	2	−1	−1	2	0	.00009
1	2	−1	−1	2	0	.00015	3	4	−3	−1	0	1	−.00006
1	2	−1	−1	2	1	−.00005	4						.00000
2						1.00759	4	2	−1	−1	2	1	−.00102
2	2	−2	0	−2	0	.00145	4	4	−3	−1	0	1	−.00029
2	2	−2	0	−2	1	−.00010	5						.00000
2	−2	1	0	3	0	.00031	5	2	−2	0	−2	1	.00007
2	−2	1	0	3	1	.00136	5	−2	1	0	3	0	.00062
2	−1	1	0	2	0	.03148	5	−2	1	0	3	1	−.00014
2	0	−1	0	1	0	.00023	5	−1	1	0	2	1	−.01686
2	0	−1	0	1	1	.00098	5	0	−1	0	1	0	.00060
2	0	0	0	1	0	−.00089	5	0	−1	0	1	1	−.00014
2	1	−1	0	0	0	.16503	5	0	0	0	1	1	.00053
2	1	1	−2	0	0	−.00062	5	1	−1	0	0	1	−.10793
2	2	−1	0	−1	0	−.00018	5	1	1	−2	0	1	.00043
2	2	−1	0	−1	1	.00081	5	2	−1	0	−1	0	.00058
2	−2	2	0	4	0	.00084	5	2	−1	0	−1	1	.00013
2	−1	0	0	3	0	.00028	5	−2	2	0	4	1	.00010
2	−1	0	0	3	1	.00122	5	−1	0	0	3	0	.00190
2	0	0	0	2	0	.02665	5	−1	0	0	3	1	−.00044
2	2	−2	0	0	0	.01351	5	0	0	0	2	1	−.02947
2	−1	1	0	4	0	.00057	5	2	−2	0	0	1	−.00882
2	1	−1	0	2	0	.00423	5	−1	1	0	4	1	−.00074
2	3	−3	0	0	0	.00107	5	1	−1	0	2	1	−.00382

5	3	−3	0	0	1	−.00068	7	2	−2	0	0	0	.01344
5	0	0	0	4	1	.00017	7	−1	1	0	4	0	.00079
5	2	−2	0	2	1	.00024	7	1	−1	0	2	0	.00417
6						2.00003	7	3	−3	0	0	0	.00107
6	2	−2	0	−2	0	−.00027	7	0	0	0	4	0	.00016
6	−2	1	0	3	0	.00035	7	2	−2	0	2	0	.00025
6	−2	1	0	3	1	.00152	8						.00251
6	−1	1	0	2	0	.03817	8	2	−2	0	−2	0	−.00024
6	0	−1	0	1	0	.00029	8	−1	1	0	2	0	.00165
6	0	−1	0	1	1	.00127	8	1	−1	0	0	0	.00054
6	0	0	0	1	0	−.00112	8	1	−1	0	0	1	−.00010
6	1	−1	0	0	0	.21765	8	−2	2	0	4	0	.00057
6	1	1	−2	0	0	−.00083	8	−2	2	0	4	1	.00017
6	2	−1	0	−1	0	−.00025	8	−1	0	0	3	0	−.00020
6	2	−1	0	−1	1	.00110	8	−1	0	0	3	1	−.00090
6	−2	2	0	4	0	.00051	8	0	0	0	2	0	−.01435
6	−1	0	0	3	0	.00064	8	2	−2	0	0	0	.00006
6	−1	0	0	3	1	.00279	8	−1	1	0	4	0	−.00018
6	0	0	0	2	0	.04294	8	−1	1	0	4	1	.00008
6	2	−2	0	0	0	.01481	8	1	−1	0	2	0	−.00161
6	−1	1	0	4	0	.00101	8	0	0	0	4	0	.00046
6	1	−1	0	2	0	.00535	8	0	0	0	4	1	−.00023
6	3	−3	0	0	0	.00104	8	2	−2	0	2	0	.00076
7						1.00474	8	2	−2	0	2	1	−.00019
7	2	−2	0	−2	0	.00173	8	2	−1	−1	2	1	.00005
7	2	−2	0	−2	1	−.00011	9						.00000
7	−2	1	0	3	0	.00030	9	−2	1	0	3	0	.00005
7	−2	1	0	3	1	.00132	9	1	−1	0	0	1	−.00020
7	−1	1	0	2	0	.03149	9	−2	2	0	4	1	.00036
7	0	−1	0	1	0	.00023	9	−1	1	0	4	1	.00017
7	0	−1	0	1	1	.00100	9	0	0	0	4	1	−.00034
7	0	0	0	1	0	−.00086	9	2	−2	0	2	0	.00006
7	1	−1	0	0	0	.16441	9	2	−2	0	2	1	−.00032
7	1	1	−2	0	0	−.00063	9	2	−1	−1	2	0	−.00009
7	2	−1	0	−1	0	−.00018	9	4	−3	−1	0	1	.00005
7	2	−1	0	−1	1	.00080	10						1.00768
7	−2	2	0	4	0	.00019	10	2	−2	0	−2	0	.00145
7	−1	0	0	3	0	.00040	10	2	−2	0	−2	1	−.00010
7	−1	0	0	3	1	.00172	10	−2	1	0	3	0	.00031
7	0	0	0	2	0	.02658	10	−2	1	0	3	1	.00136

10	−1	1	0	2	0	.03348	12	−1	0	0	3	0	−.00064
10	0	−1	0	1	0	.00023	12	−1	0	0	3	1	−.00279
10	0	−1	0	1	1	.00099	12	0	0	0	2	0	−.04294
10	0	0	0	1	0	−.00088	12	2	−2	0	0	0	−.01481
10	1	−1	0	0	0	.16504	12	−1	1	0	4	0	−.00101
10	1	1	−2	0	0	−.00062	12	1	−1	0	2	0	−.00535
10	2	−1	0	−1	0	−.00019	12	3	−3	0	0	0	−.00104
10	2	−1	0	−1	1	.00080	13						.00000
10	−2	2	0	4	0	.00084	13	2	−2	0	−2	1	.00007
10	−1	0	0	3	0	.00028	13	−2	1	0	3	0	.00062
10	−1	0	0	3	1	.00121	13	−2	1	0	3	1	−.00014
10	0	0	0	2	0	.01014	13	−1	1	0	2	1	−.01686
10	2	−2	0	0	0	.01351	13	0	−1	0	1	0	.00060
10	−1	1	0	4	0	.00057	13	0	−1	0	1	1	−.00014
10	1	−1	0	2	0	.00223	13	0	0	0	1	1	.00053
10	3	−3	0	0	0	.00107	13	1	−1	0	0	1	−.10793
10	0	0	0	4	0	.00063	13	1	1	−2	0	1	.00043
10	2	−2	0	2	0	.00109	13	2	−1	0	−1	0	.00058
10	2	−1	−1	2	1	.00007	13	2	−1	0	−1	1	.00013
10	4	−3	−1	0	0	.00007	13	−2	2	0	4	1	.00010
11						.00000	13	−1	0	0	3	0	.00190
11	−1	0	1	2	0	.00131	13	−1	0	0	3	1	−.00044
11	1	0	−1	0	0	−.00142	13	0	0	0	2	1	−.02947
11	0	0	0	4	0	.00008	13	2	−2	0	0	1	−.00882
11	0	0	0	4	1	.00006	13	−1	1	0	4	1	−.00074
11	2	−1	−1	2	0	.00391	13	1	−1	0	2	1	−.00382
11	4	−3	−1	0	0	.00116	13	3	−3	0	0	1	−.00068
12						−2.00003	13	0	0	0	4	1	.00017
12	2	−2	0	−2	0	.00027	13	2	−2	0	2	1	.00024
12	−2	1	0	3	0	−.00035	14						.00000
12	−2	1	0	3	1	−.00152	14	−1	0	1	2	1	.00144
12	−1	1	0	2	0	−.03817	14	1	0	−1	0	1	−.00145
12	0	−1	0	1	0	−.00029	14	2	−1	−1	2	1	.00203
12	0	−1	0	1	1	−.00127	14	4	−3	−1	0	1	.00058
12	0	0	0	1	0	.00112	15						.00000
12	1	−1	0	0	0	−.21765	15	−1	0	1	2	0	.00007
12	1	1	−2	0	0	.00083	15	0	0	0	4	0	−.00006
12	2	−1	0	−1	0	.00025	15	0	0	0	4	1	−.00006
12	2	−1	0	−1	1	−.00110	15	2	−1	−1	2	0	−.00339
12	−2	2	0	4	0	−.00051	15	2	−1	−1	2	1	−.00050

