UNIVERSITAT DE BARCELONA

# On Sandwiched Surface Singularities and Complete Ideals

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CERTIFICA

que la present memòria ha estat realitzada sota la seva direcció per Jesús Fernández Sánchez, i que constitueix la seva tesi per aspirar al grau de
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Signat: Eduard Casas Alvero.

*A mis padres*

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# Introduction

The original interest in sandwiched singularities comes from a natural question posed by J. Nash in the early sixties to H. Hironaka: *does a finite succession of Nash transformations or normalized Nash transformations resolve the singularities of a reduced algebraic variety?* In 1975, A. Nobile [47] proved that, in characteristic zero, a Nash transformation is an isomorphism only in case the original variety is already non-singular. It turns out, in particular, that curve singularities are resolved by a succession of Nash transformations. Rebasoo proved in his Ph. D. thesis that Nash transformations also resolve certain kinds of quasi-homogeneous hypersurface singularities in $\mathbb{C}^3$. In 1982, G. González-Sprinberg proved that normalized Nash transformations resolve rational double points and cyclic quotients singularities of surfaces [23]. Then, H. Hironaka proved that after a finite succession of normalized Nash transformations one obtains a surface $X$ which birationally dominates a non-singular surface [30]. By definition, the singularities of $X$ are sandwiched singularities. Some years later, in [58], M. Spivakovsky proves that sandwiched singularities are resolved by normalized Nash transformations, thus giving a positive answer to the original question posed by Nash for the case of surfaces over $\mathbb{C}$.

Since then, a constant interest in sandwiched singularities has been shown, and they have been deeply studied from the point of view of deformation theory by de Jong and van Straten in [13], and also by Stølen [28] and Möhring [43]. Sandwiched singularities have been also studied as a nice testing ground for the Nash and the wedge Problem by Lejeune-Jalabert and Reguera in [40], where the main idea is to extend combinatorial arguments for toric surface singularities to sandwiched ones.

Sandwiched singularities are the singularities obtained by blowing-up a complete ideal in the local ring of a regular point on a surface. They are rational surface singularities (roughly speaking, isolated singularities whose resolution has no effect on the arithmetic genus of the surface) and among them are included all cyclic quotients and minimal surface singularities. Therefore, the following chain of inclusions between classes of surface singularities does hold:

$$\text{cyclic quotients} \subsetneq \text{minimal} \subsetneq \text{sandwiched} \subsetneq \text{rational}.$$

Sandwiched singularities are Cohen-Macaulay, but are not complete intersections and in general, there are no simple equations for them. The purpose of this memoir is to study sandwiched singularities through their relationship to the infinitely near base points of the complete ideals blown-up to obtain them[1].

As said in the preface of [11], infinitely near points are a nice and old idea for describing singularities. Their use and their properties, such as proximity, satellitism, etc. give a very enlightening picture of the behaviour of singularities of plane curves and it seems to be a very promising approach in the study of singularities in a wider context. They appear in the work of M. Noether and their geometry was extensively developed by F. Enriques [16]. Besides the book of Enriques and Chisini, other classical references are the survey in the first chapter of Zariski's book on surfaces [62], chapter XI of the classical book on curves by Semple and Kneebone [57] or section 5 of Zariski's paper on saturation [63]. The theory of infinitely near points has been revised and developed in a modern account by Casas [11].

Complete ideals were introduced by Oscar Zariski twenty years after the publishing of [16]. In [61] he develops his theory of complete ideals as an arithmetic theory parallel to the geometric theory of linear systems of plane curves going through an assigned set of base points with multiplicities. One of the central points of the theory is that any complete ideal in the local ring of a surface at a regular point has a unique factorization into irreducible complete ideals. Relevant to our purposes is the fact that any complete ideal has a cluster of infinitely near base points that in turn determines the ideal.

As basic results relating a sandwiched surface $X$ and the base points of the complete ideal $I$ blown up to produce $X$, we determine the singular points of $X$, as well as their multiplicities and fundamental cycles in terms of the base points of $I$, which are also used to determine the multiplicities of the points of curves on $X$.

These facts allow us to study the existence of local equations for curves on $X$. We derive consequences relative to their orders of singularity and make explicit computations concerning the existence of Cartier divisors on $X$ with prescribed properties. In particular, we show that the tangents to the exceptional curves of $X$ going through a sandwiched singularity are linearly independent. All of this leads to studying complete ideal sheaves with finite cosupport on $X$, from which we infer results relative to the

[1] A similar viewpoint has been used by Möhring in his thesis [43] to study deformation theory and the Kollár conjecture for sandwiched singularities

factorization (or semi-factorization) of complete ideals in the local ring of a sandwiched singularity.

We also obtain some results concerning the Nash Problem of arcs through a sandwiched singularity. In a 1968 preprint later published as [46], Nash introduced arc spaces and jet schemes as a new way for understanding the singularities. The main question of [46] is to know whether each essential component of a resolution of a singular point gives rise to an irreducible component of the space of arcs through it. Nash conjectured that there is a bijection between the sets of irreducible components of the space of arcs through a singularity and the set of essential components of it. In a recent paper [33], Ishii and Kollár prove that the Nash question is true for toric singularities of any dimension, but false in general. More recently, Reguera has proved the Nash problem for a large class of singularities, including the sandwiched singularities [55].

Throughout this work and unless otherwise specified, the base field is the field $\mathbb{C}$ of complex numbers. Standard references for most of the material treated here are the book of Casas [11] and the papers of Spivakovsky [58] and Lipman [41].

Before giving an outline of the different chapters, we introduce some definitions and notations and set the framework where the study of sandwiched singularities is developed. Most of the notations concerning clusters and infinitely near points are taken from [11]. We take a smooth surface $S$, a point $O \in S$ and write $R = \mathcal{O}_{S,O}$ for the local ring at $O$ and $\mathfrak{m}_O$ for the maximal ideal of $R$. By blowing-up a complete $\mathfrak{m}_O$-primary ideal $I$ contained in $R$, we obtain a surface $X = Bl_I(S)$ with sandwiched singularities. If we write $\mathcal{K} = (K, \nu)$ for the cluster of base points of $I$ and $\pi_K : S_K \longrightarrow S$ for the blowing-up of all the points of $K$, we have a commutative diagram

$$\begin{array}{ccc} S_K & \xrightarrow{f} & X \\ & \searrow^{\pi_K} & \downarrow{\scriptstyle \pi} \\ & & S \end{array}$$

where $\pi$ is the blowing-up of $I$ and the morphism $f : S_K \to X$, given by the universal property of the blowing-up, is the minimal resolution of the singularities of $X$. We write $\{E_p\}_{p \in K}$ for the irreducible exceptional components on $S_K$ and $\mathcal{K}_+$ for the set of dicritical points of $\mathcal{K}$. Then, the exceptional irreducible components of $X$ are $\{L_p\}_{p \in \mathcal{K}_+}$, where each $L_p$ is the direct image of $E_p$ by $f$. We denote by $\mathbb{F}_K$ the set of all infinitely near points not in $K$ and which are in the first neighbourhood of some point in $K$. Given a weighted cluster $\mathcal{T} = (T, \tau)$, the equations of the curves going through $\mathcal{T}$ describe the set of non-zero elements of an ideal $H_{\mathcal{T}}$ of $R$. We say that $\mathcal{T}_1$ and $\mathcal{T}_2$ are equivalent if $H_{\mathcal{T}_1} = H_{\mathcal{T}_2}$. If $\mathcal{T} = \mathcal{K}(p)$ is the irreducible cluster determined by some point $p$ infinitely near to $O$, we write $I_p$ for the simple complete ideal $H_{\mathcal{K}(p)}$.

Now, we briefly summarize the main contents of each one of the chapters.

**Chapter I** is of preliminary nature and gives references to the literature for proofs. Concepts and well-known facts about infinitely near points, weighted clusters, complete ideals and rational and sandwiched surface singularities are reviewed and some consequences that are needed in the memoir are derived. In sections I.1 and I.2 definitions related to point blowing-ups on a regular surface and weighted clusters are recalled from [9] and [11]. In subsection I.2.1 we describe the unloading procedure due to Enriques, which will play an essential role throughout this work. Section I.3 is devoted to recall some well-known facts concerning rational surface singularities that will be used along this memoir; the main references are [3] and [41]. Section I.4 introduces sandwiched singularities and reviews the most relevant facts for our study. Finally, in section I.5 we recall some facts and elementary definitions concerning dual graphs.

In **Chapter II** we establish the main link between the study of sandwiched singularities and the theory of Enriques diagrams of weighted clusters and we derive some results on sandwiched singularities by using the unloading procedure. Enriques diagrams are combinatorial data associated with clusters of infinitely near points and proximity relations. In section II.1 and for rational surface singularities in general, we prove the following theorem.

**Theorem 1.** [Theorem II.1.7] *Let $\mathcal{O}_{S,P}$ be a local ring with a rational surface singularity, let $I \subset \mathcal{O}_{S,P}$ be a complete $\mathfrak{m}_P$-primary ideal and $X$ the surface obtained by blowing-up $I$. Given $Q \in X$, write $\mathcal{M}_Q$ for the ideal sheaf of $Q$. By associating to each point $Q$ in the exceptional locus of $X$ the complete ideal $I_Q = \pi_*(\mathcal{M}_Q I \mathcal{O}_X)$, we get a bijection between the set of points on the exceptional divisor of the surface $X$ and the set of the complete ideals $J \subset I$ of codimension one. The inverse map associates to each complete ideal $J$ the only point $Q$ all the virtual transforms of the curves $C : h = 0$, $h \in J$ are going through.*

As a consequence of this result, we recover the fundamental cycle on any resolution of a singularity on $X$ as the difference between the exceptional divisors on that resolution corresponding to $I$ and $I_Q$ (see Corollary II.1.10). These facts allow us to study the points lying in the exceptional locus of $X$ and, in particular the singularities of $X$, from the point of view of linear subsystems of codimension one in the linear system defined by $I$. This is one of the key facts of this chapter. Section II.2 uses these results to study the connection between sandwiched singularities and clusters of infinitely near points and shows how to obtain information of the surface $X$ from the Enriques diagram of $\mathcal{K}$ and its virtual multiplicities. We describe a procedure to compute the cluster $\mathcal{K}_Q$ of base points of the complete ideal $I_Q$ associated to $Q$ by the above theorem, and we characterize the singular points of $X$ in terms of the consistency of the cluster $\mathcal{K}_q$, obtained by adding to $\mathcal{K}$ some point $q \in \overline{\mathbb{F}}_K$ (depending on $Q$) counted once (Proposition II.2.5). Then, if $Q \in X$ is singular, we denote by $T_Q$ the set of (non-dicritical) points $p \in K$ such that $f$ contracts the exceptional component $E_p$ to $Q$. We show that there is only one minimal point (relative to the natural ordering) in $T_Q$, denoted by $O_Q$, and we prove

**Corollary 2.** [Corollary II.2.10] *We have that*

(a) *the number of singularities of $X$ equals the number of non-equivalent clusters $\mathcal{K}_q$, for $q \in F_p$ not already in $K$ and $p \in K$ a non-dicritical point of $\mathcal{K}$.*

(b) *If $p_1, p_2 \in T_Q$ for some singularity $Q \in X$, the exceptional components $E_{p_1}$ and $E_{p_2}$ intersect on $S_K$ if and only if $p_1$ is maximal among the points of $K$ proximate to $p_2$ or viceversa.*

(c) *Let $Q \in X$ be singular and let $q \in \overline{\mathbb{F}}_K$ be such that $H_{\mathcal{K}_q} = I_Q$. Write $Z_Q = \sum_{u \in T_Q} z_u E_u$ the fundamental cycle of $Q$. Then, for each $u \in T_Q$, $z_u$ is the number of unloading steps performed on $u$ in the unloading procedure of $\mathcal{K}_q$.*

After this, we describe the Enriques diagram of a complete $\mathfrak{m}_O$-primary ideal verifying the conditions 1. and 2. of Corollary 1.14 of [58] and we give a formula for the multiplicity of a point $Q$ in the exceptional locus of $X$ in terms of the virtual multiplicities of $\mathcal{K}$ and $\mathcal{K}_Q$:

**Theorem 3.** [Theorem II.2.14] *Let $Q$ be any point in the exceptional locus of $X$. Then the multiplicity of $X$ at $Q$ is*

$$\text{mult}_Q(X) = {\mathcal{K}_Q}^2 - \mathcal{K}^2.$$

The section II.3 is technical and devoted to study the effect of unloading multiplicities after adding some simple point $q \in \overline{\mathbb{F}}_K$ as explained above. Lemma II.3.1 shows that by unloading, the multiplicity at $O_Q$ has been increased by one and the multiplicities at the others points either remain unaffected or decrease by one. This fact suggests defining the *multiplicity relevant points (MR-points for short) of $\mathcal{K}$ relative to a sandwiched singularity* as those points of $K$ where the multiplicity decreases by one after unloading. The set of these points is denoted by $B_Q^{\mathcal{K}}$ and its interest will be clear in the forthcoming section II.5. Theorem II.3.8 relates the coefficients of the fundamental cycle of a sandwiched singularity to the proximity relations of the point in $K$. As a consequence of some of these facts, in subsection II.3.1 we give a description of the Zariski factorization of any complete ideal $J$ of codimension one in $I$ in terms of the Zariski factorization of $I$ and the point $Q$ in the exceptional locus of $X$ corresponding to $J$ by Theorem 1. Namely,

**Theorem 4.** [Theorem II.3.13] *Let $I = \prod_{p\in\mathcal{K}_+} I_p^{\alpha_p}$, with $\alpha_p \geq 1$, be the Zariski factorization of $I$ and let $J$ be a complete $\mathfrak{m}_O$-primary ideal of codimension one in $I$. Let $Q$ be the point in the exceptional locus of $X$ corresponding to $J$ by Theorem 1. Then*

$$J = H_J \prod_{p\in\mathcal{K}_+^Q} I_p^{\alpha_p-1} \prod_{p\in\mathcal{K}_+\setminus\mathcal{K}_+^Q} I_p^{\alpha_p},$$

*where $\mathcal{K}_+^Q = \{p \in \mathcal{K}_+ \mid Q \in L_p\}$ and $H_J \subset R$ is a complete $\mathfrak{m}_O$-primary ideal whose factorization shares no simple ideals with that of $I$. Moreover, $H_J$ is simple if and only if $Q$ is non-singular and in this case, $H_J = I_q$ where $q \in \overline{\mathbb{F}}_K$ is the unique point such that $H_{\mathcal{K}_q} = I_Q$. If $Q$ is singular, then the simple ideal $I_{O_Q}$ appears in the factorization of $H_J$.*

It follows in particular that the factorization of $H_J$ determines if $Q$ is singular or not. The aim of section II.4 is to study the resolution of sandwiched singularities in terms of chains of complete ideals in $R$ (Theorem II.4.5). In particular, in Proposition II.4.4 we see that the blowing-up of some singularity $Q \in X$ is the birational join of $X$ and the surface obtained by blowing-up the complete ideal $I_Q \subset I$. In section II.5 and once a curve $C$ on $S$ has been fixed, we give a formula for the multiplicity of the strict transform $\widetilde{C}$ of $C$ on $X$ at any point in the exceptional locus of $X$:

**Theorem 5.** [Theorem II.5.1 and Corollary II.5.4] *Let $Q$ be a point in the exceptional locus of $X$. If $C$ is a curve on $S$, then*

$$\operatorname{mult}_Q(\widetilde{C}) = e_{O_Q}(C) - \sum_{p\in B_Q^{\mathcal{K}}} e_p(C).$$

*In particular,* $\operatorname{mult}_Q(X) = 1 + \sharp B_Q^{\mathcal{K}}$.

From this, we obtain formulas for the number of branches of hypersurface sections of sandwiched singularities and infer the well-known fact that minimal singularities are those rational surface singularities whose fundamental cycle is reduced (this fact was already stated by Kollár in [34] without proof). In Proposition II.5.17, we prove that the number of exceptional components going through a sandwiched singularity is bounded by its embedding dimension and characterize when this bound is attained. Section II.6 is technical again and uses some previous results already seen to derive that any two exceptional curves of $X$ going through the same sandwiched singularity are not tangent.

We conclude the chapter by using techniques and results of the previous sections to derive consequences concerning adjacent complete ideals, i.e. pairs of ideals $J \subset I$ with $\dim_{\mathbb{C}} \frac{I}{J} = 1$. We answer some of the questions posed by S. Noh in the last section of [49] related to the existence of adjacent ideals right above or below of given complete ideals in a regular local two-dimensional ring

**Chapter III** deals essentially with the principality of divisors going through a sandwiched singularity. It is well known that Weil divisors going through a singularity $(X, Q)$ are not Cartier divisors in general. We investigate this fact for the case of sandwiched singularities and we obtain the following criterion.

**Theorem 6.** [Theorem III.1.1] *If $u \in \mathcal{K}_+$ denote $\mathcal{L}_u = \sum_{p \in \mathcal{K}_+} v_p(I_u) L_p$ and fixed a curve $C$ on $S$, write $L_C$ for the exceptional component of $\pi^*(C)$ on $X$. The following four assertions are equivalent:*

(i) The strict transform $\widetilde{C}$ on $X$ is a Cartier divisor;

(ii) $L_C \in \bigoplus_{u \in \mathcal{K}_+} \mathbb{Z}\mathcal{L}_u$;

(iii) There exists a curve $C_O \subset S$ such that $L_{C_O} = L_C$ and the strict transform $\widetilde{C_O}$ goes through no singularity of $X$;

(iv) If $\mathbb{H}^o_C = \{g \in R \mid v_p(g) \geq v_p(C), \forall p \in \mathcal{K}_+\}$ and if $q \in T_Q$ is not a dicritical point of $\mathcal{K}$, then $q$ is not a dicritical base point of $\mathbb{H}^o_C$ either.

As a consequence of this theorem, we see in Corollary III.1.6 that if $\widetilde{C}$ is Cartier, then

$$\mathbb{H}^o_C = \prod_{p \in \mathcal{K}_+} I_p^{a_p}$$

is the (Zariski) factorization of $\mathbb{H}^o_C$ into simple ideals, where $a_p = |\widetilde{C} \cdot L_p|_X$ for each $p \in \mathcal{K}_+$. In Remark III.1.10 we propose a procedure based on unloading to determine if the strict transform on $X$ of a curve on $S$ is Cartier or not. In section III.2 consequences are derived: writing $S_{K_C}$ for the minimal embedded resolution of $\widetilde{C} \subset X$, a formula is given for the exceptional component of the total transform of $\widetilde{C}$ on $S_{K_C}$ in terms of the values of $C$ and $\mathbb{H}^o_C$ relative to some divisorial valuations (Proposition III.2.1). From this, we derive a formula for the intersection multiplicity of effective Cartier divisors and effective Weil divisors on $X$ in terms of the intersection numbers of some curves on $S$:

**Corollary 7.** [Corollary III.2.3] *Let $C$ and $C_1$ be curves on $S$ and assume that $\widetilde{C}$ is Cartier. Then,*

$$|\widetilde{C}\cdot\widetilde{C_1}|_X = [C, C_1]_O - [\mathcal{K}_C^o, C_1]_O,$$

*where $\mathcal{K}_C^o = BP(\mathbb{H}_C^o)$.*

In the following two sections we propose algorithms to give effective Cartier divisors on $X$ with prescribed conditions. In section III.3 we give a proof of the following result:

**Theorem 8.** [Corollary III.3.5] *Let $\mathcal{Q} = \{Q_1, \ldots, Q_n\}$ be points (singular or not) in the exceptional locus of $X$ and for each $Q_i$, let $\{\alpha_p^i\}_{p\in\mathcal{K}_+^{Q_i}}$ be positive integers. Then, there exists a cluster $\mathcal{T}_{\mathcal{Q}}^\alpha$ such that if $C$ is a generic curve going through $\mathcal{T}_{\mathcal{Q}}^\alpha$, then $\widetilde{C}$ is a Cartier divisor on $X$ going through $Q_1, \ldots, Q_n$ and for each $Q_i$ and each $p \in \mathcal{K}_+^{Q_i}$,*

$$[\widetilde{C}, L_p]_{Q_i} = \alpha_p^i.$$

*Moreover, if $Q_i$ is regular or such that $O_{Q_i}$ is free or $O_{Q_i} = O$, $\widetilde{C}$ is irreducible as a principal divisor near $Q_i$.*

The proof of this result is constructive and allows us to describe a procedure to compute the cluster $\mathcal{T}_{\mathcal{Q}}^\alpha$. Moreover, from Theorem 8 we infer that the tangents at $Q$ to the exceptional curves $\{L_p\}_{p\in\mathcal{K}_+^Q}$ are linearly independent.

A $v$-minimal Cartier divisor containing a given effective Weil divisor $C$ on $X$ is a Cartier divisor such that its exceptional part in the minimal embedded resolution of $C$ is minimal relative to the order given by the coefficients of the exceptional components. In section III.4, after introducing a little modification of the unloading procedure (*partial unloading*), we give an algorithm to compute the $v$-minimal Cartier divisors containing a given Weil divisor $C$ on $X$. In Proposition III.4.15 we describe the singularities of these curves, and in section III.5 we use the tools and results developed in this chapter to compute the order of singularity of an effective Cartier divisor on $X$:

**Theorem 9.** [Theorem III.5.1 and Corollary III.5.5] *Let $C$ be a curve on $S$ such that $\widetilde{C}$ is a Cartier divisor on $X$. Then,*

$$\delta_O(C) = \delta_X(\widetilde{C}) + \delta_O(\mathcal{K}_C^o).$$

*Moreover, if for every $p \in K$ we write $\tau_p^o$ for the virtual multiplicity of $\mathcal{K}_C^o$ at $p$ and $n_p = e_p(C) - \tau_p^o$, then*

$$\delta_X(\widetilde{C}) = \sum_{p\in K_C} \frac{n_p(n_p - 1)}{2}.$$

For the general case, when $\widetilde{C}$ is not necessarily Cartier, we propose a less explicit formula for the order of singularity of $\widetilde{C}$ on $X$ in terms of some clusters, $\mathcal{T}_n^s$ and $\mathcal{T}_0^s$, obtained by applying the algorithm of section III.4 to $\widetilde{C}$.

**Proposition 10.** [Proposition III.5.8] *Let $C$ be a curve on $S$. Then,*

$$\delta_X(\widetilde{C}) = [\mathcal{T}_n^s, C]_O - [\mathcal{T}_0^s, C]_O - \dim_{\mathbb{C}}(\frac{H_{\mathcal{T}_0^s}}{H_{\mathcal{T}_n^s}}).$$

In subsection III.5.1 we use this result to compute the semigroup of a branch going through a sandwiched singularity, we give some examples and show that in general, this semigroup is not symmetric.

In **Chapter IV** we use the results of Chapter III to explore the connection between the ideal sheaves on $X$ with finite cosupport contained in the exceptional locus and the complete $\mathfrak{m}_O$-primary ideals in $R$. In virtue of Theorem 6, the principality of the strict transform on $X$ of a generic curve $C$ going through some cluster $\mathcal{T}$ only depends on $\{v_p^{\mathcal{T}}\}_{p\in\mathcal{K}_+}$. In section IV.1 we introduce the Cartier ideals for $X$ as those complete ideals for which the strict transforms on $X$ of curves defined by generic elements of them are Cartier divisors. Given a complete $\mathfrak{m}_O$-primary ideal $H \subset R$, we denote by $H^o = \{g \in R \mid v_p(g) \geq v_p(H), \forall p \in \mathcal{K}_+\}$. Then we prove the following result.

**Theorem 11.** [Theorem IV.1.3] *Let $\mathcal{J}$ be an ideal sheaf on $X$ with finite cosupport $\{Q_1, \ldots, Q_n\}$ contained in the exceptional locus of $X$ and such that for each $Q_i$, the stalk $J_i = \mathcal{J}_{Q_i}$ is a complete $\mathfrak{m}_{Q_i}$-primary ideal of $\mathcal{O}_{X,Q_i}$. There exists a complete $\mathfrak{m}_O$-primary ideal $H_{\mathcal{J}}$ in $R$ with*

$$\dim_{\mathbb{C}}(\frac{H_{\mathcal{J}}^o}{H_{\mathcal{J}}}) = \sum_{i=1}^{n} \dim_{\mathbb{C}}(\frac{\mathcal{O}_{X,Q_i}}{J_i})$$

*and such that:*

*(a) $H_{\mathcal{J}}$ is a Cartier ideal for $X$;*

*(b) the sheaf $\mathcal{H}_{\mathcal{J}} = H_{\mathcal{J}}\mathcal{O}_X$ is locally principal except precisely at the points $Q_i, i = 1, \ldots, n$ and we have*

$$\mathcal{H}_{\mathcal{J}} = \mathcal{J}\mathcal{O}_X(-L_{H_{\mathcal{J}}});$$

*(c) if $C$ is a curve defined by a generic element of $H_{\mathcal{J}}$, then its strict transform $\widetilde{C}$ on $X$ is a Cartier divisor and intersects the exceptional locus of $X$ exactly at the points $\{Q_1, \ldots, Q_n\}$.*

*Moreover, with these requirements, the ideal $H_{\mathcal{J}}$ is uniquely determined.*

This result allows us to generalize Theorem 1 to complete ideals $H \subset I$ of any codimension provided that their exceptional support on $X$ equals that of $I$, i.e. $L_H = L_I$. Namely,

**Corollary 12.** [Corollary IV.1.11] *Fixed $m \geq 0$, there is a one-to-one correspondence between the set of ideal sheaves $\mathcal{J}$ on $X$ with finite cosupport, say $\{Q_1, \ldots, Q_n\}$, contained in the exceptional locus of $X$ and such that for each $Q_i$, the stalk $J_i = \mathcal{J}_{Q_i}$ is a complete $\mathfrak{m}_{Q_i}$-primary ideal of $\mathcal{O}_{X,Q_i}$ and $\sum_i \dim_{\mathbb{C}}(\mathcal{O}_{X,Q_i}/J_i) = m$ and the set of Cartier ideals $H \subset R$ for $X$ of codimension $m$.*

*This correspondence maps each ideal sheaf $\mathcal{J}$ on $X$ to the complete $\mathfrak{m}_O$-primary ideal $H_{\mathcal{J}}$ of Theorem 11. The inverse map associates to each $H \subset R$ the ideal sheaf on $X$ generated by the equations of the virtual transforms on $X$ (relative to $H^o$) of the curves defined by elements of $H$.*

In section IV.2 we give a procedure to compute a minimal system of generators for $H_{\mathcal{J}}$. Finally, in section IV.3 we investigate the factorization and semifactorization of complete ideals in the local ring of a sandwiched singularity in terms of the factorization of complete ideals in $R$. First, we prove that the bijections of Corollary 12 define an isomorphism of semigroups (Proposition IV.3.1). Then, given $Q \in X$, if $J$ is a complete $\mathfrak{m}_Q$-primary ideal in $\mathcal{O}_{X,Q}$ and $\mathcal{J}$ is the ideal sheaf generated by it, we show that the factorization of the complete ideal $H_{\mathcal{J}}$ induces the semifactorization of $J$. Namely,

**Theorem 13.** [Theorem IV.3.5] *Let $S_{\mathcal{J}}$ be the minimal resolution of $X$ such that $J\mathcal{O}_{S_{\mathcal{J}}}$ is invertible. For an infinitely near point $p$, write $m_p$ for the least positive integer such that $I_p^{m_p}$ is a Cartier ideal for $X$. Then, if*

$$H_{\mathcal{J}} = \prod_p I_p^{\alpha_p}$$

*is the (Zariski) factorization of $H_{\mathcal{J}} \subset R$ into simple ideals, then*

$$J = \prod_p J_p^{\frac{\alpha_p}{m_p}}$$

*is the semifactorization of $J$.*

*In particular, the above factorization of $H_{\mathcal{J}}$ gives rise to a factorization of $J$ into simple complete ideals of $\mathcal{O}_{X,Q}$ if and only if $\alpha_p \in (m_p)$ for every $p$ such that $\alpha_p > 0$.*

Then, we prove that the factorizations of $J$ into simple ideals of $\mathcal{O}_{X,Q}$ induce and are induced in turn by the factorizations of $H_{\mathcal{J}}$ into irreducible Cartier ideals for $X$.

**Theorem 14.** [Theorem IV.3.7] *Given a complete $\mathfrak{m}_Q$-primary ideal $J \subset \mathcal{O}_{X,Q}$, each factorization of $J$ into complete $\mathfrak{m}_Q$-primary ideals*

$$J = \prod_{i=1}^{r} J_i^{\alpha_i}$$

*induces a factorization of $H_{\mathcal{J}}$ into Cartier ideals for $X$, $H_{\mathcal{J}} = \prod_{i=1}^{r} H_{\mathcal{J}_i}^{\alpha_i}$, and each factorization of $H_{\mathcal{J}}$ into Cartier ideals for $X$ has this form. Moreover, $H_{\mathcal{J}_i}$ is irreducible as a Cartier ideal for $X$ if and only if $J_i$ is a simple complete $\mathfrak{m}_{Q_i}$-primary ideal.*

**Chapter V** is devoted to derive consequences related to the Nash conjecture of arcs for sandwiched singularities. In section V.1 we recall some facts and set the framework for our study of the spaces of arcs. Given a rational surface singularity $(X, Q)$, write $\{E_u\}_{u \in \Delta_Q}$ the exceptional components of the minimal resolution $S'$ of $(X, Q)$. For each $u \in \Delta_Q$, write $\mathcal{F}_u^Q$ (respectively, $\mathcal{N}_u^Q$) for the space of arcs through $(X, Q)$ whose lifting on $S_K$ intersects (resp. intersects transversally) the exceptional component $E_u$. Fixed $i \geq 0$, write $Tr(i)$ for the space of $i$-truncations of arcs, and $\mathcal{F}_u^Q(i)$ and $\mathcal{N}_u^Q(i)$ for the spaces of $i$-truncations of arcs in $\mathcal{F}_u^Q$ and $\mathcal{N}_u^Q$, respectively. Sections V.2 and V.3 are aimed to prove that the reduced components of the fundamental cycle of a sandwiched singularity give rise to irreducible components of the space of arcs: section V.2 is essentially technical and in it, we prove an inequality between the intersection numbers at $O$ of the projections on $S$ of arcs of different families provided that one of them is contained in the other (Theorem V.2.1). In section V.3 an accurate and somehow tedious study of the dual graph and the proximity relations among the points of $K$ gives the wanted result:

**Theorem 15.** [Theorem V.3.1] *Let $Q$ be a sandwiched singularity. Then every reduced component of the fundamental cycle of $Q$ is associated to a Nash family of arcs. In other words, if there exists $p, q \in T_Q$ such that $\overline{\mathcal{N}_p^Q(i)} \subsetneq \overline{\mathcal{N}_q^Q(i)}$ for $i >> 0$, then $z_p > 1$.*

In particular, Theorem 15 gives an affirmative answer to the Nash conjecture for minimal singularities. This was already proved by Reguera in [53]. Finally, in section V.4 we prove that a positive answer to the Nash conjecture for sandwiched singularities would follow from a positive answer for primitive singularities. Recall that primitive singularities are those singularities that may be obtained by blowing up a simple complete ideal. The proof requires a result that is worth mentioning here:

**Proposition 16.** [Lemma V.4.1] *Let $(X, Q)$ and $(X_1, Q_1)$ be rational surface singularities, $g : X \longrightarrow (X_1, Q_1)$ a birational dominant morphism and*

*$E_p$, $p \in \Delta_Q$ an exceptional component of $Q$ such that $E_p$ also appears in the minimal resolution of $Q_1$ modulo birational equivalence. Assume that for some $i > 0$, $\overline{\mathcal{F}_p^Q(i)} \subset \overline{\mathcal{F}_q^Q(i)}$ and that the projection by $g$ of any element of $\mathcal{F}_q^Q(i)$ is in $\mathcal{F}_u^{Q_1}(i)$, for some $u \in \Delta_{Q_1}$. Then,*

$$\overline{\mathcal{F}_p^{Q_1}(i)} \subset \overline{\mathcal{F}_u^{Q_1}(i)}.$$

If a sandwiched singularity is primitive, the structure of the cluster of base points is much simpler as the set of points in the cluster $\mathcal{K}$ is completely ordered, which is not the case in general. From Proposition 16, we deduce the wanted result.

**Corollary 17.** [Corollary V.4.4] *If the Nash conjecture is true for primitive singularities, then it is also true for sandwiched singularities.*

In **Appendix A**, we provide the listings of three programs in language **C** implementing some of the algorithms proposed. These programs have been used to compute some of the examples presented throughout the memoir.

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Part of the results of this thesis has been published or will be published in

- J. Fernández-Sánchez, *On sandwiched singularities and complete ideals*, J. Pure Appl. Algebra **185** (2003), no. 1-3, 165–175. [19]
- J. Fernández-Sánchez, *Nash families of smooth arcs on a sandwiched singularity*, To appear in Math. Proc. Cambridge. Philos. Soc. [18]
- J. Fernández-Sánchez, *Equivalence of the Nash conjecture for primitive and sandwiched singularities*, To appear in Proc. Amer. Math. Soc. [17]

# Chapter I

# Preliminaries

In this chapter we review some concepts and well-known facts to be used throughout this memoir. All algebraic varieties are assumed to be defined over the field $\mathbb{C}$ of complex numbers, point means closed point and surface means irreducible algebraic surface.

## I.1 Blowing-ups

In this section and the next one, we introduce some concepts on blowing-ups and infinitely near points, and their relations with complete ideals in a local regular two-dimensional $\mathbb{C}$-algebra. The main reference is [11] and the reader is referred to it for the proofs of the facts presented here.

**Definition I.1.1.** Let $S$ be a smooth surface and fix a point $O \in S$. We will denote by $R = \mathcal{O}_{S,O}$ the complete local ring of $O$ on $S$ and by $\mathfrak{m}_O$ its maximal ideal, i.e. $R = \mathbb{C}[[x,y]]$ for any pair of generators (local coordinates) $x, y$ of $\mathfrak{m}_O$. We denote the blowing-up of $O$ on $S$ by $\pi_O : S_O \longrightarrow S$. The restriction of the morphism $\pi_O$ to $\pi_O^{-1}(S - \{O\})$ is an isomorphism onto $S - \{O\}$. $E := \pi_O^{-1}(O)$ is isomorphic to $\mathbb{P}^1$ and is called the *exceptional divisor* of $\pi_O$. It can be identified with the set of the tangent directions on $S$ at $O$.

**Definition I.1.2.** If $C$ is a curve on $S$ and $f \in R$ is a local equation for $C$ at $O$, then the *multiplicity of $C$ at $O$*, denoted by $e_O(C)$ is defined as the maximal integer $e$ such that $f \in m_O^e$.

Clearly, $e_O(C) > 0$ if and only if $O$ belongs to $C$.

**Definition I.1.3.** The pull-back $\pi_O^*(C)$ of $C$ by $\pi_O$ is called the *total transform of $C$*. It has the form

$$\pi_O^*(C) = \widetilde{C} + e_O(C)E$$

where $\widetilde{C}$ is defined as the closure of $\pi_O^{-1}(C \setminus \{O\})$ and so, it has finitely many intersections with $E$. $\widetilde{C}$ is called the *strict transform of* $C$ (after blowing up $O$).

**Definition I.1.4.** We call the exceptional divisor $E$ of blowing up $O$ on $S$ the *first (infinitesimal) neighbourhood of* $O$ (on $S$) and denote it by $F_O$. Its points are called the *points in the first neighbourhood.* If $i > 0$, we define by induction the *points in the i-th neighbourhood of* $O$ as the points in the first neighbourhood of some point in the $(i-1)$-th neighbourhood of $O$. The points which are in the $i$-th neighbourhood of $O$ for some $i > 0$ are also called *infinitely near points to O.* The point $O$ is a *proper* point of $S$ in order to distinguish it from the infinitely near ones.

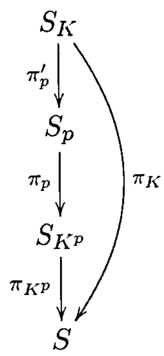
Let $q, p$ be two points in $S$ infinitely near or equal to $O$. We say that $q$ *precedes* $p$, denoted $q < p$, if and only if $p$ is infinitely near to $q$. We denote by $q \leq p$ if $p$ is infinitely near or equal to $q$. The relation $\leq$ is a partial ordering of the set of all points, proper or infinitely near; it is called the *natural ordering.* Every infinitely near point $p$ has a unique immediate predecessor, which is the point in the first neighbourhood of which $p$ lies.

Now we will deal with the blowing-up of a finite set of proper or infinitely near points in the surface $S$.

**Definition I.1.5.** A *cluster* in $S$ is a finite set $K$ of points equal or infinitely near to $O$ such that for every point $p \in K$, $K$ contains all points preceding $p$. The point $O$ is called the *origin* of the cluster. A *subcluster* $K'$ of $K$ is a subset of $K$ which is also a cluster in $S$. By a *maximal point* in $K$ we mean a maximal point in $K$ relative to the natural ordering in $K$. A total ordering $\preceq$ in $K$ is said to be *admissible* if for any pair $q, p \in K$ so that $q \leq p$, we have $q \preceq p$.

We denote the blowing-up of (all the points in) $K$ by $\pi_K : S_K \longrightarrow S$. See [11] §3.4 and §4.3 for a detailed construction of $\pi_K$ and its basic properties. An important feature is that fixed an admissible ordering $\preceq$ on $K$, $\pi_K$ is the composite of the sequence of blowing-ups of the points in $K$ following this ordering, and so the surface $S_K$ and the morphism $\pi_K$ do not depend on the order $\preceq$ chosen in $K$.

Given a point $p$ of a cluster $K$, take the subcluster $K^p = \{q \in K \mid q < p\}$. Then, $p$ is a proper point on the surface $S_{K^p}$. Let $\pi_p : S_p \longrightarrow S_{K^p}$ be the blowing-up of $p$ and write $E_p^0 \subset S_p$ for the exceptional divisor obtained. We can identify the points of $K$ not yet blown up to their corresponding ones on the surface $S_p$. Blowing up these points gives a morphism $\pi'_p : S_K \longrightarrow S_p$ and we have $\pi_K = \pi'_p \circ \pi_p \circ \pi_{K^p}$.




**Definition I.1.6.** The divisor $E_K = \pi_K^{-1}(\{O\})$ on the surface $S_K$ is called the *exceptional divisor of* $\pi_K$. Each irreducible component of $E_K$ is the strict transform of some $E_p^0$, $p \in K$ by the corresponding morphism $\pi'_p$, and hence it is isomorphic to $\mathbb{P}^1$. We denote it by $E_p^{S_K}$ and write $\overline{E_p^{S_K}}$ for the total transform of $E_p^0$ by $\pi'_p$.

**Definition I.1.7.** The *strict transform* $\widetilde{C}^{S_K}$ *of* $C$ *on* $S_K$ is the iterated strict transform of $C$ by the blowing-ups of the points in $K$. Equivalently, it is the closure of $\pi_K^{-1}(C \setminus \{O\})$ on $S_K$.

We define the *multiplicity of* $C$ *at* $p \in K$, denoted by $e_p(C)$, as the multiplicity of $\widetilde{C}^{S_{K^p}}$ at the proper point $p \in S_{K^p}$. Once an admissible total ordering on a cluster $K$ has been fixed, we may take the cluster $K$ as a set of indexes. Then, the vector $\mathbf{e}_K(C) = (e_p(C))_{p\in K}$ will be called the *vector of effective multiplicities* of $C$ at the points of $K$.

We will say that $q$ *belongs (as an infinitely near point) to* $C$ if $e_q(C) > 0$.

**Definition I.1.8.** Let $C$ be a curve on $S$. The *total transform of* $C$ *on* $S_K$ is the pull-back $\pi_K^*(C)$ of $C$ by $\pi_K$. It has the form

$$\pi_K^*(C) = \widetilde{C}^{S_K} + \sum_{p\in K} v_p(C) E_p^{S_K},$$

where each $v_p(C)$ is a non-negative integer called the *effective p-value* of $C$. We put $\mathbf{v}_K(C) = (v_p(C))_{p\in K}$ and call it the *vector of effective values of* $C$ at the points of $K$. We write

$$E_C^{S_K} = \sum_{p\in K} v_p(C) E_p^{S_K}$$

for the exceptional part of $\pi_K^*(C)$.

**Definition I.1.9.** Let $q, p$ be a pair of proper or infinitely near points on $S$. The point $p$ is said to be *proximate* to $q$ if and only if $p$ is infinitely near to $q$ and belongs as a proper or an infinitely near point to $E_q^0$. We will write $p \to q$ to mean $p$ is proximate to $q$.

Note that an infinitely near point is always proximate to its immediate predecessor. In fact, any infinitely near point $p$ is proximate to either one or two preceding points. An infinitely near point proximate to two points is called a *satellite point*; otherwise, it is called a *free point*.

**Definition I.1.10.** Let $C \subset S$ be a curve and $p$ a point of $S$, infinitely near or equal to $O$. We will say that $p$ is a *singular point of* $C$, or equivalently, $C$ *is singular at* $p$, if either

1. $e_p(C) > 1$, or
2. $e_p(C) = 1$ and $p$ is satellite, or
3. $e_p(C) = 1$ and $p$ precedes a satellite point belonging to $C$.

**Proposition I.1.11.** (Proximity equalities, [11] 3.5.3) *For every point $q$ (proper or infinitely near) and every curve $C$ on $S$,*

$$e_q(C) = \sum_{p \to q} e_p(C).$$

The next definition introduces a diagram due to Enriques that encodes the proximity relations between the points of a cluster $K$.

**Definition I.1.12.** ([16] Book 4, Ch. 1 and also [9] and [11] 3.9) Let $K$ be a cluster with origin at $O$. The *Enriques diagram $D(K)$ of $K$* is a tree-graph where each vertex represents a different point in $K$ and the vertex representing $O$ is taken as the origin of the graph. Each edge joins a pair of vertices that represent points one of which is in the first neighbourhood of the other. We name each vertex as the point it represents. The edges are of two different kinds, straight or curved, according to the following rules:

1. If $p$ is a free point in the first neighbourhood of $q$, the the edge joining $p$ and $q$ is smooth curved and if $q \neq O$, it has at $q$ the same tangent as the edge ending at $q$.
2. If $p$ is a point in the first neighbourhood of $q$, then the edges connecting all points proximate to $q$ in the successive neighbourhoods of $p$ are shaped into a line segment which starts at $p$ and is orthogonal to the edge $qp$ at $p$.

The proximity relations between the points of a cluster $K$ can also be encoded in a $K \times K$ matrix $\mathbf{P}_K$, called the *proximity matrix* of $K$ and introduced by Du Val in [14] (see also [11]). It is defined by taking $\mathbf{P}_K = (m_{q,p})$ with

$$m_{q,p} = \begin{cases} 1 & \text{if } p = q \\ -1 & \text{if } p \text{ is proximate to } q \\ 0 & \text{otherwise.} \end{cases}$$

The proximity matrix $\mathbf{P}_K$ relates the multiplicities and the values of a curve $C$ at the points of $K$.

**Lemma I.1.13.** *For any curve $C$ on $S$,*

$$\mathbf{v}_K(C) = \mathbf{P}_K^{-1}\mathbf{e}_K(C).$$

*Thus, for each $p \in K$,*

$$v_p(C) = e_p(C) + \sum_{p \to q} v_q(C).$$

**Notation I.1.14.** (see [29] V. §1 ot [5] I) If $X$ is a surface and $C, C'$ are curves on $X$ having no common irreducible components, we denote by $|C{\cdot}C'|_X$ the *intersection number of $C$ and $C'$ on $X$*. We denote by $C^2$ *the self-intersection (number)* $|C{\cdot}C|_X$ of $C$ on $X$. If $P \in X$, $[C, C']_P$ is the *intersection multiplicity of $C$ and $C'$ at $P$*.

**Proposition I.1.15.** (Proposition V.1.4 of [29]) *If $C$ and $C'$ are curves on $X$ having no common irreducible components, then*

$$|C{\cdot}C'|_X = \sum_{P \in C \cap C'} [C, C']_P.$$

**Proposition I.1.16.** *Let $K$ be a cluster with origin at $O$.*

1. *If $K = \{p_1, \ldots, p_s\}$, there is an isomorphism*

$$\begin{array}{ccc} Pic(S) \oplus \mathbb{Z}^s & \xrightarrow{\cong} & Pic(S_K) \\ (D, n_1, \ldots, n_s) & \mapsto & \pi_K^*(D) + n_1 E_{p_1}^{S_K} + \cdots + n_s E_{p_s}^{S_K}. \end{array}$$

2. *Let $C$ and $D$ be divisors on $S$. Then*

$$|\pi_K^* C{\cdot}\pi_K^* D|_{S_K} = [C, D]_O,$$

$$|\pi_K^* C{\cdot}\overline{E_p^{S_K}}|_{S_K} = 0,$$

$$|\pi_K^* C{\cdot}E_p^{S_K}|_{S_K} = 0,$$

*for any point $p$ in $K$.*

3. *For any couple $p, q \in K$, $|E_q^{S_K}{\cdot}E_p^{S_K}|_{S_K} = 1$ if $E_p^{S_K} \cap E_q^{S_K} \neq \emptyset$, i.e. if one of the points $q$ or $p$ is maximal among the points in $K$ that are proximate to the other, and $|E_q^{S_K}{\cdot}E_p^{S_K}|_{S_K} = 0$, otherwise. Moreover, $|E_p^{S_K}{\cdot}E_p^{S_K}|_{S_K} = -r_p - 1$, where $r_p$ is the number of points in $K$ proximate to $p$.*

4. *If $C$ is a curve on $S$, for all $p \in K$,*

$$|\widetilde{C}^{S_K} \cdot E_p^{S_K}|_{S_K} = e_p(C) - \sum_{q \notin K, q \to p} e_q(C).$$

5. *Projection formula: if $C$ is a curve on $S$ and $D$ is a curve on $S_K$, then*

$$|\pi_K^* C \cdot D|_{S_K} = [C, (\pi_K)_* D]_O.$$

**Definition I.1.17.** The *intersection matrix* $\mathbf{A}_K$ of the cluster $K$ is a $K \times K$ matrix defined by taking $|E_p^{S_K} \cdot E_q^{S_K}|_{S_K}$ as the entry on the $q$-th row and $p$-th column.

An easy computation using Proposition I.1.16 gives the relation:

**Lemma I.1.18.** $\mathbf{A}_K = -\mathbf{P}_K^t \mathbf{P}_K$.

**Definition I.1.19.** A pair $\mathcal{K} = (K, \nu)$ where $K$ is a cluster and $\nu : K \longrightarrow \mathbb{Z}$ is an arbitrary map will be called a *weighted cluster*. The map $\nu$ is called a *system of virtual multiplicities* for $K$ and $K$ is called the *underlying cluster* of $\mathcal{K}$. We usually write $\nu_p = \nu(p)$ and call it the virtual multiplicity of $\mathcal{K}$ at the point $p$. We take $\boldsymbol{\nu}_{\mathcal{K}} = (\nu_p)_{p \in K}$ as a column vector indexed by $K$.

**Definition I.1.20.** Let $p$ be a point of a weighted cluster $\mathcal{K} = (K, \nu)$. The *excess* $\rho_p^{\mathcal{K}}$ *of* $\mathcal{K}$ *at* $p$ is defined as

$$\rho_p^{\mathcal{K}} = \nu_p - \sum_{q \in K, q \to p} \nu_q.$$

The cluster $\mathcal{K}$ is said to be *consistent* if all the excesses $\rho_p^{\mathcal{K}}, p \in K$ are non-negative and *strictly consistent* if moreover, there are no points in $K$ having virtual multiplicity zero. We will say that $p$ is a *dicritical* point of $\mathcal{K}$ if $\rho_p^{\mathcal{K}} > 0$, and denote by $\mathcal{K}_+ = \{p \in K \mid \rho_p^{\mathcal{K}} > 0\}$ the set of dicritical points of $\mathcal{K}$.

From now on and if there is no danger of confusion, we write $\rho_p$ (instead of $\rho_p^{\mathcal{K}}$) for the excess of $\mathcal{K}$ at a point $p$.

**Definition I.1.21.** For each $p \in K$, the value $v_p$ given recursively by

$$v_p = \nu_p + \sum_{p \to q} v_q$$

is called the *virtual value of* $K$ *at* $p$. The vector $\mathbf{v}_{\mathcal{K}} = \mathbf{P}_K^{-1} \boldsymbol{\nu}_{\mathcal{K}}$ is called the *system of virtual values of* $\mathcal{K} = (K, \nu)$.

The *excess vector* $\boldsymbol{\rho}_{\mathcal{K}} = (\rho_p)_{p \in K}$ *of* $\mathcal{K} = (K, \nu)$ can be computed from the system of virtual values in the following way:

**Lemma I.1.22.** $\rho_{\mathcal{K}} = -\mathbf{A}_K \mathbf{v}_{\mathcal{K}} = \mathbf{P}_K^t \nu_{\mathcal{K}}$.

**Remark I.1.23.** Fixed a weighted cluster $\mathcal{K}$, the system of virtual multiplicities, the system of virtual values and the excess vector are equivalent data and any of them determine the others.

**Notation I.1.24.** Given a curve $C$ on $S$, we denote by $\mathcal{S}_0(C)$ the cluster of the singular points of $C$, taking the effective multiplicities of $C$ as virtual multiplicities. If $p_1, \ldots, p_m$ are the first points on the branches of $C$ which are not singular, we write $\mathcal{S}(C)$ for the cluster obtained from $\mathcal{S}_0(C)$ by adding $p_1, \ldots, p_m$ counted once.

**Definition I.1.25.** If $C$ is a curve on $S$ and $\mathcal{K} = (K, \nu)$ is a weighted cluster in $S$, we say that the curve *goes (virtually) through* $\mathcal{K}$ if the divisor on the surface $S_K$

$$\check{C}^{\mathcal{K}} := \pi_K^* C - \sum_{p \in K} \nu_p \overline{E_p^{S_K}}$$

is effective. $\check{C}^{\mathcal{K}}$ is called the *virtual transform of* $C$ *relative to the virtual multiplicities of* $\mathcal{K}$. We denote

$$H_{\mathcal{K}} = \{g \in R \mid C : g = 0 \text{ goes virtually through } \mathcal{K}\} \cup \{0\} \subset R$$

which is an $\mathfrak{m}_O$-primary ideal of $R$. Given two weighted clusters $\mathcal{K}$ and $\mathcal{K}'$, we say that they are *equivalent* if $H_{\mathcal{K}} = H_{\mathcal{K}'}$, and we write $\mathcal{K} \prec \mathcal{K}'$ and say that $\mathcal{K}'$ contains $\mathcal{K}$ if $H_{\mathcal{K}} \supset H_{\mathcal{K}'}$.

Note that the virtual transform $\check{C}^{\mathcal{K}}$ can be written in the form

$$\check{C}^{\mathcal{K}} = \widetilde{C}^{S_K} + \sum_{p \in K} u_p(C) E_p^{S_K},$$

where the vector $\mathbf{u}_K(C) = (u_p(C))_{p \in K}$ is obtained as

$$\mathbf{u}_K(C) = \mathbf{P}_K^{-1}(\mathbf{e}_K(C) - \nu_{\mathcal{K}}).$$

**Definition I.1.26.** If we have the equality of vectors

$$\mathbf{e}_K(C) = \nu_{\mathcal{K}}$$

then we say that $C$ *goes virtually* through the weighted cluster $\mathcal{K}$ *with effective multiplicities equal to the virtual ones.* If moreover, $C$ has no singular points outside of $K$ we say that $C$ *goes sharply* through $\mathcal{K}$.

**Lemma I.1.27.** *With the notation as above:*

1. *The curve $C$ goes virtually through $\mathcal{K}$ if and only if* $\mathbf{u}_K(C) \geq 0$.

2. *The following three assertions are equivalent:*

   (a) *the curve $C$ goes through $\mathcal{K}$ with effective multiplicities equal to the virtual ones.*

   (b) $\mathbf{u}_K(C) = 0$.

   (c) *the strict transform $\tilde{C}^{S_K}$ equals the virtual transform $\check{C}^{\mathcal{K}}$ relative to the virtual multiplicities of $\mathcal{K}$.*

**Theorem I.1.28.** ([11] Theorem 4.2.2) *If there is a curve $C$ going through $\mathcal{K} = (K, \nu)$ with effective multiplicities equal to the virtual ones, then $\mathcal{K}$ is consistent.*

*Conversely, assume that $\mathcal{K}$ is consistent. Then, fixed a set $T$ of points infinitely near to $O$ and not in $K$, there exists a curve $C \subset S$ going through $\mathcal{K}$ with effective multiplicities equal to the virtual ones and missing all the points in $T$.*

It follows that the curves going through $\mathcal{K}$ with effective multiplicities equal to the virtual ones share no points other than those in $K$.

**Proposition I.1.29.** ([11] Proposition 4.2.6) *Let $\mathcal{K} = (K, \nu)$ be a consistent cluster. Then*

1. *All curves going sharply through $\mathcal{K}$ are reduced.*

2. *If $C$ goes sharply through $\mathcal{K}$, then for all $p \in K$, $C$ has just $\rho_p$ branches through $p$ missing all points after $p$ in $K$. Hence, $C$ has a total of $\sum_{p \in K} \rho_p$ branches.*

3. *The germs at $O$ of any two curves going sharply through $K$ are equisingular.*

Now, we want to define a topology in $R$ to give a precise meaning to the phrase *generic element* of a linear system or ideal in $R$. To this aim, define the closed sets of this topology as the sets of the form

$$\{f = \sum_{i,j} a_{i,j} x^i y^j \mid P(a_{i,j}) = 0, P \in \mathcal{P}\}$$

where $\mathcal{P}$ is a subset of the ring of the polynomials in the (infinitely many) variables $X_{i,j}, i, j \geq 0$. This topology is called the Zariski topology of $R = \mathbb{C}[x, y]$ and its closed sets are the sets of all series whose coefficients satisfy certain given algebraic relations.

**Theorem I.1.30.** (Corollary 4.2.8 of [11]) *Let $\mathcal{K} = (K, \nu)$ be a consistent cluster and $T$ a finite set of points so that $T \cap K = \emptyset$. Then there is an non-empty Zariski-open set $U \subset H_{\mathcal{K}}$ such that for any $f \in U$, the curve $C : f = 0$ goes sharply through $\mathcal{K}$ and no point in $T$ belongs to $C$.*

The infinitely near points also give a geometrical idea of the intersection multiplicity of two curves at $O$ by means of a formula due to M. Noether.

**Theorem I.1.31.** (Noether's formula, [11] Theorem 3.3.1) *Let $C$ and $D$ be two curves on $S$. The intersection multiplicity $[C, D]_O$ is finite if and only if $C$ and $D$ share finitely many points infinitely near to $O$, and in such a case*

$$[C, D]_O = \sum_p e_p(C)e_p(D)$$

*the summation running for $p$ infinitely near or equal to $O$.*

**Definition I.1.32.** We define the *intersection multiplicity* of $\mathcal{K} = (K, \nu)$ and a curve $C$ at $O$ as being

$$[\mathcal{K}, C]_O = \sum_{p \in K} \nu_p e_p(C).$$

As it is clear, $[\mathcal{K}, C]_O$ equals the intersection multiplicity of $C$ with any curve going through $\mathcal{K}$ with effective multiplicities equal to the virtual ones and sharing no points with $C$ outside of $K$.

The *self-intersection* of $\mathcal{K}$ is

$$\mathcal{K}^2 = \sum_{p \in T} \nu_p^2.$$

Obviously, $\mathcal{K}^2 = [C, D]_O$ for any couple of curves $C, D$ going sharply through $\mathcal{K}$ and sharing no points outside of it.

## I.2 Complete ideals and weighted clusters

Next we recall some standard notions from commutative algebra. Let $R$ be a two-dimensional local domain.

**Definition I.2.1.** *(Appendix 4 of [65])* An ideal $I \subset R$ is said to be *complete* (or *integrally closed*) if it can be defined by valuations in $R$, that is, if there exists a set of valuations $\{v_l\}_{l \in \Lambda}$ of $R$ and for each $l \in \Lambda$, an element $\alpha_l$ in the value group of $v_l$ so that

$$I = \{f \in R \mid v_l(f) \geq \alpha_l, \forall l \in \Lambda\} = \cap_l (IR_l \cap R)$$

where the intersection runs over all valuations rings $R_l$ containing $R$ and with the same function field.

It is clear from the definition that the maximal ideal $\mathfrak{m}_O$ of $R$ is a complete ideal and that arbitrary intersections of complete ideals are complete too. Hence, if $I$ is an ideal in $R$, the intersection $\overline{I}$ of all complete ideals containing $I$ is also complete: it is called the *completion* or the *integral closure of* $I$. Obviously, $I$ is complete if and only if $I = \overline{I}$. The elements of $\overline{I}$ are said to be *integral over* $I$.

*Remark*: One may equivalently define an element to be integral over $I$ as an element $f \in R$ that satisfies a relation (integral dependence relation) of the form

$$f^n + a_1 f^{n-1} + \cdots + a_n = 0$$

where $n > 0$ and $a_i \in I^i$ for $i = 1, \ldots, n$ (see App. 4 in Vol.II of [65] for details). Then, the completion $\overline{I}$ of $I$ is the set of all the elements of $R$ integral over $I$.

Now, we come back to the case $R = \mathbb{C}[[x, y]]$.

**Lemma I.2.2.** ([11] Lemma 8.3.4) *For any weighted cluster* $\mathcal{K} = (K, \nu)$, *the ideal* $H_{\mathcal{K}} \subset R$ *is complete and* $\mathfrak{m}_O$*-primary.*

**Remark I.2.3.** If $K = \{p_1, \ldots, p_s\}$, then $H_{\mathcal{K}}$ is the stalk at $O$ of the ideal sheaf

$$(\pi_K)_*(-\nu_{p_1}\overline{E_{p_1}^{S_K}} - \cdots - \nu_{p_s}\overline{E_{p_s}^{S_K}}).$$

We will see in Theorem I.2.5 that if $I$ is a complete $\mathfrak{m}_O$-primary ideal, then there exists a weighted cluster $\mathcal{K}$ with origin at $O$ so that $I = H_{\mathcal{K}}$.

**Definition I.2.4.** A *linear system* is a set of curves defined by the non-zero elements of an ideal $I \subset R$. A linear system has thus the form

$$\mathfrak{L}_I = \{C : f = 0 \mid f \in I \setminus \{0\}\}$$

where $I$ is an ideal of $R$. The *irrelevant linear system* is the linear system defined by the ideal (1).

Given an ideal $I \subset R$, or its corresponding linear system $\mathfrak{L} = \mathfrak{L}_I$, set $e_O(I) = e_O(\mathcal{L}_I) = \min\{e_O(C) \mid C : h = 0, h \in I\}$. In other words, $I$ is contained in $\mathfrak{m}_O^{e_O(I)}$ but not in $\mathfrak{m}_O^{e_O(I)+1}$. Now, if $p$ is any point in the first neighbourhood of $O$, the virtual transforms of the curves of $\mathfrak{L}_I$ relative to $e_O(I)$ as virtual multiplicity of $O$ define a linear system $\mathfrak{L}_p$. Notice that if $\mathcal{O}_p$ is the local ring at $p$ and $\varphi_p : \mathcal{O} \longrightarrow \mathcal{O}_p$ the morphism induced by blowing-up, $\mathfrak{L}_p$ is the linear system defined by the ideal generated by $z^{-e_O(I)}\varphi_p(I)$. This linear system is called the *transform of* $\mathfrak{L}$ *with origin at* $p$, and we set $e_p(\mathfrak{L}) = e_p(\mathfrak{L}_p)$ and call it the *multiplicity of* $\mathfrak{L}$ *at* $p$. We extend these definitions to all points in the successive neighbourhoods by using induction.

Now, we define the (weighted) cluster of base points, $BP(I)$ by taking the points $p$ infinitely near or equal to $O$ for which $\mathfrak{L}_p$ is not irrelevant, each $p$ taken with virtual multiplicity $e_p(\mathfrak{L})$. From the definition, if $I \neq (1)$ it follows that $BP(I)$ is strictly consistent. Given an ideal $I \subset R$, a *generic curve* in $\mathfrak{L}_I$ (or in $I$) means an element in a non-empty Zariski-open set in $I$.

**Theorem I.2.5.** ([11] Theorem 8.3.5 and Corollary 8.3.6) *Let $I$ be an $\mathfrak{m}_O$-primary ideal of $R$. If $C : g = 0$ goes through $BP(I)$, then for any valuation $v$ of $R$, $v(g) \geq \min\{v(f) \mid f \in I\}$. In particular, $\overline{I} = H_{BP(I)}$.*

Now, we recall the theorem of Zariski about the factorization of complete ideals of $R$ as a product of irreducible ones. The original paper on the subject is [61] and a later version appeared in Vol. II App. 5 of [65].

**Theorem I.2.6.** (Zariski [61]) *Let $I$ be a complete $\mathfrak{m}_O$-primary ideal of $R$. Then $I$ has a unique decomposition as a product of irreducible complete $\mathfrak{m}_O$-primary ideals*

$$I = \prod_{i=1}^{n} I_i^{\alpha_i}.$$

Now, we will translate this decomposition to weighted clusters. From Theorem I.2.5, we know that any complete ideal is just the ideal defined by the equations of the curves going through a consistent cluster. Conversely, we have

**Proposition I.2.7.** *If $\mathcal{K}$ is a strictly consistent cluster, then $\mathcal{K} = BP(H_{\mathcal{K}})$.*

Given two weighted clusters $\mathcal{K} = (K, \nu)$ and $\mathcal{K}' = (K', \nu')$, define the sum $\mathcal{K} + \mathcal{K}'$ as the weighted cluster whose set of points is $K \cup K'$ and whose virtual multiplicities are $\nu_p + \nu'_p$ for $p \in K \cup K'$. The set of strictly consistent clusters with this operation is a semigroup that we will denote by $\mathbf{W}$. A strictly consistent cluster is *irreducible* if it is so as an element of the semigroup $\mathbf{W}$. It is immediate to see that any irreducible cluster consists of a point $p$ and all points preceding it and has excesses $\rho_p = 1$ and $\rho_q = 0$ for $q \neq p$. We denote such a cluster by $\mathcal{K}(p)$.

**Proposition I.2.8.** *If $\mathcal{K}$ is a strictly consistent cluster and $\rho = (\rho_p)_{p \in K}$ is its excess vector, then*

$$\mathcal{K} = \sum_{p \in K} \rho_p \mathcal{K}(p).$$

The next theorem states the main link between weighted clusters and complete ideals.

**Theorem I.2.9.** ([11] 8.4.11) *The set $\mathbf{I}$ of the complete $\mathfrak{m}_O$-primary ideals of $R$ equipped with the product of ideals is a semigroup and the maps*

$$\begin{aligned} BP \ :\ & \mathbf{I} \to \mathbf{W} \\ & I \mapsto BP(I) \end{aligned}$$

*and*

$$H \;:\; \boldsymbol{W} \longrightarrow \boldsymbol{I}$$
$$\mathcal{K} \longmapsto H_{\mathcal{K}}$$

*are reciprocal isomorphism between* $\boldsymbol{I}$ *and* $\boldsymbol{W}$.

Irreducible complete ideals are also called *simple ideals* for short. They correspond to irreducible clusters by the isomorphisms of the above theorem.

**Notation I.2.10.** We denote by $I_p$ the simple ideal corresponding to the cluster $\mathcal{K}(p)$, i.e. $I_p = H_{\mathcal{K}(p)}$.

**Notation I.2.11.** Let $J \subset R$ be a complete $\mathfrak{m}_O$-primary ideal and $\mathcal{T} = BP(J)$. Then, if $K$ is a cluster, we denote by $E_J^{S_K}$ (or $E_{\mathcal{T}}^{S_K}$) the exceptional divisor on $S_K$ given by

$$\sum_{p\in K} v_p(J) E_p^{S_K}$$

where $v_p(J) = \min\{v_p(g) \mid g \in J\}$. In particular, if $C$ goes sharply through $\mathcal{T} = BP(J)$, then

$$E_C^{S_K} = E_{\mathcal{T}}^{S_K} = E_J^{S_K}.$$

### I.2.1 Unloading

Next we will describe a procedure due to Enriques [16] called *unloading* (see also [9] or §4.6 of [11]) that given a non-consistent cluster $\mathcal{K} = (K, \nu)$ allows to compute the effective multiplicities of a generic curve going through it. At the $i$-th step of the procedure we get a new cluster $\mathcal{K}_i$ by unloading some amount of multiplicity on a point $p \in K$ at which the excess $\rho_p^{\mathcal{K}_i} < 0$ from the points that are proximate to it. Unloading may be described in terms of either the virtual multiplicities of the virtual values, and both descriptions are equivalent by means of Lemma I.1.13.

**Unloading multiplicities:** if $\mathcal{K} = (K, \nu)$ is a weighted cluster and $\rho_p^{\mathcal{K}} < 0$ for some $p \in K$, define $n$ as being the least integer non-less that $-\frac{\rho_p^{\mathcal{K}}}{r_p+1}$. Then, define a new cluster $\mathcal{K}' = (K, \nu')$ by taking

$$\nu'_q = \begin{cases} \nu_p + n & \text{if } q = p \\ \nu_q - n & \text{if } q \to p \\ \nu_q & \text{otherwise.} \end{cases}$$

We say that $\mathcal{K}'$ is the cluster obtained by *unloading multiplicities on p*.

**Unloading values:** Denote by $\mathbf{1}_p$ the column vector indexed on $K$ all whose components are zero but for the one corresponding to $p$, which is one. If $\rho_p^{\mathcal{K}} < 0$, then by Lemma I.1.22, we have

$$\mathbf{1}_p^t \mathbf{A}_K \mathbf{v}_{\mathcal{K}} > 0.$$

Denote by $n$ the least integer such that

$$\mathbf{1}_p^t \mathbf{A}_K (\mathbf{v}_{\mathcal{K}} + n\mathbf{1}_p) \leq 0.$$

Then, $\mathbf{v}'_K = \mathbf{v}_{\mathcal{K}} + n\mathbf{1}_p$ is a new system of virtual values for $K$ and defines a new weighted cluster $\mathcal{K}'$, that is called the cluster obtained by *unloading values on p*.

**Notation I.2.12.** From now on, if $\mathcal{K}$ is not a consistent cluster, we will denote by $\widetilde{\mathcal{K}}$ the consistent cluster equivalent to it obtained by unloading multiplicities.

**Definition I.2.13.** Unloading on a point of excess equal to $-1$ is called *tame* unloading.

**Remark I.2.14.** From the description of unloading using values it follows that if tame unloading is performed on $p$, then the virtual value on $p$ increases by one while that of the other points of $K$ remain unaffected.

Both procedures give rise to the same weighted cluster $\mathcal{K}'$ and we have:

**Theorem I.2.15.** (Theorem 4.6.1 of [11]) *Let $\mathcal{K}$ be a weighted cluster and assume that $\rho_p^{\mathcal{K}} < 0$ for some $p \in K$. Then, $\mathcal{K}$ and the weighted cluster $\mathcal{K}'$ obtained by unloading on p are equivalent.*

**Theorem I.2.16.** (Theorem 4.6.2 of [11]) *Assume that $\mathcal{K}$ is a non-consistent weighted cluster. Put $\mathcal{K} = \mathcal{K}^0$ and, inductively, as far as $\mathcal{K}^{i-1}$ is not consistent, define $\mathcal{K}^i$ from $\mathcal{K}^{i-1}$ by unloading on a suitable point. Then, we have:*

(a) *There is an $m$ so that $\mathcal{K}^m$ is consistent, has the same underlying cluster than $\mathcal{K}$ and $H_{\mathcal{K}} = H_{\mathcal{K}^m}$.*

(b) *$\mathcal{K}^m$ is the only consistent cluster equivalent to $\mathcal{K}$ having the same points as $\mathcal{K}$. Therefore, it does not depend on the choice of the points on which the unloading are performed.*

To close this section we state some definitions and results that will be useful throughout this memoir.

**Definition I.2.17.** [56] If $C$ is a reduced curve, the *order of singularity of $C$ at $P$* is

$$\delta_P(C) = \dim_{\mathbb{C}} \frac{\overline{\mathcal{O}_{C,P}}}{\mathcal{O}_{C,P}},$$

where $\overline{\mathcal{O}_{C,P}}$ is the integral closure of $\mathcal{O}_{C,P}$ in its full quotient ring.

If $C$ is contained on a (not necessary regular) surface $X$, we denote

$$\delta_X(C) = \sum_{P\in X} \delta_P(C)$$

and call it the *order of singularity of $C$ on $X$*.

**Theorem I.2.18.** *With the notation as above, assume that $C \subset S$ goes through $O$. Then*

$$\delta_O(C) = \sum_p \frac{e_p(C)(e_p(C)-1)}{2},$$

*the summation running on all points p infinitely near or equal to $O$.*

If $\mathcal{K} = (K, \nu)$ is a weighted cluster, we write

$$\delta_O(\mathcal{K}) = \sum_{p\in K} \frac{\nu_p(\nu_p - 1)}{2}.$$

Note that if $\mathcal{K}$ is consistent, $\delta_O(\mathcal{K})$ equals the order of singularity of any curve going sharply through $\mathcal{K}$.

**Definition I.2.19.** Let $\mathcal{K} = (K, \nu)$ be the weighted cluster on $S$. We define the *virtual codimension of $\mathcal{K}$* as

$$c(\mathcal{K}) = \sum_{p\in K} \frac{\nu_p(\nu_p + 1)}{2}.$$

**Remark I.2.20.** For any weighted cluster,

$$\mathcal{K}^2 = \delta_O(\mathcal{K}) + c(\mathcal{K}).$$

**Proposition I.2.21.** *If $\mathcal{K} = (K, \nu)$ is a consistent cluster, then*

$$c(\mathcal{K}) = \dim_{\mathbb{C}}(\frac{R}{H_{\mathcal{K}}}).$$

**Corollary I.2.22.** (Corollary 3.2 of [42]) *If $I \subset R$ is a complete $\mathfrak{m}_O$-primary ideal and $\mathcal{K} = (K, \nu)$ is the cluster of base points of $I$, then*

$$\dim_{\mathbb{C}} \frac{I}{\mathfrak{m}_O I} = \nu_O + 1.$$

**Proposition I.2.23.** (Corollary 4.7.3 of [11]) *If $\mathcal{K} = (K, \nu)$ is not consistent and $\widetilde{K}$ is the consistent cluster obtained from $\mathcal{K}$ by unloading multiplicities, then*

$$c(\mathcal{K}) \geq \dim_{\mathbb{C}}(\frac{R}{H_{\mathcal{K}}})$$

*and the equality holds if and only if all unloading steps leading from $\mathcal{K}$ to $\widetilde{\mathcal{K}}$ are tame.*

**Definition I.2.24.** [10] A sequence of weighted clusters $\{\mathcal{K}_i\}_{i=0,\dots,n}$ such that

$$H_{\mathcal{K}_0} \supset H_{\mathcal{K}_1} \supset \cdots \supset H_{\mathcal{K}_n}$$

and $\dim_{\mathbb{C}}(\frac{H_{\mathcal{K}_{i-1}}}{H_{\mathcal{K}_i}}) = 1$ for $i = 1, \dots, n$ is called a *flag of clusters* with ends $\mathcal{K}_0$ and $\mathcal{K}_n$.

## I.3 Rational singularities

The main references for this section are [3] and [41].

**Definition I.3.1.** A *normal surface singularity* (over $\mathbb{C}$) is a pair $(X, Q)$ consisting in the spectrum $X = \mathrm{Spec}(R)$ of a noetherian normal complete two-dimensional $\mathbb{C}$-algebra $R$, and the closed point $Q$.

**Definition I.3.2.** A normal surface singularity $(X, Q)$ is said to be a *rational surface singularity* if there exists a desingularization $f : S' \longrightarrow X$ such that $H^1(S', \mathcal{O}_{S'}) = 0$. We also say that $\mathcal{O}_{X,Q}$ has a rational singularity.

Note that if $\mathcal{O}_{X,Q}$ is regular, then it has a rational singularity (it is enough to take $S' = \mathrm{Spec}(\mathcal{O}_{X,Q})$). The next result says that the condition for a normal surface singularity to be rational is independent of the resolution.

**Proposition I.3.3.** (Proposition 1.2 of [41]) *Let $(X, Q)$ be a rational surface singularity and let $\pi : S' \longrightarrow X$ be a birational map of finite type.*

*(i) If $P \in S'$ is a normal point, then the local ring $\mathcal{O}_{S',P}$ has a rational singularity.*

*(ii) If $S'$ is normal and $\pi$ is proper, then $H^1(S', \mathcal{O}_{S'}) = 0$.*

Let $(X,Q)$ be a normal surface singularity and fix a desingularization $f : S' \longrightarrow X$ of $(X,Q)$. Note that by Zariski's main theorem (see Corollary III.11.4 of [29]) and since $X$ is normal, the exceptional locus $E = f^{-1}(Q)$ is connected. We will denote by $\{E_i\}_{1\leq i\leq s}$ the irreducible components of $E$.

The following result is important.

**Proposition I.3.4.** ([45] §1 or [15]) *The intersection matrix*

$$A_Q^{S'} = (|E_i{\cdot}E_j|_{S'})_{1\leq i,j\leq s}$$

*is negative definite.*

**Definition I.3.5.** An *exceptional cycle* is a non-zero effective divisor on $S'$ with exceptional support. An exceptional cycle has thus the form

$$D = \sum_{i=1}^{s} r_i E_i$$

with $r_i \geq 0$ for all $i \in \{1,\dots,s\}$ and some $r_i > 0$.

If $D$ is an exceptional cycle, we will denote by $\mathcal{O}_D^*$ the sheaf whose sections over an open set $U$ are the units in the ring $\Gamma(U,\mathcal{O}_D)$ with multiplication as the group operation. Then, we have that $H^1(S',\mathcal{O}_D^*) \simeq PicD$ (see exercise III.4.5 of [29]). If $\mathcal{L}$ is an invertible sheaf of $D$, write $d_i(\mathcal{L}) = \deg(\mathcal{L}|_{E_i})$. Then the map $\mathcal{L} \mapsto (d_1(\mathcal{L}),\dots,d_s(\mathcal{L}))$ defines a homomorphism

$$d : H^1(S',\mathcal{O}_D^*) \longrightarrow \mathbb{Z}^s$$

which is surjective if $r_i > 0$ for all $i$.

The following lemma plays a basic role in the characterization of the rational surface singularities. Recall that given a projective scheme $D$ of dimension 1, the *arithmetic genus of* $D$ is

$$g(D) = 1 - \chi(\mathcal{O}_D) = \dim H^1(D,\mathcal{O}_D).$$

**Lemma I.3.6.** ([2], Theorem 1.7) *Let* $D = \sum_{i=1}^{s} r_i E_i$ *be an exceptional cycle with* $r_i > 0$ *for all* $i$. *The following conditions are equivalent:*

*(i)* $H^1(D,\mathcal{O}_D) = 0$

*(ii) for all exceptional cycle* $D'$ *such that* $0 < D' \leq D$, $g(D') = 0$

*(iii)* $d : H^1(S',\mathcal{O}_D^*) \longrightarrow \mathbb{Z}^s$ *is an isomorphism.*

As a consequence we obtain the following proposition

**Proposition I.3.7.** ([3], Proposition 1) *Let $(X,Q)$ be a normal surface singularity and $f : S' \longrightarrow X$ a resolution of $X$. Then, $(X,Q)$ is a rational surface singularity if and only if for every exceptional cycle $D > 0$, $g(D) \leq 0$.*

The criterion of Proposition I.3.7 gives information concerning the intersection of the exceptional components of the minimal resolution of a rational surface singularity.

**Corollary I.3.8.** ([7], Theorem 1.7) *Let $f : S' \longrightarrow X$ be the minimal resolution of a rational surface singularity $(X,Q)$ and let $\{E_i\}_{1\leq i\leq s}$ be the irreducible components of the exceptional locus of $f$. Then,*

*(i) for every $i$, $E_i$ is a smooth rational curve,*

*(ii) if $i \neq j$ and $E_i \cap E_j \neq \emptyset$, $E_i$ and $E_j$ intersect transversally,*

*(iii) $E_i \cap E_j \cap E_k = \emptyset$ if $i,j,k$ are different,*

*(iv) $E$ has no cycles.*

Back to the general case, the following result allows us to introduce the fundamental cycle of a normal surface singularity $(X,Q)$.

**Proposition I.3.9.** ([3], Proposition 2) *Let $(X,Q)$ be a normal surface singularity and fix a resolution $f : S' \longrightarrow X$ of $(X,Q)$. There exist exceptional cycles $D > 0$ such that*

$$|D{\cdot}E_i|_{S'} \leq 0$$

*for all $i$. Moreover, among all these cycles, there exists a unique smallest one. We call this cycle the fundamental cycle of $(X,Q)$ on $S'$ and denote it by $Z_Q^{S'}$.*

The fundamental cycle of $(X,Q)$ on $S'$ can be computed in a recurrent way as follows (see Proposition 4.1 of [36]): let $Z_1 = E_{i_0}$, for any $E_{i_0}$. Having defined $Z_j = \sum_{i=1}^{s} \alpha_i^j E_i$, if there exists an $E_{i_j}$ such that

$$|E_{i_j}{\cdot}Z_j|_{S'} = \mathbf{1}_{i_j}^t \mathbf{A}_Q^{S'} Z_j > 0,$$

let $Z_{j+1} = Z_j + E_{i_j}$; if $|E_i{\cdot}Z_j|_{S'} \leq 0$ for all $i \in \{1,\dots,s\}$, then $Z_Q = Z_j$.

**Remark I.3.10.** Compare this procedure with the description of unloading using virtual values in I.2.1.

**Definition I.3.11.** From now on, when talking about the *fundamental cycle* of a rational surface singularity $(X,Q)$ we will mean the fundamental cycle of its minimal resolution; we will denote it by $Z_Q$.

Following the terminology of Lipman in [41], we denote by $\mathbb{E}^+_{S'}$ the set of divisors $D$ on $S'$ with exceptional support for $f$ such that $|D{\cdot}E_i|_{S'} \leq 0$ for all $i \in \{1,\dots,s\}$. Note that each $D \in \mathbb{E}^+_{S'}$ is an effective divisor: if $D = A - B$ with $A$ and $B$ effective without common components, then

$$0 \geq |D{\cdot}B| = |A{\cdot}B| - |B{\cdot}B|$$

where $|A{\cdot}B| \geq 0$ and $-|B{\cdot}B| \geq 0$ since the intersection matrix is negative definite. Therefore, $|A{\cdot}B| = 0$ and $|B{\cdot}B| = 0$, and it follows that $B = 0$.

The following theorem gives another characterization of rational surface singularities.

**Theorem I.3.12.** ([3], Theorem 3) *Keep the notation as in Proposition I.3.9. Then, $g(Z_Q) \geq 0$ and $g(Z_Q) = 0$ if and only if $(X,Q)$ is a rational surface singularity.*

The following result is due to Artin and has great importance in the study of the intersection theory on rational surface singularities.

**Proposition I.3.13.** ([3], Proof of Theorem 4) *Keep the notation of Proposition I.3.9 and assume that $(X,Q)$ is a rational surface singularity. Let $D$ be an effective divisor on $S'$ such that $|D{\cdot}E_i|_{S'} = 0$ for each $i$. Then there exists an element $h$ in the maximal ideal $\mathfrak{m}_Q$ of $\mathcal{O}_{X,Q}$ such that $D = f^*(C)$, where $f^*(C)$ is the total transform on $X$ of the divisor $C$ defined by $h = 0$ on $(X,Q)$.*

**Notation I.3.14.** If $I$ is an ideal in $\mathcal{O}_{X,Q}$ and $f : S' \longrightarrow X$ is a dominant morphism, we will denote by $I\mathcal{O}_{S'}$ the inverse image ideal sheaf of $I$ on $S'$, i.e. if $\mathcal{I} = \tilde{I}$ is the ideal sheaf generated by $I$ on $X$, $I\mathcal{O}_{S'}$ is the ideal sheaf in $\mathcal{O}_{S'}$ generated by the image $f^{-1}\mathcal{I}$ (see II §5 of [29] for details), or equivalently, $I\mathcal{O}_{S'}$ is the image of the natural map $f^*\mathcal{I} \longrightarrow \mathcal{O}_{S'}$.

