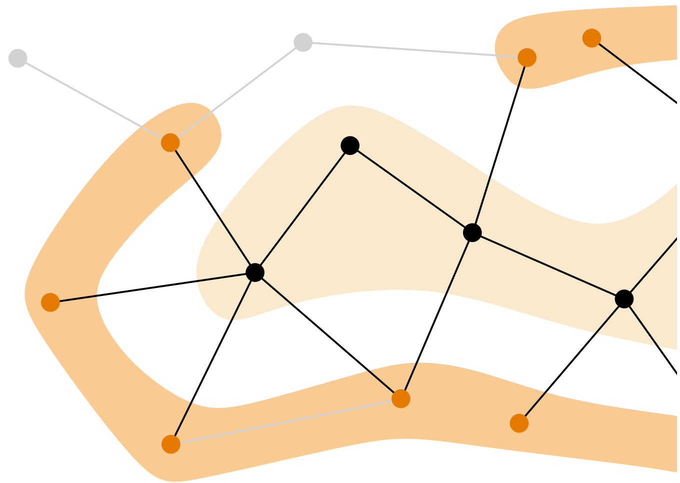


The inverse problem on finite networks

Cristina Araúz Lombardía

Universitat Politècnica de Catalunya, 2014

Thesis Advisors
Ángeles Carmona and Andrés M. Encinas

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Departament de Matemàtica Aplicada III

THE INVERSE PROBLEM ON FINITE NETWORKS

Cristina Araúz Lombardía

A Thesis submitted
for the degree of Doctor of Mathematics
in the Universitat Politècnica de Catalunya

Thesis Advisors
Ángeles Carmona and Andrés M. Encinas

Doctoral Program
APPLIED MATHEMATICS

Barcelona, June 2014

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Facultat de Matemàtiques i Estadística

Cristina Araúz Lombardía
Departament de Matemàtica Aplicada III
Universitat Politècnica de Catalunya
Edifici C2, Jordi Girona 1–3
E–08034, Barcelona
<cristina.arauz@upc.edu>

Acknowledgements

Quiero agradecer a mis directores de tesis, Ángeles Carmona y Andrés Encinas, el voto de confianza que me dieron en un principio. También la formación que me han proporcionado antes incluso de comenzar el doctorado y, durante éste, el esfuerzo que han realizado para seguir formándome y guiarme en mi trabajo. Sin ellos, esta tesis no existiría. Ambos han sido, y son, mucho más que mis directores de tesis. Su apoyo, calidez y cercanía son inigualables. El buen humor de Andrés y la capacidad de Ángeles para sacar todo adelante tampoco tienen igual! De los dos he intentado aprender como matemática y, sobre todo, como persona.

En el ámbito académico, quiero nombrar a Maria José, Margarida, Silvia, Enrique y Agustín. Junto con Ángeles y Andrés, han sido muchas las risas y buenos momentos que hemos pasado en el despacho. También agradecer al resto del departamento de MA3 estos años de convivencia.

Una mención muy especial a Javier S., quien motivó mis primeros pasos en la Teoría de Grafos, algo que siempre tengo presente, y quien me puso en contacto con los que ahora son mis directores. Y a Josep Lluís G., que hizo de nexo de unión.

Thanks to Charlie Johnson, who offered me the opportunity to stay in the College of William and Mary and help in the math REU program, as well as to work in the matrix analysis field, which was new for me and extremely interesting. I would also like to give special thanks to the members of this jury for reading this work, and to thank the referees for their helpful comments on a first version of this thesis.

En el ámbito personal, el poder escribir estas palabras se lo debo a mi familia. A mis padres, Julián y Pilar, que lo que les hace felices es la felicidad de los suyos. Gracias no sólo al cariño y apoyo de estos años de doctorado, sino al que me han dado siempre y al gran esfuerzo que han hecho toda su vida por la familia. Pocas palabras bastan, porque son buenos entendedores. Sus hijos somos muy afortunados. A mis hermanos, Ramón y Ana, siempre cuidando de la pequeña y compartiendo todo. Ahora que todos estamos un poco más crecidos (sólo un poco eh!), son mis referentes. Por nuestras tardes de risas y cachondeos.

No puedo dejar de recordar aquí a mi familia de Galicia, con quien he pasado los veranos de mi infancia, los de Castellnou–Barcelona, los fines de semana con ellos, y los de Valencia–Barcelona. Y a Maria José y su familia.

A en Marc. Els somriures, les nostres "sus cosillas", el veure les coses de la mateixa manera, l'entendre'ns sense parlar. A qui sempre hi és, de totes les maneres: gràcies. M'has acompanyat i guiat, o deixat fer; m'has animat i empès, o recolzat, atent al que he necessitat en cada moment.

I a la família d'en Marc, gràcies pel caliu.

Per últim, però no menys important, als meus amics. Gràcies per ser-hi, nois! I per compartir sopars i aventures. Com ja sabeu, per mi és important. En Marc, que també és el meu amic. La Marta, la còmplice i gran companya de fatigues i alegries, la Meri, la que sempre t'escolta, i l'Angi, la calidesa feta persona. Follonero y sus "en tu pueblo", Juanito y su gran humor, Alex y su seriedad no tan seria, Eva y su sueñito, Laura, la asilvestrada, Rubén, Yas, Marta P, Manel y Enric, nyuses todos ellos. L'Aida i les nostres malifetes, el Guillem i la nostra muy mejor amistad, en Morgan i la seva habilitat d'explicar anècdotes hilariants i l'entusiasta Berta. També la resta del cau terminal, ex–terminals i veïns: l'Elena, una gran i boníssima persona, en Vena, un home de gran sensibilitat, l'Eric, un gran conversador, i en PD, Teixi, Rubén M, Mònica, Hèctor, Víctor, Eli i Xavi. Amb els que he crescut de petita i de més gran, L'Annabel i l'Alberto. I encara em deixo amics: la toloca Vanessa, l'Aida S, en Ferran i tota la gran colla pessigolla, l'Anna R i els travessierus, la Nat, l'equip Ea Ea!, el Pascual i els del bus... a tots els que he nomenat i als que no, gràcies de nou per acompanyar-me.

Abstract

The aim of this thesis is to contribute to the understanding of discrete boundary value problems on finite networks. Boundary value problems have been considered both on the continuum and on the discrete fields. Despite working in the discrete field, we use the notations of the continuous field for elliptic operators and boundary value problems. The reason is the importance of the symbiosis between both fields, since sometimes solving a problem in the discrete setting can lead to the solution of its continuum version by a limit process. However, the relation between the discrete and the continuous settings does not work out so easily in general. Although the discrete field has softness and regular conditions on all its manifolds, functions and operators in a natural way, some difficulties that are avoided by the continuous field appear. Just to serve as an example, local behaviours in the discrete setting can be immensely different between two neighbouring points, whereas in the continuum local situations force the points in a neighbourhood to behave similarly.

Specifically, this thesis endeavors two objectives. First, we wish to deduce functional, structural or resistive data of a network taking advantage of its conductivity information. The actual goal here is to gather functional, structural and resistive information of a large network when the same specifics of the subnetworks that form it are known. The reason is that large networks are difficult to work with because of their size. The smaller the size of a network, the easier to work with it, and hence we try to break the networks into smaller parts that may allow us to solve easier problems on them. We seek the expressions of certain operators that characterize the solutions of boundary value problems on the original networks. These problems are denominated *direct boundary value problems*, on account of the direct employment of the conductivity information.

The second purpose is to recover the conductivity function or the internal configuration of a network using only boundary measurements and global equilibrium conditions. This recovery is performed using elliptic operator methods analogous to the ones of the continuous field. In fact, the resolution of this type of problem is the main objective in this thesis. For this problem is poorly arranged, at times we only target a partial reconstruction of the conductivity data or we introduce additional morphological conditions to the network in order to be able to perform a full internal reconstruction. This variety of problems is labelled as *inverse boundary value problems*, in light of the profit of boundary information to gain knowledge about the inside of the network. Inverse problems are exponentially ill–posed, since they are highly sensitive to changes in the boundary data. To sum up, our work tries to find situations where the recovery is feasible, partially or totally.

One of our ambitions regarding inverse boundary value problems is to recuperate the structure of the networks that allow the well–known Serrin's problem to have a solution in the discrete setting. Surprisingly, the answer is similar to the continuous case. On the other hand, we also aim to achieve a network characterization from a boundary operator on the network. With this end we define a new class of boundary value problems, that we call overdetermined partial boundary value problems. As a matter of fact, we can describe how the solutions of this family of problems that hold an alternating property on a part of the boundary spread through the network preserving this alternance. If we focus in a family of networks holding good structural properties, we see that the above mentioned operator on the boundary can be the response matrix of an infinite family of networks associated with different conductivity functions. Therefore, by choosing a specific extension of the positive eigenfunction associated with the lowest eigenvalue of the matrix, we get a unique network whose response matrix is equal to a previously given matrix.

Once we have characterized those matrices that are the response matrices of certain networks, we raise the problem of constructing an algorithm to recover the conductances. With this end, we characterize any solution of an overdetermined partial boundary value problem and describe its resolvent kernels. Then, we analyze two big groups of networks owning remarkable boundary properties which yield to the recovery of the conductances of certain edges near the boundary. We aim to give explicit formulae for the acquirement of these conductances. Using these formulae we are allowed to execute a full conductivity recovery under certain circumstances.

Contents

1 Introduction **9**

2 Background **15**

2.1 Network properties and subsets 15

2.2 Functions and linear operators on a network 17

2.3 Normal derivative and Schrödinger operators 19

2.4 Monotonicity and minimum principle 22

2.5 Green and Poisson operators 23

2.6 Effective resistances and Kirchhoff index 27

2.7 Circular planar networks 30

3 Effective resistances and Kirchhoff indices on composite networks **33**

3.1 Cluster networks 34

3.2 Corona networks 43

4 Product networks and applications **53**

4.1 Green function of product networks 54

4.2 Green function of spider networks 59

5 Discrete Serrin's problem **67**

5.1 Set properties and minimum principle 68

5.2 Distance layers and level sets 71

5.3 The discrete version of Serrin's problem 72

5.4 Spider networks with radial conductances 74
5.5 Regular layered networks. Characterization 78
5.6 Other networks satisfying Serrin's condition 81

6 Dirichlet–to–Robin maps on finite networks 87
6.1 The Dirichlet–to–Robin map 88
6.2 Alternating paths 97
6.3 Motivation 104
6.4 Characterization of the Dirichlet–to–Robin map 108

7 Conductivity recovery 111
7.1 Overdetermined partial boundary value problems 112
7.2 Modified Green, Poisson and Robin operators 119
7.3 Boundary spike formula 122
7.4 Networks with separated boundary 125
7.5 Recovery on rigid three dimensional grids 125
7.6 Boundary spikes on circular planar networks 132
7.7 Recovery on well–connected spider networks 135

8 Conclusions and future work 147
8.1 On Green functions in network partitioning 147
8.2 On the discrete Serrin's problem 148
8.3 On the Dirichlet–to–Robin map 149
8.4 On overdetermined partial boundary value problems 150

Bibliography 157

1 Introduction

A *graph* $G = (V, E)$ consists in a finite set of *vertices* V and a set of pairs of vertices $E \subseteq V \times V$ such that $(x, y) \in E$ if and only if $(y, x) \in E$, called *edges*. Given two vertices $x, y \in V$, they are *adjacent* or *neighbours* if and only if $(x, y) \in E$. In this case, we denote $x \sim y$ and call xy the edge dispensed by the pair (x, y). We say that x and y are the *ends* of the edge xy and that xy is *incident* on both x and y. The *set of neighbours* of a vertex $x \in V$ is denoted by $N(x) = \{x \in V : y \sim x\}$. A *loop* is an edge with both ends the same vertex and a *multiple edge* is any edge that appears more than once in E. Throughout this thesis we only consider *simple* graphs, that is, graphs with no loops nor multiple edges.

Any graph can be sketched in the plane by drawing a node for each vertex and depicting a line joining the corresponding two nodes for every edge, see Figure 1.1.

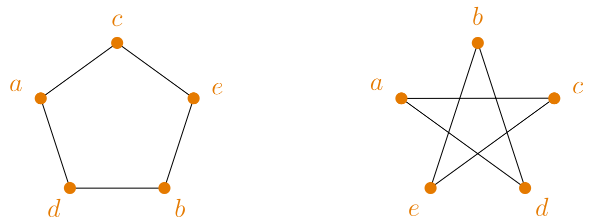


Figure 1.1 Two different representations of the graph $G = (V, E)$, where $V = \{a, b, c, d, e\}$ and $E = \{ac, ad, be, bd, ce\}$.

A *network* Γ is a graph with positive weights on the edges. These weights are called *conductances*. They are supplied by the *conductivity function*

$c\colon V \times V \longrightarrow [0, +\infty)$, which holds the symmetric property $c(x, y) = c(y, x)$ for every pair $x, y \in V$ and $c(x, y) = 0$ if and only if $xy \notin E$. Thus, $E = \{(x, y) \in V \times V : c(x, y) > 0\}$ is the set of edges of the network and any network can be fully represented by the pair $\Gamma = (V, c)$. If $c(x, y) = 1$ for every pair of vertices such that $c(x, y) > 0$, then $\Gamma = (V, c)$ can be considered a graph.

Graphs and networks can be modified in order to obtain other graphs or networks. Namely, we can remove or add vertices or edges, or even contract edges to a single vertex. Given a network $\Gamma = (V, c)$, to *remove a vertex* $x \in V$ consists in erasing the vertex x, as well as its incident edges. To *add a vertex* is to consider a new vertex different from the ones in V and, maybe, to append edges joining it to other existing vertices. It is clear what to *remove an edge* means and in a similar manner to *add an edge* is to place a new edge between two existing vertices that are not adjacent. The *contraction of an edge xy* is to erase xy and to identify x and y in a unique vertex z, transfering the neighbours of each one of them into z with the corresponding conductances. In Figure 1.2 this modification is shown.

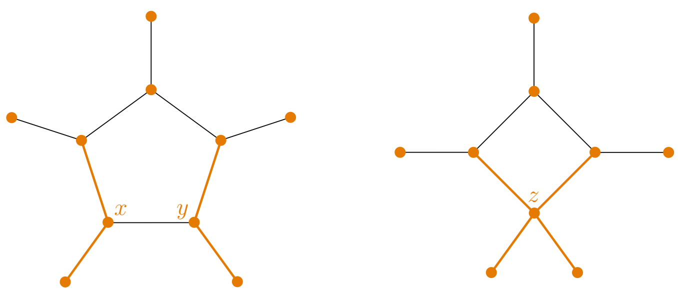


Figure 1.2 A contraction of the edge xy into a vertex z.

The discrete objects described above are the primary elements we work with throughout this thesis. Our aim is to obtain structural properties and unknown information about networks employing elliptic operator methods. Specifically, this thesis endeavors two objectives. First, we wish to obtain functional, structural or resistive data of a network taking advantage of its conductivity information. This is accomplished in some families of networks in the literature, for instance paths [19]. The goal for us is to gather functional, structural and resistive information of a large network when the same specifics of the subnetworks that form it are known. These problems are denominated *direct boundary value problems*, on account of the direct employment of the conductivity information. The second purpose is to recover the conductivity function or the internal configuration of a network using only boundary measurements and global equilibrium conditions. For this problem is poorly arranged, at times we only target a partial reconstruction of the conductivity data or we introduce additional morphological conditions to the network in order to be able to perform a full internal reconstruction. This variety of problems is labelled as *inverse boundary value problems*, in light of the profit of boundary information to gain knowledge about the inside of the network.

Despite working in the discrete field, we use the notations of the continuous field for elliptic operators and boundary value problems. The reason is the importance of the symbiosis between both fields, since sometimes solving a problem in the discrete setting can lead to the solution of its continuum version by a limit process. However, the relation between the discrete and the continuous settings does not work out so easily in general. Although the discrete field has softness and regular conditions on all of its manifolds, functions and operators in a natural way, some difficulties that are avoided by the continuous field appear. Just to serve as an example, local behaviours in the discrete setting can be immensely different between two neighbouring points, whereas in the continuum local situations force the points in a neighburhood to behave similarly.

The theoretical background needed for these objectives is described in Chapter 2. For short, we introduce several parameters of a network and present the concept of network with boundary. We also describe functions and linear operators on a network following the notations introduced by Bendito, Carmona and Encinas in [16]. In particular, Schrödinger operators on a network and the normal derivative on the boundary are presented. Both operators are related to the well–known laplacian operator on a network. These concepts being set, boundary value problems on networks using Schrödinger operators and their monotonicity properties are brought in. Moreover, we carry in two well–known functions that are the keys to describe any solution of this kind of boundary value problem, named Green and Poisson operators. Afterwards, resistive and structural parameters of a network are introduced: generalized effective resistances and Kirchhoff indices. The chapter ends with the study of circular planar networks, which have been extensively treated in [25, 36, 38].

Chapter 3 strives the study of certain direct boundary value problems. It deals with the deduction of functional, resistive and morphological data on composite networks in terms of the networks forming them. The reason is that large networks are difficult to work with because of their size. The smaller the size of a network, the easier to work with it, and hence we try to break the networks into smaller pieces that may allow us to solve easier problems on them. We seek the expressions of the orthogonal Green operator, as well as the generalized effective resistance between two vertices and the generalized Kirchhoff index of certain composite networks. Namely, generalized cluster and corona networks.

In Chapter 4 we also consider direct boundary value problems. We operate on product networks, which are the network version of the cartesian product of graphs, and we use separation of variable techniques in order to express their Green operator in terms of the Green operators of the factors. Thereafter we obtain the Green operator of another family of networks, spider networks, as an application of these results.

Chapters 5, 6 and 7 deal with the study of inverse boundary value problems in different ways. First, Chapter 5 consists in the discretization of a well–known overdetermined problem in the continuous setting, *Serrin's problem.* The continuous version deals with the characterization of those domains where a specific overdetermined boundary value problem has solution. In this event, our ambition is to recuperate the structure of the networks that allow this problem to have a solution in the discrete setting. Surprisingly, the answer is similar to the continuous case. In fact, the discrete Serrin's problem is the extreme case of a family of boundary value problems named overdetermined partial boundary value problems, which are introduced in Chapter 7.

Out of the Poisson operator, we introduce the Dirichlet–to–Robin map, which is formalized on the boundary. The intention in Chapter 6 is to achieve a network characterization from the Dirichlet–to–Robin map, which is an extension of the findings of Curtis *et al.* in [37, 38] for the response matrix associated with the laplacian. We first look upon the solutions of boundary value problems with an alternating property in a part of the boundary and show that they spread across the network in such a way that they hold a derived alternating property in another part of the boundary. In fact, these solutions spread following boundary–to–boundary paths where the sign of the solution is invariable and has opposite sign with respect to the neighbouring paths in the circular order. In a second stage we focus in circular planar networks and observe that any Dirichlet–to–Robin map can be the response matrix of an infinite family of networks associated with different conductivity functions, a phenomenon that has not been observed until now. Therefore, by choosing a specific extension of the positive eigenfunction associated with the lowest eigenvalue of the matrix, we get a unique network whose Dirichlet–to–Robin map corresponds to a certain matrix.

We give thought to a another face of inverse boundary value problems in Chapter 7. Once we have characterized those matrices that are the response matrices of certain networks, we raise the problem of constructing an algorithm to recover the conductances. With this end, we characterize the solutions of any overdetermined partial boundary value problem and describe its resolvent kernels. Then, we analyze two big groups of networks owning remarkable boundary properties which yield to the recovery of the conductances of certain edges near the boundary. We aim to give explicit formulae for the acquirement of these conductances. Using these formulae we are allowed to execute a full conductivity recovery under certain circumstances, specifically for rigid three dimensional grids and well–connected spider networks.

2
Background

The main end of this chapter is to present the basic definitions and results related to finite networks that are indispensable in the development of this thesis. We also describe functions on the sets of vertices as well as linear operators on the sets of functions, following the notations of the continuous field that were introduced in [16] for the discrete domain. Yet, now and then we put to use the matricial notation for the kernels of the linear operators.

2.1 Network properties and subsets

Let $\Gamma = (V, c)$ be a network. A *path of length* $m-1$ is a sequence of different vertices $\{x_1, \ldots, x_m\} \subseteq V$ such that $m \geq 1$ and $x_i \sim x_{i+1}$ for all $i = 1, \ldots, m-1$, together with the edges $x_i x_{i+1}$. Moreover, if $x_m \sim x_1$, then the sequence $\{x_1, \ldots, x_m\} \subseteq V$ together with the edges $x_m x_1$ and $x_i x_{i+1}$ for $i = 1, \ldots, m-1$ is a *cycle* of m vertices. Sometimes we denote them by the m*–path* P_m or the m*–cycle* C_m, respectively.

We say that $\Gamma = (V, c)$ is *connected* if any two vertices of V can be joined by a path. We abuse the notation and say that V is connected. Furthermore, given a vertex subset $F \subseteq V$ we say that F is *connected* if each pair of vertices of F is joined by a path entirely contained in F. A *connected component* of Γ is a connected subset $F \subseteq V$ such that there exists no path from any vertex $x \in F$ to any other vertex $y \notin F$. Hence, Γ is connected if and only if it has only one connected component. If Γ is connected and $k \geq 2$, a vertex $x \in V$ is k*–separating* if removing x results in breaking the network Γ into exactly k connected components.

We can define a *distance function* $d\colon V \times V \longrightarrow [0, +\infty]$ on any network. Given two vertices $x, y \in V$, the minimum length among all the paths joining

x and y is called *distance between x and y* and is denoted by $d(x, y)$. If such a path does not exist, then $d(x, y) = +\infty$. It is well–known that the distance function satisfies the triangle inequality, as well as other symmetry properties.

Lemma 2.1.1. *The distance function $d\colon V \times V \longrightarrow [0, +\infty]$ determines a distance on V, that is, it satisfies the following properties.*

(i) $d(x, y) = 0$ *if and only if* $x = y$.

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in V$.

(iii) $d(x, y) + d(y, z) \geq d(x, z)$ *for all* $x, y, z \in V$.

For any $x \in V$ we define the *distance from x to a set $F \subseteq V$* as $d(x, F) = \min_{y \in F}\{d(x, y)\}$. Notice that $x \in F$ if and only if $d(x, F) = 0$ and so $x \notin F$ if and only if $d(x, F) \geq 1$. The *external radius* of F is the value

$$r(F) = \max_{x \in V}\{d(x, F)\} = \max_{x \notin F}\{d(x, F)\} = \max_{x \notin F} \min_{y \in F}\{d(x, y)\} \geq 1.$$

Given a vertex subset $F \subseteq V$, let us introduce several sets determined by F. We denote by $F^c = V \setminus F$ the *complementary* set of F in V and call *boundary* and *closure* of F the sets $\delta(F) = \{x \in F^c : d(x, F) = 1\}$ and $\bar{F} = F \cup \delta(F)$, respectively. It is straightforward to prove that $\bar{F}$ is connected when F is. If we consider a subset in the closure of F, $S \subseteq \bar{F}$, we can denote its *neighbourhood* as $N(S) = \big\{x \in \bar{F} \setminus S : \exists y \in S \text{ with } y \sim x\big\}$. In other words, $N(S) = \delta(S) \cap \bar{F}$. When $S \subseteq \delta(F)$, its *neighbourhood in* F is $N_F(S) = \delta(S) \cap F$. We call *interior* and *exterior* of F the subsets $\overset{\circ}{F} = \{x \in F : y \in F \text{ for all } y \sim x\}$ and $\mathsf{Ext}(F) = \{x \in V : d(x, F) \geq 2\}$, respectively. Observe that $\overset{\circ}{F}$ is not necessarily connected even when F is connected.

The vertices of $\delta(F)$ are called *boundary vertices* and when a boundary vertex $x \in \delta(F)$ has a unique neighbour in $\bar{F}$ we call the edge joining them a *boundary spike*. Clearly, this unique neighbour of x is in F. When an edge has both ends in $\delta(F)$, it is called *boundary–to–boundary edge* or simply *boundary edge*. The two last concepts were introduced in [38] by Curtis, Ingerman and Morrow. Figure 2.1 illustrates all these concepts.

If all the vertices in $\delta(F)$ have a unique neighbour, then we say that $\delta(F)$ is a *separated boundary*. This concept was introduced in [46] by Friedman and

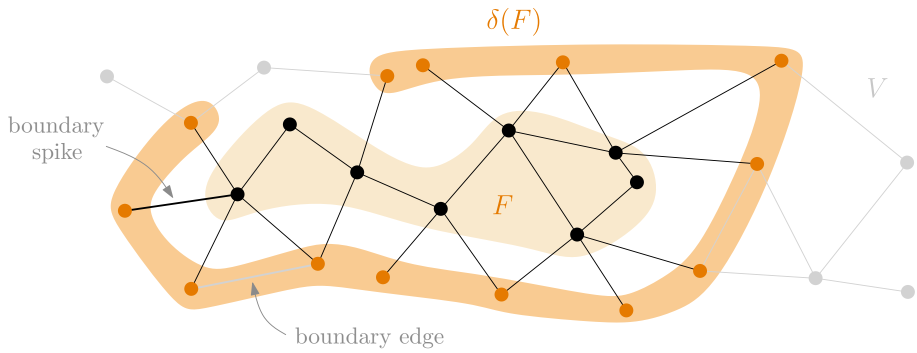


Figure 2.1 Representation of a set F and its boundary $\delta(F)$ on a network.

Tillich and will be extensively used in our work. They state that boundary separation is a property whose analogue for manifolds is always true and hence, in most practical situations, one can assume the boundary is separated. Moreover, they also say that certain boundary conditions behave bizarrely unless the boundary is separated.

2.2 Functions and linear operators on a network

Before introducing new definitions, we would like to detail the convention this thesis follows regarding functions, linear operators and other related parameters on a network. We use the stardard typeface for functions on one and two variables, working with letters in lower case or capital letters, respectively. For linear operators, we use capital caligraphical font instead. In the event of naming a matrix, the typeface put in service is capital sans serif, and when denoting a vector we use small sans serif letters.

Let $\Gamma = (V, c)$ be a network. We denote by $\mathcal{C}(V)$ the set of functions $f\colon V \longrightarrow \mathbb{R}$. For $f \in \mathcal{C}(V)$ we define its *support* as $\mathsf{supp}(f) = \{x \in V : f(x) \neq 0\} \subseteq V$. If $F \subseteq V$, the set $\mathcal{C}(F) = \{f \in \mathcal{C}(V) : \mathsf{supp}(f) \subseteq F\}$ can be identified with the *set of functions on* F given by all the functions $f\colon F \longrightarrow \mathbb{R}$. Moreover, $\mathcal{C}^+(F)$ is the *set of non–negative functions on* F. These sets are naturally identified with $\mathbb{R}^{|F|}$ and the positive cone of $\mathbb{R}^{|F|}$, respectively.

There are some functions and families of functions that will play an important role all allong this work. For instance, the *characteristic function* of a set $F \subseteq V$ is the function $\chi_F \in \mathcal{C}(F)$ defined as $\chi_F = 1$ on F. The characteristic function of a single vertex $x \in V$ is denoted by $\varepsilon_x \in \mathcal{C}(\{x\})$. Hence, $\varepsilon_x(z) = 0$ if $z \neq x$ and $\varepsilon_x(x) = 1$. The following concepts were presented in [16, 17]. Given a function $u \in \mathcal{C}(V)$ and a set $F \subseteq V$, we represent by $\int_F u(z)\,dz$ or simply by $\int_F u$ the value $\sum_{z\in F} u(z)$, for we desire to stablish a discrete notation analogous to the one of the continuous field. Putting in service this notation, we know as *weight* on F any function $\omega \in \mathcal{C}^+(F)$ such that $\mathsf{supp}(\omega) = F$ and $\int_F \omega^2 = 1$. The set of weights on F is designated by $\Omega(F)$. Another valuable function is the *total conductance* at a vertex $x \in V$, also known as the *degree* of x, which is determined by $\kappa(x) = \int_V c(x,z)\,dz$. If $F \subset V$ is a proper subset, then for any $x \in \delta(F)$ the *boundary degree* with respect to F is the function $\kappa_F \in \mathcal{C}^+(\delta(F))$, with expression $\kappa_F(x) = \int_F c(x,z)\,dz$. Also, an scalar product can be defined on a network. If $F \subseteq V$, we consider the *inner product* on F provided by $\langle\cdot,\cdot\rangle_F \colon \mathcal{C}(F)\times\mathcal{C}(F) \longrightarrow \mathbb{R}$, where $\langle u,v\rangle_F = \int_F uv$ for all $u,v \in \mathcal{C}(F)$. In particular, if $F = V$ then we denote the inner product on V simply as $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_V$. The *norm* $||\cdot||_F \colon \mathcal{C}(V)\times\mathcal{C}(V) \longrightarrow \mathbb{R}^+$ is provided by $||u||_F = \sqrt{\langle u,u\rangle_F}$ for any $u \in \mathcal{C}(V)$, and following the convention we say that $||\cdot|| = ||\cdot||_V$. Observe that $||\omega||_F = 1$ and $||\omega||_S < 1$ for any weight $\omega \in \Omega(F)$ and any proper subset $S \subset F$.

Consider two vertex sets $T,S \subseteq V$. A *linear operator* $\mathcal{K}\colon \mathcal{C}(T) \longrightarrow \mathcal{C}(S)$ is a morphism such that $\mathcal{K}(u) \in \mathcal{C}(S)$ for all $u \in \mathcal{C}(T)$. We call $\mathcal{O}(T,S)$ the *space of linear operators* on the network. The *kernel associated* with $\mathcal{K} \in \mathcal{O}(T,S)$ is the function $K \in \mathcal{C}(S\times T)$ given by $K(x,y) = \mathcal{K}(\varepsilon_y)(x)$ for all $x \in S$ and $y \in T$. K is called *kernel* for the reason that the integral operator associated with K is given by $\mathcal{K}(u)(x) = \int_T K(x,z)u(z)\,dz$ for all $u \in \mathcal{C}(T)$ and $x \in S$. We now introduce two families of functions associated with the kernel $K \in \mathcal{C}(S\times T)$ of the linear operator $\mathcal{K}\colon \mathcal{C}(T) \longrightarrow \mathcal{C}(S)$. The *first component of* K with respect to $y \in T$ is the function $K^y \in \mathcal{C}(S)$ determined by $K^y(x) = K(x,y) = \mathcal{K}(\varepsilon_y)(x)$ for all $x \in S$, whereas the *second component of* K with respect to $x \in S$ is the function $K_x \in \mathcal{C}(T)$ prescribed by $K_x(y) = K(x,y)$ for all $y \in T$.

Allow for the elements of the set V to have a certain labeling, which will be detailed when necessary. Given two vertex sets $T,S \subseteq V$ and a linear operator $\mathcal{K}\colon \mathcal{C}(T) \longrightarrow \mathcal{C}(S)$, we follow the terminology of [37] and define $\mathsf{K} \in \mathcal{M}_{|S|\times|T|}(\mathbb{R})$ as the matrix with entries provided by the values $K(x,y)$ under the ordering given by the above–mentioned labeling. Moreover, if $P \subseteq S$ and $Q \subseteq T$, then $\mathsf{K}(\mathsf{P};\mathsf{Q})$ stands for the submatrix of K given by the rows corresponding to the vertices of P and the columns corresponding to the vertices of Q. For the sake of simplicity, we write $\mathsf{K}(\mathsf{x};\mathsf{T}) = \mathsf{K}(\{\mathsf{x}\};\mathsf{T})$ and $\mathsf{K}(\mathsf{S};\mathsf{y}) = \mathsf{K}(\mathsf{S};\{\mathsf{y}\})$.

From time to time we need to express a discrete function $u \in \mathcal{C}(V)$ in vectorial notation so as to work with matricial equations. The vector $\mathsf{u}_\mathsf{S} \in \mathbb{R}^{|S|}$ is the vector whose entries are supplied by the values of u on $S \subseteq V$ under the ordering referred to.

2.3 Normal derivative and Schrödinger operators

In this section we consider a connected network $\Gamma = (V, c)$ and a non–empty connected subset $F \subseteq V$. The *laplacian operator* or *combinatorial laplacian* of Γ on F is the linear operator $\mathcal{L}\colon \mathcal{C}(\bar{F}) \longrightarrow \mathcal{C}(F)$ that assigns to each $u \in \mathcal{C}(\bar{F})$ the function

$$\mathcal{L}(u)(x) = \int_{\bar{F}} c(x,y)\big(u(x) - u(y)\big)\, dy$$

for any vertex $x \in F$. When $F \neq V$, we define the *normal derivative* of Γ on $\delta(F)$ as the operator $\dfrac{\partial}{\partial \mathsf{n}_F}\colon \mathcal{C}(\bar{F}) \longrightarrow \mathcal{C}(\delta(F))$ dispensed by

$$\frac{\partial u}{\partial \mathsf{n}_F}(x) = \int_{F} c(x,y)\big(u(x) - u(y)\big)\, dy$$

for any boundary vertex $x \in \delta(F)$ and any function $u \in \mathcal{C}(\bar{F})$. It is important to perceive that the normal derivative is an extension of the combinatorial laplacian to the boundary $\delta(F)$ with the difference that it only takes into account the neighbours in F but not those in $\delta(F)$.

On almost all occasions we work with a generalization of the combinatorial laplacian, known as Schrödinger operator. This generalization is a 0–order perturbation of the well–known laplacian operator. Given a function $q \in \mathcal{C}(\bar{F})$, the *Schrödinger operator with potential* q on Γ is the linear operator $\mathcal{L}_q\colon \mathcal{C}(\bar{F}) \longrightarrow \mathcal{C}(\bar{F})$ that ascribes to every $u \in \mathcal{C}(\bar{F})$ the function $\mathcal{L}_q(u) = \mathcal{L}(u) + qu$ on F and $\mathcal{L}_q(u) = \dfrac{\partial u}{\partial \mathsf{n}_F} + qu$ on $\delta(F)$. At times we only consider its definition on F, though. In any case, it will be clearly specified whether a Schrödinger operator is being considered only on F or on the whole closure $\bar{F}$.

The relation between the values of the Schrödinger operator with potential q on F and the values of the normal derivative on $\delta(F)$ is given by the *First*

Green Identity, proved in [16, Proposition 3.1]. If we consider the function $c_F = c \cdot \chi_{(\bar{F}\times\bar{F})\smallsetminus(\delta(F)\times\delta(F))} \in \mathcal{C}((\bar{F}\times\bar{F})\smallsetminus(\delta(F)\times\delta(F)))$, this identity is as follows:

$$\int_F v\mathcal{L}_q(u) = \frac{1}{2}\int_{\bar{F}}\int_{\bar{F}} c_F(x,y)\Big(u(x)-u(y)\Big)\Big(v(x)-v(y)\Big)\,dx\,dy + \int_F quv - \int_{\delta(F)} v\frac{\partial u}{\partial \mathsf{n}_F}$$

for all $u,v \in \mathcal{C}(\bar{F})$. A direct consequence is the *Second Green Identity*

$$\int_F \Big(v\mathcal{L}_q(u) - u\mathcal{L}_q(v)\Big) = \int_{\delta(F)}\left(u\frac{\partial v}{\partial \mathsf{n}_F} - v\frac{\partial u}{\partial \mathsf{n}_F}\right)$$

for all $u,v \in \mathcal{C}(\bar{F})$, which was proved in the same paper. In particular, if $v = \chi_{\bar{F}}$ then the Second Green Identity collapses into the well–known *Gauss Theorem*:

$$\int_F \mathcal{L}(u) = -\int_{\delta(F)}\frac{\partial u}{\partial \mathsf{n}_F}$$

for any function $u \in \mathcal{C}(\bar{F})$. If $F = V$, then $\int_V \mathcal{L}(u) = 0$ and moreover, $\mathcal{L}(u) = 0$ on V if and only if u is a constant function [18]. The following result is a direct consequence of the Second Green Identity.

Corollary 2.3.1 ([16, Corollary 3.2])**.** *Any Schrödinger operator on V is self–adjoint, that is, $\int_V u\mathcal{L}_q(v) = \int_V v\mathcal{L}_q(u)$ for all $u,v \in \mathcal{C}(V)$. Furthermore, if $u,v \in \mathcal{C}(F)$ then $\int_F u\mathcal{L}_q(v) = \int_F v\mathcal{L}_q(u)$.*

We define the symmetric bilinear form $\mathcal{E}_q^F\colon \mathcal{C}(\bar{F})\times\mathcal{C}(\bar{F}) \longrightarrow \mathbb{R}$ given by

$$\mathcal{E}_q^F(u,v) = \frac{1}{2}\int_{\bar{F}}\int_{\bar{F}} c_F(x,y)\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\,dx\,dy + \int_{\bar{F}} quv$$

for any $u,v \in \mathcal{C}(\bar{F})$. It is called *energy associated with* F, see [16], and it is inspired by the First Green Identity. In fact, using this identity, the energy associated with F is also expressed as

$$\mathcal{E}_q^F(u,v) = \int_F v\mathcal{L}_q(u) + \int_{\delta(F)} v\left(\frac{\partial u}{\partial \mathsf{n}_\mathsf{F}} + qu\right).$$

Looking back to Schrödinger operators, although any of them is self–adjoint, we are interested in those that are positive semi–definite, for they possess good properties for our interest. The characterization of this sort of operators was obtained in [18] by considering the potential determined by a weight $\omega \in \Omega(\bar{F})$, which is given by $q_\omega = -\omega^{-1}\mathcal{L}(\omega)$ on F and $q_\omega = -\omega^{-1}\dfrac{\partial \omega}{\partial \mathsf{n}_F}$ on $\delta(F)$, and 0–order perturbations of it. Furthermore, in order to study the positive–semidefiniteness of Schrödinger operators, in [21] the discrete version of the well–known *Doob transform* was introduced. The Doob transform is a worthwhile tool in the framework of Dirichlet forms.

Proposition 2.3.2 ([21, Doob Transform]). *Given a weight $\omega \in \Omega(\bar{F})$, the following identities hold for any $u \in \mathcal{C}(\bar{F})$:*

$$\mathcal{L}(u)(x) = \frac{1}{\omega(x)} \int_{\bar{F}} c(x,y)\omega(x)\omega(y) \left(\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)} \right) dy - q_\omega(x)u(x)$$

for all $x \in F$, whereas for every $x \in \delta(F)$

$$\frac{\partial u}{\partial \mathsf{n}_F}(x) = \frac{1}{\omega(x)} \int_{F} c(x,y)\omega(x)\omega(y) \left(\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)} \right) dy - q_\omega(x)u(x).$$

In addition, for any $u, v \in \mathcal{C}(\bar{F})$,

$$\begin{aligned}\mathcal{E}^F(u,v) = \frac{1}{2}\int_{\bar{F}}\int_{\bar{F}} c_F(x,y)\omega(x)\omega(y)\left(\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}\right)\left(\frac{v(x)}{\omega(x)} - \frac{v(y)}{\omega(y)}\right) dx\, dy \\ - \int_{\bar{F}} q_\omega uv,\end{aligned}$$

where $\mathcal{E}^F(u,v) = \mathcal{E}^F_q(u,v) - \int_{\bar{F}} quv$.

Proposition 2.3.3 ([21, Proposition 3.2]). *The energy $\mathcal{E}^F_q$ is positive semi–definite if and only if there exists a weight $\omega \in \Omega(\bar{F})$ such that $q \geq q_\omega$ on $\bar{F}$. Moreover, $\mathcal{E}^F_q$ is not strictly positive definite if and only if $q = q_\omega$. In this last case, $\mathcal{E}^F_q(v,v) = 0$ if and only if $v = a\omega$ for some $a \in \mathbb{R}$.*

Having fixed a weight $\omega \in \Omega(\bar{F})$, the Schrödinger operator $\mathcal{L}_{q_\omega}$ given by the potential q_ω is a very common tool in the field of Dirichlet forms and Markov processes. In fact, in the continuous field,

$$\mathcal{L}_{q_\sigma}(u) = -\mathrm{div}(\sigma \nabla u),$$

if $q_\sigma = (\sqrt{\sigma})^{-1}\Delta(\sqrt{\sigma})$, where $\mathrm{div}(\sigma\nabla u)$ is the conductivity of the terrain of a non–homogeneous but isotropic environment. This fact was observed by

Calderón in [29]. In the discrete field, $\mathcal{L}_{q_\omega}$ contains as a particular case the *normalized laplacian* introduced by Chung and Langlands in [34], which is defined as

$$\Delta(u)(x) = \frac{1}{\sqrt{\kappa(x)}} \int_V c(x,y) \left(\frac{u(x)}{\sqrt{\kappa(x)}} - \frac{u(y)}{\sqrt{\kappa(y)}} \right) dy.$$

The relation is not straightforward: the normalized laplacian with respect to a conductivity function c coincides with the Schödinger operator $\mathcal{L}_{q_\omega}$ on V with respect to a conductivity function $\widehat{c}\colon V \times V \longrightarrow [0, +\infty)$ given by $\widehat{c}(x,y) = c(x,y)k(x)^{-\frac{1}{2}}k(y)^{-\frac{1}{2}}$ and the weight $\omega(x) = k(x)^{\frac{1}{2}}(\sum_{z\in V} k(z))^{-\frac{1}{2}}$.

Proposition 2.3.4 ([18, Proposition 3.3])**.** *The Schrödinger operator $\mathcal{L}_q$ is positive semi–definite on $\bar{F}$ if and only if there exist a weight $\omega \in \Omega(\bar{F})$ and a real value $\lambda \geq 0$ such that $q = q_\omega + \lambda$. Moreover, ω and λ are uniquely determined. In addition, $\mathcal{L}_q$ is not positive definite on $\bar{F}$ if and only if $\lambda = 0$, in which case $\int_{\bar{F}} v\mathcal{L}_{q_\omega}(v) = 0$ if and only if $v = a\omega$ on $\bar{F}$ for some $a \in \mathbb{R}$. In any case, λ is the lowest eigenvalue of $\mathcal{L}_q$ and its associated eigenfunctions are multiple of ω.*

In order to develop a potential theory, see Anandam [3] and Bendito *et al.* [31]. We only work with semi–definite positive Schrödinger operators in the sequel. Therefore, we will consider that every potential $q \in \mathcal{C}(\bar{F})$ fits the expression $q = q_\omega + \lambda$ for some weight on $\bar{F}$ and real non–negative value λ, except when other setting equivalent to the last one is described for q. Moreover, we suppose that it is not simultaneously true that $F = V$ and $\lambda = 0$, unless the opposite is clearly stated. This means that we assume F to be a non empty subset of V except when $\lambda = 0$, in which case F is a proper subset.

We denote by $L_q \in \mathcal{C}(\bar{F} \times \bar{F})$ the kernel of the Schrödinger operator $\mathcal{L}_q$ on $\bar{F}$ and write as $\mathsf{L_q} \in \mathcal{M}_{|\bar{F}|\times|\bar{F}|}(\mathbb{R})$ the matrix given by this kernel, where the ordering in $\bar{F}$ is arbitrarily chosen as $\bar{F} = \{\delta(F); F\}$. Just to clarify in benefit of the reader, the matrix $\mathsf{L_q}$ has the following block structure

$$\mathsf{L_q}(\bar{\mathsf{F}};\bar{\mathsf{F}}) = \begin{bmatrix} \mathsf{L_q}(\delta(\mathsf{F});\delta(\mathsf{F})) & \mathsf{L_q}(\delta(\mathsf{F});\mathsf{F}) \\ \mathsf{L_q}(\mathsf{F};\delta(\mathsf{F})) & \mathsf{L_q}(\mathsf{F};\mathsf{F}) \end{bmatrix}.$$

2.4 Monotonicity and minimum principle

Let $u \in \mathcal{C}(V)$ be a function. We say that u is *harmonic*, *superharmonic* or *subharmonic* on F if $\mathcal{L}(u) = 0$, $\mathcal{L}(u) \geq 0$ or $\mathcal{L}(u) \leq 0$ on F, respectively. In particular, $u \in \mathcal{C}(V)$ is *strictly superharmonic* or *strictly subharmonic* on F if $\mathcal{L}(u) > 0$ or $\mathcal{L}(u) < 0$ on F, respectively. All the harmonic functions on V are multiples of χ_V because of the positive semi–definiteness of $\mathcal{L}$. In fact, if $u \in \mathcal{C}(V)$ is either superharmonic or subharmonic on V, then it is harmonic and hence constant.

The following results establish the minimum principle and the monotonicity property of the laplacian operator. They were proved in [18] in a more general context, see also [40]. These results are included here because they are the basis for the tools we work with. In the sequel, we assume that F is a proper connected subset of V.

Proposition 2.4.1 (Minimum principle for the laplacian operator)**.** *If $u \in \mathcal{C}(V)$ is superharmonic on F then*

$$\min_{x\in\delta(F)} \{u(x)\} \leq \min_{x\in F} \{u(x)\}$$

and the equality holds if and only if u coincides on $\bar{F}$ with a multiple of $\chi_{\bar{F}}$.

In Section 5.1 a generalized minimum principle is shown regarding the values on certain subsets of F of a superharmonic function on F with respect to their distance to the boundary $\delta(F)$.

Proposition 2.4.2 (Monotonicity property for the Schrödinger operator)**.** *Let $u \in \mathcal{C}(\bar{F})$. If $\mathcal{L}_q(u) \geq 0$ on F and $u \geq 0$ on $\delta(F)$, then either $u > 0$ on F or $u = 0$ on $\bar{F}$.*

2.5 Green and Poisson operators

Given a network $\Gamma = (V, c)$ and a proper connected subset $F \subset V$, let us consider boundary value problems on it. A *boundary value problem* on Γ, also written as *BVP*, is a problem that consists in finding all the functions $u \in \mathcal{C}(\bar{F})$, if there are any, such that they verify an implicit global condition on F and other different conditions on the boundary $\delta(F)$ or a part of it. These boundary conditions can be overdetermined. Along this thesis we are only interested in certain boundary value problems involving Schrödinger operators. Mainly, the boundary value problems considered in this work fit one of the following configurations:

$$\mathcal{L}_q(u) = f \ \text{ on } \ F \quad \text{and} \quad u = g \ \text{ on } \ \delta(F)$$

or

$$\mathcal{L}_q(u) = f \text{ on } F,\; u = g \text{ on } A_1 \subset \delta(F) \text{ and } \frac{\partial u}{\partial \mathsf{n}_F} = h \text{ on } A_2 \subset A_1 \subset \delta(F).$$

The first one is known as *Dirichlet boundary value problem* or *Dirichlet problem* for short. The second is an *overdetermined partial Dirichlet–Neumann boundary value problem.* Both names refer to the type of boundary conditions. The second type of problems is introduced here for the first time on finite networks and will be studied in Section 7.1.

In this section we focus in the first typology of boundary value problems. In fact, in an abuse of notation, if we cosider the non–proper subset $F = V$ then $\bar{F} = V$ and in this case the Dirichlet problem $\mathcal{L}_q(u) = f$ on F and $u = g$ on $\delta(F)$ is simply the extremal case

$$\mathcal{L}_q(u) = f \quad \text{on} \quad V,$$

which is known as *Poisson equation.*

Let us go back to the case when $F \subset V$ is a proper subset. Consider $f \in \mathcal{C}(F)$ and $g \in \mathcal{C}(\delta(F))$ two known functions. Let us consider the Dirichlet boundary value problem that consists in finding a function $u \in \mathcal{C}(\bar{F})$ such that

$$\mathcal{L}_q(u) = f \text{ on } F \quad \text{and} \quad u = g \text{ on } \delta(F). \tag{2.1}$$

Does u exist? Is it unique? The existence and uniqueness of solution for Problem (2.1) was proved in [18] by Bendito, Carmona and Encinas. Their *Dirichlet principle* tells us that for any data $f \in \mathcal{C}(F)$ and $g \in \mathcal{C}(\delta(F))$, Problem (2.1) has a unique solution, see [18, Proposition 3.3]. In the continuum, the unique solution of this kind of problems is characterized by the Green and the Poisson operators. In our setting, in order to characterize this unique solution, we need a discrete version of these two operators. So as to describe them, let us consider the following two Dirichlet boundary value problems

$$\mathcal{L}_q(u_f) = f \text{ on } F \quad \text{and} \quad u_f = 0 \text{ on } \delta(F) \tag{2.2}$$

and

$$\mathcal{L}_q(u_g) = 0 \text{ on } F \quad \text{and} \quad u_g = g \text{ on } \delta(F). \tag{2.3}$$

We call them the *Green problem* with data f on F and the *Poisson problem* with data g on F, respectively. Let $\mathcal{G}_q$ be the endomorphism of $\mathcal{C}(F)$ that assigns to each $f \in \mathcal{C}(F)$ the unique function $\mathcal{G}_q(f) \in \mathcal{C}(F)$ such that $\mathcal{L}_q\Big(\mathcal{G}_q(f)\Big) = f$ on F. We say that $\mathcal{G}_q$ is the *Green operator* of Γ on F.

On the other hand, let $\mathcal{P}_q : \mathcal{C}(\delta(F)) \longrightarrow \mathcal{C}(\bar{F})$ be the linear operator that assigns to each $g \in \mathcal{C}(\delta(F))$ the unique function $\mathcal{P}_q(g) \in \mathcal{C}(\bar{F})$ such that $\mathcal{L}_q\Big(\mathcal{P}_q(g)\Big) = 0$ on F and $\mathcal{P}_q(g) = g$ on $\delta(F)$. This operator is called the *Poisson operator* of Γ on F. Observe that $u_f = \mathcal{G}_q(f)$ and $u_g = \mathcal{P}_q(g)$ on $\bar{F}$. Thus, the Green operator $\mathcal{G}_q(f)$ on F is the unique solution of the Green problem (2.2) and the Poisson operator $\mathcal{P}_q(f)$ on F is the unique solution of the Poisson problem (2.3).

Corollary 2.5.1 ([21, Proposition 3.3])**.** *The Dirichlet boundary value problem* (2.1) *has a unique solution* $u \in \mathcal{C}(\bar{F})$ *for any data* $f \in \mathcal{C}(F)$ *and* $g \in \mathcal{C}(\delta(F))$ *and it is given by*

$$u = \mathcal{G}_q(f) + \mathcal{P}_q(g). \tag{2.4}$$

Next, we define the Green operator when the set we work on is the whole set of vertices V. This concept was also introduced in [18], as well as the following notations. First, we consider that $F = V$, denote $\mathcal{V} = \ker(\mathcal{L}_q)$ and define π as the *orthogonal projection* on $\mathcal{V}$. When $q \neq q_\omega$, then $\mathcal{V}$ is trivial and hence $\pi = 0$. Otherwise, when $q = q_\omega$, then $\mathcal{V}$ is the subspace generated by ω and therefore $\pi(f) = \langle f, \omega\rangle \omega$ is the orthogonal projection of $f \in \mathcal{C}(V)$ on $\mathcal{V}$. Recall that $\mathcal{L}_q$ is an isomorphism of $\mathcal{V}^\perp$, see [18]. Moreover, for each $f \in \mathcal{C}(V)$ there exists a function $u \in \mathcal{C}(V)$ such that $\mathcal{L}_q(u) = f - \pi(f)$ on V and then $u + \mathcal{V}$ is the set of all the functions $v \in \mathcal{C}(V)$ such that $\mathcal{L}_q(v) = f - \pi(f)$ on V.

We call *orthogonal Green operator with data* f *on* V, and denote it with the same notation $\mathcal{G}_q$, the unique endomorphism of $\mathcal{C}(V)$ that assigns to each $f \in \mathcal{C}(V)$ the unique function $u \in \mathcal{C}(V)$ such that

$$\mathcal{L}_q(u) = f - \pi(f) \quad \text{on } V \tag{2.5}$$

with $a\langle u, \omega\rangle = 0$, where $a = 0$ if $q \neq q_\omega$ and $a = 1$ if $q = q_\omega$, see [18, Definition 5.6]. In the sequel, when using a Green operator, it will be clearly indicated whether it is the Green operator on a proper subset $F \subset V$ or the orthogonal Green operator on V.

We abuse the notation and consider now a connected vertex subset $F \subseteq V$. Notice that if $F = V$, then $\bar{F} = V$ also. With this notation, the *Green kernel on* F is the function $G_q : \bar{F} \times F \longrightarrow \mathbb{R}$ given by $G_q(x, y) = \mathcal{G}_q(\varepsilon_y)(x)$ for all $x \in \bar{F}$ and $y \in F$. If $F = V$, it is the *orthogonal Green kernel on* V. We denote by $\mathsf{G_q} \in \mathcal{M}_{|\bar{F}|\times|F|}(\mathbb{R})$ the *Green matrix* given by the entries of the

Green kernel G_q on $\bar{F} \times F$. Analogously, if $F = V$ then $G_q \in \mathcal{M}_{|V|\times|V|}(\mathbb{R})$ is the *orthogonal Green matrix*.

Using these notations, the following results hold. We first fix our attention on the Green and Poisson operators on $F \subset V$ and afterwards we study the orthogonal Green operator on V.