15	4	−3	−1	0	0	−.00101	18	0	0	0	4	0	.00046
15	4	−3	−1	0	1	−.00018	18	2	−2	0	2	0	.00063
16						.00000	18	2	−1	−1	2	0	.00008
16	2	−1	−1	2	1	−.00102	18	2	−1	−1	2	1	−.00008
16	4	−3	−1	0	1	−.00029	19						.00000
17						.00000	19	−1	0	1	2	1	−.00144
17	−1	0	1	2	0	.00009	19	1	0	−1	0	1	.00145
17	0	0	0	4	0	−.00008	19	2	−1	−1	2	1	−.00203
17	0	0	0	4	1	−.00006	19	4	−3	−1	0	1	−.00058
17	2	−1	−1	2	0	−.00391	20						.00000
17	4	−3	−1	0	0	−.00116	20	2	−2	0	−2	1	.00007
18						−.00721	20	−2	1	0	3	0	.00062
18	2	−2	0	−2	0	−.00051	20	−2	1	0	3	1	−.00014
18	−2	1	0	3	0	−.00007	20	−1	1	0	2	1	−.01686
18	−2	1	0	3	1	−.00029	20	0	−1	0	1	0	.00060
18	−1	1	0	2	0	−.00805	20	0	−1	0	1	1	−.00014
18	0	−1	0	1	0	−.00006	20	0	0	0	1	1	.00053
18	0	−1	0	1	1	−.00028	20	1	−1	0	0	1	−.10793
18	0	0	0	1	0	.00024	20	1	1	−2	0	1	.00043
18	1	−1	0	0	0	−.05390	20	2	−1	0	−1	0	.00058
18	1	1	−2	0	0	.00022	20	2	−1	0	−1	1	.00013
18	2	−1	0	−1	0	.00007	20	−2	2	0	4	1	.00010
18	2	−1	0	−1	1	−.00030	20	−1	0	0	3	0	.00190
18	−2	2	0	4	0	.00010	20	−1	0	0	3	1	−.00044
18	−1	0	0	3	0	−.00038	20	0	0	0	2	1	−.02947
18	−1	0	0	3	1	−.00166	20	2	−2	0	0	1	−.00882
18	0	0	0	2	0	−.02691	20	−1	1	0	4	1	−.00074
18	2	−2	0	0	0	−.00733	20	1	−1	0	2	1	−.00382
18	−1	1	0	4	0	−.00088	20	3	−3	0	0	1	−.00068
18	1	−1	0	2	0	−.00459	20	0	0	0	4	1	.00017
18	3	−3	0	0	0	−.00078	20	2	−2	0	2	1	.00024

Table B.10. Values of $A_{i,j}$ and $B_{i,j}$. The first index gives the number, i, of the function $P(i)$. The four next indices denote the linear combination of the four basic frequencies. The fifth one is 0 for the $A_{i,j}$ coefficients (cosinus terms) and 1 for the $B_{i,j}$ ones (sinus terms). Only coefficients greater than 5×10^{-5} have been retained.

With this notation the equations of motion written in a simplified form are:

$$\begin{aligned}\ddot{x} &= P(7)\left[-\frac{x - x_E}{r_{PE}^3}(1-\mu_M) - \frac{x + x_E}{r_{PM}^3}\mu_M - x_E(1-2\mu_M)\right] + \\ &\quad + P(1) + P(2)x + P(3)y + P(4)z + P(5)\dot{x} + P(6)\dot{y},\end{aligned}$$

$$\begin{aligned}
\ddot{y} &= P(7)\left[-\frac{y-y_E}{r_{PE}^3}(1-\mu_M)-\frac{y-y_E}{r_{PM}^3}\mu_M-y_E\right]+P(8)+P(9)x+\\
&\quad +P(10)y+P(11)z+P(12)\dot{x}+P(13)\dot{y}+P(14)\dot{z},\\
\ddot{z} &= P(7)\left[-\frac{z}{r_{PE}^3}(1-\mu_M)-\frac{z}{r_{PM}^3}\mu_M\right]+P(15)+P(16)x+P(17)y+\\
&\quad +P(18)z+P(19)\dot{y}+P(20)\dot{z},
\end{aligned}$$

where r_{PE}, r_{PM} denote the distances from the particle to the Earth and Moon, respectively, given by $r_{PE}^2=(x-x_E)^2+(y-y_E)^2+z^2$, $r_{PM}^2=(x+x_E)^2+(y-y_E)^2+z^2$. We recall that $x_E=-1/2$, $y_E=-\sqrt{3}/2$ for L_4 and $x_E=-1/2$, $y_E=\sqrt{3}/2$ for L_5. Finally, μ_M is 0.012150298.

B.8 Numerical Tests

To be sure that, the simplified system is meaningful we have written a test program (FUCHE4) such that given initial conditions in normalized coordinates, transforms them to ecliptic ones. Then computes the vectorfield on the particle in ecliptic coordinates using the full solar system (JPL ephemeris) and after performs the transformation back to normalized coordinates. On the other hand we can evaluate directly the simplified vectorfield (already in normalized coordinates). Then we can select either a given point in the neighborhood of L_4 in the phase space (introducing the position and velocity) or some points from a quasiperiodic orbit (we have used the planar one mentioned in Section 3.3)

The biggest difference between the two vectorfields in normalized coordinates, is of the order of 10^{-3}. This is due to the fact that we are using a simplified version of the equations in which the time dependent coefficients are functions of an small set of frequencies (21). In the other hand the amplitudes related to these frequencies have been computed using an analytical model of the solar system and here the tests have been done against a numerical one. In any case it seems reasonable to believe that the orbits found using the simplified equations can be easely modified to satisfy the complete system of equations. Below we give some results of these tests.

First we give the Modified Julian Day (origin at 1950.0). Then the initial position and velocity in normalized units. The vectorfields computed by Newton's law using the JPL ephemeris for the full solar system, and by the simplified model are given. Finally the differences are printed.