**Theorem I.3.15.** ([3], Theorem 4) *Let $(X,Q)$ be a rational surface singularity, let $f : S' \longrightarrow X$ be the minimal resolution of $(X,Q)$ and let $Z_Q$ be the fundamental cycle of $Q$. Then, for every $r \in \mathbb{Z}_{>0}$,*

$$\mathcal{O}_{S'}(-rZ_Q) = \mathfrak{m}_Q{}^r\mathcal{O}_{S'}.$$

*Moreover,*

$$H^o(rZ_Q, \mathcal{O}_{rZ_Q}) \cong \frac{\mathcal{O}_{X,Q}}{\mathfrak{m}_Q{}^r}$$

*and*

$$\dim_k(\frac{\mathfrak{m}_Q^r}{\mathfrak{m}_Q^{r+1}}) = -r|Z_Q{\cdot}Z_Q|_{S'} + 1.$$

*Therefore, the Hilbert-Samuel polynomial of* $(X,Q)$ *is*

$$p(X) = -\frac{1}{2}|Z_Q{\cdot}Z_Q|_{S'}(X^2 - X) + X.$$

*In particular, the multiplicity of* $(X,Q)$ *is* $-Z_Q^2 := |Z_Q{\cdot}Z_Q|_{S'}$ *and the dimension of the Zariski tangent space of* $(X,Q)$ *is* $-Z_Q^2 + 1$.

**Corollary I.3.16.** *If* $m$ *is the multiplicity of* $(X,Q)$, *then the embedding dimension of* $(X,Q)$ *is* $m+1$.

Denote by $R$ be the local ring of $X$ at $Q$. An important feature of rational surface singularities is the finiteness of the divisor class group $Cl(R)$, i.e. the group of Weil divisors of $(X,Q)$ modulo linear equivalence (recall that two Weil divisors $D$ and $D'$ on $X$ are said to be linearly equivalent if $D - D'$ is principal). Although this fact is presented in Lipman [41] in a more general case, we state here for rational surface singularities.

Keep the notation as in Proposition I.3.13. Denote by $\mathbb{E}$ the group of divisors on $S'$ with exceptional support. Since no non-zero principal divisor has exceptional support, the canonical map

$$\mathbb{E} \longrightarrow Pic(S')$$

is injective, where $Pic(S') \simeq H^1(S', \mathcal{O}^*_{S'})$ is the group of divisor classes on $S'$. If $U = \mathrm{Spec}(R) \setminus \{m_Q\}$, the cokernel of this map is $Pic(U)$, which is isomorphic to $Cl(R)$ ([29] Proposition II.6.5). The restriction map $\rho : Pic(S') \longrightarrow Pic(U)$ is clearly surjective and its kernel consists of the classes of divisors $D$ on $S'$ which come principal on $U$. Thus we have an exact sequence

$$0 \longrightarrow \mathbb{E} \longrightarrow Pic(S') \longrightarrow Pic(U) \longrightarrow 0.$$

On the other hand, we have a group homomorphism

$$\theta : Pic(S') \longrightarrow \mathbb{E}^\vee = Hom(\mathbb{E}, \mathbb{Z})$$

defined by

$$(\theta(D))(E_i) = |D{\cdot}E_i|_{S'}.$$

The kernel $Pic^0(S')$ of $\theta$ is the subgroup of divisor classes whose intersection number with every $E_i$ is zero, and since we are assuming $(X,Q)$ is a rational singularity, it is zero (because of Lemma I.3.6 and Proposition I.3.7). Moreover, since $R$ is complete Lemma 14.3 of [41] says that $\theta$ is surjective. Hence,

$$\theta : Pic(S') \longrightarrow \mathbb{E}^\vee$$

is an isomorphism. Now, the cokernel $H$ of the restriction of $\theta$ to $\mathbb{E}$ is an abelian group generated by elements $e_1, \ldots, e_n$ satisfying

$$\sum_{j=i}^{s} |E_i \cdot E_j|_{S'} e_j = 0$$

for each $i \in \{1, \ldots, s\}$. Therefore, $H$ is a finite group with order equal to $det(\mathbf{A}_Q^{S'})$ and we have the following commutative diagram with exact rows and columns (see §14 of [41] for details):

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{E} & \longrightarrow & \theta(\mathbb{E}) & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Pic(S') & \xrightarrow{\theta} & \mathbb{E}^{\vee} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho} & & \downarrow & & \\
0 & \longrightarrow & Pic(U) & \longrightarrow & H & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

Since $Cl(R) \cong Pic(U)$, we deduce as already announced

**Proposition I.3.17.** (Proposition 17.1 of [41]) *Let $(X, Q)$ be a rational surface singularity. Then, $Cl(R)$ is finite and so, for any Weil divisor $C$ on $(X, Q)$, there exists an integer $r > 0$ such that $rC$ is a Cartier divisor on $(X, Q)$.*

**Definition I.3.18.** If $C$ is a Cartier divisor on $X$, the *total transform of $C$ on $S'$* is the pullback of $C$ by $f$. We write

$$f^*(C) = \widetilde{C}^{S'} + \sum_{i=1}^{s} v_i(C) E_i$$

where $\widetilde{C}^{S'}$ is the strict transform of $C$ on $S'$ and each $v_i(C)$ is a non-negative integer which will be called the *(effective) $E_i$-value of $C$.*

More in general,

**Definition I.3.19.** [Mumford [45] II.b] If $C$ is a Weil divisor on $X$, the *total transform of $C$ on $S'$* is the $\mathbb{Q}$-Cartier divisor on $S'$

$$\overline{C}^{S'} = \widetilde{C}^{S'} + \sum_{i=1}^{s} a_i E_i$$

where the $\{a_i\}_{i=1,\dots,s}$ are rational numbers such that

$$|\widetilde{C}^{S'}\cdot E_j|_{S'} + \sum_{i=1}^{s} a_i|E_i\cdot E_j|_{S'} = 0$$

for all $j \in \{1,\dots,s\}$ (by Proposition I.3.4, the intersection matrix $\mathbf{A}_Q^{S'}$ is negative definite and so the $\{a_i\}_{i=1,\dots,s}$ are unique). We will write $D_C^{S'} = \sum_{i=1}^{s} a_i E_i$ and call it the *exceptional part* of the total transform of $C$.

Note that, in general, $\overline{C}^{S'}$ is not a divisor on $S'$.

**Remark I.3.20.** If $C$ is a Cartier divisor on $(X,Q)$, the previous definition applies and we get the usual pullback $f^*(C)$. Moreover, by Proposition I.3.17 we know that if $C$ is a Weil divisor on $(X,Q)$ there exists a integer $r > 0$ such that $rC$ is Cartier. Then,

$$f^*(rC) = r\overline{C}^{S'}.$$

**Notation I.3.21.** By abuse of notation, whether $C$ is a Cartier or a Weil divisor, we will write $f^*(C)$ to mean the total transform of $C$.

In virtue of Remark I.3.20 is is possible to define an intersection multiplicity for Weil divisors on $(X,Q)$ that extends the usual intersection multiplicity defined for the intersection of a Cartier and a Weil divisor (see [45], Example 7.1.16 of [20]). If $f : S' \longrightarrow X$ is a resolution of $(X,Q)$ and $C_1, C_2$ are Weil divisors on $X$, it is enough to take

$$[C_1, C_2]_Q = |f^*C_1\cdot f^*C_2|_{S'}$$

where $f^*C_1$ and $f^*C_2$ are the total transforms of Definition I.3.19. Note that in general, $[C, D]_Q$ is a non-negative rational number and if $C_1$ or $C_2$ is a Cartier divisor, we recover the usual intersection multiplicity.

From this and Definition I.3.19, it follows that

$$|C_1\cdot C_2|_X \quad = \quad |\widetilde{C_1}^{S'}\cdot(\widetilde{C_2}^{S'} + D_{C_2}^{S'})|_{S'} \tag{3.a}$$

We call this equality the *projection formula for* $f$.

### I.3.1 Complete ideals on a rational surface singularity

Let $(X,Q)$ be a rational surface singularity and write $R = \mathcal{O}_{X,Q}$. In this section we recall some facts concerning complete ideals in $R$ (see Definition I.2.1).

**Definition I.3.22.** A coherent sheaf on $X$ is *complete* if for each open affine $U \subset X$, $\Gamma(U, \mathcal{J}) \subset \Gamma(U, \mathcal{O}_X)$ is complete.

**Lemma I.3.23.** (Lemma 5.3 of [41]) *Let $f : S' \longrightarrow X$ be a birational projective morphism (so that if $\mathcal{J}$ is a coherent sheaf on $S'$, then $f_*\mathcal{J}$ is so). Then, if $\mathcal{J}$ is a complete coherent sheaf, so is $f_*\mathcal{J}$.*

Connected with complete ideals are the contracted ideals:

**Definition I.3.24.** *(Definition 6.1 of [41])* Let $f : S' \longrightarrow X$ be a morphism of schemes such that $f_*(\mathcal{O}_{S'}) = \mathcal{O}_X$. Let $Q \in X$ and let $I$ be an ideal in $\mathcal{O}_{X,Q}$. We say that $I$ is *contracted for* $f$ if $I$ is the stalk at $O$ of $f_*(\mathcal{J})$ for some ideal sheaf $\mathcal{J}$ on $S'$.

Complete ideals may be characterized in terms of contracted ideals.

**Proposition I.3.25.** (Proposition 6.2 of [41]) *The ideal $I \subset \mathcal{O}_{X,Q}$ is complete if and only if it is contracted for every proper birational map $f : S' \longrightarrow X$.*

It can be proved that if $(X, Q)$ is a normal surface singularity, the product of contracted ideals in $\mathcal{O}_{X,Q}$ is again contracted (Theorem 7.2 of [41]). This follows from the next result that we state here for future reference:

**Lemma I.3.26.** *let $(X, Q)$ be a normal surface singularity and let $f : S' \longrightarrow X$ be a proper morphism such that $f_*(\mathcal{O}_{S'}) = \mathcal{O}_X$ and $R^1 f_*(\mathcal{O}_{S'}) = 0$. If $I$ and $J$ are contracted ideals for $f$, so is $IJ$:*

$$f_*(IJ\mathcal{O}_{S'}) = f_*(I\mathcal{O}_{S'})f_*(J\mathcal{O}_{S'}) = IJ.$$

*In particular,*

$$\Gamma(S', IJ\mathcal{O}_{S'}) = IJ.$$

From this and Proposition I.3.25, the following result for rational surface singularities is derived.

**Theorem I.3.27.** (Theorem 7.1 of [41]) *Let $(X, Q)$ be a rational surface singularity. If $I$ and $J$ are complete ideals in $\mathcal{O}_{X,Q}$, then $IJ$ is also complete.*

An important consequence of the preceding theorem is

**Corollary I.3.28.** (Proposition 8.1 of [41]) *Let $(X, Q)$ be a rational surface singularity and $I \subset \mathcal{O}_{X,Q}$ a complete ideal. Then, the blowing-up $Bl_I(X)$ of $I$ in $X$ is normal.*

Fixed a desingularization $f : S' \longrightarrow X$, to each complete $\mathfrak{m}_Q$-primary ideal $I \subset R$ such that $I\mathcal{O}_{S'}$ is invertible we can associate the unique divisor $D_I \in \mathbb{E}^+_{S'}$ such that

$$I\mathcal{O}_{S'} = \mathcal{O}_{S'}(-D_I).$$

by taking

$$D_I = \sum_{i=1}^{s} v_i(I)E_i.$$

Since $I$ is complete, $I$ is the stalk at $Q$ of the sheaf of ideals $f_*\mathcal{O}_{S'}(-D_I)$.

Conversely, given $D \in \mathbb{E}^+_{S'}$, we define the ideal $I_D$ of $R$ to be the stalk at $O$ of $f_*\mathcal{O}_X(-D)$. If $D = \sum_\alpha n_\alpha E_\alpha$, an element $g$ of $R$ is in $I_D$ if and only if $v_\alpha(g) \geq n_\alpha$ for all $\alpha$, where $v_\alpha$ is the discrete valuation defined by $E_\alpha$. Thus, $I_D$ is defined by valorative inequalities and hence, it is complete.

Using Proposition I.3.13, we have

**Proposition I.3.29.** *Let $D$ be a divisor on $\mathbb{E}^+_{S'}$. Then*

1. *For each point $P$ in $S'$, there exists an effective Cartier divisor $C$ such that $f^*(C) = \widetilde{C}^{S'} + D$ and $\widetilde{C}^{S'}$ does not go through $P$.*

2. *(Lipman) $I_D\mathcal{O}_{S'} = \mathcal{O}_{S'}(-D)$, i.e. $\mathcal{O}_{S'}(-D)$ is generated by global sections.*

Now, following Lipman [41], denote by $\mathbb{E}^\sharp_{S'}$ the set of divisors $D \in \mathbb{E}$, $D \neq 0$, such that $\mathcal{O}(-D)$ is generated by its sections over $X$. $\mathbb{E}^\sharp_{S'}$ is a semigroup (with the addition) and, for rational surface singularities, it equals $\mathbb{E}^+_{S'}$ because of Proposition I.3.29 (for normal surface singularities in general, it is contained strictly in $\mathbb{E}^+_{S'}$).

Because of the discussion above and Proposition I.3.29, the maps $D \to I_D$ and $I \to D_I$ are reciprocal bijections between $\mathbb{E}^+_{S'}$ and the set $J^{S'}_Q$ of complete $\mathfrak{m}_Q$-primary ideals $I$ in $R$ such that $I\mathcal{O}_{S'}$ is invertible. Since the product of two complete ideals is again complete (Theorem I.3.27), $J^{S'}_Q$ is a semigroup and the correspondences above are isomorphism of semigroups. Note that the sets $\{J^S_Q \mid S$ is a desingularization of $(X, Q)\}$ together with the inclusions $J^S_Q \subset J^{S'}_Q$ when $S'$ dominates $S$ form a direct system of semigroups whose direct limit is the semigroup $\mathbf{J}^*_Q$ of all complete $\mathfrak{m}_Q$-primary ideals of $R$. Hence, in order to study factorization in $\mathbf{J}^*_Q$, it suffices to study it in the semigroups $J_X$, or equivalently, in $\mathbb{E}^+_{S'}$.

**Definition I.3.30.** A complete $\mathfrak{m}_Q$-primary ideal $I \subset R$ is *simple* if it is irreducible as an element of $\mathbf{J}^*_Q$.

A well-known result due to Lipman establishes that the unique rational surface singularities for which unique factorization for complete ideals holds are the singularities of type $\mathbf{E}_8$ and the nonsingular germs of surface (see Theorems 20.1 and 25.1 of [41]). Related with factorization and as a weaker notion, Göhner introduces the *semifactorization* [22]:

**Definition I.3.31.** Let $(G, +)$ a commutative semigroup. An element $g \neq 0$ of $G$ is *extremal* if $g$ has no opposite in $G$ and if $ng = g_1 + g_2$ with

$n \in \mathbb{N}, g_1, g_2 \in G$ then $\alpha_1 g_1 = \beta_1 g$ and $\alpha_2 g_2 = \beta_2 g$ for suitable positive integers $\alpha_i, \beta_i$, $i = 1, 2$.

The semigroup $G$ is *semifactorial* if every $g \in G$ can be formally expressed in a unique way as $g = \sum_{i=1}^{r} q_i g_i$ where the $q_i$ are positive rational numbers and the $g_i$ are extremal elements of $G$.

From the fact that the divisor class group is finite (Proposition I.3.17) and the work of Göhner, it follows that $\mathbf{J}_Q^*$ is semifactorial. In fact, fixed a desingularization $S' \longrightarrow X$, the extremal elements of $\mathbb{E}_{S'}^{+}$ are all multiple integers of the divisors $D_i = m_i D_i'$, where $D_i' \in \bigoplus_{i=1}^{s} \mathbb{Q}E_i$ is the unique $\mathbb{Q}$-Cartier divisor on $S'$ such that $|D_i' \cdot E_j|_{S'} = -\delta_{i,j}$ (Kronecker $\delta$) and $m_i$ is the smallest integer such that $m_i D_i'$ is a divisor. From this, we have that the semigroup $\mathbb{E}_{S'}^{+}$ is semifactorial and also

**Theorem I.3.32.** (Corollary 1.6 of [12]) *The semigroup $\mathbf{J}_Q^*$ of $\mathfrak{m}_Q$-primary complete ideals of $R$ is semifactorial: given $I \in \mathbf{J}_Q^*$, then $I$ can be formally expressed in a unique way as*

$$I = \prod_{i=1}^{m} I_i^{q_i}$$

*with $q_i \in \mathbb{Q}_+$.*

To close this section, we give a formula for the codimension of complete ideals in a ring having a rational surface singularity. It follows directly from Corollary 23.3 of [41] and the adjunction formula (see Proposition V.1.5 of [29]).

**Proposition I.3.33.** *Let $I$ be a complete $\mathfrak{m}_O$-primary ideal in a ring $R$ with a rational surface singularity and let $f : S' \longrightarrow X$ be a birational dominant morphism such that $I\mathcal{O}_{S'}$ is invertible. Then,*

$$\dim_{\mathbb{C}} \frac{R}{I} = -\frac{1}{2} |D_I \cdot (D_I + \mathrm{K}_{S'})|_{S'}.$$

## I.4 Sandwiched singularities

In this section we introduce sandwiched singularities. From now on $R$ will mean a regular two-dimensional ring as in Definition I.1.1. The main reference for the definitions and facts of this section is [58].

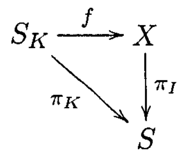
**Definition I.4.1.** *Sandwiched singularities* are normal surface singularities which birationally dominate a non-singular surface. More precisely, a normal two-dimensional complex-analytic local ring is said to have a sandwiched singularity if there exists a non-singular algebraic surface $S$ over $\mathbb{C}$, an ideal sheaf $\mathcal{J}$ on $S$ and a point $Q$ in the blowing-up $X$ of $S$ along $\mathcal{J}$

such that $(\mathcal{O}_{X,Q})_{an} \cong \mathcal{O}$. A normal local ring $\mathcal{O}$ which is a localization of a finitely generated $\mathbb{C}$-algebra is said to have a sandwiched singularity if $\mathcal{O}_{an}$ has one.

**Remark I.4.2.** If $\mathcal{O}$ has a sandwiched singularity, we may assume that the ideal sheaf $\mathcal{J}$ has cosupport at one point $O \in S$ and since $\mathcal{O}$ is normal, we may also assume that the stalk $\mathcal{J}_{S,O}$ is a complete ideal in $\mathcal{O}_{S,O}$ (see Corollary I.3.28). Note that by (i) of Proposition I.3.3, sandwiched singularities are rational surface singularities.

The following results relates sandwiched singularities to clusters of base points.

**Proposition I.4.3.** (Remark 1.4 of [58]) *Let $I \subset R$ be a complete $\mathfrak{m}_O$-primary ideal and let $\mathcal{K} = BP(I)$ be the cluster of base points of $I$. If $\pi_I : X \longrightarrow S$ is the blowing-up of $I$ ($X = Bl_I(S)$) and $\pi_K$ is the blowing-up of all the points in $\mathcal{K}$, we have a commutative diagram*




*where the morphism $f$, given by the universal property of the blowing-up, is the minimal resolution of the singularities of $X$.*

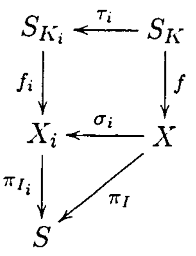
**Definition I.4.4.** Let $Z_1, \ldots, Z_n$ be reduced and irreducible algebraic varieties with the same function field $K$. The *birational join of* $Z_1, \ldots, Z_n$ is the minimal birational model of $K$ (minimal with respect to domination) which dominate $Z_1, \ldots, Z_n$. It thus consists of a reduced and irreducible algebraic variety $Z$ birational to the $Z_i$ together with birational morphisms $p_i : Z \longrightarrow Z_i$ for each $i \in \{1, \ldots, n\}$.

The construction of the birational join is as follows: if $\phi_{i,j} : Z_i \leftrightarrow Z_j$ are the birational correspondence between $Z_i$ and $Z_j$, there are open sets $U_i \subset Z_i$ and $U_j \subset Z_j$ such that $\phi_{i,j}$ induces an isomorphism between $U_i$ and $U_j$. Denote by $U$ the image of any $U_i$ in $U_1 \times \cdots \times U_n$ via the diagonal morphism, where we identify $U_i$ with $U_j$, $j \neq i$ via $\phi_{i,j}$. Then, the birational join of $Z_1, \ldots, Z_n$ is the closure of $U$ in $Z_1 \times \cdots \times Z_n$ together with the restrictions of the projections $p_i : Z_1 \times \cdots \times Z_n \longrightarrow Z_i$ to it (this closure does not depend on the choice of the $U_i$).

From Proposition 21.3 of [41] and since unique factorization for complete ideals holds in the regular ring $R$ (Theorem I.2.6), we have

**Theorem I.4.5.** *The exceptional divisor of blowing-up a simple complete ideal in $R$ is irreducible.*

Note that by the universal property of the blowing-up, if $I$ is as in Proposition I.4.3 and $I = \prod_{i=1}^{r} I_i^{\alpha_i}$ is its factorization into simple complete ideals, then $I_i\mathcal{O}_X$ is invertible and consequently, there are birational morphisms $\sigma_i : X \longrightarrow X_i = Bl_{I_i}(S)$ for each $i$. On the other hand, it is clear that $\sigma_i \circ f : S_K \longrightarrow X_i$ is a resolution of $X_i$ and hence, it factorizes through the minimal resolution of $X_i$, $f_i : S_{K_i} \longrightarrow X_i$ where $\mathcal{K}_i = BP(I_i)$. Therefore, and for each $i$, there exists a birational morphism $\tau_i : S_K \longrightarrow S_{K_i}$ doing the following diagram commutative




($\pi_{I_i} : X_{I_i} \longrightarrow S$ is the blowing-up of $I_i$). The next proposition says that in fact, $(X, \sigma_i)$ and $(S_K, \tau_i)$ are the birational join of $X_1, \ldots, X_n$ and $S_{K_1}, \ldots, S_{K_n}$, respectively.

**Proposition I.4.6.** (Corollary II.1.5 of [58]) *We have that*

*(a) $X$ together with the birational morphisms $\sigma_i : X \longrightarrow X_i$ is the birational join of $X_1, \ldots, X_r$.*

*(b) The correspondence*

$$I_i \mapsto \text{strict transform of } \pi_{I_i}^{-1}(O) \text{ on } X$$

*is a bijection between $\{I_i\}_{i=1,\ldots,r}$ and the set of irreducible components of $\pi_I^{-1}(O)$.*

*(c) $S_K$ together with the birational morphisms $\tau_i : S_K \longrightarrow S_{K_i}$ is the birational join of $S_{K_1}, \ldots, S_{K_r}$.*

**Remark I.4.7.** One may think of sandwiched singularities in the following way. We take a cluster $K$ with origin at $O$ and choose exceptional components $E_{p_i}^{S_K}$, $i = 1, \ldots, m$ on $S_K$ such that $(E_{p_i}^{S_K})^2 \leq -2$ for all $i \in \{1, \ldots, m\}$ and the exceptional cycle $\sum_{i=1}^{m} E_{p_i}^{S_K}$ on $S_K$ is connected. Then, the intersection matrix $(|E_{p_i}^{S_K} \cdot E_{p_j}^{S_K}|_{S_K})_{1 \leq i,j \leq m}$ is negative-definite and so, $\sum_{i=1}^{m} E_{p_i}^{S_K}$ can be contracted to a singularity $Q$ lying on a surface $X$. The precise meaning of "can be contracted" is the following: there exists a singularity $(X, Q)$ and a resolution $f' : S' \to (X, Q)$ such that the exceptional divisor on $S'$ is isomorphic to $\sum_{i=1}^{m} E_{p_i}^{S_K}$ (see [27]) To prove this fact, it is necessary to work in the complex-analytic category. However, as explained in Remark 1.12 of [58], the morphism $f'$ can be algebraizable (see also the proof of Théorème 4.3 of [37]).

On the other hand, since $\pi_K$ is a birational morphism and $S_K$ is normal, there exists a complete $\mathfrak{m}_O$-primary ideal $J \subset R$ such that $S_K$ is obtained by blowing-up $J$ (Theorem II. 7.17 of [29]). Then, $K$ is the set of base points of $J$. By (b) of Proposition I.4.6, the simple ideals in the factorization of $J$ correspond one-to-one with the exceptional components $E_p^{S_K}$ on $S_K$, i.e. keeping the notation of I.2.10, $I_p$ appears in the factorization of $J$ if and only if $p \in K$. By Proposition I.4.6, it follows that $X$ is isomorphic to the blowing-up of the ideal

$$I = \prod_{p \in K \setminus \{p_1, \ldots, p_m\}} I_p,$$

and by Theorem I.2.9, $p \in K$ is a dicritical point of the cluster $\mathcal{K} = BP(I)$ if and only if $p \notin \{p_1, \ldots, p_m\}$.

In particular, we see that the correspondence $p \to f_*(E_p^{S_K})$ is a bijection between the set $\mathcal{K}_+ = \{p \in K \mid \rho_p > 0\}$ and the irreducible components of the exceptional locus of $X$.

To close the present section, we fix some notation and conventions and prove a technical lemma that will be needed in the future.

**Notation I.4.8.** If $p \in \mathcal{K}_+$, we denote by $L_p$ the exceptional component on $X$ equal to $f_*(E_p^{S_K})$. Hence, $\{L_p\}_{p \in \mathcal{K}_+}$ is the set of the irreducible components of the exceptional locus of $X$.

Given a curve $C$ on $S$, we denote by $\widetilde{C}$ and $C^* = \widetilde{C} + L_C$ the strict and the total transform of $C$ on $X$. Clearly,

$$L_C = \sum_{p \in \mathcal{K}_+} v_p(C) L_p.$$

If $q$ is any point infinitely near or equal to $O$, we write $C_q$ for a curve going sharply through $\mathcal{K}(q)$ and missing the points after $q$ in $K$, and

$$\mathcal{L}_q = L_{C_q} = \sum_{p \in \mathcal{K}_+} v_p(I_q) L_p$$

for the exceptional component of $C_q^*$. If $\mathcal{T} = BP(J)$, $L_{\mathcal{T}}$ (or $L_J$) means the exceptional part of the total transform on $X$ of any curve going sharply through $\mathcal{T}$.

**Lemma I.4.9.** (Projection formula for $\pi$) *If $C_1, C_2$ are curves on $S$,*

$$[C_1, C_2]_O = |(\widetilde{C_1} + L_{C_1}) \cdot \widetilde{C_2}|_X.$$

*Proof.* Since $\pi_K = \pi \circ f$, we have that

$$\pi_K^*(C_1) = f^*(\widetilde{C_1} + L_{C_1}).$$

Hence,

$$\widetilde{C_1}^{S_K} + E_{C_1}^{S_K} = \widetilde{C_1}^{S_K} + D_{\widetilde{C_1}}^{S_K} + f^*(L_{C_1})$$

and so

$$E_{C_1}^{S_K} = D_{\widetilde{C_1}}^{S_K} + f^*(L_{C_1}) \tag{4.a}$$

Now, from the projection formula for $\pi_K$ (*5.* of Proposition I.1.16), we know that

$$\begin{aligned}[C_1, C_2]_O &= |(\widetilde{C_1}^{S_K} + E_{C_1}^{S_K})\cdot\widetilde{C_2}^{S_K}|_{S_K} = \\ &= |(\widetilde{C_1}^{S_K} + D_{\widetilde{C_1}}^{S_K})\cdot\widetilde{C_2}^{S_K}|_{S_K} + |f^*(L_{C_1})\cdot\widetilde{C_2}^{S_K}|_{S_K}\end{aligned} \tag{4.b}$$

the last equality by (4.a). Applying the projection formula for $f$ (see equality (3.a) above) we deduce that

$$|(\widetilde{C_1}^{S_K} + D_{C_1}^{S_K})\cdot\widetilde{C_2}^{S_K}|_{S_K} = |\widetilde{C_1}\cdot\widetilde{C_2}|_X \quad .$$

and

$$|f^*L_{C_1}\cdot\widetilde{C_2}^{S_K}|_{S_K} = |L_{C_1}\cdot\widetilde{C_2}|_X$$

and the claim follows from (4.b). □

**Convention** Throughout this memoir, once a complete $\mathfrak{m}_O$-primary ideal $I \subset R$ has been fixed, we denote by $\mathcal{K} = (K, \nu)$ the cluster of base points of $I$ and by $\{E_p\}_{p\in K}$ (instead of $E_p^{S_K}$) the exceptional components on $S_K$.

## I.5 Dual graphs

If $E_1, \ldots, E_s$ are irreducible curves on a surface $S$, we associate to them a weighted graph $\Gamma$ with $s$ vertices: one vertex for each curve. Two vertices are connected by an edge in $\Gamma$ if and only if the two corresponding curves on $S$ have a non-empty intersection. $\Gamma$ is called the *dual graph* of $E_1, \ldots, E_s$. The *weight* of a vertex $u$ is

$$\omega_\Gamma(u) = -E_u^2 = -|E_u\cdot E_u|_S \in \mathbb{N}.$$

A *chain* is a connected graph with no vertex belonging to more than two edges. $|\Gamma|$ denotes the set of vertices of $\Gamma$ and if $u, v \in |\Gamma|$, the *distance* $dist_\Gamma(u, v)$ *between* $u$ *and* $v$ is the number of edges in the minimal chain containing $u$ and $v$. This chain is denoted by $ch(u, v)$ and we write $ch^0(u, v)$for

$ch(u,v) \setminus \{u,v\}$. We say that $u$ and $v$ are *adjacent* if $dist_\Gamma(u,v) = 1$. For each $u \in |\Gamma|$ define

$$\gamma_\Gamma(u) = \sharp\{v \in |\Gamma| \mid dist_\Gamma(u,v) = 1\}.$$

*Dual graph of a cluster*

Let $K$ be a cluster and $D = D(K)$ its Enriques diagram (see Definition I.1.12). The *dual graph* $\Gamma_K$ *of* $K$ is the weighted dual graph associated to the exceptional components $\{E_p\}_{p\in K}$ on $S_K$. These graphs are called *non-singular*. From *3.* of Proposition I.1.16, we know that the incidence of the $\{E_p\}_{p\in K}$ is determined by the proximity relations of the points of $K$, and hence, by $D$. Conversely, from a weighted non-singular graph it is possible to recover the natural ordering and the proximity of the points of $K$ (see §4.4 of [11] for details and more facts concerning the connection between dual graphs and clusters).

**Remark I.5.1.** We have seen in *3.* of Proposition I.1.16 that

$$\omega_{\Gamma_K}(p) = r_p + 1,$$

where $r_p = \sharp\{q \in K\ midq \to p\}$.

*Dual graph of a rational surface singularity*

If $(X,Q)$ is a rational surface singularity and $f : S' \longrightarrow (X,Q)$ is a resolution of $Q$, we can take the (weighted) dual graph associated to the exceptional locus $f^{-1}(\{Q\}) \subset S'$. In case $f$ is the minimal resolution of $(X,Q)$, we denote this graph by $\Gamma_Q$. By Corollary I.3.8, the dual graph $\Gamma_Q$ is connected and simply connected.

Note that the information contained in the dual graph $\Gamma_Q$ and in the intersection matrix $\mathbf{A}_Q^{S'}$ are equivalent. Therefore, the fundamental cycle of a rational surface singularity $(X,Q)$ can be computed from the dual graph $\Gamma_Q$. From this and by using Theorem I.3.12, the condition for a graph $\Gamma$ to correspond to a rational surface singularity is purely combinatorial.

Now, let $I \subset R$ be a complete $\mathfrak{m}_O$-primary ideal in $R$, $X = Bl_I(S)$ and $S_K$ the surface obtained by blowing up all the points in $\mathcal{K} = BP(I)$. We have already pointed out that one may think of $X$ as the surface obtained by contracting the exceptional components $E_p$ of $E_K$ corresponding to the simple ideals $I_p$ not appearing in the factorization of $I$. If we denote by $E_0$ the exceptional divisor on $S_K$ composed of these components, each singularity $Q \in X$ is the contraction by $f$ of a connected component $E_Q$ of $E_0$. Hence, the dual graph $\Gamma_Q$ of $Q$ is the subgraph of $\Gamma_K$ composed of the vertices corresponding to the components of $E_Q$.

**Definition I.5.2.** A graph $\Gamma$ is said to be *sandwiched* if there exists a non-singular graph $\Gamma^*$ containing $\Gamma$ as a weighted subgraph.

**Proposition I.5.3.** (Proposition II.1.11 of [58]) *The following three conditions are equivalent:*

*(i) Any singularity having a resolution with dual graph* $\Gamma$ *is sandwiched.*

*(ii) There exists a sandwiched singularity having a resolution with dual graph* $\Gamma$.

*(iii)* $\Gamma$ *is sandwiched.*

# Chapter II

# Sandwiched singularities and the unloading procedure

This chapter essentially deals with the relationship between the (sandwiched) singularities of the blowing-up of a complete ideal in a regular local two-dimensional ring and the Enriques diagram of the cluster of base points of this ideal. We shall see that the configuration of the exceptional curves on the blown-up surface, the number of singularities on it, their multiplicities as well as the fundamental cycles can be read easily from the Enriques diagram by means of the unloading procedure. Most of the methods used here come from the equisingularity theory of plane curve singularities and the proofs given often need an accurate study of the proximity relations between the base points of the ideal.

In section II.1, once fixed a complete ideal $I$ in a local two dimensional ring having a rational singularity, we associate to every complete subideal $J \subset I$ of codimension one a point in the exceptional locus of the surface obtained by blowing-up $I$. We shall show that this correspondence is in fact a bijection and as a corollary, we shall recover the fundamental cycle of any singularity from the values of the blown-up ideal and the subideal associated to the singularity by this bijection. In section 2.2 and for the rest of the chapter, we take complete ideals in the local ring of a non-singular surface $S$ at some point $O$. Then, we relate the study of the singularities obtained by blowing-up such an ideal to the Enriques diagram of the cluster of its base points and the unloading procedure. Section II.3 gives a more accurate analysis of the unloading procedure and studies the relationship between the coefficients of the fundamental cycle of a sandwiched singularity and the proximity relations between some base points of the ideal. We introduce also the *multiplicity relevant* (infinitely near) *points* relative to a (sandwiched) singularity ($MR$-points for short). Section II.4 relates the resolution process of any sandwiched singularity to the complete ideals in $\mathcal{O}_{S,O}$ and the unloading procedure. Then, from the results seen in the preceding sections we obtain information on the singularities appearing in the resolution process. Section II.5 gives easy formulas for the multiplicity of sandwiched singularities and Weil divisors going through them in terms of the $MR$-points. A formula for the number of branches (i.e. analytically irreducible components) of a transverse hypersurface section is also given, we recover the characterization of minimal singularities as those for which the fundamental cycle is reduced and we prove that the number of exceptional components going through the same sandwiched singularity is bounded by the embedding dimension of the singularity and characterize when this bound is attained. In section II.6 we give some results concerning dual graphs of an Enriques diagram and the proximity relations between their points that will be of great use in the forthcoming chapters. As a consequence, we prove that the exceptional components going through a sandwiched singularity are not tangent. Finally, in section II.7, and as a consequence of some of the results of this chapter, we answer some questions posed by Noh in [49] relative to adjacent complete ideals.

## II.1 On the blowing-up of complete ideals on a rational surface singularity

In this section, we take a rational surface singularity $(S, O)$, i.e $S = \mathrm{Spec}(R)$ is the spectrum of a Noetherian normal complete two-dimensional local ring $(R, m_O)$ containing an algebraically closed field $k$ isomorphic to the residue field of $R$ and we assume that there exists a desingularization $g : \widetilde{S} \longrightarrow S$ such that the stalk at $O$ of $R^1 g_*(\mathcal{O}_{\widetilde{S}})$ is zero (see Definition I.3.2). We denote by $O$ the closed point of $S$. Fixed a complete $\mathfrak{m}_O$- primary ideal in $R$, denote by $X$ the surface obtained by blowing-up $I$ and by $L_I$ the exceptional divisor on $X$ relative to the blowing-up of $I$, i.e. $I\mathcal{O}_X = \mathcal{O}_X(-L_I)$.

First of all, we need a definition and a lemma.

**Definition II.1.1.** Given a curve $C$ on $S$ with equation $h = 0$, $h \in I$, the *virtual transform of $C$ relative to $I$* on $X$ is the effective Cartier divisor $\check{C} = \pi^*(C) - L_I$ obtained by removing $L_I$ from the total transform of $C$ on $X$.

**Lemma II.1.2.** *Let $J \subset I$ be a complete $\mathfrak{m}_O$-primary ideal of codimension one. Then for all $n \geq 1$, $JI^{n-1}$ has codimension one in $I^n$.*

*Proof.* Let $a \in I \backslash J$. Then, $\frac{I}{J}$ is generated by $\overline{a}$ the class of $a$ and $I = J+(a)$. It follows that

$$I^{n+1} = JI^n + aI^n \tag{1.a}$$

for all $n \in \mathbb{N}$. Now, take $\forall n \geq 1$ the morphism induced by the multiplication by $a$,

$$\psi_n : \frac{I^n}{JI^{n-1}} \longrightarrow \frac{I^{n+1}}{JI^n}.$$

$\psi_n$ is well-defined and surjective by (1.a). Thus,

$$\dim_{\mathbb{C}}(\frac{I^{n+1}}{JI^n}) \leq \dim_{\mathbb{C}}(\frac{I^n}{JI^{n-1}}), \forall n \geq 1.$$

Since $I$ and $J$ are complete ideals, they are defined by valorative inequalities and so, there exists some valuation $v$ such that

$$v(J) > v(I).$$

Thus,

$$v(a) = v(I) < v(J)$$

and

$$v(a^n) = nv(a) < v(J) + (n-1)v(I).$$

Therefore,

$$a^n \notin JI^{n-1}.$$

Hence, $\frac{I^n}{JI^{n-1}} \neq 0$ for all $n \geq 1$ and so the claim. □

**Notation II.1.3.** We will denote by $\mathcal{M}_Q$ the ideal sheaf of the point $Q \in X$.

**Proposition II.1.4.** *Let $J \subset I$ be a complete $\mathfrak{m}_O$-primary ideal of codimension one. Then there exists a point $Q$ in the exceptional locus of $X$ such that $J\mathcal{O}_X = \mathcal{M}_Q I\mathcal{O}_X$.*

*Proof.* The surface $X$ is the blowing up of a complete ideal and hence it is normal and $\pi : X \longrightarrow S$ is birational and proper (Corollary I.3.28). By Proposition I.3.3, $H^1(X, \mathcal{O}_X) = 0$ and since $S$ is affine, $R^1\pi_*(\mathcal{O}_X) = 0$.

The inclusion $J \subset I$ induces an inclusion of ideal sheaves $J\mathcal{O}_X \subset I\mathcal{O}_X$ and thus, an exact sequence of sheaves on $X$

$$0 \longrightarrow J\mathcal{O}_X \longrightarrow I\mathcal{O}_X \longrightarrow \mathcal{N} \longrightarrow 0$$

and since complete ideals are contracted for any proper birational map (see Proposition I.3.25), by applying $\pi_*$ we get the exact sequence

$$0 \longrightarrow J \longrightarrow I \longrightarrow \pi_*(\mathcal{N}) \longrightarrow R^1\pi_*(J\mathcal{O}_X).$$

The sheaf $J\mathcal{O}_X$ is the homomorphic image of $\mathcal{O}_X^t$ for some $t$. Since the fibres of $\pi$ have dimension $\leq 1$, $R^2\pi_*$ vanishes for all coherent sheaves on $X$. From this and from the fact that $R^1\pi_*(\mathcal{O}_X^t) = 0$, we infer that $R^1\pi_*(J\mathcal{O}_X)$ is zero. Now, since $J \neq I$, $\pi_*(\mathcal{N}) \neq 0$ and $\mathcal{N}$ cannot be zero and thus there exists at least one point $Q \in X$ in the support of $\mathcal{N}$. Therefore, we have

$$J\mathcal{O}_{X,Q} \subset \mathfrak{m}_Q I\mathcal{O}_{X,Q} \subset \mathcal{O}_{X,Q}$$

and the following inclusions of sheaves

$$J\mathcal{O}_X \subset \mathcal{M}_Q I\mathcal{O}_X \subset I\mathcal{O}_X.$$

This clearly induces for every $n \in \mathbb{N}$

$$JI^{n-1}\mathcal{O}_X \subset \mathcal{M}_Q I^n\mathcal{O}_X \subset I^n\mathcal{O}_X.$$

Now, as $JI^{n-1}$ and $I^n$ are complete ideals in $R$, they are contracted for $\pi$ (Proposition I.3.25) and so, by applying $\pi_*$,

$$JI^{n-1} \subset \pi_*(\mathcal{M}_Q I^n\mathcal{O}_X) \subset I^n.$$

On the other hand, by means of Theorem I.3.27, we have that $\mathcal{M}_Q I^n\mathcal{O}_X$ is a complete $\mathcal{O}_X$-module and by Lemma I.3.23, $\pi_*(\mathcal{M}_Q I^n\mathcal{O}_X)$ is complete, too. Moreover, since $I^n\mathcal{O}_X$ is invertible, $\pi_*(\mathcal{M}_Q I^n\mathcal{O}_X) \subsetneq I^n$. Now, $JI^{n-1}$ is complete and by Lemma II.1.2, it has codimension one in $I^n$. Therefore,

$$JI^{n-1} = \pi_*(\mathcal{M}_Q I^n\mathcal{O}_X)$$

and so

$$\Gamma(X, JI^{n-1}\mathcal{O}_X) = \Gamma(X, \mathcal{M}_Q I^n\mathcal{O}_X),$$

for every $n \in \mathbb{N}$. Thus, by Proposition 5.15 II of [29],

$$J\mathcal{O}_X = \mathcal{M}_Q I\mathcal{O}_X.$$

□

According to Proposition II.1.4, we can map each complete $\mathfrak{m}_O$-primary ideal $J \subset I$ of codimension one to the point $Q$ in the exceptional locus of $X$ such that $J\mathcal{O}_X = \mathcal{M}_Q I\mathcal{O}_X$. Since $J$ is complete, it is contracted for $\pi$ (see Proposition I.3.25) and

$$J = \pi_*(J\mathcal{O}_X) = \pi_*(\mathcal{M}_Q I\mathcal{O}_X),$$

so it is clear that this map in injective. The following proposition shows that it is also surjective.

**Notation II.1.5.** Throughout this chapter, we denote by $L$ the exceptional locus in $X$ of $\pi$. Given a point $Q \in L$, we write $I_Q = \pi_*(\mathcal{M}_Q I \mathcal{O}_X)$.

**Proposition II.1.6.** *If $Q \in L$, $I_Q$ is a complete ideal contained in $I$ and of codimension one. Moreover, we have $I_Q \mathcal{O}_X = \mathcal{M}_Q I \mathcal{O}_X$.*

*Proof.* First of all, according to Theorem I.3.27 and Lemma I.3.23, the ideal $I_Q \subset I$ is complete. From Theorem 3.15 of [54], we know that there exists a complete $\mathfrak{m}_O$-primary ideal $J$ such that $I_Q \subseteq J \subset I$ and $J$ has codimension one in $I$. Let $Q' \in L$ be the point associated to $J$ in Proposition II.1.4, so $J = I_{Q'}$. Then, for all $n \in \mathbb{N}$, we have that $I_Q I^n \subset I_{Q'} I^n$ and equivalently,

$$\Gamma(X, \mathcal{M}_Q I^{n+1} \mathcal{O}_X) \subset \Gamma(X, \mathcal{M}_{Q'} I^{n+1} \mathcal{O}_X).$$

Therefore, if we write $\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{N}} \Gamma(X, \mathcal{F}(n))$, then

$$\Gamma_*(\mathcal{M}_Q) \subset \Gamma_*(\mathcal{M}_{Q'}),$$

and by [29] II Proposition 5.15, there are natural isomorphism,

$$\Gamma_*(\mathcal{M}_Q)^{\sim} \cong \mathcal{M}_Q$$

and

$$\Gamma_*(\mathcal{M}_{Q'})^{\sim} \cong \mathcal{M}_{Q'}$$

where $^{\sim}$ means the sheaf associated to (see §II.5 of [29]). Thus,

$$\mathcal{M}_Q \subset \mathcal{M}_{Q'},$$

which obviously implies that $Q = Q'$ and so, that $I_Q = I_{Q'} \subset I$ has codimension one as claimed. In particular, $I_Q \mathcal{O}_X = \mathcal{M}_Q I \mathcal{O}_X$. □

By means of Propositions II.1.4 and II.1.6 we obtain the following result, as already announced.

**Theorem II.1.7.** *Let $I \subset R$ be a complete $\mathfrak{m}_O$-primary ideal, $X$ the surface obtained by blowing-up $I$ and $L$ the exceptional locus in $X$. By associating to each point $Q \in L$ the complete ideal $I_Q$, we get a bijection between the set of points on the exceptional divisor of the surface $X$ and the set of the complete ideals $J \subset I$ of codimension one. The inverse map associates to each complete ideal $J$ the only point $Q$ all the virtual transforms of the curves $C : h = 0$, $h \in J$ are going through.*

**Remark II.1.8.** A similar result has been proved by Watanabe in a more algebraic context (see Proposition 3.1 of [60]).

**Remark II.1.9.** If $Q \in X$, a *hypersurface section* of $(X, Q)$ is a Cartier divisor on $X$ given by a local equation $h = 0$ for some $h \in m_Q$. From Theorem II.1.7 we infer that if $Q$ is in the exceptional locus of $X$, the virtual transform $\check{C}$ relative to $I$ of any curve defined by an element of $I_Q$ is a hypersurface section of $(X, Q)$. Moreover, the equations of these virtual transforms generate the maximal ideal $\mathfrak{m}_Q$.

Given a singularity $Q \in X$, the following corollary states the link between its fundamental cycle and the values of $I$ and $I_Q$ by the divisorial valuations relative to the exceptional components of the minimal resolution of $Q$.

**Corollary II.1.10.** *Let $J \subset I$ be a complete ideal of codimension one and let $Q \in X$ the point associated to it by Theorem II.1.7. Let $f : S' \longrightarrow X$ be the minimal resolution of the singularities of $X$ and $\{E_\alpha\}_{\alpha \in \Delta}$ the exceptional components of $\pi \circ f : S' \longrightarrow X$ and $\{E_\alpha\}_{\alpha \in \Delta_Q}$ the exceptional components on $S'$ contracting to $Q$.*

$$\begin{array}{ccc} S' & \xrightarrow{f} & X \\ & \underset{\pi_I \circ f}{\searrow} & \downarrow \pi_I \\ & & S \end{array}$$

*For each $\alpha \in \Delta$, write $v_\alpha$ for the divisorial valuation associated to $E_\alpha$. Then:*

*(a) $J\mathcal{O}_{S'}$ is an invertible sheaf if and only if $Q$ is singular.*

*(b) If $Q$ is regular, then*

$$v_\alpha(I) = v_\alpha(J), \text{ for each } \alpha \in \Delta.$$

*(c) If $Q$ is singular, then*

$$\begin{cases} v_\alpha(J) = v_\alpha(I) + z_\alpha & \text{if } \alpha \in \Delta_Q \\ v_\alpha(J) = v_\alpha(I) & \text{otherwise} \end{cases}$$

*where $Z_Q = \sum_{\alpha \in \Delta_Q} z_\alpha E_\alpha$ is the fundamental cycle of $Q$.*

*Proof.* By Proposition II.1.4 we know that $J\mathcal{O}_X = \mathcal{M}_Q I \mathcal{O}_X$ and so, that $J\mathcal{O}_X$ is not invertible. Since the minimal resolution $f : S' \longrightarrow X$ of the singularities of $X$ induces an isomorphism in a neighborhood of any regular point in $X$, it is clear that $J\mathcal{O}_{S'}$ is not invertible if $Q$ is regular. If $Q$ is singular, then $f$ factors through the blowing-up of $Q$ and hence $J\mathcal{O}_{S'}$ is invertible. This proves (a).

Part (b) follows immediately from (a) and the equality $J\mathcal{O}_X = \mathcal{M}_Q I\mathcal{O}_X$.

Finally, if $Q$ is singular, we have seen that $J\mathcal{O}_{S'}$ is invertible. Let $E_I = \sum_{\alpha\in\Delta} v_\alpha(I)E_\alpha$ and $E_J = \sum_{\alpha\in\Delta} v_\alpha(J)E_\alpha$ be the divisors on $S'$ associated to $I\mathcal{O}_{S'}$ and $J\mathcal{O}_{S'}$, i.e. $I\mathcal{O}_{S'} \cong \mathcal{O}_X(-E_I)$ and $J\mathcal{O}_{S'} \cong \mathcal{O}_X(-E_J)$. Since $J\mathcal{O}_{S'} = \mathcal{M}_Q I\mathcal{O}_{S'}$ and $\mathcal{M}_Q\mathcal{O}_{S'} = \mathcal{O}_{S'}(-Z_Q)$ (Theorem I.3.15), we deduce that $E_J = Z_Q + E_I$ and this proves (c). □

## II.2 Enriques diagrams and sandwiched singularities

From now on, the situation we shall deal with is the following: $O \in S$ is a closed point of a non-singular algebraic surface over $\mathbb{C}$ and $\mathfrak{m}_O$ is the maximal ideal of $R = \mathcal{O}_{S,O}$. We shall take complete $\mathfrak{m}_O$-primary ideals in $R$. In this section, we begin the study of the connection between clusters of base points of complete $\mathfrak{m}_O$-primary ideals in $R$ with origin at $O$ and the sandwiched singularities obtained when blowing-up these ideals.

The aim is to show that, fixed a complete $\mathfrak{m}_O$-primary ideal $I$, the (weighted) Enriques diagram of $BP(I)$ encloses a lot of information about the configuration of the exceptional components on $X = Bl_I(S)$ and their singularities. In particular, we will be able to read from the Enriques diagram of $BP(I)$ and by means of the unloading procedure, the number of singularities of $X$ as well as their fundamental cycles and multiplicities.

Let $\mathcal{K} = (K, \nu)$ be the weighted cluster $BP(I)$ and keep the notation introduced in Chapter I. Write

$$\mathbb{F}_K = \bigsqcup_{p\in K} F_p$$

the disjoint union of the first neighbourhoods of the points of $K$ and $\overline{\mathbb{F}}_K = \mathbb{F}_K \setminus K$ the set of all points lying in the first neighbourhood of some point of $K$ and not in $K$. If $q \in \overline{\mathbb{F}}_K$, we denote by $\mathcal{K}_q$ the weighted cluster $(K \cup \{q\}, \nu^{(q)})$ where $\nu^{(q)}$ is the system of virtual multiplicities defined by

$$\nu_p^{(q)} = \begin{cases} \nu_p & \text{if } p \neq q \\ 1 & \text{if } p = q \end{cases}$$

Note that if $q \in F_p$, the excess $\rho_p = 0$ if and only if the excess of $\mathcal{K}_q$ at $p$ is $-1$ and hence, in this case, $\mathcal{K}_q$ is not consistent.

The following lemma will allow us to connect the results of the preceding section with the theory of clusters.

**Lemma II.2.1.** *If $q \in \overline{\mathbb{F}}_K$, then*

$$\dim_{\mathbb{C}} \frac{I}{H_{\mathcal{K}_q}} = 1.$$

*Moreover, every complete $\mathfrak{m}_O$-primary ideal of codimension one in $I$ has the form $H_{\mathcal{K}_q}$ for some $q \in \overline{\mathbb{F}}_K$.*

*Proof.* By computing the virtual codimension of $\mathcal{K}_q$ (Definition I.2.19), we obtain

$$c(\mathcal{K}_q) = \sum_{p \in K \cup \{q\}} \frac{\nu_p^{(q)}(\nu_p^{(q)}+1)}{2} = \sum_{p \in K} \frac{\nu_p(\nu_p+1)}{2} + 1 = c(\mathcal{K}) + 1.$$

Hence, by Proposition I.2.23,

$$\dim_{\mathbb{C}} \frac{R}{H_{\mathcal{K}_q}} \leq c(\mathcal{K}) + 1$$

and since $\mathcal{K}$ is consistent,

$$c(\mathcal{K}) = \dim_{\mathbb{C}} \frac{R}{I}.$$

Thus,

$$\dim_{\mathbb{C}} \frac{R}{I} \leq \dim_{\mathbb{C}} \frac{R}{H_{\mathcal{K}_q}} \leq \dim_{\mathbb{C}} \frac{R}{I} + 1.$$

On the other hand, since $\mathcal{K}$ is consistent, there are germs of curve going through $\mathcal{K}$ and going not through $q$ (Theorem I.1.30), and so $I \neq H_{\mathcal{K}_q}$. Because of this,

$$\dim_{\mathbb{C}} \frac{R}{I} \neq \dim_{\mathbb{C}} \frac{R}{H_{\mathcal{K}_q}}$$

and so

$$\dim_{\mathbb{C}} \frac{R}{H_{\mathcal{K}_q}} = \dim_{\mathbb{C}} \frac{R}{I} + 1.$$

This proves the first claim. The second follows from [10] §1; we sketch here the idea of the proof. If $J \subset I$ has codimension one, write $\mathcal{K}' = BP(J)$ and $\nu'_p$ for its virtual multiplicity at $p$. By Theorem I.2.9, we have $J = H_{\mathcal{K}'}$. By adding points with virtual multiplicities zero, we can assume that $\mathcal{K}$ and $\mathcal{K}'$ have the same points. Since $\mathcal{K} \prec \mathcal{K}'$ and they are consistent, there exists some point $p$ such that $\nu'_p > \nu_p$ and $\nu'_u = \nu_u$ for every $u$ preceding $p$. Then, take some $q \in F_p$ not already in $K$ and take the cluster $\mathcal{K}_q$. It is immediate to see that $H_{\mathcal{K}'} \subset H_{\mathcal{K}_q}$, and since both have codimension one in $I$, they are equal. Hence, $J = H_{\mathcal{K}_q}$. □

**Corollary II.2.2.** *If $\mathcal{K}_q$ is not consistent, all the unloading steps leading from $\mathcal{K}_q$ to $\widetilde{\mathcal{K}_q}$ are tame*

*Proof.* It follows immediately from Proposition I.2.23 and the equality $c(\mathcal{K}_q) = \dim_{\mathbb{C}} R/H_{\mathcal{K}_q}$ seen in the proof of Lemma II.2.1. □

**Remark II.2.3.** Lemma II.2.1 suggests that we can map any point $q \in \overline{\mathbb{F}}_K$ to the point in the exceptional locus of $X$ corresponding by Theorem II.1.7 to the complete ideal $H_{\mathcal{K}_q} \subset I$, and that this correspondence is surjective. Note that it is not injective, since if $p \in K$ has excess zero and $q, q' \in F_p$ are different points not in $K$, then $\mathcal{K}_q$ and $\mathcal{K}_{q'}$ are equivalent clusters (for they are not consistent and after the first unloading step, which is performed on $p$, we obtain the same cluster) and thus, $H_{\mathcal{K}_q} = H_{\mathcal{K}_{q'}}$, so $q$ and $q'$ correspond to the same point in $X$.

Before going further and for future reference, we state the following result concerning the strict, virtual and total transforms (on $X$ and on $S_K$) of curves on $S$.

**Lemma II.2.4.** *Let $C$ be a curve on $S$.*

*(a) The strict transform $\widetilde{C}^{S_K}$ equals the strict transform by $f$ of the strict transform $\widetilde{C}$.*

*(b) The total transform $\pi_K^*(C)$ equals the total transform by $f$ of the total transform $\pi^*C$.*

*(c) If $C$ goes through $\mathcal{K}$, the virtual transform $\check{C}^{\mathcal{K}} \subset S_K$ relative to the (virtual) multiplicities of $\mathcal{K}$ equals the total transform by $f$ of the virtual transform $\check{C} \subset X$ relative to $I$.*

*(d) If $C$ goes through $\mathcal{K}$, then the strict transform $\widetilde{C}$ equals the virtual transform $\check{C}$ relative to $I$ on $X$ if and only if $v_p(C) = v_p(I)$ for every dicritical point $p$ of $\mathcal{K}$. In this case, $\widetilde{C}$ goes through no singularities of $X$ if and only if the virtual transform $\check{C}^{\mathcal{K}}$ equals the strict transform $\widetilde{C}^{S_K}$, or equivalently, if and only if $C$ goes through $\mathcal{K}$ with effective multiplicities equal to the virtual ones.*

*Proof.* Since $\pi_K = \pi \circ f$, parts (a) and (b) are immediate from the definition of strict and total transform.

Now, if $C_0$ is a curve going sharply through $\mathcal{K}$, we have $\pi^*(C_0) = \widetilde{C_0} + L_I$ and $f^*(\widetilde{C_0}) = \widetilde{C_0}^{S_K} + E_I^{S_K}$. From Proposition I.1.29, we know that $\widetilde{C_0}^{S_K}$ intersects (transversally) each component $E_p$ at $\rho_p$ points and, by Remark

I.4.7, $f$ contracts $E_p$ to some singularity of $X$ if and only if $p$ is a non-dicritical point of $\mathcal{K}$. It follows that $\widetilde{C_0}$ goes through no singularity of $X$ and so, $f^*(\widetilde{C_0}) = \widetilde{C_0}^{S_K}$ and

$$f^*(L_I) = E_I^{S_K}. \tag{2.a}$$

Now, assume that $C$ goes through $\mathcal{K}$. Then, $\check{C} = \widetilde{C} + L_C - L_I$ and

$$L_C - L_I = \sum_{p \in \mathcal{K}_+} (v_p(C) - v_p(I))L_p \geq 0.$$

From this, the first claim of (d) is clear. Moreover, $f^*(\check{C}) = f^*(\widetilde{C} + L_C - L_I) = f^*(\widetilde{C} + L_C) - E_I^{S_K}$, which is the virtual transform of $C$ relative to the multiplicities of $\mathcal{K}$ (see Definition I.1.25). This proves (c).

Finally, assume that $v_p(C) = v_p(I)$ for every dicritical point $p$ of $\mathcal{K}_+$. Since $f$ is the minimal resolution of $X$, $\widetilde{C}$ goes through no singularities if and only if $f^*(\widetilde{C})$ contains no exceptional component. In our case, $\widetilde{C} = \check{C}$ and by part (c), we have

$$f^*(\check{C}) = \check{C}^{\mathcal{K}} = \widetilde{C}^{S_K} + E_C^{S_K} - E_I^{S_K}.$$

It follows that $f^*(\check{C}) = f^*(\widetilde{C})$ has no exceptional support if and only if $E_C^{S_K} = E_I^{S_K}$, i.e. if and only if $C$ goes through $\mathcal{K}$ with effective multiplicities equal to the virtual ones. □

**Proposition II.2.5.** *For each point $Q$ in the exceptional locus of $X$, there exists some $q \in \overline{\mathbb{F}}_K$ such that $I_Q = H_{\mathcal{K}_q}$.*

*Moreover, $Q$ is singular if and only if $\mathcal{K}_q$ is not consistent.*

*Proof.* We already know that $f : S_K \longrightarrow X$ is the minimal resolution of all the singularities of $X$ (Proposition I.4.3). Each component of $E_K$ is the iterated strict transform of the exceptional divisor of one of the blowing-ups composing $\pi_K$ and hence, it may be identified with some $F_p$, with $p \in K$. If $q \in F_p \setminus K$, we denote by $\varphi(q) \in S_K$ the point on $E_p$ corresponding to $q$ by the former identification. This gives a map

$$\varphi : \overline{\mathbb{F}}_K \longrightarrow E_K$$

which is exhaustive: if $P \in E_K$ let $p_0$ be the maximal point in $K$ such that $P$ is proximate to it, and let $q \in F_{p_0}$ be the point corresponding to $P$ by the identification of $F_{p_0}$ with $E_{p_0}$. By the maximality of $p_0$, $q$ is not in $K$ and $\varphi(q) = P$.

Now, if $Q$ is in the exceptional locus of $X$, let $p \in f^{-1}(Q)$ and $q \in \overline{\mathbb{F}}_K$ such that $P = \varphi(q)$. From Theorem II.1.7, we know that if $C$ is defined by an element of $I_Q$, then its virtual transform $\check{C}$ on $X$ goes through $Q$. Since the total transform of $\check{C}$ by $f$ is the virtual transform of $C$ relative to the virtual multiplicities of $K$ (see (c) of Lemma II.2.4), we see that $\check{C}^{\mathcal{K}}$ goes through $P$ and so, $C$ goes through $\mathcal{K}_q$. Therefore, $I_Q \subset H_{\mathcal{K}_q}$ and also

$$Q = f(P) = f(\varphi(q)) \tag{2.b}$$

Both ideals having codimension one in $I$ (Lemma II.2.1), they are equal.

For the second claim, recall from Remark I.4.7 that if $p \in K$, $E_p$ contracts to a point of $X$ (necessarily a singularity) if and only if $\rho_p = 0$. Let $q \in F_p$ be such that $H_{\mathcal{K}_q} = I_Q$. The excess of the cluster $\mathcal{K}_q$ at $p$ is $\rho_p - 1$, and so $\mathcal{K}_q$ is not consistent if and only if $\rho_p = 0$. Therefore, $\mathcal{K}_q$ is not consistent if and only if $E_p$ contracts to some singularity $Q_0$ of $X$. Since $P = \varphi(q) \in E_p$, it is clear that $Q_0 = f(P)$, and from equality (2.b), we infer that $E_p$ is contracted to $Q$ and we are done. □

**Remark II.2.6.** Let $Q$ be a singularity of $X$. By Proposition II.2.5, there exists some $q \in \overline{\mathbb{F}}_K$ such that $H_{\mathcal{K}_q} = I_Q$, and moreover, $\mathcal{K}_q$ is not consistent. Since there are no points in $\mathcal{K}_q$ infinitely near to $q$, the virtual multiplicity of $q$ in the consistent cluster $\widetilde{\mathcal{K}_q}$ obtained from $\mathcal{K}_q$ by unloading is zero. It follows that $q$ is not a base point of $I_Q$.

From now on, if $Q \in X$, we will write

$$T_Q = \{p \in K \mid f_*(E_p) = Q\}.$$

Hence, $\{E_p\}_{p \in T_Q}$ are the exceptional components of $S_K$ contracting to $Q$. Clearly, $T_Q = \emptyset$ if and only if $Q$ is non-singular. In particular, we have that

$$\{p \in K \mid \rho_p = 0\} = \bigcup_{Q \in Sing(X)} T_Q. \tag{2.c}$$

**Remark II.2.7.** By Zariski's Main Theorem (see for example Theorem V 5.2 of [29]), the union of the exceptional components $\{E_p\}_{p \in T_Q}$ is connected. Thus, if $p_1, p_2 \in T_Q$, then also $u \in T_Q$ for all $u \in ch(p_1, p_2)$ (see section I.5).

The following corollary says that if $Q \in X$ is singular, $p_1, p_2 \in T_Q$ and $q_1 \in F_{p_1}$ and $q_2 \in F_{p_2}$ are not in $K$, then the clusters $\mathcal{K}_{q_1}$ and $\mathcal{K}_{q_2}$ are equivalent.

**Corollary II.2.8.** *If $q_1, q_2 \in \overline{\mathbb{F}}_K$ are different, then $\widetilde{\mathcal{K}_{q_1}} = \widetilde{\mathcal{K}_{q_2}}$ if and only if $q_1 \in F_{p_1}, q_2 \in F_{p_2}$ and $p_1, p_2 \in T_Q$ for some singularity $Q \in X$.*

*In particular, $Q$ is non-singular if and only if there is only one $q \in \overline{\mathbb{F}}_K$ such that $H_{\mathcal{K}_q} = I_Q$.*

**Notation II.2.9.** If $Q$ is a singular point in the exceptional locus of $X$ and $q \in \overline{\mathbb{F}}_K$ is such that $H_{\mathcal{K}_q} = I_Q$, we denote by $\mathcal{K}_Q$ the cluster obtained by unloading $\mathcal{K}_q$ and dropping $q$; if $Q \in X$ is regular, then we write $\mathcal{K}_Q$ to mean $\mathcal{K}_q$, where $q$ is the (only) point such that $H_{\mathcal{K}_q} = I_Q$. It is clear that in any case, $H_{\mathcal{K}_Q} = I_Q$.

Note that after Corollary II.2.8, in order to obtain the cluster $\mathcal{K}_Q$ it does not matter where the point $q \in \overline{\mathbb{F}}_K$ is added, provided that $q \in F_p$ of Proposition II.2.5 and $p \in T_Q$. Note also that if $Q$ is singular, then $\mathcal{K}_Q$ is consistent but not strictly consistent in general.

The following corollary summarizes some of the information about the surface $X$ and its singularities that can be read off from the Enriques diagram of $\mathcal{K} = BP(I)$. In particular, we show that the coefficients of the fundamental cycle of any singularity $Q \in X$ at $E_p, p \in T_Q$ equals the number of unloading steps performed on $p$ in any unloading procedure of $\mathcal{K}_q$ with $H_{\mathcal{K}_q} = I_Q$, thus giving an easy way to compute the former.

**Corollary II.2.10.** *Keep the notation as above. Then:*

*(a) the number of singularities of $X$ equals the number of non-equivalent clusters $\mathcal{K}_q$, for $q \in F_p$ not already in $K$ and $p \in K$ a non-dicritical point of $\mathcal{K}$.*

*(b) If $p_1, p_2 \in T_Q$ for some singularity $Q \in X$, the exceptional components $E_{p_1}$ and $E_{p_2}$ intersect on $S_K$ if and only if $p_1$ is maximal among the points of $K$ proximate to $p_2$ or viceversa.*

*(c) Let $Q \in X$ be singular and let $q \in \overline{\mathbb{F}}_K$ be such that $H_{\mathcal{K}_q} = I_Q$. Write $Z_Q = \sum_{u \in T_Q} z_u E_u$ the fundamental cycle of $Q$. Then, for each $u \in T_Q$, $z_u$ is the number of unloading steps performed on $u$ in the unloading procedure from $\mathcal{K}_q$ to $\widetilde{\mathcal{K}_q}$.*

*Proof.* Part (a) follows immediately from the second claim of Corollary II.2.8 and part (b) follows from *3.* of Proposition I.1.16. By Remark II.2.6, we know that $q$ is not a base point of $I_Q$ and thus, the base points of $I_Q$ are contained in $K$. Hence, (c) of Corollary II.1.10 says that for each $u \in K$,

$$v_u(I_Q) = \begin{cases} v_u(I) + z_u & \text{if } u \in T_Q \\ v_u(I) & \text{otherwise.} \end{cases} \tag{2.d}$$

Now, we have already pointed out that the unloading procedure from $\mathcal{K}_q$ to $\widetilde{\mathcal{K}_q}$ is tame (Corollary II.2.2). Therefore, by Remark I.2.14, each unloading step on a point of $K$ increases by one its virtual value, while the values of the others points remain unaffected. We derive that in the unloading procedure from $\mathcal{K}_q$ to $\widetilde{\mathcal{K}_q}$, $z_u$ equals the number of unloading steps on each $u \in T_Q$ while if $u \in K \setminus T_Q$, no unloading is performed on $u$. □

**Remark II.2.11.** In Corollary II.1.14 of [58], Spivakovsky proves that once a sandwiched singularity $(\hat{X}, \hat{Q})$ is fixed, a complete $\mathfrak{m}_O$-primary ideal $I \subset R$ can be chosen so that $(\hat{X}, \hat{Q})$ is analytically isomorphic to $(X, Q)$ for some point $Q$ in $X = Bl_I(S)$ and

1. $(X, Q)$ is the only singularity of $X$
2. the strict transforms on the minimal resolution of $X$ of the irreducible components of the exceptional locus of $X$ are curves of the first kind (i.e. have self-intersection equal to $-1$).

The following lemma describes the clusters of base points of such ideals in terms of proximity and excesses.

**Lemma II.2.12.** *Assume that $I \subset R$ is a complete $\mathfrak{m}_O$-primary ideal such that $(\hat{X}, \hat{Q})$ is analytically isomorphic to some singularity in $X = Bl_I(S)$. Then, the above conditions* 1. *and* 2. *are equivalent to the following two:*

*(i) $BP(I)$ has positive excesses at its maximal points only*

*(ii) these maximal points are free*

*Proof.* By Remark I.5.1, we know that the self-intersection of the strict transform of the exceptional component $L_p$ is $-\omega(p) = -\sharp\{q \in K \mid q \to p\} - 1$. Since the exceptional components on $X$ are the direct image of the components $E_p$ where $\mathcal{K}$ has positive excess, the condition 2. above is equivalent to saying that the only points where $\mathcal{K}$ has positive excess (see Remark I.4.7) are the maximal ones. On the other hand, if $X$ has only one singularity, the union $E_0$ of the components $E_p$ such that $\mathcal{K}$ has excess 0 at $p$ is connected. If some maximal point $p \in K$ is proximate to two points, say $q_1$ and $q_2$, then *3.* of Proposition I.1.16 says that $E_p$ intersects $E_{q_1}$ and $E_{q_2}$, and hence, $E_{q_1}$ and $E_{q_2}$ are not in the same connected component of $E_0$ against the connectedness of $E_0$. □

Example II.2.16 at the end of this section presents a cluster with these requirements.

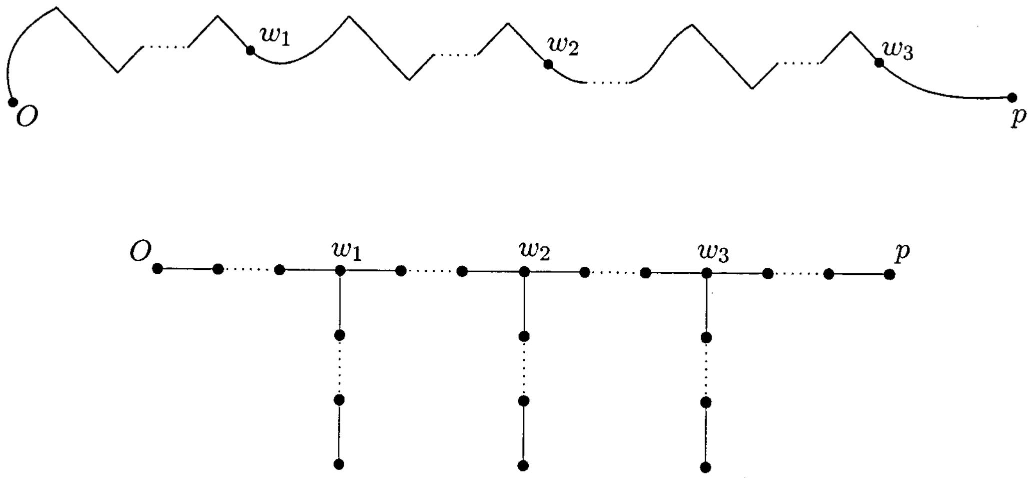



Figure II.1: Enriques diagram of $\mathcal{K}(p) = BP(I_p)$ and its corresponding (unweighted) dual graph when $p$ is free.

**Remark II.2.13.** Keep the notation as in I.2.10. Recall that by Theorem I.4.5, if $I = I_p$ is a simple ideal in $R$, the exceptional divisor on $X = Bl_{I_p}(S)$ is irreducible and the self-intersection of $E_p \subset S_{K(p)}$ (its strict transform on the minimal resolution of $X$) is $-1$, for there are no points in $K(p)$ proximate to $p$ (see *3.* of Proposition I.1.16). In fact, $X$ has one singularity if $p$ is free and two singularities if it is satellite. Figure II.1 shows how the Enriques diagram of the cluster $\mathcal{K}(p) = BP(I_p)$ looks like and also the dual graph $\Gamma$ when $p$ is free (see section I.5). Following Spivakovsky (Definition II.3.1 of [58]), *primitive* singularities are those sandwiched singularities that may be obtained by blowing-up a simple ideal verifying the assumptions of (i) and (ii) above, or equivalently, a simple ideal whose maximal base point is free.

From Remark II.2.11 and Proposition I.4.6 it follows that every sandwiched singularity is a birational join of finitely many primitive ones (Proposition II.3.6 of [58]). More precisely, once a sandwiched singularity $(X, Q)$ has been fixed, choose a complete $\mathfrak{m}_O$-primary ideal $I$ verifying (i) and (ii) of Lemma II.2.12 and let $I = \prod_{i=1}^{n} I_{p_i}$ be its decomposition into simple ideals. Since each $p_i$ is free, Lemma II.2.12 says that the blowing-up of $I_{p_i}$ gives rise to a surface $X_i$ with only one singularity, say $Q_i$. By definition, the $Q_i, 1 \leq i \leq n$ are primitive singularities, and $(X, Q)$ is the birational join of $(X_1, Q_1), \ldots, (X_n, Q_n)$.

This decomposition of a sandwiched singularity $(X, Q)$ as the birational join of primitive ones is not unique as it depends not only of $(X, Q)$ but also of the germ of the map $\pi : X \longrightarrow S$ at $Q$ (see II §3 and Example II.4.3 of [58] for details).

Now, we present a formula for the multiplicity of a sandwiched singularity $Q \in X$ in terms of the self-intersections of the clusters $\mathcal{K}$ and $\mathcal{K}_Q$ (see Definition I.2.19).

**Theorem II.2.14.** *Let $Q$ be any point in the exceptional locus of $X$. Then the multiplicity of $X$ at $Q$ is*

$$\mathrm{mult}_Q(X) = \mathcal{K}_Q{}^2 - \mathcal{K}^2.$$

*Proof.* By Proposition II.2.5, there exits some $q \in \overline{\mathbb{F}}_K$ such that $\mathcal{K}_q$ is equivalent to $\mathcal{K}_Q$. If $Q$ is regular, then $\mathcal{K}_q$ is consistent and hence, $\mathcal{K}_q = BP(I_Q)$. It is immediate to see that $\mathcal{K}_q{}^2 - \mathcal{K}^2 = 1$.

Assume now that $Q$ is singular. Then, write $\mathcal{K}_Q = (K, \nu')$ and $\overline{Z}$ for the $K$-vector whose $p$-entry is

$$\overline{z_p} = \begin{cases} z_p & \text{if } p \in T_Q \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $\mathbf{A}_K = -\mathbf{P}_K^t \mathbf{P}_K$ is the intersection matrix of $E_K$ (Lemma I.1.18) and denote by $\mathbf{A}_Q$ the intersection matrix of the exceptional divisor of the minimal resolution of $Q$. By Theorem I.3.15,

$$\mathrm{mult}_Q(X) = -Z_Q^t \mathbf{A}_Q Z_Q = -\overline{Z}^t \mathbf{A}_K \overline{Z}. \tag{2.e}$$

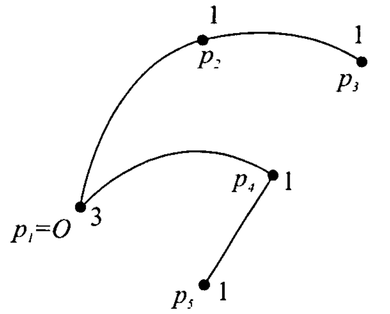
Write $\mathbf{v}_{\mathcal{K}} = \{v_p\}$ and $\mathbf{v}'_{\mathcal{K}} = \{v'_p\}$ the virtual values of $\mathcal{K}$ and $\mathcal{K}_Q$, respectively. Since $\boldsymbol{\nu}_{\mathcal{K}} = \mathbf{P}_K \mathbf{v}_{\mathcal{K}}$ (Lemma I.1.13), we have that

$$\mathcal{K}^2 = \boldsymbol{\nu}_{\mathcal{K}}^t \boldsymbol{\nu}_{\mathcal{K}} = (\mathbf{P}_K \mathbf{v}_{\mathcal{K}})^t (\mathbf{P}_K \mathbf{v}_{\mathcal{K}}) = \mathbf{v}_{\mathcal{K}}^t \mathbf{P}_K^t \mathbf{P}_K \mathbf{v}_{\mathcal{K}} = -\mathbf{v}_{\mathcal{K}}^t \mathbf{A}_K \mathbf{v}_{\mathcal{K}}$$

and similarly, $(\mathcal{K}_Q)^2 = -(\mathbf{v}'_{\mathcal{K}})^t \mathbf{A}_K \mathbf{v}'_{\mathcal{K}}$. Now, from (c) of Corollary II.1.10 we know that $\mathbf{v}'_{\mathcal{K}} = \mathbf{v}_{\mathcal{K}} + \overline{Z}$. Hence,

$$\begin{aligned} (\mathcal{K}_Q)^2 - \mathcal{K}^2 &= \mathbf{v}_{\mathcal{K}}^t \mathbf{A}_K \mathbf{v}_{\mathcal{K}} - (\mathbf{v}'_{\mathcal{K}})^t \mathbf{A}_K (\mathbf{v}'_{\mathcal{K}}) = \\ &= \mathbf{v}_{\mathcal{K}}^t \mathbf{A}_K \mathbf{v}_{\mathcal{K}} - (\mathbf{v}_{\mathcal{K}} + \overline{Z})^t \mathbf{A}_K (\mathbf{v}_{\mathcal{K}} + \overline{Z}) = \\ &= -2\overline{Z}^t \mathbf{A}_K \mathbf{v}_{\mathcal{K}} - \overline{Z}^t \mathbf{A}_K \overline{Z} = -2\overline{Z}^t \mathbf{A}_K \mathbf{v}_{\mathcal{K}} + \mathrm{mult}_Q(X), \end{aligned}$$

the last equality by (2.e) above. Since $\mathbf{A}_K \mathbf{v}_{\mathcal{K}} = -\boldsymbol{\rho}_{\mathcal{K}}$ (see Lemma I.1.22) and $T_Q \subset \{p \in K \mid \rho_p = 0\}$, we infer that $\overline{Z}^t \mathbf{A}_K \mathbf{v}_{\mathcal{K}} = 0$ and we are done. □



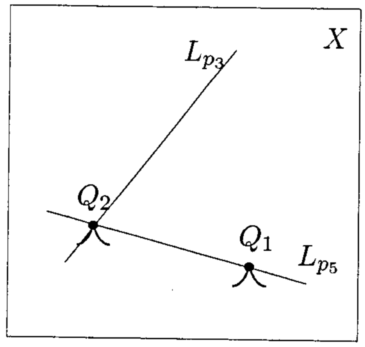




Figure II.2: On the left, the Enriques diagram of the cluster $\mathcal{K} = BP(I)$ of Example II.2.15; on the right, we represent the exceptional components and singularities of $X = Bl_I(S)$.

By means of Corollary II.2.10 and Theorem II.2.14 we obtain an easy procedure to compute the number of singularities of $X$ as well as their fundamental cycles and multiplicities from the data contained in the Enriques diagram of $\mathcal{K}$. First of all, the number of singularities equals the number of non-equivalent clusters $\mathcal{K}_q$ for $q$ in the first neighbourhood of some non-dicritical point of $\mathcal{K}$. Any of these clusters is clearly non-consistent and the number of unloading steps performed on each $p \in K$ in the unloading procedure from $\mathcal{K}_q$ to $\widetilde{\mathcal{K}_q}$ equals the coefficient of $E_p$ in the fundamental cycle of the singularity $Q \in X$ such that $I_Q = H_{\mathcal{K}_q}$. Finally, the multiplicity of $(X, Q)$ can be computed as the difference of the self-intersections of the clusters $\mathcal{K}_Q$ and $\mathcal{K}$, i.e. $\text{mult}_Q(X) = \mathcal{K}_Q{}^2 - \mathcal{K}^2$. We will come back to the fundamental cycle and the multiplicity of a sandwiched singularity in the forthcoming sections of this chapter.

Next we illustrate the results of this section by means of some examples.

**Example II.2.15.** Take $I = (x(y^2 - x^3), (x + y^2)(x + y)^3, (x + y^3)(x + y)^3, (x + y^3)(x + y)^2 y)$. We have $I = I_{p_3} I_{p_5}$, where $I_{p_3} = (x, y(x + y^3))$ and $I_{p_5} = (y^2, x^3, x^2 y)$ are simple complete ideals. The cluster $\mathcal{K}$ of base points of $I$ consists of the origin $p_1 = O$, the points $p_2 \in F_O$ and $p_3 \in F_{p_2}$ on the $x$-axis with virtual multiplicities 3,1 and 1, plus the point $p_4 \in F_O$ on the $y$-axis and the point $p_5 \in F_{p_4}$ and proximate to $O$, both with virtual multiplicity 1. The Enriques diagram of $\mathcal{K}$ is shown on the left of Figure II.2.