Lemma 2.5.2 ([18, Definition 5.2]). *If $F \subset V$ is a proper subset, the Green operator $\mathcal{G}_q$ on F is the inverse operator in $\mathcal{C}(F)$ of the Schrödinger operator $\mathcal{L}_q$ on F.*

Proof. Recall that from Proposition 2.3.4, $\mathcal{L}_q$ is positive–definite on any proper connected subset $F \subset V$. Since $\mathcal{L}_q\left(\mathcal{G}_q(u)\right) = u$ on F for all $u \in \mathcal{C}(F)$ by definition, then the inverse of the Schrödinger operator $\mathcal{L}_q$ in the corresponding space of functions is the Green operator $\mathcal{G}_q \in \mathcal{C}(F)$. □

Proposition 2.5.3 ([18, Propositions 5.1 and 5.3]). *The Green kernel on $F \subset V$ satisfies the following properties: $G_q(x,y) > 0$ for all $x, y \in F$, $G_q(x,y) = G_q(y,x)$ for all $x, y \in F$ and*

$$\mathcal{G}_q(f)(x) = \int_F G_q(x,y) f(y)\, dy$$

for any $x \in \bar{F}$.

Now we bring the attention back to the orthogonal Green operator on V. If $q \neq q_\omega$, then it is straighforward that $\mathcal{L}_q$ is invertible in $\mathcal{C}(V)$. However, when $q = q_\omega$, then the Schrödinger operator $\mathcal{L}_{q_\omega}$ is not invertible but we can consider the orthogonal Green operator as its Moore–Penrose inverse in $\mathcal{C}(V)$. Recall that a Moore–Penrose inverse $\mathsf{M}^\dagger \in \mathcal{M}_{t\times s}(\mathbb{K})$ is the pseudoinverse of a matrix $\mathsf{M} \in \mathcal{M}_{s\times t}(\mathbb{K})$ when it is either non–squared or singular. It is unique and satisfies the following properties [15]:

$$\mathsf{M} \cdot \mathsf{M}^\dagger \cdot \mathsf{M} = \mathsf{M}, \qquad \mathsf{M}^\dagger \cdot \mathsf{M} \cdot \mathsf{M}^\dagger = \mathsf{M}^\dagger,$$

$$(\mathsf{M} \cdot \mathsf{M}^\dagger)^* = \mathsf{M} \cdot \mathsf{M}^\dagger, \qquad (\mathsf{M}^\dagger \cdot \mathsf{M})^* = \mathsf{M}^\dagger \cdot \mathsf{M}.$$

Proposition 2.5.4 ([18, Proposition 5.8]). *If $q \neq q_\omega$, then the orthogonal Green operator satisfies that $\mathcal{L}_q(\mathcal{G}_q(u)) = u$ in V for all $u \in \mathcal{C}(V)$ and hence $\mathcal{G}_q$ is the inverse operator of $\mathcal{L}_q$ in $\mathcal{C}(V)$. On the other hand, if $q = q_\omega$ then $\mathcal{L}_{q_\omega}\left(\mathcal{G}_{q_\omega}(u)\right) = u - \langle u, \omega\rangle \omega$ on V and $\langle \mathcal{G}_{q_\omega}(u), \omega\rangle = 0$ for all $u \in \mathcal{C}(V)$. Moreover, G_q is a symmetric function on $V \times V$ in any case.*

Let us define the value $\lambda^\dagger = \lambda^{-1}$ when $\lambda > 0$ and $\lambda^\dagger = 0$ when $\lambda = 0$ for the following result, which is a direct consequence of Proposition 2.3.4 and the above theoretical background.

Corollary 2.5.5 ([18, Proposition 5.7]). *The lowest eigenvalue of the orthogonal Green operator $\mathcal{G}_q$ on V is $\lambda^\dagger$ and its associated eigenfunction is the weight ω. Hence, $\mathcal{G}_q(\omega) = \lambda^\dagger\omega$ on V. Moreover, $\mathcal{G}_q$ is self–adjoint, that is, $\langle g, \mathcal{G}_q(f)\rangle = \langle f, \mathcal{G}_q(g)\rangle$ for all $f, g \in \mathcal{C}(V)$.*

Finally, we fix our attention on the Poisson operator $\mathcal{P}_q$ on F and relate it to the Green operator $\mathcal{G}_q$ on F just as it was shown in [18]. The *Poisson kernel* on F is the function $P_q : \bar{F} \times \delta(F) \longrightarrow \mathbb{R}$ given by $P_q(x, y) = \mathcal{P}_q(\varepsilon_y)(x)$ for all $x \in \bar{F}$ and $y \in \delta(F)$. We denote by $\mathsf{P_q} \in \mathcal{M}_{|\bar{F}|\times|\delta(F)|}(\mathbb{R})$ the matrix given by the entries of the Poisson kernel P_q on $\bar{F} \times \delta(F)$.

Proposition 2.5.6 ([18, Propositions 5.1 and 5.3]). *The Poisson kernel satisfies the following properties: $P_q(x, y) > 0$ for any $x \in F$ and $y \in \delta(F)$, $\dfrac{\partial P_q^y}{\partial \mathsf{n}_F}(x) = \dfrac{\partial P_q^x}{\partial \mathsf{n}_F}(y)$ for all $x, y \in \delta(F)$ and*

$$\mathcal{P}_q(g)(x) = \int_{\delta(F)} P_q(x, y)g(y)\, dy$$

for any $x \in \bar{F}$.

Proposition 2.5.7 ([18, Proposition 5.5]). *For any $x \in \bar{F}$ and any $y \in \delta(F)$,*

$$P_q(x, y) = \varepsilon_y(x) - \frac{\partial G_q^x}{\partial \mathsf{n}_F}(y).$$

Proposition 2.5.8 ([18, Proposition 5.3]). *The Green and the Poisson operators of Γ on F are formally self–adjoint, that is, $\langle f, \mathcal{G}_q(g)\rangle_F = \langle \mathcal{G}_q(f), g\rangle_F$ for all $f, g \in \mathcal{C}(F)$ and $\langle f, \mathcal{P}_q(g)\rangle_{\delta(F)} = \langle \mathcal{P}_q(f), g\rangle_{\delta(F)}$ for all $f, g \in \mathcal{C}(\delta(F))$.*

2.6 Effective resistances and Kirchhoff index

In this section we set the concept of the generalized effective resistance between two vertices, which measure how difficult for an electrical current is to get from one vertex to the other one, as well as the generalized Kirchhoff index of a network, which describes its rigidity.

The generalized Kirchhoff indices and effective resistances of a network need the values of the orthogonal Green kernel on V to be known. We work with the whole set of vertices $F = V$ of the network, which in particular is a set with no boundary ($\bar{F} = V$ in this case), for the parameters we work with do not need to distinguish among interior and boundary vertices.

The *Kirchhoff index* was introduced in chemistry as a better alternative to other parameters used for the discrimination among different molecules with similar shapes and structures, see [48]. It is also known as the *total resistance* of the network, as it is a global parameter. Chemically, the Kirchhoff index of a molecular graph is the sum of the squared atomic displacements from their equilibrium positions produced by molecular vibrations [43]. Small values of the Kirchhoff index indicate that the atoms are very rigid in the molecule. Mathematically, the Kirchhoff index of a network measures its structural rigidity. This parameter has started an important and fruitful path that has carried the computation of the Kirchhoff indices in symmetrical networks such as distance–regular graphs, circulant graphs, lineal chains and some fullerenes: see for instance [14, 22, 26, 57] and the references therein. In [23, 24] a generalization of the Kirchhoff index with respect to a weight on the network and a real non–negative value was introduced. For example, this generalization is essential to obtain the expression of the classical Kirchhoff index of a composite network in terms of Kirchhoff indices of the factors, as it will be seen further in this work. Hence, we work with this generalization instead of the classical version.

The generalized Kirchhoff index needs the definition of generalized effective resistances between pairs of vertices of the network. The classical *effective resistance* between two adjacent vertices is the inverse of the conductance of the edge joining them. In general, the effective resistance between a pair of vertices –not necessarily adjacent– is a parameter that expresses how difficult for an electrical current is to go from one of these two vertices to the other one. Coppersmith et al. [35] and Ponzio [52] found the expression of the solution of certain boundary value problems in terms of the effective resistances of the network in the decade of 1990. In [23, 24] the classical effective resistance was generalized with respect to a weight on the network and a real non–negative value. We work with this generalization, too.

Let $\Gamma = (V, c)$ be a connected network. Remember that we assume the potential $q \in \mathcal{C}(V)$ to be given by $q = q_\omega + \lambda$ on V, where $\omega \in \Omega(V)$ and $\lambda \geq 0$ is a real value. Given two vertices $x, y \in V$, we define the functional

$\mathfrak{J}_{x,y} \colon \mathcal{C}(V) \longrightarrow \mathbb{R}$ as

$$\mathfrak{J}_{x,y}(u) = 2\left[\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}\right] - \langle \mathcal{L}_q(u), u\rangle,$$

for all $u \in \mathcal{C}(V)$. The *generalized effective resistance* between x and y with respect to ω and λ is the value

$$R_{\lambda,\omega}(x,y) = \max_{u\in\mathcal{C}(V)}\{\mathfrak{J}_{x,y}(u)\}.$$

In addition, we can consider the functional $\mathfrak{J}_x \colon \mathcal{C}(V) \longrightarrow \mathbb{R}$ given by

$$\mathfrak{J}_x(u) = 2\left[\frac{u(x)}{\omega(x)} - \langle u, \omega\rangle\right] - \langle \mathcal{L}_q(u), u\rangle$$

for all $u \in \mathcal{C}(V)$ and define the *generalized total resistance* at $x \in V$ with respect to ω and λ as

$$r_{\lambda,\omega}(x) = \max_{u\in\mathcal{C}(V)}\{\mathfrak{J}_x(u)\}.$$

Finally, the *generalized Kirchhoff index* of Γ with respect to ω and λ is the value

$$\mathsf{k}(\lambda,\omega) = \frac{1}{2}\sum_{x,y\in V} R_{\lambda,\omega}(x,y)\omega^2(x)\,\omega^2(y).$$

For all these new presented parameters, we omit the subscript λ when it is 0 and leave out the subfix symbol ω when it is constant. Hence, k is the normalized classical Kirchhoff index.

The role of the orthogonal Green operator on V is the key to evaluate the generalized effective resistances and Kirchhoff indices of the network. The following formulae express these different parameters in terms of orthogonal Green functions on V, see [23] for the proofs.

Proposition 2.6.1 ([23, Proposition 4.3]). *For any $x, y \in V$, it is satisfied that*

$$r_{\lambda,\omega}(x) = \frac{G_q(x,x)}{\omega^2(x)} - \lambda^\dagger \quad \textit{and} \quad R_{\lambda,\omega}(x,y) = \frac{G_q(x,x)}{\omega^2(x)} + \frac{G_q(y,y)}{\omega^2(y)} - \frac{2G_q(x,y)}{\omega(x)\omega(y)},$$

where G_q stands for the orthogonal Green kernel on V. In consequence,

$$\mathsf{k}(\lambda,\omega) = \sum_{x\in V} r_{\lambda,\omega}(x)\omega^2(x) = \sum_{x\in V} G_q(x,x) - \lambda^\dagger.$$

2.7 Circular planar networks

We present here a family of networks that plays an important role in this thesis. This family is termed circular planar networks and has been extensively used in the literature, see [25, 36, 37, 38].

A *planar network* is a network that can be drawn in the plane in such a way that the edges do not cross each other. Let $\Gamma = (V, c)$ be a connected planar network with $F \subset V$ a fixed proper connected subset such that $\bar{F} = V$ for the sake of simplicity. Γ is a *circular planar network* if it can be embedded in a closed disc D of the plane, with the vertices of F laying in $\overset{\circ}{D}$ and the boundary vertices laying on the circumference ∂D. The edges must be in D, also, with no crossings between edges. The circumference ∂D is called the *boundary circle*. If Γ is a circular planar network, the vertices in $\delta(F)$ are labelled in the clockwise circular order given by the boundary circle ∂D, see Figure 2.2. An *arc of the boundary circle* $\widehat{xy}$ is the portion of the boundary

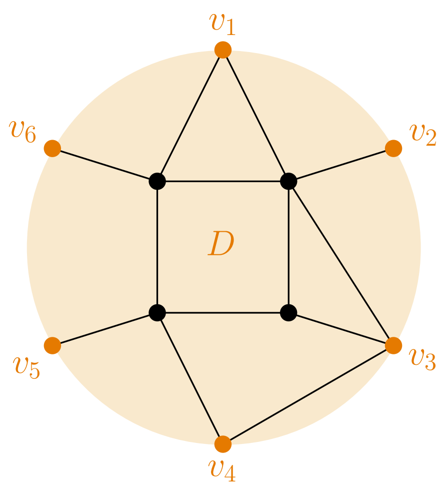


Figure 2.2 An example of circular planar network.

circle from $x \in \delta(F)$ to $y \in \delta(F)$ in the clockwise order. Two arcs are disjoint if and only if they contain no common vertices.

Curtis and Morrow defined in [37] the following concepts. Let Γ be a circular planar network and let $v_1, \ldots, v_m$ be a sequence of different vertices of $\delta(F)$. We say that the vertices $v_1, \ldots, v_m$ are in *circular order* if $\widehat{v_1 v_m}$ is an arc of the boundary circle, $v_2, \ldots, v_{m-1}$ are in the arc $\widehat{v_1 v_m}$ and $v_1 < v_2 < \ldots < v_m$

in terms of the circular order introduced by the boundary circle ∂D. Let now $P = \{p_1, \dots, p_k\}$ and $Q = \{q_1, \dots, q_k\}$ be two sequences of vertices of $\delta(F)$ in circular order. We say that $(P; Q)$ is a *circular pair* if the sequence $\{p_1, \dots, p_k, q_k, \dots, q_1\}$ is in circular order. If $\mathsf{A} \in \mathcal{M}_{|\bar{F}|\times|\bar{F}|}(\mathbb{R})$ is a matrix and $(P; Q)$ is a circular pair of vertices of $\delta(F)$, then $\mathsf{A}(\mathsf{P}; \mathsf{Q})$ is called a *circular minor of A*.

Suppose that $P = (p_1, \dots, p_k)$ and $Q = (q_1, \dots, q_k)$ are two sequences of vertices of $\delta(F)$. We say that P and Q are *k–connected through* Γ if there exist a permutation τ of the set of indices $\{1, \dots, k\}$ and k disjoint paths $\alpha_1, \dots, \alpha_k$ such that α_i connects p_i with $q_{\tau(i)}$ and goes through no other boundary vertices, for each $i = 1, \dots, k$. A *critical circular planar network* Γ is a circular planar network such that the removal of any edge breaks a connection through Γ between pairs of boundary vertices.

3
Effective resistances and Kirchhoff indices on composite networks

All allong this chapter we assume the conductivity functions of the networks to be known. A *composite network* is any network that can be expressed as a result of operations involving other networks. We want to obtain functional, resistive and morphological data of some composite networks in terms of the networks forming them. Specifically, the aim is to obtain the orthogonal Green function on V, the generalized Kirchhoff indices and the generalized effective resistances of certain composite networks in terms of their partitions. The reason is that large networks are difficult to work with because of their size. The smaller the size of a network, the easier to work with it, and hence we try to break the networks into smaller parts that may allow us to solve other problems on them that require not such an effort.

This problem is classified as a direct boundary value problem, since the conductances of the networks are always known. In general, a *direct boundary value problem* is a problem where we obtain functional, structural or resistive information of a network taking advantage of its conductivity data, among other known information, through boundary value problems.

The definitions and results given in Section 2.6 are the tools to give an expression of the generalized Kirchhoff indices and effective resistances for the generalized cluster and corona networks, which are introduced in the following sections.

The results detailed in this chapter have been published in [7, 8] and they have also been presented to a congress, see [4].

3.1 Cluster networks

In this section we follow the same techniques used in [20] for the recovery of the generalized Kirchhoff indices and effective resistances of joint networks, which is a family of composite networks. Let us consider two graphs Γ_0 and Γ_1 with vertex sets V_0 and V_1, respectively. We arbitrarily select a vertex $x \in V_1$ and call it the *distinguished vertex* of Γ_1. The *cluster graph* $\Gamma = \Gamma_0\{\Gamma_1\}$ consists in taking $m = |V_0|$ copies of the graph Γ_1 and identifying each vertex of Γ_0 with the distinguished vertex x of a different copy of Γ_1, as shown in Figure 3.1. The edges are maintained as in the original graphs. For this composite graph the expression of the classical Kirchhoff index was obtained in [59] using a very different approach.

Figure 3.1 Example of a cluster graph.

We consider the generalization of the cluster graph to the case of $m+1$ different networks. Let $\Gamma_0 = (V_0, c_0)$ be a connected network with vertex set given by $V_0 = \{x_1, \dots, x_m\}$ and let $\Gamma_i = (V_i, c_i)$, with $i = 1, \dots, m$, be m different connected networks. We call them *satellite networks* and Γ_0 is called the *basis network*. We select a vertex on each satellite network Γ_i and call it the *distinguished vertex* $x_i \in V_i$, for the sake of simplicity. Therefore, using this notation, the m distinguished vertices are already identified with a vertex of Γ_0. Let $V = \coprod_{i=1}^{m} V_i$ be the disjoint union of all the vertex sets.

We call *cluster network with basis* Γ_0 *and satellites* $\{\Gamma_i\}_{i=1}^m$ the network $\Gamma = (V, c)$ obtained by identifying each vertex x_i of Γ_0 with the distinguished vertex x_i of Γ_i. The edges are maintained as in the original networks and hence the conductances are given by $c(x,y) = c_i(x,y)$ for any $x, y \in V_i$, where $i = 0, \dots, m$, and $c(x,y) = 0$ otherwise. This network is denoted by $\Gamma = \Gamma_0\{\Gamma_1, \dots, \Gamma_m\}$ and is also called *generalized cluster network*, see Figure

3.2. A satellite network Γ_i is called *trivial* if $V_i = \{x_i\}$. Notice that the identification of a trivial satellite network Γ_i with $x_i \in V_0$ does not modify the structure nor the behaviour of the basis network Γ_0 at x_i. Taking this

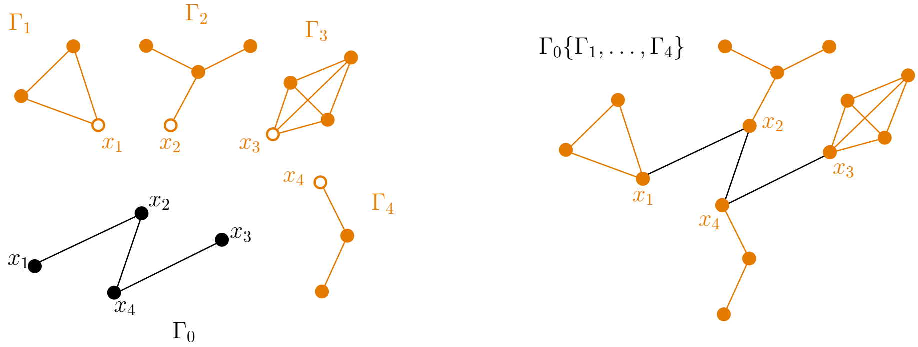


Figure 3.2 Example of a generalized cluster network.

into account, generalized cluster networks are highly relevant in chemistry applications since all composite molecules consisting of some amalgamations on a central submolecule can be understood as a cluster.

Moreover, we can say that any arbitrary connected network $\Gamma = (V, c)$ with at least one k–separating vertex for any k can be understood as a generalized cluster network. To see an easy example of this, we consider a network Γ with at least one 2–separating vertex and where there exist no k–separating vertices for $k \geq 3$. We choose a 2–separating vertex $s_1 \in V$ and take it away without removing its incident edges, obtaining exactly two connected components. Let us add a new vertex x_1 to each one of the connected components in the place where s_1 was. Name $\Gamma_0 = (V_0, c_0)$ and $\Gamma_1 = (V_1, c_1)$ these two new connected networks. Then, $x_1 \in V_0$ on Γ_0 and $x_1 \in V_1$ on Γ_1. If Γ_0 has no 2–separating vertices, then $\Gamma = \Gamma_0\{\Gamma_1, \ldots, \Gamma_m\}$, where $V_0 = \{x_1, \ldots, x_m\}$ and $\Gamma_2, \ldots, \Gamma_m$ are trivial satellite networks. Otherwise, let s_2 be a 2–separating vertex of Γ_0 and let us proceed as before. We obtain two new connected networks from the former Γ_0 and call them the new $\Gamma_0 = (V_0, c_0)$ –for the sake of simplicity– and $\Gamma_2 = (V_2, c_2)$, with $x_2 \in V_0$ on Γ_0 and $x_2 \in V_2$ on Γ_2. We can continue this process until the newest version of Γ_0 has no more 2–separating vertices. Then, $\Gamma = \Gamma_0\{\Gamma_1, \ldots, \; Gamma_m\}$ with $V_0 = \{x_1, \ldots, x_m\}$ and $\Gamma_{r+1}, \ldots, \Gamma_m$ are trivial satellite networks, where r is the number of repetitions of this process.

It is important to remark that this process does not produce a unique generalized cluster configuration for a given network and that there exist other generalized cluster configurations for a network not feasible with this algorithm. However, the point of this process is to show that a huge amount of networks are generalized cluster networks and therefore they can be broken into smaller parts, which in most of the situations are easier to work with. In Figure 3.3 we can see some examples of the reinterpretation of a network as a generalized cluster, where the shadowed areas are the satellite networks.

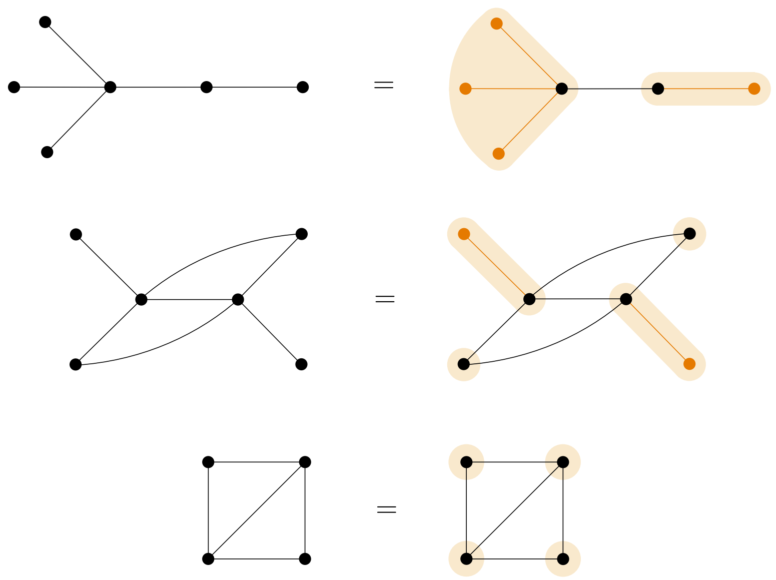


Figure 3.3 Examples of the reinterpretation of a network as a generalized cluster.

Let $\Gamma = (V, c)$ be a generalized cluster network and let $\omega \in \Omega(V)$ be a weight on Γ. Consider for any $i = 0, \dots, m$ the value $\sigma_i = \left(\int_{V_i} \omega^2\right)^{\frac{1}{2}}$ and the function $\omega_i(x) = \sigma_i^{-1}\omega(x)$ for every $x \in V_i$. It is clear that $\omega_i \in \Omega(V_i)$. If $u \in \mathcal{C}(V)$, the restriction of u to V_i is also denoted by u. Observe that if $u \in \mathcal{C}(V_i)$ and $v \in \mathcal{C}(V)$, then $\langle u, v\rangle = \langle u, v\rangle_{V_i}$. In particular, $\langle u, v\rangle = 0$ when $v \in \mathcal{C}(V_j)$ with $j \neq i$ and $\langle u, v\rangle = u(x_i)v(x_i)$ when $v \in \mathcal{C}(V_0)$ with $i \neq 0$. Notice also that $\sum_{j=1}^{m} \sigma_j^2 = 1$.

From now on, when dealing with composite networks, the superscript i on a parameter or operator stands for the one of the network Γ_i, where $i =$

$0,\dots,m$. Using this notation, we get the following result.

Proposition 3.1.1. *For any function $u \in \mathcal{C}(V)$ on the generalized cluster network, it is satisfied that*

$$\mathcal{L}_{q_\omega}(u)(x) = \mathcal{L}^i_{q_{\omega_i}}(u)(x) + \mathcal{L}^0_{q_{\omega_0}}(u)(x_i)\varepsilon_{x_i}(x)$$

for all $x \in V_i$, where $i = 1,\dots,m$.

Proof. It suffices to observe that if $x \in V_i$, then

$$\mathcal{L}(u)(x) = \mathcal{L}^i(u)(x) + \mathcal{L}^0(u)(x_i)\varepsilon_{x_i}(x) \quad \text{and} \quad q_\omega(x) = q_{\omega_i}(x) + q_{\omega_0}(x_i)\varepsilon_{x_i}(x)$$

for each $i = 1,\dots,m$. □

Our following objective is to obtain the orthogonal Green operator on V with respect to the potential q_ω of the generalized cluster network in terms of the orthogonal Green operators on V_i of the satellites. Observe that we are considering only the $\lambda = 0$ case. As a by–product, we obtain the generalized effective resistances and Kirchhoff index with respect to the weight ω. First, let us consider for any $f \in \mathcal{C}(V)$ the function $g_f \in \mathcal{C}(V_0)$ defined as

$$g_f(x_j) = \frac{\sigma_j \langle f, \omega_j\rangle}{\omega(x_j)} = \frac{\langle f, \omega_j\rangle}{\omega_j(x_j)}$$

for any $j = 1,\dots,m$. This definition is motivated by the following result.

Lemma 3.1.2. *Let $f \in \mathcal{C}(V)$ be a function on the generalized cluster network such that $\langle \omega, f\rangle = 0$. Let $u \in \mathcal{C}(V)$ be a solution of the Poisson equation $\mathcal{L}_{q_\omega}(u) = f$ on V. Then, $\mathcal{L}^0_{q_{\omega_0}}(u) = g_f$ on V_0.*

Proof. Using Proposition 3.1.1, we know that u is a solution of the equation $\mathcal{L}_{q_\omega}(u) = f$ on V if and only if u satisfies $\mathcal{L}^j_{q_{\omega_j}}(u) = f - \mathcal{L}^0_{q_{\omega_0}}(u)(x_j)\varepsilon_{x_j}$ on V_j for every $j = 1,\dots,m$. Remember that $\mathcal{L}^j_{q_{\omega_j}}(\omega_j) = 0$. As $\mathcal{L}^j_{q_{\omega_j}}$ is self–adjoint on $\mathcal{C}(V_j)$, we get that

$$0 = \langle \mathcal{L}^j_{q_{\omega_j}}(\omega_j), u\rangle = \langle \mathcal{L}^j_{q_{\omega_j}}(u), \omega_j\rangle = \langle f, \omega_j\rangle - \mathcal{L}^0_{q_{\omega_0}}(u)(x_j)\omega_j(x_j)$$

and hence the result follows. □

Proposition 3.1.3. *Let $f \in \mathcal{C}(V)$ be a function on the generalized cluster network such that $\langle \omega, f\rangle = 0$. Consider the Poisson equation $\mathcal{L}_{q_\omega}(u) = f$ on*

V. Then, the function

$$\begin{aligned} u &= \sum_{i=1}^{m} \mathcal{G}^i_{q_{\omega_i}}(f - g_f\varepsilon_{x_i}) \\ &+ \sum_{i=1}^{m} \frac{1}{\omega_i(x_i)} \Big(\mathcal{G}^i_{q_{\omega_i}}(f - g_f\varepsilon_{x_i})(x_i) - \mathcal{G}^0_{q_{\omega_0}}(g_f)(x_i) \Big) \big[\sigma_i \omega - \omega_i \big] \end{aligned}$$

is the unique solution of the Poisson equation such that $\langle \omega, u \rangle = 0$.

Proof. From Proposition 3.1.1 and Lemma 3.1.2 we know that u is a solution of the equation $\mathcal{L}_{q_\omega}(u) = f$ on V if and only if it is satisfied that $\mathcal{L}^j_{q_{\omega_j}}(u) = f - g_f\varepsilon_{x_j}$ on V_j for all $j = 1, \dots, m$. As $\lambda = 0$, then there exist $m + 1$ real values $\beta_0, \beta_1, \dots, \beta_m \in \mathbb{R}$ such that

$$u = \mathcal{G}^0_{q_{\omega_0}}(g_f) + \beta_0 \omega_0 \quad \text{on} \quad V_0$$

and

$$u = \mathcal{G}^j_{q_{\omega_j}}\big(f - g_f\varepsilon_{x_j}\big) + \beta_j \omega_j \quad \text{on} \quad V_j$$

for every $j = 1, \dots, m$. Consequently, if we define the functions $u_j = \mathcal{G}^j_{q_{\omega_j}}\big(f - g_f\varepsilon_{x_j}\big) + \beta_j \omega_j \in \mathcal{C}(V_j)$ then $u = \sum_{j=1}^m u_j$. Moreover, it satisfies that

$$\langle u, \omega \rangle = \sum_{j=1}^{m} \langle \omega, u_j \rangle = \sum_{j=1}^{m} \sigma_j \langle \omega_j, u_j \rangle = \sum_{j=1}^{m} \sigma_j \beta_j.$$

Therefore, $\langle u, \omega \rangle = 0$ if and only if $\sum_{j=1}^m \beta_j \sigma_j = 0$. On the other hand, keeping in mind that $x_j \in V_j \cap V_0$ and that $\omega_j(x_j) = \sigma_j^{-1} \sigma_0 \omega_0(x_j)$ we get that

$$\beta_j = \frac{\mathcal{G}^0_{q_{\omega_0}}(g_f)(x_j) - \mathcal{G}^j_{q_{\omega_j}}(f - g_f\varepsilon_{x_j})(x_j)}{\omega_j(x_j)} + \beta_0 \frac{\sigma_j}{\sigma_0}, \quad j = 1, \dots, m.$$

Therefore,

$$\beta_0 = \sigma_0 \sum_{j=1}^{m} \frac{\sigma_j}{\omega_j(x_j)} \Big(\mathcal{G}^j_{q_{\omega_j}}(f - g_f\varepsilon_{x_j})(x_j) - \mathcal{G}^0_{q_{\omega_0}}(g_f)(x_j) \Big),$$

and the result follows. □

Theorem 3.1.4. *The orthogonal Green function of the generalized cluster network on $V_i \times V_j$ is given by*

$$G_{q_\omega} = G^j_{q_{\omega_j}} + \frac{1}{\omega_i(x_i)}\left[\Big(\sigma_j\sigma_i G^i_{q_{\omega_i}}(\cdot,x_i) - G^j_{q_{\omega_j}}(\cdot,x_j)\Big)\otimes\omega_j\right] + \frac{1}{\omega_j(x_j)}\left[\omega_i\otimes\Big(\sigma_j\sigma_i G^j_{q_{\omega_j}}(\cdot,x_j) - G^i_{q_{\omega_i}}(\cdot,x_i)\Big)\right] + g_{ij}\,\omega_i\otimes\omega_j,$$

where

$$\begin{aligned} g_{ij} &= \frac{\sigma_i\sigma_j}{2\sigma_0^2}\left[\sum_{k=1}^m \sigma_k^2\Big(R^0_{\omega_0}(x_k,x_j)+R^0_{\omega_0}(x_k,x_i)\Big)\right] \\ &- \frac{\sigma_i\sigma_j}{2\sigma_0^2}\left[\sum_{k=1}^m\sum_{\ell=1}^m \sigma_k^2\sigma_\ell^2 R^0_{\omega_0}(x_k,x_\ell) - R^0_{\omega_0}(x_i,x_j)\right] \\ &+ \sigma_i\sigma_j\left[\sum_{k=1}^m \sigma_k^2 r^k_{\omega_k}(x_k) - r^i_{\omega_i}(x_i) - r^j_{\omega_j}(x_j)\right] + r^i_{\omega_i}(x_j). \end{aligned}$$

Proof. For any $y \in V$, the function $u = \mathcal{G}_{q_\omega}(\varepsilon_y)$ is the unique solution of the equation $\mathcal{L}_{q_\omega}(u) = \varepsilon_y - \omega(y)\omega$ on V such that $\langle u,\omega\rangle = 0$, see Equation (2.5). Then, the explicit expression of G_{q_ω} is easily deduced if we reduce the problem to some cases. Let $f = \varepsilon_y - \omega(y)\omega \in \mathcal{C}(V)$. As $y \in V_j$ for an index $j \in \{1,\dots,m\}$, then $f = \varepsilon_y - \sigma_j\omega_j(y)\omega$ and therefore

$$\begin{aligned} \langle f,\omega_j\rangle &= \omega_j(y)(1-\sigma_j^2) \\ \mathcal{G}^0_{q_{\omega_0}}(h)(x_j) &= \frac{\omega_j(y)}{\omega_j(x_j)}G^0_{q_{\omega 0}}(x_j,x_j) - \sigma_j\omega_j(y)\sum_{l=1}^m \frac{\sigma_l G^0_{q_{\omega_0}}(x_j,x_l)}{\omega_l(x_l)} \\ \mathcal{G}^j_{q_{\omega_j}}(f)(x_j) &= G^j_{q_{\omega_j}}(x_j,y) \\ \mathcal{G}^j_{q_{\omega_j}}(f)(x) &= G^j_{q_{\omega_j}}(x,y), \qquad x\in V_j \\ \langle f,\omega_k\rangle &= -\sigma_j\sigma_k\omega_j(y), \qquad k\neq j \\ \mathcal{G}^0_{q_{\omega_0}}(h)(x_k) &= \frac{\omega_j(y)}{\omega_j(x_j)}G^0_{q_{\omega 0}}(x_k,x_j) - \sigma_j\omega_j(y)\sum_{l=1}^m \frac{\sigma_l G^0_{q_{\omega_0}}(x_k,x_l)}{\omega_l(x_l)}, \quad k\neq j \\ \mathcal{G}^k_{q_{\omega_k}}(f)(x_k) &= 0, \qquad k\neq j \\ \mathcal{G}^k_{q_{\omega_k}}(f)(x) &= 0, \qquad x\in V_k,\quad k\neq j, \end{aligned}$$

where $h(x_j) = \dfrac{\langle f, \omega_j \rangle}{\omega_j(x_j)}$ for all $j = 1, \dots, m$. Notice that $G_{q_\omega}(x,y) = u(x)$ and that x can be either in V_j or in V_k for $k \neq j$. Applying Proposition 3.1.3, we get the result using the formulae corresponding to each case. □

The following result gives the expression of the generalized Kirchhoff index $\mathsf{k}(\omega)$ and the generalized effective and total resistances R_ω, r_ω in terms of the same parameters of the satellites. These expressions follow directly from the formulae in Proposition 2.6.1.

Corollary 3.1.5. *The generalized Kirchhoff index* $\mathsf{k}(\omega)$ *of the generalized cluster network is given by*

$$\mathsf{k}(\omega) = \sum_{i=1}^{m} \mathsf{k}^i(\omega_i) + \frac{1}{2\sigma_0^2} \sum_{i=1}^{m} \sum_{j=1}^{m} \sigma_i^2 \sigma_j^2 R^0_{\omega_0}(x_i, x_j) + \sum_{i=1}^{m} (1 - \sigma_i^2) r^i_{\omega_i}(x_i).$$

Moreover, for all $x \in V_j$*, the generalized total resistance at* x *is the value*

$$\begin{aligned} r_\omega(x) = {} & r^j_{\omega_j}(x) - r^j_{\omega_j}(x_j) - \frac{(\sigma_j^2 - 1)}{\sigma_j^2} R^j_{\omega_j}(x, x_j) + \sum_{k=1}^{n} \sigma_k^2 r^k_{\omega_k}(x_k) \\ & + \frac{1}{\sigma_0^2} \sum_{k,\ell=1}^{m} \sigma_k^2 \sigma_\ell^2 \left[R^0_{\omega_0}(x_k, x_j) - R^0_{\omega_0}(x_k, x_\ell) \right]. \end{aligned}$$

The generalized effective resistances are given by

$$R_\omega(x,y) = \frac{R^j_{\omega_j}(x,y)}{\sigma_j^2}$$

if $x, y \in V_j$*, whereas if* $x \in V_i$ *and* $y \in V_j$ *with* $i \neq j$ *then*

$$R_\omega(x,y) = \frac{R^i_{\omega_i}(x, x_i)}{\sigma_i^2} + \frac{R^0_{\omega_0}(x_i, x_j)}{\sigma_0^2} + \frac{R^j_{\omega_j}(x_j, y)}{\sigma_j^2}.$$

This result was obtained in [59] for the classical case, that is, for the cluster graph with not normalized constant weight. Corollary 3.1.5 clearly shows the behaviour of the effective resistances in a cluster network. First, we need to notice that all the vertices of the basis network Γ_0 are separating vertices of the generalized cluster network. In consequence, the generalized effective resistance between two vertices belonging to two different satellite networks is the weighted sum of following quantities: the resistances between each one of the vertices and the corresponding distinguished vertex of its satellite network, and the resistance between these two distinguished vertices on the basis network. In contraposition, if two vertices belong to the same satellite network, then the generalized effective resistance between them remains the same as in the original network with a weight–adaption multiplying parameter. Also, we can see that any generalized cluster network loses rigidity with respect to its parts, since its Kirchhoff index is bigger that the sum of the Kirchhoff indices of its factors.

In order to show an example, let us consider the *generalized star* $\Gamma = (V, c)$ given in Figure 3.4 with weight $\omega = 1/9$ on the central vertex, which we name $x_6 \in V$, and $\omega = 2/9$ on $V \setminus \{x_6\}$. The conductances are given by $c(x_6, z) = 2$ if z is a neighbour of the central vertex and $c(y, z) = 1$ for $y \sim z$ with $y, z \neq x_6$. Notice that Γ is a generalized cluster network with

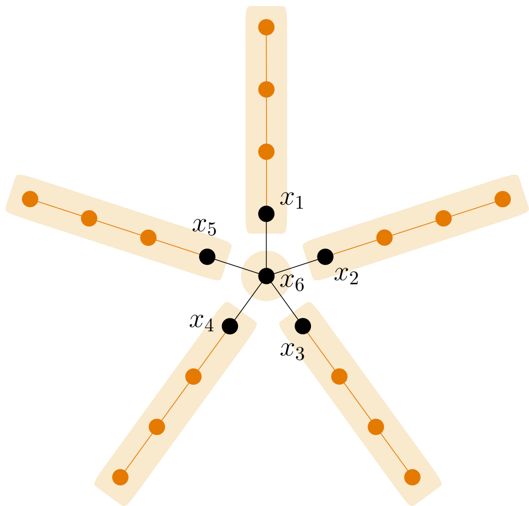


Figure 3.4 A generalized star with 5 radius of length 4.

basis network Γ_0 the 5–star with constant conductances equal to 2. The satellite networks are given by 5 copies $\Gamma_1, \dots, \Gamma_5$ of a 4–path with constant conductances equal to 1 and a trivial network Γ_6 given by the central vertex x_6, see the shadowed areas of Figure 3.4 to identify the satellite networks. Therefore, the weights are adapted as $\omega_i = 1/2$ on V_i for each $i = 1, \dots, 5$, $\omega_6 = 1$ on $V_6 = \{x_6\}$, $\omega_0(x_6) = 1/\sqrt{21}$ and $\omega_0 = 2/\sqrt{21}$ on $V_0 \setminus \{x_6\}$, since $\sigma_6 = 1/9$, $\sigma_0 = \sqrt{21}/9$ and $\sigma_i = 4/9$ for $i = 1, \dots, 5$. The orthogonal Green matrices, generalized effective resistances and Kirchhoff indices of the factor networks are easily obtained. They are given by:

$$\mathsf{G}^{\mathsf{i}}_{\mathsf{q}_{\omega_{\mathsf{i}}}} = \frac{1}{8}\begin{pmatrix} 7 & 1 & -3 & -5 \\ 1 & 3 & -1 & -3 \\ -3 & -1 & 3 & 1 \\ -5 & -3 & 1 & 7 \end{pmatrix}, \quad \mathsf{R}^{\mathsf{i}}_{\omega_{\mathsf{i}}} = \begin{pmatrix} 0 & 4 & 8 & 12 \\ 4 & 0 & 4 & 8 \\ 8 & 4 & 0 & 4 \\ 12 & 8 & 4 & 0 \end{pmatrix},$$

$$\mathsf{r}^{\mathsf{i}}_{\omega_{\mathsf{i}}} = \frac{1}{2}\begin{pmatrix} 7 \\ 3 \\ 3 \\ 7 \end{pmatrix} \quad \text{and} \quad \mathsf{k}^{i}(\omega_i) = \frac{5}{2}$$

for $i = 1, \dots, 5$, where the ordering in V_1 is given by the indices $\{1, 2, 3, 4\}$ if $\Gamma_i = \{x_1 \sim y_2 \sim y_3 \sim y_4\}$ is the 4–path. For the trivial satellite network, clearly $\mathsf{G}^{6}_{\mathsf{q}_{\omega_6}} = 0$, $\mathsf{R}^{6}_{\omega_6} = 0$, $\mathsf{r}^{6}_{\omega_6} = 0$ and $\mathsf{k}^6(\omega_6) = 0$. Finally, for the basis network,

$$\mathsf{G}^{0}_{\mathsf{q}_{\omega_0}} = \frac{1}{441}\begin{pmatrix} 353 & -88 & -88 & -88 & -88 & -2 \\ -88 & 353 & -88 & -88 & -88 & -2 \\ -88 & -88 & 353 & -88 & -88 & -2 \\ -88 & -88 & -88 & 353 & -88 & -2 \\ -88 & -88 & -88 & -88 & 353 & -2 \\ -2 & -2 & -2 & -2 & -2 & 20 \end{pmatrix}, \quad \mathsf{r}^{0}_{\omega_0} = \frac{1}{84}\begin{pmatrix} 353 \\ 353 \\ 353 \\ 353 \\ 353 \\ 80 \end{pmatrix},$$

$$\mathsf{R}^{0}_{\omega_0} = \frac{1}{4}\begin{pmatrix} 0 & 42 & 42 & 42 & 42 & 21 \\ 42 & 0 & 42 & 42 & 42 & 21 \\ 42 & 42 & 0 & 42 & 42 & 21 \\ 42 & 42 & 42 & 0 & 42 & 21 \\ 42 & 42 & 42 & 42 & 0 & 21 \\ 21 & 21 & 21 & 21 & 21 & 0 \end{pmatrix} \quad \text{and} \quad \mathsf{k}^0(\omega_0) = \frac{85}{21}.$$

Now we use the results in this section to obtain the orthogonal Green function, the generalized effective resistances and the Kirchhoff index of the generalized star Γ. First, operating, we obtain the values of the coefficients g_{ij} of Theorem 3.1.4:

$$g_{ii} = \frac{5249\left(63 + 8\sqrt{21}\right)}{118098}, \quad g_{ij} = \frac{8\left(5905\sqrt{21} - 5166\right)}{59049},$$

$$g_{i6} = g_{6i} = \frac{8\left(935\sqrt{21} - 18\right)}{59049} \quad \text{and} \quad g_{66} = \frac{40\left(63 + 8\sqrt{21}\right)}{59049}$$

for all $i, j = 1, \ldots, 5$ with $j \neq i$. Applying Theorem 3.1.4, we see that

$$\mathsf{G}_{\mathsf{q}_\omega} = \frac{1}{2187}\begin{bmatrix} \mathsf{G}_1 & -\mathsf{G}_2 & -\mathsf{G}_2 & -\mathsf{G}_2 & -\mathsf{G}_2 & \mathsf{v} \\ -\mathsf{G}_2 & \mathsf{G}_1 & -\mathsf{G}_2 & -\mathsf{G}_2 & -\mathsf{G}_2 & \mathsf{v} \\ -\mathsf{G}_2 & -\mathsf{G}_2 & \mathsf{G}_1 & -\mathsf{G}_2 & -\mathsf{G}_2 & \mathsf{v} \\ -\mathsf{G}_2 & -\mathsf{G}_2 & -\mathsf{G}_2 & \mathsf{G}_1 & -\mathsf{G}_2 & \mathsf{v} \\ -\mathsf{G}_2 & -\mathsf{G}_2 & -\mathsf{G}_2 & -\mathsf{G}_2 & \mathsf{G}_1 & \mathsf{v} \\ \mathsf{v}^\top & \mathsf{v}^\top & \mathsf{v}^\top & \mathsf{v}^\top & \mathsf{v}^\top & a \end{bmatrix},$$

where

$$\mathsf{G}_1 = \begin{pmatrix} 2123 & 1799 & 1583 & 1475 \\ 1799 & 3662 & 3446 & 3338 \\ 1583 & 3446 & 5417 & 5309 \\ 1475 & 3338 & 5309 & 7388 \end{pmatrix}, \quad \mathsf{G}_2 = \begin{pmatrix} 64 & 388 & 604 & 712 \\ 388 & 712 & 928 & 1036 \\ 604 & 928 & 1144 & 1252 \\ 712 & 1036 & 1252 & 1360 \end{pmatrix},$$

$\mathsf{v}^\top = (184, 22, -86, -140)$ and $a = 200$. Finally, by Corollary 3.1.5 we compute the generalized Kirchhoff index $\mathsf{k}(\omega) = \dfrac{1150}{27}$ and the generalized total and effective resistances, which are given by

$$\mathsf{R}_\omega = \frac{1}{4}\begin{bmatrix} \mathsf{R}_1 & \mathsf{R}_2 & \mathsf{R}_2 & \mathsf{R}_2 & \mathsf{R}_2 & \mathsf{r}_1 \\ \mathsf{R}_2 & \mathsf{R}_1 & \mathsf{R}_2 & \mathsf{R}_2 & \mathsf{R}_2 & \mathsf{r}_1 \\ \mathsf{R}_2 & \mathsf{R}_2 & \mathsf{R}_1 & \mathsf{R}_2 & \mathsf{R}_2 & \mathsf{r}_1 \\ \mathsf{R}_2 & \mathsf{R}_2 & \mathsf{R}_2 & \mathsf{R}_1 & \mathsf{R}_2 & \mathsf{r}_1 \\ \mathsf{R}_2 & \mathsf{R}_2 & \mathsf{R}_2 & \mathsf{R}_2 & \mathsf{R}_1 & \mathsf{r}_1 \\ \mathsf{r}_1^\top & \mathsf{r}_1^\top & \mathsf{r}_1^\top & \mathsf{r}_1^\top & \mathsf{r}_1^\top & 0 \end{bmatrix} \quad \text{and} \quad \mathsf{r}_\omega = \frac{1}{108}\begin{bmatrix} \mathsf{r}_2 \\ \mathsf{r}_2 \\ \mathsf{r}_2 \\ \mathsf{r}_2 \\ \mathsf{r}_2 \\ b \end{bmatrix},$$

where

$$\mathsf{R}_1 = \begin{pmatrix} 0 & 81 & 162 & 243 \\ 81 & 0 & 81 & 162 \\ 162 & 81 & 0 & 81 \\ 243 & 162 & 81 & 0 \end{pmatrix}, \quad \mathsf{R}_2 = \begin{pmatrix} 162 & 243 & 324 & 405 \\ 243 & 324 & 405 & 486 \\ 324 & 405 & 486 & 567 \\ 405 & 486 & 567 & 684 \end{pmatrix},$$

$\mathsf{r}_1^\top = (81, 162, 243, 324)$, $\mathsf{r}_2^\top = (2123, 3662, 5417, 7388)$ and $b = 800$.

3.2 Corona networks

In this section we also follow the techniques used in [20] for joint networks. Let us consider two graphs Γ_0 and Γ_1 with vertex sets V_0 and V_1, respectively. The *corona graph* $\Gamma = \Gamma_0 \circ \Gamma_1$ is the graph set up by Γ_0 and $m = |V_0|$ copies of Γ_1, where the edges are the original ones plus the edges connecting all the vertices of each copy of Γ_1 with a different vertex of Γ_0. Notice that we use each vertex of Γ_0 once, see Figure 3.5. For this composite graph the expression of the classical Kirchhoff index was obtained in [59] using a very different approach.

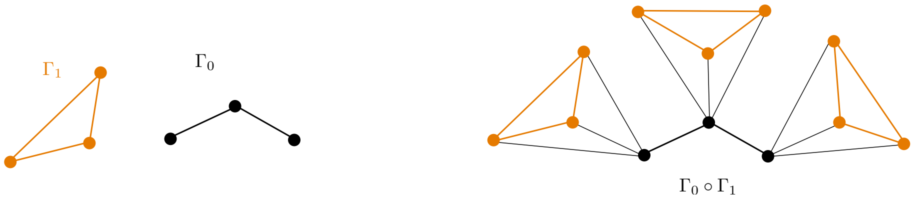


Figure 3.5 Example of a corona graph.

We consider a generalization of the corona graph to the case of $m+1$ different networks. Let $\Gamma_0 = (V_0, c_0)$ be a connected network with vertex set given by $V_0 = \{x_1, \dots, x_m\}$ and let $\Gamma_i = (V_i, c_i)$, with $i = 1, \dots, m$, be m different connected networks. We call them *satellite networks* and Γ_0 is the *basis network*. We define $V = V_0 \cup \coprod_{i=1}^{m} V_i$ as the disjoint union of all the vertex sets.

Let us consider m different positive values $a_1, \dots, a_m \in \mathbb{R}$ and call *corona network with basis* Γ_0*, satellites* $\{\Gamma_i\}_{i=1}^m$ *and conductances* $\{a_i\}_{i=1}^m$ the network $\Gamma = (V, c)$ obtained by considering the set of vertices V and all the original edges of $\Gamma_0, \Gamma_1, \dots, \Gamma_m$ plus new edges given by xx_i for all $x \in V_i$, $i = 1, \dots, m$. The conductances of the edges are given by $c(x, y) = c_i(x, y)$ if $x, y \in V_i$ for an index $i \in \{1, \dots, m\}$, $c(x, y) = a_i\omega(x)\omega(x_i)$ if $x \in V_i$ and $y = x_i$ and $c(x, y) = 0$ otherwise. This network is denoted by $\Gamma = \Gamma_0 \circ (\Gamma_1, \dots, \Gamma_m)$ and is called *generalized corona network*, see Figure 3.6. Any generalized corona network can be seen as a cluster of joint networks and this relation can be expressed by the identity $\Gamma_0 \circ (\Gamma_1, \dots, \Gamma_m) = \Gamma_0\Big\{\Gamma_1 + \{x_1\}, \dots, \Gamma_m + \{x_m\}\Big\}$, where $\Gamma_i + \{x_i\}$ is the joint network with conductance a_i, see [20].

Remember that a satellite network $\Gamma_i = (V_i, c_i)$ is called trivial if $V_i = \{x_i\}$. Also, a satellite network $\Gamma_i = (V_i, c_i)$ is called k*–clique* if $|V_i| = k$ and any two vertices of V_i are neighbours, see [50] for the definition of clique. Notice that appending a trivial satellite network to the basis network means to add a vertex and an edge joining it to the basis network. On the other

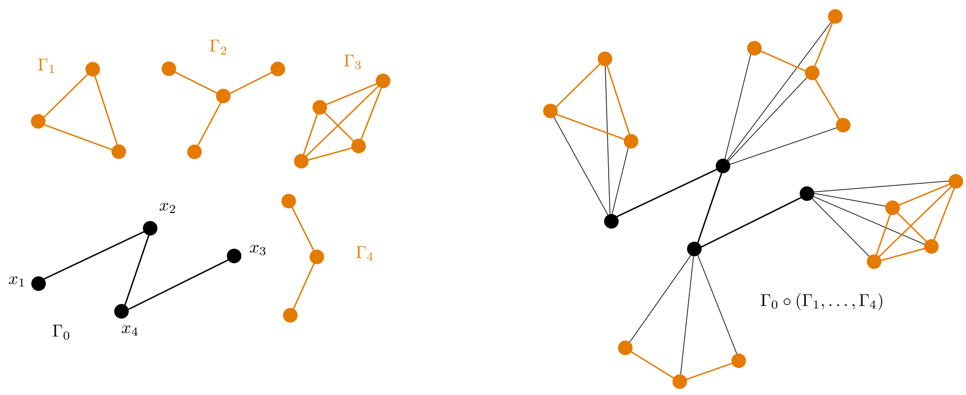


Figure 3.6 Example of a generalized corona network.

hand, to append a k–clique satellite network to Γ_0 means to have a $(k+1)$–clique in the resulting generalized corona such that only one of the vertices of the clique is a separating vertex of the generalized corona. Hence, any arbitrary connected network with at least one vertex with only one neighbour or a $(k+1)$–clique satisfying the above condition can be understood as a generalized corona network. Furthermore, there exist many networks that do not fit this description and still can be understood as a generalized corona network. In Figure 3.7 we can see some examples of the reinterpretation of a network as a generalized corona, where the shadowed areas are the satellite networks.

Let $\Gamma = (V, c)$ be a generalized corona network and let $\omega \in \Omega(V)$ be a weight on Γ. Consider for any $i = 0, \dots, m$ the value $\sigma_i = \left(\int_{V_i} \omega^2\right)^{\frac{1}{2}}$ and the function $\omega_i(x) = {\sigma_i}^{-1}\omega(x)$ for every $x \in V_i$. It is clear that $\omega_i \in \Omega(V_i)$ for all $i = 0, \dots, m$. Observe that $\langle u, v\rangle = \langle u, v\rangle_{V_i}$ if $u \in \mathcal{C}(V_i)$ and $v \in \mathcal{C}(V)$. In particular, $\langle u, v\rangle = 0$ when $v \in \mathcal{C}(V_j)$ with $j \neq i$. Finally, notice that $\sum_{j=1}^{m} \sigma_j^2 = 1 - \sigma_0^2$.