Day =18000.0

Position			Velocity		
.4331E-02	.4603E-02	.0000E+00	.7472E-02	.4518E-01	.0000E+00
JPL			Fourier		
.121E+00	-9.548E-04	8.522E-04	.123E+00	2.325E-03	4.896E-04

Differences

2.190E-03 3.280E-03 -3.625E-04

Day =18004.0

Position			Velocity		
.2308E-01	-.9956E-02	.0000E+00	.4378E-02	-.5727E-01	.0000E+00
JPL			Fourier		
-.134E+00	-7.246E-03	-1.548E-03	-.133E+00	-5.169E-03	-2.223E-03

Differences

1.309E-03 2.076E-03 -6.751E-04

Day =18008.0

Position			Velocity		
.2546E-01	-.6202E-01	.0000E+00	.8812E-02	-.3001E-01	.0000E+00
JPL			Fourier		
-.140E+00	-.113E+00	2.610E-03	-.138E+00	-.113E+00	3.198E-03

Differences

2.155E-03 7.484E-04 5.880E-04

Day =18012.0

Position			Velocity		
.1285E-01	-.2503E-01	.0000E+00	-.4244E-01	.9153E-01	.0000E+00
JPL			Fourier		
.170E+00	3.792E-02	-3.931E-03	.171E+00	4.056E-02	-3.941E-03

Differences

1.216E-03 2.634E-03 -9.854E-06

Day =18016.0

Position			Velocity		
.1858E-01	.5904E-02	.0000E+00	.2784E-01	-.2869E-01	.0000E+00
JPL			Fourier		
-3.038E-02	-3.016E-02	3.893E-03	-2.978E-02	-2.862E-02	4.030E-03

Differences
5.976E-04 1.547E-03 1.364E-04

Day =18020.0

Position			Velocity		
.2124E-01	-.4192E-01	.0000E+00	.3842E-02	-.6128E-01	.0000E+00
JPL			Fourier		
-.157E+00	-7.021E-02	-2.786E-03	-.156E+00	-6.807E-02	-3.148E-03

Differences
5.854E-04 2.135E-03 -3.624E-04

Day =18024.0

Position			Velocity		
.2540E-01	-.5558E-01	.0000E+00	-.2128E-01	.5774E-01	.0000E+00
JPL			Fourier		
5.146E-02	-2.504E-02	1.852E-03	5.331E-02	-2.191E-02	2.032E-03

Differences
1.848E-03 3.132E-03 1.802E-04

Day =18028.0

Position			Velocity		
.7192E-02	.7655E-02	.0000E+00	.2130E-01	.2310E-01	.0000E+00
JPL			Fourier		
8.352E-02	-1.359E-02	-1.498E-04	8.516E-02	-1.070E-02	-1.413E-04

Differences
1.638E-03 2.886E-03 8.434E-06

Day =18032.0

	Position			Velocity	
.2201E-01	-.1775E-01	.0000E+00	-.4279E-02	-.5752E-01	.0000E+00
	JPL			Fourier	
-.141E+00	-1.437E-02	-1.393E-03	-.140E+00	-1.164E-02	-1.664E-03

Differences

1.151E-03	2.727E-03	-2.703E-04

Day =18140.0

	Position			Velocity	
.1994E-01	-.3135E-02	.0000E+00	.7127E-01	-.3114E-01	.0000E+00
	JPL			Fourier	
-6.424E-02	1.317E-02	-1.269E-03	-6.223E-02	1.616E-02	-2.325E-03

Differences

2.012E-03	2.983E-03	-1.055E-03

Day =18144.0

	Position			Velocity	
.7461E-02	-.3875E-01	.0000E+00	-.1026E-01	-.3574E-01	.0000E+00
	JPL			Fourier	
-.116E+00	-5.530E-02	6.615E-04	-.115E+00	-5.254E-02	1.920E-03

Differences

1.311E-03	2.752E-03	1.259E-03

Day =18148.0

	Position			Velocity	
.1485E-01	-.4205E-01	.0000E+00	.3903E-02	.4640E-01	.0000E+00
	JPL			Fourier	
5.352E-02	-.101E+00	-2.899E-04	5.459E-02	-9.964E-02	-1.476E-03

Differences

1.064E-03	1.606E-03	-1.186E-03

Day =18152.0

Position			Velocity		
.1155E-01	.3235E-02	.0000E+00	.2604E-01	.1447E-03	.0000E+00
JPL			Fourier		
-4.164E-03	-3.256E-02	9.665E-04	-2.599E-03	-3.118E-02	9.921E-04

Differences

1.564E-03 1.386E-03 2.555E-05

Day =18156.0

Position			Velocity		
.1307E-01	-.2273E-01	.0000E+00	-.2854E-01	-.3552E-01	.0000E+00
JPL			Fourier		
-8.436E-02	3.492E-02	-4.509E-04	-8.362E-02	3.737E-02	-6.057E-07

Differences

7.454E-04 2.450E-03 4.503E-04

Day =18160.0

Position			Velocity		
.9541E-02	-.4964E-01	.0000E+00	.1933E-01	-.3819E-02	.0000E+00
JPL			Fourier		
-4.872E-02	-.124E+00	-2.264E-04	-4.747E-02	-.122E+00	-7.410E-04

Differences

1.251E-03 2.331E-03 -5.145E-04

Day =18164.0

Position			Velocity		
.9203E-02	-.9062E-02	.0000E+00	-.6827E-02	.5432E-01	.0000E+00
JPL			Fourier		
.111E+00	-3.098E-03	1.634E-03	.113E+00	-7.480E-04	1.453E-03

Differences

2.305E-03 2.350E-03 -1.814E-04

Day =18168.0

Position			Velocity		
.1966E-01	-.8469E-02	.0000E+00	-.5251E-02	-.3387E-01	.0000E+00

JPL			Fourier		
-8.316E-02	1.858E-02	-2.475E-03	-8.127E-02	2.203E-02	-2.419E-03

Differences		
1.891E-03	3.452E-03	5.597E-05

Day =18172.0

Position			Velocity		
.3391E-02	-.4040E-01	.0000E+00	-.5969E-02	-.2705E-01	.0000E+00

JPL			Fourier		
-.108E+00	-6.761E-02	2.435E-03	-.106E+00	-6.473E-02	2.653E-03

Differences		
1.544E-03	2.879E-03	2.186E-04

Appendix C

Proof of Theorem 2.3

Here, the reader can find the technical details concerning the proof of Theorem 2.3. As it has already been mentioned, this proof is very similar to the one of the KAM Theorem that V. I. Arnol'd did in [1]. For this reason, we have closely followed that proof, only changing those (small) parts that are different. We have used the same notation of [1], and the basic lemmas can also be found therein.

C.1 Previous Lemmas and Theorems

In this section, the reader will find the lemmas and theorems used to prove Theorem 2.3.

Lemma C.1 *Consider the function* $H(p,q) = (\varphi, p_1) + H_0(p_2) + H_1(p_2, q_1, q_2)$, *the domains* F, G, Ω, *and the positive numbers* Θ, θ, ρ, β, γ, δ, M, K. *We suppose that*

1. *In the domain* $F = \{(p,q) \,/\, p = (p_1, p_2) \in G = G^1 \times G^2,\ |\mathrm{Im}\, q| \leq \rho\}$ *the function*

$$H(p,q) = \overline{H}(p) + \widetilde{H}(p,q)$$

is analytic and

$$|\widetilde{H}(p,q)| \leq M, \qquad \oint \widetilde{H}(p,q)\, dq = 0.$$

Moreover, if A *is the diffeomorphism* $p_2 \mapsto \omega$ *of* G^2 *onto the domain* Ω *(whose points are* $\omega = \frac{\partial \overline{H}}{\partial p_2}$*), then*

$$\theta |dp_2| \leq |dA| \leq \Theta |dp_2| \quad (0 < \theta < 1 < \Theta < \infty).$$

2. *The numbers* β, γ, δ, K *satisfy the inequality*

$$\delta < \delta^{(0)}(n, \Theta) = \min\{L_1^{-1}, L_2^{-1}, L_3^{-1}\Theta^{-1}, L_4^{-1}\},$$

$$10\delta < 2\gamma < \varrho \leq 1, \quad 3\beta < 2\delta, \quad 2\beta < K,$$

where $L_i(n)$ are defined by (C.4).