The points of $\mathcal{K}$ with positive excess are $p_3$ and $p_5$. Therefore, there are two exceptional components $L_{p_3}$ and $L_{p_5}$ on the surface $X = Bl_I(S)$ and the components $E_{p_1}, E_{p_2}$ and $E_{p_4}$ on $S_K$ contract to singularities of $X$(see

Remark I.4.7). Moreover, since $p_2$ is maximal among the points proximate to $O$, the components $E_{p_2}$ and $E_O$ contract to the same singularity of $X$, say $Q_1$. Hence, $T_{Q_1} = \{O, p_2\}$. By adding a free point $q$ with virtual multiplicity one in the first neighborhood of $p_3$ or $p_5$, we get a consistent cluster $\mathcal{K}_q$ and by Corollary II.2.5, $H_{\mathcal{K}_q}$ corresponds to a regular point of $X$ lying on $L_{p_3}$ or $L_{p_5}$.

By Corollary II.2.8, if we add a free and simple point $q$ not already in $K$ in the first neighbourhood of $O$ or $p_2$ and unload multiplicities, we get a consistent cluster $\widetilde{\mathcal{K}_q}$, which does not depend on the choice of $q$ (see Corollary II.2.8). Figure II.3 shows the unloading procedure from $\mathcal{K}_q$ to $\widetilde{\mathcal{K}_q}$ when $q$ is in $F_{p_2}$. The cluster $\mathcal{K}_{Q_1}$ is obtained by dropping $q$ in $\widetilde{\mathcal{K}_q}$ and, by Corollary II.2.8 and since $I_{p_1} = \mathfrak{m}_O$ and $I_{p_2} = (x, y^2)$, the complete ideal of codimension one associated to $Q_1$ by Theorem II.1.7 is $I_{Q_1} = I_{p_1}^3 I_{p_2} \subset I$.

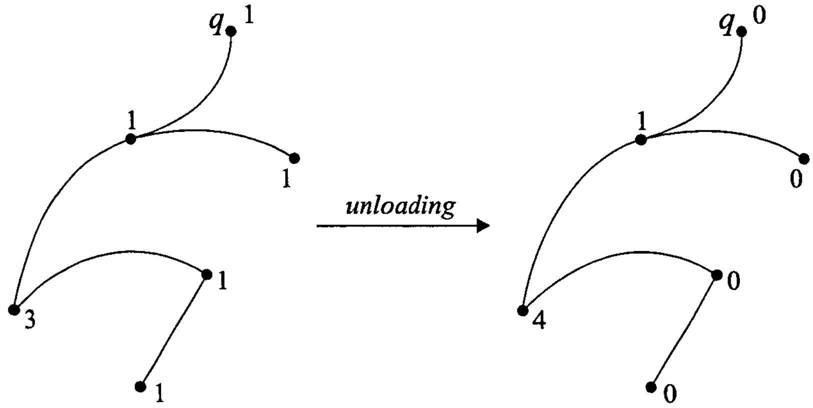



Figure II.3: Unloading from $\mathcal{K}_q$ to $\widetilde{\mathcal{K}_q}$ in Example II.2.15.

Now, Theorem II.2.14 provides the multiplicity of $Q_1$:

$$\operatorname{mult}_{Q_1}(X) = 17 - 13 = 4.$$

Analogously, by adding a free and simple point $q'$ in $F_{p_4}$ and not in $K$, we obtain a non-consistent cluster, and by Corollary II.2.8, the point of $X$ corresponding to $H_{\mathcal{K}_{q'}}$ by Theorem II.1.7 is singular. This point is just the singularity $Q_2$ is obtained by the contraction of $E_{p_4}$, and so $T_{Q_2} = \{p_4\}$.

As before, drop the point $q'$ of $\widetilde{\mathcal{K}_{q'}}$ to get $\mathcal{K}_{Q_2}$. The multiplicity of $Q_2$ is

$$\operatorname{mult}_{Q_2}(X) = 15 - 13 = 2.$$

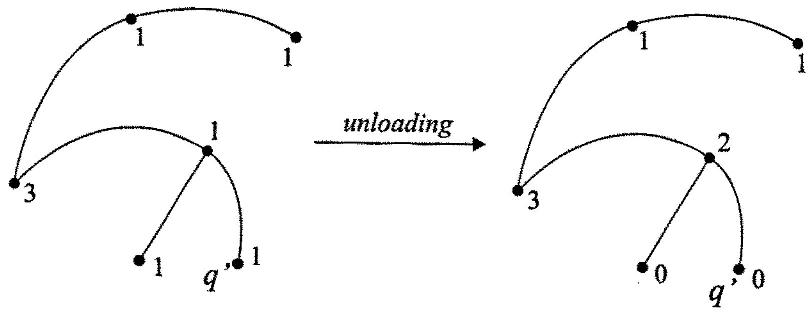



Figure II.4: Unloading procedure from $\mathcal{K}_{q'}$ to $\widetilde{\mathcal{K}_{q'}}$ in Example II.2.15.

If $I_{p_4} = (y, x^2)$, the complete ideal of codimension one associated to $Q_2$ by Theorem II.1.7 is $I_{Q_2} = I_{p_3} I_{p_4}^2 \subset I$.

**Example II.2.16.** Take a complete $\mathfrak{m}_O$-primary ideal $I = I_{p_6} I_{p_{12}}$ in $R$ with base points as shown in Figure II.5. The dicritical points of $\mathcal{K}$ are $p_6$ and $p_{12}$. The exceptional components $E_p$ for $p \neq p_6, p_{12}$ contract by $f$ to points of $X = Bl_I(S)$. Moreover, if we add some free and simple point $q$ in the first neighbourhood of any of these points and unload multiplicities, we obtain the same consistent cluster after dropping $q$. In virtue of Corollary II.2.8, this means that all the exceptional components $E_p$ for $p \neq p_6, p_{12}$ contract to the same singularity, say $Q$, and that this is the only singular point in $X$. The Enriques diagram of $\mathcal{K}_Q$ is shown in Figure II.6. As in the previous example, Theorem II.2.14 gives that

$$\mathrm{mult}_Q(X) = 187 - 179 = 8.$$

Moreover, if we compute the unloading steps performed at each point, we obtain in virtue of (c) in Corollary II.2.10,

$$Z_Q = E_{p_1} + E_{p_2} + E_{p_3} + 2E_{p_4} + E_{p_5} + 2E_{p_7} + E_{p_8} + 2E_{p_9} + 2E_{p_{10}} + E_{p_{11}}.$$

## II.3 On the fundamental cycle of a sandwiched singularity

From Corollary II.1.10, we know that each coefficient of the fundamental cycle of a sandwiched singularity $Q \in X$ is the difference between the values of $I_Q$ and $I$ by the divisorial valuation relative to the corresponding

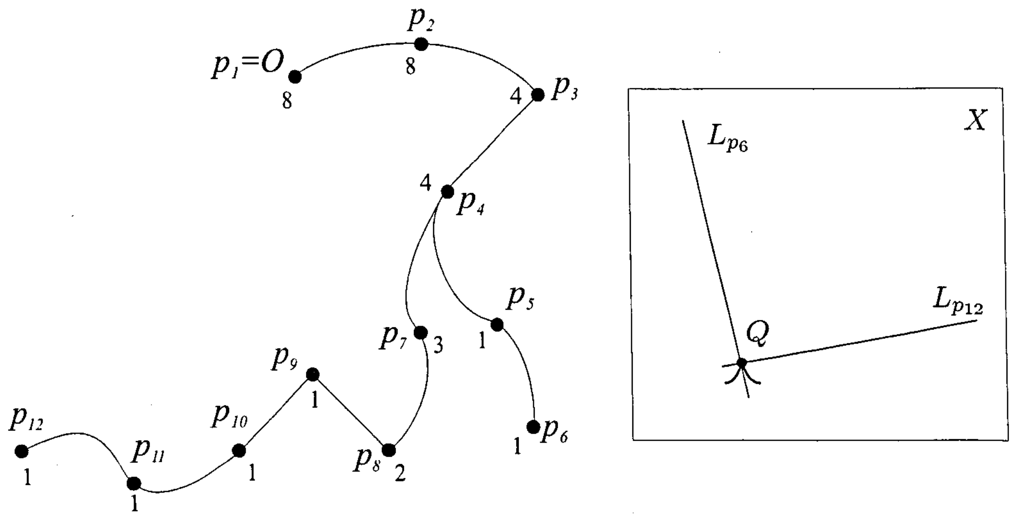



Figure II.5: On the left, the Enriques diagram of $\mathcal{K} = BP(I)$ in Example II.2.16; on the right, we represent the exceptional components and the singularity of $X = Bl_I(S)$.

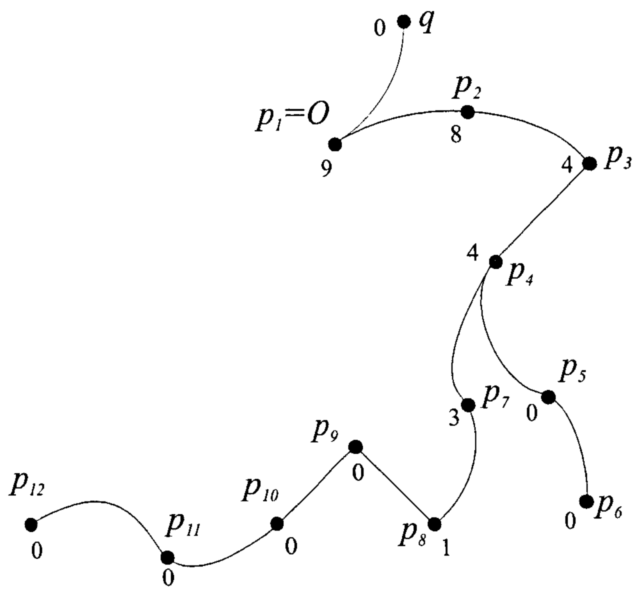



Figure II.6: Enriques diagram of the cluster $\mathcal{K}_Q$ in Example II.2.16.

exceptional component. Then, in Corollary II.2.10, we have reinterpreted these coefficients as the number of unloading steps on the points of $T_Q = \{p \in K \mid f_*(E_p) = Q\}$ in any unloading procedure giving rise to $BP(I_Q)$. In this section, we will see that these coefficients are closely related to the proximity relations of the points of $T_Q$ and we introduce the *multiplicity relevant points* (of $K$) relative to a sandwiched singularity, which will be of great use in section 2.5. The results given here will be very useful in the forthcoming chapters as they will allow us to deduce from the dual graph of a sandwiched singularity $\mathcal{O}_Q$ some proximity relations for the base points of any complete ideal giving rise to $\mathcal{O}_Q$.

Let $Q \in X$ be a sandwiched singularity and $\mathcal{K}_Q = (K, \nu')$ the cluster introduced in Notation II.2.9. Denote by $\rho'_p$ the excess of $\mathcal{K}_Q$ at $p \in K$.

**Lemma II.3.1.** *We have that*

*(a) by taking the partial order relation of being infinitely near to, there exists a unique minimal point in $T_Q$. We will denote this point by $O_Q$.*

*(b) $\nu'_{O_Q} = \nu_{O_Q} + 1$.*

*(c) if $p \in K$ and $p \neq O_Q$, then $\nu_p \geq \nu'_p \geq \nu_p - 1$.*

*Proof.* (a) Assume that $q_1, q_2 \in T_Q$ are different and minimal among the points of $T_Q$. Let $u_0 \in K$ be such that $q_1$ and $q_2$ are infinitely near to $u_0$ and maximal with this property. Then, $u_0 \in ch(q_1, q_2)$. By the minimality of $q_1$ and $q_2$, $u_0 \notin T_Q$ and Remark II.2.7 leads to contradiction.

To prove (b), keep the notation of Chapter I and denote $S^{(Q)}$ the surface obtained by blowing up the points preceding $O_Q$ ($S^{(Q)} = S$ if $O_Q = O$), so that $O_Q$ is a proper point of $S^{(Q)}$. Denote by $\varphi_{O_Q} : R \longrightarrow \mathcal{O}_{S^{(Q)}, O_Q}$ the morphism induced by blowing up. From Corollary II.1.10, we infer that $v_p(I_Q) = v_p(I)$ for each $p$ preceding $O_Q$. Therefore, the exceptional component of the total transform on $S^{(Q)}$ of curves going sharply through $\mathcal{K}$ and $\mathcal{K}_Q$ are equal. Write $z$ an equation of this exceptional component at $O_Q$. Then, the ideals generated by $z^{-1}\varphi_{O_Q}(I_Q)$ and $z^{-1}\varphi_{O_Q}(I)$ are complete $\mathfrak{m}_{O_Q}$-primary ideals $\check{I}_Q$ and $\check{I}$ in $\mathcal{O}_{S^{(Q)}, O_Q}$ as they are the stalks at $O_Q$ of the ideal sheaves $I_Q\mathcal{O}_{S^{(Q)}}$ and $I\mathcal{O}_{S^{(Q)}}$, which are complete. Since $I_Q \subset I$, we have

$$\check{I}_Q \subset \check{I}.$$

Moreover, note that the base points of $\check{I}$ (resp. $\check{I}_Q$) are the base points of $I$ (resp. $I_Q$) infinitely near or equal to $O_Q$. Since $O_Q$ is the minimal point of $T_Q$, no unloading steps are performed on any point $p$ preceding $O_Q$, and so $\nu'_p = \nu_p$. It follows that

$$\begin{aligned} \dim_{\mathbb{C}}(\frac{\check{I}}{\check{I}_Q}) &= \sum_{p \geq O_Q} (\frac{\nu'_p(\nu'_p+1)}{2} - \frac{\nu_p(\nu_p+1)}{2}) = \\ &= \sum_{p \in K} (\frac{\nu'_p(\nu'_p+1)}{2} - \frac{\nu_p(\nu_p+1)}{2}) = \dim_{\mathbb{C}}(\frac{I}{I_Q}) = 1. \end{aligned}$$

Then, from Corollary I.2.22, we know that

$$\dim_{\mathbb{C}}(\frac{\check{I}}{\mathfrak{m}_{O_Q}\check{I}}) = \nu_{O_Q} + 1$$

and

$$\dim_{\mathbb{C}}(\frac{\check{I}_Q}{\mathfrak{m}_{O_Q}\check{I}_Q}) = \nu'_{O_Q} + 1.$$

Thus, we infer that

$$\nu'_{O_Q} = \nu_{O_Q} + 1$$

and (b) is proved.

To prove (c), let $q \in \overline{\mathbb{F}}_K$ be such that $H_{\mathcal{K}_q} = BP(I_Q)$ (see Proposition II.2.5). Let $\mathcal{K}^{(0)} = \mathcal{K}_q, \mathcal{K}^{(1)}, ..., \mathcal{K}^{(m)} = \mathcal{K}'$ be any sequence of weighted clusters obtained by unloading, where $\mathcal{K}^{(n)} = (K \cup \{q\}, \nu^{(n)})$ is the cluster obtained after performing the $n$-th step. Note that by (c) of Corollary II.2.10 all the unloading steps are performed on points of $T_Q$. We will see that $\nu_u - 1 \leq \nu_u^{(n)} \leq \nu_u$, for any $u \neq O$ and any $n \in \{0, ..., m\}$. Then, part (c) follows by taking $n = m$.

First, we show that

$$\nu_u^{(n)} \leq \nu_u \tag{3.a}$$

by induction on $n$. For $n = 0$, there is nothing to prove. Assume $n > 0$ and let $u \in K$ be the point on which the $n$-th unloading is performed. By Corollary II.2.2, we know that the unloading procedure is tame and thus $\nu_u^{(n-1)} = \sum_{p \to u} \nu_p^{(n-1)} - 1$. Then,

$$\nu_p^{(n)} = \begin{cases} \nu_u^{(n-1)} + 1 & \text{if } p = u \\ \nu_p^{(n-1)} - 1 & \text{if } p \to u \\ \nu_p^{(n-1)} & \text{otherwise} \end{cases}$$

For $p \neq u$, the inequality (3.a) above is obvious by the induction hypothesis and for $u$,

$$\nu_u^{(n)} = \nu_u^{(n-1)} + 1 = \sum_{p \to u} \nu_p^{(n-1)} \leq \sum_{p \to u} \nu_p = \nu_u$$

the last equality because $u \in T_Q$ and so, the excess of $\mathcal{K}$ at $u$ is 0.

Now, to prove that $\nu_u^{(n)} \geq \nu_u - 1$ we will also use induction on $n$. Assume that there exists some $n > 0$ such that $\nu_w^{(n)} \leq \nu_w - 2$ for some point $w \in K$. We may assume that $n$ is minimal with this property and hence

$$\nu_w^{(n-1)} = \nu_w - 1 \tag{3.b}$$

and therefore, $\nu_w^{(n)} = \nu_w - 2$. Then it is clear that the $n$-th unloading step has been performed at some point $u \in K$ such that $w$ is proximate to $u$. Thus, $\rho_u^{(n-1)} = -1$. By the induction hypothesis, we have

$$\nu_u^{(n-1)} \geq \nu_u - 1. \tag{3.c}$$

Then,

$$\begin{aligned} \rho_u = \nu_u - \sum_{p \to u} \nu_p &\leq \quad \text{by (3.a) above} \\ \leq \nu_u - \nu_w - \sum_{p \to u, p \neq w} \nu_p^{(n-1)} &\leq \quad \text{by (3.c)} \\ \leq \nu_u^{(n-1)} + 1 - \nu_w - \sum_{p \to u, p \neq w} \nu_p^{(n-1)} &= \quad \text{by (3.b)} \\ = \nu_u^{(n-1)} - \sum_{p \to u} \nu_p^{(n-1)} &= \quad -1. \end{aligned}$$

against the fact that $\mathcal{K}$ is consistent. Hence the claim. □

**Remark II.3.2.** If $Q$ is non-singular in $X$, we denote by $O_Q$ the infinitely near point of $\overline{\mathbb{F}}_K$ that corresponds to $Q$ by the map $\varphi$ (see the proof of Proposition II.2.5). This is well defined by the second assertion of Corollary II.2.8.

From the above lemma, if $Q$ is singular and $q \in \overline{\mathbb{F}}_K$ is such that $H_{\mathcal{K}_q} = I_Q$, then the virtual multiplicities of $\mathcal{K}$ cannot decrease more than one unity in the unloading procedure from $\mathcal{K}_q$ to $\widetilde{\mathcal{K}_q}$.

Next we introduce subsets of points of $K$ associated to each singularity of $X$ that will play a basic role in section II.5.

**Definition II.3.3.** If $Q \in X$ is singular, we write

$$B_Q^{\mathcal{K}} = \{p \in K \mid \nu'_p = \nu_p - 1\}$$

and call it the set of the *multiplicity relevant points (of $K$) relative to $Q$* (*$MR$-points* for short). For every $p \in K$ we shall also write $\varepsilon_p^Q = \nu'_p - \nu_p$, and clearly,

$$\varepsilon_p^Q = \begin{cases} 1 & \text{if } p = O_Q \\ 0 & \text{if } p \notin B_Q^{\mathcal{K}} \\ -1 & \text{if } p \in B_Q^{\mathcal{K}} \end{cases} \tag{3.d}$$

If no confusion may arise, we will simply write $\varepsilon_p$ instead of $\varepsilon_p^Q$. In case $Q$ is non-singular, we take $B_Q^{\mathcal{K}} = \emptyset$.

**Remark II.3.4.** By (c) of Corollary II.2.10, the points on which unloading is performed are the points of $T_Q$. Thus, if $u \notin T_Q$, $u$ is proximate to some point in $T_Q$ if and only if $u \in B_Q^{\mathcal{K}}$ (see Subsection I.2.1).

**Example II.3.5.** In Example II.2.15, $O_{Q_1} = p_1$ and the $MR$-points relative to $Q_1$ are

$$B_{Q_1}^{\mathcal{K}} = \{p_3, p_4, p_5\}$$

(see Figure II.3). Similarly, $O_{Q_2} = p_4$ and $B_{Q_2}^{\mathcal{K}} = \{p_5\}$ (see Figure II.4).

**Example II.3.6.** In Example II.2.16, $O_Q = p_1$ and the $MR$-points relative to $Q$ are $B_Q^{\mathcal{K}} = \{p_5, p_6, p_8, p_9, p_{10}, p_{11}, p_{12}\}$.

We state the following corollary of Lemma II.3.1 for future reference.

**Corollary II.3.7.** $\nu_{O_Q} = \sum_{p \in B_Q^{\mathcal{K}}} \nu_p$.

*Proof.* It is clear that

$$\dim_{\mathbb{C}} \frac{R}{I_Q} = \dim_{\mathbb{C}} \frac{R}{I} + 1.$$

Then, by Proposition I.2.21,

$$\sum_{p \in K} \frac{\nu'_p(\nu'_p + 1)}{2} - \sum_{p \in K} \frac{\nu_p(\nu_p + 1)}{2} = 1. \tag{3.e}$$

From this and (3.d) above, it is immediate that

$$\sum_{p \in K} \left(\frac{\nu'_p(\nu'_p + 1)}{2} - \frac{\nu_p(\nu_p + 1)}{2}\right) = 1 + \nu_{O_Q} - \sum_{p \in B_Q^{\mathcal{K}}} \nu_p$$

and the claim follows by the equality (3.e) above. □

The main result of this section is the following theorem which will be very useful in the forthcoming Chapters III and IV. It shows that the variation of the coefficients of the fundamental cycle at the points of $T_Q$ is closely related to their proximity relations and the distribution of the $MR$-points relative to $Q$.

**Theorem II.3.8.** *Let $Q$ be a singular point of $X$. Then*

*(a) if $p \in T_Q$ verifies one of the following conditions:*

*(i) $p = O_Q$*

*(ii) $\mathcal{K}$ has positive excess at some point proximate to $p$*

*(iii) $p$ is proximate to some point not in $T_Q$*

*then, $z_p = 1$.*

*(b) For all $p \in T_Q$,*

$$z_p = \varepsilon_p + \sum_{q \in T_Q, q \to p} z_q.$$

*Proof.* Let $\overline{Z} = (\overline{z_p})_{p \in K}$ be as in the proof of Theorem II.2.14. By Lemma I.1.13, we know that

$$\nu_{\mathcal{K}} = \mathbf{P}_K \mathbf{v}_{\mathcal{K}}(I)$$

and

$$\nu'_K = \mathbf{P}_K \mathbf{v}_{\mathcal{K}}(I_Q)$$

and hence, by (c) of Corollary II.1.10,

$$\nu'_K - \nu_{\mathcal{K}} = \mathbf{P}_K(\mathbf{v}_{\mathcal{K}}(I_Q) - \mathbf{v}_{\mathcal{K}}(I)) = \mathbf{P}_K \overline{Z}.$$

Therefore, for any $u \in K$,

$$\overline{z_u} = (\nu'_u - \nu_u) + \sum_{u \to q} \overline{z_q} = \varepsilon_u + \sum_{u \to q} \overline{z_q}. \tag{3.f}$$

If $u \in T_Q$, (3.f) and the definition of $\overline{z_p}$ gives the equality claimed in (b) (see page 57). If $O_Q$ is proximate to some $q$, then $\overline{z_q} = 0$ and so, the equality (3.f) applied to $u = O_Q$ gives the case (i) of (a).

Now, if $u$ is a dicritical point of $\mathcal{K}$ proximate to $p$, then $u \notin T_Q$ and so, $\overline{z_u} = 0$ and by Remark II.3.4, $u \in B_Q^{\mathcal{K}}$. Then, $0 = \overline{z_u}$ and the equality (3.f) says that

$$\begin{aligned} \overline{z_u} &= \varepsilon_u + \sum_{u \to q} \overline{z_q} \geq \\ &\geq \varepsilon_u + 1 \end{aligned}$$

and from this, $\varepsilon_u = -1$ and also, $\sum_{u \to q} \overline{z_q} = 1$. In particular, $z_p = \overline{z_p} = 1$ and this gives (ii) of (a).

It remains to prove that if $p \in T_Q$ is proximate to some point $q \in K \backslash T_Q$, then $z_p = 1$. We have already shown that this is true for $p = O_Q$, so we assume that $p \in T_Q$ is not minimal. Then, $p$ is proximate to some $p_0 \in T_Q$ besides $q$. If $p$ is minimal among the points proximate to $q$, then $q$ is proximate to $p_0$ and we have already seen that then, $z_{p_0} = 1$. Then, we infer from (3.f) that

$$1 \leq \overline{z_p} = \varepsilon_p + \overline{z_q} + \overline{z_{p_0}} = \varepsilon_p + 1 \leq 1$$

the last inequality since $p \neq O_Q$, and hence, $\overline{z_p} = z_p = 1$.

If $p_0$ is not minimal, then it is also proximate to $q$ and we can assume by induction that $z_{p_0} = 1$. The argument used above shows similarly that $\overline{z_p} = z_p = 1$. This gives the case (iii) and completes the proof. □

The following result is a corollary of the proof of Theorem II.3.8 and we state it here for future reference.

**Corollary II.3.9.** *If $p \in T_Q$ is proximate to some point not in $T_Q$, then, $p \notin B_Q^{\mathcal{K}}$.*

*Proof.* Keep the notation of the proof of Theorem II.2.14. By (a) of Theorem II.1.10, we have that $z_p = 1$. If $p \in B_Q^{\mathcal{K}}$, then $\varepsilon_p = -1$, and by the equality (3.f), we have

$$\overline{z_p} = -1 + 1 = 0$$

against the fact $p \in T_Q$. □

**Remark II.3.10.** By (b) of Corollary II.3.8, if $p$ is satellite, proximate to $q_1$ and $q_2$, and $z_p = 1$ then $z_{q_1} = z_{q_2} = 1$ and $p \in B_Q^{\mathcal{K}}$.

### II.3.1 On the factorization of complete ideals of codimension one in $I$

Fixed a complete $\mathfrak{m}_O$-primary ideal $I$ in $R$, we study here the factorization of the complete $\mathfrak{m}_O$-primary ideals of codimension one in $I$ in terms of the factorization of $I$. To this aim, and bearing in mind Theorem I.2.9, we give a formula the excesses of the clusters $\mathcal{K}'$ obtained from $\mathcal{K} = BP(I)$ by adding some point not already in $\mathcal{K}$ counted once, and unloading multiplicities if necessary (by Lemma II.2.1, any complete $\mathfrak{m}_O$-primary ideal of codimension one in $I$ has the form $H_{\mathcal{K}'}$ for such a cluster $\mathcal{K}'$).

As before, write $X = Bl_I(S)$. The following proposition is the first result in this direction and gives a formula for the excesses of $\mathcal{K}_Q$ when $Q$ is a singular point of $X$ (see Notation II.2.9).

**Proposition II.3.11.** *Let $Q \in X$ be a sandwiched singularity. If $p \in K$, the excess of $\mathcal{K}_Q$ at $p$ is*

$$\rho'_p = \rho_p + \varepsilon_p^Q - \sum_{q \to p} \varepsilon_q^Q.$$

*If moreover $p \in T_Q$, then*

$$\rho'_p = \varepsilon_p^Q + \sharp\{q \in B_Q^{\mathcal{K}} \mid q \to p\} \tag{3.g}$$

*and, in particular, $\rho'_{O_Q} > 0$.*

*Proof.* By Definition I.1.20, $\rho'_p = \nu'_p - \sum_{q \to p} \nu'_q$ and by Lemma II.3.1, we have

$$\begin{aligned}\rho'_p &= (\nu_p + \varepsilon_p^Q) - \sum_{q \to p} (\nu_q + \varepsilon_q^Q) = \\ &= (\nu_p - \sum_{q \to p} \nu_q) + (\varepsilon_p^Q - \sum_{q \to p} \varepsilon_q^Q).\end{aligned} \tag{3.h}$$

This proves the first claim.

Now, if $p \in T_Q$, $\rho_p = \nu_p - \sum_{q \to p} \nu_q = 0$ by (2.c). Moreover, if $q$ is proximate to $p$, then $\varepsilon_q^Q \in \{-1, 0\}$. Thus, by 3.$h$,

$$\rho'_p = \varepsilon_p^Q + \sharp\{q \in B_Q^{\mathcal{K}} \mid q \to p\}$$

as claimed, and if $p = O_Q$,

$$\rho'_p = 1 + \sharp\{q \in B_Q^{\mathcal{K}} \mid q \to p\} \geq 1.$$

□

**Remark II.3.12.** Note that since $\mathcal{K}_Q$ is consistent, (3.g) says that every $MR$-point relative to $Q$ in $T_Q$ has another $MR$-point relative to $Q$ proximate to it (but this point may not belong to $T_Q$).

Recall that we write $\mathcal{K}_+$ for the set of dicritical points of $\mathcal{K}$, $\mathcal{K}_+ = \{p \in K \mid \rho_p > 0\}$, and that given a point $Q$ in the exceptional locus of $X$, $\mathcal{K}_+^Q = \{p \in \mathcal{K}_+ \mid Q \in L_p\}$. The next result relates the Zariski factorization of the complete $\mathfrak{m}_O$-primary ideals of codimension one in $I$ in terms of the factorization of $I$.

**Theorem II.3.13.** *Let $I = \prod_{p \in \mathcal{K}_+} I_p^{\alpha_p}$, with $\alpha_p \geq 1$, be the Zariski factorization of $I$ and let $J$ be a complete $\mathfrak{m}_O$-primary ideal of codimension one in $I$. Let $Q$ be the point in the exceptional locus of $X = Bl_I(S)$ corresponding to $J$ by Theorem II.1.7. Then*

$$J = H \prod_{p \in \mathcal{K}_+^Q} I_p^{\alpha_p - 1} \prod_{p \in \mathcal{K}_+ \setminus \mathcal{K}_+^Q} I_p^{\alpha_p},$$

*where $H \subset R$ is a complete $\mathfrak{m}_O$-primary ideal whose factorization shares no simple ideals with that of $I$. Moreover, $H$ is simple if and only if $Q$ is non-singular and in this case, $H = I_q$ where $q \in \overline{\mathbb{F}}_K$ is the unique point such that $H_{\mathcal{K}_q} = I_Q$ (see Corollary II.2.8). If $Q$ is singular, then the simple ideal $I_{O_Q}$ appears in the factorization of $H$.*

*Proof.* First of all, recall from Theorem I.2.9 that the exponent of a simple ideal $I_p$ in the Zariski factorization of $J = I_Q$ is the excess of $BP(I_Q)$ at $p$.

If $Q$ is regular, by Corollary II.2.8, there is only one $q \in \overline{\mathbb{F}}_K$ such that $J = H_{\mathcal{K}_q}$, and in virtue of Proposition II.2.5, $\mathcal{K}_q$ is consistent. From the definition of the excess, if $p \in K$,

$$\rho_p^{\mathcal{K}_q} = \begin{cases} \rho_p - 1 & \text{if } q \text{ is proximate to } p \\ \rho_p & \text{otherwise} \end{cases}$$

Now, if $q$ is proximate to some $p \in K$ (necessarily, $p \in \mathcal{K}_+$ since $\mathcal{K}_q$ is consistent) then $q$ is maximal among the points proximate to $p$ and so, $Q \in L_p$. Therefore, $\mathcal{K}_+^Q = \{p \in \mathcal{K}_+ \mid q \to p\}$ and the claim follows in this case.

Assume now that $Q$ is singular and let $p \in K \setminus T_Q$. Clearly, if $p$ is not infinitely near to $O_Q$, then $p \notin T_Q$ and it is proximate to no point of $T_Q$. In this case, $p \in \mathcal{K}_+^Q$ if and only if there is some point $q \in T_Q$ being maximal among the points in $K$ proximate to $p$, and by its definition, $O_Q$ is also proximate to $p$. Moreover, by Remark II.3.4, $p \notin B_Q$ and from Corollary II.3.9, no $MR$-point relative to $Q$ is proximate to $p$. From this,

$$\varepsilon_p - \sum_{q \to p} \varepsilon_q = \begin{cases} -1 & \text{if } p \in \mathcal{K}_+^Q \\ 0 & \text{otherwise.} \end{cases}$$

Now assume that $p$ is infinitely near to $O_Q$. Then, by Remark II.3.4, $p \in B_Q$ if and only if it is proximate to some point $u \in T_Q$, and so,

$$\varepsilon_p - \sum_{q \to p} \varepsilon_q = \begin{cases} -1 & \text{if } p \text{ is maximal among the points proximate to some } u \in T_Q \\ 0 & \text{otherwise.} \end{cases}$$

To sum up, we see that if $p \notin T_Q$, then

$$\rho'_p = \begin{cases} \rho_p & \text{if } p \notin \mathcal{K}_+^Q \\ \rho_p - 1 & \text{if } p \in \mathcal{K}_+^Q \end{cases}$$

and hence, we obtain that

$$J = H \prod_{p \in \mathcal{K}_+^Q} I_p^{\alpha_p - 1} \prod_{p \in \mathcal{K}_+ \setminus \mathcal{K}_+^Q} I_p^{\alpha_p},$$

where the complete ideal $H$ has a factorization

$$H = \prod_{p \in T_Q} I_p^{\beta_p},$$

with $\beta_p \geq 0$. Now, from (a) of Proposition II.3.11, we know that $I_{O_Q}$ actually appears in the factorization of $I_Q$. Since $I_{O_Q}$ is not a factor of $I$, it follows that it appears in the factorization of $H$. Finally, we know from Remark II.1.9 that the strict transform $\widetilde{C}$ of a generic curve $C$ going through $BP(I_Q)$ is a hypersurface section of $(X, Q)$. Thus, the germ of $\widetilde{C}$ at $Q$ is principal. If $H$ was simple and equal to $I_{O_Q}$, we would have that a general hyperplane section of $(X, Q)$ consists of a unique branch against the assumption that $Q$ is singular. □

## II.4 Resolution of sandwiched singularities

Keep the notation as above. We already know that if $S_K$ is the surface obtained by blowing-up all the points of $K$, then the morphism $f : S_K \longrightarrow X$ given by the universal property of the blowing-up is the minimal resolution of (all) the singularities of $X$ (Proposition I.4.3). The surface $S_K$ can also be understood as the blowing-up of the complete $\mathfrak{m}_O$-primary ideal $I' = \prod_{p \in K} I_p$ in $R$ (see Remark II.1.4 of [58]). The aim of this section is to describe the resolution process of $X$ in terms of the complete $\mathfrak{m}_O$-primary ideals of $R = \mathcal{O}_{S,O}$ and the unloading procedure. More precisely, take a sequence of blowing-ups from $X$ to $S_K$,

$$S_K = X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 = X \tag{4.a}$$

where $f_i$ is the blowing-up with center $Q_i \in X_{i-1}$, $Q_1 = Q$ and $f = f_n \circ \dots \circ f_1$. Since each surface $X_i$ is normal and birational to $S$, it is the blowing-up of some complete $\mathfrak{m}_O$-primary ideal in $R$ (Theorem II.7.17 of [29]), which is determined up to the exponents of the simple ideals appearing in its factorization provided that they are positive (i.e. the surface $X_i$ only determines the simple ideals in the factorization of any complete ideal giving rise to $X_i$).

Using the results of the previous sections, we will be able to determine the exceptional components appearing after the blowing-up of each singularity in a resolution process as (4.a) as well as their fundamental cycles and multiplicities.

First of all, we introduce some notation and definitions. The exceptional fiber $f_1^{-1}(Q) \subset X_1$ has a natural scheme structure given by the inverse image ideal sheaf $\mathfrak{m}_Q\mathcal{O}_{X_1}$. Write $Z_{X_1}$ for the codimension one component of this scheme and $|Z_{X'}|$ for its support, which is not empty since $f_1$ is not finite.

**Definition II.4.1.** We say that a hypersurface section $H : g = 0$ of $(X, Q)$ is *transverse* if $g \notin \mathfrak{m}_Q^2$ and the strict transform of $\check{C}$ on $X_1$ intersects transversally the maximal cycle $|Z_{X_1}|$ at regular points of $X_1$ and of $|Z_1|$.

REMARK: This notion of transverse hypersurface section is equivalent to that of general hypersurface section given in [26] in a more general context.

In Remark II.1.9, we have pointed out that the virtual transform $\check{C}$ on $X$ relative to $I$ of any curve $C$ going through $\mathcal{K}_Q$ is a hypersurface section of $(X, Q)$. Now, we have that

**Proposition II.4.2.** *The virtual transform on $X$ relative to $I$ of a curve on $S$ going through $\mathcal{K}$ is a hypersurface section of $(X, Q)$ if and only if $C$ goes through $\mathcal{K}_Q$. If moreover, $\widetilde{C}$ goes through $Q$, then $\check{C}$ is a transverse hypersurface section if and only if $C$ goes sharply through $\mathcal{K}_Q$, and in this case, the virtual transform $\check{C}$ equals the strict transform $\widetilde{C}$.*

*Proof.* The first claim is a direct consequence of Theorem II.1.7. Now, by Lemma II.2.4, we have that $\check{C}^{\mathcal{K}}$ is the total transform by $f$ of the virtual transform $\check{C}$ relative to $I$, and if $\widetilde{C}$ goes sharply through $\mathcal{K}'$, by Corollary II.1.10, we have that $\check{C}^{S_K} = \widetilde{C}^{S_K} + Z_Q$. This proves the "if" part.

For the converse, assume that the virtual transform of $C$ relative to $I$ is a transversal hyperplane section of $(X, Q)$ and that $\widetilde{C}$ intersects the exceptional locus at $Q$ and no other point. Then, $f^*(\check{C}) = \widetilde{C}^{S_K} + Z_Q$ and $\widetilde{C}^{S_K}$ intersects transversally $E_Q^{S_K} = \sum_{p \in T_Q} E_p$. Since $\pi_K = \pi \circ f$ and using (2.a) and (2.d), it follows that

$$\begin{aligned} \pi_K^*(C) &= f^*(\pi^*(C)) = f^*(\check{C} + L_I) = \widetilde{C}^{S_K} + Z_Q + E_I^{S_K} = \\ &= \widetilde{C}^{S_K} + E_{I_Q}^{S_K} \end{aligned}$$

So $C$ goes through $\mathcal{K}_Q$ with effective multiplicities equal to the virtual ones. Moreover, from the assumption, it is clear that $\widetilde{C}^{S_K}$ intersects (transversally) the exceptional divisor $E_I^{S_K}$ only at regular points of the exceptional locus of $f$. Only remains to show that $\widetilde{C}^{S_K}$ does not go through the points where $E_Q^{S_K}$ intersects the strict transform by $f$ of some $L_p$ for $p \in \mathcal{K}_+$.

Now, by Lemma II.2.4, we have that $\check{C}^{\mathcal{K}}$ is the total transform by $f$ of the virtual transform $\check{C}$ on $X$ relative to $I$, and if $\widetilde{C}^{S_K}$ goes sharply through $\mathcal{K}_Q$, by Corollary II.1.10, we have $\check{C}^{\mathcal{K}} = \widetilde{C}^{S_K} + Z_Q$. This proves the "if" part.

Now, for the converser, assume that the virtual transform of $C$ on $X$ relative to $I$ is a transverse hypersurface section of $(X,Q)$, $f^*(\check{C}) = \widetilde{C}^{S_K} + Z_Q$ and $\widetilde{C}^{S_K}$ intersects transversally $E_Q = \sum_{p \in T_Q} E_p$. Then, since $\pi_K = \pi \circ f$, we have

$$\pi_K^*(C) = f^*(\pi^*(C)) = f^*(\check{C} + L_I) = \widetilde{C}^{S_K} + Z_Q + E_I^{S_K},$$

the last equality by equality (2.a). Now, by 2.d, we have $Z_Q + E_I^{S_K} = E_{I_Q}^{S_K}$. Thus, we see that $C$ goes through $\mathcal{K}_Q$ with effective multiplicities equal to the virtual ones. Moreover, from the assumption, it is clear that $\widetilde{C}^{S_K}$ intersects (transversally) the exceptional divisor $E_I^{S_K}$ only at regular points of the exceptional locus of $f$. Only remains to show that $\widetilde{C}^{S_K}$ does not go through the points where $E_Q^{S_K}$ intersects the strict transform by $f$ of some $L_p$ for $p \in \mathcal{K}_+$. Assume that $\widetilde{C}^{S_K}$ goes through the point $q$ where $E_Q^{S_K}$ intersects $E_p$ with $p$ a dicritical point of $\mathcal{K}$. Then, $e_q(C) > 0$. Clearly, $q$ is proximate to $p$ and since $p \notin K$, we have from Proposition I.1.11 that

$$e_p(C) = e_q(C) + \sum_{u \to p, u \neq q} e_u(C) > \sum_{u \to p, u \in K} \nu_u = \nu_p,$$

against the fact that $C$ goes through $\mathcal{K}_Q$ with effective multiplicities equal to the virtual ones. □

To state the following proposition, we need a definition and to introduce some notation.

**Definition II.4.3.** Let $(X,Q)$ be a rational surface singularity and let $S' \longrightarrow X$ be the minimal resolution of $(X,Q)$. An irreducible component $E$ of the exceptional divisor of $S'$ is said to be *non-Tjurina* if $|Z_Q \cdot E|_{S'} < 0$, where $Z_Q$ is the fundamental cycle of $Q$ on $S'$. The *Tjurina* components of $Z_Q$ are the maximal connected components of $f^{-1}(Q)$ composed of exceptional components $E$ such that $|Z_Q \cdot E|_{S'} = 0$.

If $J = \prod_{i=1}^{s} I_j^{\alpha_j} \subset R$ is a complete $\mathfrak{m}_O$-primary ideal of $R$ (each $\alpha_j > 0$, $j = 1, \ldots, s$), we write

$$J' = \prod_{i=1}^{s} I_j.$$

Notice that by the discussion above, $Bl_J(S)$ is isomorphic to $Bl_{J'}(S)$.

**Proposition II.4.4.** *Let $Q \in X$ be a sandwiched singularity and keep the notation as in (4.a).*

*(a) Let $f^{(1)} = f_2 \circ \cdots \circ f_n : S_K \longrightarrow X_1$. The non-Tjurina components of $Z_Q$ are the components $E_p$ where $p \in T_Q$ is a dicritical point of $\mathcal{K}_Q$. In particular, $E_{O_Q}$ is a non-Tjurina components of $Z_Q$ and the exceptional component $E_{O_Q}^{X_1}$ already appears in the blowing-up of $Q$.*

*Therefore, any exceptional components of $X_1$ is either the strict transform of some $\{L_p\}_{p \in \mathcal{K}_+}$ or the direct image by $f^{(1)}$ of some non-Tjurina component of $Z_Q$.*

*(b) The surface $X' = Bl_{I_1}(S)$ obtained by blowing up $I_1 = (II_Q)'$ is isomorphic to $X_1$. Hence, $X_1$ is the birational join of $X$ and $Bl_{I_Q}(S)$ (see Definition I.4.4)*

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & X = Bl_I(S) \\ \downarrow & \searrow^{\pi_{(II_Q)'}} & \downarrow \pi_I \\ Bl_{I_Q}(S) & \xrightarrow{\pi_{I_Q}} & S \end{array}$$

*Proof.* By Proposition II.4.2, if $C$ is a curve going sharply through $\mathcal{K}_Q$, then $\widetilde{C}$ is a transverse hypersurface section of $(X, Q)$ and hence,

$$f^*(\widetilde{C}) = \widetilde{C}^{S_K} + Z_Q.$$

From this and the projection formula, we have that for each $u \in T_Q$,

$$|E_u \cdot \widetilde{C}^{S_K}|_{S_K} = -|E_u \cdot Z_Q|_{S_K}$$

and it is well known that $E_u$ does not contract by $f^{(1)}$ to a point of $X_1$ if and only if $|E_u \cdot Z_Q|_{S_K} < 0$ (see [59] Proposition 1.2). On the other hand, since $C$ goes sharply through $\mathcal{K}_Q$, part 4 of Proposition I.1.16 implies that $\rho_u^{\mathcal{K}_Q} = |\widetilde{C^{S_K}} \cdot E_u|_{S_K}$. Therefore, the non-Tjurina components of $Z_Q$ are the $E_p^{X_1}$, where $p \in T_Q$ is a dicritical point of $\mathcal{K}_Q$. This gives (a).

By (b) of Proposition I.4.6, the exceptional components on $X_{I_Q} = Bl_{I_Q}(S)$ are the strict transform of $\pi_{I_p}^{-1}(O)$ where $I_p$ are the simple ideals appearing in the factorization of $I_Q$, or equivalently by Remark I.4.7, the ideals $I_p$ where $p$ is a dicritical point of $\mathcal{K}_Q$. Now, if $p$ is a dicritical point of $\mathcal{K}_Q$ and $p \in T_Q$, part (a) says that the exceptional components $E_p$ is a non-Tjurina components of $Z_Q$ and so, it appears after blowing-up $Q$. On the other hand, if $p \in K \setminus T_Q$, then $p$ was already a dicritical point of $\mathcal{K}$. From this, it follows that $X_1$ dominates $X_{I_Q}$, and it is clear that it also dominates $X$. Therefore, $X_1$ dominates the birational join of $X$ and $X_{I_Q}$ and from part (a), we infer that actually, they are isomorphic. □

Next, in order to give a complete idea of the resolution process of the singularities of $X$ in terms of the complete ideals of $R$, take a sequence of complete $\mathfrak{m}_O$-primary ideals

$$I_0 \subset I_1 \subset ... \subset I_{n-1} \subset I_n \tag{4.b}$$

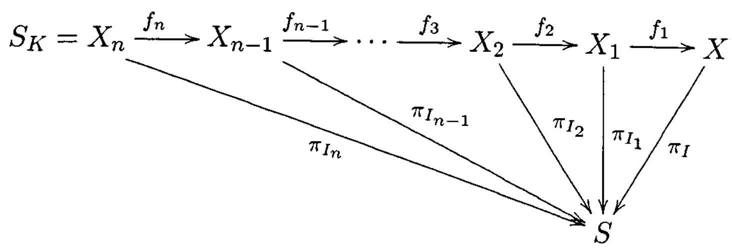
defined inductively as follows:

- for $i = 0$, take $I_0 = I'$
- once defined $I_i$ for $i \geq 0$, let $Q_i$ be any singularity of $X_i = Bl_{I_i}(S)$; then, if $(I_i)_{Q_i}$ is the complete $\mathfrak{m}_O$-primary ideal of codimension one in $I_i$ corresponding to $Q_i$ by Theorem II.1.7, define $I_{i+1} = (I_i(I_i)_{Q_i})'$.

We stop when we obtain the first ideal $I_n$ such that $X_n$ has no singularities. Note that for each $i \geq 0$, $I_i$ is a factor of $I_{i+1}$. Note also that the sequence 4.b is not unique, as it depends at each step on the choice of the singularity $Q_i$.

The following theorem follows from the previous results.

**Theorem II.4.5.** *For each $i \geq 1$, the surface $X_i$ is isomorphic to the surface obtained from $X_{i-1}$ by blowing-up the singularity $Q_{i-1}$ chosen to define $I_i$. In particular, we have that $X_n = S_K$.*




*Moreover, each $X_i$ is the birational join of $X_{i-1}$ and $Bl_{(I_{i-1})_{Q_{i-1}}}(S)$.*

**Example II.4.6.** Take the complete $\mathfrak{m}_O$-primary ideal $I = (x(y^2-x^3), (x-y^3)(y^2-x^3-x^4)(x-y), x(y^2-x^3-x^4)(x-y), x(y^2-x^3-x^5)(x-y))$ in $R$. The Enriques diagram of $\mathcal{K} = BP(I)$ is shown in Figure II.7. The dicritical points of $\mathcal{K}$ are $p_4$ and $p_9$ and the surface $X = Bl_I(S)$ has two exceptional components $L_{p_4}$ and $L_{p_9}$, and only one singularity $Q$, obtained by the contraction of the exceptional components $\{E_p\}_{p\neq p4,p9}$ of the surface $S_{\mathcal{K}}$. Hence, $T_Q = \{p_1 = O, p_2, p_3, p_5, p_6, p_7, p_8\}$. The dual graph of $Q$, $\Gamma_Q$, is shown in Figure II.7.

Note by the way that the ideal $I$ satisfies the conditions (i) and (ii) of Lemma II.2.12). In particular, the minimal point of $T_Q$, $O_Q$ equals the origin $O$ of $\mathcal{K}$.

We describe the resolution process of $Q$ in terms of a chain of complete $\mathfrak{m}_O$-primary ideals as in Theorem II.4.5. As explained in section II.2, add a free and simple point $q$ in $F_{O_Q} = F_O$, not already in $K$, and unload multiplicities to get $\mathcal{K}_Q$, see Figure II.8. $\mathcal{K}_Q$ has positive excess at $p_1 = O, p_3$ and $p_7$. Thus, in virtue of Proposition II.4.4, the exceptional divisor of blowing-up $Q_1$ has 3 components, which are the direct image by $f^{(1)}$ of $E_O, E_{p_3}$ and $E_{p_7}$.

Write $\mathcal{K}_1$ for the cluster of base points of $I_1 = (II_Q)'$. The surface $X_1 = Bl_{I_1}(S)$ has three singularities, which are resolved by blowing them up:

- Let $Q_2 \in X_1$ be the singularity obtained by the contraction of $E_{p_2}$. Thus, $T_{Q_2} = \{p_2\}$. As before, if $I_2 = (I_1(I_1)_{Q_1})'$, add a free and simple point $q_1 \in F_{p_2}$ to $\mathcal{K}_1$ and unload multiplicities in the resulting cluster $(\mathcal{K}_1)_{q_1}$. $D_1$ in Figure II.8 shows the Enriques diagram of $(\mathcal{K}_1)_{q_1}$. The unloaded cluster has positive excess at $p_2$ and thus, if $I_2 = (I_1(I_1)_{Q_2})'$, the surface $X_2 = Bl_{I_2}(S)$ is the minimal resolution of $Q_2$.

- $X_2$ has two singularities. Write $Q_3 \in X_2$ the singularity corresponding to the contraction of $E_{p_5}$ and $E_{p_6}$. The excess of the cluster obtained by unloading multiplicities after adding a new free point $q_2 \in F_{p_5}$ is positive at $p_5$ and $p_6$. Write $\mathcal{K}_2$ for the cluster of base points of $I_2$. $D_2$ in Figure II.8 is the Enriques diagram of $(\mathcal{K}_2)_{q_2}$. Hence, if $I_3 = (I_2(I_2)_{Q_3})'$ the surface $X_3 = Bl_{I_3}(S)$ is the minimal resolution of $Q_3$.

- Now, if $\mathcal{K}_3 = BP(I_3)$, $D_3$ in Figure II.8 is the Enriques diagram of $(\mathcal{K}_3)_{q_3}$. There is only one singularity $Q_4$ in $X_3$ corresponding to the contraction of $E_{p_8}$. Once again add a simple free point in $q_3 \in F_{p_8}$

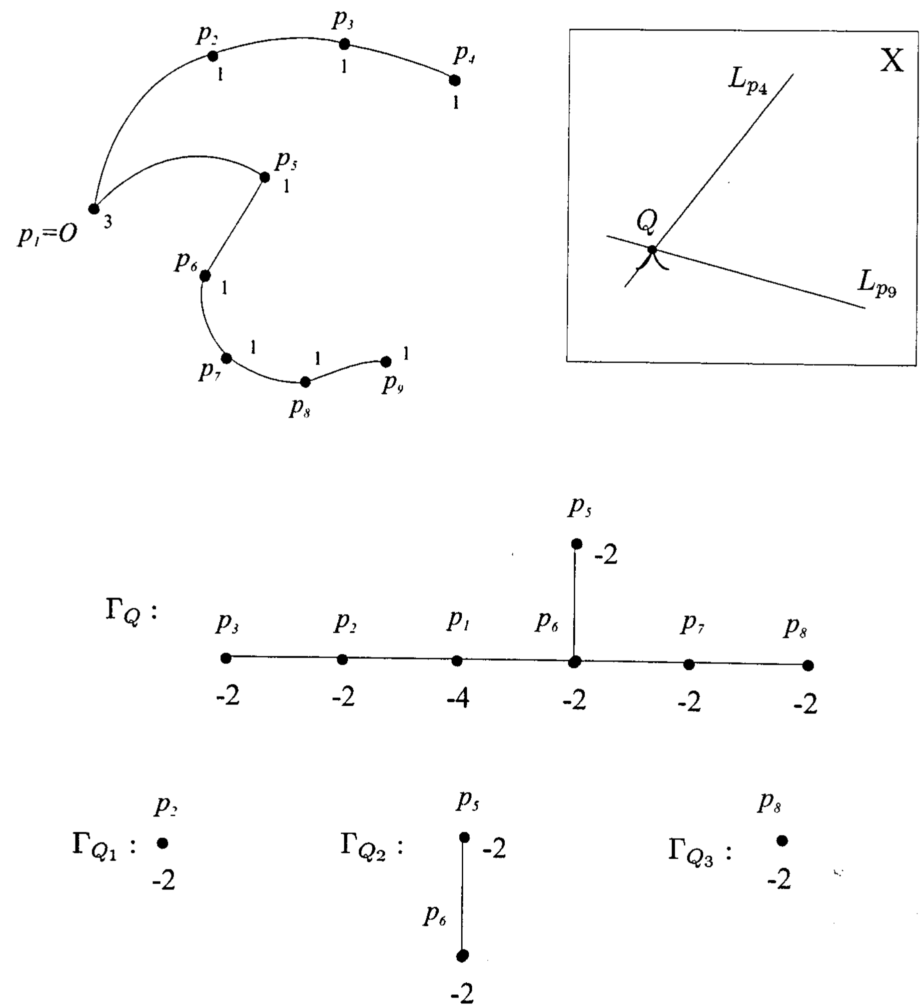



Figure II.7: On the top, the Enriques diagram of the cluster $BP(I)$ and the configuration of exceptional components and singularities of $X$. On the bottom, the resolution graphs of $Q$, $Q_1$, $Q_2$ and $Q_3$.

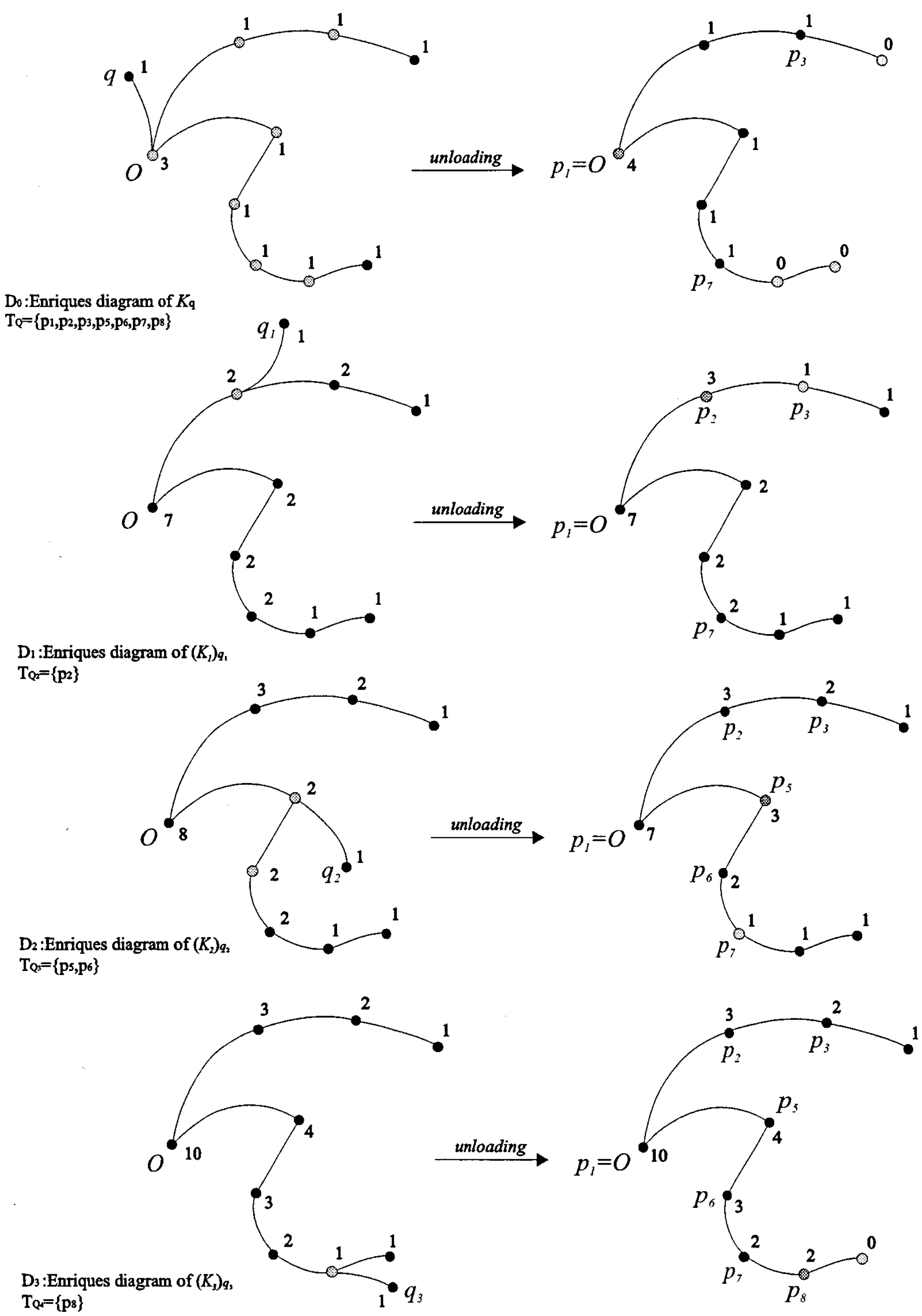



Figure II.8: Unloading in the resolution process of the singularity $Q \in X$ of Example II.4.6. In the Enriques diagrams on the left, we represent by grey-filled circles the points in $T_{Q_i}$, where $Q_i$ is the singularity to be resolved. In those on the right, we represent by dark grey-filled circles the minimal point of $T_{Q_i}$ and by light grey-filled circles the $MR$-points relative to $Q_{i-1}$.

and unload multiplicities. The resulting cluster has positive excess at $p_8$.

Finally, the blowing-up of the ideal $I_4 = (I_3(I_3)_{Q_4})'$ is (isomorphic to) $S_K$, the minimal resolution of $X$.

## II.5 On the multiplicity of curves through a sandwiched singularity

Keep the notation as in the preceding sections. Here and in the forthcoming sections, a curve will always mean an effective Weil divisor. Fixed a sandwiched singularity $Q \in X$, we give a formula for the multiplicity at $Q$ of a curve containing no exceptional components on $X$ in terms of the multiplicities of its projection on $S$ at the $MR$-points (of $K$) relative to $Q$. As a consequence, we deduce a formula for the multiplicity of a sandwiched singularity different from the formula of Theorem II.2.14.

The main result of this section is the following.

**Theorem II.5.1.** *Let $Q$ be a point in the exceptional locus of $X$. If $C$ is a curve on $S$, then*

$$\text{mult}_Q(\widetilde{C}) = e_{O_Q}(C) - \sum_{p \in B_Q^K} e_p(C).$$

**Remark II.5.2.** Note that Theorem II.5.1 gives actually the multiplicity at $Q$ of any curve on $X$ containing no exceptional components: if $C$ is such a curve, then $C = \widetilde{C_O}$ where $C_O = \pi_{I*}(C)$ and it is enough to apply Theorem II.5.1 to $C_O$.

**Remark II.5.3.** Theorem II.5.1 reveals that $O_Q$ and the $MR$-points relative to $Q$ are somehow distinguished points (hence the name!) of $K$ as the multiplicity of any curve $C$ at them determines the multiplicity of $\widetilde{C}$ at $Q$. In particular, if $e_{O_Q}(C) = 0$, then $\widetilde{C}$ does not go through $Q$.

*Proof of Theorem II.5.1* In case $Q$ is non-singular, Theorem II.5.1 gives the effective multiplicity of $C$ at $O_Q$ as one might expect, and there is nothing to prove. Hence, we assume that $Q$ is singular. By the projection formula applied to $f$, we know that

$$\text{mult}_Q(\widetilde{C}) \;=\; |\widetilde{C}^{S_K} \cdot Z_Q|_{S_K} = |\widetilde{C}^{S_K} \cdot (E_{I_Q}^{S_K} - E_I^{S_K})|_{S_K} \qquad (5.\text{a})$$

the last equality by Corollary II.1.10. Now, since the sheaves $I_Q \mathcal{O}_{S_K}$ and $I\mathcal{O}_{S_K}$ are invertible (also by Corollary II.1.10), we can take two curves $C_I$

and $C_{I_Q}$ going sharply through $\mathcal{K}$ and $\mathcal{K}_Q$, respectively and such that $\widetilde{C}_I$ and $\widetilde{C}_{I_Q}$ shares no points with $\widetilde{C}^{S_K}$ on $S_K$. Then, by the projection formula applied to $\pi_K$,

$$[C, C_I]_O = |\widetilde{C}^{S_K}, (\widetilde{C}_I^{S_K} + E_I^{S_K})|_{S_K} = |\widetilde{C}^{S_K} \cdot E_I^{S_K}|_{S_K}$$

and similarly,

$$[C, C_{I_Q}]_O = |\widetilde{C}^{S_K} \cdot (\widetilde{C}_{I_Q}^{S_K} + E_{I_Q}^{S_K})|_{S_K} = |\widetilde{C}^{S_K} \cdot E_{I_Q}^{S_K}|_{S_K}.$$

Hence, by (5.a) above,

$$\text{mult}_Q(\widetilde{C}) = [C, C_{I_Q}]_O - [C, C_I]_O,$$

and by Noether's formula (Theorem I.1.31), we have

$$[C, C_I]_O = \sum_p e_p(C)\nu_p$$

and

$$[C, C_{I_Q}]_O = \sum_p e_p(C)\nu_p'.$$

Therefore,

$$\text{mult}_Q(\widetilde{C}) = \sum_p e_p(C)(\nu_p' - \nu_p).$$

Finally, by (b) and (c) of Lemma II.3.1, we obtain

$$\text{mult}_Q(\widetilde{C}) = e_{O_Q}(C) - \sum_{p \in B_Q^{\mathcal{K}}} e_p(C)$$

as claimed.

□

As a corollary of the previous result, we get an easy formula for the multiplicity of a sandwiched singularity.

**Corollary II.5.4.** *If $Q$ is a point in the exceptional locus of $X$, then*

$$\text{mult}_Q(X) = 1 + \sharp B_Q^{\mathcal{K}}.$$

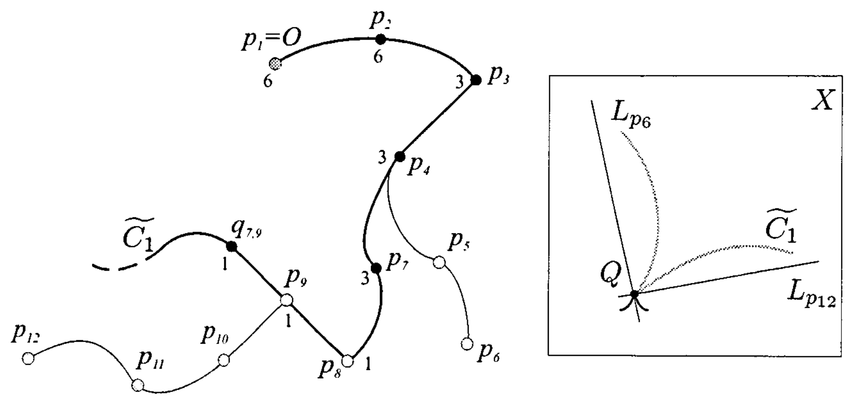



Figure II.9: On the left, we represent the Enriques diagram of $\mathcal{K}$ in Example II.5.5 and the singular points of $C_1$; $O_Q$ and the $MR$-points relative to $Q$ are represented by grey-filled circles. On the right, the exceptional components of the surface $X$ and the strict transform of $C_1$

*Proof.* To compute the multiplicity of $X$ at $Q$ it is enough to compute the multiplicity of a transverse hypersurface section of $(X, Q)$, that is, of the strict transform of a curve $C$ going sharply through $BP(I_Q)$ (see Proposition II.4.2). Then, Theorem II.5.1 says that

$$\begin{aligned}
\text{mult}_Q(X) &= e_{O_Q}(C) - \sum_{p \in B_Q^{\mathcal{K}}} e_p(C) = \nu'_{O_Q} - \sum_{p \in B_Q^{\mathcal{K}}} \nu'_p = \\
&= \nu_{O_Q} + 1 - \sum_{p \in B_Q^{\mathcal{K}}} (\nu_p - 1) = 1 + \sharp B_Q^{\mathcal{K}} + (\nu_{O_Q} - \sum_{p \in B_Q^{\mathcal{K}}} \nu_p) = \\
&= 1 + \sharp B_Q^{\mathcal{K}}
\end{aligned}$$

the last equality in virtue of Corollary II.3.7. □

**Example II.5.5.** Take again a complete $\mathfrak{m}_O$-primary ideal as in Example II.2.16. Since $B_Q^{\mathcal{K}} = \{p_5, p_6, p_8, p_9, p_{10}, p_{11}, p_{12}\}$ (see Example II.3.6), Corollary II.5.4 says that

$$\text{mult}_Q(X) = 1 + \sharp B_Q = 1 + 7 = 8$$

as we have already seen in Example II.2.16.

Now, if $p \in K$, write $C_p$ for a curve on $S$ going sharply through $\mathcal{K}(p)$ and going through no point of $K$ after $p$. Then, Theorem II.5.1 says that

$$\text{mult}_Q(\widetilde{C}_{p_8}) = e_{O_Q}(C_{p_8}) - \sum_{p \in B_Q^{\mathcal{K}}} e_p(C_{p_8}) = 2 - 1 = 1$$

Similarly,

$$\text{mult}_Q(\widetilde{C}_{p_1}) = \text{mult}_Q(\widetilde{C}_{p_2}) = \text{mult}_Q(\widetilde{C}_{p_3}) = \text{mult}_Q(\widetilde{C}_{p_5}) = \text{mult}_Q(\widetilde{C}_{p_{10}}) = 1,$$

$$\text{mult}_Q(\widetilde{C}_{p_4}) = \text{mult}_Q(\widetilde{C}_{p_7}) = \text{mult}_Q(\widetilde{C}_{p_9}) = 2$$

and clearly,

$$\text{mult}_Q(\widetilde{C}_{p_6}) = \text{mult}_Q(\widetilde{C}_{p_{12}}) = 0.$$

Now, take a curve $C_1$ on $S$ having the multiplicities shown in Figure II.9: $C_1$ is a generic branch going through the point $q_{7,9}$ where the exceptional components $E_{p_9}$ and $E_{p_7}$ on $S_K$ intersect. Again, by Theorem II.5.1, we obtain that

$$\text{mult}_Q(\widetilde{C_1}) = e_{O_Q}(C_1) - e_{p_8}(C_1) - e_{p_9}(C_1) = 6 - 1 - 1 = 4.$$

**Example II.5.6.** Take a complete $\mathfrak{m}_O$-primary ideal $I$ in $R$ having base points as shown in the Enriques diagram on the top of Figure II.10. The dicritical points of $\mathcal{K} = BP(I)$ are $p_3, p_6, p_9, p_{11}, p_{12}$ and $p_{15}$. There are two singularities in the surface $X = Bl_I(S)$, say $Q_1$ and $Q_2$, where

$$T_{Q_1} = \{p_1 = O, p_2, p_4, p_{13}, p_{14}\}$$

and

$$T_{Q_2} = \{p_5, p_7, p_8, p_{10}\}.$$

The dual graph of $Q_1$ and $Q_2$ are represented in Figure II.10. As explained in section II.2, by adding a point $q$ in the first neighbourhood of some point of $T_{Q_1}$ and $T_{Q_2}$ and unloading multiplicities, we obtain the clusters $\mathcal{K}_{Q_1}$ and $\mathcal{K}_{Q_2}$. The $IR$−points relative to $Q_1$ are $B^{\mathcal{K}}_{Q_1} = \{p_3, p_5, p_6, p_{15}\}$ and $O_{Q_1} = p_1$. Similarly, the $IR$−points relative to $Q_2$ are $B^{\mathcal{K}}_{Q_2} = \{p_6, p_9, p_{11}, p_{12}\}$ and $O_{Q_2} = p_5$. Then, by Corollary II.5.4,

$$\text{mult}_{Q_1}(X) = 1 + 4 = 5$$

$$\text{mult}_{Q_2}(X) = 1 + 4 = 5.$$

Now, if $C$ is a curve on $S$ as shown in Figure II.11, in virtue of Theorem II.5.1 the multiplicities of $\widetilde{C}$ at $Q_1$ and $Q_2$ are

$$\text{mult}_{Q_1}(\widetilde{C}) = 9 - 4 - 3 - 0 = 2$$

$$\text{mult}_{Q_2}(\widetilde{C}) = 4 - 3 - 0 - 0 - 0 = 1.$$

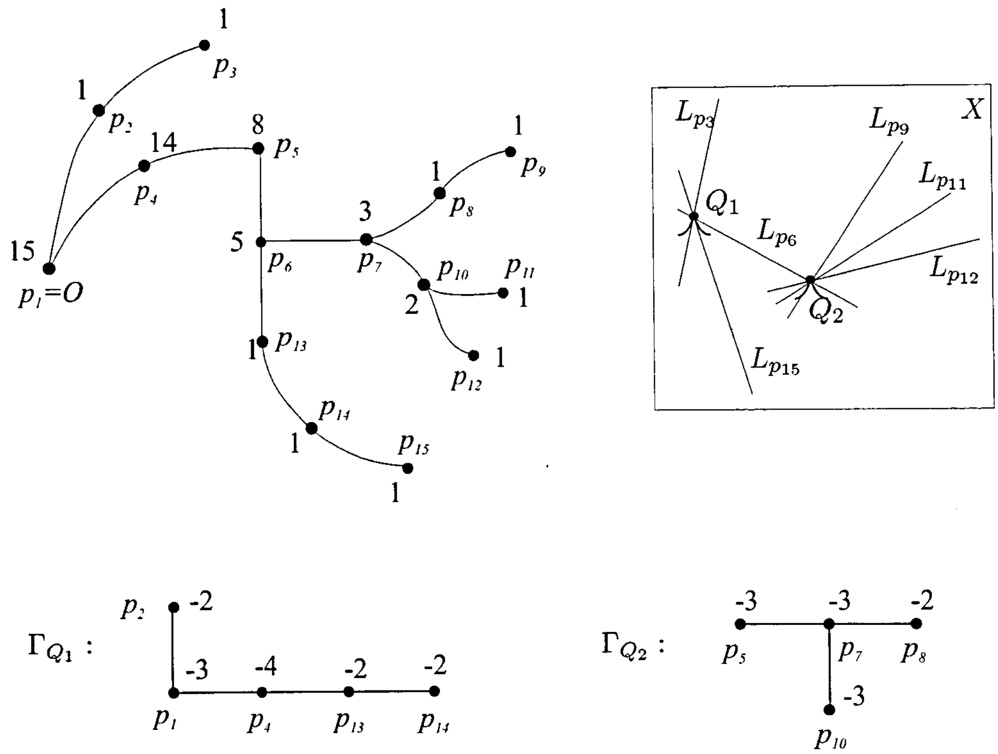



Figure II.10: On the top, the Enriques diagram of base points of $I$ in Example II.5.6 and the exceptional components and singularities if the surface $X = Bl_I(S)$. The dual graphs of $Q_1$ and $Q_2$ are represented on the bottom.

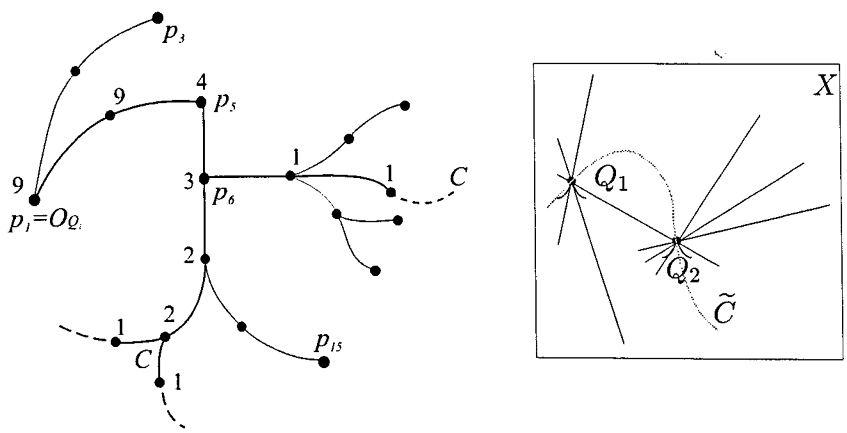



Figure II.11: On the left, the Enriques diagram of $C$ in Example II.5.6; on the right, the strict transform $\widetilde{C}$ on $X$ goes through the singularities $Q_1$ and $Q_2$ of $X$.

By means of Corollary II.5.4, we get an easy bound for the multiplicity of all the sandwiched singularities of $X$. Following [32], if $Q \in X$, we say that $R = \mathcal{O}_{S,O}$ is *maximally regular* in $\mathcal{O}_{X,Q}$ if there are no regular rings $R_0 \neq R$ such that $R \subset R_0 \subsetneq \mathcal{O}_{X,Q}$.

**Corollary II.5.7.** *If $Q$ is a point in the exceptional locus of $X$, then*

$$\mathrm{mult}_Q(X) \leq 1 + \nu_O$$

*and the equality holds if and only if $R$ is maximally regular in $\mathcal{O}_{X,Q}$ and all the $MR$-points relative to $Q$ are simple.*

*Proof.* The claim is clear if $Q$ is regular, so we assume that is singular. By Corollary II.5.4 and Corollary II.3.7, we have that

$$\mathrm{mult}_Q(X) \quad = 1 + \sharp B_Q^{\mathcal{K}} \leq 1 + \textstyle\sum_{p \in B_Q^{\mathcal{K}}} \nu_p = 1 + \nu_{O_Q} \leq 1 + \nu_O$$

and the equality holds if and only if the virtual multiplicity of all the points in $B_Q^{\mathcal{K}}$ is one and $\nu_O = \nu_{O_Q}$. Now, if $O_Q \neq O$, let $q \in K$ be the point preceding $O_Q$. If $\rho_q > 0$, then it is clear that $\nu_q > \nu_{O_Q}$ and so, $\nu_O > \nu_{O_Q}$. If $\rho_q = 0$, there exists some point $w \in K$ with $\rho_w > 0$, proximate to $O_Q$ and $q$ for otherwise, $O_Q$ would be maximal among the points proximate to $q$ and thus, $q$ would be also in $T_Q$, against the minimality of $O_Q$. Then, $\nu_O \geq \nu_q \geq \nu_{O_Q} + \nu_w > \nu_{O_Q}$. In any case, we see that $\nu_O > \nu_{O_Q}$. Therefore, $O_Q = O$ and this completes the proof. □

Now we want to compute the number of branches of a transverse hypersurface section of a sandwiched singularity in terms of the $MR$-points of $Q$. First, we need a definition.

**Definition II.5.8.** A point $q \in K$ is said to be *$T_Q$-satellite* if it is proximate to two points $p_1, p_2 \in T_Q$.

Note that any $T_Q$-satellite point is necessarily in $T_Q$.

**Proposition II.5.9.** *Let $Q$ be a singularity of $X$. Then the number of branches of a transverse hypersurface section $g = 0$ of $(X, Q)$ at $Q$ is*

$$n_{X,Q} = \sharp\{q \in B_Q^{\mathcal{K}} \mid q \text{ is } T_Q\text{-satellite}\} + \sharp(B_Q^{\mathcal{K}} \setminus T_Q) + 1.$$

*Proof.* From Proposition II.4.2, we know that if $C$ goes sharply through $\mathcal{K}' = BP(I_Q)$, its strict transform is a transverse hypersurface section of $(X, Q)$. Then, the number of branches of a transverse hypersurface section is given by

$$n_{X,Q} = |\widetilde{C}^{S_K} \cdot \sum_{p \in T_Q} E_p|_{S_K} = \sum_{p \in T_Q} |\widetilde{C}^{S_K} \cdot E_p|_{S_K}$$

and by *4.* of Proposition I.1.16 and the equality (3.g) in Proposition II.3.11,

$$n_{X,Q} = \sum_{p\in T_Q} \rho'_p = \sum_{p\in T_Q} \varepsilon_p + \sum_{p\in T_Q} \sharp\{q \in B_Q^{\mathcal{K}} \mid q \to p\}. \qquad (5.b)$$

Clearly, $\sum_{p\in T_Q} \varepsilon_p = 1 - \sharp B_Q^{\mathcal{K}} + \sharp(B_Q^{\mathcal{K}} \setminus T_Q)$ and note that

$$\sum_{p\in T_Q} \sharp\{q \in B_Q^{\mathcal{K}} \mid q \to p\} = \sum_{q\in B_Q^{\mathcal{K}}} \sharp\{p \in T_Q \mid q \to p\}.$$

Thus, by (5.b), we have

$$n_{X,Q} = 1 + \sharp B_Q^{\mathcal{K}} \setminus T_Q + \sum_{q\in B_Q^{\mathcal{K}}} (\sharp\{p \in T_Q \mid q \to p\} - 1).$$

Finally, if $q \in B_Q^{\mathcal{K}}$, then $q$ is proximate to one point or two of $T_Q$ and so, $\sharp\{p \in T_Q \mid q \to p\} - 1$ is one if $q$ is $T_Q$-satellite and zero, otherwise. This completes the proof. □

From Proposition II.5.9 we infer that

**Corollary II.5.10.** *If the ideal $I$ verifies the conditions (i) and (ii) of Lemma II.2.12, we have*

$$n_{X,Q} = \sharp\{q \in B_Q^{\mathcal{K}} \mid q \textit{ is satellite}\} + n_{\mathcal{K}} + 1 \geq n_{\mathcal{K}} + 1,$$

*where $n_{\mathcal{K}}$ is the number of irreducible subclusters of $\mathcal{K}$.*

*Proof.* It is enough to note that if $I$ satisfies (i) and (ii) of Lemma II.2.12, then $\sharp(B_Q^{\mathcal{K}} \setminus T_Q)$ equals the number $n_{\mathcal{K}}$ and that the notions of *$T_Q$-satellite* and *satellite* agree. □

**Corollary II.5.11.** *The following conditions are equivalent:*

*(i). the fundamental cycle $Z_Q$ is reduced;*

*(ii) The number of branches of a transverse hypersurface section of $(X, Q)$ equals the multiplicity of $X$ at $Q$;*

*(iii) $B_Q^{\mathcal{K}} \cap T_Q \subset \{p \in K \mid p \text{ is } T_Q\text{-satellite}\}$.*

*Moreover, if these conditions hold, the inclusion of (iii) is an equality.*

*Proof.* We will prove that (i)⇒(ii)⇒(iii).

(i)⇒(ii) By the projection formula, we have

$$\begin{aligned} n_{X,Q} &= |\widetilde{C}\cdot \sum_{p\in T_Q} E_p|_{S_K} = \sum_{p\in T_Q} |\widetilde{C}\cdot E_p|_{S_K} \leq \\ &\leq \sum_{p\in T_Q} z_p|\widetilde{C}\cdot E_p|_{S_K} = \mathrm{mult}_Q(X). \end{aligned} \tag{5.c}$$

From this, it is clear that if the fundamental cycle is reduced, then $n_{X,Q} = \mathrm{mult}_Q(X)$.

(i)⇒(ii) By (5.c) above and Corollary II.5.4, we have that $n_{X,Q} \leq 1 + \sharp B_Q$ and by Proposition II.5.9, there is an equality if and only if every point in $B_Q \cap T_Q$ is $T_Q$-satellite.

(ii)⇒(iii) In case $Z_Q$ is not reduced, take $p \in T_Q$ to be maximal among the points such that the coefficient in the fundamental cycle is bigger than one. If $q$ is the point next to $p$, then by (b) of Theorem II.3.8 applied to $q$,

$$1 = z_q = \varepsilon_q + \sum_{q\to u, u\in T_Q} z_u \geq \varepsilon_q + 2$$

and we derive that $\varepsilon_q = -1$ and that $p$ is the only point of $T_Q$ such that $q$ is proximate to. Thus, $q \in T_Q$ is a $MR$-point relative to $Q$ but it is not $T_Q$-satellite.

Finally, to prove the last claim, assume that $p$ is proximate to $q_1, q_2 \in T_Q$. Then, by (b) of Theorem II.3.8, we have that $z_p = \varepsilon_p + z_{q_1} + z_{q_2} \geq \varepsilon_p + 2$. Therefore, if $Z_Q$ is reduced, necessarily every $T_Q$-satellite point is a $MR$-point. □

**Remark II.5.12.** In Definition 3.4.1 of [34], Kollár introduces the concept of minimal singularity of a variety $V$ over $\mathbb{C}$ of any dimension: $P \in V$ has a *minimal* singularity if $\mathcal{O}_{V,P}$ is reduced, Cohen-Macaulay,

$$\mathrm{mult}_P(V) = \mathrm{emb.dim}_P(V) - \dim_P(V) + 1$$

and the tangent cone of $V$ at $P$ is reduced. Kollár himself claims that for rational surface singularities, minimal singularities are those having reduced fundamental cycle (Remark 3.4.10 of [34]).