As in the previous section, the superscript i on a parameter or operator stands for the one of the network Γ_i, where $i = 0, \dots, m$. Using this notation, we get the following result.

Proposition 3.2.1. *Given an index $j \in \{1, \dots, m\}$, for any $u \in \mathcal{C}(V)$ on the generalized corona network it is satisfied that*

$$\mathcal{L}_{q_\omega}(u)(x_j) = \mathcal{L}^0_{q_{\omega_0}}(u)(x_j) + a_j\sigma_j\Big(\sigma_j u(x_j) - \sigma_0\omega_0(x_j)\langle \omega_j, u\rangle\Big)$$

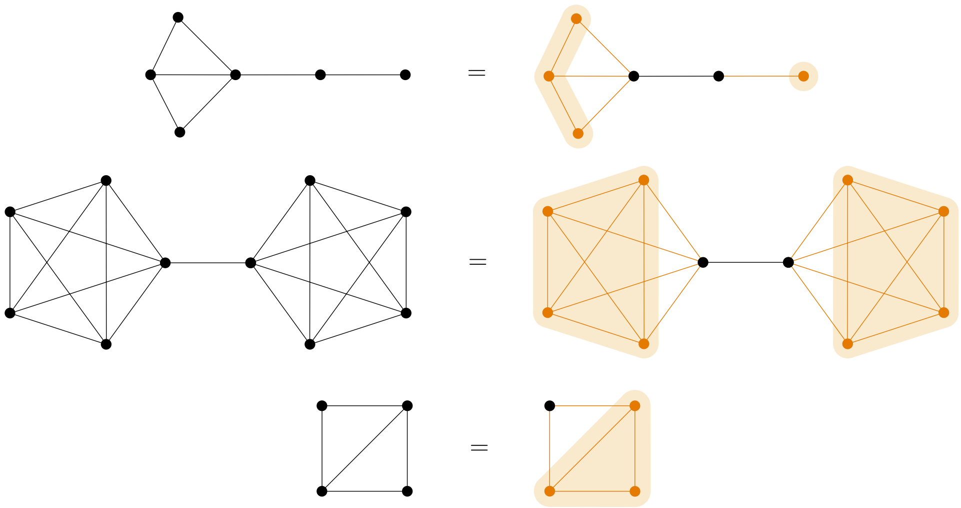


Figure 3.7 Examples of the reinterpretation of a network as a generalized corona.

and

$$\mathcal{L}_{q_\omega}(u)(x) = \mathcal{L}^j_{p_j}(u)(x) - a_j\sigma_0\omega_0(x_j)u(x_j)\sigma_j\omega_j(x)$$

for every $x \in V_j$*, where* $p_j = q_{\omega_j} + \gamma_j$ *and* $\gamma_j = a_j\sigma_0^2\omega_0^2(x_j)$.

Proof. It suffices to observe that

$$\mathcal{L}(u)(x_j) = \mathcal{L}^0(u)(x_j) + \sigma_0\omega_0(x_j)a_j\sigma_j\big(u(x_j)\langle\omega_j, 1\rangle - \langle\omega_j, u\rangle\big)$$

and

$$\mathcal{L}(u)(x) = \mathcal{L}^j(u)(x) + a_j\sigma_j\omega_j(x)\omega_0(x_j)\big(u(x) - u(x_j)\big)$$

for every $x \in V_j$, where the function $\mathbf{1}$ is no other than $\mathbf{1}(x) = 1$ for all $x \in V$. Therefore,

$$q_\omega(x_j) = q_{\omega_0}(x_j) + a_j\sigma_j\big(\sigma_j - \sigma_0\omega_0(x_j)\langle\omega_j, 1\rangle\big)$$

and

$$q_\omega(x) = q_{\omega_j}(x) - a_j\sigma_0\omega_0(x_j)\big(\sigma_j\omega_j(x) - \sigma_0\omega_0(x_j)\big)$$

for every $x \in V_j$. □

The following objective is to obtain the orthogonal Green operator on V with respect to the potential q_ω of the generalized corona network in terms of the orthogonal Green operators on V_i of $\Gamma_0,\dots,\Gamma_m$. Notice that we are considering the $\lambda = 0$ case. For the sake of simplicity, we define for any $f \in \mathcal{C}(V)$ the function $h_{_f} \in \mathcal{C}(V_0)$ given by

$$h_{_f}(x_j) = \frac{\sigma_j\langle f, \omega_j\rangle}{\omega(x_j)}$$

for any $j = 1,\dots,m$. In particular, $h_{_\omega}(x_j) = \sigma_j^2\omega^{-1}(x_j)$. The definition of $h_{_f}$ is motivated by the following result.

Lemma 3.2.2. *Let $f \in \mathcal{C}(V)$ be a function on the generalized corona network such that $\langle \omega, f\rangle = 0$. Let $u \in \mathcal{C}(V)$ be a solution of the Poisson equation $\mathcal{L}_{q_\omega}(u) = f$ on V. Then, $\mathcal{L}^0_{q_{\omega_0}}(u) = f + h_{_f}$ on V_0.*

Proof. Using Proposition 3.2.1, we know that u is a solution of the equation $\mathcal{L}_{q_\omega}(u) = f$ on V if and only if u satisfies

$$\mathcal{L}^0_{q_{\omega_0}}(u)(x_j) = f(x_j) - a_j\sigma_j\big(\sigma_j u(x_j) - \sigma_0\omega_0(x_j)\langle\omega_j, u\rangle\big) \tag{3.1}$$

and

$$\mathcal{L}^j_{p_j}(u)(x) = f(x) + a_j\sigma_0\omega_0(x_j)u(x_j)\sigma_j\omega_j(x)$$

for every $j = 1,\dots,m$ and $x \in V_j$. As $\mathcal{L}^j_{p_j}$ is self–adjoint on $\mathcal{C}(V_j)$, we get that

$$\begin{aligned} a_j\sigma_0^2\omega_0^2(x_j)\langle\omega_j, u\rangle &= \langle\mathcal{L}^j_{p_j}(\omega_j), u\rangle = \langle\omega_j, \mathcal{L}^j_{p_j}(u)\rangle \\ &= \langle\omega_j, f\rangle + a_j\sigma_j\sigma_0\omega_0(x_j)u(x_j) \end{aligned}$$

because $\mathcal{L}^j_{p_j}(\omega_j) = a_j\sigma_0^2\omega_0^2(x_j)\omega_j$. Therefore,

$$u(x_j) = \frac{\sigma_0\omega_0(x_j)\langle\omega_j, u\rangle}{\sigma_j} - \frac{h_{_f}(x_j)}{a_j\sigma_j^2}.$$

The result follows replacing the value of $u(x_j)$ in Equation (3.1). □

Proposition 3.2.3. *Let $f \in \mathcal{C}(V)$ be a function on the generalized corona network such that $\langle\omega, f\rangle = 0$. Consider the Poisson equation $\mathcal{L}_{q_\omega}(u) = f$ on V. Then, the function*

$$\begin{aligned} u &= \mathcal{G}^0_{q_{\omega_0}}(f + h_{_f}) - \left(\langle\mathcal{G}^0_{q_{\omega_0}}(f + h_{_f}), g_{_\omega}\rangle + \sum_{k=1}^m \frac{h_{_f}(x_k)}{a_k\omega(x_k)}\right)\omega \quad && \text{on } V_0 \\ u &= \mathcal{G}^j_{p_j}(f) + \frac{u(x_j)}{\omega(x_j)}\omega && \text{on } V_j \end{aligned}$$

for $j = 1,\dots,m$ is the unique solution of the Poisson equation such that $\langle\omega, u\rangle = 0$.

Proof. From Proposition 3.2.1 and the above Lemma, we know that u is a solution of the equation $\mathcal{L}_{q_\omega}(u)=f$ on V if and only if it is satisfied that $\mathcal{L}^0_{q_{\omega_0}}(u)=f+h_f$ on V_0 and $\mathcal{L}^j_{p_j}(u)=f+a_j\sigma_0\omega_0(x_j)u(x_j)\sigma_j\omega_j$ on V_j for $j=1,\dots,m$. Remember that

$$u(x_j)=\frac{\sigma_0\omega_0(x_j)\langle\omega_j,u\rangle}{\sigma_j}-\frac{h_f(x_j)}{a_j\sigma_j^2}.$$

Then, applying the corresponding orthogonal Green operators on V_j, we get that $u-\langle u,\omega_0\rangle\omega_0=\mathcal{G}^0_{q_{\omega_0}}(f+h_f)$ on V_0 and $u=\mathcal{G}^j_{p_j}(f)+\sigma_j u(x_j)\sigma_0^{-1}\omega_0^{-1}(x_j)\omega_j$ on V_j for all $j=1,\dots,m$, since Equation (2.5) applies. Notice that $\langle\omega,u\rangle=0$ if and only if $\langle\omega_0,u\rangle=-{\sigma_0}^{-1}\sum_{j=1}^m\sigma_j\langle\omega_j,u\rangle$,. If we use the two expressions obtained for u on V_0 we get that

$$\frac{\sigma_0\omega_0(x_j)}{\sigma_j}\langle\omega_j,u\rangle+\omega_0(x_j)\sum_{i=1}^m\frac{\sigma_i}{\sigma_0}\langle\omega_i,u\rangle=\mathcal{G}^0_{q_{\omega_0}}\big(f+h_f\big)(x_j)+\frac{h_f(x_j)}{a_j\sigma_j^2},$$

where $j=1,\dots,m$. This linear system is equivalent to the system

$$\sigma_0^2\langle\omega_j,u\rangle+\sum_{i=1}^m\sigma_j\sigma_i\langle\omega_i,u\rangle=\frac{\sigma_0\sigma_j}{\omega_0(x_j)}\left(\mathcal{G}^0_{q_{\omega_0}}\big(f+h_f\big)(x_j)+\frac{h_f(x_j)}{a_j\sigma_j^2}\right)$$

for every $j=1,\dots,m$. The coefficient matrix of this last system of equations is $\mathsf{H}=\sigma_0^2\,\mathsf{ID}+\sigma\otimes\sigma$, where ID is the identity matrix of size m and $\sigma=(\sigma_1,\dots,\sigma_m)$. Therefore, $\mathsf{H}^{-1}=\sigma_0^{-2}\big[\mathsf{ID}-\sigma\otimes\sigma\big]$ and so

$$\begin{aligned}\langle u,\omega_j\rangle= &\quad \frac{\sigma_j}{\omega(x_j)}\left(\mathcal{G}^0_{q_{\omega_0}}\big(f+h_f\big)(x_j)+\frac{h_f(x_j)}{a_j\sigma_j^2}\right)\\ - &\quad \sigma_j\sum_{i=1}^m\frac{\sigma_i^2}{\omega(x_i)}\left(\mathcal{G}^0_{q_{\omega_0}}\big(f+h_f\big)(x_i)+\frac{h_f(x_i)}{a_i\sigma_i^2}\right)\end{aligned}$$

for all $j=1,\dots,m$. Finally, the results follow by replacing this value on the equations. □

Theorem 3.2.4. *The orthogonal Green function on V of the generalized* *corona network is given by*

$$G_{q_\omega} = G^0_{q_\omega} - \Big[\omega \otimes \mathcal{G}^0_{q_{\omega_0}}(h_{\omega}) + \mathcal{G}^0_{q_{\omega_0}}(h_{\omega}) \otimes \omega\Big] + g_{00}\,\omega \otimes \omega \qquad \textit{on } V_0 \times V_0,$$

$$G_{q_\omega} = \frac{1}{\omega(x_k)}\mathcal{G}^0_{q_{\omega_0}}(\varepsilon_{x_k}) \otimes \omega - \mathcal{G}^0_{q_{\omega_0}}(h_{\omega}) \otimes \omega + \left(g_{00} - \frac{\mathcal{G}^0_{q_{\omega_0}}(h_{\omega})(x_k)}{\omega(x_k)} - \frac{1}{\gamma_k}\right)\omega \otimes \omega \qquad \textit{on } V_0 \times V_k,$$

$$G_{q_\omega} = \frac{1}{\omega(x_k)}\omega \otimes \mathcal{G}^0_{q_{\omega_0}}(\varepsilon_{x_k}) - \omega \otimes \mathcal{G}^0_{q_{\omega_0}}(h_{\omega}) + \left(g_{00} - \frac{\mathcal{G}^0_{q_{\omega_0}}(h_{\omega})(x_k)}{\omega(x_k)} - \frac{1}{\gamma_k}\right)\omega \otimes \omega \qquad \textit{on } V_k \times V_0,$$

$$G_{q_\omega} = G^k_{p_k} + \left(g_{00} - \frac{\mathcal{G}^0_{q_{\omega_0}}(h_{\omega})(x_k)}{\omega(x_k)} - \frac{\mathcal{G}^0_{q_{\omega_0}}(h_{\omega})(x_j)}{\omega(x_j)}\right)\omega \otimes \omega + \left(-\frac{1}{\gamma_k} - \frac{1}{\gamma_j} + \frac{G^0_{q_{\omega_0}}(x_k, x_j)}{\omega(x_k)\omega(x_j)}\right)\omega \otimes \omega \qquad \textit{on } V_k \times V_j$$

where $g_{00} = \langle h_{\omega}, \mathcal{G}^0_{q_{\omega_0}}(h_{\omega})\rangle + \sum_{i=1}^{m} \frac{\sigma_i^2}{\gamma_i}$.

Proof. For any $y \in V$, using Equation (2.5) the function $u = \mathcal{G}_{q_\omega}(\varepsilon_y)$ is the unique solution of the equation $\mathcal{L}_{q_\omega}(u) = \varepsilon_y - \omega(y)\omega$ on V such that $\langle \omega, u\rangle = 0$. Let $f = \varepsilon_y - \omega(y)\omega \in \mathcal{C}(V)$. Then, the explicit expression of G_{q_ω} is easily deduced if we reduce the problem to the following two situations.

1. Let us suppose that there exists an index $j \in \{1, \dots, m\}$ such that $y = x_j \in V_0$. Then, $f = \varepsilon_{x_j} - \sigma_0\omega_0(x_j)\omega$ and hence

$$\mathcal{G}^0_{q_{\omega_0}}(f)(x_i) = G^0_{q_{\omega_0}}(x_i, x_j)$$

$$\langle f, \omega_i\rangle = -\sigma_0\sigma_i\omega_0(x_j)$$

$$\mathcal{G}^0_{q_{\omega_0}}(g)(x_i) = -\sigma_0\omega_0(x_j)\mathcal{G}^0_{q_{\omega_0}}(\tilde{g})(x_i)$$

$$\mathcal{G}^i_{p_i}(f)(x) = -\frac{\sigma_i\omega_0(x_j)\omega_i(x)}{a_i\sigma_0\omega_0^2(x_i)}$$

for all $i = 1, \dots, m$ and $x \in V_i$, where $g, \tilde{g} \in \mathcal{C}(V_0)$ are defined as $g(x_i) = \dfrac{\sigma_i\langle f, \omega_i\rangle}{\omega_0(x_i)}$ and $\tilde{g}(x_i) = \dfrac{\sigma_i^2}{\omega_0(x_i)}$. Notice that $G_{q_\omega}(x, y) = u(x)$

and that x can be either in V_0 or in V_i for $i \in \{1,\dots,m\}$. Applying Proposition 3.2.3 we get the result using the formulae corresponding to each case.

2. Let us suppose that $y \in V_j$ for $j \in \{1,\dots,m\}$. Then, $f = \varepsilon_y - \sigma_j\omega_j(y)\omega$ and hence

$$\langle f, \omega_i\rangle = \omega_i(y)\chi_{V_i}(y) - \sigma_j\sigma_i\omega_j(y)$$

$$\mathcal{G}^0_{q_{\omega_0}}(f)(x_i) = 0$$

$$\mathcal{G}^0_{q_{\omega_0}}(g)(x_i) = \frac{\sigma_j\omega_j(y)}{\omega_0(x_j)} G^0_{q_{\omega_0}}(x_i, x_j) - \sigma_j\omega_j(y)\mathcal{G}^0_{q_{\omega_0}}(\tilde{g})(x_i)$$

$$\mathcal{G}^i_{p_i}(f)(x) = G^i_{p_i}(x,y)\chi_{V_i}(y) - \frac{\sigma_i\sigma_j\omega_i(x)\omega_j(y)}{a_i\sigma_0^2\omega_0^2(x_i)}$$

for all $i = 1,\dots,m$ and $x \in V_i$, where $g, \tilde{g} \in \mathcal{C}(V_0)$ are defined in the above case. Notice that $G_{q_\omega}(x,y) = u(x)$ and that x can be either $x = x_i \in V_0$ or $x \in V_i$, where $i \in \{1,\dots,m\}$. Applying Proposition 3.2.3 we get the result using the formulae corresponding to each x–case for this y. □

The following result gives the expression of the generalized Kirchhoff index $\mathsf{k}(\omega)$ and the generalized effective and total resistances R_ω, r_ω in terms of the same parameters of the basis and satellite networks. These expressions follow directly from the formulae in Proposition 2.6.1.

Corollary 3.2.5. *The generalized Kirchhoff index $\mathsf{k}(\omega)$ of the generalized corona network is given by*

$$\begin{aligned}\mathsf{k}(\omega) = \mathsf{k}_0(\omega_0) &+ \sum_{i=1}^m \mathsf{k}_i(\gamma_i,\omega_i) + \sum_{i=1}^m \frac{(1-\sigma_i^2)}{\gamma_i} \\ &+ \sum_{i=1}^m \sigma_i^2 r^0_{\omega_0}(x_i) + \frac{1}{2\sigma_0^2}\sum_{i,j=1}^m \sigma_i^2\sigma_j^2 R^0_{\omega_0}(x_i,x_j).\end{aligned}$$

Moreover, for all $i = 1,\dots,m$ it is satisfied that

$$\begin{aligned}r_\omega(x_i) &= r^0_{\omega_0}(x_i) - \sum_{k=1}^m \sigma_k^2 r^0_{\omega_0}(x_k) - \frac{1}{2\sigma_0^2}\sum_{k,j=1}^m \sigma_k^2\sigma_j^2 R^0_{\omega_0}(x_k,x_j) \\ &+ \frac{1}{\sigma_0^2}\sum_{j=1}^m \sigma_j^2 R^0_{\omega_0}(x_i,x_j) + \sum_{j=1}^m \frac{\sigma_j^2}{\gamma_j} \\ r_\omega(x) &= \frac{r^i_{\gamma_i,\omega_i}(x)}{\sigma_i^2} - \frac{1}{\sigma_i^2\gamma_i} + r_\omega(x_i)\end{aligned}$$

if $x \in V_i$ *and for all* $i, j = 1, \dots, m$ *the generalized effective resistances are given by*

$$\begin{aligned}
R_\omega(x_i, x_j) &= \frac{R^0_{\omega_0}(x_i, x_j)}{\sigma_0^2}, \\
R_\omega(x_i, y) &= \frac{R^0_{\omega_0}(x_i, x_j)}{\sigma_0^2} + \frac{r^j_{\gamma_j,\omega_j}(x)}{\sigma_j^2} + \frac{1}{\sigma_j^2\gamma_j}, && y \in V_j, \\
R_\omega(x, y) &= \frac{R^j_{\gamma_j,\omega_j}(x, y)}{\sigma_j^2}, && x, y \in V_j, \\
R_\omega(x, y) &= \frac{R^0_{\omega_0}(x_i, x_j)}{\sigma_0^2} + \frac{r^i_{\gamma_i,\omega_i}(x)}{\sigma_i^2} + \frac{r^j_{\gamma_j,\omega_j}(y)}{\sigma_j^2} \\
&\quad + \frac{1}{\sigma_i^2\gamma_i} + \frac{1}{\sigma_j^2\gamma_j}, && x \in V_i, y \in V_j, i \neq j.
\end{aligned}$$

Proof. It suffices to apply the formulas given in Proposition 2.6.1 and the identities

$$\begin{aligned}
\frac{2\mathcal{G}^0_{\omega_0}(h_\omega)(x_i)}{\omega(x_i)} &= \frac{1}{\sigma_0^2}\sum_{j=1}^m \sigma_j^2 \frac{2G^0_{\omega_0}(x_i, x_j)}{\omega_0(x_i)\omega_0(x_j)} \\
&= \frac{1}{\sigma_0^2}\sum_{j=1}^m \sigma_j^2 \left(r^0_{\omega_0}(x_j) + r^0_{\omega_0}(x_i) - R^0_{\omega_0}(x_i, x_j)\right) \\
&= \frac{(1-\sigma_0^2)}{\sigma_0^2} r^0_{\omega_0}(x_i) + \frac{1}{\sigma_0^2}\sum_{j=1}^m \sigma_j^2 r^0_{\omega_0}(x_j) - \frac{1}{\sigma_0^2}\sum_{j=1}^m \sigma_j^2 R^0_{\omega_0}(x_i, x_j)
\end{aligned}$$

and

$$\begin{aligned}
\langle h_\omega, \mathcal{G}^0_{\omega_0}(h_\omega)\rangle &= \sum_{i=1}^m \mathcal{G}^0_{\omega_0}(h_\omega)(x_i)\frac{\sigma_i^2}{\omega(x_i)} \\
&= \frac{1}{2}\sum_{i=1}^m \left(\frac{(1-\sigma_0^2)}{\sigma_0^2} r^0_{\omega_0}(x_i) + \frac{1}{\sigma_0^2}\sum_{j=1}^m \sigma_j^2 r^0_{\omega_0}(x_j)\right)\sigma_i^2 \\
&\quad - \frac{1}{2}\sum_{i=1}^m \left(\frac{1}{\sigma_0^2}\sum_{j=1}^m \sigma_j^2 R^0_{\omega_0}(x_i, x_j)\right)\sigma_i^2 \\
&= \frac{(1-\sigma_0^2)}{\sigma_0^2}\sum_{i=1}^m r^0_{\omega_0}(x_i)\sigma_i^2 - \frac{1}{2\sigma_0^2}\sum_{i,j=1}^m \sigma_i^2\sigma_j^2 R^0_{\omega_0}(x_i, x_j). \quad \square
\end{aligned}$$

This result was obtained in [59] for the classical case, that is, for the corona graph with not normalized constant weight. We can see that any generalized corona network loses rigidity with respect to its parts, since its Kirchhoff index is bigger that the sum of the Kirchhoff indices of its factors.

We would like to remark that the findings in Sections 3.1 and 3.2 justify the definition of the effective resistances and the Kirchhoff index with respect to a weight and a non–negative real value λ. The reason is that, even when considering a constant weight and $\lambda = 0$ on these composite networks, their generalized Kirchhoff indices and effective resistances are naturally expressed in terms of the same parameters with respect to other weights and other non–zero values on the factor networks. For example, if these generalizations had not been introduced, then we would have not been able to identify the structural parameters $\mathsf{k}_i(\gamma_i, \omega_i)$ of the factor networks of the generalized corona network in the expression of its Kirchhoff index $\mathsf{k}(\omega)$, see Corollary 3.2.5.

4
Product networks and applications

Product networks are the network version of the *cartesian product* of graphs. We use separation of variable techniques in order to work with this family of networks and express their Green operator on a subset of vertices in terms of functions and operators on the factors.

Theoretically, if we know the conductances of the network then the Green matrix on a subset of vertices is directly known because it is the inverse of a block of the Schrödinger matrix. However, for relatively large sizes it is not computationally feasible to obtain this inverse. Because of this reason, in this chapter we aim to obtain the expression of the Green operator on a subset of vertices of product networks in terms of the Green operators of the factor networks. The Green matrices of these smaller networks are computationally easier to obtain due to their smaller size. Thus, we can assume the smaller Green kernels to be known. This problem is also classified as a direct boundary value problem, since the conductances of the networks are known.

Some authors have dealt with similar problems. For example, in [33] Chung and Yau obtained the classical Green function with respect to a multiple of the combinatorial laplacian of the cartesian product of two regular graphs. Technically, they obtained the classical Green functino with respect to the normalized laplacian, but since the cartesian product of two regular graphs is a regular graph, its normalized laplacian becomes a multiple of the combinatorial laplacian. Also, in [41, 42] Ellis expressed the classical Green function with respect to a multiple of the combinatorial laplacian of the cartesian product of two regular graphs in terms of a shifted Green function of one of the factors and the eigensystem of the other one. Ellis uses separation of variables in these papers. Notice that he needs to shift the Green function, that is, he needs to consider a Green function with respect to a Schrödinger operator. He sometimes uses another technique involving methods in complex variables for these problems.

As a by–product of the findings with respect to product networks, we obtain the expression of the Green function on a subset of vertices of another family of networks, named spider networks. The difficulty to obtain this function by means of direct methods is high. However, any spider network can be seen as the modification of a certain product network. Therefore, we give the expression of the Green function of spider networks in terms of the ones of product networks.

The results given in this chapter are submitted to publication, see [13], and they have been presented in a congress as well, see [5].

4.1 Green function of product networks

Let us consider two connected networks $\Gamma_1 = (V_1, c_1)$ and $\Gamma_2 = (V_2, c_2)$ with $|V_2| \geq 3$. We call them *factor networks.* We label their vertex sets as $V_1 = \{x_1^1, x_2^1, \dots, x_n^1\}$ and $V_2 = \{x_1^2, x_2^2, \dots, x_m^2\}$, respectively, in such a way that $F_2 = \{x_2^2, \dots, x_{m-1}^2\} \subset V_2$ is connected and $\delta(F_2) = \{x_1^2, x_m^2\}$. This can always be achieved, since by [56, Proposition 1.2.29] every connected graph contains at least two vertices that are not k–separating for any k. Therefore, since V_2 is connected, there exists a non–separating vertex $x_1^2 \in V_2$. Again, $V_2 \setminus \{x_1^2\}$ is connected, so there exists another non–separating vertex $x_m^2 \in V_2 \setminus \{x_1^2\}$ and hence we can take the connected set $F_2 = V_2 \setminus \{x_1^2, x_m^2\}$. Moreover, $\delta(F_2) = \{x_1^2, x_m^2\}$ and $\bar{F}_2 = V_2$. We define mn new vertices given by the pairs $y_{ji} = (x_j^1, x_i^2)$ for all $j = 1, \dots, n$ and $i = 1, \dots, m$. Now consider the vertex set $V = \{y_{ji} : j = 1, \dots, n,\ i = 1, \dots, m\}$ and observe that $V = V_1 \times V_2$.

We define the *product network* $\Gamma = \Gamma_1 \,\Box\, \Gamma_2$ as the network $\Gamma = (V, c)$ with vertex set V and conductivity function $c\colon V \times V \longrightarrow [0, +\infty)$ given by $c(y_{ji}, y_{lk}) = c_1(x_j^1, x_l^1)$ if $i = k$, $c(y_{ji}, y_{lk}) = c_2(x_i^2, x_k^2)$ if $l = j$ and $c(y_{ji}, y_{lk}) = 0$ otherwise, where $y_{ji}, y_{lk} \in V$. Moreover, we define the set of vertices $F = \{y_{12}, \dots, y_{n2}, \dots, y_{1m-1}, \dots, y_{nm-1}\} \subset V$, the *upper boundary* set $H_1 = \{\ y_{j1} \mid j = 1, \dots, n\} \subset V$ and the *lower boundary* set $H_m = \{\ y_{jm} \mid j = 1, \dots, n\} \subset V$. Observe that $F = V_1 \times F_2$, $\delta(F) = H_1 \cup H_m$ and

$\bar{F} = V$. A simple example of a product network is shown in Figure 4.1.

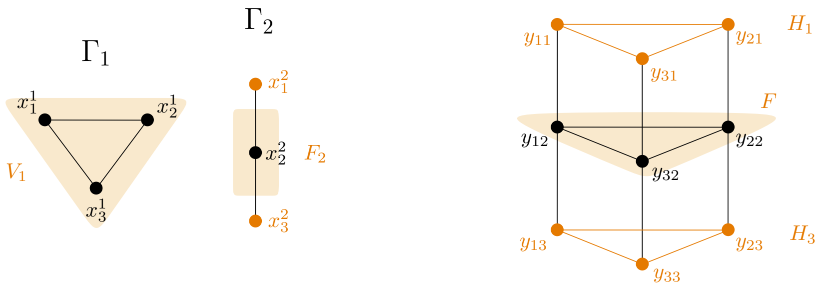


Figure 4.1 Example of a product network.

Separation of variable techniques are used both in the continuum and the discrete fields. They are used to help with those problems which are not solvable with direct methods but whose domain can be expressed as the cartesian product of two domains with smaller dimension. These techniques are useful because they split the unsolvable problem and reduce it to easier problems where the domains have smaller dimension. For instance, as we have detailed before, Ellis used separation of variables for a similar problem in [41, 42]. Hence, we use separation of variables in order to obtain the Green function on F of product networks, since their domain is the cartesian product $V = V_1 \times V_2$. The first thing to do is to study how the parameters and functions of the factor networks behave when they are put together to form the product network.

Given two functions $u_1 \in \mathcal{C}(V_1)$ and $u_2 \in \mathcal{C}(V_2)$ on each factor network, we denote by $u_1 \otimes u_2 \in \mathcal{C}(V)$ the function defined as

$$(u_1 \otimes u_2)(y_{ji}) = u_1(x_j^1)u_2(x_i^2)$$

for any $x_j^1 \in V_1$ and $x_i^2 \in V_2$. Notice that if $\omega_1 \in \Omega(V_1)$ and $\omega_2 \in \Omega(V_2)$ are two weights on the factor networks, then $\omega_1 \otimes \omega_2$ is also a weight on the product network Γ. On the other hand, if $u \in \mathcal{C}(V)$ is a function on the product network, then we can define the functions $u_{1,i} \in \mathcal{C}(V_1)$ for $i = 1, \dots, m$ and $u_{2,j} \in \mathcal{C}(V_2)$ for $j = 1, \dots, n$ given by

$$u_{1,i}(x_j^1) = u(y_{ji}) = u_{2,j}(x_i^2).$$

Observe that $u_{1,i}$ and $u_{2,j}$ are the restrictions of u to Γ_1 and Γ_2, respectively. If $\omega \in \Omega(V)$ is a weight on the product network, then neither $\omega_{1,i}$ nor $\omega_{2,j}$ are weights on the factor networks for any $i = 1, \dots, m$ and $j = 1, \dots, n$. From now on in this section, the superscript i on a parameter or operator stands for the one of the network Γ_i, where $i = 1$ or $i = 2$. Using this notation, we get the following result on the Schrödinger operator of a product network. Note that it is the sum of two Kronecker products.

Lemma 4.1.1. *Consider two weights $\omega_1 \in \Omega(V_1)$ and $\omega_2 \in \Omega(V_2)$ and a real value $\lambda \geq 0$. Given two functions $u_1 \in \mathcal{C}(V_1)$ and $u_2 \in \mathcal{C}(V_2)$, then*

$$\mathcal{L}_q(u_1 \otimes u_2) = \mathcal{L}^1_{q_1}(u_1) \otimes u_2 + u_1 \otimes \mathcal{L}^2_{q_{\omega_2}}(u_2) \quad \text{on } F,$$

where the potentials are given by $q = q_{\omega_1 \otimes \omega_2} + \lambda$ on F and $q_1 = q_{\omega_1} + \lambda$ on V_1.

Proof. It suffices to notice that $\mathcal{L}(u_1 \otimes u_2) = \mathcal{L}^1(u_1) \otimes u_2 + u_1 \otimes \mathcal{L}^2(u_2)$ and $q_{\omega_1 \otimes \omega_2} = q_{\omega_1} + q_{\omega_2}$ on F. □

Remember that the inner product $\langle \cdot, \cdot \rangle_{F_2}$ on Γ_2 is given by

$$\langle u_2, v_2 \rangle_{F_2} = \int_{F_2} u_2(x) v_2(x)\, dx = \sum_{r=2}^{m-1} u_2(x^2_r) v_2(x^2_r)$$

for all $u_2, v_2 \in \mathcal{C}(F_2)$.

Lemma 4.1.2. *If we consider an orthonormal basis $\{\phi_r\}_{r=2,\dots,m-1}$ on $\mathcal{C}(F_2)$ with respect to the inner product $\langle \cdot, \cdot \rangle_{F_2}$, then we can express any function $u_2 \in \mathcal{C}(F_2)$ as*

$$u_2 = \sum_{r=2}^{m-1} \langle u_2, \phi_r \rangle_{F_2} \phi_r. \tag{4.1}$$

Lemma 4.1.3. *Let $\{\phi_r\}_{r=2,\dots,m-1}$ be an orthonormal basis on $\mathcal{C}(F_2)$ with respect to the inner product $\langle \cdot, \cdot \rangle_{F_2}$. Let $f \in \mathcal{C}(F)$ be a function on the product network. Then,*

$$f = \sum_{r=2}^{m-1} f_r \otimes \phi_r,$$

where $f_r \in \mathcal{C}(V_1)$ is given by $f_r(x^1_j) = \langle f_{2,j}, \phi_r \rangle_{F_2}$ for all $j = 1, \dots, n$. In addition, if $g \in \mathcal{C}(F)$ is also a function on the product network, then $f = g$ if and only if $f_r = g_r$ for all $r = 2, \dots, m-1$.

Proof. Consider $y_{ji} \in V$. Using Equation (4.1),

$$f(y_{ji}) = f_{2,j}(x_i^2) = \sum_{r=2}^{m-1} \langle f_{2,j}, \phi_r \rangle_{F_2} \phi_r(x_i^2) = \sum_{r=2}^{m-1} f_r(x_j^1)\phi_r(x_i^2)$$
$$= \sum_{r=2}^{m-1} (f_r \otimes \phi_r)(y_{ji}). \quad \square$$

Lemma 4.1.4. *There exists a set of real values* $0 < \lambda_2 \leq \ldots \leq \lambda_{m-1}$ *and an orthonormal basis* $\{\phi_r\}_{r=2,\ldots,m-1}$ *of* $\mathcal{C}(F_2)$ *with respect to the inner product* $\langle \cdot, \cdot \rangle_{F_2}$ *such that* $\mathcal{L}^2_{q_{\omega_2}}(\phi_r) = \lambda_r \phi_r$ *on* F_2 *for all* $r = 2, \ldots, m-1$*. Moreover, For all* $v_2 \in \mathcal{C}(F_2)$*, the following equality holds:*

$$\mathcal{L}^2_{q_{\omega_2}}(v_2) = \sum_{r=2}^{m-1} \lambda_r \langle v_2, \phi_r \rangle_{F_2} \phi_r.$$

Proof. As $\mathcal{L}^2_{q_{\omega_2}}$ is a self–adjoint, positive semi–definite operator on $\mathcal{C}(V_2)$ then it is self–adjoint and positive definite on $\mathcal{C}(F_2)$. Hence, we can consider $0 < \lambda_2 \leq \ldots \leq \lambda_{m-1}$ and an orthonormal basis $\{\phi_r\}_{r=2,\ldots,m-1}$ of $\mathcal{C}(F_2)$ with respect to the inner product $\langle \cdot, \cdot \rangle_{F_2}$ such that $\mathcal{L}^2_{q_{\omega_2}}(\phi_r) = \lambda_r \phi_r$ for all $r = 2, \ldots, m-1$. Moreover, given a function $v_2 \in \mathcal{C}(F_2)$ we get the result applying Equation (4.1). $\square$

Now we are in the position to determine the Green function on F of product networks. It is important to mention that the following theorem expresses this Green function in terms of the eigensystem of the Schrödinger operator of one factor network and the orthogonal Green functions on the whole vertex set of the other factor network. The obtaining of both involves very different techniques. We consider this fact to be extremely interesting, since this duality can be conveniently used when there is a clear difference of difficulty between the acquirement of the eigensystem or the Green function for one of the factor networks.

Theorem 4.1.5. *Let* $\omega_1 \in \Omega(V_1)$ *and* $\omega_2 \in \Omega(V_2)$ *be two weights on the factor networks and let* $\lambda > 0$ *be a real value. The Green function on* F *of the product network with potential* $q = q_{\omega_1 \otimes \omega_2} + \lambda$ *can be written as*

$$G_q(y_{ji}, y_{lk}) = \sum_{r=2}^{m-1} \phi_r(x_k^2)\phi_r(x_i^2) G^1_{q_r}(x_j^1, x_l^1)$$

for all $y_{ji} \in \bar{F}$ *and* $y_{lk} \in F$*, where* $q_r = q_1 + \lambda_r = q_{\omega_1} + (\lambda + \lambda_r)$ *on* V_1 *and* $G^1_{q_r}$ *stands for the orthogonal Green function of* Γ_1 *on* V_1*. In particular,* $G_q(y_{j1}, y_{lk}) = G_q(y_{jm}, y_{lk}) = 0$ *for all* $j = 1, \ldots, n$*.*

Proof. Let us consider a vertex of the product network $y_{lk} \in F$. By Corollary 2.5.1, we know that the function $u = \mathcal{G}_q(\varepsilon_{y_{lk}}) \in \mathcal{C}(F)$ is the unique solution of the Green problem

$$\mathcal{L}_q(u) = \varepsilon_{y_{lk}} \text{ on } F \quad \text{and} \quad u = 0 \text{ on } \delta(F).$$

Using Lemma 4.1.4, we can consider a set of real values $0 < \lambda_2 \leq \ldots \leq \lambda_{m-1}$ and an orthonormal basis $\{\phi_r\}_{r=2,\ldots,m-1}$ of $\mathcal{C}(F_2)$ with respect to the inner product $\langle \cdot, \cdot \rangle_{F_2}$ such that $\mathcal{L}^2_{q_{\omega_2}}(\phi_r) = \lambda_r \phi_r$ on F_2 for all $r = 2, \ldots, m-1$. By Lemma 4.1.3, we can express both u and $\varepsilon_{y_{lk}}$ as

$$u = \sum_{r=2}^{m-1} u_r \otimes \phi_r \quad \text{and} \quad \varepsilon_{y_{lk}} = \sum_{r=2}^{m-1} e_r \otimes \phi_r,$$

where $u_r(x^1_j) = \langle u_{2,j}, \phi_r \rangle_{F_2}$ and $e_r(x^1_j) = \varepsilon_{x^1_l}(x^1_j)\phi_r(x^2_k)$. Hence Lemma 4.1.1 provides the following equality on F:

$$\begin{aligned}
\sum_{r=2}^{m-1} e_r \otimes \phi_r = \varepsilon_{y_{lk}} = \mathcal{L}_q(u) &= \mathcal{L}_q\left(\sum_{r=2}^{m-1} u_r \otimes \phi_r\right) \\
&= \sum_{r=2}^{m-1} \left[\mathcal{L}^1_{q_1}(u_r) \otimes \phi_r + u_r \otimes \mathcal{L}_q(\phi_r)\right] \\
&= \sum_{r=2}^{m-1} \left[\mathcal{L}^1_{q_1}(u_r) \otimes \phi_r + \lambda_r u_r \otimes \phi_r\right] = \sum_{r=2}^{m-1} \mathcal{L}^1_{q_r}(u_r) \otimes \phi_r.
\end{aligned}$$

Using Lemma 4.1.3 again, $\mathcal{L}^1_{q_r}(u_r) = e_r$ for all $r = 2, \ldots, m-1$. Therefore, $u_r(x^1_j) = \mathcal{G}^1_{q_r}(e_r)(x^1_j) = \phi_r(x^2_k) G^1_{q_r}(x^1_j, x^2_k)$ for all $j = 1, \ldots, n$ and finally

$$\begin{aligned}
G_q(y_{ji}, y_{lk}) = u(y_{ji}) &= \sum_{r=2}^{m-1} (u_r \otimes \phi_r)(x_{ji}) \\
&= \sum_{r=2}^{m-1} \phi_r(x^2_k)\phi_r(x^2_i) G^1_{q_r}(x^1_j, x^2_k). \quad \square
\end{aligned}$$

In order to show a useful application of these results, let us consider a particular case of product network: the cilindrical network. Let $C_n = (V_1, c_1)$ be a cycle with n vertices with constant conductances c_1 and let $P_m = (V_2, c_2)$ be a path with m vertices. The *cilindrical network* is the product network $C_n \,\square\, P_m$. For example, the product network in Figure 4.1 is a cilindrical network. Consider $\omega_1 \in \Omega(V_1)$ and $\omega_2 \in \Omega(V_2)$ two constant weights and the real value $\lambda > 0$. By [19, Proposition 3.7], the eigenvalues of $\mathcal{L}^2_{q_{\omega_2}}$ on F_2 are

$$\lambda_r = 2\left(1 - \cos\left(\frac{\pi(r-1)}{m-1}\right)\right)$$

for all $r = 2, \dots, m-1$. Their corresponding orthonormal basis of eigenfunctions $\{\phi_r\}_{r=2,\dots,m-1}$ on $\mathcal{C}(F_2)$ with respect to the inner product $\langle \cdot, \cdot \rangle_{F_2}$ is given by

$$\phi_r = \frac{\widetilde{\phi}_r}{\sqrt{\langle \widetilde{\phi}_r, \widetilde{\phi}_r \rangle_{F_2}}} \quad \text{on } F_2,$$

where $\widetilde{\phi}_r(x_i^2) = U_{i-2}\left(\cos\left(\frac{\pi(r-1)}{m-1}\right)\right)$ for all $i = 1, \dots, m$ and $\{U_i\}_{i=-\infty}^{+\infty}$ is the sequence of the *Second kind of Chebyshev polynomials*. On the other hand, by [19, Proposition 3.12] we know that the orthogonal Green functions $G^1_{q_r}$ of the cycle C_n on V_1 can be expressed as

$$G^1_{q_r}(x_j^1, x_l^1) = \frac{U_{n-1-|j-l|}\left(1 + \frac{\lambda+\lambda_r}{2c_1}\right) + U_{|j-l|-1}\left(1 + \frac{\lambda+\lambda_r}{2c_1}\right)}{2c_1\left[T_n\left(1 + \frac{\lambda+\lambda_r}{2c_1}\right) - 1\right]}$$

for all $r = 2, \dots, m-1$ and $x_j^1, x_l^1 \in V_1$, where $\{T_n\}_{n=-\infty}^{+\infty}$ is the sequence of the *First kind of Chebyshev polynomials*. Therefore, using Theorem 4.1.5 we can express the Green function of the cilindrical network $C_n \,\Box\, P_m$ as

$$G_q(y_{ji}, y_{lk}) = \sum_{r=2}^{m-1} \frac{U_{k-2}\left(\cos\left(\frac{\pi(r-1)}{m-1}\right)\right) U_{i-2}\left(\cos\left(\frac{\pi(r-1)}{m-1}\right)\right) P_{rjl}}{2c_1 \langle \widetilde{\phi}_r, \widetilde{\phi}_r \rangle_{\omega_2}\left[T_n\left(1 + \frac{\lambda+\lambda_r}{2c_1}\right) - 1\right]}$$

for all $y_{ji} \in \bar{F}$ and $y_{lk} \in F$, where $q = q_{\omega_1 \otimes \omega_2} + \lambda$ and

$$P_{rjl} = U_{n-1-|j-l|}\left(1 + \frac{\lambda+\lambda_r}{2c_1}\right) + U_{|j-l|-1}\left(1 + \frac{\lambda+\lambda_r}{2c_1}\right).$$

4.2 Green function of spider networks

Spider networks are a subfamily of circular planar networks and were first introduced in [37] by Curtis and Morrow. However, for these networks we adapt the notations in our interest. A *spider network with n radii and m circles* $\Gamma = (V, c)$ is a circular planar network with n boundary vertices given by $\delta(F) = \{v_1 < \ldots < v_n\}$ and labelled in the circular order provided by ∂D. The vertices in F are distributed in the following way. First, place a vertex x_{00} in the center of the boundary circle ∂D and draw a straight line from x_{00} to every boundary vertex v_j. This line is called the *radius* j, with $j = 1, \ldots, n$. Now draw m different concentric circumferences with center x_{00} and such that all of them are in $\overset{\circ}{D}$. We call each one of them *circle* i for $i = 1, \ldots, m$, where the circles are labelled from less to most diameter. Finally, place a vertex x_{ji} on each intersection of a circle i and a radius j. Then, $F = \{x_{ji}\}_{i=1,\ldots,m,\, j=1,\ldots,n} \cup \{x_{00}\}$. The edges are the ones given by the radius and the circles, see Figure 4.2 for more details. For the sake of simplicity,

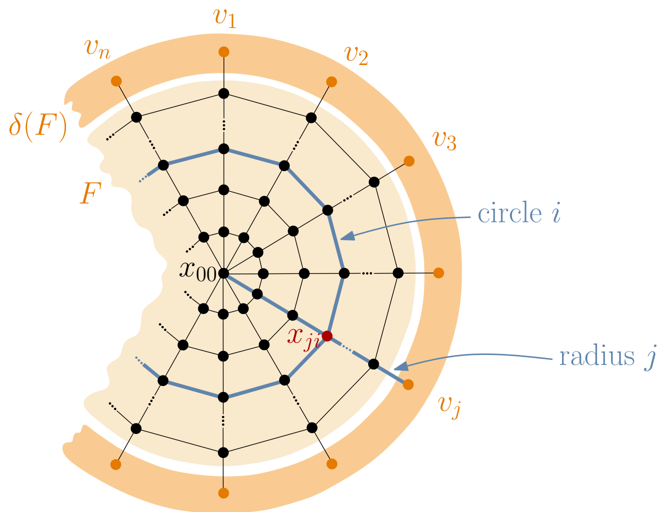


Figure 4.2 Structure of a spider network.

we define $x_{j0} = x_{00}$ and $x_{j\,m+1} = v_j$ for all $j = 1, \ldots, n$. We also use the notation $x_{ji} = x_{j-n\,i}$ for any $j > n$ and $i = 0, \ldots, m+1$. The ordering in F is given by the sequence $\{x_{11}, \ldots, x_{n1}, x_{12}, \ldots, x_{n2}, \ldots, x_{1m}, \ldots, x_{nm}, x_{00}\}$.

For we want to obtain the Green function of spider networks on F, we need to understand them as a modification of certain product networks. First, consider the product network of a cycle and a path $\Gamma_p = C_n \,\square\, P_{\widetilde{m}} = (V_p, c_p)$ with vertex set $V_p = \{y_{ji}\}_{i=1,\ldots,\widetilde{m},\, j=1,\ldots,n}$. We take the vertex subsets $F_p, H_1, H_{\widetilde{m}} \subset V_p$ as $F_p = \{y_{ji}\}_{i=2,\ldots,\widetilde{m}-1,\, j=1,\ldots,n}$, $H_1 = \{y_{j1}\}_{j=1,\ldots,n}$ and $H_{\widetilde{m}} = \{y_{j\widetilde{m}}\}_{j=1,\ldots,n}$. Notice that $\delta(F_p) = H_1 \cup H_{\widetilde{m}}$.

The next step is to transform the product network Γ_p into a spider–like network. Thus, let us add a new vertex $y_{01} \notin V_p$. We define a new network $\Gamma_a = (V_a, c_a)$ with vertex set $V_a = (V_p \setminus H_1) \cup \{y_{01}\}$ and vertex subset $F_a = F_p \cup \{y_{01}\} \subset V_a$. Then, $\delta(F_a) = H_{\widetilde{m}} \subset V_a$ and we can observe that there does not exist any vertex shaped as y_{j1} in V_a for $j = 1, \dots, n$. Hence, we can take the notation $y_{j1} = y_{01}$ in Γ_a for all $j = 1, \dots, n$ and define the conductances $c_a(y_{ji}, y_{lk}) = 0$ if $y_{ji}, y_{lk} \in \delta(F_p)$ and $c_a(y_{ji}, y_{lk}) = c_p(y_{ji}, y_{lk})$ otherwise, where $y_{ji}, y_{lk} \in \bar{F}_a$. In other words, the network Γ_a is the result of removing all the edges between boundary vertices of the product network Γ_p and identifying the vertices of $H_1 \subset \delta(F_p)$ of the product network into one unique vertex $y_{01} \in F_a$, which will not be a boundary vertex anymore. Figure 4.3 shows this transformation. Notice that Γ_a has exactly the same structure as a spider network.

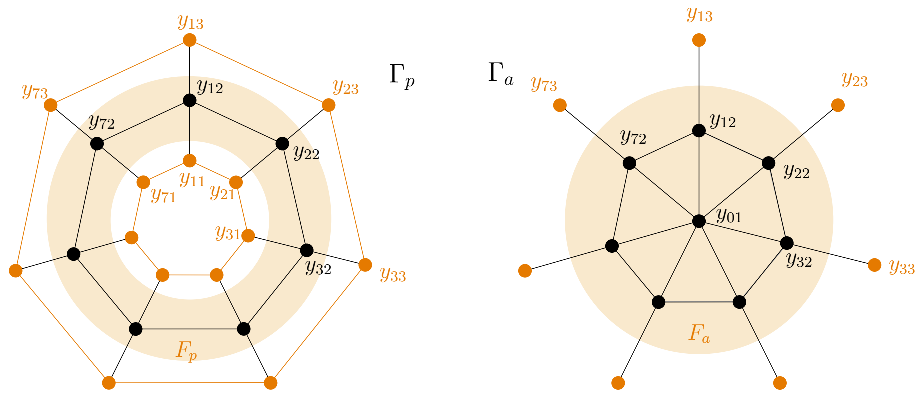


Figure 4.3 The transformation of a cilindrical network into a spider–like network.

Now we consider the spider network $\Gamma = (V, c)$ with n radii and m circles, where $m = \widetilde{m} - 2$. Its vertex set is given by $V = F \cup \delta(F)$, where $F = \{x_{ji}\}_{i=1,\dots,m,\, j=1,\dots,n} \cup \{x_{00}\}$ and $\delta(F) = \{v_j\}_{j=1,\dots,n}$. Clearly, there exists a correspondence between the vertex $x_{ji} \in \bar{F}$ of the spider network Γ and the vertex $y_{j\,i+1} \in \bar{F}_a$ of the network Γ_a. In consequence, if we take the conductances on the spider network given by $c(x_{ji}, x_{lk}) = c(y_{j\,i+1}, y_{l\,k+1})$ for all $x_{ji}, x_{lk} \in \bar{F}$, then the equivalence $\Gamma = \Gamma_a$ holds.

The next step is to express the spider network conductances, functions and operators in terms of the ones of the product network Γ_p. Obviously, $c(x_{ji}, x_{lk}) = 0$ if $x_{ji}, x_{lk} \in \delta(F)$ and $c(x_{ji}, x_{lk}) = c_p(y_{j\,i+1}, y_{l\,k+1})$ otherwise for all $x_{ji}, x_{lk} \in \bar{F}$. If $u \in \mathcal{C}(\bar{F})$ is a function on the vertices of the spider network, we define its *adaptation* $u_p \in \mathcal{C}(\bar{F}_p)$ to the product network as

$$u_p(y_{ji}) = u(x_{j\,i-1})$$

for all $y_{ji} \in \bar{F}_p$. Notice that u_p is constant on H_1, since $u_p(y_{j1}) = u_p(x_{00}) = u_p(y_{11})$ for all $j = 1, \dots, n$. If $\omega \in \Omega(\bar{F})$ is a weight on the spider network, its adaptation $\omega_p \in \mathcal{C}(\bar{F}_p)$ is not a weight on the product network anymore. We define the value $\sigma_p = \left(\int_{\bar{F}_p} \omega_p^2\right)^{\frac{1}{2}}$ and then the function $\widetilde{\omega}_p \in \mathcal{C}(\bar{F}_p)$ defined as $\widetilde{\omega}_p = \sigma_p^{-1}\omega_p$ on $\bar{F}_p$ is a weight on $\bar{F}_p$.

In the following, any superscript on a parameter or operator provides information with respect to which network it applies.

Lemma 4.2.1. *Given a function* $u \in \mathcal{C}(V)$*,*

$$\mathcal{L}_q(u)(x_{ji}) = \mathcal{L}^p_{q_p}(u_p)(y_{j\,i+1}) \quad \textit{for all} \quad x_{ji} \in F \smallsetminus \{x_{00}\}$$

and

$$\mathcal{L}_q(u)(x_{00}) = \alpha u_p(y_{11}) - \sum_{t=1}^{n} c_p(y_{t1}, y_{t2}) u_p(y_{t2}),$$

where the potentials are given by $q = q_\omega + \lambda$ *on* F *and* $q_p = q_{\widetilde{\omega}_p} + \lambda$ *on* F_p *and the real value* $\alpha > 0$ *is no other than*

$$\alpha = \lambda + \frac{1}{\widetilde{\omega}_p(y_{11})} \sum_{t=1}^{n} c_p(y_{t1}, y_{t2}) \widetilde{\omega}_p(y_{t2}).$$

Proof. It suffices to observe that $\mathcal{L}(u)(x_{ji}) = \mathcal{L}^p(u_p)(y_{j\,i+1})$ if $x_{ji} \in F \smallsetminus \{x_{00}\}$, whereas $\mathcal{L}(u)(x_{00}) = u_p(y_{11}) - \sum_{t=1}^{n} c_p(y_{t1}, y_{t2}) u_p(y_{t2})$. In consequence, $q_\omega(x_{ji}) = q_{\widetilde{\omega}_p}(y_{j\,i+1})$ for all $x_{ji} \in F \smallsetminus \{x_{00}\}$ and $q_\omega(x_{00}) = \alpha - \lambda - 1$. The result follows. $\square$

Now we are ready to determine the Green function on F of spider networks. It is worth to mention that the following result expresses it in terms of a Green function $G^p_{q_p}$ on F_p that has already been obtained in Section 4.1: the one of the product network. For the sake of readability, let us detail its value at the end of the next result.