3. $M < \delta^\nu K \beta^2$, *where* $\nu = 2n + 3$.

We set $N = \frac{1}{\gamma} \log \frac{1}{M}$ and $G^2_{KN} = A^{-1}\Omega_{KN}$, where Ω_{KN} consists of those ω for which $|(\phi, k)| \geq K|k|^{-(n+1)}$ for all integral k, $0 < |k| < N$. Finally, G_{KN} is defined as $G^1 \times G^2_{KN}$.

Then in the domain $P \in G_{KN} - 2\beta$, $|\text{Im } Q| \leq \rho - 2\gamma$ there exists a diffeomorphism $B : P, Q \to p, q$ such that

1. $|B - E| < \beta$, $|dB| < 2|dx|$ $(x = P, Q)$.
2. $H(p, q) = \overline{H}(P) + H_2(P, Q)$ *where* $(p, q) = B(P, Q)$ *and for* $P \in G_{KN} - 2\beta$, $|\text{Im } Q| \leq \varrho - 2\gamma$, $|H_2(P, Q)| < M^2 \delta^{-2\nu} \beta^{-2}$.

Proof:

1.- The canonical transformation with generating function $Pq + S(P, q)$

$$p = P + S_q, \quad Q = q + S_P, \tag{C.1}$$

takes $H(p, q)$ into the form

$$H(p, q) = \overline{H}(P) + \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4(P, Q), \tag{C.2}$$

where

$$\begin{aligned} \Sigma_1 &= (\phi(P), S_q) + [\widetilde{H}(P, q)]_N, \\ \Sigma_2 &= \overline{H}(p) - \overline{H}(P) - \left(p - P, \frac{\partial \overline{H}}{\partial P}\right), \\ \Sigma_3 &= \widetilde{H}(P, q) - [\widetilde{H}(P, q)]_N, \\ \Sigma_4 &= \widetilde{H}(p, q) - \widetilde{H}(P, q), \end{aligned}$$

$$\widetilde{H}(P, q) = \sum_{k \neq 0} h_k(P) e^{\sqrt{-1}(k, q)},$$

$$[\widetilde{H}(P, q)]_N = \sum_{0 < |k| < N} h_k(P) e^{\sqrt{-1}(k, q)},$$

and $\phi(P)$ states for the vector $\varphi, \omega(P)$, where

$$\omega(P) = \omega(P_2) = \frac{\partial \overline{H}(P)}{\partial P_2}.$$

The variables p, q are given in terms of P, Q by (C.1).

2.- In order that $\Sigma_1 \equiv 0$ we set

$$S(P,q) = \sum_{0<|k|<N} S_k(P)e^{\sqrt{-1}(k,q)},$$

where

$$S_k(P) = \frac{h_k(P)}{(\phi(P),k)}\sqrt{-1} = S_k(P_2). \tag{C.3}$$

For $P_2 \in G^2_{KN}$ we have $|(\phi,k)| \geq K|k|^{-(n+1)}$ $(0 < |k| < N)$. By 2^{O}A of 4.2 (see [1]) and hypothesis 1 of this lemma we have $|h_k| \leq Me^{-|k|\rho}$. Hence, by (8) 1^{O} of 4.2 (see [1]) we have

$$|S_k(P)| \leq \frac{Me^{-|k|\rho}}{K}\frac{L_0}{\delta^{\nu_1}}e^{|k|\delta} = M\frac{L_0}{K\delta^{\nu_1}}e^{-|k|(\rho-\delta)},$$

where $\nu_1 = n+1$, $L_0 = \nu_1^{\nu_1}e^{-\nu_1}$ and so by 2^{O}B of 4.2 (see [1]) we have, for $P_2 \in G^2_{KN}$ and $|\text{Im } q| \leq \rho - 2\delta$,

$$|S(P,q)| < \frac{ML_5}{K\delta^{\nu_2}} \quad (L_5 = 4^n L_0,\ \nu_2 = 2n+1).$$

3.- Since $M < \delta^\nu K\beta^2$, $\delta < L_2^{-1}$ $(L_2 = 16nL_5,\ \nu = 2n+3)$ we have

$$\frac{ML_5}{K\delta^{\nu_2}} < \frac{\beta^2}{16n}.$$

Thus, by 3^{O} of 4.3 (see [1]) the equations (C.1) define a canonical diffeomorphism B of the domain

$$P \in G_{KN} - 2\beta,\quad |\text{Im } Q| \leq \varrho - 5\delta < \varrho - 2\delta - 3\beta$$

(since $3\beta \leq 2\delta$), moreover

$$|B - E| < \frac{ML_5}{K\beta\delta^{\nu_2}} < \beta,\ |dB| < 2|dx|,\ |P - p| < \frac{ML_5}{K\delta^{\nu_2+1}},\ (x = P, Q).$$

4.- We estimate the quantity Σ_2 by Taylor's formula (4^{O} of 4.2 (see [1])). If

$$P_2 \in G^2_{KN} - 2\beta,\ |\text{Im } Q| \leq \varrho - 5\delta,$$

then from $\left|\frac{\partial^2 \overline{H}}{\partial P^2}\right| \leq \Theta$ (because $|\frac{dA}{dP_2}| \leq \Theta$) it follows (in view of $\delta < L_3^{-1}\Theta^{-1}$, $L_3 = \frac{1}{2}L_5^2 n^2$) that

$$|\Sigma_2| \leq \frac{M^2}{K^2}\frac{\Theta n^2 L_5^2}{2\delta^{2\nu_2+2}} < \frac{M^2}{K^2\delta^{2\nu_2+3}}.$$

5.- The estimate for Σ_3 is established in C) 2^0 4.2 (see [1]). Since $|h_k| \leq Me^{-|k|\rho}$, for $P \in G$, $|\text{Im } Q| \leq \rho - \gamma - \delta$, $N = \frac{1}{\gamma} \log \frac{1}{M}$ we have (in view of $\delta < L_4^{-1}$, $L_4 = 2(\frac{2n}{e})^n$)

$$|\Sigma_3| < \frac{M^2 L_4}{\delta^{\nu_1}} < \frac{M^2}{\delta^{\nu_1+1}}.$$

6.- The estimate for Σ_4 is obtained from the Lagrange formula (4^0 of 4.2 (see [1])). From

$$P \in G_{KN} - 2\beta, \quad |\text{Im } Q| \leq \varrho - 5\delta,$$

There follows

$$|\text{Im } q| \leq \varrho - 4\delta \text{ and } |P - p| < \beta,$$

and so

$$P_2 + \lambda(p_2 - P_2) \in G_{KN}^2 - \beta \ \ (|\lambda| \leq 1),$$

so that by Cauchy (3^0 4.2 (see [1])),

$$\left|\frac{\partial \widetilde{H}}{\partial P}\right| \leq \frac{M}{\beta}.$$

For $\delta < L_2^{-1}$, $L_2 = 16nL_5$ we have

$$|\Sigma_4| \leq \frac{M^2 L_5 n}{K\beta\delta^{\nu_2+1}} < \frac{M^2}{K\beta\delta^{\nu_2+2}}.$$