Sandwiched surface singularities are Cohen-Macaulay and reduced. Since they are rational (see (i) of Proposition I.3.3), we know that $\mathrm{mult}_Q(X) = emb.dim_Q(X) - 1$ (Theorem I.3.15). Now, from Corollary II.5.11 we infer

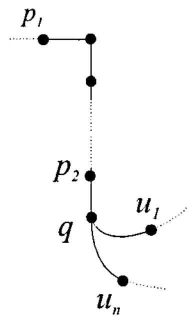



Figure II.12: Detail of an Enriques diagram: the point $q \in St_Q$ is proximate to $p_1, p_2 \in T_Q$ and all the points $u_1, \dots, u_n \in \mathcal{K}$ proximate to $q$ are free and they are also in $T_Q$.

that if $\mathcal{O}_Q$ is sandwiched, then it has reduced tangent cone if and only if its fundamental cycle is also reduced. Since any normal minimal surface singularity is sandwiched (see Proposition 2.4 of [58]), we recover the well known fact that any normal minimal surface singularity has reduced fundamental cycle.

**Corollary II.5.13.** *A normal surface singularity $(X, Q)$ is minimal if and only if is sandwiched and any of the conditions of Corollary II.5.11 holds. In this case, $\{p \in K \mid p$ is $T_Q$-satellite$\} = B_Q^{\mathcal{K}} \cap T_Q$.*

Non-minimal surface singularities may be characterized by the existence of some *special* points in $T_Q$.

**Definition II.5.14.** We say that $p \in T_Q$ is a *star of* $T_Q$ if $-\omega_{\Gamma_Q}(p) + \gamma_{\Gamma_Q}(p) = 1$, and write $St_Q$ for the set of stars of $T_Q$. of $T_Q$.

By Remark 2.3 of [58], $(X, Q)$ is minimal if and only if $St_Q = \emptyset$. The following lemma gives information concerning the proximity relations of these points and how they look like in the Enriques diagram of $K$ (see Figure II.12).

**Lemma II.5.15.** *If $q \in T_Q$, then $q \in St_Q$ if and only if*

*(i) $q$ is $T_Q$-satellite*

*(ii) all points proximate to $q$ are in $T_Q$ and free.*

*Proof.* By Remark I.5.1, $\omega(p) = \sharp\{u \in K \mid u \to p\} + 1$ and $\gamma(p)$ is the number of maximal points among those proximate to $p$ in $K$, plus the number of points of $T_Q$ to which $p$ is maximal among the points proximate to them. Since $p$ is proximate to at most two points in $T_Q$, we have

$$\begin{aligned}\gamma(p) &\leq \sharp\{u \in K \mid u \text{ maximal among the points proximate to } p\} + 2 \leq \\ &\leq \sharp\{u \in K \mid u \to p\} + 2 = \omega(p) + 1\end{aligned}$$

and the equality holds if and only if $p$ is $T_Q$-satellite and all the points proximate to $p$ are maximal. To close, it is enough to observe that this condition is equivalent to asking these points to be free. □

**Corollary II.5.16.** *Let $Q \in X$ be a sandwiched singularity such that $St_Q \neq \emptyset$. Then $\max_{p \in T_Q}\{z_p\} = \max_{p \in St_Q}\{z_p\}$*

*Proof.* By (b) of Theorem II.3.8, if $p$ is $T_Q$-satellite and proximate to $q_1$ and $q_2$, then $z_p \geq \max\{z_{q_1}, z_{q_2}\}$ while if $p$ is proximate to only one point in $T_Q$, say $q$, then $z_p \leq z_q$. From this and Lemma II.5.15, the claim easily follows. □

To close this section, we prove that the number of exceptional components going through a sandwiched singularity $(X, Q)$ is bounded by the embedding dimension of $(X, Q)$ and characterize when this bound is attained.

**Proposition II.5.17.** *Let $(X, Q)$ be a sandwiched singularity and write $m_{(X,Q)}$ for the embedding dimension of $(X, Q)$. Then,*

$$\sharp\mathcal{K}_+^Q \leq m_{(X,Q)},$$

*and the equality holds if and only if the following conditions hold:*

*(i) $O_Q$ is $T_Q$-satellite;*

*(ii) no point $p \in B_Q^{\mathcal{K}} \setminus T_Q$ has another point $q \in B_Q^{\mathcal{K}} \setminus T_Q$ proximate to it.*

*(iii) for every $q \in T_Q$, there are at most one point in $T_Q$ and one point in $B_Q^{\mathcal{K}} \setminus T_Q$ proximate to $q$ and in the same branch of $\mathcal{K}$.*

*In particular, a necessary condition for $\sharp\mathcal{K}_+^Q = m_{(X,Q)}$ is that $(X, Q)$ has a minimal surface singularity.*

*Proof.* First of all, note that if $p \in \mathcal{K}_+^Q$, then either $p$ is proximate to some $u \in T_Q$ or there is some $u \in T_Q$ maximal among the points of $K$ proximate to $p$.

In the first case and in virtue of Remark II.3.4, $p \in B_Q^{\mathcal{K}}$. Therefore,

$$\sharp\{p \in \mathcal{K}_+^Q \mid p \text{ is proximate to some } u \in T_Q\} \leq \sharp(B_Q^{\mathcal{K}} \setminus T_Q).$$

We will prove the following claim: the above inequality is an equality if and only if no point $p \in B_Q^{\mathcal{K}} \setminus T_Q$ has another point $q \in B_Q^{\mathcal{K}} \setminus T_Q$ proximate to it. Indeed, assume that $p \in B_Q^{\mathcal{K}} \setminus T_Q$ is not in $\mathcal{K}_+^Q$. Then, $p$ is proximate to some $u \in T_Q$. Now, let $q \in K$ be the point proximate to $u$ and $p$, and $w \in K$ the point proximate to $p$, infinitely near to $q$ and maximal with this property so that the exceptional components $E_p$ and $E_w$ on $S_K$ intersect. Hence, if $w \in T_Q$, we have that $p \in \mathcal{K}_+^Q$ against the assumption. Therefore, $w \notin T_Q$. Write $v_1 = q, v_2, \dots, v_n = w$ the points proximate to $p$, infinitely near to $u$ and each $v_i$ with $i \geq 2$ proximate also to $v_{i_1}$. Take $i_0 \in \{1, \dots, m\}$ such that $v_{i_0} \notin T_Q$ and minimal with this property. Then, $v_{i_0}$ is proximate to $v_{i_0-1}$ if $i_0 \geq 2$ or $u$ if $i_0 = 1$. In any case, $v_{i_0}$ is proximate to some point in $T_Q$ and by Remark II.3.4, $v_{i_0} \in B_Q$. This proves the claim.

Now, assume that $p \in \mathcal{K}_+^Q$ is not proximate to any point of $T_Q$. Then, there exists some point in $T_Q$ proximate to $p$. Let $q_0 \in T_Q$ be proximate to $p$ and minimal with this property. If $q_0 \neq O_Q$, then $q_0$ is proximate to some point, say $p'$, of $T_Q$. Then, either $p'$ is proximate to $p$ or $p$ is proximate to $p'$. The first case is not possible by the minimality of $q_0$ and neither is the second, because of the assumption. Hence, $q_0 = O_Q$. It follows that if $p \in \mathcal{K}_+^Q$ is not proximate to any point of $T_Q$, then $O_Q$ is proximate to $p$. If follows that

$$\sharp\{p \in \mathcal{K}_+^Q \mid p \text{ is not proximate to any point of } T_Q\} = \begin{cases} 2 & \text{if } O_Q \text{ is satellite} \\ 1 & \text{if } O_Q \text{ is free} \\ 0 & \text{if } O_Q = O \end{cases}$$

Therefore,

$$\sharp\mathcal{K}_+^Q \leq \sharp(B_Q^{\mathcal{K}} \setminus T_Q) + 2 \tag{5.d}$$

and the equality holds if and only if $O_Q$ is satellite and no point $p \in B_Q^{\mathcal{K}} \setminus T_Q$ has another point $q \in B_Q^{\mathcal{K}} \setminus T_Q$ proximate to it.

On the other hand, recall from Corollary I.3.16 that $\mathbf{m}_{(X,Q)} = \text{mult}_Q(X) + 1$. By Proposition II.5.9, we have that

$$\begin{aligned} \sharp(B_Q^{\mathcal{K}} \setminus T_Q) &= n_{X,Q} - \sharp\{q \in B_Q^{\mathcal{K}} \mid q \text{ is } T_Q\text{-satellite}\} - 1 \leq \\ &\leq \text{mult}_Q(X) + 1 - \sharp\{q \in B_Q^{\mathcal{K}} \mid q \text{ is } T_Q\text{-satellite}\} = \\ &= \mathbf{m}_{(X,Q)} - \sharp\{q \in B_Q^{\mathcal{K}} \mid q \text{ is } T_Q\text{-satellite}\} - 2 \leq \mathbf{m}_{(X,Q)} - 2, \end{aligned}$$

and the equality holds if and only if the following conditions holds: (1) $\text{mult}_Q(X) = n_{X,Q}$ and (2) no $MR$-point is $T_Q$-satellite. In virtue of Corollary II.5.13, these two conditions are equivalent to (1') $(X, Q)$ has a minimal singularity and (2') there are no $T_Q$-satellite points in $\mathcal{K}$.

Summing up and using (5.d) above, we infer that $\sharp\mathcal{K}_+^Q \leq \sharp(B_Q^{\mathcal{K}} \setminus T_Q) + 2 \leq \mathbf{m}_{(X,Q)}$ and the equality holds if and only if $O_Q$ is satellite, no point $p \in B_Q^{\mathcal{K}} \setminus T_Q$ has another point $q \in B_Q^{\mathcal{K}} \setminus T_Q$ proximate to it and for every $q \in T_Q$, there are at most one point in $T_Q$ and one point in $B_Q^{\mathcal{K}} \setminus T_Q$ proximate to $q$ and in the same branch of $\mathcal{K}$. This completes the proof. □

In the forthcoming section III.3, we will prove that once a sandwiched singularity $(\hat{X}, \hat{Q})$ is fixed and if $I$ is a complete $\mathfrak{m}_O$-primary ideal in $R$ such that $(\hat{X}, \hat{Q})$ is analytically isomorphic to $(X, Q)$ for some $Q \in X = Bl_I(S)$, then the tangents to the exceptional components going through $Q$ are linearly independent.

## II.6 On the exceptional components of $X = Bl_I(S)$

This section is devoted to prove that any two exceptional components on $X$ going through the same sandwiched singularity are not tangent. To prove this, some technical results concerning dual graphs of clusters are needed (see section I.5 for basic facts concerning dual graphs). This results will be used also in chapters 3 and 5.

Let $K$ be a (non-weighted) cluster with origin at $O \in S$ and $\Gamma_K$ its dual graph.

**Remark II.6.1.** Recall from section I.5 that given two points $q, p$ in a graph $\Gamma_K$, the chain $ch(q, p)$ is the subgraph of $\Gamma_K$ of all vertices and edges between $q$ and $p$, $q$ and $p$ included, and $d(q, p)$ is its length. Unless some confusion may arise, we will identify the infinitely near points $p \in K$ (or the vertices of the Enriques diagram of $K$) with the vertices in $\Gamma_K$ representing the exceptional component $\{E_p\}_{p \in K}$. Hence, we shall write for instance $K = |\Gamma_K|$ or we shall say that a point of $\Gamma_K$, say $p$, is infinitely near to some other point, say $q$ and it must be understood that the infinitely near point represented by $p$ is infinitely near to that represented by $q$.

**Proposition II.6.2.** *Let $p, q \in K$, $p$ infinitely near to $q$.*

*(a) If $u \in ch(p, q)$, $u$ is infinitely near or equal to $q$.*

*(b) All the points of $ch(q, p)$ are in the same branch of $K$.*

(c) *If $u \in ch(p,q)$ and $u \neq p$, either $p$ is infinitely near to $u$ or $u$ is infinitely near to $p$.*

(d) *Write $ch(q,p) = \{u_0 = q, u_1, \dots, u_n, u_{n+1} = p\}$. There exists some $i_0 \in \{0, \dots, n+1\}$ such that*

$$\begin{aligned} u_k \leftarrow u_{k+1} \quad & \textit{if } k \in \{0, \dots i_0 - 1\} \\ u_k \rightarrow u_{k+1} \quad & \textit{if } k \in \{i_0, \dots, n\}. \end{aligned}$$

*Moreover, if $j \geq i_0$, $u_j$ is proximate to some $u_{\sigma(j)}$ with $\sigma(j) \leq i_0 - 1$.*

*Proof.* To prove (a), we use decreasing induction on the length from the vertex $u$ to $q$. For $u = p$ there is nothing to prove. Assume $u \in ch(p,q)$ and let $u'$ be the vertex in $ch(p,u)$ adjacent to $u$. Then, $d(u',q) = d(u,q) + 1$ and by the induction hypothesis, $u'$ is infinitely near to $q$. Since $u$ and $u'$ are adjacent vertices, either $u$ is maximal among the points of $K$ proximate to $u'$ or vice versa (see *3.* of Proposition I.1.16). The first case being obvious, assume $u'$ is proximate to $u$. Notice that the set of all points in $K$ preceding $u'$ (as infinitely near points) is a totally ordered sequence and that $u$ and $q$ are in it. Hence, if $u$ is not infinitely near or equal to $q$, then $q$ is infinitely near to $u$. Since $u'$ is infinitely near to $q$ and proximate to $u$, we deduce that $q$ must be also proximate to $u$. Moreover, since $u'$ is maximal among the points proximate to $u$, $u' \in ch(u,q)$ and so, $d(u',q) < d(u,q)$, against the choice of the vertex $u'$.

Notice by the way that we have seen that all points of $ch(p,q)$ are in the same branch and hence, part (b) is proved. Part (c) follows immediately from this fact.

Now, we prove (d). If every $u_{i+1}$ is proximate to $u_i$, the first claim is obvious by taking $i_0 = n+1$. Hence, assume that there exists some $i \in \{0, \dots, n\}$ such that $u_i$ is proximate to $u_{i+1}$, and take $u_{i_0}$ to be minimal among the points of $ch(q,p)$ with this property. Now, we claim that

$$\begin{aligned} u_k \leftarrow u_{k+1} \quad & \text{if } k \in \{0, \dots i_0 - 1\} \\ u_k \rightarrow u_{k+1} \quad & \text{if } k \in \{i_0, \dots, n\}. \end{aligned}$$

To prove this, assume that there exists some $j \geq i_0 + 1$ such that $u_j \leftarrow u_{j+1}$ and take $u_{j_0}$, with $j_0 \geq i_0 + 1$ minimal. Then, both $u_{j_0 - 1}$ and $u_{j_0+1}$ are proximate to $u_{j_0}$ and, since they are adjacent to it, they are maximal among the points of $K$ proximate to $u_{j_0}$. Since they are in the same branch of the cluster ((b) of Proposition II.6.2), we derive that they must be equal which is impossible. Note by the way that $u_{i_0}$ is the maximal point of $ch(q,p)$.

Now, for each $j \in \{i_0, \dots, n\}$, write $u_{\sigma(j)}$ to mean the point maximal among the points of $ch(q, u_{i_0-1})$ $u_j$ is infinitely near to. Clearly $u_{\sigma(j)+1} \in ch(u_{\sigma(j)}, u_j)$ and, by (a) of Proposition II.6.2 again, $u_{\sigma(j)+1}$ is proximate to $u_{\sigma(j)}$. Moreover, by (c) of Proposition II.6.2 and the maximality of $u_{\sigma(j)}$, necessarily $u_{\sigma(j)+1}$ is infinitely near to $u_j$ and so, $u_j$ is proximate to $u_{\sigma(j)}$ as claimed because $u_{\sigma(j)+1}$ is so. This completes the proof. □

Now, fix a complete $\mathfrak{m}_O$-primary ideal $I \subset R$ and keep the notations introduced in the preceding sections. In particular, write $X = Bl_I(S)$ and $\mathcal{K} = BP(I)$. If $Q \in X$ is a singular point, write $\rho'_p$ for the excess of $\mathcal{K}_Q$ at $p \in K$. Write $\mathcal{K}_+^Q = \{p \in \mathcal{K}_+ \mid L_p \text{ goes through } Q\}$.

**Proposition II.6.3.** *Let $Q$ be a singular point of $X$ and $p, q \in \mathcal{K}_+^Q$. Then, there exists some $u \in ch^0(q,p)$ such that $\rho'_u > 0$.*

*Proof.* Write $ch(q,p) = \{u_0 = q, u_1, \dots, u_n, u_{n+1} = p\}$. We will distinguish two different cases.

CASE 1. $p$ is infinitely near to $q$.

In this case, we know by (d) of Proposition II.6.2 that $p$ is proximate to some $u_j$, $j \geq 0$.

*Case 1.1* If $j \geq 1$, then $u_j \in T_Q$ and hence, $p \in B_Q^{\mathcal{K}}$ by Remark II.3.4. Applying the formula (3.g) of Proposition II.3.11 to $u_i, i \in \{0, \dots, n\}$, we deduce that there exists some $i > 0$ such that $\rho'_{u_i} > 0$ or $u_j \in B_Q^{\mathcal{K}}$ for all $j \in \{1, \dots, n+1\}$, but in this case, Corollary II.3.9 applied to $u_1$ leads to contradiction.

*Case 1.2* If $j = 0$, then $p$ is proximate to $q$. Take $w$ the point in $F_p$ proximate to $q$. Then, $w \in K$ for otherwise, $p$ would be maximal among the points of $K$ proximate to $q$ and $ch(q,p) = \{q,p\}$. Moreover, $w \in ch(q,p)$ and it is proximate to $q$ and to $p$. Since $w$ cannot be proximate to any further point, is is the minimal point $O_Q$ of $T_Q$. From Proposition II.3.11, we infer that $\rho'_w > 0$.

CASE 2. $p$ is not infinitely near to $q$.

Let $x_0 \in K$ be the maximal point $p$ and $q$ are both infinitely near to. Then, $x_0 \in T_Q$ and $ch(q,p) = ch(q,x_0) \cup ch(x_0,p)$. Write

$$ch(x_0, q) = \{v_0 = x_0, \dots, v_n, v_{n+1} = q\}$$

and

$$ch(x_0, p) = \{w_0 = x_0, \dots, w_m, w_{m+1} = p\}.$$

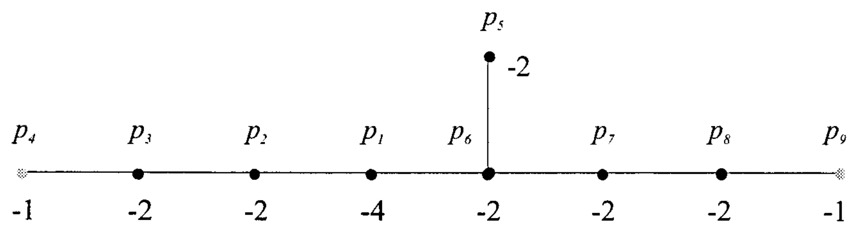



Figure II.13: Dual graph $\Gamma_K$ of $\mathcal{K} = BP(I)$ in Example II.6.4; the grey-filled dots of $\Gamma_K$, $p_4$ and $p_9$ represent the exceptional components on $S_K$ which are the strict transform by $f$ of the exceptional components of $X$.

By (a) of Proposition II.6.2, all the points $v_i$ and $w_j$ are infinitely near to $x_0$. Thus $v_1$ and $w_1$ are proximate to $x_0$. By Remark II.3.4, $p$ and $q$ are $MR$-points relative to $Q$. The same analysis of Case 1 can be done for $ch(x_0, q)$ and $ch(x_0, p)$ to deduce that if $\rho'_u = 0$ for each $u \in ch(q, x_0)$ and each $u \in ch(x_0, p)$, then $u_1$ and $v_1$ are $MR$-points relative to $Q$. Hence, by the formula (3.g) in Proposition II.3.11 applied to $x_0$, we infer that

$$\rho'_{x_0} \geq -1 + 2 = 1$$

and we are led to contradiction. □

**Example II.6.4.** In Example II.4.6, the exceptional components of $X$ are $L_{p_4}$ and $L_{p_9}$, which go through the sandwiched singularity $(X, Q)$. The exceptional divisor of the blowing-up $X_1$ of $Q$ is

$$Z_{X_1} = E_{p_1} + E_{p_3} + E_{p_7}$$

and, as shown in Figure II.13, $p_1, p_3$ and $p_7$ belong to $ch(p_4, p_9)$.

**Theorem II.6.5.** *Let $I \subset R$ be a complete $\mathfrak{m}_O$-primary ideal and let $Q$ be a singularity in $X = Bl_I(S)$. If $L_p$ and $L_q$ are two exceptional components on $X$ going through $Q$, then they are not tangent at $Q$.*

*Proof.* By Proposition II.6.3, there exists some $u \in ch^0(q, p)$ such that $\rho'_u > 0$. Therefore, by Proposition II.3.11, the simple ideal $I_u$ appears in the factorization of $I_Q$, and in virtue of Proposition II.4.4, we infer that $f_*^{(1)}(E_u)$ is an exceptional component for $f_1 : X_1 \longrightarrow X$. Since $u \in ch^0(q, p)$, it follows that the strict transforms of $L_p$ and $L_q$ on $X_1$ intersect the exceptional divisor of the blowing-up of $Q$ in different points. Hence, $L_p$ and $L_q$ are not tangent at $Q$. □

As a corollary of the forthcoming Theorem III.3.1, we will obtain a stronger result, namely that regardless of the complete $\mathfrak{m}_O$-primary ideal chosen to obtain a sandwiched singularity, the tangents to the exceptional components on $X = Bl_I(S)$ going through it are linearly independent. Nevertheless, the results of this section as well as the following technical result will be needed to prove Theorem III.3.1.

**Lemma II.6.6.** *Let* $u \in T_Q$ *and* $p_1, p_2, p_3 \in \mathcal{K}_+^Q$. *Assume that* $\rho'_w = 0$ *for* $w \in ch^0(u, p_i)$ *and* $i \in \{1, 2, 3\}$. *Then,* $\rho'_u \geq 2$.

*Proof.* First of all, note that any $w \in ch(p_i, u)$ with $w \neq p_i$ is in $T_Q$. We distinguish different cases according to the number of $p_i$'s being infinitely near to $u$.

CASE 1 Assume that $p_1, p_2$ and $p_3$ are infinitely near to $u$. Then, by (d) of Proposition II.6.2, each $p_i$ is proximate to some $u_j^i \in ch(p_i, u)$ and by Remark II.3.4, we have that $p_i \in B_Q^{\mathcal{K}}$. Applying the formula (3.g) of Proposition II.3.11, we infer that there is some $u^i \in ch(p_i, u) \cap B_Q^{\mathcal{K}}$ proximate to $u$. Therefore, $\sharp\{q \in B_Q^{\mathcal{K}} \mid q \to p\} \geq 3$ and also that $\rho'_u \geq 2$.

Now, we deal with the case where there is some $p_i$ which is not infinitely near to $u$. In this case, write $x_i \in K$ for the maximal point $p_i$ and $u$ are both infinitely near to. By the assumption, $x_i \neq u$. By (d) of Proposition II.6.2 and the argument of case 1, we see that $p \in B_Q$ and hence, that there is some $v_i \in ch(x_i, p_i) \cap B_Q^{\mathcal{K}}$ proximate to $x_i$, and also that $x_i \in B_Q^{\mathcal{K}}$.

CASE 2 Assume that $p_1, p_2$ are infinitely near to $u$, but not $p_3$. Assume also that $u \in B_Q^{\mathcal{K}}$. Using the formula (3.g) of Proposition II.3.11, that $u \in B_Q^{\mathcal{K}}$ and $\rho'_w = 0$ if $w \in ch^0(x_3, u)$, we infer that there is some $v_3 \in ch(x_3, u) \cap B_Q^{\mathcal{K}}$ proximate to $x_3$. If $x_3 \neq p_3$, the same argument used in Case 1 shows that there is some $u^3 \in ch(x_3, p_3) \cap B_Q^{\mathcal{K}}$ proximate to $x_3$. Hence, $\sharp\{q \in B_Q^{\mathcal{K}} \mid q \to x_3\} \geq 2$ and $\rho'_{x_3} \geq 1$, against the assumption. If $x_3 = p_3$, Corollary II.3.9 leads to contradiction.

CASE 3 Assume that $p_1, p_2$ are not infinitely near to $u$. We have that $u$ is infinitely near that to $x_1$ and $x_2$ and hence, $x_1, x_2$ and $u$ are in the same branch. We assume that $x_2$ is infinitely near to $x_1$ and, by $d$ of Proposition II.6.2, we infer that $x_2$ is proximate to some $w \in ch(x_1, x_2)$. We already know that if $x_2 \neq p_2$ or $x_2 = p_2$ and $x_1 \neq p_1$, then $x_2 \in B_Q^{\mathcal{K}}$. By using inductively the formula (3.g) of Proposition II.3.11, there is some $w \in ch(x_2, x_3) \cap B_Q^{\mathcal{K}}$ proximate to $x_1$. If $x_1 \neq p_1$, then we have that there are two different points of $B_Q^{\mathcal{K}}$ proximate to $x_1$, one in $ch(x_2, x_3)$ and the other in $ch(x_3, p_3)$. Hence, $\rho'_{x_1} \geq 1$, against the assumption. If $x_1 = p_1$, Corollary II.3.9 leads to contradiction.

It only remains to prove the case when $x_2 = p_2$ is proximate to $x_1 = p_1$. Write $ch(x_1, x_2) = \{u_0 = p_1, \ldots, u_n, u_{n+1} = p_2\}$. The same argument used in case 1.2 in the proof of Proposition II.6.3 shows that the minimal point $O_Q$ in $T_Q$ is in $ch(x_1, x_2)$ and so, $O_Q = u_i$ for some $i \in \{1, \ldots, n\}$. By Proposition II.3.11, $\rho'_{u_i} > 0$ and by the assumption, it follows that $u = u_i = O_Q$ and $\varepsilon_u = 1$. Now, if $p_3$ is infinitely near to $u$, the same argument used in Case 1 shows that $p_3 \in B_Q^{\mathcal{K}}$ and also that there is some $w \in ch(u, p_3) \cap B_Q^{\mathcal{K}}$, proximate to $u$. Then, by the formula (3.g) of Proposition II.3.11, we have that $\rho'_u \geq 2$. If $p_3$ is not infinitely near to $u_1$, $x_1$ and $x_3$ are in the same branch and we can repeat the argument above to show that $x_3 = p_3$ and that $u = O_Q$ and that it is proximate to $p_1$ and $p_3$. If follows that $p_2 = p_3$ against the hypothesis. This completes the proof. □

## II.7 Some consequences relative to adjacent ideals in dimension two

In this section, we derive some consequences of the results seen in the preceding ones and relative to adjacent complete ideals. To this aim, we must recall some definitions due to Zariski, Lipman and Noh concerning ideals in a two-dimensional local ring.

Following Noh [49] (page 164), two complete ideals $I \supset J$ are said to be *adjacent* if their colengths differ by one, i.e. if $\dim_{\mathbb{C}} \frac{I}{J} = 1$. In this case, we also say that $I$ is *right above* $J$ or that $J$ is *right below* $I$. Noh proves that given a complete $\mathfrak{m}_O$-primary ideal $I \subset R$, there exist adjacent complete ideals right above and below $I$ (Lemma 1.1 of [49]). In case $J$ is simple, a complete ideal $I$ right above $J$ is either simple or the product of two simple ideals (see Theorem 3.1 of [31]): keeping the notation of I.2.10 and if $J = I_p$, a right above complete ideal $I$ is simple if and only if $p$ is free.

Given an $\mathfrak{m}_O$-primary ideal $I \subset R$ of order $r$, Zariski defined the *characteristic form* $c(I)$ of $I$ to be the greatest common divisor of the elements in $\frac{I+m^r}{m^{r+1}}$ in $gr_m(R) = \frac{R}{m} \oplus \frac{m}{m^2} \oplus \cdots$ (see [65], page 363). $c(I)$ is thus composed of the principal tangents which are common to the elements of $I$ with multiplicity $r$. If $I$ is complete and $\mathcal{K} = (K, \nu)$ is the cluster of base points of $I$, we have

$$\deg(c(I)) = \sum_{p \in K, p \to O} \nu_p. \tag{7.a}$$

Following Lipman [42], Noh writes $e(I)$ for the self-intersection of $\mathcal{K}$, and in Theorem 2.1, computes the difference $e(J) - e(I)$, where $J \subset I$ are adjacent complete ideals. By Theorem II.2.14, this difference is the multiplicity of the point in the exceptional locus of $X = Bl_I(S)$ corresponding to $J$ by Theorem II.1.7.

Then, in Theorem 2.5 of [49], Noh gives the following upper bound for $e(J) - e(I)$, where $I \supset J$ are two adjacent ideals having the same order:

$$e(J) - e(I) \leq \deg(c(J)) \tag{7.b}$$

**Remark II.7.1.** If the point in the exceptional locus of $X$ associated to $J$ is singular, the above inequality is equivalent to the bound given in Corollary II.5.7.

Finally, if $I = \prod_{i=1}^{m} I_i^{\alpha_i}$ is the Zariski factorization of $I$, Noh writes $T(I)$ for the set of Rees valuations of $I$, i.e. if $I_i = I_{p_i}$ with $p_i$ some point infinitely near or equal to $O$, then $T(I)$ is the set of valuations $v_{p_i}, i \in \{1, \ldots, m\}$ and in virtue of Theorem I.2.9, it is in bijection with the set of dicritical points of $\mathcal{K}$.

Here, we use some of the results already seen to answer some of the questions concerning adjacent ideals appeared in [49]. We fix a simple $\mathfrak{m}_O$-primary ideal $J$, and keeping the notation of I.2.10, we write $p$ for the infinitely near point such that $J = I_p$.

(3) *How many complete ideals of order $o(\mathfrak{m}_O^n J) - 1$ do exist right above $\mathfrak{m}_O^n J$?*

First of all, it is clear that $o(\mathfrak{m}_O^n J) = n + o(J)$. Hence, one looks for complete $\mathfrak{m}_O$-primary ideals $I \subset R$ right above $J$ with

$$o(I) = o(J) + n - 1. \tag{7.c}$$

Since $o(I)$ equals the virtual multiplicity of $BP(I)$ at $O$, the condition (7.c) above is equivalent to ask the virtual multiplicity of $J$ at $O$ to be that of $I$ plus $n - 1$. If we write $\nu_p$ and $\nu'_p$ for the virtual multiplicities of $BP(I)$ and $BP(J)$ at some point $p$, respectively, this means $\nu'_O = \nu_O + 1$ and by Lemma II.3.1, this implies that $O$ must be a non-dicritical point of $\mathcal{K}$. Moreover, by Lemma II.2.1, we know that every complete ideal right below $I$ is of the form $H_{\mathcal{K}_q}$, for some $q \in \overline{\mathbb{F}}_K$.

Take $K$ as the cluster composed of

- all the points preceding or equal to $p$
- $v_O(I_p)$ free and consecutive points $u_1, \ldots, u_{v_O(I_p)}$, the first one in the first neighbourhood of $p$
- $n - 1$ free points $q_1, \ldots, q_{n-1}$ in the first neighbourhood of $O$, neither of them preceding or equal to $p$

and assign virtual multiplicities to $K$ by taking $\nu_{\mathcal{K}} = (\mathbf{P}_K^t)^{-1}\rho_{\mathcal{K}}$ (see Lemma I.1.22) where

$$\rho_q^{\mathcal{K}} = \begin{cases} 1 & \text{if } q \text{ is maximal in } K \text{, i.e. if } q = q_i \text{ for some } i \text{ or } w = u_{v_O(I_p)} \\ 0 & \text{otherwise.} \end{cases}$$

(See Figure II.14). Clearly, the cluster $\mathcal{K}$ verifies the conditions (i) and (ii) of Lemma II.2.12. After adding a point $q$ in the first neighbourhood of $O$ and unloading multiplicities, we obtain a cluster $\mathcal{K}'$ such that

$$\rho'_q = \begin{cases} n & \text{if } q = O \\ 1 & \text{if } q = p \\ 0 & \text{otherwise.} \end{cases}$$

(see Proposition II.3.11). After dropping the points with virtual multiplicity zero, the resulting cluster is $\mathcal{K}(p)$, the cluster of base points of $J = I_p$. Since this is true independently of the points $q_1, \dots, q_{n-1}$ and $u_1, \dots, u_{v_O(I_p)}$, we see there are infinitely many ideals right above $\mathfrak{m}_O^n J$ of order $o(\mathfrak{m}_O^n J) - 1$.

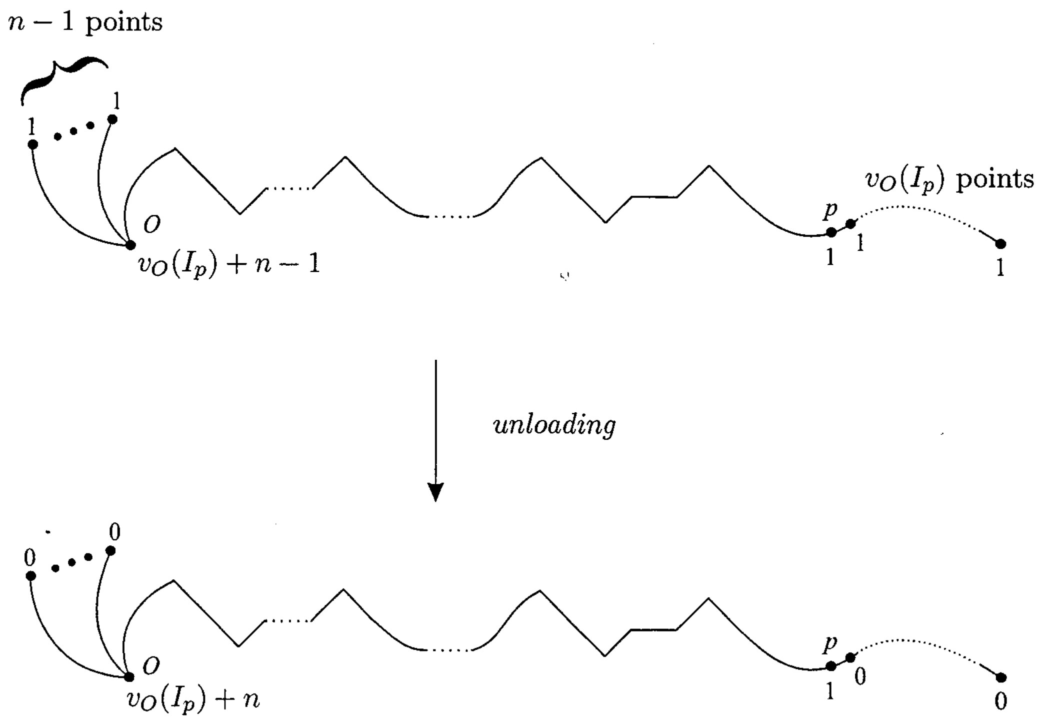



Figure II.14: Enriques diagram of $\mathcal{K}$ and $\mathcal{K}'$

(4) *How many complete ideals of order $o(\mathfrak{m}_O J^n) - 1$ do exist right above $\mathfrak{m}_O J^n$?*

Now, we take the cluster $K$ as the set composed of

- all the points that precede or equal $p$
- $n$ chains $l_1, \ldots, l_n$ of consecutive free points, each $l_i$ composed of $v_O(I_p)$ free points $u_1^i, \ldots, u_{v_O(I_p)}^i$, the first one $u_1^i$ in the first neighbourhood of $p$

and define the virtual multiplicities for $K$ by $\boldsymbol{\nu}_K = (\mathbf{P}_K^t)^{-1}\boldsymbol{\rho}_K$ where

$$\rho_q = \begin{cases} 1 & \text{if } q = u_{v_O(I_p)}^i \text{ for some } i \\ 0 & \text{otherwise.} \end{cases}$$

(see Figure II.15). By Proposition II.3.11, the excesses of the cluster obtained by adding some free point $q$ in the first neighbourhood of $O$ and unloading multiplicities are

$$\rho'_q = \begin{cases} 1 & \text{if } q = O \\ n & \text{if } q = p \\ 0 & \text{otherwise.} \end{cases}$$

Once again we see that after dropping the points with virtual multiplicity zero, the cluster $\mathcal{K}'$ obtained is independent of the points $u_j^i$ chosen and $H_{\mathcal{K}'} = \mathfrak{m}_O J^n$. Therefore, there are infinitely many complete ideals of order $o(\mathfrak{m}_O J^n) - 1$ right above $\mathfrak{m}_O J^n$.

(5) *Let $I \supset J$ be adjacent complete ideals of the same order. For $w \in T(J) \backslash T(I)$, is it true that $w(I) = w(J) - 1$? If this is true, then $e(J) - e(I) < \deg(c(J))$ in the inequality (7.b)?*

For the first question, the answer is no. Take the complete $\mathfrak{m}_O$-ideal $I = (x(y^2-x^5), (x-y)(y^2-x^5), x(x-y)(y^2+x^5), x(x-y)(y^2-x^5-x^6))$. The Enriques diagram of $\mathcal{K} = BP(I)$ is shown in Figure II.16. The dicritical points of $\mathcal{K}$ are $p_1$ and $p_6$. Write $\mathcal{K}'$ for the cluster obtained by adding some free and simple point $q$ in the first neighbourhood of $p_2$ and unloading multiplicities, and write $J = H_{\mathcal{K}'}$ (the Enriques diagram of $\mathcal{K}'$ is shown on the right of Figure II.16). The dicritical points of $\mathcal{K}'$ are $p_2$ and $p_4$. For the divisorial valuation $v_{p_4}$ corresponding to $p_4$, we have

$$v_{p_4}(J) = 14$$

$$v_{p_4}(I) = 12.$$

On the other hand, the orders of $I$ and $J$ are both equal to 3. For the

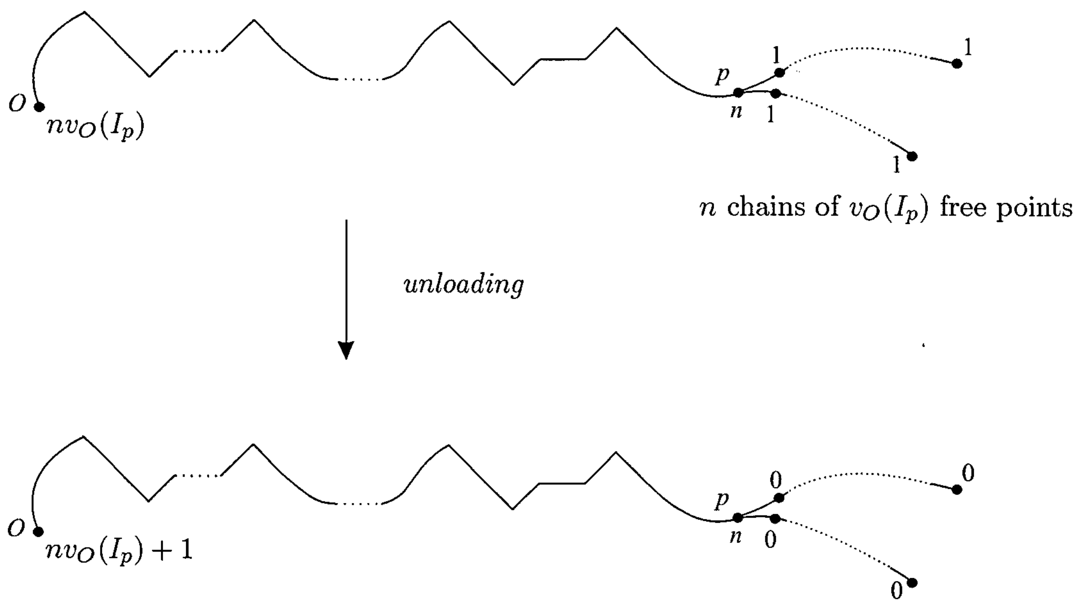



Figure II.15: Enriques diagram of $\mathcal{K}$ and $\mathcal{K}'$

second question, the answer is no again. We give an example of a couple of adjacent ideals $I \subset J$ with the same order, and a $p$-valuation $v_p \in T(J)$ such that $v_p(I) = v_p(J) - 1$ and $e(J) - e(I) = \deg(c(I))$.

Take a complete $\mathfrak{m}_O$-primary ideal $I$ in $R$ such that $\mathcal{K} = BP(I)$ has the Enriques diagram shown in Figure II.17. The dicritical points of $\mathcal{K}$ are $p_1$ and $p_3$, and there is only one singularity in $X = Bl_I(S)$, say $Q$. Take the cluster $\mathcal{K}_Q$ and write $J$ for the ideal $I_Q$, corresponding to $Q$ by Theorem II.1.7. The Enriques diagram of $\mathcal{K}_Q$ is shown on the right of Figure II.17. Clearly, both $I$ and $J$ have the same order, which is equal to two. The cluster $\mathcal{K}_Q$ has only one dicritical point, $p_2$, and we have

$$v_{p_2}(J) = 4$$

$$v_{p_2}(I) = 3.$$

However,

$$e(J) = 4 + 4 = 8$$

$$e(I) = 4 + 1 + 1 = 6$$

and so,

$$e(J) - e(I) = 8 - 6 = 2.$$

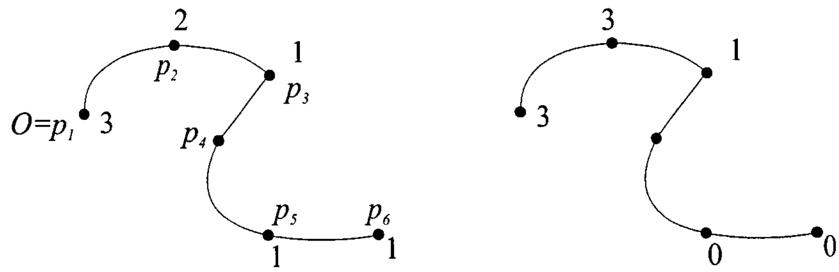



Figure II.16: Enriques diagram of $\mathcal{K}$ and $\mathcal{K}'$

(As noticed above, this is just the multiplicity of the point $Q \in X$). On the other hand, by the equality (7.a) above,

$$\deg(c(J)) = \nu'_{p2} = 2.$$

and so, there is an equality in (7.b) above.

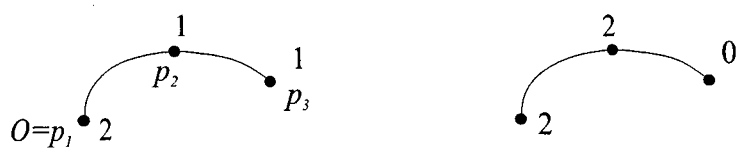



Figure II.17: Enriques diagram of $\mathcal{K}$ and $\mathcal{K}'$

(6) *Is it possible to have adjacent complete ideals* $I \supset J = J_1^{s_1} J_2^{s_2}$*, where* $T(I) \cap T(J) = \emptyset$ *and* $s_i > 1$ *for* $i = 1, 2$*?*

It is enough to take a complete ideal $I = (xy(x+y^2)), (x-y^2)(x+2y^2)(x-y)(x+y), x(x-y^2)(x-y)(x+y), x(x+y^2)(x-y)(x+y)$. The Enriques diagram of $\mathcal{K} = BP(I)$ is shown on the left of Figure II.18 and the dicritical points of $\mathcal{K}$ are $p_3, p_4$ and $p_5$. Write $\mathcal{K}'$ for the cluster obtained by adding a simple and free point in the first neighbourhood of $O$ and not already in $K$ and unloading multiplicities. The Enriques diagram of $\mathcal{K}'$ is shown on the right of Figure II.18. The dicritical points of $\mathcal{K}'$ are $p_1$ and $p_2$ and we have $\rho_{p_1}^{\mathcal{K}'} = \rho_{p_2}^{\mathcal{K}'} = 2$. Therefore, if $J = H_{\mathcal{K}'}$, we have

$$J = I_{p_1}^2 I_{p_2}^2$$

and $T(I) \cap T(J) = \emptyset$.

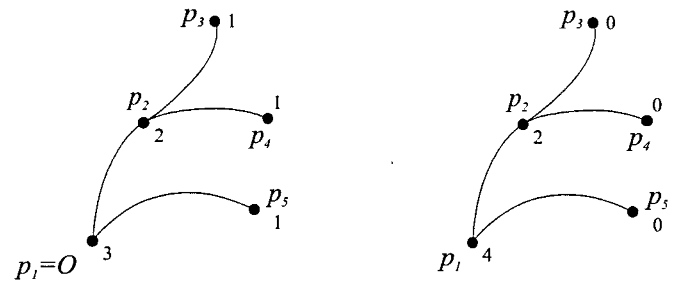



Figure II.18: Enriques diagram of $\mathcal{K}$ and $\mathcal{K}'$

# Chapter III

# Local principality of curves going through a sandwiched singularity

Fixed a complete $\mathfrak{m}_O$-primary ideal $I \subset R = \mathcal{O}_{S,O}$, this chapter deals with the problem of describing the behaviour of curves on $X = Bl_I(S)$ with non-exceptional support. These curves may be understood as strict transforms of curves on $S$ and this will be our point of view: from the cluster of singular points of these curves and their behaviour relative to $\mathcal{K} = BP(I)$, we will obtain information concerning the existence of local equations for their strict transform on $X$. Following the sort of ideas of Chapter II, we will propose also some algorithms to compute Cartier divisors on $X$ with prescribed conditions.

In section III.1 a criterion and a test are given to discern whether the strict transform on $X$ of a curve on $S$ is a Cartier divisor. As a consequence, a formula for the intersection number of a Cartier and a Weil divisor on $X$, both with non-exceptional support, is derived in section III.2. Fixed a finite set of points in the exceptional locus of $X$, the existence of locally irreducible Cartier divisors going through them with prescribed intersection numbers with the exceptional components is also proved. In section III.3 and fixed a Weil divisor on $X$ containing no exceptional components, we give an algorithm to compute Cartier divisors containing it and minimal relative to the divisorial valuations. Finally, in section III.4 and once fixed a curve $C$ on $S$, we study the order of singularity of its strict transform on $X$ and relate it to its order of singularity at $O$. Moreover, for analytically irreducible curves (branches) we connect the study of the semigroup of $\tilde{C}$ at the point where it intersects the exceptional locus of $X$ with a flag of clusters relative to $C$ and $\mathcal{K}$ constructed in section III.3 and an algorithm to compute this semigroup is derived.

## III.1 Cartier divisors and Weil divisors through a sandwiched singularity

Fix a complete $\mathfrak{m}_O$-primary ideal $I \subset R$ and $\mathcal{K} = (K, \nu)$ the cluster of base points of $I$. Write $\mathcal{K}_+ = \{p \in K \mid \rho_p > 0\}$ for the set of dicritical points of $\mathcal{K}$. In this section we give a characterization of the curves $C \subset S$ such that their strict transform $\widetilde{C}$ on $X = Bl_I(S)$ is a Cartier divisor.

The main result is the following:

**Theorem III.1.1.** *Let $C$ be a curve on $S$ and keep the notation introduced in Notation I.4.8. The following conditions are equivalent:*

*(i) $\widetilde{C}$ is a Cartier divisor on $X$*

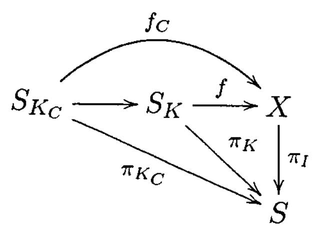
*(ii) $L_C \in \bigoplus_{u \in \mathcal{K}_+} \mathbb{Z}\mathcal{L}_u$*

*(iii) There exists a curve $C_O \subset S$ such that $L_{C_O} = L_C$ and $\widetilde{C_O}$ goes through no singularity of $X$*

*(iv) If $\mathbb{H}_C^o = \{g \in R \mid v_p(g) \geq v_p(C), \forall p \in \mathcal{K}_+\}$ and $q \in K \setminus \mathcal{K}_+$, then $q$ is a non-dicritical base point of $\mathbb{H}_C^o$*

Before proving Theorem III.1.1 we need some technical results and fixing some notation.

**Notation III.1.2.** Given a curve $C$ on $S$, let $\gamma_1, \dots, \gamma_s$ be the branches of $C$ at $O$ and for each $i$ denote by $p_i$ the first non-singular point on $\gamma_i$ and not in $K$. Define $K_C$ as being the cluster containing $K$, the points $p_i, i = 1, \dots, s$ and all the points preceding some of them. We write $\pi_{K_C} : S_{K_C} \longrightarrow S$ for the blowing-up of all the points of $K_C$. Since $K_C$ contains the points of $K$, the sheaf $I\mathcal{O}_{S_{K_C}}$ is invertible: we write $f_C : S_{K_C} \longrightarrow X$ for the morphism induced by the universal property of the blowing-up. Clearly, $\widetilde{C}^{S_{K_C}}$ is non-singular, and $f_C$ factors through $f : S_K \longrightarrow X$ as $K_C$ contains the points of $K$. Thus, we have a commutative diagram:



Note that the exceptional components on $S_{K_C}$ contracting by $f_C$ to some point of $X$ (not necessarily a singularity) are $\{E_p^{S_{K_C}}\}_{p \in K_C \setminus \mathcal{K}_+}$. If $Q \in X$, we define

$$T_Q^{S_{K_C}} = \{p \in K_C \mid (f_C)_*(E_p^{S_{K_C}}) = Q\}.$$

Then, similarly to the situation in section II.2 (see (2.c) in page 53) we have,

$$K_C \setminus \mathcal{K}_+ \quad = \quad \bigcup_{Q \in X} T_Q^{S_{K_C}} \qquad (1.a)$$

Recall from Definition I.3.19 that if $A \subset X$ is an effective Weil divisor going through a singularity $Q \in X$, there is associated to $A$ a $\mathbb{Q}$-Cartier divisor $D_A^Q$ on $S_{K_C}$ defined by the condition

$$|D_A^Q \cdot E_q^{S_{K_C}}|_{S_{K_C}} \quad = \quad -|\widetilde{A}^{S_{K_C}} \cdot E_q^{S_{K_C}}|_{S_{K_C}} \qquad (1.b)$$

for all $q \in T_Q^{S_{K_C}}$. If $A$ has no exceptional component and $Q_1, \ldots, Q_n$ are the points in the intersection of $A$ with the exceptional locus of $X$, we define

$$D_A^{S_{K_C}} = \sum_{i=1}^{n} D_A^{Q_i}.$$

Note that $D_A^{S_{K_C}}$ is not a divisor in general. By Zariski's main theorem ([29] Corollary III.11.4), the $D_A^{Q_i}, i \in \{1, \ldots, n\}$ are connected and so, they are the connected components of $D_A^{S_{K_C}}$. Moreover, in virtue of (1.a) and (1.b), we have

$$|(\widetilde{A}^{S_{K_C}} + D_A^Q) \cdot E_q^{S_{K_C}}|_{S_{K_C}} \quad = \quad 0 \qquad (1.c)$$

**Lemma III.1.3.** *If $A$ is an effective Weil divisor on $X$, $A$ is Cartier if and only if $D_A^{S_{K_C}}$ is a divisor on $S_{K_C}$.*

*Proof.* Clearly, if $A$ is a Cartier divisor on $X$, then $f_C^*(A) = \widetilde{A}^{S_{K_C}} + D_A^{S_{K_C}}$ is a divisor on $S_{K_C}$, defined by any equation of $A$ on $X$. For the converse, assume that $D_A^{S_{K_C}}$ is a divisor on $S_{K_C}$. Since the question is local, it is enough to show that for any singularity $Q \in X$, there exists an equation for $A$ in a neighbourhood of $Q$ (this is clear if $Q$ is non-singular). Clearly, the total transform $\widetilde{A}^{S_{K_C}} + D_A^{S_{K_C}}$ of $A$ on $S_{K_C}$ is a divisor and by (1.c), the claim follows from Proposition I.3.13. □

In particular and as already known (Proposition I.3.17), we see that if $A$ is not a Cartier divisor on $X$, then there exists some integer $m_A > 1$ such that $m_A A$ is: it is enough to take $m_A$ so that $m_A D_A^{S_{K_C}}$ is a divisor on $S_{K_C}$.

**Lemma III.1.4.** *For $u \in \mathcal{K}_+$, let $C_u \subset S$ be a curve going sharply through $\mathcal{K}(u)$ and missing all points after $u$ in $K$. Then $\widetilde{C_u}$ and $\mathcal{L}_u = L_{C_u}$ are Cartier divisors on $X$.*

*Proof.* Note that the strict transform $\widetilde{C_u}$ on $X$ is a Cartier divisor because it goes through no singularity of $X$. Therefore, the germs of $L_{C_u}$ and $C_u^*$ at any singularity of $X$ are equal and since the total transform of any curve $C$ on $X$ is a Cartier divisor, the first claim follows. □

**Lemma III.1.5.** *We have that:*

*(a) the exceptional divisors*

$$E_{I_p}^{S_K} = \sum_{q \in K} v_q(I_p) E_q, \text{ for } p \in K$$

*on $S_K$, are a basis of the $\mathbb{Q}$-vector space generated by $\{E_p\}_{p \in K}$. The matrix of the change of basis from $\{E_{I_p}^{S_K}\}_{p\in K}$ to $\{E_p\}_{p\in K}$ is $-\mathbf{A}_K$.*

*(b) the exceptional divisors $\{\mathcal{L}_u\}_{u \in \mathcal{K}_+}$ on $X$ are a basis of the $\mathbb{Q}$-vector space generated by $\{L_u\}_{u \in \mathcal{K}_+}$.*

*Proof.* First of all and for $p \in K$, we may assume that $\mathcal{K}$ and $\mathcal{K}(p)$ have underlying cluster $K$ (it is enough to add points with virtual multiplicity zero). It is clear from the definition of $\mathcal{K}(p)$ that the excess $\rho_q^{\mathcal{K}(p)}$ is 1 if $q = p$ and 0 otherwise. Thus, by Lemma I.1.22, we have

$$\mathbf{A}_K (v_q(I_p))_{q \in K} = -\mathbf{1}_p \tag{1.d}$$

where $\mathbf{1}_p$ is the $K$-vector having all its entries equal to 0 but the corresponding to $p$ which is 1. From this, we see that the matrix $(v_u(I_p))_{u,p\in K}$ is the inverse of $-\mathbf{A}_K$. Since (1.d) is equivalent to

$$\mathbf{A}_K E_{I_p}^{S_K} = -E_p,$$

the claim (a) follows.

(b) Clearly it is enough to see that the $\{\mathcal{L}_u\}_{u\in\mathcal{K}_+}$ are linearly independent. Assume that there exist rational numbers $\{a_u\}_{u\in\mathcal{K}_+}$ such that

$$\sum_{u \in \mathcal{K}_+} a_u \mathcal{L}_u = 0. \tag{1.e}$$

By multiplying by an integer, we may assume that the $a_u \in \mathbb{Z}$ for all $u \in \mathcal{K}_+$. Now, for each $u \in \mathcal{K}_+$ take $C_u$ a curve going sharply through $\mathcal{K}(u)$ and missing all points after $u$ in $K$, and write

$$\sum_{u \in \mathcal{K}_+} a_u C_u = C_1 - C_2$$

where $C_1 = \sum_{a_u>0} a_u C_u$ and $C_2 = \sum_{a_u<0}(-a_u)C_u$. Then, by (1.e) we have that

$$L_{C_1} = \sum_{a_u>0} a_u \mathcal{L}_u$$

and

$$L_{C_2} = \sum_{a_u<0} (-a_u)\mathcal{L}_u$$

are equal. Hence, by taking total transforms on $S_K$, we see that

$$f^*(L_{C_1}) = \sum_{a_u>0} a_u E_{C_u}^{S_K} = \sum_{a_u<0} (-a_u) E_{C_u}^{S_K} = f^*(L_{C_2}) \tag{1.f}$$

against (a). Therefore, the $\{\mathcal{L}_u\}$ are linearly independent. □

Now, we can prove Theorem III.1.1.

*Proof of Theorem III.1.1.*
We will prove that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$. Assume that $\widetilde{C}$ is Cartier on $X$ and hence, that $L_C$ is so (as the total transform $C^* = \widetilde{C} + L_C$ is always Cartier). Then, by Lemma III.1.3, $f^*(C)$ is a divisor on $S_K$ and so, the coefficients of the components $\{E_p\}_{p\in T_Q}$ in $f^*(L_C)$ are integers, i.e.

$$f^*(L_C) = \sum_{q\in K} b_q E_q \tag{1.g}$$

with $b_q \in \mathbb{Z}$ for all $q \in K$. On the other hand, by (b) of Lemma III.1.5 we can write

$$L_C = \sum_{u\in\mathcal{K}_+} a_u \mathcal{L}_u,$$

with $a_u \in \mathbb{Q}$. Now, for $u \in \mathcal{K}_+$, it is clear that $f^*(\mathcal{L}_u) = E_{C_u}^{S_K}$ and then,

$$f^*(L_C) = \sum_{u\in\mathcal{K}_+} a_u E_{C_u}^{S_K} \tag{1.h}$$

is the expression of $f^*(L_C)$ in the basis $\{E_{I_p}^{S_K}\}_{p\in K}$. Therefore, by (a) of Lemma III.1.5 and the equalities above (1.g) and (1.h),

$$(a_u)_{u\in K} = -\mathbf{A}_K (b_u)_{u\in K}$$

and hence all the $a_u$ are integers. Hence, $(i) \Rightarrow (ii)$.

$(ii) \Rightarrow (iii)$. Assume that

$$L_C = \sum_{u\in\mathcal{K}_+} a_u \mathcal{L}_u, \tag{1.i}$$

with $a_u \in \mathbb{Z}$. If $u \in \mathcal{K}_+$, the projection formula applied to $\pi$ gives that

$$|C^*{\cdot}L_u|_X = |(\widetilde{C} + L_C){\cdot}L_u|_X = 0.$$

Hence, $|\widetilde{C}{\cdot}L_u|_X = -|L_C{\cdot}L_u|_X$. In particular, for $p \in \mathcal{K}_+$, we have

$$|\mathcal{L}_p{\cdot}L_u|_X = -|\widetilde{C_p}{\cdot}L_u|_X = \begin{cases} -1, & \text{if } p = u \\ 0, & \text{otherwise.} \end{cases}$$

It follows from (1.i) that

$$|\widetilde{C}{\cdot}L_u|_X = -|L_C{\cdot}L_u|_X = -\sum_{p\in\mathcal{K}_+} a_p|\mathcal{L}_p{\cdot}L_u|_X = a_u$$

and thus, $a_u \geq 0$ for all $u \in \mathcal{K}_+$. Since the cluster $\mathcal{T} = \sum_{u\in\mathcal{K}_+} a_u\mathcal{K}(u)$ is consistent, Theorem I.1.30 says that there exists a curve $C_O$ going sharply through $\mathcal{T}$ and missing those points of $K$ not contained in the underlying cluster of $\mathcal{T}$. Then, the strict transform $\widetilde{C_O} \subset X$ cuts transversally each exceptional component $L_u$ at $a_u$ different points and goes through no singularity of $X$. Moreover, it is clear that

$$L_{C_O} = \sum_{u\in\mathcal{K}_+} a_u\mathcal{L}_u = L_C.$$

$(iii) \Rightarrow (iv)$. Take $\mathbb{H}^o_C = \{g \in R \mid v_p(g) \geq v_p(C), p \in \mathcal{K}_+\}$ which is a complete $\mathfrak{m}_O$-primary ideal as it is defined by valorative inequalities. Since $L_{C_O} = L_C$,

$$v_p(C_O) = v_p(C) = v_p(\mathbb{H}^o_C) \tag{1.j}$$

for all $p \in \mathcal{K}_+$ and hence, $C_O$ is defined by an element of $\mathbb{H}^o_C$. Therefore, $v_p(C_O) \geq v_p(\mathbb{H}^o_C)$ for all $p \in K_C$, and so

$$E^{S_{K_C}}_{C_O} \geq E^{S_{K_C}}_{\mathbb{H}^o_C} = \sum_{p\in K_C} v_p(\mathbb{H}^o_C)E^{S_{K_C}}_p \tag{1.k}$$

on $S_{K_C}$. On the other hand, since $\widetilde{C_O}$ goes through no singularities of $X$, the total transform of $\widetilde{C_O}$ by $f$ has no exceptional part, and since $\pi_K = \pi\circ f$ we have

$$\pi_K^*(C_O) = f^*(C_O^*) = f^*(\widetilde{C_O} + L_{C_O}) = \widetilde{C_O}^{S_K} + f^*(L_{C_O}).$$

Thus
$$E_{C_O}^{S_K} = f^*(L_{C_O}) = f^*(L_C).$$
Assume that these exists some $q \in K \setminus \mathcal{K}_+$ so that $BP(\mathbb{H}_C^o)$ has positive excess at it. Then $\rho_q = 0$ and by the equality (2.c) in page 53, there is some singularity $Q \in X$ such that $q \in T_Q$. Thus, if $C_1$ goes sharply through $BP(\mathbb{H}_C^o)$, $\widetilde{C_1}$ goes through $Q$ and

$$f^*(\widetilde{C_1}) = \widetilde{C_1}^{S_K} + D_{\widetilde{C_1}}^{S_K}$$

where
$$D_{\widetilde{C_1}}^{S_K} = \sum_{u \in K\setminus\mathcal{K}_+} b_u E_u.$$

Since $C_1$ goes sharply through $BP(\mathbb{H}_C^o)$, in virtue of (1.j) we have that $E_{C_1}^{S_K} = E_{\mathbb{H}_C^o}^{S_K}$ and $v_q(C_1) = v_q(\mathbb{H}_C^o) = v_q(C_O)$ for all $q \in \mathcal{K}_+$. Thus, $L_{C_1} = L_{C_O}$ and so

$$E_{C_1}^{S_K} = D_{C_1}^{S_K} + f^*(L_{C_1}) = D_{C_1}^{S_K} + f^*(L_{C_O}) = D_{C_1} + E_{C_O}^{S_K} > E_{C_O}^{S_K}$$

against (1.k). Therefore, $BP(\mathbb{H}_C^o)$ has excess 0 at every $q \in K \setminus \mathcal{K}_+$.

*(iv)* $\Rightarrow$ *(i)*. Assume that $BP(\mathbb{H}_C^o)$ has excess 0 at each point $q \in K\setminus\mathcal{K}_+$. By Theorem I.1.30, there exists a curve $C_1$ going sharply through $BP(\mathbb{H}_C^o)$ and such that no points in $K \setminus BP(\mathbb{H}_C^o)$ belong to it. Then, its strict transform $\widetilde{C_1}^{S_K}$ intersects no components $E_p$ with $p \in K\setminus\mathcal{K}_+$. Since the direct image of $\widetilde{C_1}^{S_K}$ by $f$ is the strict transform $\widetilde{C_1}$ on $X$, we see that $\widetilde{C_1}$ goes through no singularities of $X$ and hence, it is Cartier. Using that $L_{C_1} = L_C$ and that $C_1^* = \widetilde{C_1} + L_{C_1}$ and $C^* = \widetilde{C} + L_C$ are Cartier divisors, we deduce that $L_C$ and also $\widetilde{C}$, are so. □

During the proof of Theorem III.1.1 we have proved the following fact that we state separately for future reference.

**Corollary III.1.6. (of the proof of III.1.1)** *If $C$ is a curve on $S$ and $L_C = \sum_{u\in\mathcal{K}_+} a_u \mathcal{L}_u$, then*
$$a_u = |\widetilde{C}{\cdot}L_u|_X,$$
*and so, $a_u \geq 0$. Moreover, if $\widetilde{C}$ is Cartier, then*
$$\mathbb{H}_C^o = \prod_{p\in\mathcal{K}_+} I_p^{a_p}$$
*is the Zariski factorization of the ideal $\mathbb{H}_C^o$ defined in (iv) of Theorem III.1.1 and also, $L_C \in \bigoplus_{p\in\mathcal{K}_+} \mathbb{Z}_{\geq 0}\mathcal{L}_p$. In particular, $\mathbb{H}_C^o\mathcal{O}_X = \mathcal{O}_X(-L_C)$.*

**Corollary III.1.7.** *Let $I = \prod_{p\in\mathcal{K}_+} I_p^{\alpha_p}$ be the Zariski factorization of $I$ and let $Q$ be any point in the exceptional locus of $X$. If $C$ goes sharply through $\mathcal{K}_Q$, then $\widetilde{C}$ is Cartier on $X$ and $|\widetilde{C}\cdot L_p|_X = \alpha_p$.*

*Proof.* From Remark II.4.2 we already know that for such a curve $C$, $L_C = L_I$. Then, it is enough to apply Theorem III.1.1 and Corollary III.1.6. □

In the case of primitive singularities (Definition II 3.1 of [58] or Remark II.2.13), Theorem III.1.1 has a very easy formulation.

**Corollary III.1.8.** *Let $I_p \subset R$ be a simple ideal and $X = Bl_{I_p}(S)$. If $C$ is a curve on $S$, $\widetilde{C}$ is a Cartier divisor on $X$ if and only if $v_p(C)$ is a multiple of $\mathcal{K}(p)^2$. Moreover, the minimal integer $m_{\widetilde{C}}$ such that $m_{\widetilde{C}}\widetilde{C}$ is Cartier is given by*

$$m_{\widetilde{C}} = \frac{\mathrm{LCM}(v_p(C), \mathcal{K}(p)^2)}{v_p(C)}.$$

*Proof.* It is clear that $L_C = v_p(C)L_p$ and $\mathcal{L}_p = v_p(I_p)L_p$. Since $\mathcal{K}(p)^2 = v_p(I_p)$, we infer that $L_C \in \mathbb{Z}\mathcal{L}_p$ if and only if $v_p(C) \in (\mathcal{K}(p)^2)$. □

Clearly, the question of whether the curve $\widetilde{C}$ on $X$ is Cartier or not is local as it depends only on the existence of a local equation for $\widetilde{C}$ near the singularities of $X$ (recall that every Weil divisor on a neighbourhood of a regular point is also Cartier, see [29] II.6.11).

Fixed a point $Q$ in the exceptional locus of $X$, we write $\mathcal{K}_+^Q = \{p \in \mathcal{K}_+ \mid Q \in L_p\}$. From Theorem III.1.1, we get the following local criterion.

**Corollary III.1.9.** *Let $Q$ be a point in the exceptional locus of $X$. If $C \subset S$, denote by $C_Q$ the curve on $S$ composed of the branches $\gamma$ of $C$ whose strict transform $\widetilde{\gamma}$ on $X$ goes through $Q$. Then, $\widetilde{C}$ is locally principal in a neighbourhood of $Q$ if and only if*

$$L_{C_Q} \in \bigoplus_{u\in\mathcal{K}_+^Q} \mathbb{Z}\mathcal{L}_u.$$

*Proof.* First of all, from the definition of $C_Q$ the germs of $\widetilde{C}$ and $\widetilde{C_Q}$ at $Q$ are equal and so, $\widetilde{C}$ is locally principal near $Q$ if and only if $\widetilde{C_Q}$ is Cartier. Now, by Theorem III.1.1, we know that this is the case if and only if $L_{C_Q} = \sum_{p\in\mathcal{K}_+} a_p\mathcal{L}_p$, with $a_p \in \mathbb{Z}$ for each $p \in \mathcal{K}_+$, and by Corollary III.1.6, we have that $a_p = |\widetilde{C_Q}\cdot L_p|_X \geq 0$, which is zero if $p \notin \mathcal{K}_+^Q$. The claim follows. □

**Remark III.1.10.** From Theorem III.1.1, we deduce an algorithm based on the unloading procedure that provides a test to verify if the strict transform on $X$ of a given curve $C \subset S$ is Cartier or not. Take the weighted cluster $\mathcal{T}_C = (K, \sigma)$ with virtual values given by

$$v_p = \begin{cases} v_p(C) & \text{if } p \in \mathcal{K}_+ \\ 0 & \text{otherwise.} \end{cases}$$

By definition we have

$$H_{\mathcal{T}_C} = \mathbb{H}^o_C,$$

so in order to know the dicritical points of $BP(\mathbb{H}^o_C)$, it is enough to unload values in $\mathcal{T}_C$ if it is not consistent (see subsection I.2.1). By (iv) of Theorem III.1.1, $\widetilde{C}$ is Cartier if and only if the unloaded cluster $\widetilde{\mathcal{T}_C}$ has excess 0 at every point in $K \setminus \mathcal{K}_+$. Note that in case the ideal $I$ verifies the conditions (i) and (ii) of Lemma II.2.12, then $\mathcal{T}_C$ has an easy description in terms of its virtual multiplicities: for each $p \in K$,

$$\sigma_p = \begin{cases} v_p(C) & \text{if } p \in \mathcal{K}_+ \\ 0 & \text{otherwise.} \end{cases}$$

Before going on, we introduce some notation to be used in the following sections.

**Notation III.1.11.** Given a curve $C$ on $S$, we write $\mathcal{K}_C = (K_C, \tau^C)$ for the cluster defined by taking $\tau^C_p = e_p(C)$ for each $p \in K_C$. Note that $\mathcal{K}_C$ is consistent, but not strictly consistent in general.

Next we illustrate the results of this section with a couple of examples.

**Example III.1.12.** Take a complete $\mathfrak{m}_O$-primary ideal $I \subset R$ with base points as shown in the Enriques diagram on the left of Figure III.1. Then, there is only one singularity in $X = Bl_I(S)$, say $Q$, and there are three exceptional components on $X$, $L_{p_4}, L_{p_9}$ and $L_{p_{12}}$ corresponding to the dicritical points of $\mathcal{K} = BP(I)$. Take a curve $C$ with singular points as in Figure III.2. Then, by computing the values of $C$ relative to $p_4$, $p_9$ and $p_{12}$, we obtain that (see Notation I.4.8)

$$L_C = 25L_{p_4} + 90L_{p_9} + 111L_{p_{12}} \tag{1.1}$$

In this example, the ideal $I$ verifies the conditions (i) and (ii) of Lemma

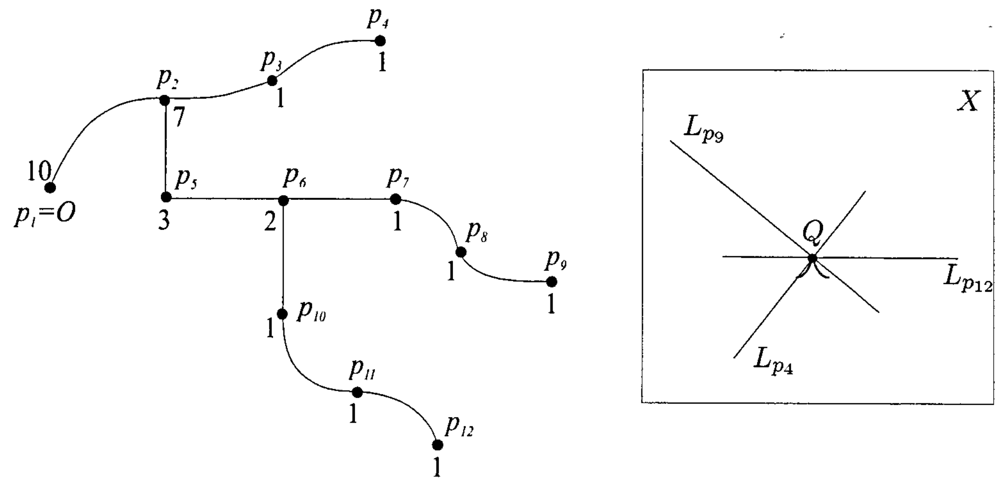



Figure III.1: On the left, the Enriques diagram of the cluster $\mathcal{K} = BP(I)$ of Example III.1.12. On the right, we represent the exceptional components $L_{p_4}$, $L_{p_9}$ and $L_{p_{12}}$ of $X = Bl_I(S)$; these three components intersect at the only singularity $Q$ of $X$.

II.2.12 and so, the virtual multiplicities of the cluster $\mathcal{T}_C$ defined in Remark III.1.10 are

$$\sigma_p = \begin{cases} 25 & \text{if } p = p_4 \\ 90 & \text{if } p = p_9 \\ 111 & \text{if } p = p_{12} \\ 0 & \text{otherwise.} \end{cases}$$

Figure III.3 shows the Enriques diagram of the cluster $\widetilde{\mathcal{T}_C}$ obtained by applying the algorithm described in Remark III.1.10. Since it has positive excess at the points $p_1, p_2$ and $p_6$, Theorem III.1.1 says that the strict transform $\widetilde{C}$ is not a Cartier divisor on $X$.

Note that

$$\begin{aligned} \mathcal{L}_{p_4} &= 4L_{p_4} + 7L_{p_9} + 8L_{p_{12}} \\ \mathcal{L}_{p_9} &= 7L_{p_4} + 30L_{p_9} + 32L_{p_{12}} \\ \mathcal{L}_{p_{12}} &= 8L_{p_4} + 32L_{p_9} + 42L_{p_{12}}. \end{aligned}$$

Thus, the matrix of the change of basis from $\{\mathcal{L}_{p_i}\}_{i=4,9,12}$ to $\{L_{p_i}\}_{i=4,9,12}$ is

$$A_L = \begin{pmatrix} 4 & 7 & 8 \\ 7 & 30 & 32 \\ 8 & 32 & 42 \end{pmatrix}$$

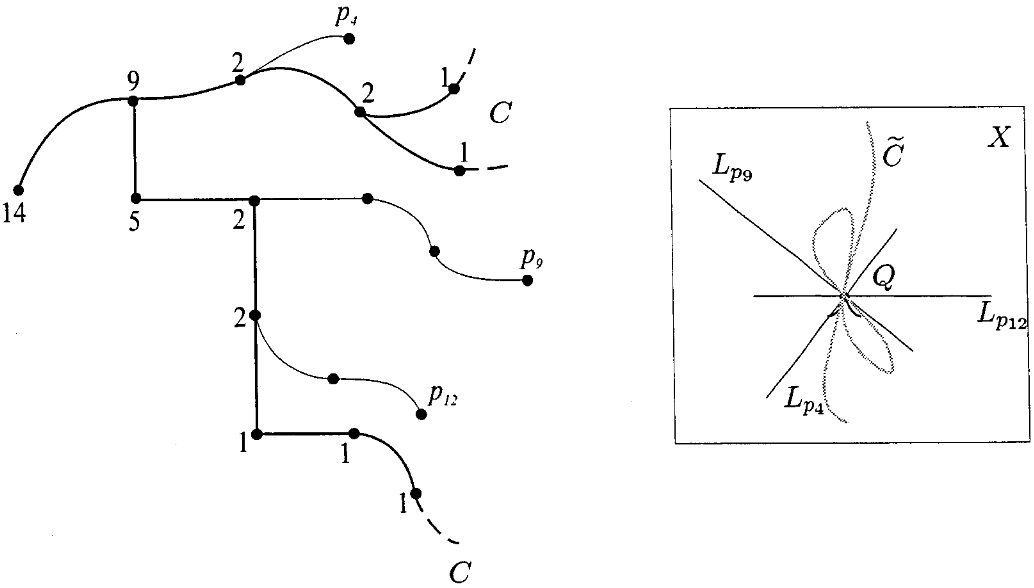



Figure III.2: On the left, the Enriques diagram of the singular points of $C \subset S$ in Example III.1.12 is represented with bold lines; on the right, the strict transform $\widetilde{C}$ goes through the only singularity $Q$ of $X$.

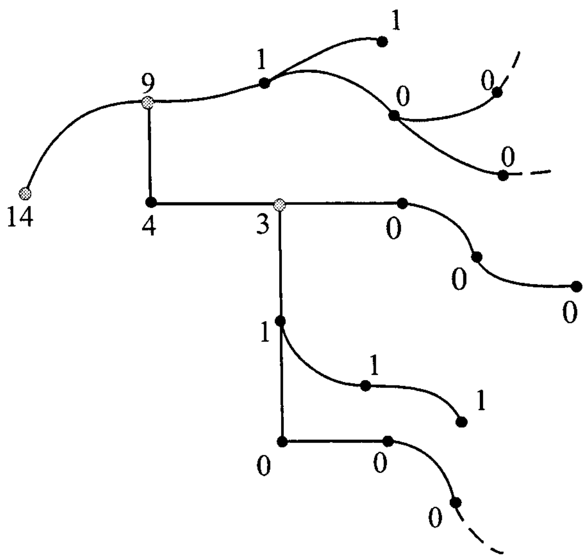



Figure III.3: Enriques diagram obtained by means of the algorithm described in Remark III.1.10. The grey-filled circles represent the dicritical points of the unloaded cluster.

Then, by the equality (1.1) above,

$$L_C = \frac{32}{25}\mathcal{L}_{p_4} + \frac{19}{25}\mathcal{L}_{p_9} + \frac{91}{50}\mathcal{L}_{p_{12}}$$

and by Corollary III.1.6, we infer that

$$\begin{aligned} |\widetilde{C}\cdot L_{p_4}|_X &= \frac{32}{25} \\ |\widetilde{C}\cdot L_{p_9}|_X &= \frac{19}{25} \\ |\widetilde{C}\cdot L_{p_{12}}|_X &= \frac{91}{50}. \end{aligned}$$

**Example III.1.13.** Take a complete $\mathfrak{m}_O$-primary ideal $I$ with base points as on the top of Figure III.4. The dicritical points of $\mathcal{K} = BP(I)$ are $p_2, p_4$ and $p_8$ and so, the surface $X = Bl_I(S)$ has thee exceptional components $L_{p_2}, L_{p_4}$ and $L_{p_8}$. Moreover, $X$ has two singularities: $Q_1$ in the intersection of $L_{p_2}$ and $L_{p_4}$ with $T_{Q_1} = \{p_3\}$ and $Q_2$ in the intersection of $L_{p_2}$ and $L_{p_8}$ with $T_{Q_2} = \{p_1, p_5, p_6, p_7\}$. We have

$$\begin{aligned} \mathcal{L}_{p_2} &= 2L_{p_2} + 2L_{p_4} + 2L_{p_8} \\ \mathcal{L}_{p_4} &= 2L_{p_2} + 4L_{p_4} + 2L_{p_8} \\ \mathcal{L}_{p_8} &= 2L_{p_4} + 2L_{p_4} + 9L_{p_8}. \end{aligned}$$

Now, if $C$ is a curve on $S$ with singular points as represented on the bottom of Figure III.4, then,

$$\begin{aligned} L_C &= 6L_{p_2} + 8L_{p_4} + 11L_{p_8} = \\ &= \frac{7}{6}\mathcal{L}_{p_2} + \mathcal{L}_{p_4} + \frac{5}{6}\mathcal{L}_{p_8}. \end{aligned}$$

By Theorem III.1.1, $\widetilde{C}$ is not a Cartier divisor on $X$. However, if $C_{Q_1}$ is the curve composed of the branches of $C$ whose strict transform on $X$ go through $Q_1$, then

$$\begin{aligned} L_{C_{Q_1}} &= 4L_{p_2} + 6L_{p_4} + 4L_{p_8} = \\ &= \mathcal{L}_{p_2} + \mathcal{L}_{p_4}, \end{aligned}$$

and by Corollary III.1.9, $\widetilde{C}$ is principal in a neighbourhood of $Q_1$.