Theorem 4.2.2. *Let* $\lambda > 0$ *be a real value. The Green function on* F *of the spider network with potential* $q = q_\omega + \lambda$ *can be written in the following ways, depending on the nature of the vertices.*

1. *If* $x_{lk} \in F \setminus \{x_{00}\}$, *then*

$$G_q(x_{00}, x_{lk}) = \frac{\sum\limits_{t=1}^{n} G^p_{q_p}(y_{t2}, y_{l\,k+1})}{\alpha - \sum\limits_{t=1}^{n}\sum\limits_{r=1}^{n} c_p(y_{r1}, y_{r2}) G^p_{q_p}(y_{t2}, y_{r2})},$$

whereas

$$\begin{aligned} G_q(x_{ji}, x_{lk}) &= G^p_{q_p}(y_{j\,i+1}, y_{l\,k+1}) \\ &\quad + G_q(x_{00}, x_{lk}) \sum_{t=1}^{n} c_p(y_{t1}, y_{t2}) G^p_{q_p}(y_{j\,i+1}, y_{t2}) \end{aligned}$$

for all $x_{ji} \in F \setminus \{x_{00}\}$.

2. *If* $x_{lk} = x_{00} \in F$, *then*

$$G_q(x_{00}, x_{00}) = \frac{1}{\alpha - \sum\limits_{t=1}^{n}\sum\limits_{r=1}^{n} c_p(y_{r1}, y_{r2}) G^p_{q_p}(y_{t2}, y_{r2})}$$

whereas

$$G_q(x_{ji}, x_{00}) = G_q(x_{00}, x_{00}) \sum_{t=1}^{n} c_p(y_{t1}, y_{t2}) G^p_{q_p}(y_{j\,i+1}, y_{t2})$$

for all $x_{ji} \in F \setminus \{x_{00}\}$.

The values of the function $G^p_{q_p}$ *are given by*

$$G^p_{q_p}(y_{t2}, y_{de}) = \sum_{r=2}^{m+1} \frac{U_{e-2}\left(\cos\left(\frac{\pi(r-1)}{m+1}\right)\right)}{2c_1 \langle \widetilde{\phi}_r, \widetilde{\phi}_r \rangle_{F_2} \left[T_n\left(1 + \frac{\lambda+\lambda_r}{2c_1}\right) - 1\right]}$$

for all $y_{de} \in F_p$, *where and*

$$P_{rtd} = U_{n-1-|t-d|}\left(1 + \frac{\lambda + \lambda_r}{2c_1}\right) + U_{|t-d|-1}\left(1 + \frac{\lambda + \lambda_r}{2c_1}\right).$$

Proof. Let us consider a vertex of the spider network $x_{lk} \in F$. By Corollary 2.5.1 we know that the function $u = \mathcal{G}_q(\varepsilon_{x_{lk}}) \in \mathcal{C}(F)$ is the unique solution of the Green problem

$$\mathcal{L}_q(u) = \varepsilon_{x_{lk}} \text{ on } F \quad \text{and} \quad u = 0 \text{ on } \delta(F).$$

We define the function $v = \varepsilon_{x_{lk}} \in \mathcal{C}(F)$ and consider its adaptation to the product network $v_p \in \mathcal{C}(F \cup H_1)$. Observe that $v_p = \varepsilon_{y_{l\,k}} \in \mathcal{C}(F_p)$ if $x_{lk} \in F \setminus \{x_{00}\}$ and $v_p = \chi_{H_1} \in \mathcal{C}(H_1)$ if $x_{lk} = x_{00}$. By Lemma 4.2.1,

$$\mathcal{L}_{q_p}^p(u_p) = v_p \ \text{ on } \ F_p, \qquad u_p = 0 \ \text{ on } \ H_{\widetilde{m}}$$

and

$$\alpha u_p(y_{11}) - \sum_{t=1}^{n} c_p(y_{t1}, y_{t2}) u_p(y_{t2}) = v_p \ \text{ on } \ H_1.$$

This problem is expressed in an equivalent way as

$$\mathcal{L}_{q_p}^p(u_p) = v_p \ \text{ on } \ F_p \quad \text{and} \quad u_p = \beta\chi_{H_1} \ \text{ on } \ \delta(F_p)$$

with the additional condition

$$\beta = \frac{1}{\alpha}\left(v_p(y_{11}) + \sum_{t=1}^{n} c_p(y_{t1}, y_{t2}) u_p(y_{t2}). \right).$$

By Corollary 2.5.1 again, the unique solution of the last problem is $u_p = \mathcal{G}_{q_p}^p(v_p) + \mathcal{P}_{q_p}^p(\beta\chi_{H_1})$ on $\bar{F}_p$. Since $\mathcal{P}_{q_p}^p(\beta\chi_{H_1})$ can be written as

$$\begin{aligned}\mathcal{P}_{q_p}^p(\beta\chi_{H_1})(y) &= \beta \sum_{t=1}^{n} P_{q_p}^p(y, y_{t1}) = \beta \sum_{t=1}^{n} \left(\varepsilon_{y_{t1}}(y) - \frac{\partial G_{q_p}^p}{\partial \mathsf{n}_{y_{t1}}}(y, y_{t1}) \right) \\ &= \beta\chi_{H_1}(y) + \beta \sum_{t=1}^{n} c_p(y_{t1}, y_{t2}) G_{q_p}^p(y, y_{t2})\end{aligned}$$

for all $y \in \bar{F}_p$, then

$$u_p = \mathcal{G}_{q_p}^p(v_p) + \beta\chi_{H_1} + \beta \sum_{t=1}^{n} c_p(y_{t1}, y_{t2}) \mathcal{G}_{q_p}^p(\varepsilon_{y_{t2}}) \ \text{ on } \bar{F}_p. \tag{4.2}$$

Remember that there exists an additional condition on u_p given by the following equation

$$\beta\alpha = v_p(y_{11}) + \sum_{t=1}^{n} c_p(y_{t1}, y_{t2}) u_p(y_{t2}). \tag{4.3}$$

Using Equations (4.2) and (4.3), we get that

$$\beta = \frac{v_p(y_{11}) + \sum\limits_{t=1}^{n} \sum\limits_{y \in F_p} G_{q_p}^p(y_{t2}, y) v_p(y)}{\alpha - \sum\limits_{t=1}^{n} \sum\limits_{r=1}^{n} c_p(y_{r1}, y_{r2}) G_{q_p}^p(y_{t2}, y_{r2})}.$$

Finally, we consider different situations depending on the nature of the vertex $x_{lk} \in F$. If $x_{lk} \in F \setminus \{x_{00}\}$ then $v_p = \varepsilon_{y_{l\,k+1}}$ on $\bar{F}_p$ and therefore $v_p(y_{11}) = 0$. In consequence,

$$G_q(x_{00}, x_{lk}) = u(x_{00}) = u_p(y_{11}) = \beta = \frac{\sum\limits_{t=1}^{n} G^p_{q_p}(y_{t2}, y_{l\,k+1})}{\alpha - \sum\limits_{t=1}^{n}\sum\limits_{r=1}^{n} c_p(y_{r1}, y_{r2}) G^p_{q_p}(y_{t2}, y_{r2})},$$

whereas

$$\begin{aligned}G_q(x_{ji}, x_{lk}) &= u(x_{ji}) = u_p(y_{j\,i+1}) = \mathcal{G}^p_{q_p}(v_p)(y_{j\,i+1}) + \beta\chi_{H_1}(y_{j\,i+1})\\ &\quad + \beta \sum_{t=1}^{n} c_p(y_{t1}, y_{t2}) G^p_{q_p}(y_{j\,i+1}, y_{t2})\\ &= G^p_{q_p}(y_{j\,i+1}, y_{l\,k+1}) + G_q(x_{00}, x_{lk}) \sum_{t=1}^{n} c_p(y_{t1}, y_{t2}) G^p_{q_p}(y_{j\,i+1}, y_{t2})\end{aligned}$$

for all $x_{ji} \in F \setminus \{x_{00}\}$. On the other hand, if $x_{lk} = x_{00} \in F$ then $v_p = \chi_{H_1}$ on $\bar{F}_p$. Therefore, $v_p(y_{11}) = 1$ and then

$$G_q(x_{00}, x_{00}) = u(x_{00}) = u_p(y_{11}) = \beta = \frac{1}{\alpha - \sum\limits_{t=1}^{n}\sum\limits_{r=1}^{n} c_p(y_{r1}, y_{r2}) G^p_{q_p}(y_{t2}, y_{r2})}$$

whereas

$$\begin{aligned}G_q(x_{ji}, x_{00}) &= u(x_{ji}) = u_p(y_{j\,i+1}) = \mathcal{G}^p_{q_p}(v_p)(y_{j\,i+1}) + \beta\chi_{H_1}(y_{j\,i+1})\\ &\quad + \beta \sum_{t=1}^{n} c_p(y_{t1}, y_{t2}) G^p_{q_p}(y_{j\,i+1}, y_{t2})\\ &= G_q(x_{00}, x_{00}) \sum_{t=1}^{n} c_p(y_{t1}, y_{t2}) G^p_{q_p}(y_{j\,i+1}, y_{t2})\end{aligned}$$

for all $x_{ji} \in F \setminus \{x_{00}\}$. □

To show an example, let us consider a particular case of spider network: the spider network on $n = 7$ radii and $m = 1$ circles given in Figure 4.3. This network can be obtained from the product network on the same figure using the results of this section. Consider the weight $\omega \in \Omega(V)$ given by $\omega = 1/6$ on $\delta(F) \cup \{x_{00}\}$ and $\omega = 1/3$ on $F \setminus \{x_{00}\}$, as well as the value $\lambda = 1$.

Then, $\widetilde{\omega}_p = 1/\sqrt{42}$ on $\delta(F_p)$ and $\widetilde{\omega}_p = 2/\sqrt{42}$ on F_p is a weight on the product network. We assume the Green operator $\mathcal{G}^p_{q_p}$ to be known:

$$G^p_{q_p}(F_p;F_p) = \frac{1}{1131}\begin{pmatrix} 197 & 62 & 20 & 8 & 8 & 20 & 62 \\ 62 & 197 & 62 & 20 & 8 & 8 & 20 \\ 20 & 62 & 197 & 62 & 20 & 8 & 8 \\ 8 & 20 & 62 & 197 & 62 & 20 & 8 \\ 8 & 8 & 20 & 62 & 197 & 62 & 20 \\ 20 & 8 & 8 & 20 & 62 & 197 & 62 \\ 62 & 20 & 8 & 8 & 20 & 62 & 197 \end{pmatrix},$$

where the ordering in F_p is given by $F_p = \{y_{12}, \dots, y_{72}\}$. Also, the conductances of the vertices of the product network are known: $c_p(y_{ji}, y_{j\,i+1}) = 1$ and $c_p(y_{ji}, y_{j+1\,i}) = 2$. Then, we can easily compute the values of the Green operator $\mathcal{G}_q$ of the spider network for the potential $q = q_\omega + \lambda$ using the formulae given in Theorem 4.2.2: $\alpha = 15$, $\beta = 3/38$ and therefore

$$G_q(F;F) = \frac{1}{14326}\begin{pmatrix} 2621 & 911 & 379 & 227 & 227 & 379 & 911 & 377 \\ 911 & 2621 & 911 & 379 & 227 & 227 & 379 & 377 \\ 379 & 911 & 2621 & 911 & 379 & 227 & 227 & 377 \\ 227 & 379 & 911 & 2621 & 911 & 379 & 227 & 377 \\ 227 & 227 & 379 & 911 & 2621 & 911 & 379 & 377 \\ 379 & 227 & 227 & 379 & 911 & 2621 & 911 & 377 \\ 911 & 379 & 227 & 227 & 379 & 911 & 2621 & 377 \\ 377 & 377 & 377 & 377 & 377 & 377 & 377 & 1131 \end{pmatrix},$$

where the ordering in F is given by $F = \{x_{11}, \dots, x_{71}, x_{00}\}$.

5

Discrete Serrin's problem

In 1971, J. Serrin stated the following overdetermined problem in the continuum field. If $\Omega \subset \mathbb{R}^n$ is a connected open bounded domain with smooth boundary $\delta(\Omega)$ and u is a smooth function on Ω such that

$$\begin{cases} -\Delta(u) = 1 & \text{on } \Omega \\ \quad\;\; u = 0 & \text{on } \delta(\Omega), \end{cases}$$

then the normal derivative $\dfrac{\partial u}{\partial \mathsf{n}}$ is constant on $\delta(\Omega)$ if and only if Ω is a ball on $\mathbb{R}^n$, see [54]. Furthermore, the unique solution u of this problem is radial. Now, this problem is known in the literature as *Serrin's problem*.

The main tools used in [54] in order to solve the problem were the moving planes arrangement, also known as Alexandroff–Serrin method, and a refinement of the maximum principle. Weinberger gave in [55] an alternative proof using elementary arguments. Mainly, he described the laplacian in polar coordinates and applied the minimum principle and the Green identity. During the last decade many generalizations of Serrin's problem have been done. Namely, for the case when the laplacian is replaced by a quasilinear or nonlinear elliptic operator, the case when the elliptic problem is stated on an exterior domain, or the case when the overdetermined boundary condition takes place only in a part of the boundary, see [2, 27, 44, 45, 47] and the references therein.

In this chapter, our aim is to solve this problem in the discrete field. Specifically, we consider a connected network $\Gamma = (V, c)$ and a proper connected subset $F \subset V$. We call $\nu^F \in \mathcal{C}^+(F)$ the unique function such that

$$\mathcal{L}(\nu^F) = 1 \;\text{ on }\; F \quad \text{and} \quad \nu^F = 0 \;\text{ on }\; \delta(F),$$

see Proposition 2.4.2 and Corollary 2.5.1. The function ν^F is called *equilibrium measure* on F, see [16] for an extended study of its properties. Notice that we are working with the combinatorial laplacian $\mathcal{L}$, which is a particular case of the Schrödinger operator for a constant weight and the real value $\lambda = 0$, since the original problem is posed in this setting.

Therefore, the *discrete Serrin's problem* (or *DSP* for short) consists in the characterization of those networks with boundary such that the normal derivative of ν^F is constant on $\delta(F)$. We formulate the question in terms of the structure of the network and the properties of the solution of the problem.

We would like to remark that the discrete Serrin's problem is the extreme case of the overdetermined partial Dirichlet–Neumann boundary value problem, which we will define and study in Section 7.1.

The reader can find the publication of the works in this chapter in [10]. They have been presented in congress as well, see [6].

5.1 Set properties and minimum principle

First, we detail some useful set identities. Observe that $\delta(F) \cap \delta(\overset{\circ}{F}) = \emptyset$ if $F \neq \emptyset$. This property is discordant with the topological situation. Similarly, $\bar{F} = F$ and $\overset{\circ}{F} = F$ are conditions that hold exclusively in the extremal case $F = V$. Nevertheless, some properties resembling to the ones of the topological case are fulfilled when the situation is not extremal.

Lemma 5.1.1. *Let $F \subset V$ be a proper subset. Then, $\delta(\overset{\circ}{F}) \subseteq \delta(F^c)$ and the following equalities remain true:*

$$F \cap \delta(F) = \delta(F^c) \cap \overset{\circ}{F} = \emptyset, \qquad F = \delta(F^c) \cup \overset{\circ}{F},$$

$$\mathsf{Ext}(F) = \left(\bar{F}\right)^c = (\overset{\circ}{F^c}) \qquad \textit{and} \qquad (\overset{\circ}{F})^c = (\bar{F^c}).$$

Proof. Clearly, $\delta(\overset{\circ}{F}) \subseteq F$. Let $x \in \delta(\overset{\circ}{F})$. Then, $x \notin \overset{\circ}{F} \cup F^c$ and there exists a vertex $z \notin F$ such that $x \sim z$. That is, $z \in F^c$ whereas $x \notin F^c$ and $x \sim z$, which means that $x \in \delta(F^c)$.

By definition, the sets F and $\delta(F)$ share no elements. If we take into account that $\delta(F^c) = \{x \in F : \exists y \in F^c \text{ with } x \sim y\}$, then we see that $F = \overset{\circ}{F} \cup \delta(F^c)$

with $\overset{\circ}{F} \cap \delta(F^c) = \emptyset$. On the other hand,

$$\big(\bar{F}\big)^c = \big\{x \in V \,:\, x \notin \bar{F}\big\} = \big\{x \in V \,:\, d(x, \bar{F}) \geq 1\big\}$$
$$= \{x \in V \,:\, d(x, F) \geq 2\} = \mathsf{Ext}(F).$$

Finally, from equality $V = F^c \cup F = F^c \cup \overset{\circ}{F} \cup \delta(F^c)$, which is a disjoint union, we get that $(\overset{\circ}{F})^c = V \setminus \overset{\circ}{F} = F^c \cup \delta(F^c)$. □

Now we suggest the following subdivision of V into layers with respect to a set $F \subseteq V$. For each $i = 0, \ldots, r(F)$, we designate the sets $B_i(F) = \{x \in V \,:\, d(x, F) \leq i\}$ as the *i–th ball*, $B^i(F) = \{x \in V \,:\, d(x, F) \geq i\}$ as the *i–th crown* and $S_i(F) = \{x \in V \,:\, d(x, F) = i\}$ as the *i–th sphere* of F. Undoubtedly, $B_0(F) = S_0(F) = F$, $B^0(F) = V$, $B_1(F) = \bar{F}$, $B^1(F) = F^c$, $S_1(F) = \delta(F)$ and $B_{r(F)}(F) = V$, whereas $B^i(F) = (B_{i-1}(F))^c$ for all $i = 1, \ldots, r(F)$.

Lemma 5.1.2. *Let $F \subset V$ be a proper subset. Then, for every index $i = 0, \ldots, r(F) - 1$, it is satisfied that*

$$\delta(B_i(F)) = S_{i+1}(F), \quad \bar{B}_i(F) = B_{i+1}(F) \quad \textit{and} \quad B_i(F) \subseteq \overset{\circ}{B}_{i+1}(F).$$

In particular, $(\delta(B_{i+1}(F))^c) \subseteq S_{i+1}(F)$.

Proof. Let $i \in \{0, \ldots, r(F) - 1\}$ and let $x \in \delta(B_i(F))$. Then, $d(x, F) > i$ and there exists a vertex $y \in B_i(F)$ such that $d(x, y) = 1$. Ergo, $i + 1 \leq d(x, F) \leq d(x, y) + d(y, F) \leq i + 1$, which means that $d(x, F) = i + 1$. Conversely, let $x \in S_{i+1}(F)$. Then, there exists a path of length i given by $\{x_1 \sim \ldots \sim x_i \sim x\} \subseteq V$ with $x_1 \in F$ and $x_j \in B_j(F)$ for all $j = 1, \ldots, i$. As a result, $x \in \delta(B_i(F))$. On the other hand, it is straightforward that $B_{i+1}(F) = B_i(F) \cup S_{i+1}(F)$ and in consequence $\bar{B}_i(F) = B_i(F) \cup \delta(B_i(F)) = B_i(F) \cup S_{i+1}(F) = B_{i+1}(F)$. Let $x \in B_i(F) \subset B_{i+1}(F)$ and consider $y \in V$ such that $d(y, x) = 1$. Along these lines, $d(y, F) \leq d(x, y) + d(x, F) \leq i + 1$ and hence $y \in B_{i+1}(F)$. Finally, let us mention that

$$\delta\left((B_{i+1}(F))^c\right) = B_{i+1}(F) \setminus \overset{\circ}{B}_{i+1}(F) \subseteq B_{i+1}(F) \setminus B_i(F) = S_{i+1}(F)$$

and so the last claim holds. □

Lemma 5.1.3. *Let $F \subset V$ be a proper subset. For any $i = 1, \ldots, r(F)$ the following properties hold:*

$$\delta((B^i(F))^c) = S_i(F), \quad \overset{\circ}{B^i}(F) = B^{i+1}(F) \quad \textit{and} \quad \bar{B}^{i+1}(F) \subseteq B^i(F).$$

In particular, $\delta(B^i(F)) \subseteq S_{i-1}(F)$.

Proof. It is analogue to the proof of Lemma 5.1.2. □

Regarding the minimum principle for the laplacian operator, now we can state a more detailed property: the values of the superharmonic functions on F increase with the distance from $\delta(F)$.

Theorem 5.1.4 (General minimum principle). *Let* $u \in \mathcal{C}(\bar{F})$ *be a superharmonic function on* F*. Then,*

$$\min_{x\in\delta(F)}\{u(x)\} \le \min_{x\in S_i(F^c)}\{u(x)\} \le \min_{x\in S_{i+1}(F^c)}\{u(x)\}$$

for any $i = 1,\dots,r(F^c)-1$*. Moreover, if the left inequality is an equality for some* i*, then* u *is constant on* $\bar{F}$*. On the other hand, if the right inequality is an equality for some* i*, then* u *is constant on* $\bar{B}^{i+1}(F^c)$*.*

Proof. Notice that for any $i = 1,\dots,r(F^c)-1$ it is satisfied that

$$\min_{x\in\delta(F)}\{u(x)\} \le \min_{x\in F}\{u(x)\} \le \min_{x\in B^i(F^c)}\{u(x)\},$$

since $B^i(F^c) \subset B^1(F^c) = F$ and we have applied Proposition 2.4.1 to obtain the first inequality. Now fix $i = 1,\dots,r(F^c)-1$. Then, from Lemma 5.1.3 we know that

$$B^i(F^c) = \overset{\circ}{B^i}(F^c) \cup \delta\big((B^i(F^c))^c\big) = B^{i+1}(F^c) \cup S_i(F^c).$$

Therefore, it suffices to prove that $\min_{x\in S_i(F^c)}\{u(x)\} \le \min_{x\in B^{i+1}(F^c)}\{u(x)\}$. If we consider the set $H = B^{i+1}(F^c)$ and define the function $v = u\chi_{\bar{H}} \in \mathcal{C}(\bar{H})$, then $\mathcal{L}(v) \ge 0$ on H. Keeping in mind that $\delta(H) \subseteq S_i(F^c)$, from Proposition 2.4.1, we obtain

$$\min_{x\in S_i(F^c)}\{u(x)\} \le \min_{x\in\delta(H)}\{u(x)\} \le \min_{x\in H}\{u(x)\} \le \min_{x\in S_{i+1}(F^c)}\{u(x)\}.$$

Finally, if $\min_{x\in S_i(F^c)}\{u(x)\} = \min_{x\in S_{i+1}(F^c)}\{u(x)\}$ then the equality

$$\min_{x\in\delta(H)}\{u(x)\} = \min_{x\in H}\{u(x)\}$$

also holds and hence u is constant on $\bar{H}$. □

From the above results we conclude that there exist strictly superharmonic functions on F that are null on $\delta(F)$ and strictly positive on F, see also [18, Corollary 4.3]. The next result shows that strictly superharmonic functions cannot have a local minimum on F.

Lemma 5.1.5. *If $u \in \mathcal{C}^+(F)$ is a strictly superharmonic function on F, then for any $x \in F$ there exists a vertex $y \in \bar{F}$ such that $c(x,y) > 0$ and $u(y) < u(x)$.*

Proof. Let $x \in F$. Suppose that for all $y \in \bar{F}$ such that $c(x,y) > 0$, $u(y) \geq u(x)$. Then,

$$0 < \mathcal{L}(u)(x) = \sum_{y \in V} c(x,y)\big(u(x) - u(y)\big) \leq 0,$$

which is a contradiction. □

5.2 Distance layers and level sets

We first study some vertex sets of a network that will be useful all allong this chapter. Remember that given a proper connected subset $F \subset V$, the parameter $r(F^c)$ is given by $r(F^c) = \max_{x \in F^c}\{d(x, F^c)\}$, see Section 2.1. For simplicity of notation, let $r = r(F^c)$ be the *radius* of F and we denote by $D_0 = \delta(F)$ or $D_i = S_i(F^c)$ *the layer with distance i to the boundary of* F for $i = 1, \ldots, r$. Observe that if $x \in D_i$ with $i \geq 1$, then its neighbours belong to $D_{i-1} \cup D_i \cup D_{i+1}$.

Consider now a strictly superharmonic function $u \in \mathcal{C}^+(F)$ on F, see Section 2.4. Let $s+1$ be the number of different values $u_0, \ldots, u_s$ that u takes on $\bar{F}$. We assume these values to be ordered as $0 = u_0 < u_1 < \ldots < u_s$ and define *the i–th level set of* u, denoted by $U_i = \{\, x \in \bar{F} \mid u(x) = u_i \,\}$ for $i = 0, \ldots, s$. Observe that $U_0 = \delta(F)$ because u is strictly positive on F and that $U_i \cap U_j = \emptyset$ for $i \neq j$.

Proposition 5.2.1. *Let $u \in \mathcal{C}^+(F)$ be a strictly superharmonic function on F. Then, $U_0 = D_0$ and $U_i \subset \bigcup_{j=1}^{i} D_i$ for any $i = 1, \ldots, s$.*

Proof. Let $i \in \{0, \ldots, s\}$. It is enough to prove that if $x \in U_i$ then $d(x, \delta(F)) \leq i$. We prove this by mathematical induction. First, we see that it is true for the case $i = 0$ because of Proposition 2.4.1. Now we assume that for every $j < i$, if $x \in U_j$ then $d(x, \delta(F)) \leq j$. We want to see what happens for $x \in U_i$. By Lemma 5.1.5, there exists a vertex $y \in \bar{F}$ such that $c(x,y) > 0$ and $y \in U_j$ with $j \leq i - 1$. Hence, the result follows by induction hypothesis. □

Notice that $U_1 \subseteq D_1$ is always satisfied. However, it is not true in general that $U_i \subseteq D_i$ for $i \geq 2$, as the example on Figure 5.1 shows. Despite that,

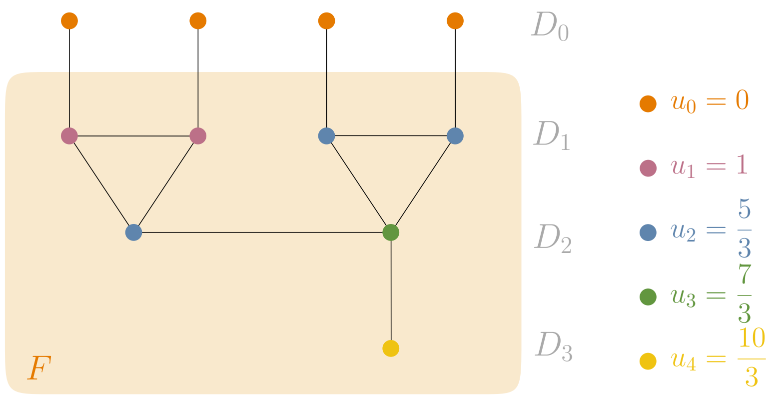


Figure 5.1 A graph Γ with u strictly superharmonic such that $U_2 \not\subseteq D_2$.

if the subsets U_i and D_i are the same until a certain layer, then the above inclusion is satisfied for the next layer, as the following result shows.

Corollary 5.2.2. *Let $u \in \mathcal{C}^+(F)$ be a strictly superharmonic function on F. If $U_j = D_j$ for all $j = 0, \dots, i$, then $U_{i+1} \subseteq D_{i+1}$.*

This behaviour inspires the following definition. A strictly superharmonic function $u \in C^+(F)$ on F is called *radial* if $U_i = D_i$ for all $i = 0, \dots, s$. In this case, $s = r$. In addition, for all $i = 1, \dots, s$ and $x \in D_i$ it is satisfied that

$$\mathcal{L}(u)(x) = k_{i+1}(x)\big(u_i - u_{i+1}\big) + k_{i-1}(x)\big(u_i - u_{i-1}\big) > 0, \qquad (5.1)$$

where $k_j(x) = \int_{D_j} c(x, y)\, dy$ is defined as the *degree with respect to the layer* D_j of x for all $j = 0, \dots, s$. Moreover, for all $x \in \delta(F) = D_0$,

$$\frac{\partial u}{\partial \mathsf{n}_F}(x) = -k_1(x) u_1 < 0.$$

5.3 The discrete version of Serrin's problem

Now we study to what extent the discrete and the continuous version of Serrin's problem are analogous, as well as the differences between both cases. We focus in the difficulties that appear in the discrete setting.

The *discrete Serrin's problem* consists in the characterization of those networks with boundary such that $\dfrac{\partial \nu^F}{\partial \mathsf{n}_F}$ is constant on $\delta(F)$. This boundary condition is known as the *Serrin's condition*. Notice that if the equilibrium measure ν^F satisfies the Serrin's condition then

$$\frac{\partial \nu^F}{\partial \mathsf{n}_F} = -\frac{|F|}{|\delta(F)|}$$

on $\delta(F)$ because of the Gauss Theorem. Hence, if a network satisfies the Serrin's condition then the value of the constant only depends on the ratio between the number of vertices of F and the number of vertices of its boundary $\delta(F)$. This property is exactly the same as in the continuous case, where the constant is the ratio between the volume of Ω and the area of its boundary.

It is important to observe that the equilibrium measure ν^F does not depend on the arrangement of the boundary–to–boundary edges but on the total conductance flowing from a boundary vertex. However, its normal derivative on the boundary is indeed affected by the structure of the boundary edges and therefore the Serrin's condition depends on its composition. This fact creates a lack of unicity when determining the structure of a network that fits the Serrin's problem premises in the discrete setting, as we can see in the examples of Figure 5.2. In this figure, both graphs Γ_1 and Γ_2 have

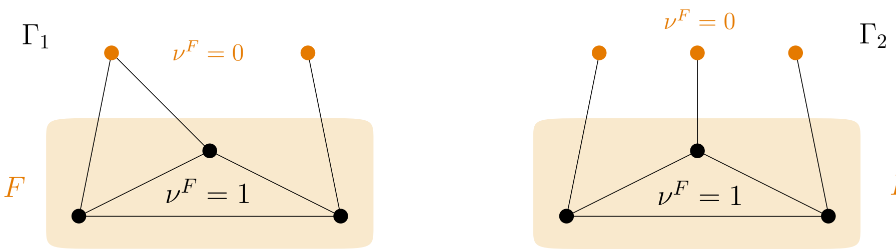


Figure 5.2 The equilibrium measure is not affected by the boundary edges.

the same equilibrium measure, which is given by $\nu^F = 1$ on F. However, Γ_2 satisfies the Serrin's condition but Γ_1 does not. In order to avoid this kind of ambiguities, from now on in this section we assume that given a connected network $\Gamma = (V, c)$ with a proper connected subset $F \subset V$, it has separated boundary $\delta(F)$. Then, every boundary vertex has a unique adjacent vertex, which clearly is in D_1. This choice is in correspondence with the continuous concept of normal derivative on the boundary, since it is a directional derivative.

For any $x \in \delta(F)$ we arbitrarily denote its unique neighbour in $\bar{F}$ as $\hat{x} \in D_1$. Notice that $x\hat{x}$ is a boundary spike by definition and that given two different vertices $x, y \in \delta(F)$ it can occur that $\hat{x} = \hat{y}$. We also assume that $|\delta(F)| \geq 2$.

Lemma 5.3.1. *Given a network with separated boundary, if $U_1 = D_1$ then the minimum non–null value of ν^F satisfies that $u_1 > k_0(\widehat{x})^{-1}$ for all $\widehat{x} \in D_1$.*

Proof. Notice that $1 = \mathcal{L}(\nu^F)(\widehat{x}) = u_1 k_0(\widehat{x}) + \sum_{y \in D_2} c(\widehat{x}, y)(u_1 - \nu^F(y))$ for all $\widehat{x} \in D_1$. Since $\nu^F(y) > u_1$ for all $y \in D_2$ by hypothesis, then $1 - u_1 k_0(\widehat{x}) < 0$ and the result holds for all $\widehat{x} \in D_1$. □

Proposition 5.3.2. *Any two of the following conditions imply the third one in a network with separated boundary.*

(i) ν^F *satisfies the Serrin's condition.*

(ii) $U_1 = D_1$.

(iii) $c(x, \hat{x})$ *is constant for any* $x \in \delta(F)$.

Proof. Notice that

$$\frac{\partial \nu^F}{\partial \mathsf{n}_F}(x) = -c(x, \hat{x})\nu^F(\hat{x})$$

for all $x \in \delta(F)$. If ν^F satisfies the Serrin's condition, then the product $-c(x, \hat{x})\nu^F(\hat{x})$ is constant and hence (ii) and (iii) are equivalent. On the other hand, (ii) and (iii) clearly imply (i). □

By the above result, if the Serrin's condition is satisfied then the boundary spike conductances are constant if and only if $D_1 = U_1$. That is, if and only if the solution is radial on the first layer. Figure 5.3 shows that in the discrete setting the Serrin's condition is not enough to guarantee that the equilibrium measure is a radial function. We remark that a similar situation takes place in the continuous case when the considered operator is non–linear, see [28].

5.4 Spider networks with radial conductances

In this section we study whether spider networks, defined in Section 4.2, satisfy Serrin's condition. In particular, we perform research on the discrete

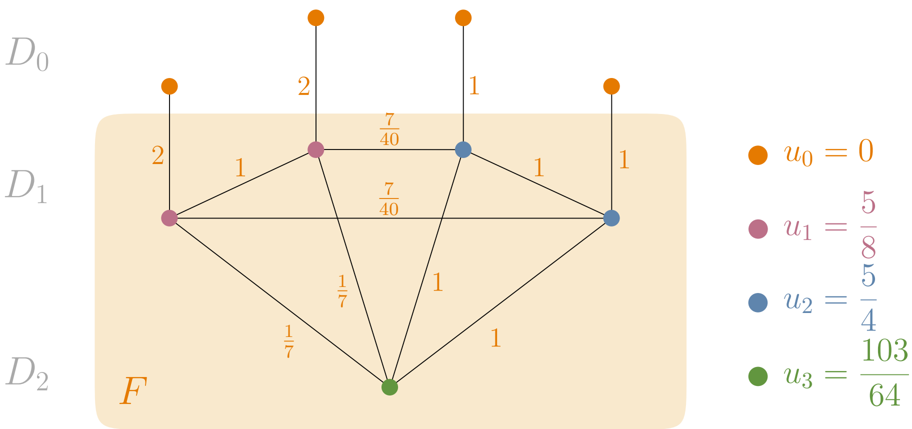


Figure 5.3 A network satisfying Serrin's condition with non radial equilibrium measure

Serrin's problem when the conductances of the spider network are radial. This subfamily of spider networks is important because they have a two–dimensional ball–like structure and hence they fit our interest.

Let $m, n \in \mathbb{N}$ and let $\{a_i\}_{i=0,\dots,m}$ be $m+1$ real positive values. Let $\Gamma = (V, c)$ be a spider network with n radii and m circles and we assume that the conductivity function is given by $c(x_{ji}, x_{ji-1}) = a_{m-i+1}$ for all $i = 1, \dots, m+1$ and $j = 1, \dots, n$ on the radial edges. We call these networks *spider networks with radial conductances.* Notice that there is no condition on the conductances of the circular edges $x_{ji}x_{j-1\,i}$.

Lemma 5.4.1. *The layer with distance i to the boundary $\delta(F)$ is given by $D_i = \{x_{j\,m-i+1}\}_{j=1,\dots,n}$ for all $i = 0, \dots, m+1$.*

Proposition 5.4.2. *The equilibrium measure ν^F of a spider network with radial conductances is radial. It is given by*

$$\nu^F = \frac{1}{n}\sum_{k=0}^{m-i} \frac{n(m-k)+1}{a_k} \qquad \text{on } D_{m-i+1}$$

for all $i = 0, \dots, m+1$. Moreover, it satisfies the Serrin's condition.

Proof. We know that the equilibrium measure ν^F exists and is unique. Let us assume that it is radial. Then, $U_i = D_i$ are the levels of the equilibrium measure for all $i = 0, \dots, m+1$ and we can define the function $q(i) = \nu^F(x_{ji})$ for any $j = 1, \dots, n$ and $i = 0, \dots, m+1$. Notice that $\nu^F = q(m-i+1)$ on D_i by Lemma 5.4.1. In particular, $q(m+1) = 0$.

The equilibrium measure is superharmonic on F because $\mathcal{L}(\nu^F) = 1$ on F. Since $F = \bigcup_{i=1}^{m+1} D_i$, from Equation (5.1) and Lemma 5.4.1 we obtain the following recurrence equation

$$1 = a_{m-i+1}\big(q(i) - q(i-1)\big) - a_{m-i}\big(q(i+1) - q(i)\big)$$

for all $i = 1, \dots, m$ and

$$1 = -na_m\big(q(1) - q(0)\big).$$

Let us define the function $\psi(i) = a_{m-i+1}\big(q(i) - q(i-1)\big)$ for all $i = 1, \dots, m+1$. Then, we have the system

$$1 = \psi(i) - \psi(i+1)$$

for all $i = 1, \dots, m$ with initial condition $\psi(1) = -\dfrac{1}{n}$. The above recurrence system has a unique solution, which is given by

$$\psi(i) = -\frac{n(i-1)+1}{n}$$

for all $i = 1, \dots, m+1$. Therefore, it determines the equilibrium measure ν^F because it is unique, and it is given by $q(i) - q(i-1) = -\dfrac{n(i-1)+1}{na_{m-i+1}}$ for all $i = 1, \dots, m+1$. We solve this recurrence keeping in mind that $q(m+1) = 0$. The solution is

$$q(i) = \frac{1}{n}\sum_{k=i}^{m} \frac{nk+1}{a_{m-k}} = \frac{1}{n}\sum_{k=0}^{m-i} \frac{n(m-k)+1}{a_k}$$

for all $s = 0, \dots, m+1$. Moreover, on $\delta(F)$ it is satisfied that

$$\frac{\partial \nu^F}{\partial \mathsf{n}_F} = -a_0 q(m) = -m - \frac{1}{n}. \quad \square$$

Corollary 5.4.3. *In particular, when all the radial conductances are the same real value $a > 0$, then*

$$\nu^F = \frac{(m-i+1)(nm+in+2)}{2na} \qquad \text{on } D_{m-i+1}$$

for all $i = 0, \dots, m+1$.

If we consider a spider network that not fulfills the radial conductances property, then the Serrin's condition may not be satisfied on this network. In order to show this, we consider the following example. Let $a_1, a_2, b_1, b_2 > 0$ be four real values. Consider the spider network on n radii and $m = 1$ circles with conductances $c(x_{j1}, v_j) = a_1$, $c(x_{00}, x_{j1}) = a_2$ for all $j = 1, \dots, n-1$, $c(x_{n1}, v_n) = b_1$ and $c(x_{00}, x_{n1}) = b_2$, see Figure 5.4. If the Serrin's con-

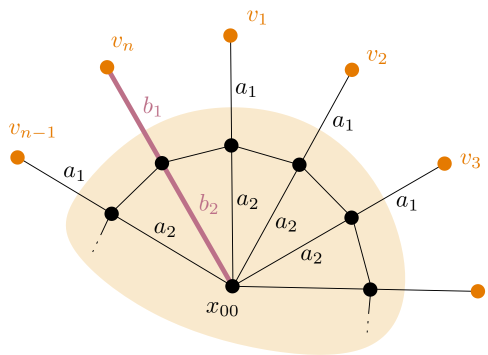


Figure 5.4 A spider network with $m = 1$ and almost radial conductances.

dition holds then $C = \dfrac{\partial \nu^F}{\partial \mathsf{n}_F}(x_{j1}) = -a_1 \nu^F(x_{j1})$ for all $j = 1, \dots, m-1$ and $\nu^F(x_{n1}) = -b_1^{-1} C$ proceeding in the same way. On the other hand, $\mathcal{L}(\nu^F)(x_{j1}) = 1$ for all $j = 2, \dots, n-2$ and we get that

$$\nu^F(x_{00}) = -\frac{1}{a_2} - \frac{(a_1 + a_2)}{a_1 a_2} C .$$

Also, $\mathcal{L}(\nu^F)(x_{11}) = 1$ and working analogously we obtain the equality

$$c(x_{n1}, x_{11}) \left(\frac{1}{a_1} - \frac{1}{b_1} \right) C = 0.$$

Since $c(x_{n1}, x_{11}) \neq 0$ and $C \neq 0$, necessarily $a_1 = b_1$. Doing the same for $\mathcal{L}(\nu^F)(x_{n1}) = 1$, we see that $a_2 = b_2$. Hence, this network satisfies the Serrin's condition if and only if $a_i = b_i$ for $i = 1, 2$. That is, it satisfies the Serrin's condition if and only if the conductances are radial. This example shows how important is to have, in addition to a ball–like physical structure, a ball–like conductivity behaviour.

5.5 Regular layered networks. Characterization

In this section we study the discrete Serrin's problem on another family of networks, termed regular layered networks. In 1973, Delsarte introduced the concept of *completely regular codes* as certain sets of vertices of a *distance–regular graph*, see [39, 49]. We adapt this concept considering the boundary set of vertices of any network as a completely regular code of the network itself: this readjustment is defined in the following. Notice that in this section we do not need to assume the networks to have separated boundary.

Let $\Gamma = (V, c)$ be a connected network with $F \subset V$ a proper connected subset and consider the value $m = r(F^c)$. Let $\{b_i\}_{i=0,\dots,m}$ and $\{c_i\}_{i=0,\dots,m}$ be two sequences of real positive numbers with $c_0 = b_m = 0$. We say that Γ is a *regular layered network with sequences* $\{b_i\}$ *and* $\{c_i\}$ if for any $x \in D_i$ it is satisfied that $k_{i-1}(x) = c_i$ and $k_{i+1}(x) = b_i$ for all $i = 1, \dots, m$. Remember that $k_j(x) = \int_{D_j} c(x, y)\, dy$ for all $x \in \bar{F}$. In Figure 5.5 we show the general representation of a regular layered graph. Observe that any spider network

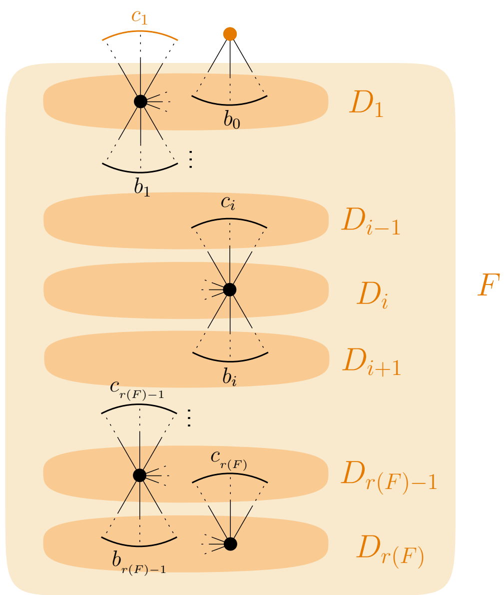


Figure 5.5 A regular layered graph.

with radial conductances is a regular layered network with $b_i = a_{i-1}$ and $c_i = a_i$. Futhermore, any distance–regular graph with vertex set V is also a regular layered network if we consider $F = V \setminus \{x\}$ for an arbitrary vertex $x \in V$.

Lemma 5.5.1. *If* Γ *is a regular layered network then*

$$\frac{|F|}{|\delta(F)|} = \sum_{i=1}^{m} \left(\prod_{\ell=0}^{i-1} \frac{b_\ell}{c_{\ell+1}} \right).$$

Proof. First, observe that $|F| = \sum_{i=1}^{m} |D_i|$. Moreover, for all $i = 0, \dots, m-1$, clearly

$$b_i|D_i| = \sum_{x\in D_i} \sum_{y \in D_{i+1}} c(x,y) = \sum_{y\in D_{i+1}} \sum_{x\in D_i} c(x,y) = c_{i+1}|D_{i+1}|.$$

The solution of this recurrence equation on the values $|D_i|$ is given by

$$|D_i| = |D_0| \prod_{\ell=0}^{i-1} \frac{b_\ell}{c_{\ell+1}}$$

for all $i = 0, \dots, m+1$ and since $|D_0| = |\delta(F)|$ the result follows. □

Proposition 5.5.2. *Let* Γ *be a regular layered network. Then, its equilibrium measure is given by*

$$\nu^F = \sum_{j=1}^{i} \frac{1}{b_{j-1}} \sum_{k=j}^{m} \left(\prod_{\ell=j-1}^{k-1} \frac{b_\ell}{c_{\ell+1}} \right) \qquad \text{on } D_i$$

for all $i = 0, \dots, m$. *Therefore, it is radial. In addition,* ν^F *satisfies the Serrin's condition.*

Proof. We know that the equilibrium measure $\nu^F(x)$ exists and is unique. Let us assume that it is radial. Then, $U_i = D_i$ are the levels of the equilibrium measure for all $i = 0, \dots, m$ and we can define the function $q(i) = \nu^F(x)$ for any $x \in D_i$. Notice that $q(0) = 0$.

On the other hand, $\nu^F(x)$ is superharmonic on F because $\mathcal{L}(\nu^F) = 1$ on F. Since $F = \bigcup_{i=1}^{m} D_i$, from Equation (5.1) we obtain the recurrence equation

$$1 = c_i\big(q(i) - q(i-1)\big) - b_i\big(q(i+1) - q(i)\big)$$

for all $i = 1, \ldots, m$. If the above recurrence system has solution, then it determines the equilibrium measure because it is unique. Let us designate the values

$$\rho_i = \prod_{j=0}^{i-1} \frac{b_j}{c_{j+1}}$$

for all $i = 1, \ldots, m$. Multiplying the above equation by ρ_i we get

$$\rho_s = \rho_{i-1} b_{i-1}\big(q(i) - q(i-1)\big) - \rho_i b_i \big(q(i+1) - q(i)\big)$$

for all $i = 1, \ldots, m$. If we consider the function $\psi(i) = \rho_{i-1} b_{i-1}\big(q(i) - q(i-1)\big)$ for any $i = 1, \ldots, m+1$, then the recurrence equation becomes

$$\rho_i = \psi(i) - \psi(i+1)$$

for all $i = 1, \ldots, m$, where $\psi(m) = \rho_m$ because $b_m = 0$. This last recurrence has solution

$$\psi(i) = \sum_{k=i}^{m} \rho_k$$

for all $i = 1, \ldots, m$ and hence

$$q(i) - q(i-1) = \frac{1}{\rho_{i-1} b_{i-1}} \sum_{k=i}^{m} \rho_k .$$

Remember that $q(0) = 0$. Then,

$$q(i) = \sum_{j=1}^{i} \frac{1}{b_{j-1}\rho_{j-1}} \sum_{k=j}^{m} \rho_k = \sum_{j=1}^{i} \frac{1}{b_{j-1}} \sum_{k=j}^{m} \prod_{\ell=j-1}^{k-1} \frac{b_\ell}{c_{\ell+1}}$$

for all $i = 0, \ldots, m$. □

Finally, we can provide a characterization in a class of regular layered networks of certain networks that satisfy the Serrin's condition.

Theorem 5.5.3. *Let $\Gamma = (V, c)$ be a connected network and $F \subset V$ a proper connected subset. Consider $m = r(\delta(F))$ and the value s as the number of different values that ν^F takes on F. If $U_i = D_i$ for all $i = 1, \ldots, m-1$, $s = m$ and $k_{i+1}(x) + k_{i-1}(x) = d_i$ for all $i = 1, \ldots, m-1$ and $x \in D_i$, then Γ satisfies Serrin's condition if and only if Γ is a regular layered network.*

Proof. The necessary condition follows from Proposition 5.5.2. In order to prove the sufficient condition, we need Corollary 5.2.2. This result states that $U_m \subseteq D_m$ and hence $U_m = D_m$, since $s = m$. On the other hand, let $i = 1, \dots, m-1$. From Equation (5.1), we get that

$$1 = k_{i+1}(x)\big(u_i - u_{i+1}\big) + k_{i-1}(x)\big(u_i - u_{i-1}\big)$$

for all $x \in D_i$. In particular, if $x, y \in D_i$ are two different vertices, then

$$k_{i-1}(x) - k_{i-1}(y) = k_{i+1}(y) - k_{i+1}(x)$$

and therefore

$$\begin{aligned} 0 &= \Big(k_{i+1}(y) - k_{i+1}(x)\Big)(u_{i+1} - u_i) + \Big(k_{i-1}(x) - k_{i-1}(y)\Big)(u_i - u_{i-1}) \\ &= \Big(k_{i+1}(y) - k_{i+1}(x)\Big)(u_{i+1} - u_{i-1}). \end{aligned}$$

We know that $u_{i+1} - u_{i-1} > 0$ because of the ordering of the level sets. Hence, the last equality holds if a only if $k_{i+1}(y) = k_{i+1}(x)$ for all $x, y \in D_i$, that is, if and only if k_{i+1} is constant on D_i. In consequence, also k_{i-1} is constant on D_i. Consider now the case $i = m$ and let $x \in D_m$. From Equation (5.1) we get that

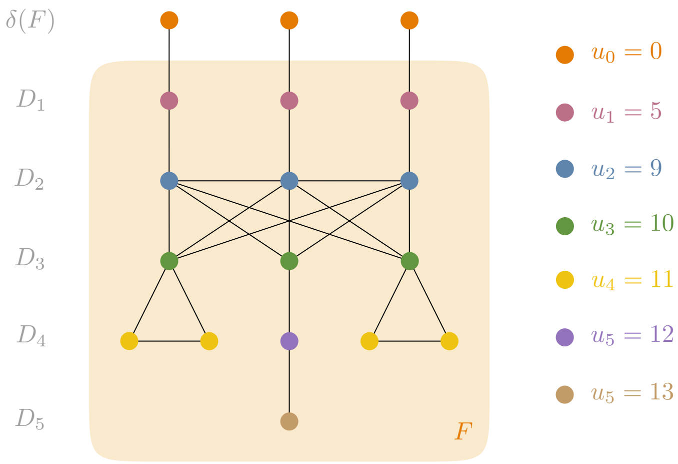
$$1 = k_{m-1}(x)\big(u_m - u_{m-1}\big)$$

and therefore $k_{m-1}(x)$ does not depend on x: it is constant on D_m. Finally, since Γ satisfies Serrin's condition, then $k_1(x) = -\dfrac{C}{u_1}$ for all $x \in D_0$, which means that k_1 is constant on D_0. Thus, Γ is a regular layered network. □

The last result shows that asking a relatively regular network to satisfy the Serrin's condition forces the network to be a regular layered network, that is, forces the network to have stronger regularity properties. Notice that regular layered networks are, somehow, ball–like discrete domains, since their behaviour between distance layers is regular and does not depend on the election of the vertex in the layer.

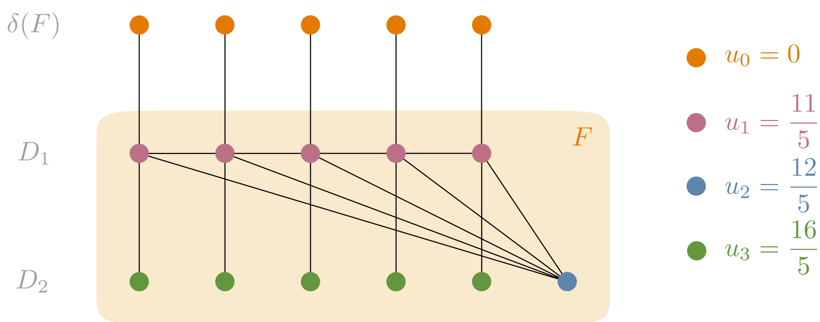
5.6 Other networks satisfying Serrin's condition

In benefit of the reader, it is important to know that there exist networks that are not regular layered and Serrin's condition holds on them. For instance, see Figures 5.6, 5.7, 5.8, 5.9, where the conductances of the edges always equal 1 and the level sets of the equilibrium measure are detailed.



$C = -5$

Figure 5.6 A non–regular layered network satisfying Serrin's condition.



$C = -\frac{11}{5}$

Figure 5.7 A non–regular layered network satisfying Serrin's condition.

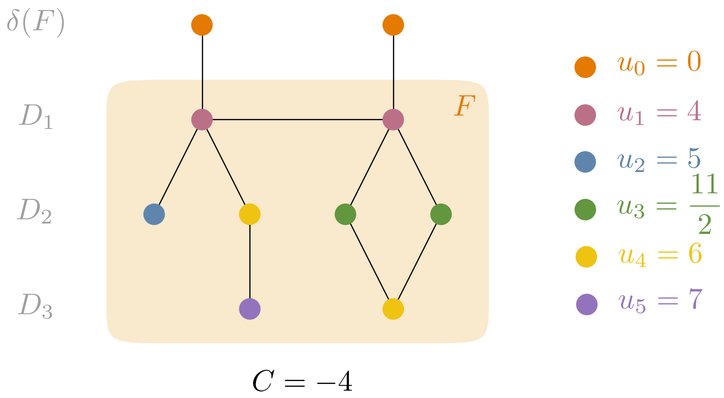


Figure 5.8 A non–regular layered network satisfying Serrin's condition.