7.- Combining the estimates for Σ_2, Σ_3, Σ_4 and using the conditions $2\beta < K$, $\delta < L_1^{-1} = 1/12$, $\gamma > 3\delta$, $\nu = 2n+3$, $\nu_1 = n+1$, $\nu_2 = 2n+1$ we obtain, for $P_2 \in G_{KN}^2 - 2\beta$, $|\text{Im } Q| < \rho - 2\delta \leq \rho - 5\delta - \gamma$ the inequalities

$$|\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4| < M^2 \left[\frac{\delta^{-(2\nu_2+3)}}{K^2} + \delta^{-(\nu_1+1)} + \frac{\delta^{-(\nu_2+2)}}{K\beta}\right] < M^2\delta^{-2\nu}\beta^{-2},$$

as required. The constants L_i are given by

$$\begin{gathered} L_1 = 12, \ L_2 = 4^{n+2} n e^{-(n+1)}(n+1)^{n+1}, \\ L_3 = 2^{4n-1} n^2 e^{-(2n+2)}(n+1)^{2n+2}, \ L_4 = 2^{n+1} e^{-n} n^n. \end{gathered} \tag{C.4}$$

■

Lemma C.2 *We consider the function $H(p,q)$, domains F, G, Ω, and positive numbers Θ, θ, ρ, β, γ, δ, M, K. We suppose that*

1. *In the domain* $F = \{(p,q)\,/\,p = (p_1,p_2) \in G = G^1 \times G^2,\ |\mathrm{Im}\ q| \le \rho\}$ *the function*

$$H(p,q) = H_0(p) + H_1(p,q)$$

is analytic and

$$|H_1(p,q)| \le M,\ \ \theta|dp_2| \le |dA| \le \Theta|dp_2| \quad (0 < \theta < 1 < \Theta < \infty),$$

where A *is a diffeomorphism* $p_2 \mapsto \omega = \frac{\partial H_0}{\partial p_2}$ *of the domain* G^2 *onto the domain* Ω.

2. *The numbers* β, γ, δ, K *satisfy the inequalities*

$$\delta < \delta^{(1)}(n,\theta,\Theta) = \min\{\delta^{(0)}(n,2\Theta), 2^{-1}n^{-1}\theta\},$$

$$10\delta < 2\gamma < \varrho \le 1, \quad 3\beta < 2\delta, \quad 2\beta < K,$$

where $\delta^{(0)}$ *is defined in Lemma C.1.*

3. $M < \delta^{\nu} K \beta^2$, *where* $\nu = 2n+3$.

Then, there exists a domain $F' = \{(P,Q)\,/\,P \in G_1 = G_1^1 \times G_1^2 \subset G,\ |\mathrm{Im}\ Q| \le \rho' = \rho - 3\gamma\}$ *and a canonical diffeomorphism* $B : P,Q \to p,q$ *of the domain* F' *into* F, *such that*

1. $|B - E| < \beta$, $|dB| < 2|dx|$, $F' \subset F - \beta$, $(x = P,Q)$.

2. $H(p,q) = H(p) + H_2(P,Q)$ *where in the domain* $P,Q \in F'$

$$|H_2| < M' = M^2\delta^{-2\nu}\beta^{-2}, \qquad \left|\frac{\partial H_2}{\partial x}\right| < \frac{M'}{\beta}, \qquad \left|\frac{\partial^2 H_2}{\partial x^2}\right| < \frac{2M'}{\beta^2}.$$

3. *The mapping* $A' : P_2 \to \frac{\partial \overline{H}}{\partial p_2}$ *is a diffeomorphism of* G_1^2 *onto* Ω_1, *and* $\theta'|dP_2| < |dA'| < \Theta'|dP_2|$, *where* $\theta' = \theta(1-\delta)$, $\Theta' = \Theta(1+\delta)$ *and* $|A' - A| < \beta\delta$, *and in the notation of Lemma C.1*

$$\Omega_1 = \Omega_{KN} - d,$$

where

$$d = (5 + 7\Theta)\beta, \quad N = \frac{1}{\gamma}\log\frac{1}{2M}.$$

4. $\mathrm{mes}\ (G - G_1) < \theta^{-n}\mathrm{mes}\ (\Omega - \overline{\Omega}_1)$, *where*

$$\overline{\Omega}_1 = \Omega_{KN} - \overline{d}, \quad \overline{d} = (6 + 7\Theta)\beta.$$

Proof:
1.- We set
$$H_1(p,q) = \overline{H}_1(p) + \widetilde{H}_1(p,q),$$
where
$$\oint \widetilde{H}_1(p,q)\, dq = 0.$$
Then
$$H(p,q) = \overline{H}(p) + \widetilde{H}_1(p,q),$$
where
$$\overline{H}(p) = H_0(p) + \overline{H}_1(p).$$
Consider the mapping $A' : p_2 \to \frac{\partial \overline{H}}{\partial p_2} = A(p_2) + \Delta(p_2)$. By hypothesis 2 and 3 we have $M < \frac{\delta\theta}{2n}\beta^2$, $\delta < 1$, $\theta < 1$. Consequently, by Cauchy (3^{o} of 4.2 (see [1])), for $p \in G - \beta$ we have $|\Delta| < \frac{M}{\beta} < \beta\delta$, $|d\Delta| < \delta\theta|dp_2|$. We now apply the lemma of 5^{o} 4.3 (see [1]) concerning the variation of the frequency, putting $b = 3\beta$. By 5^{o} 4.3 (see [1]) A' maps $G^2 = (A')^{-1}\Omega_1$ diffeomorphically onto $\Omega_1 = \Omega_{KN} - d$ where $d = (5 + 7\theta)\beta$ and $G_1^2 + 3\beta$ is mapped into Ω_{KN}. Furthermore, the conclusions 3 and 4 of the present lemma are also valid.

2.- We now apply Lemma C.1 to the function $H(p,q) = \overline{H}(p) + \widetilde{H}_1(p,q)$ in the domain F. Since, by **1.-**, $\widetilde{H}_1 \leq 2M$ and $\theta'|dp_2| < |dA'| < \Theta'|dp_2|$, the hypothesis of this lemma imply that we may apply Lemma C.1. This latter gives us a diffeomorphism $B : P,Q \to p,q$ of the domain $P \in G_{KN} - 2\beta$, $|\text{Im}\, Q| < \rho - 2\gamma$ and the inequality $|H_2(P,Q)| < M'$. Hence, by Cauchy (3^{o} 4.2 (see [1])) for $P \in G_{KN} - 2\beta$, $|\text{Im}\, Q| \leq \rho - 3\gamma < \rho - 2\gamma - \beta$ we obtain the estimates of conclusion 2.