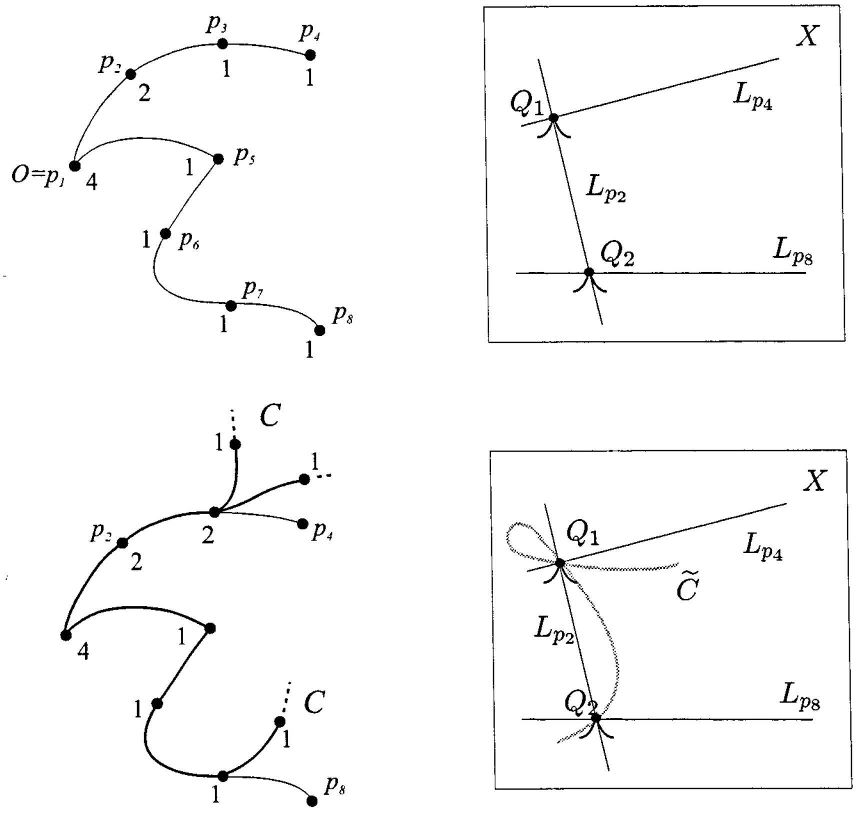



Figure III.4: On the top: on the left, the Enriques diagram of $BP(I)$ and on the right, we represent the exceptional components $L_{p_2}, L_{p_4}$ and $L_{p_8}$; on the bottom and left, the singular points of the curve $C$ and on the right, the we represent the strict transform $\widetilde{C}$ on $X$ in Example III.1.13.

## III.2 On Cartier divisors going through a sandwiched singularity

Throughout this section we fix a curve $C$ on $S$ such that the strict transform $\widetilde{C}$ is a Cartier divisor on $X$, and write $\mathcal{K}_C^o = (K, \tau^{C,o})$ for the cluster of base points of the complete ideal $\mathbb{H}_C^o = \{g \in R \mid v_p(g) \geq v_p(C), \forall p \in \mathcal{K}_+\}$. With this assumption, we know by Corollary III.1.6 that the ideal sheaf $\mathbb{H}_C^o \mathcal{O}_X$ is invertible and so, the base points of $\mathbb{H}_C^o$ are contained in $K$. Moreover, it is clear from the definition of $\mathbb{H}_C^o$ that $v_p(C) \geq v_p(\mathbb{H}_C^o)$ for all $p \in K$, and so $\mathcal{K}_C^o \prec \mathcal{K}_C$. By Theorem III.1.1, we have that every $q \in K \setminus \mathcal{K}_+$ is a non-dicritical base point of $\mathbb{H}_C^o$. Moreover, since $C$ goes sharply through $\mathcal{K}_C$, we have

$$E_C^{S_{K_C}} = E_{\mathcal{K}_C}^{S_{K_C}}$$

and also

$$L_C = L_{\mathcal{K}_C^o} \tag{2.a}$$

The following proposition allows to compute the exceptional part $D_{\widetilde{C}}^{S_{K_C}}$ of the total transform of $\widetilde{C}$ on $S_{K_C}$ as the difference of the exceptional parts of the total transforms of $C$ on $S_K$ and that of a curve defined by a generic element of $H_{\mathcal{K}_C^o}$. Precisely,

**Proposition III.2.1.** *We have*

$$D_{\widetilde{C}}^{S_{K_C}} = E_C^{S_{K_C}} - E_{\mathcal{K}_C^o}^{S_{K_C}}.$$

*Proof.* By definition $\pi_I^*(C) = \widetilde{C} + L_C$ and $\pi_{K_C}^*(C) = \widetilde{C}^{S_{K_C}} + E_C^{S_{K_C}}$. Since $\pi_{K_C} = f_C \circ \pi_I$ and $f_C^*(L_C)$ has exceptional support for $\pi_{K_C}$, we infer that

$$E_C^{S_{K_C}} = D_{\widetilde{C}}^{S_{K_C}} + f_C^*(L_C). \tag{2.b}$$

Now, let $C_O$ be a curve going sharply through $\mathcal{K}_C^o$ and missing all points in $K_C \setminus K$. Then, $\widetilde{C_O}$ goes through no singularities of $X$ and shares no points with $\widetilde{C}$ in $X$. Thus, by (2.a),

$$\pi^*(C_O) = \widetilde{C_O} + L_C$$

and again from $\pi_{K_C} = f_C \circ \pi_I$,

$$\pi_{K_C}^*(C_O) = f_C^*(\widetilde{C_O} + L_C) = \widetilde{C_O}^{S_{K_C}} + f_C^*(L_C)$$

and we deduce that

$$E_{\mathcal{K}_C^o}^{S_{K_C}} = f_C^*(L_C).$$

Now, from (2.b) above, we infer that

$$E_C^{S_{K_C}} = D_C^{S_{K_C}} + E_{\mathcal{K}_C^o}^{S_{K_C}}$$

as claimed. □

The following result is technical and we state it here for future reference.

**Lemma III.2.2.** *If $A \subset S$ is a curve such that $\widetilde{A}$ shares no points on $X$ with $\widetilde{C}$, then*

$$[C, A]_O = [\mathcal{K}_C^o, A]_O.$$

*In particular,*

$$\sum_p e_p(C)e_p(A) = \sum_p \tau_p^{C,o} e_p(A).$$

*Proof.* First of all, by the projection formula applied to $\pi$ (Lemma I.4.9), we have

$$[C, A]_O = |(\widetilde{C} + L_C)\cdot\widetilde{A}|_X = |L_C\cdot\widetilde{A}|_X, \tag{2.c}$$

the last equality since $\widetilde{A}$ and $\widetilde{C}$ share no points on $X$. Now, in virtue of Theorem I.1.30, we take a curve $C_O \subset S$ going sharply through $\mathcal{K}_C^o$ and sharing no points with $A$ outside $K$. Then, $L_{C_O} = L_{\mathcal{K}_C^o} = L_C$ and $\widetilde{A}$ and $\widetilde{C_O}$ share no points on $X$. Hence, and using again the projection formula for $\pi$, we get

$$\begin{aligned}[\mathcal{K}_C^o, A]_O &= [C_O, A]_O = |(\widetilde{C_O} + L_{C_O})\cdot\widetilde{A}|_X = \\ &= |L_{C_O}\cdot\widetilde{A}|_X = |L_C\cdot\widetilde{A}|_X.\end{aligned}$$

Now, the first claim follows from (2.c) and the second follows directly from the first one by applying the Noether's formula (see Theorem I.1.31). □

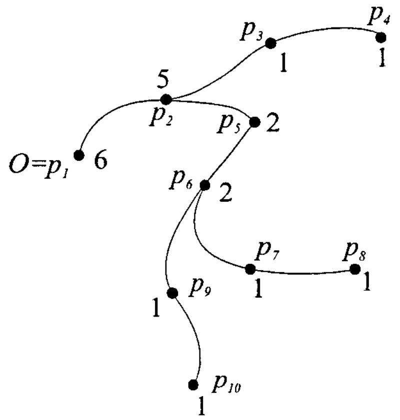
From Proposition III.2.1 we get the following formula for the intersection number of the strict transforms on $X$ of a couple of curves on $S$.

**Corollary III.2.3.** *Let $C$ and $C_1$ be curves on $S$ and assume that $\widetilde{C}$ is Cartier. Then,*

$$|\widetilde{C}\cdot\widetilde{C_1}|_X = [C, C_1]_O - [\mathcal{K}_C^o, C_1]_O.$$

*Proof.* By the projection formula applied to $f_C : S_{K_C} \longrightarrow X$ and Proposition III.2.1, we have

$$\begin{aligned}|\widetilde{C}\cdot\widetilde{C_1}|_X &= |(\widetilde{C}^{S_{K_C}} + D_{\widetilde{C}}^{S_{K_C}})\cdot\widetilde{C_1}^{S_{K_C}}|_{S_{K_C}} = \\ &= |(\widetilde{C}^{S_{K_C}} + E_C^{S_{K_C}} - E_{\mathcal{K}_C^o}^{S_{K_C}})\cdot\widetilde{C_1}^{S_{K_C}}|_{S_{K_C}} = \\ &= |(\widetilde{C} + E_C^{S_{K_C}})\cdot\widetilde{C_1}^{S_{K_C}}|_{S_{K_C}} - |E_{\mathcal{K}_C^o}^{S_{K_C}}\cdot\widetilde{C_1}^{S_{K_C}}|_{S_{K_C}}\end{aligned} \tag{2.d}$$



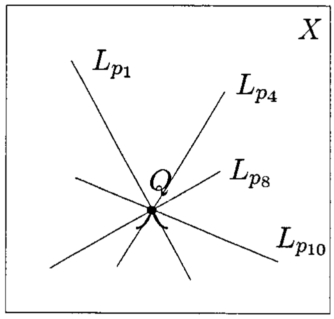




Figure III.5: Enriques diagram of $\mathcal{K} = BP(I)$ in Example III.2.4. On the right, we represent the exceptional components $L_{p_1}, L_{p_4}, L_{p_8}$ and $L_{p_{10}}$ of $X$; these four components intersect at the only singularity $Q$ of $X$.

Now, let $C_O$ be a curve going sharply through $\mathcal{K}_C^o$ and such that $\widetilde{C_O}^{S_{K_C}}$ shares no point with $\widetilde{C_1}^{S_{K_C}}$ on $S_{K_C}$. Then, by the projection formula applied to $\pi_{K_C}$, we have

$$\begin{aligned}[\mathcal{K}_C^o, C_1]_O &= [C_O, C_1]_O = |(\widetilde{C_O}^{S_{K_C}} + E_{\mathcal{K}_C^o})\cdot\widetilde{C_1}^{S_{K_C}}|_{S_{K_C}} = \\ &= |E_{\mathcal{K}_C^o}\cdot\widetilde{C_1}^{S_{K_C}}|_{S_{K_C}}\end{aligned}$$

and also,

$$[C, C_1]_O = |(\widetilde{C}^{S_{K_C}} + E_C^{S_{K_C}})\cdot\widetilde{C_1}^{S_{K_C}}|_{S_{K_C}}$$

The claim is derived immediately from (2.d). □

We discuss an example to illustrate the results of this section.

**Example III.2.4.** Let $I \subset R$ be a complete $\mathfrak{m}_O$-primary ideal with base points as on the left of Figure III.5. Then, the dicritical points of $\mathcal{K} = BP(I)$ are $p_1, p_4, p_8$ and $p_{10}$ and so, the surface $X = Bl_I(S)$ has exceptional components $L_{p_1}, L_{p_4}, L_{p_8}$ and $L_{p_{10}}$. Moreover, there is only one singularity on $X$, say $Q$, and we have,

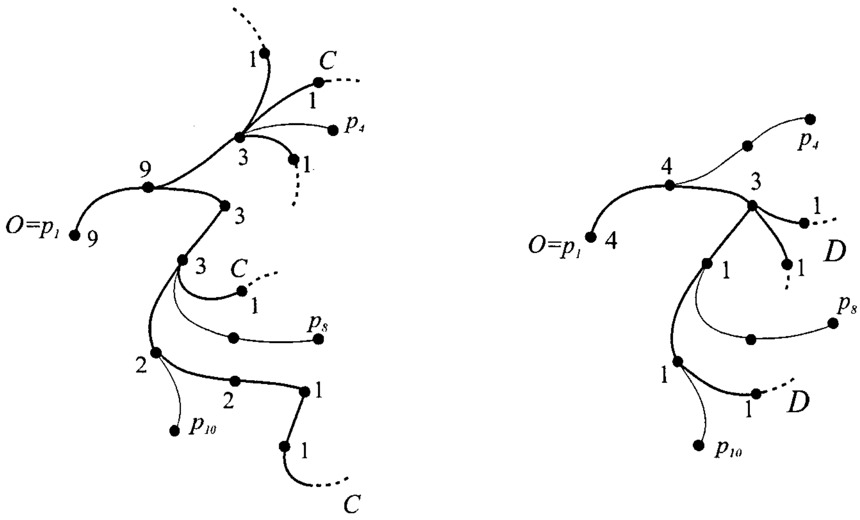



Figure III.6: The singular points of the curves $C$ (left) and $D$ (right) in Example III.2.4.

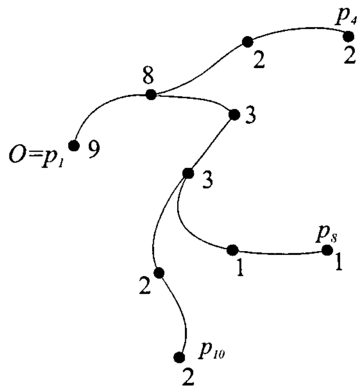



Figure III.7: Enriques Diagram of the cluster $\mathcal{K}_C^o$ in Example III.2.4

$$\begin{aligned}\mathcal{L}_{p_1} &= L_{p_1} + L_{p_4} + 2L_{p_8} + 2L_{p_{10}}\\ \mathcal{L}_{p_4} &= L_{p_1} + 4L_{p_4} + 4L_{p_8} + 4L_{p_{10}}\\ \mathcal{L}_{p_8} &= 2L_{p_1} + 4L_{p_4} + 12L_{p_8} + 10L_{p_{10}}\\ \mathcal{L}_{p_{10}} &= 2L_{p_1} + 4L_{p_4} + 10L_{p_8} + 12L_{p_{10}}.\end{aligned}$$

Now, take the curves $C$ and $D$ having singularities as in Figure III.6. Then, an easy computation shows that

$$\begin{aligned}L_C &= 9L_{p_1} + 21L_{p_4} + 42L_{p_8} + 44L_{p_{10}} =\\ &= \mathcal{L}_{p_1} + 2\mathcal{L}_{p_4} + \mathcal{L}_{p_8} + 2\mathcal{L}_{p_{10}}\end{aligned}$$

and so, by Theorem III.1.1, $\widetilde{C}$ is a Cartier divisor on $X$. The cluster $\mathcal{K}_C^o$ is represented in Figure III.7. We have that $[C, D]_O = 88$ and $[\mathcal{K}_C^o, D]_O = 82$. Thus, Corollary III.2.3 says that

$$|\widetilde{C}{\cdot}\widetilde{D}|_X = 88 - 82 = 6.$$

Moreover, in virtue of Corollary III.1.6, we have

$$\begin{array}{ll}|\widetilde{C}{\cdot}L_{p_1}|_X = 1 & |\widetilde{C}{\cdot}L_{p_4}|_X = 2\\ |\widetilde{C}{\cdot}L_{p_8}|_X = 1 & |\widetilde{C}{\cdot}L_{p_{12}}|_X = 2\end{array}$$

## III.3 On Cartier divisors and exceptional components

In this section, we show that once finitely many points in the exceptional locus of $X$ have been fixed, there exist Cartier divisors $C$ on $X$ intersecting the exceptional locus of $X$ exactly at these points, with prefixed intersection multiplicities with the exceptional components. Precisely, given points $\{Q_1, \ldots, Q_n\}$ in the exceptional locus of $X$, and for each $Q_i$, positive integers $\alpha_p^i$ for $p \in \mathcal{K}_+^{Q_i}$, there exist curves $C \subset S$ such that $\widetilde{C}$ is Cartier and $[\widetilde{C}, L_p]_{Q_i} = \alpha_p^i$ for each $p \in \mathcal{K}_+^{Q_i}$. From this, we infer that once a sandwiched singularity $Q \in X$ has been fixed, the exceptional components $\{L_p\}_{p \in \mathcal{K}_+^Q}$ going through $Q$ span a linear space of dimension equal to $\sharp\mathcal{K}_+^Q$.

The key result is the following theorem.

**Theorem III.3.1.** *Let $Q$ be a singular point on $X$ and for each $p \in \mathcal{K}_+^Q$, let $\alpha_p \in \mathbb{Z}_{>0}$. Then there exists a consistent cluster $\mathcal{T}_Q^\alpha$ such that the strict* *transform $\widetilde{C}$ on $X$ of any curve $C$ going sharply through it is a Cartier divisor that intersects the exceptional locus of $X$ only at $Q$ and*

$$[\widetilde{C}, L_p]_Q = \alpha_p$$

*for all $p \in \mathcal{K}_+^Q$. Moreover, if $O_Q$ is free or $O_Q$ is equal to $O \in S$, $\widetilde{C}$ is irreducible as a Cartier divisor on $X$.*

*Proof.* Write $\mathcal{K}_+^Q = \{p_1, \dots, p_m\}$, and $\mathcal{T}^{(\alpha)}$ for the cluster obtained from the cluster of base points of $I^{(\alpha)} = \prod_{i=1}^m I_{p_i}^{\alpha_{p_i}}$ by adding the points of $K$ and not in it (if any) with virtual multiplicities zero.

For each $i \in \{1, \dots, m\}$ write $q_i$ for the only point of $T_Q$ such that the components $E_{p_i}$ and $E_{q_i}$ intersect on $S_K$ (the uniqueness of this point follows from the fact that $E_K$ has no cycles, see Corollary I.3.8: if $q, q' \in T_Q$ are two such points, then $p_i \in ch(q, q')$ and by Remark II.2.7, $p_i \in T_Q$ against the hypothesis). Put $\mathcal{T}_0 = \mathcal{T}^{(\alpha)}$ and as far as $\rho_{p_i}^{\mathcal{T}_{j-1}} > 0$ for some $i \in \{1, \dots, m\}$, write $w_i^{j-1} \notin T_{j-1}$ for the point proximate to $p_i$, infinitely near to $q_i$ and minimal with this property, and take $\mathcal{T}_j = (T_j, \tau^j)$ to be the strictly consistent cluster obtained from $(\mathcal{T}_{j-1})_{w_i^{j-1}}$ by unloading (if necessary) and dropping the points with virtual multiplicity zero. We claim that there exists some $n$ such that for each $i \in \{1, \dots, m\}$, $\rho_{p_i}^{\mathcal{T}_n} = 0$, and also $v_{p_i}^{\mathcal{T}_n} = v_{p_i}^{\mathcal{T}^{(\alpha)}}$. To show this, we need a technical lemma that we state separately for clarity.

First, we fix some notation. Take $j \geq 1$ and assume that $\mathcal{T}_j$ is obtained by unloading (if necessary) from $(\mathcal{T}_{j-1})_{w_i^{j-1}}$ with $w_i^{j-1}$ proximate to $p_i$. Write $T_{j-1}^0 = \{p \in T_j \mid v_p^{\mathcal{T}_j} > v_p^{\mathcal{T}_{j-1}}\}$, which in virtue of Remark I.2.14 is the set of points where some unloading has been performed to obtain $\mathcal{T}_j$ from $(\mathcal{T}_{j-1})_{w_i^{j-1}}$. Note that $\mathcal{T}_{j-1}$ has excess 0 at the points of $T_{j-1}^0$. Write $T_{j-1}^+ = \{p \in (\mathcal{T}_{j-1})_+ \mid ch^0(p_i, p) \subset T_{j-1}^0\}$ and note that $p_i \in T_{j-1}^+$, so we put $T_{j-1}^+ = \{p_1^{j-1} = p_i, \dots, p_{m_{j-1}}^{j-1}\}$.

**Lemma III.3.2.** *If $j \geq 1$ and $p_m \neq p_n$, there exists some $u \in ch^0(p_m, p_m)$ such that $\rho_u^{\mathcal{T}_j} > 0$.*

*Proof.* We use induction on $j$. The case $j = 1$ is Proposition II.6.3, so there is nothing to prove. Now, assume that we reach a cluster $\mathcal{T}_{j-1}$ with the wanted property, and that $\rho_{p_i}^{\mathcal{T}_{j-1}} > 0$ so that $\mathcal{T}_j$ is obtained by unloading (if necessary) after adding a point $w_i^{j-1}$ proximate to $p_i$ as described above.

Now, by Proposition II.6.3, for each $t \in \{2, \dots, m_{j-1}\}$, there exists some $u_t^j \in ch^0(p_i, p_t^{j-1})$ such that $\mathcal{T}_j$ has positive excess at $u_t^j$. From this, it follows that if $p_k \neq p_i$, then there is some $u \in ch^0(p_k, p_i)$ such that $\rho_u^{\mathcal{T}_j} > 0$. If $p_m, p_n$ are different from $p_i$, by the induction hypothesis, there exists some $u \in ch^0(p_m, p_n)$ so that $\rho_u^{\mathcal{T}_{j-1}} > 0$. Hence, $u \notin T_{j-1}^0$, and by Proposition II.3.11, $\rho_u^{\mathcal{T}_j} = \rho_u^{\mathcal{T}_{j-1}} - 1$ if $u \in T_j^+$, and remains unchanged, otherwise. In the second case, there is nothing to prove, so we assume that $u \in T_j^+$. Note that if $w \in ch(p_m, p_n)$, $w \neq u$, then $w \notin T_j^+$ as the exceptional divisor $E_Q$ has no cycles. Therefore, we can assume that $u$ is the only point of $ch^0(p_m, p_n)$ at which $\mathcal{T}_j$ has positive excess. Thus, we have $\rho_w^{\mathcal{T}_{j-1}} = 0$ for any $w \in ch^0(p_m, u) \cup ch^0(p_n, u)$ and any $w \in ch^0(p_i, u)$ because $ch^0(p_i, u) \subset T_j^0$. From Lemma II.6.6 we infer that $\rho_u^{\mathcal{T}_{j-1}} \geq 2$. Hence, $\rho_u^{\mathcal{T}_j} \geq 1$. This completes the proof of the lemma. □

From the previous lemma we infer that if $j \geq 1$, neither of the $p_1, \ldots, p_m$ belongs to $T_j^0$. Moreover, we have that if $\mathcal{T}_j$ is equivalent to $(\mathcal{T}_{j-1})_{w_i^j}$, then $\rho_{p_i}^{\mathcal{T}_j} = \rho_{p_i}^{\mathcal{T}_{j-1}} - 1 \geq 0$ and if $k \neq i$, then $\rho_{p_k}^{\mathcal{T}_j} = \rho_{p_k}^{\mathcal{T}_{j-1}}$. Therefore, after finitely many steps we reach a cluster $\mathcal{T}_n$ such that $\rho_{p_i}^{\mathcal{T}_n} = 0$ for each $i$. Hence, we obtain a sequence of clusters

$$\mathcal{T}_0 = \mathcal{T}^{(\alpha)} \prec \mathcal{T}_1 \prec \ldots \prec \mathcal{T}_n.$$

Furthermore, since no unloading steps are performed on any $p \in \mathcal{K}_+$, we have that $v_p^{\mathcal{T}_n} = v_p^{\mathcal{T}^{(\alpha)}}$, and in particular, $v_{p_i}^{\mathcal{T}_n} = v_{p_i}^{\mathcal{T}}$ for $i \in \{1, \ldots, m\}$. Define $\mathcal{T}_Q^\alpha$ as the cluster obtained from $\mathcal{T}_n$ by adding those points in $K$ and not in $T_n$ (if any) with virtual multiplicities equal to zero. If $C$ goes sharply through $\mathcal{T}_Q^\alpha$, then $\widetilde{C}^{S_K}$ intersects no component $E_p$ with $\rho_p^{\mathcal{T}_Q^\alpha} = 0$, and hence the strict transform $\widetilde{C}$ of $C$ on $X$ does not intersect any exceptional component of $X$ at some point besides $Q$. Moreover, we have that $L_C = L_{\mathcal{T}_n} = \sum_{p \in \mathcal{K}_+^Q} \alpha_p \mathcal{L}_p$ and by Corollary III.1.6, we infer that if $p \in \mathcal{K}_+^Q$,

$$[\widetilde{C}, L_p]_Q = |\widetilde{C}, L_p|_X = \alpha_p.$$

Notice by the way that $\mathbb{H}_C^o = \{g \in R \mid v_p(g) \geq \alpha_p, \forall p \in \mathcal{K}_+\}$ and so, $\mathcal{K}_C^o = \mathcal{T}^{(\alpha)}$ (see page 114). From this and in virtue of Proposition III.2.1, we have that

$$D_{\widetilde{C}}^{S_{K_C}} = E_C^{S_{K_C}} - E_{\mathcal{T}^{(\alpha)}}^{S_{K_C}}. \tag{3.a}$$

Now, we prove the second claim. By Proposition II.3.11, we have that $\rho_{p_1}^{\mathcal{T}_1} = \alpha_{p_1} - 1$ and also $\rho_{O_Q}^{\mathcal{T}_1} > 0$. If $\alpha_p = 1$ or $O_Q$ is a proper point of $S$, there is nothing to prove, so we assume that $\alpha_p \geq 2$ and $O_Q$ is free.

Write $p_1 \in \mathcal{K}_+^Q$ for the point $O_Q$ is proximate to. Then, at some step of the above procedure, say $j$, we add to $\mathcal{T}_j$ the point $w_1^j$. Since $\rho_{O_Q}^{\mathcal{T}_1} > 0$, the resulting cluster $(\mathcal{T}_j)_{w_1^j}$ is consistent. If fact, as long as the excess at $p_1$ is positive, the clusters obtained by adding points proximate to $p_1$, infinitely near to $O_Q$ and minimal with this property will be consistent and hence, no unloading will be necessary. From Remark I.2.14 it follows that $v_{O_Q}^{\mathcal{T}_j} = v_{O_Q}^{\mathcal{T}_1}$ for every $j \geq 1$ and, in particular, $v_{O_Q}^{\mathcal{T}_n} = v_{O_Q}^{\mathcal{T}_1}$. Now, by (b) of Lemma II.3.1 applied to $\mathcal{T}^{(\alpha)}$, we have $\tau_{O_Q}^1 = \tau_{O_Q} + 1$ and $\tau_p^1 = \tau_p$ for every $p$ preceding $O_Q$. Therefore, by using Lemma I.1.13, we see that $v_{O_Q}^{\mathcal{T}_1} + 1 = v_{O_Q}^{\mathcal{T}^{(\alpha)}} + 1$ and hence,

$$v_{O_Q}^{\mathcal{T}_Q^\alpha} = v_{O_Q}^{\mathcal{T}^{(\alpha)}} + 1.$$

If $O_Q = O$ is a proper point of $S$, $O_Q$ is proximate to no point and the equality $v_{O_Q}^{\mathcal{T}_Q^\alpha} = v_{O_Q}^{\mathcal{T}^{(\alpha)}} + 1$ easily follows.

In any case, we infer from (3.a) that the coefficient of $E_{O_Q}$ in $D_{\widetilde{C}}^{S_{K_C}}$ is one and hence, that $\widetilde{C}$ is irreducible as a Cartier divisor near $Q$. This completes the proof. □

**Remark III.3.3.** The assumption that $O_Q$ is free or proper is necessary for the irreducibility of Cartier divisors through $Q$ intersecting all the exceptional components with prefixed intersection multiplicities at $Q$.

**Remark III.3.4.** Note that, in general, the Cartier divisors constructed in Theorem III.3.1 are not analytically irreducible as they are composed of several branches.

Now, we obtain the following corollary, as already claimed.

**Corollary III.3.5.** *Let $\mathcal{Q} = \{Q_1, \dots, Q_n\}$ be points (singular or not) in the exceptional locus of $X$ and for each $Q_i$, let $\{\alpha_p^i\}_{p\in\mathcal{K}_+^{Q_i}}$ be positive integers. Then, there exists a cluster $\mathcal{T}_\mathcal{Q}^\alpha$ such that if $C$ is a generic curve going through $\mathcal{T}_\mathcal{Q}^\alpha$, then $\widetilde{C}$ is a Cartier divisor on $X$ going through $Q_1, \dots, Q_n$ and for each $Q_i$ and for each $p \in \mathcal{K}_+^{Q_i}$,*

$$[\widetilde{C}, L_p]_{Q_i} = \alpha_p^i.$$

*Furthermore, if $Q_i$ is regular or such that $O_{Q_i}$ is free or equal to $O$, then $\widetilde{C}$ is irreducible as a principal divisor near $Q_i$.*

*Proof.* If $Q_i$ is singular, define $\mathcal{T}^{\alpha}_{Q_i}$ as the cluster obtained by Theorem III.3.1 applied to $Q_i$ and $\{\alpha_p\}_{p\in\mathcal{K}^{Q_i}_+}$.

Now, if $Q_i$ is not singular, there are one or two exceptional components going through $Q_i$ as there exists an open neighbourhood $U$ of $Q_i$ such that the restriction $f_{|f^{-1}(U)}: f^{-1}(U) \longrightarrow U$ is an isomorphism.

Assume first that $\sharp\mathcal{K}^{Q_i}_+ = 2$ and write $\mathcal{K}^{Q_i}_+ = \{p_1, p_2\}$. Take local coordinates $\bar{x}, \bar{y}$ in a neighbourhood of $Q_i$ such that $\bar{x}, \bar{y}$ are equations of the germs of $L_{p_1}$ and $L_{p_2}$, respectively. If $n' = \gcd(\alpha_{p_1}, \alpha_{p_2})$ is equal to one, take $\gamma$ as the curve having Puiseux series

$$s(\bar{x}) = \bar{x}^{\alpha_{p_2}/\alpha_{p_1}}.$$

(Puiseux series of germs of curves at regular points are explained for instance in Chapters 1 and 5 of [11]). Otherwise, write $m_1 = \frac{\alpha_{p_1}}{n'}$ and take $\gamma$ as the curve having Puiseux series

$$s(\bar{x}) = \bar{x}^{\alpha_{p_2}/\alpha_{p_1}} + \bar{x}^{(\alpha_{p_2}+m_1)/\alpha_{p_1}}.$$

In any case, $\gamma$ is analytically irreducible and $[\gamma, L_{p_j}]_{Q_i} = \alpha_{p_j}$ for $j = 1, 2$. Moreover, if $\gamma_O$ is the direct image of $\gamma$ by $\pi : X \longrightarrow S$, $\gamma_O$ is also analytically irreducible. Define $\mathcal{T}^{\alpha}_{Q_i}$ as the (irreducible) cluster obtained by taking the first non-singular point of $\gamma_O$ and all the points preceding it, each point with virtual multiplicity equal to the effective multiplicity of $\gamma_O$. Then, by Proposition I.1.29, any curve $C$ going sharply through $\mathcal{T}^{\alpha}_{Q_i}$ is also analytically irreducible and hence, so is its strict transform $\widetilde{C}$ on $X$ and $\widetilde{C}$ intersects the exceptional locus only at $Q_i$. Moreover, as any such curve has no singular points outside of $\mathcal{T}^{\alpha}_{Q_i}$, any point $p$ of $C$ being proximate to $p_1$ or $p_2$ is in $\mathcal{T}^{\alpha}_{Q_i}$, and so $e_p(C) = e_p(\gamma)$. Therefore and for $j = 1, 2$, we have that

$$\sum_{p\notin K, p\to p_j} e_p(C) = \sum_{p\notin K, p\to p_j} e_p(\gamma)$$

and by 4. of Proposition I.1.16, we infer that

$$[\widetilde{C}, L_{p_j}]_{Q_i} = [\widetilde{\gamma_O}, L_{p_j}]_{Q_i} = \alpha_{p_j}.$$

If $\sharp\mathcal{K}^{Q_i}_+ = 1$, write $\mathcal{K}^{Q_i}_+ = \{p_1\}$. In this case, take local coordinates $\bar{x}, \bar{y}$ in a neighbourhood of $Q_i$ such that $\bar{y}$ is an equation of the germ of $L_{p_1}$ at $Q_i$. Take $\gamma$ as the curve having equation $\bar{y} - \bar{x}^{\alpha_p} = 0$. Clearly, $\gamma$ is smooth and so, analytically irreducible. As above, write $\gamma_O$ for the direct image of $\gamma$ by $\pi$ and define $\mathcal{T}^{\alpha}_{Q_i}$ in the same way. Then, if $C$ goes sharply through

$\mathcal{T}^{\alpha}_{Q_i}$, $\widetilde{C}$ is analytically irreducible and the same argument used there shows that $\widetilde{C}$ intersects the exceptional locus only at $Q_i$ and $[\widetilde{C}, L_{p_1}]_{Q_i} = \alpha_{p_1}$.

Now, it is enough to define $\mathcal{T}^{\alpha}_Q$ as the sum of the clusters $\{\mathcal{T}^{\alpha}_{Q_i}\}_{i=1,\dots,n}$, $\mathcal{T}^{\alpha}_Q = \sum_{i=1}^{n} \mathcal{T}^{\alpha}_{Q_i}$ to obtain the wanted cluster. □

**Remark III.3.6.** It is worth noting that an easy procedure for computing Cartier divisors with the required intersection multiplicities with the exceptional components can be derived from the proof of Theorem III.3.1: we assume that $m = 1$, as for the general case it is enough to take the sum of the clusters obtained for each point $Q_i$. The regular case being clear, we assume also that $Q$ is singular. Keeping the notation as in the proof of Theorem III.3.1, take $\mathcal{T}^{(\alpha)} = BP(I^{(\alpha)})$ and take also the consistent cluster $\mathcal{T}_1$ which is obtained by adding some simple point in the first neighbourhood of some point in $T_Q$ and unloading multiplicities. Then, if $\alpha_p > 0$, write $q$ for the point of $T_Q$ such that $E_p$ and $E_q$ intersect on $S_K$, and while $\rho_p^{\mathcal{T}_i} > 0$ for some $p \in \mathcal{K}_+^Q$, add points proximate to $p$, infinitely near to $q$ and minimal with this property, and unload multiplicities if necessary. After finitely many steps, we reach the cluster $\mathcal{T}^{\alpha}_Q$ of Theorem III.3.1.

Next we give an easy example to illustrate this procedure.

**Example III.3.7.** Take the singularity of Example III.2.4 (see Figure III.5) and take

$$\alpha_{p_1} = 4, \quad \alpha_{p_4} = 2, \quad \alpha_{p_8} = 4, \quad \alpha_{p_{10}} = 1.$$

Keeping the notation as above, write $I^{(\alpha)} = I^4_{p_1} I^2_{p_4} I^4_{p_8} I_{p_{10}}$. Figure III.8 shows the Enriques diagrams of the clusters $\mathcal{T}^{\alpha}$ and $\mathcal{T}_1$, and Figure III.9 gives the Enriques diagram of the cluster $\mathcal{T}^{\alpha}_Q$. By Theorem III.3.1, for any curve $C$ going sharply through $\mathcal{T}^{\alpha}_Q$, $\widetilde{C}$ is a Cartier divisor on $X$ locally irreducible near $Q$ and

$$[\widetilde{C}, L_{p_1}]_Q = 4$$
$$[\widetilde{C}, L_{p_4}]_Q = 2$$
$$[\widetilde{C}, L_{p_8}]_Q = 4$$
$$[\widetilde{C}, L_{p_{10}}]_Q = 1.$$

To close this section, we obtain as a corollary of Theorem III.3.1 that the tangents to the exceptional components going through some sandwiched singularity $Q \in X$ are linearly independent. To this aim, we need first an easy lemma.

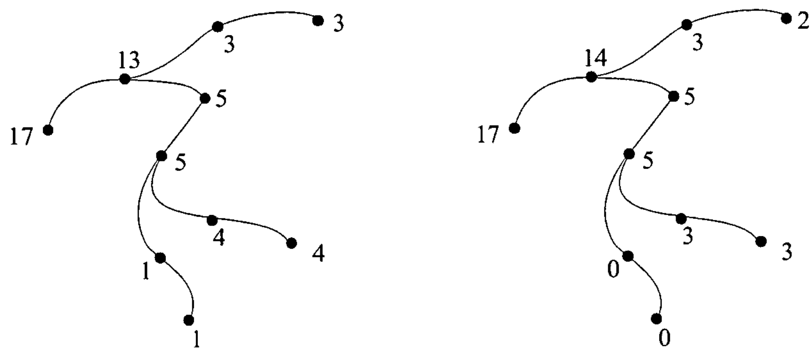



Figure III.8: On the left, the Enriques diagram of the cluster $\mathcal{T}^{(\alpha)} = BP(I^{(\alpha)})$ in Example III.3.7; on the right, the Enriques diagram of $\mathcal{T}_1$.

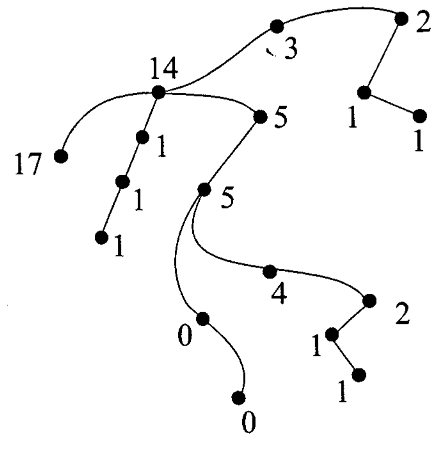



Figure III.9: Enriques diagram of the cluster $\mathcal{T}_Q^{\alpha}$ in Example III.3.7.

**Lemma III.3.8.** *Let $\xi_1, \dots, \xi_m$ be smooth curves in $\mathbb{C}^N$ ($m \leq N$) going thought the point $O = (0, \dots, 0)$ and for each $i$, write $l_i$ for the tangent to $\xi$ at $O$. Assume there is a hypersurface $H$ of $\mathbb{C}^N$ such that $[H, \xi_1]_O = 1$ and $[H, \xi_i]_O \geq 2$ for $i \in \{2, \dots, m\}$. Then, $l_1$ does not belong to the linear space generated by $l_2, \dots, l_m$ at $O$.*

*Proof.* If $H$ is a hypersurface of $\mathbb{C}^N$ such that $[H, \xi_1]_O = 1$, it is necessarily smooth at $O$. If moreover $[H, \xi_i]_O \geq 2$ for $i \in \{2, \dots, m\}$, the tangent space to $H$ at $O$ contains $l_2, \dots, l_m$. Therefore, if $l_1$ is contained in the linear space generated by $l_2, \dots, l_m$, it is also contained in the tangent space to $H$ at $O$ and $[H, \xi_1]_O \geq 2$ against the assumption. □

**Corollary III.3.9.** *Let $I \subset R$ be a complete $\mathfrak{m}_O$-primary ideal, $X = Bl_I(S)$ and $Q \in X$ a (sandwiched) singularity. Then, the tangent lines to the exceptional components on $X = Bl_I(S)$ going through $Q$ are linearly independent. In particular, the curve $L = \sum_{p \in \mathcal{K}_+^Q} L_p$ has a minimal singularity at $Q$.*

*Proof.* Write $\mathcal{K}_+^Q = \{p_1, \dots, p_m\}$. For each $i \in \{1, \dots, m\}$, and in virtue of Theorem III.3.1, there is a Cartier divisor $C_i$ going through $Q$ such that

$$[C_i, L_{p_j}]_Q = \begin{cases} 1 & \text{if } i = j \\ 2 & \text{if } i \neq j \end{cases}$$

Consider an embedding of $(X, Q)$ in an ambiental nonsingular variety $A_{\mathbb{C}}^{(\mathbf{m}_{X,Q})}$ of dimension $\mathbf{m}_{X,Q} = \dim_{\mathbb{C}} \frac{\mathfrak{m}_Q}{\mathfrak{m}_Q^2}$. Since $C_i$ is a Cartier divisor, there is some hypersurface $H_i : g_i = 0$ in $A_{\mathbb{C}}^{(\mathbf{m}_{X,Q})}$ such that

$$[H_i, L_{p_j}]_O = \begin{cases} 1 & \text{if } i = j \\ 2 & \text{if } i \neq j \end{cases}$$

From Lemma III.3.8 it follows that the lines $\{l_{p_j}\}_{j=1,\dots,m}$ span a linear space of dimension equal to $m = \sharp \mathcal{K}_+^Q$ and the first claim is proved. Now, the second claim follows directly from the definition of minimal singularity in dimension one (see Remark II.5.12) and the fact that the multiplicity at $Q$ of $L = \sum_{p \in \mathcal{K}_+^Q} L_p$ is equal to $\sharp \mathcal{K}_+^Q$. □

## III.4 On the Cartier divisors containing a given Weil divisor on $X$

Keep the notation as in the preceding sections. In this one, we fix a curve $C$ on $S$ but do not assume that the strict transform $\widetilde{C}$ on $X$ is a Cartier divisor. Hence, as already pointed out in section III.1, if $Q_1, \ldots, Q_m$ are the points where $\widetilde{C}$ intersects the exceptional locus of $X$, $D_{\widetilde{C}} = \sum_{i=1}^{n} D_{\widetilde{C}}^{Q_i}$ is a $\mathbb{Q}$-Cartier divisor on $S_{K_C}$ but not necessarily a Weil divisor (see Lemma III.1.3). However, for each singularity $Q \in X$, we can take a divisor $\overline{D}_{\widetilde{C}}^{Q}$ on $S_{K_C}$ having exceptional support contracting by $f_C$ to $Q$ and such that:

(a) for all $p \in T_Q^{S_{K_C}}$, $|E_p^{S_{K_C}} \cdot \overline{D}_{\widetilde{C}}^{Q}|_{S_{K_C}} \leq |E_p^{S_{K_C}} \cdot D_{\widetilde{C}}^{Q}|_{S_{K_C}}$

(b) $\overline{D}_{\widetilde{C}}^{Q}$ is the minimal divisor verifying (a) i.e. if $D$ is a divisor on $S_{K_C}$ contracting by $f_C$ to $Q$ and for all $p \in T_Q^{S_{K_C}}$, $|E_p^{S_{K_C}} \cdot D|_{S_{K_C}} \leq |E_p^{S_{K_C}} \cdot D_{\widetilde{C}}^{Q}|_{S_{K_C}}$, then $D \geq \overline{D}_{\widetilde{C}}^{Q}$.

Moreover, it can be seen that $\overline{D}_{\widetilde{C}}^{Q} \geq D_{\widetilde{C}}^{Q}$, and the equality holds if and only if $D_{\widetilde{C}}^{Q}$ is a divisor on $S_{K_C}$ or, in virtue of Lemma III.1.3, if and only if $\widetilde{C}$ is principal near $Q$ (see [21] §1 for details).

**Definition III.4.1.** We say that an effective Cartier divisor $C'$ on $X$ containing $\widetilde{C}$ is *v-minimal* if

$$D_{C'}^{S_{K_C}} = \overline{D}_{\widetilde{C}}^{S_{K_C}} := \sum_{i=1}^{n} \overline{D}_{\widetilde{C}}^{Q_i}.$$

and the strict transform $\widetilde{C'}^{S_{K_C}}$ intersects transversally the exceptional divisor of $f_C$.

The aim of this section is to describe an algorithm to construct curves on $S$ such that their strict transforms on $X$ are $v$-minimal Cartier divisors containing a given effective Weil divisor without exceptional component. This procedure provides also a formula for $\overline{D}_{\widetilde{C}}^{Q_i}$ as a difference of exceptional divisors as in Proposition III.2.1.

First of all, some remarks are in order.

**Remark III.4.2.** Given an effective Cartier divisor $C'$ containing $\widetilde{C}$, whether $C'$ is $v$-minimal or not is a local question, as it depends only on the germ of $C'$ at the singularities of $X$.

**Remark III.4.3.** If $\widetilde{C}$ is already Cartier, then $\overline{D}_{\widetilde{C}}^{S_{K_C}} = D_{\widetilde{C}}^{S_{K_C}}$ and there is only one $v$-minimal Cartier divisor containing $\widetilde{C}$, which is $\widetilde{C}$ itself.

**Remark III.4.4.** Let $C'$ be an effective Cartier divisor on $X$ containing $\widetilde{C}$ and assume that it has an exceptional component $L' = \sum_{p \in \mathcal{K}_+} a_p L_p$, with $a_p \in \mathbb{Z}_{\geq 0}$. For each $p \in \mathcal{K}_+$, let $u_1^p, \dots, u_{n_p}^p \in K_C \setminus \mathcal{K}_+$ be the points whose corresponding vertices in the dual graph $\Gamma_{K_C}$ are adjacent to $p$ (clearly, $n_p \leq \gamma_{\Gamma_{K_C}}(p)$; see section I.5). Then, if $q \in K_C \setminus \mathcal{K}_+$, the strict transform $\widetilde{L'}^{S_{K_C}}$ of $L'$ on $S_{K_C}$ satisfies that

$$|\widetilde{L'}^{S_{K_C}} \cdot E_q^{S_{K_C}}|_{S_{K_C}} = \begin{cases} a_p & \text{if } q \in \{u_1^p, \dots, u_{n_p}^p\} \text{ for some } p \in \mathcal{K}_+ \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, since $\widetilde{C}$ contains no exceptional component, $\widetilde{C}$ is also contained in the effective divisor $C' - L'$, which may not be Cartier. However, if $C_1$ is a generic curve going through the consistent cluster $\mathcal{T} = \sum_{p \in \mathcal{K}_+} a_p (\sum_{i=1}^{n_p} \mathcal{K}(u_i^p))$, then it has $a_p$ branches through each $u_i^p$, all of them missing the points in $K_C$ after it. Hence, for all $q \in K_C \setminus \mathcal{K}_+$,

$$|\widetilde{C_1}^{S_{K_C}} \cdot E_q^{S_{K_C}}|_{S_{K_C}} = \begin{cases} a_p & \text{if } q \in \{u_1^p, \dots, u_{n_p}^p\} \text{ for some } p \in \mathcal{K}_+ \\ 0 & \text{otherwise.} \end{cases}$$

By Definition I.3.19, we deduce that $D_{\widetilde{C_1}}^{S_{K_C}} = D_{L'}^{S_{K_C}}$, and by Lemma III.1.3, we infer that $C' - L' + \widetilde{C_1}$ is an effective Cartier divisor on $X$. Therefore, in order to compute $\overline{D}_{\widetilde{C}}^{S_{K_C}}$ it is enough to take Cartier divisors without exceptional component.

**Remark III.4.5.** Assume that $C'$ is an effective Cartier divisor containing $\widetilde{C}$ and having no exceptional component. Then, the image by $\pi$ of this curve has the form $C + C_O$ for some curve $C_O \subset S$. Equivalently, $C'$ is the strict transform on $X$ of $C'_O = C + C_O$. Then, by Corollary III.1.6, the complete $\mathfrak{m}_O$-primary ideal

$$\mathbb{H}^o_{C'_O} = \{g \in R \mid v_p(g) \geq v_p(C'_O), \forall p \in \mathcal{K}_+\}$$

has factorization

$$\mathbb{H}^o_{C'_O} = \prod_{p \in \mathcal{K}_+} I_p^{m_p},$$

where $m_p = |C' \cdot L_p|_X$. This fact suggests the idea of the algorithm stated below: by means of a procedure based on unloading, we compute the minimal integers $m_p$ for which there exist curves $C_O \subset S$ so that $C + C_O$ goes virtually through $\sum_{p\in\mathcal{K}_+} m_p\mathcal{K}(p)$. The procedure provides a flag of clusters that allows us to give a complete description of such curves $C_O$ and also, of the $v$-minimal Cartier divisors (with no exceptional component) containing $\widetilde{C}$.

Before describing the algorithm, we need some technical results that we state separately for clarity. The fist one introduces a variation of the unloading procedure. Recall from Notation I.2.12 that given a non-consistent cluster $\mathcal{K}$, we write $\widetilde{\mathcal{K}}$ for the consistent cluster obtained from $\mathcal{K}$ by unloading multiplicities.

**Lemma III.4.6.** *Let $\mathcal{K} = (K, \nu)$ be a non-consistent cluster and let $K_0 \subset K$. There exists a cluster $\mathcal{K}' = (K, \nu')$ equivalent to $\mathcal{K}$ and such that:*

*(i) $\rho_p^{\mathcal{K}'} \geq 0$ if $p \in K \setminus K_0$;*

*(ii) $v_p^{\mathcal{K}'} = v_p^{\mathcal{K}}$ if $p \in K_0$.*

*Moreover, if we take the partial order relation given by the virtual values, there is a unique minimal cluster with these assumptions.*

*Proof.* Put $\mathcal{K}^0 = \mathcal{K}$ and, inductively, as far as $\mathcal{K}^{i-1}$ has negative excess at some point $p \in K \setminus K_0$, define $\mathcal{K}^i$ from $\mathcal{K}^{i-1}$ by unloading on $p$. We claim that there is an $m$ so that $\mathcal{K}^m$ has non-negative excess at each point in $K \setminus K_0$, and $v_p^{\mathcal{K}^m} = v_p^{\mathcal{K}}$ if $p \in K_0$. To show this, note that the steps on the above procedure are part of an unloading sequence giving rise to a consistent cluster as described in Theorem I.2.16. From (b) of Theorem I.2.16, we reach this cluster after finitely many steps, independently on the choice of the points on which unloadings are performed. This shows that after finitely many steps we reach a cluster $\mathcal{K}^m$ satisfying the condition (i). By Remark I.2.14, the condition (ii) is clear, as no unloading is performed on a point of $K_0$.

Now, to prove the uniqueness of a cluster with the above assumptions and minimal relative to the virtual values, assume that we have clusters $\mathcal{K}_{(1)} = (K, \nu^{(1)})$ and $\mathcal{K}_{(2)} = (K, \nu^{(2)})$ equivalent to $\mathcal{K}$ and verifying the conditions (i) and (ii). Then, for every $p \in K$,

$$v_p^{\mathcal{K}} \leq v_p^{\mathcal{K}^{(i)}} \leq v_p^{\widetilde{\mathcal{K}}} \quad i = 1, 2 \tag{4.a}$$

Put $\mathcal{K}^{(0)} = (K, \nu^{(0)})$ the cluster defined by taking values $v_p^{\mathcal{K}^{(0)}} = \min\{v_p^{\mathcal{K}^{(1)}}, v_p^{\mathcal{K}^{(2)}}\}$ for every $p \in K$. Now, for $p \in K$, write $\omega(p) = \sharp\{q \in K \mid q \to p\} + 1$.

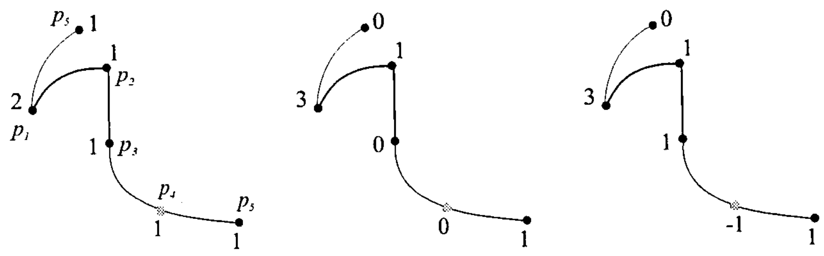



Figure III.10: The Enriques diagrams of $\mathcal{K}$, $\widetilde{\mathcal{K}}^{K_0}$ and $\mathcal{K}'$ in Example III.4.8.

Then, if $p \in K \setminus K_0$ and say $v_p^{\mathcal{K}^{(1)}} \leq v_p^{\mathcal{K}^{(2)}}$, we have (Artin's trick [3])

$$\begin{aligned} \rho_p^{\mathcal{K}^{(0)}} &= -\mathbf{1}_p^t \mathbf{A}_K v^{\mathcal{K}^{(0)}} = \omega(p) v_p^{\mathcal{K}^{(0)}} - \sum_{d(p,q)=1} v_q^{\mathcal{K}^{(0)}} \geq \\ &\geq \omega(p) v_p^{\mathcal{K}^{(1)}} - \sum_{d(p,q)=1} v_q^{\mathcal{K}^{(1)}} = -\mathbf{1}_p^t \mathbf{A}_K v^{\mathcal{K}^{(1)}} = \rho_p^{\mathcal{K}^{(1)}} \geq 0 \end{aligned}$$

the last inequality by assumption. Moreover, if $p \in K_0$, then $v_p^{\mathcal{K}^{(0)}} = v_p^{\mathcal{K}}$ and from (4.a) above, we infer that $\mathcal{K}^{(0)}$ is equivalent to $\mathcal{K}$. Therefore, $\mathcal{K}^{(0)}$ is equivalent to $\mathcal{K}$ and verifies the conditions (i) and (ii). From this, we derive the claimed uniqueness. □

**Notation III.4.7.** From now on, we will denote by $\widetilde{\mathcal{K}}^{K_0}$ the minimal cluster equivalent to $\mathcal{K}$ and verifying the conditions (i) and (ii) of Lemma III.4.6.

In the following example, we show that the hypothesis of minimality relative to the virtual values is necessary for the uniqueness of $\widetilde{\mathcal{K}}^{K_0}$.

**Example III.4.8.** Consider a cluster $\mathcal{K}$ having Enriques diagram as shown on the left of Figure III.10 and put $K_0 = \{p_4\}$. The Enriques diagram in the middle shows the virtual multiplicities of $\widetilde{\mathcal{K}}^{K_0}$, while that on the right shows the virtual multiplicities of another cluster $\mathcal{K}'$ verifying the conditions (i) and (ii) of Lemma III.4.6. Moreover, we have that

$$\mathbf{v}_{\mathcal{K}} = \{2,3,6,7,8,3\} \quad \mathbf{v}_{\widetilde{\mathcal{K}}^{K_0}} = \{3,4,7,7,8,3\}$$

$$\mathbf{v}_{\mathcal{K}'} = \{3,4,8,7,8,3\} \quad \mathbf{v}_{\widetilde{\mathcal{K}}} = \{3,4,8,8,8,3\}$$

and so, both $\widetilde{\mathcal{K}}^{K_0}$ and $\mathcal{K}'$ are equivalent to $\mathcal{K}$.

In order to obtain $\widetilde{\mathcal{K}}^{K_0}$ in practice, it is enough to perform usual unloading on the points not in $K_0$ with negative excess. To show this, consider a sequence of clusters $\mathcal{K}^i$ as in the proof of Lemma III.4.6, and assume that for some $i \geq 1$, there is some point in $K$, say $p$, such that $v_p^{\mathcal{K}^i} > v_p^{\widetilde{\mathcal{K}}^{K_0}}$. We can assume that $i$ is minimal with this property and so, $v_q^{\mathcal{K}^{i-1}} \leq v_q^{\widetilde{\mathcal{K}}^{K_0}}$ for each $q \in K$. Put $n_p^0 = v_p^{\widetilde{\mathcal{K}}^{K_0}} - v_p^{\mathcal{K}^{i-1}} > 0$. Using the minimality of $i$, we have that

$$\begin{aligned} \mathbf{1}_p^t \mathbf{A}_K(\mathbf{v}_{\mathcal{K}^{i-1}} + n_p^0 \mathbf{1}_p) &= -\omega(p)(v_p^{\mathcal{K}^{i-1}} + n_p^0) + \sum_{d(p,q)=1} v_q^{\mathcal{K}^{i-1}} \leq \\ &\leq -\omega(p) v_p^{\widetilde{\mathcal{K}}^{K_0}} + \sum_{d(p,q)=1} v_q^{\widetilde{\mathcal{K}}^{K_0}} = \\ &= \mathbf{1}_p^t \mathbf{A}_K \mathbf{v}_{\widetilde{\mathcal{K}}^{K_0}} = -\rho_p^{\widetilde{\mathcal{K}}^{K_0}} \end{aligned}$$

the last equality by Lemma I.1.22. From the definition of $\widetilde{\mathcal{K}}^{K_0}$, we infer that

$$\mathbf{1}_p^t \mathbf{A}_K(\mathbf{v}_{\mathcal{K}^{i-1}} + n_p^0 \mathbf{1}_p) \leq 0 \tag{4.b}$$

Now, as explained in the subsection 1.2.1, the unloading step on $p$ giving rise to $\mathcal{K}^i$ increases the virtual value at $p$ by an amount $n$, which is the least integer such that

$$\mathbf{1}_p^t \mathbf{A}_K(\mathbf{v}_{\mathcal{K}} + n\mathbf{1}_p) \leq 0,$$

while the virtual values at the other points of $K$ remain unchanged. By (4.b), we have that $n \leq n_p^0$ and so

$$v_p^{\mathcal{K}^i} = v_p^{\mathcal{K}^{i-1}} + n \leq v_p^{\mathcal{K}^{i-1}} + n_p^0 = v_p^{\widetilde{\mathcal{K}}^{K_0}}$$

against the assumption. Since any unloading step increases the value at some point by some positive quantity, we derive that after finitely many unloading steps, we reach $\widetilde{\mathcal{K}}^{K_0}$.

**Definition III.4.9.** We will say that $\widetilde{\mathcal{K}}^{K_0}$ has been obtained from $\mathcal{K}$ by *partial unloading relative to* $K_0$.

**Remark III.4.10.** Let $\mathcal{K} = (K, \nu)$ be a consistent cluster and $q \in \overline{\mathbb{F}}_K$. Since all unloading steps leading from $\mathcal{K}_q$ to $\widetilde{\mathcal{K}_q}$ are tame (Corollary II.2.2), so are the unloading steps from $\mathcal{K}_q$ to $\widetilde{\mathcal{K}_q}^{K_0}$, and from $\widetilde{\mathcal{K}_q}^{K_0}$ to $\widetilde{\mathcal{K}_q}$. In particular, we have $c(\mathcal{K}_q) = c(\widetilde{\mathcal{K}_q}^{K_0}) = c(\widetilde{\mathcal{K}_q})$.

Note also that if $K_0 = \emptyset$, partial unloading and usual unloading agree.

**Lemma III.4.11.** *Let $K$ be a cluster, $K_0 \subset K$ a proper subset and $q \in \overline{\mathbb{F}}_K$, not proximate to any point of $K_0$. Let*

$$\mathcal{T}^1 = \sum_{p \in K_0} m_p^1 \mathcal{K}(p) + \sum_{p \in K \setminus K_0} \alpha_p \mathcal{K}(p)$$

$$\mathcal{T}^2 = \sum_{p \in K_0} m_p^2 \mathcal{K}(p) + \sum_{p \in K \setminus K_0} \alpha_p \mathcal{K}(p)$$

*be (weighted) clusters with $\alpha_p \geq 0$ for each $p \in K \setminus K_0$. If $\mathcal{T}_q^1$ is not consistent, then $\mathcal{T}_q^2$ is not consistent, either, and we have that*

$$\widetilde{\mathcal{T}_q^1}^{K_0} = \sum_{p \in K_0} (m_p^1 - s_p) \mathcal{K}(p) + \sum_{p \in K \setminus K_0} (\alpha_p - s_p) \mathcal{K}(p)$$

$$\widetilde{\mathcal{T}_q^2}^{K_0} = \sum_{p \in K_0} (m_p^2 - s_p) \mathcal{K}(p) + \sum_{p \in K \setminus K_0} (\alpha_p - s_p) \mathcal{K}(p),$$

*where $\alpha_p \geq s_p$ if $p \in K \setminus K_0$ and $s_p \geq 0$ if $p \in K_0$.*

*Moreover, if $p \in K$ is such that no point of $K_0$ is infinitely near to it, and $\tau^1$, $\tau^2$ are the virtual multiplicities at $p$ of $\widetilde{\mathcal{T}_q^1}^{K_0}$, $\widetilde{\mathcal{T}_q^2}^{K_0}$, respectively, then $\tau_p^1 = \tau_p^2$.*

*Proof.* First of all, notice that if $p \in K \setminus K_0$ and for $i \in \{1, 2\}$, we have

$$\rho_p^{\mathcal{T}^i} = -\mathbf{1}_p^t \mathbf{A}_K \mathbf{v}_{\mathcal{T}^i} = \alpha_p. \tag{4.c}$$

Write $T = K \cup \{q\}$ and for $i = 1, 2$, write $\widehat{\mathcal{T}^i}$ for the cluster $\widetilde{\mathcal{T}_q^i}^{K_0}$. Then, by definition of partial unloading, we have $\mathbf{v}_{\widehat{\mathcal{T}^i}} = \mathbf{v}_{\mathcal{T}_q^i} + \overline{n}^i$, where $\overline{n}^i = (n_p^i)_{p \in T}$, and

$$n_p^i \geq 0 \text{ if } p \in T \quad \text{and} \quad n_p^i = 0 \text{ if } p \in K_0.$$

Now, denote $\overline{n}^0 = (n_p^0)_{p \in T}$, $n_p^0 = \min\{n_p^1, n_p^2\}$, and for $i = 1, 2$, write $\mathcal{T}_0^i$ for the cluster with set of points $T$ and system of values given by

$$\mathbf{v}_{\mathcal{T}_0^i} = \mathbf{v}_{\mathcal{T}_q^i} + \overline{n}^0. \tag{4.d}$$

For $p \in T$, let $j \in \{1, 2\}$ such that $n_p^j = \min\{n_p^1, n_p^2\}$. Then,

$$-\mathbf{1}_p^t \mathbf{A}_T \overline{n}^0 = \omega(p)\, n_p^0 - \sum_{d(p,q)=1} n_q^0 \geq \omega(p)\, n_p^j - \sum_{d(p,q)=1} n_q^j = -\mathbf{1}_p^t \mathbf{A}_T \overline{n}^j.$$

Therefore, if $p \in T \setminus K_0$ and $i \in \{1,2\}$,

$$\begin{aligned} \rho_p^{\mathcal{T}_0^i} &= -\mathbf{1}_p^t \mathbf{A}_T \mathbf{v}_{\mathcal{T}_0^i} = -\mathbf{1}_p^t \mathbf{A}_T \mathbf{v}_{\mathcal{T}^i} - \mathbf{1}_p^t \mathbf{A}_T \overline{n}^0 \geq \\ &\geq -\mathbf{1}_p^t \mathbf{A}_T \mathbf{v}_{\mathcal{T}^i} - \mathbf{1}_p^t \mathbf{A}_T \overline{n}^j = \\ &= \alpha_p - \mathbf{1}_p^t \mathbf{A}_T \overline{n}^j = \rho_p^{\widehat{\mathcal{T}^j}} \geq 0, \end{aligned}$$

the last inequality by definition of $\widehat{\mathcal{T}}^j$. On the other hand, if $p \in K_0$, we have that $n_p^0 = 0$ and by (4.d), $v_p^{\mathcal{T}_0^i} = v_p^{\mathcal{T}^i}$. Hence, $\mathcal{T}_0^i$ verifies the conditions (i) and (ii) of Lemma III.4.6 and by the minimality of $\widehat{\mathcal{T}^i}$, we deduce that $\overline{n}^0 = \overline{n}^i$ and therefore, $n_p^1 = n_p^2$ for each $p \in T$. In particular, for every $p \in T$, $\rho_p^{\widehat{\mathcal{T}^i}} = -\mathbf{1}_p^t \mathbf{A}_T(\mathbf{v}_{\mathcal{T}^i} + \overline{n}^0)$. Now, if we write $s_p = \mathbf{1}_p^t \mathbf{A}_T \overline{n}^0$ for each $p \in K$, we have that for $i = 1, 2$,

$$\rho_p^{\widehat{\mathcal{T}^i}} = -\mathbf{1}_p^t \mathbf{A}_T \mathbf{v}_{\mathcal{T}^i} - s_p = \rho_p^{\mathcal{T}^i} - s_p.$$

Now, if $p \in K \setminus K_0$, by (4.c), we know that $\rho_p^{\mathcal{T}^i} = \alpha_p$ and so, we have $\rho_p^{\widehat{\mathcal{T}^1}} = \rho_p^{\widehat{\mathcal{T}^2}} = \alpha_p - s_p$. By definition of partial unloading relative to $K_0$, it is clear that $\alpha_p \geq s_p$. On the other hand, if $p \in K_0$, we have $\rho_p^{\widehat{\mathcal{T}^1}} = m_p^1 - s_p$ and $\rho_p^{\widehat{\mathcal{T}^2}} = m_p^2 - s_p$. Moreover, since $n_p^0 = 0$, we infer that $s_p = \mathbf{1}_p^t \mathbf{A}_T \overline{n}^0 = \sum_{d(p,q)=1} n_q^0 \geq 0$. This proves the first claim. The second one follows immediately from it. □

**Lemma III.4.12.** *Let $\mathcal{Q} = (K, \sigma)$ be a consistent cluster and let $p_1, \dots, p_m$ be points infinitely near to $O$ and not in $K$. Write $\overline{\mathcal{Q}} = (T, \overline{\tau})$ for the cluster $\mathcal{Q} + \sum_{i=1}^m \mathcal{K}(p_i)$. Take the chain of clusters*

$$\mathcal{T}_0 \prec \mathcal{T}_1 \prec \dots \prec \mathcal{T}_j \prec \dots \quad \text{with } \mathcal{T}_j = (T, \tau^j) \tag{4.e}$$

*defined as follows: put $\mathcal{T}_0 = \mathcal{Q}$ and for $j \geq 0$, as far as there exists some $p_i$ such that $\tau_{p_i}^j = 0$, take $\mathcal{T}_{j+1}$ as the cluster obtained from $(\mathcal{T}_j)_{p_i}$ by partial unloading relative to $\mathcal{Q}_+ = \{p \in K \mid \rho_p^{\mathcal{Q}} > 0\}$ and dropping the points with virtual multiplicity zero. Then, after finitely many steps, this procedure stops and we reach a cluster $\mathcal{T}_n$ such that*

*(i) $\tau_p^n = \overline{\tau}_p$, if $p \in T \setminus K$,*

*(ii) if $\mathcal{T} = (T, \tau)$ is such that $H_{\mathcal{T}} \subset H_{\mathcal{Q}}$ and $\tau_p = \overline{\tau}_p$ for each $p \notin K$, then $H_{\mathcal{T}} \subset H_{\mathcal{T}_n}$.*

*(iii) $\mathcal{T}_n$ has non-negative excess at every point $p \in T \setminus \mathcal{Q}_+$; if $p \in \mathcal{Q}_+$, then $\rho_p^{\mathcal{T}_n} = \rho_p^{\mathcal{Q}} - s_p$, where $s_p \geq 0$ does not depend on the excesses of $\mathcal{Q}$ at*

*the points in $\mathcal{Q}_+$. In particular, if the excesses of $\mathcal{Q}$ at these points are big enough, then all the clusters $\mathcal{T}_i$ in the flag (4.e) are consistent.*

*Proof.* Clearly, $\overline{\mathcal{Q}}$ is consistent and $H_{\overline{\mathcal{Q}}} \subset H_{\mathcal{Q}}$. Moreover, the clusters $\mathcal{T}_j$ of the chain above verify that

$$H_{\overline{\mathcal{Q}}} \subset H_{\mathcal{T}_j}. \tag{4.f}$$

To show this, we use induction on $j$. The above inclusion is clear for $j=0$, so assume it is true for $j \geq 1$. Write $\widetilde{\mathcal{T}_j}$ for the strictly consistent cluster obtained by unloading multiplicities (if necessary) and dropping the points with virtual multiplicity zero, and write $\widetilde{\tau}_p$ for its virtual multiplicity at a point $p$. The cluster $\mathcal{T}_{j+1}$ is then equivalent to $(\widehat{\mathcal{T}_j})_{p_i}$, where $p_i$ is such that $\tau^j_{p_i}=0$, and also $\widetilde{\tau}^j_{p_i}=0$ for there are no points in $T_j$ infinitely near to $p_i$. Hence, $p_i$ is not a point of $\widetilde{T}_j$. Now, to prove that $\overline{\mathcal{Q}}$ contains $\mathcal{T}_{j+1}$, it is enough to show that the virtual transform $\check{C}^{\widetilde{\mathcal{T}_j}}$ of any curve $C$ going sharply through $\overline{\mathcal{Q}}$ goes through the point $p_i$, and this is clear as such a curve goes effectively through $p_i$.

Now, in virtue of (4.f), it is clear that after finitely many steps, we reach a (not necessarily consistent) cluster $\mathcal{T}_n$ such that $\tau^n_{p_i}=1$ for each $i \in \{1,\dots,m\}$. Since the excesses of the clusters $\mathcal{T}_j$ at the points in $T \setminus Q_+$ are non-negative, we infer that

$$\tau^n_p \geq \overline{\tau}_p \quad \text{for } p \in T \setminus K. \tag{4.g}$$

Now, assume that we have a couple of (not necessarily consistent) clusters $\mathcal{T}^{(1)}=(T^1,\tau^{(1)})$ and $\mathcal{T}^{(2)}=(T^2,\tau^{(2)})$ such that $H_{\mathcal{T}^{(1)}}, H_{\mathcal{T}^{(2)}} \subset H_{\mathcal{Q}}$ and $\tau^{(i)}_p \geq \overline{\tau}_p$ for $p \in T \setminus K$. Define a new cluster $\mathcal{T}^{(0)}=(T,\tau^{(0)})$ by taking

$$\tau^{(0)}_p = \begin{cases} \mathbf{1}^t_p \mathbf{P}_K \mathbf{v}_{\mathcal{K}^{(0)}} & \text{if } p \in K \\ \overline{\tau}_p & \text{if } p \in T \setminus K \end{cases}$$

where $\mathbf{v}_{\mathcal{K}^{(0)}} = (v^0_p)_{p\in K}$ and $v^0_p = \min\{v^{\mathcal{T}^{(1)}}_p, v^{\mathcal{T}^{(2)}}_p\}$. Since $H_{\mathcal{T}^{(1)}}, H_{\mathcal{T}^{(2)}} \subset H_{\mathcal{T}^{(0)}}$, we have that if $p \in K$, then $v^{\mathcal{T}^{(0)}}_p = \min\{v^{\mathcal{T}^{(1)}}_p, v^{\mathcal{T}^{(2)}}_p\} \geq v^{\mathcal{T}_0}_p$ and so, $H_{\mathcal{T}^{(0)}} \subset H_{\mathcal{Q}}$. That $\mathcal{T}^{(0)} \prec \mathcal{T}^{(1)}, \mathcal{T}^{(2)}$ and $\tau^{(0)}_p = \overline{\tau}_p$ for $p \in T \setminus K$ follows from the definition of $\mathcal{T}^{(0)}$. Now, by using Artin's trick as in the proof of Lemma III.4.6, it is easy to see that $\rho^{\mathcal{T}^{(0)}}_p \geq 0$ for $p \in T \setminus \mathcal{K}_+$. This proves the uniqueness of a minimal cluster $\mathcal{T}'=(T,\tau')$ containing $\mathcal{Q}$ and satisfying (4.g), and shows that in fact, $\tau'_p = \overline{\tau}_p$ if $p \in T \setminus K$. From the way the cluster $\mathcal{T}_n$ has been constructed, necessarily $\mathcal{T}'=\mathcal{T}_n$, and so $\mathcal{T}_n$ does not depend on the choices done when constructing the chain (4.e).

In virtue of the last assertion of Lemma III.4.11 we infer that if $j \geq 1$ and if $p \in T \setminus \mathcal{Q}_+$, $\tau_p^j$ does not depend on the excesses $\{\rho_q^{\mathcal{T}_j}\}_{q \in \mathcal{Q}_+}$ and inductively, not on $\{\rho_q^{\mathcal{Q}}\}_{q \in \mathcal{Q}_+}$. Moreover, we the variation between the excesses of $\mathcal{T}_{j+1}$ and $\mathcal{T}_j$ at each $p \in T$,

$$s_p^j = \rho_p^{\mathcal{T}_{j-1}} - \rho_p^{\mathcal{T}_j},$$

does not depend on the excesses of $\mathcal{T}_{j-1}$ at the points of $\mathcal{Q}_+$, and inductively, nor on the excesses of $\mathcal{T}_0 = \mathcal{Q}$ at these points. Therefore, if for each $p \in \mathcal{Q}_+$, we put

$$s_p = \sum_{j=1}^{n} s_p^j,$$

we have that $\rho_p^{\mathcal{T}_n} = \rho_p^{\mathcal{Q}} - s_p$, where $s_p \geq 0$ is independent from the excesses of $\mathcal{Q}$ at the points of $\mathcal{Q}_+$. In particular, if $\rho_p^{\mathcal{Q}} \geq s_p$ for every $p \in \mathcal{Q}_+$, the resulting cluster $\mathcal{T}_n$ (and in fact, every $\mathcal{T}_i$) is consistent. □

*Description of the algorithm*

Let $\mathbf{m} = \{m_p\}_{p \in \mathcal{K}_+}$ be a vector of integers and denote

$$\mathcal{T}_0^{\mathbf{m}} = \sum_{p \in \mathcal{K}_+} m_p \mathcal{K}(p).$$

Although $\mathcal{T}_0^{\mathbf{m}}$ depends on $\mathbf{m}$, for simplicity in the notation, we write $\tau_p$ for its virtual multiplicity at $p$. Let $q_1, \ldots, q_m$ be the first non-singular points in the branches of $C$ and not in $K$. The procedure explained here is a direct implementation of Lemma III.4.12 applied to the cluster $\mathcal{T}_0^{\mathbf{m}}$ and the points $q_1, \ldots, q_m$.

STEP 0 Take any $q_i$ and define $\mathcal{T}_1^{\mathbf{m}} = (K_C, \tau^1)$ as the cluster obtained from $(\mathcal{T}_0^{\mathbf{m}})_{q_i}$ by partial unloading relative to $\mathcal{K}_+$.

STEP $j$ Once $\mathcal{T}_{j-1}^{\mathbf{m}}$ is defined, assume that there is some $q_i$ such that $\tau_{q_i}^{j-1} = 0$ (otherwise, the algorithm stops here) and define $\mathcal{T}_j^{\mathbf{m}}$ to be the cluster obtained from $(\mathcal{T}_{j-1}^{\mathbf{m}})_{q_i}$ by partial unloading relative to $\mathcal{K}_+$.

As already pointed out, at each step of the above procedure, the excesses at the points in $\mathcal{K}_+$ decrease by one or remain unaffected. In virtue of Lemma III.4.12, this procedure stops after finitely many steps, say $n$, and we obtain a cluster $\mathcal{T}_n^{\mathbf{m}}$ such that:

(A) $\tau_p^n = e_p(C)$ if $p \in K_C \setminus K$,

(B) for each $p \in K_C \setminus K$, the excess $\rho_p^{\mathcal{T}_n^{\mathbf{m}}}$ is non-negative and if $p \in \mathcal{K}_+$, $\rho_p^{\mathcal{T}_n^{\mathbf{m}}} = m_p - s_p$, where $s_p$ is a non-negative integer not depending on $\mathbf{m}$.

In particular, if we choose $\mathbf{m} = \{m_p\}_{p\in\mathcal{K}_+}$ such that $m_p \geq s_p$ for each $p \in \mathcal{K}_+$, we obtain a flag of *consistent* clusters

$$\mathcal{T}_0^{\mathbf{m}} \prec \mathcal{T}_1^{\mathbf{m}} \prec \ldots \prec \mathcal{T}_n^{\mathbf{m}}. \tag{4.h}$$

**Remark III.4.13.** In virtue of Lemma III.4.11, the value of $n$ depends on the points of $\mathcal{K}_+$ and the position of $C$ relative to $K$, but not on $\mathbf{m}$. Note also that if $\mathcal{T}_n^{\mathbf{m}}$ is consistent, then

$$E_{\mathcal{T}_n^{\mathbf{m}}} - E_{\mathcal{T}_0^{\mathbf{m}}} = \sum_{p\in K_C} n_p E_p^{S_{K_C}}$$

where $n_p$ is the number of unloadings performed on each $p \in K_C$ (in particular, $n_p = 0$ if $p \in \mathcal{K}_+$, see Remark I.2.14). Therefore, the difference $E_{\mathcal{T}_n^{\mathbf{m}}} - E_{\mathcal{T}_0^{\mathbf{m}}}$ does not depend on $\mathbf{m}$, either.

**Remark III.4.14.** As in section III.2, write $\mathcal{K}_C = (K_C, e(C))$ where $e(C)$ is the system of virtual multiplicities given by the effective multiplicities of $C$. Clearly, if the strict transform $\widetilde{C}$ on $X$ is Cartier, then $\mathcal{K}_C = \mathcal{T}_n^{\mathbf{s}}$ and also, $\mathcal{K}_C^o = \mathcal{T}_0^{\mathbf{s}}$.

The following proposition is the main result of this section. It gives a complete description of the $v$-minimal Cartier divisors containing $\widetilde{C}$ without exceptional support (see Remark III.4.4) and a geometric interpretation of the integers $s_p$ above. It also presents a formula for the divisor $\overline{D}_{\widetilde{C}}^{S_{K_C}}$ introduced at the beginning of this section.

**Proposition III.4.15.** *Let $C \subset S$ and write $\mathcal{K}_C = (K_C, e(C))$ for the weighted cluster of its singular points taken with virtual multiplicities equal to the effective ones. Keep the notations as above.*

*(a) Assume that $\mathbf{m} \geq \mathbf{s}$ and write $\mathcal{T}_n^{m} = (K_C, \omega^{m})$ for the cluster obtained by the algorithm above described. Then, the cluster*

$$\mathcal{K}_{\mathrm{Cart}(C)}^{m} = (K_C, \omega^{m} - e(C))$$

*is consistent, and if $C_O$ is a curve going through it with effective multiplicities equal to the virtual ones and having, for each $p \in K \setminus \mathcal{K}_+$, $\rho_p^{\mathcal{K}_{\mathrm{Cart}(C)}^{m}}$ branches through $p$ and missing all points after $p$ in $K_C$, then $\widetilde{C} + \widetilde{C_O}$ is a $v$-minimal Cartier divisor containing $\widetilde{C}$. Moreover, every $v$-minimal Cartier divisor $C'$ containing $\widetilde{C}$ and without exceptional components has this form.*

*(b)* $\overline{D}_{\widetilde{C}}^{S_{K_C}} = E_{\mathcal{T}_n^{m}}^{S_{K_C}} - E_{\mathcal{T}_0^{m}}^{S_{K_C}}$.

(c) *If* $p \in K_C$ *is a dicritical point of* $\mathcal{K}^{\mathbf{m}}_{\mathrm{Cart}(C)}$, *then* $p \in K \setminus \mathcal{K}_+$. *Therefore, the curves* $C_O$ *as in (a) share no points outside of* $K$, *the set of base points of* $I$.

After Proposition III.4.15, one should think of the curves $\widetilde{C_O}$ as the Weil divisors to be added to $\widetilde{C}$ to obtain a $v$-minimal Cartier divisor on $X$.

*Proof.* Since we are assuming that $\mathbf{m} \geq \mathbf{s}$, the cluster $\mathcal{T}_n^{\mathbf{m}}$ is consistent, and from its construction, it has virtual multiplicity one at $q_1, \ldots, q_m$ and these points are maximal in it. It follows that $\rho_{q_i}^{\mathcal{T}_n^{\mathbf{m}}} = 1$ for $i \in \{1, \ldots, m\}$. On the other hand, from the definition of the points $\{q_i\}_{i=1,\ldots,m}$, the cluster $\mathcal{K}_C$ has excess equal to one at them and zero at the remaining points. Since the excess of $\mathcal{K}^{\mathbf{m}}_{\mathrm{Cart}(C)}$ at any point is the difference between the excesses of $\mathcal{T}_n^{\mathbf{m}}$ and $\mathcal{K}_C$, we infer that the excesses of $\mathcal{K}^{\mathbf{m}}_{\mathrm{Cart}(C)}$ are non-negative and hence, that $\mathcal{K}^{\mathbf{m}}_{\mathrm{Cart}(C)}$ is consistent.

