Topological dualities and completions for (distributive) partially ordered sets

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A mi padre Juan, a mi madre Alicia y
a mi hermana Fanny

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Abstract

This PhD thesis is the result of our research on duality theory and completions for partially ordered sets. A first main aim of this dissertation is to propose different kind of topological dualities for some classes of partially ordered sets and a second aim is to try to use these dualities to obtain completions with nice properties. To this end, we intend to follow the line of the classical dualities for bounded distributive lattices due to Stone and Priestley. Thus, we will need to consider a notion of distributivity on partially ordered sets. Also we propose a topological duality for the class of all partially ordered sets and we use this duality to study some properties of partially ordered sets like its canonical extension, order-preserving maps and the extensions of n-ary maps that are order-preserving in each coordinate. Moreover, to attain these aims we will study the partially ordered sets from an algebraic point of view.

Resumen

Esta tesis doctoral es el resultado de nuestra investigación sobre la teoría de la dualidad y completaciones de conjuntos parcialmente ordenados. Un primer objetivo general de este trabajo es proponer diferentes tipos de dualidades topológicas para algunas clases de conjuntos parcialmente ordenados y un segundo objetivo es tratar de utilizar estas dualidades para obtener diferentes completaciones con buenas propiedades. Para este fin, nos proponemos seguir la línea de las dualidades clásicas para retículos distributivos acotados debidas a Stone y a Priestley. Por lo tanto, necesitaremos considerar una noción de distributividad sobre conjuntos parcialmente ordenados. También proponemos una dualidad topológica para la clase de todos los conjuntos parcialmente ordenados y usamos esta dualidad para estudiar algunas propiedades de los conjuntos parcialmente ordenados como su extensión canónica, funciones que preservan orden y las extensiones de funciones n-arias que preservan orden en cada coordenada. Por otra parte, para alcanzar estos objetivos vamos a estudiar los conjuntos parcialmente ordenados desde un punto de vista algebraico.

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General introduction

Many algebraic structures that appear in mathematics have associated in a very natural way a partial order. For instance, ordered groups, ordered rings, ordered vector spaces, the collection of open or closed subsets of a topological space, etc. In particular and this is more interesting for us, almost all classes of algebras associated to logics are classes of ordered algebras. For instance, the class of Boolean algebras associated to classical propositional logic, the class of Heyting algebras associated to propositional intuitionistic logic, the class of modal algebras associated to propositional modal logic, the class of MV-algebras associated to the Lukasiewicz's infinite-valued logic, etc.

An important and useful tool to study many abstract mathematical objects in mathematics, among them ordered algebraic structures, is Duality Theory. Roughly speaking, we can say that if two categories are dually equivalent then they are the two sides of the same coin, each bringing a different but equivalent perspective. In particular, if one category is a category of algebras and the other a category of topological spaces (perhaps with some extra structure), the duality brings a geometric view into the picture. The existence of a categorical duality between two categories is very useful for translating questions from one to other and return with the answer. Some problems are easy to handle in one category, and others in the other.

The topological dualities for classes of algebras associated with logics arose mainly with M.H. Stone's work [**54**] in the mid-thirties of the twentieth century when he developed a duality between Boolean algebras and a class of topological spaces, later known as Stone spaces. In the subsequent paper [**55**] Stone generalizes the previous duality for Boolean algebras to show that the category of bounded distributive lattices and lattice homomorphisms is dually equivalent to the category of spectral spaces and spectral maps. Both topological categories, Stone spaces and spectral spaces, are subcategories of the category of all topological spaces and continuous maps. Another classical duality, related to Stone's, is given by H.A. Priestley in [**52**] between the category of bounded distributive lattices and certain ordered topological spaces, which are known as Priestley spaces. Unlike Stone's duality, Priestley spaces are equipped with an additional partial order on the points in the space. These two kind of dualities, Stone and Priestley, are important to obtain other dualities for some other algebras associated to logics. For instance, in the spirit of Stone's work we can mentioned the duality between modal algebras and descriptive general frames [**33**] and in the spirit of Priestley's work we can mentioned the duality between Heyting algebras and Esakia spaces [**22**].

An ordered algebra can be considered as a partially ordered set with additional operations, see for instance [**17, 51**]. This point of view is useful to find complete relational semantics (Kripke-style semantics) for some non-classical logics which have associated classes of ordered algebras, see [**17, 25**]. Relational semantics have been a powerful tool to study and understand intuitionistic and modal logics. And it was shown that they are very closely related to topological dualities for the classes of Heyting algebras and modal algebras. In the literature there is a number of papers that get complete relational semantics for several non-classical logics [**17, 25, 1, 43, 45**]. One way to find complete relational semantics for non-classical logics is by means of completions of the ordered algebraic structures of classes associated to the logics, see [**17, 25**].

A *monotone poset expansion* is a tupla $\langle P, (f_i)_{i\in I}\rangle$ where P is a partially ordered set and for each $i \in I$, f_i is an n_i-ary operation on P such that is order-preserving or order-reversing in each argument. In general, a *completion* of an ordered algebraic structure is a pair consisting of a complete ordered algebraic structure and an embedding that maps the original structure into the complete one. In particular, it can be seen that the completions of several classes of monotone poset expansions do not depend of the n-ary operations, but only on the underlying partially ordered set, see [**28, 26, 17**].

An important and very well known class of monotone poset expansions is the class of Boolean algebras with operators, where an operator is an n-ary map defined on a Boolean algebra that preserves finite joins in each argument. In their 1951 papers [**41, 42**], Jónsson and Tarski introduced the notion of canonical extensions for Boolean algebras with operators and to attain this they used the Stone's classical representation theorem for Boolean algebras. Moreover, they proved that every identity not involving negation that holds in a Boolean algebra with operators also holds in its canonical extension. In [**28**] Gehrke and Jónsson proved that every bounded distributive lattice with operators can be embedded in a completely distributive algebraic lattice with operators, in such a way that every identity that holds in the original lattice also holds in its completion. To attain this, they used classical Priestley duality for distributive lattice to obtain the completions of bounded distributive lattices they were interested in. So, they extended the results of Jónsson and Tarski [**41, 42**] to the distributive lattice setting and this completion of a bounded distributive lattice was also called the *canonical extension*. In a more general setting, Gehrke and Jónsson [**29, 30**] studied which identities that are preserved in a distributive lattice expansion (a distributive lattice with additional operations, not necessarily operators) are also preserved in its canonical extension.

In their 2001 paper [**26**], Gehrke and Harding generalized the notion of canonical extension to non-distributive lattices setting. They obtained the canonical extension of a bounded lattice by means of purely lattice-theoretic tools and then they pointed out that the canonical extension can be obtained through Urquhart duality [**56**] in an analogous way as was obtained the canonical extension of a distributive lattice using the Priestley duality. Moreover, we can mention that it has been recognized in the literature that canonical extensions play an important role in completeness theorems for various expansions of classical logic such as modal logic, and various expansions of other logics.

The notion of canonical extension was also generalized to arbitrary partially ordered sets by Dunn, Gehrke and Palmigiano [**17**] by purely algebraic means. Then, they applied the canonical extension for partially ordered sets to particular monotone poset expansions to obtain complete relational semantics for some substructural logics. Other completions of partially ordered sets have been considered in the literature for different purposes [**46, 32, 47, 19**]; a uniform treatment of completions of partially ordered sets has been given in [**27**].

As we can observe, completions, topological dualities and relational semantics are very closely related. In particular, and this is more close to our research in this dissertation, we notice that the canonical extension is closely related to duality theory and provides an algebraic approach to topological duality. For this reason we think that it is important to study and investigate possible topological dualities and completions for partially ordered sets in general or for some classes of them in such a way that these dualities and completions generalize and extend those for Boolean algebras and distributive lattices. We believe that this is the first step to extend the results of Dunn et. al. [**17**] to other classes of monotone poset expansions related to some non-classical logics.

In this dissertation we study several topological dualities and completions for partially ordered sets. To attain this, first we need to study the partially ordered sets from an algebraic point of view. To this end, we consider in the setting of partially ordered sets three notions that are natural in Lattice Theory: filter and ideal, homomorphism and a distributivity condition. The notions of filter and ideal in Lattice Theory are generalized to partially ordered sets in at least three forms. Lattice homomorphisms are also generalized to partially ordered sets. We study these generalizations and we introduce other new possible generalizations that play an important role in this dissertation. We consider a distributivity condition that generalizes the distributivity condition in Lattice Theory due to David and Erné [**16**].

We believe that the topological dualities and completions for some classes of partially ordered sets that we develop may serve as tools to find and study some possible complete relational semantics for broader classes of non-classical logics following the line of Dunn et. al. [**17**]. This dissertation does not contain any result about this subject. We think that two previous steps are necessary before achieving this aim: (1) to try to apply the topological dualities to obtain representation theorems for some classes of monotone poset expansions; and (2) to try to extend the operations of a monotone poset expansion to its completions we consider in this dissertation in such a way that as many as possible equations or inequalities that hold in the monotone poset expansion also hold in its completion. Unfortunately, we did not have time to work properly on these subjects and we left it for a future work.

Now we will present the main contents of this dissertation. **Chapter 1** is about the concepts and results the reader is supposed to know in order to understand the rest of the chapters of this dissertation. We also introduce the conventional notations that are needed for the next chapters.

In **Chapter 2** we study partially ordered sets from an algebraic point of view. In Section 2.1 we consider three different classes of up-sets and down-sets on partially ordered sets that are well-known in the literature and we study their properties and the relations between them. These three classes of up-sets (down-sets) are natural generalizations of the notion of filter (ideal) in the setting of lattices and because of this they are called "filters" ("ideals") with some adjective. The notion of filter for lattices has useful applications in other branches of mathematics such as topology and logic and the notion of ideal is important for instance in Ring Theory. So, it is important to study their possible generalizations to a broader framework such as that of partially ordered sets. Two of the three notions of "filter" ("ideal") on partially ordered sets that we study give on any poset an algebraic closure system and hence they form complete lattices. The third one, despite not giving a closure system on every poset and so not providing a lattice, is interesting and will be central in Chapter 5 to develop a topological duality for the full class of partially ordered sets; it is also central for the theory of canonical extensions of posets developed in [**17**]. It is the notion of order-filter (order-ideal), being the order-filters (order-ideals) the down-directed up-sets (up-directed down-sets).

An important class of lattices, as we mentioned before, is the variety of distributive lattices. The condition of distributivity is defined by means of an identity: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (or equivalently, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$) and it is characterized by the distributivity property of the lattice of all its filters (ideals).

The distributivity condition on a lattice plays a fundamental role to obtain the topological dualities of Stone and Priestley. So, if we are interested in developing topological dualities for partially ordered sets that generalize the classical dualities by Stone and Priestley for bounded distributive lattices, it will be important to consider a notion of "distributivity" on partially ordered sets which generalizes the usual notion of distributivity on lattices. Hence, in Section 2.2 we consider a first-order condition on partially ordered sets due to David and Erné [**16**] that does this and that can be characterized by the distributivity of the lattice formed by certain up-sets of the poset. We will prove some new characterizations of this first-order condition that play a central rôle in the chapters to come. Next in Section 2.3 we study some kinds of morphisms between partially ordered sets that generalize the notion of lattice homomorphism. We study their properties, the relation between them and also the relation with the different kinds of "filters" and "ideals".

In Sections 2.4 and 2.5 we will study two extensions for partially ordered sets to respectively distributive meet-semilattices and distributive lattices. In §2.4 we extend certain partially ordered sets to distributive meet-semilattices and we investigate the internal structural relation between them such as the correspondence between certain "filters" of the partially ordered set and the filters of its extension. In §2.5 we obtain extensions of certain partially ordered sets to distributive lattices and we study the relation between this extension and the previous one.

Chapters 3, 4 and **5** are devoted to develop three topological dualities for partially ordered sets and used these dualities to obtain three completions of partially ordered sets. In **Chapter 3** we develop a *Stone-like* topological duality for *meet-order distributive* partially ordered sets (Definition 2.2.1). The central notion to obtain this duality is that of *prime Frink-filter* (Definitions 2.1.4 and 2.1.16). We characterize topologically the classes of Frink-filters, finitely generated Frink-filters and prime Frink-filters. In Section 3.4 we compare our duality with the topological duality developed by David and Erné [**16**]. In Section 3.5 we introduce a new completion of a partially ordered set and we apply the topological duality developed in this chapter to show its existence and that it has very nice properties.

In **Chapter 4**, the topological duality for partially ordered sets, unlike to the previous one, is developed in the spirit of Priestley duality. In this case the notion of *s-optimal Frink-filter* (Definition 2.4.23) plays a key role. In §4.3.2 we derive the Priestley duality for bounded distributive lattices from our duality, showing that our duality is a generalization of that of Priestley. In Section 4.4 we use the Priestley-style duality that we have obtained in this chapter to characterize topologically the Frink-filters and we also show that we can obtain the completion defined in Chapter 3. In Section 4.5 we define a new completion for partially ordered sets. We prove, using our Priestley-style duality, that this completion exists and we show that it has very nice properties. We also show, with an example, that this new completion for partially ordered sets is different from the completion that we obtained in Chapter 3. Furthermore, we show that the two new completions (of Chapters 3 and 4) and the canonical extension are different.

Finally, in **Chapter 5** we develop the third topological duality considered in this dissertation. This duality is developed for the class of all partially ordered sets and a fundamental concept to build it is the notion of *order-filter* (Definition 2.1.1). We intend that the dual category of the partially ordered sets with their order-preserving maps that in addition satisfy that the inverse image of an order-filter is an order-filter form a subcategory of the category of topological spaces, and that our duality generalizes the duality given by Moshier and Jipsen for bounded lattices [**48**]. In Section 5.3 we apply this duality to obtain a topological proof of the existence of the canonical extension of a partially ordered set as defined in [**17**]. This is the parallel result, but with a different kind of proof, to the topological proof of the existence of the canonical extension of a lattice provided in [**48**]. In Section 5.4 given a poset P we will see how to obtain, from the duality for P, the dual space of the dual order poset P^{∂}. In other words, from the duality of a poset we will characterize the dual space of the dual order poset. Section 5.5 deals with the topological representation of *quasi-monotone maps* (see definition on page 180) between partially ordered sets by maps between their duals, and with related issues. Finally, in Section 5.7 we specialize our duality to a duality for meet-semilattices and characterize the dual spaces. In this way we obtain by further specializing to meet-semilattices with a top element the duality obtained in [**48**].

CHAPTER 1

Preliminaries and Notational Conventions

In this first chapter we introduce the background that is necessary for this dissertation. We also fix and introduce some notational conventions that should be kept in mind.

1.1. Set Theory

This first brief section is dedicated only to fix some set-theoretical notations. We assume that the reader is familiar with the basic concepts of Set Theory.

We denote by ω the set of all natural numbers. Let X be a set and let $A \subseteq X$. We denote the complement of A relative to X as $X \setminus A$ or also, when confusion is unlikely, we write A^c. Given a set X, $A \subseteq_\omega X$ says that A is a (possibly empty) finite subset A of X. We denote by $\mathcal{P}(X)$ the set of all subsets of X and, $\mathcal{P}_\omega(X) = \{A \subseteq X : A \text{ is finite}\}$.

Let X, Y be sets and $f\colon X \to Y$ be a function. Then, for every $A \subseteq X$ and for every $B \subseteq Y$, let us denote, respectively, the image of A by f and the inverse image of B by f by $f[A]$ and $f^{-1}[B]$; thus

$$f[A] := \{f(a) : a \in A\} \quad \text{and} \quad f^{-1}[B] := \{x \in X : f(x) \in B\}.$$

Let X, Y be sets and $R \subseteq X \times Y$ be a binary relation. For every element $x_0 \in X$ and for every element $y_0 \in Y$, let us define the following sets:

$$R[x_0] := \{y \in Y : x_0 R y\} \quad \text{and} \quad R^{-1}[y_0] := \{x \in X : x R y_0\}.$$

Moreover, for every $A \subseteq X$ and for every $B \subseteq Y$ we define the sets:

$$R[A] := \bigcup \{R[x] : x \in A\} = \{y \in Y : xRy \text{ for some } x \in A\}$$

and

$$R^{-1}[B] := \bigcup \{R^{-1}[y] : y \in B\} = \{x \in X : xRy \text{ for some } y \in B\}.$$

Given sets X, Y and Z and binary relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, we recall that the set-theoretical composition $R \circ S \subseteq X \times Z$ is defined as follows:

$$x(R \circ S)z \iff (\exists y \in Y)(xRy \ \text{ and } \ ySz)$$

for every $x \in X$ and $z \in Z$.

1.2. Partially ordered sets

In this section we introduce the main notions that will be important in this dissertation. We fix also some notations and conventions that will be useful. The main references for this section are [**15, 34**].

Definition 1.2.1. A *partially ordered set* (for short *poset*) is a pair $\langle P, \leq \rangle$ where P is a non-empty set and $\leq$ is a binary relation on P satisfying for all $a, b, c \in P$ the following conditions:

(1) $a \leq a$ (reflexive law);
(2) if $a \leq b$ and $b \leq a$, then $a = b$ (anti-symmetric law);
(3) if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitive law).

When confusion is unlikely, we use simply the symbol P to denote a poset $\langle P, \leq \rangle$. Let $\langle P, \leq_1 \rangle$ and $\langle Q, \leq_2 \rangle$ be posets. We call $\langle Q, \leq_2 \rangle$ a *subposet* of $\langle P, \leq_1 \rangle$ if $Q \subseteq P$ and for all $x, y \in Q$, $x \leq_2 y \iff x \leq_1 y$.

Let P be a poset and let A be a subset of P. An element $x \in P$ is called a *lower bound* of A if $x \leq a$ for all $a \in A$. An element $y \in P$ is called an *upper bound* of A if $a \leq y$ for all $a \in A$. We denote the set of all lower bounds of A by A^{l}, that is, $A^{\mathrm{l}} = \{x \in P : \ x \leq a \text{ for all } a \in A\}$. Similarly, $A^{\mathrm{u}} = \{y \in P : \ a \leq y \text{ for all } a \in A\}$ is the set of all upper bounds of A. Notice that if $A = \emptyset$, then trivially we have that $\emptyset^{\mathrm{l}} = P$ and $\emptyset^{\mathrm{u}} = P$.

A lower bound x of A is the *meet* (or *greatest lower bound*) of A if and only if, for any lower bound b of A, we have $b \leq x$. If the meet of A exists, then we denote it by $\bigwedge A$ and when we write $x = \bigwedge A$ we mean that the meet of A exists and it is equal to x. Similarly, an upper bound y of A is the *join* (or *least upper bound*) of A if and only if, for any upper bound b of A, we have $y \leq b$. If the join of A exists, then we denote it by $\bigvee A$ and when we write $y = \bigvee A$ we mean that the join of A exists and it is equal to y. If A is finite and non-empty, say $A = \{a_1, \ldots, a_n\}$, we write $a_1 \wedge \cdots \wedge a_n$ for $\bigwedge A$ and $a_1 \vee \cdots \vee a_n$ for $\bigvee A$.

We say that a poset P has *top element* if there exists $a \in P$ such that $x \leq a$ for all $x \in P$ and we denote it by $\top$, if it exists. A poset P has a *bottom element* if there exists an element $b \in P$ such that $b \leq x$ for all $x \in P$, and if it exists we denote it by $\bot$. A *bounded* poset is a poset with top and bottom element.

An element $x \in P$ is said to be *join-irreducible* if it is not the bottom element (if this exists) and for every non-empty finite subset A of P, $x = \bigvee A$ implies $x \in A$, and x is said to be *completely join-irreducible* if it is not the bottom element (if this exists) and for every non-empty subset A of P, $x = \bigvee A$ implies $x \in A$. We denote by $\mathcal{J}(P)$ and $\mathcal{J}^{\infty}(P)$ the collections of all join-irreducible elements of P and all completely join-irreducible elements of P, respectively. Dually, an element $y \in P$, is said to be *meet-irreducible* if it is not the top element (if this exists) and for every non-empty finite subset A of P, $y = \bigwedge A$ implies $y \in A$. Also, y is said to be *completely meet-irreducible* if it is not the top (in case P has a top) element and for every non-empty subset A of P, $y = \bigwedge A$ implies $y \in A$. We denote by $\mathcal{M}(P)$ the set of all meet-irreducible elements of P and we denote the set of all completely meet-irreducible elements of P by $\mathcal{M}^\infty(P)$. Now, an element $x \in P$ is called *meet-prime* if it is not the top element (if this exists) and for every finite A of P, if $\bigwedge A \leq x$, then there is $a \in A$ such that $a \leq x$. Dually, an element $y \in P$ is said to be *join-prime* if it is not the bottom element (if this exists) and for every finite A of P, if $y \leq \bigvee A$, then there is $a \in A$ such that $y \leq a$.

A subset A of a poset P is said to be a *down-set* of P if for all $a, b \in P$ such that $a \in A$ and $b \leq a$, then $b \in A$. Dually, a subset A of P is called an *up-set* if $a \in A$ and $a \leq b$ implies $b \in A$. Let A be a subset of a poset P. We define the following sets

$$\uparrow A := \{x \in P : \ a \leq x \text{ for some } a \in A\} \text{ and } \downarrow A := \{x \in P : \ x \leq a \text{ for some } a \in A\}.$$

$\uparrow A$ is called the *up-set generated by* A and $\downarrow A$ is called the *down-set generated by* A. If $A = \{a\}$, we simply write $\uparrow a$ for $\uparrow \{a\}$ and, similarly for $\downarrow a$. Notice that $\uparrow A$ and $\downarrow A$ are up-set and down-set, respectively.

Definition 1.2.2. Let P be a poset.

(1) A subset A of P is an *up-directed subset* if for all $a, b \in A$ there exists $c \in A$ such that $a \leq c$ and $b \leq c$.
(2) A subset A of P is a *down-directed subset* if for all $a, b \in A$ there exists $c \in A$ such that $c \leq a$ and $c \leq b$.

It is obvious that for every element a of a poset P, $\uparrow a$ is a down-directed subset of P and $\downarrow a$ is an up-directed subset of P. We say that a subset A of a poset P is *inaccessible by up-directed joins* if for every up-directed subset $D \subseteq X$ such that $\bigvee D$ exists and $\bigvee D \in A$, then $A \cap D \neq \emptyset$.

Given a partially ordered set $\langle P, \leq \rangle$ we can form a new partially ordered set $\langle P, \leq^\partial \rangle$ by defining $x \leq^\partial y$ iff $x \leq y$. We denote the poset $\langle P, \leq^\partial \rangle$ by P^∂ and the order $\leq^\partial$ simply by $\geq$. P^∂ is called *the dual poset of* P. Given any statement Φ about partially ordered sets, the dual statement Φ^∂ is obtained by replacing $\leq$ by $\geq$ everywhere. For example, if x is a join-irreducible element of the poset P, then x is a meet-irreducible element of the dual poset P^∂. The *duality principle* is used to prove two statement at once.

Lemma 1.2.3 (The Duality Principle). *Given a statement Φ about partially ordered sets which is true in all partially ordered sets, then the dual statement Φ^∂ is true in all partially ordered sets.*

Let P and Q be posets and let $h\colon P \to Q$ be a map. We say that h is:

(1) an *order-preserving map* if for all $a, b \in P$ satisfies the following condition:

$$a \leq b \implies h(a) \leq h(b);$$

(2) an *order-reversing map* if for all $a, b \in P$ satisfies the following condition:

$$a \leq b \implies h(b) \leq h(a);$$

(3) an *order-embedding map* if for all $a, b \in P$ satisfies the following condition:

$$a \leq b \iff h(a) \leq h(b).$$

(4) a *dual order-embedding map* if for all $a, b \in P$ satisfies the following condition:

$$a \leq b \iff h(b) \leq h(a).$$

It is not hard to check that every order-embedding is an injective map, but the reverse is not true. A map $h\colon P \to Q$ between posets is called an *order-isomorphism map* if h is an onto order-embedding and h is called a *dual order-isomorphism map* if h is an onto dual order-embedding map. It is straightforward to show that every order-isomorphism $h\colon P \to Q$ preserves all existing finite meets and joins. That is, if $a_1, \dots, a_n \in P$ are such that $a_1 \wedge \cdots \wedge a_n$ ($a_1 \vee \cdots \vee a_n$) exists in P, then $h(a_1) \wedge \cdots \wedge h(a_n)$ ($h(a_1) \vee \cdots \vee h(a_n)$) exists in Q and $h(a_1 \wedge \cdots \wedge a_n) = h(a_1) \wedge \cdots \wedge h(a_n)$ ($h(a_1 \vee \cdots \vee a_n) = h(a_1) \vee \cdots \vee h(a_n)$). Moreover, if $h\colon P \to Q$ is a dual order-isomorphism, then existing finite meets (joins) in P correspond to finite joins (meets) in Q.

An order-preserving map $f\colon P \to Q$ behaves well with the up-sets and down-sets, in the sense that the inverse image of any up-set (down-set) of Q by f is an up-set (down-set) of P. More specifically,

Lemma 1.2.4. *Let $f\colon P \to Q$ be a map between the posets P and Q. Then, the following conditions are equivalent:*

(1) *f is order-preserving;*
(2) *for every up-set (down-set) U of Q, $f^{-1}[U]$ is an up-set (down-set) of P.*

From a family of posets we can construct two new posets. Let $\{P_i\}_{i \in I}$ be a family of posets. Let us denote the elements of the Cartesian product $P = \prod_{i \in I} P_i$ as $\overline{x}$, where for every $i \in I$, $\overline{x}(i) = x_i$. Thus, it is defined a partial order on the Cartesian product $P = \prod_{i \in I} P_i$ as

$$\overline{x} \leq \overline{y} \text{ if and only if } x_i \leq_i y_i \text{ for all } i \in I.$$

for every $\overline{x}, \overline{y} \in P$ and, where $\leq_i$ is the partial order of P_i for every $i \in I$. We call the poset $\langle P, \leq \rangle$ the *product poset* of the family $\{P_i\}_{i\in I}$ and we denote it, simply, by $\prod_{i\in I} P_i$

Now, we assume that the posets in the family $\{P_i\}_{i\in I}$ are pairwise disjoint and we suppose that I is a poset. Thus, the binary relation $\leq$ on $P = \bigcup_{i\in I} P_i$ defined for all $x, y \in P$ as: $x \leq y$ if and only if

- $x, y \in P_i$ and $x \leq_i y$ for some $i \in I$, or
- $x \in P_i$ and $y \in P_j$ with $i \leq_I j$

is a partial order on P. The poset $\langle P, \leq \rangle$ is called the *I-linear sum* of the family $\{P_i\}_{i\in I}$ and we denote it by $\bigoplus_{i\in I} P_i$.

1.3. Closure operators and systems

Closure operators and closure systems are closely related to complete lattices (see, for instance [**7**]). In logic, the notion of closure operator is also very important as Tarski showed in his study of "consequence" in logic during the 1930's. Here we give the main definitions and facts that we need throughout this dissertation.

Definition 1.3.1. Let X be a non-empty set. A map $C\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is a *closure operator on* X when it satisfies the following conditions for all $A, B \subseteq X$:

(CO1) $A \subseteq C(A)$;
(CO2) if $A \subseteq B$, then $C(A) \subseteq C(B)$;
(CO3) $C(C(A)) = A$.

A subset A of X is a *C-closed subset of* X when $C(A) = A$. We denote by $\mathcal{C}_C$ the collection of all C-closed subsets of X. It is clear that $\mathcal{C}_C$ with the inclusion order is a poset.

Some abbreviations have become standard, such as $C(A, B)$ for $C(A \cup B)$ or $C(A, x)$ for $C(A \cup \{x\})$, when $A, B \subseteq X$ and $x \in X$. Some basic properties of closure operators are summarized in the following lemma and they should be kept in mind since we will use them without mention:

Lemma 1.3.2. *Let X be a non-empty set and let C be a closure operator on X. Then:*

(1) $C\left(\bigcup_{i\in I} A_i\right) = C\left(\bigcup_{i\in I} C(A_i)\right)$ *whenever $A_i \subseteq X$ for each $i \in I$;*
(2) $C(A, B) = C(A, B_0)$ *whenever $A, B, B_0 \subseteq X$ are such that $B \subseteq B_0 \subseteq C(B)$;*
(3) $C(A, B) = C(A, C(B))$ *whenever $A, B \subseteq X$.*

Next, we give an important example of closure operator for further work.

EXAMPLE 1.3.3. Let P be a poset. We define two maps $(.)^{\mathrm{l}}\colon \mathcal{P}(P) \to \mathcal{P}(P)$ and $(.)^{\mathrm{u}}\colon \mathcal{P}(P) \to \mathcal{P}(P)$, respectively, as follows:

$$A^{\mathrm{l}} = \{x \in P : \ x \leq a \text{ for all } a \in A\} \quad \text{and} \quad A^{\mathrm{u}} = \{x \in P : \ a \leq x \text{ for all } a \in A\}$$

for each $A \subseteq P$. That is, we take for each $A \subseteq P$ the set of all lower bounds of A and the set of all upper bounds of A, respectively. Then, both compositions of these maps

$$(.)^{\mathrm{lu}}\colon \mathcal{P}(P) \to \mathcal{P}(P) \quad \text{and} \quad (.)^{\mathrm{ul}}\colon \mathcal{P}(P) \to \mathcal{P}(P)$$

define two closure operators on P. It is not hard to check this, and we leave the details to the reader. Other properties of the maps $(.)^{\mathrm{l}}$ and $(.)^{\mathrm{u}}$ that may be useful are the following:

(1) $A^{\mathrm{lul}} = A^{\mathrm{l}}$ and $A^{\mathrm{ulu}} = A^{\mathrm{u}}$, for all $A \subseteq P$;
(2) the maps $(.)^{\mathrm{l}}$ and $(.)^{\mathrm{u}}$ are order-reversing maps (w.r.t the inclusion order on $\mathcal{P}(P)$);
(3) for all $a \in P$, $\{a\}^{\mathrm{lu}} = {\uparrow} a$ and $\{a\}^{\mathrm{ul}} = {\downarrow} a$.
(4) for every family $\{A_i : \ i \in I\}$ of subsets of P,

$$\bigcup_{i \in I} A_i^{\mathrm{lu}} \subseteq \left(\bigcup_{i \in I} A_i \right)^{\mathrm{lu}}$$

and

$$\bigcup_{i \in I} A_i^{\mathrm{ul}} \subseteq \left(\bigcup_{i \in I} A_i \right)^{\mathrm{ul}}.$$

DEFINITION 1.3.4. A *closure system* on a non-empty set X is a collection $\mathcal{C} \subseteq \mathcal{P}(X)$ such that

(CS1) $X \in \mathcal{C}$;
(CS2) $\mathcal{C}$ is closed under arbitrary intersection. That is, for all $\{A_i : \ i \in I\} \subseteq \mathcal{C}$, $\bigcap_{i \in I} A_i \in \mathcal{C}$.

Thus, given a closure operator C on a non-empty set X, $\mathcal{C}_C$ is a closure system on X and it is called the *closure system associated with* C. Conversely, if $\mathcal{C}$ is a closure system on a non-empty set X and we define the map $C_{\mathcal{C}}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ by

$$C_{\mathcal{C}}(A) = \bigcap \{B \in \mathcal{C} : \ A \subseteq B\}$$

for all $A \subseteq X$, then $C_{\mathcal{C}}$ is a closure operator on X and it is called the *closure operator associated with* $\mathcal{C}$. Moreover, the mappings

$$C \mapsto \mathcal{C}_C \quad \text{and} \quad \mathcal{C} \mapsto C_{\mathcal{C}}$$

are inverse to one another. That is,

$$C = C_{\mathcal{C}_C} \quad \text{and} \quad \mathcal{C} = \mathcal{C}_{C_{\mathcal{C}}}.$$

Let X be a non-empty set and let $\mathcal{B}$ be a collection of subsets of X. Then, we can take the *closure system generated* by $\mathcal{B}$, which is denoted by $\mathcal{C}(\mathcal{B})$, defined in the following way

$$A \in \mathcal{C}(\mathcal{B}) \iff A = \bigcap\{B \in \mathcal{B} : \ A \subseteq B\} \text{ or } A = X.$$

Definition 1.3.5. Let X be a non-empty set.

(1) A closure operator C on X is said to be *finitary* (or *algebraic*) if for every $A \subseteq X$

$$C(A) = \bigcup\{C(B) : \ B \subseteq_\omega A\}.$$

(2) A closure system $\mathcal{C}$ on X is said to be *inductive* (or *algebraic*) if $\mathcal{C}$ is closed under unions of non-empty subfamilies of $\mathcal{C}$ that are up-directed (with respect to the inclusion order). That is, if $\mathcal{B} \subseteq \mathcal{C}$ is up-directed, then $\bigcup \mathcal{B} \in \mathcal{C}$.

Lemma 1.3.6. *Let X be a non-empty set and let C be a closure operator on X. Then, C is finitary if and only if the closure system associated $\mathcal{C}_C$ of C is inductive.*

It is clear, from the correspondence between closure operators and closure systems, that a closure system $\mathcal{C}$ is inductive if and only if the closure operator associated $C_{\mathcal{C}}$ of $\mathcal{C}$ is finitary.

Lemma 1.3.7. *Let C be a finitary closure operator on a non-empty set X. Let $A, B \subseteq X$ and $x, y \in X$. Then:*

(1) *if $B \subseteq C(A)$ and B is finite, then there exists a finite $A_0 \subseteq A$ such that $B \subseteq C(A_0)$;*

(2) *if $C(A, x) = C(A, y)$, then there exists a finite $A_0 \subseteq A$ such that $C(A_0, x) = C(A_0, y)$.*

1.4. Semilattices

Here we introduce the fundamental notions and known properties about meet-semilattices and join-semilattices. Since meet-semilattice and join-semilattice are dual notions, we only present the theory of meet-semilattice and we mention that the dual statements for join-semilattices hold. The main references for this section are [**15, 34**]. It is also interesting to see [**11**].

Definition 1.4.1. A partially ordered set $\langle M, \leq\rangle$ is a *meet-semilattice* if for each $a, b \in M$, the meet of a and b exists. As outlined above, we denote the meet of a and b by $a \wedge b$.

The dual of a meet-semilattice is called a *join-semilattice*. That is, a poset $\langle J, \leq\rangle$ is a join-semilattice if for all $a, b \in J$, the join of a and b exists.

Given a meet-semilattice $\langle M, \leq \rangle$, we can define the binary operation meet $\wedge\colon M \times M \to M$. Notice that $\wedge$ is an order-preserving map in both arguments and it is clear that for each $a, b \in M$, $a \leq b \iff a \wedge b = a$. Now we present the properties of $\wedge$.

Lemma 1.4.2. *Let $\langle M, \leq \rangle$ be a meet-semilattice. Then, for all $a, b, c \in M$ we have*

(1) $a \wedge a = a$*;*
(2) $a \wedge b = b \wedge a$*;*
(3) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$*.*

Meet-semilattices and join-semilattices can be defined as algebraic structures, in the sense of Universal Algebra (see [**7**]), as follows.

Definition 1.4.3. An algebra $\langle M, * \rangle$ of type (2) is a *semilattice* if the following identities hold:

(1) $a * a = a$;
(2) $a * b = b * a$;
(3) $a * (b * c) = (a * b) * c$.

By Lemma 1.4.2 it is clear that for each meet-semilattice $\langle M, \leq \rangle$, the algebra $\langle M, \wedge \rangle$ is a semilattice. Conversely, let $\langle M, * \rangle$ be a semilattice. Define the binary relation $\leq_*$ on M as follows: $a \leq_* b \iff a * b = a$. Then, $\langle M, \leq_* \rangle$ is a meet-semilattice and $a * b = a \wedge b$. Dually, if $\langle M, * \rangle$ is a semilattice and we define $\leq_*$ as: $a \leq_* b \iff a * b = b$, then $\langle M, \leq_* \rangle$ is a join-semilattice and $a * b = a \vee b$.

Hence, the meet-semilattices can be completely characterized in terms of the meet operation. We pointed out that when we say that $\langle M, \wedge \rangle$ is a meet-semilattice we mean that $\langle M, \wedge \rangle$ is a semilattice and it is ordered by the order $\leq$: $a \leq b \iff a \wedge b = a$.

Let $\langle M_1, \wedge_1 \rangle$ and $\langle M_2, \wedge_2 \rangle$ be meet-semilattices and let $h\colon M_1 \to M_2$ be a map. We will say that h is a *meet-homomorphism* from M_1 to M_2 if for all $a, b \in M_1$,

$$h(a \wedge_1 b) = h(a) \wedge_2 h(b).$$

We say that h is a *meet-embedding* if h is a meet-homomorphism and it is one-to-one. If h is a meet-embedding form M_1 onto M_2, we say that h is a *meet-isomorphism*. Dually, the notions of *join-homomorphism*, *join-embedding* and *join-isomorphism* can be defined between join-semilattices.

Given a meet-semilattice M, there exists an important subcollection of up-sets of M, which is defined below.

Definition 1.4.4. Let M be a meet-semilattice. A non-empty subset F of M is called a *filter* of M if it satisfies the following conditions: for every $a, b \in M$,

(1) if $a \in F$ and $a \leq b$, then $b \in F$ (up-set);
(2) if $a, b \in M$, then $a \wedge b \in F$.

We denote the set of all filters of M by $\mathsf{Fi}(M)$.

Lemma 1.4.5. *Let M be a meet-semilattice. Then, $\mathsf{Fi}(M) \cup \{\emptyset\}$ is an algebraic closure system on M. If M has a top element, then $\mathsf{Fi}(M)$ is an algebraic closure system.*

We denote by $\mathrm{Fi}(.)\colon \mathcal{P}(M) \to \mathcal{P}(M)$ the closure operator associated with the closure system $\mathsf{Fi}(M) \cup \{\emptyset\}$, if M has no top element, and with the closure system $\mathsf{Fi}(M)$, if M has a top element. In any case, for every non-empty $X \subseteq M$, $\mathrm{Fi}(X)$ is the smallest filter of M containing X and it is called the *filter generated by* X. It is straightforward to check that for each $a \in M$, $\mathrm{Fi}(a) = {\uparrow}a$. More generally, it is not hard to show that for every non-empty $X \subseteq M$

$$\mathrm{Fi}(X) = \{a \in M : \ x_1 \wedge \cdots \wedge x_n \leq a \text{ for some } n \in \omega \text{ and some } x_1, \ldots, x_n \in X\}.$$

Lemma 1.4.6. *Let M_1 and M_2 be meet-semilattices with top element and let $h\colon M_1 \to M_2$ be an order-preserving map that preservers top, i.e., $h(\top_1) = \top_2$. Then, the following are equivalent:*

(1) *h is a meet-homomorphism;*
(2) *if $G \in \mathsf{Fi}(M_2)$, then $h^{-1}[G] \in \mathsf{Fi}(M_1)$.*

The dual notion of filter is that of ideal. That is:

Definition 1.4.7. Let J be a join-semilattice. A non-empty subset I of J is called an *ideal* of J if satisfies the following conditions: for every $a, b \in J$,

(1) if $a \in I$ and $b \leq a$, then $b \in I$ (down-set);
(2) if $a, b \in I$, then $a \vee b \in I$.

The set of all ideals of J is denoted by $\mathsf{Id}(J)$.

Lemma 1.4.8. *Let J be a join-semilattice. Then, $\mathsf{Id}(J) \cup \{\emptyset\}$ is an algebraic closure system on J. If J has a bottom element, then $\mathsf{Id}(J)$ is an algebraic closure system.*

Given a join-semilattice J, we denote by $\mathrm{Id}(.)$ the closure operator associated with $\mathsf{Id}(J) \cup \{\emptyset\}$, if J has not bottom element and the closure operator associated with the closure system $\mathsf{Id}(J)$, if J has a bottom element. In any case, we have for every non-empty $X \subseteq J$,

$$\mathrm{Id}(X) = \{a \in J : \ a \leq x_1 \vee \cdots \vee x_n \text{ for some } n \in \omega \text{ and for some } x_1, \ldots, x_n \in X\}$$

and is called the *ideal of J generated by* X.

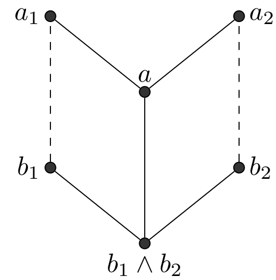


FIGURE 1.1. Distributive condition in meet-semilattice

The notion of distributivity is natural for lattices (see the next section), but the concept can be, in fact, generalized to meet-semilattices and join-semilattices. The distributive condition for join-semilattices was introduced by Grätzer in [**34**] (also see [**11**]). By the Duality Principle, the dual notion of distributivity can be considered for meet-semilattices. The class of distributive meet-semilattice is studied, for instance, in [**8, 4, 5, 9**].

DEFINITION 1.4.9. A meet-semilattice M is said to be *distributive* if $a, b_1, b_2 \in M$ are such $b_1 \wedge b_2 \leq a$, then there exist $a_1, a_2 \in M$ such that $b_1 \leq a_1$, $b_2 \leq a_2$ and $a = a_1 \wedge a_2$ (see Figure 1.1).

1.5. Lattices

The main references for this section are [**34, 15**].

DEFINITION 1.5.1. A partially ordered set $\langle L, \leq \rangle$ is a *lattice* if for all $a, b \in L$ there exist the meet and join of a and b. That is, for all $a, b \in L$, $a \wedge b$ and $a \vee b$ exist in L.

It should be noted that a lattice L is both a meet-semilattice and a join-semilattice; the relation between both is given by the following equivalences

$$a \wedge b = a \iff a \leq b \iff a \vee b = b$$

for all $a, b \in L$.

As in the case of meet-semilattices, a lattice can be defined as an algebraic structure. An algebra $\langle L, \wedge, \vee \rangle$ of type (2,2) is a *lattice* if the following identities are satisfied:

(1) $a \wedge a = a$;
(2) $a \wedge b = b \wedge a$;
(3) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$;
(4) $a \wedge (a \vee b) = a$;
(5) $a \vee a = a$;
(6) $a \vee b = b \vee a$;
(7) $a \vee (b \vee c) = (a \vee b) \vee c$;
(8) $a \vee (a \wedge b) = a$;

where the order $\leq$ on L is defined as:

$$a \leq b \Longleftrightarrow a = a \wedge b \Longleftrightarrow b = a \vee b$$

for all $a, b \in L$. So, we can observe that the reduct $\langle L, \wedge \rangle$ is a meet-semilattice and the reduct $\langle L, \vee \rangle$ a join-semilattice.

A lattice L is a *complete lattice* if for every subset A of L, the meet of A and the join of A exist in L. That is, for all $A \subseteq L$, there exist $x, y \in L$ such that $x = \bigwedge A$ and $y = \bigvee A$. Every complete lattice is bounded, where $\top = \bigwedge \emptyset$ and $\bot = \bigvee \emptyset$.

Let X be a non-empty set. Let $C \colon \mathcal{P}(X) \to \mathcal{P}(X)$ be a closure operator on X and let $\mathcal{C}$ be the associated closure system of C. Hence, $\mathcal{C}$ is a complete lattice where the meet and the join of $\{A_i : \ i \in I\} \subseteq \mathcal{C}$ are given by:

$$\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i \quad \text{and} \quad \bigvee_{i \in I} A_i = C\left(\bigcup_{i \in I} A_i\right).$$

Since each lattice is both a meet-semilattice and join-semilattice we can consider the notion of filter and ideal.

Definition 1.5.2. Let L be a lattice.

(1) A non-empty subset F of L is a *filter* of L if it is a filter of the meet-semilattice reduct $\langle L, \wedge \rangle$ of L.
(2) A non-empty subset I of L is an *ideal* of L if it is an ideal of the join-semilattice reduct $\langle L, \vee \rangle$ of L.

Let us denote by $\mathsf{Fi}(L)$ and $\mathsf{Id}(L)$ the set of all filters of L and the set of all ideals of L, respectively.

Lemma 1.5.3. *Let L be a lattice. Then, $\mathsf{Fi}(L) \cup \{\emptyset\}$ and $\mathsf{Id}(L) \cup \{\emptyset\}$ are algebraic closure systems. If L has top element (bottom element), then $\mathsf{Fi}(L)$ ($\mathsf{Id}(L)$) is an algebraic closure system.*

Definition 1.5.4. A lattice L is called *distributive* if for all $a, b, c \in L$ the following identities are satisfied:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \text{and} \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

The previous identities are, actually, equivalent. Moreover, in every lattice L the the following inequalities

$$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c) \quad \text{and} \quad (a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$$

hold. Hence, if we want to prove that a lattice L is distributive it is only necessary to show that one of the inequalities

$$a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c) \quad \text{or} \quad (a \vee b) \wedge (a \vee c) \leq a \vee (b \wedge c)$$

M_5 N_5

FIGURE 1.2. The non-distributive lattices M_5 and N_5.

is satisfied. The following lemma is a characterization of distributive lattice through of the notions of filter and ideal.

LEMMA 1.5.5. *Let L be a lattice. Then, the following are equivalent*

(1) *L is distributive;*
(2) *$\mathsf{Fi}(L)$ is a distributive lattice;*
(3) *$\mathsf{Id}(L)$ is a distributive lattice.*

Another important characterization of distributive lattice is by means of the existence or not of certain kinds of sublattices. It is useful to identify non-distributive lattices.

LEMMA 1.5.6. *Let L be a lattice. Then, L is non-distributive if and only if it contains a sublattice isomorphic to the lattice M_5 or to the lattice N_5 give in Figure 1.2.*

Let $\langle L_1, \wedge_1, \vee_1\rangle$ and $\langle L_2, \wedge_2, \vee_2\rangle$ be lattices and let $h\colon L_1 \to L_2$ be a map. We say that h is a *homomorphism* if for all $a, b \in L$ the following identities are satisfied

$$h(a \wedge_1 b) = h(a) \wedge_2 h(b) \quad \text{and} \quad h(a \vee_1 b) = h(a) \vee_2 h(b).$$

A homomorphism $h\colon L_1 \to L_2$ is called an *embedding* if it is one-to-one and, we say that h is an *isomorphism* if h is an embedding from L_1 onto L_2.

LEMMA 1.5.7. *Let L_1 and L_2 be lattices and let $h\colon L_1 \to L_2$ be an order-preserving map. Then, h is a homomorphism if and only if the following two conditions are satisfied:*

(1) *if $G \in \mathsf{Fi}(L_2)$, then $h^{-1}[G] \in \mathsf{Fi}(L_1)$;*
(2) *if $J \in \mathsf{Id}(L_2)$, then $h^{-1}[J] \in \mathsf{Id}(L_1)$.*

1.6. Topology

In this section we introduce the basic notions of general topology that will be needed in later chapters. We also fix some conventional notations. The main references for this section are: [**18, 39, 40**].

Definition 1.6.1. A *topological space* is a pair $\langle X, \tau\rangle$ consisting of a set X and a family τ of subsets of X satisfying the following conditions:

(1) $\emptyset \in \tau$ and $X \in \tau$;
(2) if $U_1, U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$;
(3) if $\{U_i :\ i \in I\} \subseteq \tau$, then $\bigcup_{i\in I} U_i \in \tau$.

The set X is called a *space*, the family τ is called a *topology* and, the elements of the topology τ are called *open subsets of* X. Often we simply say that X is a topological space and sometimes we will aslo use $\mathsf{O}(X)$ to refer to the collection of all open subsets of X.

Remark 1.6.2. Let X be a topological space, then $\langle \mathsf{O}(X), \cap, \cup, \Rightarrow, \emptyset, X\rangle$ is a complete Heyting algebra (see [**3**, p. 177]), where

$$U \Rightarrow V := \mathrm{int}(U^c \cup V)$$

for every $U, V \in \mathsf{O}(X)$.

A family $\mathcal{B} \subseteq \tau$ is called a *base* for a topological space $\langle X, \tau\rangle$ if every non-empty open subset of X can be represented as the union of a subfamily of $\mathcal{B}$. One can easily check that a family $\mathcal{B}$ of subsets of X is a base for the topological space $\langle X, \tau\rangle$ if and only if $\mathcal{B} \subseteq \tau$ and, for every point $x \in X$ and any $V \in \tau$ such that $x \in V$ there exists $U \in \mathcal{B}$ such that $x \in U \subseteq V$.

A family $\mathcal{A} \subseteq \tau$ is called a *subbase* for a topological space $\langle X, \tau\rangle$ if the family of all finite intersections $U_1 \cap U_2 \cap \cdots \cap U_n$, where $U_i \in \mathcal{A}$ for $i = 1, 2, \ldots, n$ is a base for $\langle X, \tau\rangle$. Let X be an arbitrary set and let $\mathcal{A} \subseteq \mathcal{P}(X)$. Then, $\mathcal{A}$ is a subbase for a topological space. Indeed, let $\tau_{\mathcal{A}}$ be the collection of all unions of finite intersections of elements of $\mathcal{A}$, i.e.,

$$\tau_{\mathcal{A}} = \left\{ \bigcup \bigcap \mathcal{A}_0 :\ \mathcal{A}_0 \subseteq_\omega \mathcal{A} \right\}.$$

Hence, it is not hard to prove $\tau_{\mathcal{A}}$ is a topology on X and $\mathcal{A}$ is a subbase for the topological space $\langle X, \tau_{\mathcal{A}}\rangle$. The topology $\tau_{\mathcal{A}}$ on X is called the *topology generated by* $\mathcal{A}$ and $\langle X, \tau_{\mathcal{A}}\rangle$ is called the *topological space generated by* $\mathcal{A}$.

Let $\langle X, \tau\rangle$ be a topological space. A subset F of X is called a *closed subset* of X if F^c is an open subset of X. We denote by $\mathsf{C}(X)$ the collection of all closed subsets of X. Then, $\mathsf{C}(X)$ has the following properties:

(1) $\emptyset, X \in \mathsf{C}(X)$;

(2) if $F_1, F_2 \in \mathsf{C}(X)$, then $F_1 \cup F_2 \in \mathsf{C}(X)$;
(3) if $\mathcal{C}_0 \subseteq \mathsf{C}(X)$, then $\bigcap \mathcal{C}_0 \in \mathsf{C}(X)$.

Let $A \subseteq X$. Since X is a closed subset of itself, there exists the smallest closed subset of X containing A; this set is called the *closure of* A and we denote it by $\mathrm{cl}(A)$. We also write $\mathrm{cl}(x)$ instead of $\mathrm{cl}(\{x\})$.

Notice that a subset A of a topological space X can be simultaneously open and closed, if this is the case we say that A is a *clopen subset of* X. By $\mathsf{CL}(X)$ we denote the collection of all clopen subsets of X.

Let $\langle X_1, \tau_1 \rangle$ and $\langle X_2, \tau_2 \rangle$ be topological spaces and let $f\colon X_1 \to X_2$ be a map. We say that f is a *continuous map* if for each $V \in \tau_2$, $f^{-1}[V] \in \tau_1$. The map f is called *open* if for each $U \in \tau_1$ it holds $f[U] \in \tau_2$. And f is called a *homeomorphism* if it is a bijective continuous and open map.

Lemma 1.6.3. *Let $f\colon X_1 \to X_2$ be a map form a topological spaces X_1 to a topological space X_2. Then, the following conditions are equivalent:*

(1) *f is continuous;*
(2) *the inverse image of every member of any base $\mathcal{B}$ for X_2 is an open in X_1;*
(3) *the inverse image of every member of any subbase $\mathcal{A}$ for X_2 is an open in X_1.*

A subset A of a topological space X is *compact* if every open cover of A has a finite subcover. We denote by $\mathsf{K}(X)$ the family of all compact subsets of the space X. Another important family to keep in mind is the family of all compact open subsets of X, which is denoted by $\mathsf{KO}(X)$.

If $\mathcal{B}$ is a base for a space X, $A \subseteq X$ is compact if and only if every open basic cover of A has a finite subcover of A. Similarly, the last claim is also true if we replace the base $\mathcal{B}$ by a subbase $\mathcal{A}$ of X. That is,

Lemma 1.6.4 (Alexander Subbase Lemma). *Let X be a topological space and let $\mathcal{A}$ be a subbase of X. A subset A of X is compact if and only if every open cover of A by members of $\mathcal{A}$ has a finite subcover.*

Let $\langle X, \tau \rangle$ be a topological space. The space X is said to be T_0 if for every pair of distinct points $x, y \in X$ there exists an open subset of X containing one of these two points and not the other. We define the binary relation $\preceq$ on X as follows: for every $x, y \in X$

$$x \preceq y \iff (\forall U \in \tau)(x \in U \implies y \in U) \iff x \in \mathrm{cl}(y).$$

This relation is transitive and reflexive. It is straightforward to show the relation $\preceq$ is a partially ordered if and only if the space X is T_0. In this case we say that $\preceq$ is the *specialization order of the space* X. Therefore, if X is T_0, then $\langle X, \preceq \rangle$ is a

$$
\begin{tikzcd}
 & T_1 \arrow[dr, Rightarrow] & \\
T_2 \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & T_0 \\
 & T_S \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

FIGURE 1.3. T_S in the separation hierarchy.

poset. Notice that for every element x of a topological space X, $\mathrm{cl}(x) = \{y \in X : y \preceq x\} = {\downarrow}x$.

LEMMA 1.6.5. *Let $\langle X_1, \tau_1\rangle$ and $\langle X_2, \tau_2\rangle$ be T_0-spaces and let $f\colon X_1 \to X_2$ be a map. If f is continuous, then f is order-preserving with respect to the specialization orders.*

A space X is said to be T_1 if for every pair of distinct points $x, y \in X$ there exists an open subset U of X such that $x \in U$ and $y \notin U$. A topological space X is called T_2 (or a *Hausdorff space*) if for every pair of distinct elements $x, y \in X$ there exist open subsets U and V of X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Next, we present the definition of another axiom, less simple than the separation axioms T_0, T_1 and T_2, but that will play an important role in later chapters. First we introduce the concept of irreducible subset. Let X be a topological space and let F be a subset of X. We say F is an *irreducible subset of* X if $F \subseteq A \cup B$ with A and B closed subsets of X, then $F \subseteq A$ or $F \subseteq B$.

DEFINITION 1.6.6. A topological space $\langle X, \tau\rangle$ is called *sober* if for all irreducible subset F of X, there exists a unique point $x \in X$ such that $F = \mathrm{cl}(x)$.

The sober condition (T_S) for topological spaces is a kind of separation axiom (as are T_0, T_1 and T_2 axioms) where its position in the separation hierarchy is as shown in Figure 1.3.

The following lemmas give a characterization and some properties of sober space that can be useful when we working with sober spaces.

LEMMA 1.6.7. *A topological space $\langle X, \tau\rangle$ is sober if and only if X is T_0 and for every completely prime filter $\mathcal{F}$ in the lattice of open subsets of X there exists an element $x \in X$ such that $\mathcal{F} = \{U \in \tau : x \in U\}$.*

LEMMA 1.6.8. *Let X be a sober space. Then,*

(1) *each up-directed subset D of X has join $\bigvee D$;*
(2) *if U is an open subset of X, then U is inaccessible by up-directed joins;*

(3) *every continuous function f between sober spaces preservers up-directed joins, that is, for every up-directed subset D of X such that $\bigvee D$ exists,*

$$f\left(\bigvee D\right) = \bigvee f[D].$$

1.7. Category Theory

In this section we introduce the basic notions and definitions about of the theory of categories that we will need in this dissertation. Our main references for Category Theory are [**45, 50**].

Definition 1.7.1. A *category* $\mathbb{C}$ consists:

(1) a collection of *objects*;
(2) a collection of *morphisms*; to each morphism f corresponds exactly an object $dom(f)$, its domain, and exactly an object $cod(f)$, its codomain. We write $f\colon A \to B$ to show that $dom(f) = A$ and $cod(f) = B$; the collection of all morphisms with domain A and codomain B is denote by $\mathbb{C}(A, B)$;
(3) a *composition* that assigns to each morphisms $f\colon A \to B$ and $g\colon B \to C$, a *composite* morphism $g \circ f\colon A \to C$, satisfying the following: for any morphisms $f\colon A \to B$, $g\colon B \to C$ and $h\colon C \to D$,

$$h \circ (g \circ f) = (h \circ g) \circ f;$$

(4) for each object A, an *identity* morphism $\mathrm{id}_A\colon A \to A$ satisfying the following: for each morphism $f\colon A \to B$,

$$\mathrm{id}_B \circ f = f \quad \text{and} \quad f \circ \mathrm{id}_A = f.$$

Let $\mathbb{C}$ be a category. A morphism $f\colon A \to B$ in $\mathbb{C}$ is called an *isomorphism* of $\mathbb{C}$ if there exists another morphism $g\colon B \to A$ in $\mathbb{C}$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Definition 1.7.2. Let $\mathbb{C}$ be a category. A category $\mathbb{B}$ is a *subcategory* of $\mathbb{C}$ if:

(1) each object of $\mathbb{B}$ is an object of $\mathbb{C}$;
(2) for all objects A and B of $\mathbb{B}$, $\mathbb{B}(A, B) \subseteq \mathbb{C}(A, B)$;
(3) composition and identity morphisms are the same in $\mathbb{B}$ as in $\mathbb{C}$.

A subcategory $\mathbb{B}$ of $\mathbb{C}$ is called *full* if for all objects A and B of $\mathbb{B}$, it follows that $\mathbb{B}(A, B) = \mathbb{C}(A, B)$.

Now we introduce the definition that establishes a relation between two categories.

Definition 1.7.3. Let $\mathbb{C}$ and $\mathbb{D}$ be categories. A (*contravariant*) *functor* $\mathbf{F}\colon \mathbb{C} \to \mathbb{D}$ is a function, which assigns to each object A of $\mathbb{C}$ an object $\mathbf{F}(A)$ of $\mathbb{D}$ and to each morphism $f\colon A \to B$ in $\mathbb{C}$ a morphism $\mathbf{F}(f)\colon \mathbf{F}(A) \to F(B)$ ($\mathbf{F}(f)\colon \mathbf{F}(B) \to \mathbf{F}(A)$) in $\mathbb{D}$, in such a way that the following conditions are satisfied:

(1) for each object A of $\mathbb{C}$, $\mathbf{F}(\mathrm{id}_A) = \mathrm{id}_{\mathbf{F}(A)}$;
(2) for each morphisms $f\colon A \to B$ and $g\colon B \to C$ in $\mathbb{C}$, $\mathbf{F}(g \circ f) = \mathbf{F}(g) \circ \mathbf{F}(f)$ ($\mathbf{F}(g \circ f) = \mathbf{F}(f) \circ \mathbf{F}(g)$).

Given two (contravariant) functors $\mathbf{F}\colon \mathbb{C} \to \mathbb{D}$ and $\mathbf{G}\colon \mathbb{D} \to \mathbb{E}$, the *composite functor* $\mathbf{G} \circ \mathbf{F}\colon \mathbb{C} \to \mathbb{E}$ is defined as:

(1) for each object A of $\mathbb{C}$, $(\mathbf{G} \circ \mathbf{F})(A) := \mathbf{G}(\mathbf{F}(A))$;
(2) for each morphism $f\colon A \to B$ in $\mathbb{C}$, $(\mathbf{G} \circ \mathbf{F})(f) := \mathbf{G}(\mathbf{F}(f))\colon \mathbf{G}(\mathbf{F}(A)) \to \mathbf{G}(\mathbf{F}(B))$.

It is straightforward to check directly that $\mathbf{G} \circ \mathbf{F}$ is a functor.

Definition 1.7.4. Let $\mathbb{C}$ and $\mathbb{D}$ be categories and let $\mathbf{F}$ and $\mathbf{G}$ be functors from $\mathbb{C}$ to $\mathbb{D}$. A *natural transformation* η from $\mathbf{F}$ to $\mathbf{G}$ is a function that assigns to every object A of $\mathbb{C}$ a morphism $\eta(A)\colon \mathbf{F}(A) \to \mathbf{G}(A)$ in $\mathbb{D}$ such that for every morphism $f\colon A \to B$ in $\mathbb{C}$ the following diagram commutes in $\mathbb{D}$:

$$\begin{tikzcd}
\mathbf{F}(A) \arrow[r, "\mathbf{F}(f)"] \arrow[d, "\eta(A)"'] & \mathbf{F}(B) \arrow[d, "\eta(B)"] \\
\mathbf{G}(A) \arrow[r, "\mathbf{G}(f)"'] & \mathbf{G}(B)
\end{tikzcd}$$

If each component $\eta(A)$ of η is an isomorphism in $\mathbb{D}$ then η is called a *natural isomorphism* (or *natural equivalence*) and we denote it by $\eta\colon \mathbf{F} \cong \mathbf{G}$.

Definition 1.7.5. Let $\mathbb{C}$ and $\mathbb{D}$ be categories. An *adjunction* from $\mathbb{C}$ to $\mathbb{D}$ is a triple $(\mathbf{F}, \mathbf{G}, \eta)$ where

(1) $\mathbf{F}\colon \mathbb{C} \to \mathbb{D}$ and $\mathbf{G}\colon \mathbb{D} \to \mathbb{C}$ are functors
(2) $\eta\colon \mathbf{F} \to \mathbf{G}$ is a natural transformation;

such that for each object A of $\mathbb{C}$ and each morphism $f\colon A \to \mathbf{G}(B)$ in $\mathbb{C}$, there is a unique morphism $\widehat{f}\colon \mathbf{F}(A) \to B$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\eta(A)"] \arrow[dr, "f"'] & \mathbf{G}(\mathbf{F}(A)) \arrow[d, "\mathbf{G}(\widehat{f})"] \\ & \mathbf{G}(B) \end{tikzcd}$$

We say that $\mathbf{F}$ is the *left adjoint* of $\mathbf{G}$ and $\mathbf{G}$ is the *right adjoint* of $\mathbf{F}$.

Let $\mathbb{C}$ be a category. A subcategory $\mathbb{B}$ of $\mathbb{C}$ is called *reflective* in $\mathbb{C}$ when the inclusion functor $\mathbf{I}\colon \mathbb{B} \to \mathbb{C}$ has a left adjoint $\mathbf{F}\colon \mathbb{C} \to \mathbb{B}$.

Let $\mathbb{C}$ and $\mathbb{D}$ be categories. A functor $\mathbf{F}\colon \mathbb{C} \to \mathbb{D}$ is an *isomorphism* of categories if there is a functor $\mathbf{G}\colon \mathbb{D} \to \mathbf{C}$ such that $\mathbf{G} \circ \mathbf{F} = \mathbf{Id}_{\mathbb{C}}$ and $\mathbf{F} \circ \mathbb{G} = \mathbf{Id}_{\mathbb{D}}$, where $\mathbf{Id}_{\mathbb{C}}$ and $\mathbf{Id}_{\mathbb{D}}$ are the corresponding identity functors.

Now we present another important notion that is more general and useful than isomorphisms. Two categories $\mathbb{C}$ and $\mathbb{D}$ are (*dually*) *equivalent* if there exist two (contravariant) functors $\mathbf{F}\colon \mathbb{C} \to \mathbb{D}$ and $\mathbf{G}\colon \mathbb{D} \to \mathbb{C}$ such that there are two natural isomorphisms $\varphi\colon (\mathbf{G} \circ \mathbf{F}) \cong \mathbf{Id}_{\mathbb{C}}$ and $\psi\colon (\mathbf{F} \circ \mathbf{G}) \cong \mathbf{Id}_{\mathbb{D}}$.

CHAPTER 2

A Study of Partially Ordered Sets

Lattice Theory, mainly developed by the work of G. Birkhoff in the mid-thirties of last century is fundamental in the study of many ordered algebraic structures and also with regards to the classes of algebras that are associated with certain logics. Moreover, Lattice Theory is also important in other branches of mathematics like, for instance, Algebra, Computer Science, Domain Theory, etc.

Partially ordered sets form a large and general class of ordered structures which encompasses that of lattices. As we saw in the previous chapter, a lattice is a poset in which the greatest lower bound and the least upper bound exist for every pair of elements. From this point of view we can observe the importance of studying posets in general and trying to develop for them analogous results to results obtained in Lattice Theory. This quest has been pursued by many; to name a few we can highlight the works of M. Erné and recently the extension to posets given in [**17**] of the theory of the canonical extension of a lattice.