3.- According to **1.-**, for $P, Q \in F'$ (that is, $P \in G_1$, $|\text{Im}\, Q| \leq \rho - 3\gamma$) there follows $P \in G_{KN} - 3\beta$. But then, by **2.-**, the conclusions 1 and 2 hold, and this completes the proof. ■

Theorem C.1 *Consider the function $H(p,q)$, the domains F, G, Ω, and positive numbers D, M, Θ, θ, ρ, β, γ, δ, κ. We suppose that:*

1. *In the domain $F = \{(p,q) \,/\, p = (p_1,p_2) \in G = G^1 \times G^2,\ |\text{Im}\, q| \leq \rho \leq 1\}$ the function*
$$H(p,q) = H_0(p) + H_1(p,q)$$
is analytic and
$$|H_1(p,q)| \leq M, \quad \theta|dp_2| \leq |dA| \leq \Theta|dp_2| \quad (0 < \theta < 1 < \Theta < \infty),$$
where A is a diffeomorphism $p_2 \mapsto \omega = \frac{\partial H_0}{\partial p_2}$ of the domain G^2 onto the domain Ω of the type D (see §4, 4.1 *of* [1]*).*

2. *The inequality*

$$\delta < \delta^{(5)}(n, \theta, \Theta, \varrho, \varkappa, D) = \\ = \min\{\delta^{(1)}(n, \frac{1}{2}\theta, 2\Theta), \delta^{(2)}(n, \varkappa, \varrho), \delta^{(3)}(n, \theta, \Theta, \varkappa, D), \delta^{(4)}(\varkappa, \theta)\}$$

is satisfied, where $\delta^{(1)}$ *is defined in Lemma C.2,*

$$\delta^{(2)} = \min\{10^{-4n}\varrho^{4n}, 4^{-4n}, \varkappa\}, \quad \delta^{(4)} = \varkappa(2 + \theta^{-1})^{-1},$$

$$\delta^{(3)} = \min\{e^{2n}(32n^2 + 100n)^{-2n},\ (6 + 14\Theta)^{-1},\ 4^{-n-2}\varkappa\theta^n\Theta^{-n}D^{-1}n^{-1}\}.$$

3. *Let* $\beta = \delta^3$, $\gamma = \delta^{\frac{1}{4}n}$, $T = 8n + 24$. *Suppose also that* $\delta_1 = \delta$, *and for* $s \geq 1$ *put*

$$\delta_{s+1} = \delta_{s-1}^{3/2}, \quad \beta_s = \delta_s^3, \quad \gamma_s = \delta_s^{\frac{1}{4}n}, \quad M_s = \delta_s^T.$$

Then there exists a sequence of domains $F_0 = F$, F_1, F_2, ... *of the form* $F_s = \{(P_s, Q_s) / P_s \in G_s = G_s^1 \times G_s^2, |\text{Im } Q_s| \leq \rho_s\}$ *and a sequence of canonical diffeomorphisms* $B_s : P_s, Q_s \to P_{s-1}, Q_{s-1}$ *of the domains* F_s *into* F_{s-1} *such that for* $s \geq 1$

1. $|B_s - E| < \beta_s$, $|dB_s| < 2|dx_s|$, $F_s \subset F_{s-1} - \beta$, $\varrho_s > \frac{\varrho}{3}$.

2. *For* $p, q = B_1 B_2 \dots B_s(P_s, Q_s)$ *where* $P_s, Q_s \in F_s$ *we have*

$$H(p, q) = H^{(s)}(P_s, Q_s) = H_0^{(s)}(P_s) + H_1^{(s)}(P_s, Q_s),$$

$$|H_1^{(s)}| \leq M_{s+1}, \quad \left|\frac{\partial H_1^{(s)}}{\partial x_s}\right| < \delta_s \beta_{s+1}, \quad \left|\frac{\partial^2 H_1^{(s)}}{\partial x_s^2}\right| < \delta_s \quad (x_s = P_s, Q_s).$$

3. *The mapping* $A_s : (P_s)_2 \mapsto \frac{\partial H_0^{(s)}}{\partial (P_s)_2}$ *is a diffeomorphism of the domain* G_s, *such that*

$$\underline{\theta}|d(P_s)_2| \leq |dA_s| \leq \overline{\Theta}|d(P_s)_2|, \quad |A_s - A_{s-1}| < \beta_s \delta_s,$$

where $\underline{\theta} = \frac{1}{2}\theta$ *and* $\overline{\Theta} = 2\Theta$.

4. mes $(G - G_s) \leq \varkappa$ mes G.

Proof:

1.- Put $K = \kappa\delta_1$. We shall show that under the conditions of this theorem, Lemma C.2 is applicable. In fact, hypothesis 1 of Lemma C.2 follows from hypothesis 1 of this theorem. The condition $\delta < \delta^{(1)}$ of that lemma follows from the inequality $\delta < \delta^{(5)}$. Since $\delta < \delta^{(2)}$, hypothesis 2 of Lemma C.2 is satisfied because

$$\gamma < \frac{1}{10}\varrho, \quad \gamma < \frac{1}{4},$$

$$10\delta < 10\delta^{1/2}\delta^{n/4} < \frac{10}{4^{2n}}\gamma < 2\gamma, \quad 3\delta^3 < \frac{3}{4}\delta < 2\delta, \quad 2\delta^3 < \delta^2 < \delta\varkappa = K.$$

Finally, hypothesis 3 follows from the inequality $8n + 24 = T > \nu + 2 + 6 = 2n + 11$.

2.- Since $\delta < \delta^{(2)}$, we easily obtain for $s \geq 1$ the inequalities

$$\delta_s + \delta_{s+1} + \cdots < 2\delta_s, \quad 3(\gamma_s + \gamma_{s+1} + \cdots) < 6\gamma_s \leq 6\gamma_1 < 2\frac{\varrho}{3}. \tag{C.5}$$

By (C.5), if we put $\theta_0 = \theta$, $\Theta_0 = \Theta$, $\rho_0 = \rho$, $\theta_s = \theta_{s-1}(1 - \delta_s)$, $\Theta_s = \Theta_{s+1}(1 + \delta_s)$, $\rho_s = \rho_{s-1} - 3\gamma_s$, $(s = 1, 2, \ldots)$, we obtain the inequalities

$$\theta_s > \underline{\theta} = \frac{\theta}{2}, \quad \Theta_s < \overline{\Theta} = 2\Theta, \quad \varrho_s > \varrho_\infty = \frac{\varrho}{3}. \tag{C.6}$$

It is easy to verify that for $s \geq 1$ the numbers β_s, γ_s, δ_s, M_s, K satisfy the inequalities of hypothesis 2 of Lemma C.2 with the constants θ_{s-1}, Θ_{s-1}, ρ_{s-1}. This was established in **1.**- for $s = 1$.

3.- From $\delta < \delta^{(3)}$ it follows in view of the inequality 1° 4.2 of [1], for $N_s = \frac{1}{\gamma_s}\log\frac{1}{2M_s}$ that

$$\begin{aligned} \delta_s N_s^n &\leq \delta_s \left(\delta_s^{\frac{1}{4}n} \log \delta_s^{-(T+1)}\right)^n \leq \delta_s \left(4n(T+1)e^{-1}\delta_s^{\frac{1}{2}n}\right)^n \leq \\ &\leq \delta_s^{\frac{1}{2}}(4n(T+1)e^{-1})^n < 1. \end{aligned} \tag{C.7}$$

Put

$$\sigma_s = \sum_{N_{s-1} \leq m < N_s} m^{-2}.$$

Since $\sum \sigma_s < 2$, $\delta < \delta^{(3)}$ and (C.7) imply the inequalities

$$\sum_{s=1}^{\infty} [K\sigma_s + (6 + 7\Theta_s)\beta_s N_s^n] < \sum_{s=1}^{\infty} [K\sigma_s + \delta_s] < 4\delta_1 < \varkappa \overline{D}^{-1}, \tag{C.8}$$

where

$$D = \left(2\frac{\Theta}{\theta}\right)^n LD, \quad L = n2^{n+2}.$$

4.- Suppose that the quantities

$$A_{s-1},\ F_{s-1},\ G_{s-1},\ H^{(s-1)}(P_{s-1}, Q_{s-1}),\ \Omega_{s-1},\ \theta_{s-1},\ \Theta_{s-1},\ \varrho_{s-1}, \tag{C.9}$$

and β_s, γ_s, δ_s, M_k satisfy the conditions of Lemma C.2. Then that lemma defines the quantities

$$A',\ B,\ F',\ G_1,\ \overline{H}(P) + H_2(P, Q),\ \Omega_1,\ \overline{\Omega}_1,\ \theta',\ \Theta',\ \varrho',$$

which we denote (in the notation of **2.-**), respectively, by

$$A_s,\ B_s,\ F_s,\ G_s,\ H^{(s)}(P_s,Q_s) = H_0^{(s)}(P_s) + H_1^{(s)}(P_s,Q_s),\ \Omega_s,\ \overline{\Omega}_s,\ \theta_s,\ \Theta_s,\ \varrho_s. \quad \text{(C.10)}$$