Let $A'$ be an effective Cartier divisor on $X$ containing $\widetilde{C}$ and without exceptional components, so that there exists some curve $A_O$ such that $A' = \widetilde{C} + \widetilde{A_O}$ (see Remark III.4.5). By Theorem III.1.1, we have that

$$L_{C+A_O} = \sum_{p \in \mathcal{K}_+} m_p \mathcal{L}_p \qquad (4.i)$$

for some $m_p \in \mathbb{Z}_{\geq 0}$. We claim that $m_p \geq s_p$ for each $p \in \mathcal{K}_+$. To show this, write $A'_O = C + A_O$ and construct the chain (4.e) of Lemma III.4.12 applied to $\mathcal{K}^o_{A'_O}$ and $q_1, \ldots, q_m$,

$$\mathcal{T}_0^{\mathbf{m}} = \mathcal{K}^o_{A'_O} \prec \cdots \prec \mathcal{T}_n^{\mathbf{m}}.$$

Since $e_p(A'_O) \geq e_p(C)$ for each $p \in K_C$, we infer from the condition (ii) of Lemma III.4.12 that $A'_O$ goes through $\mathcal{T}_n^{\mathbf{m}}$ and so,

$$v_p(A'_O) \geq v_p^{\widetilde{\mathcal{T}_n^{\mathbf{m}}}}. \qquad (4.j)$$

By condition (B) above, if $\mathcal{T}_n^{\mathbf{m}}$ is not consistent, then there exists some $p \in \mathcal{K}_+$ such that $m_p < s_p$. In this case, some unloading must be performed on $p$ when unloading multiplicities to reach $\widetilde{\mathcal{T}_n^{\mathbf{m}}}$ from $\mathcal{T}_n^{\mathbf{m}}$. Hence, in virtue of Remark I.2.14 and (4.j) above, we have $v_p(A'_O) > v_p^{\mathcal{T}_n^{\mathbf{m}}}$ against (4.i). Therefore, $\mathcal{T}_n^{\mathbf{m}}$ is consistent and $m_p \geq s_p$ for each $p \in K_+$. Now, by Proposition III.2.1, we have that

$$D_{A'}^{S_{K_C}} = E_{A'_O}^{S_{K_C}} - E_{\mathcal{T}_0^{\mathbf{m}}}^{S_{K_C}} \geq E_{\mathcal{T}_n^{\mathbf{m}}}^{S_{K_C}} - E_{\mathcal{T}_0^{\mathbf{m}}}^{S_{K_C}}$$

and the equality holds if and only if $A'_O$ goes through $\mathcal{T}_n^{\mathbf{m}}$ with effective multiplicities equal to the virtual ones, or equivalently, if and only if $A_O$ goes through $\mathcal{K}^{\mathbf{m}}_{\mathrm{Cart}(C)}$ with effective multiplicities equal to the virtual ones. By definition of the cluster $K_C$, $\widetilde{C}^{S_{K_C}}$ intersects transversally the exceptional divisor of $f_C$. Therefore, the strict transform $\widetilde{A'}^{S_{K_C}}$ of $A'$ on $S_{K_C}$ intersects transversally the exceptional divisor of $f_C$ if and only if for each $p \in K \setminus \mathcal{K}_+$, $A'_O$ has $\rho_p^{\mathcal{T}_n^{\mathbf{m}}}$ branches through $p$ and misses all points after $p$ in $K$. Therefore, we see that $A'$ is a $v$-minimal Cartier divisor containing $\widetilde{C}$ if and only if $A_O$ goes through $\mathcal{K}^{\mathbf{m}}_{\mathrm{Cart}(C)}$ with effective multiplicities equal to the virtual ones and has $\rho_p^{\mathcal{T}_n^{\mathbf{m}}}$ branches through $p$ and misses all points after $p$ in $K$. This completes the proof of (a). Moreover, if $A'$ is $v$-minimal, then $D_{A'}^{S_{K_C}} = E_{\mathcal{T}_n^{\mathbf{m}}}^{S_{K_C}} - E_{\mathcal{T}_0^{\mathbf{m}}}^{S_{K_C}}$ and in particular, by Definition III.4.1,

$$\overline{D}_{\widetilde{C}}^{S_{K_C}} = E_{\mathcal{T}_n^{\mathbf{m}}}^{S_{K_C}} - E_{\mathcal{T}_0^{\mathbf{m}}}^{S_{K_C}}.$$

This proves (b).

Finally, from the definition of $\mathcal{T}_n^{\mathbf{m}}$, we know that $\omega_p^{\mathbf{m}} = e_p(C)$ for every $p \in K_C \setminus K$. Therefore, $p \notin K$ is a dicritical point of $\mathcal{T}_n^{\mathbf{m}}$ if and only if $p = q_i$ for some $i$. From its own definition, it follows that $\mathcal{K}^{\mathbf{m}}_{\mathrm{Cart}(C)}$ has no dicritical points out of $K$. This gives (c) and completes the proof of the proposition. □

**Example III.4.16.** Take the complete $\mathfrak{m}_O$-primary ideal $I$ and the curve $C$ of Example III.1.13, see Figure III.4. We have already seen that $\widetilde{C}$ is locally principal near $Q_1$ but not near $Q_2$ (see Example III.1.13). In fact, we have that $D_{\widetilde{C}}^{Q_1} = E_{p_3}^{S_{K_C}}$ while $D_{\widetilde{C}}^{Q_2} = \frac{1}{6}E_{p_1}^{S_{K_C}} + \frac{1}{3}E_{p_5}^{S_{K_C}} + \frac{2}{3}E_{p_6}^{S_{K_C}} + \frac{5}{6}E_{p_7}^{S_{K_C}}$, so $\overline{D}_{\widetilde{C}}^{S_{K_C}} = D_{\widetilde{C}}^{Q_1} + \overline{D}_{\widetilde{C}}^{Q_2}$. In Figure III.11, we represent $\mathcal{T}_0^{\mathbf{m}}$ and the cluster $\mathcal{T}_n^{\mathbf{m}}$ obtained by applying the algorithm described in this section. The excesses of $\mathcal{T}_n^{\mathbf{m}}$ at $p_2, p_4$ and $p_8$ are

$$\rho_{p_2}^{\mathcal{T}_n^{\mathbf{m}}} = m_2 - 2$$

$$\rho_{p_4}^{\mathcal{T}_n^{\mathbf{m}}} = m_4 - 1$$

$$\rho_{p_8}^{\mathcal{T}_n^{\mathbf{m}}} = m_8 - 2$$

and so, $s_2 = 2$, $s_4 = 1$ and $s_8 = 2$. Figure III.12 represents the clusters $\mathcal{T}_n^{\mathbf{s}}$ and $\mathcal{K}^{\mathbf{s}}_{Cart(C)}$. The virtual values of $\mathcal{T}_n^{\mathbf{s}}$ and $\mathcal{T}_0^{\mathbf{s}}$ are

$$\mathbf{v}_{\mathcal{T}_n^{\mathbf{s}}} = \{8, 10, 12, 12, 10, 20, 22, 22\}$$

$$\mathbf{v}_{\mathcal{T}_0^{\mathbf{s}}} = \{7, 10, 11, 12, 9, 18, 20, 22\}$$

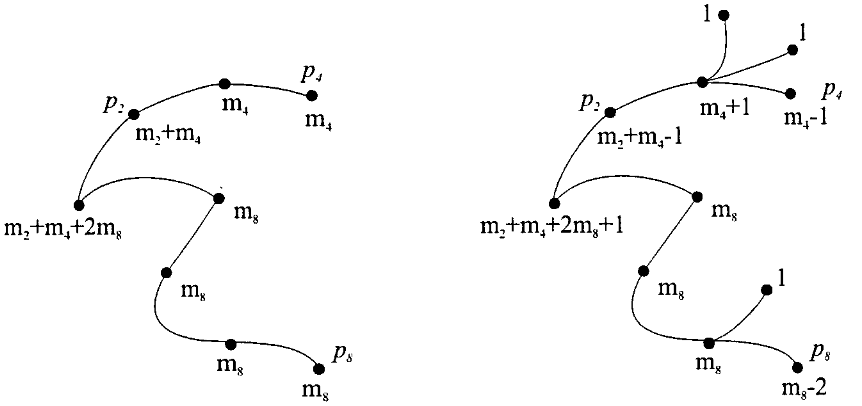



Figure III.11: On the left, we represent the Enriques diagram of the cluster $\mathcal{T}_0^{\mathbf{m}}$ corresponding to Example III.4.16; on the right, the Enriques diagram of $\mathcal{T}_n^{\mathbf{m}}$.

respectively. Hence, by (b) of Proposition III.4.15, we have that

$$\begin{aligned}\overline{D}_{\widetilde{C}}^{S_{K_C}} &= E_{\mathcal{T}_n^{\mathrm{s}}}^{S_{K_C}} - E_{\mathcal{T}_0^{\mathrm{s}}}^{S_{K_C}} = \\ &= E_{p_3}^{S_{K_C}} + E_{p_1}^{S_{K_C}} + E_{p_5}^{S_{K_C}} + 2E_{p_6}^{S_{K_C}} + E_{p_7}^{S_{K_C}}.\end{aligned}$$

**Example III.4.17.** Take the sandwiched singularity $(X, Q)$ and the curve $C$ of Example III.1.12 (see Figure III.1). By applying the algorithm seen in this section we get the cluster $\mathcal{T}_n^{\mathrm{s}}$ shown on the left of Figure III.14.

By Proposition III.4.15, the $v$-minimal Cartier divisors on $X$ containing $\widetilde{C}$ are the curves $\widetilde{C} + \widetilde{C_O}$, where $C_O$ is a curve going sharply through the cluster $\mathcal{K}_{\mathrm{Cart}(C)}^{\mathrm{s}}$. The Enriques diagram of $\mathcal{K}_{\mathrm{Cart}(C)}^{\mathrm{s}}$ is shown on the right of Figure III.14.

We have that the virtual multiplicities of $\mathcal{T}_0^{\mathrm{s}}$ are

$$\nu_{\mathcal{T}_0^{\mathrm{s}}} = \{16, 11, 2, 2, 5, 3, 1, 1, 1, 2, 2, 2\}.$$

and the virtual values of $\mathcal{T}_n^{\mathrm{s}}$ and $\mathcal{T}_0^{\mathrm{s}}$ are, respectively

$$\mathrm{v}_{\mathcal{T}_n^{\mathrm{s}}} = \{17, 28, 31, 31, 50, 80, 108, 108, 108, 132, 132, 132\}$$

$$\mathrm{v}_{\mathcal{T}_0^{\mathrm{s}}} = \{16, 27, 29, 31, 48, 78, 106, 107, 108, 128, 130, 132\}$$

and again, by (b) of Proposition III.4.15,

$$\begin{aligned}\overline{D}_{\widetilde{C}}^{S_{K_C}} &= E_{\mathcal{T}_n^{\mathrm{s}}}^{S_{K_C}} - E_{\mathcal{T}_0^{\mathrm{s}}}^{S_{K_C}} = \\ &= E_{p_1}^{S_{K_C}} + E_{p_2}^{S_{K_C}} + E_{p_3}^{S_{K_C}} + 2E_{p_5}^{S_{K_C}} + 2E_{p_6}^{S_{K_C}} + \\ &+ 2E_{p_7}^{S_{K_C}} + E_{p_8}^{S_{K_C}} + 4E_{p_{10}}^{S_{K_C}} + 2E_{p_{11}}^{S_{K_C}}.\end{aligned}$$

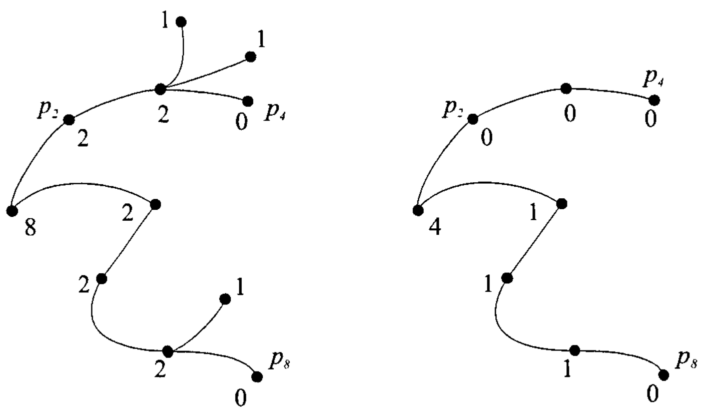


Figure III.12: On the left, the Enriques diagram of $\mathcal{T}_n^{\mathrm{s}}$ of Example III.4.16; on the right, the Enriques diagram of $\mathcal{K}^{\mathrm{s}}_{Cart(C)}$.

## III.5 The order of singularity of curves through a sandwiched singularity

Our aim in this section is to apply some of the results of the former ones in order to get some formulas for the orders of singularity $\delta$ of curves on $X$ without exceptional support.

Our first result is the following theorem.

**Theorem III.5.1.** *Let $C$ be a curve on $S$ such that $\widetilde{C}$ is a Cartier divisor on $X$. Then,*

$$\delta_O(C) = \delta_X(\widetilde{C}) + \delta_O(\mathcal{K}^o_C),$$

*where $\delta_X(\widetilde{C}) = \sum_{Q \in X} \delta_Q(\widetilde{C})$.*

**Remark III.5.2.** Recall from Definition I.1.1 that $R$ is

Theorem III.5.1 says that the difference between the order of singularity of $C$ at $O$ and that of its strict transform $\widetilde{C}$ on $X$ equals the order of singularity at $O$ of a generic curve going through $\mathcal{K}^o_C$. In other words, if $f = 0$ is an equation of $C$ at $O$, $R_f = \frac{R}{(f)}$ and $\{Q_1, \ldots, Q_n\}$ are the points where $\widetilde{C}$ intersects the exceptional locus of $X$, then

$$\dim_{\mathbb{C}} \frac{R^I_f}{R_f} = \delta_O(\mathcal{K}^o_C),$$

where

$$R^I_f = \prod_{i=1}^{n} \frac{\mathcal{O}_{X,Q_i}}{J_{\widetilde{C},Q_i}}$$

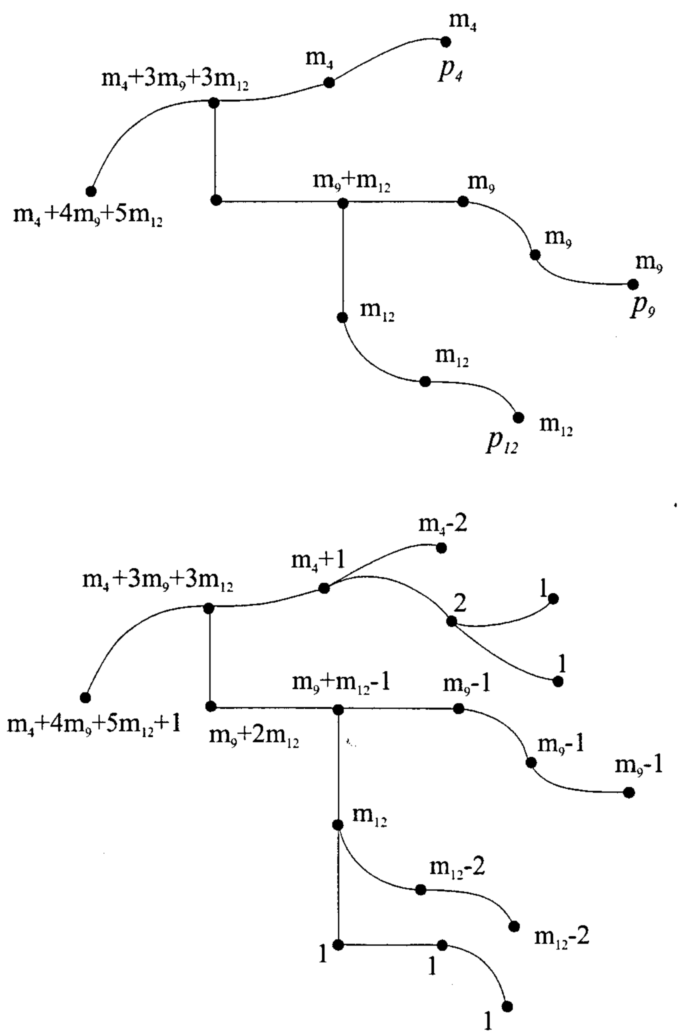



Figure III.13: On the top, we represent the Enriques diagram of the cluster $\mathcal{T}_0^{\mathbf{m}}$ corresponding to Example III.4.17; on the bottom, the Enriques diagram of $\mathcal{T}_n^{\mathbf{m}}$.

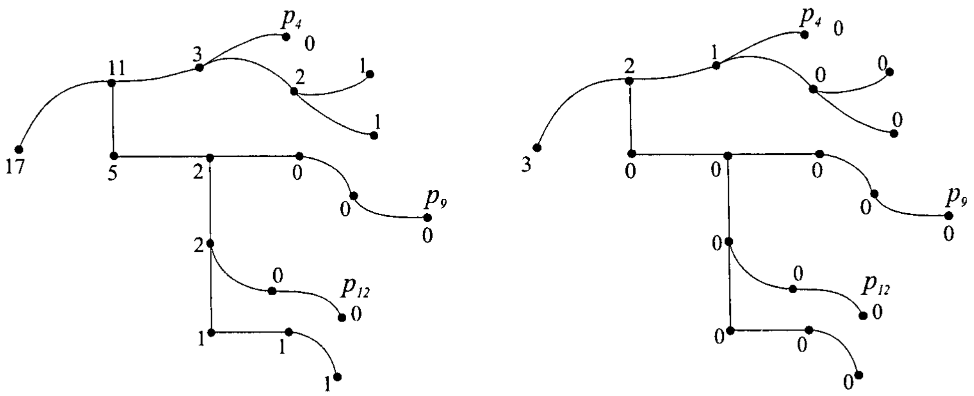



Figure III.14: On the left, Enriques diagram of the cluster $\mathcal{T}_n^{\mathrm{s}}$ obtained by means of the algorithm described in section III.4 and corresponding to Example III.4.17; on the right, the Enriques diagram of $\mathcal{K}^{\mathrm{s}}_{\mathrm{Cart}(C)}$.

and for each $i \in \{1, \ldots, n\}$, $J_{\widetilde{C},Q_i} \subset \mathcal{O}_{X,Q_i}$ is the principal ideal defining $\widetilde{C}$ near $Q_i$.

**Remark III.5.3.** If $I = m_O$, $R_f^I$ is the ring of $C$ in the first neighbourhood of $O$ introduced by Northcott (see [50, 51]). In this case, Theorem III.5.1 and Theorem I.2.18 say that

$$\delta_X(\widetilde{C}) - \delta_O(C) = \frac{e_O(C)(e_O(C)-1)}{2}$$

which is known to be the variation in the order of singularity of $C$ after blowing up $O$.

To prove Theorem III.5.1, we need a technical lemma.

**Lemma III.5.4.** *Let $T$ be a cluster and take $\tau$ a system of virtual multiplicities for $T$ so that $\mathcal{T} = (T, \tau)$ is strictly consistent. Denote by $\mathcal{T}_{adj}$ the cluster obtained by taking $\tau_p - 1$ as the virtual multiplicity for $p \in T$. If $C$ and $C_{adj}$ are curves going sharply through $\mathcal{T}$ and $\mathcal{T}_{adj}$, respectively, and sharing no points out of $T$, then $\widetilde{C_{adj}}^{S_T} - \widetilde{C}^{S_T}$ is a canonical divisor of $S_T$.*

*Proof.* First of all, if $\mathrm{K}_{S_T}$ is a canonical divisor on $S_T$, the adjunction formula (see [29] V.1.5) says that

$$|\mathrm{K}_{S_T} \cdot E_p^{S_T}|_{S_T} = -2 - (E_p^{S_T})^2$$

for each $p \in T$. On the other hand, recall from section I.3 that if $\mathbb{E}^{S_T} = \bigoplus_{p\in T} \mathbb{Z}E_p^{S_T}$, the map

$$\theta : Pic(S_T) \rightarrow Hom(\mathbb{E}^{S_T}, \mathbb{Z})$$

defined by $\theta(D)(E_p^{S_T}) = [D\cdot E_p^{S_T}]_{S_T}$ is an isomorphism of groups. Therefore, it is enough to show that

$$[(\widetilde{C_{adj}}^{S_T} - \widetilde{C}^{S_T})\cdot E_p^{S_T}]_{S_T} = -2 - (E_p^{S_T})^2$$

for every $p \in T$. By *4.* of Proposition I.1.16, we have

$$\begin{aligned} [(\widetilde{C_{adj}}^{S_T} - \widetilde{C}^{S_T})\cdot E_p^{S_T}]_{S_T} &= (e_p(C_{adj}) - e_p(C)) - \sum_{q\rightarrow p, q\in T} (e_q(C_{adj}) - e_q(C)) = \\ &= (\tau_p - 1 - \tau_p) - \sum_{q\rightarrow p, q\in T} (\tau_q - 1 - \tau_q) = \\ &= -1 + \sharp\{q \in T \mid q \rightarrow p\} = -2 - (E_p^{S_T})^2, \end{aligned}$$

the last equality by Remark I.5.1. □

Write $\mathrm{K}_{S_{K_C}}$ for the canonical divisor of $S_{K_C}$. Then, using Lemma III.5.4, we can prove now Theorem III.5.1.

*Proof of Theorem III.5.1* By Corollary 2.1.2. of [44], we have the following formula for the order of singularity of $\widetilde{C}$ at some point $Q$:

$$\begin{aligned} \delta_Q(\widetilde{C}) &= \frac{1}{2}[D_{\widetilde{C}}^Q\cdot(\mathrm{K}_{S_{K_C}} - D_{\widetilde{C}}^Q)]_{S_{K_C}} + \frac{1}{2}[(D_{\widetilde{C}}^Q - \overline{D}_{\widetilde{C}}^Q)\cdot(D_{\widetilde{C}}^Q - \overline{D}_{\widetilde{C}}^Q - \mathrm{K}_{S_{K_C}})]_{S_{K_C}} = \\ &= \frac{1}{2}[D_{\widetilde{C}}^Q\cdot\overline{D}_{\widetilde{C}}^Q]_{S_{K_C}} - \frac{1}{2}[\overline{D}_{\widetilde{C}}^Q\cdot(D_{\widetilde{C}}^Q - \overline{D}_{\widetilde{C}}^Q - \mathrm{K}_{S_{K_C}})]_{S_{K_C}} = \\ &= \frac{1}{2}[\overline{D}_{\widetilde{C}}^Q\cdot(\overline{D}_{\widetilde{C}}^Q - 2D_{\widetilde{C}}^Q + \mathrm{K}_{S_{K_C}})]_{S_{K_C}} \end{aligned} \qquad (5.a)$$

By the assumption, $\widetilde{C}$ is locally principal near each $Q_i$ and by Remark III.4.3, $\overline{D}_{\widetilde{C}}^Q = D_{\widetilde{C}}^Q$. Therefore,

$$\delta_Q(\widetilde{C}) = \frac{1}{2}[D_{\widetilde{C}}^Q\cdot(-D_{\widetilde{C}}^Q + \mathrm{K}_{S_{K_C}})]_{S_{K_C}} = \frac{1}{2}[D_{\widetilde{C}}^Q\cdot(-D_{\widetilde{C}}^{S_{K_C}} + \mathrm{K}_{S_{K_C}})]_{S_{K_C}}, \qquad (5.b)$$

the last equality because the divisors $D_{\widetilde{C}}^Q$ are the connected components of $D_{\widetilde{C}}^{S_{K_C}}$.

Now, let $C_O$ be a curve going sharply through $\mathcal{K}$ and sharing no points with $C$ out of $K$ (Theorem I.1.30). Then, the curve $A = C + C_O$ goes sharply through the (strictly) consistent cluster $\Delta = \mathcal{K}_C + \mathcal{K}$. Moreover, since $\widetilde{C_O}$ shares no points with $\widetilde{C}$ and goes through no singularities of $X$, its total transform by $f_C$ equals the strict transform $\widetilde{C_O}^{S_{K_C}}$ and so,

$$f_C^*(\widetilde{A}) = \widetilde{C_O}^{S_{K_C}} + \widetilde{C}^{S_{K_C}} + D_{\widetilde{C}}^{S_{K_C}}.$$

Therefore, by the projection formula applied to $f_C$, we deduce that

$$|D_{\widetilde{C}}^Q \cdot (\widetilde{A}^{S_{K_C}} + D_{\widetilde{C}}^{S_{K_C}})|_{S_{K_C}} = 0.$$

Now, if $A_{adj}$ is a curve on $S$ going sharply through $\Delta_{adj}$ and sharing no points with $C$ out of $K_C$, we infer from Lemma III.5.4 and the above equality that

$$\begin{aligned} |D_{\widetilde{C}}^Q \cdot (-D_{\widetilde{C}}^{S_{K_C}} + \mathrm{K}_{S_{K_C}})|_{S_{K_C}} &= |D_{\widetilde{C}}^Q \cdot (-D_{\widetilde{C}}^{S_{K_C}} + \widetilde{A_{adj}}^{S_{K_C}} - \widetilde{A}^{S_{K_C}})|_{S_{K_C}} = \\ &= |D_{\widetilde{C}}^Q \cdot \widetilde{A_{adj}}^{S_{K_C}}|_{S_{K_C}}. \end{aligned}$$

Therefore, by (5.b),

$$\delta_Q(\widetilde{C}) = \frac{1}{2} |D_{\widetilde{C}}^Q \cdot \widetilde{A_{adj}}^{S_{K_C}}|_{S_{K_C}},$$

and summing-up for the points on $X$,

$$\sum_{Q \in X} \delta_Q(\widetilde{C}) = \frac{1}{2} |D_{\widetilde{C}}^{S_{K_C}} \cdot \widetilde{A_{adj}}^{S_{K_C}}|_{S_{K_C}}. \tag{5.c}$$

Now, by Proposition III.2.1, we have $D_{\widetilde{C}}^{S_{K_C}} = E_C^{S_{K_C}} - E_{\mathcal{K}_C^o}^{S_{K_C}}$. By the projection formula for $\pi_{K_C}$ (*3.* of Proposition I.1.16) and the assumption on the curve $A_{adj}$, we have

$$|E_C^{S_{K_C}} \cdot \widetilde{A_{adj}}^{S_{K_C}}|_{S_{K_C}} = [C, A_{adj}]_O.$$

Similarly,

$$|E_{\mathcal{K}_C^o}^{S_{K_C}} \cdot \widetilde{A_{adj}}^{S_{K_C}}|_{S_{K_C}} = [\mathcal{K}_C^o, A_{adj}]_O,$$

since the strict transform on $S_{K_C}$ of a generic curve going through $\mathcal{K}_C^o$ shares no points with $\widetilde{A_{adj}}^{S_{K_C}}$. From this and (5.c) above,

$$\sum_{Q \in X} \delta_Q(\widetilde{C}) = \frac{1}{2}([C, A_{adj}]_O - [\mathcal{K}_C^o, A_{adj}]_O) =$$

and by using the Noether's formula,

$$\begin{aligned} &= \frac{1}{2} \sum_{p \in K_C} e_p(C)(e_p(C) + \nu_p - 1) - \sum_{p \in K_C} \tau_p^o(e_p(C) + \nu_p - 1) = \\ &= \frac{1}{2} \sum_{p \in K_C} e_p(C)(e_p(C) - 1) + \frac{1}{2} \sum_{p \in K_C} e_p(C)\nu_p - \\ &- \frac{1}{2} \sum_{p \in K_C} \tau_p^{C,o} \nu_p - \frac{1}{2} \sum_{p \in K_C} \tau_p^{C,o}(e_p(C) - 1). \end{aligned} \tag{5.d}$$

Finally, in virtue of Lemma III.2.2, we have

$$\sum_p e_p(C)\nu_p \quad = \quad \sum_p \tau_p^{C,o}\nu_p, \tag{5.e}$$

and also, $\sum_p e_p(C)\tau_p^{C,o} = \sum_p (\tau_p^{C,o})^2$. From (5.d) and using Theorem I.2.18, we infer

$$\sum_{Q\in X} \delta_Q(\widetilde{C}) = \delta_O(C) + \delta_O(\mathcal{K}_C^o)$$

as claimed. □

From Theorem III.5.1 we deduce the following formula for the order of singularity of a Cartier divisor on $X$.

**Corollary III.5.5.** *Let $C \subset S$ be a curve such that $\widetilde{C}$ is Cartier on $X$. For every $p \in K_C$, write $n_p = e_p(C) - \tau_p^{C,o}$. Then*

$$\delta_X(\widetilde{C}) = \sum_{p\in K_C} \frac{n_p(n_p-1)}{2}.$$

*Proof.* By Theorem III.5.1 and Theorem I.2.18, we have

$$\delta_X(\widetilde{C}) \quad = \quad \delta_O(C) - \delta_O(\mathcal{K}_C^o) = \sum_{p\in K_C} \frac{e_p(C)(e_p(C)-1)}{2} - \sum_{p\in K_C} \frac{\tau_p^{C,o}(\tau_p^{C,o}-1)}{2}.$$

Now, by (5.e) above, we have $\sum_{p\in\mathcal{N}_O} n_p\tau_p^{C,o} = 0$. Then, a direct computation shows that

$$\delta_X(\widetilde{C}) = \sum_p \frac{n_p(n_p-1)}{2}.$$

□

**Remark III.5.6.** In virtue of Corollary III.5.5, we can write

$$\delta_X(\widetilde{C}) = \delta_O(\mathcal{K}_C - \mathcal{K}_C^o),$$

where $\mathcal{K}_C - \mathcal{K}_C^o = (K_C, e(C) - \tau^{C,o})$ is, in general, a non-consistent cluster.

With the notation of section II.3, we obtain also the following corollary.

**Corollary III.5.7.** *If $\check{C}$ is a transverse hypersurface section of $(X,Q)$, then*

$$\delta_Q(\check{C}) = \delta_O(\mathcal{K}_Q) - \delta_O(\mathcal{K}) = \sharp B_Q^{\mathcal{K}},$$

*where $\mathcal{K}_Q$ is the cluster of base points of the ideal $I_Q \subset I$ corresponding to $Q$ by Theorem II.1.7.*

*Proof.* The first equality is derived from Theorem III.5.1 applied to a curve $C$ going sharply through $\mathcal{K}_Q$ (see Proposition II.4.2 and Corollary III.1.7). Denote by $\nu'_p$ the virtual multiplicity of $\mathcal{K}_Q$ at $p$. If $Q$ is singular, by Lemma II.3.1, we have

$$n_p = \nu'_p - \nu_p = \begin{cases} 1 & \text{if } p = O_Q \\ -1 & \text{if } p \in B_Q \\ 0 & \text{otherwise.} \end{cases}$$

(see Definition II.3.3) and in virtue of Corollary III.5.5, we obtain

$$\delta_Q(\widetilde{C}) = \sum_p \frac{n_p(n_p-1)}{2} = \sharp B_Q^{\mathcal{K}}.$$

If $Q$ is regular, $\check{C}$ is not singular and hence, $\delta_Q(\check{C}) = 0$. In this case, $B_Q^{\mathcal{K}} = \emptyset$ by definition, so the second equality also holds in this case. □

Next, we deal with the case where the strict transform $\widetilde{C}$ is not necessarily a Cartier divisor. The following proposition gives a formula for the order of singularity of $\widetilde{C}$ on $X$ in this case.

**Proposition III.5.8.** *Let $C$ be a curve on $S$ and keep the notation as in section III.3. Then,*

$$\delta_X(\widetilde{C}) = [\mathcal{T}_n^s, C]_O - [\mathcal{T}_0^s, C]_O - \dim_{\mathbb{C}}(\frac{H_{\mathcal{T}_0^s}}{H_{\mathcal{T}_n^s}}).$$

**Remark III.5.9.** Notice that if $\widetilde{C}$ is Cartier, by Remark III.4.14, we have

$$[\mathcal{T}_n^s, C]_O = [\mathcal{K}_C, C]_O = \mathcal{K}_C^2$$

and similarly,

$$[\mathcal{T}_0^s, C]_O = [\mathcal{K}_C^o, C]_O = (\mathcal{K}_C^o)^2.$$

In this case, and using Lemma III.2.2, Proposition III.5.8 says that

$$\begin{aligned} \delta_X(\widetilde{C}) &= \mathcal{K}_C^2 - (\mathcal{K}_C^o)^2 - (\dim_{\mathbb{C}}(\frac{R}{H_{\mathcal{K}_C}}) - \dim_{\mathbb{C}}(\frac{R}{H_{\mathcal{K}_C^o}})) = \\ &= \delta_O(C) - \delta_O(\mathcal{K}_C^o) \end{aligned}$$

the last equality by Remark I.2.20. Therefore, we see that the formula in Proposition III.5.8 can be understood as a generalization of the formula given in Theorem III.5.1.

*Proof.* From (5.a) above and the fact that the divisors $D^{Q}_{\widetilde{C}}$ and $D^{Q'}_{\widetilde{C}}$ are disjoint if $Q \neq Q'$, we have

$$\begin{aligned}\delta_Q(\widetilde{C}) &= \frac{1}{2}|\overline{D}^{S_{K_C}}_{\widetilde{C}Q} \cdot (\overline{D}^{S_{K_C}}_{\widetilde{C}Q} - 2D^{Q}_{\widetilde{C}} + \mathrm{K}_{S_{K_C}})|_{S_{K_C}} = \\ &= \frac{1}{2}|\overline{D}^{S_{K_C}}_{\widetilde{C}Q} \cdot (\overline{D}^{S_{K_C}}_{\widetilde{C}} - 2D^{S_{K_C}}_{\widetilde{C}} + \mathrm{K}_{S_{K_C}})|_{S_{K_C}}.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{Q\in X} \delta_Q(\widetilde{C}) &= \frac{1}{2}|\overline{D}^{S_{K_C}}_{\widetilde{C}} \cdot (\overline{D}^{S_{K_C}}_{\widetilde{C}} - 2\overline{D}^{S_{K_C}}_{\widetilde{C}} + \mathrm{K}_{S_{K_C}})|_{S_{K_C}} = \\ &= \frac{1}{2}|\overline{D}^{S_{K_C}}_{\widetilde{C}} \cdot (\overline{D}^{S_{K_C}}_{\widetilde{C}} + \mathrm{K}_{S_{K_C}})|_{S_{K_C}} - |\overline{D}^{S_{K_C}}_{\widetilde{C}} \cdot D^{S_{K_C}}_{\widetilde{C}}|_{S_{K_C}}.\end{aligned} \tag{5.f}$$

Now, let $C'$ be a curve going sharply through $\mathcal{T}^{\mathrm{s}}_n$. Then, by (b) of Proposition III.4.15, $D_{\widetilde{C'}} = \overline{D_{\widetilde{C}}}$. Therefore, by the projection formula applied to $f_C$,

$$|(\widetilde{C'} + \overline{D}^{S_{K_C}}_{\widetilde{C}}) \cdot \overline{D}^{S_{K_C}}_{\widetilde{C}}|_{S_{K_C}} = 0 \tag{5.g}$$

On the other hand, in virtue of (a) of Proposition III.4.15, we have

$$\overline{D}^{S_{K_C}}_{\widetilde{C}} = E^{S_{K_C}}_{\mathcal{T}^{\mathrm{s}}_n} - E^{S_{K_C}}_{\mathcal{T}^{\mathrm{s}}_0} \tag{5.h}$$

Now, if $A$ and $A_{adj}$ as in the proof of Lemma III.5.1, by Lemma III.5.4 and the equalities (5.g) and (5.h) above

$$\begin{aligned}&|\overline{D}^{S_{K_C}}_{\widetilde{C}} \cdot (\overline{D}^{S_{K_C}}_{\widetilde{C}} + \mathrm{K}_{S_{K_C}})|_{S_{K_C}} = \\ =\; &|(E^{S_{K_C}}_{\mathcal{T}^{\mathrm{s}}_n} - E^{S_{K_C}}_{\mathcal{T}^{\mathrm{s}}_0}) \cdot (\overline{D}^{S_{K_C}}_{\widetilde{C}} + \widetilde{A_{adj}}^{S_{K_C}} - \widetilde{A}^{S_{K_C}})|_{S_{K_C}} = \\ =\; &|E^{S_{K_C}}_{\mathcal{T}^{\mathrm{s}}_n} \cdot (-\widetilde{C'}^{S_{K_C}} + \widetilde{A_{adj}}^{S_{K_C}} - \widetilde{A}^{S_{K_C}})|_{S_{K_C}} - \\ -\; &|E^{S_{K_C}}_{\mathcal{T}^{\mathrm{s}}_0} \cdot (-\widetilde{C'}^{S_{K_C}} + \widetilde{A_{adj}}^{S_{K_C}} - \widetilde{A}^{S_{K_C}})|_{S_{K_C}}.\end{aligned} \tag{5.i}$$

Now, since a generic curve through $\mathcal{T}^{\mathrm{s}}_n$ (resp. through $\mathcal{T}^{\mathrm{s}}_0$) goes sharply through it and shares no points outside $K_C$ with $\widetilde{C'}^{S_{K_C}}$, $\widetilde{A_{adj}}^{S_{K_C}}$ or $\widetilde{A}^{S_{K_C}}$ (see Theorem I.1.30), we have

$$|E^{S_{K_C}}_{\mathcal{T}^{\mathrm{s}}_n} \cdot (-\widetilde{C}^{S_{K_C}} + \widetilde{A_{adj}}^{S_{K_C}} - \widetilde{A}^{S_{K_C}})|_{S_{K_C}} = [\mathcal{T}^{\mathrm{s}}_n, (-C' + A_{adj} - A)]_O$$

and similarly,

$$|E^{S_{K_C}}_{\mathcal{T}^{\mathrm{s}}_0} \cdot (-\widetilde{C}^{S_{K_C}} + \widetilde{A_{adj}}^{S_{K_C}} - \widetilde{A}^{S_{K_C}})|_{S_{K_C}} = [\mathcal{T}^{\mathrm{s}}_0, (-C' + A_{adj} - A)]_O.$$

It is clear that for each $p \in K_C$, $e_p(A_{adj}) - e_p(A) = -1$. Therefore, from (5.i) and using Noether's formula (see Theorem I.1.31), we infer that

$$\begin{aligned} |\overline{D}_{\widetilde{C}}^{S_{K_C}} \cdot (\overline{D}_{\widetilde{C}}^{S_{K_C}} + K_{S_C})|_{S_{K_C}} &= \sum_p \omega_p^{C,o}(\omega_p^C - 1) - \sum_p \omega_p^C(\omega_p^C - 1) = \\ &= \sum_p \omega_p^{C,o}(\omega_p^{C,o} - 1) - \sum_p \omega_p^C(\omega_p^C - 1) \end{aligned} \tag{5.j}$$

the last equality in virtue of Lemma III.2.2. Hence, by Proposition I.2.21,

$$\begin{aligned} \frac{1}{2}|\overline{D}_{\widetilde{C}}^{S_{K_C}} \cdot (\overline{D}_{\widetilde{C}}^{S_{K_C}} + \mathrm{K}_{S_{K_C}})|_{S_{K_C}} &= \frac{1}{2}\sum_p \omega_p^{C,o}(\omega_p^{C,o} - 1) - \frac{1}{2}\sum_p \omega_p^C(\omega_p^C - 1) = \\ &= \dim_{\mathbb{C}}(\frac{H_{\mathcal{T}_0^{\mathrm{s}}}}{H_{\mathcal{T}_n^{\mathrm{s}}}}). \end{aligned}$$

On the other hand, since $D_{\widetilde{C}'}^{S_{K_C}} = \overline{D}_{\widetilde{C}}^{S_{K_C}}$, the projection formula for $f_C$ says that

$$|\overline{D}_{\widetilde{C}}^{S_{K_C}} \cdot D_{\widetilde{C}}^{S_{K_C}}|_{S_{K_C}} = -|\overline{D}_{\widetilde{C}}^{S_{K_C}} \cdot \widetilde{C}^{S_{K_C}}|_{S_{K_C}} =$$

and using (5.h) above, we infer that

$$= [\mathcal{T}_0^{\mathrm{s}}, C]_O - [\mathcal{T}_n^{\mathrm{s}}, C]_O.$$

Finally, the claim follows from (5.f). □

Before going further we discuss an example.

**Example III.5.10.** Take the cluster and the curve $C$ of Example III.1.12 (see Figure III.1 and III.2). We have already seen that $\widetilde{C}$ is not a Cartier divisor and in Example III.4.17, we have computed the clusters $\mathcal{T}_n^{\mathrm{s}}$ and $\mathcal{T}_0^{\mathrm{s}}$. From this, we have that

$$[\mathcal{T}_n^{\mathrm{s}}, C]_O = 385$$

and

$$[\mathcal{T}_0^{\mathrm{s}}, C]_O = 362.$$

On the other hand,

$$\dim_{\mathbb{C}}(\frac{H_{\mathcal{T}_0^{\mathrm{s}}}}{H_{\mathcal{T}_n^{\mathrm{s}}}}) = \sum_{p \in K_C} \frac{\omega_p^C(\omega_p^C - 1)}{2} - \sum_{p \in K_C} \frac{\omega_p^{C,o}(\omega_p^{C,o} - 1)}{2} = 254 - 241 = 13.$$

Therefore, by Proposition III.5.8, we obtain

$$\delta_X(\widetilde{C}) = \sum_{Q \in X} \delta_Q(\widetilde{C}) = 385 - 362 - 13 = 10$$

Finally, if $C'$ goes sharply through $\mathcal{T}_n^{\mathrm{s}}$, then $\widetilde{C'}^{S_{K_C}}$ is Cartier and by Theorem III.5.1, we have also

$$\delta_X(\widetilde{C'}) = \sum_{Q \in X} \delta_Q(\widetilde{C'}) = \delta_O(C') - \delta_O(\mathcal{T}_0^{\mathrm{s}}) = 207 - 193 = 14.$$

## III.5.1 The semigroup of a branch going through a sandwiched singularity

Here we want to derive from results already seen some facts concerning the semigroup of a (non-exceptional) branch with origin at some point in the exceptional locus of $X$.

Let $C$ be a branch on $S$ and let $Q \in X$ be the point where $\widetilde{C}$ intersects the exceptional divisor. Then, since $(\widetilde{C}, Q)$ is analytically irreducible, $\overline{\mathcal{O}_{\widetilde{C},Q}} \cong \mathbb{C}\{t\}$ is a discrete valuation ring. We can take the semigroup $\Sigma_Q(\widetilde{C})$ of $\widetilde{C}$ at $Q$ defined as

$$\Sigma_Q(\widetilde{C}) = \{v_t(g) \mid g \in \mathcal{O}_{\widetilde{C},Q}\}$$

where $v_t$ is the valuation corresponding to $\overline{\mathcal{O}_{\widetilde{C},Q}}$ (see appendix of [64] for details). It is known that $\delta_Q(\widetilde{C})$ measures the number of elements in $\overline{\mathbb{N}} \setminus \Sigma_Q(\widetilde{C})$, where $\overline{\mathbb{N}} = \mathbb{N} \cup \{0\}$ (see Lemma 2.11.1 of [64]). Take a flag of clusters with ends $\mathcal{T}_0^{\mathrm{s}}$ and $\mathcal{T}_n^{\mathrm{s}}$ as in (4.h):

$$\mathcal{T}_0 = \mathcal{T}_0^{\mathrm{s}} \prec \ldots \prec \mathcal{T}_i^{\mathrm{s}} \prec \ldots \prec \mathcal{T}_n = \mathcal{T}_n^{\mathrm{s}}. \tag{5.k}$$

The following proposition relates the semigroup $\Sigma_Q(\widetilde{C})$ of the branch $\widetilde{C}$ at $Q$ to the values $[\mathcal{T}_i^{\mathrm{s}}, C]_O$, for $i \in \{0, \ldots, n\}$.

**Proposition III.5.11.** *For each $i \in \{0, \ldots, n\}$,*

$$[\mathcal{T}_i^{s}, C]_O - [\mathcal{T}_0^{s}, C]_O \in \Sigma_Q(\widetilde{C}).$$

*Moreover, if $j \geq [\mathcal{T}_n^{s}, C]_O - [\mathcal{T}_0^{s}, C]_O$, then $j \in \Sigma_Q(\widetilde{C})$. In particular, if $c$ is the conductor of $\Sigma_Q(\widetilde{C})$, then*

$$c \leq [\mathcal{T}_n^{s}, C]_O - [\mathcal{T}_0^{s}, C]_O.$$

*Proof.* We already know that if $C_i$ is a generic curve going through $\mathcal{T}_i^{\mathrm{s}}$, then

$$L_{C_i} \in \bigoplus_{u \in \mathcal{K}_+} \mathbb{Z}\mathcal{L}_u$$

and so, the strict transform $\widetilde{C_i}$ on $X$ is a Cartier divisor. Let $g_i \in \mathfrak{m}_Q$ be a local equation for $\widetilde{C_i}$ near $Q$, and $\overline{g_i}$ its class in $\mathcal{O}_{\widetilde{C},Q}$. Then, $|\widetilde{C_i}{\cdot}\widetilde{C}|_X = [\widetilde{C_i}, \widetilde{C}]_Q = v_t(\overline{g_i})$ and by Corollary III.2.3,

$$[C_i, C]_O - [\mathcal{T}_0^{\mathrm{s}}, C]_O = |\widetilde{C_i}{\cdot}\widetilde{C}|_X \in \Sigma_Q(\widetilde{C}).$$

Finally, if $j \geq [\mathcal{T}_n^{\mathrm{s}}, C]_O - [\mathcal{T}_0^{\mathrm{s}}, C]_O$, take a curve $C'$ going sharply through $\mathcal{T}_n^{\mathrm{s}}$ and sharing exactly $j + [\mathcal{T}_0^{\mathrm{s}}, C]_O - [\mathcal{T}_n^{\mathrm{s}}, C]_O$ points with $C$ outside $K_C$. Then, $L_{C'} = L_{\mathcal{T}_n^{\mathrm{s}}} \in \oplus_{p \in \mathcal{K}_+} \mathbb{Z}\mathcal{L}_p$ and by Theorem III.1.1, $\widetilde{C'}$ is Cartier. On the other hand, by Noether's formula,

$$\begin{aligned} [C', C]_O &= \sum_{p \in K_C} e_p(C') e_p(C) + (j + [\mathcal{T}_0^{\mathrm{s}}, C]_O - [\mathcal{T}_n^{\mathrm{s}}, C]_O) = \\ &= j + [\mathcal{T}_0^{\mathrm{s}}, C]_O, \end{aligned}$$

the last equality since $C'$ goes sharply through $\mathcal{T}_n^{\mathrm{s}}$. The argument used above shows that $j = [C', C]_O - [\mathcal{T}_0^{\mathrm{s}}, C]_O \in \Sigma_Q(\widetilde{C})$ and the claim follows.

□

From Proposition III.5.11 we detail that any element of $\Sigma_Q(\widetilde{C})$ has the form $[C', C]_O - [\mathcal{T}_0^{\mathrm{s}}, C]_O$ for some curve $C'$ going sharply through some $\mathcal{T}_i^{\mathrm{s}}$, $i \in \{0, \ldots, n\}$. Note that although the flag of (5.k) is not unique, we infer that the integers $[\mathcal{T}_i^{\mathrm{s}}, C]_O - [\mathcal{T}_0^{\mathrm{s}}, C]_O$ are completely determined (as they are the first elements of $\Sigma_Q(\widetilde{C})$). It follows also that the gaps in $\Sigma_Q(\widetilde{C})$ are the integers between $[\mathcal{T}_i^{\mathrm{s}}, C]_O - [\mathcal{T}_0^{\mathrm{s}}, C]_O$ and $[\mathcal{T}_{i+1}^{\mathrm{s}}, C]_O - [\mathcal{T}_0^{\mathrm{s}}, C]_O$, for $i \in \{0, \ldots, n-1\}$, and thus, we obtain a complete description of the semigroup $\Sigma_Q(\widetilde{C})$ in terms of differences of intersection multiplicities of curves at $O$.

This also provides an easy way to compute the semigroup of $\widetilde{C}$ once the flag (4.h) has been computed: the differences

$$[\mathcal{T}_i^{\mathrm{s}}, C]_O - [\mathcal{T}_0^{\mathrm{s}}, C]_O$$

for $i = 0, \ldots, N-1$ are the first elements of $\Sigma_Q(\widetilde{C})$, and the remaining elements of $\Sigma_Q(\widetilde{C})$ are all the integers $j$ such that $j \geq [\mathcal{T}_n^{\mathrm{s}}, C]_O - [\mathcal{T}_0^{\mathrm{s}}, C]_O$.

**Remark III.5.12.** As we will see in the following examples, the bound for the conductor of Proposition III.5.11 is far from being sharp. We will see also that in general the semigroup $\Sigma_Q(\widetilde{C})$ is not symmetric (and therefore, the curve $\widetilde{C}$ needs not to be Gorenstein, see [35]).

In Appendix $A$ we include a program written in **C** which implements the algorithm for the computation of the semigroup of a branch $\widetilde{C}$ on $X$ at the point of intersection with the exceptional locus.

**Example III.5.13.** Take a cluster $\mathcal{K}$ and a curve $C$ as shown in Figure III.15. By means of the algorithm explained in section II.4, we compute the clusters $\mathcal{T}_n^{\mathrm{s}}$ and $\mathcal{T}_0^{\mathrm{s}}$. The virtual multiplicities of $\mathcal{T}_n^{\mathrm{s}}$ and $\mathcal{T}_0^{\mathrm{s}}$ are respectively,

$$\nu^{\mathcal{T}_n^{\mathrm{s}}} = \{12, 1, 0, 8, 1, 1, 0, 0, 0, 5, 2, 2, 1, 1, 1\}$$

$$\nu^{\mathcal{T}_0^{\mathrm{s}}} = \{11, 1, 1, 4, 4, 2, 2, 2, 2\}.$$

The cluster $\mathcal{T}_n^{\mathrm{s}}$ is shown on the right of Figure III.16. Then,

$$[\mathcal{T}_n^{\mathrm{s}}, C]_O = 176$$

and

$$[\mathcal{T}_0^{\mathrm{s}}, C]_O = 105.$$

Also, by using Proposition I.2.21, we have

$$\dim_{\mathbb{C}}(\frac{H_{\mathcal{T}_0^{\mathrm{s}}}}{H_{\mathcal{T}_n^{\mathrm{s}}}}) = \dim_{\mathbb{C}}(\frac{R}{H_{\mathcal{T}_n^{\mathrm{s}}}}) - \dim_{\mathbb{C}}(\frac{R}{H_{\mathcal{T}_0^{\mathrm{s}}}}) = 141 - 100 = 41.$$

Thus, by Proposition III.5.8,

$$\begin{aligned} \delta_Q(\widetilde{C}) &= [\mathcal{T}_n^{\mathrm{s}}, C]_O - [\mathcal{T}_0^{\mathrm{s}}, C]_O - \dim_{\mathbb{C}}(\frac{H_{\mathcal{T}_0^{\mathrm{s}}}}{H_{\mathcal{T}_n^{\mathrm{s}}}}) = \\ &= 176 - 105 - 41 = 30. \end{aligned}$$

In virtue of Proposition III.5.11, we see that

$$\begin{aligned} \Sigma_Q(\widetilde{C}) = \{&0, 7, 12, 14, 19, 21, 24, 26, 28, 31, 33, 35, 36, 38, 40, 41, \ldots \\ &\ldots, 42, 43, 45, 47, 48, 49, 50, 52, 53, 54, 55, 56, 57, 59, 60, \ldots\} \end{aligned}$$

Therefore, the conductor of the semigroup is 59. In this case, the semigroup is symmetric (see Figure III.16) and so, the curve $\widetilde{C}$ is Gorenstein (see [1, 6]; see also [8]).

**Example III.5.14.** Take a cluster $\mathcal{K}$ and a curve $C$ as shown in Figure III.17. Note that this sandwiched singularity is a singularity $\mathbf{A}_2$. As before, we apply the algorithm of section II.4 to compute the clusters $\mathcal{T}_n^{\mathrm{s}}$ and $\mathcal{T}_0^{\mathrm{s}}$. The virtual multiplicities of $\mathcal{T}_n^{\mathrm{s}}$ and $\mathcal{T}_0^{\mathrm{s}}$ are respectively,

$$\nu^{\mathcal{T}_n^{\mathrm{s}}} = \{6, 0, 2, 2, 1, 1, 1, \}$$

$$\nu^{\mathcal{T}_0^{\mathrm{s}}} = \{6, 6\}.$$

The Enriques diagram of $\mathcal{T}_n^{\mathrm{s}}$ is shown on the right of Figure III.17. Then,

$$[\mathcal{T}_n^{\mathrm{s}}, C]_O = 41$$

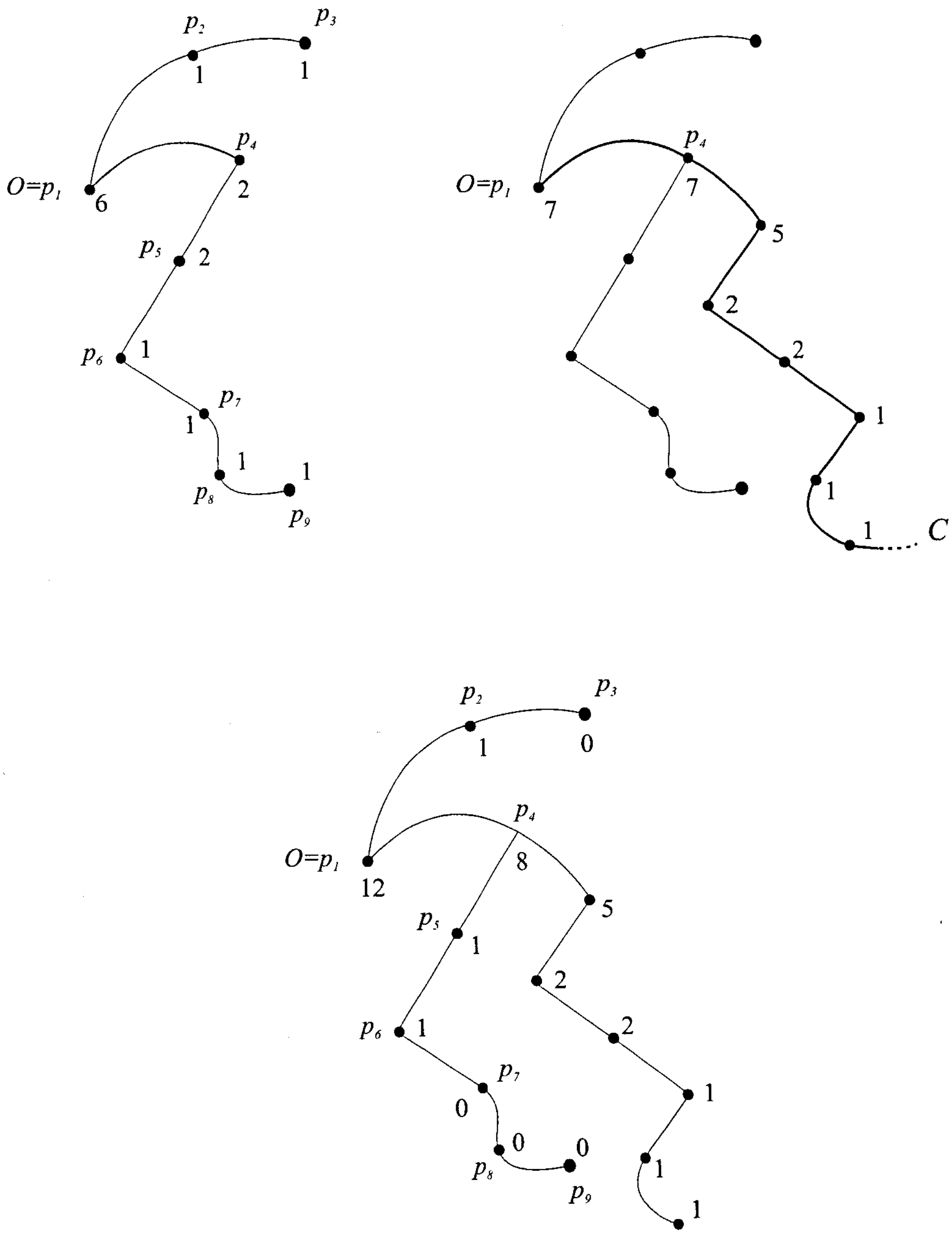



Figure III.15: On the top and left, the Enriques diagram of the cluster $\mathcal{K}$ of Example III.5.13; on the right,the bold edges represent the Enriques diagram of singular points of $C$; on the bottom, the Enriques diagram of the cluster $\mathcal{T}_n^s$ obtained by means of the algorithm described in section III.3.

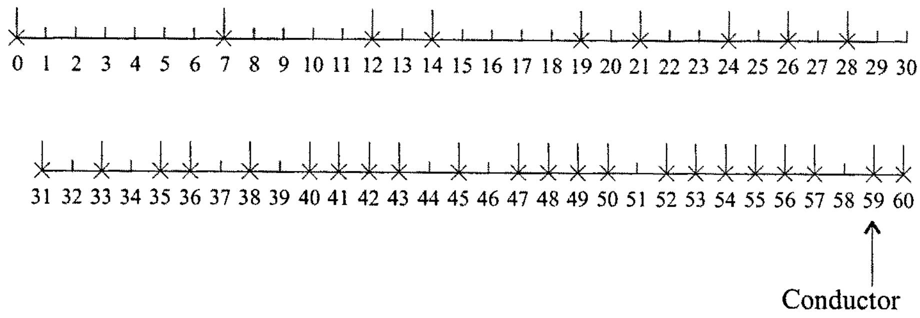



Figure III.16: Semigroup of the branch $\widetilde{C}$ at $Q$ corresponding to Example III.5.13. We mark the elements of $\Sigma_Q(\widetilde{C})$ with a cross.

and

$$[\mathcal{T}_0^{\mathrm{s}}, C]_O = 15.$$

Moreover, we have

$$\dim_{\mathbb{C}}(\frac{H_{\mathcal{T}_0^{\mathrm{s}}}}{H_{\mathcal{T}_n^{\mathrm{s}}}}) = 18.$$

Thus, By Proposition III.5.8,

$$\begin{aligned} \delta_Q(\widetilde{C}) &= [\mathcal{T}_n^{\mathrm{s}}, C]_O - [\mathcal{T}_0^{\mathrm{s}}, C]_O - \dim_{\mathbb{C}}(\frac{H_{\mathcal{T}_0^{\mathrm{s}}}}{H_{\mathcal{T}_n^{\mathrm{s}}}}) = \\ &= 41 - 15 - 18 = 8. \end{aligned}$$

The semigroup of $\widetilde{C}$ at $Q$ is

$$\Sigma_Q(\widetilde{C}) = \{0, 5, 7, 9, 10, 12, 14, 15, 16, 17, \ldots\}$$

and the conductor of the semigroup is 14. In this case, $\Sigma_Q(\widetilde{C})$ is not symmetric as the integers 2 or $14 - 2 - 1 = 11$ do not belong to the semigroup (see Figure III.18). In particular, $\widetilde{C}$ is not Gorenstein.

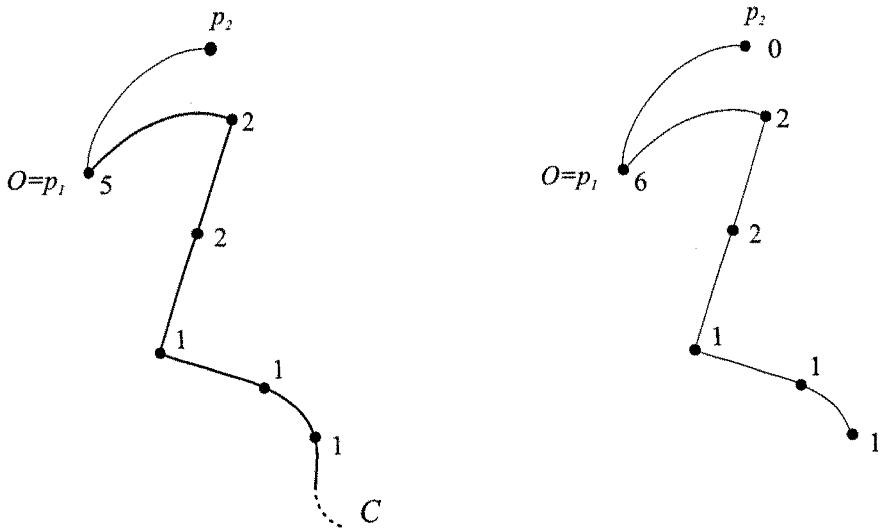



Figure III.17: On the left, we represent the singular points of the branch $C$ and the cluster $\mathcal{K}$ (the bold edges represent the Enriques diagram of singular points of $C$); on the right, the Enriques diagram of the cluster $\mathcal{T}_n^s$ obtained by means of the algorithm described in section III.3.

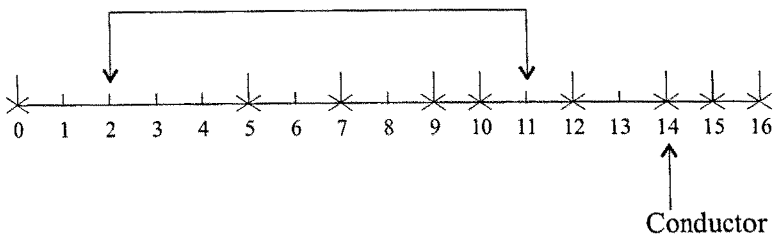



Figure III.18: Semigroup of the branch $\widetilde{C}$ at $Q$ corresponding to Example III.5.14. We mark the elements of $\Sigma_Q(\widetilde{C})$ with a cross. Since both 2 and 11 are not in $\Sigma_Q(\widetilde{C})$, we see that $\Sigma_Q(\widetilde{C})$ is not symmetric.

# Chapter IV

# Complete ideals on sandwiched singularities

Fixed a complete $\mathfrak{m}_O$-primary ideal $I \subset R$, this chapter deals with the relationship between complete ideal sheaves on $X$ with finite cosupport in the exceptional locus of $X$ and complete ideals in the local regular ring $R$. Precisely, in section IV.1, we associate to such an ideal sheaf $\mathcal{J}$ a complete $\mathfrak{m}_O$-primary ideal $H_{\mathcal{J}} \subset R$, from which $\mathcal{J}$ can be recovered. Then, we prove that fixed $m \geq 0$, there is a bijection between these ideal sheaves and the complete $\mathfrak{m}_O$-primary ideals of codimension $m$ in $R$. In section IV.2 we give an algorithm to compute minimal systems of global sections generating some ideal sheaves on $X$. In section IV.3 we use the results of section IV.1 to prove that there is an isomorphism of semigroups between the semigroup of ideal sheaves on $X$ and that of complete ideals in $R$. Then, we relate the factorization of complete ideals in $R$ to the factorization and semifactorization of complete ideals in the local ring of a sandwiched singularity in $X$.

## IV.1 Complete ideals on the local ring of a sandwiched singularity

In this section, we deal with ideal sheaves $\mathcal{J}$ on $X$ satisfying the following condition

(†) $\mathcal{J}$ has finite cosupport, say $\{Q_1, \ldots, Q_n\}$, contained in the exceptional locus of $X$ and for each $Q_i$, the stalk $J_i = \mathcal{J}_{Q_i}$ is a complete $\mathfrak{m}_{Q_i}$-primary ideal of $\mathcal{O}_{X,Q_i}$.

The goal is to show that such an ideal sheaf may be associated to some complete $\mathfrak{m}_O$-primary ideal in $R$, and conversely. To this aim, we need a definition.

**Definition IV.1.1.** We say that a complete $\mathfrak{m}_O$-primary ideal $H \subset R$ is *Cartier for $X$* if the curves defined by generic elements of $H$ satisfy any of (and thus, all) the conditions of Theorem III.1.1.

**Notation IV.1.2.** Given a complete $\mathfrak{m}_O$-primary ideal $H \in R$, we denote

$$H^o = \Gamma(X, \mathcal{O}_X(-L_H)) = \{g \in R \mid v_p(g) \geq v_p(H), \forall p \in \mathcal{K}_+\}.$$

Clearly, $H^o \supset H$ and $L_H = L_{H^o}$. Note that $H$ is Cartier for $X$ if and only if the ideal sheaf $H^o\mathcal{O}_X$ generated by the elements of $H^o$ on $X$ is invertible.

The main result is the following theorem.

**Theorem IV.1.3.** *Let $\mathcal{J}$ be an ideal sheaf on $X$ satisfying (†). There exists a complete $\mathfrak{m}_O$-primary ideal $H_\mathcal{J}$ in $R$ with*

$$\dim_\mathbb{C}(\frac{H^o_\mathcal{J}}{H_\mathcal{J}}) = \sum_{i=1}^{n} \dim_\mathbb{C}(\frac{\mathcal{O}_{X,Q_i}}{J_i})$$

*and such that:*

*(a) $H_\mathcal{J}$ is a Cartier ideal for $X$, i.e. the sheaf $H^o_\mathcal{J}\mathcal{O}_X$ is invertible;*

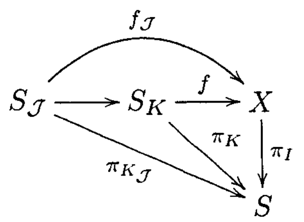
*(b) the sheaf $\mathcal{H}_\mathcal{J} = H_\mathcal{J}\mathcal{O}_X$ is locally principal except precisely at the points $Q_i, i = 1, \ldots, n$, and we have*

$$\mathcal{H}_\mathcal{J} = \mathcal{J}\mathcal{O}_X(-L_{H_\mathcal{J}});$$

*(c) if $C$ is a curve defined by a generic element of $H_\mathcal{J}$, then its strict transform $\widetilde{C}$ on $X$ is a Cartier divisor and intersects the exceptional locus of $X$ exactly in the points $\{Q_1, \ldots, Q_n\}$.*

*Moreover, with these requirements, the ideal $H_\mathcal{J}$ is unique.*

Before proving Theorem IV.1.3, we must fix some notation. Given an ideal sheaf $\mathcal{J}$ as in (†), let $f_\mathcal{J} : S_\mathcal{J} \longrightarrow X$ be the minimal resolution of the singularities of $X$ such that the sheaf $\mathcal{J}\mathcal{O}_{S_\mathcal{J}}$ is invertible (equivalently, $S_\mathcal{J}$ is the minimal resolution of the surface obtained by blowing-up $\mathcal{J}$ on $X$). Since $\pi_I \circ f_\mathcal{J} : S_\mathcal{J} \longrightarrow S$ is a birational morphism between regular surfaces, there exists some cluster $K_{\mathcal{J}}$ containing $K$ so that $S_{\mathcal{J}}$ is obtained by blowing up all the points in $K_{\mathcal{J}}$. Thus, we have a commutative diagram



For each $p \in K_{\mathcal{J}}$, write $E_p^{S_{\mathcal{J}}}$ for the irreducible component being the strict transform on $S_{\mathcal{J}}$ of the first neighbourhood $F_p$ of $p$. If $Q$ is any point in the exceptional locus of $X$, define

$$T_Q^{S_{\mathcal{J}}} = \{p \in K_{\mathcal{J}} \mid (f_{\mathcal{J}})_*(E_p^{S_{\mathcal{J}}}) = Q\}.$$

Note that $T_Q^{S_{\mathcal{J}}} \neq \emptyset$ if and only if either $Q \in Sing(X)$ or $Q$ is in the cosupport of $\mathcal{J}$. Note also that

$$K_{\mathcal{J}} \setminus \mathcal{K}_+ = \bigcup_{Q \in X} T_Q^{S_{\mathcal{J}}}.$$

We write $D_{\mathcal{J}}^{S_{\mathcal{J}}}$ for the exceptional divisor on $S_{\mathcal{J}}$ associated to $\mathcal{J}\mathcal{O}_{S_{\mathcal{J}}}$, that is, $\mathcal{J}\mathcal{O}_{S_{\mathcal{J}}} = \mathcal{O}_{S_{\mathcal{J}}}(-D_{\mathcal{J}}^{S_{\mathcal{J}}})$. For every $p \in \mathcal{K}_+$, we denote

$$a_p^{\mathcal{J}} = |D_{\mathcal{J}}^{S_{\mathcal{J}}} \cdot E_p^{S_{\mathcal{J}}}|_{S_{\mathcal{J}}} \in \mathbb{Z}_{\geq 0}$$

and by abuse of notation, we write $L_{\mathcal{J}}$ for the exceptional divisor on $X$ given by

$$L_{\mathcal{J}} = \sum_{p \in \mathcal{K}_+} a_p^{\mathcal{J}} \mathcal{L}_p \in \bigoplus_{p \in \mathcal{K}_+} \mathbb{Z}\mathcal{L}_p. \tag{1.a}$$

Similarly, if $D_{J_i}^{S_{\mathcal{J}}}$ is the exceptional divisor on $S_{\mathcal{J}}$ associated to the invertible sheaf $J_i\mathcal{O}_{S_{\mathcal{J}}}$, *i.e.* $J_i\mathcal{O}_{S_{\mathcal{J}}} = \mathcal{O}_{S_{\mathcal{J}}}(-D_{J_i}^{S_{\mathcal{J}}})$, then for $p \in \mathcal{K}_+$, we denote $a_p^{J_i} = |D_{J_i}^{S_{\mathcal{J}}} \cdot E_p^{S_{\mathcal{J}}}|_{S_{\mathcal{J}}}$. Clearly,

$$D_{\mathcal{J}}^{S_{\mathcal{J}}} = \sum_{i=1}^{n} D_{J_i}^{S_{\mathcal{J}}} \tag{1.b}$$

and

$$a_p = \sum_{i=1}^{n} a_p^i. \tag{1.c}$$

We shall make use of:

**Lemma IV.1.4.** *The sheaf $\mathcal{J}\mathcal{O}_X(-L_{\mathcal{J}})$ is generated by global sections, i.e. if $H_{\mathcal{J}} = \Gamma(X, \mathcal{J}\mathcal{O}_X(-L_{\mathcal{J}}))$, then $H_{\mathcal{J}}\mathcal{O}_X = \mathcal{J}\mathcal{O}_X(-L_{\mathcal{J}})$.*

*Proof.* First, since $\mathcal{J}$ is complete and $\mathcal{O}_X(-L_{\mathcal{J}})$ is invertible, we have that $\mathcal{J}\mathcal{O}_X(-L_{\mathcal{J}})$ is complete, and by Lemma I.3.23, the ideal

$$H_{\mathcal{J}} = \Gamma(X, \mathcal{J}\mathcal{O}_X(-L_{\mathcal{J}})) \subset R \tag{1.d}$$

is complete and $\mathfrak{m}_O$-primary. From this, $H_{\mathcal{J}}\mathcal{O}_X$ is a complete ideal sheaf on $X$ and, in particular, both $H_{\mathcal{J}}\mathcal{O}_X$ and $\mathcal{J}\mathcal{O}_X(-L_{\mathcal{J}})$ are contracted for $f_{\mathcal{J}}$ (see Proposition I.3.25). Hence

$$H_{\mathcal{J}}\mathcal{O}_X = (f_{\mathcal{J}})_*(H_{\mathcal{J}}\mathcal{O}_{S_{\mathcal{J}}}) \tag{1.e}$$
$$\mathcal{J}\mathcal{O}_X(-L_{\mathcal{J}}) = (f_{\mathcal{J}})_*(\mathcal{J}\mathcal{O}_{S_{\mathcal{J}}}(-(f_{\mathcal{J}})^*(L_{\mathcal{J}}))). \tag{1.f}$$

Now, by the definition of $S_{\mathcal{J}}$, $\mathcal{J}\mathcal{O}_{S_{\mathcal{J}}}(-(f_{\mathcal{J}})^*(L_{\mathcal{J}}))$ is invertible and clearly,

$$\mathcal{J}\mathcal{O}_{S_{\mathcal{J}}}(-(f_{\mathcal{J}})^*(L_{\mathcal{J}})) = \mathcal{O}_{S_{\mathcal{J}}}(-D_{\mathcal{J}}^{S_{\mathcal{J}}} - (f_{\mathcal{J}})^*(L_{\mathcal{J}})).$$

Moreover, from the way $L_{\mathcal{J}}$ has been defined, we have that for all $p \in K_{\mathcal{J}}$,

$$|E_p^{S_{\mathcal{J}}} \cdot (D_{\mathcal{J}}^{S_{\mathcal{J}}} + f_{\mathcal{J}}^*(L_{\mathcal{J}}))|_{S_{\mathcal{J}}} \leq 0.$$

Therefore, by 2. of Proposition I.3.29, $\mathcal{J}\mathcal{O}_{S_{\mathcal{J}}}(-(f_{\mathcal{J}})_*(L_{\mathcal{J}}))$ is generated by global sections. Now, note that

$$\Gamma(S_{\mathcal{J}}, \mathcal{J}\mathcal{O}_{S_{\mathcal{J}}}(-(f_{\mathcal{J}})_*(L_{\mathcal{J}}))) = \Gamma(X, \mathcal{J}\mathcal{O}_X(-L_{\mathcal{J}})) = H_{\mathcal{J}}$$

and so,

$$H_{\mathcal{J}}\mathcal{O}_{S_{\mathcal{J}}} = \mathcal{J}\mathcal{O}_{S_{\mathcal{J}}}(-(f_{\mathcal{J}})_*(L_{\mathcal{J}}))).$$

From this and the equalities (1.e) and (1.f) above, we deduce that $H_{\mathcal{J}}\mathcal{O}_X = \mathcal{J}\mathcal{O}_X(-L_{\mathcal{J}})$. □

**Notation IV.1.5.** From now on, given an ideal sheaf $\mathcal{J}$ satisfying (†), we will write $H_{\mathcal{J}} = \Gamma(X, \mathcal{J}\mathcal{O}_X(-L_{\mathcal{J}}))$.

**Corollary IV.1.6.** *We have*

$$E_{H_{\mathcal{J}}}^{S_{\mathcal{J}}} = E_{H_{\mathcal{J}}^{\circ}}^{S_{\mathcal{J}}} + D_{\mathcal{J}}^{S_{\mathcal{J}}}.$$

*Proof.* In the proof of Lemma IV.1.4, we have seen

$$H_{\mathcal{J}}\mathcal{O}_{S_{\mathcal{J}}} = \mathcal{O}_{S_{\mathcal{J}}}(-D_{\mathcal{J}}^{S_{\mathcal{J}}} - f_{\mathcal{J}}^*(L_{\mathcal{J}})),$$

and also,

$$H^o_{\mathcal{J}}\mathcal{O}_{S_{\mathcal{J}}} = \mathcal{O}_{S_{\mathcal{J}}}(-f^*_{\mathcal{J}}(L_{\mathcal{J}})).$$

As the strict transform on $X$ of any curve going sharply through $\mathcal{K}^o_{\mathcal{J}}$ goes through no point of the cosupport of $\mathcal{J}$ or singularity of $X$, from the definition of $E^{S_{\mathcal{J}}}_{H^o_{\mathcal{J}}}$ we infer that $E^{S_{\mathcal{J}}}_{H^o_{\mathcal{J}}} = f^*_{\mathcal{J}}(L_{\mathcal{J}})$ and so,

$$E^{S_{\mathcal{J}}}_{H_{\mathcal{J}}} = D^{S_{\mathcal{J}}}_{\mathcal{J}} + E^{S_{\mathcal{J}}}_{H^o_{\mathcal{J}}}$$

as claimed. □

**Remark IV.1.7.** In virtue of Lemma IV.1.4, if $C$ is a curve defined by a generic element of $H_{\mathcal{J}}$, then $L_C = L_{H_{\mathcal{J}}} = L_{\mathcal{J}}$, and so, $\mathbb{H}^o_C = H^o_{\mathcal{K}}$. Moreover, since $L_{\mathcal{J}} = \sum_{p\in\mathcal{K}_+} a_p\mathcal{L}_p$ (see (1.a) above), in virtue of Theorem III.1.1 we have the strict transform of $C$ on $X$ is Cartier. It follows that $H_{\mathcal{J}}$ and $H^o_{\mathcal{J}} = \Gamma(X, \mathcal{O}_X(-L_{\mathcal{J}}))$ are Cartier ideals for $X$, and that

$$H^o_{\mathcal{J}} \;=\; \prod_{p\in\mathcal{K}_+} I_p^{a_p} \subset R. \tag{1.g}$$

is the Zariski factorization of $H^o_{\mathcal{J}}$.

Note that the set of base points of $H^o_{\mathcal{J}}$ and $H_{\mathcal{J}}$ are contained in $K$ and $K_{\mathcal{J}}$, respectively. We write

- $\mathcal{K}^o_{\mathcal{J}} = (K, \sigma^o)$ for the cluster of base points of $H^o_{\mathcal{J}}$;
- $\mathcal{K}_{\mathcal{J}} = (K_{\mathcal{J}}, \sigma)$ for the cluster of base points of $H_{\mathcal{J}} \subset R$.

Note that $\mathcal{K}_{\mathcal{J}}$ has excess 0 at every point of $\mathcal{K}_+$.

*Proof of Theorem IV.1.3* First of all, as a direct consequence of Lemma IV.1.4, we have that if $C$ goes sharply through $\mathcal{K}_{\mathcal{J}}$, then $\widetilde{C}$ is defined locally near each $Q_i$ by an element of $J_i \subset \mathcal{O}_{X,Q_i}$. Hence, keeping the notation of section III.4, $D^{Q_i}_{\widetilde{C}} \geq D^{S_{\mathcal{J}}}_{J_i}$ and so,

$$[\widetilde{C}, L_p]_{Q_i} \geq \alpha^i_p \tag{1.h}$$

for each $p \in \mathcal{K}_+$. Therefore, we have

$$\sum_{i=1}^{n} \alpha^i_p \leq \sum_{i=1}^{n} [\widetilde{C}, L_p]_{Q_i} \leq |\widetilde{C}\cdot L_p|_X = a_p$$

the last equality by Corollary III.1.6. By (1.c), we infer that $|\widetilde{C}{\cdot}L_p|_X = \sum_{i=1}^n \alpha_p^i$ and by (1.h), $[\widetilde{C}, L_p]_{Q_i} = \alpha_p^i$. It follows that $\widetilde{C}$ cannot intersect $L_p$ at some point besides those $Q_i$ lying in $L_p$.

To prove the uniqueness of the ideal $H_{\mathcal{J}}$, we assume that $H \subset R$ is a complete $\mathfrak{m}_O$-primary ideal verifying the conditions of Theorem IV.1.3. Then, $H^o\mathcal{O}_X$ is invertible and we can write $H^o\mathcal{O}_X = \mathcal{O}_X(-L^{(\alpha)})$ with $L^{(\alpha)} = \sum_{p\in\mathcal{K}_+} \alpha_p \mathcal{L}_p$. The same argument used in the proof of Lemma IV.1.4 shows that $\mathcal{J}\mathcal{O}_X(-L^{(\alpha)})$ is generated by global sections if and only if $\alpha_p \geq a_p$ for all $p \in \mathcal{K}_+$. In fact, if $\alpha_p < a_p$ for some $p \in \mathcal{K}_+$, then $|E_p^{S_\mathcal{J}}{\cdot}(D_\mathcal{J} + f_\mathcal{J}^*(L^{(\alpha)}))|_{S_\mathcal{J}} > 0$ and in this case, there are no curves $C \subset S$ such that $\pi_{K_\mathcal{J}}^*(C) = \widetilde{C} + D_\mathcal{J}^{S_\mathcal{J}} + f_\mathcal{J}^*(L^{(\alpha)})$ for otherwise, by the projection formula,

$$|(D_\mathcal{J}^{S_\mathcal{J}} + f_\mathcal{J}^*(L^\alpha)){\cdot}E_p^{S_\mathcal{J}}|_{S_\mathcal{J}} = -|\widetilde{C}^{S_\mathcal{J}}{\cdot}E_p^{S_\mathcal{J}}|_{S_\mathcal{J}} < 0.$$

Therefore, the ideals $H^o = \prod_{p\in\mathcal{K}_+} I_p^{\alpha_p}$ and $H = \Gamma(X, \mathcal{J}\mathcal{O}_X(-L^{(\alpha)}))$ verify (a) and (b) in Theorem IV.1.3. However, the strict transform on $X$ of a curve defined by a generic elements of $H$ intersects $L_p$ at $\alpha_p - a_p \geq 0$ points besides the points $Q_i$ lying in $L_p$, and hence, the condition (c) in Theorem IV.1.3 is not satisfied unless $\alpha_p = a_p$. Therefore, condition (c) determines the divisor $L_\mathcal{J}$ and by the equalities (1.g) and (1.d) above, the ideal $H_\mathcal{J}$ is unique.