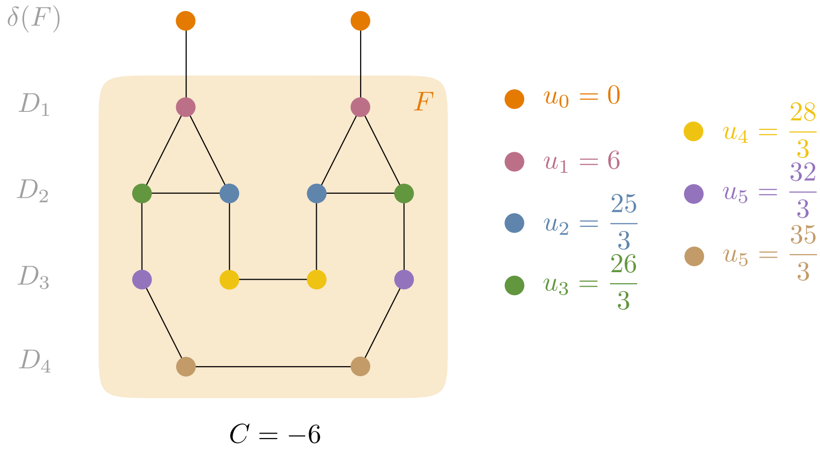


Figure 5.9 A non–regular layered network satisfying Serrin's condition.

Moreover, in some of them the equilibrium measure is a radial function, as Figure 5.10 shows. For the network in this figure, all the conductances of the edges also equal 1 for the sake of simplicity.

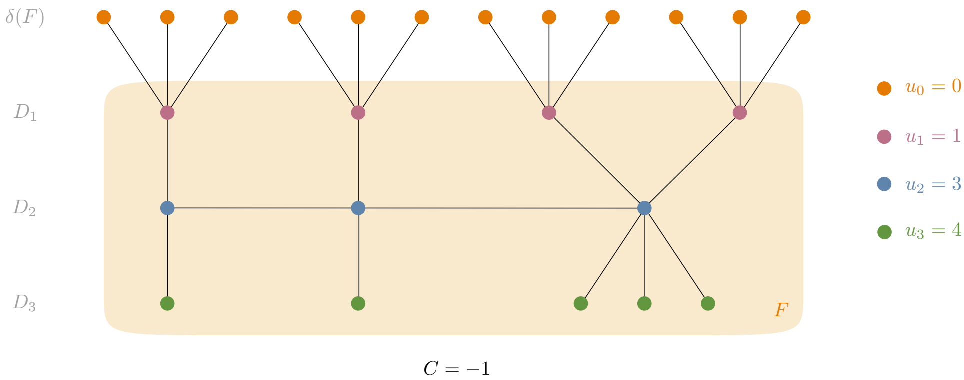


Figure 5.10 A non–regular layered network with radial ν^F.

The reader may find interesting the network in Figure 5.11, where all the conductances of the edges equal 1: the level set U_3 is entirely contained in D_3, whereas U_4 is entirely contained in D_2. That is, it is not true in general that the equilibrium measure ν^F grows with the distance to $\delta(F)$.

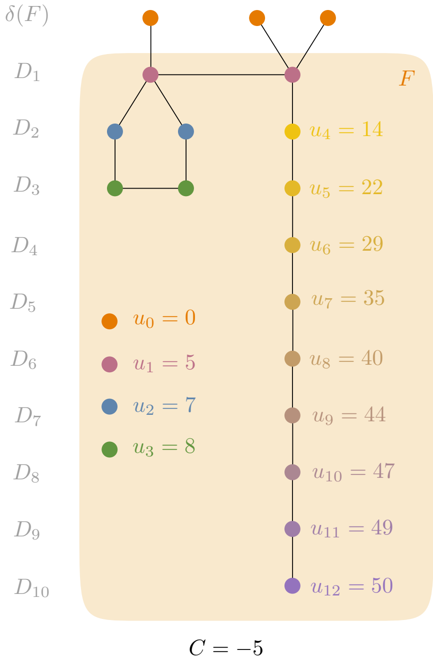


Figure 5.11 An example with $U_3 \subseteq D_3$ and $U_4 \subseteq D_2$.

All these examples show why Theorem 5.5.3 in last section needs to be restrictive in order to perform a characterization.

6

Dirichlet–to–Robin maps on finite networks

The concept of inverse boundary value problem was introduced in 1950 by Alberto Calderón. However, he did not publish any work on the subject until 1980, when his paper *On an inverse boundary–value problem* [29] appeared. The first applications of inverse boundary value problems are found in geophysical electrical prospection and electrical impedance tomography. The objective in physical electrical prospection is to deduce internal terrain properties from surface electrical measurements, which is of great interest in the engineering field. Electrical impedance tomography is a medical imaging technique where the aim is to obtain visual information of the body densities from some electrodes placed on the skin of the pacient. Hence, its usefulness in organ monitoring or bone–like cancer detection are remarkable [32].

The corresponding mathematical problem is whether it is possible to determine the conductivity of a body from boundary measurements and global equilibrium conditions. That is, an *inverse boundary value problem* consists in the recovery of the internal structure or conductivity information of a body using only external data and general conditions on the body. This problem is non–linear and it is exponentially ill–posed [1, 51], since its solution is highly sensitive to changes in the boundary data. For this problem is poorly arranged, at times the target is only a partial reconstruction of the conductivity data or the addition of morphological conditions so as to perform a full internal recovery.

In this chapter we work with discrete inverse boundary value problems on finite networks. Thus, we assume the conductances of the network to be unknown. In a first stage we introduce the Dirichlet–to–Robin map. It is a map naturally associated with the Schrödinger operator of the network and generalizes the definition of the *Dirichlet–to–Neumann* map Λ associated with the combinatorial laplacian, see the literature [36, 37, 38] for more details.

Afterwards, the second step is to focus in general networks and look upon the solution of certain boundary value problems having an alternating property in a part of the boundary. We show that these solutions spread across the network to end holding a derived alternating property in another part of the boundary. In fact, they spread following boundary–to–boundary paths where the sign of the solution is invariant and has opposite sign with respect to the neighbouring paths in the circular order. They are called *alternating paths*. Then, we achieve a circular planar network characterization through the Dirichlet–to–Robin map, which is an extension of the characterization given in [38].

The results detailed in this chapter have been submitted to publication, see [9].

6.1 The Dirichlet–to–Robin map

Consider a connected network $\Gamma = (V, c)$ and a proper connected subset $F \subset V$. Then, the following result holds.

Lemma 6.1.1. *If there exist a weight $\omega \in \Omega(\bar{F})$ and a real value $\lambda \geq 0$ such that $q = q_\omega + \lambda \chi_{\delta(F)}$ on $\bar{F}$, then the energy $\mathcal{E}_q^F$ is positive semi–definite. In particular, it is not strictly positive definite if and only if $\lambda = 0$, in which case $\mathcal{E}_q^F(v, v) = 0$ if and only if $v = a\omega$ for some $a \in \mathbb{R}$.*

Proof. It is a direct consequence of Proposition 2.3.3. □

In this chapter we work with potentials shaped as $q = q_\omega + \lambda \chi_{\delta(F)} \in \mathcal{C}(\bar{F})$ for a weight $\omega \in \Omega(\bar{F})$ and a real value $\lambda \geq 0$. The reason is that the corresponding Schrödinger operator is positive semi–definite and in this case, as we will see, λ and $\omega \chi_{\delta(F)}$ become the lowest eigenvalue and its associated eigenfunction of the Dirichlet–to–Robin map.

In benefit of the reader, let us remember here that for any $g \in \mathcal{C}(\delta(F))$ the Poisson operator $\mathcal{P}_q(g)$ is the unique solution $u \in \mathcal{C}(\bar{F})$ of the Dirichlet problem

$$\mathcal{L}_q(u) = 0 \text{ on } F \quad \text{and} \quad u = g \text{ on } \delta(F),$$

see Section 2.5. Moreover, as the potential is given by $q = q_\omega + \lambda\chi_{\delta(F)}$ on $\bar{F}$, in this case the following holds.

Lemma 6.1.2. *When* $q = q_\omega + \lambda\chi_{\delta(F)}$ *on* $\bar{F}$*, then* $\mathcal{L}_q(\omega) = \lambda\omega\chi_{\delta(F)}$ *and* $\mathcal{P}_q(\omega\chi_{\delta(F)}) = \omega$ *on* $\bar{F}$*.*

Proof. The first equality is direct from the definitions of $\mathcal{L}_q$ and q. The second equality is straightforward, since by the first equality the weight $\omega \in \Omega(\bar{F})$ is a solution of the Dirichlet problem

$$\mathcal{L}_q(u) = 0 \ \text{ on } \ F \quad \text{and} \quad u = \omega\chi_{\delta(F)} \ \text{ on } \ \delta(F)$$

and this problem has a unique solution given by $u = \mathcal{P}_q(\omega\chi_{\delta(F)}) \in \mathcal{C}(\bar{F})$. □

Now we are ready to introduce a boundary–to–boundary operator of great importance in this thesis: the Dirichlet–to–Robin map. This map measures the difference of voltages between boundary vertices when electrical currents are applied to them. In fact, it can be also called the *response matrix* of the network because of this property, see [38] for the case of the combinatorial laplacian. Notice that the boundary is the unique part of the network from where direct information can be obtained and also provided. Hence, as we can obtain the Dirichlet–to–Robin map by means of boundary measurements, it is reasonable to suppose that the values of this map are known. Therefore, the Dirichlet–to–Robin map is the key to obtain the conductances of the network.

The *Dirichlet–to–Robin map* is the linear form $\Lambda_q\colon \mathcal{C}(\delta(F)) \longrightarrow \mathcal{C}(\delta(F))$ that assigns to any function $g \in \mathcal{C}(\delta(F))$ the function $\Lambda_q(g) = \dfrac{\partial \mathcal{P}_q(g)}{\partial \mathsf{n}_F} + qg \in \mathcal{C}(\delta(F))$. The kernel associated with the Dirichlet–to–Robin map Λ_q is denoted by $N_q : \delta(F) \times \delta(F) \longrightarrow \mathbb{R}$, where $N_q(x,y) = \Lambda_q(\varepsilon_y)(x)$ for all $x, y \in \delta(F)$. Notice that

$$\Lambda_q(g)(x) = \int_{\delta(F)} N_q(x,y)g(y)\,dy$$

for all $g \in \mathcal{C}(\delta(F))$ and $x \in \delta(F)$. In the sequel, we will also denote by $\mathsf{N_q} \in \mathcal{M}_{|\delta(F)|\times|\delta(F)|}(\mathbb{R})$ the matrix given by the values of the kernel of the Dirichlet–to–Robin map on $\delta(F)\times\delta(F)$. In an equivalent way, $\mathsf{N} \in \mathcal{M}_{|\delta(F)|\times|\delta(F)|}(\mathbb{R})$ is the matrix given by the kernel of the Dirichlet–to–Neumann map associated with the combinatorial laplacian, that is, when ω is constant and $\lambda = 0$.

Proposition 6.1.3. *The Dirichlet–to–Robin map Λ_q is a self–adjoint, positive semi–definite operator whose associated quadratic form is given by*

$$\int_{\delta(F)} g\Lambda_q(g) = \mathcal{E}_q^F\Big(\mathcal{P}_q(g), \mathcal{P}_q(g)\Big).$$

In addition, λ is the lowest eigenvalue of Λ_q and its associated eigenfunctions are multiple of $\omega\chi_{\delta(F)}$.

Proof. Let $f, g \in \delta(F)$. From the definition of the energy associated with F and the Poisson operator,

$$\mathcal{E}_q^F\Big(\mathcal{P}_q(f), \mathcal{P}_q(g)\Big) = \int_{\delta(F)} g\Lambda_q(f).$$

By Proposition 2.3.3,

$$\int_{\delta(F)} g\Lambda_q(f) = \mathcal{E}_q^F\Big(\mathcal{P}_q(f), \mathcal{P}_q(g)\Big) = \mathcal{E}_q^F\Big(\mathcal{P}_q(g), \mathcal{P}_q(f)\Big) = \int_{\delta(F)} f\Lambda_q(g).$$

and hence Λ_q is self–adjoint and positive semi–definite. Moreover,

$$\Lambda_q(\omega\chi_{\delta(F)}) = \lambda\omega\chi_{\delta(F)} \quad \text{on} \quad \delta(F).$$

On the other hand, from Proposition 2.3.2 we get that

$$\mathcal{E}_q^F\Big(\mathcal{P}_q(g), \mathcal{P}_q(g)\Big) = \frac{1}{2}\int_{\bar{F}}\int_{\bar{F}} c_F(x,y)\omega(x)\omega(y)\left(\frac{\mathcal{P}_q(g)(x)}{\omega(x)} - \frac{\mathcal{P}_q(g)(y)}{\omega(y)}\right)^2 dx\,dy$$
$$+ \lambda\int_{\delta(F)} g^2 \geq \lambda\int_{\delta(F)} g^2$$

and the equality holds if and only if $\mathcal{P}_q(g) = a\omega$ on $\delta(F)$, that is, if and only if $g = a\omega$ on $\delta(F)$. Finally, suppose that g is a non–null eigenfunction corresponding to the eigenvalue α. Therefore,

$$\alpha\int_{\delta(F)} g^2 = \mathcal{E}_q^F\Big(\mathcal{P}_q(g), \mathcal{P}_q(g)\Big) \geq \lambda\int_{\delta(F)} g^2,$$

which implies that $\alpha \geq \lambda$. □

Let us remark here that, as the last result shows, the equality $\Lambda_q(\omega\chi_{\delta(F)}) = \lambda\omega\chi_{\delta(F)}$ is always satisfied on $\delta(F)$.

Once the Dirichlet–to–Robin map Λ_q and its properties have been set, we want to show that the matrix $\mathsf{N_q}$ given by its kernel is closely related to the Green and the Schrödinger matrices. Before doing this, however, we need some notation and previous results. On the first place, we extend the definition of normal derivative and apply it to functions on two variables instead of one. Let $\mathcal{K}\colon \mathcal{C}(\bar F) \longrightarrow \mathcal{C}(\bar F)$ be a linear operator and let $K \in \mathcal{C}(\bar F \times \bar F)$ be its associated kernel. Then, for any $x \in \delta(F)$ and $y \in \bar F$, in [18] the values

$$\frac{\partial K}{\partial \mathsf{n}_x}(x,y) = \frac{\partial K^y}{\partial \mathsf{n}_F}(x) = \frac{\partial \mathcal{K}(\varepsilon_y)}{\partial \mathsf{n}_F}(x),$$

for each $x \in \bar F$ and $y \in \delta(F)$ and

$$\frac{\partial K}{\partial \mathsf{n}_y}(x,y) = \frac{\partial K_x}{\partial \mathsf{n}_F}(y)$$

were defined. For the sake of plainness, we denote by $\dfrac{\partial^2 K}{\partial \mathsf{n}_{x,y}}$ and $\dfrac{\partial^2 K}{\partial \mathsf{n}_{y,x}}$ the values $\dfrac{\partial}{\partial \mathsf{n}_x}\left(\dfrac{\partial K}{\partial \mathsf{n}_y}\right)$ and $\dfrac{\partial}{\partial \mathsf{n}_y}\left(\dfrac{\partial K}{\partial \mathsf{n}_x}\right)$, respectively. Clearly, $\dfrac{\partial K}{\partial \mathsf{n}_x} \in \mathcal{C}(\delta(F) \times \bar F)$, $\dfrac{\partial K}{\partial \mathsf{n}_y} \in \mathcal{C}(\bar F \times \delta(F))$ and $\dfrac{\partial^2 K}{\partial \mathsf{n}_{x,y}}, \dfrac{\partial^2 K}{\partial \mathsf{n}_{y,x}} \in \mathcal{C}(\delta(F) \times \delta(F))$ are all kernels on $\bar F \times \bar F$.

Lemma 6.1.4. *If K is a kernel on $\bar F \times \bar F$, then $\dfrac{\partial^2 K}{\partial \mathsf{n}_{x,y}} = \dfrac{\partial^2 K}{\partial \mathsf{n}_{y,x}}$. Moreover, $\dfrac{\partial^2 K}{\partial \mathsf{n}_{x,y}}$ is a symmetric kernel when K is.*

Proof. It suffices to see that

$$\begin{aligned}\frac{\partial^2 K}{\partial \mathsf{n}_{x,y}}(x,y) = {} & \kappa_F(x)\kappa_F(y)K(x,y) - \kappa_F(x)\int_F c(y,z)K(x,z)\,dz \\ & - \kappa_F(y)\int_F c(x,z)K(z,y)\,dz \\ & + \int_F\int_F c(x,t)c(y,z)K(t,z)\,dt\,dz. \quad \square\end{aligned}$$

Regarding to the Schrödinger operator, we can deduce certain properties from the results in Section 2.3.

Corollary 6.1.5. *The matrix* $\mathsf{L_q}$ *is symmetric and positive semi–definite. Moreover, if* $S \subset \bar{F}$ *is a proper connected subset then* $\mathsf{L_q(S;S)}$ *is positive definite.*

Proof. This result is due to Proposition 2.3.4. □

We now bring in four matrices that will be serviceable for the matricial formulae of this thesis. First, consider as the *identity matrix of size* k the diagonal matrix $\mathsf{ID} \in \mathcal{M}_{k\times k}(\{0,1\})$ with diagonal entries equal to 1. Also, we define the *matrix of conductances* $\mathsf{C} \in \mathcal{M}_{|\bar{F}|\times|\bar{F}|}([0,+\infty))$, where the entries are given by the conductances between all the pairs of vertices of $\bar{F}$. Next in order, let $\mathsf{k_F} \in \mathcal{M}_{|\delta(F)|\times|\delta(F)|}((0,+\infty))$ be the diagonal *matrix of boundary degrees* with diagonal entries provided by κ_F for all $x \in \delta(F)$. Finally, $\mathsf{Q} \in \mathcal{M}_{|\bar{F}|\times|\bar{F}|}(\mathbb{R})$ is another diagonal matrix called the *matrix of potentials*, with diagonal values brought by the potential $q(x)$ for all $x \in \bar{F}$.

Lemma 6.1.6. *For the matrix* $\mathsf{L_q(\bar{F};\bar{F})}$*, the following block identites hold:*

$$\mathsf{L_q}(\delta(\mathsf{F});\delta(\mathsf{F})) = \mathsf{k_F}(\delta(\mathsf{F});\delta(\mathsf{F})) + \mathsf{Q}(\delta(\mathsf{F});\delta(\mathsf{F})),$$

$$\mathsf{L_q}(\delta(\mathsf{F});\mathsf{F}) = -\mathsf{C}(\delta(\mathsf{F});\mathsf{F}) \quad \textit{and} \quad \mathsf{L_q}(\mathsf{F};\delta(\mathsf{F})) = -\mathsf{C}(\mathsf{F};\delta(\mathsf{F})).$$

Proof. Since the block $\mathsf{L_q}(\delta(\mathsf{F});\mathsf{F})$ is given by the entries $L_q(x,y)$ for all $x \in \delta(F)$ and $y \in F$, we easily see that

$$L_q(x,y) = \mathcal{L}_q(\varepsilon_y)(x) = -\int_{\bar{F}} c(x,z)\varepsilon_y(z)\,dz = -c(x,y).$$

Analogously, if $x,y \in \delta(F)$ then

$$L_q(x,y) = \mathcal{L}_q(\varepsilon_y)(x) = \frac{\partial \varepsilon_y}{\partial \mathsf{n}_F}(x) + q(x)\varepsilon_y(x) = (\kappa_F(x) + q(x))\varepsilon_y(x)$$

and therefore $\mathsf{L_q}(\delta(\mathsf{F});\delta(\mathsf{F})) = \mathsf{k_F}(\delta(\mathsf{F});\delta(\mathsf{F})) + \mathsf{Q}(\delta(\mathsf{F});\delta(\mathsf{F}))$. □

The following matricial results on the Green and Poisson kernels will be handy all allong this chapter and, in fact, all allong this work.

Lemma 6.1.7. *The inverse of the Schrödinger matrix on* $F \times F$ *is given by* $\mathsf{G_q(F;F)} = \mathsf{L_q(F;F)}^{-1}$*. On the other hand, if* $\lambda = 0$ *then the matrix of the orthogonal Green operator on* V *satisfies the following matricial equations:*

$$\mathsf{ID(F;F)} - \omega_{\mathsf{V}} \cdot \omega_{\mathsf{V}}^\top = \mathsf{L}_{\mathsf{q}_\omega} \cdot \mathsf{G}_{\mathsf{q}_\omega}$$

with

$$\mathsf{G}_{\mathsf{q}_\omega} \cdot \omega_{\mathsf{V}} = 0.$$

Lemma 6.1.8. *The matrix of the double derivatives of the Green kernel is given by*

$$\left[\frac{\partial^2 \mathsf{G_q}}{\partial \mathsf{n}_{\mathsf{x,y}}}(\mathsf{x},\mathsf{y})\right]_{\mathsf{x,y}\in\delta(\mathsf{F})} = \mathsf{C}(\delta(\mathsf{F});\mathsf{F})\cdot\mathsf{G_q}(\mathsf{F};\mathsf{F})\cdot\mathsf{C}(\mathsf{F};\delta(\mathsf{F})).$$

Proof. For all $x, y \in \delta(F)$,

$$\begin{aligned}\mathsf{C}(\mathsf{x};\mathsf{F})\cdot\mathsf{G_q}(\mathsf{F};\mathsf{F})\cdot\mathsf{C}(\mathsf{F};\mathsf{y}) &= \mathsf{C}(\mathsf{x};\mathsf{F})\cdot\sum_{\mathsf{t}\in\mathsf{F}}\mathsf{G_q}(\mathsf{F};\mathsf{t})\cdot\mathsf{C}(\mathsf{t};\mathsf{y})\\ &= \sum_{z\in F}\mathsf{C}(\mathsf{x};\mathsf{z})\cdot\left(\sum_{t\in F}\mathsf{G_q}(\mathsf{z};\mathsf{t})\cdot\mathsf{C}(\mathsf{t};\mathsf{y})\right)\\ &= \sum_{z\in F}c(x,z)\left(\sum_{t\in F}G_q(z,t)c(t,y)\right)\\ &= -\sum_{z\in F}c(x,z)\frac{\partial G_q}{\partial \mathsf{n}_y}(z,y)\\ &= \frac{\partial^2 G_q}{\partial \mathsf{n}_{x,y}}(x,y). \qquad \square\end{aligned}$$

Lemma 6.1.9. *The matrices of the Green and Poisson kernels on F are related by the equality*

$$\mathsf{P_q}(\mathsf{F};\delta(\mathsf{F})) = \mathsf{G_q}(\mathsf{F};\mathsf{F})\cdot\mathsf{C}(\mathsf{F};\delta(\mathsf{F})).$$

Proof. Let $x \in F$ and $y \in \delta(F)$. Then,

$$\begin{aligned}\mathsf{P_q}(\mathsf{x},\mathsf{y}) = P_q(x,y) = \mathcal{P}_q(\varepsilon_y)(x) = \varepsilon_y(x) - \frac{\partial G_q^x}{\partial \mathsf{n}_F}(y)\\ = -\sum_{z\in F}c(y,z)\Big(G_q^x(y) - G_q^x(z)\Big) = \sum_{z\in F}c(y,z)G_q(x,z)\\ = \sum_{z\in F}G_q(x,z)c(z,y) = \mathsf{G_q}(\mathsf{x};\mathsf{F})\cdot\mathsf{C}(\mathsf{F};\mathsf{y}). \quad \square\end{aligned}$$

Now we are ready to proceed with the study of the Dirichlet–to–Robin matrix $\mathsf{N_q}$ and relate it to the Green matrix $\mathsf{G_q}$ on $F \times F$ and the Schrödinger matrices $\mathsf{L_q}(\bar{\mathsf{F}};\bar{\mathsf{F}})$ and $\mathsf{L_q}(\mathsf{F};\mathsf{F})$.

Lemma 6.1.10. *The matrix given by the kernel of* Λ_q *can be written as*

$$\mathsf{N_q} = \mathsf{k_F}(\delta(\mathsf{F});\delta(\mathsf{F})) + \mathsf{Q}(\delta(\mathsf{F});\delta(\mathsf{F})) - \mathsf{C}(\delta(\mathsf{F});\mathsf{F})\cdot \mathsf{G_q}(\mathsf{F};\mathsf{F})\cdot \mathsf{C}(\mathsf{F};\delta(\mathsf{F})).$$

Moreover, $\mathsf{N_q}$ *is a symmetric matrix with negative off–diagonal entries and positive on the diagonal.*

Proof. For all $x, y \in \delta(F)$, using Proposition 2.5.7, we get that

$$\begin{aligned} N_q(x,y) = \Lambda_q(\varepsilon_y)(x) &= \frac{\partial \mathcal{P}_q(\varepsilon_y)}{\partial \mathsf{n}_F}(x) + q(x)\mathcal{P}_q(\varepsilon_y)(x) \\ &= \frac{\partial \varepsilon_y}{\partial \mathsf{n}_F}(x) - \frac{\partial}{\partial \mathsf{n}_x}\left(\frac{\partial G_q}{\partial \mathsf{n}_y}(x,y)\right) + q(x)\varepsilon_y(x) \\ &= \varepsilon_y(x)\kappa_F(x) - \frac{\partial^2 Gq}{\partial \mathsf{n}_{x,y}}(x,y) + q(x)\varepsilon_y(x). \end{aligned}$$

In matricial terms, using Lemma 6.1.8,

$$\mathsf{N_q} = \mathsf{k_F}(\delta(\mathsf{F});\delta(\mathsf{F})) + \mathsf{Q}(\delta(\mathsf{F});\delta(\mathsf{F})) - \mathsf{C}(\delta(\mathsf{F});\mathsf{F})\cdot \mathsf{G_q}(\mathsf{F};\mathsf{F})\cdot \mathsf{C}(\mathsf{F};\delta(\mathsf{F})).$$

Moreover, since $G_q(x,y) > 0$ for any $x, y \in F$ then for $x, y \in \delta(F)$ with $y \neq x$ it is satisfied that $N_q(x,y) = -\dfrac{\partial^2 G_q}{\partial \mathsf{n}_{x,y}}(x,y) = -\mathsf{C}(\mathsf{x};\mathsf{F})\cdot\mathsf{G_q}(\mathsf{F};\mathsf{F})\cdot\mathsf{C}(\mathsf{F};\mathsf{y}) < 0$. Therefore, $\mathsf{N_q}$ is negative off–diagonal. What is more, as $\Lambda_q(\omega) = \lambda\omega$ on $\delta(F)$, then for any $y \in \delta(F)$

$$\int_{\delta(F)} \Lambda_q(\varepsilon_y)(x)\omega(x)\,dx = \lambda\omega(y)$$

and hence

$$N_q(y,y) = \Lambda_q(\varepsilon_y)(y) = \lambda - \omega(y)^{-1}\int_{\delta(F)\smallsetminus\{y\}} N_q(x,y)\omega(x)\,dx > 0. \quad \square$$

Corollary 6.1.11. *Let* $P, Q \subset \delta(F)$ *be two boundary sets such that* $P \cap Q = \emptyset$. *Then,*

$$\mathsf{N_q}(\mathsf{P};\mathsf{Q}) = -\mathsf{C}(\mathsf{P};\mathsf{F})\cdot\mathsf{G_q}(\mathsf{F};\mathsf{F})\cdot\mathsf{C}(\mathsf{F};\mathsf{Q}).$$

The kernel of the Dirichlet–to–Robin map is directly related to the Schur complement of $\mathsf{L_q}(\mathsf{F};\mathsf{F})$ in $\mathsf{L_q}(\bar{\mathsf{F}};\bar{\mathsf{F}})$, as the next result shows. In [38, Theorem 3.2] an analogous result was given for the case of the Dirichlet–to–Neumann map and a Schur complement of the combinatorial laplacian matrix. The reader may redirect to [30, 58] and the references therein for the Schur complement definition and properties.

Proposition 6.1.12. *The matrix of the kernel of the Dirichlet–to–Robin map Λ_q is given by*

$$\mathsf{N_q} = {}^{\mathsf{L_q}(\bar{\mathsf{F}};\bar{\mathsf{F}})}\Big/_{\mathsf{L_q}(\mathsf{F};\mathsf{F})}.$$

Proof. Consider the boundary value problem that consists in finding $u \in \mathcal{C}(\bar{F})$ such that

$$\mathcal{L}_q(u) = f \ \text{ on } \ F \quad \text{and} \quad \frac{\partial u}{\partial \mathsf{n}_F} + qu = g \ \text{ on } \ \delta(F).$$

Using Lemma 6.1.6, this problem has the following matricial expression

$$\mathsf{L_q}(\bar{\mathsf{F}};\bar{\mathsf{F}}) \cdot \mathsf{u}_{\bar{\mathsf{F}}} = \begin{bmatrix} \mathsf{k_F}(\delta(\mathsf{F});\delta(\mathsf{F})) + \mathsf{Q}(\delta(\mathsf{F});\delta(\mathsf{F})) & -\mathsf{C}(\delta(\mathsf{F});\mathsf{F}) \\ -\mathsf{C}(\mathsf{F};\delta(\mathsf{F})) & \mathsf{L_q}(\mathsf{F};\mathsf{F}) \end{bmatrix} \cdot \begin{bmatrix} \mathsf{u}_{\delta(\mathsf{F})} \\ \mathsf{u_F} \end{bmatrix}$$
$$= \begin{bmatrix} \mathsf{g} \\ \mathsf{f} \end{bmatrix}.$$

We know that $\mathsf{L_q}(\mathsf{F};\mathsf{F})$ is invertible because F is a proper subset by definition and hence $\mathsf{L_q}(\mathsf{F};\mathsf{F})^{-1} = \mathsf{G_q}(\mathsf{F};\mathsf{F})$ by Lemma 6.1.7. Thus, the Schur complement of $\mathsf{L_q}(\mathsf{F};\mathsf{F})$ in $\mathsf{L_q}(\bar{\mathsf{F}};\bar{\mathsf{F}})$ is given by

$$\begin{aligned} {}^{\mathsf{L_q}(\bar{\mathsf{F}};\bar{\mathsf{F}})}\Big/_{\mathsf{L_q}(\mathsf{F};\mathsf{F})} &= \mathsf{k_F}(\delta(\mathsf{F});\delta(\mathsf{F})) + \mathsf{Q}(\delta(\mathsf{F});\delta(\mathsf{F})) \\ &\quad - \mathsf{C}(\delta(\mathsf{F});\mathsf{F}) \cdot \mathsf{G_q}(\mathsf{F};\mathsf{F}) \cdot \mathsf{C}(\mathsf{F};\delta(\mathsf{F})) = \mathsf{N_q}, \end{aligned}$$

where we have used Lemmas 6.1.8 and 6.1.10. □

The following result is analogous to the result given by Curtis and Morrow for the combinatorial laplacian, see [37] for the proof.

Corollary 6.1.13. *If $P, Q \subseteq \delta(F)$, then the block $\mathsf{N_q}(\mathsf{P};\mathsf{Q})$ of the Dirichlet–to–Robin matrix $\mathsf{N_q}$ is given by the Schur complement*

$$\mathsf{N_q}(\mathsf{P};\mathsf{Q}) = {}^{\mathsf{L_q}(\mathsf{P}\cup\mathsf{F};\mathsf{Q}\cup\mathsf{F})}\Big/_{\mathsf{L_q}(\mathsf{F};\mathsf{F})}.$$

We show now some examples of Dirichlet–to–Robin matrices. Consider the four networks displayed in Figure 6.1, the potential on them $q = q_\omega + \lambda\chi_{\delta(F)}$ on $\bar{F}$, the real value $\lambda = 2$ and the weight given by $\omega = 1/6$ on $\delta(F) = \{z_1, \ldots, z_4\}$ and $\omega = 1/3$ on $F = \{x, y\}$, except for the network in Figure (6.1c), for which $\omega(z_1) = \omega(z_4) = \omega(x) = \omega(y) = 1/6$ and $\omega(z_2) = \omega(z_3) = 1/3$.

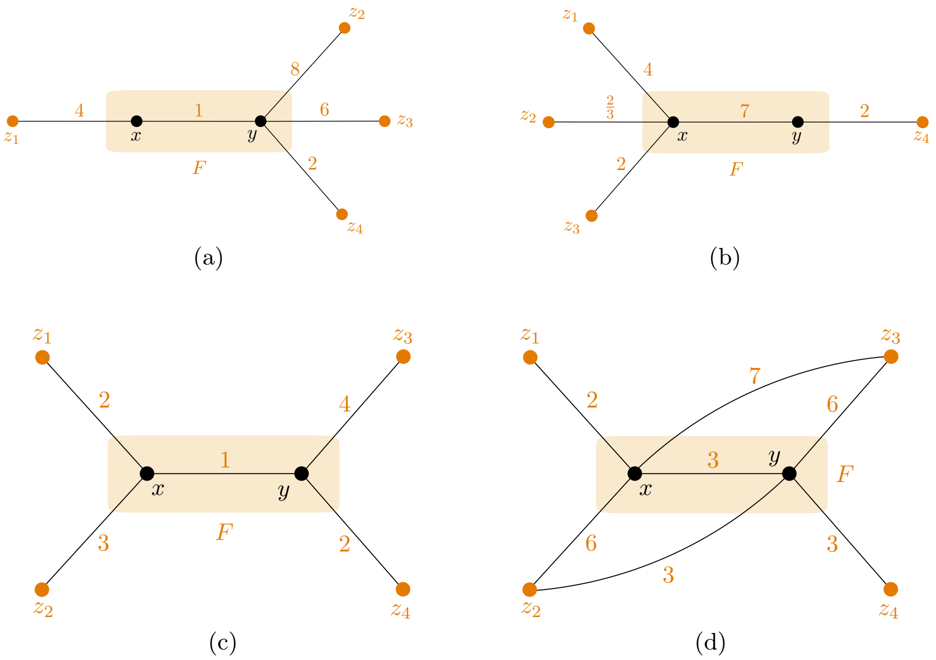


Figure 6.1 Some examples of networks for which we compute their Dirichlet–to–Robin matrices.

Their associated Dirichlet–to–Robin matrices are given by

$$\text{(a)} \quad \mathsf{N}_\mathsf{q} = \frac{1}{9}\begin{pmatrix} 82 & -32 & -24 & -8 \\ -32 & 178 & -96 & -32 \\ -24 & -96 & 162 & -24 \\ -8 & -32 & -24 & 82 \end{pmatrix}.$$

$$\text{(b)} \quad \mathsf{N}_\mathsf{q} = \frac{1}{24}\begin{pmatrix} 252 & -30 & -90 & -84 \\ -30 & 107 & -15 & -14 \\ -90 & -15 & 195 & -42 \\ -84 & -14 & -42 & 188 \end{pmatrix}.$$

$$\text{(c)} \quad \mathsf{N}_\mathsf{q} = \frac{1}{142}\begin{pmatrix} 530 & -57 & -4 & -2 \\ -57 & 305 & -6 & -3 \\ -4 & -6 & 306 & -60 \\ -2 & -3 & -60 & 538 \end{pmatrix}.$$

$$\text{(d)} \quad \mathsf{N}_\mathsf{q} = \frac{1}{42}\begin{pmatrix} 388 & -120 & -160 & -24 \\ -120 & 1083 & -726 & -153 \\ -160 & -726 & 1216 & -246 \\ -24 & -153 & -246 & 507 \end{pmatrix}.$$

6.2 Alternating paths

In [38], the existence of disjoint alternating paths between sets of boundary vertices in a class of graphs with conductivity was shown. In this section we study the existence of alternating paths in general networks. Namely, we show that the Dirichlet–to–Robin map has the alternating property, which may be considered as a generalization of the monotonicity property. We use the terminology following the guidelines of [38] and the notations and definitions in Section 2.7.

In the forthcoming results, $\Gamma = (V, c)$ is a network with $F \subset V$ a proper connected subset and $A, B \subset \delta(F)$ are two boundary sets such that $\delta(F) = A \cup B$ with $A \cap B = \emptyset$.

Theorem 6.2.1 (Alternating paths). *Let $g \in \mathcal{C}(B)$ be such that there exists a sequence of vertices $\{p_1, \ldots, p_k\} \in A$ with*

$$(-1)^{i+1}\Lambda_q(g)(p_i) > 0 \tag{6.1}$$

for all $i = 1, \ldots, k$. Then, there is a sequence of vertices $\{q_1, \ldots, q_k\} \in B$ such that

$$\Lambda_q(g)(p_i)g(q_{k-i+1}) < 0.$$

Moreover, p_i and q_{k-i+1} are connected through Γ by a path γ_i and it is satisfied that

$$g(q_{k-i+1})\mathcal{P}_q(g) > 0 \quad \text{on } \gamma_i \smallsetminus p_i$$

for all $i = 1, \ldots, k$.

Proof. We assume that $(-1)^{i+1}\Lambda_q(g)(p_i) > 0$ for all $i = 1, \ldots, k$. Now, if there exists a sequence of points $\{q_1, \ldots, q_k\} \in B$ such that $(-1)^i g(q_{k-i+1}) > 0$ for all $i = 1, \ldots, k$, then clearly

$$\Lambda_q(g)(p_i)g(q_{k-i+1}) < 0.$$

Hence, we only need to prove that this sequence of vertices of B such that $(-1)^i g(q_{k-i+1}) > 0$ exists. Notice that if we change indices, the last inequality is equivalent to the inequality

$$(-1)^{k-i+1}g(q_i) > 0.$$

We prove the existence of this sequence by construction. First, we describe how to pick the point $q_k \in B$. Let $u_g = \mathcal{P}_q(g) \in \mathcal{C}(F \cup B)$. By Equation (6.1), since

$$\Lambda_q(g) = \frac{\partial u_g}{\partial \mathsf{n}_F} + q u_g = \frac{\partial u_g}{\partial \mathsf{n}_F} \quad \text{on } A,$$

we get that

$$0 < \Lambda_q(g)(p_1) = \frac{\partial u_g}{\partial \mathsf{n}}(p_1) = -\int_F c(p_1, y) u_g(y)\, dy.$$

Then, there exists a vertex $y \in F \cap N(p_1)$ such that $u_g(y) < 0$. Let $W \subset F$ be the connected component of $\{z \in F : u_g(z) < 0\}$ such that $y \in W$. Notice that $W \neq \emptyset$ since $y \in W$. Suppose that $\bar{W} \cap B = \emptyset$, that is, $\bar{W} \subset F \cup A$. Then we can define the function $v = u_g \chi_{\bar{W}} \in \mathcal{C}(\bar{W} \cap F)$. Given a vertex $x \in W$, as $c(x, y) = 0$ for all $y \in F \setminus \bar{W}$, we get that

$$\mathcal{L}_q(v)(x) = \int_{\bar{W}} c(x, y)\Big(u_g(x) - u_g(y)\Big) dy = \int_{\bar{F}} c(x, y)\Big(u_g(x) - u_g(y)\Big) dy$$

$$= \mathcal{L}_q(u_g)(x) = 0.$$

Moreover, let us take into account that $W \subset F$ and $\bar{W} \cap B = \emptyset$. Then, given a vertex $x \in \delta(W)$ we know that $u_g(x) \geq 0$ and so we have the inequalities $\mathcal{L}_q(v) = 0$ on W and $v \geq 0$ on $\delta(W)$. Using Proposition 2.4.2 we conclude that $v \geq 0$ on W, which is a contradiction. Therefore, $\bar{W} \cap B \neq \emptyset$

From the definition of $W \subset F$ we deduce that $u_g \geq 0$ on $\delta(W) \cap F$. Let us suppose that it is also true that $u_g \geq 0$ on $\delta(W) \cap B$. Then, $\mathcal{L}_q(u_g) = 0$ on W and $u_g \geq 0$ on $\delta(W)$. Hence, applying Proposition 2.4.2 again, we see that $u_g \geq 0$ on $\bar{W}$, which is a contradiction. In conclusion, there exists a vertex $q_k \in \delta(W) \cap B \subset B$ such that $g(q_k) = u_g(q_k) < 0$. Since $q_k \in \delta(W)$, there exists a vertex $\widetilde{y} \in W$ with $\widetilde{y} \sim q_k$. Observe that $u_g(\widetilde{y}) < 0$. Finally, we can join $\widetilde{y}$ with y by a path $\widetilde{\gamma}_1 = \{y \sim \ldots \sim \widetilde{y}\} \subset W \subset F$ because W is connected and therefore the path $\gamma_1 = \{p_1 \sim y \sim \ldots \sim \widetilde{y} \sim q_k\}$ connects p_1 and q_k through Γ. Notice that $u_g(z) < 0$ for all $z \in \widetilde{\gamma}_1$ and hence $g(q_k)\mathcal{P}_q(g) = g(q_k)u_g > 0$ on $\gamma_1 \setminus p_1$.

We can repeat this argument to produce paths γ_j such that γ_j connects $p_j \in A$ with a vertex $q_{k-j+1} \in B$ through Γ and $(-1)^j u_g(z) < 0$ for all $z \in \gamma_j \setminus p_j$. □

Corollary 6.2.2. *Suppose that now Γ is a circular planar network. Let $g \in \mathcal{C}(B)$ be such that there exists a sequence of different vertices $\{p_1, \ldots, p_k\} \in A$ in circular order with*

$$(-1)^{i+1}\Lambda_q(g)(p_i) > 0$$

for all $i = 1, \ldots, k$. Then, there is a sequence of different vertices in circular order $\{q_1, \ldots, q_k\} \in B$ such that

$$\Lambda_q(g)(p_i)g(q_{k-i+1}) < 0.$$

Moreover, p_i and q_{k-i+1} are connected through Γ by a path γ_i with

$$g(q_{k-i+1})\mathcal{P}_q(g) > 0 \quad on\ \gamma_i \setminus p_i$$

for all $i = 1, \ldots, k$ and the paths $\{\gamma_i\}_{i=1,\ldots,k}$ are disjoint. In other words, $\{p_1, \ldots, p_k\}$ and $\{q_1, \ldots, q_k\}$ are connected through Γ.

Proof. The paths built in the proof of Theorem 6.2.1 do not intersect each other if the network is planar and the vertices $p_j \in A$ are in circular order, since the values of the function u_g on γ_j have different sign than the ones on γ_{j-1} and γ_{j+1}. Therefore, $q_j \in B$ are the points given by the last theorem and are also in circular order. □

Theorem 6.2.3 (Strong alternating paths)**.** *Let $g \in \mathcal{C}(B)$ be a function such that there exists a sequence of points $\{p_1, \ldots, p_k\} \in A$ with*

$$\Lambda_q(g)(p_i) = 0$$

for all $i = 1, \ldots, k$. Then, there is a sequence of points $\{q_1, \ldots, q_k\} \in B$ such that

$$(-1)^i g(q_i) \geq 0.$$

Moreover, p_i and q_i are connected through Γ by a path $\gamma_i = \{p_i \sim x_1^i \sim \ldots \sim x_{n_i}^i \sim q_i\}$ for all $i = 1, \ldots, k$ and there exists an index $j_i \in \{1, \ldots, n_i + 1\}$ such that $\mathcal{P}_q(g)(x_\ell) = 0$ for all $\ell = 0, \ldots, j_i - 1$ and $g(q_i)\mathcal{P}_q(g)(x_\ell) > 0$ for all $\ell = j_i, \ldots, n_i$.

Proof. We assume that $\Lambda_q(g)(p_i) = 0$ for all $i = 1, \ldots, k$. In other words, if $u_g = \mathcal{P}_q(g) \in \mathcal{C}(F \cup B)$ then

$$0 = \frac{\partial u_g}{\partial \mathsf{n}_F}(p_1) = -\int_F c(p_1, y) u_g(y)\, dy.$$

Hence, there are only two possible situations.

1. There exists a vertex $y \in N(p_1) \cap F$ such that $u_g(y) < 0$. Let $W \subset F$ be the connected component of $\{z \in F : u_g(z) < 0\}$ such that $y \in W$. Proceeding as in the proof of Theorem 6.2.1 we get the result.

2. $u_g = 0$ on $N(p_1) \cap F$. In this case, for any $x \in N(p_1) \cap F$ it is satisfied that
$$0 = \mathcal{L}_q(u_g)u(x) = -\int_{F \cup B} c(x, y) u_g(y)\, dy.$$
Again, there are only two possible situations.

2.1. There exists a vertex $y \in N(x) \cap (F \cup B)$ such that $u_g(y) < 0$. If $y \in B$, then we take $q_1 = y$ and we are done. Otherwise, let $W \subset F$ be the connected component of $\{z \in F : u_g(z) < 0\}$ such that $y \in W$. Proceeding as in the proof of Theorem 6.2.1 we get the result.

2.2. $u_g = 0$ on $N(x) \cap (F \cup B)$. If $N(x) \cup B \neq \emptyset$, we are done. Otherwise, the situation is analogous to the one of point 2. □

The following result deals with networks that may not be circular planar, but still needs the concept of ordered pair. Hence, we need to introduce a generalization of this concept. When the network is not circular planar, we can label the boundary nodes, say $\delta(F) = \{t_1, \dots, t_n\}$, where $n = |\delta(F)|$. In this case, a subset $\{p_1, \dots, p_k\}$ of boundary nodes is called an *ordered set* if there exists a non decreasing function $\sigma : \{1, \dots, k\} \longrightarrow \{1, \dots, n\}$ such that $p_j = t_{\sigma(j)}$. The pair $(S;T)$ is called *ordered pair* if $(p_1, \dots, p_k; q_1, \dots, q_k)$ is an ordered set. Notice that in the definition of ordered set we are not assuming that the vertices in S nor T are different, but $S \cap T = \emptyset$.

Now, having fixed a label in the boundary, we say that a network has the *alternating property* if for any ordered set $\{p_1, \dots, p_k\}$ satisfying the hypothesis of Theorem 6.2.1, the vertices $\{q_1, \dots, q_k\}$ given by the Theorem are also in order. The next result also tells us a property of the Dirichlet–to–Robin map of networks having the alternating property which is related with totally nonnegative matrices.

Theorem 6.2.4. *Let Γ be a network having the alternating property and let the pair $(p_1, \dots, p_k; q_1, \dots, q_k)$ be an ordered pair on $\delta(F)$. If we consider the matrix $\mathsf{M} = (m_{ij}) \in \mathcal{M}_{k\times k}(\mathbb{R})$ given by the entries $m_{ij} = \dfrac{\partial P_q(p_i, q_j)}{\partial \mathsf{n}_x}$, then*

$$(-1)^{\frac{k(k+1)}{2}} \det(\mathsf{M}) \geq 0.$$

Proof. Clearly we can suppose that the vertices $(p_1, \dots, p_k)$ are different and $(q_1, \dots, q_k)$ also. We prove the result by induction on k. If $k = 1$, by Proposition 6.1.3 the result holds. Let $\ell \leq k$ and let us assume now that the result is true for all the sizes $i < \ell$. If the result is not true for ℓ, then there is a sequence of different boundary vertices $\{p_1, \dots, p_\ell; q_1, \dots, q_\ell\} \subseteq \delta(F)$ such that

$$(-1)^{\frac{\ell(\ell+1)}{2}} \det(\mathsf{L}) < 0, \tag{6.2}$$

where $\mathsf{L} = (\ell_{ij}) \in \mathcal{M}_{\ell\times\ell}(\mathbb{R})$ with $\ell_{ij} = \dfrac{\partial P_q(p_i,q_j)}{\partial \mathsf{n}_x}$. Notice that L is invertible by assumption. Let us consider now its inverse $\mathsf{L}^{-1} = (h_{ij}) \in \mathcal{M}_{\ell\times\ell}(\mathbb{R})$. Hence,

$$h_{ij} = (-1)^{i+j}\,\frac{\det(\hat{\mathsf{L}}_{\mathsf{ij}})}{\det(\mathsf{L})}, \tag{6.3}$$

where $\hat{\mathsf{L}}_{\mathsf{ij}}$ is the minor of L resulting from the removal of row i and column j. By induction hypothesis and using Equations (6.2) and (6.3), we get that

$$(-1)^{i+j+\ell+1}\,h_{ij} = (-1)^{i+j-\frac{\ell(\ell-1)}{2}+\frac{\ell(\ell+1)}{2}+1}\,h_{ij} \geq 0. \tag{6.4}$$

Since L is a nonsingular matrix, for a fixed index i there must exist an index j such that

$$(-1)^{i+j+\ell+1}\,h_{ij} > 0. \tag{6.5}$$

Consider now the vectors $\mathsf{w},\mathsf{z} \in \mathbb{R}^\ell$ defined as $\mathsf{w} = (w_i)$ with $w_i = (-1)^{i+1}$ and $\mathsf{z} = \mathsf{L}^{-1}\cdot\mathsf{w}$. Using (6.4) and (6.5) it is easy to see that

$$(-1)^{i+\ell} z_i > 0. \tag{6.6}$$

Therefore, since $\mathsf{w} = \mathsf{L}\cdot\mathsf{z}$, we get that $w_i = \displaystyle\sum_{j=1}^{\ell} \frac{\partial P_q(p_i,q_j)}{\partial \mathsf{n}_x} z_j$ and so

$$(-1)^{\ell+1} z_i w_i > 0 \tag{6.7}$$

because of Equation (6.6) and the definition of w_i.

We define the function $f \in \delta(F)$ as $f(q_j) = z_j$ for all $j = 1,\dots,\ell$ and $f = 0$ otherwise. Then, for all $i = 1,\dots,\ell$,

$$\begin{aligned}\Lambda_q(f)(p_i) &= \frac{\partial \mathcal{P}_q(f)}{\partial \mathsf{n}_F}(p_i) + q(p_i) f(p_i) = \int_{\delta(F)} \frac{\partial P_q(p_i,z)}{\partial \mathsf{n}_x} f(z)\,dz \\ &= \sum_{j=1}^{\ell} \frac{\partial P_q(p_i,q_j)}{\partial \mathsf{n}_x} z_j = w_i.\end{aligned}$$

Therefore, $(-1)^{\ell+1}\Lambda_q(f)(p_i) z_i > 0$ for all $i = 1,\dots,\ell$ and by Equation (6.6) this occurs if and only if $(-1)^{i+1}\Lambda_q(f)(p_i) > 0$. Using Theorem 6.2.1, there exists a sequence of vertices $\{y_1,\dots,y_\ell\} \in \delta(F)$ different from $\{p_1,\dots,p_\ell\}$ such that the pair $\{p_1,\dots,p_\ell; y_1,\dots,y_\ell\}$ is an ordered pair and $w_i f(y_{\ell-i+1}) < 0$ or, equivalenty, such that

$$(-1)^\ell w_i f(y_i) > 0.$$

Therefore, for any $i = 1, \dots, \ell$, necessarily $y_i \in \{q_1, \dots, q_\ell\}$, since otherwise we get $f(y_i) = 0$, which is a contradiction with the above inequality. Moreover, $\{q_1, \dots, q_\ell\}$ and $\{y_1, \dots, y_\ell\}$ are ordered subsets and hence there exists i such that $y_i = q_i$. This means that

$$(-1)^\ell w_i z_i = (-1)^\ell \Lambda_q(f)(p_i) f(q_i) > 0$$

and therefore it is a contradiction with Equation (6.7). Hence,

$$(-1)^{\frac{\ell(\ell+1)}{2}} \det(\mathsf{L}) \geq 0$$

and we get the result. □

The following examples show the behavior of the paths described in the above results. First, consider the spider network displayed in Figure 6.2 where the conductances equal 1 on the edges of the radii and equal 2 on the edges of the circles. See Section 4.2 for the definition and notations of spider networks. We choose $\lambda = 2$ and the weight $\omega = 1/10$ on $\delta(F) \cup \{x_{00}\}$ and $\omega = 1/5$ on $F \setminus \{x_{00}\}$. Its Dirichlet–to–Robin matrix $\mathsf{N_q}$ is given by the values

$$\mathsf{N_q}(\mathsf{v_i};\mathsf{v_i}) = \frac{395732805366}{110384474959}, \qquad \mathsf{N_q}(\mathsf{v_i};\mathsf{v_{i\pm1}}) = -\frac{28317414524}{110384474959},$$

$$\mathsf{N_q}(\mathsf{v_i};\mathsf{v_{i\pm2}}) = -\frac{19609504324}{110384474959}, \qquad \mathsf{N_q}(\mathsf{v_i};\mathsf{v_{i\pm3}}) = -\frac{15073456676}{110384474959},$$

$$\mathsf{N_q}(\mathsf{v_i};\mathsf{v_{i\pm4}}) = -\frac{12739926180}{110384474959}, \qquad \mathsf{N_q}(\mathsf{v_i};\mathsf{v_{i\pm5}}) = -\frac{11741626020}{110384474959}$$

for all $i = 1, \dots, 11$. Then, for the function $g \in \mathcal{C}(B)$ given by $\mathsf{g} = (-4, 21, -37.2, 26.38, -6.29519)^\top$ on B, we get the sign pattern for $u = \mathcal{P}_q(g)$ shown in Figure 6.2, as Corollary 6.2.2 states.

Now consider the spider network displayed in Figure 6.3 where the conductances equal 1 on the edges of the radii and equal 2 on the edges of the circles. We choose $\lambda = 2$ again and the weight $\omega = 1/6$ on $\delta(F) \cup \{x_{00}\}$ and $\omega = 1/3$ on $F \setminus \{x_{00}\}$. For this network, the Dirichlet–to–Robin matrix $\mathsf{N_q}$ is given by

$$\mathsf{N_q}(\mathsf{v_i};\mathsf{v_i}) = \frac{3128}{889}, \qquad \mathsf{N_q}(\mathsf{v_i};\mathsf{v_{i\pm1}}) = -\frac{281}{889},$$

$$\mathsf{N_q}(\mathsf{v_i};\mathsf{v_{i\pm2}}) = -\frac{211}{889}, \qquad \mathsf{N_q}(\mathsf{v_i};\mathsf{v_{i\pm3}}) = -\frac{183}{889}$$

Taking the function $g \in \mathcal{C}(B)$ whose vector is $\mathsf{g} = (-1, 3.5, -3.5, 1)^\top$ on B, the sign pattern corresponding to $u = \mathcal{P}_q(g)$ is shown in Figure 6.3, as expected from Theorem 6.2.3.