From conclusion 2 of Lemma C.2 in the form $T = 8n + 24$ we obtain, in F_s,

$$|H_1^{(s)}| < M_s^2\delta_s^{-2\nu}\beta_s^{-2} = \delta_s^{2T-4n-12} = \delta_s^{\frac{3}{2}T} = \delta_{s+1}^T = M_{s+1}. \quad \text{(C.11)}$$

The conclusions 1, 2, 3 of Lemma C.2 with **2.-** and (C.11) imply that if the quantities (C.9) and β_s, γ_s, δ_s, M_s, K satisfy the hypothesis of Lemma C.2, then so do the quantities (C.9) when $s-1$ is replaced by s, and also β_{s+1}, γ_{s+1}, δ_{s+1}, M_{s+1}, K.

5.- But by **1.-** the quantities (C.9) when $s = 0$ (where $A_0 = A$, $F_0 = F$, $G_0 = G$, $H^{(0)} = H$, $P_0 = p$, $Q_0 = q$) and β_1, γ_1, δ_1, M_1, K satisfy the hypothesis of Lemma C.2. Consequently it can be applied for all s, and so we obtain the quantities (C.9) for $s = 1, 2, \ldots$. The conclusions 1, 2 and 3 of the theorem now follow from conclusions 1, 2 and 3 of Lemma C.2 in the form (C.6). We have, however, not yet proved that F_s is nonempty. This follows from conclusion 4 of the theorem, which we shall now prove.

6.- By conclusions 3 and 4 of Lemma C.2 we have

$$\text{mes }(G_{s-1} - G_s) \leq \underline{\theta}^{-n}\text{mes }(\Omega_{s-1} - \overline{\Omega}_s), \quad \text{(C.12)}$$

where $\overline{\Omega}_s = (\Omega_{s-1})_{KN_s} - \overline{d}_s$, $\overline{d}_s = (6 + 7\Theta_s)\beta_s$ and Ω_{s-1} are obtained from Ω by means of the formulae

$$\Omega_0 = \Omega, \quad \Omega_m = (\Omega_{m-1})_{KN_m} - d_m,$$

$$d_m = (5 + 7\Theta_s)\beta_s, \quad N_m = \frac{1}{\gamma_m}\log\frac{1}{2M_m}\ (m = 1, 2, \ldots, s-1).$$

Since $d_m > 0$, $1 < N_1 < N_2 < \cdots$, and the domain $\Omega = \Omega_0$ is of type D by hypothesis 1, we have by the aritmetical lemma (we refer to the lemma contained in 3^{O} of 4.1 in [1], that can be proved for this case using the ideas contained in Lemma 2.12)

$$\text{mes }(\Omega_{s-1} - \Omega_s) \leq DL[K\sigma_s + (6 + 7\Theta_s)\beta_s N_s^n]\text{mes }\Omega. \quad \text{(C.13)}$$

But since mes $\Omega \leq \Theta^n$mes G, (C.12), (C.13), (C.8) lead to

$$\begin{aligned}\text{mes }(G - G_s) &= \sum_{m=1}^{s}\text{mes }(G_{m-1} - G_m) \leq \\ &\leq \sum_{m=1}^{s}[K\sigma_m + (6 + 7\Theta_m)\beta_m N_m^n]\overline{D}\text{mes } G \leq \varkappa\,\text{mes } G.\end{aligned} \quad \text{(C.14)}$$

Consequently conclusion 4 is established. ■

C.2 Proof of Theorem 2.3

In this proof all the variables are to be taken as real, unless otherwise stated.

1.- Because of inequality (2.12) of Theorem 2.3, for any $\kappa > 0$ we may find positive numbers θ, Θ, D, m depending only on κ, G, H_0 such that the domain G can be represented in the form $G^{(1)} \cup \ldots \cup G^{(m)} \cup \overline{G}$, where mes $\overline{G} < \kappa/2$ and each domain $G^{(i)}$ is transformed diffeomorphically by the mapping $A : p_2 \mapsto \frac{\partial H_0}{\partial p_2}$ into a domain $\Omega^{(i)}$ of type D (see 1° of 4.1 (see [1])); moreover the inequalities

$$\theta|dp_2| \leq |dA| \leq \Theta|dp_2| \quad (0 < \theta < 1 < \Theta < \infty).$$

are satisfied in each of the domains $G^{(i)}$.

2.- If we can find $M(\kappa, \rho, G^{(i)}, H_0)$ in each of the domains $G^{(i)}$, then

$$M(\kappa, \varrho, G, H_0) = \min_i M\left(\frac{\varkappa}{2m}, \varrho, G^{(i)}, H^{(0)}\right),$$

gives the proof of Theorem 2.3, We shall therefore assume henceforth that hypothesis 1 of Theorem C.1 is satisfied in the domain G. We shall prove Theorem 2.3 assuming $M = \delta_1^T$, $T = 8n+24$, $\delta_1 < \delta^{(5)}(n, \theta, \Theta, \rho, \kappa, D)$, where the constant $\delta^{(5)}$ is defined in Theorem C.1. Now, the conditions of Theorem C.1 are satisfied and so its conclusion holds.

3.- *Convergence.* Theorem C.1 yields the sequences F_s and B_s. From $\beta_s = \delta_s^3 < 4^{-s}$ and the conclusion 1 of Theorem C.1 all the conditions of the lemma on convergence of 1° of 4.4 (see [1]) follow. According to this lemma the sequence of diffeomorphisms $S_s = B_1 B_2 \cdots B_s$ $(s = 1, 2, \ldots)$ on the compact set

$$F_\infty = \bigcap_{s \geq 0} F_s \quad \left(P_\infty \in G_\infty,\ |\text{Im } G_\infty| \leq \varrho_\infty,\ \text{where } G_\infty = \bigcap_{s \geq 0} G_s,\ \varrho_\infty \geq \frac{\varrho}{3}\right),$$

converge to a certain mapping S_∞. From conclusion 4 of Theorem C.1 mes $G_\infty \geq (1 - \kappa)$mes G. But S_s are canonical transformations and so preserve measure, so that mes $S_s F_\infty =$ mes F_∞. By 4° 4.4 (see [1]) it follows that

$$\begin{aligned} \text{mes } S_\infty F_\infty &\geq \lim_{s\to\infty} \text{mes } S_s F_\infty = (2\pi)^n \text{mes } G_\infty \geq \\ &\geq (2\pi)^n (1 - \varkappa) \text{ mes } G = (1 - \varkappa) \text{ mes } F. \end{aligned} \tag{C.15}$$

We set $F' = S_\infty F_\infty$ and prove the assertions 1–4 of Theorem 2.3

4.- *Invariance.* It follows from conclusion 3 of Theorem C.1 that the sequence of diffeomorphisms A_s converges on G_∞ to a mapping A_∞, where

$$|A_s - A_\infty| \leq \sum_{m=s+1}^{\infty} \beta_m \delta_m < \frac{1}{2}\beta_{s+1}. \tag{C.16}$$

Let us write the canonical equations with Hamiltonian $H^{(s)}(P_s, Q_s)$ in the form

$$x_s = X_s(x_s), \text{ where } x_s = P_s, Q_s. \tag{C.17}$$

The transformations S_s are canonical and so if $x_s(t)$ is the solution of the equations (C.17), then $x(t) = S_s x_s(t)$ satisfies (C.17) when $s = 0$. We shall show that if $x_\infty = P_\infty$, $Q_\infty \in F_\infty$ putting $X_\infty = 0$, $A_\infty(P_\infty)$ and $x_\infty(t)$ is the solution of the equation (C.17) when $s = \infty$ with initial conditions in F_∞, then $x_\infty(t) \in F_\infty$ and $S_\infty x_\infty(t)$ belongs to F and satisfies (C.17) when $s = 0$.