From the definition of $S_\mathcal{J}$, the sheaf $J_i\mathcal{O}_{S_\mathcal{J}}$ is invertible and we can write $J_i\mathcal{O}_{S_\mathcal{J}} = \mathcal{O}_{S_\mathcal{J}}(-D_{J_i}^{S_\mathcal{J}})$, where $D_{J_i}^{S_\mathcal{J}}$ has support $\bigcup_{p\in T_{Q_i}^{S_\mathcal{J}}} E_p^{S_\mathcal{J}}$. According to Proposition I.3.33, we have

$$\dim_{\mathbb{C}}(\frac{\mathcal{O}_{X,Q_i}}{J_i}) = -\frac{1}{2}|D_{J_i}^{S_\mathcal{J}}{\cdot}(D_{J_i}^{S_\mathcal{J}} + \mathrm{K}_{S_\mathcal{J}})|_{S_\mathcal{J}} = -\frac{1}{2}|D_{J_i}^{S_\mathcal{J}}{\cdot}(D_\mathcal{J}^{S_\mathcal{J}} + \mathrm{K}_{S_\mathcal{J}})|_{S_\mathcal{J}}$$

the last equality for the divisors $D_{J_i}^{S_\mathcal{J}}$ are the connected components of $D_\mathcal{J}^{S_\mathcal{J}}$. Therefore,

$$\begin{aligned}\sum_{i=1}^n \dim_{\mathbb{C}}(\frac{\mathcal{O}_{X,Q_i}}{J_i}) &= -\frac{1}{2}|D_\mathcal{J}^{S_\mathcal{J}}{\cdot}(D_\mathcal{J}^{S_\mathcal{J}} + \mathrm{K}_{S_\mathcal{J}})|_{S_\mathcal{J}} = \\ &= -\frac{1}{2}|D_\mathcal{J}^{S_\mathcal{J}}{\cdot}D_\mathcal{J}^{S_\mathcal{J}}|_{S_\mathcal{J}} - \frac{1}{2}|D_\mathcal{J}^{S_\mathcal{J}}{\cdot}K_{S_J}|_{S_\mathcal{J}} = \\ &= -\frac{1}{2}|(E_{H_\mathcal{J}}^{S_\mathcal{J}} - E_{H_\mathcal{J}^o}^{S_\mathcal{J}}){\cdot}D_\mathcal{J}^{S_\mathcal{J}}|_{S_\mathcal{J}} - \frac{1}{2}|(E_{H_\mathcal{J}}^{S_\mathcal{J}} - E_{H_\mathcal{J}^o}^{S_\mathcal{J}}){\cdot}\mathrm{K}_{S_J})|_{S_\mathcal{J}}\end{aligned} \tag{1.i}$$

by Corollary IV.1.6. Now, by the projection formula applied to $\pi_{K_{\mathcal{J}}}$ and Corollary IV.1.6 again, we have that

$$\begin{aligned}|(E^{S_{\mathcal{J}}}_{H_{\mathcal{J}}} - E^{S_{\mathcal{J}}}_{H^o_{\mathcal{J}}})\cdot D^{S_{\mathcal{J}}}_{\mathcal{J}}|_{S_{\mathcal{J}}} &= |E^{S_{\mathcal{J}}}_{H_{\mathcal{J}}}\cdot D^{S_{\mathcal{J}}}_{\mathcal{J}}|_{S_{\mathcal{J}}} - |E^{S_{\mathcal{J}}}_{H^o_{\mathcal{J}}}\cdot D^{S_{\mathcal{J}}}_{\mathcal{J}}|_{S_{\mathcal{J}}} = \\ &= |E^{S_{\mathcal{J}}}_{H_{\mathcal{J}}}\cdot E^{S_{\mathcal{J}}}_{H_{\mathcal{J}}}|_{S_{\mathcal{J}}} - 2|E^{S_{\mathcal{J}}}_{H_{\mathcal{J}}}\cdot E^{S_{\mathcal{J}}}_{H^o_{\mathcal{J}}}|_{S_{\mathcal{J}}} + |E^{S_{\mathcal{J}}}_{H^o_{\mathcal{J}}}\cdot E^{S_{\mathcal{J}}}_{H^o_{\mathcal{J}}}|_{S_{\mathcal{J}}} = \\ &= \mathcal{K}^2_{\mathcal{J}} - 2[\mathcal{K}_{\mathcal{J}}, \mathcal{K}^o_{\mathcal{J}}]_O + (\mathcal{K}^o_{\mathcal{J}})^2,\end{aligned}$$

and from Corollary III.2.2, we infer that $[\mathcal{K}_{\mathcal{J}}, \mathcal{K}^o_{\mathcal{J}}]_O = (\mathcal{K}^o_{\mathcal{J}})^2$ and hence,

$$|(E^{S_{\mathcal{J}}}_{H_{\mathcal{J}}} - E^{S_{\mathcal{J}}}_{H^o_{\mathcal{J}}})\cdot D^{S_{\mathcal{J}}}_{\mathcal{J}}|_{S_{\mathcal{J}}} = [(\mathcal{K}_{\mathcal{J}} - \mathcal{K}^o_{\mathcal{J}}), \mathcal{K}_{\mathcal{J}}]_O.$$

On the other hand, from Lemma III.5.4 and the projection formula again, we have

$$|(E^{S_{\mathcal{J}}}_{H_{\mathcal{J}}} - E^{S_{\mathcal{J}}}_{H^o_{\mathcal{J}}})\cdot \mathrm{K}_{S_{\mathcal{J}}})|_{S_{\mathcal{J}}} = [(\mathcal{K}_{\mathcal{J}} - \mathcal{K}^o_{\mathcal{J}}), (C_{adj} - C)]_O$$

where $C$ and $C_{adj}$ can be chosen to go sharply through $K_{\mathcal{J}}$ and $(K_{\mathcal{J}})_{adj}$, respectively. Thus, from the equality (1.i) above and the Noether formula, we have that

$$\begin{aligned}\sum_{i=1}^{n} \dim_{\mathbb{C}}(\frac{\mathcal{O}_{X,Q_i}}{\mathcal{J}_i}) &= -\frac{1}{2}[(\mathcal{K}_{\mathcal{J}} - \mathcal{K}^o_{\mathcal{J}}), \mathcal{K}_{\mathcal{J}}]_O - \frac{1}{2}[(\mathcal{K}_{\mathcal{J}} - \mathcal{K}^o_{\mathcal{J}}), (C_{adj} - C)]_O = \\ &= -\frac{1}{2}[(\mathcal{K}_{\mathcal{J}} - \mathcal{K}^o_{\mathcal{J}}), C_{adj}]_O = -\frac{1}{2}\sum_p (\sigma_p - \sigma^o_p)(\sigma_p - 1) = \\ &= -\frac{1}{2}\sum_p \sigma_p(\sigma_p - 1) + \frac{1}{2}\sum_p \sigma^o_p(\sigma_p - 1) = \\ &= \dim_{\mathbb{C}}(\frac{H^o_{\mathcal{J}}}{H_{\mathcal{J}}})\end{aligned}$$

the last equality by Corollary III.2.2 and Proposition I.2.21. This completes the proof. □

The following corollary is a direct consequence of Theorem IV.1.3.

**Corollary IV.1.8.** *Let $J'$ be a complete $\mathfrak{m}_O$-primary ideal such that $H_{\mathcal{J}} \subset J' \subset H^o_{\mathcal{J}}$. Then, the ideal sheaf $\mathcal{J}' = J'\mathcal{O}_X(L_{\mathcal{J}})$ satisfies (†) and its cosupport is contained in $\{Q_1, \ldots, Q_n\}$. Moreover,*

$$\sum_{i=1}^{n} \dim_{\mathbb{C}}(\frac{\mathcal{O}_{X,Q_i}}{\mathcal{J}'_{Q_i}}) = \dim_{\mathbb{C}}(\frac{H^o_{\mathcal{J}}}{J'}).$$

To close this section, we state another consequence of Theorem IV.1.3, which expresses the relationship between complete $\mathfrak{m}_O$-primary ideals in $R$ and ideal sheaves on $X$ satisfying (†). To this aim, we need a definition, which is a generalization of Definition II.1.1, and to fix some notation.

**Definition IV.1.9.** If $H \subset R$ is a Cartier ideal for $X$ and $C$ is a curve on $S$ defined by some element of $H$, the *virtual transform of $C$ relative to $H$ (or $H^o$) on $X$, $\check{C}^{H^o}$*, is the effective divisor given by $\pi^*(C) - L_{H^o}$.

Note that if $\mathcal{J}$ is an ideal sheaf on $X$ satisfying (†), then $L_{\mathcal{J}}$ is a Cartier divisor on $X$ (see (1.a) and Lemma III.1.4) and so, the virtual transform $\check{C}^{\mathcal{K}^o_{\mathcal{J}}}$ is an effective Cartier divisor on $X$, too.

**Notation IV.1.10.** For each $m \geq 0$, denote by

$$\mathbf{S}_X^m = \{\text{ideal sheaves } \mathcal{J} \text{ on } X \text{ satisfying } (\dagger) \text{ and } \textstyle\sum_{i=1}^n \dim_{\mathbb{C}}(\frac{\mathcal{O}_{X,Q_i}}{J_i}) = m\}$$

and

$$\mathbf{I}_R^m = \left\{ \text{ Cartier ideals } H \text{ for } X \;\mid\; \dim_{\mathbb{C}}(\tfrac{H^o}{H}) = m \right\}.$$

**Corollary IV.1.11.** *For each $m \geq 0$, the maps*

$$\begin{array}{ccc} \mathbf{S}_X^m & \longrightarrow & \mathbf{I}_R^m \\ \mathcal{J} & \mapsto & H_{\mathcal{J}} \end{array}$$

*and*

$$\begin{array}{ccc} \mathbf{I}_R^m & \longrightarrow & \mathbf{S}_X^m \\ H & \mapsto & H\mathcal{O}_X(L_H) \end{array}$$

*are reciprocal bijections between $\mathbf{S}_X^m$ and $\mathbf{I}_R^m$.*

*Therefore, fixed $m \geq 0$, by associating to each ideal sheaf $\mathcal{J}$ on $X$ verifying (†) the complete $\mathfrak{m}_O$-primary ideal $H_{\mathcal{J}}$, we get a bijection between the set of ideal sheaves on $X$ satisfying (†) with $\sum \dim_{\mathbb{C}}(\mathcal{O}_{X,Q_i}/J_i) = m$ and the set of Cartier ideals $H \subset R$ for $X$ of codimension $m$. The inverse map associates to each $H \subset R$, the ideal sheaf on $X$ obtained by removing the exceptional part from $H\mathcal{O}_X$.*

Note that the cosupport of $H\mathcal{O}_X(L_H)$ is composed of the points on $X$ the virtual transforms relative to $H^o$ of all the curves defined by elements of $H$ are going through.

*Proof.* Fix $m \geq 0$ and let $\mathcal{J} \in \mathbf{S}_X^m$. Then, from Lemma IV.1.4 we know that $H_{\mathcal{J}}\mathcal{O}_X = \mathcal{J}\mathcal{O}_X(-L_{\mathcal{J}})$ and so, $H_{\mathcal{J}}\mathcal{O}_X(L_{\mathcal{J}}) = \mathcal{J}$. On the other hand, by Remark IV.1.7, we have that $L_H = L_{\mathcal{J}}$. Therefore, $H_{\mathcal{J}}\mathcal{O}_X(L_{H_{\mathcal{J}}}) = \mathcal{J}$.

Now, if $H \subset R$ is a Cartier ideal for $X$, the strict transform on $X$ of any curve going sharply through $BP(H)$ is a Cartier divisor (Theorem III.1.1) and $\mathcal{O}_X(L_H)$ is invertible. Hence, the cosupport of the sheaf $\mathcal{J} = H\mathcal{O}_X(L_H)$ is composed of the points shared by the virtual transforms on $X$ relative to $H^o$ of the curves defined by elements of $H$, and hence it is finite. Moreover, for each point $Q$ in the cosupport of $\mathcal{J}$, the stalk $\mathcal{J}_Q$ is a complete $\mathfrak{m}_Q$-primary ideal in $\mathcal{O}_{X,Q}$, and so $\mathcal{J}$ verifies (†). Since $H$ satisfies the conditions of Theorem IV.1.3, we infer from the uniqueness of $H_{\mathcal{J}}$ that $H_{\mathcal{J}} = H$. □

**Remark IV.1.12.** Note that by taking $m = 1$ and identifying each point in the exceptional locus of $X$ with its corresponding ideal sheaf, the preceding theorem generalizes Theorem II.1.7: any ideal sheaf $\mathcal{J}$ of $\mathbf{S}_X^1$ is $\mathcal{M}_Q$, for some $Q$ in the exceptional locus of $X$, and the corresponding ideal $H_{\mathcal{M}_Q} \subset R$ by Corollary IV.1.11 is $I_Q$.

## IV.2 Systems of generators for a complete ideal in the local ring of a sandwiched singularity

Fixed an ideal sheaf $\mathcal{J}$ on $X$ satisfying (†), we have seen in Lemma IV.1.4 that $\mathcal{J}\mathcal{O}_X(-L_{\mathcal{J}})$ is generated by global sections. This section is devoted to construct an algorithm to describe a minimal system of global sections generating $\mathcal{J}\mathcal{O}_X(-L_{\mathcal{J}})$. The algorithm suggested here follows the idea of [10] and constructs a flag of clusters

$$\mathcal{T}^0 = \mathcal{K}_{\mathcal{J}} \prec \mathcal{T}^1 \prec \ldots \prec \mathcal{T}^N = BP(\mathfrak{m}_O H_{\mathcal{J}})$$

with $N = \dim_{\mathbb{C}}(\frac{H_{\mathcal{J}}}{\mathfrak{m}_O H_{\mathcal{J}}})$ and $H_{\mathcal{J}} = \Gamma(X, \mathcal{J}\mathcal{O}_X(-L_{\mathcal{J}}))$.

As above, we denote by $\{Q_1, \ldots, Q_n\}$ the cosupport of $\mathcal{J}$ and for each point $Q_i$, $T_{Q_i}^{S_{\mathcal{J}}} = \{p \in K_{\mathcal{J}} \mid (f_{\mathcal{J}})_*(E_p^{S_{\mathcal{J}}}) = Q_i\}$.

We make use of the following easy lemma.

**Lemma IV.2.1.** *Let $\mathcal{O}_{X,Q}$ be a rational surface singularity and let $H_1, H_2 \subset \mathcal{O}_{X,Q}$ be complete $\mathfrak{m}_Q$-primary ideals. Then,*

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_{X,Q}}{H_1 H_2} = \dim_{\mathbb{C}} \frac{\mathcal{O}_{X,Q}}{H_1} + \dim_{\mathbb{C}} \frac{\mathcal{O}_{X,Q}}{H_2} + [H_1, H_2]_Q,$$

*where* $[H_1, H_2]_Q = -|D^{S'}_{H_1} \cdot D^{S'}_{H_2}|_{S'}$ *and* $f' : S' \longrightarrow X$ *is any resolution of* $Q$ *such that both* $H_1\mathcal{O}_{S'}$ *and* $H_2\mathcal{O}_{S'}$ *are invertible.*

*In particular, if* $R = \mathcal{O}_{S,O}$ *is regular and* $\mathcal{Q}_1, \mathcal{Q}_2$ *are the clusters of base points of* $H_1, H_2 \subset R$*, respectively, then*

$$c(\mathcal{Q}_1 + \mathcal{Q}_2) = c(\mathcal{Q}_1) + c(\mathcal{Q}_2) + [\mathcal{Q}_1, \mathcal{Q}_2]_O.$$

*Proof.* The first equality follows directly from Proposition I.3.33. If $R = \mathcal{O}_{S,0}$ is regular, the claim follows by applying Proposition I.2.21 and the Noether's formula (see Theorem I.1.31). □

Now in order to describe the algorithm, we need to introduce some notation.

**Notation IV.2.2.** If $p$ is a point infinitely near to $O$ and not in $K$, we write $\sigma_K(p)$ for the first point of $\mathcal{K}(p)$ which is not in $K$.

*Description of the Algorithm*

STEP 0 Put $\mathcal{T}_0 = (K_J, \tau^0)$ the cluster $\mathcal{K}_{\mathcal{J}}$ and let $p \in T^{S_{\mathcal{J}}}_{Q_1}$ be a dicritical point of $\mathcal{T}_0$ (for instance, take $p$ any maximal point in $T^{S_{\mathcal{J}}}_{Q_1}$). Clearly, $v_p^{\mathcal{K}_{\mathcal{J}}} = v_p(H_{\mathcal{J}}) < v_p(\mathfrak{m}_O H_{\mathcal{J}})$. Choose any point $q$ in the first neighbourhood of $p$ and not already in $\mathcal{T}_0$, and define $\mathcal{T}_1$ as the cluster $(\mathcal{T}_0)_q$ obtained by adding $q$ to $\mathcal{T}_0$ counted once. Note that $\mathcal{T}_1$ is consistent since $p$ is dicritical in $\mathcal{T}_0$.

STEP $j$ Once defined $\mathcal{T}_{j-1} = (T_{j-1}, \tau^{j-1})$, assume that there exists some maximal point $p$ in $\mathcal{T}_{j-1}$ such that

$$v_p^{\mathcal{T}_{j-1}} < v_p(\mathfrak{m}_O H_{\mathcal{J}}).$$

Note that $\sigma_K(p)$ is proximate or equal to some point of $T^{S_{\mathcal{J}}}_{Q_i}$ for some $i \in \{1, \ldots, n\}$. Choose $p$ in such a way that $i$ is minimal and define $\mathcal{T}_i$ to be the cluster $(\mathcal{T}_{j-1})_q$, where $q$ in the first neighbourhood of $p$ and not in $T_{i-1}$ (as before, $(\mathcal{T}_{j-1})_q$ is consistent because $p$ is dicritical in $\mathcal{T}_{j-1}$).

It is clear that the clusters $\mathcal{T}_j, j \geq 0$ are contained in $BP(\mathfrak{m}_O H_{\mathcal{J}})$ and hence, after finitely many steps, the procedure stops, giving rise to a cluster $\mathcal{T}_{N-1}$ with no dicritical points satisfying $v_p^{\mathcal{T}_{N-1}} < v_p(\mathfrak{m}_O H_{\mathcal{J}})$. Moreover, as no unloading is performed during the above procedure, we have $\tau_p^j = \tau_p^0$ for every $p \in K_{\mathcal{J}}$. Since the virtual multiplicity of $BP(\mathfrak{m}_O H_{\mathcal{J}})$ is $\sigma_O + 1$, we infer that, for all $i \in \{0, \ldots, N-1\}$,

$$\mathcal{T}_i \subsetneq BP(\mathfrak{m}_O H_{\mathcal{J}}).$$

Note that $O$ is a non-dicritical point of $\mathcal{T}_{N-1}$. Define $\mathcal{T}_N$ as the consistent cluster obtained by adding to $\mathcal{T}_{N-1}$ some $q$ counted once in the first neighbourhood of $O$, not in $\mathcal{T}_{N-1}$, and unloading multiplicities.

**Lemma IV.2.3.** *We have that* $\mathcal{T}^N = BP(\mathfrak{m}_O H_{\mathcal{J}})$ *and so,*

$$N = \dim_{\mathbb{C}} \frac{H_{\mathcal{J}}}{\mathfrak{m}_O H_{\mathcal{J}}}.$$

*Proof.* We already know that for each $j \geq 0$, $\mathcal{T}_j \subsetneq BP(\mathfrak{m}_O H_{\mathcal{J}})$ and that there are no dicritical points in $\mathcal{T}_{N-1}$ satisfying $v_p^{\mathcal{T}_{N-1}} < v_p(\mathfrak{m}_O H_{\mathcal{J}})$. First, we prove that $\mathcal{T}_N \prec BP(\mathfrak{m}_O H_{\mathcal{J}})$. To this aim, it is enough to show that a generic curve going through $BP(\mathfrak{m}_O H_{\mathcal{J}})$ goes virtually through $\mathcal{T}_N$, and this is clear for any such curve $C$ has effective multiplicity $\sigma_O + 1$, strictly bigger than the virtual multiplicity of $(\mathcal{T}_N)_q$, so that its virtual transform relative to $\mathcal{T}_{N-1}$ contains the exceptional divisor of blowing-up $O$, and hence has multiplicity at least one at $q$.

On the other hand, we have $\dim_{\mathbb{C}}(H_{\mathcal{J}}/\mathfrak{m}_O H_{\mathcal{J}}) = \sigma_O + 1$ and since $\dim_{\mathbb{C}}(H_{\mathcal{T}_j}/H_{\mathcal{T}_{j+1}}) = 1$ for all $j \geq 0$, $\dim_{\mathbb{C}}(H_{\mathcal{T}_O}/H_{\mathcal{T}_N}) = N$ and $H_{\mathcal{T}_O} = H_{\mathcal{J}}$. Therefore, in order to see that $H_{\mathcal{T}_N} = \mathfrak{m}_O H_{\mathcal{J}}$, it is enough to see that $N = \sigma_O + 1$. To this aim, note that $\mathcal{T}_{N-1}$ is composed of the points and the multiplicities of $\mathcal{K}_{\mathcal{J}}$ plus, for each point $p \in K_{\mathcal{J}}$, $\rho_p^{\mathcal{K}_{\mathcal{J}}} > 0$ chains of $v_p(\mathfrak{m}_O)$ free and consecutive points, all of them with virtual multiplicity one and infinitely near to $p$. Then, in virtue of Lemmas I.1.22 and I.1.13, we have

$$\begin{aligned} N - 1 &= \sum_{p \in K_{\mathcal{J}}} \rho_p^{\mathcal{K}_{\mathcal{J}}} v_p(\mathfrak{m}_O) = \rho_{\mathcal{K}_{\mathcal{J}}}^t \mathbf{v}_{\mathcal{K}_{\mathcal{J}}}(\mathfrak{m}_O) = \boldsymbol{\sigma}_{K_{\mathcal{J}}}^t \mathbf{P}_{K_{\mathcal{J}}} \mathbf{v}_{K_{\mathcal{J}}}(\mathfrak{m}_O) = \\ &= \boldsymbol{\sigma}_{K_{\mathcal{J}}}^t \mathbf{e}_{K_{\mathcal{J}}}(C_O) = \sum_{p \in K_{\mathcal{J}}} \sigma_p e_p(C_O) = \sigma_O \end{aligned}$$

where $C_O$ is a generic curve going through $O$. Hence, $N = \sigma_O + 1$ as wanted. □

Note that we have constructed a flag

$$\mathcal{T}_0 = \mathcal{K}_{\mathcal{J}} \prec \mathcal{T}_1 \prec \ldots \prec \mathcal{T}_N = BP(\mathfrak{m}_O H_{\mathcal{J}}) \tag{2.a}$$

such that for all $j = 0, \ldots, N-1$,

$$L_{\mathcal{T}_j} = L_{H_{\mathcal{J}}}, \tag{2.b}$$

and hence, the strict transform $\widetilde{C}$ of any curve $C$ going sharply through some $\mathcal{T}_j$ is a Cartier divisor on $X$ (see Theorem III.1.1).

From this, it follows

**Proposition IV.2.4.** *By picking, for $j = 0, \dots, N-1$ an equation $h_j$ of a curve $C_j$ going through $\mathcal{T}_j$ but going not through $\mathcal{T}_{j+1}$, we get a minimal system of generators of $H_{\mathcal{J}} = \Gamma(X, \mathcal{J}\mathcal{O}_X(-L_{\mathcal{J}}))$.*

The algorithm suggested here allows also to give, for each $Q_i$ in the cosupport of $\mathcal{J}$, a minimal system of generators for the complete $\mathfrak{m}_{Q_i}$-primary ideal $J_i \subset \mathcal{O}_{X,Q_i}$. For each $i \in \{1, \dots, n\}$, write $N_i = \dim_{\mathbb{C}}(\frac{J_i}{\mathfrak{m}_{Q_i} J_i})$ and $M_i = \sum_{j=1}^{i-1} N_i$ (if $i = 1$, take $M_1 = 0$).

**Lemma IV.2.5.** *If $\widetilde{h}_j$ is an equation for $\widetilde{C}_j$ near $Q_i$, then $\{\widetilde{h}_{M_i}, \dots, \widetilde{h}_{M_i+N_i-1}\}$ is a minimal system of generators of $J_i \subset \mathcal{O}_{X,Q_i}$.*

*Proof.* Let $C$ be a generic curve going through $\mathcal{K}_{\mathcal{J}}$. By Lemma IV.2.1 and the projection formula applied to $f_{\mathcal{J}}$, we have

$$\begin{aligned} N_i &= \dim_{\mathbb{C}}(\frac{J_i}{\mathfrak{m}_{Q_i} J_i}) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{X,Q_i}}{\mathfrak{m}_{Q_i} J_i} - \dim_{\mathbb{C}} \frac{\mathcal{O}_{X,Q_i}}{J_i} = \\ &= \dim_{\mathbb{C}} \frac{\mathcal{O}_{X,Q_i}}{\mathfrak{m}_{Q_i}} - |D_{\mathcal{J}}^{S_{\mathcal{J}}} \cdot Z_{Q_i}^{S_{\mathcal{J}}}|_{S_{\mathcal{J}}} = \\ &= 1 + |\widetilde{C}^{S_{\mathcal{J}}} \cdot Z_{Q_i}^{S_{\mathcal{J}}}|_{S_{\mathcal{J}}} = 1 + \sum_{p \in T_{Q_i}^{S_{\mathcal{J}}}} \rho_p^{\mathcal{K}_{\mathcal{J}}} z_p^{Q_i}. \end{aligned}$$

Take the subflag of (2.a) composed of all the clusters $\mathcal{T}_{M_i+j}$ for some $j \in \{0, \dots, N_i\}$:

$$\mathcal{T}_{M_i} \prec \mathcal{T}_{M_i+1} \prec \dots \prec \mathcal{T}_{M_i+N_i}.$$

If we write $H_j$ for the ideal $H_{\mathcal{T}_{M_i+j}}$ the above flag of clusters, gives a filtration of ideals in $R$

$$H_0 \supset H_1 \supset \dots \supset H_{N_i} \tag{2.c}$$

and $\dim_{\mathbb{C}} \frac{H_j}{H_{j+1}} = 1$ for each $j \in \{0, \dots, N_i - 1\}$. Since we know that $L_{H_j} = L_{H_{\mathcal{J}}}$ (see (2.b) above) we can apply Corollary IV.1.8 to $H_j$ to infer that $\mathcal{H}'_j = H_j \mathcal{O}_X(L_{\mathcal{J}})$ has finite cosupport contained in $\{Q_1, \dots, Q_n\}$ and

$$\sum_{k=1}^{n} \dim_{\mathbb{C}}(\frac{\mathcal{O}_{X,Q_k}}{\mathcal{H}'_j \mathcal{O}_{X,Q_k}}) = \dim_{\mathbb{C}}(\frac{H_{\mathcal{J}}^o}{H_j}).$$

Therefore, if $j \in \{0, \dots, N_i - 1\}$,

$$\sum_{k=1}^{n} \dim_{\mathbb{C}}(\frac{\mathcal{H}'_j \mathcal{O}_{X,Q_k}}{\mathcal{H}'_{j+1} \mathcal{O}_{X,Q_k}}) = \dim_{\mathbb{C}}(\frac{H_j}{H_{j+1}}) = 1.$$

Note that if $Q_k \neq Q_i$ and $j \in \{0, \ldots, N_i - 1\}$, then $\mathcal{H}'_j\mathcal{O}_{X,O_k} = \mathcal{H}'_{j+1}\mathcal{O}_{X,O_k}$ and hence,

$$\dim_{\mathbb{C}}(\frac{\mathcal{H}'_j\mathcal{O}_{X,Q_i}}{\mathcal{H}'_{j+1}\mathcal{O}_{X,Q_i}}) = 1.$$

It follows that the filtration (2.c) induces a filtration of ideals

$$J_i \supset \mathcal{H}'_1\mathcal{O}_{X,Q_i} \supset \ldots \supset \mathcal{H}'_{N_i}\mathcal{O}_{X,Q_i},$$

all of them containing $\mathfrak{m}_{Q_i}J_i$. Since $N_i = \dim_{\mathbb{C}}(\frac{J_i}{\mathfrak{m}_{Q_i}J_i})$, we see that $H_{N_i}\mathcal{O}_{X,Q_i} = \mathfrak{m}_{Q_i}J_i$. Moreover, for each $0 \leq j \leq N_i$, $\widetilde{C_j}$ is defined locally at $Q_i$ by an element in $H_j\mathcal{O}_{X,Q_i}$ not in $H_{j+1}\mathcal{O}_{X,Q_i}$. This completes the proof. □

To close this section, we illustrate this procedure for getting global sections of $\mathcal{J}\mathcal{O}_X(-L_{\mathcal{J}})$ with an example.

**Example IV.2.6.** Take the complete ideal $I$ of Example II.2.15. On $X = Bl_I(S)$, take the ideal sheaf $\mathcal{J}$ defined in the following way: the stalk at $Q_1$ is the maximal ideal, $\mathcal{J}_{Q_1} = \mathfrak{m}_{Q_1}$; the stalk at $Q_2$ is the complete ideal of $\mathcal{O}_{X,Q_2}$ generated by the curves $C$ with two smooth branches, one of them tangent to $L_{p_5}$; finally, if $P_3$ is the point of $L_{p_3}$ where the strict transform of the branch $x = 0$ intersects the exceptional locus of $X$, the stalk of $\mathcal{J}$ at $P_3$ is the complete ideal generated by the smooth curves tangent to $L_{p_3}$. Then, we have that

$$L_{\mathcal{J}} = 15L_{p_3} + 24L_{p_5} = 3\mathcal{L}_{p_3} + 3\mathcal{L}_{p_5}$$

and hence, $H^o_{\mathcal{J}} = I^3_{p_3}I^3_{p_5}$. Figures IV.2 and IV.3 represent the clusters obtained by the algorithm above described. It turns out that we can take the curves $C_i : h_i = 0$ as a minimal system of global sections generating $\mathcal{J}\mathcal{O}_X(-L_{\mathcal{J}})$, where

$$\begin{aligned}
h_0 &= (x^2+y^7+y^8+y^9)(y^3+x^5+x^6+x^7+x^8)(y+x^2)(y-x)(y+3x) \\
&\quad (y-2x)(x+y^2) \\
h_1 &= (x^2+y^7+y^8+y^9)(y^3+x^5+x^6+x^7+x^8)(y+x^2)(y+x)(y+3x) \\
&\quad (y-2x)(x+y^2) \\
h_2 &= (x^2+y^7+y^8+y^9)(y^3+x^5+x^6+x^7+x^8)(y+x^2)(y+x)(y-x) \\
&\quad (y-2x)(x+y^2)
\end{aligned}$$

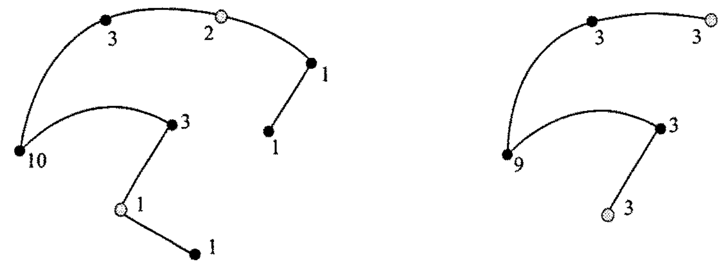



Figure IV.1: On the left Enriques diagram of the cluster $\mathcal{K}_{\mathcal{J}}$; on the right, the Enriques diagram of the cluster $\mathcal{K}^{\circ}_{\mathcal{J}}$ of Example IV.2.6.

$$\begin{aligned}
h_3 &= (x^2+y^7+y^8+y^9)(y^3+x^5+x^6+x^7+x^8)(y+x^2)(y+x)(y-x)\\
&\quad (y+2x)(x+y^2)\\
h_4 &= (x^2+y^7+y^8+y^9)(y^3+x^5+x^6+x^7+x^8)(y+x^2)(y+x)(y-x)\\
&\quad (y+2x)(x-y^2)\\
h_5 &= (x^2+y^7+y^9)(y^3+x^5+x^6+x^7+x^8)(y+x^2)(y+x)(y-x)\\
&\quad (y+2x)(x-y^2)\\
h_6 &= (x^2+y^7)(y^3+x^5+x^6+x^7+x^8)(y+x^2)(y+x)(y-x)(y+2x)(x-y^2)\\
h_7 &= (x^2+y^7)(y^3+x^5+x^6+x^7+x^8)(y-x^2)(y+x)(y-x)(y+2x)(x-y^2)\\
h_8 &= (x^2+y^7)(y^3+x^5+x^7+x^8)(y-x^2)(y+x)(y-x)(y+2x)(x-y^2)\\
h_9 &= (x^2+y^7)(y^3+x^5+x^8)(y-x^2)(y+x)(y-x)(y+2x)(x-y^2)\\
h_{10} &= (x^2+y^7)(y^3+x^5)(y-x^2)(y+x)(y-x)(y+2x)(x-y^2)
\end{aligned}$$

## IV.3 Factorization of complete ideals in the local ring of a sandwiched singularity

In this section, we establish an isomorphism of semigroups between the semigroup of ideal sheaves on $X$ satisfying the condition (†) and that of Cartier ideals for $X$. Then, we relate the factorization and semifactorization of complete $\mathfrak{m}_Q$-primary ideals in $\mathcal{O}_{X,Q}$ to the factorization of complete $\mathfrak{m}_O$-primary ideals in $R$.

We begin by showing that the maps defined in Corollary IV.1.11 are isomorphisms of semigroups.

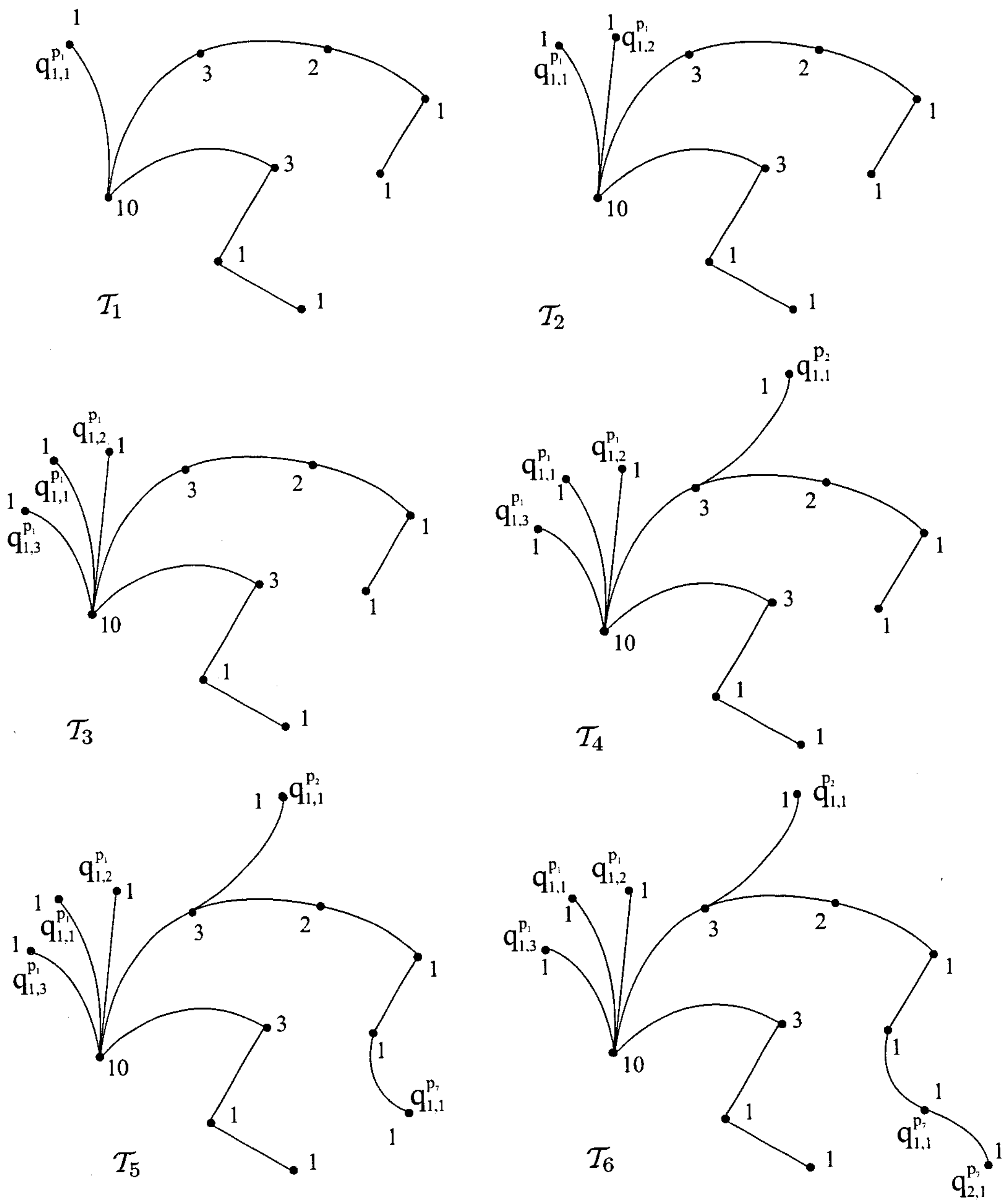



Figure IV.2: Enriques diagrams of the clusters $\mathcal{T}_i, i = 1, \ldots, 6$ obtained by means of the algorithm described for getting generators in Example IV.2.6.

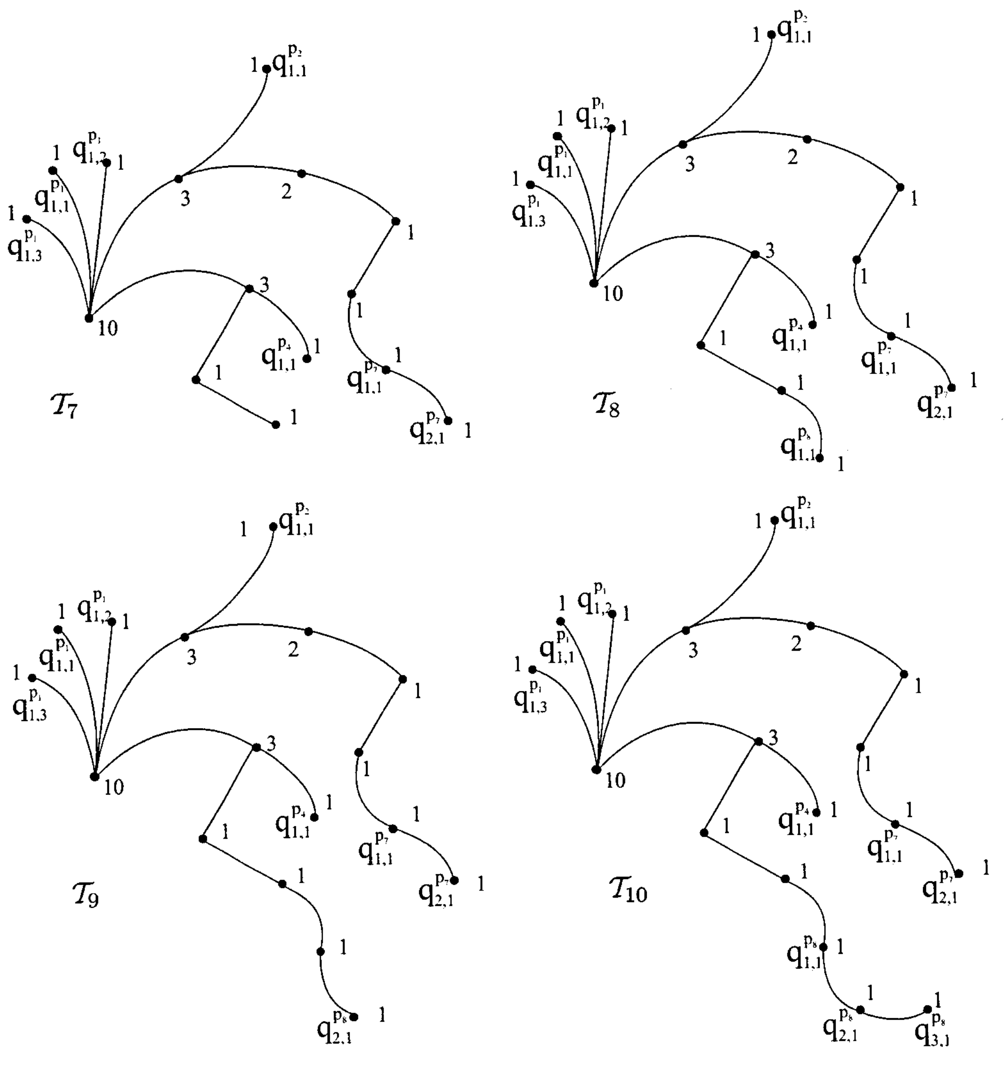



Figure IV.3: Enriques diagrams of the clusters $\mathcal{T}_i, i = 7, \ldots, 10$ obtained by means of the algorithm described for getting generators in Example IV.2.6.

Define the product $\mathcal{J}_1\mathcal{J}_2$ of two ideal sheaves on $X$ satisfying (†) as the image of the natural map $\mathcal{J}_1 \otimes \mathcal{J}_2 \to \mathcal{O}_X$. The ideal sheaf $\mathcal{J}_1\mathcal{J}_2$ has cosupport equal to the union of the cosupport of $\mathcal{J}_1$ and $\mathcal{J}_2$ and for each $Q \in X$, the stalk is the product $(\mathcal{J}_1)_Q(\mathcal{J}_2)_Q \subset \mathcal{O}_{X,Q}$. This product is associative and commutative, thus making the set

$$\mathbf{S}_X = \{\text{ideal sheaves } \mathcal{J} \text{ on } X \text{ satisfying } (\dagger)\}$$

a semigroup. Now, note that if $H_1$ and $H_2$ are Cartier ideals for $X$, then the product $H_1H_2$ is also a Cartier ideal for $X$, as

$$L_{H_1H_2} = L_{H_1} + L_{H_2} \in \bigoplus_{p\in\mathcal{K}_+} \mathbb{Z}\mathcal{L}_p.$$

Therefore, the set

$$\mathbf{I}_R = \{\text{Cartier ideals for } X\}$$

with the natural product of ideals is also a semigroup. Note also that if $H_1, H_2 \in \mathbf{I}_R$, then

$$L_{H_1H_2} = L_{H_1} + L_{H_2} = L_{H_1^o} + L_{H_2^o} = L_{(H_1H_2)^o},$$

and so,

$$(H_1H_2)^o = H_1^oH_2^o. \tag{3.a}$$

It follows in particular that $(H_1H_2)^o\mathcal{O}_X = H_1^oH_2^o\mathcal{O}_X$ is invertible, $H_1^oH_2^o\mathcal{O}_X = \mathcal{O}_X(-L_{H_1} - L_{H_2})$.

**Proposition IV.3.1.** *We have that*

*(a) if $\mathcal{J}_1$ and $\mathcal{J}_2$ are ideal sheaves on $X$ satisfying (†), then $H_{\mathcal{J}_1\mathcal{J}_2} = H_{\mathcal{J}_1}H_{\mathcal{J}_2}$ and $H^o_{\mathcal{J}_1\mathcal{J}_2} = H^o_{\mathcal{J}_1}H^o_{\mathcal{J}_2}$*

*(b) the maps defined in Corollary IV.1.11 give reciprocal isomorphisms of semigroups between $\mathbf{S}_X$ and $\mathbf{I}_R$.*

*Proof.* Clearly, we have

$$L_{\mathcal{J}_1\mathcal{J}_2} = L_{\mathcal{J}_1} + L_{\mathcal{J}_2}. \tag{3.b}$$

By definition (see 1.d), we have

$$H_{\mathcal{J}_1} = \Gamma(X, \mathcal{J}_1\mathcal{O}_X(-L_{\mathcal{J}_1})) \quad \text{and} \quad H_{\mathcal{J}_2} = \Gamma(X, \mathcal{J}_2\mathcal{O}_X(-L_{\mathcal{J}_2})),$$

and so, by Lemma I.3.26,

$$\begin{aligned} H_{\mathcal{J}_1}H_{\mathcal{J}_2} &= \Gamma(X, \mathcal{J}_1\mathcal{O}_X(-L_{\mathcal{J}_1}))\Gamma(X, \mathcal{J}_2\mathcal{O}_X(-L_{\mathcal{J}_2})) = \\ &= \Gamma(X, \mathcal{J}_1\mathcal{J}_2\mathcal{O}_X(-L_{\mathcal{J}_1\mathcal{J}_2})) = H_{\mathcal{J}_1\mathcal{J}_2}. \end{aligned}$$

Now, by (3.a) above, $(H_{\mathcal{J}_1\mathcal{J}_2})^o = H^o_{\mathcal{J}_1}H^o_{\mathcal{J}_2}$. This proves (a). Now, to prove (b), it only remains to show that if $H_1$ and $H_2$ are in $\mathbf{I}_R$, then

$$(H_1H_2\mathcal{O}_X)(H_1H_2\mathcal{O}_X)^{-1} = (H_1\mathcal{O}_X)(H_1^o\mathcal{O}_X)^{-1}(H_2\mathcal{O}_X)(H_2^o\mathcal{O}_X)^{-1}.$$

Clearly, $(H_1\mathcal{O}_X)(H_2)\mathcal{O}_X = H_1H_2\mathcal{O}_X$ and using (3.b), the wanted equality follows. □

Now, in order the study the factorization and semifactorization of complete ideals in the local ring of a sandwiched singularity, we need to fix some notation.

**Notation IV.3.2.** Once a sandwiched singularity $Q \in X$ and a complete $\mathfrak{m}_Q$-primary ideal $J \subset \mathcal{O}_{X,Q}$ are fixed, we write $\mathcal{J}$ for the ideal sheaf on $X$ generated by $J$, $\mathcal{J} = J\mathcal{O}_X$. Clearly, $\mathcal{J}$ satisfies the condition (†) and its cosupport is reduced to the point $Q$.

For any resolution $f' : S' \longrightarrow X$ of $Q$, recall that $J_Q^{S'}$ is the semigroup of all complete $\mathfrak{m}_Q$-primary ideals $J$ such that $J\mathcal{O}_{S'}$ is invertible with the natural product (see subsection I.3.1) and that $\mathbf{J}_Q^*$ is the semigroup of the complete $\mathfrak{m}_Q$-primary ideals in $\mathcal{O}_{X,Q}$. Write $K'$ for the cluster with origin at $O$ so that $S'$ is the surface obtained by blowing-up all the points in $K'$, $S' = S_{K'}$, and let $\{E_p^{S'}\}_{p\in K'}$ be the exceptional components on $S'$ for the blowing-up $\pi_{K'}$ of the points in $K'$,

$$\begin{array}{ccc} S' & \xrightarrow{f'} & X \\ & \searrow^{\pi_{K'}} & \downarrow \pi_I \\ & & S \end{array}$$

As usually, if $Q$ is any point in the exceptional locus of $X$, write

$$T_Q^{S'} = \{p \in K' \mid (f')_*(E_p^{S'}) = Q\}.$$

For each $p \in T_Q^{S'}$, let $D_p^{S'} = \sum_{q\in T_Q^{S'}} a_q^{(p)}$ be the $\mathbb{Q}$-Cartier divisor on $S'$ defined by $|D_p^{S'}\cdot E_q^{S'}|_{S'} = -\delta_{p,q}$ (this is well-defined because the intersection matrix $\mathbf{A}_Q^{S'}$ is negative-definite, see Proposition I.3.4). Equivalently, $D_p^{S'}$ is the exceptional part of the total transform of $\widetilde{C_p}$, where $C_p$ is a curve going sharply through $\mathcal{K}(p)$ and missing all points after $p$ in $K'$.

For each $p \in T_Q^{S'}$, let $m_p \in \mathbb{Z}_{>0}$ be the least integer such that $m_pD_p^{S'}$ is a divisor on $S'$. In virtue of Lemma III.1.3 and Theorem III.1.1, $m_p$ is the least integer $n$ such that one of the following equivalent conditions holds:

(a) $n\widetilde{C_p}$ is Cartier on $X$

(b) $n\mathcal{L}_p \in \bigoplus_{q\in\mathcal{K}_+} \mathbb{Z}\mathcal{L}_q$.

In particular we see that $m_p$ is independent of the resolution $S'$ of $Q$.

**Remark IV.3.3.** If $p \in T_Q^{S'}$, $\mathcal{L}_p$ is not a Cartier divisor in general and hence, $m_p > 1$.

Recall from subsection I.3.1 that if $\mathbb{E}_{S'}^+$ is the semigroup of all exceptional divisors $D$ on $S'$ such that

$$[D\cdot E_q^{S'}]_{S'} \leq 0 \quad \text{for all } q \in T_Q^{S'}$$

then the extremal elements of $\mathbb{E}_{S'}^+$ are all integer multiples of $\{m_p D_p^{S'}\}_{p\in S'}$. Since the sum of divisors in $\mathbb{E}_{S'}^+$ corresponds to the product of complete ideals in $\mathcal{O}_{X,Q}$, we infer that the complete $\mathfrak{m}_Q$-primary ideals $J_p \subset \mathcal{O}_{X,Q}$ defined as the stalks at $Q$ of the ideal sheaves

$$\mathcal{J}_p = (f')_*(\mathcal{O}_{S'}(-m_p D_p^{S'}))$$

are the extremal elements of $J_Q^{S'}$ (see Proposition 1.4 of [12]). Because all of this, the following lemma needs no proof.

**Lemma IV.3.4.** *For each $p \in T_Q^{S'}$, the ideal sheaf $I_p^{m_p}\mathcal{O}_X(m_p\mathcal{L}_p)$ is locally principal except at $Q$, and its stalk at $Q$ equals the complete $\mathfrak{m}_Q$-primary ideal $J_p \subset \mathcal{O}_{X,Q}$.*

The following theorem relates the factorization of complete $\mathfrak{m}_O$-primary ideals in $R$ to the semifactorization and factorization of complete $\mathfrak{m}_Q$-primary ideals in $\mathcal{O}_{X,Q}$. Keep the notation as in section IV.1.

**Theorem IV.3.5.** *Let $J \subset \mathcal{O}_{X,Q}$ be a complete $\mathfrak{m}_Q$-primary ideal. If*

$$H_{\mathcal{J}} = \prod_{p\in T_Q^{S_{\mathcal{J}}}} I_p^{\alpha_p} \tag{3.c}$$

*is the (Zariski) factorization of $H_{\mathcal{J}} \subset R$ into simple ideals, then*

$$J = \prod_{p\in T_Q^{S_{\mathcal{J}}}} J_p^{\frac{\alpha_p}{m_p}}$$

*is the semifactorization of $J$ (in the sense of Theorem I.3.32).*

*In particular, the factorization (3.c) of $H_{\mathcal{J}}$ gives rise to a factorization of $J$ into simple complete ideals of $\mathcal{O}_{X,Q}$ if and only if $\alpha_p \in (m_p)$, for each $p \in T_Q^{S_{\mathcal{J}}}$.*

*Proof.* Write $m = \mathrm{LCM}_{\alpha_p>0}(m_p)$. From the equality (3.c) above, we have that

$$mL_{H_\mathcal{J}} = \sum_{p\in T_Q^{S_\mathcal{J}}} m\alpha_p \mathcal{L}_p. \tag{3.d}$$

and by Theorem IV.1.3, $\mathcal{J} = H_\mathcal{J}\mathcal{O}_X(L_{H_\mathcal{J}})$. Therefore,

$$\begin{aligned}\mathcal{J}^m &= H_\mathcal{J}^m\mathcal{O}_X(mL_{H_\mathcal{J}}) = (\prod_{p\in T_Q^{S_\mathcal{J}}} I_p^{m\alpha_p})\mathcal{O}_X(mL_{H_\mathcal{J}}) = \\ &= \prod_{p\in T_Q^{S_\mathcal{J}}} (I_p^{m\alpha_p}\mathcal{O}_X(m\alpha_p\mathcal{L}_p))\end{aligned}$$

Since the ideals $J_p \subset \mathcal{O}_{X,Q}$ are the stalks at $Q$ of the ideals sheaves given by $I_p^{m_p}\mathcal{O}_X(m_p\mathcal{L}_p)$ (see Lemma IV.3.4), it follows from (3.d) above that the stalk of $\mathcal{J}^m$ at $Q$ is

$$\mathcal{J}_Q^m = \prod_{p\in T_Q^{S_\mathcal{J}}} J_p^{\frac{m\alpha_p}{m_p}},$$

and by allowing rational exponents,

$$J = \prod_{p\in T_Q^{S_\mathcal{J}}} J_p^{\frac{\alpha_p}{m_p}}.$$

This completes the proof of the first claim. The second assertion follows immediately from this one. □

In virtue of Remark IV.3.3 and Theorem IV.3.5, it is clear that one may not expect the factorization of a complete $\mathfrak{m}_O$-primary ideal $H \subset R$ to give rise to a factorization into simple complete ideals in $\mathcal{O}_{X,Q}$. However, from Proposition IV.3.1 we know that fixed a complete $\mathfrak{m}_O$-primary ideal $J \subset \mathcal{O}_{X,Q}$ and if $\mathcal{J}$ is the ideal sheaf generated by $J$ on $X$, each factorization of $J$ induces a factorization of the ideals $H_\mathcal{J}$ and $H_\mathcal{J}^o$ into complete $\mathfrak{m}_O$-primary ideals of $R$ (which are not simple in general). In particular, it gives a necessary condition for a factorization of $H_\mathcal{J}$

$$H_\mathcal{J} = \prod_{j=1}^{s} I_j^{\beta_j}$$

to give rise to a factorization of $J$: for each $j \in \{1, \dots, s\}$,

$$L_{I_j} = \sum_{p\in\mathcal{K}_+} v_p(I_j)L_p \in \bigoplus_{p\in\mathcal{K}_+} \mathbb{Z}\mathcal{L}_p,$$

or equivalently, by Theorem III.1.1, $L_{I_j}$ must be a Cartier divisor on $X$. In Theorem IV.3.7 we will show that this condition is somehow sufficient. First, we need a definition.

**Definition IV.3.6.** We say that a complete $\mathfrak{m}_O$-primary ideal $H \subset R$ is an *irreducible Cartier ideal for* $X$ if it is Cartier for $X$ and $L_H$ is irreducible as an element of $\bigoplus_{p\in\mathcal{K}_+} \mathbb{Z}\mathcal{L}_p$.

**Theorem IV.3.7.** *Given a complete* $\mathfrak{m}_Q$*-primary ideal* $J \subset \mathcal{O}_{X,Q}$*, each factorization of* $J$ *into complete* $\mathfrak{m}_Q$*-primary ideals*

$$J = \prod_{i=1}^{r} J_i^{\alpha_i} \tag{3.e}$$

*induces a factorization of* $H_{\mathcal{J}}$ *into Cartier ideals for* $X$

$$H_{\mathcal{J}} = \prod_{i=1}^{r} H_{\mathcal{J}_i}^{\alpha_i},$$

*and each factorization of* $H_{\mathcal{J}}$ *into Cartier ideals for* $X$ *has this form. Moreover,* $H_{\mathcal{J}_i}$ *is irreducible as a Cartier ideal for* $X$ *if and only if* $J_i$ *is a simple complete* $\mathfrak{m}_Q$*-primary ideal.*

*Proof.* By (a) of Proposition IV.3.1, each factorization of $J$ as (3.e) gives rise to a factorization

$$H_{\mathcal{J}} = \prod_{i=1}^{r} H_{\mathcal{J}_i}^{\alpha_i},$$

where $\mathcal{J}_i$ are the ideal sheaves generated by $J_i$ on $X$. By Remark IV.1.7, each $H_{\mathcal{J}_i}$ is a Cartier ideal for $X$ (see Corollary IV.1.11).

Conversely, assume that

$$H_{\mathcal{J}} = \prod_{j=1}^{s} H_j^{\beta_j}$$

where each $H_j$ is a Cartier ideal for $X$. Then, $L_{I_j} \in \bigoplus_{p\in\mathcal{K}_+} \mathbb{Z}\mathcal{L}_p$ and by Corollary IV.1.11, the ideal sheaves given by $H_j\mathcal{O}_X(L_{H_j})$ satisfy the condition (†). Therefore, their stalk $J_i$ at $Q$ are complete $\mathfrak{m}_{Q_i}$-primary ideals in $\mathcal{O}_{X,Q}$. Since $H_{\mathcal{J}}\mathcal{O}_X = \prod_{i=1}^{s} I_j^{\beta_j}\mathcal{O}_X$ and $L_{\mathcal{J}} = L_{H_{\mathcal{J}}} = \sum_{j=1}^{s} \beta_j L_{i_j}$, we deduce

$$H_{\mathcal{J}}\mathcal{O}_X(L_{H_{\mathcal{J}}}) = \sum_{i=1}^{s} I_j^{\beta_j}\mathcal{O}_X(L_{I_j}).$$

By taking the stalks at $Q$, we deduce that

$$J = \prod_{i=1}^{s} J_i^{\beta_i}$$

and the claim follows. □

**Remark IV.3.8.** In Example IV.3.10, we show that in general, it is not true that if $J$ is a simple ideal in $\mathcal{O}_{X,Q}$, then $L_{\mathcal{J}}$ is irreducible in $\bigoplus_{p\in\mathcal{K}_+} \mathbb{Z}\mathcal{L}_p$.

**Remark IV.3.9.** In virtue of Theorem IV.3.7, if a complete $\mathfrak{m}_Q$-primary ideal $J \subset \mathcal{O}_{X,Q}$ has unique factorization into simple $\mathfrak{m}_Q$-primary ideals, then $H_{\mathcal{J}}$ has unique factorization into irreducible Cartier ideals for $X$. Of course, this is the case if $J$ is simple. Next we show an example of this fact.

**Example IV.3.10.** Take the sandwiched singularity $(X, Q)$ obtained when blowing-up a complete $\mathfrak{m}_O$-primary ideal $I$ whose cluster of base points has the Enriques Diagram shown in Figure IV.4. Take the complete $\mathfrak{m}_Q$-primary ideal $J = \mathfrak{m}_Q^3$ of $\mathcal{O}_{X,Q}$. Then,

$$H_{\mathcal{J}} = I_Q^3 = I_{p_1}^3 I_{p_2}^3.$$

By Theorem IV.3.7, each factorization of $H_{\mathcal{J}}$ into irreducible Cartier ideals for $X$ induces a factorization of $J$ into simple ideals of $\mathcal{O}_{X,Q}$. Now, note that the ideals $I_Q$, $I_{p_1}^3$ and $I_{p_2}^3$ are Cartier ideals for $X$, since

$$L_{I_Q} = L_I = 3L_{p_3} \qquad L_{I_{p_1}^3} = 3L_{p_3} \qquad L_{I_{p_2}^3} = 6L_{p_3},$$

and in fact, $I_Q$ and $I_{p_1}^3$ are irreducible as Cartier ideals for $X$. Moreover, although $I_{p_2}^3$ is not an irreducible Cartier ideal for $X$, the stalk at $Q$ of the ideal sheaf $I_{p_2}^3\mathcal{O}_X(6L_{p_3})$ is simple. Hence, the factorizations of $H_{\mathcal{J}}$

$$H_{\mathcal{J}} = I_Q^3 = I_{p_1}^3 I_{p_2}^3$$

into Cartier ideals for $X$ induce two different factorizations of $J$ into simple ideals of $\mathcal{O}_{X,Q}$:

$$J = \mathfrak{m}_Q^3 = J_1 J_2,$$

where $J_1, J_2$ are the stalks at $Q$ of the ideal sheaves on $X$ given by $I_{p_1}^3\mathcal{O}_X(3L_{p_3})$ and $I_{p_2}^3\mathcal{O}_X(6L_{p_3})$.

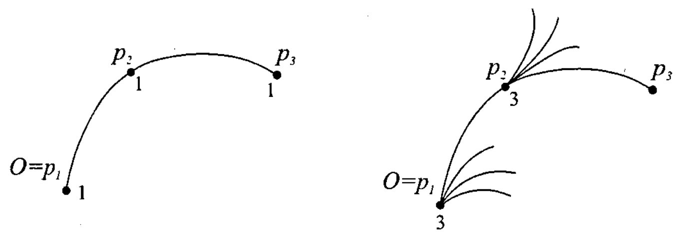



Figure IV.4: On the left, the Enriques diagram of the cluster of base points of the ideal $I$ in Example obtained by means of the algorithm described in section III.3 and corresponding to Example IV.3.10; on the right, the Enriques diagram of the cluster $\mathcal{K}_J$.

# Chapter V

# Some consequences on the Nash conjecture for sandwiched singularities

The study of the space of arcs of a singular variety is originally motivated by a preprint of Nash, later published as [46], where he posed the question whether each essential component over a singular variety is associated to some arc family. This question has been studied by Bouvier, González-Sprinberg, Lejeune-Jalabert, Nobile, Reguera-López among others [24, 25, 26, 38, 39, 48, 53] and more recently, by Ishii and Kollár [33], who proved the question to be true for toric varieties but false in general. However, the Nash conjecture remains open for surface singularities in general.

Related with the question proposed by Nash and for normal surface singularities, Lejeune-Jalabert posed in [38] an apparently simpler question: whether a wedge centered at a general arc on the singularity lifts to its minimal desingularization. In [38] an affirmative answer is given for toric surface singularities and some cases of quasi-homogeneous surface singularities, and in [40], for sandwiched singularities. A criterion for the existence of smooth curves through any surface singularity is given in [25] and from it, a partition of the space of smooth curves into disjoint families, each family corresponding to a reduced component of the maximal cycle in case the singularity is normal. As a consequence, it can be seen that the wedges centered at a smooth curve lift to the minimal resolution of the singularity.

The aim of this chapter is to apply some of the results already seen in the previous chapters of this memoir to infer some facts concerning the spaces of Nash arcs on a sandwiched singularity. In sections V.2 and V.3 we prove that the reduced components of the fundamental cycle correspond to Nash families of arcs and so, we reprove in particular that the Nash conjecture for minimal surface singularities is true (see Corollary 2.6 of [53]). In section V.4, we show that the Nash conjecture for sandwiched and for primitive singularities are equivalent.

REMARK While completing the results of this chapter, A. Reguera has proved that if the problem of wedges has an affirmative answer, then also does the Nash conjecture (see [55]). As a consequence of the results of [40], it turns out that the Nash conjecture is true for sandwiched singularities.

## V.1 On Nash families of arcs on a sandwiched singularity

Let $(X,Q)$ be a normal surface singularity and $f : S' \longrightarrow (X,Q)$ its minimal resolution. An *arc* on $(X,Q)$ is a formal parametrized curve, i.e. a $\mathbb{C}$-morphism (Spec $\mathbb{C}[[t]], O) \longrightarrow (X,Q)$ or equivalently, a $\mathbb{C}$-morphism of local rings

$$\varphi : \mathcal{O}_{X,Q} \longrightarrow \mathbb{C}[[t]].$$

We write $Z(\varphi)$ for its image in $X$, i.e. $Z(\varphi) = s\ C$, where $C$ is the prime divisor defined by $\varphi$ and $s$ is the ramification index of the morphism (Spec $\mathbb{C}[[t]], 0) \longrightarrow (\overline{C}, Q)$, $\overline{C}$ being the the normalization of $C$. We denote by $\mathcal{H}_Q$ the set of arcs on $(X,Q)$.

For each $i > 0$, an *arc of order* $i$ *on* $(X,Q)$ is a $\mathbb{C}$-morphism

$$\varphi_i : \mathcal{O}_{X,Q} \longrightarrow \frac{\mathbb{C}[[t]]}{(t^{i+1})}$$

and we denote by $\mathcal{H}_Q(i)$ the set of arcs of order $i$ on $(X,Q)$. The projection $\pi_i : \mathbb{C}[[t]] \longrightarrow \frac{\mathbb{C}[[t]]}{(t^{i+1})}$ induces a natural map $\rho_i : \mathcal{H}_Q \longrightarrow \mathcal{H}_Q(i)$ and if $Q$ is a singular point, $\rho_i$ is not surjective in general. We write $Tr(i)$ for the image $\rho_i(\mathcal{H}_Q)$ and we call the elements of $Tr(i)$ the *i-truncations* of arcs on $(X,Q)$. The $\mathcal{H}_Q(i)$ together with the natural projections

$$\rho_{i,j} : \mathcal{H}_Q(j) \longrightarrow \mathcal{H}_Q(i)$$

for $j \geq i$, form a projective system of affine algebraic sets and we have $\mathcal{H}_Q = \varprojlim \mathcal{H}_Q(i)$. Therefore, $\mathcal{H}_Q$ has a proalgebraic structure.

In [46], J. Nash proved that $Tr(i)$ is a constructible subset of $\mathcal{H}_Q(i)$. In fact, it can be seen that for all $i \geq 0$, there exists some $\beta(i) \geq i$ such that $Tr(i) = \rho_{i,\beta(i)}(\mathcal{H}_Q(\beta(i)))$ and $Tr(i)$ being the image of an algebraic set by a morphism, it is constructible (see [4, 39]). We write $\overline{Tr(i)}$ for the Zariski closure of $Tr(i)$ in $\mathcal{H}_Q(i)$.

If $\{E_u\}_{u \in \Delta_Q}$ are the irreducible exceptional components of $S'$, let $\mathcal{F}_u^Q$ denote the set of arcs on $(X,Q)$ such that the lifted arc $\widetilde{\varphi}$ on $S'$ intersects $E_u$.

Then, if $\mathcal{F}_u(i) = \rho_i(\mathcal{F}_u^Q)$, the minimal resolution $f : S' \longrightarrow (X, Q)$ induces a decomposition of the space of $i$-truncations $Tr(i) = \bigcup_{u\in\Delta_Q} \mathcal{F}_u(i)$ and, by taking the Zariski closure in $\mathcal{H}_Q(i)$,

$$\overline{Tr(i)} = \bigcup_{u\in\Delta_Q} \overline{\mathcal{F}_u(i)}. \tag{1.a}$$

Similarly, for each $u \in \Delta_Q$, write $\mathcal{N}_u^Q$ for the set of arcs $\varphi$ on $(X, Q)$ such that the lifted arc on $S'$ intersects transversally $E_u$ and does not intersect the other exceptional components (i.e. $|\widetilde{Z(\varphi)}^{S'} \cdot E_v|_{S'} = \delta_{uv}$, the Kronecker delta). For each $i > 0$, write $\mathcal{N}_u(i)$ for the set of $i$-truncations of arcs in $\mathcal{N}_u^Q$, $\mathcal{N}_u(i) = \rho_i(\mathcal{N}_u(i))$, and $\overline{\mathcal{N}_u(i)}$ its Zariski closure in $\mathcal{H}_Q(i)$. Clearly, for each $u$ and each $i > 0$, we have $\mathcal{N}_u(i) \subset \mathcal{F}_u(i)$. In Proposition 1.2 of [53], it is shown that in fact, $\overline{\mathcal{N}_u(i)} = \overline{\mathcal{F}_u(i)}$, and by (1.a),

$$\overline{Tr(i)} = \cup_{u\in\Delta_Q}\overline{\mathcal{N}_u(i)}.$$

It is not hard to see that the $\mathcal{F}_u(i), u \in \Delta_Q$ are irreducible, and consequently, so are $\overline{\mathcal{N}_u(i)} = \overline{\mathcal{F}_u(i)}$. Hence, it is clear that the number of irreducible components of $\overline{Tr(i)}$ is bounded by the number $\sharp\Delta_Q$ of exceptional components in the minimal resolution of $Q$. The Nash conjecture for normal surface singularities says that for $i \gg 0$, the number of the irreducible components of $\overline{Tr(i)}$ is exactly $\sharp\Delta_Q$. Equivalently, the question is to decide if for $i \gg 0$, there are no inclusions $\overline{\mathcal{N}_u(i)} \subset \overline{\mathcal{N}_v(i)}$ for $u \neq v$.

Since the number of the irreducible components $C_\lambda(i)$ of $\overline{Tr(i)}$ increases with $i$ and is bounded, it becomes constant for $i$ big enough. Thus, for $j > i \gg 0$, each component of $\overline{Tr(j)}$ is projected into a dense subset of a component of $\overline{Tr(i)}$ and thus, the components $C_\lambda(j)$ of $\overline{Tr(j)}$ may be identified with those of $\overline{Tr(i)}$. A Nash family of arcs consists of those arcs whose truncation are projected by $\rho_{j,i}$ in $C_\lambda(j)$ for some $\lambda$ all $j > i \gg 0$ (see [46] p.32).

## V.2 Some results for sandwiched singularities

From now on, we assume that $(X, Q)$ is a sandwiched singularity. Keeping the notation introduced in Chapters I and II, we assume that a complete $\mathfrak{m}_O$-primary ideal $I \subset R$ has been chosen, and write $X = Bl_I(S)$ and $\mathcal{K} = (K, \nu)$ for the cluster of base points of $I$. If $f : S_K \longrightarrow X$ is the minimal resolution of $X$, then $T_Q = \{p \in K \mid f_*(E_p) = Q\}$ ($T_Q = \Delta_Q$ with the notation of the preceding section).

In this section, we give a necessary condition in order to have an inclusion $\overline{\mathcal{N}_p(i)} \subset \overline{\mathcal{N}_q(i)}$ for $p \neq q$ in $T_Q$ which is formulated in terms of the intersection multiplicity of generic curves (on $S$) going virtually through the clusters $\mathcal{K}(p)$ and $\mathcal{K}(q)$.

For each $p \in T_Q$, write

$$D_p^Q = \sum_{u \in T_Q} a_u^{(p)} E_u \in \bigoplus_{u \in T_Q} \mathbb{Q} E_u,$$

for the $\mathbb{Q}$-Cartier divisor on $S_K$ being the exceptional part of the total transform of any curve on $X$ whose strict transform on $S_K$ intersects transversally $E_p$ and does not intersect the other exceptional components (see Definition I.3.19). In Theorem 1.10 of [53] and for rational surface singularities, it is seen that an inclusion $\overline{\mathcal{N}_p(i)} \subset \overline{\mathcal{N}_q(i)}$ implies that $D_p^Q - D_q^Q$ is a non-zero effective $\mathbb{Q}$-Cartier divisor. We write $D_p^Q > D_q^Q$ in this situation.

The main result of this section is the following.

**Theorem V.2.1.** *If $D_p^Q > D_q^Q$ and $C$ is any curve on $S$, then*

$$[C, C_p]_O > [C, C_q]_O$$

*for a generic curve $C_q$ going through the cluster $\mathcal{K}(q)$ and any curve $C_p$ going through $\mathcal{K}(p)$. In particular, $v(I_p) > v(I_q)$ for any divisorial valuation $v$ of $R$, and so $I_p \subset I_q$.*

*Proof.* Let $C$ be any curve through $O$. Clearly it is enough to prove the first assertion for generic curves $C_q$ and $C_p$ going through $\mathcal{K}(q)$ and $\mathcal{K}(p)$, respectively. If $C_u, u \in T_Q$ goes sharply through $\mathcal{K}(u)$ and misses all points after $u$ in $K$ (see Theorem I.1.30), in virtue of 4. of Proposition I.1.16, we have that for every $v \in T_Q$,

$$|\widetilde{C_u}^{S_K} \cdot E_v|_{S_K} = \delta_{uv},$$

and so, with the notation of section III.2, $D_{\widetilde{C_u}}^{S_K} = D_u^Q$. If moreover we assume that $C_u$ shares no points with $C$ outside of $K$ (we can do it again by Theorem I.1.30), then $|\widetilde{C}^{S_K} \cdot \widetilde{C_u}^{S_K}|_{S_K} = 0$ and by the projection formula applied to $\pi_K$ (5. of Proposition I.1.16)

$$\begin{aligned} [C, C_u]_O &= |(\widetilde{C}^{S_K} + E_C^{S_K}) \cdot \widetilde{C_u}^{S_K}|_{S_K} = |E_C^{S_K} \cdot \widetilde{C_u}^{S_K}|_{S_K} = \\ &= |D_{\widetilde{C}} \cdot \widetilde{C_u}^{S_K}|_{S_K} + |f^*(L_C) \cdot \widetilde{C_u}^{S_K}|_{S_K}, \end{aligned}$$

the last equality since $E_C^{S_K} = D_{\widetilde{C}} + f^*(L_C)$. Again by the projection formula and the symmetry of the intersection number (see Definition I.3.19) we have

$$|D_{\widetilde{C}} \cdot \widetilde{C_u}^{S_K}|_{S_K} = |\widetilde{C}^{S_K} \cdot D_u^Q|_{S_K}$$

and similarly,

$$|f^*(L_C)\cdot\widetilde{C_u}^{S_K}|_{S_K} = |\widetilde{L_C}^{S_K}\cdot D_u^Q|_{S_K}.$$

Thus, for any $u \in T_Q$,

$$[C, C_u]_O \;=\; |\widetilde{C}^{S_K}\cdot D_u^Q|_{S_K} + |\widetilde{L_C}^{S_K}\cdot D_u^Q|_{S_K}. \tag{2.a}$$

Now, since we are assuming that $D_p^Q > D_q^Q$, it is clear that

$$\begin{aligned} |\widetilde{C}^{S_K}\cdot D_p^Q|_{S_K} &\geq |\widetilde{C}^{S_K}\cdot D_q^Q|_{S_K} \\ |\widetilde{L_C}^{S_K}\cdot D_p^Q|_{S_K} &\geq |\widetilde{L_C}^{S_K}\cdot D_q^Q|_{S_K} \end{aligned} \tag{2.b}$$

and from the equality (2.a) above, we have

$$[C, C_p]_O \;\geq\; [C, C_q]_O. \tag{2.c}$$

**Remark V.2.2.** Notice by the way that $q$ cannot be infinitely near to $p$, for otherwise

$$e_v(C_q) \geq e_v(C_p), \quad \text{for all } v \in \mathcal{K}(q)$$

and hence, if $C$ goes sharply through $\mathcal{K}(q)$, the Noether formula (Theorem I.1.31) would say that

$$\begin{aligned} [C, C_p]_O &= \sum_{u\in\mathcal{K}(p)} e_u(C)e_u(C_p) \leq \sum_{u\in\mathcal{K}(p)} e_u(C)e_u(C_q) < \\ &< \sum_{u\in\mathcal{K}(q)} e_u(C)e_u(C_p) = [C, C_q]_O \end{aligned}$$

against (2.c).

To complete the proof it will be sufficient to prove that the inequality (2.b) is strict. To this aim, we need a couple of easy lemmas concerning chains in the dual graph $\Gamma_Q$ of $Q$ (see section I.5). We state them separately for clarity.

**Remark V.2.3.** Unless some confusion may arise, we will identify the infinitely near points of $T_Q$ and the vertices in the dual graph of $Q$, $\Gamma_Q$, representing the exceptional component of $S_K$ associated to them (see §1.5). Hence, we may write for instance $T_Q = |\Gamma_Q|$.

**Lemma V.2.4.** *Let $\Gamma'$ be the connected component of $\Gamma_Q - \{q\}$ the vertex $p$ is belonging to. We have*

(a) *Let* $\alpha, \beta, u \in |\Gamma_Q|$ *such that* $ch(\alpha, u) \cap ch(\beta, u) = \{u\}$. *Then* $ch(\alpha, \beta) = ch(\alpha, u) \cup ch(\beta, u)$ *and so,* $u \in ch(\alpha, \beta)$.

(b) *If* $\alpha, \beta \in |\Gamma'|$, *then* $ch(\alpha, \beta) \subset |\Gamma'|$.

(c) *If* $\alpha \in |\Gamma'|$ *and* $\beta \notin |\Gamma'|$, *then* $q \in ch(\alpha, \beta)$.

*Proof.* To prove (a), assume that $u \notin ch(\alpha, \beta)$ and let $w \in ch(\alpha, \beta)$ be such that $d(w, u) = \min_{v \in ch(\alpha,\beta)} d(v, u)$. Then, $w \neq u$ and $w \in ch(\alpha, u) \cap ch(\beta, u)$, against the assumption. Part (b) and (c) follow directly from the definitions. □

**Lemma V.2.5.** *Assume as above that* $D_p^Q > D_q^Q$. *Then we have*
(a) $a_u^{(p)} > a_u^{(q)}$ *for all* $u \in |\Gamma'|$.
(b) $p$ *and all the points of* $T_Q$ *infinitely near to* $p$ *are in* $\Gamma'$.

*Proof.* By the definition of $\Gamma'$, it is clear that $p \in |\Gamma'|$. Take $u \in T_Q$ any point infinitely near to $p$. If $u \notin |\Gamma'|$, from (c) of Lemma V.2.4 we know that $q \in ch(p, u)$ and by (a) of Proposition II.6.2, $q$ is infinitely near to $p$, which is impossible by Remark V.2.2. This proves (b).

Now, we use induction on the length of $ch(p, u)$ to prove (a). First, since $|D_u^Q.E_v|_{S_K} = -\delta_{u,v}$ for all $u, v \in T_Q$, we have

$$\mathbf{A}_Q D_u^Q = -\mathbf{1}_u$$

and hence

$$\begin{aligned} -1 &= \mathbf{1}_p^t \mathbf{A}_Q (D_p^Q - D_q^Q) \\ &= -\omega(p)(a_p^{(p)} - a_p^{(q)}) + \Sigma_{d(u,p)=1}(a_u^{(p)} - a_u^{(q)}) \end{aligned}$$

Since $a_u^{(p)} \geq a_u^{(q)}$ for all $u \in T_Q$, we deduce from the above equality that $a_p^{(p)} > a_p^{(q)}$. Now, if $u \in |\Gamma'|$ is not $p$ or $q$, take $u'$ the vertex adjacent to $u$ such that $d(p, u') = d(p, u) - 1$. By the induction hypothesis, $a_{u'}^{(p)} > a_{u'}^{(q)}$. Thus

$$\begin{aligned} \mathbf{1}_u^t \mathbf{A}_Q (D_p^Q - D_q^Q) &= -\omega(u)(a_u^{(p)} - a_u^{(q)}) + \Sigma_{d(u,v)=1}(a_v^{(p)} - a_v^{(q)}) \\ &= -\omega(u)(a_u^{(p)} - a_u^{(q)}) + (a_{u'}^{(p)} - a_{u'}^{(q)}) + \Sigma_{d(u,v)=1, v\neq u'}(a_v^{(p)} - a_v^{(q)}) \\ &> -\omega(u)(a_u^{(p)} - a_u^{(q)}) \end{aligned}$$

Since the first member of this inequality is zero, we deduce that

$$a_u^{(p)} > a_u^{(q)}$$

as claimed. □

*End of the proof of Theorem V.2.1.* Take $u_0 \in T_Q$ any point infinitely near to $p$ and maximal. Thus, by *3.* of Proposition I.1.16, $E_{u_0}$ intersects $\widetilde{L_C}^{S_K}$. By Lemma V.2.5, $u_0 \in |\Gamma'|$ and $a_{u_0}^{(p)} > a_{u_0}^{(q)}$. Since $|\widetilde{L_C}^{S_K}.E_{u_0}|_{S_K} > 0$, it follows that

$$\begin{aligned} |\widetilde{L_C}^{S_K} \cdot D_p^Q|_{S_K} &= a_{u_0}^{(p)} |\widetilde{L_C}^{S_K} \cdot E_{u_0}|_{S_K} + \sum_{u \in T_Q, u \neq u_0} a_u^{(p)} |\widetilde{L_C}^{S_K} \cdot E_u|_{S_K} \\ &> a_{u_0}^{(q)} |\widetilde{L_C}^{S_K} \cdot E_{u_0}|_{S_K} + \sum_{u \in T_Q, u \neq u_0} a_u^{(q)} |\widetilde{L_C}^{S_K} \cdot E_u|_{S_K} \\ &= |\widetilde{L_C}^{S_K} \cdot D_q^Q|_{S_K} \end{aligned}$$

Hence, by the equality (2.a),

$$[C, C_p]_O > [C, C_q]_O$$

as claimed. In particular, by taking $C$ a smooth curve through $O$ and not tangent to $C_p$ or $C_q$, we have that

$$e_O(C_p) > e_O(C_q). \qquad (2.d)$$

Finally, we have that for any divisorial valuation $v_u$ and any complete ideal $J \subset R$, $v_u(J)$ is the intersection multiplicity of a curve defined by a generic element of $J$ with a generic curve going through $\mathcal{K}(u)$. Hence, in virtue of (2.d), we have $v_u(I_p) > v_u(I_Q)$ and since the complete $\mathfrak{m}_O$-primary ideals are defined by divisorial valuations (see Definition I.2.1), the inclusion of simple ideals

$$I_p \subset I_q$$

follows. □

## V.3 On Nash families of smooth arcs

This section is aimed to prove the following theorem.

**Theorem V.3.1.** *Let $Q$ be a sandwiched singularity. Then every reduced component of the fundamental cycle of $Q$ is associated to a Nash family of arcs. In other words, if there exists $p, q \in T_Q$ such that $\overline{\mathcal{N}_p(i)} \subsetneq \overline{\mathcal{N}_q(i)}$ for $i >> 0$, then $z_p > 1$.*

The proof will follow from an accurate analysis of the graph $\Gamma_Q$ and the coefficients of the fundamental cycle in connection with the proximity relations of the points of the cluster.

Throughout this section, we assume that $D_p^Q > D_q^Q$ for $p, q \in T_Q$ and keep the notation of chapter II. In particular, $\mathcal{K}_Q = (K, \nu')$ is the cluster introduced in Notation II.2.9 and $\rho'_u$ means its excess at any $u \in K$. The following results are technical and we state them separately for clarity.

**Lemma V.3.2.** *If there exists some $u_0 \in |\Gamma'|$ such that $\rho'_{u_0} > 0$, then $z_p > z_q$.*

*Proof.* Let $C \subset S$ be a curve going sharply through the cluster $\mathcal{K}_Q$. Then, if $u \in T_Q$, $\rho'_u$ is the number of branches of $C$ missing all points after $u$ in $K$ (Proposition I.1.29) and by 4. of Proposition I.1.16, we have $\rho'_u = |\widetilde{C}^{S_K} \cdot E_u|_{S_K}$. On the other hand, the strict transform on $X$ of $C$ is a generic hypersurface section of $(X, Q)$ and not contained in the exceptional locus of $X$ (see Proposition II.4.2). Thus, $\rho'_u$ is also the number of branches meeting $E_u$ on $S_K$ and no one is not transverse to $E_u$. From this, $\rho'_u = -\mathbf{1}_u^t \mathbf{A}_Q Z_Q$.