In this chapter we will study the class of partially ordered sets from an algebraic point of view through several algebraic concepts like filter and ideal, homomorphisms and a distributivity condition. The notions of filter, ideal, homomorphism and distributivity are natural of Lattice Theory and they are important to understand the inner algebraic structure of lattices and moreover they are also useful to develop topological dualities (see for instance [**55, 52, 56, 48**]). The previously mentioned notions were generalized to broader classes than lattices, for instance to meet-semilattices (or dually to join-semilattice) as can be seen in [**34, 11, 8, 4, 2, 37, 13**] and also different generalizations of filters, ideals, homomorphisms and distributivity conditions were defined for the bigger class of partially ordered sets, see for instance [**16, 24, 38, 47, 17**].

The different notions of filter (ideal), homomorphism and distributivity condition that we study in this chapter are important to obtain the topological dualities that we present in this dissertation. We will develop in this dissertation three topological dualities, two of them use the notion of Frink-filter and they need a kind of separation theorem (as in the setting of distributive lattices a topological duality, like Stone or like Priestley, for them use the notion of filter and need a separation theorem, often called prime filter theorem). The third topological duality for posets that we develop use the notion of order-filter.

In the first section of this chapter we introduce three notions of filter and ideal, known in the literature, that generalize the notions of filter and ideal for lattices. In §2.2 we study a class of posets that satisfy a kind of distributivity condition, the posets in that class are called meet-order distributive. We give several characterizations of this distributivity condition, one of which is the following: a poset is meet-order distributive if and only if the lattice of all Frink-filters of the poset is distributive. In §2.3 we present several definitions of functions between posets that intend to generalize the notion of homomorphism for lattices. We study the relation of these notions of homomorphisms between posets with the notions of filters and ideals and also with regard to the distributivity condition given in §2.2. In Section 2.4 we define the distributive meet-semilattice envelope of a poset. The distributive meet-semilattice envelope of a poset is a distributive meet-semilattice, in which the poset is embedded in a very nice way. We establish a correspondence between certain filters of a poset and the filters of its distributive meet-semilattice envelope. In Section 2.5 we introduce the notion of distributive lattice envelope of a poset; this concept will be important in Chapter 4 to develop a Priestley-style duality for a class of posets. The distributive lattice envelope of a poset is a distributive lattice, in which the poset is embedded. We present two abstract characterizations of the distributive lattice envelope and we show a correspondence between certain filters of a poset and the filters of its distributive lattice envelope.

2.1. Filters and ideals

The notions of filter and ideal in Lattice Theory are very important for understanding the internal structure of a lattice. For instance, a lattice is distributive if and only if the lattice of its filters (ideals) is distributive (see for instance [**34**] and [**15**]). But the notions of filter and ideal are not only important for characterizations of the structure of a lattice, they play a central role in the topological dualities for lattices. For instance, in the classical topological dualities due Stone for Boolean algebras [**54**] and for distributive lattices [**55**] and Priestley's duality for distributive lattices [**52**]. Other references where the notion of filter (ideal) plays a fundamental role for a topological duality for lattices or semilattices are [**55, 35, 36, 56, 48, 4, 5, 8**]. We also mention that the notion of filter has several application in logic.

We will study three different notions of filter and ideal for posets that are known in the literature. The different definitions of filter and ideal for posets that we consider are natural generalizations of the notions of filter and ideal for lattices.

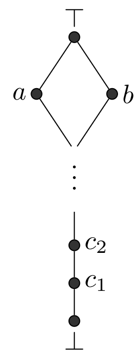


FIGURE 2.1. Example of a poset P where $\mathsf{Fi}_{\mathsf{or}}(P)$ is not a closure system.

2.1.1. Order-filters and order-ideals. The generalization of filter (ideal) on partially ordered sets that we consider in this subsection has an advantage, namely filters (ideals) are down-directed (up-directed) and has a weakness, the collection of all filters (ideals) is not a closure system. The notion of filter (ideal) that we introduce in this subsection is the stronger of the three kind of filter (ideal) that we consider in this section.

DEFINITION 2.1.1. Let P be a poset.

- A non-empty subset $F \subseteq P$ is called an *order-filter of* P if it is a down-directed up-set.
- A non-empty subset $I \subseteq P$ is called an *order-ideal of* P if it is an up-directed down-set.

We denote by $\mathsf{Fi}_{\mathsf{or}}(P)$ the family of all order-filters of P and by $\mathsf{Id}_{\mathsf{or}}(P)$ the family of all order-ideals of P. From §1.2, it is clear that for every element $a \in P$, $\uparrow a$ is an order-filter of P and $\downarrow a$ is an order-ideal of P.

It should be noted that the families $\mathsf{Fi}_{\mathsf{or}}(P)$ and $\mathsf{Id}_{\mathsf{or}}(P)$ are not necessarily closure systems, because they are not necessarily closed under arbitrary intersections. For instance, consider the poset P given in Figure 2.1. Let us consider the collection $\{\uparrow c_i : \ i \geq 1\}$ of order-filters of P. Then, it is not hard to see that $\bigcap_{i=1}^{n} \uparrow c_i = \{a, b, \top\}$, where $\{a, b, \top\}$ is not an order-filter of P, because for a and b there is no $x \in \{a, b, \top\}$ such that $x \leq a$ and $x \leq b$. Thus, $\mathsf{Fi}_{\mathsf{or}}(P)$ is not closed under intersection and, consequently it is not a closure system on P. The dual poset P^{∂} of P given in Figure 2.1, can be used to show that $\mathsf{Id}_{\mathsf{or}}(P^{\partial})$ is not a closure system.

It is straightforward to check that if $\langle M, \wedge\rangle$ is a meet-semilattice, then the collection of all filters of M, $\mathsf{Fi}(M)$, coincide with the collection of all order-filters, $\mathsf{Fi}_{\mathsf{or}}(M)$, of the poset associated with M (see §1.4). That is, $\mathsf{Fi}(M) = \mathsf{Fi}_{\mathsf{or}}(M)$. Dually, if J is a join-semilattice, then $\mathsf{Id}(J) = \mathsf{Id}_{\mathsf{or}}(J)$. In particular, we have that if M is a meet-semilattice with top element, then $\mathsf{Fi}_{\mathsf{or}}(M) = \mathsf{Fi}(M)$ is a closure system. The following lemma expresses the converse of the previous statement, providing a characterization of when a poset P is a meet-semilattice with top element.

Lemma 2.1.2. *Let P be a poset. Then, $\mathsf{Fi}_{\mathsf{or}}(P)$ is a closure system if and only if P is a meet-semilattice with top element.*

Proof. Let P be a poset. We assume that $\mathsf{Fi}_{\mathsf{or}}(P)$ is a closure system on P. We denote by $\mathrm{Fi}_{\mathrm{or}}(.)$ the closure operator associated with the closure system $\mathsf{Fi}_{\mathsf{or}}(P)$. Let $a, b \in P$. Since $\mathrm{Fi}_{\mathrm{or}}(\uparrow a \cup \uparrow b)$ is an order-filter of P and $a, b \in \mathrm{Fi}_{\mathrm{or}}(\uparrow a \cup \uparrow b)$, there exists $c \in \mathrm{Fi}_{\mathrm{or}}(\uparrow a \cup \uparrow b)$ such that $c \leq a$ and $c \leq b$. So, we have $\uparrow c \subseteq \mathrm{Fi}_{\mathrm{or}}(\uparrow a \cup \uparrow b)$ and $\uparrow a \cup \uparrow b \subseteq \uparrow c$. Then, $\mathrm{Fi}_{\mathrm{or}}(\uparrow a \cup \uparrow b) = \uparrow c$. Now, we show $c = a \wedge b$. We know that c is a lower bound of a and b. Let $d \in P$ such that $d \leq a$ and $d \leq b$. So, $\uparrow a \cup \uparrow b \subseteq \uparrow d$, which implies that $\mathrm{Fi}_{\mathrm{or}}(\uparrow a \cup \uparrow b) \subseteq \uparrow d$. Then, $\uparrow c \subseteq \uparrow d$ and, thus $d \leq c$. Therefore, $c = a \wedge b$. To prove that P has a top element, consider the set

$$F = \bigcap \{G : \ G \in \mathsf{Fi}_{\mathsf{or}}(P)\}.$$

Since $\mathsf{Fi}_{\mathsf{or}}(P)$ is closure system, it is closed under arbitrary intersection. Then, $F \in \mathsf{Fi}_{\mathsf{or}}(P)$. So, $F \neq \emptyset$. Let $a \in F$. We want to show a is the top element of P. Let $b \in P$. Since $\uparrow b$ is an order-filter of P, $a \in \uparrow b$. Whereupon, $b \leq a$. Hence, a is the top element of P. Therefore, we have proved that P is a meet-semilattice with top element.

The reverse implication was shown in the previous paragraph. □

Lemma 2.1.3. *Let P be a poset. Then, $\mathsf{Id}_{\mathsf{or}}(P)$ is a closure system if and only if P is a join-semilattice with bottom element.*

2.1.2. Frink-filters and Frink-ideals. The notion of filter (ideal) on a partially ordered set that we consider in this part is due to O. Frink in [**24**], which also generalizes the usual notion of filter (ideal) in Lattice Theory.

Definition 2.1.4 ([**24**]). Let P be a poset.

(1) A subset F of P is said to be a *Frink-filter of* P if for every $A \subseteq_\omega F$, we have $A^{\mathrm{lu}} \subseteq F$. Let us denote the collection of all Frink-filters of P by $\mathsf{Fi}_{\mathsf{F}}(P)$.
(2) A subset I of P is said to be a *Frink-ideal of* P if for every $A \subseteq_\omega I$, we have $A^{\mathrm{ul}} \subseteq I$. We denote the collection of all Frink-ideals of P by $\mathsf{Id}_{\mathsf{F}}(P)$.

Notice that the empty set may be a Frink-filter or a Frink-ideal of a poset P. In fact, we have that for a poset P, the empty set is a Frink-filter (Frink-ideal) of P if and only if P has no top (bottom) element. This is consequence of the fact $\emptyset^{\mathrm{lu}} = P^{\mathrm{u}}$ ($\emptyset^{\mathrm{ul}} = P^{\mathrm{l}}$).

The following lemma, that it is not hard to prove, allows us to give a characterization of the Frink-filters and the Frink-ideals of a poset P; the properties in this lemma should be kept in mind, because they will be repeatedly used later on, without explicit mention.

Lemma 2.1.5. *Let P be a poset and let $X \subseteq P$ and $a \in P$. Then,*

$$a \in X^{\mathrm{lu}} \quad \textit{iff} \quad \bigcap_{x \in X} {\downarrow} x \subseteq {\downarrow} a$$

and

$$a \in X^{\mathrm{ul}} \quad \textit{iff} \quad \bigcap_{x \in X} {\uparrow} x \subseteq {\uparrow} a.$$

Thus, $F \subseteq P$ ($I \subseteq P$) is a Frink-filter (Frink-ideal) of a poset P if for any $X \subseteq_\omega F$ ($X \subseteq_\omega I$) and $a \in P$, if $\bigcap_{x \in X} {\downarrow} x \subseteq {\downarrow} a$ ($\bigcap_{x \in X} {\uparrow} x \subseteq {\uparrow} a$), then $a \in F$ ($a \in I$).

Lemma 2.1.6 ([**24**]). *Given an arbitrary poset P, $\mathsf{Fi}_{\mathsf{F}}(P)$ and $\mathsf{Id}_{\mathsf{F}}(P)$ are closure systems.*

Notice that for every non-empty finite subset A of a lattice L, it follows that $A^{\mathrm{lu}} = {\uparrow}(\bigwedge A)$ and $A^{\mathrm{ul}} = {\downarrow}(\bigvee A)$. It is helpful to keep these equalities in mind. The next lemma says that the notions of Frink-filter and Frink-ideal on posets can be considered nice generalizations of the concepts of filter and ideal on lattices. Its proof is not hard and thus we leave the details to the reader. Moreover, as every lattice is a meet-semilattice and a join-semilattice, we can formulate the following lemma in a broader setting.

Lemma 2.1.7. *If M is a meet-semilattice, then $\mathsf{Fi}_{\mathsf{F}}(M) \setminus \{\emptyset\} = \mathsf{Fi}(M)$. If J is a join-semilattice, then $\mathsf{Id}_{\mathsf{F}}(J) \setminus \{\emptyset\} = \mathsf{Id}(J)$. Therefore, if L is a lattice, then $\mathsf{Fi}_{\mathsf{F}}(L) \setminus \{\emptyset\} = \mathsf{Fi}(L)$ and $\mathsf{Id}_{\mathsf{F}}(L) \setminus \{\emptyset\} = \mathsf{Id}(L)$.*

It is easy to check that each Frink-filter of a poset is an up-set and each Frink-ideal is a down-set of the poset. Moreover, given a poset P, we have that for every $a \in P$, ${\uparrow} a$ is a Frink-filter of P and ${\downarrow} a$ is a Frink-ideal of P. More in general, we have the following connection between order-filters and Frink-filters (order-ideals and Frink-ideals).

Lemma 2.1.8. *Let P be a poset. Then, $\mathsf{Fi}_{\mathsf{or}}(P) \subseteq \mathsf{Fi}_{\mathsf{F}}(P)$ and $\mathsf{Id}_{\mathsf{or}}(P) \subseteq \mathsf{Id}_{\mathsf{F}}(P)$.*

Proof. Let $F \in \mathsf{Fi}_{\mathsf{or}}(P)$ and let $A \subseteq_\omega F$. Suppose that $A = \emptyset$. So $A^{\mathrm{lu}} = \{\top\}$ if P has a top element $\top$ and $A^{\mathrm{lu}} = \emptyset$ if P has no top element. In any case, since F is non-empty and it is an up-set, $A^{\mathrm{lu}} \subseteq F$. Assume that $A \neq \emptyset$ and let $x \in A^{\mathrm{lu}}$. Since F is a down-directed subset of P and $A \subseteq_\omega F$ is non-empty, there exists $b \in F$ such that $b \in A^{\mathrm{l}}$. So, $b \leq x$. Then, $x \in F$. Therefore, $F \in \mathsf{Fi}_{\mathsf{F}}(P)$. Dually it can be proved that $\mathsf{Id}_{\mathsf{or}}(P) \subseteq \mathsf{Id}_{\mathsf{F}}(P)$. $\square$

In the previous lemma, the inclusions can not be an equality. For instance, consider the poset P in Figure 2.1. The set $\{a, b, \top\}$ is a Frink-filter but it is not an order-filter.

Lemma 2.1.9. *Let P be a poset with top element. Then, P is a meet-semilattice if and only if $\mathsf{Fi}_{\mathsf{or}}(P) = \mathsf{Fi}_{\mathsf{F}}(P)$.*

Proof. We assume first that P is a meet-semilattice. By the previous lemma, we have $\mathsf{Fi}_{\mathsf{or}}(P) \subseteq \mathsf{Fi}_{\mathsf{F}}(P)$. Let $F \in \mathsf{Fi}_{\mathsf{F}}(P)$. So, it is clear that F is a non-empty up-set of P. Let $a, b \in F$. Then, $\{a, b\}^{\mathrm{lu}} \subseteq F$. Since $a \wedge b$ exists in P, $a \wedge b \in \{a, b\}^{\mathrm{lu}}$. Hence, $a \wedge b \in F$. Whereupon, $F \in \mathsf{Fi}_{\mathsf{or}}(P)$.

Conversely, we suppose $\mathsf{Fi}_{\mathsf{or}}(P) = \mathsf{Fi}_{\mathsf{F}}(P)$. Then, $\mathsf{Fi}_{\mathsf{or}}(P)$ is a closure system. Hence, by Lemma 2.1.2, P is a meet-semilattice. $\square$

Lemma 2.1.10. *Let P be a poset with bottom element. Then, P is a join-semilattice if and only if $\mathsf{Id}_{\mathsf{or}}(P) = \mathsf{Id}_{\mathsf{F}}(P)$.*

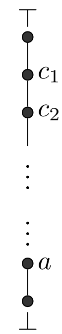
Let P be a poset and $X \subseteq P$. We denote by $\mathrm{Fi}_{\mathrm{F}}(X)$ and $\mathrm{Id}_{\mathrm{F}}(X)$ the *Frink-filter generated by X* and the *Frink-ideal generated by X*, respectively. Then, by Lemma 2.1.6, we have that for every poset P, $\mathsf{Fi}_{\mathsf{F}}(P)$ and $\mathsf{Id}_{\mathsf{F}}(P)$ are complete lattices, where for every family $\mathcal{F} \subseteq \mathsf{Fi}_{\mathsf{F}}(P)$ the meet and join are given by

$$\bigwedge \mathcal{F} = \bigcap \mathcal{F} \quad \text{and} \quad \bigvee \mathcal{F} = \mathrm{Fi}_{\mathrm{F}}\left(\bigcup \mathcal{F}\right).$$

Similarly, for the meet and join in $\mathsf{Id}_{\mathsf{F}}(P)$. A Frink-filter (Frink-ideal) is *finitely generated* if it is a Frink-filter (Frink-ideal) generated by a finite subset of P. For every $x_1, \ldots, x_n \in P$, we write $\mathrm{Fi}_{\mathrm{F}}(x_1, \ldots, x_n)$ and $\mathrm{Id}_{\mathrm{F}}(x_1, \ldots, x_n)$ instead of $\mathrm{Fi}_{\mathrm{F}}(\{x_1, \ldots, x_n\})$ and $\mathrm{Id}_{\mathrm{F}}(\{x_1, \ldots, x_n\})$, respectively. Let us denote by $\mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$ and $\mathsf{Id}^{\mathsf{f}}_{\mathsf{F}}(P)$ the collections of all finitely generated Frink-filters and all finitely generated Frink-ideals of P, respectively. Notice that $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \vee\rangle$ and $\langle \mathsf{Id}^{\mathsf{f}}_{\mathsf{F}}(P), \vee\rangle$ are sub-join-semilattices of the join-reduct of the lattices $\mathsf{Fi}_{\mathsf{F}}(P)$ and $\mathsf{Id}_{\mathsf{F}}(P)$, respectively, because for every $X, Y \subseteq_\omega P$,

$$\mathrm{Fi}_{\mathrm{F}}(X) \vee \mathrm{Fi}_{\mathrm{F}}(Y) = \mathrm{Fi}_{\mathrm{F}}(X \cup Y) \quad \text{and} \quad \mathrm{Id}_{\mathrm{F}}(X) \vee \mathrm{Id}_{\mathrm{F}}(Y) = \mathrm{Id}_{\mathrm{F}}(X \cup Y)$$

with $X \cup Y \subseteq_\omega P$.



P

FIGURE 2.2. A poset P and $X \subseteq P$ such that $\mathrm{Fi}_{\mathrm{F}}(P) \neq X^{\mathrm{lu}}$.

If P has top element $\top$, then the finitely generated Frink-filter $\mathrm{Fi}_{\mathrm{F}}(\top) = \uparrow\top = \{\top\}$, is the bottom element of $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \vee\rangle$ and if P has bottom element $\bot$, then $P = \mathrm{Fi}_{\mathrm{F}}(\bot)$ is the top element of $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \vee\rangle$. Dually, $\langle \mathsf{Id}^{\mathsf{f}}_{\mathsf{F}}(P), \vee\rangle$ has top or bottom element if P has top or bottom element, respectively.

It is clear that for every $a \in P$, $\mathrm{Fi}_{\mathrm{F}}(a) = \uparrow a = \{a\}^{\mathrm{lu}}$ and $\mathrm{Id}_{\mathrm{F}}(a) = \downarrow a = \{a\}^{\mathrm{ul}}$. Moreover, one can check that $\mathrm{Fi}_{\mathrm{F}}(a_1, \ldots, a_n) = \{a_1, \ldots, a_n\}^{\mathrm{lu}}$ and $\mathrm{Id}_{\mathrm{F}}(a_1, \ldots, a_n) = \{a_1, \ldots, a_n\}^{\mathrm{ul}}$, for all $a_1, \ldots, a_n \in P$. For an arbitrary subset X of a poset P, it is straightforward to show the set X^{lu} is a Frink-filter of P and $X \subseteq X^{\mathrm{lu}}$ (see Example 1.3.3), but it is not necessarily the smallest Frink-filter of P containing X, i.e., X^{lu} is not necessarily the Frink-filter generated by X (dually, X^{ul} is not necessarily the Frink-ideal generated by the set X). This is shown in the following example.

EXAMPLE 2.1.11. Consider the poset P give in Figure 2.2 and consider the set $X = \{\top, c_1, c_2, \ldots\}$. Notice that X is a Frink-filter of P and, so $\mathrm{Fi}_{\mathrm{F}}(X) = X$. Now, since $X^{\mathrm{l}} = \{\bot, a\}$, we have $X^{\mathrm{lu}} = \{\top, a, c_1, c_2, \ldots\}$. Hence, $\mathrm{Fi}_{\mathrm{F}}(X) = X \neq X^{\mathrm{lu}}$.

Despite this, there is a good characterization of the Frink-filter generated by a set X. The following lemma express this and, dually, the reader can obtain the characterization for the Frink-ideal generated by a set X.

LEMMA 2.1.12. *Let P be a poset and let $X \subseteq P$. Then,*

$$\mathrm{Fi}_{\mathrm{F}}(X) = \bigcup \left\{ X_0^{\mathrm{lu}} : \ X_0 \subseteq_\omega X \right\}.$$

PROOF. Let X be a subset of the poset P. We denote the set referred to by right hand side of the last equality by U. It is clear that $X \subseteq U$. Now, we prove that U is a Frink-filter of P. Let $A \subseteq_\omega U$. So, for every $a \in A$, there exists

$X_a \subseteq_\omega X$ such that $a \in X_a^{\mathrm{lu}}$. Then,

$$A \subseteq \bigcup_{a\in A} X_a^{\mathrm{lu}} \subseteq \left(\bigcup_{a\in A} X_a\right)^{\mathrm{lu}},$$

which implies

$$A^{\mathrm{lu}} \subseteq \left(\bigcup_{a\in A} X_a\right)^{\mathrm{lu}}$$

and it also holds that $\bigcup\limits_{a\in A} X_a \subseteq_\omega X$. Thus, $A^{\mathrm{lu}} \subseteq U$ and, therefore U is a Frink-filter of P. Lastly, let G be Frink-filter of P such that $X \subseteq G$. Let $u \in U$. So, there is $X_0 \subseteq_\omega X$ such that $u \in X_0^{\mathrm{lu}}$. Since $X \subseteq G$ and G is a Frink-filter, $u \in G$. Therefore, U is the smallest Frink-filter of P containing X, i.e., U is the Frink-filter of P generated by X. □

Corollary 2.1.13. *For every poset P, $\mathsf{Fi}_\mathsf{F}(P)$ and $\mathsf{Id}_\mathsf{F}(P)$ are algebraic closure systems.*

We have shown that for every poset P, $\mathsf{Fi}_\mathsf{or}(P) \subseteq \mathsf{Fi}_\mathsf{F}(P)$ and they are not necessarily equal. Now we will see how we can reach the Frink-filters from the order-filters. Let P be a poset. Let $\mathsf{C}_\mathsf{or}(P) := \mathcal{C}(\mathsf{Fi}_\mathsf{or}(P))$ be the closure system generated by $\mathsf{Fi}_\mathsf{or}(P)$ and we denote by $C_\mathsf{or} \colon \mathcal{P}(P) \to \mathcal{P}(P)$ the closure operator associated with $\mathsf{C}_\mathsf{or}(P)$. Thus, for every $A \subseteq P$,

$$C_\mathsf{or}(A) = \bigcap\{F \in \mathsf{Fi}_\mathsf{or}(P) :\ A \subseteq F\}.$$

Now we consider the operator $C^\mathsf{f}_\mathsf{or} \colon \mathcal{P}(P) \to \mathcal{P}(P)$ defined as

$$C^\mathsf{f}_\mathsf{or}(A) = \bigcup\{C_\mathsf{or}(A_0) :\ A_0 \subseteq_\omega A\}$$

for each $A \subseteq P$. Then, C^f_or has the following properties:

(1) $C^\mathsf{f}_\mathsf{or}(A) = C_\mathsf{or}(A)$ if $A \subseteq_\omega P$;
(2) C^f_or is an algebraic closure operator and $C^\mathsf{f}_\mathsf{or} \leq C_\mathsf{or}$ (that is, $C^\mathsf{f}_\mathsf{or}(A) \subseteq C_\mathsf{or}(A)$ for $A \subseteq P$);
(3) C^f_or is the strongest of all algebraic closure operators C on P such that $C \leq C_\mathsf{or}$.

The closure operator C^f_or is sometime called *the algebraic companion of* C_or. Let us denote by $\mathsf{C}^\mathsf{f}_\mathsf{or}(P)$ the closure system associated with C^f_or. Since $\mathsf{Fi}_\mathsf{or}(P) \subseteq \mathsf{Fi}_\mathsf{F}(P)$ and for every $A_0 \subseteq_\omega P$, $C^\mathsf{f}_\mathsf{or}(A_0) = C_\mathsf{or}(A_0)$ is an intersection of order-filters, it follows that $C^\mathsf{f}_\mathsf{or}(A_0) \in \mathsf{Fi}_\mathsf{F}(P)$ for all $A_0 \subseteq_\omega P$. Then, given that $\mathsf{Fi}_\mathsf{F}(P)$ is an algebraic closure system and since for every $A \subseteq P$ the set $\{C_\mathsf{or}(A_0) :\ A_0 \subseteq_\omega A\}$ is up-directed, it follows that

$$C^\mathsf{f}_\mathsf{or}(A) = \bigcup\{C_\mathsf{or}(A_0) :\ A_0 \subseteq_\omega A\} \in \mathsf{Fi}_\mathsf{F}(P)$$

for all $A \subseteq P$.

LEMMA 2.1.14. *Let P be a poset and $A_0 \subseteq_\omega P$. Then $C_{\mathsf{or}}(A_0) = A_0^{\mathrm{lu}}$.*

PROOF. Let $A_0 \subseteq_\omega P$. Since $C_{\mathsf{or}}(A_0) \in \mathsf{Fi}_{\mathsf{F}}(P)$ and $A_0 \subseteq C_{\mathsf{or}}(A_0)$, it follows that $A_0^{\mathrm{lu}} \subseteq C_{\mathsf{or}}(A_0)$. Let $a \in C_{\mathsf{or}}(A_0)$ and let $b \in A_0^{\mathrm{l}}$. So $A_0 \subseteq {\uparrow} b$. Because $a \in C_{\mathsf{or}}(A_0)$, a belongs to each order-filter that contains A_0. Then, $a \in {\uparrow} b$ and thus $b \leq a$. So $a \in A_0^{\mathrm{lu}}$ and hence $C_{\mathsf{or}}(A_0) \subseteq A_0^{\mathrm{lu}}$. Therefore $C_{\mathsf{or}}(A_0) = A_0^{\mathrm{lu}}$. □

LEMMA 2.1.15. *Let P be a poset. Then $\mathsf{C}^{\mathsf{f}}_{\mathsf{or}}(P) = \mathsf{Fi}_{\mathsf{F}}(P)$.*

PROOF. Already we know that $\mathsf{C}^{\mathsf{f}}_{\mathsf{or}}(P) \subseteq \mathsf{Fi}_{\mathsf{F}}(P)$. Now let $F \in \mathsf{Fi}_{\mathsf{F}}(P)$. Let us show that $C^{\mathsf{f}}_{\mathsf{or}}(F) = F$. Let $a \in C^{\mathsf{f}}_{\mathsf{or}}(F)$. So, there is $A_0 \subseteq_\omega F$ such that $a \in C_{\mathsf{or}}(A_0)$. So $A_0^{\mathrm{lu}} \subseteq F$ and then, by Lemma 2.1.14, we have $C_{\mathsf{or}}(A_0) \subseteq F$. Thus $a \in F$. Hence $C^{\mathsf{f}}_{\mathsf{or}}(F) = F$. □

In the next definition we introduce the notions of irreducible and prime Frink-filter. These are the usual definitions of meet-irreducible and meet-prime element of a lattice, applied to the lattices of Frink-filters and Frink-ideals.

DEFINITION 2.1.16. Let P be a poset and let $F \in \mathsf{Fi}_{\mathsf{F}}(P)$ be proper.

(I) F is called *irreducible* when for every $F_1, F_2 \in \mathsf{Fi}_{\mathsf{F}}(P)$ if $F_1 \cap F_2 = F$, then $F_1 = F$ or $F_2 = F$.

(P) F is called *prime* when for every $F_1, F_2 \in \mathsf{Fi}_{\mathsf{F}}(P)$ if $F_1 \cap F_2 \subseteq F$, then $F_1 \subseteq F$ or $F_2 \subseteq F$.

We denote by $\mathsf{Fi}^{\mathsf{irr}}_{\mathsf{F}}(P)$ the family of all irreducible Frink-filters of P and by $\mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$ the family of all prime Frink-filters of P.

Dually, the reader can give the notions of *irreducible* and *prime* Frink-ideal in the spirit of the previous Definition. Put in another words, we can say that a proper Frink-ideal I of a poset P is called *irreducible* if it is a meet-irreducible element of the lattice $\mathsf{Id}_{\mathsf{F}}(P)$ and, it is called *prime* if is a meet-prime element of the lattice $\mathsf{Id}_{\mathsf{F}}(P)$. Similarly, we write $\mathsf{Id}^{\mathsf{irr}}_{\mathsf{F}}(P)$ and $\mathsf{Id}^{\mathsf{pr}}_{\mathsf{F}}(P)$ to denote the families of all irreducible Frink-ideals and all prime Frink-ideals of P, respectively. It is straightforward to check directly that $\mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P) \subseteq \mathsf{Fi}^{\mathsf{irr}}_{\mathsf{F}}(P)$ and $\mathsf{Id}^{\mathsf{pr}}_{\mathsf{F}}(P) \subseteq \mathsf{Id}^{\mathsf{irr}}_{\mathsf{F}}(P)$.

The following lemmas are useful to show that a Frink-filter is prime or irreducible. The first lemma establishes a connection between prime Frink-filters and order-ideals, and Lemma 2.1.19 is a generalization of a characterization of irreducible filter in the framework of semilattices give in [**8**] by Celani.

LEMMA 2.1.17. *Let P be a poset and let $F \in \mathsf{Fi}_{\mathsf{F}}(P)$ be proper. Then, F is prime if and only if F^c is an order-ideal of P.*

PROOF. We assume that F is a prime Frink-filter of P. Since F is a proper Frink-filter, F^c is a non-empty down-set of P. Let $a, b \in F^c$. So, ${\uparrow} a \nsubseteq F$ and

$\uparrow b \nsubseteq F$. Then, as F is prime, $\uparrow a \cap \uparrow b \nsubseteq F$. That is, there is $c \in \uparrow a \cap \uparrow b$ and $c \notin F$. We thus obtain $a \leq c$, $b \leq c$ and $c \in F^c$. Hence, F^c is an order-ideal of P.

We now assume F^c is an order-ideal of P. Let $F_1, F_2 \in \mathsf{Fi}_\mathsf{F}(P)$ be such that $F_1 \cap F_2 \subseteq F$. We suppose $F_1 \nsubseteq F$ and $F_2 \nsubseteq F$. So, there exist $a \in F_1 \setminus F$ and $b \in F_2 \setminus F$. Since F^c is an order-ideal, there exists $c \in F^c$ such that $a \leq c$ and $b \leq c$. Then, $c \in F_1 \cap F_2 \subseteq F$, which is a contradiction. Hence, $F_1 \subseteq F$ or $F_2 \subseteq F$. Therefore, F is a prime Frink-filter. □

Lemma 2.1.18. *Let P be a poset and let $I \in \mathsf{Id}_\mathsf{F}(P)$ be proper. Then, I is prime if and only if I^c is an order-filter of P.*

Lemma 2.1.19. *Let P be a poset and let $F \in \mathsf{Fi}_\mathsf{F}(P)$ be proper. Then, F is irreducible if and only if for every $a, b \notin F$ there exist $c \notin F$ and $C \subseteq_\omega F$ such that $c \in (C \cup \{a\})^{\mathrm{lu}}$ and $c \in (C \cup \{b\})^{\mathrm{lu}}$.*

Proof. Let F be a proper Frink-filter of P. We assume that F is irreducible and let $a, b \notin F$. We take the following Frink-filters of P, $F_a = \mathrm{Fi}_\mathrm{F}(F \cup \{a\})$ and $F_b = \mathrm{Fi}_\mathrm{F}(F \cup \{b\})$. It is clear that $F \neq F_a$ and $F \neq F_b$ and since F is irreducible, it follows that $F \subsetneq F_a \cap F_b$. So, let $c \in F_a \cap F_b$ be such that $c \notin F$. As $c \in F_a$, it follows that there exists $A \subseteq_\omega F \cup \{a\}$ such that $c \in A^{\mathrm{lu}}$ and, since also $c \in F_b$, there exists $B \subseteq_\omega F \cup \{b\}$ such that $c \in B^{\mathrm{lu}}$. Taking $C = (A \cap F) \cup (B \cap F) \subseteq_\omega F$ we get $c \in (C \cup \{a\})^{\mathrm{lu}}$ and $c \in (C \cup \{b\})^{\mathrm{lu}}$.

Now assume that the condition on the right hand side of the "if and only if" of the lemma is satisfied. Let F_1 and F_2 be Frink-filters such that $F = F_1 \cap F_2$. Suppose $F \neq F_1$ and $F \neq F_2$. So, there are $a \in F_1 \setminus F$ and $b \in F_2 \setminus F$. Then, there exist $c \notin F$ and $C \subseteq_\omega F$ such that $c \in (C \cup \{a\})^{\mathrm{lu}}$ and $c \in (C \cup \{b\})^{\mathrm{lu}}$. Notice that $(C \cup \{a\})^{\mathrm{lu}} \subseteq F_1$ and $(C \cup \{b\})^{\mathrm{lu}} \subseteq F_2$. Then, $c \in F_1 \cap F_2 = F$, which is a contradiction. Thus, $F_1 = F$ or $F_2 = F$ and therefore F is irreducible. □

Theorem 2.1.20. *Let P be a poset. If $F \in \mathsf{Fi}_\mathsf{F}(P)$ and $I \in \mathsf{Id}_{\mathsf{or}}(P)$ are such that $F \cap I = \emptyset$, then there exists $U \in \mathsf{Fi}_\mathsf{F}^{\mathsf{irr}}(P)$ such that $F \subseteq U$ and $U \cap I = \emptyset$.*

Proof. Consider the following set

$$\mathcal{G} = \{G \in \mathsf{Fi}_\mathsf{F}(P) : \ F \subseteq G \text{ and } G \cap I = \emptyset\}$$

ordered by the inclusion order. Notice that $\mathcal{G} \neq \emptyset$ because $F \in \mathcal{G}$ and, it is straightforward to show that the union of any chain of elements of $\mathcal{G}$ is in $\mathcal{G}$. Then, by Zorn's Lemma, there exists a maximal element U of $\mathcal{G}$. We now prove that U is irreducible using Lemma 2.1.19. Let $a, b \notin U$. So, it is clear that $U \subsetneq F_a = \mathrm{Fi}_\mathrm{F}(U \cup \{a\})$ and $U \subsetneq F_b = \mathrm{Fi}_\mathrm{F}(U \cup \{b\})$. By the maximality of U, we have $F_a, F_b \notin \mathcal{G}$. So, $F_a \cap I \neq \emptyset$ and $F_b \cap I \neq \emptyset$. Let $x \in F_a \cap I$ and $y \in F_b \cap I$. Then, there are $A, B \subseteq_\omega U$ such that $x \in (A \cup \{a\})^{\mathrm{lu}}$ and $y \in (B \cup \{b\})^{\mathrm{lu}}$. Let $C := A \cup B$.

We thus obtain $x \in (C \cup \{a\})^{\mathrm{lu}}$, $y \in (C \cup \{b\})^{\mathrm{lu}}$ and $C \subseteq_\omega U$. Since $x, y \in I$ and I is an order-ideal, it follows that there exists $c \in I$ such that $x \leq c$ and $y \leq c$. Hence $c \notin U$, $c \in (C \cup \{a\})^{\mathrm{lu}}$ and $c \in (C \cup \{b\})^{\mathrm{lu}}$. Therefore, by Lemma 2.1.19, U is an irreducible Frink-filter of P. □

Corollary 2.1.21. *Let P be a poset. If $a, b \in P$ are such that $a \nleq b$, then there exists $U \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{irr}}(P)$ such that $a \in U$ and $b \notin U$.*

Corollary 2.1.22. *Let P be a poset. If F is a Frink-filter of P and $a \notin F$, then there exists $U \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{irr}}(P)$ such that $F \subseteq U$ and $a \notin U$.*

It is clear that the notion of prime Frink-filter of a poset is a generalization of the notion of prime filter in Lattice Theory. But, it is not the only one possible. We consider other two definitions that generalize the notion of prime filter in Lattice Theory that can be found in the literature. One is the notion of optimal filter for meet-semilattices, see for instance [**4, 5**].

Definition 2.1.23. Let P be a poset. Let $F \in \mathsf{Fi}_{\mathsf{F}}(P)$ be proper. We say that F is an *optimal* Frink-filter if F^c is a Frink-ideal. Let us denote by $\mathsf{Opt}(P)$ the class of all optimal Frink-filters of P.

The dual definition of optimal Frink-filter for Frink-ideal can be given, but in this dissertation we will not use it. In the next lemma there are several useful characterizations of the condition of being optimal and, since they are easy to check, we leave the proofs as an exercise.

Lemma 2.1.24. *Let P be a poset and let F be a proper Frink-filter of P. Then, the following conditions are equivalent:*

(1) *F is optimal;*
(2) *if $a_1, \ldots, a_n \notin F$ and $\bigcap_{i=1}^{n} \uparrow a_i \subseteq \uparrow c$, then $c \notin F$;*
(3) *if $c \in F$, then for all $A \subseteq_\omega P$ $\left(A \cap F = \emptyset \text{ implies } \bigcap_{a \in A} \uparrow a \nsubseteq \uparrow c\right)$;*
(4) *if $c \in F$, then for all $A \subseteq_\omega P$ $\left(A \cap F = \emptyset \text{ implies } c \notin A^{\mathrm{ul}}\right)$;*
(5) *if $a_1, \ldots, a_n \notin F$ and $c \in \{a_1, \ldots, a_n\}^{\mathrm{ul}}$, then $c \notin F$;*
(6) *if $a_1, \ldots, a_n \notin F$, then $\{a_1, \ldots, a_n\}^{\mathrm{ul}} \cap F = \emptyset$;*
(7) *if $\{a_1, \ldots, a_n\}^{\mathrm{ul}} \cap F \neq \emptyset$, then $a_i \in F$ for some $i \in \{1, \ldots, n\}$.*

Lemma 2.1.25. *Let P be a poset. If F is a prime Frink-filter of P, then F is an optimal Frink-filter of P. That is,*

$$\mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P) \subseteq \mathsf{Opt}(P).$$

Proof. It is straightforward from Lemma 2.1.8 and Lemma 2.1.17. □

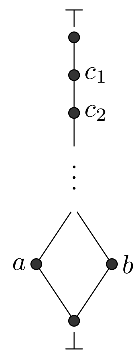


FIGURE 2.3. Example of a poset P such that $\mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P) \subset \mathsf{Opt}(P)$.

EXAMPLE 2.1.26. Consider the poset P in Figure 2.3 (observe that it is the dual poset of Figure 2.1). It is clear that $F = \{\top, c_1, c_2, \dots, c_n, \dots\}$ is a non-empty proper Frink-filter of P. Notice that $F^c = \{\bot, a, b\}$ is a Frink-ideal of P, but it is not an order-ideal of P because for $a, b \in F^c$ there is no $x \in F^c$ such that $a \leq x$ and $b \leq x$. Hence, we have F is an optimal Frink-filter of P and is not a prime Frink-filter of P.

DEFINITION 2.1.27. Let P be a poset and let F be a proper Frink-filter of P. We say F is $\vee$-*prime* Frink-filter of P if for all $a_1, \dots, a_n \in P$ such that $a_1 \vee \cdots \vee a_n$ exists in P and $a_1 \vee \cdots \vee a_n \in F$, then $a_i \in F$ for some $i = 1, \dots, n$.

LEMMA 2.1.28. *Let P be a poset. If F is an optimal Frink-filter of P, then F is a $\vee$-prime Frink-filter of P.*

PROOF. Let F be an optimal Frink-filter of P. Let $a_1, \dots, a_n \in P$ and assume $a_1 \vee \cdots \vee a_n$ exists in P and $a_1 \vee \cdots \vee a_n \in F$. Suppose $a_i \notin F$ for all $i = 1, \dots, n$. Since $\uparrow a_1 \cap \cdots \cap \uparrow a_n = \uparrow(a_1 \vee \cdots \vee a_n)$, by Lemma 2.1.24, we have $a_1 \vee \cdots \vee a_n \notin F$; which is a contradiction. Thus, $a_i \in F$ for some $i = 1, \dots, n$. Hence, F is a $\vee$-prime Frink-filter. □

EXAMPLE 2.1.29. In this example we show that not every $\vee$-prime Frink-filter is an optimal Frink-filter. Consider the poset P in Figure 2.4. The Frink-filter $\uparrow a$ is clearly $\vee$-prime. Notice that $(\uparrow a)^c = \{\bot, b, c\}$ and $\{b, c\}^{\mathrm{ul}} = \{\top, d_1, d_2, \dots\}^{\mathrm{l}} = \{\bot, a, b, c\} \nsubseteq (\uparrow a)^c$. Then $(\uparrow a)^c$ is not a Frink-ideal of P and hence $\uparrow a$ is not an optimal Frink-filter of P.

2.1.3. Meet-filters and join-ideals. In this part we introduce the third notion of filter and ideal on posets known in the literature. Some papers that consider

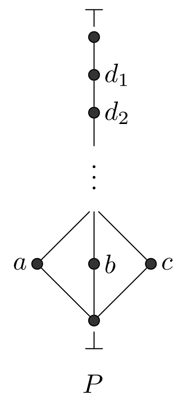


FIGURE 2.4. The Frink-filter $\uparrow a$ is $\vee$-prime, but it is not optimal.

the notion of join-ideal in the framework of meet-semilattice are, for instance [**13**] and [**2**]. The notions of filter and ideal on posets that we discuss in this subsection are weaker than Frink-filter and Frink-ideal, respectively.

DEFINITION 2.1.30. Let P be a poset and let $F, I \subseteq P$.

- F is said to be *meet-filter* if:
  (1) F is an up-set;
  (2) if $a_1, \ldots, a_n \in F$ and $a_1 \wedge \cdots \wedge a_n$ exists in P, then $a_1 \wedge \cdots \wedge a_n \in F$.
- I is said to be *join-ideal* if:
  (1) I is a down-set;
  (2) if $a_1, \ldots, a_n \in I$ and $a_1 \vee \cdots \vee a_n$ exists in P, then $a_1 \vee \cdots \vee a_n \in I$.

Observe that for any poset P, the empty set is a meet-filter (join-ideal), even if P has top (bottom) element. For this reason we consider the following notation, depending if P has or not top (bottom) element. Let P be a poset. If P has no top element, $\mathsf{Fi_m}(P)$ denotes the collection of all meet-filters of P including the empty set and if P has a top element, then $\mathsf{Fi_m}(P)$ denotes the collection of all non-empty meet-filters of P. Dually, $\mathsf{Id_j}(P)$ denotes all join-ideals of P if P has no bottom element and, $\mathsf{Id_j}(P)$ denotes all non-empty join-ideals of P, if P has bottom element.

LEMMA 2.1.31. *Let P be a poset. Then,*

$$\mathsf{Fi_F}(P) \subseteq \mathsf{Fi_m}(P) \quad \textit{and} \quad \mathsf{Id_F}(P) \subseteq \mathsf{Id_j}(P).$$

The following example shows that the above inclusions are not necessarily e-qualities.

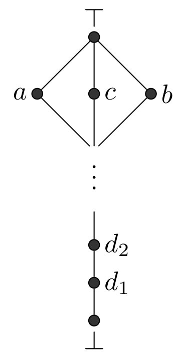


FIGURE 2.5. A poset P such that $\mathsf{Fi}_{\mathsf{F}}(P) \subset \mathsf{Fi}_{\mathsf{m}}(P)$.

EXAMPLE 2.1.32. Let us consider the poset P presented in Figure 2.5. Let $F = \{a, c, \top\}$. It is straightforward to check that F is a meet-filter. But F is not a Frink-filter because $a, c \in F$ and $\{a, c\}^{\mathrm{lu}} = \{a, b, c, \top\} \nsubseteq F$.

LEMMA 2.1.33. *Let P be an arbitrary poset. Then, $\mathsf{Fi}_{\mathsf{m}}(P)$ and $\mathsf{Id}_{\mathsf{j}}(P)$ are algebraic closure systems on P.*

Let us denote by $\mathrm{Fi}_{\mathrm{m}}(.)$ and $\mathrm{Id}_{\mathrm{j}}(.)$ the closure operators associated with the closure systems $\mathsf{Fi}_{\mathsf{m}}(P)$ and $\mathsf{Id}_{\mathsf{j}}(P)$, respectively. The following result provides a description of the meet-filter generated by a non-empty subset of a poset P.

LEMMA 2.1.34. *Let P be a poset and let a non-empty $X \subseteq P$. Consider the following sets X_n for every $n \in \omega$ defined by induction as follows:*

$$X_0 = X$$
$$X_{n+1} = \{a \in P : \ x_1 \wedge \cdots \wedge x_k \leq a \text{ for some } x_1, \ldots, x_k \in X_n\}^1.$$

Then,

$$\mathrm{Fi}_{\mathrm{m}}(X) = \bigcup_{n \in \omega} X_n.$$

PROOF. We put $U = \bigcup_{n \in \omega} X_n$ and we show that U is the smallest meet-filter of P containing the set X. First it is clear that $X \subseteq U$. Notice that $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \ldots$ and, for each $n \geq 1$, X_n is an up-set of P. Whereupon, U is an up-set of P. Let $a_1, \ldots, a_m \in U$ be such that $a_1 \wedge \cdots \wedge a_m$ exists in P. Thus, there are $n_1, \ldots, n_m \in \omega$ such that $a_i \in X_{n_i}$ for every $i \in \{1, \ldots, m\}$. Let $n = \max\{n_1, \ldots, n_m\}$. Then, $a_1, \ldots, a_m \in X_n$. Thus, since $a_1 \wedge \cdots \wedge a_m$ exists, $a_1 \wedge \cdots \wedge a_m \in X_{n+1}$. Hence, $a_1 \wedge \cdots \wedge a_m \in U$. Now, let $F \in \mathsf{Fi}_{\mathsf{m}}(P)$ be such that

[1] Recall that when we write $x_1 \wedge \cdots \wedge x_k \leq a$ for some elements $x_1, \ldots, x_k, a$ of a poset P, we mean that the meet of $\{x_1, \ldots, x_k\}$ exists in P and it is less than or equal to a.

$X \subseteq F$. We prove, by induction on n, that $X_n \subseteq F$ for all $n \in \omega$. It is obvious for $n = 0$. Suppose $X_n \subseteq F$ and let $a \in X_{n+1}$. Then, there exist $x_1, \ldots, x_k \in X_n$ such that $x_1 \wedge \cdots \wedge x_k \leq a$. Since F is a meet-filter and $x_1, \ldots, x_k \in F$, $x_1 \wedge \cdots \wedge x_k \in F$. Thus, $a \in F$ and, hence $X_{n+1} \subseteq F$. Then, $U \subseteq F$. $\square$

Lemma 2.1.35. *Let P be a poset and let $Y \subseteq P$ be non-empty. Consider the following sets Y_n for every $n \in \omega$ defined by induction as follows:*

$$Y_0 = Y$$
$$Y_{n+1} = \{a \in P : \ a \leq y_1 \vee \cdots \vee y_k \text{ for some } y_1, \ldots, y_k \in Y_n\}.$$

Then,

$$\mathrm{Id_j}(X) = \bigcup_{n \in \omega} Y_n.$$

2.2. Distributive posets

In this section we consider a notion of distributivity for posets. The condition of distributivity for posets we discuss in this dissertation is due to David and Erné in [**16**] and it is a generalization of the notion of distributivity in semilattices and so, it is also a generalization of the usual distributive condition in Lattice Theory. We also have to say that there are other possible generalizations on posets of the concept of distributivity in Lattice Theory, for instance see [**13**] and [**38**].

An important feature of the property of distributivity in lattices and semilattices is that it plays a fundamental role to find some kind of topological representations and dualities. Moreover, not only in lattices and semilattices the property of distributivity is important to find topological representations, in any ordered algebraic structures an adequate notion of distributivity is fundamental, see for instance the PhD. Thesis in [**23**] by M. Esteban on dualities in the framework of Abstract Algebraic Logic.

Definition 2.2.1. Let P be a poset.

(1) We say that P is *meet-order distributive* (*mo-distributive* for short) if for every $b_1, \ldots, b_n, a \in P$ the following condition is satisfied:

$$a \in \{b_1, \ldots, b_n\}^{\mathrm{lu}} \implies \text{there are } a_1, \ldots, a_k \in {\uparrow} b_1 \cup \cdots \cup {\uparrow} b_n \text{ such that } a = a_1 \wedge \cdots \wedge a_k. \tag{2.1}$$

(2) We say that P is *join-order distributive* (*jo-distributive* for short) if for every $b_1, \ldots, b_n, a \in P$ the following condition is satisfied:

$$a \in \{b_1, \ldots, b_n\}^{\mathrm{ul}} \implies \text{there are } a_1, \ldots, a_k \in {\downarrow} b_1 \cup \cdots \cup {\downarrow} b_n \text{ such that } a = a_1 \vee \cdots \vee a_k. \tag{2.2}$$

Remark 2.2.2. In the above definition, recall that when we write $a = a_1 \wedge \cdots \wedge a_k$ ($a = a_1 \vee \cdots \vee a_k$) we claim two things: first that the meet (join) of $a_1, \ldots, a_k$ exists and second it is equals to a. Moreover, it is straightforward to show that in each poset P the reciprocal conditions of (2.1) and (2.2) always hold.

The two first consequences of the conditions of mo-distributivity and jo-distributivity on posets are that the collection of Frink-filters (Frink-ideal) and the collection of meet-filters (join-ideal) coincide on mo-distributive (jo-distributive) posets and the notions of optimal Frink-filter (Definition 2.1.23) and $\vee$-prime Frink-filter (Definition Definition 2.1.27) are equivalent on jo-distributive posets.

Lemma 2.2.3. *Let P be a poset.*

(1) *If P is mo-distributive, then $\mathsf{Fi}_{\mathsf{F}}(P) = \mathsf{Fi}_{\mathsf{m}}(P)$.*
(2) *If P is jo-distributive, then $\mathsf{Id}_{\mathsf{F}}(P) = \mathsf{Id}_{\mathsf{j}}(P)$.*

Proof.

(1) Assume that P is a mo-distributive poset. We already know, by Lemma 2.1.31, $\mathsf{Fi}_{\mathsf{F}}(P) \subseteq \mathsf{Fi}_{\mathsf{m}}(P)$. Now let $F \in \mathsf{Fi}_{\mathsf{m}}(P)$. Let $a_1, \ldots, a_n \in F$ and let $a \in \{a_1, \ldots, a_n\}^{\mathrm{lu}}$. Since P is mo-distributive, there exist $b_1, \ldots, b_k \in {\uparrow}a_1 \cup \cdots \cup {\uparrow}a_n$ such that $a = b_1 \wedge \cdots \wedge b_k$. Thus, since F is an up-set, $b_1, \ldots, b_k \in F$. So, $a = b_1 \wedge \cdots \wedge b_k \in F$. Then, F is a Frink-filter and therefore $\mathsf{Fi}_{\mathsf{m}}(P) \subseteq \mathsf{Fi}_{\mathsf{F}}(P)$.
(2) It can be shown dually.

□

Lemma 2.2.4. *Let P be a jo-distributive poset and let $F \in \mathsf{Fi}_{\mathsf{F}}(P)$. Then, F is an optimal Frink-filter if and only if F is a $\vee$-prime Frink-filter.*

Proof. From Lemma 2.1.28, we know that each optimal Frink-filter is $\vee$-prime. Now, let F be a $\vee$-prime Frink-filter of P. We need to prove that F^c is a Frink-ideal of P. Let $A \subseteq_\omega F^c$. If $A = \emptyset$ then, since F is a proper up-set of P, $A^{\mathrm{ul}} \subseteq F^c$. So, we suppose that $A \neq \emptyset$ and let $b \in A^{\mathrm{ul}}$. Since P is jo-distributive, there exist $b_1, \ldots, b_k \in \bigcup_{a \in A} {\downarrow}a$ such that $b = b_1 \vee \cdots \vee b_k$. If $b \in F$, since F is $\vee$-prime, then there is $i = 1, \ldots, k$ such that $b_i \in F$. So, there exists $a \in A$ such that $b_i \leq a$, whereupon $a \in F$. This is a contradiction. Hence, $b \in F^c$. Therefore, F^c is a Frink-ideal and, then F is an optimal Frink-filter of P. □

The next two lemmas are straightforward to prove, so we omit their proofs, and they help us to see that the conditions of mo-distributivity and jo-distributivity are preserved by order-isomorphisms and, in the presence of a dual order-isomorphism between posets the condition of mo-distributivity on one poset is transferred to the condition of jo-distributivity on the other poset and vice versa.

LEMMA 2.2.5. *Let P and Q be posets. If $h\colon P \to Q$ is an order-isomorphism, then for all $a, a_1, \ldots, a_n \in P$, the following conditions*

$$a \in \{a_1, \ldots, a_n\}^{\mathrm{lu}} \iff h(a) \in \{h(a_1), \ldots, h(a_n)\}^{\mathrm{lu}} \tag{2.3}$$

and

$$a \in \{a_1, \ldots, a_n\}^{\mathrm{ul}} \iff h(a) \in \{h(a_1), \ldots, h(a_n)\}^{\mathrm{ul}} \tag{2.4}$$

hold.

LEMMA 2.2.6. *Let P and Q be posets. If $h\colon P \to Q$ is a dual order-isomorphism, then for all $a, a_1, \ldots, a_n \in P$, the following conditions*

$$a \in \{a_1, \ldots, a_n\}^{\mathrm{lu}} \iff h(a) \in \{h(a_1), \ldots, h(a_n)\}^{\mathrm{ul}} \tag{2.5}$$

and

$$a \in \{a_1, \ldots, a_n\}^{\mathrm{ul}} \iff h(a) \in \{h(a_1), \ldots, h(a_n)\}^{\mathrm{lu}} \tag{2.6}$$

hold.

LEMMA 2.2.7. *Let P and Q be posets and let $h\colon P \to Q$ be an order-isomorphism. Then, P is mo-distributive (jo-distributive) if and only if Q is mo-distributive (jo-distributive).*

LEMMA 2.2.8. *Let P and Q be posets and let $h\colon P \to Q$ be a dual order-isomorphism. Then, P is mo-distributive (jo-distributive) if and only if Q is jo-distributive (mo-distributive).*

The condition (2.2) in Definition 2.2.1 was given by David and Erné in [**16**], where they proved that it is a first order characterization of the fact that the lattice of all Frink-ideals on a quasi-ordered set is distributive. Here we present a direct proof of the dual of this fact for a poset, that is, we show that the condition (2.1) is a first order characterization of the fact that the lattice of all Frink-filters is distributive. First we need the following lemma.

LEMMA 2.2.9. *Let P be a poset and $a_1, \ldots, a_n \in P$. If $a \in \{a_1, \ldots, a_n\}^{\mathrm{lu}} \cap \{a_1, \ldots, a_n\}^{\mathrm{l}}$, then $a = a_1 \wedge \cdots \wedge a_n$.*

PROOF. Let $a \in \{a_1, \ldots, a_n\}^{\mathrm{l}}$, $a \leq a_i$ for all $i = 1, \ldots, n$. Let $x \in P$ such that $x \leq a_i$ for every $i = 1, \ldots, n$. So, $x \in \{a_1, \ldots, a_n\}^{\mathrm{l}}$ and, since $a \in \{a_1, \ldots, a_n\}^{\mathrm{lu}}$, $x \leq a$. Then, a is the greatest lower bound of $a_1, \ldots, a_n$, i.e., $a = a_1 \wedge \cdots \wedge a_n$. □

THEOREM 2.2.10. *Let P be a poset. Then, P is mo-distributive if and only if the lattice $\mathsf{Fi}_{\mathsf{F}}(P)$ is distributive.*

Proof. We assume first that $\mathsf{Fi}_{\mathsf{F}}(P)$ is a distributive lattice and we prove that P is mo-distributive. For this, let $a, b_1, \ldots, b_n \in P$ be such that $a \in \{b_1, \ldots, b_n\}^{\mathrm{lu}}$. So, we have

$$\begin{aligned} \uparrow a \cap \{b_1, \ldots, b_n\}^{\mathrm{lu}} &= \uparrow a \cap (\uparrow b_1 \vee \cdots \vee \uparrow b_n) \\ &= (\uparrow a \cap \uparrow b_1) \vee \cdots \vee (\uparrow a \cap \uparrow b_n) . \end{aligned}$$

Since $a \in \uparrow a \cap \{b_1, \ldots, b_n\}^{\mathrm{lu}}$, it follows that $a \in (\uparrow a \cap \uparrow b_1) \vee \cdots \vee (\uparrow a \cap \uparrow b_n)$. Then, by Lemma 2.1.12, there exist $a_1, \ldots, a_k \in (\uparrow a \cap \uparrow b_1) \cup \cdots \cup (\uparrow a \cap \uparrow b_n)$ such that $a \in \{a_1, \ldots, a_k\}^{\mathrm{lu}}$. Moreover, as $a_1, \ldots, a_k \in \uparrow a$, $a \in \{a_1, \ldots, a_k\}^{\mathrm{l}}$. Thus, from the previous lemma, $a = a_1 \wedge \cdots \wedge a_k$ with $a_1, \ldots, a_k \in \uparrow b_1 \cup \cdots \cup \uparrow b_n$. Therefore, P is mo-distributive.

Now, we suppose that P is mo-distributive. Let $F_1, F_2, F_3 \in \mathsf{Fi}_{\mathsf{F}}(P)$. We need to prove only that $F_1 \cap (F_2 \vee F_3) \subseteq (F_1 \cap F_2) \vee (F_1 \cap F_3)$. Let $a \in F_1 \cap (F_2 \vee F_3)$. So, $a \in F_1$ and there exist $a_1, \ldots, a_n \in F_2 \cup F_3$ such that $a \in \{a_1, \ldots, a_n\}^{\mathrm{lu}}$. Then, since P is mo-distributive, there exist $a'_1, \ldots, a'_k \in \uparrow a_1 \cup \cdots \cup \uparrow a_n$ such that $a = a'_1 \wedge \cdots \wedge a'_k$. Clearly, $a \in \{a'_1, \ldots, a'_k\}^{\mathrm{lu}}$. Given that $a \in F_1$ and $a \leq a'_i$ for all $i = 1, \ldots, k$, we have $a'_1, \ldots, a'_k \in F_1$. It also holds $a'_1, \ldots, a'_k \in \uparrow a_1 \cup \cdots \cup \uparrow a_n \subseteq F_2 \cup F_3$. Thus, $a'_1, \ldots, a'_k \in F_1 \cap (F_2 \cup F_3) = (F_1 \cap F_2) \cup (F_1 \cap F_3)$. Hence, since $a \in \{a'_1, \ldots, a'_k\}^{\mathrm{lu}}$, $a \in (F_1 \cap F_2) \vee (F_1 \cap F_3)$. Therefore, $\mathsf{Fi}_{\mathsf{F}}(P)$ is a distributive lattice. □

Theorem 2.2.11. ([**16**, Proposition 3.1]). *Let P be a poset. Then, P is jo-distributive if and only if the lattice $\mathsf{Id}_{\mathsf{F}}(P)$ is distributive.*

The following corollary shows that the condition of mo-distributivity (jo-distributivity) is a good generalization of the condition of distributivity in a meet-semilattice (join-semilattice). Before, we give the following result that was proved by Grätzer in [**34**] for distributive join-semilattices.

Lemma 2.2.12. *Let M be a meet-semilattice and let J be a join-semilattice. Then,*

(1) *M is distributive if and only if $\mathsf{Fi}(M)$ is a distributive lattice;*
(2) *J is distributive if and only if $\mathsf{Id}(J)$ is a distributive lattice.*

Then, by the previous lemma and from lemmas 2.1.9 and 2.1.10, we have:

Corollary 2.2.13. *Let M be a meet-semilattice and let J be a join-semilattice. Then,*

(1) *M is distributive (as a meet-semilattice) if and only if M is mo-distributive;*
(2) *J is distributive (as a join-semilattice) if and only if J is jo-distributive.*

Example 2.2.14. Figure 2.6 shows a poset P and its lattice of Frink-filters. Since the lattice $\mathsf{Fi}_{\mathsf{F}}(P)$ is distributive we can conclude, by Theorem 2.2.10, that P

FIGURE 2.6. A mo-distributive and jo-distributive poset.

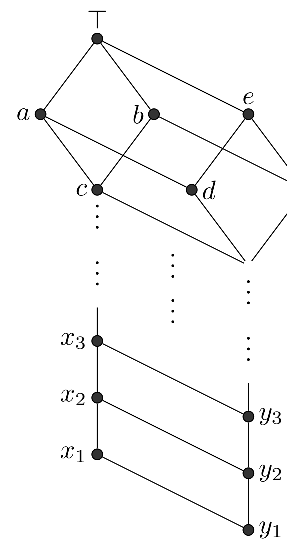


FIGURE 2.7. The poset Q is mo-distributive, but it is not jo-distributive.

is mo-distributive. Moreover, it is not hard to check that P and its dual poset, P^∂, are order-isomorphic and dual order-isomorphic, hence P is also jo-distributive.

Now, consider the poset Q given in Figure 2.7. In Figure 2.8 we display the lattices of Frink-filters and Frink-ideals, respectively, of Q. The lattice $\mathsf{Fi_F}(Q)$ is distributive because it is isomorphic to the product of two distributive lattices: $\mathsf{Fi_F}(Q) \cong ((\mathbf{2} \times \mathbf{2}) \oplus \mathbb{Z}^-) \times \mathbf{2}$, where $\mathbf{2}$ is the distributive lattice of two elements. Hence, the poset Q is mo-distributive. In the lattice $\mathsf{Id_F}(Q)$ of Figure 2.8, $I = \bigcup_{i \geq 1} {\downarrow} x_i$ and $J = \bigcup_{i \geq 1} {\downarrow} y_i$. Then, taking the Frink-ideals I, ${\downarrow}c$, ${\downarrow}b$, ${\downarrow}f$ and J of the poset Q we can see, by Lemma 1.5.6, that the lattice $\mathsf{Id_F}(Q)$ is not distributive. Therefore, Q is not jo-distributive.

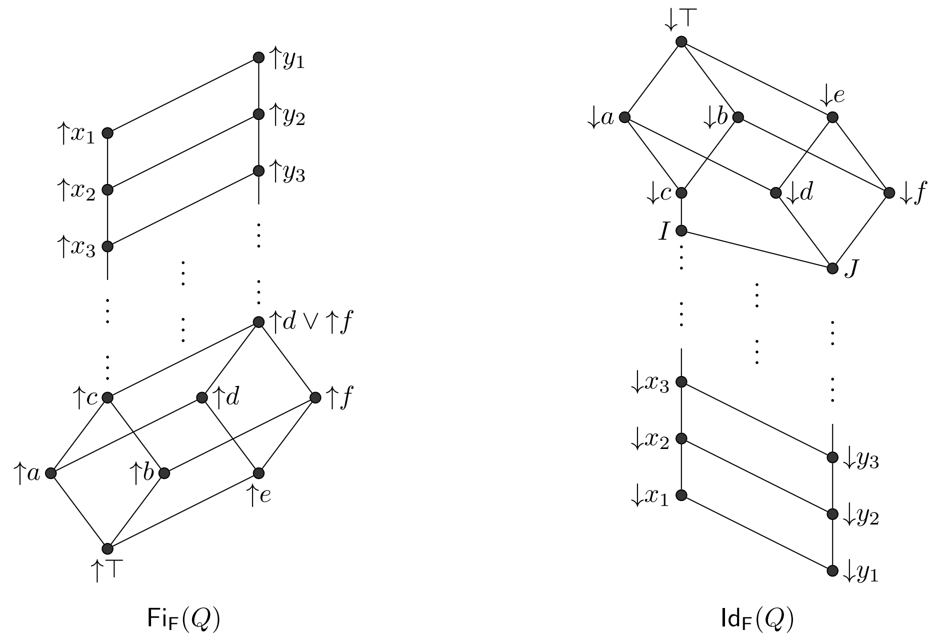


FIGURE 2.8. The lattices of Frink-filters and Frink-ideals, respectively, of the poset Q given in Figure 2.7.