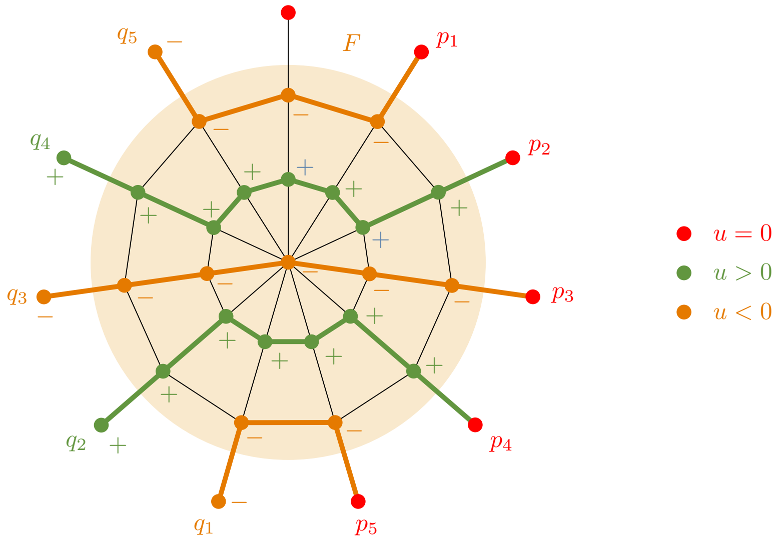


Figure 6.2 The sign pattern of $\mathcal{P}_q(g)$ on the spider network.

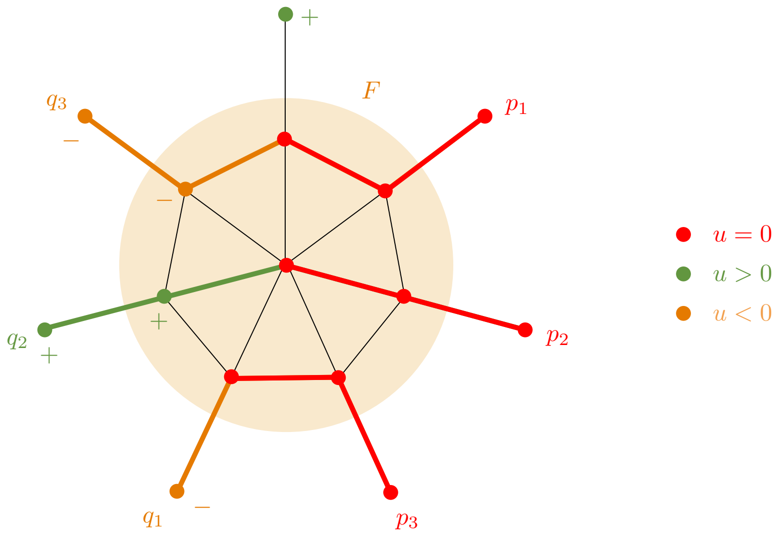


Figure 6.3 The sign pattern of $\mathcal{P}_q(g)$ on the spider network.

Finally, let us consider the network displayed in Figure 6.4 with the following parameters: $\lambda = 2$, $\omega = 1/4$ on $A \cup F$ and $\omega = 1/2$ on B, conductances 2 on the edges of $F \times F$ and 1 on the other edges. Its Dirichlet–to–Robin matrix $\mathsf{N_q}$ is

$$\mathsf{N_q} = \frac{1}{24}\begin{pmatrix} 105 & -13 & -15 & -13 & -3 & -5 \\ -13 & 105 & -13 & -15 & -5 & -3 \\ -15 & -13 & 105 & -13 & -3 & -5 \\ -13 & -15 & -13 & 105 & -5 & -3 \\ -3 & -5 & -3 & -5 & 57 & -1 \\ -5 & -3 & -5 & -3 & -1 & 57 \end{pmatrix}.$$

For $\mathsf{g} = (1, -1.5)^\top$ on B, we get the following sign pattern for $u = \mathcal{P}_q(g)$. Notice that it fits the conclusions of Theorem 6.2.1.

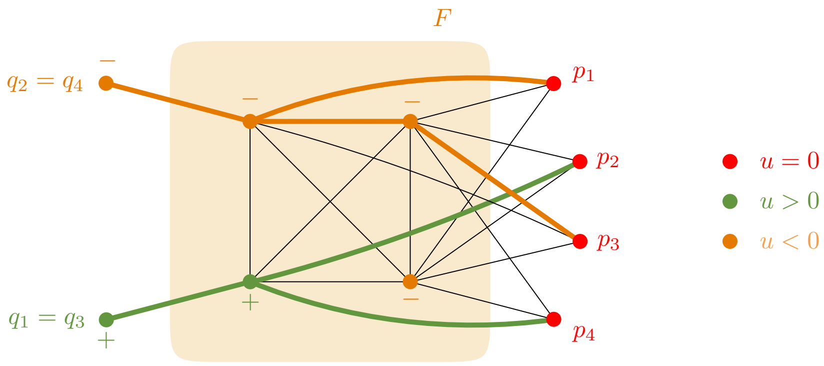


Figure 6.4 The sign pattern of $\mathcal{P}_q(g)$ on this network.

6.3 Motivation

First, let us clarify that a *symmetric and diagonally dominant M–matrix* is a symmetric and diagonally dominant matrix with non–positive off–diagonal entries and positive eigenvalues.

For the case of the combinatorial laplacian, Curtis et *al.* characterized in [38] those singular, symmetric and diagonally dominant M–matrices that are the response matrix of a circular planar network. This case corresponds to $\lambda = 0$, a constant weight ω and Λ the associated Dirichlet–to–Neumann map. Our purpose is to infer this result to the general case in order to include a wider class of M–matrices. Thus, the objective is to characterize the kernels on $\delta(F) \times \delta(F)$ that are the kernels associated with a Dirichlet–to–Robbin map.

However, this generalization is not straightforward. One of the most surprising facts of this extension is that any response matrix $\mathsf{N_q}$ can be the response matrix of an infinite family of Schrödinger operators associated with different conductivity functions and potentials. This occurs even with the response matrices associated with the combinatorial laplacian. The reason is that the Dirichlet–to–Robin matrices do not identify unambiguously the associated difference operator. In other words, a given symmetric and diagonally dominant M–matrix can be the response matrix associated with multiple Schrödinger operators of a circular planar network. This lack of uniqueness is due to the fact that the eigenfunction correlated with the lowest eigenvalue of the response matrix can be extended to the network as a weight in infinite ways. Therefore, by choosing a specific extension of the positive eigenfunction associated with the lowest eigenvalue of the matrix, we get a unique Schrödinger operator whose Dirichlet–to–Robin map corresponds to the initial matrix. The following example shows these difficulties.

Let $\Gamma = (V, c)$ be the star network on $n \geq 3$ vertices with *central vertex* x_0 and *peripherical vertices* $x_1, \dots, x_n$, see Figure 6.5. Consider the set

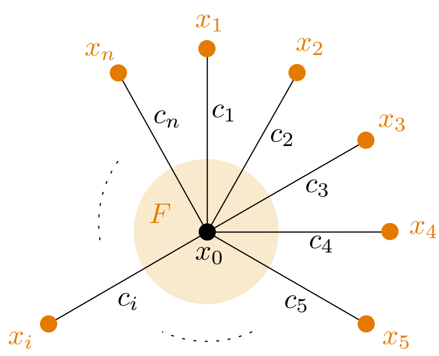


Figure 6.5 The star network on n vertices.

$F = \{x_0\} \subset V$ and its boundary, given by $\delta(F) = \{x_1, \dots, x_n\} \subset V$. The conductances of the network are given by $c(x_0, x_i) = c_i$ for all $i = 1, \dots, n$, where $c_1, \dots, c_n > 0$ are unknown real values. Then, if $\sigma \in \Omega(\bar{F})$ is a weight, it is easy to see that for any function $u \in \mathcal{C}(\bar{F})$

$$\mathcal{L}_{q_\sigma}(u)(x_0) = u(x_0) \sum_{k=1}^{n} c_k \frac{\sigma(x_k)}{\sigma(x_0)} - \sum_{k=1}^{n} c_k u(x_k) \tag{6.8}$$

and

$$\frac{\partial u}{\partial \mathsf{n}_F}(x_i) + q_\sigma(x_i) u(x_i) = \frac{c_i \sigma(x_0)}{\sigma(x_i)} u(x_i) - c_i u(x_0) \tag{6.9}$$

for all $i = 1, \ldots, n$. Let us consider a function on the boundary $g \in \mathcal{C}(\delta(F))$. From Corollary 2.5.1 we know that the Poisson problem $\mathcal{L}_{q_\sigma}(u) = 0$ on F and $u = g$ on $\delta(F)$ has a unique solution given by $u = \mathcal{P}_{q_\sigma}(g) \in \mathcal{C}(\bar{F})$. Then, $u = g$ on $\delta(F)$ and using Equation (6.8) we easily see that

$$u(x_0) = \frac{\sum\limits_{k=1}^{n} c_k g(x_k)}{\sum\limits_{k=1}^{n} c_k \sigma(x_k)}\, \sigma(x_0). \tag{6.10}$$

We now want to obtain the expression of the Dirichlet–to–Robin map with respect to the Schrödinger operator $\mathcal{L}_{q_\sigma}$. Although Λ_{q_σ} is the usual notation, here we use the notation $\Lambda_{c,\sigma}$ instead. The reason is that we will consider other conductances and weights for this same star network and will compare their associated Dirichlet–to–Robin maps, and so we need to tell them apart clearly. Since $\Lambda_{c,\sigma}(g) = \dfrac{\partial \mathcal{P}_{q_\sigma}(g)}{\partial \mathsf{n}_F} + q_\sigma \mathcal{P}_{q_\sigma}(g)$ on $\delta(F)$ for all $g \in \mathcal{C}(\delta(F))$, using Equations (6.9) and (6.10) we get that

$$\Lambda_{c,\sigma}(g)(x_i) = \frac{g(x_i)}{\sigma(x_i)}\, c_i \sigma(x_0) - \frac{\sum\limits_{k=1}^{n} c_k g(x_k)}{\sum\limits_{k=1}^{n} c_k \sigma(x_k)}\, c_i \sigma(x_0) \tag{6.11}$$

for all $i = 1, \ldots, n$. Clearly, $\Lambda_{c,\sigma}(g) = 0$ on $\delta(F)$ if and only if g is a multiple of $\sigma \chi_{\delta(F)}$.

We define a weight $\hat{\sigma} \in \Omega(\delta(F))$ on the boundary given by

$$\hat{\sigma}(x_i) = \frac{\sigma(x_i)}{\sqrt{1 - \sigma(x_0)^2}}$$

for $i = 1, \ldots, n$. Then, $\hat{\sigma}$ is the unique eigenfunction of $\Lambda_{c,\sigma}$ corresponding to the null eigenvalue that is also a weight on $\delta(F)$. Now, for any real value $0 < \omega_0 < 1$, we consider a new weight $\omega \in \Omega(\bar{F})$ on the star network defined as $\omega = \omega_0$ on F and $\omega = \sqrt{1 - \omega_0^2}\, \hat{\sigma}$ on $\delta(F)$. Notice that ω has been built by normalizing an arbitrary extension of $\hat{\sigma}$ to $\bar{F}$. We wonder if there exists a conductivity function $d \colon V \times V \longrightarrow [0, +\infty)$ on Γ such that $\Lambda_{d,\omega} = \Lambda_{c,\sigma}$, where $d(x_0, x_i) = d_i > 0$ for all $i = 1, \ldots, n$. From Equation (6.11) this occurs if and only if

$$\frac{g(x_i)}{\sigma(x_i)}\, c_i \sigma(x_0) - \frac{\sum\limits_{k=1}^{n} c_k g(x_k)}{\sum\limits_{k=1}^{n} c_k \sigma(x_k)}\, c_i \sigma(x_0) = \frac{g(x_i)}{\alpha \sigma(x_i)}\, d_i \omega_0 - \frac{\sum\limits_{k=1}^{n} d_k g(x_k)}{\alpha \sum\limits_{k=1}^{n} d_k \sigma(x_k)}\, d_i \omega_0$$

for all $i = 1, \ldots, n$, where $\alpha = \sqrt{\dfrac{1-\omega_0^2}{1-\sigma^2(x_0)}}$. If $g = \varepsilon_{x_j} \in \mathcal{C}(\delta(F))$ for some $j \in \{1, \ldots, n\}$, then

$$\frac{c_i c_j \sigma(x_0)}{\sum\limits_{k=1}^{n} c_k \sigma(x_k)} = \frac{d_i d_j \omega_0}{\alpha \sum\limits_{k=1}^{n} d_k \sigma(x_k)}$$

for all $i \neq j$. If we define the value $\beta = \dfrac{\alpha\sigma(x_0) \sum\limits_{k=1}^{n} d_k \sigma(x_k)}{\omega_0 \sum\limits_{k=1}^{n} c_k \sigma(x_k)}$, then it is satisfied that $\beta = \dfrac{d_i d_j}{c_i c_j}$ for all $i, j = 1, \ldots, n$ with $i \neq j$ and therefore

$$\frac{d_i d_j}{c_i c_j} = \frac{d_k d_j}{c_k c_j}$$

for all $i, k = 1, \ldots, n$ with $i, k \neq j$. Since $n \geq 3$, we deduce that $\dfrac{d_i}{c_i} = \dfrac{d_j}{c_j}$ for all $i, j = 1, \ldots, n$. Hence,

$$d_j = \frac{\alpha\sigma(x_0)}{\omega_0} c_j = \sqrt{\frac{\sigma^2(x_0)(1-\omega_0^2)}{\omega_0^2(1-\sigma^2(x_0))}}\, c_j$$

for all $j = 1, \ldots, n$. Notice that $d = c$ if and only if $\omega = \sigma$.

In conclusion, given a star network on $n \geq 3$ vertices with conductances c and a weight $\sigma \in \Omega(\bar{F})$ on it, there exists a conductivity function d and a modification of the weight $\omega \in \Omega(\bar{F})$ such that the Dirichlet–to–Robin operator $\Lambda_{d,\omega}$ remains the same as $\Lambda_{c,\sigma}$. That is, two different Schrödinger operators of a same network have the same associated response matrix.

In particular, if σ is a constant weight then $q_\sigma = 0$ on $\bar{F}$ and hence the corresponding Schrödinger operator is the combinatorial laplacian. This means that in this case the Dirichlet–to–Robin map becomes the classical Dirichlet–to–Neumann map. Thus, the above results indicate that the Dirichlet–to–Neumann map does not identify uniquely the Schrödinger operator of a circular planar network. In fact, it appears as the Dirichlet–to–Robin map of an infinite family of Schrödinger operators.

This motivational example is a hint of the difficulty when solving inverse boundary value problems on finite networks associated with an Schrödinger operator.

6.4 Characterization of the Dirichlet–to–Robin map

Now we are prepared to generalize the results of [38] and give a characterization for circular planar networks through the Dirichlet–to–Robin map. We follow the terminology in this paper.

Given a network $\Gamma = (V, c)$ and a proper connected subset $F \subset V$ such that $|\delta(F)| \geq 2$, let us consider a weight $\sigma \in \Omega(\delta(F))$ on the boundary and an real value $\lambda \geq 0$. Let $\Phi_{\lambda,\sigma}$ be the set of irreducible and symmetric matrices $\mathsf{M} \in \mathcal{M}_{|\delta(F)|\times|\delta(F)|}(\mathbb{R})$ for which λ is the lowest eigenvalue, with eigenvector σ, and such that if $\mathsf{M}(\mathsf{P};\mathsf{Q})$ is a $k \times k$ circular minor, then $(-1)^k \det(\mathsf{M}(\mathsf{P};\mathsf{Q})) \geq 0$. This condition says that if $\mathsf{M} \in \Phi_{\lambda,\sigma}$ and $(P;Q)$ is a circular pair of indices, then the matrix $-\mathsf{M}(\mathsf{P};\mathsf{Q})$ is totally non–negative. In particular, if $\mathsf{M} \in \Phi_{\lambda,\sigma}$, then M is an M–matrix.

When σ is a constant weight on the boundary and $\lambda = 0$, we denote $\Phi = \Phi_{0,\sigma}$, for which the next results were obtained in [38].

Lemma 6.4.1 ([38, Theorem 3]). *Let $\mathsf{M} \in \Phi$. Then, there exists a circular planar graph with conductivity function c such that $\mathsf{M} = \mathsf{N}$, where N is the Dirichlet–to–Neumann matrix.*

Notice that the equality $\Phi = D_\sigma \Phi_{0,\sigma} D_\sigma$ holds for any arbitrary non–constant weight $\sigma \in \Omega(\delta(F))$

Consider a circular planar network $\Gamma = (V, c)$ with n boundary vertices. We define the set $\pi = \pi(\Gamma)$ as the set of circular pairs $(P;Q)$ that are connected through Γ. Then, the subset $\Phi_{\lambda,\sigma}(\pi) \subset \Phi_{\lambda,\sigma}$ is defined as the set of matrices $\mathsf{M} \in \Phi_{\lambda,\sigma}$ such that satisfy the following condition: if $(P;Q)$ is a circular pair of indices, then $(P;Q) \in \pi$ if and only if $(-1)^k \det(\mathsf{M}(\mathsf{P};\mathsf{Q})) > 0$.

Lemma 6.4.2 ([38, Theorem 4]). *Let $\Gamma = (V, c)$ be a critical circular planar graph with m edges and $\pi = \pi(\Gamma)$. Then, the map that sends c to Λ is a diffeomorphism of $(\mathbb{R}^+)^m$ onto $\Phi(\pi)$.*

Given a matrix $\mathsf{M} \in \Phi_{\lambda,\sigma}$, we say that a function $u \in \mathcal{C}(\bar{F})$ is M*–harmonic* if $\mathcal{L}(u) = 0$ on F, where $\mathcal{L}$ is the combinatorial laplacian whose conductivity function is the only one associated with the matrix $\mathsf{N} = \mathsf{D}_\sigma \cdot (\mathsf{M} - \lambda\mathsf{ID}) \cdot \mathsf{D}_\sigma$ given in Lemma 6.4.1.

Theorem 6.4.3 (Characterization through the Dirichlet–to–Robin map). *Let $n \geq 2$, $\sigma \in \Omega(\delta(F))$ and $\lambda \geq 0$. Suppose that $\mathsf{M} \in \Phi_{\lambda,\sigma}$. Then, there is a circular planar network with conductivity function c such that for* *any* $\omega \in \Omega(\bar{F})$ *satisfying* $\omega = k\sigma$ *on* $\delta(F)$*, then* $\mathsf{M} = \mathsf{N_q}$*, where* Λ_q *is the Dirichlet–to–Robin map associated with the operator* $\mathcal{L}_q$ *for* $q = q_\omega + \lambda\chi_{\delta(F)}$ *and conductances* $c_\omega = \dfrac{c}{\omega \otimes \omega}$*. Moreover, if* $\mathsf{M} \in \Phi_{\lambda,\sigma}(\pi)$*, then there is a unique critical circular planar network with conductivity function* c *and a unique* $\omega \in \Omega(\bar{F})$*,* M*–harmonic weight such that* $\mathsf{M} = \mathsf{N_q}$*.*

Proof. Let $\mathsf{M} \in \Phi_{\lambda,\sigma}$. Then, the matrix $\hat{\mathsf{M}} = \mathsf{D}_\sigma \cdot (\mathsf{M} - \lambda \mathsf{ID}) \cdot \mathsf{D}_\sigma$ is in Φ. Applying Lemma 6.4.1, there exists a circular planar network with a conductivity function c such that $\hat{\mathsf{M}} = \mathsf{N}$, where Λ is the Dirichlet–to–Neumann map associated with the combinatorial laplacian $\mathcal{L}^c$ of the network. The superindex c is necessary here because we need to distinguish this operator from the laplacians of the same network associated with other conductivity functions.

Consider now a weight $\widetilde{\omega} \in \mathcal{C}(\bar{F})$ such that $\widetilde{\omega} = \sigma$ on $\delta(F)$ and $\widetilde{\omega} > 0$ on F, that is, an extension of σ to $\bar{F}$. If we define $\omega = ||\widetilde{\omega}||_{\bar{F}}^{-1}\widetilde{\omega}$, then $\omega \in \Omega(\bar{F})$ is a normalized extension of σ to F, since $\omega = k\sigma$ on $\delta(F)$ for $k = (\int_{\delta(F)} \omega^2)^{\frac{1}{2}}$. Consider now the conductivity function $c_\omega = \dfrac{c}{\omega \otimes \omega} \in \mathcal{C}(\bar{F} \times \bar{F})$ and its associated combinatorial laplacian $\mathcal{L}^{c_\omega}$. Then, applying the Doob transform, we obtain the identities $\mathcal{D}_\omega \circ \mathcal{L}^{c_\omega}_{q_\omega} \circ \mathcal{D}_\omega = \mathcal{L}^c$ and $\mathcal{D}_\omega \circ \Lambda_{q_\omega} \circ \mathcal{D}_\omega = \Lambda$. Hence, $\mathsf{M} - \lambda\mathsf{ID}$ is the matrix associated with Λ_{q_ω}.

On the other hand, if $\mathsf{M} \in \Phi_\sigma(\pi)$, then by Lemma 6.4.2 there exists a unique critical circular planar network with conductivity function c such that $\mathsf{N} = \hat{\mathsf{M}} = \mathsf{D}_\sigma \cdot (\mathsf{M} - \lambda\mathsf{ID}) \cdot \mathsf{D}_\sigma$. In addition, if we choose $\widetilde{\omega}$ the unique solution of the Dirichlet problem

$$\mathcal{L}^c(\widetilde{\omega}) = 0 \text{ on } F \quad \text{and} \quad \widetilde{\omega} = \sigma \text{ on } \delta(F),$$

then by the minimum principle $\widetilde{\omega} > 0$ on F and hence $\omega = ||\widetilde{\omega}||_{\bar{F}}^{-1}\widetilde{\omega} \in \Omega(\bar{F})$ is a M–harmonic function with $\mathsf{M} = \mathsf{N}_{\mathsf{q}_\omega}$.

Finally, let $\tau \in \Omega(\bar{F})$ such that it is M–harmonic and $\mathsf{N}_{\mathsf{q}_\tau} = \mathsf{M} = \mathsf{N}_{\mathsf{q}_\omega}$. Then, $0 = \Lambda_{q_\tau}(\tau) = \Lambda_{q_\omega}(\tau)$, which implies that $\tau = \alpha\omega$ on $\bar{F}$ because τ is M–harmonic. For both τ and ω are weights, necessarily $\alpha = 1$. □

The reader is maybe interested in the type of characterization that Theorem 6.4.3 provides. Curtis and Morrow show in [37] that, under certain transformations that preserve the electrical behaviour of the network (and therefore its response matrix), all the critical circular planar networks with n boundary vertices are equal to G_n, see the above reference for its technical definition.

Basically, the networks G_n are defined depending on the value of n modulus 4. In Figure 6.6 we show the structure of these networks for $n = 8, \ldots, 11$. Note that for $n \equiv 3 \pmod 4$, G_n is what we call a well–connected spider network, which will be defined in Section 7.7.

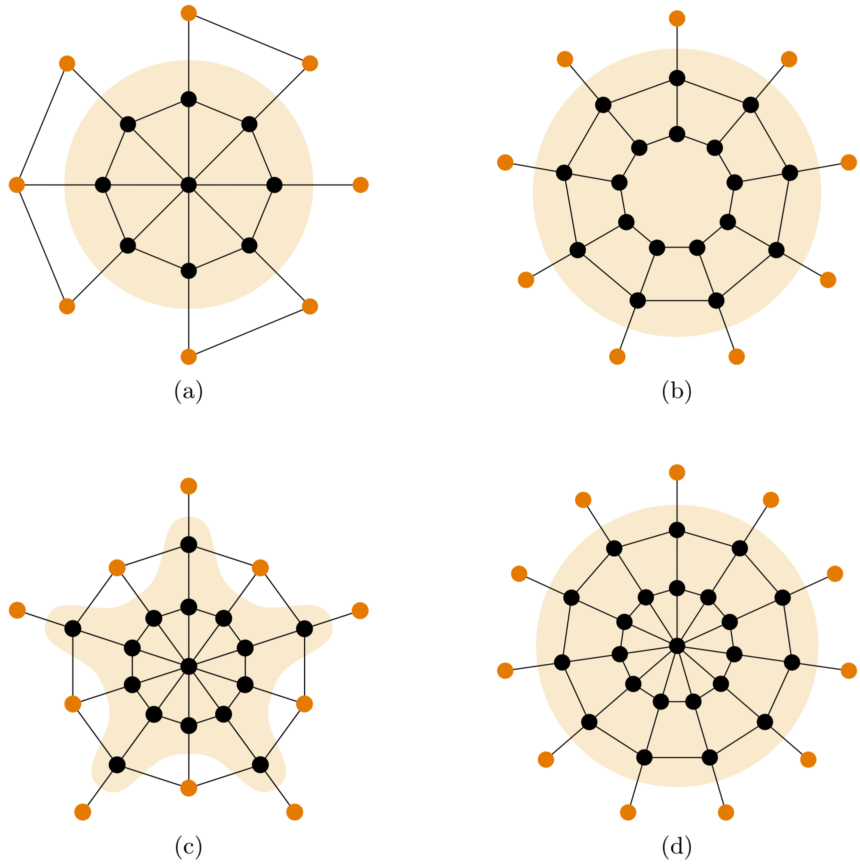


Figure 6.6 Some examples of G_n for $n = 8, \ldots, 11$.

Hence, Theorem 6.4.3 tells us that the unique critical circular planar network with Dirichlet–to–Robin matrix an arbitrary given matrix $\mathsf{M} \in \Phi_{\lambda,\sigma}(\pi)$ is the network G_n with an appropiate conductivity function and an appropiate weight. If the critical property is not imposed, then Theorem 6.4.3 states that there exists an infinite family of networks with equivalent electrical behaviour and satisfying that $\mathsf{N_q} = \mathsf{M}$.

7

Conductivity recovery

In this chapter we also work with inverse boundary value problems on finite networks. Thus, we assume the conductances of the networks to be unknown. The aim is to obtain, totally or partially, the conductivity function of as many families of networks as attainable using overdetermined partial Dirichlet–Neumann boundary value problems. However, it is not always feasible to recover the conductances of a network due to the ill–posing of the problem. For this reason we focus in two big families of networks with properties that yield to a better posing of the problem and look upon the solutions of these overdetermined partial problems on them. Networks with separated boundary and circular planar networks are the two families we operate with.

First, we perform a detailed study of the overdetermined partial Dirichlet–Neumann boundary value problems and characterize their solutions through three operators. Afterwards, we give explicit formulae for the acquirement of boundary spike conductances on these networks and execute a full conductance recovery for certain subfamilies of them.

Depending on the nature of the family of networks we work with, we use two different methods for the recovery of boundary spike conductances. Both methods are necessary, for each family of networks we work with finds utility in one of the methodologies but not in the other one.

Like in the previous chapter, in this chapter we work with potentials shaped as $q = q_\omega + \lambda\chi_{\delta(F)} \in \mathcal{C}(\bar{F})$ for a weight $\omega \in \Omega(\bar{F})$ and a real value $\lambda \geq 0$. Thus, from the lowest eigenvalue of the Dirichlet–to–Robin map and its associated eigenfunction we can recover λ and $\omega\chi_{\delta(F)}$.

The reader will find the results given in this chapter in two different publications, [11, 12].

7.1 Overdetermined partial boundary value problems

Let $\Gamma = (V, c)$ be a network and $F \subset V$ a proper connected subset. Here we consider a new type of boundary value problems on Γ in which the values of the functions and their normal derivatives are known at the same part of the boundary, which represents an overdetermined problem, and there exists another part of the boundary where no data is known.

Let $A, B \subset \delta(F)$ be two non–empty boundary subsets such that $A \cap B = \emptyset$. We denote by $R = \delta(F) \setminus (A \cup B)$ the other part of the boundary, so $\delta(F) = A \cup B \cup R$ is a partition of $\delta(F)$. Notice that R can be an empty set. For any three known functions $f \in \mathcal{C}(F)$, $g \in \mathcal{C}(A \cup R)$ and $h \in \mathcal{C}(A)$, the *overdetermined partial Dirichlet–Neumann boundary value problem on F with data f, g, h* consists in finding a function $u \in \mathcal{C}(\bar{F})$ such that

$$\mathcal{L}_q(u) = f \text{ on } F, \quad u = g \text{ on } A \cup R \quad \text{and} \quad \frac{\partial u}{\partial \mathsf{n}_F} = h \text{ on } A, \qquad (7.1)$$

where $q = q_\omega + \lambda \chi_{\delta(F)}$ on $\bar{F}$ for a weight $\omega \in \Omega(\bar{F})$ and a real value $\lambda \geq 0$.

For $B = \emptyset$, this kind of problem has been considered in the continuous case as an extension of Serrin's problem, see [45].

We would like to remark here that the discrete Serrin's problem is the extreme case of the overdetermined partial Dirichlet–Neumann boundary value problem if we allow A to be the whole boundary set. Specifically, the discrete Serrin's problem consists in determining whether a function $u \in \mathcal{C}(F)$ such that

$$\mathcal{L}(u) = 1 \text{ on } F, \quad u = 0 \text{ on } A \cup R \quad \text{and} \quad \frac{\partial u}{\partial \mathsf{n}_F} = C \text{ on } A$$

exists, where $A = \delta(F)$ and $B = R = \emptyset$ is the above–mentioned extreme situation, see Section 5. However, the existence of this function u is not always satisfied. If it exists, then we know that u is unique and it is given by $u = \nu^F \in \mathcal{C}(F)$.

Going back to the general overdetermined partial Dirichlet–Neumann boundary value problem (7.1), an extended study of the existence and uniqueness of its solution is performed in this section. There are two important details that must be observed. The first one is that the boundary conditions fix the values of any solution u and of its normal derivative $\dfrac{\partial u}{\partial \mathsf{n}_F}$ on A. Hence, A is the overdetermined set of the boundary, whereas there exist no requirements on B. The second detail is related to the potential q on A. The reader may ask why the problem does not consider the condition $\dfrac{\partial u}{\partial \mathsf{n}_F} + qu = \hat{h}$ on A and considers $\dfrac{\partial u}{\partial \mathsf{n}_F} = h$ instead. The answer is that both conditions are equivalent taking $h = \hat{h} - qg$, as we already know that $u = g$ on A. For the sake of simplicity, we choose the second condition.

The *homogeneous overdetermined partial Dirichlet–Neumann boundary value problem on F* consists in finding a function $u \in \mathcal{C}(\bar{F})$ such that

$$\mathcal{L}_q(u) = 0 \text{ on } F, \quad u = 0 \text{ on } A \cup R \text{ and } \frac{\partial u}{\partial \mathsf{n}_F} = 0 \text{ on } A \tag{7.2}$$

It is clear that the set of solutions of this problem is a subspace of $\mathcal{C}(F \cup B)$, which we denote by $\mathcal{V}_B$. The following results are straightforward.

Lemma 7.1.1. *If the overdetermined partial Dirichlet–Neumann boundary value problem (7.1) has solutions and u is a particular one, then $u + \mathcal{V}_B$ describes the set of all of its solutions.*

Lemma 7.1.2. *If u is a solution of the overdetermined partial Dirichlet–Neumann boundary value problem (7.1), then for any $x \in A$ we get that*

$$\int_F c(x,y)u(y)dy = g(x)\kappa_F(x) - h(x).$$

Corollary 7.1.3. *If u is a solution of the homogeneous overdetermined partial Dirichlet–Neumann boundary value problem (7.2), then for any $x \in A$ we get that*

$$\int_F c(x,y)u(y)dy = 0.$$

On the other hand, the *adjoint overdetermined problem* of the homogeneous overdetermined partial Dirichlet–Neumann boundary value problem consists in finding a function $v \in \mathcal{C}(\bar{F})$ such that

$$\mathcal{L}_q(v) = 0 \text{ on } F, \quad v = 0 \text{ on } B \cup R \text{ and } \frac{\partial v}{\partial \mathsf{n}_F} = 0 \text{ on } B. \tag{7.3}$$

The subspace of solutions of this problem is denoted by $\mathcal{V}_A \subseteq \mathcal{C}(F \cup A)$.

Proposition 7.1.4. *The homogeneous overdetermined partial Dirichlet–Neumann boundary value problem (7.2) and its Adjoint problem (7.3) are mutually adjoint, that is*

$$\int_F v\mathcal{L}_q(u) = \int_F u\mathcal{L}_q(v)$$

for any pair of functions $u, v \in \mathcal{C}(\bar{F})$ *such that* $\dfrac{\partial u}{\partial \mathsf{n}_\mathsf{F}} = u = 0$ *on* A, $\dfrac{\partial v}{\partial \mathsf{n}_\mathsf{F}} = v = 0$ *on* B *and* $u = v = 0$ *on* R.

Proof. By the Second Green Identity we get that

$$\int_F \Big(v\mathcal{L}_q(u) - u\mathcal{L}_q(v)\Big) = \int_{\delta(F)} \Big(u\frac{\partial v}{\partial \mathsf{n}_\mathsf{F}} - v\frac{\partial u}{\partial \mathsf{n}_\mathsf{F}}\Big) = \int_B u\frac{\partial v}{\partial \mathsf{n}_\mathsf{F}} - \int_A v\frac{\partial u}{\partial \mathsf{n}_\mathsf{F}} = 0,$$

obtaining the result. □

Proposition 7.1.5 (Fredholm alternative)**.** *Given* $f \in \mathcal{C}(F)$, $g \in \mathcal{C}(A \cup R)$ *and* $h \in \mathcal{C}(A)$, *the overdetermined partial Dirichlet–Neumann boundary value problem*

$$\mathcal{L}_q(u) = f \ \text{ on } \ F, \quad u = g \ \text{ on } \ A \cup R \quad \text{and} \quad \frac{\partial u}{\partial \mathsf{n}_F} = h \ \text{ on } \ A \tag{7.4}$$

has solution if and only if

$$\int_F fv + \int_A hv = \int_{A\cup R} g\frac{\partial v}{\partial \mathsf{n}_\mathsf{F}}$$

for every $v \in \mathcal{V}_A$. *When the above condition holds, there exists a unique solution of Problem* (7.4) *in* $\mathcal{V}_B^{\perp}$, *that is, a unique solution* u *such that*

$$\int_{F\cup B} uz = 0$$

for any $z \in \mathcal{V}_B$.

Proof. Let us consider the following problem

$$\mathcal{L}_q(\tilde{v}) = f - \mathcal{L}(g) \ \text{ on } \ F, \quad \tilde{v} = 0 \ \text{ on } \ A \cup R \quad \text{and} \quad \frac{\partial \tilde{v}}{\partial \mathsf{n}_F} = h - g\kappa_F \ \text{ on } \ A. \tag{7.5}$$

Observe that solving this problem is equivalent to solve Problem (7.4), since $\tilde{v}$ is a solution of Problem (7.5) if and only if $u = \tilde{v} + g$ is a solution of Problem (7.4).

Consider now the following linear operators $\mathcal{F}\colon \mathcal{C}(F\cup B)\longrightarrow \mathcal{C}(F\cup A)$ and $\mathcal{F}^*\colon \mathcal{C}(F\cup A)\longrightarrow \mathcal{C}(F\cup B)$ defined as $\mathcal{F}(u)=\mathcal{L}_q(u)$ on F, $\mathcal{F}(u)=\dfrac{\partial u}{\partial \mathsf{n}_F}$ on A, $\mathcal{F}^*(v)=\mathcal{L}_q(v)$ on F and $\mathcal{F}^*(v)=\dfrac{\partial v}{\partial \mathsf{n}_F}$ on B. Clearly, $\ker(\mathcal{F}^*)=\mathcal{V}_A$ and $\ker(\mathcal{F})=\mathcal{V}_B$. Moreover, using the Second Green Identity, we see that

$$\begin{aligned}\int_{F\cup A} v\mathcal{F}(u) &= \int_F v\mathcal{L}_q(u)+\int_{\delta(F)} v\frac{\partial u}{\partial \mathsf{n}_\mathsf{F}}=\\ &=\int_F u\mathcal{L}_q(v)+\int_{\delta(F)} u\frac{\partial v}{\partial \mathsf{n}_\mathsf{F}}=\int_{F\cup B} u\mathcal{F}^*(v)\end{aligned}\tag{7.6}$$

for any $u\in\mathcal{C}(F\cup B)$ and $v\in\mathcal{C}(F\cup A)$. Let $\tilde f\in\mathcal{C}(F\cup A)$ be a new function given by $\tilde f=f-\mathcal{L}(g)$ on F and $\tilde f=h-g\kappa_F$ on A. Then, Problem (7.4) has solution if and only if $\tilde f\in \operatorname{Img}(\mathcal{F})$ or, equivalently, if and only if there exists a function $\tilde v\in\mathcal{C}(F\cup B)$ such that $\tilde f=\mathcal{F}(\tilde v)$. Hence, using the last equality, Problem (7.4) has solution if and only if

$$\begin{aligned}0&=\int_{F\cup B}\tilde v\mathcal{F}^*(v)=\int_{F\cup A}\mathcal{F}(\tilde v)v=\int_{F\cup A}\tilde f v=\int_F fv+\int_A hv\\ &-\int_F v\mathcal{L}(g)-\int_A vg\kappa_F=\int_F fv+\int_A hv-\int_{R\cup A} g\frac{\partial v}{\partial \mathsf{n}_\mathsf{F}}\end{aligned}$$

for every $v\in\mathcal{V}_A=\ker(\mathcal{F}^*)$. Finally, assume that this necessary and sufficient condition holds. Then, the classical Fredholm Alternative for linear operators establishes that there exists a unique $w\in(\ker(\mathcal{F}))^\perp=\mathcal{V}_B^\perp$ such that $\mathcal{F}(w)=\tilde f$. Observe that w and $\widetilde v$ fulfill the same properties taken into use until now and hence $u=w+g$ is the unique solution of Problem (7.4) such that for any $z\in\ker(\mathcal{F})=\mathcal{V}_B$

$$\int_{F\cup B} uz=\int_{F\cup B} wz+\int_{F\cup B} gz=0.\quad\square$$

Proposition 7.1.6. *Given the Dirichlet–Neumann boundary value problem* (7.1) *with data $f\in\mathcal{C}(F)$, $g\in\mathcal{C}(A\cup R)$ and $h\in\mathcal{C}(A)$, the following formula holds*

$$\dim(\mathcal{V}_A)-\dim(\mathcal{V}_B)=|A|-|B|.$$

Furthermore, the existence of solution of Problem (7.1) *for any data is equivalent to the condition $\mathcal{V}_A=\{0\}$ and the uniqueness of solution for any data is equivalent to the condition $\mathcal{V}_B=\{0\}$. In particular, if $|A|=|B|$, the existence of solution for any data is equivalent to the uniqueness of solution for any data and hence it is equivalent to the condition $\mathcal{V}_A=\mathcal{V}_B=\{0\}$.*

Proof. Consider again the linear operators $\mathcal{F}\colon \mathcal{C}(F\cup B)\longrightarrow \mathcal{C}(F\cup A)$ and $\mathcal{F}^*\colon \mathcal{C}(F\cup A)\longrightarrow \mathcal{C}(F\cup B)$ defined in the proof of the last result. Observe that $\mathcal{F}^*$ is the adjoint operator of $\mathcal{F}$ because of the Identity (7.6). Therefore, as the classical Fredholm Alternative establishes that $\operatorname{Img}(\mathcal{F}) = \ker(\mathcal{F}^*)^{\perp}$, we get that

$$\dim(\operatorname{Img}(\mathcal{F})) = \dim(\ker(\mathcal{F}^*)^{\perp}) = \dim(\mathcal{V}_A^{\perp}) = |F| + |A| - \dim(\mathcal{V}_A).$$

On the other hand,

$$\dim(\operatorname{Img}(\mathcal{F})) = |F| + |B| - \dim(\ker(\mathcal{F})) = |F| + |B| - \dim(\mathcal{V}_B).$$

Finally, the rest of assertions are a direct consequence of Proposition 7.1.5. □

Observe that if $|A| > |B|$ then Problem (7.1) does not have existence of solution in general. On the other hand, if $|B| > |A|$ then Problem (7.1) does not hold uniqueness of solution in general. Finally, if $|A| = |B|$ then the last result says that there is existence and uniqueness of solution for any data if and only if the unique solution of the homogeneous overdetermined Problem (7.2) is $u = 0$.

This result motivates the definition of two maps derived from the Dirichlet–to–Robin map: the partial Dirichlet–to–Robin maps, since it is of interest to possess a partial tool instead of a global one. Let us consider the linear operators $\Lambda_{A,B}\colon \mathcal{C}(A)\longrightarrow \mathcal{C}(B)$ and $\Lambda_{B,A}\colon \mathcal{C}(B)\longrightarrow \mathcal{C}(A)$ defined as

$$\Lambda_{A,B}(f) = \frac{\partial \mathcal{P}_q(f)}{\partial \mathsf{n}_F}\chi_B \quad \text{on } B \qquad \text{and} \qquad \Lambda_{B,A}(g) = \frac{\partial \mathcal{P}_q(g)}{\partial \mathsf{n}_F}\chi_A \quad \text{on } A$$

for all $f \in \mathcal{C}(A)$ and $g \in \mathcal{C}(B)$. They are the *partial Dirichlet–to–Robin maps*. Notice that $\Lambda_{A,B}(f) = \Lambda_q(f)\chi_B$ and $\Lambda_{B,A}(g) = \Lambda_q(g)\chi_A$.

Proposition 7.1.7. *The partial Dirichlet–to–Robin maps satisfy that* $\Lambda^*_{A,B} = \Lambda_{B,A}$. *In addition,* $\ker(\Lambda_{A,B}) = \mathcal{V}_A\chi_A$ *and* $\ker(\Lambda_{B,A}) = \mathcal{V}_B\chi_B$.

Proof. Let $v \in \mathcal{C}(A)$ and $w \in \mathcal{C}(B)$. Then, $\mathcal{L}_q(\mathcal{P}_q(v)) = \mathcal{L}_q(\mathcal{P}_q(w)) = 0$ on F, $v = \mathcal{P}_q(v)$ on $\delta(F)$ and $w = \mathcal{P}_q(w)$ on $\delta(F)$. Using the Second Green Identity we get that

$$\int_B w\Lambda_{A,B}(v) = \int_{\delta(F)} \mathcal{P}_q(w)\frac{\partial \mathcal{P}_q(v)}{\partial \mathsf{n}_F} = \int_{\delta(F)} \mathcal{P}_q(v)\frac{\partial \mathcal{P}_q(w)}{\partial \mathsf{n}_F} = \int_A v\Lambda_{B,A}(w).$$

Now, let $v \in \mathcal{C}(A)$ be such that $v \in \ker(\Lambda_{A,B})$. Then, clearly $v = \mathcal{P}_q(v)\chi_A$ on A and $\mathcal{P}_q(v) \in \mathcal{V}_A$. Conversely, if $u \in \mathcal{C}(A)$ is such that $u \in \mathcal{V}_A$ then

$\mathcal{L}_q(u) = 0$ on F and $\dfrac{\partial u}{\partial \mathsf{n}_\mathsf{F}} = 0$ on B. Therefore, $u \in \ker(\Lambda_{A,B})$. That is, $\ker(\Lambda_{A,B}) = \mathcal{V}_A \chi_A$. The result for $\ker(\Lambda_{B,A})$ follows analogously. □

Let $N_{A,B} \colon B \times A \longrightarrow \mathbb{R}$ and $N_{B,A} \colon A \times B \longrightarrow \mathbb{R}$ be the kernels of $\Lambda_{A,B}$ and $\Lambda_{B,A}$, respectively, and let $\mathsf{N}_{\mathsf{A,B}}$ and $\mathsf{N}_{\mathsf{B,A}}$ be the matrices provided by them. The next result is a characterization of the existence and uniqueness of solution of Problem (7.1) for any data in terms of necessary and sufficient conditions of the Dirichlet–to–Robin matrix N_q. It is a direct consequence of Propositions 7.1.6 and 7.1.7.

Corollary 7.1.8. *The overdetermined partial Dirichlet–Neumann boundary value problem* (7.1) *has solution for any data f, g, h if and only if $\mathsf{N}_{\mathsf{A,B}}$ has maximum rank. It has uniqueness of solution for any data if and only if $\mathsf{N}_{\mathsf{B,A}}$ has maximum rank. In particular, when $|A| = |B|$, then $\mathsf{N}_{\mathsf{A,B}}$ is non–singular if and only if $\mathsf{N}_{\mathsf{B,A}}$ is non–singular, and in this case Problem* (7.1) *has a unique solution for any data.*

In the following, we will use the matricial equivalences $\mathsf{N}_{\mathsf{A,B}} = \mathsf{N}_\mathsf{q}(\mathsf{B};\mathsf{A})$ and $\mathsf{N}_{\mathsf{B,A}} = \mathsf{N}_\mathsf{q}(\mathsf{A};\mathsf{B})$, which are straighforward. We present some examples that help to understand all the existing settings that can be found when solving Problem (7.1) on a network. We consider four different networks, shown in Figure 7.1. Notice that they all have $R = \emptyset$ in the interest of unadornment. The sets $A, B \subset \delta(F)$ and the conductances of each edge are detailed in the pictures. Also, we take the real value $\lambda = 2$ on the four networks. First, we work with Network (7.1a). It is easily seen that the adjoint overdetermined Problem (7.3) on this network has 0 as its unique solution. Hence, $\mathcal{V}_A = \{0\}$ and by Proposition 7.1.6 we get that $\dim(\mathcal{V}_B) = \dim(\mathcal{V}_A) - |A| + |B| = 2$. This means that Problem (7.1) has at least one solution for all data on Network (7.1a), but it is never unique. A deeper inspection shows that all the solutions of Problem (7.1) are given by a particular solution plus all the solutions on $\mathcal{V}_B$, which means that the set of solutions of Problem (7.1) has also dimension two for this network. As expected, by Corollary 7.1.8, $\mathsf{N}_\mathsf{q}(\mathsf{B};\mathsf{A})$ has maximum rank. Working analogously, Network (7.1b) has $\mathcal{V}_B = \{0\}$ and $\dim(\mathcal{V}_A) = 2$, which means that there is uniqueness of solution for any data but not necessarily existence, since the space of solutions of Problem (7.1) is the space given by a particular solution –if it exists. Also, $\mathsf{N}_\mathsf{q}(\mathsf{A};\mathsf{B})$ has maximum rank. Network (7.1c) satisfies that $|A| = |B|$. Nevertheless,

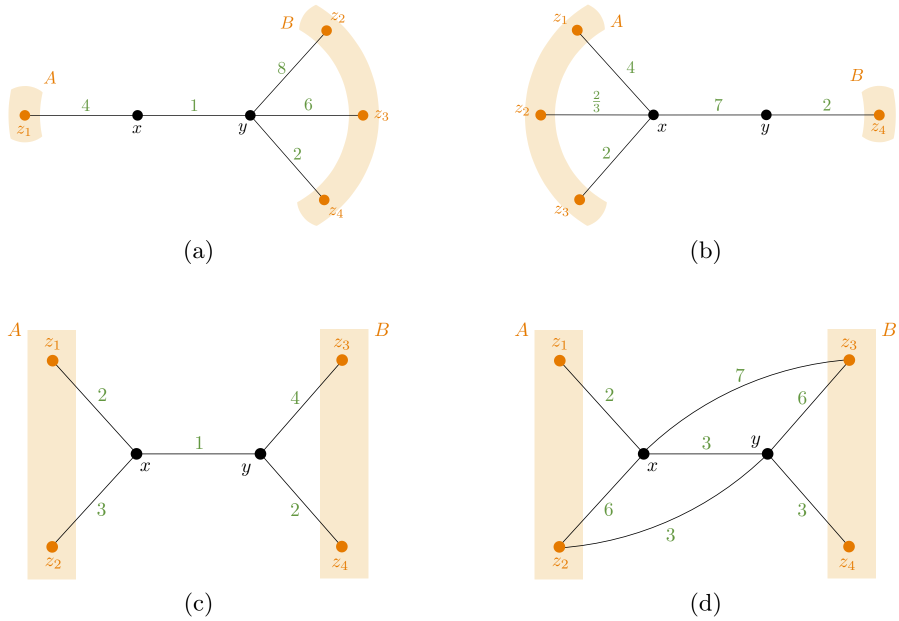


Figure 7.1 Some examples of networks on which we solve overdetermined partial boundary value problems.

$\mathcal{V}_A, \mathcal{V}_B \neq \{0\}$, which means that $\dim(\mathcal{V}_B) = \dim(\mathcal{V}_A)$. For

$$\mathsf{N_q}(\mathsf{z}_3, \mathsf{z}_1) = -\frac{2}{71}, \qquad \mathsf{N_q}(\mathsf{z}_3, \mathsf{z}_2) = -\frac{3}{71},$$

$$\mathsf{N_q}(\mathsf{z}_4, \mathsf{z}_1) = -\frac{1}{71}, \qquad \mathsf{N_q}(\mathsf{z}_4, \mathsf{z}_2) = -\frac{3}{142},$$

then $\mathsf{N_q}(\mathsf{B};\mathsf{A})$ and $\mathsf{N_q}(\mathsf{A};\mathsf{B})$ are singular. Therefore, Problem (7.1) can have either no solution at all or more than one solution on Network (7.1c), depending on the data. Finally, Network (7.1d) has $|A| = |B|$ and $\mathcal{V}_A = \mathcal{V}_B = \{0\}$, which means that Problem (7.1) has a unique solution for any data. As awaited, in this case $\mathsf{N_q}(\mathsf{B};\mathsf{A})$ and $\mathsf{N_q}(\mathsf{A};\mathsf{B})$ are non–singular because

$$\mathsf{N_q}(\mathsf{z}_3, \mathsf{z}_1) = -\frac{80}{21}, \qquad \mathsf{N_q}(\mathsf{z}_3, \mathsf{z}_2) = -\frac{121}{7},$$

$$\mathsf{N_q}(\mathsf{z}_4, \mathsf{z}_1) = -\frac{4}{7}, \qquad \mathsf{N_q}(\mathsf{z}_4, \mathsf{z}_2) = -\frac{51}{14}.$$

The next table shows some particular examples of these conclusions. The ordering on $\bar{F}$ is given by $\bar{F} = \{\delta(F); F\} = \{z_1, z_2, z_3, z_4; x, y\}$.

Γ	$\omega_{\mathsf{F}}^{\top}$	$\mathsf{g}_{\mathsf{A}}^{\top}$, $\mathsf{h}_{\mathsf{A}}^{\top}$, $\mathsf{f}_{\mathsf{F}}^{\top}$	$\mathsf{u}_{\mathsf{F}}^{\top}$
(7.1a)	$\big(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{2}{3},\frac{2}{3}\big)$	(3), (1), $(0,2)$	$\big(3,a,b,-\frac{149}{8}-4a-3b,\frac{11}{4},-\frac{13}{2}\big)$, with $a,b\in\mathbb{R}$
(7.1b)	$\big(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{2}{3},\frac{2}{3}\big)$	$(0,1,-1)$, $(2,1,-1)$, $(4,3)$	$\big(0,1,-1,-\frac{7}{2},-\frac{1}{2},-1\big)$
	$\big(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{2}{3},\frac{2}{3}\big)$	$(3,0,2)$, $(0,1,1)$, $(3,1)$	*none*
(7.1c)	$\big(\frac{1}{6},\frac{2}{3},\frac{2}{3},\frac{1}{6},\frac{1}{6},\frac{1}{6}\big)$	$(-8,2)$, $(16,54)$, $(0,1)$	$\big(-8,2,a,-\frac{4355+4a}{2},-16,-230\big)$, with $a\in\mathbb{R}$
	$\big(\frac{1}{6},\frac{2}{3},\frac{2}{3},\frac{1}{6},\frac{1}{6},\frac{1}{6}\big)$	$(1,2)$, $(0,1)$, $(-2,1)$	*none*
(7.1d)	$\big(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{2}{3},\frac{2}{3}\big)$	$(1,-2)$, $(2,0)$, $(7,-21)$	$(1,-2,3,-9,0,-6)$

7.2 Modified Green, Poisson and Robin operators

Let $\Gamma=(V,c)$ be a network with $F\subset V$ a proper connected subset. As before, consider a partition of the boundary given by $A,B\subset\delta(F)$ two non–empty subsets such that $A\cap B=\emptyset$ and $R=\delta(F)\setminus(A\cup B)$.

In this section we characterize the solution of any overdetermined partial Dirichlet–Neumann boundary value problem when it is unique. Therefore, from now on we assume that problem

$$\mathcal{L}_q(u)=f \;\text{ on }\; F,\quad u=g \;\text{ on }\; A\cup R \;\text{ and }\; \frac{\partial u}{\partial \mathsf{n}_F}=h \;\text{ on }\; A \tag{7.7}$$

has a unique solution $u\in\mathcal{C}(\bar F)$ for any data $f\in\mathcal{C}(F)$, $g\in\mathcal{C}(A\cup R)$ and $h\in\mathcal{C}(A)$. Equivalently, by Corollary 7.1.8 it is enough to assume that $|A|=|B|$ and that $\mathsf{N}_{\mathsf{q}}(\mathsf{A};\mathsf{B})$ is invertible.