We use Lemma 3° of 4.4 (see [1]). Suppose that for $x_s = P_s, Q_s$, $\overline{X}_s(x_s) = 0$, $A_s(P_s)$. By conclusion 2 of Theorem C.1 we have $|X_s - \overline{X}_s| < \frac{1}{2}\beta_{s+1}$ in F_s. From (C.16) we obtain $|X_\infty - X_s| < \beta_{s+1}$ in F_∞. Also, from the conclusions 2 and 3 of Theorem C.1 we obtain

$$\left|\frac{\partial X_s}{\partial x_s}\right| < 2n\delta_s + \overline{\Theta} < n + \overline{\Theta}.$$

Lemma 3° of 4.4 (see [1]) now shows that $S_\infty x_\infty(t)$ satisfies (C.17) when $s = 0$, that is, (2.14).

5.- Let us introduce the notation $p_\omega = A^{-1}A_\infty(P_\infty)_2$, where $(P_\infty)_2 \in G^2_\infty$. Since (see **4.-**) $|A(P_\infty)_2 - A_\infty(P_\infty)_2| < \beta$, we have $|A(P_\infty)_2 - Ap_\omega| < \beta_1$ and by Lemma 4° of 4.3 (see [1]) $|(P_\infty)_2 - p_\omega| < \beta_1\theta^{-1}$. Also, by the lemma of 3° (see [1]) on convergence we have $|S_\infty - E| < 2\beta$. Thus, for $P_\infty, Q_\infty \in F_\infty$ we have (from the condition $\delta < \delta^{(4)}$ of Theorem C.1)

$$|S_\infty(P_\infty, Q_\infty) - (p_\omega, Q_\infty)| < \beta_1(2 + \theta^{-1}) < \varkappa. \tag{C.18}$$

6.- The equations $p, q = S_\infty(p_1, A^{-1}_\infty\omega, Q)$ for each fixed $(P_\infty)_2 = A^{-1}_\infty\omega \in G^2_\infty$ can be written in the form (2.15). They define the torus T_ϕ and the coordinates $Q = Q_\infty$ on it. The invariance of I_ϕ is proved in **4.-** and also the equation (2.17). The analyticity of S_∞ with respect to Q_∞ follows from the uniform convergence of S_s for each fixed $P_\infty \in G_\infty$ in the complex domain $|\text{Im } Q_\infty| \leq \rho_\infty$. Conclusion 3 of this theorem follows from (C.18). This completes the proof. ∎

Bibliography

[1] Arnol'd V. I.: "Proof of a Theorem of A. N. Kolmogorov on the Invariance of Quasi-Periodic Motions Under Small Perturbations of the Hamiltonian", *Russ. Math. Surveys* **18:5** (1963) pp. 9-36.

[2] Arnol'd V. I.: Chapitres Supplémentaires de la Théorie des Équations Différentielles Ordinaires, Mir, Moscow, 1980.

[3] Bogoljubov N. N., Mitropoliski Ju. A. and Samoilenko A. M.: Methods of Accelerated Convergence in Nonlinear Mechanics, Springer-Verlag, New York, 1976.

[4] Coppel W. A.: Stability and Asymptotic Behavior of Differential Equations, Heath Mathematical Monographs, Heath, Boston, 1965.

[5] Celletti A. and Giorgilli A.: "On the Stability of the Lagrangian Points in the Spatial Restricted Problem of Three Bodies", *Celestial Mechanics*, **50** (1991) pp. 31-58.

[6] Coppel W. A.: Dichotomies in Stability Theory, Lecture Notes in Mathematics, no. 629, Springer-Verlag, Berlin, 1978.

[7] Díez C., Jorba A. and Simó C.: "A Dynamical Equivalent to the Equilateral Libration Points of the Earth-Moon System", *Celestial Mechanics*, **50** (1991), pp. 13-29.

[8] Eliasson L. H.: "Perturbations of Stable Invariant Tori", *Ann. Scuola Norm. Su. Pisa*, **15**, (1988) pp. 115-148.

[9] Fink A. M.: Almost Periodic Differential Equations, Lecture Notes in Mathematics, no. 377, Springer-Verlag, Berlin, 1974.

[10] Giorgilli A., Delshams A., Fontich E., Galgani L. and Simó C.: "Effective Stability for a Hamiltonian System near an Elliptic Equilibrium Point, with an Application to the Restricted Three Body Problem", *Journal of Differential Equations*, **77** (1989), pp. 167-198.

[11] Gómez G., Jorba A., Masdemont J. and Simó C.: "Quasiperiodic Orbits as a Substitute of Libration Points in the Solar System", Proceedings of the NATO/ASI held at Cortina d'Ampezzo, 1990. Plenum Press (to appear).

[12] Gómez G., Jorba A., Masdemont J. and Simó C.: Study Refinement of Semi-Analytical Halo Orbit Theory, ESOC Contract 8625/89/D/MD(SC), Final Report, (1991).

[13] Gómez G., Llibre J., Martínez R. and Simó C.: Study on Orbits near the Triangular Libration Points in the perturbed Restricted Three-Body Problem, ESOC Contract 6139/84/D/JS(SC), Final Report, (1987).

[14] R. A. Johnson and G. R. Sell: "Smoothness of Spectral Subbundles and Reducibility of Quasi-Periodic Linear Differential Systems", *Journal of Differential Equations* **41** (1981), pp. 262-288.

[15] Jorba A. and Simó C.: "On the Reducibility of Linear Differential Equations with Quasiperiodic Coefficients", to appear in *Journal of Differential Equations.*

[16] Moser J.: "Convergent Series Expansions for Quasiperiodic Motions", *Math. Annalen* **169** (1967), pp 136-176.

[17] Schultz B. E. and Tapley B. D.: "Numerical Studies of Solar Influenced Particle Motion near Triangular Earth-Moon Libration Points", in Periodic Orbits, Stability and Resonances, Ed G. E. O. Giacaglia, 82-90, Reidel (1970).

[18] Shampine L. F. and Gordon M. K.: Computer Solution of Ordinary Differential Equations. The Initial Value Problem. Freeman, 1975.

[19] Simó C.: "Estabilitat de Sistemes Hamiltonians", *Memorias de la Real Academia de Ciencias y Artes de Barcelona*, **48** (1989), pp. 303-348.

[20] Siegel C. L. and Moser J. K.: Lectures on Celestial Mechanics, Springer-Verlag (1971).

[21] Stoer J. and Bulirsch R.: Introduction to Numerical Analysis, Springer-Verlag, 1983 (second printing).

[22] Szebehely V.: Theory of Orbits, Academic Press (1967).