The following lemma shows, for a mo-distributive poset P, that $\mathsf{Fi_F}(P)$ is more than a complete distributive lattice, indeed it is a complete Heyting algebra. To see this fact, let P be a poset and for every $F_1, F_2 \in \mathsf{Fi_F}(P)$ we define the following set

$$F_1 \to F_2 := \{a \in P : \ {\uparrow}a \cap F_1 \subseteq F_2\}.$$

LEMMA 2.2.15. *Let P be a mo-distributive poset. Then $\langle \mathsf{Fi_F}(P), \cap, \vee, \to, F_0, P\rangle$ is a complete Heyting algebra, where F_0 is the least element of $\mathsf{Fi_F}(P)$.*

PROOF. We only need to show that $\to$ is well define and satisfies the following condition

$$F_1 \cap F_2 \subseteq F_3 \iff F_1 \subseteq F_2 \to F_3. \tag{2.7}$$

Let $F_1, F_2, F_3 \in \mathsf{Fi_F}(P)$. To prove that $F_1 \to F_2$ belongs to $\mathsf{Fi_F}(P)$ let $A \subseteq_\omega F_1 \to F_2$ and $b \in A^{lu}$. Notice that ${\uparrow}b \subseteq A^{lu}$. Let $x \in {\uparrow}b \cap F_1$. Since ${\uparrow}a \cap F_1 \subseteq F_2$ for all $a \in A$, it follows that $\bigvee_{a\in A}({\uparrow}a \cap F_1) \subseteq F_2$ and, since $\mathsf{Fi_F}(P)$ is a distributive lattice, we have that $(\bigvee_{a\in A}{\uparrow}a) \cap F_1 \subseteq F_2$. Then $A^{lu} \cap F_1 \subseteq F_2$. We thus obtain ${\uparrow}b \cap F_1 \subseteq F_2$ and hence $x \in F_2$. Therefore $b \in F_1 \to F_2$.

Now, we show (2.7). Assume first that $F_1 \cap F_2 \subseteq F_3$ and let $a \in F_1$. Let $x \in {\uparrow}a \cap F_2$. So, $x \in F_1 \cap F_2$ and then, $x \in F_3$. Hence, $a \in F_2 \to F_3$ and therefore, $F_1 \subseteq F_2 \to F_3$. Reciprocally, assume that $F_1 \subseteq F_2 \to F_3$. Let $a \in F_1 \cap F_2$. So,

$a \in F_2 \to F_3$ and whereupon $\uparrow a \cap F_2 \subseteq F_3$. Since $a \in \uparrow a \cap F_2$, $a \in F_3$. Therefore, $F_1 \cap F_2 \subseteq F_3$. This complete the proof. $\square$

Recall that $\mathsf{Fi}_{\mathsf{F}}^{\mathsf{irr}}(P)$ and $\mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)$ denote the collections of all irreducible and prime Frink-filters of a poset P, respectively. In the following lemma we obtain a characterization of the mo-distributivity condition similar to a well-known characterization in the setting of lattice and also similar to a characterization in the setting of meet-semilattice (see for instance [**8**, Theorem 10]).

Lemma 2.2.16. *Let P be a poset. Then, P is mo-distributive if and only if $\mathsf{Fi}_{\mathsf{F}}^{\mathsf{irr}}(P) = \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)$.*

Proof. First assume that the poset P is mo-distributive. We know that $\mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P) \subseteq \mathsf{Fi}_{\mathsf{F}}^{\mathsf{irr}}(P)$. Let $F \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{irr}}(P)$ and let $F_1, F_2 \in \mathsf{Fi}_{\mathsf{F}}(P)$ be such that $F_1 \cap F_2 \subseteq F$. So, $F = F \vee (F_1 \cap F_2)$. Since P is a mo-distributive poset, it follows by Theorem 2.2.10 that $\mathsf{Fi}_{\mathsf{F}}(P)$ is a distributive lattice. Then, $F = (F \vee F_1) \cap (F \vee F_2)$. As F is irreducible, we have that $F = F \vee F_1$ or $F = F \vee F_2$. Hence, $F_1 \subseteq F$ or $F_2 \subseteq F$. Therefore, $F \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)$.

Conversely, assume that $\mathsf{Fi}_{\mathsf{F}}^{\mathsf{irr}}(P) = \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)$. We prove that $\mathsf{Fi}_{\mathsf{F}}(P)$ is a distributive lattice. Let $F_1, F_2, F_3 \in \mathsf{Fi}_{\mathsf{F}}(P)$. It is only necessary to show that $F_1 \cap (F_2 \vee F_3) \subseteq (F_1 \cap F_2) \vee (F_1 \cap F_3)$. Suppose towards a contradiction that $F_1 \cap (F_2 \vee F_3) \nsubseteq (F_1 \cap F_2) \vee (F_1 \cap F_3)$. So, there exists $a \in F_1 \cap (F_2 \vee F_3) \setminus (F_1 \cap F_2) \vee (F_1 \cap F_3)$. Then, we obtain that $\downarrow a \cap ((F_1 \cap F_2) \vee (F_1 \cap F_3)) = \emptyset$. By Theorem 2.1.20, there is $G \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{irr}}(P)$ such that $(F_1 \cap F_2) \vee (F_1 \cap F_3) \subseteq G$ and $\downarrow a \cap G = \emptyset$. We thus obtain that $F_1 \cap F_2 \subseteq G$ and $F_1 \cap F_3 \subseteq G$. Now, since $G \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{irr}}(P) = \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)$, it follows that ($F_1 \subseteq G$ or $F_2 \subseteq G$) and ($F_1 \subseteq G$ or $F_3 \subseteq G$). As $a \in F_1 \setminus G$, we have that $F_2 \subseteq G$ and $F_3 \subseteq G$. Then, $F_2 \vee F_3 \subseteq G$ and hence $F_1 \cap (F_2 \vee F_3) \subseteq G$. Since $a \in F_1 \cap (F_2 \vee F_3)$, it follows that $a \in G$, which is a contradiction. $\square$

The following theorem is another characterization of the mo-distributivity condition and it will be useful in the next chapters to develop topological dualities.

Theorem 2.2.17. *Let P be a poset. Then, the following conditions are equivalent:*

(1) *P is mo-distributive;*
(2) *If F is a Frink-filter and I is an order-ideal of P such that $F \cap I = \emptyset$, then there exists a prime Frink-filter U of P such that $F \subseteq U$ and $U \cap I = \emptyset$.*

Proof. (1) $\Rightarrow$ (2) It is a consequence of Lemma 2.2.16 and Theorem 2.1.20.

(2) $\Rightarrow$ (1) We prove that the lattice of all Frink-filters is distributive. Let $F_1, F_2, F_3 \in \mathsf{Fi}_{\mathsf{F}}(P)$. We need only to show that $F_1 \cap (F_2 \vee F_3) \subseteq (F_1 \cap F_2) \vee (F_1 \cap F_3)$ because the other inclusion always holds. We suppose that $F_1 \cap (F_2 \vee F_3) \nsubseteq$

$(F_1 \cap F_2) \vee (F_1 \cap F_3)$. So, let $a \in F_1 \cap (F_2 \vee F_3) \setminus (F_1 \cap F_2) \vee (F_1 \cap F_3)$. Since $a \notin (F_1 \cap F_2) \vee (F_1 \cap F_3)$, it follows that there exists $U \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)$ such that $a \notin U$ and $(F_1 \cap F_2) \vee (F_1 \cap F_3) \subseteq U$. Then, $F_1 \cap F_2 \subseteq U$ and $F_1 \cap F_3 \subseteq U$. As U is prime, we have

$$(F_1 \subseteq U \text{ or } F_2 \subseteq U) \text{ and } (F_1 \subseteq U \text{ or } F_3 \subseteq U).$$

Since $a \in F_1$ and $a \notin U$, we have $F_2 \subseteq U$ and $F_3 \subseteq U$. Then, $F_2 \vee F_3 \subseteq U$. We thus get $a \in U$, which is a contradiction. Hence, $F_1 \cap (F_2 \vee F_3) \subseteq (F_1 \cap F_2) \vee (F_1 \cap F_3)$. Therefore, by Theorem 2.2.10, P is mo-distributive. □

Corollary 2.2.18. *Let P be a mo-distributive poset. If F is a Frink-filter of P and $a \notin F$, then there exists a prime Frink-filter U of P such that $F \subseteq U$ and $a \notin U$.*

Corollary 2.2.19. *Let P be a mo-distributive poset. Then, every Frink-filter of P is the intersection of all prime Frink-filters of P that include it.*

Let P be a poset. Recall that $\langle \mathsf{Fi}_{\mathsf{F}}^{\mathsf{f}}(P), \vee \rangle$ and $\langle \mathsf{Id}_{\mathsf{F}}^{\mathsf{f}}(P), \vee \rangle$ are the join-semilattices of all finitely generated Frink-filters and all finitely generated Frink-ideals of P, respectively. The following lemmas are characterizations of mo-distributive and jo-distributive posets by means of $\mathsf{Fi}_{\mathsf{F}}^{\mathsf{f}}(P)$ and $\mathsf{Id}_{\mathsf{F}}^{\mathsf{f}}(P)$, respectively.

Lemma 2.2.20. *Let P be a poset. P is mo-distributive if and only if the join-semilattice $\langle \mathsf{Fi}_{\mathsf{F}}^{\mathsf{f}}(P), \vee \rangle$ is distributive.*

Proof. Let P be a poset. We assume first that P is mo-distributive. To prove that $\langle \mathsf{Fi}_{\mathsf{F}}^{\mathsf{f}}(P), \vee \rangle$ is a distributive join-semilattice, let $F_1 = A_1^{\mathrm{lu}}$, $F_2 = A_2^{\mathrm{lu}}$ and $G = B^{\mathrm{lu}}$ with non-empty $A_1, A_2, B \subseteq_\omega P$ and assume that $G \subseteq F_1 \vee F_2$. We observe that $B \subseteq G \subseteq F_1 \vee F_2 = (A_1 \cup A_2)^{\mathrm{lu}}$. So, since P is mo-distributive, for every $b \in B$ there exists $A_b \subseteq_\omega \uparrow(A_1 \cup A_2)$ such that $b = \bigwedge A_b$. Since $G = B^{\mathrm{lu}}$, we have $A_b \subseteq G$ for all $b \in B$. For every $b \in B$, we consider the sets

$$A'_b = \{x \in A_b : (\exists y \in A_1)(y \leq x)\} \quad \text{and} \quad A''_b = \{x \in A_b : (\exists y \in A_2)(y \leq x)\}$$

and, let $A' = \bigcup_{b \in B} A'_b$ and $A'' = \bigcup_{b \in B} A''_b$. Then, since $A' \subseteq F_1$ and $A'' \subseteq F_2$, we obtain that

$$G_1 = A'^{\mathrm{lu}} \subseteq F_1 \quad \text{and} \quad G_2 = A''^{\mathrm{lu}} \subseteq F_2.$$

We only remain to show that $G = G_1 \vee G_2$. Notice the following: for every $b \in B$, $A_b = A'_b \cup A''_b$. Then,

$$\begin{aligned} G_1 \vee G_2 &= (A' \cup A'')^{\mathrm{lu}} \\ &= \left(\bigcup_{b \in B} A'_b \cup \bigcup_{b \in B} A''_b \right)^{\mathrm{lu}} \end{aligned}$$

$$= \left(\bigcup_{b\in B} (A'_b \cup A''_b)\right)^{\mathrm{lu}}$$
$$= \left(\bigcup_{b\in B} A_b\right)^{\mathrm{lu}} \subseteq G.$$

Hence, $G_1 \vee G_2 \subseteq G$. On the other hand, observe that for every $b \in B$, $A_b \subseteq_\omega \bigcup_{b\in B} A_b$. So, $b = \bigwedge A_b \in (\bigcup_{b\in B} A_b)^{\mathrm{lu}}$ for all $b \in B$. Then, $B \subseteq G_1 \vee G_2$ and this implies $G \subseteq G_1 \vee G_2$. Hence, $G = G_1 \vee G_2$ for $G_1, G_2 \in \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$ such that $G_1 \subseteq F_1$ and $G_2 \subseteq F_2$. Therefore, $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \vee\rangle$ is a distributive join-semilattice.

Reciprocally, assume that the join-semilattice $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \vee\rangle$ is distributive. Let $a, a_1, \ldots, a_n \in P$ such that $a \in \{a_1, \ldots, a_n\}^{\mathrm{lu}}$. So, $\uparrow a \subseteq \uparrow a_1 \vee \cdots \vee \uparrow a_n$. Then, there exist $F_1, \ldots, F_n \in \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$ such that $\uparrow a = F_1 \vee \cdots \vee F_n$ and $F_i \subseteq \uparrow a_i$ for all $i = 1, \ldots, n$. For every $i = 1, \ldots, n$, let $F_i = A_i^{\mathrm{lu}}$ for some non-empty $A_i \subseteq_\omega P$ and let $B = \bigcup_{i=1}^{n} A_i$. Then,

$$\begin{aligned}\uparrow a &= F_1 \vee \cdots \vee F_n\\ &= A_1^{\mathrm{lu}} \vee \cdots \vee A_n^{\mathrm{lu}}\\ &= B^{\mathrm{lu}}.\end{aligned}$$

Hence, $a = \bigwedge B$ and $B \subseteq F_1 \cup \cdots \cup F_n \subseteq \uparrow a_1 \cup \cdots \cup \uparrow a_n$. Therefore, P is mo-distributive. □

Lemma 2.2.21. *Let P be a poset. P is jo-distributive if and only if the join-semilattice $\langle \mathsf{Id}^{\mathsf{f}}_{\mathsf{F}}(P), \vee\rangle$ is distributive.*

We end this section by proving that the condition of mo-distributivity (jo-distributivity) behaves well with the formation of finite products and finite ordinal sums of mo-distributive (jo-distributive) posets.

Lemma 2.2.22. *Let P_1 and P_2 be bounded mo-distributive (jo-distributive) posets. Then, the poset $P_1 \times P_2$ is a bounded mo-distributive (jo-distributive) poset.*

Proof. Let P_1 and P_2 be bounded mo-distributive posets. We put $\top_i$ and $\bot_i$ for the top and bottom element of P_i, respectively, with $i = 1, 2$. Notice that $(\top_1, \top_2)$ and $(\bot_1, \bot_2)$ are the top and the bottom elements of $P_1 \times P_2$, respectively. Moreover, it is straightforward to show that if the meet of $a_1, \ldots, a_n$ exists in P_1 and the meet of $b_1, \ldots, b_n$ exists in P_2, then the meet of $(a_1, b_1), \ldots, (a_n, b_n)$ exists in $P_1 \times P_2$ and equals $(a_1 \wedge \cdots \wedge a_n, b_1 \wedge \cdots \wedge b_n)$.

Let $(a, b), (c_1, d_1), \ldots, (c_n, d_n) \in P_1 \times P_2$ and we assume

$$(a, b) \in \{(c_1, d_1), \ldots, (c_n, d_n)\}^{\mathrm{lu}}. \tag{2.8}$$

First, let us see that $a \in \{c_1, \dots, c_n\}^{\mathrm{lu}}$ and $b \in \{d_1, \dots, d_n\}^{\mathrm{lu}}$. Let $x \in P_1$ such that $x \le c_i$ for all $i \in \{1, \dots, n\}$. Then, $(x, \bot_2) \le (c_i, d_i)$ for all $i \in \{1, \dots, n\}$. So, by (2.8), we have $(x, \bot_2) \le (a, b)$. Whereupon, $x \le a$ and, therefore $a \in \{c_1, \dots, c_n\}^{\mathrm{lu}}$. Similarly, we can obtain $b \in \{d_1, \dots, d_n\}^{\mathrm{lu}}$. Now, since P_1 and P_2 are mo-distributive posets, there exist $c'_1, \dots, c'_k \in \uparrow c_1 \cup \dots \cup \uparrow c_n$ and $d'_1, \dots, d'_h \in \uparrow d_1 \cup \dots \cup \uparrow d_n$ such that

$$a = c'_1 \wedge \dots \wedge c'_k \quad \text{and} \quad b = d'_1 \wedge \dots \wedge d'_h. \tag{2.9}$$

Let $m = k + h$. We define the following finite sequences $(a_i)_{i=1}^m$ and $(b_i)_{i=1}^m$ of P_1 and P_2, respectively, as follows:

$$a_i = \begin{cases} c'_i & \text{if } 1 \le i \le k \\ \top_1 & \text{if } k+1 \le i \le m \end{cases} \quad \text{and} \quad b_i = \begin{cases} \top_2 & \text{if } 1 \le i \le k \\ d'_i & \text{if } k+1 \le i \le m. \end{cases}$$

Let $i \in \{1, 2, \dots, m\}$. If $1 \le i \le k$, then $a_i = c'_i \in \uparrow c_1 \cup \dots \cup \uparrow c_n$ and $b_i = \top_2$. So, it is clear that

$$(a_i, b_i) \in \uparrow(c_1, d_1) \cup \dots \cup \uparrow(c_n, d_n).$$

If $k + 1 \le i \le m$, then $b_i = d'_i \in \uparrow d_1 \cup \dots \cup \uparrow d_n$ and $a_i = \top_1$. So,

$$(a_i, b_i) \in \uparrow(c_1, d_1) \cup \dots \cup \uparrow(c_n, d_n).$$

Hence, we have that for all $i \in \{1, 2, \dots, m\}$,

$$(a_i, b_i) \in \uparrow(c_1, d_1) \cup \dots \cup \uparrow(c_n, d_n).$$

Now, by (2.9) and from definitions of $(a_i)_{i=1}^m$ and $(b_i)_{i=1}^m$, it is clear that

$$a = a_1 \wedge \dots \wedge a_m \quad \text{and} \quad b = b_1 \wedge \dots \wedge b_m.$$

Then, we obtain

$$(a, b) = (a_1 \wedge \dots \wedge a_m, b_1 \wedge \dots \wedge b_m) = (a_1, b_1) \wedge \dots \wedge (a_m, b_m)$$

with $(a_1, b_1), \dots, (a_m, b_m) \in \uparrow(c_1, d_1) \cup \dots \cup \uparrow(c_n, d_n)$. Therefore, $P_1 \times P_2$ is mo-distributive. □

Lemma 2.2.23. *Let P_1 and P_2 be disjoint mo-distributive (jo-distributive) posets. Then, $P_1 \oplus P_2$ is mo-distributive (jo-distributive).*

Proof. Let P_1 and P_2 be disjoint mo-distributive posets and let $a, b_1, \dots, b_n \in P_1 \oplus P_2$ such that

$$a \in \{b_1, \dots, b_n\}^{\mathrm{lu}}. \tag{2.10}$$

We consider several cases.

(1) If $a, b_1, \dots, b_n \in P_i$ for some $i = 1, 2$, then, by the mo-distributivity of P_i, we have $a = a_1 \wedge \dots \wedge a_m$ for some $a_1, \dots, a_m \in \uparrow b_1 \cup \dots \cup \uparrow b_n$.

(2) If $a \in P_1$ and $b_1, \dots, b_n \in P_2$, then $\{b_1, \dots, b_n\}^{\mathrm{l}} \cap P_2 = \emptyset$. So, by (2.10), a is the top element of P_1 and, hence $a = b_1 \wedge \cdots \wedge b_n$.

(3) We assume $a \in P_1$ and there is $k \in \{1, \dots, n\}$ such that $b_1, \dots, b_k \in P_1$ and $b_{k+1}, \dots, b_n \in P_2$. Then, we have $\{b_1, \dots, b_n\}^{\mathrm{l}} = \{b_1, \dots, b_k\}^{\mathrm{l}}$ and so, by (2.10), $a \in \{b_1, \dots, b_k\}^{\mathrm{lu}}$ with $a, b_1, \dots, b_k \in P_1$. Hence, by the mo-distributivity of P_1, we obtain that $a = a_1 \wedge \cdots \wedge a_m$ for some $a_1, \dots, a_m \in {\uparrow} b_1 \cup \cdots \cup {\uparrow} b_n$.

(4) We suppose that $a \in P_2$ and there exists $k \in \{1, \dots, n\}$ such that $b_k \in P_1$. Since $b_k \leq a$, this case is obvious.

Therefore, we can conclude $P_1 \oplus P_2$ is mo-distributive. $\square$

By an inductive argument, we extend the previous lemmas to arbitrary finite products and ordinal sums.

Corollary 2.2.24. *Let $P_1, \dots, P_n$ be bounded mo-distributive (jo-distributive) posets (with $n \geq 1$). Then, the poset $\prod_{i=1}^{n} P_i$ is mo-distributive (jo-distributive).*

Corollary 2.2.25. *Let $P_1, \dots, P_n$ be mo-distributive (jo-distributive) posets such that $P_i \cap P_j = \emptyset$ if $i \neq j$. Then, $\bigoplus_{i=1}^{n} P_i$ is a mo-distributive (jo-distributive) poset.*

2.3. Homomorphisms between posets

In this part we introduce the definitions of certain morphisms between posets that intend to be a generalization of the notion of homomorphism in Lattice Theory. The definition of sup-homomorphism can be found in the work of Bezhanishvili and Jansana [**4**] for meet-semilattices and the definition of inf-homomorphism is obtained dually from the notion of sup-homomorphism in the setting of posets. We will see these notions behave well with Frink-filters and Frink-ideals.

Definition 2.3.1. Let P and Q be two posets. A map $h\colon P \to Q$ is said to be

(1) an *inf-homomorphism* if for every $A \subseteq_\omega P$, we have
$$a \in A^{\mathrm{lu}} \text{ implies } h(a) \in h[A]^{\mathrm{lu}};$$

(2) a *sup-homomorphism* if for every $A \subseteq_\omega P$, we have
$$a \in A^{\mathrm{ul}} \text{ implies } h(a) \in h[A]^{\mathrm{ul}};$$

(3) an *inf-sup-homomorphism* if h is inf-homomorphism and sup-homomorphism.

Remark 2.3.2. Notice that if $h\colon P \to Q$ is an inf-homomorphism or a sup-homomorphism, then h is order-preserving. This is a consequence of the fact that in every poset the following equivalences are

$$x \leq y \text{ if and only if } y \in {\uparrow}x = \{x\}^{\mathrm{lu}}$$

and

$$x \leq y \text{ if and only if } x \in {\downarrow}y = \{y\}^{\mathrm{ul}}.$$

LEMMA 2.3.3. *Let P and Q be posets and, let $h\colon P \to Q$ be a map.*

(1) *If h is an inf-homomorphism and P has a top element $\top_P$, then $h(\top_P)$ is the top element of Q;*
(2) *if h is a sup-homomorphism and P has a bottom element $\bot_P$, then $h(\bot_P)$ is the bottom element of Q.*

PROOF. (1) Assume $h\colon P \to Q$ is an inf-homomorphism and $\top_P$ is the top element of P. So we have $\top_P \in \emptyset^{\mathrm{lu}}$ and, since h is an inf-homomorphism, $h(\top_P) \in h[\emptyset]^{\mathrm{lu}}$. Then, $h(\top_P) \in \emptyset^{\mathrm{lu}} = Q^{\mathrm{u}}$. Hence, $h(\top_P)$ is the top of Q. (2) By a dual argument. □

LEMMA 2.3.4.

(1) *Let M_1 and M_2 be meet-semilattices and let $h\colon M_1 \to M_2$ be a map. If h is an inf-homomorphism, then h is a meet-homomorphism.*
(2) *Let J_1 and J_2 be join-semilattices and let $h\colon J_1 \to J_2$ be a map. If h is a sup-homomorphism, then h is a join-homomorphism.*

PROOF. (1) We assume that h is an inf-homomorphism. Let $a, b \in M_1$. Since h is order-preserving, we have $h(a \wedge b) \leq h(a) \wedge h(b)$. Because $a \wedge b \in {\uparrow}(a \wedge b) = \{a \wedge b\}^{\mathrm{lu}} = \{a, b\}^{\mathrm{lu}}$, we obtain $h(a \wedge b) \in \{h(a), h(b)\}^{\mathrm{lu}} = {\uparrow}(h(a) \wedge h(b))$. Then, $h(a) \wedge h(b) \leq h(a \wedge b)$. Hence, $h(a \wedge b) = h(a) \wedge h(b)$ and, therefore h is a meet-homomorphism. (2) It is obtained dually. □

LEMMA 2.3.5.

(1) *Let M_1 and M_2 be meet-semilattices with top element and let $h\colon M_1 \to M_2$ be a map. Then, h is an inf-homomorphism if and only if h is a meet-homomorphism preserving top.*
(2) *Let J_1 and J_2 be join-semilattices with bottom element and let $h\colon J_1 \to J_2$ be a map. Then, h is a sup-homomorphism if and only if h is a join-homomorphism preserving bottom.*

PROOF. (1) Let $\top_1$ and $\top_2$ be the top elements of M_1 and M_2 respectively. If h is an inf-homomorphism then, by the previous lemma and Lemma 2.3.3, h is a meet-homomorphism preserving top. Conversely, we assume that h is a meet-homomorphism preserving top. Let $A \subseteq_\omega P$ and let $a \in A^{\mathrm{lu}}$. If $A = \emptyset$, then $a = \top_1$. So, $h(a) = h(\top_1) = \top_2 \in h[A]^{\mathrm{lu}}$. Suppose that $A \neq \emptyset$. Since M_1 is a meet-semilattice, $A^{\mathrm{lu}} = {\uparrow}(\bigwedge A)$. So, $\bigwedge A \leq a$, which implies $\bigwedge h[A] = h(\bigwedge A) \leq h(a)$.

Hence, $h(a) \in \uparrow(\bigwedge h[A]) = h[A]^{\mathrm{lu}}$. Therefore, h is an inf-homomorphism. (2) It is obtained dually. $\square$

From the above lemmas we can see that the notion of inf-homomorphism is a stronger notion than of the meet-homomorphism. This is a consequence of the fact that in the definition of inf-homomorphism we took finite subsets $A \subseteq_\omega P$ to be possibly empty. If we restrict the definition of inf-homomorphism only for non-empty $A \subseteq_\omega P$, then the notion of inf-homomorphism is a direct generalization of meet-homomorphism, in the sense that for every $h\colon M_1 \to M_2$ map from a meet-semilattice M_1 to a meet-semilattice M_2, h is a inf-homomorphism if and only if h is a meet-homomorphism. The definition of inf-homomorphism that we choose is because for our interests several results are proved in a more easy and elegant way; it also plays an important role in Chapters 3 and 4. The following two lemmas are characterizations of inf-homomorphism and sup-homomorphism by means of Frink-filters and Frink-ideals, respectively.

LEMMA 2.3.6. *Let P and Q be posets and let $h\colon P \to Q$ be a map. Then, h is an inf-homomorphism if and only if $h^{-1}[G] \in \mathsf{Fi}_{\mathsf{F}}(P)$ for all $G \in \mathsf{Fi}_{\mathsf{F}}(Q)$.*

PROOF. Assume that h is an inf-homomorphism and let $G \in \mathsf{Fi}_{\mathsf{F}}(Q)$. Let $A \subseteq_\omega h^{-1}[G]$ and $a \in A^{\mathrm{lu}}$. Since h is an inf-homomorphism, $h(a) \in h[A]^{\mathrm{lu}}$. As $h[A] \subseteq_\omega G$ and $G \in \mathsf{Fi}_{\mathsf{F}}(Q)$, it follows that $h[A]^{\mathrm{lu}} \subseteq G$. Then, $h(a) \in G$ and hence $a \in h^{-1}[G]$. Therefore, $h^{-1}[G] \in \mathsf{Fi}_{\mathsf{F}}(P)$. Reciprocally, suppose that $h^{-1}[G] \in \mathsf{Fi}_{\mathsf{F}}(P)$ for all $G \in \mathsf{Fi}_{\mathsf{F}}(Q)$. Let $A \subseteq_\omega P$ and let $a \in A^{\mathrm{lu}}$. By hypothesis, $h^{-1}\left[h[A]^{\mathrm{lu}}\right] \in \mathsf{Fi}_{\mathsf{F}}(P)$. Moreover, notice that $A \subseteq h^{-1}\left[h[A]\right] \subseteq h^{-1}\left[h[A]^{\mathrm{lu}}\right]$, consequently $A^{\mathrm{lu}} \subseteq h^{-1}\left[h[A]^{\mathrm{lu}}\right]$. Thus, $a \in h^{-1}\left[h[A]^{\mathrm{lu}}\right]$ and hence, $h(a) \in h[A]^{\mathrm{lu}}$. Therefore, h is an inf-homomorphism. $\square$

LEMMA 2.3.7. *Let P and Q be posets and let $h\colon P \to Q$ be a map. Then, h is a sup-homomorphism if and only if $h^{-1}[J] \in \mathsf{Id}_{\mathsf{F}}(P)$ for all $J \in \mathsf{Id}_{\mathsf{F}}(Q)$.*

DEFINITION 2.3.8. Let P and Q be posets and let $h\colon P \to Q$ be a map. The map h is called an *inf-embedding* (*sup-embedding*) if h is an inf-homomorphism (a sup-homomorphism) and an order-embedding. Moreover, h is said to be an *inf-sup-embedding* if h is an inf-embedding and a sup-embedding.

LEMMA 2.3.9. *Let P and Q be posets and, let $h\colon P \to Q$ be a map. Then, the following are equivalent:*

(1) *h is an inf-embedding;*
(2) *for every $A \subseteq_\omega P$ and $a \in P$, $a \in A^{\mathrm{lu}}$ if and only if $h(a) \in h[A]^{\mathrm{lu}}$.*

PROOF. (1) $\Rightarrow$ (2) We only need to prove that $h(a) \in h[A]^{\mathrm{lu}}$ implies $a \in A^{\mathrm{lu}}$. So, let $A \subseteq_\omega P$ and let $a \in P$ be such that $h(a) \in h[A]^{\mathrm{lu}}$ and $b \in A^{\mathrm{l}}$. Thus,

$b \leq a'$ for all $a' \in A$. Since h is order-preserving, $h(b) \leq h(a')$ for all $a' \in A$. So, $h(b) \in h[A]^{\mathrm{l}}$ and, whereupon $h(b) \leq h(a)$. Hence, since h is order-embedding, $b \leq a$. Therefore $a \in A^{\mathrm{lu}}$.

(2) $\Rightarrow$ (1) From (2) it is clear that h is an inf-homomorphism and so, it is also order-preserving. Let $a, b \in P$. Suppose that $h(a) \leq h(b)$. So, $h(b) \in \{h(a)\}^{\mathrm{lu}}$. Then, $b \in \{a\}^{\mathrm{lu}}$. Thus, $a \leq b$. Hence, h is an order-embedding. □

LEMMA 2.3.10. *Let P and Q be posets and let $h\colon P \to Q$ be a map. Then, the following are equivalent:*

(1) *h is a sup-embedding;*
(2) *for every $A \subseteq_\omega P$ and $a \in P$, $a \in A^{\mathrm{ul}}$ if and only if $h(a) \in h[A]^{\mathrm{ul}}$.*

The following definition, known in the literature, introduces another kind of homomorphism that we can consider between posets (see for instance [**37**]).

DEFINITION 2.3.11. Let P and Q be posets and let $h\colon P \to Q$ be a map. We say that h is a $\wedge$-*homomorphism* if h preserves all existing finite meets. That is, h is a $\wedge$-homomorphism if and only if for each $a_1, \dots, a_n \in P$ such that $a_1 \wedge \cdots \wedge a_n$ exists in P, then $h(a_1) \wedge \cdots \wedge h(a_n)$ exists in Q and $h(a_1 \wedge \cdots \wedge a_n) = h(a_1) \wedge \cdots \wedge h(a_n)$. Dually, h is called $\vee$-*homomorphism* if h preserves all existing finite joins.

Notice that every $\wedge$-homomorphism ($\vee$-homomorphism) $h\colon P \to Q$ is order-preserving. Because for every $a, b \in P$, if $a \leq b$ then $a = a \wedge b$. So, $h(a) = h(a \wedge b) = h(a) \wedge h(b)$ and, hence $h(a) \leq h(b)$.

LEMMA 2.3.12. *Let P and Q be posets with top element and let $h\colon P \to Q$ be a map. Then, h is a $\wedge$-homomorphism that preserves top element if and only if $h^{-1}[G] \in \mathsf{Fi}_{\mathsf{m}}(P)$ for all $G \in \mathsf{Fi}_{\mathsf{m}}(Q)$.*

PROOF. We denote by $\top_P$ and $\top_Q$ the top elements of P and Q, respectively. Assume h is a $\wedge$-homomorphism such that $h(\top_P) = \top_Q$ and let $G \in \mathsf{Fi}_{\mathsf{m}}(Q)$. As $h(\top_P) = \top_Q \in G$, $\top_P \in h^{-1}[G]$. Since h is order-preserving and G is an up-set of Q, $h^{-1}[G]$ is an up-set of P. Let $a_1, \dots, a_n \in h^{-1}[G]$ be such that $a_1 \wedge \cdots \wedge a_n$ exists in P. Then, $h(a_1 \wedge \cdots \wedge a_n) = h(a_1) \wedge \cdots \wedge h(a_n)$ and $h(a_1), \dots, h(a_n) \in G$. So, since G is a meet-filter, $h(a_1 \wedge \cdots \wedge a_n) = h(a_1) \wedge \cdots \wedge h(a_n) \in G$. Hence, $a_1 \wedge \cdots \wedge a_n \in h^{-1}[G]$. Therefore, $h^{-1}[G] \in \mathsf{Fi}_{\mathsf{m}}(P)$. Conversely, suppose that $h^{-1}[G] \in \mathsf{Fi}_{\mathsf{m}}(P)$ for all $G \in \mathsf{Fi}_{\mathsf{m}}(Q)$. It is clear that $h(\top_P) = \top_Q$, because $\{\top_Q\} \in \mathsf{Fi}_{\mathsf{m}}(Q)$. Let $a_1, \dots, a_n \in P$ be such that $a_1 \wedge \cdots \wedge a_n$ exists in P. We show that h is order-preserving. Let $a, b \in P$ such that $a \leq b$. Since $a \in h^{-1}[{\uparrow} h(a)] \in \mathsf{Fi}_{\mathsf{m}}(P)$, $b \in h^{-1}[{\uparrow} h(a)]$. Then, $h(a) \leq h(b)$. Now, using that h is order-preserving, we have $h(a_1 \wedge \cdots \wedge a_n) \leq h(a_i)$ for all $i \in \{1, \dots, n\}$. Let $y \in Q$ be such that $y \leq h(a_i)$ for all $i \in \{1, \dots, n\}$. So, $h(a_1), \dots, h(a_n) \in {\uparrow} y \in \mathsf{Fi}_{\mathsf{m}}(Q)$. Then,

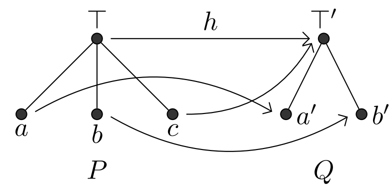


FIGURE 2.9. A $\wedge$-homomorphism that is not an inf-homomorphism.

$a_1, \ldots, a_n \in h^{-1}[\uparrow y] \in \mathsf{Fi_m}(P)$ and this implies $a_1 \wedge \cdots \wedge a_n \in h^{-1}[\uparrow y]$. Hence, $y \leq h(a_1 \wedge \cdots \wedge a_n)$. That is, we proved that $h(a_1 \wedge \cdots \wedge a_n)$ is the meet of $\{h(a_1), \ldots, h(a_n)\}$, i.e., $h(a_1 \wedge \cdots \wedge a_n) = h(a_1) \wedge \cdots \wedge h(a_n)$. Therefore, h is a $\wedge$-homomorphism. □

LEMMA 2.3.13. *Let P and Q be posets with bottom element and let $h\colon P \to Q$ be a map. Then, h is a $\vee$-homomorphism that preserves bottom element if and only if $h^{-1}[J] \in \mathsf{Id_j}(P)$ for all $J \in \mathsf{Id_j}(Q)$.*

Next, we will see the connection between inf-homomorphisms (sup-homomorphisms) and $\wedge$-homomorphisms ($\vee$-homomorphisms) for arbitrary posets and for mo-distributive (jo-distributive) posets.

LEMMA 2.3.14. *Let P and Q be posets and let $h\colon P \to Q$ be a map. If h is an inf-homomorphism (sup-homomorphism), then h is a $\wedge$-homomorphism ($\vee$-homomorphism).*

PROOF. We assume $h\colon P \to Q$ is an inf-homomorphism. Let $a_1, \ldots, a_n \in P$ be such that $a_1 \wedge \cdots \wedge a_n$ exists in P. Since h is order-preserving, we have $h(a_1 \wedge \cdots \wedge a_n) \leq h(a_i)$ for all $i \in \{1, \ldots, n\}$. Let $y \in Q$ such that $y \leq h(a_i)$ for all $i \in \{1, \ldots, n\}$. So, $y \in \{h(a_1), \ldots, h(a_n)\}^{\mathrm{l}}$. Since $a_1 \wedge \cdots \wedge a_n \in \{a_1, \ldots, a_n\}^{\mathrm{lu}}$ and h is an inf-homomorphism, $h(a_1 \wedge \cdots \wedge a_n) \in \{h(a_1), \ldots, h(a_n)\}^{\mathrm{lu}}$, whereupon $y \leq h(a_1 \wedge \cdots \wedge a_n)$. Hence, we have shown that $h(a_1 \wedge \cdots \wedge a_n)$ is the greatest lower bound of $\{h(a_1), \ldots, h(a_n)\}$, i.e., $h(a_1 \wedge \cdots \wedge a_n) = h(a_1) \wedge \cdots \wedge h(a_n)$. Therefore, h is a $\wedge$-homomorphism. □

EXAMPLE 2.3.15. In this example we show that the converse of the statement of the previous lemma does not holds. Consider the posets P and Q depicted in Figure 2.9 and the map $h\colon P \to Q$ as is also shown in Figure 2.9. Notice that P is non-mo-distributive poset and Q is a mo-distributive poset and it is also clear that h preserves top element. The map h is not an inf-homomorphism because $a \in \{b, c\}^{\mathrm{lu}}$ and $h(a) \notin \{h(b), h(c)\}^{\mathrm{lu}}$. It is straightforward check directly that h preserves all existing finite meets and hence h is a $\wedge$-homomorphism.

Lemma 2.3.16. *Let P be a mo-distributive poset with top element and let Q be an arbitrary poset with top element. Let $h\colon P \to Q$ be a map. Then, h is an inf-homomorphism if and only if h is a $\wedge$-homomorphism preserving top element.*

Proof. The implication from left to right is by the previous lemma and by Lemma 2.3.3. For the reverse implication, assume that h is a $\wedge$-homomorphism preserving top. Let $A \subseteq_\omega P$ and $b \in A^{\mathrm{lu}}$. If $A = \emptyset$, then $b = \top_P$. Then, $h(b) = h(\top_P) = \top_Q \in h[A]^{\mathrm{lu}}$. Now, suppose $A \neq \emptyset$ and let $A = \{a_1, \ldots, a_n\}$. So, $b \in \{a_1, \ldots, a_n\}^{\mathrm{lu}}$. From the mo-distributive condition for P, there exist $b_1, \ldots, b_k \in {\uparrow}a_1 \cup \cdots \cup {\uparrow}a_n$ such that $b = b_1 \wedge \cdots \wedge b_k$. Then, by hypothesis, we have that $h(b) = h(b_1) \wedge \cdots \wedge h(b_k)$. Since h is order-preserving, we obtain $h(b_1), \ldots, h(b_k) \in {\uparrow}h(a_1) \cup \cdots \cup {\uparrow}h(a_n)$. Let $y \in \{h(a_1), \ldots, h(a_n)\}^{\mathrm{l}}$. So, $y \leq h(a_i)$ for all $i \in \{1, \ldots, n\}$ and, whereupon $y \leq h(b_j)$ for all $j \in \{1, \ldots, k\}$. Thus, $y \leq h(b)$. Hence, $h(b) \in \{h(a_1), \ldots, h(a_n)\}^{\mathrm{lu}}$ and, therefore h is an inf-homomorphism. □

Lemma 2.3.17. *Let P be a jo-distributive poset with bottom element and let Q be an arbitrary poset with bottom element. Let $h\colon P \to Q$ be a map. Then, h is a sup-homomorphism if and only if h is a $\vee$-homomorphism preserving bottom.*

Lemma 2.3.18. *Let P and Q be bounded posets and let $h\colon P \to Q$ be an inf-sup-homomorphism. Then $h^{-1}[G] \in \mathsf{Opt}(P)$ for all $G \in \mathsf{Opt}(Q)$.*

Proof. Let G be an optimal Frink-filter of Q. By Lemma 2.3.6, $h^{-1}[G]$ is a Frink-filter of P. Since $\left(h^{-1}[G]\right)^c = h^{-1}[G^c]$ and using again Lemma 2.3.7, it follows that $\left(h^{-1}[G]\right)^c$ is a Frink-ideal of P. Moreover, since $G \in \mathsf{Opt}(Q)$, by Lemma 2.3.3 we have that $h^{-1}[G]$ is non-empty and proper. Hence, $h^{-1}[G]$ is an optimal Frink-filter of P. □

Let $h\colon P \to Q$ be a map from a poset P to a poset Q and consider the map $h^{-1}\colon \mathcal{P}(Q) \to \mathcal{P}(P)$. We collect in Table 2.1 all the relations between the different kind of homomorphisms of posets and the different notions of filters and ideals on posets that we established in this section.

2.4. The distributive meet-semilattice envelope

In order to make the concepts and results that we expound in this part of the dissertation with respect to the conditions of mo-distributivity and jo-distributivity to be clear to understand and in order to avoid any confusion, we restrict our attention to the mo-distributive case and we leave the dual results for the jo-distributive posets to the reader; they can be obtained directly using the Duality Principle (see Lemma 1.2.3).

In this section we show that every mo-distributive poset can be extended to a distributive meet-semilattice enjoying a universal property. Apart from the intrinsic

$h\colon P \to Q$		$h^{-1}\colon \mathcal{P}(Q) \to \mathcal{P}(P)$
inf-homomorphism	$\equiv$	$h^{-1}[\mathsf{Fi_F}(Q)] \subseteq \mathsf{Fi_F}(P)$
sup-homomorphism	$\equiv$	$h^{-1}[\mathsf{Id_F}(Q)] \subseteq \mathsf{Id_F}(P)$
$\wedge$-homomorphism preserving top	$\equiv$	$h^{-1}[\mathsf{Fi_m}(Q)] \subseteq \mathsf{Fi_m}(P)$
$\vee$-homomorphism preserving bottom	$\equiv$	$h^{-1}[\mathsf{Id_j}(Q)] \subseteq \mathsf{Id_j}(P)$
inf-sup-homomorphism, with P and Q bounded	$\Rightarrow$	$h^{-1}[\mathsf{Opt}(Q)] \subseteq \mathsf{Opt}(P)$

TABLE 2.1. Relations between homomorphisms, filters and ideals on posets.

interest of having a distributive meet-semilattice extension of a mo-distributive poset, a consequence of having these extensions is that the category of distributive meet-semilattices with top element and meet-homomorphisms preserving top is a reflective subcategory of the category of mo-distributive posets with top element and inf-homomorphisms.

2.4.1. Existence and uniqueness.

DEFINITION 2.4.1. Let P be a poset and let $\langle M, \wedge\rangle$ be a distributive meet-semilattice. We say that M is a *distributive meet-semilattice envelope of P* if there is a map $e\colon P \to M$ such that:

(DE1) $e[P]$ is finitely meet-dense on M (that is, for every $x \in M$ there is a non-empty $A \subseteq_\omega P$ such that $x = \bigwedge e[A]$);
(DE2) e is an inf-sup-embedding;

We also say that the pair $\langle M, e\rangle$ is a distributive meet-semilattice envelope of P, if M is a distributive meet-semilattice and e is a map satisfying Conditions (DE1)-(DE2). Firstly, we show that if there exists the distributive meet-semilattice envelope of a poset, then the poset is mo-distributive.

LEMMA 2.4.2. *Let P be a poset. If there exists a distributive meet-semilattice envelope $\langle M, e\rangle$ of P, then P is mo-distributive.*

PROOF. Let $a, a_1, \ldots, a_n \in P$ be such that $a \in \{a_1, \ldots, a_n\}^{\mathrm{lu}}$. Since e is an inf-homomorphism, it follows that $e(a) \in {\uparrow}(e(a_1) \wedge \cdots \wedge e(a_n))$. So $e(a_1) \wedge \cdots \wedge e(a_n) \leq e(a)$. Thus, because M is a distributive meet-semilattice, we have that there exist $x_1, \ldots, x_n \in M$ such that $e(a) = x_1 \wedge \cdots \wedge x_n$ and $e(a_i) \leq x_i$ for all $i = 1, \ldots, n$. Now, by Condition (DE1) of Definition 2.4.1 we have that for each

$i = 1, \ldots, n$ there exists a non-empty $A_i \subseteq_\omega P$ such that $x_i = \bigwedge e[A_i]$. Then $e(a) = \bigwedge e[A_1] \wedge \cdots \wedge \bigwedge e[A_n] = \bigwedge e[\bigcup_{i=1}^{n} A_i]$. We thus obtain that $e(a) \in {\uparrow}(\bigwedge e[\bigcup_{i=1}^{n} A_i])$ and then, since e is an inf-embedding, it follows that $a \in (\bigcup_{i=1}^{n} A_i)^{\mathrm{lu}}$. We also have that $e(a) \leq \bigwedge e[\bigcup_{i=1}^{n} A_i] \leq e(b)$ for all $b \in \bigcup_{i=1}^{n} A_i$. As e is an order-embedding, $a \leq b$ for all $b \in \bigcup_{i=1}^{n} A_i$ and then $a \in (\bigcup_{i=1}^{n} A_i)^{\mathrm{l}}$. So, we have obtained that $a \in (\bigcup_{i=1}^{n} A_i)^{\mathrm{lu}}$ and $a \in (\bigcup_{i=1}^{n} A_i)^{\mathrm{l}}$. Then, by Lemma 2.2.9, we obtain that $a = \bigwedge(\bigcup_{i=1}^{n} A_i)$. Moreover, for each $i = 1, \ldots, n$ we have $e(a_i) \leq x_i = \bigwedge e[A_i] \leq e(b)$ for all $b \in A_i$ and thus for each $i = 1, \ldots, n$, $a_i \leq b$ for all $b \in A_i$. Therefore, P is mo-distributive. $\square$

Now we prove a very nice property of the distributive meet-semilattice envelope of a poset.

Lemma 2.4.3. *Let P be a poset and let $\langle M, e\rangle$ be a distributive meet-semilattice envelope of P. If $\langle M', \wedge\rangle$ is a distributive meet-semilattice and $f\colon P \to M'$ is an inf-sup-embedding, then there is a unique meet-embedding $h\colon M \to M'$ such that $h \circ e = f$ (see Figure 2.10).*

Proof. Since $\langle M, e\rangle$ is a distributive meet-semilattice envelope of P, by (DE1) we have that for each $x \in M$ there is a non-empty $A \subseteq_\omega P$ such that $x = \bigwedge e[A]$. So, we define $h\colon M \to M'$ as follows: for every $x \in M$,

$$h(x) = \bigwedge f[A]$$

where $x = \bigwedge e[A]$ for some non-empty $A \subseteq_\omega P$. First we show that h is well-defined. Let $A, B \subseteq_\omega P$ be non-empty and suppose that $\bigwedge e[A] = \bigwedge e[B]$. So $\bigwedge e[A] \leq e(b)$ for all $b \in B$ and then $e(b) \in {\uparrow}(\bigwedge e[A])$ for all $b \in B$. Since e is an inf-embedding, it follows that $b \in A^{\mathrm{lu}}$ for all $b \in B$. Since f is an inf-homomorphism, we obtain that $f(b) \in {\uparrow}(\bigwedge f[A])$ for all $b \in B$. Then $\bigwedge f[A] \leq \bigwedge f[B]$. Similarly, we can obtain that $\bigwedge f[B] \leq \bigwedge f[A]$ and thus $\bigwedge f[A] = \bigwedge f[B]$. Hence h is well-defined. With a similar argument to above we can prove that h is injective. It is straightforward to prove directly that h is a meet-homomorphism and $h \circ e = f$. Now we show that h is unique. Suppose that $g\colon M \to M'$ is a meet-embedding such that $g \circ e = f$. Let $x \in M$. Then, there is a non-empty $A \subseteq_\omega P$ such that $x = \bigwedge e[A]$. Thus, we have

$$\begin{aligned} h(x) &= \bigwedge f[A] \\ &= \bigwedge g[e[A]] \\ &= g(\bigwedge e[A]) \\ &= g(x). \end{aligned}$$

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "e"'] & M' \\ M \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

FIGURE 2.10. An universal property of the distributive meet-semilattice envelope

Hence, $h = g$. This finishes the proof. $\square$

Recall that an inf-sup-embedding $h\colon P \to Q$ between posets preserves all finite existing meets and joins, see Lemma 2.3.14. Next, we show that the distributive meet-semilattice envelope of a poset P, if it exists, is unique up to isomorphism.

LEMMA 2.4.4. *Let P be a poset. If $\langle M, e\rangle$ and $\langle M', e'\rangle$ are distributive meet-semilattice envelopes of P, then M and M' are isomorphic.*

PROOF. Let $\langle M, e\rangle$ and $\langle M', e'\rangle$ be distributive meet-semilattice envelopes of P. Since $e'\colon P \to M'$ and $e\colon P \to M$ are inf-sup-embeddings, by Lemma 2.4.3 we obtain that there exist meet-embeddings $h_1\colon M \to M'$ and $h_2\colon M' \to M$ such that $h_1 \circ e = e'$ and $h_2 \circ e' = e$. Let $x \in M$. By (DE1) for $\langle M, e\rangle$, we have that $x = e(a_1) \wedge \cdots \wedge e(a_n)$ for some $a_1, \ldots, a_n \in P$. Then,

$$\begin{aligned} (h_2 \circ h_1)(x) &= h_2\left(h_1(e(a_1)) \wedge \cdots \wedge h_1(e(a_n))\right) \\ &= h_2\left(e'(a_1) \wedge \cdots \wedge e'(a_n)\right) \\ &= h_2(e'(a_1)) \wedge \cdots \wedge h_2(e'(a_n)) \\ &= e(a_1) \wedge \cdots \wedge e(a_n) \\ &= x. \end{aligned}$$

Similarly, we can show that if $y \in M'$, then $(h_1 \circ h_2)(y) = y$. Therefore, $h_1 : M \cong M' : h_2$. $\square$

We consider the meet-semilattice $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \wedge_d\rangle$ as the dual of the join-semilattice $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \vee\rangle$. That is, for all $F_1, F_2 \in \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$,

$$F_1 \wedge_d F_2 = F_1 \vee F_2.$$

We also have the dual order of $\subseteq$ associated to $\wedge_d$ on $\mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$, given by

$$F_1 \leq_d F_2 \iff F_2 \subseteq F_1.$$

LEMMA 2.4.5. *Let P be a mo-distributive poset. Then, $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \wedge_d\rangle$ is the distributive meet-semilattice envelope of P.*

Proof. Since P is mo-distributive, by Lemma 2.2.20 we have that $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \vee\rangle$ is a distributive join-semilattice. Then the meet-semilattice $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \wedge_d\rangle$ is distributive. We show that Conditions (DE1) and (DE2) in Definition 2.4.1 for the distributive meet-semilattice $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \wedge_d\rangle$ hold.

Let $e\colon P \to \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$ be defined by $e(a) = {\uparrow}a$ for each $a \in P$. Let $F \in \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$. So, $F = \{a_1, \dots, a_n\}^{\mathrm{lu}}$ for some $a_1, \dots, a_n \in P$. Then, we have

$$F = {\uparrow}a_1 \vee \cdots \vee {\uparrow}a_n = e(a_1) \wedge_d \cdots \wedge_d e(a_n).$$

Thus, $e[P]$ is finitely meet-dense on $\mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$ and hence Condition (DE1) holds. Let $a, b \in P$. So,

$$a \leq b \iff {\uparrow}b \subseteq {\uparrow}a \iff {\uparrow}a \leq_d {\uparrow}b \iff e(a) \leq_d e(b).$$

Then, e is an order-embedding. To show that e is an inf-sup-homomorphism, let $A \subseteq_\omega P$ and $b \in P$. First we assume $b \in A^{\mathrm{lu}}$ and we want to prove that $e(b) \in e[A]^{\mathrm{lu}}$. So, let $F \in \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$ be such that $F \leq_d e(a)$ for all $a \in A$. So, $e(a) \subseteq F$ for all $a \in A$; this implies that $A \subseteq F$. Then, since F is a Frink-filter, $b \in F$. Thus $e(b) \subseteq F$, whereupon $F \leq_d e(b)$. Then, $e(b) \in e[A]^{\mathrm{lu}}$ and hence e is an inf-homomorphism. Now, we show e is a sup-homomorphism. So, assume $b \in A^{\mathrm{ul}}$ and let $F \in \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$ be such that $e(a) \leq_d F$ for all $a \in A$. Then, we have $F \subseteq \bigcap_{a\in A} {\uparrow}a \subseteq {\uparrow}b$; that is, $e(b) \leq_d F$. Thus, $e(b) \in e[A]^{\mathrm{ul}}$. This proves that e is a sup-homomorphism. Hence, we have proved that e is an inf-sup-embedding and thus Condition (DE2) holds. Therefore $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \wedge_d\rangle$ is the distributive meet-semilattice envelope of P.

□

Hereinafter we use the following notation. If P is a mo-distributive poset, then we denote by $\langle M(P), e_P\rangle$ or simply by $M(P)$ its distributive meet-semilattice envelope. As usual, we omit the subscript on e_P whenever confusion is unlikely.

Lemma 2.4.6. *Let P be a mo-distributive poset and $\langle M(P), e\rangle$ its distributive meet-semilattice envelope. Then, P has a top element if and only if $M(P)$ has a top element. Moreover, e preserves the top element, if it exists.*

Proof. It is straightforward by the fact that e is an inf-sup-embedding and because $M(P)$ satisfies Condition (DE1). □

Lemma 2.4.7. *Let P be a mo-distributive poset. If P is a join-semilattice, then $\mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$ is a sub-lattice of $\mathsf{Fi}_{\mathsf{F}}(P)$.*

Proof. We know that $\mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$ is closed under finite joins taken in $\mathsf{Fi}_{\mathsf{F}}(P)$. Now, let $F_1 = A^{\mathrm{lu}}$ and $F_2 = B^{\mathrm{lu}}$ with non-empty $A, B \subseteq_\omega P$. Then, since P is mo-distributive, we have

$$F_1 \cap F_2 = A^{\mathrm{lu}} \cap B^{\mathrm{lu}}$$

$$
\begin{aligned}
&= \left(\bigvee_{a\in A} {\uparrow} a\right) \cap \left(\bigvee_{b\in B} {\uparrow} b\right) \\
&= \bigvee_{(a,b)\in A\times B} ({\uparrow} a \cap {\uparrow} b) \\
&= \bigvee_{(a,b)\in A\times B} {\uparrow}(a \vee b) \\
&= \{a \vee b :\ (a,b) \in A \times B\}^{\mathrm{lu}}.
\end{aligned}
$$

As $A \times B$ is finite, then $F_1 \cap F_2 \in \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$. Therefore, $\langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \cap, \vee\rangle$ is a sub-lattice of $\mathsf{Fi}_{\mathsf{F}}(P)$. □

A dual result of the previous lemma, that is, for jo-distributive meet-semilattices[2], can be found in the paper [**37**] due to Hickman. The previous lemma and the uniqueness and existence of the distributive meet-semilattice envelope of a mo-distributive poset allow us to obtain the following corollary, whose proof we omit.

Corollary 2.4.8. *If P is a mo-distributive join-semilattice, then the distributive meet-semilattice envelope of P is a distributive lattice.*

Recall the following two equivalences, that are used in the rest of this subsection and the next. Let P be a mo-distributive poset, $\langle M(P), e\rangle$ its distributive meet-semilattice envelope and let $a, a_1, \dots, a_n \in P$, then

$$
a \in \{a_1, \dots, a_n\}^{\mathrm{lu}} \quad \text{if and only if} \quad e(a_1) \wedge \cdots \wedge e(a_n) \le e(a) \tag{2.11}
$$

and

$$
a \in \{a_1, \dots, a_n\}^{\mathrm{ul}} \quad \text{if and only if} \quad e(a) \in \{e(a_1), \dots, e(a_n)\}^{\mathrm{ul}}. \tag{2.12}
$$

Lemma 2.4.9. *Let P and Q be mo-distributive posets. If $h\colon P \to Q$ is an inf-homomorphism, then there is a unique meet-homomorphism $M(h)\colon M(P) \to M(Q)$ such that $e_Q \circ h = M(h) \circ e_P$. Moreover, if h is an inf-embedding, then $M(h)$ is a meet-embedding.*

Proof. We have, by (DE1), that for every $x \in M(P)$ there are $a_1, \dots, a_n \in P$ such that $x = e_P(a_1) \wedge \cdots \wedge e_P(a_n)$. So, we define $M(h)\colon M(P) \to M(Q)$ for all $x \in M(P)$ as follows:

$$
M(h)(x) = e_Q\left(h(a_1)\right) \wedge \cdots \wedge e_Q\left(h(a_n)\right)
$$

if $x = e_P(a_1) \wedge \cdots \wedge e_P(a_n)$ for some $a_1, \dots, a_n \in P$. First we show that $M(h)$ is well defined. Suppose that $x = e_P(a_1) \wedge \cdots \wedge e_P(a_n) = e_P(b_1) \wedge \cdots \wedge e_P(b_m)$. Notice that this equality implies that $e_P(a_1) \wedge \cdots \wedge e_P(a_n) \le e_P(b_j)$ for all $j \in \{1, \dots, m\}$.

[2]In [**37**] the jo-distributive meet-semilattices are called mildly-distributive meet-semilattices.

So, by (2.11), $b_j \in \{a_1, \dots, a_n\}^{\mathrm{lu}}$ for all $j \in \{1, \dots, m\}$. Using that h is an inf-homomorphism we have $h(b_j) \in \{h(a_1), \dots, h(a_n)\}^{\mathrm{lu}}$ for all $j \in \{1, \dots, m\}$. Then, again by (2.11), we obtain that $e_Q(h(a_1)) \wedge \cdots \wedge e_Q(h(a_n)) \leq e_Q(h(b_j))$ for all $j \in \{1, \dots, m\}$. Thus,

$$e_Q(h(a_1)) \wedge \cdots \wedge e_Q(h(a_n)) \leq e_Q(h(b_1)) \wedge \cdots \wedge e_Q(h(b_m)).$$

In a similar way, we can obtain the inverse inequality of the previous one. Hence,

$$e_Q(h(a_1)) \wedge \cdots \wedge e_Q(h(a_n)) = e_Q(h(b_1)) \wedge \cdots \wedge e_Q(h(b_m))$$

and thus, $M(h)$ is well defined. It is straightforward to show that $M(h)$ is a meet-homomorphism and satisfies $e_Q \circ h = M(h) \circ e_P$. If $k\colon M(P) \to M(Q)$ is a meet-homomorphism such that $e_Q \circ h = k \circ e_P$, then

$$\begin{aligned} M(h)(e_P(a_1) \wedge \cdots \wedge e_P(a_n)) &= e_Q(h(a_1)) \wedge \cdots \wedge e_Q(h(a_n)) \\ &= k(e_P(a_1)) \wedge \cdots \wedge k(e_P(a_n)) \\ &= k(e_P(a_1) \wedge \cdots \wedge e_P(a_n)). \end{aligned}$$

Hence, $k = M(h)$.

Lastly, assume that h is an inf-embedding. Let $x, y \in M(P)$ and suppose $M(h)(x) = M(h)(y)$. By (DE1), $x = e_P(a_1) \wedge \cdots \wedge e_P(a_n)$ and $y = e_P(b_1) \wedge \cdots \wedge e_P(b_m)$ for some $a_1, \dots, a_n, b_1, \dots, b_m \in P$. So, we have

$$e_Q(h(a_1)) \wedge \cdots \wedge e_Q(h(a_n)) = e_Q(h(b_1)) \wedge \cdots \wedge e_Q(h(b_m)),$$

whereupon $e_Q(h(a_1)) \wedge \cdots \wedge e_Q(h(a_n)) \leq e_Q(h(b_j))$ for all $j \in \{1, \dots, m\}$. From (2.11), we obtain $h(b_j) \in \{h(a_1), \dots, h(a_n)\}^{\mathrm{lu}}$ for all $j \in \{1, \dots, m\}$. Since h is an inf-embedding, $b_j \in \{a_1, \dots, a_n\}^{\mathrm{lu}}$ for all $j \in \{1, \dots, m\}$. Again by (2.11), $e_P(a_1) \wedge \cdots \wedge e_P(a_n) \leq e_P(b_j)$ for all $j \in \{1, \dots, m\}$, thus $e_P(a_1) \wedge \cdots \wedge e_P(a_n) \leq e_P(b_1) \wedge \cdots \wedge e_P(b_m)$. By a similar reasoning, we can obtain the inverse inequality. Hence, $x = e_P(a_1) \wedge \cdots \wedge e_P(a_n) = e_P(b_1) \wedge \cdots \wedge e_P(b_m) = y$. Therefore, $M(h)$ is a meet-embedding. □

Remark 2.4.10. It is clear that if P and Q have top elements $\top_P$ and $\top_Q$, respectively, and $h\colon P \to Q$ is an inf-homomorphism, then the meet-homomorphism $M(h)\colon M(P) \to M(Q)$ preserves top element. Indeed, since $e_P(\top_P)$ and $e_Q(\top_Q)$ are, respectively, the top elements of $M(P)$ and $M(Q)$ (see Lemma 2.4.6), we have that $M(h)(e_P(\top_P)) = e_Q(h(\top_P)) = e_Q(\top_Q)$.

Lemma 2.4.11. *Let P, Q and R be mo-distributive posets and let $h\colon P \to Q$ and $j\colon Q \to R$ be inf-homomorphisms. Then, $M(j \circ h) = M(j) \circ M(h)$. Moreover, if $\mathrm{id}_P\colon P \to P$ is the identity map, then $M(\mathrm{id}_P) = \mathrm{id}_{M(P)}$.*

Proof. We know that the composition $j \circ h \colon P \to R$ is an inf-homomorphism. Then by Lemma 2.4.9, $M(j\circ h) \colon M(P) \to M(R)$ is the unique meet-homomorphism such that $e_R \circ (j \circ h) = M(j \circ h) \circ e_P$. By Lemma 2.4.9 again we have that $M(h) \colon M(P) \to (Q)$ and $M(j) \colon M(Q) \to M(R)$ are meet-homomorphisms such that $e_Q \circ h = M(h) \circ e_P$ and $e_R \circ j = M(j) \circ e_Q$. Then, we have

$$\begin{aligned} e_R \circ (j \circ h) &= (e_R \circ j) \circ h \\ &= (M(j) \circ e_Q) \circ h \\ &= M(j) \circ (e_Q \circ h) \\ &= M(j) \circ (M(h) \circ e_P) \\ &= (M(j) \circ M(h)) \circ e_P. \end{aligned}$$

Hence $M(j \circ h) = M(j) \circ M(h)$. Moreover, since $e_P \circ \mathrm{id}_P = \mathrm{id}_{M(P)} \circ e_P$, it follows that $\mathrm{id}_{M(P)} = M(\mathrm{id}_P)$. $\square$

Let us denote by $\mathbb{MODP}$ the category formed by all mo-distributive posets and all inf-homomorphisms between mo-distributive posets. It should be clear that the composition between morphisms in this category is the usual set-theoretical composition of functions and the identity morphism for an object of $\mathbb{MODP}$ is the identity map. Let $\mathbb{MODP}^{\top}$ be the full subcategory of $\mathbb{MODP}$ of all mo-distributive posets with top element. We consider also the category of all distributive meet-semilattices and all meet-homomorphisms. We denote this category by $\mathbb{DMSL}$. Let $\mathbb{DMSL}^{\top}$ be the subcategory of $\mathbb{DMSL}$ of all distributive meet-semilattices with top element and all meet-homomorphisms that preserve top element. Therefore, by Lemmas 2.4.9 and 2.4.11, the map $M(-)$ sending every mo-distributive poset P to its distributive meet-semilattice envelope $M(P)$ extends to a functor $\mathbf{M} \colon \mathbb{MODP} \to \mathbb{DMSL}$ from the category $\mathbb{MODP}$ to the category $\mathbb{DMSL}$. Moreover, by Lemma 2.4.6 and Remark 2.4.10, we have that the functor $\mathbf{M}$ restricts to a functor from the category $\mathbb{MODP}^{\top}$ to the category $\mathbb{DMSL}^{\top}$.