Overdetermined partial Dirichlet–Neumann boundary value problems require working with other linear operators different from the Green and Poisson operators defined in Section 2.5. Our goal is to describe them. First, given three functions $f\in\mathcal{C}(F)$, $g\in\mathcal{C}(A\cup R)$ and $h\in\mathcal{C}(A)$, let us consider the following problems

$$\mathcal{L}_q(v_f)=f \;\text{ on }\; F,\quad v_f=0 \;\text{ on }\; A\cup R \;\text{ and }\; \frac{\partial v_f}{\partial \mathsf{n}_F}=0 \;\text{ on }\; A, \tag{7.8}$$

$$\mathcal{L}_q(v_g)=0 \;\text{ on }\; F,\quad v_g=g \;\text{ on }\; A\cup R \;\text{ and }\; \frac{\partial v_g}{\partial \mathsf{n}_F}=0 \;\text{ on }\; A,$$

$$\mathcal{L}_q(v_h) = 0 \text{ on } F, \quad v_h = 0 \text{ on } A \cup R \quad \text{and} \quad \frac{\partial v_h}{\partial \mathsf{n}_F} = h \text{ on } A.$$

We call them *modified Green problem* with data f, *modified Poisson problem* with data g and *modified Robin problem* with data h, respectively. By assumption, all of them have a unique solution. Hence, we are ready to define the *modified Green operator* $\widetilde{\mathcal{G}}_q\colon \mathcal{C}(F) \longrightarrow \mathcal{C}(F \cup B)$, where $\widetilde{\mathcal{G}}_q(f) = v_f$ for all $f \in \mathcal{C}(F)$, as well as the *modified Poisson operator* $\widetilde{\mathcal{P}}_q\colon \mathcal{C}(A \cup R) \longrightarrow \mathcal{C}(\bar{F})$ with $\widetilde{\mathcal{P}}_q(g) = v_g$ for all $g \in \mathcal{C}(A \cup R)$ and the *modified Robin operator* $\widetilde{\mathcal{R}}_q\colon \mathcal{C}(A) \longrightarrow \mathcal{C}(F \cup B)$ given by $\widetilde{\mathcal{R}}_q(h) = v_h$ for all $h \in \mathcal{C}(A)$.

Corollary 7.2.1. *The overdetermined partial Dirichlet–Neumann boundary value problem* (7.7) *has a unique solution $u \in \mathcal{C}(\bar{F})$ for any data $f \in \mathcal{C}(F)$, $g \in \mathcal{C}(A \cup R)$ and $h \in \mathcal{C}(A)$. This solution can be written as*

$$u = \widetilde{\mathcal{G}}_q(f) + \widetilde{\mathcal{P}}_q(g) + \widetilde{\mathcal{R}}_q(h).$$

We need to know the properties of these modified operators. Thus, what we can do is to express them in terms of the Green operator $\mathcal{G}_q$ and the Dirichlet–to–Robin map Λ_q, for whom we already know important attributes. Let $\widetilde{G}_q\colon (F \cup B) \times F \longrightarrow \mathbb{R}$, $\widetilde{P}_q\colon \bar{F} \times (A \cup R) \longrightarrow \mathbb{R}$ and $\widetilde{R}_q\colon (F \cup B) \times A \longrightarrow \mathbb{R}$ be the modified Green, Poisson and Robin *kernels*, respectively. We denote by $\widetilde{\mathsf{G}}_\mathsf{q} \in \mathcal{M}_{|F\cup B|\times|F|}(\mathbb{R})$, $\widetilde{\mathsf{P}}_\mathsf{q} \in \mathcal{M}_{|\bar{F}|\times|A\cup R|}(\mathbb{R})$ and $\widetilde{\mathsf{R}}_\mathsf{q} \in \mathcal{M}_{|F\cup B|\times|A|}(\mathbb{R})$ their associated matrices.

Proposition 7.2.2. *The blocks of the modified Green matrix $\widetilde{\mathsf{G}}_\mathsf{q}$ can be expressed in terms of the conductances, the Green and the Dirichlet–to–Robin matrices as*

$$\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{F};\mathsf{F}) = \mathsf{G}_\mathsf{q}(\mathsf{F};\mathsf{F}) + \mathsf{G}_\mathsf{q}(\mathsf{F};\mathsf{F}) \cdot \mathsf{C}(\mathsf{F};\mathsf{B}) \cdot \mathsf{N}_\mathsf{q}(\mathsf{A};\mathsf{B})^{-1} \cdot \mathsf{C}(\mathsf{A};\mathsf{F}) \cdot \mathsf{G}_\mathsf{q}(\mathsf{F};\mathsf{F}),$$

$$\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{B};\mathsf{F}) = \mathsf{N}_\mathsf{q}(\mathsf{A};\mathsf{B})^{-1} \cdot \mathsf{C}(\mathsf{A};\mathsf{F}) \cdot \mathsf{G}_\mathsf{q}(\mathsf{F};\mathsf{F}),$$

$$\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A} \cup \mathsf{R};\mathsf{F}) = 0.$$

Proof. Let $y \in F$ and let $v = v_{\varepsilon_y} = \widetilde{\mathcal{G}}_q(\varepsilon_y) \in \mathcal{C}(F \cup B)$ be the unique solution of Problem (7.8) for $f = \varepsilon_y \in \mathcal{C}(F)$. We denote its restriction to B by $v_B = v\chi_B \in \mathcal{C}(B)$. Notice that Problem (7.8) with $f = \varepsilon_y$ is equivalent to problem

$$\mathcal{L}_q(v) = \varepsilon_y \text{ on } F \quad \text{and} \quad v = v_B \text{ on } \delta(F)$$

with the additional condition

$$\frac{\partial v}{\partial \mathsf{n}_F} = 0 \text{ on } A.$$

Thus, by Identity (2.4) we know that $v = \mathcal{G}_q(\varepsilon_y) + \mathcal{P}_q(v_B)$ on $\bar{F}$ with the additional condition $\dfrac{\partial v}{\partial \mathsf{n}_F} = 0$ on A. We use the double information on A in order to obtain the expression of v_B in terms of $\mathcal{G}_q$ and Λ_q. Given a boundary vertex $x \in A$,

$$0 = \frac{\partial v}{\partial \mathsf{n}_F}(x) = \frac{\partial \mathcal{G}_q(\varepsilon_y)}{\partial \mathsf{n}_F}(x) + \frac{\partial \mathcal{P}_q(v_B)}{\partial \mathsf{n}_F}(x) = \frac{\partial \mathcal{G}_q(\varepsilon_y)}{\partial \mathsf{n}_F}(x) + \Lambda_q(v_B)(x)$$

$$- q(x)v_B(x) = -\int_F c(x,z)G_q(z,y)\,dz + \int_B N_q(x,z)v_B(z)\,dz$$

$$= -\mathsf{C}(\mathsf{x};\mathsf{F}) \cdot \mathsf{G_q}(\mathsf{F};\mathsf{y}) + \mathsf{N_q}(\mathsf{x};\mathsf{B}) \cdot \mathsf{v_B}.$$

If we consider all the vertices in A we obtain the matricial equation $\mathsf{C}(\mathsf{A};\mathsf{F}) \cdot \mathsf{G_q}(\mathsf{F};\mathsf{y}) = \mathsf{N_q}(\mathsf{A};\mathsf{B}) \cdot \mathsf{v_B}$. For $\mathsf{N_q}(\mathsf{A};\mathsf{B})$ is invertible, clearly

$$\mathsf{v_B} = \mathsf{N_q}(\mathsf{A};\mathsf{B})^{-1} \cdot \mathsf{C}(\mathsf{A};\mathsf{F}) \cdot \mathsf{G_q}(\mathsf{F};\mathsf{y}).$$

On the other hand, Lemma 6.1.9 tells us that for all $x \in F$,

$$v(x) = \mathcal{G}_q(\varepsilon_y)(x) + \mathcal{P}_q(v_B)(x) = \mathsf{G_q}(\mathsf{x};\mathsf{y}) + \mathsf{P_q}(\mathsf{x};\mathsf{B}) \cdot \mathsf{v_B}$$

$$= \mathsf{G_q}(\mathsf{x};\mathsf{y}) + \mathsf{G_q}(\mathsf{x};\mathsf{F}) \cdot \mathsf{C}(\mathsf{F};\mathsf{B}) \cdot \mathsf{v_B}$$

and therefore $\mathsf{v_F} = \mathsf{G_q}(\mathsf{F};\mathsf{y}) + \mathsf{G_q}(\mathsf{F};\mathsf{F}) \cdot \mathsf{C}(\mathsf{F};\mathsf{B}) \cdot \mathsf{v_B}$. Finally, take into account that by definition $\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{F};\mathsf{y}) = \mathsf{v_F}$ and $\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{B};\mathsf{y}) = \mathsf{v_B}$. Regarding all the possible vertices $y \in F$, we deduce the matricial equations

$$\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{B};\mathsf{F}) = \mathsf{N_q}(\mathsf{A};\mathsf{B})^{-1} \cdot \mathsf{C}(\mathsf{A};\mathsf{F}) \cdot \mathsf{G_q}(\mathsf{F};\mathsf{F}),$$

$$\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{F};\mathsf{F}) = \mathsf{G_q}(\mathsf{F};\mathsf{F}) + \mathsf{G_q}(\mathsf{F};\mathsf{F}) \cdot \mathsf{C}(\mathsf{F};\mathsf{B}) \cdot \widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{B};\mathsf{F})$$

and the result follows. □

The next Propositions show analogous results for the modified Poisson and Robin operators. The proofs are analogous to the last demonstration, so we let them to the reader.

Proposition 7.2.3. *The modified Poisson matrix* $\widetilde{\mathsf{P}}_\mathsf{q}$ *is expressed in blocks as*

$$\widetilde{\mathsf{P}}_\mathsf{q}(\mathsf{F};\mathsf{A}\cup\mathsf{R}) = \mathsf{G_q}(\mathsf{F};\mathsf{F}) \cdot \Big(\mathsf{C}(\mathsf{F};\mathsf{A}\cup\mathsf{R}) - \mathsf{C}(\mathsf{F};\mathsf{B}) \cdot \mathsf{N_q}(\mathsf{A};\mathsf{B})^{-1} \cdot \mathsf{N_q}(\mathsf{A};\mathsf{A}\cup\mathsf{R})\Big)$$

$$\widetilde{\mathsf{P}}_\mathsf{q}(\mathsf{B};\mathsf{A}\cup\mathsf{R}) = -\mathsf{N_q}(\mathsf{A};\mathsf{B})^{-1} \cdot \mathsf{N_q}(\mathsf{A};\mathsf{A}\cup\mathsf{R})$$

$$\widetilde{\mathsf{P}}_\mathsf{q}(\mathsf{A}\cup\mathsf{R};\mathsf{A}\cup\mathsf{R}) = \mathsf{ID}(\mathsf{A}\cup\mathsf{R};\mathsf{A}\cup\mathsf{R})$$

in terms of the conductances, the Green and the Dirichlet–to–Robin matrices, where ID *is the identity matrix.*

Proposition 7.2.4. *The modified Robin matrix* $\widetilde{\mathsf{R}}_\mathsf{q}$ *is given by the blocks*

$$\widetilde{\mathsf{R}}_\mathsf{q}(\mathsf{F};\mathsf{A}) = \mathsf{G}_\mathsf{q}(\mathsf{F};\mathsf{F})\cdot\mathsf{C}(\mathsf{F};\mathsf{B})\cdot\mathsf{N}_\mathsf{q}(\mathsf{A};\mathsf{B})^{-1}$$

$$\widetilde{\mathsf{R}}_\mathsf{q}(\mathsf{B};\mathsf{A}) = \mathsf{N}_\mathsf{q}(\mathsf{A};\mathsf{B})^{-1}$$

$$\widetilde{\mathsf{R}}_\mathsf{q}(\mathsf{A}\cup\mathsf{R};\mathsf{A}) = 0,$$

all in terms of the conductances, the Green and the Dirichlet–to–Robin matrices.

Hence, the last results provide the matricial expression of the solution of the overdetermined partial Dirichlet–Neumann boundary value problem (7.7) in terms of the classical Green operator and the Dirichlet–to–Robin map.

Corollary 7.2.5. *The unique solution* $u\in\mathcal{C}(\bar{F})$ *of the overdetermined partial Dirichlet–Neumann boundary value problem* (7.7) *is given by the matricial equations*

$$\mathsf{u}_\mathsf{B} = \mathsf{N}_\mathsf{q}(\mathsf{A};\mathsf{B})^{-1}\cdot\Big(\mathsf{C}(\mathsf{A};\mathsf{F})\cdot\mathsf{G}_\mathsf{q}(\mathsf{F};\mathsf{F})\cdot\mathsf{f}_\mathsf{F} - \mathsf{N}_\mathsf{q}(\mathsf{A};\mathsf{A}\cup\mathsf{R})\cdot\mathsf{g}_{\mathsf{A}\cup\mathsf{R}} + \mathsf{h}_\mathsf{A}\Big),$$

$$\mathsf{u}_\mathsf{F} = \mathsf{G}_\mathsf{q}(\mathsf{F};\mathsf{F})\cdot\Big(\mathsf{f}_\mathsf{F} + \mathsf{C}(\mathsf{F};\mathsf{B})\cdot\mathsf{u}_\mathsf{B} + \mathsf{C}(\mathsf{F};\mathsf{A}\cup\mathsf{R})\cdot\mathsf{g}_{\mathsf{A}\cup\mathsf{R}}\Big),$$

$$\mathsf{u}_{\mathsf{A}\cup\mathsf{R}} = \mathsf{g}_{\mathsf{A}\cup\mathsf{R}}.$$

7.3 Boundary spike formula

We work under the same premises of the last section. Remember that, by assumption, $\mathsf{N}_\mathsf{q}(\mathsf{A};\mathsf{B})$ is invertible. Then, we can deduce an invertibility property for a submatrix of the Schrödinger matrix $\mathsf{L}_\mathsf{q}(\bar{\mathsf{F}};\bar{\mathsf{F}})$ different from the ones given in Corollary 6.1.5, and using the results of the last section we can explicitly express its inverse.

Lemma 7.3.1. *The matrix* $\mathsf{L}_\mathsf{q}(\mathsf{A}\cup\mathsf{F};\mathsf{B}\cup\mathsf{F})$ *is invertible and its inverse is given in block form by*

$$\mathsf{L}_\mathsf{q}(\mathsf{A}\cup\mathsf{F};\mathsf{B}\cup\mathsf{F})^{-1} = \begin{bmatrix} \widetilde{\mathsf{R}}_\mathsf{q}(\mathsf{B};\mathsf{A}) & \widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{B};\mathsf{F}) \\ \widetilde{\mathsf{R}}_\mathsf{q}(\mathsf{F};\mathsf{A}) & \widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{F};\mathsf{F}) \end{bmatrix}.$$

Proof. Notice that since $\mathsf{N_q}(\mathsf{A};\mathsf{B})$ is invertible then problem

$$\mathcal{L}_q(u) = f \text{ on } F, \quad u = 0 \text{ on } A \cup R \text{ and } \frac{\partial u}{\partial \mathsf{n}_F} = h \text{ on } A$$

has a unique solution $u \in \mathcal{C}(F \cup B)$ for any $f \in \mathcal{C}(F)$ and $h \in \mathcal{C}(A)$. The matricial expression of this problem can be written as

$$\mathsf{L_q}(\mathsf{A}\cup\mathsf{F};\mathsf{B}\cup\mathsf{F})\cdot \mathsf{u}_{\mathsf{F}\cup\mathsf{B}} = \begin{bmatrix} 0 & \mathsf{L_q}(\mathsf{A};\mathsf{F}) \\ \mathsf{L_q}(\mathsf{F};\mathsf{B}) & \mathsf{L_q}(\mathsf{F};\mathsf{F}) \end{bmatrix} \cdot \begin{bmatrix} \mathsf{u_F} \\ \mathsf{u_B} \end{bmatrix} = \begin{bmatrix} \mathsf{f} \\ \mathsf{h} \end{bmatrix}$$

and therefore $\mathsf{L_q}(\mathsf{A}\cup\mathsf{F};\mathsf{B}\cup\mathsf{F})$ is invertible because it is the matrix associated with this problem with a unique solution for any data. Now we consider the matrix

$$\mathsf{M}(\mathsf{B}\cup\mathsf{F};\mathsf{A}\cup\mathsf{F}) = \begin{bmatrix} \widetilde{\mathsf{R}}_\mathsf{q}(\mathsf{B};\mathsf{A}) & \widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{B};\mathsf{F}) \\ \widetilde{\mathsf{R}}_\mathsf{q}(\mathsf{F};\mathsf{A}) & \widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{F};\mathsf{F}) \end{bmatrix} \in \mathcal{M}_{|\mathsf{A}\cup\mathsf{F}|\times|\mathsf{A}\cup\mathsf{F}|}(\mathbb{R}).$$

We want to check that it is the inverse of $\mathsf{L_q}(\mathsf{A}\cup\mathsf{F};\mathsf{B}\cup\mathsf{F})$. Observe that the product $\mathsf{L_q}(\mathsf{A}\cup\mathsf{F};\mathsf{B}\cup\mathsf{F})\cdot\mathsf{M}(\mathsf{B}\cup\mathsf{F};\mathsf{A}\cup\mathsf{F})$ is given by the matrix

$$\begin{bmatrix} \mathsf{L_q}(\mathsf{A};\mathsf{F})\cdot\widetilde{\mathsf{R}}_\mathsf{q}(\mathsf{F};\mathsf{A}) & \mathsf{L_q}(\mathsf{A};\mathsf{F})\cdot\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{F};\mathsf{F}) \\ \mathsf{L_q}(\mathsf{F};\mathsf{B})\cdot\widetilde{\mathsf{R}}_\mathsf{q}(\mathsf{B};\mathsf{A}) + \mathsf{L_q}(\mathsf{F};\mathsf{F})\cdot\widetilde{\mathsf{R}}_\mathsf{q}(\mathsf{F};\mathsf{A}) & \mathsf{L_q}(\mathsf{F};\mathsf{B})\cdot\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{B};\mathsf{F}) + \mathsf{L_q}(\mathsf{F};\mathsf{F})\cdot\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{F};\mathsf{F}) \end{bmatrix}.$$

Then, it is enough to study the four blocks of this product.

1. By Proposition 7.2.4 and Corollary 6.1.11 we see that the equality $\mathsf{L_q}(\mathsf{A};\mathsf{F})\cdot\widetilde{\mathsf{R}}_\mathsf{q}(\mathsf{F};\mathsf{A}) = \mathsf{ID}(\mathsf{A};\mathsf{A})$ holds.

2. The block $\mathsf{L_q}(\mathsf{F};\mathsf{B})\cdot\widetilde{\mathsf{R}}_\mathsf{q}(\mathsf{B};\mathsf{A}) + \mathsf{L_q}(\mathsf{F};\mathsf{F})\cdot\widetilde{\mathsf{R}}_\mathsf{q}(\mathsf{F};\mathsf{A})$ is a non–squared null matrix from Proposition 7.2.4 and Lemma 6.1.7.

3. The same occurs for the block $\mathsf{L_q}(\mathsf{A};\mathsf{F})\cdot\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{F};\mathsf{F})$ because of Proposition 7.2.2 and Corollary 6.1.11.

4. Finally, by Proposition 7.2.2 and Lemma 6.1.7 we get that the block $\mathsf{L_q}(\mathsf{F};\mathsf{B})\cdot\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{B};\mathsf{F}) + \mathsf{L_q}(\mathsf{F};\mathsf{F})\cdot\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{F};\mathsf{F})$ is the identity matrix $\mathsf{ID}(\mathsf{F};\mathsf{F})$. □

The invertibility of $\mathsf{L_q}(\mathsf{A}\cup\mathsf{F};\mathsf{B}\cup\mathsf{F})$ allows us to obtain a formula for the values of the boundary spike conductances of any network.

Proposition 7.3.2 (Boundary spike formula)**.** *If $x \in R$ is a boundary vertex and $y \in F$ is its unique neighbour, then*

$$\mathsf{N_q}(\mathsf{x};\mathsf{x}) - \mathsf{N_q}(\mathsf{x};\mathsf{B}) \cdot \mathsf{N_q}(\mathsf{A};\mathsf{B})^{-1} \cdot \mathsf{N_q}(\mathsf{A};\mathsf{x}) = \lambda + \frac{\omega(y)}{\omega(x)} c(x,y) - \widetilde{G}_q(y,y) c(x,y)^2.$$

Proof. The matrices $\mathsf{L_q}(\mathsf{F};\mathsf{F})$ and $\mathsf{L_q}(\mathsf{A}\cup\mathsf{F};\mathsf{B}\cup\mathsf{F})$ are invertible by Corollary 6.1.5 and Lemma 7.3.1, respectively. Applying the properties of the Schur complement we see that

$$\begin{aligned}
&\mathsf{L_q}(\mathsf{x};\mathsf{x}) - \mathsf{L_q}(\mathsf{x};\mathsf{B}\cup\mathsf{F}) \cdot \mathsf{L_q}(\mathsf{A}\cup\mathsf{F};\mathsf{B}\cup\mathsf{F})^{-1} \cdot \mathsf{L_q}(\mathsf{A}\cup\mathsf{F};\mathsf{x}) \\
&= \mathsf{L_q}(\{\mathsf{x}\}\cup\mathsf{A}\cup\mathsf{F};\{\mathsf{x}\}\cup\mathsf{B}\cup\mathsf{F}) \Big/ \mathsf{L_q}(\mathsf{A}\cup\mathsf{F};\mathsf{B}\cup\mathsf{F}) \\
&= \mathsf{L_q}(\{\mathsf{x}\}\cup\mathsf{A}\cup\mathsf{F};\{\mathsf{x}\}\cup\mathsf{B}\cup\mathsf{F}) / \mathsf{L_q}(\mathsf{F};\mathsf{F}) \Big/ \mathsf{L_q}(\mathsf{A}\cup\mathsf{F};\mathsf{B}\cup\mathsf{F}) / \mathsf{L_q}(\mathsf{F};\mathsf{F}).
\end{aligned}$$

By Corollary 6.1.13, the last equality can be written as

$$\begin{aligned}
&\mathsf{L_q}(\mathsf{x};\mathsf{x}) - \mathsf{L_q}(\mathsf{x};\mathsf{B}\cup\mathsf{F}) \cdot \mathsf{L_q}(\mathsf{A}\cup\mathsf{F};\mathsf{B}\cup\mathsf{F})^{-1} \cdot \mathsf{L_q}(\mathsf{A}\cup\mathsf{F};\mathsf{x}) \\
&= \mathsf{N_q}(\{\mathsf{x}\}\cup\mathsf{A};\{\mathsf{x}\}\cup\mathsf{B}) \Big/ \mathsf{N_q}(\mathsf{A};\mathsf{B}) \qquad (7.9) \\
&= \mathsf{N_q}(\mathsf{x};\mathsf{x}) - \mathsf{N_q}(\mathsf{x};\mathsf{B}) \cdot \mathsf{N_q}(\mathsf{A};\mathsf{B})^{-1} \cdot \mathsf{N_q}(\mathsf{A};\mathsf{x}).
\end{aligned}$$

On the other hand, we clearly discern that

$$L_q(x,x) = \mathcal{L}_q(\varepsilon_x)(x) = c(x,y) - \frac{\mathcal{L}(\omega)(x)}{\omega(x)} + \lambda = \frac{\omega(y)}{\omega(x)} c(x,y) + \lambda \qquad (7.10)$$

and

$$\begin{aligned}
&\mathsf{L_q}(\mathsf{x};\mathsf{B}\cup\mathsf{F}) \cdot \mathsf{L_q}(\mathsf{A}\cup\mathsf{F};\mathsf{B}\cup\mathsf{F})^{-1} \cdot \mathsf{L_q}(\mathsf{A}\cup\mathsf{F};\mathsf{x}) \\
&= \mathsf{C}(\mathsf{x};\mathsf{y}) \cdot \Big[\mathsf{L_q}(\mathsf{A}\cup\mathsf{F};\mathsf{B}\cup\mathsf{F})\Big]^{-1}(\mathsf{y};\mathsf{y}) \cdot \mathsf{C}(\mathsf{y};\mathsf{x}) \qquad (7.11) \\
&= c(x,y)^2 \Big[\mathsf{L_q}(\mathsf{A}\cup\mathsf{F};\mathsf{B}\cup\mathsf{F})\Big]^{-1}(\mathsf{y};\mathsf{y})
\end{aligned}$$

because the unique neighbour of x is y. The result follows when we properly join Equations (7.9), (7.10), (7.11) and use the equality

$$\Big[\mathsf{L_q}(\mathsf{A}\cup\mathsf{F};\mathsf{B}\cup\mathsf{F})\Big]^{-1}(\mathsf{y};\mathsf{y}) = \widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{y},\mathsf{y})$$

from Lemma 7.3.1. □

What remains left in order to recover the boundary spike conductance $c(x,y)$ is the value $\widetilde{G}_q(y,y)$, where $y \in N(R)$. In general, this value is not known for an arbitrary network. However, there exist some families of networks for which this value can be obtained, for instance rigid three dimensional grids, as we will see in Section 7.5.

7.4 Networks with separated boundary

First, let us work with networks having separated boundary, defined in Section 2.1. From now on $\Gamma = (V,c)$ is a connected network and $F \subset V$ is a proper connected subset such that $\delta(F)$ is a separated boundary. This means that for any $x \in \delta(F)$ there exists a unique vertex $y \in F$ such that $y \sim x$. Notice that xy is a boundary spike, the class of edges from which we want to recover the conductances. Subordinated to the above premises, we can say a little bit more about the modified Green kernel.

Lemma 7.4.1. *The blocks* $\widetilde{\mathsf{G}}_{\mathsf{q}}(\mathsf{N(A)};\mathsf{F})$ *and* $\widetilde{\mathsf{G}}_{\mathsf{q}}(\mathsf{F};\mathsf{N(B)})$ *of the modified Green matrix are always* $\mathsf{0}$.

Proof. Consider two vertices $x \in A$ and $z \in F$. Then, if $y \in F$ is the unique neighbour of x,

$$0 = \frac{\partial \widetilde{\mathcal{G}}_q(\varepsilon_z)}{\partial \mathsf{n}_F}(x) = -c(x,y)\widetilde{G}_q(y,z).$$

For $c(x,y) > 0$, then $\widetilde{G}_q(y,z) = 0$ for all $y \in N(A)$ and $z \in F$. On the other hand, by considering the adjoint overdetermined problem of the modified Green problem, we analogously see that $\widetilde{G}_q(y,z) = 0$ for all $y \in F$ and $z \in N(B)$. □

7.5 Recovery on rigid three dimensional grids

The starting point are the results obtained in Section 7.3, which allow us to perform the recovery of boundary spike conductances if we know certain entries of the modified Green kernel. Yet, we ask ourselves whether further conductivity information can be recovered. In some cases, the answer is affirmative, namely rigid three dimensional grids. For these networks, we are able to recuperate the whole conductances function.

Nevertheless, before studying the process of the obtaining of the conductances on a three dimensional grid, it is important to know what occurs in the two dimensional case. In the continuous field, the problem of determining the conductivity of a two dimensional rectangular domain using only information on the perimeter and global equilibrium conditions has been solved in certain situations, see [53] and the references therein. The discrete version of this problem is also solvable for networks, see the works of Curtis, Ingerman and Morrow in [37, 38] for a type of planar networks, or the works of Borcea, Druskin and Mamonov in [25] for the numerical conductivity recovery of circular discretized domains.

The three dimensional case holds a very different behaviour. If we consider the cuboid in the continuous field $\mathbb{R}^3$, then it is already known that its conductivity is not recoverable in a non–numerical approaching way. Nevertheless, in the discrete setting the problem is solvable under certain restrictions, as it will be shown in this paper.

A three dimensional grid is the discretization of any cuboid in $\mathbb{R}^3$. Let $a, p, \ell \in \mathbb{N}$. We define the *three dimensional grid* with height a, width p and length ℓ as the network $\Gamma = (V, c)$ with vertex set

$$V = \{x_{ijk} : i = 0, \dots, a+1, j = 0, \dots, p+1, k = 0, \dots, \ell+1\}$$

and conductances c on the set of edges. This set of edges is given by the adjacencies

$$x_{ijk} \sim x_{pqr} \quad \Leftrightarrow \quad \begin{cases} p = i \pm 1, \;\; q = j \text{ and } r = k, \\ p = i, \;\; q = j \pm 1 \text{ and } r = k, \\ p = i, \;\; q = j \text{ and } r = k \pm 1. \end{cases}$$

Let $F = \{x_{ijk} : i = 1, \dots, a, j = 1, \dots, p, k = 1, \dots, \ell\} \subseteq V$. Then, its boundary is given by $\delta(F) = A \cup B \cup R$, where the sets that provide this partition are arbitrarily defined as $A = \{x_{ij0} : i = 1, \dots, a, j = 1, \dots, p\}$, $B = \{x_{ij\ell+1} : i = 1, \dots, a, j = 1, \dots, p\}$ and $R = \delta(F) \smallsetminus (A \cup B)$. On account of simplicity, we also define the sets

$$A_k = \{x_{ijk} : i = 1, \dots, a, j = 1, \dots, p\} \subset V$$

for all $k = 0, \dots, \ell+1$. In particular, $A_0 = A$, $A_1 = N(A)$, $A_\ell = N(B)$ and $A_{\ell+1} = B$. See Figure 7.2 for an illustration of a three dimensional grid.

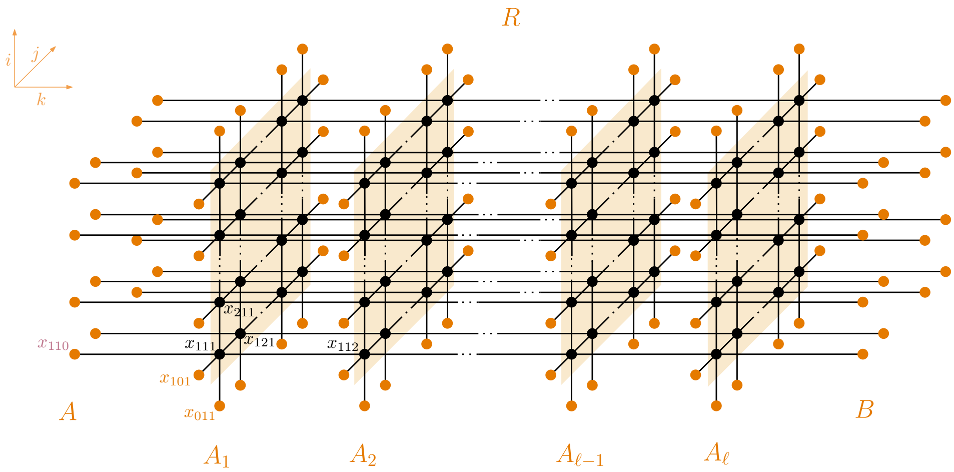


Figure 7.2 Graphical representation of a three dimensional grid.

From now on, whenever an ordering in V needs to be considered, we take the one given by $\{A; R; A_1; \dots; A_\ell; B\}$. Even more, the ordering inside the subsets A_k is given by the sequence

$$\{x_{11k}, \dots, x_{1pk}; x_{21k}, \dots, x_{2pk}; \dots; x_{a1k}, \dots, x_{apk}\}.$$

In order to perform a full conductance recovery, we work with a subfamily of three dimensional grids. First, we detail the properties of their modified Green kernels and afterwards we proceed to the complete conductivity recovery. A *rigid three dimensional grid* $\Gamma = (V, c)$ is a three dimensional grid where the conductances are constant in each direction, that is, $c(x_{ijk}, x_{ijk+1}) = c_h > 0$, $c(x_{ijk}, x_{i+1jk}) = c_v > 0$ and $c(x_{ijk}, x_{ij+1k}) = c_d > 0$ are the conductances with horizontal, vertical and depth direction, respectively. We also assume for every rigid three dimensional grid to have a constant weight on the vertices $\omega \in \Omega(V)$ given by the value $\omega \equiv w$ on V.

The family of rigid three dimensional grids has good structural properties, which are detailed in the following results.

Lemma 7.5.1. *Given a rigid three dimensional grid,* $\mathsf{C}(\mathsf{A_k}; \mathsf{A_{k+1}}) = c_h \cdot \mathsf{ID}(\mathsf{A}; \mathsf{A})$ *for all* $k = 0, \dots, \ell$. *Moreover, if* $s \geq 2$, *then* $\mathsf{L_q}(\mathsf{A_k}; \mathsf{A_{k+s}}) = 0$.

For the sake of simplicity we denote by $\mathsf{I} \in \mathcal{M}_{ap\times ap}(\mathbb{R})$ the matrix $\mathsf{ID}(\mathsf{A};\mathsf{A})$. We also define the matrix $\mathsf{H} = (\mathsf{h_{ij}})_{\mathsf{ij}} \in \mathcal{M}_{p\times p}(\mathbb{R})$ given by the entries $\mathsf{h_{ii}} = 2c_h + 2c_d + 2c_v$, $\mathsf{h_{ii\pm 1}} = -c_d$ and $\mathsf{h_{ij}} = 0$ otherwise for all $i, j = 1, \dots, p$.

Lemma 7.5.2. *Given a rigid three dimensional grid, for all $k \in \{1,\dots,\ell\}$ the matrix $\mathsf{L_q}(\mathsf{A_k};\mathsf{A_k})$ has the following block structure:*

$$\mathsf{L_q}(\mathsf{A_k};\mathsf{A_k}) = \begin{bmatrix} \mathsf{H} & -c_v \cdot \mathsf{I} & \cdots & \cdots & 0 \\ -c_v \cdot \mathsf{I} & \mathsf{H} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \mathsf{H} & -c_v \cdot \mathsf{I} \\ 0 & \cdots & \cdots & -c_v \cdot \mathsf{I} & \mathsf{H} \end{bmatrix}.$$

Notice that the block $\mathsf{L_q}(\mathsf{A_k};\mathsf{A_k})$ does not depend on k. In consequence, we can define the matrix $\mathsf{Z} = \mathsf{L_q}(\mathsf{A_k};\mathsf{A_k})$ for all $k = 1,\dots,\ell$.

Proposition 7.5.3. *Given a rigid three dimensional grid, its modified Green matrix has many* 0 *blocks, which are given by*

$$\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A_k};\mathsf{A_s}) = 0$$

for all $k = 2,\dots,\ell+1$ and $s \geq k$. Moreover, if $s = k-1$ then

$$\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A_k};\mathsf{A_{k-1}}) = -\frac{1}{c_h} \cdot \mathsf{I}.$$

Proof. Observe that a three dimensional grid is a network with separated boundary. Then, by Lemma 7.4.1

$$\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A_1};\mathsf{F}) = 0. \tag{7.12}$$

Now, given two indices $k, s \in \{1,\dots,\ell\}$, let us consider two vertices $x \in A_k$ and $y \in A_s$. By definition, x only has neighbours in A_{k-1}, A_k and A_{k+1}. Therefore,

$$\begin{aligned}\varepsilon_y(x) = \mathcal{L}_q\Big(\widetilde{\mathcal{G}}_q(\varepsilon_y)\Big)(x) &= \int_{\bar{F}} L_q(x,z)\widetilde{G}_q(z,y)\,dz = \mathsf{L_q}(\mathsf{x};\bar{\mathsf{F}}) \cdot \widetilde{\mathsf{G}}_\mathsf{q}(\bar{\mathsf{F}};\mathsf{y}) \\ &= \sum_{r=k-1}^{k+1} \mathsf{L_q}(\mathsf{x};\mathsf{A_r}) \cdot \widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A_r};\mathsf{y}).\end{aligned}$$

Considering all the vertices $x \in A_k$ and $y \in A_s$, we obtain the following matricial identity

$$\mathsf{ID}(\mathsf{A_k};\mathsf{A_s}) = \mathsf{Z} \cdot \widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A_k};\mathsf{A_s}) - c_h \cdot \widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A_{k-1}};\mathsf{A_s}) - c_h \cdot \widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A_{k+1}};\mathsf{A_s}). \tag{7.13}$$

Using this identity, we prove the result by induction on k, with $s \geq k-1$. First, let us consider the case $k=1$, with $s \geq 0$. By Equations (7.12) and (7.13) we get that $\mathsf{ID}(\mathsf{A_1}; \mathsf{A_s}) = -\, c_h \cdot \widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A_2}; \mathsf{A_s})$ and hence

$$\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A_2}; \mathsf{A_s}) = \begin{cases} -\dfrac{1}{c_h} \cdot \mathsf{I} & \text{if } s = 1 \\ \mathsf{0} & \text{if } s > 1. \end{cases}$$

Let us assume that the result is true for any index $i < k$, with $s \geq i-1$. We want to see that the result is also true for k. If $s \geq k-1$, then by Equation (7.13) we get that $\mathsf{ID}(\mathsf{A_{k-1}}; \mathsf{A_s}) = -c_h \cdot \widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A_k}; \mathsf{A_s})$ and so

$$\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A_k}; \mathsf{A_s}) = \begin{cases} -\dfrac{1}{c_h} \cdot \mathsf{I} & \text{if } s = k-1 \\ \mathsf{0} & \text{if } s \geq k, \end{cases}$$

completing the induction. Therefore, the result holds for every $k = 2, \dots, \ell+1$. □

Proposition 7.5.3 shows that the diagonal entries of the modified Green matrix of a rigid three dimensional grid are always 0. Therefore, now we are equipped to determine some of the unknown conductances of a rigid three dimensional grid. However, first it is of interest to continue with the study of the modified Green matrix of this family of networks, for its structure is fascinating and may be useful to the reader.

Lemma 7.5.4. *The block* $\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A_k}; \mathsf{A_{k-s}})$ *does not depend on* k *for all* $s = 0, \dots, \ell+1$ *and* $k = s+1, \dots, \ell+1$.

Proof. We prove the result by induction on k, with $s \leq k-1$. For $k=2$, with $s \leq 1$, in Proposition 7.5.3 it has been proved that $\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A_2}; \mathsf{A_1}) = \mathsf{0}$ and hence it does not depend on k. Now we assume that the result is true for any index $i < k$, with $s \leq i-1$. We want to see that the result is also true for k. If $s \leq k-1$ then by Equation (7.13) and rearranging indices we get that

$$\mathsf{0} = \mathsf{Z} \cdot \widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A_{k-1}}; \mathsf{A_{(k-1)-(s-1)}}) - c_h \cdot \widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A_{k-2}}; \mathsf{A_{(k-2)-(s-2)}}) - c_h \cdot \widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A_k}; \mathsf{A_{k-s}}).$$

Notice that Z, $\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A_{k-1}}; \mathsf{A_{(k-1)-(s-1)}})$ and $\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A_{k-2}}; \mathsf{A_{(k-2)-(s-2)}})$ do not depend on k. Then, the block $\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A_k}; \mathsf{A_{k-s}})$ neither does. □

Therefore, for all $s = 0, \dots, \ell+1$ we can define the matrices

$$\mathsf{T_s} = \widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A_k}; \mathsf{A_{k-s}}),$$

where $k \in \{s+1,\dots,\ell+1\}$. In particular, $\mathsf{T}_0 = 0$ and $\mathsf{T}_1 = -\dfrac{1}{c_h}\cdot\mathsf{I}$. From all the previous results in this section we deduce that the modified Green matrix of a rigid three dimensional grid is block–triangular, since $\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A}_\mathsf{k};\mathsf{A}_\mathsf{s}) = 0$ for all $k = 1,\dots,\ell+1$ and $s = k,\dots,\ell+1$. Moreover, $\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A}_\mathsf{k};\mathsf{A}_\mathsf{s}) = \mathsf{T}_{\mathsf{k-s}}$ for all $k, s = 1,\dots,\ell+1$ if we take the notation $\mathsf{T}_\mathsf{r} = 0$ for $r \leq 0$. Hence, the modified Green matrix can be written as

$$\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{F}\cup\mathsf{B};\mathsf{F}) = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 0 \\ \mathsf{T}_1 & 0 & \cdots & \cdots & 0 \\ \vdots & \mathsf{T}_1 & \ddots & & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ \mathsf{T}_{\ell-1} & \cdots & \cdots & \mathsf{T}_1 & 0 \\ \mathsf{T}_\ell & \mathsf{T}_{\ell-1} & \cdots & \cdots & \mathsf{T}_1 \end{bmatrix},$$

which means that if we managed to determine the entries of $\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{B};\mathsf{F})$ then we automatically would know all the entries of $\widetilde{\mathsf{G}}_\mathsf{q}(\bar{\mathsf{F}};\mathsf{F})$.

Proposition 7.5.5. *Given a rigid three dimensional grid, the following matricial recurrence formula is satisfied:*

$$\mathsf{T}_\mathsf{s} = -\frac{1}{c_h}\cdot\mathsf{Z}\cdot\mathsf{T}_{\mathsf{s-1}} - \mathsf{T}_{\mathsf{s-2}}$$

for all $s = 2,\dots,\ell$, where the initial matrices of the recurrence are $\mathsf{T}_0 = 0$ and $\mathsf{T}_1 = -\dfrac{1}{c_h}\cdot\mathsf{I}$.

Proof. Consider a family of matrices given by $\mathsf{M}_\mathsf{ks} = \widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{A}_\mathsf{k};\mathsf{A}_\mathsf{s})$ for all $k, s = 1,\dots,\ell$ and notice that $\mathsf{M}_{\mathsf{kk-s}} = \mathsf{T}_\mathsf{s}$. Since T_s does not depend on k, then $\mathsf{M}_{\mathsf{k-ik-i-s}} = \mathsf{T}_\mathsf{s} = \mathsf{M}_{\mathsf{kk-s}}$ for all $i = 1,\dots,k-s-1$. Using Equation (7.13), we get that

$$0 = \mathsf{Z}\cdot\mathsf{M}_{\mathsf{kk-s}} - c_h\cdot\mathsf{M}_{\mathsf{k-1k-s}} - c_h\cdot\mathsf{M}_{\mathsf{k+1k-s}} = \mathsf{Z}\cdot\mathsf{T}_\mathsf{s} - c_h\cdot\mathsf{T}_{\mathsf{s-1}} - c_h\cdot\mathsf{T}_{\mathsf{s+1}}. \quad \square$$

After this detailed study of the modified Green matrix, we redirect our efforts to the recovery of the conductivity function of rigid three dimensional grids. The next result provides the values of c_d and c_v.

Proposition 7.5.6 (Boundary spike formula for rigid three dimensional grids)**.** *Let $\Gamma = (V, c)$ be a rigid three dimensional grid. Let $x \in R$ and let $y \in F$ be its unique neighbour. Then,*

$$c(x,y) = \mathsf{N}_\mathsf{q}(\mathsf{x};\mathsf{x}) - \mathsf{N}_\mathsf{q}(\mathsf{x};\mathsf{B})\cdot\mathsf{N}_\mathsf{q}(\mathsf{A};\mathsf{B})^{-1}\cdot\mathsf{N}_\mathsf{q}(\mathsf{A};\mathsf{x}) - \lambda.$$

Proof. From Proposition 7.3.2, we know that

$$\mathsf{N_q}(\mathsf{x};\mathsf{x}) - \mathsf{N_q}(\mathsf{x};\mathsf{B}) \cdot \mathsf{N_q}(\mathsf{A};\mathsf{B})^{-1} \cdot \mathsf{N_q}(\mathsf{A};\mathsf{x}) = \lambda + c(x,y) - \widetilde{G}_q(y,y)c(x,y)^2.$$

Since $\widetilde{G}_q(y,y)$ is a diagonal term of the matrix $\widetilde{\mathsf{G}}_\mathsf{q}(\mathsf{F};\mathsf{F})$, by Proposition 7.5.3 it is clear that $\widetilde{G}_q(y,y) = 0$. □

With this last result we obtain the conductances c_d and c_v, for the matrix $\mathsf{N_q}$ is already known. The remaining unknown conductances, given by c_h, can be obtained in the same way by switching the sets of vertices of the boundary, as we show in the following example.

Let $\Gamma = (V,c)$ be a rigid three dimensional grid with height $a=2$, depth $p=2$ and width $\ell=3$, see Figure 7.3. Let $\lambda = 1$ be the non–negative real value. As the entries of the matrix $\mathsf{N_q}$ can be known by means of boundary

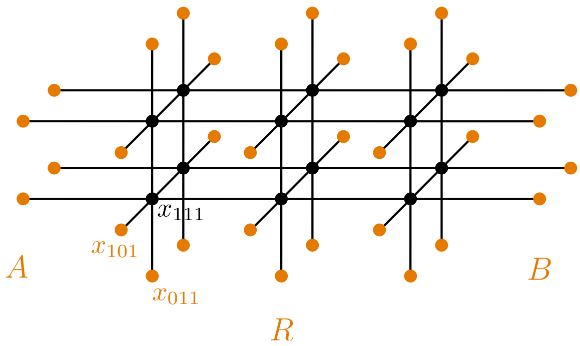


Figure 7.3 A rigid three dimensional grid with height $a=2$, depth $p=2$ and width $\ell=3$.

measurements, we assume that they are given. These are the needed values for our purposes:

$$\mathsf{N_q}(\mathsf{A};\mathsf{B}) = -\frac{1}{70079272263}\begin{pmatrix} 1762824160 & 641638160 & 1043929504 & 458466560 \\ 641638160 & 1762824160 & 458466560 & 1043929504 \\ 1043929504 & 458466560 & 1762824160 & 641638160 \\ 458466560 & 1043929504 & 641638160 & 1762824160 \end{pmatrix},$$

$$\mathsf{N_q}(\mathsf{x}_{011};\mathsf{B}) = -\frac{1}{70079272263}\begin{pmatrix} 1762824160 & 641638160 & 1043929504 & 458466560 \end{pmatrix},$$

$$\mathsf{N_q}(\mathsf{x}_{101};\mathsf{B}) = \frac{1}{2}\mathsf{N_q}(\mathsf{x}_{011};\mathsf{B}),$$

$$\mathsf{N_q}(\mathsf{A};\mathsf{x}_{011}) = -\frac{1}{70079272263}\begin{pmatrix} 31322290760 \\ 3846506644 \\ 7080283736 \\ 1703075680 \end{pmatrix} = 2\,\mathsf{N_q}(\mathsf{A};\mathsf{x}_{101}),$$

$$\mathsf{N_q}(\mathsf{x}_{011};\mathsf{x}_{011}) = \frac{178915526029}{70079272263}, \qquad \mathsf{N_q}(\mathsf{x}_{101};\mathsf{x}_{101}) = \frac{132327971836}{123493440}.$$

Therefore, we use Proposition 7.5.6 to get the following conductances:

$$c(x_{011}, x_{111}) = \mathsf{N_q}(\mathsf{x}_{011};\mathsf{x}_{011}) - \mathsf{N_q}(\mathsf{x}_{011};\mathsf{B}) \cdot \mathsf{N_q}(\mathsf{A};\mathsf{B})^{-1} \cdot \mathsf{N_q}(\mathsf{A};\mathsf{x}_{011}) - \lambda = 2,$$

$$c(x_{101}, x_{111}) = \mathsf{N_q}(\mathsf{x}_{101};\mathsf{x}_{101}) - \mathsf{N_q}(\mathsf{x}_{101};\mathsf{B}) \cdot \mathsf{N_q}(\mathsf{A};\mathsf{B})^{-1} \cdot \mathsf{N_q}(\mathsf{A};\mathsf{x}_{101}) - \lambda = 1,$$

and hence $c_v = 2$, $c_d = 1$. Finally, in order to obtain the conductances c_h, we consider now the partition of the boundary given by $\widetilde{A} = \{x_{i0k}\}_{ik}$, $\widetilde{B} = \{x_{ip+1k}\}_{ik}$ and $\widetilde{R} = \delta(F) \setminus (\widetilde{A} \cup \widetilde{B})$ for $i = 1, \dots, a$ and $k = 1, \dots, \ell$. Working in the same way as before, we get that $c_h = 2$. Therefore, we have recovered all the conductances for this example of rigid three dimensional grid.

Now that all the conductances are known, we can determine the modified Green kernel of this rigid three dimensional grid. We give here the matrices $\mathsf{T_k}$ for $k = 1, \dots, 3$, for they determine the modified Green matrix as we have seen above. We use the recursive formula given in Proposition 7.5.5:

$$\widetilde{\mathsf{G}}_{\mathsf{q}}(\mathsf{F} \cup \mathsf{B}; \mathsf{F}) = \begin{bmatrix} 0 & 0 & 0 \\ \mathsf{T_1} & 0 & 0 \\ \mathsf{T_2} & \mathsf{T_1} & 0 \\ \mathsf{T_3} & \mathsf{T_2} & \mathsf{T_1} \end{bmatrix},$$

where

$$\mathsf{T_1} = -\frac{1}{2} \cdot \mathsf{I}, \quad \mathsf{T_2} = \frac{1}{4}\begin{pmatrix} 10 & -1 & -2 & 0 \\ -1 & 10 & 0 & -2 \\ -2 & 0 & 10 & -1 \\ 0 & -2 & -1 & 10 \end{pmatrix}$$

and

$$\mathsf{T_3} = \frac{1}{8}\begin{pmatrix} -101 & 20 & 40 & -4 \\ 20 & -101 & -4 & 40 \\ 40 & -4 & -101 & 20 \\ -4 & 40 & 20 & -101 \end{pmatrix}.$$

7.6 Boundary spikes on circular planar networks

In this section we work with circular planar networks, which have been defined in Section 2.7. From now on, $\Gamma = (V, c)$ is a circular planar network with proper connected subset $F \subset V$ and known Dirichlet–to–Robin map.

The purpose is to recover boundary spike conductances using only the information provided by this map. Our findings show that this can be done for boundary spikes having a certain structural property.

We need to recall the results in Chapter 6. Theorem 6.4.3 shows that given a matrix $\mathsf{M} \in \Phi_{\lambda,\sigma}$ then there exists a circular planar network with a weight $\omega \in \Omega(\bar{F})$ such that $\omega = k\sigma$ on $\delta(F)$ and $\mathsf{N_q} = \mathsf{M}$ is the Dirichlet–to–Robin matrix associated with the operator $\mathcal{L}_q$ for $q = q_\omega + \lambda\chi_{\delta(F)}$, where λ is the smallest eigenvalue of M and $\sigma_{\delta(\mathsf{F})}$ is its associated eigenvector. Therefore, given a circular planar network Γ and its Dirichlet–to–Robin map $\mathsf{N_q}$, we know that we can recover its unique conductivity function, the weight and the real value that compound its Schrödinger operator $\mathcal{L}_q$. Remember the assumption in this chapter that any potential fits the expression $q = q_\omega + \lambda\chi_{\delta(F)}$ on $\bar{F}$.

The definitions and results in this section are inspired in the works of [37] for the combinatorial laplacian, although we labour with Schrödinger operators and extend the tools to this case. In [37, Theorem 3.13] it has been shown that if $(A;B)$ is a circular pair of size k of $\delta(F)$, with A and B laying in disjoint arcs of the boundary circle, then it is satisfied that $(A;B)$ are not connected through Γ if and only if $\det(\mathsf{N}(\mathsf{A};\mathsf{B})) = 0$. Moreover, (A,B) are connected through Γ if and only if $(-1)^k \det(\mathsf{N}(\mathsf{A};\mathsf{B})) > 0$. By a straightforward extension, the same occurs for the matrix $\mathsf{N_q}(\mathsf{A};\mathsf{B})$.

Corollary 7.6.1. *Let (A,B) be a circular pair of size k of $\delta(F)$, where A and B are in disjoint arcs of the boundary circle. Then, (A,B) are not connected through Γ if and only if* $\det(\mathsf{N_q}(\mathsf{A};\mathsf{B})) = 0$. *Moreover, (A,B) are connected through Γ if and only if* $(-1)^k \det(\mathsf{N_q}(\mathsf{A};\mathsf{B})) > 0$.