Since $\mathbf{A}_Q D_u^Q = -\mathbf{1}_u$ for all $u \in T_Q$, we have

$$z_u = \mathbf{1}_u^t Z_Q = -(D_u^Q)^t \mathbf{A}_Q Z_Q$$

and so,

$$\begin{aligned} z_p - z_q &= -(D_p^Q - D_q^Q)^t \mathbf{A}_Q Z_Q = (D_p^Q - D_q^Q)^t (\rho'_u)_{u \in T_Q} \\ &= (a_{u_0}^{(p)} - a_{u_0}^{(q)}) \rho'_{u_0} + \sum_{u \in T_Q, u \neq u_0} (a_u^{(p)} - a_u^{(q)}) \rho'_u > 0 \end{aligned}$$

the last inequality by (a) of Lemma V.2.5 applied to $u_0$. □

**Lemma V.3.3.** *Assume that $z_p = z_q$.*

*(a) If $u$ is any point in $|\Gamma'|$ such that all the points in $T_Q$ infinitely near to $u$ are also in $|\Gamma'|$, then $\nu'_u = 0$ and $\nu_u = 1$.*

*(b) If $u$ is any point in $|\Gamma'|$, there is at most one point $v$ in $K$ proximate to $u$ such that $\nu'_v = \nu_v - 1$. If $\nu'_u = \nu_u$, there is not such a point.*

*Proof.* By (c) of Lemma II.3.1, it is enough to prove that $\nu'_u = 0$ and this in turn follows by induction on the number of points infinitely near to $u$, using that by Lemma V.3.2 all these points have excess in $\mathcal{K}'$ equal to zero. This gives (a).

For (b), write $s_u = \sharp\{v \in \mathcal{K}(\omega) \mid v \to u, \nu'_v = \nu_v - 1\}$. By Lemma II.3.1 and since $\rho_u = 0$ we have that

$$\rho'_u = \nu'_u - \sum_{v \to u} \nu'_v \geq (\nu_u - 1) - \sum_{v \to u} \nu_v + s_u = s_u - 1.$$

Since $u \in |\Gamma'|$, $\rho'_u = 0$ and so $s_u \leq 1$. If $\nu'_u = \nu_u$, the same argument shows that $\rho'_u \geq s_u$ and so, $s_u = 0$. Hence the claim. □

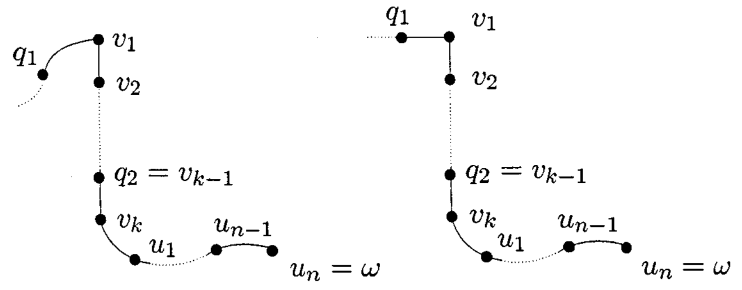



Figure V.1: Enriques Diagrams of the cluster $\mathcal{K}(\omega)$. On the left it is shown the case when $v_1$ is free; on the right, when $v_1$ is satellite.

In particular, if $z_p = z_q$ by (b) of Lemma V.2.5 we have that $\nu'_p = 0$ and $\nu_p = 1$, and so there is only one point in $\mathcal{K}$ infinitely near to $p$, say $\omega$, having positive excess. Let $\mathcal{K}(\omega)$ be the irreducible cluster determined by $\omega$. Denote by $x_p$ the last satellite point in $\mathcal{K}(\omega)$ and $q_1, q_2$ the points $x_p$ is proximate to, $q_2$ proximate to $q_1$. Write $v_1, ..., v_{k-1} = q_2, v_k = x_p$ the points of $\mathcal{K}(\omega)$ proximate to $q_1$ and $u_1, ..., u_n = \omega$ the free points in $\mathcal{K}(\omega)$ following $x_p$. Notice that $\nu_{q_1} > 1$ and that $p$ is infinitely near to $q_1$ since it has virtual multiplicity one in $\mathcal{K}$. The next figure shows the points infinitely near to $q_1$ as they appear in the Enriques Diagram of $\mathcal{K}(\omega)$.

*Proof of Theorem V.3.1* For any $u \in T_Q$, $z_u$ is the multiplicity at $Q$ of any arc in $\mathcal{N}_u$. Hence, if $\overline{\mathcal{N}_p(i)} \subset \overline{\mathcal{N}_q(i)}$, $z_p \geq z_q \geq 1$. Obviously, if $z_p > z_q$ there is nothing to prove. So, we will assume that $D_p^Q > D_q^Q$ and $z_p = z_q$.

Since $x_p$ is the last satellite point in $\mathcal{K}(\omega)$, we have that $v_O(I_{x_p}) \geq v_O(I_u)$ for any point $u$ in $\mathcal{K}(\omega)$. In particular, $v_O(I_{x_p}) \geq v_O(I_p)$. Let $\widetilde{u_0}$ be the last point in $\mathcal{K}(p) \cap \mathcal{K}(q)$. Since $q$ is infinitely near or equal to $\widetilde{u_0}$, $v_O(I_q) \geq v_O(I_{\widetilde{u_0}})$. Now, by Theorem V.2.1, we know that $v_O(I_p) > v_O(I_q)$, so we get a chain of inequalities

$$v_O(I_{x_p}) \geq v_O(I_p) > v_O(I_q) \geq v_O(I_{\widetilde{u_0}}) \tag{3.a}$$

and from it, we see that $q$ is not infinitely near to $x_p$. Moreover, since $\widetilde{u_0}$ and $x_p$ are both in $\mathcal{K}(\omega)$, $x_p$ is infinitely near to $\widetilde{u_0}$.

Now, if $x_p \notin |\Gamma'|$, by (c) of Lemma V.2.4, we have that $q \in ch(p, x_p)$. Since $p$ and $x_p$ are in $\mathcal{K}(\omega)$, either $p$ is infinitely near to $x_p$ or vice versa. In any case, by (a) of Proposition II.6.2 $q$ must be infinitely near to one of them, against (3.a). Hence, $x_p \in |\Gamma'|$. The same argument shows that if $u$ is any point of $T_Q$ infinitely near to $x_p$, then $u \in |\Gamma'|$. Thus, we can apply (a) of Lemma V.3.3 to get that $\nu'_{x_p} = 0$ and $\nu_{x_p} = 1$. In particular, there is just one point in $\mathcal{K}$ and in the first neighborhood of $x_p$ and so, it is $u_1$, which is free. Thus, $x_p$ is maximal among the points of $\mathcal{K}$ proximate to $q_1$ and so, the corresponding vertices in $\Gamma_Q$ are adjacent (see *3.* of Proposition I.1.16).

To complete the proof, we will distinguish if $q$ is in $\mathcal{K}(\omega)$ or not. It is worth notice that the assumption $z_p = z_q$ will lead to contradiction in all cases but when $q = q_1$ and $p$ is infinitely near to it: then we need also to assume that $z_q = 1$. Notice that as $\mathcal{K}(\omega)$ is irreducible, all the points in it are ordered.

*Case 1:* $q$ is not in $\mathcal{K}(\omega)$.

As the vertices of $q_1$ and $x_p$ are adjacent, we deduce that $q_1$ is in $\Gamma'$, and by the same reason, $v_{k-1}$ is in $\Gamma'$ too. Moreover, we know that $\nu'_{x_p} = \nu_{x_p} - 1$ hence, by (b) of Lemma V.3.3, $\nu'_{v_j} = \nu_{v_j}$ for $j = 1, ..., k-1$. Applying again (b) of Lemma V.3.3 to $v_{k-1}$, we are led to contradiction.

*Case 2:* $q$ is in $\mathcal{K}(\omega)$.

If $q \neq q_1$ and $v_{k-1} \in |\Gamma'|$, the same argument used in *Case 1* works. For the remaining cases, we assume that $z_q = 1$.

*Case 2.1:* Assume that $q \neq q_1$ and that $v_{k-1} \notin |\Gamma'|$. As in *Case 1*, $q_1 \in |\Gamma'|$ and $\nu'_{v_j} = \nu_{v_j}$ for $j = 1, ..., k-1$ and $\nu'_{q_1} = \nu_{q_1} - 1$. Moreover, $v_{k-1} = q$. In particular, $\nu'_q = \nu_q$ and by the Remark II.3.10 and using that $z_q = 1$, we deduce that $q$ is free. Since $x_p$ is also proximate to $q_1$, the vertices of $q$ and $q_1$ are not adjacent in the graph $\Gamma_Q$, and thus, the only point of $\mathcal{K}(w)$ adjacent to $q$ is $x_p$. It follows that all the points of $\mathcal{K}(w)$ but $q$ are in $|\Gamma'|$. In particular, $O \in |\Gamma'|$ and (b) of Lemma II.3.1 and Lemma V.3.2 lead to contradiction.

*Case 2.2:* Assume now that $q = q_1$. If $u \in T_Q$ is any point infinitely near to $v_1$, then $u$ is in $|\Gamma'|$ for otherwise, $q \in ch(v_1, u)$ hence, $q$ is infinitely near to $v_1$ against the assumption. Moreover, by (a) of Lemma V.3.3, $\nu_u = 1$ and $\nu'_u = 0$.

Now, by (a) and (b) of Theorem II.3.8 and using induction backwards, $z_{u_i} = n - i$, for $i = 1, ..., n-1$ and $z_{x_p} = n$. By (b) of Theorem II.3.8 and for $i \in \{2, ..., k\}$, we see that

$$z_{v_i} = z_q + z_{v_{i-1}} - 1 = z_{v_{i-1}}$$

and by taking $i = k$, we have that $z_{v_i} = n$, $i \in \{1, ..., k\}$. Notice that the point $v_1$ must be satellite for otherwise, (b) of Theorem II.3.8 says that $z_{v_1} = 0$ which is impossible. Thus, $v_1$ is proximate to $q$ and another point, say $q'$, and again (b) of Theorem II.3.8 gives that

$$n = z_{v_1} = z_{q'} + z_q - 1 = z_{q'}. \tag{3.b}$$

We will see that this situation leads to contradiction, no matter the value of $n$. First, if $n > 1$, (3.b) says that $z_{q'} > 1$ and since $q$ is proximate to $q'$,

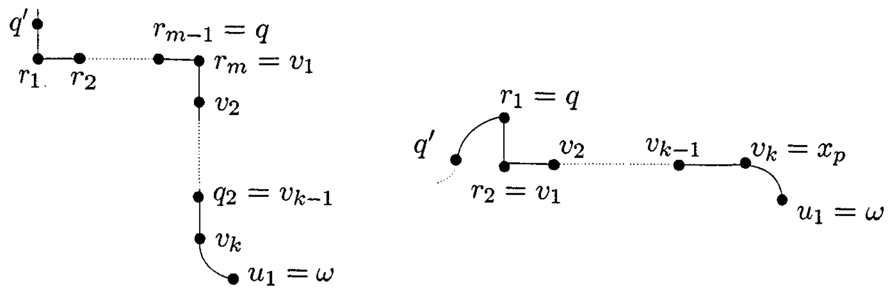



Figure V.2: Enriques Diagrams of $\mathcal{K}(\omega)$ when $p$ is infinitely near to $q = q_1$ and $n = 1$. The case when $m = 2$ and $r_1$ is free is shown on the right.

by (b) of Theorem II.3.8 we have that $z_q \geq z_{q'} > 1$ against the assumption. Hence, $n = 1$. Write $r_1, ..., r_{m-1} = q, r_m = v_1$ the points of $\mathcal{K}(\omega)$ proximate to $q'$.

Since $z_q = 1$ by iterated use of Remark II.3.10, we deduce that if $z_{r_i} = 1$ for all $i$. Moreover, $\nu'_{r_i} = \nu_{r_i} - 1$ for $i \geq 2$, and also for $i = 1$ if $r_1$ is satellite. Hence, if $m > 2$ or if $r_1$ is satellite, we are done by applying (b) of Lemma V.3.3 to the point $q'$.

If $m = 2$ and $r_1$ is free, then $r_1 = q$ and $x_p$ is the only point of $\mathcal{K}(\omega)$ whose vertex is adjacent to that of $q$. The same argument used at the end of *Case 2.1* completes the proof. □

In the proof of Theorem V.3.1 we have also seen:

**Corollary V.3.4. (of the proof of V.3.1)** *If* $\overline{\mathcal{N}_p(i)} \subset \overline{\mathcal{N}_q(i)}$ *and* $z_p = z_q$, *then* $p$ *is infinitely near to* $q$.

From Theorem V.3.1, the positive answer of the Nash problem for minimal singularities, which was already proved in Corollary 2.6 of [53], also follows.

**Corollary V.3.5.** *Let* $(X, Q)$ *be a minimal singularity. If* $p, q \in T_Q$ *are different and* $i \gg 0$, *then* $\overline{\mathcal{N}_p(i)} \not\subset \overline{\mathcal{N}_q(i)}$.

## V.4 The Nash conjecture for primitive singularities

In this section, we show that in order to prove the Nash conjecture for sandwiched singularities it is enough to prove it for primitive singularities.

A similar simplification was already pointed out by Lejeune-Jalabert and Reguera in [40], where they gave a positive answer to the wedge problem for sandwiched singularities by proving it first for primitive singularities.

First, we need a lemma.

**Lemma V.4.1.** *Let $(X_1, Q_1)$ be a rational surface singularity, $g : X \longrightarrow (X_1, Q_1)$ a birational dominant morphism and $E_p$ an exceptional component of $Q$ such that $E_p$ also appears in the minimal resolution of $Q_1$ modulo birational equivalence. Assume that for some $i > 0$, $\overline{\mathcal{F}_p^Q(i)} \subset \overline{\mathcal{F}_q^Q(i)}$ and that the projection by $g$ of any element of $\mathcal{F}_q^Q(i)$ is in $\mathcal{F}_u^{Q_1}(i)$, for some $u \in T_{Q_1}$. Then,*

$$\overline{\mathcal{F}_p^{Q_1}(i)} \subset \overline{\mathcal{F}_u^{Q_1}(i)}.$$

*Proof.* The morphism $g : X \longrightarrow (X_1, Q_1)$ is the blow-up of a complete ideal $J = (g_1, ..., g_m) \subset A = \mathcal{O}_{X_1,Q_1}$, and we may assume that $Q \in U_0$, where $U_0$ is an affine open set of $X$ of the form $\operatorname{Spec} A[g_1/g_0, ..., g_m/g_0] \subset \mathbb{A}_A^m$. Now, if $\operatorname{Spec}(A) \subset \mathbb{A}_{\mathbb{C}}^n$, any arc $\gamma$ on $(X_1, Q_1)$ is written in the form $\gamma = (x_1, ..., x_n)$, $x_k \in \mathbb{C}[[t]]$. Thus, the lifting $\widetilde{\gamma}$ of $\gamma$ on $X$ is given by

$$(x_1(t), ..., x_n(t), \overline{g_1}(t)/\overline{g_0}(t), ..., \overline{g_m}(t)/\overline{g_0}(t))$$

where $\overline{g_k}(t) = g_k(x_1(t), ..., x_n(t))$, $k = 1, ..., m$.

If $\overline{\mathcal{F}_p^Q(i)} \subset \overline{\mathcal{F}_q^Q(i)}$, the $i$-truncation of any arc of $\mathcal{F}_p^Q$ can be approximated by the $i$-truncations of arcs of $\mathcal{F}_q^Q$. By taking the projections of these $i$-truncations on $(X, Q)$, we see that $\mathcal{F}_p^{Q_1}(i) \subset \overline{g_*(\mathcal{F}_q^Q(i))} \subset \overline{\mathcal{F}_u^{Q_1}(i)}$ and hence, $\overline{\mathcal{F}_p^{Q_1}(i)} \subset \overline{\mathcal{F}_u^{Q_1}(i)}$. □

**Remark V.4.2.** A similar result has been proved independently by Camille Plénat [52].

Now, we come back to the situation where $(X, Q)$ is a sandwiched singularity and $X = Bl_I(S)$, where $I$ is a complete $\mathfrak{m}_O$-primary ideal in $R$. Since the exponents of the simple ideals in the Zariski factorization of $I$ are irrelevant when blowing it up, we may assume that

$$I = \prod_{j=1}^{N} I_j$$

and that $I_j \neq I_k$ for $j \neq k$.

Take the notation as in section I.4. In particular, write $\mathcal{K}$ and $\mathcal{K}_j$ for the clusters of base points of $I$ and the simple ideals $I_j$, $j = 1, \ldots, N$. Moreover, the surfaces $X_j = Bl_{I_j}(S)$ have only one singularity, that is denoted by $Q_j$.

In particular, we have $X_j = Bl_{I_j}(S)$ and the morphisms $f : S_K \longrightarrow X$ and $f_j : S_{K_j} \longrightarrow X_j$ induced by the universal property of the blowing up are the minimal resolutions of $X$ and $X_j, j = 1, \ldots, N$ respectively. Moreover, $X$ and $S_K$ are the birational join of the surfaces $X_j$ and $S_j$ for $j = 1, \ldots, N$ (see Proposition I.4.6). If we write $\sigma_j : X \longrightarrow X_j$ for the blowing-up of $I\mathcal{O}_{X_j}$ in $X_j$ and $\tau_j : S_K \longrightarrow S_{\mathcal{K}_j}$ the induced morphism, we have commutative diagrams of birational morphisms:

$$\begin{array}{ccc} S_K & \xrightarrow{f} & X \\ {\scriptstyle \tau_i}\downarrow & & \downarrow{\scriptstyle \sigma_i} \\ S_{K_j} & \xrightarrow[f_j]{} & X_j \end{array}$$

We write $\{E_u^{S_{K_j}}\}_{u \in T_{Q_j}}$ for the exceptional components of $f_j : S_{K_j} \longrightarrow X_j$.

**Proposition V.4.3.** *Let $p, q \in T_Q$, $p \neq q$ be such that $\overline{\mathcal{F}_p^Q(i)} \subset \overline{\mathcal{F}_q^Q(i)}$. Then there exists some $j \in \{1, \ldots, N\}$ with $p \in T_{Q_j}$ and some $u \in T_{Q_j}, u \neq p$, such that $\overline{\mathcal{F}_p^{Q_j}(i)} \subset \overline{\mathcal{F}_u^{Q_j}(i)}$.*

*Proof.* Assume that $p, q \in T_Q$ are points such that $\overline{\mathcal{F}_p^Q(i)} \subset \overline{\mathcal{F}_q^Q(i)}$ for $i \gg 0$. From Theorem V.2.1 we know that $q$ is not infinitely near to $p$. Let $\mathcal{K}_j$ be any irreducible subcluster of $\mathcal{K}$ containing $p$.

If $p$ is infinitely near to $q$, then $q$ is also in $\mathcal{K}_j$ and hence, the exceptional components $E_p$ and $E_q$ also appear in the minimal resolution $S_j$ of $Q_j$ modulo birational equivalence. By Lemma V.4.1 applied to $\alpha_j : X \longrightarrow X_j$, we deduce that $\overline{\mathcal{F}_p^{Q_j}(i)} \subset \overline{\mathcal{F}_q^{Q_j}(i)}$.

If $p$ is not infinitely near to $q$, let $u_0$ be the maximal point of $\mathcal{K}_j$ to which $q$ is infinitely near. Then the projection by $\alpha_j$ of any element in $\mathcal{F}_q^Q$ gives an arc $\gamma$ on $(X_j, Q_j)$ whose lifting to $S_j$ intersects (transversally or not) the exceptional component $E_{u_0}$ and therefore, $\gamma \in \mathcal{F}_{u_0}^{Q_j}$. Thus, the projection of any element of $\overline{\mathcal{F}_q^Q(i)}$ is in $\mathcal{F}_{u_0}^{Q_1}(i)$ and by Lemma V.4.1 again, we deduce that $\overline{\mathcal{F}_p^{Q_j}(i)} \subset \overline{\mathcal{F}_{u_0}^{Q_j}(i)}$.

In any case, we see that an inclusion of spaces of arcs on the sandwiched singularity $(X, Q)$ implies a non-trivial inclusion of some spaces of arcs on the primitive singularity $(X_j, Q_j)$. The claim follows. □

As a direct consequence, we obtain the wanted result.

**Corollary V.4.4.** *If the Nash conjecture is true for primitive singularities, then it is also true for sandwiched singularities.*

**Remark V.4.5.** Notice that if $(X, Q)$ is a primitive singularity and we have $\overline{\mathcal{F}_p^Q(i)} \subset \overline{\mathcal{F}_q^Q(i)}$ for $p, q \in T_Q$ with $p \neq q$, then by Theorem V.2.1, $p$ must be infinitely near to $q$ and moreover, $e_O(I_p) > e_O(I_q)$ where $e_O(J)$ is the multiplicity at $O$ of the linear system $\mathfrak{L}_J$ defined by $J$.

# Appendix A

# Implementation of some algorithms

In this appendix we provide the listings of three programs written in **C** which implement some algorithms suggested throughout this memoir. The reader will notice that they are written in type `typewriter`, while the comments are in the usual roman type. These programs are available, on request, from the author.

Keeping the notation already used, the input data for these programs consist of

(i) the proximity relations among the base points of a complete ideal $I$ (which is assumed to verify the conditions of Remark II.2.11) and among the singular points of a curve $C$ on $S$ whose strict transform on $X = Bl_I(S)$ goes through the singular point $Q \in X$,

(ii) the virtual multiplicities of $\mathcal{K} = BP(I)$ at these points,

(iii) the effective multiplicities of $C$ at these points.

These data are introduced as follows:

- the number `k` of points which are either the base points of $I$ or the singular points of $C$,

  These points will be treated as the entries of a vector of length `k` with the convention that the first entries correspond to the base points of $I$ and the remaining ones correspond to singular points of $C$ not in $K$, all of them given with an admissible order,

- the proximity relations among all these points are introduced by means of the elements below the diagonal of a `k`×`k` matrix `P` (the proximity matrix), whose $(i, j)$-entry is denoted by `P[i][j]`,

- the virtual multiplicities of the cluster $\mathcal{K} = BP(I)$ at these points are introduced as the entries of a k-vector nu (note that nu[i] is 0 if i corresponds to a singular point of $C$, which is not a base point of $I$),
- the effective multiplicities of the curve $C$ at these points are introduced as the entries of another k-vector e.

REMARK. Note that if some point is a base point of $I$ or not is inferred from the value of the virtual multiplicity nu at it.

## A.1 The semigroup of a branch through a sandwiched singularity

The following program implements the algorithm for the computation of the semigroup of a branch going through a sandwiched singularity, see subsection III.5.1.

```
/*****************************************************/
/*    Algorithm for the computation of the semigroup   */
/*  of a branch going through a sandwiched singularity */
/*****************************************************/

#include <stdio.h>
int k; int P[30][30];

void unloading(int s, int *mult[300]) {

    int i,j,sum,exp[30];
    for (i=0;i<k;i++) {
        sum=0;
        for (j=0;j<k;j++) {
            sum=sum+P[j][i]*mult[s][j];
        }
        exp[i]=sum;
    }
    for(i=0;i<k;i++) {
        if(exp[i]<0) {
            mult[s][i]=mult[s][i]+1;
            for(j=i+1;j<k;j++) {
                if ((P[j][i]==-1)&&(mult[s][j]>0)) {
                    mult[s][j]=mult[s][j]-1;
                }
            }
```

```
                unloading(s,mult);
                break;
            }
        }
}

void main ()
    {
    int i,j,s,aux,sum,cond;
    int nu[30],e[30],exc[30],inter[300],sem[300];
    int *mult[300];

    void unloading(int,int*[]);

    printf ("\n Introduce the number of points of the cluster");
    scanf("%d",&k);
    printf ("Introduce the proximity matrix \n");
    for(i=0;i<k;i++) {
        for (j=0;j<k;j++) {
            if(j>i) {
                P[i][j]=0;
            } else {
                scanf("%d",&P[i][j]);
            }
        }
    }
    printf ("\n Introduce the virtual multiplicities of the cluster");
    for(i=0;i<k;i++) {
        scanf("%d",&nu[i]);
    }
    mult[0]=nu;
    printf ("\n Introduce the effective multiplicities of the curve C");
    for(i=0;i<k;i++) {
        scanf("%d",&e[i]);
    }

    s=0;
    aux=0;
    for(i=0;i<k;i++) {
        aux=aux+nu[i]*e[i];
    }
    inter[0]=0;
    while (mult[s][k-1]<e[k-1]) {
        sum=0;
```

```
        for(i=0;i<k-1;i++) {
            mult[s+1][i]=mult[s][i];
        }
        mult[s+1][k-1]=mult[s][k-1]+1;

        s=s+1;
        unloading(s,mult);
        for(i=0;i<k;i++) {
            sum=sum+mult[s][i]*e[i];
        }
        inter[s]=sum-aux;
    }

    printf("\n The number of unloading steps is : %d",s);
    printf("\n The semigroup is : \n");
    for(i=1;i<=s;i++) {
        printf(" %d,",inter[i]);
    }
    for(i=0;i<=inter[s];i++) {  sem[i]=0; }
    for(i=0;i<=s;i++) {
        sem[inter[i]]=1;
    }
    i=inter[s];
    while (sem[i]==1) { i--; }
    cond=i+1;
    printf("\n The conductor of the semigroup is %d",cond);
    sum=0;
    for (i=0;i<cond;i++) {
        sum=sum+sem[i];
    }
    printf("\n The value of delta is %d",cond-sum);
}
```

## A.2 On the factorization of complete ideals in the local ring of a sandwiched singularity

The following program implements an algorithm to know, once a curve $C$ on $S$ is fixed, all the subcurves of $C$ whose strict transform on $X$ is a Cartier divisor. The algorithm is based on the procedure suggested in Remark III.1.10 and the output consists of a list of the wanted subcurves, where each one is presented by giving which branches of $C$ it is composed of.

REMARK: Notice that once a complete $\mathfrak{m}_Q$-primary ideal $J \subset \mathcal{O}_{X,Q}$ is fixed, this algorithm allows to know all the factorizations of $J$ into complete ideal in $\mathcal{O}_{X,Q}$: with the notation of section IV.1, we introduce the number of base points of the ideal $IH_{\mathcal{J}}$ and the proximity relations among these points. Then, the output is a listing of all the Cartier ideals $H$ for $X$ (see Definition IV.1.1) being a factor of $H_{\mathcal{J}}$. In virtue of Proposition IV.3.1, this gives all the factorizations of the ideal $J \subset \mathcal{O}_{X,Q}$.

```
/****************************************************/
/*  Algorithm for the computation of the subcurves   */
/*       whose strict transform on X is Cartier      */
/****************************************************/

#include <stdio.h>
#include <stdlib.h>
#define N 5000
int k; int cardK; int w; int
P[30][30]; int tt[30];

void unloading(int s, int **mult) {
    int i,j,u,sum,exp[30],aux,b;
    for (i=0;i<cardK;i++) {
        sum=0;
        for (j=0;j<cardK;j++) {
            sum=sum+P[j][i]*mult[s][j];
        }
        exp[i]=sum;
    }

    for(i=0;i<cardK;i++) {
        if(exp[i]<0) {
            aux=-exp[i];
            u=aux/tt[i];
```

```
            b=tt[i]*u;
            if(b<aux) { u=u+1; }
            mult[s][i]=mult[s][i]+u;
            for(j=i+1;j<cardK;j++) {
                if(P[j][i]==-1) { mult[s][j]=mult[s][j]-u; }
            }
            unloading(s,mult);
            break;
        }
    }
}

void curves(int i,int exp0[30], int **m) {
    int j,r;
    for (j=0;j<=exp0[i];j++) {
        m[w][i]=j;
        if (i<k-1) { curves(i+1,exp0,m);    } else {
            w=w+1;
            m[w]=(int*)calloc(30,sizeof(int));
            for(r=0;r<k;r++) {
                m[w][r]=m[w-1][r];
            }
        }
    }
}

void main ()
    {
    int i,j,u,v,s,t,aux,branchesC,sum,r;
    int **mult,**m,**expau,*Cartier,*brC;
    int nu[30],e[30],exp0[30],expK[30],valC[30],exc[30];
    int Q[30][30],can[30][30],InvA[30][30],val[30][30];

    void unloading(int,int*[]);
    void curves(int,int[],int*[]);

    m=(int**)malloc(N*sizeof(int*));
    mult=(int**)malloc(N*sizeof(int*));
    expau=(int**)malloc(N*sizeof(int*));

    printf ("\n Introduce the number of poitns of the cluster");
    scanf("%d",&k);

    printf ("Introduce the  proximity matrix\n");
```

```
for(i=0;i<k;i++) {
    for (j=0;j<k;j++) {
        if(j>i) {
            P[i][j]=0;
        } else {
            scanf("%d",&P[i][j]);
        }
    }
}

mult[0]=(int*)calloc(30,sizeof(int));
printf ("\n Introduce the virtual multiplicities of the cluster");
for(i=0;i<k;i++) {
    scanf("%d",&nu[i]);
    mult[0][i]=nu[i];
    if(nu[i]!=0) {cardK=i+1;}
}
for(i=0;i<cardK;i++) {
    t=1;
    for(j=i+1;j<cardK;j++) {
        if(P[j][i]==-1) {   t=t+1; }
    }
    tt[i]=t;
}

printf("\n Introduce the effective multiplicities of the curve C");
for(i=0;i<k;i++) {
    scanf("%d",&e[i]);
}

/**********************************************************/
/*    Computation of the excesses of the cluster and the    */
/*              cardinal of branches of C                   */
/**********************************************************/

branchesC=0;
for (i=0;i<k;i++) {
    sum=0;
    aux=0;
    for (j=0;j<k;j++) {
        sum=sum+P[j][i]*e[j];
        aux=aux+P[j][i]*nu[j];
        if (i==j) {
            Q[i][j]=1;
```

```
            can[i][j]=1;
        } else { Q[i][j]=0; can[i][j]=0;}
    }
    exp0[i]=sum;
    expK[i]=aux;
    branchesC=branchesC+exp0[i];
}

/*  Computation of the inverse of the intersection matrix  */
for(i=0;i<k;i++) {
    for(j=0;j<i;j++) {
        if (P[i][j]==-1) {
            for (r=0;r<k;r++) {
                Q[i][r]=Q[i][r]+Q[j][r];
            }
        }
    }
}
for(i=0;i<k;i++) {
    for(j=0;j<k;j++) {
        sum=0;
        for (r=0;r<k;r++) {
            sum=sum+Q[i][r]*Q[j][r];
        }
        InvA[i][j]=-sum;
    }
}
for(t=0;t<k;t++) {
    for(j=0;j<k;j++) {
        val[t][j]=-InvA[t][j];
    }
}

/*  We generate the different subcurves and verify  */
/*  if their strict transform on X are Cartier      */
w=1;
m[1]=(int*)calloc(30,sizeof(int));
curves(0,exp0,m);
w=w-1;

Cartier=(int*)calloc(w,sizeof(int));
brC=(int*)calloc(w,sizeof(int));

mult=(int**)malloc(w*sizeof(int*));
```

```
expau=(int**)malloc(w*sizeof(int*));

for(i=1;i<=w;i++) {
    brC[i]=0;
    for (j=0;j<k;j++) {
        brC[i]=brC[i]+m[i][j];
    }

    for(t=0;t<k;t++) {
        if(expK[t]>0) {
            sum=0;
            for(u=0;u<k;u++) {
                sum=sum+m[i][u]*val[u][t];
            }
            valC[t]=sum;
        } else { valC[t]=0;}
    }

    mult[i]=(int*)calloc(30,sizeof(int));
    expau[i]=(int*)calloc(30,sizeof(int));
    for(t=0;t<cardK;t++) {
        mult[i][t]=valC[t];
    }

    unloading(i,mult);

    /* Cartier[i]=1 if the strict transform of C[i] is Cartier */
    Cartier[i]=1;
    for (u=0;u<k;u++) {
        sum=0;
        for (v=0;v<k;v++) {
            sum=sum+P[v][u]*mult[i][v];
        }
        expau[i][u]=sum;
        if ((expK[u]==0)&&(sum>0)) {
            Cartier[i]=0;
        }
    }
}

printf("\n The subcurves of C giving Cartier divisors are: \n");
for(i=1;i<=w;i++) {
    if (Cartier[i]==1) {
        printf("\n (%d): ",i);
```

```
            for(j=0;j<k;j++) {
                printf("%d ",m[i][j]);
            }
        }
    }
}
```

## A.3 On $v$-minimal Cartier divisors containing a Weil divisor

The following listing is a program in **C** which implements the algorithm of section III.3 for the computation of the $v$-minimal Cartier divisors containing a Weil divisor going through a sandwiched singularity.

```
/*****************************************************************/
/* Algorithm for the computation of the v-minimal Cartier divisors */
/*   containing a Weil divisor through a sandwiched singularity    */
/*****************************************************************/

#include <stdio.h>
#define N 5000
#define M 50
int k; int branchesC; int cardK; int P[30][30]; int tt[30];

void unloading(int *mult, int tt[30]) {

    int i,j,b,aux,u,sum;
    int exp[30];
    for (i=0;i<k;i++) {
        sum=0;
        for (j=0;j<k;j++) { sum=sum+P[j][i]*mult[j]; }
        exp[i]=sum;
    }
    for(i=0;i<k;i++) {
        if(exp[i]<0) {
            aux=-exp[i];
            u=aux/tt[i];
            b=tt[i]*u;
            if(b<aux) { u=u+1; }
            mult[i]=mult[i]+u;
            for(j=i+1;j<k;j++) {
                if(P[j][i]==-1) {
                    mult[j]=mult[j]-u;
                }
            }
            unloading(mult,tt);
            break;
        }
    }
}
```

```
void algorithm(int *mult, int branC[30]) {
    int b;
    void unloading(int*,int[]);
    for (b=0;b<branchesC;b++) {
         if(mult[ branC[b] ]==0) {
            mult[branC[b]]=1;
            unloading(mult,tt);
            algorithm(mult,branC);
         }
    }
}

void main ()
    {
    int b,i,j,t,aux,sum;
    int excess[30];
    int nu[30],e[30],exp0[30],expK[30],branC[30];
    int *mult;
    void algorithm(int*,int[]);
    mult=(int*)malloc(N*sizeof(int));

    printf ("\n Introduce the number of points of the cluster");
    scanf("%d",&k);
    printf ("Introduce the proximity matrix \n");
    for(i=0;i<k;i++) {
        for (j=0;j<k;j++) {
            if(j>i) {
                P[i][j]=0;
            } else {
                scanf("%d",&P[i][j]);
            }
        }
    }
    printf ("\n Introduce the virtual multiplicities of the cluster");
    for(i=0;i<k;i++) {
        scanf("%d",&nu[i]);
        mult[i]=M*nu[i];
        if(nu[i]!=0) {cardK=i+1;}
    }

    for(i=0;i<k;i++) {
        t=1;
        for(j=i+1;j<k;j++) {
            if(P[j][i]==-1) { t=t+1; }
```

```
        }
        tt[i]=t;
    }
    printf("\n Introduce the effective multiplicities of the curve C");
    for(i=0;i<k;i++) {
        scanf("%d",&e[i]);
    }

    branchesC=0;
    b=0;
    for (i=0;i<k;i++) {
        sum=0; aux=0;
        for (j=0;j<k;j++) {
            sum=sum+P[j][i]*e[j];
            aux=aux+P[j][i]*nu[j];
        }
        exp0[i]=sum; expK[i]=aux;
        branchesC=branchesC+exp0[i];
        if(exp0[i]>0) {
            branC[b]=i; b=b+1;
        }
    }
    algorithm(mult,branC);

    for(i=0;i<k;i++) {
        if(expK[i]>0) {
            excess[i]=0;
        } else {
            sum=0;
            for(t=0;t<k;t++) {
                sum=sum+P[t][i]*mult[t];
            }
            excess[i]=sum;
        }
    }
    printf("\n The branches of the v-minimal Cartier divisors of C are:");
    for(i=0;i<k;i++) {
        printf(" %d",excess[i]);
    }
}
```

# Appendix B

# Resum en català

## B.1 Introducció

L'interès original en les singularitats sandwiched prové d'una pregunta natural de J. Nash a H. Hironaka a principis dels anys seixanta: *és possible resoldre les singularitats d'una varietat algebraica reduïda mitjançant una successió finita de transformacions de Nash o de transformacions de Nash normalitzades?* En 1975, A. Nobile [47] prova que, en característica zero, una transformació de Nash és un isomorfisme si i només si la varietat original no és singular. D'aquí es dedueix que les singularitats de corba es poden resoldre mitjançant una successió de transformacions de Nash. En la seva tesi doctoral, Rebasoo demostra que mitjançant transformacions de Nash també és possible resoldre alguns tipus de singularitats de hipersuperfície quasi-homogènies en $\mathbb{C}^3$. En 1982, G. González-Sprinberg prova que les transformacions de Nash normalitzades resolen els punts dobles racionals i els quocients cíclics de superfícies [23], i un any més tard, H. Hironaka demostra que mitjançant una successió finita de transformacions de Nash normalitzades aplicades a una superfície s'obté una superfície $X$ que domina biracionalment una superfície no singular [30]. Per definició, les singularitats de la superfície $X$ són *singularitats sandwiched.* Alguns anys més tard, en [58], M. Spivakovsky demostra que les singularitats sandwiched també es poden resoldre mitjançant transformacions de Nash normalitzades, donant així una resposta afirmativa a la pregunta inicial de Nash pel cas de superfícies sobre $\mathbb{C}$.

Des de llavors, hi ha hagut un interès constant en les singularitats sandwiched des del punt de vista de la teoria de deformacions per de Jong i van Straten en [13], i també per Stølen [28] i Möhring [43]. També han rebut una atenció especial com a camp de proves per la conjectura de Nash i el problema dels wedges per part de Lejeune-Jalabert i Reguera en [40], on la idea principal consisteix en estendre alguns arguments combinatoris propis de les singularitats de superfícies tòriques per les singularitats sandwiched.

De la seva definició se segueix que les singularitats sandwiched són aquelles singularitats que s'obtenen per l'explosió d'un ideal complet (és a dir, íntegrament tancat) en l'anell local d'un punt regular en una superfície, i per tant, són singularitats racionals. Les singularitats racionals de superfície són singularitats aïllades la resolució de les quals no altera el gènere aritmètic de la superfície. Entre les singularitats sandwiched es troben els quocients cíclics i les singularitats minimals i constitueixen doncs, una àmplia classe de singularitats racionals,

$$\text{quocients cíclics} \subsetneq \text{minimals} \subsetneq \text{sandwiched} \subsetneq \text{racionals}.$$

Les singularitats sandwiched són Cohen-Macaulay, però no són intersecció completa en general, i no existeixen equacions simples o fàcils per elles. L'objectiu d'aquesta memòria és estudiar les singularitats sandwiched a través dels punts base infinitament propers dels ideals complets explotats per obtenir-les[1].

Tal com s'explica en la Introducció del llibre de Casas [11], els punts infinitament propers són una eina antiga per descriure singularitats, i ja apareixen en el treball de M. Noether. El seu ús i les seves propietats, com la proximitat, el satel·litisme, etc. permeten, per exemple, una descripció molt entenedora del comportament de les singularitats de corbes planes i, en general, proporcionen un estudi molt acurat de les singularitats de varietats en un context més ampli. A més del llibre d'Enriques i Chisini [16], on F. Enriques desenvolupa àmpliament la geometria dels punts infinitament propers, altres referències clàssiques pel seu estudi són el resum del primer capítol del llibre de Zariski sobre superfícies [62], el capítol XI del llibre clàssic sobre corbes de Semple i Kneebone [57] o la secció 5 de l'article de Zariski sobre saturació [63]. La teoria de punts infinitament propers ha estat revisada i desenvolupada en llenguatge modern per Casas [9, 11].

Els ideals complets són introduïts per Oscar Zariski vint anys després de l'aparició de [16]. En [61], Zariski desenvolupa una teoria aritmètica paral·lela a la teoria geomètrica de sistemes lineals de corbes planes que passen a

[1]En la seva tesi [43], Möhring fa servir un punt de vista semblant per estudiar teoria de deformacions i la conjectura de Kollár per singularitats sandwiched.

través d'un conjunt de punts amb multiplicitats assignades (multiplicitats virtuals). Un dels fets clau d'aquesta teoria és que tot ideal complet en un anell local regular de dimensió dos té una factorització única en ideals complets irreductibles; aquests ideals irreductibles s'anomenen ideals simples. Un fet rellevant pels nostres propòsits és el resultat que estableix que tot ideal complet té un clúster de punts base infinitament propers i que aquest clúster determina el propi ideal.

Com resultats base relacionant les singularitats sandwiched en una superfície $X$ i els punts base de l'ideal complet $I$ explotat per obtenir aquesta superfície, determinem els punts singulars de $X$, les seves multiplicitats i els seus cicles fonamentals en termes dels punts base de $I$, i donem una fórmula explícita per la multiplicitat dels punts de les corbes en $X$ també en funció d'aquests punts base.

Aquests fets ens permeten estudiar l'existència d'equacions locals per les corbes de $X$. Deduïm conseqüències relatives als seus ordres de singularitat i fem càlculs explícits relacionats amb l'existència de divisors de Cartier sobre $X$ passant per singularitats sandwiched i amb propietats prefixades. En particular, provem que les tangents a les components excepcionals de $X$ passant per una mateixa singularitat sandwiched són linealment independents. Tot això ens porta a l'estudi dels feixos d'ideals complets amb cosuport finit en $X$, i a deduir resultats relatius a la factorització i semi-factorització d'ideals complets en l'anell local d'una singularitat sandwiched.

També obtenim alguns resultats relacionats amb la conjectura de Nash relativa als arcs per una singularitat sandwiched. En una pre-publicació de 1968, que va aparèixer publicada més tard com [46], Nash introduïa l'estudi dels espais d'arcs com una nova via per entendre les singularitats. La principal qüestió de [46], coneguda més endavant com la *conjectura de Nash*, és saber si cada component essencial de la resolució d'un punt singular dóna lloc a una component irreductible de l'espai d'arcs que passen per aquest punt. La Conjectura de Nash estableix que efectivament és així, i per tant, que hi ha una bijecció entre el conjunt de components irreductibles de l'espai d'arcs i el conjunt de components essencials d'una singularitat. En un article recent [33], Ishii i Kollár donen una resposta afirmativa a la qüestió de Nash per singularitats tòriques de qualsevol dimensió, però proven que la conjectura és falsa en general. Més recentment, A. Reguera ha demostrat que la conjectura és certa per una àmplia gamma de singularitats, incloses les singularitats sandwiched [55].

## B.2 Resultats i Conclusions

Al llarg d'aquest treball i si no s'especifica una cosa diferent, el cos base és el cos $\mathbb{C}$ dels nombres complexos. Les principals referències per la major part del material tractat aquí són el llibre de Casas [11] i els articles de Spivakovsky [58] i de Lipman [41].

Abans de passar a resumir els resultats presents en els diferents capítols, introduirem algunes definicions i notacions per tal de fixar el marc on el nostre estudi de les singularitats sandwiched es desenvoluparà. La major part de les notacions relatives a clústers i punts infinitament propers provenen de [11]. Sigui $S$ una superfície llisa, $O \in S$ un punt i notem $R = \mathcal{O}_{S,O}$ l'anell local de $O$ i $\mathfrak{m}_O$ el seu ideal maximal. Explotant un ideal complet $\mathfrak{m}_O$-primari $I$ contingut en $R$, obtenim una superfície $X = Bl_I(S)$ amb singularitats sandwiched. Si denotem per $\mathcal{K} = (K, \nu)$ el clúster dels punts base de $I$ i $\pi_K : S_K \longrightarrow S$ l'explosió de tots els punts de $K$, tenim un diagrama commutatiu

$$\begin{array}{ccc} S_K & \xrightarrow{f} & X \\ & \searrow^{\pi_K} & \downarrow \pi \\ & & S \end{array}$$

on $\pi$ és l'explosió de $I$ i el morfisme $f : S_K \to X$, donat per la propietat universal de l'explosió, és la resolució minimal de les singularitats de $X$. Denotem per $\{E_p\}_{p \in K}$ les components excepcionals irreductibles de $S_K$ i per $\mathcal{K}_+$ el conjunt dels punts dicrítics de $\mathcal{K}$. Les components irreductibles excepcionals de $X$ són $\{L_p\}_{p \in \mathcal{K}_+}$, i cada $L_p$ és la imatge directa de la component $E_p$ per $f$. Denotem per $\overline{\mathbb{F}}_K$ el conjunt dels punts en el primer entorn d'algun punt de $K$ que no estan a $K$. Donat un clúster amb pesos $\mathcal{T} = (T, \tau)$, les equacions de les corbes que passen per $\mathcal{T}$ descriuen el conjunt dels elements diferents de zero d'un ideal complet $H_{\mathcal{T}}$ de $R$. Diem que $\mathcal{T}_1$ i $\mathcal{T}_2$ són equivalents si $H_{\mathcal{T}_1} = H_{\mathcal{T}_2}$. Si $\mathcal{T} = \mathcal{K}(p)$ és el clúster irreductible determinat per un un punt $p$ infinitament proper a $O$, denotem per $I_p$ l'ideal complet simple donat per $H_{\mathcal{K}(p)}$.

A continuació descriuré els continguts principals de cadascun dels capítols d'aquesta memòria.

El **Capítol I** és introductori i dóna referències a la literatura per les demostracions. Es revisen conceptes i fets ben coneguts relatius a punts infinitament propers, clúster amb pesos, ideals complets i singularitats racionals i sandwiched, i es deriven algunes conseqüències que seran necessàries a la memòria. A les seccions I.1 i I.2 es recorden definicions relatives a l'explosió de punts i clústers en una superfície regular; les referències bàsiques són [9] i [11]. A la subsecció I.2.1 descrivim el procés de descàrrega d'Enriques (*principio di scaricamento*), que jugarà un paper essencial al llarg d'aquest treball. La secció I.3 està dedicada a recordar alguns fets relatius a les singularitats racionals de superfície que també seran necessaris més endavant; les referències principals són [3] i [41]. La secció I.4 introdueix les singularitats sandwiched i repassa els resultats més rellevants pel nostre estudi. Finalment, a la secció I.5 recordem alguns fets i definicions elementals relatius a grafs duals.

En el **Capítol II** estudiem el lligam entre les singularitats sandwiched i la teoria dels diagrames d'Enriques de clústers amb pesos i deduïm alguns resultats relatius a les singularitats sandwiched fent servir el procés de descàrrega abans esmentat. Els diagrames d'Enriques codifiquen dades combinatòries associades als clústers i a les relacions de proximitat entre els seus punts. A la secció II.1 i per singularitats racionals de superfície en general, provem el següent teorema.

**Teorema 1.** [Teorema II.1.7] *Sigui $\mathcal{O}_{S,P}$ un anell local amb una singularitat racional de superfície. Sigui $I \subset \mathcal{O}_{S,P}$ un ideal complet $\mathfrak{m}_P$-primari i $X$ la superfície obtinguda en explotar $I$. Associant a cada punt $Q$ del lloc excepcional de $X$ l'ideal complet $I_Q = \pi_*(\mathcal{M}_Q I \mathcal{O}_X)$, obtenim una bijecció entre el conjunt de punts en el lloc excepcional de $X$ i el conjunt dels ideals complets $J \subset I$ de codimensió 1. L'aplicació inversa associa a cada ideal complet $J$ l'únic punt $Q$ tal que totes les transformades virtuals de les corbes $C : h = 0$, $h \in J$ passen per ell.*

Com a conseqüència d'aquest resultat, recuperem el cicle fonamental en qualsevol resolució d'una singularitat de $X$ com la diferència entre els divisors excepcionals en aquella resolució corresponents a $I$ i $I_Q$ (vegeu el Corol·lari II.1.10). Aquests fets ens permeten estudiar els punts del lloc excepcional de $X$, i en particular les singularitats de $X$, a partir dels subsistemes lineals de codimensió 1 del sistema lineal definit per $I$. Aquest és un dels fets fonamentals d'aquest capítol. La secció II.2 fa servir aquests resultats per estudiar la connexió entre les singularitats sandwiched i els clústers de punts infinitament propers i mostra com obtenir informació de la superfície $X$ a partir del diagrama d'Enriques de $\mathcal{K}$ i les seves multiplicitats virtuals. Descrivim un procés per calcular el clúster $\mathcal{K}_Q$ de punts base de l'ideal $I_Q$ associat a $Q$, i caracteritzem els punts singulars de $X$ en termes de la consistència d'un cert clúster $\mathcal{K}_q$, obtingut afegint a $\mathcal{K}$ un punt $q \in \overline{\mathbb{F}}_K$ (que depèn de $Q$) amb multiplicitat virtual 1 (Proposició II.2.5). En el cas que $Q \in X$ sigui singular, denotem per $T_Q$ el conjunt de punts $p \in K$ tal que $f$ contrau la component excepcional $E_p$ a $Q$. Tots aquests punts són no-dicrítics a $\mathcal{K}$, i existeix un únic punt minimal en $T_Q$ relatiu a l'ordre natural, que denotem per $O_Q$. Aleshores, provem que

**Corol·lari 2.** [Corol·lari II.2.10]

*(a) El nombre de singularitats de $X$ és igual al nombre de clústers no equivalents $\mathcal{K}_q$, per $q \in F_p$ no pertanyent a $K$ i $p \in K$ un punt no-dicrític de $\mathcal{K}$.*

*(b) Si $p_1, p_2 \in T_Q$ per algun singularitat $Q \in X$, les components excepcionals $E_{p_1}$ i $E_{p_2}$ es tallen en $S_K$ si i només si $p_1$ és maximal entre els punts de $K$ que són pròxims a $p_2$ o viceversa.*

*(c) Sigui $Q \in X$ un punt singular i sigui $q \in \overline{\mathbb{F}}_K$ tal que $H_{\mathcal{K}_q} = I_Q$. Denotem el cicle fonamental de $Q$ per $Z_Q = \sum_{u \in T_Q} z_u E_u$. Llavors, per cada $u \in T_Q$, $z_u$ és igual al nombre de passos de descàrrega realitzats al punt $u$ en el procés de descàrrega de $\mathcal{K}_q$.*

A continuació, caracteritzem els diagrames d'Enriques dels clústers de punts base d'ideals complets $\mathfrak{m}_O$-primaris satisfent les condicions 1. i 2. del Corol·lari 1.14 de [58] i finalment, donem una fórmula per la multiplicitat d'un punt $Q$ en el lloc excepcional de $X$ en termes de les multiplicitats virtuals de $\mathcal{K}$ i $\mathcal{K}_Q$.

**Teorema 3.** [Teorema II.2.14] *Sigui $Q$ un punt en el lloc excepcional de $X$. Aleshores, la multiplicitat de $X$ a $Q$ és*

$$\text{mult}_Q(X) = {\mathcal{K}_Q}^2 - \mathcal{K}^2.$$

La secció II.3 és tècnica i està dedicada a l'estudi de l'efecte de la descàrrega després d'afegir un punt simple $q \in \overline{\mathbb{F}}_K$, tal com s'explica a la secció II.2. El Lema II.3.1 mostra que si $\mathcal{K}_q$ és equivalent a $\mathcal{K}_Q$ per $Q \in X$ singular, la multiplicitat del punt minimal $O_Q \in T_Q$ després de descarregar $\mathcal{K}_q$ augmenta en una unitat, i les multiplicitats als altres punts de $K$ o bé decreixen en una unitat o bé queden inalterades. Aquest fet suggereix la definició dels $MR$-punts relatius a la singularitat sandwiched $(X, Q)$: són aquells punts de $K$ pels quals la multiplicitat decreix en una unitat després de descarregar les multiplicitats de $\mathcal{K}_q$. Escriurem $B_Q^{\mathcal{K}}$ per referir-nos al conjunt d'aquests punts. El Teorema II.3.8 relaciona els coeficients del cicle fonamental d'una singularitat sandwiched amb les relacions de proximitat dels punts $K$. Com a conseqüència d'alguns d'aquests fets, en la subsecció II.3.1 donem una descripció de la factorització de Zariski dels ideals complets $J$ de codimensió 1 en $I$ en termes de la factorització de Zariski de $I$ i del punt $Q$ de $X$ corresponent a $J$ pel Teorema 1. Concretament,

**Teorema 4.** [Teorema II.3.13] *Sigui* $I = \prod_{p \in \mathcal{K}_+} I_p^{\alpha_p}$, *amb* $\alpha_p \geq 1$, *la factorització de Zariski de* $I$ *i sigui* $J$ *un ideal complet* $\mathfrak{m}_O$*-primari de codimensió 1 en* $I$. *Sigui* $Q$ *el punt del lloc excepcional de* $X$ *corresponent a* $J$ *pel Teorema 1. Aleshores,*

$$J = H_J \prod_{p \in \mathcal{K}_+^Q} I_p^{\alpha_p - 1} \prod_{p \in \mathcal{K}_+ \setminus \mathcal{K}_+^Q} I_p^{\alpha_p},$$

*on* $\mathcal{K}_+^Q = \{p \in \mathcal{K}_+ \mid Q \in L_p\}$ *i* $H_J \subset R$ *és un ideal complet* $\mathfrak{m}_O$*-primari la factorització del qual no té ideals simples en comú amb la factorització de* $I$. *A més,* $H_J$ *és simple si i només si* $Q$ *és no-singular, i en aquest cas,* $H_J = I_q$ *on* $q \in \overline{\mathbb{F}}_K$ *és l'únic punt tal que* $H_{\mathcal{K}_q} = I_Q$. *Si* $Q$ *és singular, llavors l'ideal simple* $I_{O_Q}$ *apareix a la factorització de* $H_J$.

Es dedueix en particular que la factorització de $H_J$ determina si $Q$ és singular o no. L'objectiu de la secció II.4 és estudiar la resolució de les singularitats sandwiched en termes de cadenes d'ideals complets en $R$ (Teorema II.4.5). En particular, en la Proposició II.4.4 veiem que l'explosió d'una singularitat $Q \in X$ és la unió biracional de $X$ i la superfície obtinguda explotant l'ideal complet $I_Q \subset I$. A la secció II.5, un cop fixada una corba $C$ en $S$, donem una fórmula per la multiplicitat de la transformada estricta de $C$ en $X$ en qualsevol punt del lloc excepcional de $X$:

**Teorema 5.** [Teorema II.5.1 i Corol·lari II.5.4] *Sigui* $Q$ *un punt en el lloc excepcional de* $X$. *Si* $C$ *és una corba en* $S$, *llavors*

$$\text{mult}_Q(\widetilde{C}) = e_{O_Q}(C) - \sum_{p \in B_Q^{\mathcal{K}}} e_p(C).$$

*En particular,* $\text{mult}_Q(X) = 1 + \sharp B_Q^{\mathcal{K}}$.

Com a conseqüència, obtenim fórmules pel nombre de branques d'una secció d'hipersuperfície d'una singularitat sandwiched i deduïm el fet ben conegut que estableix que les singularitats minimals són aquelles singularitats racionals de superfície amb el cicle fonamental reduït (aquest fet ja havia estat anunciat per Kollár en [34] sense demostració). A continuació, en la Proposició II.5.17 provem que el nombre de components excepcionals que passen per una singularitat sandwiched està afitat per la dimensió d'immersió de la singularitat i caracteritzem quan s'assoleix aquesta fita. La secció II.6 és tècnica i fa servir alguns resultats ja vistos per deduir que les components excepcional en $X$ passant per una mateixa singularitat sandwiched no són tangents. Més endavant, a la secció III.3, provarem un resultat més fort.

Acabem el capítol II fent servir les tècniques i alguns resultats de les seccions anteriors per provar algunes conseqüències relatives a ideals complets adjacents, i.e. parelles d'ideals $J \subset I$ amb $\dim_{\mathbb{C}} \frac{I}{J} = 1$, i responem algunes de les preguntes plantejades per S. Noh en la darrera secció de [49] referents a l'existència d'ideals adjacents contenint o continguts en ideals complets d'un cert tipus en un anell local regular de dimensió dos.

El **Capítol III** tracta essencialment de la principalitat de divisors efectius que passen per una singularitat sandwiched. És un fet ben conegut que els divisors de Weil per una singularitat $(X, Q)$ no són divisors de Cartier en general. Investiguem aquest fet pel cas de singularitats sandwiched i obtenim el següent criteri.

**Teorema 6.** [Teorema III.1.1] *Per $u \in \mathcal{K}_+$, notem $\mathcal{L}_u = \sum_{p\in\mathcal{K}_+} v_p(I_u)L_p$ i fixada una corba $C$ en $S$, notem $L_C$ per la component excepcional de $\pi^*(C)$ en $X$. Aleshores, les quatre afirmacions següents són equivalents*

*(i) la transformada estricta $\widetilde{C}$ en $X$ és un divisor de Cartier;*

*(ii) $L_C \in \bigoplus_{u\in\mathcal{K}_+} \mathbb{Z}\mathcal{L}_u$;*

*(iii) existeix una corba $C_O \subset S$ tal que $L_{C_O} = L_C$ i la transformada estricta $\widetilde{C_O}$ no passa per cap singularitat de $X$;*

*(iv) si $\mathbb{H}^o_C = \{g \in R \mid v_p(g) \geq v_p(C), \forall p \in \mathcal{K}_+\}$ i si $q \in T_Q$ no és un punt dicrític de $\mathcal{K}$, llavors tampoc no és un punt base dicrític de $\mathbb{H}^o_C$.*

Com a conseqüència d'aquest teorema, en el Corol·lari III.1.6 veiem que si $\widetilde{C}$ és Cartier, llavors

$$\mathbb{H}^o_C = \prod_{p\in\mathcal{K}_+} I_p^{a_p}$$

és la factorització (de Zariski) de $\mathbb{H}_C^o$ en ideals simples, on $a_p = |\widetilde{C}\cdot L_p|_X$ per cada $p \in \mathcal{K}_+$. Proposem un algoritme basat en el procés de descàrrega per saber si la transformada estricta en $X$ d'una corba en $S$ és Cartier o no. A la secció III.2 deduïm algunes conseqüències: si $f_C : S_{K_C} \longrightarrow X$ és la resolució immersa minimal de $\widetilde{C} \subset X$, donem una fórmula per la component excepcional de la transformada total de $\widetilde{C}$ en $S_{K_C}$ en termes dels valors de $C$ i $\mathbb{H}_C^o$ relatius a certes valoracions divisorials (Proposició III.2.1). Com a conseqüència, deduïm una fórmula per la multiplicitat d'intersecció de divisors efectius de Cartier i de Weil en $X$ a partir dels nombres d'intersecció de certes corbes en $S$.

**Corol·lari 7.** [Corol·lari III.2.3] *Siguin $C$ i $C_1$ corbes en $S$ i suposem que $\widetilde{C}$ és Cartier. Llavors,*

$$|\widetilde{C}\cdot\widetilde{C_1}|_X = [C, C_1]_O - [\mathcal{K}_C^o, C_1]_O,$$

*on $\mathcal{K}_C^o = BP(\mathbb{H}_C^o)$.*

En les dues seccions següents proposem algoritmes pel càlcul de divisors efectius de Cartier en $X$ amb condicions prefixades. A la secció III.3 donem una prova del següent resultat:

**Teorema 8.** [Corol·lari III.3.5] *Siguin $\mathcal{Q} = \{Q_1, \ldots, Q_m\}$ punts (singulars o no) en el lloc excepcional de $X$ i per cada $Q_i$, siguin $\{\alpha_p^i\}_{p\in\mathcal{K}_+^{Q_i}}$ enters positius. Llavors, existeix un clúster $\mathcal{T}_{\mathcal{Q}}^{\alpha}$ tal que si $C$ és una corba genèrica passant per $\mathcal{T}_{\mathcal{Q}}^{\alpha}$, llavors $\widetilde{C}$ és un divisor de Cartier en $X$ passant per $Q_1, \ldots, Q_m$ i per cada $Q_i$ i cada $p \in \mathcal{K}_+^{Q_i}$,*

$$[\widetilde{C}, L_p]_{Q_i} = \alpha_p^i.$$

*A més, si $Q_i$ és regular o bé si $O_{Q_i}$ és lliure o $O_{Q_i} = O$, $\widetilde{C}$ és irreductible com divisor principal a prop de $Q_i$.*

La demostració del resultat anterior és constructiva i ens permet descriure un algoritme per calcular el clúster $\mathcal{T}_{\mathcal{Q}}^{\alpha}$. Com a conseqüència del Teorema 8, deduïm també que les tangents de les components excepcionals $\{L_p\}_{p\in\mathcal{K}_+^Q}$ en una singularitats sandwiched $(X, Q)$ són linealment independents.

Fixat un divisor efectiu de Weil $C$ en $X$, un divisor de Cartier $v$-minimal contenint $C$ és un divisor (efectiu) de Cartier tal que la seva transformada estricta en $S_{K_C}$ talla transversalment el divisor excepcional de $f_C$ i els coeficients de la component excepcional de la seva transformada total per

$f_C$ són minimals. A la secció III.4, després d'introduir una variant de l'algoritme de descàrrega, donem un algoritme per calcular els divisors de Cartier $v$-minimals que contenen un divisor de Weil en $X$ donat. En la Proposició III.4.15 descrivim les singularitats d'aquestes corbes. Finalment, ala secció III.5 fem servir les eines i resultats desenvolupats en aquest capítol per calcular l'ordre de singularitat d'un divisor efectiu de Cartier en $X$.

**Teorema 9.** [Teorema III.5.1 i Corol·lari III.5.5] *Sigui $C$ una corba en $S$ tal que $\widetilde{C}$ és un divisor de Cartier en $X$. Llavors,*

$$\delta_O(C) = \delta_X(\widetilde{C}) + \delta_O(\mathcal{K}_C^o).$$

*A més, si per cada $p \in K_C$ denotem per $\tau_p^o$ la multiplicitat virtual de $\mathcal{K}_C^o$ a $p$ i $n_p = e_p(C) - \tau_p^o$, llavors*

$$\delta_X(\widetilde{C}) = \sum_{p \in K_C} \frac{n_p(n_p - 1)}{2}.$$

Pel cas general, quan $\widetilde{C}$ no és necessàriament un divisor de Cartier, presentem una fórmula menys explícita per l'ordre de singularitat de $\widetilde{C}$ en $X$ en termes dels clústers $\mathcal{T}_n^s$ i $\mathcal{T}_0^s$ obtinguts en aplicar l'algoritme de la secció III.4 a $\widetilde{C}$.

**Proposició 10.** [Proposició III.5.8] *Sigui $C$ una corba en $S$. Llavors,*

$$\delta_X(\widetilde{C}) = [\mathcal{T}_n^s, C]_O - [\mathcal{T}_0^s, C]_O - \dim_{\mathbb{C}}(\frac{H_{\mathcal{T}_0^s}}{H_{\mathcal{T}_n^s}}).$$

En la subsecció III.5.1 utilitzem aquest resultat per calcular el semigrup d'una branca passant per una singularitat sandwiched, donem alguns exemples i mostrem que, en general, la transformada estricta en $X$ d'una branca en $S$ no és Gorenstein.

En el **Capítol IV** utilitzem els resultats del Capítol III per estudiar la connexió entre els feixos d'ideals en $X$ amb cosuport finit contingut en el lloc excepcional i els ideals complets $\mathfrak{m}_O$-primaris en $R$. En virtut del Teorema 6, la principalitat de les transformades estrictes en $X$ de corbes genèriques passant per un clúster fixat $\mathcal{T}$ només depèn de $\{v_p^{\mathcal{T}}\}_{p \in \mathcal{K}_+}$. A la secció IV.1 introduïm els ideals de Cartier per $X$ com aquells ideals complets pels quals un element genèric defineix una corba tal que la seva transformada estricta és un divisor de Cartier en $X$. Donat un ideal complet $\mathfrak{m}_O$-primari $H \subset R$, denotem $H^o = \{g \in R \mid v_p(g) \geq v_p(H), \forall p \in \mathcal{K}_+\}$. Llavors, provem el següent resultat:

**Teorema 11.** [Teorema IV.1.3] *Sigui $\mathcal{J}$ un feix d'ideals en $X$ amb cosuport finit $\{Q_1, \ldots, Q_n\}$ contingut en el lloc excepcional de $X$ i tal que per tot $Q_i$, la fibra $J_i = \mathcal{J}_{Q_i}$ és un ideal complet $\mathfrak{m}_{Q_i}$-primari de $\mathcal{O}_{X,Q_i}$. Llavors existeix un ideal complet $\mathfrak{m}_O$-primari $H_{\mathcal{J}}$ en $R$ amb*

$$\dim_{\mathbb{C}}(\frac{H^o_{\mathcal{J}}}{H_{\mathcal{J}}}) = \sum_{i=1}^{n} \dim_{\mathbb{C}}(\frac{\mathcal{O}_{X,Q_i}}{J_i})$$

*i tal que:*

*(a) $H_{\mathcal{J}}$ és un ideal de Cartier per $X$;*

*(b) el feix $\mathcal{H}_{\mathcal{J}} = H_{\mathcal{J}}\mathcal{O}_X$ és localment principal, tret d'en els punts $Q_i, i = 1, \ldots, n$ i tenim*

$$\mathcal{H}_{\mathcal{J}} = \mathcal{J}O_X(-L_{H_{\mathcal{J}}});$$

*(c) si $C$ és una corba definida per un element genèric de $H_{\mathcal{J}}$, llavors la seva transformada estricta $\widetilde{C}$ en $X$ és un divisor de Cartier i interseca el lloc excepcional de $X$ exactament en els punts $\{Q_1, \ldots, Q_n\}$.*

*A més, amb aquestes condicions, l'ideal $H_{\mathcal{J}}$ està completament determinat.*

Aquest resultat ens permet generalitzar el Teorema 1 per ideals complets $H \subset I$ de qualsevol codimensió amb la condició que el seu suport excepcional en $X$ sigui igual al suport excepcional de $I$, i.e. $L_H = L_I$. Concretament,

**Corol·lari 12.** [Corol·lari IV.1.11] *Fixat $m \geq 0$, existeix una correspondència bijectiva entre el conjunt de feixos d'ideals $\mathcal{J}$ en $X$ amb cosuport finit $\{Q_1, \ldots, Q_n\}$ contingut en el lloc excepcional de $X$ i tal que per cada $Q_i$, la fibra $J_i = \mathcal{J}_{Q_i}$ és un ideal complet $\mathfrak{m}_{Q_i}$-primari de $\mathcal{O}_{X,Q_i}$ i $\sum \dim_{\mathbb{C}}(\mathcal{O}_{X,Q_i}/J_i) = m$, i el conjunt dels ideals de Cartier $H \subset R$ per $X$ de codimensió $m$.*

*Aquesta correspondència associa a cada feix d'ideals $\mathcal{J}$ en $X$ l'ideal complet $\mathfrak{m}_O$-primari $H_{\mathcal{J}}$ del Teorema 11. L'aplicació inversa fa correspondre a cada $H \subset R$ el feix d'ideals en $X$ determinat per les transformades virtuals en $X$ relatives a $H^o$ de totes les corbes definides per elements de $H$.*

A la secció IV.2 donem un algoritme per calcular un sistema minimal de generadors per $H_{\mathcal{J}}$. Finalment, a la secció IV.3 investiguem la semifactorització i la factorització d'ideals complets en l'anell local d'una singularitat sandwiched en termes de la factorització d'ideals complets en $R$. Primer de tot, demostrem que les bijeccions del Corol·lari 12 donen lloc a un isomorfisme de semigrups (Proposició IV.3.1). Llavors, si $J$ és un ideal complet $\mathfrak{m}_Q$-primari en l'anell local $\mathcal{O}_{X,Q}$ d'una singularitat sandwiched $Q \in X$ i $\mathcal{J}$ és el feix d'ideals generat per ell, provem que la factorització de l'ideal complet $H_{\mathcal{J}}$ indueix la semifactorització de $J$. Concretament,

**Teorema 13.** [Teorema IV.3.5] *Sigui $S_{\mathcal{J}}$ la resolució minimal de $X$ tal que $J\mathcal{O}_{S_{\mathcal{J}}}$ és invertible. Per tot punt infinitament proper $p$, denotem per $m_p$ l'enter positiu més petit tal que $I_p^{m_p}$ és un ideal de Cartier per $X$. Llavors, si*

$$H_{\mathcal{J}} = \prod_p I_p^{\alpha_p} \tag{2.a}$$

*és la factorització (de Zariski) de $H_{\mathcal{J}} \subset R$ en ideals simples, llavors*

$$J = \prod_p J_p^{\frac{\alpha_p}{m_p}}$$

*és la factorització de $J$ en el sentit del Teorema I.3.32.*

*En particular, la factorització (2.a) de $H_{\mathcal{J}}$ dóna lloc a una factorització de $J$ en ideals simples de $\mathcal{O}_{X,Q}$ si i només si $\alpha_p \in (m_p)$ per tot $p$ tal que $\alpha_p > 0$.*

També provem que les factoritzacions de $J$ en ideals simples de $\mathcal{O}_{X,Q}$ indueixen i són induïdes per les factoritzacions de $H_{\mathcal{J}}$ en ideals irreductibles de Cartier per $X$. Concretament,

**Teorema 14.** [Teorema IV.3.7] *Donat un ideal complet $\mathfrak{m}_Q$-primari $J \subset \mathcal{O}_{X,Q}$, cada factorització de $J$ en ideals complets $\mathfrak{m}_Q$-primari ideals*

$$J = \prod_{i=1}^{r} J_i^{\alpha_i} \tag{2.b}$$

*indueix una factorització de $H_{\mathcal{J}}$ en ideals de Cartier per $X$*

$$H_{\mathcal{J}} = \prod_{i=1}^{r} H_{\mathcal{J}_i}^{\alpha_i},$$

*i cada factorització de $H_{\mathcal{J}}$ en ideals de Cartier per $X$ és d'aquest tipus. A més, $H_{\mathcal{J}_i}$ és irreductible com ideal de Cartier per $X$ si i només si $J_i$ és un ideal $\mathfrak{m}_{Q_i}$-primari i simple.*

En la secció V.1 recordem alguns resultats i fixem el marc pel nostre estudi dels espais d'arcs. Donada una singularitat racional de superfície $(X,Q)$, notem $\{E_u\}_{u\in\Delta_Q}$ les components excepcionals de la resolució minimal $S'$ de $(X,Q)$. Per cada $u \in \Delta_Q$, notem $\mathcal{F}_u^Q$ (respectivament, $\mathcal{N}_u^Q$) l'espai d'arcs per $(X,Q)$ tals que la seva elevació a $S_K$ interseca (resp. interseca transversalment) la component excepcional $E_u$. Fixat $i \geq 0$, notem $Tr(i)$ l'espai de les $i$-truncacions d'arcs, i $\mathcal{F}_u^Q(i)$ i $\mathcal{N}_u^Q(i)$ els espais de $i$-truncacions d'arcs en $\mathcal{F}_u^Q$ i $\mathcal{N}_u^Q$, respectivament. Les seccions V.2 i V.3 estan dedicades a demostrar que les components reduïdes del cicle fonamental d'una singularitat sandwiched donen lloc a components irreductibles de l'espai d'arcs: la secció V.2 és essencialment tècnica i provem una desigualtat entre el nombre d'intersecció a $O$ de les projeccions en $S$ d'arcs de famílies diferents en el cas que una d'elles estigui continguda en l'altra (Teorema V.2.1). A la secció V.3, l'estudi acurat i un xic tediós del graf dual i les relacions de proximitat entre els punts de $K$ donen el resultat desitjat. Concretament,

**Teorema 15.** [Teorema V.3.1] *Sigui $Q$ una singularitat sandwiched. Llavors cada component reduïda del cicle fonamental de $Q$ està associada a una família de Nash d'arcs. Equivalentment, si existeixen $p, q \in T_Q$ tal que $\overline{\mathcal{N}_p^Q(i)} \subsetneq \overline{\mathcal{N}_q^Q(i)}$ per $i >> 0$, llavors $z_p > 1$.*

En particular, el Teorema 15 dóna una resposta afirmativa a la conjectura de Nash pel cas de singularitats minimals de superfície. Aquest fet ja va ser demostrar per Reguera en [53]. Finalment, la secció V.4 està dedicada a demostrar que una resposta afirmativa a la conjectura de Nash per singularitats primitives implicaria la resposta afirmativa per singularitats sandwiched. Recordem que les singularitats primitives són aquelles singularitats que es poden obtenir explotant un ideal simple. La demostració d'aquest fet requereix un següent resultat:

**Proposició 16.** [Lema V.4.1] *Sigui $(X,Q)$ i $(X_1,Q_1)$ singularitats racionals de superfície, $g: X \longrightarrow (X_1,Q_1)$ un morfisme biracional dominant i $E_p$, $p \in \Delta_Q$ una component excepcional de $Q$ tal que $E_p$ apareix en la resolució minimal de $Q_1$ modulo equivalència biracional. Suposem que per algun $i > 0$, $\overline{\mathcal{F}_p^Q(i)} \subset \overline{\mathcal{F}_q^Q(i)}$ i que la projecció per* *g de qualsevol element de* $\mathcal{F}_q^Q(i)$ *està a* $\mathcal{F}_u^{Q_1}(i)$*, per algun* $u \in \Delta_{Q_1}$. *Llavors,*

$$\overline{\mathcal{F}_p^{Q_1}(i)} \subset \overline{\mathcal{F}_u^{Q_1}(i)}.$$

Si una singularitat sandwiched és primitiva, l'estructura dels clúster de punts base és molt més simple donat que el conjunt de $K$ està completament ordenat, fet que és fals en el cas general. A partir de la Proposició 16, deduïm el resultat desitjat

**Corol·lari 17.** [Corol·lari V.4.4] *Si la Conjectura de Nash és certa per singularitats primitives, també ho és per singularitats sandwiched.*

A l'**Appendix A**, presentem els llistats de tres programes en llenguatge **C** que implementen alguns dels algoritmes proposats. Aquests programes han estat fets servir per calcular alguns dels exemples presentats en aquesta memòria.

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# Glossary

| | |
|---|---|
| $\mathbf{1}_p$ | see page 25 |
| $\mathbf{A}_Q^{S'}$ | intersection matrix of the surface singularity $Q$, 28 |
| $\mathbf{A}_T$ | intersection matrix of the cluster $T$, 18 |
| $BP(I)$ | cluster of base points of $I$, 22 |
| $B_Q^{\mathcal{K}}$ | set of the $MR$-points of $\mathcal{K}$ relative to $Q$, 65 |
| $\widetilde{C}$ | strict transform of $C$ on $X$, 39 |
| $\widetilde{C}^{S_K}$ | strict transform of $C$ on $S_K$, 15 |
| $C^*$ | total transform of $C$ on $X$, 39 |
| $\check{C}^{H^o}$ | virtual transform of $C$ relative to $H^o$ on $X$, 162 |
| $\check{C}$ | virtual transform of $C$ relative to $I$ on $X$, 44 |
| $\check{C}^{\mathcal{K}}$ | virtual transform of $C$ relative to the virtual multiplicities of $\mathcal{K}$, 19 |
| $C_q$ | curve going sharply through $\mathcal{K}(q)$ and missing the points after $q$ in $K$, 39 |
| $c(\mathcal{T})$ | virtual codimension of $\mathcal{T}$, 26 |
| $[C, C']_P$ | intersection multiplicity at $P$, 17 |
| $\lvert C \cdot C' \rvert_{S'}$ | intersection number on $S'$, 17 |
| $ch^0(u, v)$ | see page 40 |
| $ch(u, v)$ | chain in a dual graph determined by $u$ and $v$, 40 |
| $\delta_P$ | order of singularity at $P$, 26 |
| $\delta_X$ | order of singularity on $X$, 26 |
| $D_p^Q$ | see page 182 |
| $E_K$ | exceptional divisor of $\pi_K : S_K \to S$, 15 |
| $e_p(C)$ | multiplicity of $C$ at $p$, 15 |
| $e_p(I)$ | multiplicity of $\mathfrak{L}_I$ at $p$, 22 |
| $\mathbb{E}_{S'}$ | group of divisors on $S'$ with exceptional support, 31 |
| $\mathbb{E}_{S'}^{\sharp}$ | set of divisors $D \in \mathbb{E}_{S'}$ such that $\mathcal{O}_{S'}(-D)$ is generated by global sections, 35 |

| | |
|---|---|
| $f : S_K \to X$ | minimal resolution of $X$, 37 |
| $f_C : S_{K_C} \to X$ | embedded resolution of $\widetilde{C}$, 102 |
| $\mathbb{F}_K, \overline{\mathbb{F}}_K$ | see page 49 |
| $F_p$ | first (infinitesimal) neighbourhood of $p$, 14 |
| $\mathcal{F}_u^Q$, $\mathcal{F}_u(i)$ | see page 180 |
| $\Gamma_K, \Gamma_Q$ | dual graph, 41 |
| $H_{\mathcal{K}}$ | ideal of curves going through $\mathcal{K}$, 19 |
| $H^o$ | see page 156 |
| $\mathbb{H}_C^o$ | see page 102 |
| $\mathcal{H}_Q$, $\mathcal{H}_Q(i)$ | see page 180 |
| $I_p$ | simple ideal in $R$ with maximal base point $p$, 24 |
| $I_Q$ | see page 47 |
| $\mathbf{I}_R^m$ | see page 162 |
| $\mathbf{J}_Q^*$ | semigroup of all complete $\mathfrak{m}_Q$-primary ideals in the local ring $\mathcal{O}_Q$, 35 |
| $K_C$ | see page 102 |
| $\mathcal{K}_C$ | see page 109 |
| $\mathcal{K}_C^o$ | see page 114 |
| $\mathcal{K}(p)$ | irreducible cluster with maximal point $p$, 23 |
| $\mathcal{K}_Q$ | see page 54 |
| $\mathcal{K}_+^Q$ | see page 68 |
| $\mathrm{K}_{S_T}$ | canonical divisor on $S_T$, 142 |
| $L_p$ | irreducible exceptional component on $X$, 39 |
| $\mathcal{L}_p$ | exceptional component of $C_p^*$ on $X$, 39 |
| $L_C, L_J, L_T$ | see page 39 |
| $\mathfrak{L}_I$ | linear system defined by $I$, 22 |
| $\mathfrak{m}_P$ | maximal ideal of $\mathcal{O}_P$, 13 |
| $\mathcal{M}_P$ | ideal sheaf of $P$, 45 |
| $\mathcal{N}_u^Q$, $\mathcal{N}_u(i)$ | see page 181 |
| $n_{\mathcal{T}}$ | number of irreducible subclusters of $\mathcal{T}$, 84 |
| $O_Q$ | minimal point of $T_Q$, 62 |
| $\pi_I : X \to S$ | blowing-up of the ideal $I \subset R$, 37 |
| $\pi_K^*(C)$ | total transform of $C$ on $S_K$, 15 |
| $\pi_T : S_T \to S$ | blowing-up of the points of $T$, 14 |
| $\mathbf{P}_K$ | proximity matrix of $K$, 16 |
| $R$ | regular local two-dimensional ring, 13 |
| $\rho_p^{\mathcal{K}}$ | excess of $\mathcal{K}$ at $p$, 18 |
| $r_p$ | number of points in $K$ proximate to $p$ plus one, 17 |
| $S_{\mathcal{J}}$ | see page 156 |

| | |
|---|---|
| $S_T$ | surface obtained by blowing-up the points of $T$, 14 |
| $\mathbf{S}_X^m$ | see page 162 |
| $\mathcal{T}_+$ | set of dicritical points of $\mathcal{T}$, 18 |
| $\mathcal{T}^2$ | self-intersection of $\mathcal{T}$, 21 |
| $T_Q$ | see page 53 |
| $\mathcal{T}_q$ | cluster obtained by adding $q$ to $\mathcal{T}$ with virtual multiplicity one, 49 |
| $Tr(i)$ | see page 180 |
| $\widetilde{\mathcal{T}}$ | consistent cluster obtained by unloading $\mathcal{T}$, 25 |
| $\widetilde{\mathcal{T}}^{K_0}$ | cluster obtained by partial unloading from $\mathcal{T}$ and relative to $K_0$, 129 |
| $v_p(C)$ | (effective) $p$-value of $C$, 15 |
| $\omega_\Gamma(u)$ | weight of $\Gamma$ at $u$, 40 |
| $Z_Q$ | fundamental cycle of the singularity $Q$, 29 |

# Index