It is clear that if M is a distributive meet-semilattice, then the distributive meet-semilattice envelope of M is, up to isomorphism, M. Thus, we have an immediate consequence of Lemma 2.4.9.

Corollary 2.4.12. *Let P be a mo-distributive poset and let L be a distributive meet-semilattice. If $h \colon P \to L$ is an inf-homomorphism, then there exists a unique meet-homomorphism $M(h) \colon M(P) \to L$ such that $h = M(h) \circ e_P$. Moreover, if h is an inf-embedding, then $M(h)$ is a meet-embedding.*

Recall that every meet-semilattice M can be considered as a poset such that the meet exists for every pair of elements of M. Then by Corollary 2.2.13 and Lemma 2.3.5, we can consider to the category $\mathbb{DMSL}^{\top}$ as a full subcategory of $\mathbb{MODP}^{\top}$ and thus we can define the inclusion functor $\mathbf{U}\colon \mathbb{DMSL}^{\top} \to \mathbb{MODP}^{\top}$. Then, Corollary 2.4.12 implies that the functor $\mathbf{M}\colon \mathbb{MODP}^{\top} \to \mathbb{DMSL}^{\top}$ is left adjoint to $\mathbf{U}$ and therefore the category $\mathbb{DMSL}^{\top}$ is a reflective subcategory of the category $\mathbb{MODP}^{\top}$.

The following is an important property of the distributive meet-semilattice envelope of a mo-distributive poset that we will use in the next subsection.

LEMMA 2.4.13. *Let P be a mo-distributive poset and $\langle M, e\rangle$ its distributive meet-semilattice envelope. Then, for every $A, B \subseteq_\omega P$, we have*

$$A^{\mathrm{u}} \subseteq B^{\mathrm{lu}} \quad \textit{if and only if} \quad e[A]^{\mathrm{u}} \subseteq e[B]^{\mathrm{lu}}.$$

PROOF. Let $A, B \subseteq_\omega P$. We assume first that $A^{\mathrm{u}} \subseteq B^{\mathrm{lu}}$. Let $m \in e[A]^{\mathrm{u}}$. So, $e(a) \leq m$ for all $a \in A$. By (DE1), there are $a_1, \dots, a_k \in P$ such that $m = e(a_1) \wedge \dots \wedge e(a_k)$. Then, for every $a \in A$, we have $e(a) \leq e(a_1) \wedge \dots \wedge e(a_k) \leq e(a_i)$ for all $i \in \{1, \dots, k\}$. Since e is an inf-sup-embedding, for every $i \in \{1, \dots, k\}$, $a \leq a_i$ for all $a \in A$. Thus $a_1, \dots, a_k \in A^{\mathrm{u}}$, whereupon $a_1, \dots, a_k \in B^{\mathrm{lu}}$. Then, $e(a_1), \dots, e(a_k) \in e[B]^{\mathrm{lu}}$. Hence, $m = e(a_1) \wedge \dots \wedge e(a_k) \in e[B]^{\mathrm{lu}}$. Therefore, $e[A]^{\mathrm{u}} \subseteq e[B]^{\mathrm{lu}}$.

Now, we assume that $e[A]^{\mathrm{u}} \subseteq e[B]^{\mathrm{lu}}$. Let $x \in A^{\mathrm{u}}$ and let $y \in B^{\mathrm{l}}$. So, $a \leq x$ for all $a \in A$ and $y \leq b$ for all $b \in B$. Then, $e(a) \leq e(x)$ for all $a \in A$ and $e(y) \leq e(b)$ for all $b \in B$. That is, $e(x) \in e[A]^{\mathrm{u}}$ and $e(y) \in e[B]^{\mathrm{l}}$. By hypothesis, we obtain $e(y) \leq e(x)$, whereupon $y \leq x$. Hence, $x \in B^{\mathrm{lu}}$ and therefore $A^{\mathrm{u}} \subseteq B^{\mathrm{lu}}$. $\square$

2.4.2. The relation between the Frink-filters (Frink-ideals) of a mo-distributive poset and the filters (Frink-ideals) of its distributive meet-semilattice envelope. Without loss of generality, we can to establish the following convention. This allows us to make the exposition about the relation between a mo-distributive poset P and its distributive meet-semilattice envelope $M(P)$ more clear.

Let P be a mo-distributive poset. Then, the distributive meet-semilattice envelope of P is up to isomorphism the unique distributive meet-semilattice $M(P)$ such that $P \subseteq M(P)$ and the following conditions are satisfied:

- (E1) P is a sub-poset of $M(P)$;
- (E2) P is finitely meet-dense in $M(P)$;
- (E3) for every $A \subseteq_\omega P$ and $a \in P$,
  - (E3.1) $a \in A^{\mathrm{lu}}$ in P if and only if $a \in A^{\mathrm{lu}}$ en $M(P)$;
  - (E3.2) $a \in A^{\mathrm{ul}}$ in P if and only if $a \in A^{\mathrm{ul}}$ in $M(P)$;

For instance, under this consideration Lemma 2.4.13 is expressed as follows: for every $A, B \subseteq_\omega P$, we have

$$A^{\mathrm{u}} \subseteq B^{\mathrm{lu}} \text{ in } P \quad \text{if and only if} \quad A^{\mathrm{u}} \subseteq B^{\mathrm{lu}} \text{ in } M(P). \tag{2.13}$$

Since $M(P)$ is a meet-semilattice, it should be remembered that $X^{\mathrm{lu}} = {\uparrow}(\bigwedge X)$ for every non-empty $X \subseteq_\omega M(P)$. Then, for a non-empty $A \subseteq_\omega P$, Condition (E3.1) becomes in: $a \in A^{\mathrm{lu}}$ in P if and only if $a \in {\uparrow}(\bigwedge A)$.

Let P be a mo-distributive poset and $M(P)$ its distributive meet-semilattice envelope. We recall that for a subset $X \subseteq P$, $\mathrm{Fi}_{\mathrm{F}}(X)$ denotes the Frink-filter of P generated by X. If $M(P)$ has a top element, then $\mathrm{Fi}_{M(P)}(.)$ denotes the closure operator associated with the closure system $\mathsf{Fi}(M(P))$ of the filters of $M(P)$, and if $M(P)$ has no top element then $\mathrm{Fi}_{M(P)}(.)$ denotes the closure operator associated with the closure system $\mathsf{Fi}(M(P)) \cup \{\emptyset\}$ (see Lemma 1.4.5).

LEMMA 2.4.14. *Let P be a mo-distributive poset and $M(P)$ its distributive meet-semilattice envelope. If F is a Frink-filter of P, then* $\mathrm{Fi}_{M(P)}(F) \cap P = F$.

PROOF. Let F be a Frink-filter of P. If F is empty, then P has no top. So, by Lemma 2.4.6, we have $\mathrm{Fi}_{M(P)}(F) = \mathrm{Fi}_{M(P)}(\emptyset) = \emptyset = F$. We assume that F is non-empty. Since $F \subseteq \mathrm{Fi}_{M(P)}(F)$, we obtain $F = F \cap P \subseteq \mathrm{Fi}_{M(P)}(F) \cap P$. Now, let $a \in \mathrm{Fi}_{M(P)}(F) \cap P$. So, there are $a_1, \ldots, a_n \in F$ such that $a_1 \wedge \cdots \wedge a_n \leq a$. Then, by (E3.1), $a \in \{a_1, \ldots, a_n\}^{\mathrm{lu}}$ and, since F is a Frink-filter of P, it follows that $a \in F$. Therefore, $\mathrm{Fi}_{M(P)}(F) \cap P \subseteq F$. This completes the proof. $\square$

LEMMA 2.4.15. *Let P be a mo-distributive poset and $M(P)$ its distributive meet-semilattice envelope. If G is a filter of $M(P)$, then $G \cap P$ is a Frink-filter of P and $G = \mathrm{Fi}_{M(P)}(G \cap P)$.*

PROOF. Let G be a filter of $M(P)$. By Condition (E3.1), it is straightforward to show that $G \cap P$ is a Frink-filter of P. Now, it is also clear that $\mathrm{Fi}_{M(P)}(G \cap P) \subseteq G$. Let $x \in G$. By (E2), $x = a_1 \wedge \cdots \wedge a_n$ for some $a_1, \ldots, a_n \in P$. Since G is an up-set, $a_i \in G$ for all $i \in \{1, \ldots, n\}$. Then, $a_i \in G \cap P$ for all $i \in \{1, \ldots, n\}$, which implies $a_i \in \mathrm{Fi}_{M(P)}(G \cap P)$ for all $i \in \{1, \ldots, n\}$. Hence, we have $x = a_1 \wedge \cdots \wedge a_n \in \mathrm{Fi}_{M(P)}(G \cap P)$. Therefore, $G = \mathrm{Fi}_{M(P)}(G \cap P)$. $\square$

From the two previous lemmas, we obtain the first result with regard to the connection between the Frink-filters of a mo-distributive poset and the filters of its distributive meet-semilattice envelope. Before, we consider the following convention. Let M be an arbitrary meet-semilattice. Then,

$$\mathsf{Fi}(M)^* := \begin{cases} \mathsf{Fi}(M) & \text{if } M \text{ has top element} \\ \mathsf{Fi}(M) \cup \{\emptyset\} & \text{if } M \text{ has no top element} \end{cases}$$

Hence, notice that if P is a poset and $M(P)$ its distributive meet-semilattice envelope, then

$$\mathsf{Fi}(M(P))^* := \begin{cases} \mathsf{Fi}(M(P)) & \text{if } P \text{ has top element} \\ \mathsf{Fi}(M(P)) \cup \{\emptyset\} & \text{if } P \text{ has no top element.} \end{cases}$$

Now we can consider the maps $\alpha\colon \mathsf{Fi}_{\mathsf{F}}(P) \to \mathsf{Fi}(M(P))^*$ and $\beta\colon \mathsf{Fi}(M(P))^* \to \mathsf{Fi}_{\mathsf{F}}(P)$ defined as follows:

$$\alpha(F) = \mathrm{Fi}_{M(P)}(F) \quad \text{and} \quad \beta(G) = G \cap P$$

for every $F \in \mathsf{Fi}_{\mathsf{F}}(P)$ and for every $G \in \mathsf{Fi}(M(P))^*$, respectively.

Theorem 2.4.16. *Let P be a mo-distributive poset and $M(P)$ its distributive meet-semilattice envelope. Then, the map $\alpha\colon \mathsf{Fi}_{\mathsf{F}}(P) \to \mathsf{Fi}(M(P))^*$ establishes a lattice isomorphism from the lattice of Frink-filters of P onto the lattice of filters of $M(P)$, whose inverse is the map $\beta\colon \mathsf{Fi}(M(P))^* \to \mathsf{Fi}_{\mathsf{F}}(P)$.*

Proof. Let $F_1, F_2 \in \mathsf{Fi}_{\mathsf{F}}(P)$. Then, by Lemma 2.4.14, we have

$$F_1 \subseteq F_2 \iff \mathrm{Fi}_{M(P)}(F_1) \subseteq \mathrm{Fi}_{M(P)}(F_2) \iff \alpha(F_1) \subseteq \alpha(F_2).$$

Thus, we see that α is an order-embedding. By Lemma 2.4.15, it is clear that α is an onto map. Hence, α is an order-isomorphism and therefore a lattice isomorphism. Moreover, from Lemmas 2.4.14 and 2.4.15, we obtain β is the inverse map of α. $\square$

Given that any isomorphism between lattices sends meet-prime elements to meet-prime elements, we can conclude from the previous theorem that α and β restrict to order-isomorphisms between the prime Frink-filters of P and the prime elements of $\mathsf{Fi}(M(P))^*$. That is,

$$\alpha : \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P) \cong \mathsf{Fi}^{\mathsf{pr}}(M(P))^* : \beta,$$

where $\mathsf{Fi}^{\mathsf{pr}}(M(P))^*$ denotes the collection of all prime elements of $\mathsf{Fi}(M(P))^*$. It should be noted that $\mathsf{Fi}^{\mathsf{pr}}(M(P)) \subseteq \mathsf{Fi}^{\mathsf{pr}}(M(P))^*$ and if $H \in \mathsf{Fi}^{\mathsf{pr}}(M(P))^*$ is non-empty, then $H \in \mathsf{Fi}^{\mathsf{pr}}(M(P))$. It follows that $U \subseteq P$ is a non-empty prime Frink-filter of P if and only if there is a prime filter H of $M(P)$ such that $U = H \cap P$.

Next, we want to investigate what kind of filters on the distributive meet-semilattice envelop of a mo-distributive poset P correspond to the optimal Frink-filters of P. First we study the relation between the Frink-ideals of P and the Frink-ideals of its distributive meet-semilattice envelop.

Lemma 2.4.17. *Let P be a mo-distributive poset and $M(P)$ its distributive meet-semilattice envelope. If I is a Frink-ideal of P, then the Frink-ideal $\mathrm{Id}_{\mathrm{F}}(I)$ of $M(P)$ generated by I is such that $I = \mathrm{Id}_{\mathrm{F}}(I) \cap P$.*

Proof. It is similar in spirit to the proof of Lemma 2.4.14 and so, we leave the details to the reader. $\square$

This Lemma implies that the maps

$$\eta\colon \mathsf{Id}_{\mathsf{F}}(M(P)) \to \mathsf{Id}_{\mathsf{F}}(P) \quad \text{and} \quad \mu\colon \mathsf{Id}_{\mathsf{F}}(P) \to \mathsf{Id}_{\mathsf{F}}(M(P))$$

given by

$$\eta(J) = J \cap P \quad \text{and} \quad \mu(I) = \mathrm{Id_F}\,(I)$$

for every $J \in \mathsf{Id_F}(M(P))$ and $I \in \mathsf{Id_F}(P)$, respectively, are such that

$$\eta \circ \mu = \mathrm{id}_{\mathsf{Id_F}(P)}.$$

But, unlike the statement in Theorem 2.4.16, the lattices of the Frink-ideals of P and of the Frink-ideals of $M(P)$ are not necessarily order-isomorphic. In fact, if the mo-distributive poset P is not jo-distributive, we can claim that $\mathsf{Id_F}(P)$ is not isomorphic to $\mathsf{Id_F}(M(P))$. This is a consequence of the following fact. Since $M(P)$ is a distributive meet-semilattice, it follows that $M(P)$ is jo-distributive (a proof of this statement can be found in [**37**], see Theorem 3.5). Then, $\mathsf{Id_F}(M(P))$ is a distributive lattice. If $\mathsf{Id_F}(M(P)) \cong \mathsf{Id_F}(P)$, then the lattice $\mathsf{Id_F}(P)$ is distributive and hence P is jo-distributive, which is a contradiction if we started from a non-jo-distributive mo-distributive poset P. The following example shows that the lattice of the Frink-ideals of a mo-distributive poset is not necessarily isomorphic to the lattice of the Frink-ideals of its distributive meet-semilattice envelope even if the poset is jo-distributive.

Example 2.4.18. In Figure 2.11 we show (on the left) a mo-distributive and jo-distributive poset P and its distributive meet-semilattice envelope (on the right) $M(P) = \langle \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P), \wedge_d \rangle$. Then we construct the lattice of all Frink-ideals of P and the lattice of all Frink-ideals of the distributive meet-semilattice envelope $M(P)$ of P, and they are depicted in Figure 2.12, where $I = \bigcup\limits_{i \geq 0} {\downarrow} e(a_i)$. Hence we can see, from Figure 2.12, that $\mathsf{Id_F}(P)$ and $\mathsf{Id_F}(M(P))$ are not isomorphic.

Now, we need to introduce a stronger notion that of Frink-ideal in a mo-distributive poset, that will be useful to define another kind of "prime" Frink-filters; they will correspond to the optimal filters of the distributive meet-semilattice envelope of the poset. The stronger notion that of Frink-ideal that we use here is a generalization of the definition given by Celani and Jansana in [**10**]. A similar generalization was also given by Esteban in her PhD Tesis [**23**] in the context of Abstract Algebraic Logic.

Definition 2.4.19. Let P be a poset and let $I \subseteq P$ be non-empty. We say that I is a *strong Frink-ideal* if I is a down-set and for every non-empty $X \subseteq_\omega I$ and every non-empty $Y \subseteq_\omega P$,

$$X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}} \quad \text{implies} \quad Y^{\mathrm{lu}} \cap I \neq \emptyset.$$

Let us denote by $\mathsf{Id_{sF}}(P)$ the family of all strong Frink-ideals of P.

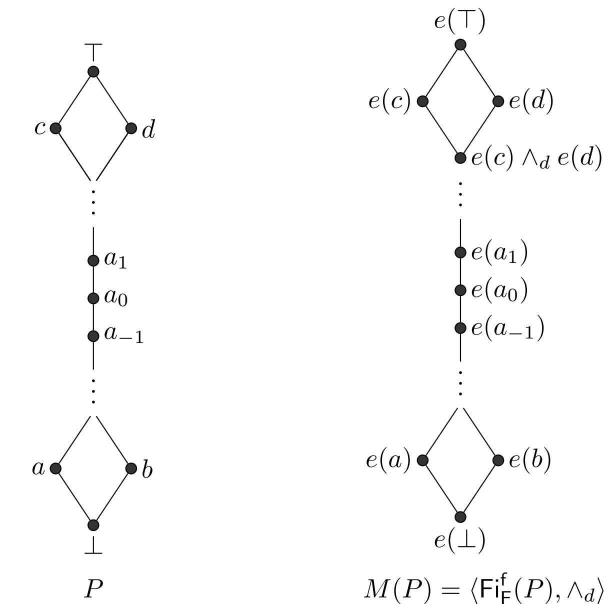


FIGURE 2.11. A mo-jo-distributive poset and its distributive meet-semilattice envelope.

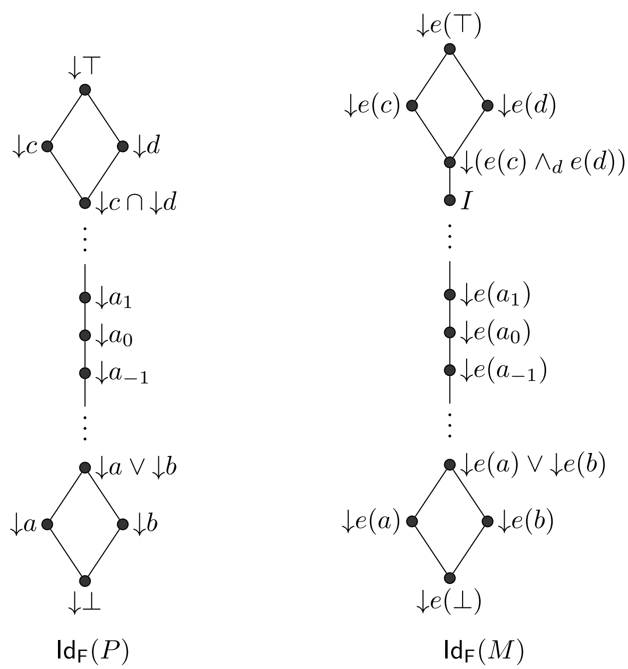


FIGURE 2.12. The lattices of the Frink-ideals, respectively, of the poset P and its distributive meet-semilattice envelope M depicted in Figure 2.11.

It should be noted that P is a strong Frink-ideal of itself and moreover that for every $a \in P$, $\downarrow a$ is a strong Frink-ideal of P. The following lemma shows that, in fact, the notion of strong Frink-ideal is stronger than Frink-ideal and it is weaker than the notion of order-ideal. Moreover, these facts are easy to prove and so we omit their proof.

LEMMA 2.4.20. *Let P be a poset. Then,* $\mathsf{Id}_{\mathsf{or}}(P) \subseteq \mathsf{Id}_{\mathsf{sF}}(P) \subseteq \mathsf{Id}_{\mathsf{F}}(P)$.

The inclusions given in the previous lemma are not necessarily equalities. For the first inclusion, consider the poset given in Figure 2.3. Then, $I = \{\perp, a, b\}$ is a strong Frink-ideal but it is not an order-ideal of the poset. For the second inclusion, consider the poset P given in Figure 2.6. Take $I = \{\perp, a, c\}$. It is clear that I is a Frink-ideal of P. We show I is not a strong Frink-ideal of P. Let $X = \{a, c\} \subseteq_\omega I$ and let $Y = \{b, d\} \subseteq_\omega P$. Thus, $X^{\mathrm{u}} = \{b, d, \top\}$ and $Y^{\mathrm{lu}} = \{b, d, \top\}$ whereupon, $X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}}$. But, $Y^{\mathrm{lu}} \cap I = \emptyset$. Hence, I is not a strong Frink-ideal.

LEMMA 2.4.21. *Let P be a poset. Then,* $\mathsf{Id}_{\mathsf{F}}^{\mathsf{pr}}(P) \subseteq \mathsf{Id}_{\mathsf{sF}}(P)$.

PROOF. Let I be a prime Frink-ideal of P. Let $X \subseteq_\omega I$ and $Y \subseteq_\omega P$ be non-empty sets such that $X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}}$. We suppose that $Y^{\mathrm{lu}} \cap I = \emptyset$. So, $Y \subseteq Y^{\mathrm{lu}} \subseteq I^c$. Since I is a prime Frink-ideal, I^c is an order-filter of P. Then, there exists $b \in I^c$ such that $b \in Y^{\mathrm{l}}$. Thus, since $X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}}$, we have $b \in X^{\mathrm{ul}}$. Then, as I is a Frink-ideal, $b \in I$. Which is a contradiction. Hence, $Y^{\mathrm{lu}} \cap I \neq \emptyset$ and therefore I is a strong Frink-ideal. $\square$

The following lemma shows that the notions of Frink-ideal and strong Frink-ideal on a meet-semilattice coincide.

LEMMA 2.4.22. *Let M be a meet-semilattice. Then* $\mathsf{Id}_{\mathsf{F}}(M) = \mathsf{Id}_{\mathsf{sF}}(M)$.

PROOF. By Lemma 2.4.20, we know that $\mathsf{Id}_{\mathsf{sF}}(M) \subseteq \mathsf{Id}_{\mathsf{F}}(M)$. Now let $I \in \mathsf{Id}_{\mathsf{F}}(M)$. Let $X \subseteq_\omega I$ and $Y \subseteq_\omega P$ be non-empty. Assume that $X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}}$. Since M is a meet-semilattice, it follows that $Y^{\mathrm{lu}} = \uparrow(\bigwedge Y)$. Thus, $\uparrow(\bigwedge Y) \in Y^{\mathrm{lu}}$ and $\uparrow(\bigwedge Y) \in I$ and hence $Y^{\mathrm{lu}} \cap I \neq \emptyset$. Then I is a strong Frink-ideal of M. We thus obtain $\mathsf{Id}_{\mathsf{F}}(M) \subseteq \mathsf{Id}_{\mathsf{sF}}(M)$ and therefore $\mathsf{Id}_{\mathsf{F}}(M) = \mathsf{Id}_{\mathsf{sF}}(M)$. $\square$

DEFINITION 2.4.23. Let P be a poset. A Frink-filter F of P is said to be *s-optimal* if F^c is a strong Frink-ideal of P. Let us denote by $\mathsf{Opt}_{\mathsf{s}}(P)$ the collection of all s-optimal Frink-filters of P.

When a poset P has no top element, the empty set is an s-optimal Frink-filter, because $\emptyset$ is a Frink-filter and $\emptyset^c = P$ is a strong Frink-ideal of P. The proof of the following lemma is straightforward by Lemmas 2.4.20 and 2.4.22.

Lemma 2.4.24. *Let P be a poset. Then $\mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P) \subseteq \mathsf{Opt}_{\mathsf{s}}(P) \subseteq \mathsf{Opt}(P)$. If M is a meet-semilattice, then $\mathsf{Opt}_{\mathsf{s}}(M) = \mathsf{Opt}(M)$.*

Let P be a mo-distributive poset and $M(P)$ its distributive meet-semilattice envelope. Note that $\mathsf{Opt}(M(P))$ denotes the collection of all optimal Frink-filters of $M(P)$ considered as a poset and, by Lemma 2.4.6, if P has no top element, then the empty set is an optimal Frink-filter of $M(P)$. Moreover, since $M(P)$ is a meet-semilattice, it follows by Lemma 2.1.7 that $\mathsf{Fi}(M(P)) \cap \mathsf{Opt}(M(P)) = \mathsf{Opt}(M(P)) \setminus \{\emptyset\}$.

Lemma 2.4.25. *Let P be a mo-distributive poset and $M(P)$ its distributive meet-semilattice envelope. Then,*

1. *for every $G \in \mathsf{Opt}(M(P))$, we have $G \cap P \in \mathsf{Opt}_{\mathsf{s}}(P)$;*
2. *for every $F \in \mathsf{Opt}_{\mathsf{s}}(P)$, we have $\mathrm{Fi}_{M(P)}(F) \in \mathsf{Opt}(M(P))$.*

Proof.

(1) Let $G \in \mathsf{Opt}(M(P))$. Since $M(P)$ is the distributive meet-semilattice envelope of P, it follows that $G \cap P$ is a proper Frink-filter of P. To prove that $G \cap P$ is an s-optimal Frink-filter of P, let $X \subseteq_\omega (G \cap P)^c$ (where the complement is taken with respect to P, that is, $(G \cap P)^c = P \setminus (G \cap P))$ and $Y \subseteq_\omega P$ be non-empty and such that $X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}}$. Suppose towards a contradiction that

$$(G \cap P)^c \cap Y^{\mathrm{lu}} = \emptyset.$$

So $Y \subseteq Y^{\mathrm{lu}} \subseteq G \cap P$, whereupon $Y \subseteq G$. Since G is a filter of $M(P)$, it follows that $\bigwedge Y \in G$. As $X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}}$ in P, by (2.13), we have $X^{\mathrm{u}} \subseteq {\uparrow}(\bigwedge Y)$ in $M(P)$ and this implies that $\bigwedge Y \in X^{\mathrm{ul}}$. Then, since $X \subseteq G^c$ and G^c is a Frink-ideal of $M(P)$, we have $X^{\mathrm{ul}} \subseteq G^c$. We thus obtain $\bigwedge Y \in G^c$, which is a contradiction. Then, $(G \cap P)^c \cap Y^{\mathrm{lu}} \neq \emptyset$ and we thus obtain that $(G \cap P)^c$ is a strong Frink-ideal of P. Therefore, $G \cap P \in \mathsf{Opt}_{\mathsf{s}}(P)$.

(2) Let $F \in \mathsf{Opt}_{\mathsf{s}}(P)$. Since F is proper, it follows by Lemma 2.4.14 that $\mathrm{Fi}_{M(P)}(F)$ is proper. Notice that if $F = \emptyset$, then P has no top element and thus we obtain that $\mathrm{Fi}_{M(P)}(F) = \emptyset \in \mathsf{Opt}(M(P))$. Now we assume that $F \neq \emptyset$. Let $X \subseteq_\omega \mathrm{Fi}_{M(P)}(F)^c$ and let $y \in X^{\mathrm{ul}}$. We suppose that $y \in \mathrm{Fi}_{M(P)}(F)$. Thus, there is a non-empty $A \subseteq_\omega F$ such that $\bigwedge A \leq y$. By (E2), there exists a non-empty $B \subseteq_\omega P$ such that $\bigwedge B = y$ and for every $x \in X$ there exists non-empty $B_x \subseteq_\omega P$ such that $\bigwedge B_x = x$. As $x \notin \mathrm{Fi}_{M(P)}(F)$ for every $x \in X$, we have that for every $x \in X$ there exists $b_x \in B_x$ such that $b_x \notin F$. Let $C := \{b_x : \ x \in X\}$. We show that $C^{\mathrm{u}} \subseteq B^{\mathrm{lu}}$. Let $c \in C^{\mathrm{u}}$ and let $b \in B^{\mathrm{l}}$. Then, $b \leq \bigwedge B$ and $b_x \leq c$ for all $x \in X$. Thus, for every $x \in X$ we obtain $x = \bigwedge B_x \leq b_x \leq c$. That

is, $x \leq c$ for all $x \in X$. Then, $y \leq c$ and so $\bigwedge B \leq c$; this implies $b \leq c$. Hence $c \in B^{\mathrm{lu}}$.

Since F^c is a strong Frink-ideal of P and $C \subseteq_\omega F^c$ is non-empty and since moreover $C^{\mathrm{u}} \subseteq B^{\mathrm{lu}}$, it follows that

$$F^c \cap B^{\mathrm{lu}} \neq \emptyset.$$

Thus, there exists $b \in B^{\mathrm{lu}}$ such that $b \in F^c$. By Condition (E3.1), we have $\bigwedge B \leq b$, whereupon $\bigwedge A \leq y \leq b$. That is, $b \in \uparrow(\bigwedge A) = A^{\mathrm{lu}}$ and by Condition (E3.1) again, we have $b \in A^{\mathrm{lu}}$ in P. Observe that $A \subseteq_\omega F$ and since F is a Frink-filter, it follows that $b \in F$. This is a contradiction. Hence, $x \notin \mathrm{Fi}_{M(P)}(F)$ and so we conclude that $\mathrm{Fi}_{M(P)}(F)^c$ is a Frink-ideal of $M(P)$. Therefore, $\mathrm{Fi}_{M(P)}(F) \in \mathsf{Opt}(M(P))$.

□

Corollary 2.4.26. *Let P be a mo-distributive poset and $M(P)$ its distributive meet-semilattice envelope. Then, the lattice isomorphisms α and β given in Theorem 2.4.16 restrict to order-isomorphisms between $\mathsf{Opt}_{\mathsf{s}}(P)$ and $\mathsf{Opt}(M(P))$, respectively. That is,*

$$\alpha : \mathsf{Opt}_{\mathsf{s}}(P) \cong \mathsf{Opt}(M(P)) : \beta.$$

We summarize in Table 2.2 the correspondence between the Frink-filters of a mo-distributive poset P and the filters of its distributive meet-semilattice envelope $M(P)$. It should be noted that the star over $\mathsf{Fi}(M(P))$ and $\mathsf{Fi}^{\mathsf{pr}}(M(P))$ only means that when necessary the empty set belongs to the classes these expressions denote.

$\mathsf{Fi}_{\mathsf{F}}(P) = \{G \cap P : \ G \in \mathsf{Fi}(M(P))^*\}$
$\mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P) = \{H \cap P : \ H \in \mathsf{Fi}^{\mathsf{pr}}(M(P))^*\}$
$\mathsf{Opt}_{\mathsf{s}}(P) = \{U \cap P : \ U \in \mathsf{Opt}(M(P))\}$
$\mathsf{Opt}(P) = \{\mathrm{Fi}_{M(P)}(U) \cap P : \ U \in \mathsf{Opt}(P)\}$

Table 2.2. Correspondence between the Frink-filters of a mo-distributive poset P and the filters of its distributive meet-semilattice envelope $M(P)$.

Now we can use the Corollary 2.4.26 to prove the following theorem that will play a central role in Chapter 4 to develop a topological duality.

Theorem 2.4.27. *Let P be a mo-distributive poset and let $F \in \mathsf{Fi}_{\mathsf{F}}(P)$ and $I \in \mathsf{Id}_{\mathsf{sF}}(P)$. If $F \cap I = \emptyset$, then there exists $U \in \mathsf{Opt}_{\mathsf{s}}(P)$ such that $F \subseteq U$ and $I \cap U = \emptyset$.*

Proof. Let $F \in \mathsf{Fi}_{\mathsf{F}}(P)$ and $I \in \mathsf{Id}_{\mathsf{sF}}(P)$ be such that $F \cap I = \emptyset$. Notice that if $F = \emptyset$, then P has no top element and $F = \emptyset \in \mathsf{Opt}_{\mathsf{s}}(P)$. Thus the theorem holds. So, we can assume that $F \neq \emptyset$. Let $G := \mathsf{Fi}(F)$ be the filter of $M(P)$ generated by F and let $J := \mathrm{Id}_{\mathrm{F}}(I)$ be the Frink-filter of $M(P)$ generated by I. Suppose that $G \cap J \neq \emptyset$. So there is $x \in G \cap J$. Then, there are $a_1, \dots, a_n \in F$ such that $a_1 \wedge \cdots \wedge a_n \leq x$ and there are $b_1, \dots, b_m \in I$ such that $x \in \{b_1, \dots, b_m\}^{\mathrm{ul}}$. We thus obtain $a_1 \wedge \cdots \wedge a_n \in \{b_1, \dots, b_m\}^{\mathrm{ul}}$, which implies $\{b_1, \dots, b_m\}^{\mathrm{u}} \subseteq {\uparrow}(a_1 \wedge \cdots \wedge a_n)$. Then, by (2.13), we have $\{b_1, \dots, b_m\}^{\mathrm{u}} \subseteq \{a_1, \dots, a_n\}^{\mathrm{lu}}$ in P. Then, since I is a strong Frink-ideal of P and $\{b_1, \dots, b_m\} \subseteq I$, it follows that $\{a_1, \dots, a_n\}^{\mathrm{lu}} \cap I \neq \emptyset$ and this implies that $F \cap I \neq \emptyset$, a contradiction. Hence $G \cap J = \emptyset$. Thus, given that $M(P)$ is a distributive meet-semilattice, we obtain by [**5**, Lemma 4.7] that there exists $H \in \mathsf{Opt}(M(P))$ such that $G \subseteq H$ and $H \cap J = \emptyset$. By Corollary 2.4.26, $U := H \cap P$ is an s-optimal Frink-filter of P and it is clear that $F \subseteq U$ and $U \cap I = \emptyset$. □

We finish this section by introducing a new kind of homomorphism between posets, which is motivated by the property given in Lemma 2.4.13 that has the distributive meet-semilattice envelope of a mo-distributive poset. We also establish some properties of this new notion that will be important in Chapter 4.

Definition 2.4.28. Let P and Q be posets. A map $h\colon P \to Q$ is said to be a *strong inf-homomorphism* if for all subsets $X, Y \subseteq_\omega P$, we have

$$X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}} \text{ implies } h[X]^{\mathrm{u}} \subseteq h[Y]^{\mathrm{lu}}.$$

Lemma 2.4.29. *Let P and Q be posets. If $h\colon P \to Q$ is a strong inf-homomorphism, then h is an inf-sup-homomorphism.*

Proof. Let $a \in P$ and let $A \subseteq_\omega P$. Then, we have

$$\begin{aligned} a \in A^{\mathrm{lu}} &\implies {\uparrow}a \subseteq A^{\mathrm{lu}} \\ &\implies {\uparrow}h(a) \subseteq h[A]^{\mathrm{lu}} \\ &\implies h(a) \in h[A]^{\mathrm{lu}} \end{aligned}$$

and

$$\begin{aligned} a \in A^{\mathrm{ul}} &\implies A^{\mathrm{u}} \subseteq {\uparrow}a \\ &\implies h[A]^{\mathrm{u}} \subseteq {\uparrow}h(a) \\ &\implies h(a) \in h[A]^{\mathrm{ul}}. \end{aligned}$$

Therefore, h is an inf-sup-homomorphism. □

In the following example we show that the converse of the previous lemma does not hold, even if P and Q are bounded mo-distributive and jo-distributive posets.

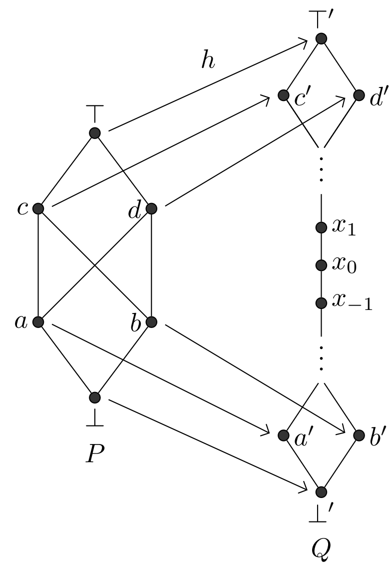


FIGURE 2.13. An inf-sup-homomorphism that is not a strong inf-homomorphism.

EXAMPLE 2.4.30. Consider the posets P and Q depicted in Figure 2.13. As we seen in Example 2.2.14 on page 36, the poset P is mo-distributive and jo-distributive. It is also clear that the poset Q is mo-distributive and jo-distributive (Figure 2.12 on page 60 shows the lattice of the Frink-ideals of Q, and it is clear that the lattice of the Frink-filters of Q is dual to the Frink-ideals). Consider the map $h\colon P \to Q$ as is defined in Figure 2.13. Notice that h preserves bounds. It is straightforward to show directly that for every $G \in \mathsf{Fi}_{\mathsf{F}}(Q)$, $h^{-1}[G] \in \mathsf{Fi}_{\mathsf{F}}(P)$ and it is also straightforward to show that $h^{-1}[J] \in \mathsf{Id}_{\mathsf{F}}(P)$ for all $J \in \mathsf{Id}_{\mathsf{F}}(Q)$. Then, by Lemmas 2.3.6 and 2.3.7, we have that h is an inf-sup-homomorphism. Now we show that h is not a strong inf-homomorphism. Notice that $\{a,b\}^{\mathrm{u}} = \{c,d,\top\} = \{c,d\}^{\mathrm{lu}}$. In Q we have that $\{h(a),h(b)\}^{\mathrm{u}} = \{a',b'\}^{\mathrm{u}} = \{\dots,x_{-1},x_0,x_1,\dots,c',d',\top'\}$ and $\{h(c),h(d)\}^{\mathrm{lu}} = \{c',d'\}^{\mathrm{lu}} = \{c',d',\top'\}$. Hence, we have that $\{a,b\}^{\mathrm{u}} \subseteq \{c,d\}^{\mathrm{lu}}$ and $\{h(a),h(b)\}^{\mathrm{u}} \nsubseteq \{h(c),h(d)\}^{\mathrm{lu}}$. Therefore, h is not a strong inf-homomorphism.

LEMMA 2.4.31. *Let P and Q be posets and let $h\colon P \to Q$ be a strong inf-homomorphism. Let $V \in \mathsf{Opt}_{\mathsf{s}}(Q)$. If $h^{-1}[V] \neq P$, then $h^{-1}[V] \in \mathsf{Opt}_{\mathsf{s}}(P)$.*

PROOF. Assume h is a strong inf-homomorphism and let $V \in \mathsf{Opt}_{\mathsf{s}}(Q)$ be such that $h^{-1}[V] \neq P$. From Lemmas 2.4.29 and 2.3.6, we have $h^{-1}[V]$ is a Frink-filter of P. To show $h^{-1}[V]$ is an s-optimal Frink-filter, we prove that $h^{-1}[V]^c$ is a strong Frink-ideal. It is clear that $h^{-1}[V]^c$ is a non-empty down-set of P. Let $X \subseteq_\omega h^{-1}[V]^c$ be non-empty and let $Y \subseteq_\omega P$ be non-empty such that $X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}}$.

Then, since h is a strong inf-homomorphism,

$$h[X]^{\mathrm{u}} \subseteq h[Y]^{\mathrm{lu}}. \tag{2.14}$$

Suppose that $Y^{\mathrm{lu}} \cap h^{-1}[V]^c = \emptyset$. Then, $Y \subseteq Y^{\mathrm{lu}} \subseteq h^{-1}[V]$. So,

$$h[Y]^{\mathrm{lu}} \subseteq V. \tag{2.15}$$

Since $X \subseteq_\omega h^{-1}[V]^c$, it follows that $h[X] \subseteq_\omega V^c$ and moreover as V is an s-optimal Frink-filter of Q, we obtain that V^c is a strong Frink-ideal of Q. Then, by (2.14), we have $h[Y]^{\mathrm{lu}} \cap V^c \neq \emptyset$ and thus $h[Y]^{\mathrm{lu}} \nsubseteq V$. This contradicts (2.15). Hence, $Y^{\mathrm{lu}} \cap h^{-1}[V]^c \neq \emptyset$. We thus obtain $h^{-1}[V]^c$ is a strong Frink-ideal of P and therefore $h^{-1}[V] \in \mathsf{Opt_s}(P)$. □

The next lemma shows that the strong inf-homomorphisms between bounded mo-distributive posets are characterized by the s-optimal Frink-filters.

Lemma 2.4.32. *Let P and Q be bounded mo-distributive posets and $h\colon P \to Q$ a map. Then, h is a strong inf-homomorphism if and only if $h^{-1}[V] \in \mathsf{Opt_s}(P)$ for all $V \in \mathsf{Opt_s}(Q)$.*

Proof. The implication from left to right is a consequence of the previous lemma and Lemmas 2.4.29 and 2.3.3. Now we assume that $h^{-1}[V] \in \mathsf{Opt_s}(P)$ for all $V \in \mathsf{Opt_s}(Q)$. Let $X, Y \subseteq_\omega P$ be such that $X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}}$. Without loss of generality we can assume that X and Y are non-empty, because if $X = \emptyset$, then $X^{\mathrm{u}} = P = \{\bot\}^{\mathrm{u}}$ and if $Y = \emptyset$, then $Y^{\mathrm{lu}} = \{\top\} = \{\top\}^{\mathrm{lu}}$. We suppose towards a contradiction that $h[X]^{\mathrm{u}} \nsubseteq h[Y]^{\mathrm{lu}}$. So there is $q \in h[X]^{\mathrm{u}}$ such that $q \notin h[Y]^{\mathrm{lu}}$. Then, by Theorem 2.4.27, there exists $V \in \mathsf{Opt_s}(Q)$ such that $h[Y]^{\mathrm{lu}} \subseteq V$ and $q \notin V$. Thus $h[Y] \subseteq V$ and then $Y \subseteq h^{-1}[V]$. Since $h^{-1}[V]$ is an s-optimal Frink-filter, it follows that $Y^{\mathrm{lu}} \subseteq h^{-1}[V]$. As $q \notin V$ and $q \in h[X]^{\mathrm{u}}$, we have $X \subseteq h^{-1}[V]^c$. Hence, since $X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}}$, $X \subseteq h^{-1}[V]^c$ and $h^{-1}[V]^c$ is a strong Frink-ideal, it follows that $Y^{\mathrm{lu}} \cap h^{-1}[V]^c \neq \emptyset$. So $Y^{\mathrm{lu}} \nsubseteq h^{-1}[V]$, which is a contradiction. Hence, $h[X]^{\mathrm{u}} \subseteq h[Y]^{\mathrm{lu}}$ and therefore h is a strong inf-homomorphism. □

Definition 2.4.33. We say that a map $h\colon P \to Q$ from a poset P to a poset Q is a *strong inf-embedding* if for all subsets $X, Y \subseteq_\omega P$ we have

$$X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}} \text{ if and only if } h[X]^{\mathrm{u}} \subseteq h[Y]^{\mathrm{lu}}.$$

The next lemma follows straightforwardly from Lemma 2.4.13.

Lemma 2.4.34. *Let P be a mo-distributive poset and $\langle M(P), e\rangle$ its distributive meet-semilattice envelope. Then, $e\colon P \to M(P)$ is a strong inf-embedding.*

Lemma 2.4.35. *Let P and Q be posets and, let $h\colon P \to Q$ be a map. Then, the following conditions are equivalent:*

(1) *h is a strong inf-embedding;*

(2) *h is a strong inf-homomorphism and an order-embedding.*

Proof. (1) ⇒ (2) We only need to prove that $h(a) \leq h(a)$ implies $a \leq b$, for all $a, b \in P$. Let $a, b \in P$. We suppose $h(a) \leq h(b)$. Then, $\{h(b)\}^{\mathrm{u}} \subseteq \{h(a)\}^{\mathrm{lu}}$. By hypothesis, $\{b\}^{\mathrm{u}} \subseteq \{a\}^{\mathrm{lu}}$ and hence $a \leq b$.

(2) ⇒ (1) Let $X, Y \subseteq_\omega P$ be non-empty. We assume $h[X]^{\mathrm{u}} \subseteq h[Y]^{\mathrm{lu}}$. We want to prove that $X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}}$. So, let $a \in X^{\mathrm{u}}$ and $b \in Y^{\mathrm{l}}$. Thus, $x \leq a$ for all $x \in X$ and $b \leq y$ for all $y \in Y$. Then, $h(x) \leq h(a)$ for all $x \in X$ and $h(b) \leq h(y)$ for all $y \in Y$. This implies $h(a) \in h[X]^{\mathrm{u}}$ and $b \in h[Y]^{\mathrm{l}}$. Thus, by hypothesis, $h(b) \leq h(a)$. Since h is an order-embedding, $b \leq a$. Therefore, $a \in Y^{\mathrm{lu}}$. This completes the proof. □

Lemma 2.4.36. *Let P and Q be mo-distributive posets and $h\colon P \to Q$ a strong inf-homomorphism. Then, for every non-empty $X, X_1, \ldots, X_n \subseteq_\omega P$, we have*

$$\bigcap_{i=1}^{n} X_i^{\mathrm{lu}} \subseteq X^{\mathrm{lu}} \implies \bigcap_{i=1}^{n} h[X_i]^{\mathrm{lu}} \subseteq h[X]^{\mathrm{lu}}.$$

Moreover, if h is an order-embedding, then the inverse implication holds.

Proof. Let $X, X_1, \ldots, X_n \subseteq_\omega P$ be non-empty such that $\bigcap_{i=1}^{n} X_i^{\mathrm{lu}} \subseteq X^{\mathrm{lu}}$. Suppose towards a contradiction that $\bigcap_{i=1}^{n} h[X_i]^{\mathrm{lu}} \nsubseteq h[X]^{\mathrm{lu}}$. So, there is $b \in \bigcap_{i=1}^{n} h[X_i]^{\mathrm{lu}}$ such that $b \notin h[X]^{\mathrm{lu}}$. Then, there exists $V \in \mathsf{Opt_s}(Q)$ such that $h[X]^{\mathrm{lu}} \subseteq V$ and $b \notin V$. Thus $h[X] \subseteq V$ and this implies that $X^{\mathrm{lu}} \subseteq h^{-1}[V]$. Moreover, we have $h[X_i]^{\mathrm{lu}} \nsubseteq V$ for all $i \in \{1, \ldots, n\}$ and so $h[X_i] \nsubseteq V$ for all $i \in \{1, \ldots, n\}$. Hence, for every $i \in \{1, \ldots, n\}$ there is $x_i \in X_i$ such that $h(x_i) \notin V$. Then, $\{x_1, \ldots, x_n\} \subseteq h^{-1}[V]^c$. Note that $h^{-1}[V] \neq P$. Since h is a strong inf-homomorphism and by Lemma 2.4.31, it follows that $h^{-1}[V]$ is an s-optimal Frink-filter of P, whereupon $h^{-1}[V]^c$ is a strong Frink-ideal. Now, notice that $\{x_1, \ldots, x_n\}^{\mathrm{u}} \subseteq X^{\mathrm{lu}}$. Then, as $\{x_1, \ldots, x_n\} \subseteq h^{-1}[V]^c$ and $\{x_1, \ldots, x_n\}^{\mathrm{u}} \subseteq X^{\mathrm{lu}}$, we have $X^{\mathrm{lu}} \cap h^{-1}[V]^c \neq \emptyset$. We thus obtain $X^{\mathrm{lu}} \nsubseteq h^{-1}[V]$ and this is a contradiction. Hence, $\bigcap_{i=1}^{n} h[X_i]^{\mathrm{lu}} \subseteq h[X]^{\mathrm{lu}}$.

Now, moreover, assume that h is an order-embedding. So, h is a strong inf-embedding. Suppose that $\bigcap_{i=1}^{n} h[X_i]^{\mathrm{lu}} \subseteq h[X]^{\mathrm{lu}}$. Let $a \in \bigcap_{i=1}^{n} X_i^{\mathrm{lu}}$. Then ${\uparrow}a \subseteq \bigcap_{i=1}^{n} X_i^{\mathrm{lu}}$ and, since h is a strong inf-homomorphism, it follows that ${\uparrow}h(a) \subseteq \bigcap_{i=1}^{n} h[X_i]^{\mathrm{lu}}$. Thus, by hypothesis, ${\uparrow}h(a) \subseteq h[X]^{\mathrm{lu}}$. As h is a strong inf-embedding, we have that ${\uparrow}a \subseteq X^{\mathrm{lu}}$. Hence, $\bigcap_{i=1}^{n} X_i^{\mathrm{lu}} \subseteq X^{\mathrm{lu}}$. □

In Figure 2.14 we summarize the different relations between the several notions of homomorphism between posets that we considered throughout this chapter. Moreover, we complete the diagram with the following notions of homomorphism.

The first and the second are the dual definitions of a strong inf-homomorphism and strong inf-embedding, respectively. Let $h\colon P \to Q$ be a map from a poset P to a poset Q. We say that h is a *strong sup-homomorphism* if for every $X, Y \subseteq_\omega P$, we have

$$X^{\mathrm{l}} \subseteq Y^{\mathrm{ul}} \text{ implies } h[X]^{\mathrm{l}} \subseteq h[Y]^{\mathrm{ul}},$$

moreover if the reverse implication also holds, h is called *strong sup-embeding*. We say that a map $h\colon P \to Q$ is a *strong inf-sup-homomorphism* if h is a strong inf-homomorphism and strong sup-homomorphism. Lastly, we say that h is a *strong inf-sup-embedding* if it is a strong inf-embedding and strong sup-embedding.

2.5. The distributive lattice envelope

In this last section we consider an extension of a mo-distributive poset to a distributive lattice. This extension of a mo-distributive poset is obtained by means of its distributive meet-semilattice envelope. Such an extension will be the main tool used in Chapter 4 to develop a Priestley-style topological duality for bounded mo-distributive posets.

Since we will need the theory of the distributive envelope of a distributive meet-semilattice developed in [**5, 4**] we present its main definitions and facts in the following subsection.

2.5.1. The distributive envelope of a distributive meet-semilattice.

Here we prefer to give an abstract definition of the distributive envelope, unlike in [**5, 4**].

Definition 2.5.1. Let L be a distributive meet-semilattice. A distributive lattice D is a *distributive envelope* of L if there is a meet-embedding $\sigma\colon L \to D$ such that

(1) for each $a, a_1, \ldots, a_n \in L$,

$$a \in \{a_1, \ldots, a_n\}^{\mathrm{ul}} \quad \text{implies} \quad \sigma(a) \in \{\sigma(a_1), \ldots, \sigma(a_n)\}^{\mathrm{ul}};$$

(2) $\sigma[L]$ is finitely join-dense in D.

Now we introduce the following definition. Let P and Q be posets. A map $h\colon P \to Q$ is called *almost sup-homomorphism* if for each $a, a_1, \ldots, a_n \in P$ such that $a \in \{a_1, \ldots, a_n\}^{\mathrm{ul}}$, then $h(a) \in \{h(a_1), \ldots, h(a_n)\}^{\mathrm{ul}}$. It should be noted that every sup-homomorphism (Definition 2.3.1) is an almost sup-homomorphism. But, it is clear that an almost sup-homomorphism may not preserve bottom element and thus it may not be a sup-homomorphism. However, it is not hard to show that a map $h\colon P \to Q$ is a sup-homomorphism if and only if it is an almost sup-homomorphism and preserves bottom, if it exists.

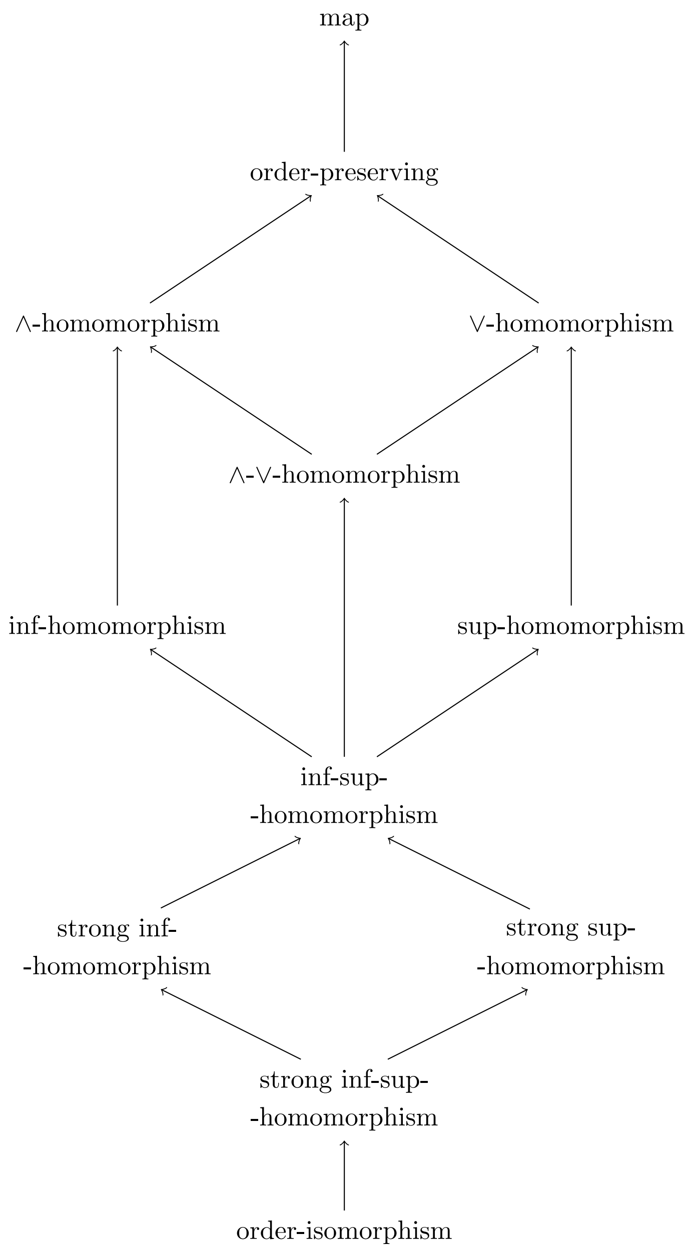


FIGURE 2.14. Relations between different kind of homomorphisms for posets.

THEOREM 2.5.2. ([**5**, Theorem 3.9]). *Let L be a distributive meet-semilattice. Then, there exists a unique up to isomorphism distributive envelope of L.*

We denote by $D(L)$ the distributive envelope of a distributive meet-semilattice L and by σ is the corresponding meet-embedding almost sup-homomorphism from L to $D(L)$.

REMARK 2.5.3. Let L be a distributive meet-semilattice and $\langle D(L), \sigma\rangle$ its distributive envelope. Since $\sigma[L]$ is finitely join-dense in $D(L)$, we obtain easily that:

(1) if L has top $\top_L$, then $\sigma(\top_L)$ is the top element of $D(L)$;
(2) L has bottom element if and only if $D(L)$ has bottom element.

LEMMA 2.5.4. *Let L be a distributive meet-semilattice and $\langle D(L), \sigma\rangle$ its distributive envelope. Then σ is a sup-homomorphism.*

PROOF. It is consequence of Condition (1) in Definition 2.5.1 and Remark 2.5.3. □

THEOREM 2.5.5. ([**5**, Theorem 3.8]). *Let L be a distributive meet-semilattice. The distributive envelope $D(L)$ of L is up to isomorphism the unique distributive lattice E for which there is a meet-embedding almost sup-homomorphism $f\colon L \to E$ such that for each distributive lattice D and a meet-embedding almost sup-homomorphism $h\colon L \to D$, there is a unique lattice embedding $K\colon E \to D$ with $K \circ f = h$.*

LEMMA 2.5.6. ([**5**, Lemma 3.2]). *Let L be a distributive meet-semilattice and $\langle D(L), \sigma\rangle$ its distributive envelope. Then, for every $a, a_1, \dots, a_n \in L$, we have*

$$\bigcap_{i=1}^{n} {\uparrow} a_i \subseteq {\uparrow} a \quad \textit{iff} \quad \sigma(a) \subseteq \bigvee_{i=1}^{n} \sigma(a_i) \quad \textit{iff} \quad \bigcap_{i=1}^{n} {\uparrow}\sigma(a_i) \subseteq {\uparrow}\sigma(a).$$

For what follows, we need to make some clarifications. In [**4, 5**] the Frink-ideals of a meet-semilattice L are considered non-empty (see [**5**, Definition 4.1]), even if L has no bottom element. But, if we consider the meet-semilattice L as a poset and it has no bottom element, then the empty set is a Frink-ideal under our definition of Frink-ideal on posets (see Definition 2.1.4). Thus to be consistent with the previous and future considerations in this dissertation, we establish that we will follow our definition. That is, if L is a meet-semilattice, then a Frink-ideal is taken as in Definition 2.1.4. So, if L has no bottom, then the empty set is a Frink-ideal of L.

Now we proceed to present the correspondence between the Frink-ideals of a distributive meet-semilattice L and the ideals of its distributive envelope $D(L)$. First, we introduce the following notation. Let L be a distributive meet-semilattice and $D(L)$ its distributive envelope. Recall that $\mathsf{Id}(D(L))$ denotes the lattice of ideals of $D(L)$. We denote the collection of all prime ideals of $D(L)$ by $\mathsf{Id}^{\mathsf{pr}}(D(L))$. Then we let

$$\mathsf{Id}(D(L))^* := \begin{cases} \mathsf{Id}(D(L)) & \text{if } L \text{ has bottom} \\ \mathsf{Id}(D(L)) \cup \{\emptyset\} & \text{if } L \text{ has no bottom} \end{cases}$$

and

$$\mathsf{Id}^{\mathsf{pr}}(D(L))^* := \begin{cases} \mathsf{Id}^{\mathsf{pr}}(D(L)) & \text{if } L \text{ has bottom} \\ \mathsf{Id}^{\mathsf{pr}}(D(L)) \cup \{\emptyset\} & \text{if } L \text{ has no bottom.} \end{cases}$$

Having [**5**, Theorem 4.3] and Remark 2.5.3 in mind, we define the map $\kappa\colon \mathsf{Id}(D(L))^* \to \mathsf{Id}_{\mathsf{F}}(L)$ as $\kappa(I) = \sigma^{-1}[I]$ for each $I \in \mathsf{Id}(D(L))^*$. Moreover, if $I \in \mathsf{Id}^{\mathsf{pr}}(D(L))$, then $\kappa(I) \in \mathsf{Id}_{\mathsf{F}}^{\mathsf{pr}}(L)$. Now observe the following. If L has no bottom element, we know that $\emptyset \in \mathsf{Id}_{\mathsf{F}}(L)$. Suppose that the empty set is not a prime Frink-ideal. Then there are Frink-ideals I_1 and I_2 of L such that $I_1 \cap I_2 = \emptyset$, $I_1 \neq \emptyset$ and $I_2 \neq \emptyset$. So there are $a \in I_1$ and $b \in I_2$ such that ${\downarrow}a \cap {\downarrow}b = \emptyset$, which is impossible because $a \wedge b \in {\downarrow}a \cap {\downarrow}b$. Hence, we have proved that if L has no bottom element then $\emptyset \in \mathsf{Id}_{\mathsf{F}}^{\mathsf{pr}}(L)$. Therefore, we obtain that for each $I \in \mathsf{Id}^{\mathsf{pr}}(D(L))^*$, $\kappa(I) \in \mathsf{Id}_{\mathsf{F}}^{\mathsf{pr}}(L)$. Now we are ready to present the following theorem.

Theorem 2.5.7. ([**5**, Corollary 4.4]). *Let L be a distributive meet-semilattice and $\langle D(L), \sigma\rangle$ its distributive envelope. Then, $\kappa\colon \langle \mathsf{Id}(D(L))^*, \subseteq\rangle \to \langle \mathsf{Id}_{\mathsf{F}}(L), \subseteq\rangle$ is an order-isomorphism. Moreover, κ restricts to an order-isomorphism between the ordered sets $\langle \mathsf{Id}^{\mathsf{pr}}(D(P))^*, \subseteq\rangle$ and $\langle \mathsf{Id}_{\mathsf{F}}^{\mathsf{pr}}(L), \subseteq\rangle$.*

The following theorem shows the correspondence between the optimal Frink-filters of a distributive meet-semilattice L and the prime filters of its distributive envelope $D(L)$. We need the following notation:

$$\mathsf{Fi}^{\mathsf{pr}}(D(L))^* := \begin{cases} \mathsf{Fi}^{\mathsf{pr}}(D(L)) & \text{if } L \text{ has top} \\ \mathsf{Fi}^{\mathsf{pr}}(D(L)) \cup \{\emptyset\} & \text{if } L \text{ has no top.} \end{cases}$$

Then, by [**4**, Proposition 4.21] and Remark 2.5.3, we can define the map

$$\lambda\colon \mathsf{Fi}^{\mathsf{pr}}(D(L))^* \to \mathsf{Opt}(L)$$

as follows: for every $H \in \mathsf{Fi}^{\mathsf{pr}}(D(L))^*$,

$$\lambda(H) = \sigma^{-1}[H].$$

Theorem 2.5.8. ([**4**, Corollary 4.22]). *Let L be a distributive meet-semilattice and $\langle D(L), \sigma\rangle$ its distributive envelope. Then, the map $\lambda\colon \langle \mathsf{Fi}^{\mathsf{pr}}(D(L))^*, \subseteq\rangle \to \langle \mathsf{Opt}(L), \subseteq\rangle$ is an order-isomorphism whose inverse is the map given by $U \mapsto \mathrm{Fi}_{D(L)}(\sigma[U])$.*

2.5.2. The distributive lattice envelope of a mo-distributive poset. Now in this subsection we use the theory of the distributive meet-semilattice envelope introduced in §2.4 and the theory of the distributive envelope presented in the previous subsection to embed a mo-distributive poset into a distributive lattice. To attain this, a first important step is to show that if L is a distributive meet-semilattice and $\langle D(L), \sigma\rangle$ is its distributive envelope, then σ is more that a meet-embedding sup-homomorphism, namely it is a strong inf-embedding.