Corollary 7.6.2 (Boundary spike formula for circular planar networks)**.** *Suppose that Γ has a boundary spike xy with $x \in \delta(F)$ and $y \in F$. If contracting xy to a unique boundary vertex results in breaking the connection through Γ between a circular pair (A,B), then*

$$c(x,y) = \frac{\omega(x)}{\omega(y)} \Big(\mathsf{N_q}(\mathsf{x};\mathsf{x}) - \mathsf{N_q}(\mathsf{x};\mathsf{B}) \cdot \mathsf{N_q}(\mathsf{A};\mathsf{B})^{-1} \cdot \mathsf{N_q}(\mathsf{A};\mathsf{x}) - \lambda\Big).$$

Proof. This result is proved analogously to [37, Corollary 3.16]. However, it is interesting to see the details and the tools of the proof. First, observe that $x \notin A, B$ by definition. Since xy is a boundary spike, the unique neighbour of x is y. We arbitrarily choose the ordering in $F \cup A \cup B \cup \{x\}$ supplied by $\{\{x\}; A; B; \{y\}; F \setminus \{y\}\}$. Then, the matrix $\mathsf{L_q}(\{\mathsf{x}\} \cup \mathsf{A} \cup \mathsf{F}; \{\mathsf{x}\} \cup \mathsf{B} \cup \mathsf{F})$

owns the following block structure:

$$\mathsf{L_q}\Big(\{\mathsf{x}\}\cup\mathsf{A}\cup\mathsf{F};\{\mathsf{x}\}\cup\mathsf{B}\cup\mathsf{F}\Big)=\begin{bmatrix} c(x,y)+q(x) & 0 \quad -c(x,y) \quad 0 \\ 0 & \\ -c(x,y) & \mathsf{L_q}\left(\mathsf{A}\cup\mathsf{F};\mathsf{B}\cup\mathsf{F}\right) \\ 0 & \end{bmatrix}.$$

In the interest of readability, we define the matrices

$$\mathsf{L_1}=\mathsf{L_q}\left(\{\mathsf{x}\}\cup\mathsf{A}\cup\mathsf{F};\{\mathsf{x}\}\cup\mathsf{B}\cup\mathsf{F}\right),$$
$$\mathsf{L_2}=\mathsf{L_q}\left(\{\mathsf{x}\}\cup\mathsf{A}\cup(\mathsf{F}\setminus\{\mathsf{y}\});\mathsf{B}\cup\mathsf{F}\right),$$
$$\mathsf{L_3}=\mathsf{L_q}\left(\mathsf{A}\cup(\mathsf{F}\setminus\{\mathsf{y}\});\mathsf{B}\cup(\mathsf{F}\setminus\{\mathsf{y}\})\right),$$
$$\mathsf{L_4}=\mathsf{L_q}\left(\mathsf{A}\cup\mathsf{F};\mathsf{B}\cup\mathsf{F}\right).$$

The determinant of $\mathsf{L_1}$ is given by

$$\det(\mathsf{L_1})=\Big(c(x,y)+q(x)\Big)\det(\mathsf{L_4})+(-1)^{k+4}c(x,y)\det(\mathsf{L_2}),$$

where $\det(\mathsf{L_2})=(-1)^{k+3}c(x,y)\det(\mathsf{L_3})$ using the same technique. Hence,

$$\det(\mathsf{L_1})=\Big(c(x,y)+q(x)\Big)\det(\mathsf{L_4})-c(x,y)^2\det(\mathsf{L_3}). \tag{7.14}$$

On the other hand, let us consider a new network $\widetilde{\Gamma}=(\widetilde{V},\widetilde{c})$ that is the result of contracting the boundary spike xy to a boundary vertex z. By hypothesis, we break the connection between A and B. Observe that $\widetilde{V}=(V\setminus\{x,y\})\cup\{z\}$, whereas $\widetilde{c}=c$ in $(\widetilde{V}\setminus z)\times(\widetilde{V}\setminus z)$ and $\widetilde{c}(z,t)=c(x,t)+c(y,t)$ for all $t\in\widetilde{V}$. We consider the subset $\widetilde{F}\subset\widetilde{V}$ given by $\widetilde{F}=F\setminus\{y\}$ so that $z\in\delta(\widetilde{F})$. Notice that $A,B\subset\delta(\widetilde{F})$ and $z\notin A,B$. Moreover, it is satisfied that

$$\widetilde{\mathsf{L}}_\mathsf{q}(\mathsf{A}\cup\widetilde{\mathsf{F}};\mathsf{B}\cup\widetilde{\mathsf{F}})=\mathsf{L_q}(\mathsf{A}\cup(\widetilde{\mathsf{F}}\setminus\{\mathsf{y}\});\mathsf{B}\cup(\widetilde{\mathsf{F}}\setminus\{\mathsf{y}\})).$$

Therefore, by Corollary 7.6.1 and Proposition 6.1.12,

$$0=\det\Big(\widetilde{\mathsf{N}}_\mathsf{q}(\mathsf{A};\mathsf{B})\Big)=\frac{\det\Big(\widetilde{\mathsf{L}}_\mathsf{q}(\mathsf{A}\cup\widetilde{\mathsf{F}};\mathsf{B}\cup\widetilde{\mathsf{F}})\Big)}{\det\Big(\widetilde{\mathsf{L}}_\mathsf{q}(\widetilde{\mathsf{F}};\widetilde{\mathsf{F}})\Big)}=\frac{\det(\mathsf{L_3})}{\det\Big(\widetilde{\mathsf{L}}_\mathsf{q}(\widetilde{\mathsf{F}};\widetilde{\mathsf{F}})\Big)},$$

from where we deduce that $\det(\mathsf{L_3})=0$. Going back to the original network Γ, by Equation (7.14) we get that

$$c(x,y)+q(x)=\frac{\det(\mathsf{L_1})}{\det(\mathsf{L_4})}.$$

Finally, since A and B are connected through Γ by hypothesis, we can consider the following Schur complement

$$\mathsf{N_q(\{x\}\cup A;\{x\}\cup B)}\Big/\mathsf{N_q(A;B)} = \mathsf{N_q(x;x)} - \mathsf{N_q(x;B)}\cdot \mathsf{N_q(A;B)}^{-1}\cdot \mathsf{N_q(A;x)}.$$

It is a matrix of size 1. Hence, taking determinants and using Proposition 6.1.12,

$$\begin{aligned}\mathsf{N_q(x;x)} - \mathsf{N_q(x;B)}\cdot \mathsf{N_q(A;B)}^{-1}\cdot \mathsf{N_q(A;x)} &= \frac{\det\left(\mathsf{L_1}\big/\mathsf{L_q(F;F)}\right)}{\det\left(\mathsf{L_4}\big/\mathsf{L_q(F;F)}\right)} = \frac{\det(\mathsf{L_1})}{\det(\mathsf{L_4})}\\ &= c(x,y)+q(x)\end{aligned}$$

Since $q(x) = \left(\dfrac{\omega(y)}{\omega(x)} - 1\right) c(x,y) + \lambda$ because $x \in \delta(F)$, we get the result. $\square$

This result allows us to recover the conductances of all the boundary spikes that disconnect two boundary sets through Γ. However, not all the boundary spikes hold this property, and this is the reason why in the sequel we restrict ourselves to a family of circular planar networks that grant such a commending feature on their boundary spikes.

7.7 Recovery on well–connected spider networks

The goal now is the complete recovery of the conductivity function of spider networks when possible. The forthcoming results are also inspired in the methodology for the combinatorial laplacian in [37]. Since we work with Schrödinger operators, we are involved in situations of a different nature than the ones they dealt with in the referred paper. Hence, we need other tools and techniques.

We work with spider networks because they have circular simmetry and good connectivity properties, which mixed together is useful for conductances recovery purposes, see Section 4.2 for their definition. In fact, we work with a subfamily of spider networks, known as well–connected spider networks.

Let $\Gamma = (V, c)$ be a spider network with n radii and m circles. We say that Γ is a *well–connected spider network* if $n \equiv 3 (\text{mod } 4)$ and $m = \frac{n-3}{4}$. For each $j = 1, \dots, n$ we consider the boundary sets $A_j = \{v_{1+j}, \dots, v_{\frac{n-1}{2}+j}\} \subset \delta(F)$, $B_j = \{v_{\frac{n+1}{2}+j}, \dots, v_{n-1+j}\} \subset \delta(F)$ and $R_j = \{v_j\} \subset \delta(F)$. Notice that

$|A_j| = |B_j| = \dfrac{n-1}{2}$. Moreover, these boundary configurations on a well–connected spider network guarantee that A_j and B_j are always connected through Γ. What is more, if we contract the boundary spike $v_j x_{jm}$ to a single boundary vertex then we break the connection between A_j and B_j, see [37] and Figure 7.4 for more details. Given an index $i \in \{0, \ldots, m+1\}$,

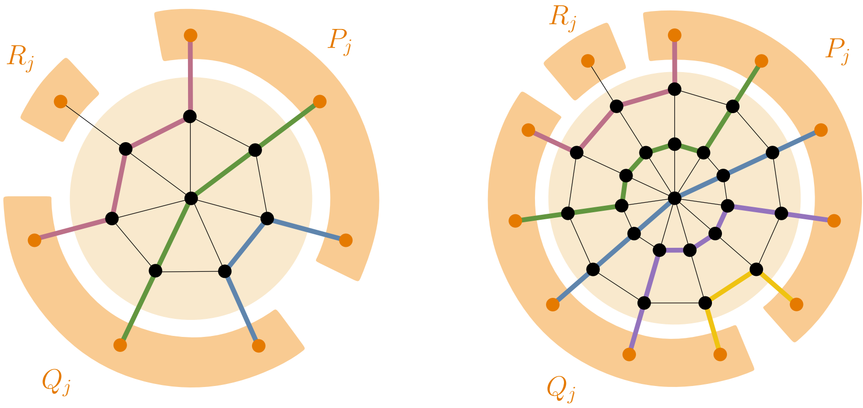


Figure 7.4 The well–connected spider networks with $n = 7$ and $n = 11$ boundary vertices.

we consider the *circular layers* of vertices $D_i = \{x_{li} \in V : l = 1, \ldots, n\} \subset \bar{F}$. In particular, $D_0 = \{x_{00}\}$ and $D_{m+1} = \delta(F)$.

The recovery of conductances on a well–connected spider network is an iterative process, for we are not able to give explicit formulae for all the conductances at the same time but we can give a recovery algorithm instead. Hence, we describe the algorithm in steps, each of them requiring the information obtained in the last one.

To start with, let $\mathsf{N_q}$ be an irreducible and symmetric M–matrix of order n satisfying that if $\mathsf{N_q}(\mathsf{A};\mathsf{B})$ is a $k \times k$ circular minor of $\mathsf{N_q}$, then $-\mathsf{N_q}(\mathsf{A};\mathsf{B})$ is totally positive. Let $\lambda \geq 0$ be the lowest eigenvalue of $\mathsf{N_q}$ and $\sigma \in \Omega(\delta(F))$ the eigenvector associated with λ. In addition, we choose $\omega \in \Omega(\bar{F})$ such that $\omega = k\sigma$ on $\delta(F)$, $0 < k < 1$.

Step 0

In this step we do not recover any conductance. However, we set the necessary tools to obtain them in future steps. Having fixed an index

$j \in \{1, \dots, n\}$, we consider the overdetermined partial Dirichlet–Neumann boundary value problem that consists in finding $u \in \mathcal{C}(\bar{F})$ such that

$$\mathcal{L}_q(u) = 0 \text{ on } F, \quad u = \varepsilon_{v_j} \text{ on } A_j \cup R_j \quad \text{and} \quad \frac{\partial u}{\partial \mathsf{n}_F} = 0 \text{ on } A_j. \quad (7.15)$$

There exists a large set of vertices of the well–connected spider network Γ where $u = 0$. We denote this set by

$$Z(u) = \big\{x \in \bar{F} \, : \, u(x) = 0\big\} = \bar{F} \setminus \mathsf{supp}(u).$$

Clearly, $A_j \subseteq Z(u)$. The size of $Z(u)$, however, is much bigger than the size of A_j.

Proposition 7.7.1. *It is satisfied that*

$$Z(u) = \{x_{li} \in V \, : \, i = 0, \dots, m+1, \; l = i + j - m, \dots, 3m + 2 + j - i\}.$$

Proof. We divide the proof into different stages that lead to the result. So, we have to prove the following claims:

(i) $H_k = \big\{x_{i\,m-k} \in V \, : \, i = 1 + j + k, \dots, \frac{n-1}{2} + j - k\big\}$ is a subset of $Z(u)$ for all $k = 0, \dots, m$,

(ii) $T_1 = \big\{x_{j+k-i\,m-k} \in V \, : \, i = 0, \dots, 2m-2, \; k = \lceil \frac{i}{2} \rceil, \dots, m-1\big\}$ is a subset of $Z(u)$,

(iii) $T_2 = \Big\{x_{\frac{n+1}{2}+j-k+i\,m-k} \in V \, : \, i = 0, \dots, 2m-2, \; k = \lceil \frac{i}{2} \rceil, \dots, m-1\Big\}$ is also a subset of $Z(u)$,

and let $Z_0 = T_1 \cup T_2 \cup \bigcup_{k=0}^{m} H_k$. To prove (i), we perform induction on k. For $k = 0$ and $i = 1 + j, \dots, \frac{n-1}{2} + j$, it is satisfied that $v_i \in A_j$ and hence

$$0 = \frac{\partial u}{\partial \mathsf{n}_F}(v_i) = -c(v_i, x_{im}) u(x_{im}),$$

which means that $u(x_{im}) = 0$ for all $i = 1+j, \dots, \frac{n-1}{2}+j$. Then, $H_0 \subseteq Z(u)$. Let us assume that (i) is true for any index $l < k$ and we want to see that the result it holds for k. If $i \in \{1 + j + k, \dots, \frac{n-1}{2} + j - k\}$, then by induction hypothesis $0 = \mathcal{L}_q(u)(x_{i\,m-k+1}) = -c(x_{i\,m-k+1}, x_{i\,m-k}) u(x_{i\,m-k})$, which means that $u(x_{i\,m-k}) = 0$ and so (i) follows.

To demonstrate (ii) we use double induction on i and k. For $i = 0$ and $k = m-1$, using (i) we get that

$$0 = \mathcal{L}_q(u)(x_{j+m\,1}) = -c(x_{j+m\,1}, x_{j+m-1\,1})u(x_{j+m-1\,1})$$

and hence $u(x_{j+m-1\,1}) = 0$. Now we assume that the result holds for $i = 0$ and any index $l > k$ and we want to see that it also holds for k. Using (i) and the induction hypothesis,

$$0 = \mathcal{L}_q(u)(x_{j+k+1\,m-k}) = -c(x_{j+k+1\,m-k}, x_{j+k\,m-k})u(x_{j+k\,m-k})$$

and so $u(x_{j+k\,m-k}) = 0$. Therefore, the case $i = 0$ holds. The next phase is to suppose that (ii) holds for any index $l < i$ and any $k = \left\lceil \frac{l}{2} \right\rceil, \dots, m-1$, and to prove that in this case it also holds for i and $k \in \left\{\left\lceil \frac{i}{2} \right\rceil, \dots, m-1\right\}$. By induction hypothesis,

$$0 = \mathcal{L}_q(u)(x_{j+k-i+1\,m-k}) = -c(x_{j+k-i+1\,m-k}, x_{j+k-i\,m-k})u(x_{j+k-i\,m-k})$$

and hence $u(x_{j+k-i\,m-k}) = 0$, completing the double induction. In consequence, $T_1 \subseteq Z(u)$. The result in (iii) is proved analogously.

The inclusion $Z_0 \subseteq Z(u)$ is a direct consequence of (i), (ii) and (iii) if we rearrange the indices. Moreover, suppose that there exists a vertex $x \in \delta(Z_0) \cap Z(u)$. Then, using the same techniques for equation $\mathcal{L}_q(u) = 0$ on F as in the proofs of (i) and (ii), we see that $u = 0$ on $\bar{F}$. This is a contradiction with $u(v_j) = 1$ and hence $\delta(Z_0) \subset \mathsf{supp}(u)$.

Finally, notice that the k–connection between A_j and B_j covers any vertex of $\bar{F} \setminus \{v_j\}$ and hence, keeping in mind the strong alternating property proved in Theorem 6.2.3, we conclude that $\bar{F} \setminus Z_0 \subseteq \mathsf{supp}(u)$, which means that $Z_0 = Z(u)$. $\square$

Actually, the set $Z(u)$ has a very characteristic shape. In Figure 7.5(a) we show this pattern. In particular, there are exactly $n-2$ vertices in D_1 for which $u = 0$ and exactly two vertices in D_1 for which $u \neq 0$.

Step 1

Let us fix the index $j \in \{1, \dots, n\}$ again for this step and let us consider the unique solution $u \in \mathcal{C}(\bar{F})$ of Problem (7.15). We already know that $u = 0$ on A_j and $u = 1$ on R_j. Moreover, the values of u on B_j are given by the matricial equation

$$\mathsf{u}_{\mathsf{B_j}} = -\mathsf{N_q}(\mathsf{A_j}; \mathsf{B_j})^{-1} \cdot \mathsf{N_q}(\mathsf{A_j}; \mathsf{v_j})$$

because of Corollary 7.2.5. Notice that all the values of u on B_j are known, for the Dirichlet–to–Robin map is known. In consequence, we know u on all the boundary $\delta(F)$. In Figure 7.5(b) we show all the information obtained at the end of this step.

Step 2

In this step recover the conductances of all the boundary spikes of the well–connected by means of the *boundary spike formula for circular planar networks*:

$$c(v_j, x_{jm}) = \frac{\omega(v_j)}{\omega(x_{jm})}\Big(\mathsf{N_q}(\mathsf{v_j};\mathsf{v_j}) - \mathsf{N_q}(\mathsf{v_j};\mathsf{B_j})\cdot \mathsf{N_q}(\mathsf{A_j};\mathsf{B_j})^{-1}\cdot \mathsf{N_q}(\mathsf{A_j};\mathsf{v_j}) - \lambda\Big)$$

for all $j = 1, \ldots, n$. That is, now we know the values of the conductances of all the edges joining vertices from D_{m+1} and D_m. In Figure 7.5(c) we show all the information obtained at the end of this step.

Step 3

Again, let us fix the index $j \in \{1, \ldots, n\}$ in this step and let us consider the unique solution $u \in \mathcal{C}(\bar{F})$ of Problem (7.15). Then, we know all the values of u on D_m, as the following result shows.

Lemma 7.7.2. *The values of u on D_m are known. They are given by*

$$u(x_{km}) = \frac{1}{c(v_k, x_{km})}\Big(\lambda u(v_k) - \mathsf{N_q}(\mathsf{v_k};\mathsf{v_j}) - \mathsf{N_q}(\mathsf{v_k};\mathsf{B_j})\cdot \mathsf{u_{B_j}}\Big) + \frac{\omega(x_{km})}{\omega(v_k)}u(v_k)$$

for all $k = 1, \ldots, n$.

Proof. We can express Problem (7.15) as the Dirichlet problem

$$\mathcal{L}_q(u) = 0 \text{ on } F \quad \text{and} \quad u = \varepsilon_{v_j} + u_{B_j} \text{ on } \delta(F)$$

with the additional condition $\dfrac{\partial u}{\partial \mathsf{n}_F} = 0$ on A_j. Therefore, by the definition of the Dirichlet–to–Robin map, for all $v_k \in \delta(F)$ it is satisfied that

$$\begin{aligned}\mathsf{N_q}(\mathsf{v_k};\mathsf{v_j}) + \mathsf{N_q}(\mathsf{v_k};\mathsf{B_j})\cdot \mathsf{u_{B_j}} &= \Lambda_q\Big(\varepsilon_{v_j} + u_{B_j}\Big)(v_k) = \frac{\partial u}{\partial \mathsf{n}_F}(v_k) + q(v_k)u(v_k) \\ &= \left(\lambda + \frac{\omega(x_{km})}{\omega(v_k)}c(v_k, x_{km})\right)u(v_k) - c(v_k, x_{km})u(x_{km}).\end{aligned}$$

Observe that all the terms of this equality, except the value $u(x_{km})$, are already known. Therefore, we get the result. □

In Figure 7.5(d) we show all the data gathered from the well–connected spider network at the end of this step.

Step 4

Let us define the linear operator $\wp\colon \mathcal{C}(\bar{F}) \longrightarrow \mathcal{C}(F \setminus \{x_{00}\})$ given by the values

$$\wp(v)(x_{lk}) = c(x_{lk}, x_{l\,k+1})v(x_{l\,k+1}) + c(x_{lk}, x_{l+1\,k})v(x_{l+1\,k}) + c(x_{lk}, x_{l-1\,k})v(x_{l-1\,k})$$

for all $v \in \mathcal{C}(\bar{F})$ and $x_{lk} \in F \setminus \{x_{00}\}$. This operator will be useful in this and also in the following steps.

Here we find the conductances of all the edges with both ends in D_m. However, we state a more general result.

Proposition 7.7.3. *Let $i \in \{0,\dots,m-1\}$. For every $j = 1,\dots,n$, let us suppose that we know the values of u on D_{i+2} and D_{i+1}. Also, we suppose that the conductances of all the edges joining vertices from D_{i+2} and D_{i+1} are known. Now fix the index $j \in \{1,\dots,n\}$. Then, the conductances $c(x_{i+j-m+1\,i+1}, x_{i+j-m\,i+1})$ are also known. They are given by*

$$c(x_{i+j-m+1\,i+1}, x_{i+j-m\,i+1}) = -\frac{u(x_{i+j-m+1\,i+2})}{u(x_{i+j-m\,i+1})}\, c(x_{i+j-m+1\,i+1}, x_{i+j-m+1\,i+2}).$$

Proof. We fix the indices $i \in \{0,\dots,m-1\}$ and $j \in \{1,\dots,n\}$. Then, by Corollary 7.7.1,

$$u(x_{i+j-m+1\,i+1}) = u(x_{i+j-m+2\,i+1}) = u(x_{i+j-m+1\,i}) = 0.$$

In consequence, we get

$$\begin{aligned}0 &= \mathcal{L}_q(u)(x_{i+j-m+1\,i+1})\\ &= -c(x_{i+j-m+1\,i+1}, x_{i+j-m+1\,i+2})u(x_{i+j-m+1\,i+2})\\ &\quad - c(x_{i+j-m+1\,i+1}, x_{i+j-m\,i+1})u(x_{i+j-m\,i+1}).\end{aligned}$$

The value $c(x_{i+j-m+1\,i+1}, x_{i+j-m\,i+1})$ is the only unknown term of this equality and by Proposition 7.7.1 we know that $u(x_{i+j-m\,i+1}) \neq 0$. □

When $i = m-1$, Propositions 7.7.1 and 7.7.3 show that $c(x_{jm}, x_{j-1\,m})$ is known for all $j = 1, \ldots, n$. See Figure 7.5(e) in order to see all the known information at the end of this step.

Step 5

In this step we give the conductances of all the edges joining the vertices from D_m and D_{m-1}. Furthermore, we state a more general result.

Proposition 7.7.4. *Let $i \in \{0, \ldots, m-1\}$. For every $j = 1, \ldots, n$, let us suppose that we know the values of u on D_{i+2} and D_{i+1}. Also, let us suppose that we know the conductances of all the edges joining vertices from D_{i+2} and D_{i+1}, and the ones of the edges with both ends in D_{i+1}. Now fix the index $j \in \{1, \ldots, n\}$. Then, the conductances $c(x_{i+j-m\,i}, x_{i+j-m\,i+1})$ are also known. They are given by*

$$c(x_{i+j-m\,i}, x_{i+j-m\,i+1}) = \frac{\wp(u)(x_{i+j-m\,i+1})\omega(x_{i+j-m\,i+1})}{u(x_{i+j-m\,i+1})\omega(x_{i+j-m\,i})} - \frac{\wp(\omega)(x_{i+j-m\,i+1})}{\omega(x_{i+j-m\,i})}.$$

Proof. We fix the indices $i \in \{0, \ldots, m-1\}$ and $j \in \{1, \ldots, n\}$. Observe that $\wp(\omega)(x_{i+j-m\,i+1})$ and $\wp(u)(x_{i+j-m\,i+1})$ are already known. Then,

$$0 = \mathcal{L}_q(u)(x_{i+j-m\,i+1}) = \frac{u(x_{i+j-m\,i+1})}{\omega(x_{i+j-m\,i+1})}\wp(\omega)(x_{i+j-m\,i+1}) - \wp(u)(x_{i+j-m\,i+1}) + \frac{\omega(x_{i+j-m\,i})}{\omega(x_{i+j-m\,i+1})}c(x_{i+j-m\,i+1}, x_{i+j-m\,i})u(x_{i+j-m\,i+1})$$

and hence $c(x_{i+j-m\,i+1}, x_{i+j-m\,i})$ is the only unknown term of this equality. Notice that $u(x_{i+j-m\,i+1}) \neq 0$ because of Proposition 7.7.1. $\square$

In particular, when $i = m-1$, Propositions 7.7.1 and 7.7.4 show that $c(x_{j-1\,m}, x_{j-1\,m-1})$ is known for all $j = 1, \ldots, n$. See Figure 7.5(f) in order to see all the information gathered at the end of this step.

Step 6

In this step we are able to obtain the values of u on D_{m-1} for all $j = 1, \ldots, n$. In fact, let us state a more general result.

Proposition 7.7.5. *Let $i \in \{0,\dots,m-1\}$. For every $j = 1,\dots,n$, let us suppose that we know the values of u on D_{i+2} and D_{i+1}. Also, let us suppose that we know the conductances of all the edges joining vertices from D_{i+2} and D_{i+1}, from D_{i+1} and D_i and the ones of the edges with both ends in D_{i+1}. Now fix the index $j \in \{1,\dots,n\}$. Then, the values of u on D_i are also known. They are given by*

$$u(x_{k\,i}) = -\frac{\wp(\omega)(x_{k\,i+1})}{\omega(x_{k\,i+1})c(x_{k\,i+1},x_{k\,i})}u(x_{k\,i+1}) - \frac{\wp(u)(x_{k\,i+1})}{c(x_{k\,i+1},x_{k\,i})} - \frac{\omega(x_{k\,i})}{\omega(x_{k\,i+1})}u(x_{k\,i+1})$$

for all $k = 1,\dots,n$.

Proof. Fixed two indices $i \in \{0,\dots,m-1\}$ and $j \in \{1,\dots,n\}$, let $x_{ki} \in D_i$ with $k \in \{1,\dots,n\}$. Observe that $\wp(\omega)(x_{k\,i+1})$ and $\wp(u)(x_{k\,i+1})$ are known. Then,

$$\begin{aligned} 0 = \mathcal{L}_q(u)(x_{k\,i+1}) &= -\frac{u(x_{k\,i+1})}{\omega(x_{k\,i+1})}\wp(\omega)(x_{k\,i+1}) - \wp(u)(x_{k\,i+1}) \\ &- c(x_{k\,i+1},x_{k\,i})u(x_{k\,i}) - \frac{\omega(x_{k\,i})}{\omega(x_{k\,i+1})}c(x_{k\,i+1},x_{k\,i})u(x_{k\,i+1}) \end{aligned}$$

and hence $u(x_{k\,i})$ is the only unknown term of this equality. □

In particular, when $i = m-1$, Propositions 7.7.1 and 7.7.5 show that u is known on D_{m-1} for all $j = 1,\dots,n$. Observe that we already knew some of the values of u on D_{m-1}, which are those of the vertices in $Z(u)$. Figure 7.5(g) shows the information obtained until this step.

Step 7 and beyond

We keep repeating the same process to obtain more conductances, that is, we keep applying Proposition 7.7.3 from Step 4, then Proposition 7.7.4 from Step 5 and then Proposition 7.7.5 from Step 6 for each $i = m-2,\dots,0$. We stop when applying Proposition 7.7.5 from Step 6 for $i = 0$. In fact, we obtain the value $u(x_{00}) = 0$ for all $j = 1,\dots,n$, which is already known because $x_{00} \in Z(u)$. This is the last step of the process, for all the conductances are known at this point.

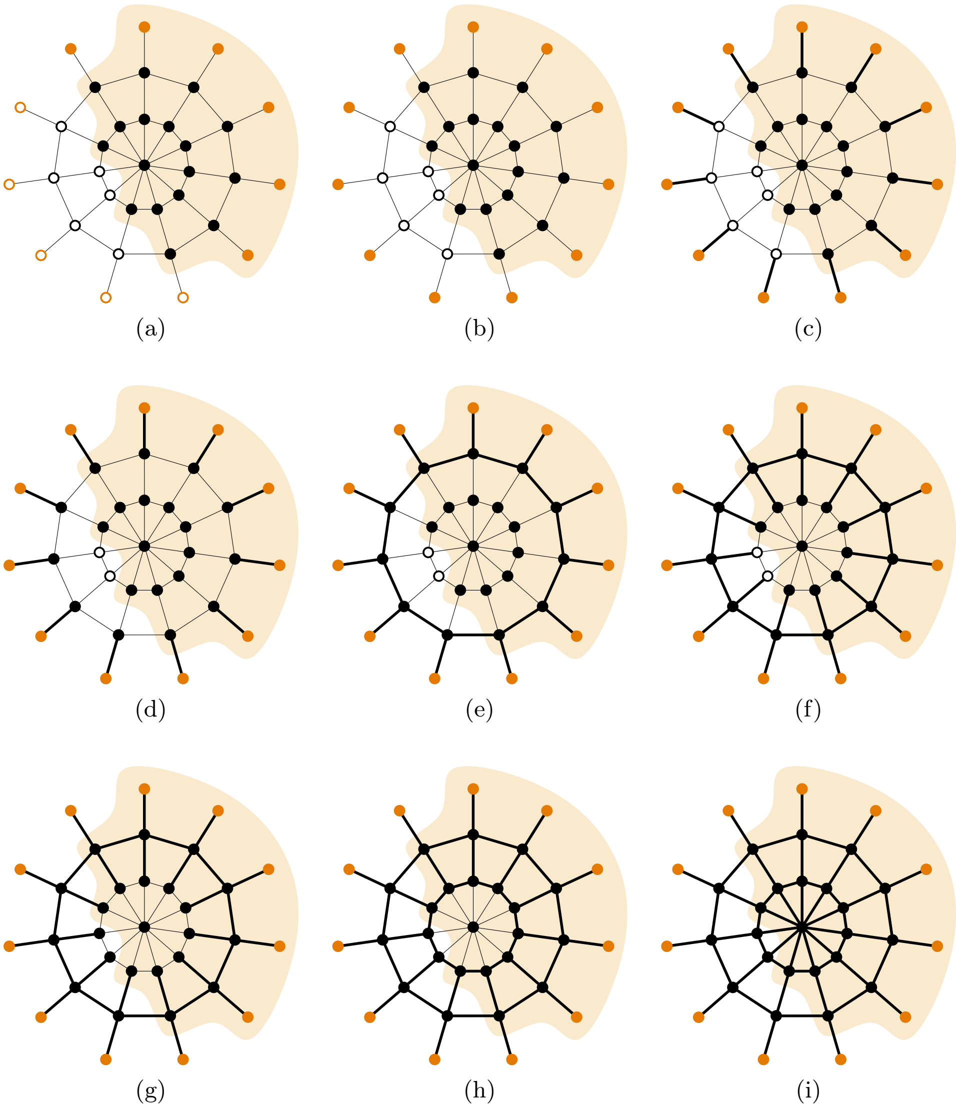


Figure 7.5 The bold items are the ones known at the end of each step for the case $n = j = 11$.

Example

Let us consider the well–connected spider network on $n = 7$ radii, with associated Dirichlet–to–Robin matrix

$$\mathsf{N_q}(\delta(\mathsf{F});\delta(\mathsf{F})) = \frac{1}{889}\begin{pmatrix} 2239 & -281 & -211 & -183 & -183 & -211 & -281 \\ -281 & 2239 & -281 & -211 & -183 & -183 & -211 \\ -211 & -281 & 2239 & -281 & -211 & -183 & -183 \\ -183 & -211 & -281 & 2239 & -281 & -211 & -183 \\ -183 & -183 & -211 & -281 & 2239 & -281 & -211 \\ -211 & -183 & -183 & -211 & -281 & 2239 & -281 \\ -281 & -211 & -183 & -183 & -211 & -281 & 2239 \end{pmatrix}$$

where the ordering in $\delta(F)$ is $v_1 < \ldots < v_7$. We want to determine the unknown conductances of this network following the above algorithm. First, we deduce the weight $\omega \in \delta(F)$ and the real value $\lambda \geq 0$ such that $q = q_\omega + \lambda\chi_{\delta(F)}$ on $\bar{F}$. Operating we see that the first eigenvalue of $\mathsf{N_q}(\delta(\mathsf{F});\delta(\mathsf{F}))$ is $\lambda = 1$ and its associated normalized eigenvector is the constant vector $\sigma_{\delta(\mathsf{F})} = (7^{-\frac{1}{2}}, \ldots, 7^{-\frac{1}{2}})$, which is a weight on the boundary. By an arbitrary extension of it to F and normalizing, we can consider the weight $\omega = 1/6$ on $\delta(F) \cup \{x_{00}\}$ and $\omega = 1/3$ on $F \smallsetminus \{x_{00}\}$. By Step 0, we fix the index $j = 7$ and see that $u = 0$ in $Z(u) = \{v_1, v_2, v_3, x_{71}, x_{11}, \ldots, x_{41}, x_{00}\}$. Moreover, $u(v_7) = 1$ by hypothesis and $u(x_{51}), u(x_{61}) \neq 0$. Using Step 1,

$$\begin{aligned} \mathsf{u}_{\mathsf{B}_7} &= -\mathsf{N_q}(\mathsf{A}_7;\mathsf{B}_7)^{-1} \cdot \mathsf{N_q}(\mathsf{A}_7;\mathsf{v}_7) \\ &= -\frac{1}{3556}\begin{pmatrix} -366 & 788 & -366 \\ 915 & -1843 & 788 \\ -366 & 915 & -366 \end{pmatrix} \cdot \begin{pmatrix} -281 \\ -211 \\ -183 \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{7}{2} \\ -\frac{7}{2} \end{pmatrix} \end{aligned}$$

and so $u(v_4) = -1$, $u(v_5) = \frac{7}{2}$ and $u(v_6) = -\frac{7}{2}$. Regarding Step 2, we easily get that

$$\begin{aligned} c(v_7, x_{71}) &= \frac{\omega(v_7)}{\omega(x_{71})}\Big(\mathsf{N_q}(\mathsf{v}_7;\mathsf{v}_7) - \mathsf{N_q}(\mathsf{v}_7;\mathsf{B}_7) \cdot \mathsf{N_q}(\mathsf{A}_7;\mathsf{B}_7)^{-1} \cdot \mathsf{N_q}(\mathsf{A}_7;\mathsf{v}_7) - \lambda\Big) \\ &= 1. \end{aligned}$$

Therefore, considering the last steps for all he values $j \in \{1, \ldots, 7\}$, we see that all the edges joining vertices from D_2 and D_1 have conductance 1. Having fixed again the index $j = 7$, by Step 3 we get the values

$$u(x_{k1}) = u(v_k) - \mathsf{N_q}(\mathsf{v_k};\mathsf{v}_7) - \mathsf{N_q}(\mathsf{v_k};\mathsf{B}_7) \cdot \mathsf{u}_{\mathsf{B}_7} + 2u(v_k)$$

for all $k = 1, \dots, 7$ and so $u(x_{51}) = -u(x_{61}) = \frac{1}{2}$, which were still unknown. Step 4 shows that

$$c(x_{71}, x_{61}) = \frac{u(v_7)}{u(x_{61})} c(x_{71}, v_7) = 2.$$

Considering the last steps for all $j = 1, \dots, 7$ we easily see that $c(x_{k1}, x_{k+1\,1})$ equals 2 for all $k = 1, \dots, 7$ and then all the edges with both ends in D_1 have conductance 2. Finally, we only need to obtain the conductances $c(x_{k1}, x_{00})$ for all $k = 1, \dots, 7$. Step 5 provides them: fixing $j = 7$,

$$c(x_{00}, x_{61}) = \left(\frac{\wp(u)(x_{61})}{u(x_{61})} - \frac{\wp(\omega)(x_{61})}{\omega(x_{61})} - \lambda \right) \frac{\omega(x_{61})}{\omega(x_{00})} = 1,$$

where $\wp(u)(x_{61}) = c(x_{61}, v_6)u(v_6) + c(x_{61}, x_{71})u(x_{71}) + c(x_{61}, x_{51})u(x_{51})$ and $\wp(\omega)(x_{61}) = c(x_{61}, v_6)\omega(v_6) + c(x_{61}, x_{71})\omega(x_{71}) + c(x_{61}, x_{51})\omega(x_{51})$. Moreover, doing the same for all $j = 1, \dots, 7$, we get that $c(x_{00}, x_{k1})$ for all $k = 1, \dots, 7$. Hence, we have performed a complete conductivity recovery in a network where only boundary information was provided.

8

Conclusions and future work

In this chapter we want to summarize the participation of this thesis in the subjects we have dealt with, as well as future works suggested by these findings.

8.1 On Green functions in network partitioning

In Chapters 3 and 4 we provide the necessary tools and techniques for the division of networks into pieces or their transformations in order to achieve results on the original network from similar results on the pieces or the transformations. In particular, we deal with orthogonal Green functions, effective resistances, Kirchhoff indices and Green functions on a subset of vertices so as to show useful applications of these techniques. However, since the Green functions allow us to obtain the solution of Poisson equations on a network, the variety of problems that can be solved reducing them to smaller problems and using these techniques is almost unlimited. Clearly, the underlying philosophy is defined by the sentence "divide and conquer".

We perform network divisions on generalized cluster, corona and product networks. All of them are of great importance in the network field. For example, any network having at least one separating vertex can be seen as a generalized cluster network. Hence, any Poisson equation on such a network can be reduced to two or more smaller problems, which are easier to solve. Similarly, any network having at least one vertex with only one neighbour or at least one clique can be considered a generalized corona network. Therefore, the three studied families cover a wide range of networks. In addition, we also perform network transformations with a view to cover even more families of networks: ball–like networks, for instance spider networks, can be seen as the transformation of a product network.

Hence, the main contribution of our works to this field is to set the foundations for network breakage and to show its usefulness with some families of composite networks. Several future works could be extracted from here by considering other families of composite networks or other functions and parameters subject to recovery on them.

Effective resistances and Kirchhoff indices are global parameters associated with a network that describe structural properties and are high sensitive to local perturbations. Our main contribution is to show the behaviour of these parameters when networks are appended to form bigger networks. Does the network get stiffer? Is the resistance between two vertices reduced? We have seen the answers in generalized cluster and corona networks (see Corollaries 3.1.5 and 3.2.5), but future works could perform the same study for other composite networks. We would like to remark that the findings in Sections 3.1 and 3.2 justify the definition of the effective resistances and the Kirchhoff index with respect to a weight and a non–negative real value λ. The reason is that, even when considering a constant weight and $\lambda = 0$ on these composite networks, their generalized Kirchhoff indices and effective resistances are naturally expressed in terms of the same parameters with respect to other weights and other non–zero values on the factor networks.

Since generalized effective resistances and Kirchhoff indices are resistive and structural parameters that have only been defined on the whole set of vertices of the network, it would be of great interest in the future to define similar concepts with similar properties on a subset of vertices of the network and relate them to the Green function with respect to this subset. Specifically, this study will require the definition of Kirchhoff Index associated with a boundary value problem and the analysis of its properties.

In [7, 4, 8, 5, 13] the reader can find the publications related to these results.

8.2 On the discrete Serrin's problem

Our main contribution in this area has been to define the discrete version of a classical problem in the study of the symmetries of the solution of elliptic boundary value problems; that is, the discrete Serrin's problem. Our main result is the characterization of regular networks that satisfy the Serrin's condition given in Theorem 5.5.3. We have proved that the verification of Serrin's condition forces a relatively regular network to be a regular layered network; that is, forces the network to have stronger regularity properties. Notice that regular layered networks are, somehow, ball–like discrete domains, since their behaviour between distance layers is regular and does not depend on the election of the vertex in the layer. Therefore, we can conclude that our participation in Serrin's problem is the definition of the structures on the finite network framework that resemble balls and possess their conductivity spreading properties.

On the other hand, we have found some networks that fit the Serrin's condition and are not regular layered networks. Future work could consist in the study of these networks and the attempt to achieve a more general characterization that includes them.

Regarding other Serrin–type problems, an interesting future work could be to consider the following overdetermined partial Dirichlet–Neumann boundary value problem

$$\mathcal{L}(u) = 1 \text{ on } F, \quad u = 0 \text{ on } A \quad \text{and} \quad \frac{\partial u}{\partial \mathsf{n}_F} = C \text{ on } A$$

for a boundary subset $A \subseteq \delta(F)$ and try to deduce the networks and the sets A in them that allow this problem to have a unique solution. Observe that the extreme case (when $A = \delta(F)$) is the discrete Serrin's problem.

The works of this thesis on the discrete Serrin's problem appear in [6, 10].

8.3 On the Dirichlet–to–Robin map

The Dirichlet–to–Neumann map of the laplacian operator has been proved to be the key for conductance recovery on finite networks [38]. Here, we have shown that the same occurs for the Dirichlet–to–Robin map related to a positive semi–definite Schrödinger operator. Moreover, in addition to the conductances, it is also the key to recover the associated potential.

In Theorem 6.2.3 we look upon the solutions of boundary value problems with an alternating property in a part of the boundary and show that they spread across the network in such a way that they hold a derived alternating property in another part of the boundary. In fact, these solutions spread following boundary–to–boundary paths where the sign of the solution is invariable and has opposite sign with respect to the neighbouring paths in the circular order. This is an important contribution of this thesis to the literature, for it explains the behaviour of the distribution in the network of alternating currents applied on the boundary.

However, our main contribution in this field is the characterization through the Dirichlet–to–Robin map of circular planar networks, shown in Theorem 6.4.3. First, we focus in circular planar networks and observe a ground–breaking fact: any Dirichlet–to–Robin map can be the response matrix of an infinite family of networks associated with different conductivity functions. This phenomenon has not been observed in the standard treatment of the Dirichlet–to–Neumann map, since it was implicitly assumed that the recovered operator is the Laplacian one. Despite that, by choosing a specific extension of the positive eigenfunction associated with the lowest eigenvalue of this matrix, we get a unique network whose Dirichlet–to–Robin map corresponds to the given matrix. Therefore, this represents an extension since the response matrices that have been considered do not have to be singular nor diagonally dominant.

The publication related to the above–mentioned results is [9].

8.4 On overdetermined partial boundary value problems

When constructing an algorithm for the recovery of the conductances in a network, we see the necessity of considering some overdetermined partial boundary value problems. Our contribution to this field is the definition and the study of this type problems in the context of networks. In particular, in Corollary 7.1.8 we characterize the existence and uniqueness of solution of these problems for any data. In addition, we introduce the overdetermined partial Dirichlet–to–Neumann maps, whose properties characterize the existence and uniqueness of solution of the overdetermined partial boundary value problems. We also have introduced the resolvent kernels associated with this type of problems and have studied their interrelation.

We use the above mentioned results in order to give formulae for the conductances of certain edges. One of the formulas, given in Corollary 7.6.2, is a generalization of a similar formula given in [37] for the response matrix associated with the laplacian operator. The other formula, presented in

Proposition 7.3.2, is ground–breaking and provides the values of separated boundary conductances when the modified Green matrix of a network is known. In fact, only certain diagonal values of this matrix are necessary.

Both formulae allow us to recover the conductivity function of two families of networks. Specifically, we perform a full conductivity recovery on three dimensional grids, which is the discrete analogous to a cuboid in $\mathbb{R}^3$ with constant flow in each direction. This problem is not solvable on the continuum but we have solved it in the discrete setting. We also recover completely the conductances of a spider network in a recursive way. It is important to notice that from minimal data on the boundary we recover plenty of values in the interior of the networks that can have a much bigger cardinality.

The future work in this area can be the characterization of response matrices for other networks different from circular planar networks. In addition, we can tackle the construction of an algorithm for the recovery of the conductances of three–dimensional networks assuming symmetry properties on the conductances.

The results of this thesis in the above–mentioned fields can be found in [11, 12].

Bibliography

[1] G. Alessandrini. Stable determination of conductivity by boundary measurements. *Appl. Anal.*, 27(1–3):153–172, 1988.

[2] T. Amdeberhan. Two symmetry problems in potential theory. *Electron. J. Differential Equations*, 43: 5 pp. (electronic), 2001.

[3] V. Anandam. *Harmonic functions and potentials on finite or infinite networks*, volume 12 of *Lecture Notes of the Unione Matematica Italiana*. Springer, Heidelberg; UMI, Bologna, 2011.

[4] C. Araúz, E. Bendito, A. Carmona, and A. M. Encinas. La función de Green y el Índice de Kirchhoff de redes cluster (in congress). *VII Jornadas de Matemática Discreta y Algorítmica, Universidad de Cantabria*, pages 79–88, 2010.

[5] C. Araúz, A. Carmona, and A. M. Encinas. Green function on product networks (in congress, in Spanish). *VIII Jornadas de Matemática Discreta y Algorítmica, Universidad de Almería*, pages 159–166, 2012.

[6] C. Araúz, A. Carmona, and A. M. Encinas. El Problema de Serrin Discreto (in congress). *Segundo Congreso de Jóvenes Investigadores RSME. Libro de Resúmenes. Sevilla.*, page 94, 2013.

[7] C. Araúz. The Kirchhoff indexes of some composite networks. *Discrete Appl. Math.*, 160(10–11):1429–1440, 2012.

[8] C. Araúz, E. Bendito, A. Carmona, and A. M. Encinas. The Kirchhoff index of cluster networks. *Electronic Notes in Discrete Mathematics*, 38:57–62, 2011.

[9] C. Araúz, A. Carmona, and A. M. Encinas. Dirichlet–to–Robin maps on finite networks. 2014. *Submitted.*

[10] C. Araúz, A. Carmona, and A. M. Encinas. Discrete Serrin's Problem. *Linear Algebra and its Applications*, 2014. *To appear.* http://dx.doi.org/10.1016/j.laa.2014.01.038.

[11] C. Araúz, A. Carmona, and A. M. Encinas. Overdetermined partial boundary value problems on finite networks. 2014. *Submitted.*

[12] C. Araúz, A. Carmona, and A. M. Encinas. Recovering the conductance in 3–dimensional grids. 2014. *Submitted.*

[13] C. Araúz, A. Carmona, and A. M. Encinas. Separation of variables on finite networks. 2014. *Submitted.*

[14] R. B. Bapat and S. Gupta. Resistance distance in wheels and fans. *Indian J. Pure Appl. Math.*, 41(1):1–13, 2010.

[15] A. Ben-Israel and T. N. E. Greville. *Generalized inverses*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 15. Springer-Verlag, New York, second edition, 2003. Theory and applications.

[16] E. Bendito, A. Carmona, and A. M. Encinas. Solving boundary value problems on networks using equilibrium measures. *J. Funct. Anal.*, 171(1):155–176, 2000.

[17] E. Bendito, A. Carmona, and A. M. Encinas. Equilibrium measures on finite networks: Effective resistance and hitting time. *Electronic Notes in Discrete Math.*, 10:68–71, 2001.

[18] E. Bendito, A. Carmona, and A. M. Encinas. Potential theory for Schrödinger operators on finite networks. *Rev. Mat. Iberoamericana*, 21(3):771–818, 2005.

[19] E. Bendito, A. Carmona, and A. M. Encinas. Eigenvalues, eigenfunctions and Green's functions on a path via Chebyshev polynomials. *Appl. Anal. Discrete Math.*, 3(2):282–302, 2009.

[20] E. Bendito, A. Carmona, and A. M. Encinas. The Kirchhoff indices of join networks. *Discrete Appl. Math.*, 160(1–2):24–37, 2012.

[21] E. Bendito, A. Carmona, A. M. Encinas, and J. M. Gesto. Potential theory for boundary value problems on finite networks. *Appl. Anal. Discrete Math.*, 1(1):299–310, 2007.

[22] E. Bendito, A. Carmona, A. M. Encinas, and J. M. Gesto. A formula for the Kirchhoff index. *Int. J. Quantum Chem.*, 108:1200–1206, 2008.

[23] E. Bendito, A. Carmona, A. M. Encinas, and J. M. Gesto. Characterization of symmetric M–matrices as resistive inverses. *Linear Algebra Appl.*, 430(4):1336–1349, 2009.

[24] E. Bendito, A. Carmona, A. M. Encinas, J. M. Gesto, and M. Mitjana. Kirchoff indexes of a network. *Linear Algebra Appl.*, 432(9):2278–2292, 2010.

[25] L. Borcea, V. Druskin, and A. V. Mamonov. Circular resistor networks for electrical impedance tomography with partial boundary measurements. *Inverse Problems*, 26(4):045010, 30, 2010.

[26] S. Boyd, A. Ghosh, and A. Saleri. Minimizing the effective resistance of a graph. *SIAM Rev.*, 50:37–66, 2008.

[27] B. Brandolini, C. Nitsch, P. Salani, and C. Trombetti. Serrin–type overdetermined problems: an alternative proof. *Arch. Ration. Mech. Anal.*, 190(2):267–280, 2008.

[28] F. Brock and A. Henrot. A symmetry result for an overdetermined elliptic problem using continuous rearrangement and domain derivative. *Rend. Circ. Mat. Palermo (2)*, 51(3):375–390, 2002.

[29] A. P. Calderón. On an inverse boundary value problem. In *Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980)*, pages 65–73. Soc. Brasil. Mat., Rio de Janeiro, 1980.

[30] D. Carlson. What are Schur complements, anyway? *Linear Algebra Appl.*, 74:257–275, 1986.

[31] A. Carmona, A. M. Encinas, and M. Mitjana. On the Moore–Penrose inverse of singular, symmetric and periodic Jacobi M–matrices. *SIAM 2012 (personal communication).*

[32] M. Cheney, D. Isaacson, and J. C. Newell. Electrical impedance tomography. *SIAM Rev.*, 41(1):85–101 (electronic), 1999.

[33] F. Chung and S. T. Yau. Discrete Green's functions. *J. Combin. Theory Ser. A*, 91(1–2):191–214, 2000. In memory of Gian–Carlo Rota.

[34] F. R. K. Chung and R. P. Langlands. A combinatorial Laplacian with vertex weights. *J. Combin. Theory Ser. A*, 75(2):316–327, 1996.

[35] D. Coppersmith, P. Doyle, P. Raghavan, and M. Snir. Random walks on weighted graphs and applications to on-line algorithms. *J. Assoc. Comput. Mach.*, 40(3):421–453, 1993.

[36] E. Curtis, E. Mooers, and J. Morrow. Finding the conductors in circular networks from boundary measurements. *RAIRO Modél. Math. Anal. Numér.*, 28(7):781–814, 1994.

[37] E. Curtis and J. Morrow. *Inverse Problems for Electrical Networks*, volume 13 of *Series on Applied Mathematics*. 2000.

[38] E. B. Curtis, D. Ingerman, and J. A. Morrow. Circular planar graphs and resistor networks. *Linear Algebra Appl.*, 283(1–3):115–150, 1998.

[39] P. Delsarte. An algebraic approach to the association schemes of coding theory. *Philips Res. Rep. Suppl.*, (10):vi+97, 1973.

[40] J. Dodziuk. Difference equations, isoperimetric inequality and transience of certain random walks. *Trans. Amer. Math. Soc.*, 284(2):787–794, 1984.

[41] R. B. Ellis. Discrete Green's functions for products of regular graphs. *ArXiv Mathematics e–prints*, Sept. 2003.

[42] R. B. Ellis, III. *Chip–firing games with Dirichlet eigenvalues and discrete Green's functions*. ProQuest LLC, Ann Arbor, MI, 2002. Thesis (Ph.D.)–University of California, San Diego.

[43] E. Estrada and N. Hatano. Topological atomic displacement and resistance distance in molecules. *In the book: Novel Molecular Descriptors. Theory and Applications I. Mathematical Chemistry Monographs*, 8:3–28, 2010.

[44] A. Farina and B. Kawohl. Remarks on an overdetermined boundary value problem. *Calc. Var. Partial Differential Equations*, 31(3):351–357, 2008.

[45] I. Fragalà and F. Gazzola. Partially overdetermined elliptic boundary value problems. *J. Differential Equations*, 245(5):1299–1322, 2008.

[46] J. Friedman and J. P. Tillich. Wave equations for graphs and the edge–based Laplacian. *Pacific J. Math.*, 216(2):229–266, 2004.

[47] N. Garofalo and J. L. Lewis. A symmetry result related to some overdetermined boundary value problems. *Amer. J. Math.*, 111(1):9–33, 1989.

[48] D. J. Klein and M. Randić. Resistance distance. *J. Math. Chem.*, 12(1-4):81–95, 1993. Applied graph theory and discrete mathematics in chemistry (Saskatoon, SK, 1991).

[49] W. J. Martin. Completely regular codes: a viewpoint and some problems. *Proceedings of 2004 Com2MaC Workshop on Distance–Regular Graphs and Finite Geometry*, pages 43–56, 2004.

[50] J. W. Moon and L. Moser. On cliques in graphs. *Israel J. Math.*, 3:23–28, 1965.

[51] A. I. Nachman. Reconstructions from boundary measurements. *Ann. of Math. (2)*, 128(3):531–576, 1988.

[52] S. Ponzio. The combinatorics of effective resistances and resistive inverses. *Inform. and Comput.*, 147(2):209–223, 1998.

[53] M. J. Rodriguez, M. D. Roselló, S. Jerez, and L. Jódar. Solving Dirichlet's problem in a rectangle for separate variable coefficient elliptic equations. *Appl. Math. Lett.*, 15(6):693–696, 2002.

[54] J. Serrin. A symmetry problem in potential theory. *Arch. Rational Mech. Anal.*, 43:304–318, 1971.

[55] H. F. Weinberger. Remark on the preceding paper of Serrin. *Arch. Rational Mech. Anal.*, 43:319–320, 1971.

[56] D. B. West. *Introduction to graph theory.* Prentice Hall, Inc., Upper Saddle River, NJ, 1996.

[57] Y. Yang and H. Zhang. Kirchhoff index of linear hexagonal chains. *Int. J. Quantum Chem.*, 108:503–512, 2008.

[58] F. Zhang, editor. *The Schur complement and its applications*, volume 4 of *Numerical Methods and Algorithms.* Springer–Verlag, New York, 2005.

[59] H. Zhang, Y. Yang, and C. Li. Kirchhoff index of composite graphs. *Discrete Appl. Math.*, 157(13):2918–2927, 2009.