LEMMA 2.5.9. *Let L be a distributive meet-semilattice and $\langle D(L), \sigma\rangle$ its distributive envelope. Then, $\sigma\colon L \to D(L)$ is a strong inf-embedding.*

PROOF. Let $X, Y \subseteq_\omega L$. By Definition 2.4.33, we need to prove that

$$X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}} \quad \text{if and only if} \quad \sigma[X]^{\mathrm{u}} \subseteq \sigma[Y]^{\mathrm{lu}}.$$

Given that σ is an order-embedding and Lemma 2.4.35, it is only necessary prove that

$$X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}} \quad \text{implies} \quad \sigma[X]^{\mathrm{u}} \subseteq \sigma[Y]^{\mathrm{lu}}.$$

So, we assume that $X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}}$. If $X = \emptyset$, then $X^{\mathrm{u}} = L$ and $\sigma[X]^{\mathrm{u}} = D(L)$. Let $u \in D(L)$. So, there are $a_1, \dots, a_n \in L$ such that $x = \sigma(a_1) \vee \cdots \vee \sigma(a_n)$. Then, $a_1, \dots, a_n \in L = X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}}$. Since σ is a meet-homomorphism, it follows that $\sigma(a_1), \dots, \sigma(a_n) \in \sigma[Y]^{\mathrm{lu}}$. Thus $x = \sigma(a_1) \vee \cdots \vee \sigma(a_n) \in \sigma[Y]^{\mathrm{lu}}$ and hence $\sigma[X]^{\mathrm{u}} = D(L) \subseteq \sigma[Y]^{\mathrm{lu}}$. Now we suppose that $X \neq \emptyset$. If $Y = \emptyset$, then

$$Y^{\mathrm{lu}} = \begin{cases} \{\top_L\} & \text{if } L \text{ has top } \top_L \\ \emptyset & \text{if } L \text{ has no top.} \end{cases} \tag{2.16}$$

Let $u \in \sigma[X]^{\mathrm{u}}$. Thus $\sigma(x) \leq u$ for all $x \in X$. Since $\sigma[L]$ is join-dense in $D(L)$, it follows that there are $a_1, \dots, a_n \in L$ such that $u = \sigma(a_1) \vee \cdots \vee \sigma(a_n)$. Then $\sigma(x) \leq \sigma(a_1) \vee \cdots \vee \sigma(a_n)$ for all $x \in X$. By Lemma 2.5.6, we have $\bigcap\limits_{i=1}^{n} {\uparrow} a_i \subseteq {\uparrow} x$ for all $x \in X$. So $\bigcap\limits_{i=1}^{n} {\uparrow} a_i \subseteq \bigcap\limits_{x \in X} {\uparrow} x = X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}}$. We claim that u is the top element of $D(L)$. Let $v \in D(L)$ and let $b_1, \dots, b_m \in L$ be such that $v = \sigma(b_1) \vee \cdots \vee \sigma(b_m)$. By (2.16), we have $\bigcap\limits_{i=1}^{n} {\uparrow} a_i \subseteq \bigcap\limits_{j=1}^{m} {\uparrow} b_j$. Notice that, by Lemma 2.5.6, we have

$$\begin{aligned} \bigcap_{i=1}^{n} {\uparrow} a_i \subseteq \bigcap_{j=1}^{m} {\uparrow} b_j \iff & \bigcap_{i=1}^{n} {\uparrow} a_i \subseteq {\uparrow} b_j \ \text{ for all } j \in \{1, \dots, m\} \\ \iff & \sigma(b_j) \leq \sigma(a_1) \vee \cdots \vee \sigma(a_n) \ \text{ for all } j \in \{1, \dots, m\} \\ \iff & \sigma(b_1) \vee \cdots \vee \sigma(b_m) \leq \sigma(a_1) \vee \cdots \vee \sigma(a_n). \end{aligned}$$

Then, $v = \sigma(b_1) \vee \cdots \vee \sigma(b_m) \leq \sigma(a_1) \vee \cdots \vee \sigma(a_n) = u$ and hence u is the top of $D(L)$. We thus obtain $\sigma[X]^{\mathrm{u}} \subseteq \sigma[Y]^{\mathrm{lu}}$. Finally, we suppose that $Y \neq \emptyset$. Since X and Y are non-empty and since L is a meet-semilattice, it follows that $\bigcap_{x\in X} \uparrow x = X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}} = \uparrow(\bigwedge Y)$. Then, by Lemma 2.5.6 and since σ is a meet-homomorphism and moreover since $D(L)$ is a lattice, we have that $\sigma[X]^{\mathrm{u}} = \bigcap_{x\in X} \uparrow\sigma(x) \subseteq \uparrow(\bigwedge \sigma[Y]) = \sigma[Y]^{\mathrm{lu}}$. This completes the proof. □

Let P be a mo-distributive poset and $\langle M(P), e\rangle$ its distributive meet-semilattice envelope. By Lemma 2.4.34, we have that e is a strong inf-embedding. Since $M(P)$ is a distributive meet-semilattice, it follows that there exists its distributive envelope $\langle D(M(P)), \sigma\rangle$ and by the previous lemma we know that σ is a strong inf-embedding. It is not hard to check, in general, that the composition of two strong inf-embeddings between posets is a strong inf-embedding. Hence, $\widetilde{\sigma} := \sigma \circ e \colon P \to D(M(P))$ is a strong inf-embedding. Moreover, by Condition (DE1) in Definition 2.4.1 and by Condition (2) in Definition 2.5.1, we have that for each $u \in D(M(P))$, there are non-empty $A_1, \ldots, A_n \subseteq_\omega P$ such that $u = \bigvee_{i=1}^{n} \bigwedge \widetilde{\sigma}[A_i]$. Our next purpose is to obtain an abstract characterization of the distributive lattice $D(M(P))$ with respect to P. To this end, we need the following lemma. Recall, by Lemma 2.4.29, that if $h \colon P \to Q$ is a strong inf-embedding from a poset P to a poset Q, then h is an inf-sup-embedding.

Lemma 2.5.10. *Let P be a mo-distributive poset and $\langle M(P), e\rangle$ its distributive meet-semilattice envelope. If M is a distributive meet-semilattice and $f \colon P \to M$ is a strong inf-embedding, then the unique meet-embedding $h \colon M(P) \to M$ such that $h \circ e = f$, given by Lemma 2.4.3, is such that for each $x, x_1, \ldots, x_n \in M(P)$, if $x \in \{x_1, \ldots, x_n\}^{\mathrm{ul}}$, then $h(x) \in \{h(x_1), \ldots, h(x_n)\}^{\mathrm{ul}}$.*

Proof. Let $x, x_1, \ldots, x_n \in M(P)$ and assume that $x \in \{x_1, \ldots, x_n\}^{\mathrm{ul}}$. By Condition (DE1), there are non-empty $A, A_1, \ldots, A_n \subseteq_\omega P$ such that $x = \bigwedge e[A]$ and $x_i = \bigwedge e[A_i]$ for each $i \in \{1, \ldots, n\}$. Recall that $x \in \{x_1, \ldots, x_n\}^{\mathrm{ul}}$ is equivalent to $\bigcap_{i=1}^{n} \uparrow x_i \subseteq \uparrow x$. Then, we have $\bigcap_{i=1}^{n} \uparrow(\bigwedge e[A_i]) \subseteq \uparrow(\bigwedge e[A])$ and this is equivalent to $\bigcap_{i=1}^{n} e[A_i]^{\mathrm{lu}} \subseteq e[A]^{\mathrm{lu}}$, because $M(P)$ is a meet-semilattice. We know, by Lemma 2.4.34, that e is a strong inf-embedding and then, by Lemma 2.4.36, we obtain that $\bigcap_{i=1}^{n} A_i^{\mathrm{lu}} \subseteq A^{\mathrm{lu}}$. By Lemma 2.4.36 again, we have $\bigcap_{i=1}^{n} f[A_i]^{\mathrm{lu}} \subseteq f[A]^{\mathrm{lu}}$ and this is equivalent to $\bigcap_{i=1}^{n} \uparrow(\bigwedge f[A_i]) \subseteq \uparrow(\bigwedge f[A])$. Since $h \circ e = f$ and h is a meet-homomorphism, it follows that $\bigcap_{i=1}^{n} \uparrow(h(\bigwedge e[A_i])) \subseteq \uparrow(h(\bigwedge e[A]))$ and hence $\bigcap_{i=1}^{n} \uparrow(h(x_i)) \subseteq \uparrow h(x)$. Therefore, $h(x) \in \{h(x_1), \ldots, h(x_n)\}^{\mathrm{ul}}$. □

Theorem 2.5.11. *Let P be a mo-distributive poset and $\langle M(P), e\rangle$ its distributive meet-semilattice envelope. Let $\langle D(M(P)), \sigma\rangle$ the distributive envelope of $M(P)$. Then, $D(M(P))$ is up to isomorphism the unique distributive lattice D for which there is a strong inf-embedding $\widetilde{\sigma}\colon P \to D$ such that for each distributive lattice L and a strong inf-embedding $f\colon P \to L$, there is a unique lattice embedding $K\colon D \to L$ such that $K \circ \widetilde{\sigma} = f$.*

Proof. We know that the map $\widetilde{\sigma} = \sigma \circ e\colon P \to D(M(P))$ is a strong inf-embedding. Let L be a distributive lattice and let $f\colon P \to L$ be a strong inf-embedding. Then, by Lemma 2.4.3, we have that there exists a unique meet-embedding $h\colon M(P) \to L$ such that $f = h \circ e$. By Lemma 2.5.10, we have that h is an almost sup-homomorphism (see page 68). Then, by Theorem 2.5.5, there exists a unique lattice embedding $K\colon D(M(P)) \to L$ such that $h = K \circ \sigma$. We thus obtain $f = h \circ e = (K \circ \sigma) \circ e = K \circ (\sigma \circ e) = K \circ \widetilde{\sigma}$ (see Figure 2.15).

Now assume that D is a distributive lattice and $\widetilde{\sigma}'\colon P \to D$ is a strong inf-embedding such that for each distributive lattice L and a strong inf-embedding $f\colon P \to L$, there is a unique lattice embedding $K\colon D \to L$ such that $K \circ \widetilde{\sigma}' = f$. We need to prove that $D(M(P))$ and D are isomorphic. Since D is a distributive lattice and $\widetilde{\sigma}'\colon P \to D$ is a strong inf-embedding, it follows that there exists a lattice embedding $K\colon D(M(P)) \to D$ such that $\widetilde{\sigma}' = K \circ \widetilde{\sigma}$. Similarly, since $D(M(P))$ is a distributive lattice and $\widetilde{\sigma}\colon P \to D(M(P))$ is a strong inf-embedding, it follows that there exists a lattice embedding $K'\colon D \to D(M(P))$ such that $\widetilde{\sigma} = K' \circ \widetilde{\sigma}'$. We will show that $K \circ K' = \mathrm{id}_D$ and $K' \circ K = \mathrm{id}_{D(M(P))}$. Let $u \in D$. Since $K'(u) \in D(M(P))$, it follows that there exist non-empty $A_1, \dots, A_n \subseteq_\omega P$ such that $K'(u) = \bigvee_{i=1}^{n} \bigwedge \widetilde{\sigma}[A_i]$. So, using that $\widetilde{\sigma} = K' \circ \widetilde{\sigma}'$, we have $K'(u) = \bigvee_{i=1}^{n} \bigwedge K'[\widetilde{\sigma}'[A_i]]$. Since K' is a lattice embedding, it follows that $u = \bigvee_{i=1}^{n} \bigwedge \widetilde{\sigma}'[A_i]$. Hence,

$$\begin{aligned}
K(K'(u)) &= K(\bigvee_{i=1}^{n} \bigwedge \widetilde{\sigma}[A_i]) \\
&= \bigvee_{i=1}^{n} \bigwedge K[\widetilde{\sigma}[A_i]] \\
&= \bigvee_{i=1}^{n} \bigwedge \widetilde{\sigma}'[A_i] \\
&= u.
\end{aligned}$$

Thus, $K \circ K' = \mathrm{id}_D$. Now let $u \in D(M(P))$. So, there are non-empty $A_1, \dots, A_n \subseteq_\omega P$ such that $u = \bigvee_{i=1}^{n} \bigwedge \widetilde{\sigma}[A_i]$. Since K is a lattice embedding, it follows that

$$\begin{tikzcd}
P \arrow[r, "e"] \arrow[rr, bend left, "\widetilde{\sigma} = \sigma \circ e"] \arrow[d, "f"'] & M(P) \arrow[r, "\sigma"] \arrow[dl, dashed, "h"'] & D(M(P)) \arrow[dll, dashed, "K"] \\
L & &
\end{tikzcd}$$

FIGURE 2.15. Commutative diagram of the distributive meet-semilattice envelope and the distributive envelope with respect to a mo-distributive poset.

$K(u) = \bigvee\limits_{i=1}^{n} \bigwedge K[\widetilde{\sigma}[A_i]] = \bigvee\limits_{i=1}^{n} \bigwedge \widetilde{\sigma}'[A_i]$. Since K' is a lattice embedding, we have

$$\begin{aligned}
K'(K(u)) &= K'(\bigvee_{i=1}^{n} \bigwedge \widetilde{\sigma}'[A_i]) \\
&= \bigvee_{i=1}^{n} \bigwedge K'[\widetilde{\sigma}'[A_i]] \\
&= \bigvee_{i=1}^{n} \bigwedge \widetilde{\sigma}[A_i] \\
&= u.
\end{aligned}$$

Thus, $K' \circ K = \mathrm{id}_{D(M(P))}$. Hence, K and K' are lattice isomorphisms, one invers of the other, and therefore $D(M(P))$ and D are isomorphic. This finishes the proof. □

By the previous theorem, we obtain another abstract characterization of the distributive lattice $D(M(P))$ with respect to P.

THEOREM 2.5.12. *Let P be a mo-distributive poset and $\langle M(P), e\rangle$ its distributive meet-semilattice envelope. Let $\langle D(M(P)), \sigma\rangle$ be the distributive envelope of $M(P)$. Then, $D(M(P))$ is up to isomorphism the unique distributive lattice D for which there is a strong inf-embedding $\widetilde{\sigma}\colon P \to D$ such that for every $u \in D$, there are non-empty $A_1, \ldots, A_n \subseteq_\omega P$ such that $u = \bigvee\limits_{i=1}^{n} \bigwedge \widetilde{\sigma}[A_i]$.*

PROOF. We know that $D(M(P)$ and $\widetilde{\sigma} = \sigma \circ e\colon P \to D(M(P))$ have the property established in the theorem (see page 73). Assume that D is a distributive lattice and $h\colon P \to D$ is a strong inf-embedding such that for each $u \in D$, there are non-empty $A_1, \ldots, A_n \subseteq_\omega P$ such that $u = \bigvee\limits_{i=1}^{n} \bigwedge h[A_i]$. We need to prove that $D(M(P))$ and D are isomorphic. To this end, we show that D has the property established in Theorem 2.5.11, that is, we prove that for each distributive lattice L and a strong inf-embedding $f\colon P \to L$, there is a unique lattice embedding $K\colon D \to L$ such that $f = K \circ h$. Let $A_1, \dots, A_n, B_1, \dots, B_m \subseteq_\omega P$ be non-empty. Suppose that $\bigvee_{i=1}^{n} \bigwedge h[A_i] = \bigvee_{j=1}^{m} \bigwedge h[B_j]$. So, $\bigwedge h[A_i] \leq \bigvee_{j=1}^{m} \bigwedge h[B_j]$ for all $i \in \{1, \dots, n\}$. This is equivalent to $\uparrow(\bigvee_{j=1}^{m} \bigwedge h[B_j]) \subseteq \uparrow(\bigwedge h[A_i])$ for all $i \in \{1, \dots, n\}$. Since D is a lattice, it follows that $\bigcap_{j=1}^{m} h[B_j]^{\mathrm{lu}} \subseteq h[A_i]^{\mathrm{lu}}$ for all $i \in \{1, \dots, n\}$. By Lemma 2.4.36, we obtain that $\bigcap_{j=1}^{m} B_j^{\mathrm{lu}} \subseteq A_i^{\mathrm{lu}}$ for all $i \in \{1, \dots, n\}$. Then, by Lemma 2.4.36 again, we have $\bigcap_{j=1}^{m} f[B_j]^{\mathrm{lu}} \subseteq f[A_i]^{\mathrm{lu}}$ for all $i \in \{1, \dots, n\}$ and, since L is a lattice, it follows that $\bigwedge f[A_i] \leq \bigvee_{j=1}^{m} \bigwedge f[B_j]$ for all $i \in \{1, \dots, n\}$. Then, $\bigvee_{i=1}^{n} \bigwedge f[A_i] \leq \bigvee_{j=1}^{m} \bigwedge f[B_j]$. Similarly we obtain $\bigvee_{j=1}^{m} \bigwedge f[B_j] \leq \bigvee_{i=1}^{n} \bigwedge f[A_i]$ and hence $\bigvee_{i=1}^{n} \bigwedge f[A_i] = \bigvee_{j=1}^{m} \bigwedge f[B_j]$. Therefore, we can define the map $K\colon D \to L$ as follows: for every $u \in D$,

$$K(u) = \bigvee_{i=1}^{n} \bigwedge f[A_i]$$

for some non-empty $A_1, \dots, A_n \subseteq_\omega P$ such that $u = \bigvee_{i=1}^{n} \bigwedge h[A_i]$. By a similar argument used to show that K is well-defined we obtain that K is an injective map. Lastly, it is straightforward to show directly that K is the unique lattice homomorphism and $f = K \circ h$. Therefore, by Theorem 2.5.11, we have that $D(M(P))$ and D are isomorphic. $\square$

Now it makes sense to introduce the following definition.

Definition 2.5.13. Let P be a mo-distributive poset and $M(P)$ its distributive meet-semilattice envelope. Let $D(M(P))$ be the distributive envelope of $M(P)$. We will call the distributive lattice $D(M(P))$ the *distributive lattice envelope* of P and denote it by $D(P)$.

Now we will show a correspondence between the s-optimal Frink-filters of a mo-distributive poset and the prime filters of its distributive lattice envelope. To attain this, without loss of generality we establish the following convention, as we did in §2.4.2. Let P be a mo-distributive poset. Then, the distributive meet-semilattice envelope of P is up to isomorphism the unique distributive meet-semilattice $M(P)$ such that $P \subseteq M(P)$ and Conditions (E1)-(E3) on page 56 hold. The distributive lattice envelope of P is up to isomorphism the unique distributive lattice $D(P)$ such that $M(P) \subseteq D(P)$ and the following conditions are satisfied:

(D1) $M(P)$ is a sub-meet-semilattice of $D(P)$;

(D2) for each $x, x_1, \dots, x_n \in M(P)$,

$$x \in \{x_1, \dots, x_n\}^{\mathrm{ul}} \text{ in } M(P) \quad \text{if and only if} \quad x \in {\downarrow}(x_1 \vee \cdots \vee x_n) \text{ in } D(P);$$

(D3) $M(P)$ is finitely join-dense in $D(P)$.

Let P be a mo-distributive poset and $M(P)$ its distributive meet-semilattice envelope and let $D(P)$ be the distributive lattice envelope of P. Then, by Corollary 2.4.26, we have that the map $\beta\colon \mathsf{Opt}(M(P)) \to \mathsf{Opt}_{\mathsf{s}}(P)$ defined as $\beta(U) = U \cap P$ for each $U \in \mathsf{Opt}(M(P))$ is an order-isomorphism. And, by Theorem 2.5.8, we have that the map $\lambda\colon \mathsf{Fi}^{\mathsf{pr}}(D(P))^* \to \mathsf{Opt}(M(P))$ defined as $\lambda(H) = H \cap M(P)$ for each $H \in \mathsf{Fi}^{\mathsf{pr}}(D(P))^*$ is an order-isomorphism. We thus have proved the following lemma.

Lemma 2.5.14. *Let P be a mo-distributive poset and $D(P)$ its distributive lattice envelope. Then, the map $\mu\colon \mathsf{Fi}^{\mathsf{pr}}(D(P))^* \to \mathsf{Opt}_{\mathsf{s}}(P)$ defined as $\mu(H) = H \cap P$ for each $H \in \mathsf{Fi}^{\mathsf{pr}}(D(P))^*$ is an order-isomorphism.*

Corollary 2.5.15. *Let P be a mo-distributive poset and $D(P)$ its distributive lattice envelope. Then,*

$$\mathsf{Opt}_{\mathsf{s}}(P) = \{H \cap P : \ H \in \mathsf{Fi}^{\mathsf{pr}}(D(P))^*\}.$$

The fact that the ordered sets $\langle \mathsf{Fi}^{\mathsf{pr}}(D(P))^*, \subseteq \rangle$ and $\langle \mathsf{Opt}_{\mathsf{s}}(P), \subseteq \rangle$ are order-isomorphic will be important in Chapter 4 to develop a topological duality for bounded mo-distributive posets.

Notice that, by Theorem 2.5.12, we have an equivalent definition of the distributive lattice envelope of a mo-distributive poset. Let P be a mo-distributive poset. The distributive lattice envelope of P is up to isomorphism the unique distributive lattice $D(P)$ such that $P \subseteq D(P)$ and the following conditions are satisfied:

(D1') for each $A, B \subseteq_\omega P$,

$$A^{\mathrm{u}} \subseteq B^{\mathrm{lu}} \text{ in } P \quad \text{if and only if} \quad A^{\mathrm{u}} \subseteq B^{\mathrm{lu}} \text{ in } D(P);$$

(D2') for each $u \in D(P)$, there are non-empty $A_1, \dots, A_n \subseteq_\omega P$ such that $u = \bigvee_{i=1}^{n} \bigwedge A_i$.

By Condition (D1'), it is easy to show that if P has top (bottom) element $\top$ ($\bot$), then $\top$ ($\bot$) is also the top (bottom) element of $D(P)$.

CHAPTER 3

A spectral-style duality for mo-distributive posets

The famous duality developed in [**54**] by Stone for Boolean algebras was first generalized to bounded distributive lattices by himself [**55**]. The dual topological spaces of distributive lattices given by Stone in [**55**] are the spectral spaces, that is, the sober topological spaces in which the compact open subsets form a base that is closed under finite intersection. Unlike Stone spaces, the spectral spaces are not Hausdorff, not even T_1-spaces (in fact, a spectral space is a Stone space if and only if it is T_1). This is a disadvantage to handle these spaces, but the way in which they are obtained from distributive lattices is considered by some authors the most natural way to obtain a duality for distributive lattices.

In [**34**] Grätzer introduces the class of distributive join-semilattices with bottom element that is a class of ordered algebraic structures larger than the bounded distributive lattices and develops a topological representation for this class of ordered structures that extends the representation given by Stone for bounded distributive lattices. In [**8**] Celani and in [**9**] Celani and Calomino obtained a full topological duality for distributive meet-semilattices with top element (which are dual to join-semilattices with bottom element) and meet-homomorphisms preserving the top element. Actually, in [**8**] Celani used ordered topological spaces for his duality and then in [**9**] Celani and Calomino provided a simplification by means of sober spaces of the topological duality given in [**8**]. Hence, the duality provided by Celani (and by Celani and Calomino) for distributive meet-semilattices with top element can be considered a spectral duality in the spirit of the duality introduced by Stone.

The main purpose of this chapter is to present a spectral-style topological duality for mo-distributive posets and inf-homomorphisms. We intend that this duality be a generalization of the duality due to Celani and so it be a generalization of the duality provided by Stone for distributive lattices.

It is clear that dual results to ours for mo-distributive posets and inf-homomorphisms can be obtained for jo-distributive posets and sup-homomorphisms. In this chapter the topological duality that we develop for mo-distributive posets and inf-homomorphisms uses the notion of prime Frink-filter; thus a topological duality for jo-distributive posets and sup-homomorphisms should use the notion of prime Frink-ideal. Another more interesting possibility to investigate is to look for a spectral-style topological duality for meet-order and join-order distributive posets and inf-sup-homomorphisms. We have decided not to explore this ideas here and leave it for future work, where the duality obtained in this chapter may be helpful to attain this aim.

In Section 3.1 we present a topological representation for mo-distributive posets by means of a certain kind of sober topological spaces. Moreover, we obtain the necessary properties to introduce in Section 3.2 an adequate definition of the dual topological spaces for mo-distributive posets. We prove that the Frink-filters and finitely generated Frink-filters of a mo-distributive poset correspond, respectively, to the open subsets and compact open subsets of the topological space constructed by means of the mo-distributive poset. In Section 3.2 we define the topological spaces, that we call DP-spaces, that will be the dual topological spaces of mo-distributive posets and we study them.

In §3.3 we establish a full topological duality for mo-distributive posets. The morphisms in the category of mo-distributive posets that we consider are the inf-homomorphisms between mo-distributive posets. So, we need to find the morphisms between DP-spaces that correspond to the inf-homomorphisms. In §3.3.1 we introduce the definition of DP-morphism, which is a binary relation between DP-spaces and we prove that the DP-spaces together with the DP-morphisms form a category. Then in §3.3.2 we establish the full categorical duality between the category of mo-distributive posets and inf-homomorphisms, and the category of DP-spaces and DP-morphisms.

In [**16**] David and Erné presented a topological duality for mo-distributive posets[1] where their dual spaces are the same as ours, but the kind of morphism considered in the category of mo-distributive posets by David and Erné is much stronger than the notion of inf-homomorphism that we used in our duality. With this stronger notion of morphism they obtained in the category of topological spaces that the morphisms were functions between the spaces unlike us that we need binary relations between the DP-spaces to represent the inf-homomorphisms. In Section 3.4, we establish a connection between our duality and the duality given by David and Erné and we show how derive the duality of David and Erné from our duality.

Lastly, in Section 3.5 of this chapter we study a completion of a mo-distributive poset that is a particular Δ_1-completion as it is defined in [**27**] and we show that such a completion has very nice properties.

[1] In [**16**], actually, the topological duality is developed for jo-distributive posets. There the jo-distributive posets are called ideal-distributive posets.

3.1. A spectral topological representation

Let P be a fixed but arbitrary mo-distributive poset. We recall that $\mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)$ and $\mathsf{Fi}_{\mathsf{F}}^{\mathsf{irr}}(P)$ denote the collections of all prime Frink-filters and all irreducible Frink-filters of P, respectively. Moreover, since P is mo-distributive, we have $\mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P) = \mathsf{Fi}_{\mathsf{F}}^{\mathsf{irr}}(P)$ and this should be kept in mind for the rest of the chapter because we will use it without mention.

We consider the map $\varphi\colon P \to \mathcal{P}\left(\mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)\right)$ defined by

$$\varphi(a) = \{F \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P) :\ a \in F\}$$

for every $a \in P$. We also consider for every $a \in P$ the set

$$\varphi(a)^c = \{F \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P) :\ a \notin F\}.$$

It is obvious, because each prime Frink-filter of P is proper, that

$$\mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P) = \bigcup_{a\in P} \varphi(a)^c. \tag{3.1}$$

Lemma 3.1.1. *For every $a, b \in P$ and every $F \in \varphi(a)^c \cap \varphi(b)^c$, there exists $c \in P$ such that $F \in \varphi(c)^c \subseteq \varphi(a)^c \cap \varphi(b)^c$.*

Proof. Let $a, b \in P$ and $F \in \varphi(a)^c \cap \varphi(b)^c$. So, $a, b \notin F$. By Lemma 2.1.17, there exists $c \in F^c$ such that $a \leq c$ and $b \leq c$. Then, $F \in \varphi(c)^c$ and $\varphi(c)^c \subseteq \varphi(a)^c \cap \varphi(b)^c$. $\square$

We now consider the topology on $\mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)$ generated by the family $\mathcal{B}_P = \{\varphi(a)^c :\ a \in P\}$, which is denoted by τ_P and we denote the topological space $\langle \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P), \tau_P\rangle$ by $\mathbf{X}(P)$.

Lemma 3.1.2. *The family $\mathcal{B}_P = \{\varphi(a)^c :\ a \in P\}$ is a basis for the space $\boldsymbol{X}(P)$.*

Proof. It is straightforward from (3.1) and the previous lemma. $\square$

We define $P_{\mathbf{X}(P)} := \{\varphi(a) :\ a \in P\}$ and we consider the poset $\left\langle P_{\mathbf{X}(P)}, \subseteq\right\rangle$. Then, we obtain the following theorem:

Theorem 3.1.3 (Representation theorem). *The map $\varphi\colon P \to P_{\boldsymbol{X}(P)}$ is an order-isomorphism.*

Proof. It is clear that φ is an onto map. Let $a, b \in P$. Assume $a \leq b$ and let $F \in \varphi(a)$. Since F is an up-set, it follows that $b \in F$ and so $F \in \varphi(b)$. Hence, $\varphi(a) \subseteq \varphi(b)$. Reciprocally, assume $\varphi(a) \subseteq \varphi(b)$. If $a \nleq b$, then $\uparrow a \cap \downarrow b = \emptyset$. So, by Theorem 2.2.17, there exists $F \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)$ such that $a \in F$ and $b \notin F$. We thus obtain $F \in \varphi(a)$ and $F \notin \varphi(b)$, which is a contradiction. Hence, $a \leq b$. We have proved that φ is an order-embedding. Therefore, φ is an order-isomorphism. $\square$

The following corollary states a useful property of the poset $P_{\mathbf{X}(P)}$.

COROLLARY 3.1.4. *For every $a, a_1, \ldots, a_n \in P$, we have*

$$\varphi(a) \in \{\varphi(a_1), \ldots, \varphi(a_n)\}^{\mathrm{lu}} \quad \textit{if and only if} \quad \varphi(a_1) \cap \cdots \cap \varphi(a_n) \subseteq \varphi(a).$$

PROOF. Let $a, a_1, \ldots, a_n \in P$. Then, by Corollary 2.2.19 and Lemma 2.2.5, we have

$$\begin{aligned}
\varphi(a) \in \{\varphi(a_1), \ldots, \varphi(a_n)\}^{\mathrm{lu}} \ &\text{iff} \ \ a \in \{a_1, \ldots, a_n\}^{\mathrm{lu}} \\
&\text{iff} \ \ (\forall F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)) \big(\text{if } \{a_1, \ldots, a_n\}^{\mathrm{lu}} \subseteq F, \text{ then } a \in F\big) \\
&\text{iff} \ \ (\forall F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)) \, (\text{if } \{a_1, \ldots, a_n\} \subseteq F, \text{ then } a \in F) \\
&\text{iff} \ \ \varphi(a_1) \cap \cdots \cap \varphi(a_n) \subseteq \varphi(a).
\end{aligned}$$

□

Given that the family $\mathcal{B}_P = \{\varphi(a)^c : \ a \in P\}$ is a base for the space $\mathbf{X}(P)$, the specialization quasi-order $\preceq$ of the space $\mathbf{X}(P)$ is the dual of the inclusion order. Indeed: for every $F, G \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$,

$$\begin{aligned}
F \preceq G \ \ &\text{iff} \ \ (\forall a \in P) \, (F \in \varphi(a)^c \text{ implies } G \in \varphi(a)^c) \\
&\text{iff} \ \ (\forall a \in P) \, (a \notin F \text{ implies } a \notin G) \\
&\text{iff} \ \ (\forall a \in P) \, (a \in G \text{ implies } a \in F) \\
&\text{iff} \ \ G \subseteq F.
\end{aligned}$$

Hence, it follows that $\preceq$ is a partial order and thus the next lemma is an immediate consequence:

LEMMA 3.1.5. *The space $\boldsymbol{X}(P)$ is a T_0-space.*

LEMMA 3.1.6. *For every $a \in P$, $\varphi(a)^c$ is a compact subset of $\boldsymbol{X}(P)$.*

PROOF. Let $a \in P$. Because $\mathcal{B}_P$ is a base for the space $\mathbf{X}(P)$ we can use it to prove the compactness of $\varphi(a)^c$. We suppose that there is a subset $\{a_i : \ i \in I\}$ of P such that $\varphi(a)^c \subseteq \bigcup\limits_{i \in I} \varphi(a_i)^c$. So, it is clear that $\bigcap\limits_{i \in I} \varphi(a_i) \subseteq \varphi(a)$. Let us consider the Frink-filter $F = \mathrm{Fi}_{\mathrm{F}}(\{a_i : \ i \in I\})$. Then, by Theorem 2.2.17, we have that $a \in F$. Consequently, there exist $i_1, \ldots, i_n \in I$ such that $a \in \{a_{i_1}, \ldots, a_{i_n}\}^{\mathrm{lu}}$ and so, since φ is an order-isomorphism, we have $\varphi(a) \in \{\varphi(a_{i_1}), \ldots, \varphi(a_{i_n})\}^{\mathrm{lu}}$. Now, by Corollary 3.1.4, we obtain $\varphi(a_{i_1}) \cap \ldots \cap \varphi(a_{i_n}) \subseteq \varphi(a)$ and this implies that $\varphi(a)^c \subseteq \varphi(a_{i_1})^c \cup \ldots \cup \varphi(a_{i_n})^c$. Therefore, $\varphi(a)^c$ is compact. □

So far we have that the space $\mathbf{X}(P)$ is T_0 and has the collection $\mathcal{B}_P = \{\varphi(a)^c : a \in P\}$ as a base of compact open subsets. The next step is to prove that the space $\mathbf{X}(P)$ is sober. For this we need to obtain a characterization of all open subsets and all compact open subsets of the space $\mathbf{X}(P)$. Let us introduce notations that will be used later on. Let $A \subseteq P$, we write

$$\widehat{\varphi}(A) := \{F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P) : \ A \subseteq F\} \quad \text{and} \quad \widehat{\varphi}(A)^c := \{F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P) : \ A \nsubseteq F\}.$$

Notice that for every $A \subseteq P$, we have that $\widehat{\varphi}(A) = \widehat{\varphi}(\mathrm{Fi_F}(A))$.

Lemma 3.1.7. *A subset $\mathcal{U} \subseteq \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$ is an open subset of $\boldsymbol{X}(P)$ if and only if there exists a Frink-filter F of P such that $\widehat{\varphi}(F)^c = \mathcal{U}$.*

Proof. Let $\mathcal{U} \subseteq \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$ be an open subset of $\mathbf{X}(P)$. Since $\mathcal{B}_P$ is a base, it follows that there exists $A \subseteq P$ such that $\mathcal{U} = \bigcup\limits_{a \in A} \varphi(a)^c$. It is not hard to see that $\mathcal{U} = \widehat{\varphi}(A)^c$. Hence, by the previous observation, $\mathcal{U} = \widehat{\varphi}(F)^c$ where $F = \mathrm{Fi_F}(A)$. Conversely, let F be a Frink-filter of P. Notice that $\widehat{\varphi}(F) = \bigcap\limits_{a \in F} \varphi(a)$. Then, $\widehat{\varphi}(F)^c = \bigcup\limits_{a \in F} \varphi(a)^c$ and therefore $\widehat{\varphi}(F)^c$ is an open subset of $\mathbf{X}(P)$. $\square$

Lemma 3.1.8. *A subset $\mathcal{U} \subseteq \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$ is a compact open subset of $\boldsymbol{X}(P)$ if and only if there exists $A \subseteq_\omega P$ such that $\mathcal{U} = \widehat{\varphi}\left(A^{\mathrm{lu}}\right)^c$.*

Proof. Let $\mathcal{U}$ be a compact open subset of the space $\mathbf{X}(P)$. By the previous lemma, there is $F \in \mathsf{Fi}_{\mathsf{F}}(P)$ such that $\mathcal{U} = \widehat{\varphi}(F)^c = \bigcup\limits_{a \in F} \varphi(a)^c$. Since U is compact, it follows that there is $A \subseteq_\omega F$ such that $\mathcal{U} = \bigcup\limits_{a \in A} \varphi(a)^c = \widehat{\varphi}(A)^c = \widehat{\varphi}\left(A^{\mathrm{lu}}\right)^c$. Reciprocally, we assume that $A \subseteq_\omega P$ and we want to prove $\widehat{\varphi}(A)^c$ is a compact open subset of the space $\mathbf{X}(P)$. Since $\widehat{\varphi}\left(A^{\mathrm{lu}}\right)^c = \widehat{\varphi}(A)^c = \bigcup\limits_{a \in A} \varphi(a)^c$ and each $\varphi(a)^c$ is a compact open subset, we have $\widehat{\varphi}\left(A^{\mathrm{lu}}\right)^c$ is a finite union of compact open subsets. Hence, $\widehat{\varphi}\left(A^{\mathrm{lu}}\right)^c$ is a compact open subset of the space $\mathbf{X}(P)$. $\square$

Corollary 3.1.9. *The map $\widehat{\varphi}(.)^c \colon \mathsf{Fi}_{\mathsf{F}}(P) \to \mathsf{O}(\boldsymbol{X}(P))$ is an order-isomorphism. Moreover, the restriction $\widehat{\varphi}(.)^c \colon \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P) \to \mathsf{KO}^*(\boldsymbol{X}(P))$ is also an order-isomorphism, where $\mathsf{KO}^*(\boldsymbol{X}(P)) = \mathsf{KO}(\boldsymbol{X}(P))$, if P has top, and $\mathsf{KO}^*(\boldsymbol{X}(P)) = \mathsf{KO}(\boldsymbol{X}(P)) \setminus \{\emptyset\}$, if P has no top element.*

Proof. By Lemma 3.1.7, we have that $\widehat{\varphi}(.)^c$ is a map from $\mathsf{Fi}_{\mathsf{F}}(P)$ onto $\mathsf{O}\left(\mathbf{X}(P)\right)$. It is straightforward to check that $\widehat{\varphi}(.)^c$ is order-preserving. Let $F, G \in \mathsf{Fi}_{\mathsf{F}}(P)$ and assume $\widehat{\varphi}(F)^c \subseteq \widehat{\varphi}(G)^c$. Suppose that there is $a \in F$ such that $a \notin G$. By Corollary 2.2.18, there exists $H \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$ such that $G \subseteq H$ and $a \notin H$. So $H \notin \widehat{\varphi}(G)^c$, whereupon $H \notin \widehat{\varphi}(F)^c$. We thus get $F \subseteq H$, which implies $a \in H$; this is a contradiction. Hence, $F \subseteq G$. We have proved $\widehat{\varphi}(.)^c$ is an order-embedding. Therefore, $\widehat{\varphi}(.)^c$ is an order-isomorphism. Lastly, by Lemma 3.1.8, we can conclude that the restriction $\widehat{\varphi}(.)^c : \mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P) \to \mathsf{KO}^*(\mathbf{X}(P))$ is an order-isomorphism. $\square$

As outlined above the specialization order of the space $\mathbf{X}(P)$ coincides with the dual of the inclusion order. So, it follows that in the poset $\langle \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P), \preceq \rangle$ we have ${\downarrow}F = \widehat{\varphi}(F)$ for every $F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$. In fact,

$$\begin{aligned} G \in {\downarrow}F \quad &\text{iff} \quad G \preceq F \\ &\text{iff} \quad F \subseteq G \\ &\text{iff} \quad G \in \widehat{\varphi}(F). \end{aligned}$$

LEMMA 3.1.10. *The space* $\boldsymbol{X}(P) = \langle \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P), \tau_P \rangle$ *is sober.*

PROOF. From Lemma 3.1.5 we know that $\mathbf{X}(P)$ is a T_0-space. Let Z be an irreducible closed subset of $\mathbf{X}(P)$. Since Z^c is an open subset, it follows by Lemma 3.1.7 that there exists a Frink-filter F of P such that $Z^c = \widehat{\varphi}(F)^c$. So, $Z = \widehat{\varphi}(F)$. We now show that F is a prime Frink-filter of P. Since $Z \neq \emptyset$, we have $F \neq P$. Let $F_1, F_2 \in \mathsf{Fi}_{\mathsf{F}}(P)$ be such that $F_1 \cap F_2 \subseteq F$. By Corollary 3.1.9, we have that $\widehat{\varphi}(F_1)$ and $\widehat{\varphi}(F_2)$ are closed subsets of $\mathbf{X}(P)$ and $\widehat{\varphi}(F) \subseteq \widehat{\varphi}(F_1) \cup \widehat{\varphi}(F_2)$. Since $\widehat{\varphi}(F) = Z$ is an irreducible closed set, we obtain $\widehat{\varphi}(F) \subseteq \widehat{\varphi}(F_1)$ or $\widehat{\varphi}(F) \subseteq \widehat{\varphi}(F_2)$; which implies, by Corollary 3.1.9, $F_1 \subseteq F$ or $F_2 \subseteq F$. Hence, F is a prime Frink-filter of P and $Z = \widehat{\varphi}(F) = {\downarrow}F$. Therefore, $\mathbf{X}(P)$ is sober. □

3.2. DP-spaces

The main aim of this section is to introduce the definition of the topological spaces that will be dual to the mo-distributive posets and to study these topological spaces by showing their principal properties. We also characterize topologically the Frink-filters, finitely generated Frink-filters and prime Frink-filters of a mo-distributive poset by means of its dual topological space, see Table 3.1.

This sort of topological spaces were considered by David and Erné [**16**] to develop a topological duality for jo-distributive posets and certain particular morphisms (see [**16**, pp. 108]) between jo-distributive posets. In this chapter, the topological spaces discussed here are used to develop a topological duality for a category of mo-distributive posets where the morphisms are the inf-homomorphisms; these morphisms are different from the morphisms considered by David and Erné. Moreover, in Section 3.3 (see on page 101) we show that the kind of topological spaces that is defined in this section is a generalization of the topological spaces considered in [**8**] by Celani to develop a topological duality for distributive meet-semilattices.

We start with the main definition of this section. Recall that $\mathsf{KO}(X)$ denotes de collection of all compact open subsets of a topological space X.

DEFINITION 3.2.1. A triple $\mathbf{X} = \langle X, \tau, \mathcal{B} \rangle$ is a DP-*space* if:

(DP1) $\langle X, \tau \rangle$ is a sober topological space;

(DP2) $\mathcal{B}$ is a base for $\langle X, \tau\rangle$ of compact open subsets that satisfies the following condition:

$$\text{for every } C \in \mathsf{KO}(\mathbf{X}), \text{ there exists } \mathcal{A} \subseteq \mathcal{B} \text{ such that } C = \bigcap \mathcal{A}. \tag{3.2}$$

We often denote a DP-space $\langle X, \tau, \mathcal{B}\rangle$ by $\langle \mathbf{X}, \mathcal{B}\rangle$ or simply by $\mathbf{X}$, if the confusion is unlikely; in this case the base $\mathcal{B}$ is denoted by $\mathcal{B}(\mathbf{X})$. It should be noted, from (3.2), that if $\mathbf{X}$ is a DP-space and there is $U \in \mathcal{B}(\mathbf{X})$ such that $U \subseteq V$ for all $V \in \mathcal{B}(\mathbf{X})$, then $U = \emptyset$. This fact should be kept in mind, because it will be used later on.

Lemma 3.2.2. *Let $\boldsymbol{X}$ be a topological space and let $\mathcal{B}$ be a base for $\boldsymbol{X}$ of compact open subsets. Then, the following conditions are equivalent:*

(1) *for every $U \in \mathcal{B}$ and every $\mathcal{A} \subseteq_\omega \mathcal{B}$,*
$$(\forall V \in \mathcal{B}) \left(\bigcup \mathcal{A} \subseteq V \implies U \subseteq V\right) \implies U \subseteq \bigcup \mathcal{A};$$
(2) *for every $U \in \mathcal{B}$ and every $C \in \mathsf{KO}(\boldsymbol{X})$, if $U \nsubseteq C$ then there is $U_0 \in \mathcal{B}$ such that $U \nsubseteq U_0$ and $C \subseteq U_0$;*
(3) *for every $C \in \mathsf{KO}(\boldsymbol{X})$, there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $C = \bigcap \mathcal{A}$.*

Proof. It is straightforward to show the equivalence between (1) and (2) and, the equivalence between (2) and (3) is given in [**16**, p. 110]. □

Recall that in the previous section for a mo-distributive poset P we build the space $\mathbf{X}(P) = \langle \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P), \tau_P\rangle$ with the base $\mathcal{B}_P = \{\varphi(a)^c : a \in P\}$.

Lemma 3.2.3. *Let P be a mo-distributive poset. Then, $\langle \boldsymbol{X}(P), \mathcal{B}_P\rangle$ is a DP-space.*

Proof. By Lemma 3.1.10, we have $\mathbf{X}(P)$ is sober. By Lemmas 3.1.2 and 3.1.6, $\mathcal{B}_P$ is a base of compact open subsets for the space $\mathbf{X}(P)$. It only remains to prove that the base $\mathcal{B}_P$ satisfies Property (3.2) of Condition (DP2). To attain this, we show that $\mathcal{B}_P$ satisfies Condition (1) in Lemma 3.2.2. Let $b \in P$ and $A \subseteq_\omega P$ be such that

$$(\forall x \in P) \left(\bigcup_{a\in A} \varphi(a)^c \subseteq \varphi(x)^c \implies \varphi(b)^c \subseteq \varphi(x)^c\right). \tag{3.3}$$

We need to prove $\varphi(b)^c \subseteq \bigcup_{a\in A} \varphi(a)^c$. If $A = \emptyset$, by (3.3), we have $\varphi(b)^c \subseteq \varphi(x)^c$ for all $x \in P$. Which implies that b is the top element of P and whereupon $\varphi(b)^c = \emptyset = \bigcup_{a\in A} \varphi(a)^c$. If now $A \neq \emptyset$, then Condition (3.3) implies that $\varphi(b) \in \varphi[A]^{\mathrm{lu}}$ in the poset $\langle P_{\mathbf{X}(P)}, \subseteq\rangle$. So, by Corollary 3.1.4, we have $\bigcap_{a\in A} \varphi(a) \subseteq \varphi(b)$. Then, taking complements we obtain $\varphi(b)^c \subseteq \bigcup_{a\in A} \varphi(a)^c$. Therefore, $\langle \mathbf{X}(P), \mathcal{B}_P\rangle$ is a DP-space. □

Let $\langle \mathbf{X}, \mathcal{B}\rangle$ be a DP-space. We define the set $P_{\mathbf{X}} := \{U^c : U \in \mathcal{B}\}$ and we consider the poset $\langle P_{\mathbf{X}}, \subseteq\rangle$. Since $\mathcal{B}$ is a base for $\mathbf{X}$, it follows that for every $x, x' \in X$

$$\begin{aligned} x \preceq_{\mathbf{X}} x' &\iff (\forall U \in \mathcal{B})(x \in U \implies x' \in U) \\ &\iff (\forall A \in P_{\mathbf{X}})(x' \in A \implies x \in A) \\ &\iff x \in \mathrm{cl}(x'). \end{aligned}$$

These equivalences should be kept in mind, because they will be used in the rest of the chapter without mention. The following corollary is a consequence of Theorem 3.1.3 and Lemma 3.2.3:

Corollary 3.2.4. *For every mo-distributive poset P, there exists a* DP*-space $\langle \boldsymbol{X}, \mathcal{B}\rangle$ such that P is isomorphic to $P_{\boldsymbol{X}}$.*

Let $\langle \mathbf{X}, \mathcal{B}\rangle$ be a DP-space. Notice that the top element of $P_{\mathbf{X}}$ is, if it exists, X. Indeed, if A is the top element of $P_{\mathbf{X}}$, then $A^c \in \mathcal{B}$ and $A^c \subseteq U$ for all $U \in \mathcal{B}$. So, $A^c = \emptyset$ and hence $A = X$. In the rest of this section, unless stated otherwise, we will denote the elements of the poset $P_{\mathbf{X}}$ by $A, B, \dots$ and the elements of $\mathcal{B}$ by $U, V, \dots$. First of all, we show that the property given in Corollary 3.1.4 is satisfied in general on a DP-space. That is:

Lemma 3.2.5. *Let $\langle \boldsymbol{X}, \mathcal{B}\rangle$ be a* DP*-space and let $A, A_1, \dots, A_n \in P_{\boldsymbol{X}}$. Then,*

$$A \in \{A_1, \dots, A_n\}^{\mathrm{lu}} \quad \textit{if and only if} \quad A_1 \cap \dots \cap A_n \subseteq A.$$

Proof. Let $A, A_1, \dots, A_n \in P_{\mathbf{X}}$. Assume first that $A \in \{A_1, \dots, A_n\}^{\mathrm{lu}}$. Let $U \in \mathcal{B}$ such that $A_1^c \cup \ldots \cup A_n^c \subseteq U$. Then, $U^c \subseteq A_1 \cap \dots \cap A_n$ and $U^c \in P_{\mathbf{X}}$, whereupon $U^c \in \{A_1, \dots, A_n\}^{\mathrm{l}}$. So, $U^c \subseteq A$ and then $A^c \subseteq U$. Hence, by (3.2), we have $A^c \subseteq A_1^c \cup \dots \cup A_n^c$. Thus, $A_1 \cap \dots \cap A_n \subseteq A$. Reciprocally, assume that $A_1 \cap \dots \cap A_n \subseteq A$. Let $B \in \{A_1, \dots, A_n\}^{\mathrm{l}}$. So $B \subseteq A_1 \cap \dots \cap A_n$ and this implies that $B \subseteq A$. Hence, $A \in \{A_1, \dots, A_n\}^{\mathrm{lu}}$. □

Theorem 3.2.6. *Let $\langle \boldsymbol{X}, \mathcal{B}\rangle$ be a DP-space. Then the poset $\langle P_{\boldsymbol{X}}, \subseteq\rangle$ is mo-distributive.*

Proof. To prove that the poset $P_{\mathbf{X}}$ is mo-distributive, let $A, B_1, \dots, B_n \in P_{\mathbf{X}}$ such that $A \in \{B_1, \dots, B_n\}^{\mathrm{lu}}$. By the previous lemma, we obtain $B_1 \cap \dots \cap B_n \subseteq A$. Then,

$$\begin{aligned} A &= A \cup (B_1 \cap \dots \cap B_n) \\ &= (A \cup B_1) \cap \dots \cap (A \cup B_n) \\ A^c &= (A^c \cap B_1^c) \cup \dots \cup (A^c \cap B_n^c). \end{aligned}$$

Notice that for every $i \in \{1, \dots, n\}$, we have that $A^c \cap B_i^c \in \mathsf{O}(\mathbf{X})$. Then, for every $i \in \{1, \dots, n\}$ there exists $\{U_j^i : \ j \in J_i\} \subseteq \mathcal{B}$ such that

$$A^c \cap B_i^c = \bigcup_{j \in J_i} U_j^i.$$

So, for every $i \in \{1, \dots, n\}$ it follows that

$$U_j^i \subseteq \bigcup_{j \in J_i} U_j^i \subseteq B_i^c$$

for all $j \in J_i$. On the other hand, we have

$$A^c = \left(\bigcup_{j \in J_1} U_j^1 \right) \cup \cdots \cup \left(\bigcup_{j \in J_n} U_j^n \right).$$

Then, since A^c is compact, it follows that there exist $i_1, \dots, i_m \in \{1, \dots, n\}$ and $k_1 \in J_{i_1}, \dots, k_m \in J_{i_m}$ such that

$$A^c = U_{k_1}^{i_1} \cup \cdots \cup U_{k_m}^{i_m}.$$

Hence,

$$A = \left(U_{k_1}^{i_1}\right)^c \cap \cdots \cap \left(U_{k_m}^{i_m}\right)^c$$

with $\left(U_{k_1}^{i_1}\right)^c, \dots, \left(U_{k_m}^{i_m}\right)^c \in {\uparrow}B_1 \cup \cdots \cup {\uparrow}B_n$. Therefore, $P_{\mathbf{X}}$ is a mo-distributive poset. $\square$

Let $\langle \mathbf{X}, \mathcal{B} \rangle$ be a DP-space and consider the mo-distributive poset $\langle P_{\mathbf{X}}, \subseteq \rangle$ induced by this space. From the previous section and by Theorem 3.2.6 and Lemma 3.2.3, we can consider the DP-space $\mathbf{X}(P_{\mathbf{X}})$. We want to prove that the DP-spaces $\mathbf{X}$ and $\mathbf{X}(P_{\mathbf{X}})$ are homeomorphic. To this end, we will first show a correspondence between the closed subsets of the space $\mathbf{X}$ and the Frink-filters of the poset $P_{\mathbf{X}}$. We start with the following lemma.

Lemma 3.2.7. *Let $\langle \boldsymbol{X}, \mathcal{B} \rangle$ be a DP-space. If C is a closed subset of $\boldsymbol{X}$, then $F_C := \{A \in P_{\boldsymbol{X}} : \ C \subseteq A\}$ is a Frink-filter of $P_{\boldsymbol{X}}$.*

Proof. Let C be a closed subset of the DP-space $\mathbf{X}$. If $C = X$, then $F_C = \{A \in P_{\mathbf{X}} : \ X \subseteq A\}$. If $P_{\mathbf{X}}$ has top element, then $X \in P_{\mathbf{X}}$ and thus $F_C = \{X\} \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathbf{X}})$. If $P_{\mathbf{X}}$ has no top element, then $X \notin P_{\mathbf{X}}$ and so $F_C = \emptyset \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathbf{X}})$. Now, it is assumed that $C \neq X$. So, since $\mathcal{B}$ is a base for $\mathbf{X}$, we have $F_C \neq \emptyset$. To show that F_C is a Frink-filter of the poset $P_{\mathbf{X}}$, let $A_1, \dots, A_n \in F_C$ and $B \in \{A_1, \dots, A_n\}^{\mathrm{lu}}$. So, by Lemma 3.2.5, $A_1 \cap \cdots \cap A_n \subseteq B$. Then $C \subseteq B$, which implies $B \in F_C$. Therefore, F_C is a Frink-filter of $P_{\mathbf{X}}$. $\square$

Recall that $\mathsf{C}(\mathbf{X})$ denotes the lattice of all closed subsets of a topological space $\mathbf{X}$. For every DP-space $\mathbf{X}$, we define the maps $\Phi\colon \mathsf{C}(\mathbf{X}) \to \mathsf{Fi}_{\mathsf{F}}(P_{\mathbf{X}})$ and $\Psi\colon \mathsf{Fi}_{\mathsf{F}}(P_{\mathbf{X}}) \to \mathsf{C}(\mathbf{X})$ as follows:

$$\Phi(C) = F_C \quad \text{and} \quad \Psi(F) = C_F = \bigcap F$$

for every $C \in \mathsf{C}(\mathbf{X})$ and $F \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathbf{X}})$, respectively. From the previous lemma, we have Φ is well-defined and it is clear that Ψ is also well-defined, because for every $F \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathbf{X}})$, $\Psi(F)$ is an intersection of closed subsets of $\mathbf{X}$.

Lemma 3.2.8. *For every DP-space $\boldsymbol{X}$, the maps Φ and Ψ are dual lattice isomorphisms, one inverse of the other. Moreover, if $\mathsf{C}_{\mathsf{irr}}(\boldsymbol{X})$ denotes the collection of all irreducible closed subsets of $\boldsymbol{X}$, then $\mathsf{C}_{\mathsf{irr}}(\boldsymbol{X})$ and $\mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\boldsymbol{X}})$ are dual order-isomorphic under the corresponding restrictions of the maps Φ and Ψ.*

Proof. First we show that Φ and Ψ are one inverse of the other. Let $C \in \mathsf{C}(\mathbf{X})$. We need to show that $C = C_{F_C}$. Let $x \in C$ and let $A \in F_C$. So, $x \in C \subseteq A$ and hence $x \in C_{F_C}$. Let $x \in C_{F_C}$. Thus, $x \in A$ for all $A \in F_C$. Since C is a closed subset of $\mathbf{X}$ and $\mathcal{B}(\mathbf{X})$ is a base for $\mathbf{X}$, it follows that $C = \bigcap \mathcal{A}$ for some $\mathcal{A} \subseteq P_{\mathbf{X}}$. It is clear that $A \in F_C$ for all $A \in \mathcal{A}$. Then, $x \in A$ for all $A \in \mathcal{A}$. This implies $x \in C$. Hence, $C = C_{F_C}$. Now, let $F \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathbf{X}})$. We need to show that $F = F_{C_F}$. Let $A \in F$. It is clear that $C_F \subseteq A$ and so $A \in F_{C_F}$. Let now $A \in F_{C_F}$. Then, $C_F = \bigcap F \subseteq A$. We thus get $A^c \subseteq \bigcup\{B^c : B \in F\}$. Since A^c is compact, we have that there exist $B_1, \dots, B_n \in F$ such that $A^c \subseteq B_1^c \cup \dots \cup B_n^c$. So, $B_1 \cap \dots \cap B_n \subseteq A$. Hence, by Lemma 3.2.5 and since F is a Frink-filter of $P_{\mathbf{X}}$, it follows that $A \in \{B_1, \dots, B_n\}^{\mathrm{lu}} \subseteq F$, which implies $A \in F$. Hence, $F = F_{C_F}$. Therefore, this completes the proof of the fact that Φ and Ψ are one inverse of the other. By the definitions of the maps Φ and Ψ, it is straightforward to check that

$$A \subseteq B \iff \Phi(B) \subseteq \Phi(A) \quad \text{and} \quad F \subseteq G \iff \Psi(G) \subseteq \Psi(F)$$

for every $A, B \in \mathsf{C}(\mathbf{X})$ and for every $F, G \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathbf{X}})$, respectively. Hence, we have proved that Φ and Ψ are dual order-isomorphisms one inverse of the other and therefore they are dual lattice isomorphisms.

Lastly, we prove that the restrictions of Φ and Ψ to $\mathsf{C}_{\mathsf{irr}}(\mathbf{X})$ and $\mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}})$ have, respectively, as range $\mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}})$ and $\mathsf{C}_{\mathsf{irr}}(\mathbf{X})$. Let $C \in \mathsf{C}_{\mathsf{irr}}(\mathbf{X})$. Since $C \neq \emptyset$, we have $F_C \neq P_{\mathbf{X}}$. Let $F_1, F_2 \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathbf{X}})$ be such that $F_1 \cap F_2 \subseteq F_C$. Then, $C \subseteq A_{F_1} \cup A_{F_2}$. Since C is irreducible, we have $C \subseteq A_{F_1}$ or $C \subseteq A_{F_2}$. Thus, $F_1 \subseteq F_C$ or $F_2 \subseteq F_C$. Hence, $\Phi(C) = F_C \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}})$. Now, let $F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}})$. Suppose towards a contradiction that $C_F = \emptyset$. So, by definition of C_F, we have $X = \bigcup_{A \in F} A^c$. As $F \neq P_{\mathbf{X}}$, there is $B \in P_{\mathbf{X}}$ such that $B \notin F$. Since $B^c \subseteq X = \bigcup_{A \in F} A^c$ and B^c is a compact subset of $\mathbf{X}$, it follows that there exist $A_1, \dots, A_n \in F$ such that

$B^c \subseteq A_1^c \cup \cdots \cup A_n^c$. We thus get $A_1 \cap \cdots \cap A_n \subseteq B$. By Lemma 3.2.5, we have $B \in \{A_1, \ldots, A_n\}^{\mathrm{lu}}$, which implies that $B \in F$. This is a contradiction, hence $C_F \neq \emptyset$. We now suppose $C_F \subseteq C_1 \cup C_2$ for some closed subsets C_1 and C_2. So $F_{C_1} \cap F_{C_2} \subseteq F$ and, since F is a prime Frink-filter, we obtain $F_{C_1} \subseteq F$ or $F_{C_2} \subseteq F$. Hence, $C_F \subseteq C_1$ or $C_F \subseteq C_2$. That is, $\Psi(F) = C_F \in \mathsf{C}_{\mathsf{irr}}(\mathbf{X})$. This completes the proof. □

Let $\mathbf{X}$ be a DP-space. We define $\theta_{\mathbf{X}} \colon \mathbf{X} \to \mathcal{P}(P_{\mathbf{X}})$ as follows:

$$\theta_{\mathbf{X}}(x) = \{A \in P_{\mathbf{X}} : \; x \in A\}$$

for every $x \in X$. As usual, we omit the subscript on θ whenever confusion is unlikely. Next lemma shows that the range of this map is included in $\mathbf{X}(P_{\mathbf{X}})$.

Lemma 3.2.9. *Let $\mathbf{X}$ be a DP-space. For every $x \in X$, $\theta(x)$ is a prime Frink-filter of $P_{\mathbf{X}}$.*

Proof. Let $\mathbf{X}$ be a DP-space and $x \in X$. Let $\mathcal{A} \subseteq_\omega \theta(x)$. If $\mathcal{A} = \emptyset$, then $\mathcal{A}^{\mathrm{lu}} = \emptyset$, if $P_{\mathbf{X}}$ has not top element or $\mathcal{A}^{\mathrm{lu}} = \{X\}$, if $P_{\mathbf{X}}$ has top element. In both cases we have $\mathcal{A}^{\mathrm{lu}} \subseteq \theta(x)$. Suppose $\mathcal{A}$ is non-empty and $B \in \mathcal{A}^{\mathrm{lu}}$. Then, by Lemma 3.2.5, we obtain $\bigcap \mathcal{A} \subseteq B$. Since $x \in \bigcap \mathcal{A}$, it follows that $x \in B$ and so $B \in \theta(x)$. Thus $\theta(x)$ is a Frink-filter of $P_{\mathbf{X}}$ and, since $\mathcal{B}(\mathbf{X})$ is a base for $\mathbf{X}$, we have that $\theta(x) \neq P_{\mathbf{X}}$. We show now that the Frink-filter $\theta(x)$ is prime. To prove this, we show that $\theta(x)^c$ is an order-ideal of $P_{\mathbf{X}}$. Given that $\theta(x)$ is a Frink-filter of $P_{\mathbf{X}}$, we have that $\theta(x)^c$ is a down-set of $P_{\mathbf{X}}$. Let $A, B \in \theta(x)^c$. So, $x \in A^c \cap B^c$. Since $A^c \cap B^c$ is an open subset, it follows that there is $U \in \mathcal{B}(\mathbf{X})$ such that $x \in U \subseteq A^c \cap B^c$. Then, $U^c \in \theta(x)^c$ and $A, B \subseteq U^c$. Hence, $\theta(x)^c$ is an order-ideal and therefore $\theta(x)$ is a prime Frink-filter of $P_{\mathbf{X}}$. □

Theorem 3.2.10. *Let $\mathbf{X}$ be a DP-space. Then, θ is a homeomorphism and therefore $\mathbf{X}$ and $\mathbf{X}(P_{\mathbf{X}})$ are homeomorphic DP-spaces.*

Proof. From the previous lemma we know that θ is well defined. Notice that the basic open subsets of the DP-space $\mathbf{X}(P_{\mathbf{X}})$ are of the form

$$\varphi(A)^c = \{F \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P_{\mathbf{X}}) : \; A \notin F\}$$

for each $A \in P_{\mathbf{X}}$. So, to prove that θ is continuous, let $A \in P_{\mathbf{X}}$ and $x \in X$. Then,

$$\begin{aligned} x \in \theta^{-1}[\varphi(A)^c] &\iff \theta(x) \in \varphi(A)^c \\ &\iff A \notin \theta(x) \\ &\iff x \notin A \\ &\iff x \in A^c \end{aligned}$$

P		$\mathbf{X}(P)$	$\mathbf{X}$		$P_{\mathbf{X}}$
$\mathsf{Fi}_{\mathsf{F}}(P)$	$\cong$	$\mathsf{O}(\mathbf{X}(P))$	$\mathsf{C}(\mathbf{X})$	$\overset{D}{\cong}$	$\mathsf{Fi}_{\mathsf{F}}(P_{\mathbf{X}})$
$\mathsf{Fi}^{\mathsf{f}}_{\mathsf{F}}(P)$	$\cong$	$\mathsf{KO}^*(\mathbf{X}(P))$			
			$\mathsf{C}_{\mathsf{irr}}(\mathbf{X})$	$\overset{D}{\cong}$	$\mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}})$

TABLE 3.1. Correspondences between Frink-filters of P and open subsets of $\mathbf{X}(P)$, and between closed subsets of $\mathbf{X}$ and Frink-filters of $P_{\mathbf{X}}$.

and since $A^c \in \mathcal{B}(\mathbf{X})$, it follows that $\theta^{-1}[\varphi(A)^c]$ is an open subset of $\mathbf{X}$. Hence, θ is continuous. Next, we show θ is an onto map. Let $F \in \mathbf{X}(P_{\mathbf{X}}) = \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}})$. By Lemma 3.2.8, $C_F = \bigcap F$ is an irreducible closed subset of $\mathbf{X}$ and given that $\mathbf{X}$ is a sober space, we have that there exists a unique $x \in X$ such that $\mathrm{cl}(x) = C_F$. Let $A \in P_{\mathbf{X}}$, then we have

$$\begin{aligned} A \in \theta(x) &\iff x \in A \\ &\iff \mathrm{cl}(x) \subseteq A \\ &\iff C_F \subseteq A \\ &\iff A \in F. \end{aligned}$$

We thus obtain $\theta(x) = F$ and therefore θ is onto. Now, let us show that θ is an open map. Let $U \in \mathcal{B}(\mathbf{X})$. So, we have

$$\begin{aligned} F \in \theta[U] &\iff \exists x \in U(F = \theta(x)) \\ &\iff \exists x \in X(U^c \notin \theta(x) = F) \\ &\iff F \in \varphi(U^c)^c. \end{aligned}$$

Hence, θ is an open map. It should be noted that in the last equivalence it is necessary to use the fact that θ is an onto map. Finally, since $\mathbf{X}$ is a T_0-space, it is clear that θ is an injective map. Therefore, $\theta\colon \mathbf{X} \to \mathbf{X}(P_{\mathbf{X}})$ is a homeomorphism. □

In Table 3.1 we summarize the relations between the different classes of Frink-filters of a mo-distributive poset P and the different classes of open subsets of its dual DP-space $\mathbf{X}(P)$ that we obtained in the previous section. Table 3.1 shows also relations between the different classes of closed subsets of a DP-space $\mathbf{X}$ and the different classes of Frink-filters of the mo-distributive poset $P_{\mathbf{X}}$.

3.3. Spectral topological duality

The main purpose of this section is to extend the results obtained in the previous section to a full categorical duality between a certain category of mo-distributive posets and a certain categoriy of DP-spaces. The first step to achieve this goal is to give an adequate definition of morphism between DP-spaces.

3.3.1. DP-morphisms. The kind of morphism that we define between DP-spaces in this subsection is motivated by Celani in [**8**] (see also [**9**]) and therefore this kind of morphism is a generalization to that given by Celani.

Let $\mathbf{X}$ and $\mathbf{Y}$ be DP-spaces and let $R \subseteq X \times Y$ be a binary relation. We define the map $h_R \colon \mathcal{P}(Y) \to \mathcal{P}(X)$ as follows:

$$h_R(Z) := \{x \in X : \ R[x] \subseteq Z\} \quad \text{for every } Z \subseteq Y.$$

Definition 3.3.1. Let $\mathbf{X}$ and $\mathbf{Y}$ be DP-spaces. A binary relation $R \subseteq X \times Y$ is said to be a DP-*morphism* if

(DPM1) $h_R(B) \in P_{\mathbf{X}}$ for all $B \in P_{\mathbf{Y}}$;

(DPM2) $R[x] = \bigcap\{B \in P_{\mathbf{Y}} : \ R[x] \subseteq B\}$ for all $x \in X$.

In this case, we write $R \subseteq \mathbf{X} \times \mathbf{Y}$.

Since $\mathcal{B}(\mathbf{Y})$ is a base for $\mathbf{Y}$, it follows that Condition (DPM2) in the above definition of DP-morphism is equivalent to the following condition:

(DPM2') $R[x]$ is a closed subset of $\mathbf{Y}$ for all $x \in X$.

Notice that Condition (DPM1) tells us that the restriction of h_R to $P_{\mathbf{Y}}$ is a map from the poset $P_{\mathbf{Y}}$ to the poset $P_{\mathbf{X}}$. Moreover, it is not hard to check that for every $Z_1, Z_2 \subseteq Y$

$$h_R(Z_1 \cap Z_2) = h_R(Z_1) \cap h_R(Z_2) \quad \text{and} \quad h_R(Y) = X. \tag{3.4}$$

Lemma 3.3.2. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be* DP*-spaces and let $R \subseteq \boldsymbol{X} \times \boldsymbol{Y}$ be a* DP*-morphism. Then, the map $h_R \colon P_{\mathbf{Y}} \to P_{\mathbf{X}}$ is an inf-homomorphism.*

Proof. Observe that if $B_1, B_2 \in P_{\mathbf{Y}}$ and $B_1 \wedge B_2$ exists in $P_{\mathbf{Y}}$, then $B_1 \wedge B_2 = B_1 \cap B_2$. We show that the map $h_R \colon P_{\mathbf{Y}} \to P_{\mathbf{X}}$ is a $\wedge$-homomorphism. So, let $B_1, B_2 \in P_{\mathbf{Y}}$ and suppose that $B_1 \cap B_2 \in P_{\mathbf{Y}}$. By Condition (DPM1) and (3.4), we have that $h_R(B_1) \cap h_R(B_2) = h_R(B_1 \cap B_2) \in P_{\mathbf{X}}$. Then, h_R is a $\wedge$-homomorphism. Moreover, notice that from (3.4) and (DPM1) we have that if $P_{\mathbf{Y}}$ has top element, that is, if $Y \in P_{\mathbf{Y}}$ (see on page 86), then $h_R(Y) = X \in P_{\mathbf{X}}$; in other words, h_R preserves top element, if it exists. Hence, since the poset $P_{\mathbf{Y}}$ is mo-distributive, by Lemma 2.3.16 it follows that h_R is an inf-homomorphism. $\square$

Unfortunately, the usual set-theoretical composition of two DP-morphisms may not be a DP-morphism. So, we need to introduce an adequate notion of composition between DP-morphisms. To this end, we follow the same approach as Celani and Calomino follow in [**9**].

Definition 3.3.3. Let $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ be DP-spaces and let $R \subseteq \mathbf{X} \times \mathbf{Y}$ and $S \subseteq \mathbf{Y} \times \mathbf{Z}$ be DP-morphisms. We define the binary relation $S * R \subseteq X \times Z$ as follows:

$$(x, z) \in S * R \iff (\forall C \in P_{\mathbf{Z}})((R \circ S)[x] \subseteq C \implies z \in C) \tag{3.5}$$

for every pair $(x, z) \in X \times Z$.

It should be noted, from the previous definition and from the fact that $\mathcal{B}(\mathbf{Z})$ is a base for the DP-space $\mathbf{Z}$, that for every $x \in X$,

$$(S * R)[x] = \mathrm{cl}((R \circ S)[x]). \tag{3.6}$$

This implies that the binary relation $S * R \subseteq X \times Z$ satisfies Condition (DPM2) (or equivalently (DPM2')).

Lemma 3.3.4. *Let $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ be* DP*-spaces and let $R \subseteq \boldsymbol{X} \times \boldsymbol{Y}$ and $S \subseteq \boldsymbol{Y} \times \boldsymbol{Z}$ be* DP*-morphisms. Then, for every $C \in P_{\boldsymbol{Z}}$,*

$$h_{S*R}(C) = (h_R \circ h_S)(C).$$

Proof. Let $C \in P_{\mathbf{Z}}$ and let $x \in h_{S*R}(C)$. Then $(S * R)[x] \subseteq C$, which implies that $(R \circ S)[x] \subseteq C$. We need to show that $R[x] \subseteq h_S(C)$. Let $y \in R[x]$ and let $z \in S[y]$. So $z \in (R \circ S)[x]$, whereupon $z \in C$. This implies that $R[x] \subseteq h_S(C)$ and thus $x \in (h_R \circ h_S)(C)$. Therefore, $h_{S*R}(C) \subseteq (h_R \circ h_S)(C)$.

Let now $x \in (h_R \circ h_S)(C)$. Then $R[x] \subseteq h_S(C)$, which implies that $(R \circ S)[x] \subseteq C$. Since C is a closed subset of $\mathbf{Z}$, we have $\mathrm{cl}((R \circ S)[x]) \subseteq C$. Then, by (3.6), it follows that $(S * R)[x] \subseteq C$. Thus, $x \in h_{S*R}(C)$ and hence $(h_R \circ h_S)(C) \subseteq h_{S*R}(C)$. This completes the proof. □

From this lemma and by the previous observation, the next corollary easily follows.

Corollary 3.3.5. *Let $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ be* DP*-spaces. If $R \subseteq \boldsymbol{X} \times \boldsymbol{Y}$ and $S \subseteq \boldsymbol{Y} \times \boldsymbol{Z}$ are* DP*-morphisms, then $S * R \subseteq X \times Z$ is a* DP*-morphism.*

Lemma 3.3.6. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be* DP*-spaces and let $R_1, R_2 \subseteq \boldsymbol{X} \times \boldsymbol{Y}$ be* DP*-morphisms. If the restrictions of h_{R_1} and h_{R_2} to $P_{\mathbf{Y}}$ are the same, then $R_1 = R_2$.*

Proof. Given $x \in X$, we want to prove that $R_1[x] = R_2[x]$. So, let $y \in R_1[x]$ and let $B \in P_{\mathbf{Y}}$ be such that $R_2[x] \subseteq B$. By definition we have that $x \in h_{R_2}(B)$ and then, by the hypothesis, we obtain $x \in h_{R_1}(B)$. Then, $R_1[x] \subseteq B$ and so $y \in B$. Hence, by (DPM2), we have that $y \in R_2[x]$. We thus get $R_1[x] \subseteq R_2[x]$.

With a similar argument, we obtain $R_2[x] \subseteq R_1[x]$. Then, $R_1[x] = R_2[x]$ for all $x \in X$. Therefore, $R_1 = R_2$. $\square$

Using now the previous lemma and the fact that the usual set-theoretical composition of functions is associative, it is easy to show that the binary relation $*$ is associative.

LEMMA 3.3.7. *Let* $\boldsymbol{X}$*,* $\boldsymbol{Y}$*,* $\boldsymbol{Z}$ *and* $\boldsymbol{W}$ *be* DP*-spaces. If* $R \subseteq \boldsymbol{X} \times \boldsymbol{Y}$*,* $S \subseteq \boldsymbol{Y} \times \boldsymbol{Z}$ *and* $T \subseteq \boldsymbol{Z} \times \boldsymbol{W}$ *are* DP*-morphisms, then* $T * (S * R) = (T * S) * R$*.*

PROOF. Observe that $T * (S * R), (T * S) * R \subseteq \mathbf{X} \times \mathbf{W}$. Let $D \in \mathbf{W}$. By Lemma 3.3.4, we have

$$\begin{aligned} h_{T*(S*R)}(D) &= (h_{S*R} \circ h_T)(D) \\ &= ((h_R \circ h_S) \circ h_T)(D) \\ &= (h_R \circ (h_S \circ h_T))(D) \\ &= (h_R \circ (h_{T*S}))(D) \\ &= \left(h_{(T*S)*R}\right)(D). \end{aligned}$$

Then, by the previous lemma, we obtain $T * (S * R) = (T * S) * R$. $\square$

LEMMA 3.3.8. *Let* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *be* DP*-spaces and* $R \subseteq \boldsymbol{X} \times \boldsymbol{Y}$ *a* DP*-morphism. Let* $x, x' \in X$*. If* $x \preceq_{\mathbf{X}} x'$*, then* $R[x] \subseteq R[x']$*.*

PROOF. Assume $x \preceq_{\mathbf{X}} x'$ and let $y \in R[x]$. Let $B \in P_{\mathbf{Y}}$ be such that $R[x'] \subseteq B$. So, $x' \in h_R(B)$. Since $h_R(B)$ is a closed subset of $\mathbf{X}$ and $x \preceq_{\mathbf{X}} x'$, it follows that $x \in h_R(B)$. Then, $R[x] \subseteq B$ and we thus get $y \in B$. Hence, by (DPM2), we have $y \in R[x']$. Therefore, $R[x] \subseteq R[x']$. $\square$

LEMMA 3.3.9. *Let* $\boldsymbol{X}$ *be a* DP*-space. Then, the dual of the specialization order of the space* $\boldsymbol{X}$ *is a* DP*-morphism.*

PROOF. Let $x \in X$. Then,

$$\begin{aligned} \succeq [x] &= \{x' \in X : \ x \succeq x'\} \\ &= \{x' \in X : \ x' \preceq x\} \\ &= {\downarrow} x \\ &= \mathrm{cl}(x). \end{aligned}$$

Then, $\succeq [x]$ is a closed subset of $\mathbf{X}$ and hence Condition (DPM2) holds. Let now $A \in P_{\mathbf{X}}$. So, using the previous equality we have

$$\begin{aligned} h_{\succeq}(A) &= \{x \in X : \succeq [x] \subseteq A\} \\ &= \{x \in X : \ \mathrm{cl}(x) \subseteq A\} \end{aligned}$$

$$\begin{aligned} &= \{x \in X : \ x \in A\} \\ &= A. \end{aligned}$$

Then, $h_{\succeq}(A) \in P_{\mathbf{X}}$ and thus $\succeq$ satisfies Condition (DPM1). Therefore, $\succeq$ is a DP-morphism. $\square$

LEMMA 3.3.10. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be DP-spaces and let $R \subseteq \boldsymbol{X} \times \boldsymbol{Y}$ be a DP-morphism. Then,*

$$R * \succeq_{\boldsymbol{X}} = R \quad \text{and} \quad \succeq_{\boldsymbol{Y}} * R = R.$$

PROOF. Let us see first that $\succeq_{\mathbf{X}} \circ R = R$. Let $(x, y) \in \succeq_{\mathbf{X}} \circ R$. So, there is $x' \in X$ such that $x' \preceq_{\mathbf{X}} x$ and $y \in R[x']$. By Lemma 3.3.8, we have $R[x'] \subseteq R[x]$ and we thus obtain $y \in R[x]$. Hence, $\succeq_{\mathbf{X}} \circ R \subseteq R$. The inverse inclusion is a consequence of the fact $\succeq_{\mathbf{X}}$ is reflexive. Hence, $\succeq_{\mathbf{X}} \circ R = R$. Now, for every $x \in X$ we have

$$\begin{aligned} (R * \succeq_{\mathbf{X}})\,[x] &= \mathrm{cl}\,((\succeq_{\mathbf{X}} \circ R)\,[x]) \\ &= \mathrm{cl}\,(R[x]) \\ &= R[x]. \end{aligned}$$

Then, $R * \succeq_{\mathbf{X}} = R$. Similarly, we can get $\succeq_{\mathbf{Y}} * R = R$. $\square$

Collecting all the results we have obtained so far, we can define the category of all DP-spaces and all DP-morphisms, where the composition between DP-morphisms is $*$ and for every DP-space $\mathbf{X}$ the identity DP-morphism is $\succeq_{\mathbf{X}}$. We denote this category by $\mathbb{DPS}$.

3.3.2. Duality. Recall that $\mathbb{MODP}$ denotes the category of all mo-distributive posets and all inf-homomorphisms between mo-distributive posets where the composition of morphisms is the usual set-theoretical composition of functions and the identity morphism for every object of $\mathbb{MODP}$ is the identity map.

Now, we can define $\Delta\colon \mathbb{DPS} \to \mathbb{MODP}$ as follows:

- for every DP-space $\mathbf{X}$,

$$\Delta(\mathbf{X}) := \langle P_{\mathbf{X}}, \subseteq \rangle;$$

- for every morphism $R \subseteq \mathbf{X} \times \mathbf{Y}$ of $\mathbb{DPS}$,

$$\Delta(R) := h_R\colon P_{\mathbf{Y}} \to P_{\mathbf{X}}.$$

From Theorem 3.2.6 and Lemma 3.3.2, it follows that Δ sends objects and morphisms of the category $\mathbb{DPS}$ to objects and morphisms of the category $\mathbb{MODP}$, respectively. Moreover, by Lemma 3.3.4 and using the second part in the proof of Lemma 3.3.9, we have that for all DP-spaces $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ and for all DP-morphisms

$R \subseteq \mathbf{X}\times\mathbf{Y}$ and $S \subseteq \mathbf{Y}\times\mathbf{Z}$, it follows that $\Delta(S*R) = h_{S*R} = h_R \circ h_S = \Delta(R)\circ\Delta(S)$ and $\Delta(\succeq_X) = h_{\succeq_{\mathbf{X}}} = \mathrm{id}_{P_{\mathbf{X}}}$. Therefore, we have proved the following lemma:

Lemma 3.3.11. *$\Delta\colon \mathbb{DPS} \to \mathbb{MODP}$ is a contravariant functor.*

Now we want to find a contravariant functor from $\mathbb{MODP}$ to $\mathbb{DPS}$. To this end, by Lemma 3.2.3, we only need to define the image of morphisms of the category $\mathbb{MODP}$. So, let P and Q be mo-distributive posets and let $h\colon P \to Q$ be an inf-homomorphism. We define a binary relation $R_h \subseteq \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(Q) \times \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$ as follows:

$$G\, R_h\, F \iff h^{-1}[G] \subseteq F$$

for every pair $(G, F) \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(Q) \times \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$.

Recall that for a mo-distributive poset P, the map $\varphi\colon P \to P_{\mathbf{X}(P)} = \{\varphi(a) : a \in P\}$ defined by $\varphi(a) = \{F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P) : \ a \in F\}$ for every $a \in P$ is an order-isomorphism from the poset P to the poset $\langle P_{\mathbf{X}(P)}, \subseteq\rangle$. Moreover, recall that the distinguished base of the DP-space $\mathbf{X}(P)$ is $\mathcal{B}(\mathbf{X}(P)) = \mathcal{B}_P = \{\varphi(a)^c : \ c \in P\}$ (see Lemma 3.2.3 on page 85).

Lemma 3.3.12. *If P and Q are mo-distributive posets and $h\colon P \to Q$ is an inf-homomorphism, then for every $a \in P$ we have*

$$h_{R_h}(\varphi(a)) = \varphi(h(a)).$$

Proof. Let $a \in P$ and let $G \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(Q)$. Then, by Lemma 2.3.6 and by Corollary 2.2.19, we have

$$\begin{aligned}
G \in \varphi(h(a)) &\iff h(a) \in G\\
&\iff a \in h^{-1}[G]\\
&\iff (\forall F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P))(h^{-1}[G] \subseteq F \implies a \in F)\\
&\iff (\forall F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P))(G R_h F \implies F \in \varphi(a))\\
&\iff R_h[G] \subseteq \varphi(a)\\
&\iff G \in h_{R_h}(\varphi(a)).
\end{aligned}$$

Hence, $h_{R_h}(\varphi(a)) = \varphi(h(a))$ for all $a \in P$. □

Lemma 3.3.13. *Let P and Q be mo-distributive posets and let $h\colon P \to Q$ be an inf-homomorphism. Then, $R_h \subseteq \boldsymbol{X}(Q) \times \boldsymbol{X}(P)$ is a DP-morphism.*

Proof. We need to prove the two conditions of Definition 3.3.1. By Lemma 3.3.12, Condition (DPM1) holds. To prove Condition (DPM2), let $G \in \mathbf{X}(Q)$. It is clear that

$$R_h[G] \subseteq \bigcap\{\varphi(a) \in P_{\mathbf{X}(P)} : \ R_h[G] \subseteq \varphi(a)\}.$$

Let $F \in \bigcap\{\varphi(a) \in P_{\mathbf{X}(P)} : \ R_h[G] \subseteq \varphi(a)\}$ and let $b \in h^{-1}[G]$. So, by the previous lemma, we have $R_h[G] \subseteq \varphi(b)$. Then $F \in \varphi(b)$, which implies $b \in F$. We thus obtain $h^{-1}[G] \subseteq F$. Then, $F \in R_h[G]$. Hence,

$$R_h[G] = \bigcap\{\varphi(a) \in P_{\mathbf{X}(P)} : \ R_h[G] \subseteq \varphi(a)\}.$$

Therefore, R_h is a DP-morphism. ☐

We now can define $\Gamma\colon \mathbb{MODP} \to \mathbb{DPS}$ as follows:

- for every mo-distributive poset P,

$$\Gamma(P) := \langle \mathbf{X}(P), \mathcal{B}_P \rangle;$$

- for every morphism $h\colon P \to Q$ of the category $\mathbb{MODP}$,

$$\Gamma(h) := R_h \subseteq \mathbf{X}(Q) \times \mathbf{X}(P).$$

Thus, by Lemmas 3.2.3 and 3.3.13, Γ sends objects and morphisms from the category $\mathbb{MODP}$ to objects and morphisms of $\mathbb{DPS}$, respectively.

Lemma 3.3.14. *$\Gamma\colon \mathbb{MODP} \to \mathbb{DPS}$ is a contravariant functor.*

Proof. By the previous paragraph, we only need to prove two things: (1) Γ sends all identity morphisms of $\mathbb{MODP}$ to the corresponding identity morphisms of $\mathbb{DPS}$ and (2) Γ respects adequately the composition of morphisms between the categories.

Let P be a mo-distributive poset and let $\mathrm{id}_P\colon P \to P$ be the identity map. We must to prove that $\Gamma(\mathrm{id}_P) = \succeq_{\mathbf{X}(P)}$. So, let $F_1, F_2 \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$. Then, we have

$$\begin{aligned} F_1 R_{\mathrm{id}_P} F_2 &\iff \mathrm{id}^{-1}[F_1] \subseteq F_2 \\ &\iff F_1 \subseteq F_2 \\ &\iff F_1 \succeq_{\mathbf{X}(P)} F_2. \end{aligned}$$

Hence, $\Gamma(\mathrm{id}_P) = R_{\mathrm{id}_P} = \succeq_{\mathbf{X}(P)}$.

Now, let P_1, P_2 and P_3 be mo-distributive posets and let $h_1\colon P_1 \to P_2$ and $h_2\colon P_2 \to P_3$ be inf-homomorphisms. We must to prove that $\Gamma(h_2 \circ h_1) = \Gamma(h_1) * \Gamma(h_2)$. Observe that $\Gamma(h_1) = R_{h_1} \subseteq \mathbf{X}(P_2) \times \mathbf{X}(P_1)$, $\Gamma(h_2) = R_{h_2} \subseteq \mathbf{X}(P_3) \times \mathbf{X}(P_2)$ and $\Gamma(h_2 \circ h_1) = R_{(h_2 \circ h_1)} \subseteq \mathbf{X}(P_3) \times \mathbf{X}(P_1)$. So, we want to prove that $R_{(h_2 \circ h_1)} = R_{h_1} * R_{h_2}$. By Lemma 3.3.6, it is enough to show that $h_{R_{(h_2 \circ h_1)}} = h_{R_{h_1} * R_{h_2}}$. Let $a \in P_1$. Then, using Lemma 3.3.12 we have

$$\begin{aligned} h_{R_{(h_2 \circ h_1)}}(\varphi(a)) &= \varphi\left((h_2 \circ h_1)(a)\right) \\ &= \varphi\left(h_2(h_1(a))\right) \\ &= h_{R_{h_2}}\left(\varphi(h_1(a))\right) \\ &= h_{R_{h_2}}\left(h_{R_{h_1}}(\varphi(a))\right) \end{aligned}$$

$$= \left(h_{R_{h_2}} \circ h_{R_{h_1}}\right)(\varphi(a)).$$

We thus obtain $h_{R_{(h_2 \circ h_1)}} = h_{R_{h_2}} \circ h_{R_{h_1}}$. Now, by Lemma 3.3.4, we know that $h_{R_{h_2}} \circ h_{R_{h_1}} = h_{(R_{h_1} * R_{h_2})}$. We thus get $h_{R_{(h_2 \circ h_1)}} = h_{(R_{h_1} * R_{h_2})}$ and this implies that $R_{(h_2 \circ h_1)} = R_{h_1} * R_{h_2}$. This completes the proof. $\square$

Our purpose is to prove that the categories $\mathbb{MODP}$ and $\mathbb{DPS}$ are dually equivalent via the functors $\Delta\colon \mathbb{DPS} \to \mathbb{MODP}$ and $\Gamma\colon \mathbb{MODP} \to \mathbb{DPS}$. So, we need to define natural equivalences $\eta\colon \mathrm{Id}_{\mathbb{DPS}} \cong \Gamma \circ \Delta$ and $\mu\colon \mathrm{Id}_{\mathbb{MODP}} \cong \Delta \circ \Gamma$, where $\mathrm{Id}_{\mathbb{MODP}}$ and $\mathrm{Id}_{\mathbb{DPS}}$ are the identity functors on the categories $\mathbb{MODP}$ and $\mathbb{DPS}$, respectively.

Let $\mathbf{X}$ be a DP-space. Recall from §3.1, applied to the mo-distributive poset $P_{\mathbf{X}}$, that $\mathbf{X}(P_{\mathbf{X}}) = \langle \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}}), \tau_{P_{\mathbf{X}}}, \mathcal{B}_{P_{\mathbf{X}}}\rangle$ where $\mathcal{B}_{P_{\mathbf{X}}} = \{\varphi(A)^c : \ A \in P_{\mathbf{X}}\}$ with $\varphi(A) = \{F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}}) : \ A \in F\}$ for every $A \in P_{\mathbf{X}}$ and thus, $P_{\mathbf{X}(P_{\mathbf{X}})} = \{\varphi(A) : A \in P_{\mathbf{X}}\}$. Recall also from Theorem 3.2.10 that $\theta\colon \mathbf{X} \to \mathbf{X}(P_{\mathbf{X}})$ defined by $\theta(x) = \{A \in P_{\mathbf{X}} : \ x \in A\}$ is a homeomorphism. It should be kept in mind that for every mo-distributive poset P and every DP-space $\mathbf{X}$, we have $(\Delta \circ \Gamma)(P) = P_{\mathbf{X}(P)}$ and $(\Gamma \circ \Delta)(\mathbf{X}) = \mathbf{X}(P_{\mathbf{X}})$.

Let $\mathbf{X}$ be a DP-space. We define the binary relation $R_\theta \subseteq \mathbf{X} \times \mathbf{X}(P_{\mathbf{X}})$ as follows: for every $x \in X$ and every $F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}})$

$$xR_\theta F \iff \theta(x) \subseteq F.$$

We can easily observe that for every $x \in X$, $xR_\theta\theta(x)$ because $\theta(x) \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}})$.

Lemma 3.3.15. *For every DP-space $\mathbf{X}$, the relation R_θ is a DP-morphism.*

Proof. We prove Conditions (DPM1) and (DPM2) of Definition 3.3.1. To prove Condition (DPM1), let $A \in P_{\mathbf{X}}$. We show that $h_{R_\theta}(\varphi(A)) = A$. Let $x \in h_{R_\theta}(\varphi(A))$. So, by definition, $R_\theta[x] \subseteq \varphi(A)$. Since $\theta(x) \in R_\theta[x]$, we have $A \in \theta(x)$. That is, $x \in A$. Reciprocally, let $x \in A$. So, $A \in \theta(x)$. Let $F \in R_\theta[x]$. Thus, $\theta(x) \subseteq F$; this implies that $A \in F$. Then, $F \in \varphi(A)$. Hence, we have proved that $R_\theta[x] \subseteq \varphi(A)$ and thus, $x \in h_{R_\theta}(\varphi(A))$. Therefore, Condition (DPM1) holds. To prove (DPM2), let $x \in X$. We need to show that

$$R_\theta[x] = \bigcap\{\varphi(A) : \ A \in P_{\mathbf{X}} \text{ and } R_\theta[x] \subseteq \varphi(A)\}.$$

It should be noted that this is equivalent to prove that

$$R_\theta[x] = \bigcap\{\varphi(A) : \ A \in P_{\mathbf{X}} \text{ and } x \in A\},$$

which can be checked directly. Hence, Condition (DPM2) holds. Therefore, R_θ is a DP-morphism. $\square$

In this way, our next step is to show that for every DP-space $\mathbf{X}$ the DP-morphism $R_\theta \subseteq \mathbf{X} \times \mathbf{X}(P_{\mathbf{X}})$ is an isomorphism of the category $\mathbb{DPS}$. So we must find a DP-morphism $S \subseteq \mathbf{X}(P_{\mathbf{X}}) \times \mathbf{X}$ such that $S * R_\theta = \succeq_{\mathbf{X}}$ and $R_\theta * S = \succeq_{\mathbf{X}(P_{\mathbf{X}})}$.

Let $\mathbf{X}$ be a DP-space. Since $\theta\colon \mathbf{X} \to \mathbf{X}(P_{\mathbf{X}})$ is a homeomorhism, we can consider its inverse homeomorphism $\theta^{-1}\colon \mathbf{X}(P_{\mathbf{X}}) \to \mathbf{X}$. So, we have that for every $x \in X$ and every $F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}})$

$$\theta^{-1}(F) = x \iff \theta(x) = F,$$

or in other words,

$$\theta(\theta^{-1}(F)) = F \quad \text{and} \quad \theta^{-1}(\theta(x)) = x.$$

We now define the binary relation $R_{\theta^{-1}} \subseteq \mathbf{X}(P_{\mathbf{X}}) \times \mathbf{X}$ as follows: for every $x \in X$ and every $F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}})$,

$$F R_{\theta^{-1}} x \iff \theta^{-1}(F) \succeq_{\mathbf{X}} x.$$

Lemma 3.3.16. *For every* DP*-space* $\boldsymbol{X}$*, the relation* $R_{\theta^{-1}} \subseteq \boldsymbol{X}(P_{\boldsymbol{X}}) \times \boldsymbol{X}$ *is a* DP*-morphism.*

Proof. To show that the relation $R_{\theta^{-1}}$ is a DP-morphism we have to prove that Conditions (DPM1) and (DPM2) of Definition 3.3.1, hold. To this end, we first show that for every $A \in P_{\mathbf{X}}$ and $F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}})$

$$R_{\theta^{-1}}[F] \subseteq A \text{ if and only if } A \in F. \tag{3.7}$$

Let $A \in P_{\mathbf{X}}$ and $F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}})$. Suppose $R_{\theta^{-1}}[F] \subseteq A$. Notice that $F R_{\theta^{-1}} \theta^{-1}(F)$; whereupon $\theta^{-1}(F) \in R_{\theta^{-1}}[F]$. So, $\theta^{-1}(F) \in A$ and then $A \in \theta(\theta^{-1}(F)) = F$. Now assume $A \in F$. Since $F = \theta(\theta^{-1}(F))$, it follows that $\theta^{-1}(F) \in A$. Let $x \in R_{\theta^{-1}}[F]$. Then $\theta^{-1}(F) \succeq_{\mathbf{X}} x$ and therefore $x \in A$. Hence, $R_{\theta^{-1}}[F] \subseteq A$.

Let $A \in P_{\mathbf{X}}$. We prove that $h_{R_{\theta^{-1}}}(A) = \varphi(A)$. Let $F \in h_{R_{\theta^{-1}}}(A)$. So, $R_{\theta^{-1}}[F] \subseteq A$ and, by (3.7), then $A \in F$. Hence, $F \in \varphi(A)$. We now assume that $F \in \varphi(A)$. So, by (3.7), we have that $R_{\theta^{-1}}[F] \subseteq A$; this implies that $F \in h_{R_{\theta^{-1}}}(A)$. Hence, we have proved that $h_{R_{\theta^{-1}}}(A) = \varphi(A) \in P_{\mathbf{X}(P_{\mathbf{X}})}$ for all $A \in P_{\mathbf{X}}$. Therefore, Condition (DPM1) holds. To prove Condition (DPM2), let $F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}})$. We need to prove that

$$R_{\theta^{-1}}[F] = \bigcap \{A \in P_{\mathbf{X}} :\ R_{\theta^{-1}}[F] \subseteq A\}.$$

By (3.7), it is equivalent to show that

$$R_{\theta^{-1}}[F] = \bigcap \{A \in P_{\mathbf{X}} :\ A \in F\}.$$

It is immediate that $R_{\theta^{-1}}[F] \subseteq \bigcap\{A \in P_{\mathbf{X}} :\ A \in F\}$. Let $x \in \bigcap\{A \in P_{\mathbf{X}} :\ A \in F\}$ and let $A \in P_{\mathbf{X}}$ be such that $\theta^{-1}(F) \in A$. So, $A \in \theta(\theta^{-1}(F)) = F$ and then $x \in A$. Thus, $\theta^{-1}(F) \succeq_{\mathbf{X}} x$. That is, $x \in R_{\theta^{-1}}[F]$. Hence, Condition (DPM2) holds. Therefore, $R_{\theta^{-1}}$ is a DP-morphism. $\square$

Lemma 3.3.17. *Let $\boldsymbol{X}$ be a* DP*-space. Then,*

$$R_{\theta^{-1}} * R_\theta = \succeq_{\boldsymbol{X}} \quad \text{and} \quad R_\theta * R_{\theta^{-1}} = \succeq_{\boldsymbol{X}(P_{\boldsymbol{X}})}.$$

Therefore, R_θ is an isomorphism of the category $\mathbb{DPS}$.

Proof. We first show that $R_{\theta^{-1}} * R_\theta = \succeq_{\mathbf{X}}$. So, let $x, x' \in X$. We assume $x(R_{\theta^{-1}} * R_\theta)x'$. Thus, for every $A \in P_{\mathbf{X}}$, if $(R_\theta \circ R_{\theta^{-1}})[x] \subseteq A$ then $x' \in A$. To prove that $x \succeq_{\mathbf{X}} x'$, let $A \in P_{\mathbf{X}}$ be such that $x \in A$. Let $x'' \in (R_\theta \circ R_{\theta^{-1}})[x]$. So, there exists $F \in \mathbf{X}(P_{\mathbf{X}})$ such that $xR_\theta F$ and $FR_{\theta^{-1}}x''$. That is, $\theta(x) \subseteq F$ and $\theta^{-1}(F) \succeq_{\mathbf{X}} x''$. As $x \in A$, we have $A \in \theta(x)$, which implies that $A \in F = \theta(\theta^{-1}(F))$. Then $\theta^{-1}(F) \in A$ and, since $\theta^{-1}(F) \succeq_{\mathbf{X}} x''$, it follows that $x'' \in A$. Thus, $(R_\theta \circ R_{\theta^{-1}})[x] \subseteq A$ and then, by hypothesis, $x' \in A$. Hence, we have proved that for every $A \in P_{\mathbf{X}}$, if $x \in A$ then $x' \in A$; which implies that $x \succeq_{\mathbf{X}} x'$. Reciprocally, we suppose that $x \succeq_{\mathbf{X}} x'$. We need to prove that $x(R_{\theta^{-1}} * R_\theta)x'$. Let $A \in P_{\mathbf{X}}$ be such that $(R_\theta \circ R_{\theta^{-1}})[x] \subseteq A$. Notice that $x(R_\theta \circ R_{\theta^{-1}})x$, because $xR_\theta\theta(x)$ and $\theta(x)R_{\theta^{-1}}x$. We thus obtain that $x \in A$ and, since $x \succeq_{\mathbf{X}} x'$, we have $x' \in A$. Hence, $x(R_{\theta^{-1}} * R_\theta)x'$. Therefore,

$$R_{\theta^{-1}} * R_\theta = \succeq_{\mathbf{X}}.$$

We now prove that $R_\theta * R_{\theta^{-1}} = \succeq_{\mathbf{X}(P_{\mathbf{X}})}$. So, let $F_1, F_2 \in \mathbf{X}(P_{\mathbf{X}})$. We assume first that $F_1(R_\theta * R_{\theta^{-1}})F_2$. Then, we have

$$(\forall A \in P_{\mathbf{X}})\,((R_{\theta^{-1}} \circ R_\theta)[F_1] \subseteq \varphi(A) \implies F_2 \in \varphi(A)).$$

Recall from §3.1 that

$$F_1 \succeq_{\mathbf{X}(P_{\mathbf{X}})} F_2 \iff (\forall A \in P_{\mathbf{X}})\,(F_1 \in \varphi(A) \implies F_2 \in \varphi(A)).$$

Let $A \in P_{\mathbf{X}}$ be such that $F_1 \in \varphi(A)$. We show that $(R_{\theta^{-1}} \circ R_\theta)[F_1] \subseteq \varphi(A)$. Let $F \in (R_{\theta^{-1}} \circ R_\theta)[F_1]$. Then, there exists $x \in X$ such that $F_1 R_{\theta^{-1}} x$ and $xR_\theta F$. That is, $\theta^{-1}(F_1) \succeq_{\mathbf{X}} x$ and $\theta(x) \subseteq F$. Notice that $\theta(x) \subseteq F$ is equivalent to $\theta(x) \succeq_{\mathbf{X}(P_{\mathbf{X}})} F$. Since $\theta^{-1}\colon \mathbf{X}(P_{\mathbf{X}}) \to \mathbf{X}$ is a homeomorphism, it follows that is order-preserving with respect to the specialization order (see §1.6). We thus obtain $\theta^{-1}(\theta(x)) \succeq_{\mathbf{X}} \theta^{-1}(F)$ and then $x \succeq_{\mathbf{X}} \theta^{-1}(F)$. By the transitivity of $\succeq_{\mathbf{X}}$, we obtain $\theta^{-1}(F_1) \succeq_{\mathbf{X}} \theta^{-1}(F)$. Using the fact that θ is order-preserving, because it is a homeomorphism, we have that $F_1 \succeq_{\mathbf{X}(P_{\mathbf{X}})} F$ and, since $F_1 \in \varphi(A)$, it follows that $F \in \varphi(A)$. Thus, $(R_{\theta^{-1}} \circ R_\theta)[F_1] \subseteq \varphi(A)$. Hence, by hypothesis, $F_2 \in \varphi(A)$. Therefore, $F_1 \succeq_{\mathbf{X}(P_{\mathbf{X}})} F_2$. Now, reciprocally, we assume that $F_1 \succeq_{\mathbf{X}(P_{\mathbf{X}})} F_2$. Then, it follows that

$$(\forall A \in P_{\mathbf{X}})\,(F_1 \in \varphi(A) \implies F_2 \in \varphi(A)).$$

$$\begin{tikzcd} P_1 \arrow[r, "h"] \arrow[d, "\varphi_1"'] & P_2 \arrow[d, "\varphi_2"] \\ P_{\mathbf{X}(P_1)} \arrow[r, "h_{R_h}"'] & P_{\mathbf{X}(P_2)} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{X}_1 \arrow[r, "R"] \arrow[d, "R_{\theta_1}"'] & \mathbf{X}_2 \arrow[d, "R_{\theta_2}"] \\ \mathbf{X}(P_{\mathbf{X}_1}) \arrow[r, "R_{h_R}"'] & \mathbf{X}(P_{\mathbf{X}_2}) \end{tikzcd}$$

FIGURE 3.1. Commutative diagrams of morphisms in the categories $\mathbb{MODP}$ and $\mathbb{DPS}$.

Let $A \in P_{\mathbf{X}}$ be such that $(R_{\theta^{-1}} \circ R_\theta)[F_1] \subseteq \varphi(A)$. It should be noted that $F_1 \in (R_{\theta^{-1}} \circ R_\theta)[F_1]$ because $F_1 R_{\theta^{-1}} \theta^{-1}(F_1)$ and $\theta^{-1}(F_1) R_\theta F_1$. Then, $F_1 \in \varphi(A)$ and so, $F_2 \in \varphi(A)$. Hence, $F_1(R_\theta * R_{\theta^{-1}})F_2$. This finishes the proof. □

We are now ready to establish the main result of this chapter.

THEOREM 3.3.18. *The categories* $\mathbb{MODP}$ *and* $\mathbb{DPS}$ *are dually equivalent via the functors* $\Delta\colon \mathbb{DPS} \to \mathbb{MODP}$ *and* $\Gamma\colon \mathbb{MODP} \to \mathbb{DPS}$.

PROOF. As outlined above, it only remains to define the natural equivalences $\mu\colon \mathrm{Id}_{\mathbb{MODP}} \cong \Delta \circ \Gamma$ and $\eta\colon \mathrm{Id}_{\mathbb{DPS}} \cong \Gamma \circ \Delta$. We consider the following definitions:

- for every mo-distributive poset P,

$$\mu(P) = \varphi\colon P \to P_{\mathbf{X}(P)};$$

- for every DP-space $\mathbf{X}$,

$$\eta(\mathbf{X}) = R_\theta \subseteq \mathbf{X} \times \mathbf{X}(P_{\mathbf{X}}).$$

By Lemmas 3.1.3 and 3.3.17 we have that for every mo-distributive poset P and every DP-space $\mathbf{X}$, $\mu(P) = \varphi$ and $\eta(\mathbf{X}) = R_\theta$ are isomorphisms of the categories $\mathbb{MODP}$ and $\mathbb{DPS}$, respectively. Lastly, we show that for every morphism $h\colon P_1 \to P_2$ of the category $\mathbb{MODP}$ and every morphism $R \subseteq \mathbf{X}_1 \times \mathbf{X}_2$ of the category $\mathbb{DPS}$ the diagrams in Figure 3.1 commute.

That the diagram on the left hand side of Figure 3.1 commutes is consequence of Lemma 3.3.12. For the diagram on the right hand side of Figure 3.1, we must to show that $R_{h_R} * R_{\theta_1} = R_{\theta_2} * R$. To this end, we first show that for every $x \in X_1$

$$(R_{\theta_1} \circ R_{h_R})[x] = (R \circ R_{\theta_2})[x]. \tag{3.8}$$

Let $G \in (R_{\theta_1} \circ R_{h_R})[x]$. So, there exists $F \in \mathbf{X}(P_{\mathbf{X}_1})$ such that $xR_{\theta_1}F$ and $FR_{h_R}G$. That is, $\theta_1(x) \subseteq F$ and $h_R^{-1}[F] \subseteq G$. Since $G \in \mathbf{X}(P_{\mathbf{X}_2})$, it follows that there is $x_2 \in \mathbf{X}_2$ such that $G = \theta_2(x_2)$ and thus it is clear that $x_2 R_{\theta_2} G$. Now, we want to show that xRx_2. Let $B \in P_{\mathbf{X}_2}$ be such that $R[x] \subseteq B$. Then, we have the following implications:

$$\begin{aligned} R[x] \subseteq B &\implies x \in h_R(B) \\ &\implies h_R(B) \in \theta_1(x) \\ &\implies h_R(B) \in F \\ &\implies B \in h_R^{-1}[F] \\ &\implies B \in \theta_2(x_2) \\ &\implies x_2 \in B. \end{aligned}$$

Hence, by (DPM2) of Definition 3.3.1, $x_2 \in R[x]$. We thus obtain xRx_2 and $x_2 R_{\theta_2} G$. Hence, $G \in (R \circ R_{\theta_2})[x]$. Reciprocally, let $G \in (R \circ R_{\theta_2})[x]$. So, there is $x_2 \in \mathbf{X}_2$ such that xRx_2 and $x_2 R_{\theta_2} G$. Then, $x_2 \in R[x]$ and $\theta_2(x_2) \subseteq G$. Given that $xR_{\theta_1}\theta_1(x)$, we want to show that $\theta_1(x) R_{h_R} G$. Let $B \in h_R^{-1}(\theta_1(x))$. So, $h_R(B) \in \theta_1(x)$ and this implies that $x \in h_R(B)$. Then, $R[x] \subseteq B$ and thus $x_2 \in B$. That is, $B \in \theta_2(x_2)$ and whereupon $B \in G$. Thus, $h_R^{-1}(\theta_1(x)) \subseteq G$ and hence $\theta_1(x) R_{h_R} G$. Therefore, we have $xR_{\theta_1}\theta_1(x)$ and $\theta_1(x) R_{h_R} G$, i.e., $G \in (R_{\theta_1} \circ R_{h_R})[x]$. Hence, (3.8) holds. Now let $x \in X_1$ and $G \in \mathbf{X}(P_{\mathbf{X}_2})$. Then, by (3.8), we have

$$\begin{aligned} x(R_{h_R} * R_{\theta_1})G &\iff (\forall A \in P_{\mathbf{X}_2})\,((R_{\theta_1} \circ R_{h_R})[x] \subseteq \varphi(A) \implies G \in \varphi(A)) \\ &\iff (\forall A \in P_{\mathbf{X}_2})\,((R \circ R_{\theta_2})[x] \subseteq \varphi(A) \implies G \in \varphi(A)) \\ &\iff x(R_{\theta_2} * R)G. \end{aligned}$$

Hence, $R_{h_R} * R_{\theta_1} = R_{\theta_2} * R$. Therefore, the categories $\mathbb{MODP}$ and $\mathbb{DPS}$ are dually equivalent. □

As we mentioned in the introduction to this chapter, our topological duality for mo-distributive posets is a generalization of the topological duality for distributive meet-semilattices with top element obtained in [**8**] by Celani (see also [**9**]). Recall that a topological space $\langle X, \tau \rangle$ is called a *DS-space* ([**9**]) if:

(1) $\langle X, \tau \rangle$ is a sober space, and
(2) $\mathsf{KO}(X)$ is a base for the topology τ.

Let $\langle X, \tau \rangle$ be a DS-space. Let $\mathcal{D}(X) := \{U^c : U \in \mathsf{KO}(X)\}$. Then, we know ([**8**, pp. 46]) that $\langle \mathcal{D}(X), \cap, X \rangle$ is the dual distributive meet-semilattice with top element of X. Recall also that for DS-spaces $\langle X_1, \tau_1 \rangle$ and $\langle X_2, \tau_2 \rangle$ a binary relation $R \subseteq X_1 \times X_2$ is called a *meet-relation* ([**9**]) when:

(1) for every $A \in \mathcal{D}(X_2)$, $h_R(A) = \{x \in X_1 : R[x] \subseteq A\} \in \mathcal{D}(X_1)$;
(2) $R[x] = \bigcap\{A \in \mathcal{D}(X_2) : R[x] \subseteq A\}$, for all $x \in X_1$,

Let us denote by $\mathbb{DSS}$ the category of all DS-spaces and all meet-relations (see [**9, 8**]). Let $\langle X, \tau \rangle$ be a DS-space. It is straightforward to show that $\langle X, \tau, \mathsf{KO}(X) \rangle$ is a DP-space and then its dual mo-distributive poset is $P_X = \mathcal{D}(X) = \{U^c : U \in$

$\mathsf{KO}(X)\}$. Moreover, if X and Y are DS-spaces and $R \subseteq X \times Y$ is a binary relation, then by Definition 3.3.1 it is easy to check that R is a meet-relation if and only if R is a DP-morphism. Therefore, it is clear that the category $\mathbb{DSS}$ is a full subcategory of $\mathbb{DPS}$.

Let M be a distributive meet-semilattice with top element and consider its dual DP-space $\mathbf{X}(M) = \langle \mathsf{Fi}^{\mathsf{pr}}(M), \tau_M, \mathcal{B}_M \rangle$. Since M is a meet-semilattice, by Theorem 3.1.3 it follows that $\mathcal{B}_M = \mathsf{KO}(\mathbf{X}(M))$. Hence $\langle \mathsf{Fi}^{\mathsf{pr}}(M), \tau_M \rangle$ is the dual DS-space of M ([**9, 8**]). Let M_1 and M_2 be distributive meet-semilattices with top element and let $h \colon M_1 \to M_2$ be a map. By Lemma 2.3.5 we know that h is a meet-homomorphism preserving top if and only if h is an inf-homomorphism. Recall that $\mathbb{DMSL}^{\top}$ denotes the category of distributive meet-semilattices with top element and meet-homomorphisms preserving top. Hence, $\mathbb{DMSL}^{\top}$ is a full subcategory of the category $\mathbb{MODP}^{\top}$ of all mo-distributive posets with top element and all inf-homomorphisms.

Now we are ready to show that we can obtain the dual equivalence developed in [**8**] between the categories $\mathbb{DMSL}^{\top}$ and $\mathbb{DSS}$. Thus, by the previous observations and Theorem 3.3.18, we have the following theorem. We leave the details to the reader.

Theorem 3.3.19. *The categories $\mathbb{DMSL}^{\top}$ and $\mathbb{DSS}$ are dually equivalent via the corresponding restrictions of the functors Γ and Δ of Theorem 3.3.18.*

3.4. Connection with the work of David and Erné

In this section we study the relations between our duality, developed in the previous section, and the duality presented in [**16**] by David and Erné. The main difference between the two dualities lies on the definitions of the morphisms considered in the respective categories of both dualities. The morphisms considered by David and Erné in [**16**] on the two categories are stronger than our morphisms. For instance, the morphisms between the topological spaces considered by David and Erné are functions while our morphisms are binary relations between DP-spaces. The aim of the definition of the morphisms considered by David and Erné between mo-distributive posets was that the extension map to the lattices of Frink-filters preserve not only arbitrary joins but also finite meets. That is, for an inf-homomorphism $h \colon P \to Q$ between mo-distributive posets they looked for necessary and sufficient conditions for that the extension map $\widehat{h} \colon \mathsf{Fi}_{\mathsf{F}}(P) \to \mathsf{Fi}_{\mathsf{F}}(Q)$ defined by $\widehat{h}(F) = \mathrm{Fi}_{\mathrm{F}}(h[F])$ preserves finite meets.

3.4.1. $\vee$-stable maps. Here we present a kind of homomorphism from a poset P to a poset Q due to David and Erné and we show that this kind of homomorphism (inf-homomorphisms that are $\vee$-stable maps, see Definition 3.4.1 below) can be characterized through our duality in a very nice way. The sup-homomorfisms between posets in [**16**] are called ideal-continuous maps and the Frink-ideals are called ideals.

Definition 3.4.1 ([**16**]). Let P and Q be posets. A map $h\colon P \to Q$ is called $\vee$-*stable* if for every $A \subseteq_\omega P$ we have

$$h[A]^{\mathrm{u}} = \mathsf{Fi}_{\mathsf{F}}\left(h[A^{\mathrm{u}}]\right).$$

Lemma 3.4.2. ([**16**, Proposition 2.3]) *Let P and Q be posets. If $h\colon P \to Q$ is a $\vee$-stable map, then h is a sup-homomorphism.*

This lemma seems to show that the notion of $\vee$-stability can be considered as a homomorphism between posets in the sense that on the join-semilattice setting the condition of $\vee$-stability on a map implies that the map is a join-homomorphism (actually, a map between join-semilattices is $\vee$-stable if and only if it is a join-homomorphism), but the notion of $\vee$-stable has a problem and it is that the composition of two $\vee$-stable maps on posets need not be a $\vee$-stable map. An example of this can be found in [**20**]. However, as shown in [**16**] by David and Erné, the maps between posets that are $\vee$-stable and inf-homomorphism can be considered as a good generalization of notion of lattice homomorphism.

Lemma 3.4.3. ([**16**, Proposition 3.2]) *Let P and Q be posets and let $h\colon P \to Q$ be a map. Then, the following conditions are equivalent:*

(1) *h is a $\vee$-stable map and an inf-homomorphism;*
(2) *for every $G \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(Q)$, we have $h^{-1}[G] \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$.*

The following lemma shows that the inf-homomorphisms between mo-distributive posets that are $\vee$-stables are strong inf-homomorphisms and Example 3.4.5 shows that the converse is not true. Thus the notion of being a $\vee$-stable inf-homomorphism is stronger than the notion of being a strong inf-homomorphism.

Lemma 3.4.4. *Let P and Q be mo-distributive posets. If $h\colon P \to Q$ is a $\vee$-stable map and an inf-homomorphism, then h is a strong inf-homomorphism.*

Proof. Let $h\colon P \to Q$ be a $\vee$-stable map and an inf-homomorphism. Let $X, Y \subseteq_\omega P$. We assume that $X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}}$ and we must to show that $h[X]^{\mathrm{u}} \subseteq h[Y]^{\mathrm{lu}}$. Suppose towards a contradiction that there is $b \in h[X]^{\mathrm{u}}$ such that $b \notin h[Y]^{\mathrm{lu}}$. Then, by Corollary 2.2.18, there exists $G \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(Q)$ such that $h[Y]^{\mathrm{lu}} \subseteq G$ and $b \notin G$. We thus get $h[Y] \subseteq G$ and then $Y \subseteq h^{-1}[G]$. Since h is an inf-homomorphism, it follows that $h^{-1}[G]$ is a Frink-filter of P. So, we obtain

$$\bigcap_{x\in X} {\uparrow} x = X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}} \subseteq h^{-1}[G]. \tag{3.9}$$

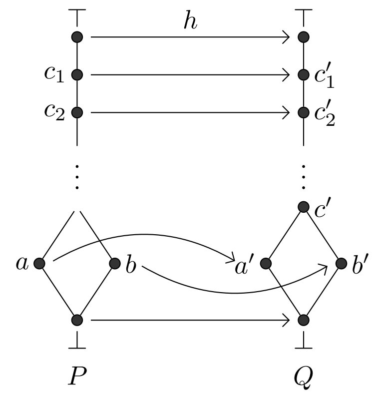


FIGURE 3.2. $h\colon P \to Q$ is a strong inf-homomorphism but it is not a $\vee$-stable map.

Since h is a $\vee$-stable map, by Lemma 3.4.3 we have $h^{-1}[G] \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)$. Then, by (3.9), there exists $x_0 \in X$ such that $\uparrow x_0 \subseteq h^{-1}[G]$. We thus get $h(x_0) \in G$ and since $b \in h[X]^{\mathrm{u}}$, it follows that $h(x_0) \leq b$. Then $b \in G$, which is a contradiction. Hence, $h[X]^{\mathrm{u}} \subseteq h[Y]^{\mathrm{lu}}$ and therefore h is a strong inf-homomorphism. $\square$

EXAMPLE 3.4.5. We consider the posets P and Q given in Figure 3.2. It should be noted that P and Q are mo-distributive posets, because the lattices $\mathsf{Fi}_{\mathsf{F}}(P)$ and $\mathsf{Fi}_{\mathsf{F}}(Q)$ are distributive. Moreover, notice that Q is a lattice. We define the map $h\colon P \to Q$ as follows:

$$h(\bot) = \bot, \quad h(\top) = \top, \quad \text{and for each } x \in \{a, b, c_1, c_2, \dots\}, h(x) = x'.$$

A glance at the definition of h in Figure 3.2 shows that h is an order-embedding. We claim that h is a strong inf-homomorphism. The proof of this fact is not hard but it is a bit tedious, so we leave the details to the reader. To check that h is not a $\vee$-stable map we use Lemma 3.4.3. Let $G = \uparrow c' \setminus \{c'\} = \{\top, c_1', c_2', \dots\}$. It is clear that G is a filter of Q and, since $G^c = \{\bot, a', b', c'\}$ is an ideal of Q, it is a prime filter of Q. That is, $G \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(Q)$. The set $h^{-1}[G] = \{\top, c_1, c_2, \dots\}$ is a Frink-filter of P. But, since $h^{-1}[G]^c = \{\bot, a, b\}$ is not an order-ideal of P, it follows that $h^{-1}[G] \notin \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)$. Hence, by Lemma 3.4.3, h is not a $\vee$-stable map.

Our purpose now is to characterize the inf-homomorphisms that are $\vee$-stable by properties of their topological duals. That is, given an inf-homomorphism $h\colon P \to Q$ from a mo-distributive poset P to a mo-distributive poset Q we look for which conditions the DP-morphism $R_h \subseteq \mathbf{X}(Q) \times \mathbf{X}(P)$ must satisfy so that h be a $\vee$-stable map. Let us remember that for a mo-distributive poset P, $\langle \mathbf{X}(P), \mathcal{B}_P \rangle$ is its dual DP-space (see §3.1) with $\mathbf{X}(P) = \langle \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P), \tau_P \rangle$ and, for short, we write $\mathbf{X}(P) = \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)$.

Lemma 3.4.6. *Let P and Q be mo-distributive posets and let $h\colon P \to Q$ be an inf-homomorphism.*

(1) *Assume that P and Q have bottom elements. Then, h preserves bottom if and only if $R_h^{-1}[\boldsymbol{X}(P)] = \boldsymbol{X}(Q)$;*
(2) *h is a $\vee$-stable map if and only if for every $G \in \boldsymbol{X}(Q)$, $R_h[G]$ has a least element with respect to $\subseteq$ in $\boldsymbol{X}(P)$.*

Proof. (1) Assume that P and Q have bottom elements. It is clear (see §3.1) that $\varphi(\bot) = \emptyset$. Then, using Lemma 3.3.12, we have

$$\begin{aligned} h \text{ preserves bottom} \quad & \text{iff} \quad h(\bot) = \bot \\ & \text{iff} \quad \varphi(h(\bot)) = \varphi(\bot) \\ & \text{iff} \quad h_{R_h}(\varphi(\bot)) = \varphi(\bot) \\ & \text{iff} \quad h_{R_h}(\emptyset) = \emptyset \\ & \text{iff} \quad R_h^{-1}[\mathbf{X}(P)] = \mathbf{X}(Q). \end{aligned}$$

(2) We prove first that for each $G \in \mathbf{X}(Q)$, $R_h[G]$ has a least element (w.r.t $\subseteq$) if and only if $h^{-1}[G] \in \mathbf{X}(P)$. Let $G \in \mathbf{X}(Q)$. Suppose first that $R_h[G]$ has a least element $F \in R_h[G]$. So, $h^{-1}[G] \subseteq F$. If $F \neq h^{-1}[G]$, then there exists $a \in F$ such that $a \notin h^{-1}[G]$. Since h is an inf-homomorphism, we have $h^{-1}[G]$ is a Frink-filter of P. Then, by Corollary 2.2.18, there exists $F' \in \mathbf{X}(P)$ such that $h^{-1}[G] \subseteq F'$ and $a \notin F'$. We thus get $F' \in R_h[G]$ and $F \nsubseteq F'$, which is a contradiction because F is the least element of $R_h[G]$. Hence, $h^{-1}[G] = F \in \mathbf{X}(P)$. Reciprocally, if $h^{-1}[G] \in \mathbf{X}(P)$ then $h^{-1}[G]$ is the least element of $R_h[G]$ in $\mathbf{X}(P)$. Hence, by Lemma 3.4.3, it follows that h is a $\vee$-stable map if and only if $R_h[G]$ has a least element for every $G \in \mathbf{X}(Q)$. □

From §3.1 we know that for every mo-distributive poset P, the specialization order $\preceq$ of the DP-space $\mathbf{X}(P)$ is the inverse of the inclusion order. So, our next definition is motivated by the previous lemma.

Definition 3.4.7. Let $\mathbf{X}$ and $\mathbf{Y}$ be DP-spaces and let $R \subseteq \mathbf{X} \times \mathbf{Y}$ be a DP-morphism.

(T) R is called *total* if $R^{-1}[Y] = X$;
(F) R is called *functional* if for every $x \in X$ there exists $y \in Y$ such that $R[x] = {\downarrow}y$, where ${\downarrow}y = \{y' \in Y : \ y' \preceq y\}$.

As a consequence of Lemma 3.4.6 we have the following corollary.

Corollary 3.4.8. *Let P and Q be mo-distributive posets and let $h\colon P \to Q$ be an inf-homomorphism.*

(1) *It is assumed that P and Q has bottom elements. Then, h preserves bottom if and only if the* DP*-morphism $R_h \subseteq \boldsymbol{X}(Q) \times \boldsymbol{X}(P)$ is total.*

(2) *The inf-homomorphism h is $\vee$-stable if and only if the* DP*-morphism R_h is functional*

From Theorem 3.3.18 and the previous corollary, it is straightforward to show the proof of the next corollary.

COROLLARY 3.4.9. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be* DP*-spaces and let $R \subseteq \boldsymbol{X} \times \boldsymbol{Y}$ be a* DP*-morphism.*

(1) *If $\boldsymbol{X} \in \mathcal{B}(\boldsymbol{X})$ and $\boldsymbol{Y} \in \mathcal{B}(\boldsymbol{Y})$, then $h_R \colon P_{\boldsymbol{X}} \to P_{\boldsymbol{Y}}$ preserves bottom if and only if R is total.*

(2) *h_R is $\vee$-stable if and only if R is functional.*

The following lemma shows that the composition $*$ between functional DP-morphisms is the usual set-theoretical composition of relations.

LEMMA 3.4.10. *Let $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ be* DP*-spaces and let $R \subseteq \boldsymbol{X} \times \boldsymbol{Y}$ and $S \subseteq \boldsymbol{Y} \times \boldsymbol{Z}$ be functional* DP*-morphisms. Then $S * R = R \circ S$.*

PROOF. Let $x \in X$. So, there is $y \in Y$ such that $R[x] = {\downarrow} y$. Then, there exists $z \in Z$ such that $S[y] = {\downarrow} z$. We prove that $(R \circ S)[x] = {\downarrow} z$. Let $z' \in (R \circ S)[x]$. Thus, there is $y' \in Y$ such that $y' \in R[x]$ and $z' \in S[y']$. We thus obtain $y' \preceq y$ and, by Lemma 3.3.8, we have $S[y'] \subseteq S[y]$. Then, $z' \in S[y] = {\downarrow} z$. Hence, $(R \circ S)[x] \subseteq {\downarrow} z$. To show the other inclusion, let $z' \in {\downarrow} z = S[y]$. Since xRy and ySz', it follows that $z' \in (R \circ S)[x]$. Hence, ${\downarrow} z \subseteq (R \circ S)[x]$. Then, $(R \circ S)[x] = {\downarrow} z$ and so it is a closed subset of $\mathbf{Z}$. By (3.6) on page 92, we have

$$(S * R)[x] = \operatorname{cl}\left((R \circ S)[x]\right) = (R \circ S)[x].$$

Thus we obtain $(S * R)[x] = (R \circ S)[x]$ for all $x \in X$ and therefore $S * R = R \circ S$. $\square$

Next, we prove that the composition $*$ between functional DP-morphisms is a functional DP-morphism and the composition $*$ between total DP-morphisms is a total DP-morphism.

LEMMA 3.4.11. *Let $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ be* DP*-spaces and let $R \subseteq \boldsymbol{X} \times \boldsymbol{Y}$ and $S \subseteq \boldsymbol{Y} \times \boldsymbol{Z}$ be* DP*-morphisms.*

(1) *If R and S are total, then $S * R$ is total;*

(2) *if R and S are functional, then $S * R$ is functional.*

PROOF. (1) We assume that R and S are total. We need to show that $(S * R)^{-1}[Z] = X$. Let $x \in X$. Since R is total, it follows that $R^{-1}[Y] = X$ and so there is $y \in Y$ such that xRy. As S is total, we have that $S^{-1}[Z] = Y$ and thus there is $z \in Z$ such that ySz. We thus obtain $x(R \circ S)$ and then $z \in (R \circ S)[x]$. By definition of $*$ (see §3.3.1), we have $(R \circ S)[x] \subseteq (S * R)[x]$. Thus $z \in (S * R)[x]$, which implies $x \in (S * R)^{-1}[z]$. Hence, $X \subseteq (S * R)^{-1}[Z]$ and, since the other inclusion is obvious, it follows that $(S * R)^{-1}[Z] = X$.

(2) We suppose that R and S are functional. Let $x \in X$. As R is functional, there exists $y \in Y$ such that $R[x] = {\downarrow} y$ and, since S is functional, it follows that there exists $z \in Z$ such that $S[y] = {\downarrow} z$. By Lemma 3.4.10, we have proved that $(S * R)[x] = (R \circ S)[x] = {\downarrow} z$. Hence, $S * R$ is a functional DP-morphism. □

It should be noted that for every DP-space $\mathbf{X}$, the identity DP-morphism $\succeq_{\mathbf{X}}$ is total and functional. The proof of this facts is straightforward. Then, using the previous lemma, we can consider the following categories:

- let $\mathbb{DPS}^{\mathrm{T}}$ be the category of all DP-spaces $\mathbf{X}$ such that $\mathbf{X} \in \mathcal{B}(\mathbf{X})$ and all total DP-morphisms;
- let $\mathbb{DPS}^{\mathrm{F}}$ be the category of all DP-spaces and all functional DP-morphisms.

Notice that the categories $\mathbb{DPS}^{\mathrm{T}}$ and $\mathbb{DPS}^{\mathrm{F}}$ are subcategories of $\mathbb{DPS}$. It is not hard to check that for a mo-distributive poset P the identity inf-homomorphism $\mathrm{id}_P \colon P \to P$ preserves bottom, if P has bottom element and it is a $\vee$-stable map. Moreover, the composition of two $\vee$-stable inf-homomorphisms is a $\vee$-stable inf-homomorphism (see [**16**, p. 105]). Thus, we can consider the following categories:

- the category of all mo-distributive posets with bottom element and all inf-homomorphisms preserving the bottom element, that we denote by $\mathbb{MODP}^{\perp}$;
- the category of all mo-distributive posets and all inf-homomorphisms between mo-distributive posets that are $\vee$-stable, that we denote by $\mathbb{MODP}^{\mathrm{sta}}$.

Hence, we have that the categories $\mathbb{MODP}^{\perp}$ and $\mathbb{MODP}^{\mathrm{sta}}$ are subcategories of $\mathbb{MODP}$.

Remark 3.4.12. Let $\mathbf{X}$ be a DP-space. Recall that the DP-morphism $R_\theta \subseteq \mathbf{X} \times \mathbf{X}(P_{\mathbf{X}})$ is defined by

$$xR_\theta F \iff \theta(x) \subseteq F$$

for every $x \in X$ and $F \in \mathbf{X}(P_{\mathbf{X}})$, where $\theta \colon \mathbf{X} \to \mathbf{X}(P_{\mathbf{X}})$ is the homeomorphism given in Theorem 3.2.10. As noted in the previous section, we have $xR_\theta\theta(x)$ for every $x \in X$ (see on page 97). It follows that $R_\theta^{-1}[\mathbf{X}(P_{\mathbf{X}})] = \mathbf{X}$ and therefore R_θ is a total DP-morphism. Moreover, for every $x \in X$ and every $F \in \mathbf{X}(P_{\mathbf{X}})$ we have that

$$\begin{aligned} F \in R_\theta[x] &\iff \theta(x) \subseteq F \\ &\iff F \preceq_{\mathbf{X}(P_{\mathbf{X}})} \theta(x) \end{aligned}$$

$$\iff F \in {\downarrow}\theta(x).$$

Then, $R_\theta[x] = {\downarrow}\theta(x)$ for all $x \in X$. Hence, R_θ is a functional DP-morphism.

It should be noted that if P is a mo-distributive poset with bottom element $\perp$ then $\varphi(\perp) = \emptyset$, which implies that $\mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P) = \varphi(\perp)^c \in \mathcal{B}_P$. That is, the DP-space $\langle \mathbf{X}(P), \mathcal{B}_P\rangle$ satisfies that $\mathbf{X}(P) \in \mathcal{B}_P$ and hence $\langle \mathbf{X}(P), \mathcal{B}_P\rangle$ is an object of the category $\mathbb{DPS}^{\mathrm{T}}$. And conversely, if $\langle \mathbf{X}, \mathcal{B}\rangle$ is a DP-space such that $X \in \mathcal{B}$ then $\emptyset \in P_{\mathbf{X}}$ and therefore, $P_{\mathbf{X}}$ is an object of the category $\mathbb{MODP}^{\perp}$. Therefore, by the previous remark and by Corollaries 3.4.6 and 3.4.9, and using moreover the dual equivalences $\Gamma\colon \mathbb{MODP} \to \mathbb{DPS}$ and $\Delta\colon \mathbb{DPS} \to \mathbb{MODP}$ given by Theorem 3.3.18 restricted to the corresponding subcategories, we have proved the following result.

Theorem 3.4.13.

(1) *The categories $\mathbb{MODP}^{\perp}$ and $\mathbb{DPS}^{\mathrm{T}}$ are dually equivalent via the functors*

$$\Gamma\colon \mathbb{MODP}^{\perp} \to \mathbb{DPS}^{\mathrm{T}} \quad \textit{and} \quad \Delta\colon \mathbb{DPS}^{\mathrm{T}} \to \mathbb{MODP}^{\perp}.$$

(2) *The categories $\mathbb{MODP}^{\mathrm{sta}}$ and $\mathbb{DPS}^{\mathrm{F}}$ are dually equivalent via the functors*

$$\Gamma\colon \mathbb{MODP}^{\mathrm{sta}} \to \mathbb{DPS}^{\mathrm{F}} \quad \textit{and} \quad \Delta\colon \mathbb{DPS}^{\mathrm{F}} \to \mathbb{MODP}^{\mathrm{sta}}.$$

3.4.2. Functional morphisms between DP-spaces. In this subsection we show that the category $\mathbb{DPS}^{\mathrm{F}}$ and the category of the dual spaces of mo-distributive posets given by David and Erné (see [**16**, p. 110]) are isomorphic. Then, we use this categorical isomorphism to derive the duality established in [**16**]. Our first definition is due to David and Erné.

Definition 3.4.14 (See p. 110 in [**16**]). Let $\mathbf{X}$ and $\mathbf{Y}$ be DP-spaces. A map $f\colon X \to Y$ is called a DP-*function* if for every $V \in \mathcal{B}(\mathbf{Y})$, $f^{-1}[V] \in \mathcal{B}(\mathbf{X})$.

Notice that every DP-function $f\colon \mathbf{X} \to \mathbf{Y}$ is a continuous map, because $\mathcal{B}(\mathbf{X})$ and $\mathcal{B}(\mathbf{Y})$ are bases for the DP-spaces $\mathbf{X}$ and $\mathbf{Y}$, respectively. Moreover, since f is a continuous map, it follows that f is order-preserving with regard to the specialization order. It is straightforward to show that the usual set-theoretical composition of two DP-functions is a DP-function and it is also clear that the identity map $\mathrm{id}_{\mathbf{X}}\colon X \to X$ for a DP-space $\mathbf{X}$ is a DP-function. Then, we have the category of all DP-spaces and all DP-functions. We denote this category by $\mathbb{DPS}^{\mathrm{sta}}$.

Let $\mathbf{X}$ and $\mathbf{Y}$ be DP-spaces and let $R \subseteq \mathbf{X} \times \mathbf{Y}$ be a functional DP-morphism. We define the map $f^R\colon X \to Y$ by setting

$$f^R(x) := \text{ the greatest element of } R[x] \text{ (with respect to the specialization order)}$$

for every $x \in X$.

Lemma 3.4.15. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be* DP*-spaces and let $R \subseteq \boldsymbol{X} \times \boldsymbol{Y}$ be a functional* DP*-morphism. Then, the map $f^R\colon X \to Y$ is a* DP*-function. Moreover, if $\boldsymbol{Z}$ is a* DP*-space and $S \subseteq \boldsymbol{Y} \times \boldsymbol{Z}$ is a* DP*-morphism, then we have $f^{S*R} = f^S \circ f^R$.*

Proof. To prove that f^R is a DP-function, let $V \in \mathcal{B}(\mathbf{Y})$ and let $x \in X$. Then, we have

$$\begin{aligned} x \in f^{R^{-1}}[V^c] &\iff f^R(x) \in V^c \\ &\iff {\downarrow} f^R(x) \subseteq V^c \\ &\iff R[x] \subseteq V^c \\ &\iff x \in h_R(V^c). \end{aligned}$$

So, $f^{R^{-1}}[V^c] = h_R(V^c) \in P_{\mathbf{X}}$ and hence $f^{R^{-1}}[V] \in \mathcal{B}(\mathbf{X})$. Therefore, f^R is a DP-function.

Now, let $\mathbf{Z}$ be a DP-space and let $S \subseteq \mathbf{Y} \times \mathbf{Z}$ be a functional DP-morphism. By Lemma 3.4.10, we have $S * R = R \circ S$. We thus obtain $f^{S*R} = f^{R\circ S}$. Let now $x \in X$. Then,

$$\begin{aligned} f^{R\circ S}(x) &= \text{the greatest element of } (R \circ S)[x] \\ &= \text{the greatest element of } S[y], \text{ where } y \text{ is the greatest element of } R[x] \\ &= f^S(y), \text{ where } y \text{ is the greatest element of } R[x] \\ &= f^S(f^R(x)) \\ &= (f^S \circ f^R)(x). \end{aligned}$$

Therefore, $f^{S*R} = f^{R\circ S} = f^S \circ f^R$. $\square$

We now consider the converse construction. Let $\mathbf{X}$ and $\mathbf{Y}$ be DP-spaces and let $f\colon \mathbf{X} \to \mathbf{Y}$ be a DP-function. We define the binary relation $R^f \subseteq X \times Y$ as follows:

$$xR^f y \iff y \preceq f(x)$$

for every pair $(x, y) \in X \times Y$.

Lemma 3.4.16. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be* DP*-spaces and let $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ be a* DP*-function. Then, the relation $R^f \subseteq X \times Y$ is a functional* DP*-morphism. Moreover, if $\boldsymbol{Z}$ is a* DP*-space and $g\colon \boldsymbol{Y} \to \boldsymbol{Z}$ is a* DP*-function, then $R^{g\circ f} = R^g * R^f$.*

Proof. We need to prove Conditions (DPM1) and (DPM2) of Definition 3.3.1. Let $B \in P_{\mathbf{Y}}$. Since $R^f[x] = {\downarrow} f(x)$ for all $x \in X$, we have

$$\begin{aligned} h_{R^f}(B) &= \{x \in X :\ R^f[x] \subseteq B\} \\ &= \{x \in X :\ {\downarrow} f(x) \subseteq B\} \\ &= \{x \in X :\ f(x) \in B\} \end{aligned}$$

$$= f^{-1}[B] \in P_{\mathbf{X}}.$$

Then, Condition (DPM1) holds. Let now $x \in X$. Since $R^f[x] = {\downarrow} f(x) = \mathrm{cl}(f(x))$, we obtain that $R^f[x]$ is a closed subset of $\mathbf{Y}$. Then, R^f satisfies Condition (DPM2'). Hence, $R^f \subseteq \mathbf{X} \times \mathbf{Y}$ is a DP-morphism. Since $R^f[x] = {\downarrow} f(x)$ for all $x \in X$, it follows that R^f is functional. This completes the proof of first part of the lemma.

Let $\mathbf{Z}$ be a DP-space and let $g\colon \mathbf{Y} \to \mathbf{Z}$ be a DP-function. Since R^f and R^g are functional DP-morphisms, by Lemma 3.4.10, it follows that $R^g * R^f = R^f \circ R^g$. Then to prove the second part of the lemma is equivalent to show that $R^{g\circ f} = R^f \circ R^g$. Let $x \in X$ and $z \in Z$. Then,

$$\begin{aligned} xR^{g\circ f}z &\iff z \preceq (g \circ f)(x) \\ &\iff z \preceq g(f(x)) \\ &\iff f(x)R^g z \\ &\iff xR^f f(x) \text{ and } f(x)R^g z \\ &\iff x(R^f \circ R^g)z. \end{aligned}$$

Hence, $R^{g\circ f} = R^f \circ R^g = R^g * R^f$. □

Lemma 3.4.17. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be DP-spaces. Let $R \subseteq \boldsymbol{X} \times \boldsymbol{Y}$ be a functional DP-morphism and let $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ be a DP-function. Then, $R^{f^R} = R$ and $f^{R^f} = f$.*

Proof. Let $x \in X$. By definitions of R^{f^R} and f^R, we have

$$\begin{aligned} R^{f^R}[x] &= {\downarrow} f^R(x) \\ &= R[x]. \end{aligned}$$

Hence, $R^{f^R} = R$. In a similar fashion, we have

$$\begin{aligned} f^{R^f}(x) &= \text{the greatest element of } R^f[x] = {\downarrow} f(x) \\ &= f(x). \end{aligned}$$

Hence, $f^{R^f} = f$. This completes the proof. □

Putting these results together we obtain the following theorem, whose proof we omit.

Theorem 3.4.18. *The categories $\mathbb{DPS}^{\mathrm{F}}$ and $\mathbb{DPS}^{\mathrm{sta}}$ are isomorphic.*

Then, by the previous theorem and Theorem 3.4.13, we can directly derive the duality established by David and Erné in [**16**, Theorem 4.2].

Theorem 3.4.19. *The categories $\mathbb{MODP}^{\mathrm{sta}}$ and $\mathbb{DPS}^{\mathrm{sta}}$ are dually equivalent.*

Finally, we want to establish the explicit construction of the functors that give the dual equivalence between the categories $\mathbb{MODP}^{\mathrm{sta}}$ and $\mathbb{DPS}^{\mathrm{sta}}$. These functors are those defined in [**16**] by David and Erné to establish the dual equivalence. To this end, the following lemma is central.

LEMMA 3.4.20.

(1) *For every* DP*-space* $\boldsymbol{X}$, $f^{R_\theta} = \theta$.

(2) *Let* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *be* DP*-spaces and let* $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ *be a* DP*-function. Then, for every* $B \in P_{\mathbf{Y}}$, $h_{R^f}(B) = f^{-1}[B]$.

(3) *Let* P *and* Q *be mo-distributive posets and let* $h\colon P \to Q$ *be a* $\vee$*-stable inf-homomorphism. Then,* $f^{R_h}(G) = h^{-1}[G]$ *for all* $G \in \boldsymbol{X}(Q)$.

PROOF. (1) Let $\mathbf{X}$ be a DP-space. We recall that $\theta\colon \mathbf{X} \to \mathbf{X}(P_{\mathbf{X}})$ is the homeomorphism given in Theorem 3.2.10 and $f^{R_\theta}\colon \mathbf{X} \to \mathbf{X}(P_{\mathbf{X}})$ is defined by $f^{R_\theta}(x) =$ the greatest element of $R_\theta[x]$. Let $x \in X$. By Remark 3.4.12, we have $R_\theta[x] = {\downarrow}\theta(x)$. Then, $f^{R_\theta}(x) = \theta(x)$. Therefore, $f^{R_\theta} = \theta$. (2) Let $B \in P_{\mathbf{Y}}$ and let $x \in X$. Then,

$$\begin{aligned} x \in h_{R^f}(B) &\iff R^f[x] \subseteq B \\ &\iff {\downarrow}f(x) \subseteq B \\ &\iff f(x) \in B \\ &\iff x \in f^{-1}[B]. \end{aligned}$$

Hence, $h_{R^f}(B) = f^{-1}[B]$ for all $B \in P_{\mathbf{Y}}$. (3) Given that $h\colon P \to Q$ is a $\vee$-stable inf-homomorphism, we have that $R_h \subseteq \mathbf{X}(Q) \times \mathbf{X}(P)$ is a functional DP-morphism where $h^{-1}[G]$ is the greatest element of $R_h[G]$ in $\mathbf{X}(P)$ (with respect to the specialization order $\preceq$ of the space $\mathbf{X}(P)$) for every $G \in \mathbf{X}(Q)$. By definition of $f^{R_h}\colon \mathbf{X}(Q) \to \mathbf{X}(P)$, we have

$$f^{R_h}(G) = h^{-1}[G]$$

for every $G \in \mathbf{X}(Q)$. $\square$

We now can define the corresponding functors:

- $\Gamma^*\colon \mathbb{MODP}^{\mathrm{sta}} \to \mathbb{DPS}^{\mathrm{sta}}$ is defined as follows:
  - for every mo-distributive poset P, $\Gamma^*(P) := \langle \mathbf{X}(P), \mathcal{B}_P\rangle = \Gamma(P)$;
  - for every morphism $h\colon P \to Q$ of the category $\mathbb{MODP}^{\mathrm{sta}}$, $\Gamma^*(h) := h^{-1}\colon \mathbf{X}(Q) \to \mathbf{X}(P)$.
- $\Delta^*\colon \mathbb{DPS}^{\mathrm{sta}} \to \mathbb{MODP}^{\mathrm{sta}}$ is defined as follows:
  - for every DP-space $\mathbf{X}$, $\Delta^*(\mathbf{X}) := P_{\mathbf{X}} = \Delta(\mathbf{X})$;
  - for every morphism $f\colon \mathbf{X} \to \mathbf{Y}$ of the category $\mathbb{DPS}^{\mathrm{sta}}$, $\Delta^*(f) := f^{-1}\colon P_{\mathbf{Y}} \to P_{\mathbf{X}}$.

Therefore, it is not hard to show that the functors Γ^* and Δ^* establish a dual equivalence between the categories $\mathbb{MODP}^{\mathrm{sta}}$ and $\mathbb{DPS}^{\mathrm{sta}}$ where the natural equivalences $\eta^*\colon \mathrm{Id}_{\mathbb{DPS}^{\mathrm{sta}}} \cong \Gamma^* \circ \Delta^*$ and $\mu^*\colon \mathrm{Id}_{\mathbb{MODP}^{\mathrm{sta}}} \cong \Delta^* \circ \Gamma^*$ are given by

$$\eta^*(\mathbf{X}) := \theta\colon \mathbf{X} \to \mathbf{X}(P_{\mathbf{X}}) \quad \text{and} \quad \mu^*(P) := \varphi\colon P \to P_{\mathbf{X}(P)}$$

for every DP-space $\mathbf{X}$ and every mo-distributive poset P.

We conclude this section with Table 3.2 that summarizes all dual equivalences that we obtained throughout this chapter.

Categories of posets			**Categories of topological spaces**	
mo-distributive posets and inf-homomorphisms	$\mathbb{MODP}$	dually equivalent to	$\mathbb{DPS}$	DP-spaces and DP-morphisms
mo-distributive posets with bottom element and inf-homomorphisms preserving bottom	$\mathbb{MODP}^{\perp}$	dually equivalent to	$\mathbb{DPS}^{\mathrm{T}}$	DP-spaces X such that $X \in \mathcal{B}(X)$ and total DP-morphisms
mo-distributive posets and $\vee$-stable inf-homomorphisms	$\mathbb{MODP}^{\mathrm{sta}}$	dually equivalent to	$\mathbb{DPS}^{\mathrm{F}}$	DP-spaces and functional DP-morphisms
	$\mathbb{MODP}^{\mathrm{sta}}$	dually equivalent to	$\mathbb{DPS}^{\mathrm{sta}}$	DP-spaces and DP-functions

Table 3.2. Dual equivalences between categories of mo-distributive posets and DP-spaces.

3.5. The Frink completion

The aim of this section is to study a completion of a mo-distributive poset obtained using its dual DP-space. It will be a Δ_1-completion in the sense of [**27**].

A Δ_1-*completion* of a poset is a completion for which each element can be obtained both as a join of meets of elements of the original poset and as a meet of joins of elements of the original poset. A nice and important way to obtain Δ_1-completions of a poset P is by means of *polarities* $(\mathcal{F}, \mathcal{I}, R)$ where $\mathcal{F}$ is a collection of up-sets of P such that all principal up-sets of P belong to $\mathcal{F}$, $\mathcal{I}$ is a collection of down-sets of P such that all principal down-sets of P belong to $\mathcal{I}$ and $R \subseteq \mathcal{F} \times \mathcal{I}$ is the binary relation defined by: FRI if and only if $F \cap I \neq \emptyset$, for every $F \in \mathcal{F}$ and $I \in \mathcal{I}$. Let P be a poset and $(\mathcal{F}, \mathcal{I}, R)$ a polarity of the defined type. The complete lattice $L = \mathcal{G}(\mathcal{F}, \mathcal{I}, R)$ of Galois closed subsets of $\mathcal{F}$ is a Δ_1-completion of P that has two important properties: compactness and density (see [**27**, Theorem 5.10]). In [**27**] the completion $L = \mathcal{G}(\mathcal{F}, \mathcal{I}, R)$ is called the $(\mathcal{F}, \mathcal{I})$-*completion of* P. For further details and background on polarities see [**27, 25**], [**15**, Chapters 3 and 7] and [**21, 6**].

Many of the notions and results that we consider in this section are particular cases of the notions and results presented in [**27**]. We direct the reader to [**27, 47, 21**] for a more general discussion about completions and extensions of posets.

Let P be a poset. A complete lattice L is called a *completion* of P if there is an order-embedding $e\colon P \hookrightarrow L$. We also say that the pair $\langle L, e\rangle$ is a completion of P if L is a complete lattice and $e\colon P \hookrightarrow L$ is an order-embedding.

Let P be a poset and recall that $\mathsf{Fi}_{\mathsf{F}}(P)$ and $\mathsf{Id}_{\mathsf{or}}(P)$ denote the collections of all Frink-filters and all order-ideals of P, respectively. Let $\langle L, e\rangle$ be a completion of P. An element $x \in L$ is called *Frink-closed* if there exists $F \in \mathsf{Fi}_{\mathsf{F}}(P)$ such that $x = \bigwedge_L e[F]$ and an element $y \in L$ is called *order-open* if there exits $I \in \mathsf{Id}_{\mathsf{or}}(P)$ such that $y = \bigvee_L e[I]$. Let us denote by $\mathsf{K}_{\mathsf{F}}(L)$ the set of all Frink-closed elements of L and by $\mathsf{O}_{\mathsf{or}}(L)$ we denote the set of all order-open elements of L. In the sequel, we omit the subscript L when denoting joins and meets in the lattice L and only use it when we need to indicate which lattice is under consideration.

Theorem 3.5.1. ([**27**, Theorem 5.10]). *Let P be a poset. Then, there exists a unique up to isomorphism completion $\langle L, e\rangle$ of P such that the following conditions are satisfied:*

(C) *for every $F \in \mathsf{Fi}_{\mathsf{F}}(P)$ and $I \in \mathsf{Id}_{\mathsf{or}}(P)$ if $\bigwedge e[F] \leq \bigvee e[I]$, then $F \cap I \neq \emptyset$.*

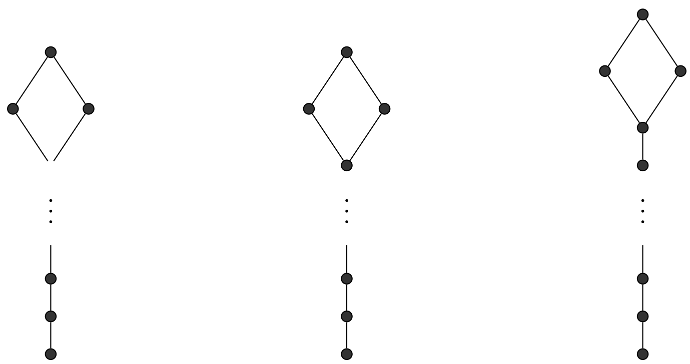
(D) *each element of L is both the join of all the Frink-closed elements below it and the meet of all the order-open elements above it. That is, for all $a \in L$, we have*

$$a = \bigvee\{x \in \mathsf{K}_{\mathsf{F}}(L) : \ x \leq a\} \quad \textit{and} \quad a = \bigwedge\{y \in \mathsf{O}_{\mathsf{or}}(L) : \ a \leq y\}.$$

Definition 3.5.2. Let P be a poset. The *Frink completion* of P is the unique up to isomorphism completion of P such that Conditions (C) and (D) hold.

For every poset P, we denote the Frink completion of P by $\langle P^{\mathrm{F}}, e\rangle$ or simply by P^{F}. We note that the Frink completion of a poset P is the $(\mathsf{Fi}_{\mathsf{F}}(P), \mathsf{Id}_{\mathsf{or}}(P))$-completion of P ([**27**]). Another important completion of a poset considered in the literature is the canonical extension as defined in [**17**]. The canonical extension of a poset P is the $(\mathsf{Fi}_{\mathsf{or}}(P), \mathsf{Id}_{\mathsf{or}}(P))$-completion of P and it is denoted by P^{σ}, see [**27**] (also [**47**]). In the following example we show that the Frink completion and the canonical extension of a poset may be different, even if the poset is a mo-distributive poset.

Example 3.5.3. We consider the poset P given on the right hand side in Figure 3.3. The canonical extension P^{σ} and the Frink completion P^{F} of P are also shown in Figure 3.3. Thus we observe that P^{σ} and P^{F} are not isomorphic. Moreover, it is clear that the poset P is mo-distributive.



P P^{σ} P^{F}

FIGURE 3.3. A mo-distributive poset P and its canonical extension P^{σ} and Frink completion P^{F}.

LEMMA 3.5.4. *Let M be a meet-semilattice. Then the canonical extension of M coincides with the Frink completion of M. That is, $M^{\sigma} \cong M^{\mathrm{F}}$.*

PROOF. It is an immediate consequence of the fact $\mathsf{Fi}_{\mathsf{or}}(M) = \mathsf{Fi}_{\mathsf{F}}(M)$, because M is a meet-semilattice. □

Some of the following results are consequence of the fact that $\mathsf{Fi}_{\mathsf{F}}(P)$ is an algebraic closure system and $\mathsf{Id}_{\mathsf{or}}(P)$ is closed under unions of up-directed families and they are obtained by a direct application of the results in [**27**] and thus we omit their proofs leaving the details to the reader.

LEMMA 3.5.5. ([**27**, Proposition 6.4]). *Let P be a poset and P^{F} its Frink completion. Then:*

(1) $\mathcal{J}^{\infty}(P^{\mathrm{F}}) \subseteq \mathsf{K}_{\mathsf{F}}(P^{\mathrm{F}})$;
(2) $\mathcal{M}^{\infty}(P^{\mathrm{F}}) \subseteq \mathsf{O}_{\mathsf{or}}(P^{\mathrm{F}})$.

LEMMA 3.5.6. ([**27**, Proposition 6.5]). *Let P be a poset and P^{F} its Frink completion. Then, $\mathcal{J}^{\infty}(P^{\mathrm{F}})$ is join-dense in P^{F} and $\mathcal{M}^{\infty}(P^{\mathrm{F}})$ is meet-dense in P^{F}.*

Let P be a poset and let $F \in \mathsf{Fi}_{\mathsf{F}}(P)$ and $I \in \mathsf{Id}_{\mathsf{or}}(P)$. We say that $\langle F, I\rangle$ is a *maximal pair* of P provided F is maximal in the set $\{G \in \mathsf{Fi}_{\mathsf{F}}(P) : \ G \cap I = \emptyset\}$ and I is maximal in the set $\{J \in \mathsf{Id}_{\mathsf{or}}(P) : \ J \cap F = \emptyset\}$. Given $F \in \mathsf{Fi}_{\mathsf{F}}(P)$, we will say that F is in a maximal pair if there is an order-ideal I such that $\langle F, I\rangle$ is a maximal pair. In [**27**] the maximal pairs are called $(\mathsf{Fi}_{\mathsf{F}}(P), \mathsf{Id}_{\mathsf{or}}(P))$-optimal, but in this dissertation such terminology can generate confusion.

LEMMA 3.5.7. *Let P be a mo-distributive poset and let $F \in \mathsf{Fi}_{\mathsf{F}}(P)$. Then, F is in a maximal pair if and only if F is prime.*

Proof. Let $F \in \mathsf{Fi_F}(P)$. Assume that F is in a maximal pair. So there is $I \in \mathsf{Id_{or}}(P)$ such that $\langle F, I\rangle$ is a maximal pair. Since $F \cap I = \emptyset$ and P is mo-distributive, it follows that there exists $H \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$ such that $F \subseteq H$ and $H \cap I = \emptyset$. Then, by the maximality of F, we have $F = H \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$. Conversely, assume that F is a prime Frink-filter of P. By Lemma 2.1.17, we get that F^c is an order-ideal of P. Hence, as it is not hard to check, $\langle F, F^c\rangle$ is a maximal pair. $\square$

Corollary 3.5.8. *Let P be a mo-distributive poset and let $F \in \mathsf{Fi_F}(P)$ and $I \in \mathsf{Id_{or}}(P)$. Then, $\langle F, I\rangle$ is a maximal pair of P if and only if $I = F^c$.*

Proof. Let $F \in \mathsf{Fi_F}(P)$ and $I \in \mathsf{Id_{or}}(P)$. Assume that $\langle F, I\rangle$ is a maximal pair. So it is clear that $I \subseteq F^c$. By the previous lemma, we have that F is prime and consequently it follows that F^c is an order-ideal of P. Then, since $F \cap F^c = \emptyset$ and by the maximality of I, we obtain that $I = F^c$. To the converse, assume that $I = F^c$. Then F is prime and hence, by the previous lemma, it follows that $\langle F, I\rangle$ is a maximal pair. $\square$

Lemma 3.5.9. ([**27**, Propositions 5.4 and 6.9]). *Let P be a mo-distributive poset and $\langle P^{\mathrm{F}}, e\rangle$ its Frink completion. Let $\Phi\colon \mathsf{Fi_F}(P) \to \mathsf{K_F}(P^{\mathrm{F}})$ and $\Psi\colon \mathsf{Id_{or}}(P) \to \mathsf{O_{or}}(P^{\mathrm{F}})$ be the maps defined by*

$$\Phi(F) = \bigwedge e[F] \quad \text{and} \quad \Psi(I) = \bigvee e[I]$$

for every $F \in \mathsf{Fi_F}(P)$ and every $I \in \mathsf{Id_{or}}(P)$, respectively. Then, Φ is a dual order-isomorphism and Ψ is an order-isomorphism. Moreover, Φ restricts to a dual order-isomorphism from $\mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$ onto $\mathcal{J}^\infty(P^{\mathrm{F}})$ and Ψ restricts to an order-isomorphism from $\{F^c : F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)\}$ onto $\mathcal{M}^\infty(P^{\mathrm{F}})$.

Lemma 3.5.10. ([**27**, Proposition 6.10]). *Let P be a poset and P^{F} its Frink completion. Then, the finite meets and joins existing in P are preserved in P^{F}.*

We now turn our attention to study the Frink completion of a finite product of posets. To this, it is important that the posets in the product are bounded. We will show that the Frink completion of a finite product of bounded posets is the product of the Frink completions of the corresponding bounded posets. If at least one of the factor posets is not bounded, then the previous statement can be not true as showed in Example 3.5.14. The same was showed in a more general framework, namely in the Δ_1-completion setting, see Section 6.5 in [**27**]. In [**47**] Morton studied three particular Δ_1-completions, among them the canonical extension, of a finite product of posets and he showed that the condition that the posets are bounded is necessary.

Lemma 3.5.11. *Let P_1 and P_2 be bounded posets. Then,*

$$\mathsf{Fi_F}(P_1 \times P_2) = \mathsf{Fi_F}(P_1) \times \mathsf{Fi_F}(P_2) \quad \text{and} \quad \mathsf{Id_F}(P_1 \times P_2) = \mathsf{Id_F}(P_1) \times \mathsf{Id_F}(P_2).$$

Proof. Let P_1 and P_2 be bounded posets. We only prove the first equality, the second one can be proved dually. Let us denote by $\top_i$ and $\bot_i$ the top and bottom element of the poset P_i, respectively, for $i = 1, 2$. Let $F \in \mathsf{Fi}_{\mathsf{F}}(P_1 \times P_2)$. We consider the sets

$$F_1 = \{a \in P_1 :\ (a, \top_2) \in F\} \quad \text{and} \quad F_2 = \{b \in P_2 :\ (\top_1, b) \in F\}.$$

We show that F_i is a Frink-filter of P_i, for $i = 1, 2$. Notice that $\top_1 \in F_1$, because $(\top_1, \top_2) \in F$. Let $a_1, \ldots, a_n \in F_1$ and $a \in \{a_1, \ldots, a_n\}^{\mathrm{lu}}$. So, $(a_i, \top_2) \in F$ for all $i = 1, 2, \ldots, n$ and it is not hard to check that $(a, \top_2) \in \{(a_1, \top_2), \ldots, (a_n, \top_2)\}^{\mathrm{lu}}$. Then, since F is a Frink-filter of $P_1 \times P_2$, it follows that $(a, \top_2) \in F$ and thus $a \in F_1$. Hence, F_1 is a Frink-filter of P_1. Analogously, F_2 is a Frink-filter of P_2. We now show that $F = F_1 \times F_2$. Let $(a, b) \in F$. Since F is an up-set and $(a, b) \leq (a, \top_2), (\top_1, b)$, it follows that $a \in F_1$ and $b \in F_2$. Thus $(a, b) \in F_1 \times F_2$ and hence, $F \subseteq F_1 \times F_2$. Let $(a, b) \in F_1 \times F_2$. So, $(a, \top_2), (\top_1, b) \in F$. It is clear that $(a, b) \in \{(a, \top_2), (\top_1, b)\}^{\mathrm{lu}}$ and, since F is a Frink-filter, we have $(a, b) \in F$. Then $F_1 \times F_2 \subseteq F$ and hence, $F = F_1 \times F_2$. Therefore, $\mathsf{Fi}_{\mathsf{F}}(P_1 \times P_2) \subseteq \mathsf{Fi}_{\mathsf{F}}(P_1) \times \mathsf{Fi}_{\mathsf{F}}(P_2)$. Now we show the inclusion $\mathsf{Fi}_{\mathsf{F}}(P_1) \times \mathsf{Fi}_{\mathsf{F}}(P_2) \subseteq \mathsf{Fi}_{\mathsf{F}}(P_1 \times P_2)$. Let $F_1 \in \mathsf{Fi}_{\mathsf{F}}(P_1)$ and $F_2 \in \mathsf{Fi}_{\mathsf{F}}(P_2)$ and let $A \subseteq_\omega F_1 \times F_2$. We consider the sets

$$A_1 := \{a \in P_1 :\ (a, y) \in A \text{ for some } y \in P_2\} \text{ and}$$
$$A_2 := \{b \in P_2 :\ (x, b) \in A \text{ for some } x \in P_1\}.$$

It is clear that $A_1 \subseteq_\omega F_1$ and $A_2 \subseteq_\omega F_2$. Then $A_1^{\mathrm{lu}} \subseteq F_1$ and $A_2^{\mathrm{lu}} \subseteq F_2$. Let $(a, b) \in A^{\mathrm{lu}}$. We show that $a \in A_1^{\mathrm{lu}}$. Let $x \in A_1^{\mathrm{l}}$. So $x \leq a'$ for all $a' \in A_1$ and then $(x, \bot_2) \leq (a', b')$ for all $(a', b') \in A$. Thus $(x, \bot_2) \in A^{\mathrm{l}}$ and hence $(x, \bot_2) \leq (a, b)$. So $x \leq a$. Therefore $a \in A_1^{\mathrm{lu}}$. With a similar argument we have that $b \in A_2^{\mathrm{lu}}$. Hence $(a, b) \in F_1 \times F_2$. We thus obtain that $F_1 \times F_2 \in \mathsf{Fi}_{\mathsf{F}}(P_1 \times P_2)$. Hence $\mathsf{Fi}_{\mathsf{F}}(P_1) \times \mathsf{Fi}_{\mathsf{F}}(P_2) \subseteq \mathsf{Fi}_{\mathsf{F}}(P_1 \times P_2)$. Therefore,

$$\mathsf{Fi}_{\mathsf{F}}(P_1 \times P_2) = \mathsf{Fi}_{\mathsf{F}}(P_1) \times \mathsf{Fi}_{\mathsf{F}}(P_2).$$

□

Lemma 3.5.12. *Let P_1 and P_2 be arbitrary posets. Then,*

$$\mathsf{Fi}_{\mathsf{or}}(P_1 \times P_2) = \mathsf{Fi}_{\mathsf{or}}(P_1) \times \mathsf{Fi}_{\mathsf{or}}(P_2) \quad \textit{and} \quad \mathsf{Id}_{\mathsf{or}}(P_1 \times P_2) = \mathsf{Id}_{\mathsf{or}}(P_1) \times \mathsf{Id}_{\mathsf{or}}(P_2).$$

Proof. Here let us prove only the second equality and the first one can be proved dually. Let $I_1 \in \mathsf{Id}_{\mathsf{or}}(P_1)$ and $I_2 \in \mathsf{Id}_{\mathsf{or}}(P_2)$. We prove $I_1 \times I_2 \in \mathsf{Id}_{\mathsf{or}}(P_1 \times P_2)$. Since I_1 and I_2 are non-empty down-sets of P_1 and P_2, respectively, it follows clearly that $I_1 \times I_2$ is a non-empty down-set of $P_1 \times P_2$. Let $(a, b), (c, d) \in I_1 \times I_2$. So $a, c \in I_1$ and $b, d \in I_2$ and then there exist $x \in I_1$ and $y \in I_2$ such that $a, c \leq x$ and $b, d \leq y$. Then $(a, b), (c, d) \leq (x, y) \in I_1 \times I_2$. Hence $I_1 \times I_2 \in \mathsf{Id}_{\mathsf{or}}(P_1 \times P_2)$.

Therefore, $\mathsf{Id}_{\mathsf{or}}(P_1) \times \mathsf{Id}_{\mathsf{or}}(P_2) \subseteq \mathsf{Id}_{\mathsf{or}}(P_1 \times P_2)$. Now let $I \in \mathsf{Id}_{\mathsf{or}}(P_1 \times P_2)$. We consider the sets

$$I_1 := \{a \in P_1 : (a, y) \in I \text{ for some } y \in P_2\} \quad \text{and}$$
$$I_2 := \{b \in P_2 : (x, b) \in I \text{ for some } x \in P_1\}.$$

Given that $I \neq \emptyset$, it is clear that $I_1 \neq \emptyset$ and $I_2 \neq \emptyset$. We show that I_1 is an order-ideal of P_1. Let $a \in I_1$ and $x \in P_1$ be such that $x \leq a$. By definition of I_1, we have that there is $y \in P_2$ such that $(a, y) \in I$. Then $(x, y) \leq (a, y)$ and thus $(x, y) \in I$. So $x \in I_1$. Let $a, a' \in I_1$. Thus there are $y, y' \in P_2$ such that $(a, y), (a', y') \in I$. Then there exists $(c, d) \in I$ such that $(a, y), (a', y') \leq (c, d)$. Hence $a, a' \leq c$ and $c \in I_1$. Therefore, $I_1 \in \mathsf{Id}_{\mathsf{or}}(P_1)$. With an analogous argument we have that $I_2 \in \mathsf{Id}_{\mathsf{or}}(P_2)$. Now we need to prove that $I = I_1 \times I_2$. First it is clear that $I \subseteq I_1 \times I_2$. Let $(a, b) \in I_1 \times I_2$. So $a \in I_1$ and $b \in I_2$. Then there exist $y \in P_2$ and $x \in P_1$ such that $(a, y), (x, b) \in I$. Thus there exists $(c, d) \in I$ such that $(a, y), (x, b) \leq (c, d)$. Hence $(a, b) \leq (c, d)$ and then $(a, b) \in I$. Therefore $I = I_1 \times I_2$. This finishes the proof. □

Lemma 3.5.13. *Let P_1 and P_2 be bounded posets. Then, $(P_1 \times P_2)^{\mathrm{F}} = P_1^{\mathrm{F}} \times P_2^{\mathrm{F}}$.*

Proof. It is a consequence of Lemmas 3.5.11 and 3.5.12 and Proposition 6.12 in [**27**]. □

Example 3.5.14. In this example we show that the previous lemma can not be true if one of the posets is not bounded. Let P_1 and P_2 be the posets depicted in Figure 3.4. To obtain the Frink completions of P_1 and P_2 observe first that $P_1^{\mathrm{F}} \cong P_1$. On the other hand, we have $\mathsf{Fi}_{\mathsf{F}}(P_2) = \{\emptyset, {\uparrow}a, {\uparrow}b, {\uparrow}0\}$ and $\mathsf{Id}_{\mathsf{or}}(P_2) = \{{\downarrow}0, {\downarrow}a, {\downarrow}b\}$. From [**27**, Theorem 5.10] we get that $P_2^{\mathrm{F}} = \{{}^{R}\mathcal{A} : \mathcal{A} \subseteq \mathsf{Id}_{\mathsf{or}}(P_2)\}$ where ${}^{R}\mathcal{A} = \{F \in \mathsf{Fi}_{\mathsf{F}}(P_2) : F \cap I \neq \emptyset \text{ for all } I \in \mathcal{A}\}$. Then, $P_2^{\mathrm{F}} = \{\{{\uparrow}0\}, \{{\uparrow}a, {\uparrow}0\}, \{{\uparrow}b, {\uparrow}0\}, \mathsf{Fi}_{\mathsf{F}}(P_2)\}$ with the inclusion order. The Frink completions P_1^{F} and P_2^{F} are given in Figure 3.4. The products $P_1 \times P_2$ and $P_1^{\mathrm{F}} \times P_2^{\mathrm{F}}$ are also represented in Figure 3.4. Now, to find the Frink completion of $P_1 \times P_2$ we observe that $\mathsf{Fi}_{\mathsf{F}}(P_1 \times P_2) = \{\emptyset, {\uparrow}x, {\uparrow}y, {\uparrow}z, {\uparrow}u, {\uparrow}v, {\uparrow}0\}$ and $\mathsf{Id}_{\mathsf{or}}(P_1 \times P_2) = \{{\downarrow}0, {\downarrow}x, {\downarrow}y, {\downarrow}z, {\downarrow}u, {\downarrow}v\}$. Then, by [**27**, Theorem 5.10], it follows that $(P_1 \times P_2)^{\mathrm{F}} = \{{}^{R}\mathcal{A} : \mathcal{A} \subseteq \mathsf{Id}_{\mathsf{or}}(P_1 \times P_2)\} = \{\{{\uparrow}0\}, \{{\uparrow}0, {\uparrow}x\}, \{{\uparrow}0, {\uparrow}y\}, \{{\uparrow}0, {\uparrow}z\}, \{{\uparrow}0, {\uparrow}x, {\uparrow}z, {\uparrow}u\}, \{{\uparrow}0, {\uparrow}y, {\uparrow}z, {\uparrow}v\}, \mathsf{Fi}_{\mathsf{F}}(P_1 \times P_2)\}$. The Frink completion $(P_1 \times P_2)^{\mathrm{F}}$ is depicted in Figure 3.5 and hence, comparing it with $P_1^{\mathrm{F}} \times P_2^{\mathrm{F}}$, we can easily conclude that $(P_1 \times P_2)^{\mathrm{F}} \ncong P_1^{\mathrm{F}} \times P_2^{\mathrm{F}}$.

Now we will use the topological duality for mo-distributive posets developed in this chapter to prove the existence of the Frink completion of mo-distributive posets. In another words, for mo-distributive posets, the Frink completion will

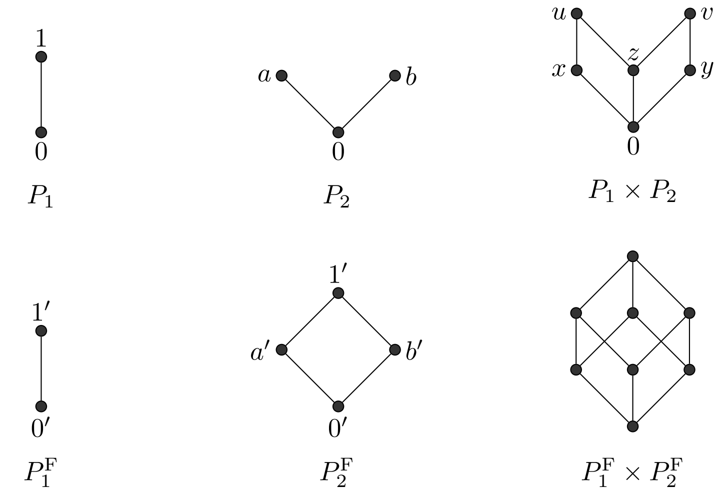


FIGURE 3.4. Example of two posets, their Frink completions and the corresponding products.

FIGURE 3.5. Example showing that the Frink completion $(P_1 \times P_2)^{\mathrm{F}}$ is not necessarily isomorphic to $P_1^{\mathrm{F}} \times P_2^{\mathrm{F}}$ if the posets are not bounded, where the posets P_1 and P_2 are given in Figure 3.4.

be obtained via the topological duality proved in Theorem 3.3.18 in an analogous fashion as the canonical extension for bounded distributive lattices is obtained via the Priestley duality [**28**]. This allows us to show that the Frink completion of a mo-distributive poset has very nice properties (see Corollary 3.5.16).

Let P be a fixed but arbitrary mo-distributive poset and $\mathbf{X}(P) = \langle \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P), \tau_P, \mathcal{B}_P \rangle$ the dual DP-space of P. To simplify the notation we let $\mathbf{X} := \mathbf{X}(P)$ and $P_{\mathbf{X}} := P_{\mathbf{X}(P)}$. Recall that the specialization order of $\mathbf{X}$ is given by:

$$F_1 \preceq F_2 \iff F_2 \subseteq F_1$$

for every $F_1, F_2 \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$. Let $\mathsf{Down}(\mathbf{X})$ be the collection of all down-sets of the poset $\langle \mathbf{X}, \preceq \rangle$. That is, for every $D \subseteq \mathbf{X}$,

$$D \in \mathsf{Down}(\mathbf{X}) \iff (\forall F, G \in \mathbf{X})(F \in D \text{ and } G \preceq F \implies G \in D).$$

It is well known, in Order Theory, that $\langle \mathsf{Down}(\mathbf{X}), \cap, \cup \rangle$ is a completely distributive algebraic lattice where $\mathcal{J}^{\infty}(\mathsf{Down}(\mathbf{X})) = \{{\downarrow}F : F \in \mathbf{X}\}$ and $\mathcal{M}^{\infty}(\mathsf{Down}(\mathbf{X})) = \{({\uparrow}F)^c : F \in \mathbf{X}\}$ with ${\downarrow}F = \{G \in \mathbf{X} : G \preceq F\}$ and ${\uparrow}F = \{G \in \mathbf{X} : F \preceq G\}$. It is also clear that $P_{\mathbf{X}} \subseteq \mathsf{C}(\mathbf{X}) \subseteq \mathsf{Down}(\mathbf{X})$ where $\mathsf{C}(\mathbf{X})$ is a sub-lattice of $\mathsf{Down}(\mathbf{X})$. Hence $\mathsf{Down}(\mathbf{X})$ is a completion of $P_{\mathbf{X}}$ and therefore it is a completion of P.

Theorem 3.5.15. *Let P be a mo-distributive poset and $\boldsymbol{X} = \boldsymbol{X}(P)$ its dual* DP*-space. Then, $\langle \mathsf{Down}(\boldsymbol{X}), \cap, \cup \rangle$ is the Frink completion of P.*

Proof. Since P and $P_{\mathbf{X}}$ are isomorphic, by Theorem 3.5.1 it is enough to prove that $\mathsf{Down}(\mathbf{X})$ is the Frink completion of $P_{\mathbf{X}}$. Thus we need to show that the completion $\mathsf{Down}(\mathbf{X})$ of $P_{\mathbf{X}}$ is such that Conditions (C) and (D) in Theorem 3.5.1 hold.

To prove Condition (D), let $D \in \mathsf{Down}(\mathbf{X})$. Notice that

$$\bigcup \left\{ \bigcap \mathcal{F} : \mathcal{F} \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathbf{X}}) \text{ and } \bigcap \mathcal{F} \subseteq D \right\} \subseteq D.$$

Let now $F \in D$. Since $F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$ and $\varphi \colon P \to P_{\mathbf{X}}$ is an order-isomorphism, it follows that $\mathcal{F} = \varphi[F] \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathbf{X}})$. Let $G \in \bigcap \mathcal{F}$. So, $G \in \varphi(a)$ for all $a \in F$ and then $F \subseteq G$. Thus $G \preceq F$ and since D is a down-set, we have that $G \in D$. Hence $\bigcap \mathcal{F} \subseteq D$ and it is clear that $F \in \bigcap \mathcal{F}$. Then, $F \in \bigcup \{\bigcap \mathcal{F} : \mathcal{F} \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathbf{X}})$ and $\bigcap \mathcal{F} \subseteq D\}$ and therefore

$$D = \bigcup \left\{ \bigcap \mathcal{F} : \mathcal{F} \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathbf{X}}) \text{ and } \bigcap \mathcal{F} \subseteq D \right\}.$$

To prove the second part of Condition (D), we note that

$$D \subseteq \bigcap \left\{ \bigcup \mathcal{I} : \mathcal{I} \in \mathsf{Id}_{\mathsf{or}}(P_{\mathbf{X}}) \text{ and } D \subseteq \bigcup \mathcal{I} \right\}.$$

To prove the other inclusion, let $F \in \bigcup \mathcal{I}$ for all $\mathcal{I} \in \mathsf{Id}_{\mathsf{or}}(P_{\mathbf{X}})$ such that $D \subseteq \bigcup \mathcal{I}$. As $F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$, we have $F^c \in \mathsf{Id}_{\mathsf{or}}(P)$ and since φ is an order-isomorphism, it follows that $\mathcal{I} = \varphi[F^c] \in \mathsf{Id}_{\mathsf{or}}(P_{\mathbf{X}})$. We suppose that $F \notin D$. So, $F \npreceq G$ for all $G \in D$ and this implies that for every $G \in D$ there exists $a_G \in G \setminus F$. Then, by the definition of $\mathcal{I}$, we have that $D \subseteq \bigcup \mathcal{I}$ and $F \notin \bigcup \mathcal{I}$; which is a contradiction. Thus, $F \in D$ and hence

$$D = \bigcap \left\{ \bigcup \mathcal{I} : \mathcal{I} \in \mathsf{Id}_{\mathsf{or}}(P_{\mathbf{X}(P)}) \text{ and } D \subseteq \bigcup \mathcal{I} \right\}.$$

Therefore, the completion $\mathsf{Down}(\mathbf{X})$ of $P_{\mathbf{X}}$ satisfies Condition (D).

Now to prove Condition (C), let $\mathcal{F} \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathbf{X}})$ and $\mathcal{I} \in \mathsf{Id}_{\mathsf{or}}(P_{\mathbf{X}})$. It is assumed that $\bigcap \mathcal{F} \subseteq \bigcup \mathcal{I}$. Suppose towards a contradiction that $\mathcal{F} \cap \mathcal{I} = \emptyset$. Let $F = \varphi^{-1}[\mathcal{F}]$ and $I = \varphi^{-1}[\mathcal{I}]$. It is clear that $F \cap I = \emptyset$ and because $\varphi \colon P \to P_{\mathbf{X}}$ is an order-isomorphism, we have $F \in \mathsf{Fi}_{\mathsf{F}}(P)$ and $I \in \mathsf{Id}_{\mathsf{or}}(P)$. Since P is mo-distributive, it follows that there is $H \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$ such that $F \subseteq H$ and $H \cap I = \emptyset$. We thus get $H \in \bigcap \mathcal{F}$ and $H \notin \bigcup \mathcal{I}$, which is a contradiction. Then, $\mathcal{F} \cap \mathcal{I} \neq \emptyset$. Hence, the completion $\mathsf{Down}(\mathbf{X})$ of $P_{\mathbf{X}}$ satisfies Condition (C). Therefore, by Theorem 3.5.1 we have proved that $\mathsf{Down}(\mathbf{X})$ is the Frink completion of $P_{\mathbf{X}}$ and thus it is the Frink completion of P. $\square$

Therefore we have proved the following properties of the Frink completion of a mo-distributive poset:

COROLLARY 3.5.16. *Let P be a mo-distributive poset and P^{F} its Frink completion. Then,*

(1) *P^{F} is a completely distributive algebraic lattice;*
(2) *$\mathcal{J}^{\infty}(P^{\mathrm{F}})$ and $\mathcal{M}^{\infty}(P^{\mathrm{F}})$ are isomorphic posets.*

These properties are the same that hold in the canonical extension of a distributive lattice, see [**28**].

CHAPTER 4

A Priestley-style duality for mo-distributive posets

In [**52**] Priestley developed a topological duality for the variety of bounded distributive lattices different from Stone duality. Priestley used ordered topological spaces to develop her duality. To be more specific, her dual spaces to the bounded distributive lattices are those which are compact totally order-disconnected ordered topological spaces (see [**15**, Chapter 11]), later known as *Priestley spaces*, and the bounded distributive lattices are represented by the lattice of clopen up-sets of such spaces.

Priestley's duality has found many applications in the theory of distributive lattices and also in Universal Algebra. For instance, an easy application of the duality is to describe the congruences of a bounded distributive lattice by means of closed subsets of its dual Priestley space ([**15**]). Another is the description of the free bounded distributive lattice, that is an important concept in Universal Algebra, in a nice way using the duality ([**53**, Theorem 3.2]). Priestley duality is also used to obtain topological dualities for subcategories of the category of bounded distributive lattices, for instance to develop a topological duality for Heyting algebras. Moreover, Priestley duality is important to develop topological dualities for some classes of algebras associated to some non-classical logics and these classes are formed by bounded distributive lattices with additional operations. For instance, Stone algebras, De Morgan algebras, Lukasiewicz algebras, etc. For more information and references about the possible applications of Priestley's duality we refer the reader to [**53**] due to Priestley. Priestley duality is also useful to obtain completions of bounded distributive lattices, for instance the canonical extension ([**28**],) and completions of bounded distributive lattices with operators ([**28**]).

As was pointed out on page 79, Stone's duality for Boolean algebras extends to a topological duality for bounded distributive lattices by means of the category of spectral spaces ([**3**] and [**55**]). Thus we can conclude that the categories of Priestley spaces and spectral spaces are equivalent. But Cornish proved in [**12**] in fact the categories are isomorphic.

Priestley duality has been used as a starting point and as a motivation to develop topological dualities for some broader classes of ordered algebraic structures than the class of bounded distributive lattices. In [**5**] (see also [**4**]) Bezhanishvili and Jansana developed a duality for bounded distributive meet-semilattices. Their dual spaces to the bounded distributive meet-semilattices, known as *generalized Priestley spaces*, are Priestley spaces augmented with a dense subset satisfying some conditions. This duality can be applied to bounded distributive lattices to obtain the Priestley duality and so the duality obtained by Bezhanishvili and Jansana for bounded distributive meet-semilattices is a generalization of the classical Priestley duality. Bezhanishvili and Jansana also generalize the notion of Esakia space (dual spaces to the Heyting algebras) to that of *generalized Esakia space* and developed a duality between the category of bounded implicative meet-semilattices and the category of generalized Esakia spaces.

In [**23**] Esteban used the classical Priestley duality theory to develop, in a uniform way, topological dualities for categories formed by ordered algebraic structures associated to certain non-classical logics. To obtain her abstract Priestley duality theory she follows the Abstract Algebraic Logic point of view. The approach by Esteban applies to several classes of algebras that are canonically associated with non-classical logics, for instance, to the class of Hilbert algebras that is the algebraic counterpart of the implicative fragment of intuitionistic logic and to several classes of algebras associated with different expansions (modal, with disjunction and with conjunction) of the implicative fragment of intuitionistic logic.

Many of the ordered algebraic structures (or ordered algebras) previously mentioned correspond to the kind of structures $\langle P, \{f_i : i \in I\}\rangle$ where P is a poset and each f_i is an n_i-ary operation on P such that f_i is either order-preserving or order-reversing in each coordinate. Such ordered algebras $\langle P, \{f_i : i \in I\}\rangle$ are called *monotone poset expansions* (MPEs) in [**17**]. The class of MPEs includes all the classes of algebras mentioned before, and in many of them the poset P is in particular a distributive lattice. Some classes of MPEs are closely linked with relational semantics for certain non-classical logics [**17, 25, 26**] and thus it is important to study these classes of structures in the more uniform and genera way. Since several of the above ordered algebras are based on distributive lattices, it can be of interest to consider the class of MPEs $\langle P, \{f_i : i \in I\}\rangle$ such that P is a mo-distributive poset.

In the distributive lattice setting the importance of obtaining (canonical) extensions of distributive lattice expansions DLEs (*distributive lattice expansions* are MPEs $\langle P, \{f_i : i \in I\}\rangle$ such that P is a distributive lattice) is because, from a point of view of logic, they provide a natural way to develop complete relational semantics for many propositional logics. A classic example is modal logic, which is also a motivating example. As is outlined in [**14**] the canonical extensions of a DLE are, traditionally, obtained in two stages. At the first stage, it is considered the canonical extension of the distributive lattice reduct of the DLE obtained by means of the Priestley duality. The canonical extension of a (bounded) distributive lattice is, up to isomorphism, the lattice of all down-sets of its Priestley dual space. At the second stage, the non-lattice operations f_i are extended to the extension. The way that the non-lattice operations are extended is naturally treated as in the modal logic case. If the non-lattice operations are join-preserving in each coordinate, then the extension of them works very well and allows to obtain relational frames (see [**33**] and [**31**]). But if the non-lattice operations are not necessarily join-preserving in each coordinate the situation is much more complicated and it is no clear how the extensions should be defined.

It is the purpose of this chapter to develop a Priestley-style topological duality for the class of bounded mo-distributive posets that extends and generalizes the duality for bounded distributive meet-semilattices obtained by Bezhanishvili and Jansana [**5, 4**]. This duality can serve as a starting point to obtain Priestley-style dualities for ordered algebraic structures based on mo-distributive posets, that is, for certain classes of MPEs where the poset reduct is mo-distributive.

This chapter is organized as follows. In Section 4.1 we define the ordered topological spaces that will be the duals to the bounded mo-distributive posets and we prove a representation theorem. Then in Section 4.2 we extend the representation theorem to a full duality between the category of the bounded mo-distributive posets and inf-homomorphisms and the category of the spaces defined in the previous section. To this, we introduce an adequate notion of morphism between the spaces that corresponds to the inf-homomorphisms. In Sections 4.3 and 4.4 we use Priestley-style duality to obtain several results about mo-distributive posets. In Section 4.3 we develop several dualities for some subcategories of the category of mo-distributive posets and then we show how we can derive the classical Priestley duality for bounded distributive lattices from our duality for mo-distributive posets. In Section §4.4 we apply Priestley-style duality to derive some results about of bounded mo-distributive posets. In §4.4.1 we use the duality to obtain the distributive meet-semilattice envelope of a bounded mo-distributive poset. In §4.4.2 we prove a correspondence between the Frink-filters of a bounded mo-distributive poset and the closed up-sets of its dual space and in §4.4.3 we derive the Frink completion (defined in Section 3.5) of a bounded mo-distributive poset through its dual space. In the last section, §4.5, we consider a new completion of a poset that it is a particular Δ_1-completion ([**27**]) and we use our Priestley-style duality for mo-distributive poset to give a topological proof of the existence of this completion for mo-distributive poset. We also show that this new completion has very nice properties.

4.1. A Priestley-style representation theorem

The broad main aim of this section is to obtain a Priestley-style representation theorem for bounded mo-distributive posets with the help of the distributive lattice envelope introduced in §2.5.2. In this section and the next we work with bounded mo-distributive posets. The boundedness restriction makes the results to be much clear, as it happens in the case of the Priestley representation theorem for bounded distributive lattices (see [**15**, Chapter 11]) and also in the case of the Priestley-style representation theorem for bounded distributive meet-semilattices (see [**4, 5**]). The results developed in this section and in the next are motivated by the papers [**5, 4**] due to Bezhanishvili and Jansana.

Let P be a fixed but arbitrary bounded mo-distributive poset. We denote the top and bottom of P, respectively, by $\top_P$ and $\bot_P$ and when confusion is unlikely we omit the subscript. Let $D(P)$ be the distributive lattice envelope of P. Recall that $D(P)$ is up to isomorphism the unique distributive lattice such that $P \subseteq D(P)$ and the following conditions are satisfied:

(D1') for each $A, B \subseteq_\omega P$,

$$A^{\mathrm{u}} \subseteq B^{\mathrm{lu}} \text{ in } P \quad \text{if and only if} \quad A^{\mathrm{u}} \subseteq B^{\mathrm{lu}} \text{ in } D(P);$$

(D2') for each $x \in D(P)$, there are non-empty $A_1, \ldots, A_n \subseteq_\omega P$ such that $x = \bigvee_{i=1}^{n} \bigwedge A_i$.

Moreover recall that $\top_P$ and $\bot_P$ are also the top and bottom element of $D(P)$. Now, given that $D(P)$ is a bounded distributive lattice, we can consider its dual Priestley space $D(P)_* = \langle \mathsf{Fi}^{\mathsf{pr}}(D(P)), \tau_D, \subseteq \rangle$. Recall, from Lemma 2.5.14, that the order-isomorphism $\mu \colon \mathsf{Fi}^{\mathsf{pr}}(D(P)) \to \mathsf{Opt_s}(P)$ is defined as: $\mu(H) = H \cap P$ for every $H \in \mathsf{Fi}^{\mathsf{pr}}(D(P))$. Then $\tau_P := \{\mu[\mathcal{U}] : \mathcal{U} \in \tau_D\}$ is a topology on $\mathsf{Opt_s}(P)$. Notice that $\mu[\mathcal{U}] = \{H \cap P : H \in \mathcal{U}\}$ for each $\mathcal{U} \in \tau_D$. We let $P_* = \langle \mathsf{Opt_s}(P), \tau_P, \subseteq \rangle$. We use frequently P_* to denote the set $\mathsf{Opt_s}(P)$. Now the following lemma is obvious.

Lemma 4.1.1. *Let P be a bounded mo-distributive poset and $D(P)$ its distributive lattice envelope. Then, $\mu \colon D(P)_* \to P_*$ is an order-isomorphism and a homeomorphism. Therefore, $P_* = \langle \mathsf{Opt_s}(P), \tau_P, \subseteq \rangle$ is a Priestley space.*

Define $\varphi_D \colon D(P) \to \mathcal{P}(\mathsf{Fi}^{\mathsf{pr}}(D(P)))$ by $\varphi_D(x) = \{H \in \mathsf{Fi}^{\mathsf{pr}}(D(P)) : x \in H\}$. Then, recall that $\{\varphi_D(x) : x \in D(P)\} \cup \{\varphi_D(y)^c : y \in D(P)\}$ is a subbase for $D(P)_*$.

Lemma 4.1.2. *Let P be a bounded mo-distributive poset and $D(P)$ its distributive lattice envelope. Then the family $\{\varphi_D(a) : a \in P\} \cup \{\varphi_D(b)^c : b \in P\}$ is a subbase for $D(P)_*$.*

Proof. Let $x \in D(P)$. By Condition (D2'), we have that there are non-empty $A_1, \dots, A_n \subseteq_\omega P$ such that $x = \bigvee_{i=1}^{n} \bigwedge A_i$. So, $\varphi_D(x) = \bigcup_{i=1}^{n} \bigcap \varphi_D[A_i]$. Hence, since the family $\{\varphi_D(x) : x \in D(P)\} \cup \{\varphi_D(y)^c : y \in D(P)\}$ is a subbase for $D(P)_*$, it follows that $\{\varphi_D(a) : a \in P\} \cup \{\varphi_D(b)^c : b \in P\}$ is a subbase for $D(P)_*$. □

We define the map $\varphi_P \colon P \to \mathcal{P}(\mathsf{Opt_s}(P))$ by $\varphi_P(a) = \{U \in \mathsf{Opt_s}(P) : a \in U\}$ for each $a \in P$. We thus obtain, for every $a \in P$,

$$\begin{aligned} \mu[\varphi_D(a)] &= \{H \cap P : H \in \varphi_D(a)\} \\ &= \{U \in \mathsf{Opt_s}(P) : a \in U\} \qquad (4.1) \\ &= \varphi_P(a). \end{aligned}$$

Corollary 4.1.3. *Let P be a bounded mo-distributive poset. Then the family $\{\varphi_P(a) : a \in P\} \cup \{\varphi_P(b)^c : b \in P\}$ is a subbase for P_*.*

Proof. It is an immediate consequence of Lemmas 4.1.1 and 4.1.2 and (4.1). □

We let $\mathcal{B}_P := \{\varphi_P(a) : a \in P\}$ and we consider the poset $P_{\mathcal{B}_P} = \langle \mathcal{B}_P, \subseteq \rangle$.

Lemma 4.1.4. *Let P be a bounded mo-distributive poset. Then $\varphi_P \colon P \to P_{\mathcal{B}_P}$ is an order-isomorphism.*

Proof. It is clear that φ_P is an onto map. Moreover, since each s-optimal Frink-filter is an up-set, it follows that φ_P is order-preserving. Let $a, b \in P$. Assume that $\varphi_P(a) \subseteq \varphi_P(b)$. Suppose towards a contradiction that $a \nleq b$. Then, by Theorem 2.4.27, there exists $U \in \mathsf{Opt_s}(P)$ such that $a \in U$ and $b \notin U$. So $U \in \varphi_P(a)$ and $U \notin \varphi_P(b)$, which is a contradiction. Thus $a \leq b$ and hence φ_P is an order-embedding. Therefore φ_P is an order-isomorphism. □

Let $\langle X, \tau, \leq \rangle$ be an arbitrary ordered topological space. We denote by $\mathsf{CLUp}(X)$ the collection of all clopen up-sets of X. By Priestley duality we know that $D(P) \cong \mathsf{CLUp}(D(P)_*)$ and by Lemma 4.1.1 we have that $\mathsf{CLUp}(D(P)_*) \cong \mathsf{CLUp}(P_*)$. Hence, we have

$$D(P) \cong \mathsf{CLUp}(P_*).$$

The following lemma shows a characterization of the clopen up-sets of P_*.

Lemma 4.1.5. *Let P be a bounded mo-distributive poset. Then, for each clopen up-set $\mathcal{U}$ of P_* there are non-empty $A_1, \dots, A_n \subseteq_\omega P$ such that $\mathcal{U} = \bigcup_{i=1}^{n} \bigcap \varphi_P[A_i]$.*

Proof. Let P be a bounded mo-distributive poset and $D(P)$ its distributive lattice envelope. Let $\mathcal{U}$ be a clopen up-set of P_*. By Lemma 4.1.1, we have that there is a clopen up-set $\mathcal{V}$ of $D(P)_*$ such that $\mu[\mathcal{V}] = \mathcal{U}$. Thus there is $x \in D(P)$ such that $\varphi_D(x) = \mathcal{V}$. Now, by Condition (D2'), there are non-empty $A_1, \dots, A_n \subseteq_\omega P$ such that $x = \bigvee_{i=1}^{n} \bigwedge A_i$. Then $\mathcal{V} = \varphi_D(x) = \bigcup_{i=1}^{n} \bigcap \varphi_D[A_i]$. Hence,

$$\begin{aligned}
\mathcal{U} &= \mu[\mathcal{V}] \\
&= \mu\left[\bigcup_{i=1}^{n} \bigcap \varphi_D[A_i]\right] \\
&= \bigcup_{i=1}^{n} \bigcap \mu[\varphi_D[A_i]] \\
&= \bigcup_{i=1}^{n} \bigcap \varphi_p[A_i].
\end{aligned}$$

□

The next lemma will be useful for what follows.

Lemma 4.1.6. *Let P be a bounded mo-distributive poset and $A, A_1, \dots, A_n \subseteq_\omega P$. Then,*

$$\bigcap \varphi_P[A] \subseteq \bigcup_{i=1}^{n} \bigcap \varphi_P[A_i] \iff \bigcap_{i=1}^{n} A_i^{\mathrm{lu}} \subseteq A^{\mathrm{lu}}.$$

Proof. First we assume that $\bigcap \varphi_P[A] \subseteq \bigcup_{i=1}^{n} \bigcap \varphi_P[A_i]$. Let $a \in \bigcap_{i=1}^{n} A_i^{\mathrm{lu}}$ and we suppose that $a \notin A^{\mathrm{lu}}$. Then, by Theorem 2.4.27, there exists $U \in \mathsf{Opt}_{\mathsf{s}}(P)$ such that $A^{\mathrm{lu}} \subseteq U$ and $a \notin U$. So, $U \in \bigcap \varphi_P[A]$ and thus there is $i \in \{1, \dots, n\}$ such that $A_i^{\mathrm{lu}} \subseteq U$. Then $a \in U$, which is a contradiction. Thus $a \in A^{\mathrm{lu}}$ and hence $\bigcap_{i=1}^{n} A_i^{\mathrm{lu}} \subseteq A^{\mathrm{lu}}$. Now we assume that $\bigcap_{i=1}^{n} A_i^{\mathrm{lu}} \subseteq A^{\mathrm{lu}}$. Let $U \in \bigcap \varphi_P[A]$. So, $A^{\mathrm{lu}} \subseteq U$ and we thus obtain $\bigcap_{i=1}^{n} A_i^{\mathrm{lu}} \subseteq U$. Suppose that $U \notin \bigcup_{i=1}^{n} \bigcap \varphi_P[A_i]$. Then, for every $i \in \{1, \dots, n\}$, there is $a_i \in A_i$ such that $a_i \notin U$. Notice that $\{a_1, \dots, a_n\}^{\mathrm{u}} \subseteq A^{\mathrm{lu}}$ and $\{a_1, \dots, a_n\} \subseteq_\omega U^c$. Now we consider two cases: $A = \emptyset$ or $A \neq \emptyset$. If $A = \emptyset$, then $A^{\mathrm{lu}} = \{\top\}$. So, $\{a_1, \dots, a_n\}^{\mathrm{u}} \subseteq \{\top\}$. Since U is an s-optimal Frink-filter, it follows that $U^c \cap \{\top\} \neq \emptyset$ and this is impossible. If $A \neq \emptyset$, then using again that U is an s-optimal Frink-filter, we have $U^c \cap A^{\mathrm{lu}} \neq \emptyset$. So, $A^{\mathrm{lu}} \nsubseteq U$ and this is a contradiction. Hence, $U \in \bigcup_{i=1}^{n} \bigcap \varphi_P[A_i]$. Therefore, $\bigcap \varphi_P[A] \subseteq \bigcup_{i=1}^{n} \bigcap \varphi_P[A_i]$. □

Now we show some properties that hold in the space P_*. These properties are important because they allow us to define abstractly the dual spaces to the bounded mo-distributive posets. Before stating the following lemma we recall that for every poset P we have that $\mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P) \subseteq \mathsf{Opt}_{\mathsf{s}}(P)$.

Lemma 4.1.7. *Let P be a bounded mo-distributive poset. Then $\mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)$ is dense in P_*.*

Proof. Given that $\{\varphi_P(a) : a \in P\} \cup \{\varphi_P(b)^c : b \in P\}$ is a subbase for P_*, it follows that a basic open of P_* is of the form $\bigcap \varphi_P[A] \cap \bigcap_{b\in B} \varphi_P(b)^c$ for some non-empty $A, B \subseteq_\omega P$. So, let $A, B \subseteq_\omega P$ be non-empty and such that $\bigcap \varphi_P[A] \cap \bigcap_{b\in B} \varphi_P(b)^c \neq \emptyset$. We thus have $\bigcap \varphi_P[A] \nsubseteq \bigcup_{b\in B} \varphi_P(b)$ and, by Lemma 4.1.6, we obtain that $\bigcap_{b\in B} {\uparrow} b \nsubseteq A^{\mathrm{lu}}$. Then, there is $a \in \bigcap_{b\in B} {\uparrow} b$ and $a \notin A^{\mathrm{lu}}$. By Theorem 2.2.17, there exists $F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$ such that $A^{\mathrm{lu}} \subseteq F$ and $a \notin F$. We thus obtain $A \subseteq F$ and $F \cap B = \emptyset$ and this implies that $F \in \bigcap \varphi_P[A]$ and $F \notin \bigcup_{b\in B} \varphi_P(b)$. Hence $F \in \bigcap \varphi_P[A] \cap \bigcap_{b\in B} \varphi_P(b)^c$. Therefore $\mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$ is dense in P_*. $\square$

Lemma 4.1.8. *Let P be a bounded mo-distributive poset. Then, for every $U \in \mathsf{Opt}_{\mathsf{s}}(P)$, there exists $F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$ such that $U \subseteq F$.*

Proof. Let $U \in \mathsf{Opt}_{\mathsf{s}}(P)$. Since U is proper, it follows that there is $a \notin U$. So, by Corollary 2.2.18, there is $F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$ such that $U \subseteq F$ and $a \notin F$. $\square$

Lemma 4.1.9. *Let P be a bounded mo-distributive poset. Let $\mathcal{U}$ be a clopen up-set of P_*. Then the following conditions are equivalent:*

(1) *$\mathcal{U} = \bigcap \varphi_P[A]$ for some non-empty $A \subseteq_\omega P$;*
(2) *$P_* \setminus \mathcal{U} = {\downarrow}(\mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P) \setminus \mathcal{U})$;*
(3) *$\max(P_* \setminus \mathcal{U}) \subseteq \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$.*

Proof. (1) $\Rightarrow$ (2) We assume that $\mathcal{U} = \bigcap \varphi_P[A]$ for some non-empty $A \subseteq_\omega P$. It is clear that $P_* \setminus \mathcal{U}$ is a down-set of P_* and $\mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P) \setminus \mathcal{U} \subseteq P_* \setminus \mathcal{U}$. Thus ${\downarrow}(\mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P) \setminus \mathcal{U}) \subseteq P_* \setminus \mathcal{U}$. Let now $U \in P_* \setminus \mathcal{U}$. So, $U \notin \mathcal{U} = \bigcap \varphi_P[A]$. Then, there is $a \in A$ such that $a \notin U$. By Corollary 2.2.18, we have that there exists $F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$ such that $U \subseteq F$ and $a \notin F$. So $F \notin \bigcap \varphi_P[A] = \mathcal{U}$ and we thus obtain that $F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P) \setminus \mathcal{U}$ and $U \subseteq F$. Then, $U \in {\downarrow}(\mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P) \setminus \mathcal{U})$. Hence, $P_* \setminus \mathcal{U} \subseteq {\downarrow}(\mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P) \setminus \mathcal{U})$ and therefore $P_* \setminus \mathcal{U} = {\downarrow}(\mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P) \setminus \mathcal{U})$.

(2) $\Rightarrow$ (3) Let $U \in \max(P_* \setminus \mathcal{U})$. Since $U \in P_* \setminus \mathcal{U}$ and by (2), it follows that there exists $F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P) \setminus \mathcal{U}$ such that $U \subseteq F$. By Lemma 2.4.24, we have that $F \in P_* \setminus \mathcal{U}$ and then, by the maximality of U, we obtain $U = F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$. Therefore, $\max(P_* \setminus \mathcal{U}) \subseteq \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$.

(3) $\Rightarrow$ (1). Since $\mathcal{U}$ is a clopen up-set, by Lemma 4.1.5 it follows that there are non-empty $A_1, \ldots, A_n \subseteq_\omega P$ such that $\mathcal{U} = \bigcup_{i=1}^{n} \bigcap \varphi_P[A_i]$. Now we consider the filter $F := \mathrm{Fi}_{D(P)}(\bigcap_{i=1}^{n} A_i^{\mathrm{lu}})$ of $D(P)$ generated by $\bigcap_{i=1}^{n} A_i^{\mathrm{lu}}$ (notice that $\bigcap_{i=1}^{n} A_i^{\mathrm{lu}}$ is taken in P). Suppose towards a contradiction that $\bigvee_{i=1}^{n} \bigwedge A_i \notin F$. So, there exists $H \in \mathsf{Fi}^{\mathsf{pr}}(D(P))$ such that $F \subseteq H$ and $\bigvee_{i=1}^{n} \bigwedge A_i \notin H$. Let $U := H \cap P$. By Corollary

2.5.15, we have that $U \in \mathsf{Opt_s}(P) = P_*$. Now on the one hand, since H is an up-set, it follows that $\bigwedge A_i \notin H$ for all $i \in \{1, \dots, n\}$ and this implies that $A_i \nsubseteq H \cap P = U$ for all $i \in \{1, \dots, n\}$. Then $U \notin \bigcup\limits_{i=1}^{n} \bigcap \varphi_P[A_i] = \mathcal{U}$. Since $P_* \setminus \mathcal{U}$ is a closed subset of the Priestley space P_* and moreover since $U \in P_* \setminus \mathcal{U}$, it follows that there exists $V \in \max(P_* \setminus \mathcal{U})$ such that $U \subseteq V$. It is clear that $V \notin \mathcal{U}$ and, by Condition (3), we have that $V \in \mathsf{Fi^{pr}_F}(P)$. On the other hand, since $F \subseteq H$, we obtain that $\bigcap\limits_{i=1}^{n} A_i^{\mathrm{lu}} \subseteq H \cap P = U$ and then $\bigcap\limits_{i=1}^{n} A_i^{\mathrm{lu}} \subseteq V$. Given that V is a prime Frink-filter of P, we have that there is $i_0 \in \{1, \dots, n\}$ such that $A_{i_0}^{\mathrm{lu}} \subseteq V$, which implies that $V \in \bigcup\limits_{i=1}^{n} \bigcap \varphi_P[A_i] = \mathcal{U}$, a contradiction. Hence, we have that $\bigvee\limits_{i=1}^{n} \bigwedge A_i \in F$. So, there are $a_1, \dots, a_k \in \bigcap\limits_{i=1}^{n} A_i^{\mathrm{lu}}$ such that $a_1 \wedge \cdots \wedge a_k \leq \bigvee\limits_{i=1}^{n} \bigwedge A_i$. Then $\varphi_D(a_1 \wedge \cdots \wedge a_k) \subseteq \varphi_D(\bigvee\limits_{i=1}^{n} \bigwedge A_i)$ and thus $\varphi_D(a_1) \cap \cdots \cap \varphi_D(a_k) \subseteq \bigcup\limits_{i=1}^{n} \bigcap \varphi_D[A_i]$. Using the fact $\mu\colon \mathsf{Fi^{pr}}(D(P)) \to \mathsf{Opt_s}(P)$ is an order-isomorphism and by (4.1), we obtain that $\varphi_P(a_1) \cap \cdots \cap \varphi_P(a_k) \subseteq \bigcup\limits_{i=1}^{n} \bigcap \varphi_P[A_i] = \mathcal{U}$. Moreover, as $a_1, \dots, a_k \in \bigcap\limits_{i=1}^{n} A_i^{\mathrm{lu}}$, it is straightforward to check that $\mathcal{U} = \bigcup\limits_{i=1}^{n} \bigcap \varphi_P[A_i] \subseteq \varphi_P(a_1) \cap \cdots \cap \varphi_P(a_k)$. Hence, $\mathcal{U} = \varphi_P(a_1) \cap \cdots \cap \varphi_P(a_k) = \bigcap \varphi_P[\{a_1, \dots, a_k\}]$. This completes the proof. $\square$

LEMMA 4.1.10. *Let P be a bounded mo-distributive poset and $U \in \mathsf{Opt_s}(P)$. Then, $U \in \mathsf{Fi^{pr}_F}(P)$ if and only if the set $\{\varphi_P(a) : a \notin U\}$ is up-directed w.r.t. the inclusion order.*

PROOF. It is a consequence of the fact that $\varphi_P\colon P \to P_{\mathcal{B}_P}$ is an order-isomorphism and by Lemma 2.1.17. $\square$

Next we present the main definition of this section, namely we present the definition of the spaces that will be the dual to the bounded mo-distributive posets. Firstly, we need to introduce some notations. Let $\langle X, \tau, \leq\rangle$ be a Priestley space and let $\mathcal{B}$ be a family of open subsets of $\langle X, \tau\rangle$. For every element $x \in X$ we consider the set

$$I_x^{\mathcal{B}} := \{U \in \mathcal{B} : x \notin U\}.$$

Now we define the following subset of X:

$$X_0^{\mathcal{B}} := \{x \in X : I_x^{\mathcal{B}} \text{ is up-directed}\}$$

where $I_x^{\mathcal{B}}$ is an up-directed subset with respect to inclusion order. Finally, we denote by $X_*^{\mathcal{B}}$ the set of all clopen up-sets U of X such that $\max(U^c) \subseteq X_0^{\mathcal{B}}$. Now we are ready to present the following definition.

DEFINITION 4.1.11. A structure $X = \langle X, \tau, \leq, \mathcal{B}\rangle$ is a *poset Priestley space* if the following conditions are satisfied:

(P1) $\langle X, \tau, \leq \rangle$ is a Priestley space;
(P2) $X_0^{\mathcal{B}}$ is a dense subset of X;
(P3) for each $x \in X$ there exists $x_0 \in X_0^{\mathcal{B}}$ such that $x \leq x_0$;
(P4) $\mathcal{B} \subseteq \tau$ such that:
  (P4.1) $\emptyset \in \mathcal{B}$;
  (P4.2) $\mathcal{B} \cup \{U^c : U \in \mathcal{B}\}$ is a subbase for X;
  (P4.3) for every $x, y \in X$;

$$x \leq y \iff (\forall U \in \mathcal{B})(x \in U \implies y \in U);$$

  (P4.4) for each $U, U_1, \ldots, U_n \in \mathcal{B}$,

$$(\forall V \in \mathcal{B})(V \subseteq U_1 \cap \cdots \cap U_n \implies V \subseteq U) \implies U_1 \cap \cdots \cap U_n \subseteq U;$$

(P5) $A \in X_*^{\mathcal{B}}$ if and only if $A = U_1 \cap \cdots \cap U_n$ for some $U_1, \ldots, U_n \in \mathcal{B}$.

Remark 4.1.12. Let $X = \langle X, \tau, \leq, \mathcal{B} \rangle$ be a poset Priestley space. Let us consider the poset $P_{\mathcal{B}} = \langle \mathcal{B}, \subseteq \rangle$. Then, it should be noted that Condition (P4.4) is equivalent to the following condition: for each $U, U_1, \ldots, U_n \in P_{\mathcal{B}}$,

$$U \in \{U_1, \ldots, U_n\}^{\mathrm{lu}} \implies U_1 \cap \cdots \cap U_n \subseteq U. \tag{4.2}$$

The equivalence between (P4.4) and (4.2) should be kept in mind, because it will be repeatedly used later on. Moreover, notice that the inverse implication in (4.2) holds. We also note that every $U \in \mathcal{B}$ is an up-set of X.

Lemma 4.1.13. *Let P be a bounded mo-distributive poset. Then $P_* = \langle \mathsf{Opt_s}(P), \tau_P, \subseteq, \mathcal{B}_P \rangle$ is a poset Priestley space.*

Proof. Let P be a bounded mo-distributive poset. We need to show that Conditions (P1)-(P5) of Definition 4.1.11 hold. By Lemma 4.1.1, we have that $\langle \mathsf{Opt_s}(P), \tau_P, \subseteq \rangle$ is a Priestley space and then (P1) holds. To prove (P2) notice that $X_0^{\mathcal{B}_P} = \{U \in \mathsf{Opt_s}(P) : I_U^{\mathcal{B}_P} \text{ is up-directed}\}$ and for every $U \in \mathsf{Opt_s}(P)$, $I_U^{\mathcal{B}_P} = \{\varphi_P(a) \in \mathcal{B}_P : U \notin \varphi_P(a)\} = \{\varphi_P(a) \in \mathcal{B}_P : a \notin U\}$. Then, by Lemma 4.1.10, we obtain that $X_0^{\mathcal{B}_P} = \mathsf{Fi_F^{pr}}(P)$ and hence, by Lemma 4.1.7, we have that $X_0^{\mathcal{B}_P}$ is dense in P_*. Condition (P3) follows from Lemma 4.1.8 and the fact that $X_0^{\mathcal{B}_P} = \mathsf{Fi_F^{pr}}(P)$. Condition (P4.1) is consequence of the fact that $\varphi_P(\perp) = \emptyset$. Conditions (P4.2) and (P4.3) are straightforward to check. Let us show (P4.4). By the previous remark, it is equivalent to prove Condition (4.2). Let $a, a_1, \ldots, a_n \in P$. Assume that $\varphi_P(a) \in \{\varphi_P(a_1), \ldots, \varphi_P(a_n)\}^{\mathrm{lu}}$ in $P_{\mathcal{B}_P}$. Then, since $\varphi_P \colon P \to P_{\mathcal{B}_P}$ is an order-isomorphism and by Lemma 2.2.5, we obtain that $a \in \{a_1, \ldots, a_n\}^{\mathrm{lu}}$. Let $U \in \varphi_P(a_1) \cap \cdots \cap \varphi_P(a_n)$. So $a_1, \ldots, a_n \in U$ and, since U is a Frink-filter of P, it follows that $a \in U$. Hence $\varphi_P(a_1) \cap \cdots \cap \varphi_P(a_n) \subseteq \varphi_P(a)$. Lastly, Condition (P5) follows from Lemma 4.1.9. Therefore, we have proved that $P_* = \langle \mathsf{Opt_s}(P), \tau_P, \subseteq, \mathcal{B}_p \rangle$ is a poset Priestley space. $\square$

Now, by the previous lemma and moreover by Lemma 4.1.4, it is straightforward to obtain directly one of the main results of this section. We leave the details to the reader.

THEOREM 4.1.14 (Priestley-style representation theorem). *For every bounded mo-distributive poset P, there exists a poset Priestley space $X = \langle X, \tau, \leq, \mathcal{B}\rangle$ such that P is isomorphic to $P_{\mathcal{B}} = \langle \mathcal{B}, \subseteq\rangle$.*

We end this section with another important result that is consequence of the definition of poset Priestley space. Before, we need to establish a connection between our definition of poset Priestley space and the definition of generalized Priestley space due to Bezhanishvili and Jansana [**4, 5**]. Thus we present the definition of generalized Priestley space and for more details and information about generalized Priestley spaces we address the reader to [**5, 4**]. Let $\langle X, \tau, \leq\rangle$ be a Priestley space and let X_0 be a dense subset of X. Let X^* denote the collection of all clopen up-sets U of X such that $\max(U^c) \subseteq X_0$ and for every $x \in X$ let $I_x := \{U \in X^* : \ x \notin U\}$.

DEFINITION 4.1.15. ([**5**, Definition 5.5]). A quadruple $X = \langle X, \tau \leq, X_0\rangle$ is a *generalized Priestley space* if:

(1) $\langle X, \tau, \leq\rangle$ is a Priestley space;
(2) X_0 is a dense subset of X;
(3) for each $x \in X$ there is $y \in X_0$ such that $x \leq y$;
(4) $x \in X_0$ iff I_x is up-directed;
(5) for all $x, y \in X$, we have $x \leq y$ iff $(\forall U \in X^*)(x \in U \implies y \in U)$.

LEMMA 4.1.16. ([**5**, Proposition 5.9]). *Let $X = \langle X, \tau \leq, X_0\rangle$ be a generalized Priestley space. Then $\langle X^*, \cap, \emptyset, X\rangle$ is a bounded distributive meet-semilattice.*

Now we are ready to show that every poset Priestley space becomes in a generalized Priestley space.

LEMMA 4.1.17. *Let $\langle X, \tau, \leq, \mathcal{B}\rangle$ be a poset Priestley space. Then $\langle X, \tau, \leq, X_0^{\mathcal{B}}\rangle$ is a generalized Priestley space.*

PROOF. Let $\langle X, \tau, \leq, \mathcal{B}\rangle$ be a poset Priestley space and consider the quadruple $\langle X, \tau, \leq, X_0^{\mathcal{B}}\rangle$. By Conditions (P1)-(P3) of Definition 4.1.11, it is clear that conditions (1)-(3) hold. Now, notice that $X^* = X_*^{\mathcal{B}}$. So for every $x \in X$ we have $I_x = \{U \in X_*^{\mathcal{B}} : \ x \notin U\}$. Thus to prove Condition (4), first we show that for every $x \in X$, I_x is up-directed if and only if $I_x^{\mathcal{B}}$ is up-directed. Let $x \in X$. Suppose that I_x is up-directed and let $U_1, U_2 \in I_x^{\mathcal{B}}$. Then, by Condition (P5), we have that $U_1, U_2 \in I_x$. So there is $A \in I_x$ such that $U_1 \cup U_2 \subseteq A$. Since $A \in X^* = X_*^{\mathcal{B}}$, by Condition (P5) again it follows that $A = V_1 \cap \cdots \cap V_n$ for some $V_1, \ldots, V_n \in \mathcal{B}$. As $x \notin A$, there is $i \in \{1, \ldots, n\}$ such that $x \notin V_i$. Then $U_1 \cup U_2 \subseteq A \subseteq V_i \in I_x^{\mathcal{B}}$.

Hence $I_x^{\mathcal{B}}$ is up-directed. Now assume that $I_x^{\mathcal{B}}$ is up-directed. Let $A_1, A_2 \in I_x$. So $A_1, A_2 \in X^* = X_*^{\mathcal{B}}$ and using Condition (P5) we obtain that $A_1 = U_1 \cap \cdots \cap U_n$ and $A_2 = V_1 \cap \cdots \cap V_m$ for some $U_1, \ldots, U_n, V_1, \ldots, V_m \in \mathcal{B}$. Since $x \notin A_1 \cup A_2$, it follows that there exist $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$ such that $x \notin U_i \cup V_j$. It is clear that $U_i, V_j \in I_x^{\mathcal{B}}$. So there is $W \in I_x^{\mathcal{B}}$ such that $U_i \cup V_j \subseteq W$. By (P5), we have $W \in I_x$. We thus obtain that $A_1 \cup A_2 \subseteq W$ and $W \in I_x$ and hence I_x is up-directed. Hence, for every $x \in X$, we have that $x \in X_0^{\mathcal{B}}$ if and only if $I_x^{\mathcal{B}}$ is up-directed if and only if I_x is up-directed. Then Condition (4) holds. Now let $x, y \in X$. Suppose $x \leq y$ and let $A \in X^* = X_*^{\mathcal{B}}$ be such that $x \in A$. By Condition (P5), we have $A = U_1 \cap \cdots \cap U_n$ for some $U_1, \ldots, U_n \in \mathcal{B}$. So $x \in U_i$ for all $i \in \{1, \ldots, n\}$ and then, by (P4.3), we obtain $y \in U_i$ for all $i \in \{1, \ldots, n\}$. Hence $y \in A$. Conversely, suppose that $(\forall A \in X^* = X_*^{\mathcal{B}})(x \in A \implies y \in A)$. Since $\mathcal{B} \subseteq X_*^{\mathcal{B}} = X^*$ and by (P4.3), it follows that $x \leq y$. Hence Condition (5) holds. This completes the proof. □

Lemma 4.1.18. *Let $X = \langle X, \tau, \leq, \mathcal{B} \rangle$ be a poset Priestley space. Then $P_{\mathcal{B}}$ is a bounded mo-distributive poset.*

Proof. First we show that the poset $P_{\mathcal{B}}$ is bounded. By Condition (P4.1), we have that $\emptyset \in \mathcal{B}$ and thus $\emptyset$ is the bottom of $P_{\mathcal{B}}$. Notice that $X \in X_*^{\mathcal{B}}$. So, by Condition (P5), we obtain that $X = U_1 \cap \cdots \cap U_n$ for some $U_1, \ldots, U_n \in \mathcal{B}$ and this implies that $X = U_1 = \cdots = U_n \in \mathcal{B}$. Then X is the top element of $P_{\mathcal{B}}$. Hence $P_{\mathcal{B}}$ is bounded. Next we prove that $P_{\mathcal{B}}$ is mo-distributive. Let $U, U_1, \ldots, U_n \in P_{\mathcal{B}}$ be such that $U \in \{U_1, \ldots, U_n\}^{\mathrm{lu}}$. By (P4.4), we have $U_1 \cap \cdots \cap U_n \subseteq U$. From Lemmas 4.1.17 and 4.1.16 we know that $\langle X_*^{\mathcal{B}}, \cap \rangle$ is a distributive meet-semilattice and from (P5) we have $\mathcal{B} \subseteq X_*^{\mathcal{B}}$. Then there exist $A_1, \ldots, A_n \in X_*^{\mathcal{B}}$ such that $U_i \subseteq A_i$ for all $i \in \{1, \ldots, n\}$ and $U = A_1 \cap \cdots \cap A_n$. Now, using Condition (P5) again, we have that for every $i \in \{1, \ldots, n\}$ there exists a non-empty $\mathcal{X}_i \subseteq_\omega \mathcal{B}$ such that $A_i = \bigcap \mathcal{X}_i$. Notice that for every $i \in \{1, \ldots, n\}$ we have that $U_i \subseteq A_i \subseteq V$ for all $V \in \mathcal{X}_i$. Then $U = \bigcap \mathcal{X}_1 \cap \cdots \cap \bigcap \mathcal{X}_n$ with $V \in {\uparrow}U_1 \cup \cdots \cup {\uparrow}U_n$ for all $V \in \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_n$. Hence $P_{\mathcal{B}} = \langle \mathcal{B}, \subseteq \rangle$ is a mo-distributive poset. □

4.2. Duality

Our main aim in this section is to extend the Priestley-style representation theorem for bounded mo-distributive posets obtained in the previous section to a categorical duality. Let us denote by $\mathbb{BMODP}$ the category of all bounded mo-distributive posets and all inf-homomorphisms. We want that the dual objects corresponding to the category dually equivalent to $\mathbb{BMODP}$ are the poset Priestley spaces. Thus, it only remains to consider an adequate notion of morphisms between poset Priestley spaces that correspond to the inf-homomorphisms. To this end, we extend the notion of *generalized Priestley morphism* due to Bezhanishvili and Jansana [**5, 4**].

From the previous section we have that if P is a bounded mo-distributive poset, then $P_* = \langle \mathsf{Opt_s}(P), \tau_P, \subseteq, \mathcal{B}_P\rangle$ is a poset Priestley space such that P and $P_{\mathcal{B}_P} = \langle \mathcal{B}_P, \subseteq\rangle$ are order-isomorphic via the order-isomorphism $\varphi\colon P \to P_{\mathcal{B}_P}$. Our first step in this section is to prove a dual result in the following sense. We will prove that every poset Priestley space $X = \langle X, \tau, \leq, \mathcal{B}\rangle$ is order-isomorphic and homeomorphic to the poset Priestley space $(P_{\mathcal{B}})_* = \langle \mathsf{Opt_s}(P_{\mathcal{B}}), \tau_{P_{\mathcal{B}}}, \subseteq, \mathcal{B}_{P_{\mathcal{B}}}\rangle$ dual to the poset $P_{\mathcal{B}} = \langle \mathcal{B}, \subseteq\rangle$. We begin with the following technical lemma that is similar to Lemma 4.1.6.

Lemma 4.2.1. *Let $X = \langle X, \tau, \leq, \mathcal{B}\rangle$ be a poset Priestley space and $\mathcal{X}, \mathcal{Y}_1, \ldots, \mathcal{Y}_n \subseteq_\omega P_{\mathcal{B}}$. Then*

$$\bigcap \mathcal{X} \subseteq \bigcup_{i=1}^{n} \bigcap \mathcal{Y}_i \iff \bigcap_{i=1}^{n} \mathcal{Y}_i^{\mathrm{lu}} \subseteq \mathcal{X}^{\mathrm{lu}}.$$

Proof. Suppose that $\bigcap \mathcal{X} \subseteq \bigcup_{i=1}^{n} \bigcap \mathcal{Y}_i$ and let $U \in \bigcap_{i=1}^{n} \mathcal{Y}_i^{\mathrm{lu}}$. So $U \in \mathcal{Y}_i^{\mathrm{lu}}$ for all $i \in \{1, \ldots, n\}$. By Condition (P4.4) of Definition 4.1.11, we have that $\bigcap \mathcal{Y}_i \subseteq U$ for all $i \in \{1, \ldots, n\}$ and then $\bigcup_{i=1}^{n} \bigcap \mathcal{Y}_i \subseteq U$. Let now $W \in \mathcal{X}^{\mathrm{l}}$. So $W \subseteq \bigcap \mathcal{X}$ and, by hypothesis, we obtain that $W \subseteq \bigcup_{i=1}^{n} \bigcap \mathcal{Y}_i$. Then $W \subseteq U$, which implies that $U \in \mathcal{X}^{\mathrm{lu}}$. Hence $\bigcap_{i=1}^{n} \mathcal{Y}_i^{\mathrm{lu}} \subseteq \mathcal{X}^{\mathrm{lu}}$. Now, conversely, assume that $\bigcap_{i=1}^{n} \mathcal{Y}_i^{\mathrm{lu}} \subseteq \mathcal{X}^{\mathrm{lu}}$. First we show that $\bigcap \mathcal{X} \cap X_0^{\mathcal{B}} \subseteq \bigcup_{i=1}^{n} \bigcap \mathcal{Y}_i$. To this, let $x \in \bigcap \mathcal{X} \cap X_0^{\mathcal{B}}$. If $x \notin \bigcup_{i=1}^{n} \bigcap \mathcal{Y}_i$, then for every $i \in \{1, \ldots, n\}$ there exists $V_i \in \mathcal{Y}_i$ such that $x \notin V_i$. We thus obtain $V_1, \ldots, V_n \in I_x^{\mathcal{B}}$. Since $x \in X_0^{\mathcal{B}}$, it follows that $I_x^{\mathcal{B}}$ is up-directed and then we have that there is $V \in I_x^{\mathcal{B}}$ such that $V_1 \cup \cdots \cup V_n \subseteq V$ and this implies that $V \in \bigcap_{i=1}^{n} \mathcal{Y}_i^{\mathrm{lu}}$. We thus obtain, by hypothesis, that $V \in \mathcal{X}^{\mathrm{lu}}$. By Condition (P4.4), we have $\bigcap \mathcal{X} \subseteq V$ and then $x \in V$, which is a contradiction. So $x \in \bigcup_{i=1}^{n} \bigcap \mathcal{Y}_i$ and hence $\bigcap \mathcal{X} \cap X_0^{\mathcal{B}} \subseteq \bigcup_{i=1}^{n} \bigcap \mathcal{Y}_i$. Now, since $X_0^{\mathcal{B}}$ is a dense subset of X and $\bigcap \mathcal{X}$ is an open subset of X, it follows that $\bigcap \mathcal{X} \cap X_0^{\mathcal{B}}$ is a dense subset of $\bigcap \mathcal{X}$. Hence $\bigcap \mathcal{X} \subseteq \bigcup_{i=1}^{n} \bigcap \mathcal{Y}_i$. This completes the proof. $\square$

Let $X = \langle X, \tau, \leq, \mathcal{B}\rangle$ be a poset Priestley space. We define the map $\varepsilon\colon X \to \mathcal{P}(\mathcal{B})$ as follows:

$$\varepsilon(x) = \{U \in \mathcal{B} :\ x \in U\}$$

for every $x \in X$.

Lemma 4.2.2. *Let $X = \langle X, \tau, \leq, \mathcal{B}\rangle$ be a poset Priestley space. Then for every $x \in X$ we have that $\varepsilon(x)$ is an s-optimal Frink-filter of $P_{\mathcal{B}}$. Therefore the map $\varepsilon\colon X \to \mathsf{Opt_s}(P_{\mathcal{B}})$ is well-defined.*

Proof. Let $x \in X$. First we show that $\varepsilon(x) = \{U \in \mathcal{B} : x \in U\}$ is a Frink-filter of $P_{\mathcal{B}}$. Notice that the top element of $P_{\mathcal{B}}$, which is X, clearly belongs to $\varepsilon(x)$. Let $U_1, \ldots, U_n \in \varepsilon(x)$ and let $U \in P_{\mathcal{B}}$ be such that $U \in \{U_1, \ldots, U_n\}^{\mathrm{lu}}$. We thus obtain $x \in U_1 \cap \cdots \cap U_n$ and, by Condition (P4.4) of Definition 4.1.11, $U_1 \cap \cdots \cap U_n \subseteq U$. Then $x \in U$, that is, $U \in \varepsilon(x)$. Hence $\varepsilon(x)$ is a Frink-filter of $P_{\mathcal{B}}$. Now we show that $\varepsilon(x)$ is an s-optimal. To this end, we need to show that $\varepsilon(x)^c$ is a strong Frink-ideal of $P_{\mathcal{B}}$ (see Definition 2.4.19 on page 59). Since $\emptyset \in \mathcal{B} \setminus \varepsilon(x)$ and $\varepsilon(x)$ is an up-set, it clearly follows that $\varepsilon(x)^c$ is a non-empty down-set of $P_{\mathcal{B}}$. Let $\mathcal{X} \subseteq_\omega \varepsilon(x)^c$ be non-empty and let $\mathcal{Y} \subseteq_\omega P_{\mathcal{B}}$ be non-empty. Assume that $\mathcal{X}^{\mathrm{u}} \subseteq \mathcal{Y}^{\mathrm{lu}}$. Suppose towards a contradiction that $\mathcal{Y}^{\mathrm{lu}} \cap \varepsilon(x)^c = \emptyset$. So $\mathcal{Y}^{\mathrm{lu}} \subseteq \varepsilon(x)$, which implies that $x \in \bigcap \mathcal{Y}$. Notice that the inclusion $\mathcal{X}^{\mathrm{u}} \subseteq \mathcal{Y}^{\mathrm{lu}}$ is equivalent to the inclusion $\bigcap_{U \in \mathcal{X}} {\uparrow} U \subseteq \mathcal{Y}^{\mathrm{lu}}$. Then, by Lemma 4.2.1, we obtain that $\bigcap \mathcal{Y} \subseteq \bigcup \mathcal{X}$. Thus $x \in \bigcup \mathcal{X}$ and this is a contradiction. Hence $\mathcal{Y}^{\mathrm{lu}} \cap \varepsilon(x)^c \neq \emptyset$. Therefore $\varepsilon(x)^c$ is a strong Frink-ideal. □

Lemma 4.2.3. *Let $X = \langle X, \tau, \leq, \mathcal{B}\rangle$ be a poset Priestley space. If $x \in X_0^{\mathcal{B}}$ then $\varepsilon(x)$ is a prime Frink-filter of $P_{\mathcal{B}}$.*

Proof. Let $x \in X_0^{\mathcal{B}}$. By the previous lemma we know that $\varepsilon(x)$ is a proper Frink-filter of $P_{\mathcal{B}}$. To prove that $\varepsilon(x)$ is prime, we show that $\varepsilon(x)^c$ is an order-ideal (see Lemma 2.1.17 on page 27). Notice that $\varepsilon(x)^c = I_x^{\mathcal{B}}$ and, since $x \in X_0^{\mathcal{B}}$, it follows that $I_x^{\mathcal{B}}$ is up-directed. Hence $\varepsilon(x)^c = I_x^{\mathcal{B}}$ is an order-ideal of $P_{\mathcal{B}}$. Therefore $\varepsilon(x)$ is a prime Frink-filter of $P_{\mathcal{B}}$. □

Before starting next lemma we recall that $\mathsf{CLUp}(X)$ denotes the collection of all clopen up-sets for a topological space X.

Lemma 4.2.4. *For every poset Priestley space $X = \langle X, \tau, \leq, \mathcal{B}\rangle$, the map $\varepsilon\colon X \to \mathsf{Opt_s}(P_{\mathcal{B}})$ is onto.*

Proof. Let $\mathcal{U} \in \mathsf{Opt_s}(P_{\mathcal{B}})$. Notice that $\langle \mathsf{CLUp}(X), \cap, \cup\rangle$ is a distributive lattice and moreover we have that $P_{\mathcal{B}} \subseteq \mathsf{CLUp}(X)$. So we can consider the filter $F = \mathrm{Fi}(\mathcal{U})$ of $\mathsf{CLUp}(X)$ generated by $\mathcal{U}$ and the ideal $I = \mathrm{Id}(\mathcal{U}^c)$ of $\mathsf{CLUp}(X)$ generated by $\mathcal{U}^c$ (notice that the complement of $\mathcal{U}$ is consider with respect to $P_{\mathcal{B}}$, that is, $\mathcal{U}^c = P_{\mathcal{B}} \setminus \mathcal{U}$). Suppose that $F \cap I \neq \emptyset$. So there is $A \in F \cap I$ and then there exist $U_1, \ldots, U_n \in \mathcal{U}$ and $V_1, \ldots, V_m \in \mathcal{U}^c$ such that $U_1 \cap \cdots \cap U_n \subseteq A \subseteq V_1 \cup \cdots \cup V_m$. We thus obtain, by Lemma 4.2.1, that $\bigcap_{j=1}^{m} {\uparrow} V_j \subseteq \{U_1, \ldots, U_n\}^{\mathrm{lu}}$. As $\mathcal{U}$ is an s-optimal Frink-filter of $P_{\mathcal{B}}$, we have that $\mathcal{U}^c$ is a strong Frink-ideal of $P_{\mathcal{B}}$. Then

$\{U_1, \ldots, U_n\}^{\mathrm{lu}} \cap \mathcal{U}^c \neq \emptyset$, which implies that $\mathcal{U} \cap \mathcal{U}^c \neq \emptyset$, a contradiction. Hence $F \cap I = \emptyset$. Now, we have that there exists a prime filter H of the distributive lattice $\mathsf{CLUp}(X)$ such that $F \subseteq H$ and $H \cap I = \emptyset$. Then, by Priestley duality for distributive lattice, there exists $x \in X$ such that $H = \{A \in \mathsf{CLUp}(X) : x \in A\}$. Clearly $\mathcal{U} \subseteq H \cap \mathcal{B}$. Let $U \in H \cap \mathcal{B}$. We thus have $U \in \mathcal{B}$ and $x \in U$. Since $U \in H$ and $H \cap I = \emptyset$, it follows that $U \notin I$. Thus $U \notin \mathcal{U}^c$, that is, $U \in \mathcal{U}$. Hence, we have proved that $\mathcal{U} = H \cap \mathcal{B} = \{U \in \mathcal{B} : x \in U\} = \varepsilon(x)$. Therefore, the map ε is onto. □

Let $X = \langle X, \tau, \leq, \mathcal{B} \rangle$ be a poset Priestley space. Since $P_{\mathcal{B}} = \langle \mathcal{B}, \subseteq \rangle$ is a bounded mo-distributive poset, we can consider the poset Priestly space $(P_{\mathcal{B}})_* = \langle \mathsf{Opt}_{\mathsf{s}}(P_{\mathcal{B}}), \tau_{P_{\mathcal{B}}}, \subseteq, \mathcal{B}_{P_{\mathcal{B}}} \rangle$ as defined in the previous section. Recall that $\mathcal{B}_{P_{\mathcal{B}}} = \{\varphi(U) : U \in P_{\mathcal{B}}\}$ where for every $U \in P_{\mathcal{B}}$, $\varphi(U) = \{\mathcal{U} \in \mathsf{Opt}_{\mathsf{s}}(P_{\mathcal{B}}) : U \in \mathcal{U}\}$. Now we are ready to prove that the spaces X and $(P_{\mathcal{B}})_*$ are order-isomorphic and homeomorphic.

Theorem 4.2.5. *Let $X = \langle X, \tau, \leq, \mathcal{B} \rangle$ be a poset Priestley space. Then the map $\varepsilon \colon X \to \mathsf{Opt}_{\mathsf{s}}(P_{\mathcal{B}})$ is an order-isomorphism and a homeomorphism between the poset Priestley spaces X and $(P_{\mathcal{B}})_*$.*

Proof. By Lemma 4.2.4, we have that ε is onto and, by Condition (P4.3) of Definition 4.1.11, it is clear that ε is an order-embedding. Then ε is an order-isomorphism. To prove that ε is a continuous map we observe that a subbasic open subset of the space $(P_{\mathcal{B}})_*$ is of the form $\varphi(U)$ or $\varphi(U)^c$ for some $U \in P_{\mathcal{B}}$. So, let $U \in P_{\mathcal{B}}$ and let $x \in X$. Then, we have

$$\begin{aligned} x \in \varepsilon^{-1}[\varphi(U)] &\iff \varepsilon(x) \in \varphi(U) \\ &\iff U \in \varepsilon(x) \\ &\iff x \in U. \end{aligned}$$

Thus $\varepsilon^{-1}[\varphi(U)] = U$, which also implies that $\varepsilon^{-1}[\varphi(U)^c] = U^c$. We thus obtain that $\varepsilon^{-1}[\varphi(U)]$ and $\varepsilon^{-1}[\varphi(U)^c]$ are open subsets of $(P_{\mathcal{B}})_*$ and hence ε is continuous. Since X and $(P_{\mathcal{B}})_*$ are Priestley spaces, it follows that ε is a homeomorphism. This finishes the proof. □

Lemma 4.2.6. *Let $X = \langle X, \tau, \leq, \mathcal{B} \rangle$ be a poset Priestley space. Then $X_0^{\mathcal{B}} = \{\varepsilon^{-1}(\mathcal{F}) : \mathcal{F} \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P_{\mathcal{B}})\}$.*

Proof. Let $\mathcal{F} \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P_{\mathcal{B}})$ and $x_0 := \varepsilon^{-1}(\mathcal{F})$. So $\varepsilon(x_0) = \mathcal{F}$. We need to prove that the set $I_{x_0}^{\mathcal{B}}$ is up-directed. It is clear that $\mathcal{F}^c = \varepsilon(x_0)^c = I_{x_0}^{\mathcal{B}}$. Since $\mathcal{F}$ is prime, it follows that $\mathcal{F}^c$ is a order-ideal of $P_{\mathcal{B}}$ and then $\mathcal{F}^c = I_{x_0}^{\mathcal{B}}$ is up-directed. Hence, $\varepsilon^{-1}(\mathcal{F}) = x_0 \in X_0^{\mathcal{B}}$. Thus, $\{\varepsilon^{-1}(\mathcal{F}) : \mathcal{F} \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P_{\mathcal{B}})\} \subseteq X_0^{\mathcal{B}}$ and, by Lemma 4.2.3, we obtain that $X_0^{\mathcal{B}} \subseteq \{\varepsilon^{-1}(\mathcal{F}) : \mathcal{F} \in \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P_{\mathcal{B}})\}$. □

Recall that $\mathbb{BMODP}$ denotes the category of all bounded mo-distributive posets and all inf-homomorphisms. The next step in this section is introduce a definition of morphism between poset Priestley spaces that correspond to the morphisms in the category $\mathbb{BMODP}$.

Let X and Y be non-empty sets and let $R \subseteq X \times Y$ be a binary relation. Define the map $h_R \colon \mathcal{P}(Y) \to \mathcal{P}(X)$ as follows:

$$h_R(B) = \{x \in X : R[x] \subseteq B\} \tag{4.3}$$

for every $B \subseteq Y$.

Remark 4.2.7. It is easy to check that $h_R(B_1 \cap B_2) = h_R(B_1) \cap h_R(B_2)$ for all $B_1, B_2 \subseteq Y$ and $h_R(Y) = X$.

Let P be a bounded mo-distributive poset. Recall that the dual poset Priestley space of P is $P_* = \langle \mathsf{Opt_s}(P), \tau_P, \subseteq, \mathcal{B}_P \rangle$ and we use also P_* to denote $\mathsf{Opt_s}(P)$. Let now P and Q be bounded mo-distributive posets and let $h \colon P \to Q$ be an inf-homomorphism. Let us define the binary relation $R_h \subseteq Q_* \times P_*$ as follows: for every $U \in P_*$ and $V \in Q_*$,

$$V R_h U \iff h^{-1}[V] \subseteq U. \tag{4.4}$$

Next we show several properties that satisfies the binary relation R_h.

Lemma 4.2.8. *Let P and Q be bounded mo-distributive posets and let $h \colon P \to Q$ be an inf-homomorphism. Then, $h_{R_h}(\varphi(a)) = \varphi(h(a))$ for all $a \in P$.*

Proof. Let $a \in P$. By definition of the map h_{R_h}, see (4.3), we have that $h_{R_h}(\varphi(a)) = \{V \in Q_* : R_h[V] \subseteq \varphi(a)\}$. Let now $V \in Q_*$. By Theorem 2.4.27, it follows that

$$\begin{aligned}
V \in h_{R_h}(\varphi(a)) &\iff R_h[V] \subseteq \varphi(a) \\
&\iff (\forall U \in P_*)(V R_h U \implies U \in \varphi(a)) \\
&\iff (\forall U \in P_*)(h^{-1}[V] \subseteq U \implies a \in U) \\
&\iff a \in h^{-1}[V] \\
&\iff h(a) \in V \\
&\iff V \in \varphi(h(a)).
\end{aligned}$$

We thus obtain that $h_{R_h}(\varphi(a)) = \varphi(h(a))$. □

Lemma 4.2.9. *Let P and Q be bounded mo-distributive posets and let $h \colon P \to Q$ be an inf-homomorphism. Then,*

(1) *for every $a \in P$, $h_{R_h}(\varphi(a)) \in \mathcal{B}_Q$;*
(2) *for every $V \in Q_*$, $R_h[V] = \bigcap\{\varphi(a) \in \mathcal{B}_P : R_h[V] \subseteq \varphi(a)\}$.*

Proof. (1) It follows straightforwardly from Lemma 4.2.8. To prove (2), let $V \in Q_*$. Clearly $R_h[V] \subseteq \bigcap\{\varphi(a) \in \mathcal{B}_P : R_h[V] \subseteq \varphi(a)\}$. Let now $U \in \bigcap\{\varphi(a) \in \mathcal{B}_P : R_h[V] \subseteq \varphi(a)\}$ and we suppose that $V \not\!\!R_h U$. So, by (4.4), we have $h^{-1}[V] \nsubseteq U$. Then there is $a \in h^{-1}[V]$ such that $a \notin U$. Notice that the following equivalences are valid

$$\begin{aligned}(\forall U' \in P_*)(h^{-1}[V] \subseteq U' \implies a \in U') &\iff (\forall U' \in P_*)(V R_h U' \implies a \in U') \\ &\iff R_h[V] \subseteq \varphi(a).\end{aligned}$$

Thus, since $a \in h^{-1}[V]$, it follows that $R_h[V] \subseteq \varphi(a)$ and then $U \in \varphi(a)$. So $a \in U$, which is a contradiction. Hence $U \in R_h[V]$ and therefore (2) holds. $\square$

Now we are ready to present an adequate definition of morphisms between poset Priestley spaces and prove that they correspond to the inf-homomorphisms. The following definition is an extension of the definition of *generalized Priestley morphism* in the setting of generalized Priestley space due to Bezhanishvili and Jansana [**5, 4**].

Definition 4.2.10. Let $X = \langle X, \tau_X, \leq_X, \mathcal{B}_X\rangle$ and $Y = \langle Y, \tau_Y, \leq_Y, \mathcal{B}_Y\rangle$ be poset Priestley spaces. A binary relation $R \subseteq X \times Y$ is said to be a *poset Priestley morphism* when:

(PM1) for every $V \in \mathcal{B}_Y$, $h_R(V) \in \mathcal{B}_X$;
(PM2) for every $x \in X$, $R[x] = \bigcap\{V \in \mathcal{B}_Y : R[x] \subseteq V\}$.

The following lemma is an immediate consequence of Lemma 4.2.9 and thus we leave the details to the reader.

Lemma 4.2.11. *Let P and Q be bounded mo-distributive posets and let $h\colon P \to Q$ be an inf-homomorphism. Then $R_h \subseteq Q_* \times P_*$ is a poset Priestley morphism.*

Recall that for a poset Priestley space $X = \langle X, \tau_X, \leq_X, \mathcal{B}_X\rangle$ the bounded mo-distributive poset $\langle \mathcal{B}_X, \subseteq\rangle$ is denoted by $P_{\mathcal{B}_X}$. Notice that Condition (P4.4) implies that in the poset $P_{\mathcal{B}_X}$ the meet of two element, if it exists, is the intersection. That is, if $U_1, U_2 \in P_{\mathcal{B}_X}$ and $U_1 \wedge U_2$ exists in $P_{\mathcal{B}_X}$, then $U_1 \wedge U_2 = U_1 \cap U_2$.

Lemma 4.2.12. *Let $X = \langle X, \tau_X, \leq_X, \mathcal{B}_X\rangle$ and $Y = \langle Y, \tau_Y, \leq_Y, \mathcal{B}_Y\rangle$ be poset Priestley spaces and let $R \subseteq X \times Y$ be a poset Priestley morphism. Then $h_R\colon P_{\mathcal{B}_Y} \to P_{\mathcal{B}_X}$ is an inf-homomorphism.*

Proof. Since $P_{\mathcal{B}_X}$ and $P_{\mathcal{B}_Y}$ are bounded mo-distributive posets, by Lemma 2.3.16 (see page 47) it follows that h_R is an inf-homomorphism if and only if it is a $\wedge$-homomorphism preserving top. Then, by Remark 4.2.7, we have h_R is a $\wedge$-homomorphism preserving top and hence it is an inf-homomorphism. $\square$

Recall that for every poset Priestley space $X = \langle X, \tau_X, \leq_X, \mathcal{B}_X\rangle$ we have the poset Priestley space $(P_{\mathcal{B}_X})_* = \langle \mathsf{Opt_s}(P_{\mathcal{B}_X}), \tau_{P_{\mathcal{B}_X}}, \subseteq, \mathcal{B}_{P_{\mathcal{B}_X}}\rangle$ and the order-isomorphism and homeomorphism $\varepsilon_X \colon X \to (P_{\mathcal{B}_X})_*$ defined by $\varepsilon_X(x) = \{U \in P_{\mathcal{B}_X} : x \in U\}$.

Lemma 4.2.13. *Let $X = \langle X, \tau_X, \leq_X, \mathcal{B}_X\rangle$ and $Y = \langle Y, \tau_Y, \leq_Y, \mathcal{B}_Y\rangle$ be poset Priestley spaces and let $R \subseteq X \times Y$ be a poset Priestley morphism. Then, for every $x \in X$ and every $y \in Y$, we have*

$$xRy \iff \varepsilon_X(x) R_{h_R} \varepsilon_Y(y).$$

Proof. By Condition (PM2) of Definition 4.2.10, (4.3) and the definition of ε, we have that

$$\begin{aligned} xRy &\iff y \in R[x] \\ &\iff (\forall V \in \mathcal{B}_Y)(R[x] \subseteq V \implies y \in V) \\ &\iff (\forall V \in \mathcal{B}_Y)(x \in h_R(V) \implies V \in \varepsilon_Y(y)) \\ &\iff (\forall V \in \mathcal{B}_Y)(h_R(V) \in \varepsilon_X(x) \implies V \in \varepsilon_Y(y)) \\ &\iff (\forall V \in \mathcal{B}_Y)(V \in h_R^{-1}[\varepsilon_X(x)] \implies V \in \varepsilon_Y(y)) \\ &\iff h_R^{-1}[\varepsilon_X(x)] \subseteq \varepsilon_Y(y) \\ &\iff \varepsilon_X(x) R_{h_R} \varepsilon_Y(y). \end{aligned}$$

□

Now we want to define a category formed by the poset Priestley spaces and the poset Priestley morphisms. But unfortunately, the set-theoretical composition of two poset Priestley morphisms may not be a poset Priestley morphism. Hence we need to introduce a new definition of composition between poset Priestley morphisms. The following definition is similar to Definition 6.2 in[**5**].

Definition 4.2.14. Let X, Y and Z be poset Priestley spaces and let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be poset Priestley morphisms. We define $S * R \subseteq X \times Z$ as follows: for every $x \in X$ and $z \in Z$,

$$x(S * R)z \iff (\forall W \in \mathcal{B}_Z)(x \in (h_R \circ h_S)(W) \implies z \in W).$$

Next we will prove some properties of $*$ with the aim to show that $*$ can be used as composition between poset Priestley morphisms in order to define a category.

Lemma 4.2.15. *Let X, Y and Z be poset Priestley spaces and let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be poset Priestley morphisms. Then, for every $x \in X$ and $z \in Z$,*

$$x(S * R)z \iff \varepsilon_X(x) R_{(h_R \circ h_S)} \varepsilon_Z(z).$$

PROOF. Since $h_R\colon P_{\mathcal{B}_Y} \to P_{\mathcal{B}_X}$ and $h_S\colon P_{\mathcal{B}_Z} \to P_{\mathcal{B}_Y}$ are inf-homomorphisms, it follows that $h_R \circ h_S\colon P_{\mathcal{B}_Z} \to P_{\mathcal{B}_X}$ is an inf-homomorphism and then $R_{(h_R \circ h_S)} \subseteq (P_{\mathcal{B}_X})_* \times (P_{\mathcal{B}_Z})_*$ given by (4.4) is a poset Priestley morphism. We thus obtain

$$\begin{aligned}
\varepsilon_X(x) R_{(h_R\circ h_S)} \varepsilon_Z(z) &\iff (h_R \circ h_S)^{-1}[\varepsilon_X(x)] \subseteq \varepsilon_Z(z)\\
&\iff (\forall W \in \mathcal{B}_Z)(W \in (h_R \circ h_S)^{-1}[\varepsilon_X(x)] \implies W \in \varepsilon_Z(z))\\
&\iff (\forall W \in \mathcal{B}_Z)((h_R \circ h_S)(W) \in \varepsilon_X(x) \implies z \in W)\\
&\iff (\forall W \in \mathcal{B}_Z)(x \in (h_R \circ h_S)(W) \implies z \in W)\\
&\iff x(S * R)z.
\end{aligned}$$

$\square$

LEMMA 4.2.16. *Let P_1, P_2 and P_3 be bounded mo-distributive posets and let $h_1\colon P_1 \to P_2$ and $h_2\colon P_2 \to P_3$ be inf-homomorphisms. Then $R_{(h_2\circ h_1)} = R_{h_1} * R_{h_2}$.*

PROOF. Notice that $R_{h_1} \subseteq (P_2)_* \times (P_1)_*$ and $R_{h_2} \subseteq (P_3)_* \times (P_2)_*$ and then $R_{h_1} * R_{h_2} \subseteq (P_3)_* \times (P_1)_*$. Recall that $\mathcal{B}_{P_1} = \{\varphi(a) :\ a \in P_1\}$ where for every $a \in P_1$, $\varphi(a) = \{U \in (P_1)_* = \mathsf{Opt_s}(P_1) :\ a \in U\}$. Let $U \in (P_1)_*$ and $W \in (P_3)_*$. Then, using Lemma 4.2.8, we obtain that

$$\begin{aligned}
W(R_{h_1} * R_{h_2})U &\iff (\forall a \in P_1)(W \in (h_{R_{h_2}} \circ h_{R_{h_1}})(\varphi(a)) \implies U \in \varphi(a))\\
&\iff (\forall a \in P_1)(W \in (h_{R_{h_2}}(h_{R_{h_1}}(\varphi(a)))) \implies a \in U)\\
&\iff (\forall a \in P_1)(W \in (h_{R_{h_2}}(\varphi(h_1(a)))) \implies a \in U)\\
&\iff (\forall a \in P_1)(W \in \varphi(h_2(h_1(a))) \implies a \in U)\\
&\iff (\forall a \in P_1)((h_2 \circ h_1)(a) \in W \implies a \in U)\\
&\iff (\forall a \in P_1)(a \in (h_2 \circ h_1)^{-1}[W] \implies a \in U)\\
&\iff (h_2 \circ h_1)^{-1}[W] \subseteq U\\
&\iff W R_{(h_2\circ h_1)} U.
\end{aligned}$$

Hence $R_{(h_2\circ h_1)} = R_{h_1} * R_{h_2}$. $\square$

LEMMA 4.2.17. *Let X, Y and Z be poset Priestley spaces and let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be poset Priestley morphisms. Then, for every $W \in \mathcal{B}_Z$, we have that $(h_R \circ h_S)(W) = h_{(S*R)}(W)$.*

PROOF. Let $W \in \mathcal{B}_Z$. Since $\varepsilon_Z\colon Z \to (P_{\mathcal{B}_Z})_*$ is an order-isomorphism, it follows that $(P_{\mathcal{B}_Z})_* = \mathsf{Opt_s}(P_{\mathcal{B}_Z}) = \{\varepsilon_Z(z) :\ z \in Z\}$. Using this fact and by Theorem 2.4.27 applied to the mo-distributive poset $P_{\mathcal{B}_Z}$ and moreover by Lemma 4.2.15, we obtain that for every $x \in X$,

$$x \in (h_R \circ h_S)(W) \iff (h_R \circ h_S)(W) \in \varepsilon_X(x)$$

$$\begin{aligned}
&\iff W \in (h_R \circ h_S)^{-1}[\varepsilon_X(x)]\\
&\iff (\forall z \in Z)(\varepsilon_X(x) R_{(h_R \circ h_S)} \varepsilon_Z(z) \implies z \in W)\\
&\iff (\forall z \in Z)(x(S * R)z \implies z \in W)\\
&\iff (S * R)[x] \subseteq W\\
&\iff x \in h_{(S*R)}(W).
\end{aligned}$$

Hence $(h_R \circ h_S)(W) = h_{(S*R)}(W)$. □

Now we can prove that $S*R$ is a poset Priestley morphism for all poset Priestley morphisms R and S.

LEMMA 4.2.18. *Let X, Y and Z be poset Priestley spaces and let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be poset Priestley morphisms. Then the relation $S * R \subseteq X \times Z$ is a poset Priestley morphism.*

PROOF. By Lemma 4.2.17, we have that $h_{(S*R)}(W) = (h_R \circ h_S)(W) \in \mathcal{B}_X$ for every $W \in \mathcal{B}_Z$. Then, Condition (PM1) of Definition 4.2.10 holds. Now to prove Condition (PM2), let $x \in X$. It is clear that $(S*R)[x] \subseteq \bigcap\{W \in \mathcal{B}_Z : (S*R)[x] \subseteq W\}$. For the reverse inclusion, let $z \in \bigcap\{W \in \mathcal{B}_Z : (S*R)[x] \subseteq W\}$. Let $W' \in \mathcal{B}_Z$ be such that $x \in (h_R \circ h_S)(W')$. So $W' \in (h_R \circ h_S)^{-1}[\varepsilon_X(x)]$. Note that

$$\begin{aligned}
W' \in (h_R \circ h_S)^{-1}[\varepsilon_X(x)] &\iff (\forall z' \in Z)(\varepsilon_X(x) R_{(h_R \circ h_S)} \varepsilon_Z(z') \implies z' \in W')\\
&\iff (\forall z' \in Z)(x(S * R)z' \implies z' \in W')\\
&\iff (S * R)[x] \subseteq W'.
\end{aligned}$$

Then $z \in W'$. By definition of $*$, we obtain that $x(S * R)z$. Hence $(S * R)[x] = \bigcap\{W \in \mathcal{B}_Z : (S * R)[x] \subseteq W\}$. Then Condition (PM2) holds. Therefore $S * R$ is a poset Priestley morphism. □

LEMMA 4.2.19. *Let X_1, X_2, X_3 and X_4 be poset Priestley spaces and let $R \subseteq X_1 \times X_2$, $S \subseteq X_2 \times X_3$ and $T \subseteq X_3 \times X_4$ be poset Priestley morphisms. Then $T * (S * R) = (T * S) * R$.*

PROOF. Let $x_1 \in X_1$ and $x_4 \in X_4$. Then, by Lemma 4.2.17, we have

$$\begin{aligned}
x_1(T * (S * R))x_4 &\iff (\forall U \in \mathcal{B}_{X_4})(x_1 \in (h_{(S*R)} \circ h_T)(U) \implies x_4 \in U)\\
&\iff (\forall U \in \mathcal{B}_{X_4})(x_1 \in ((h_R \circ h_S) \circ h_T)(U) \implies x_4 \in U)\\
&\iff (\forall U \in \mathcal{B}_{X_4})(x_1 \in (h_R \circ (h_{(T*S)}))(U) \implies x_4 \in U)\\
&\iff x_1((T * S) * R)x_4.
\end{aligned}$$

Hence $T * (S * R) = (T * S) * R$. □

LEMMA 4.2.20. *Let X and Y be poset Priestley spaces and let $R \subseteq X \times Y$ be a poset Priestley morphism. Then,*

(1) *the order $\leq_X$ is a poset Priestley morphism;*
(2) $\leq_X \circ R = R$*;*
(3) $R \circ \leq_Y = R$*;*
(4) $R * \leq_X = R$*;*
(5) $\leq_Y * R = R$*.*

PROOF.

(1) Let $U \in \mathcal{B}_X$. By (4.3) and by Condition (P4.3) of Definition 4.1.11, we have

$$\begin{aligned} h_{\leq_X}(U) &= \{x \in X : \ \leq_X[x] \subseteq U\} \\ &= \{x \in X : \ \uparrow x \subseteq U\} \\ &= \{x \in X : \ x \in U\} \\ &= U. \end{aligned}$$

We thus obtain $h_{\leq_X}(U) = U \in \mathcal{B}_X$ and hence Condition (PM1) holds. To prove Condition (PM2), let $x \in X$. By again Condition (P4.3), we obtain that

$$\begin{aligned} \bigcap\{U \in \mathcal{B}_X : \ \leq_X[x] \subseteq U\} &= \bigcap\{U \in \mathcal{B}_X : \ \uparrow x \subseteq U\} \\ &= \bigcap\{U \in \mathcal{B}_X : \ x \in U\} \\ &= \uparrow x. \end{aligned}$$

Then $\leq_X[x] = \uparrow x = \bigcap\{U \in \mathcal{B}_X : \ \leq_X[x] \subseteq U\}$ and hence (PM2) holds. Therefore $\leq_X$ is a poset Priestley space.

(2) By the reflexivity of $\leq_X$, it is clear that $R \subseteq \leq_X \circ R$. Now let $x \in X$ and $y \in Y$. Assume that $x(\leq_X \circ R)y$. So there is $x' \in X$ such that $x \leq x'$ and $x'Ry$. Let $V \in \mathcal{B}_Y$ be such that $R[x] \subseteq V$. Thus $x \in h_R(V)$ and, since $h_R(V)$ is an up-set of $\langle X, \leq_X\rangle$, it follows that $x' \in h_R(V)$. Then $R[x'] \subseteq V$ and hence $y \in V$. We thus obtain $y \in R[x]$. Therefore $\leq_X \circ R = R$.

(3) It is similar to the proof of (2).

(4) Let $x \in X$ and $y \in Y$. Notice that in the proof of (1) we obtained that $h_{\leq_X}(U) = U$ for every $U \in \mathcal{B}_X$. Then,

$$\begin{aligned} x(R * \leq_X)y &\iff (\forall V \in \mathcal{B}_Y)(x \in (h_{\leq_X} \circ h_R)(V) \implies y \in V) \\ &\iff (\forall V \in \mathcal{B}_Y)(x \in h_R(V) \implies y \in V) \\ &\iff (\forall V \in \mathcal{B}_Y)(R[x] \subseteq V \implies y \in V) \end{aligned}$$

$$\iff y \in R[x].$$

Hence $R * \leq_X = R$.

(5) It is similar to the proof of (4).

$\square$

Now, by Lemmas 4.2.18, 4.2.19 and 4.2.20, we can define the category formed by all poset Priestley spaces and all poset Priestley morphisms, in which $*$ is the composition of two morphisms and $\leq_X$ is the identity morphism for each poset Priestley space X. We denote this category by $\mathbb{PPS}$. We show that $\mathbb{BMODP}$ is dually equivalent to $\mathbb{PPS}$. To this end, we define the following functors. The functor $\Phi\colon \mathbb{BMODP} \to \mathbb{PPS}$ is defined as follows:

- for every bounded mo-distributive poset P,

$$\Phi(P) := P_* = \langle \mathsf{Opt}_\mathsf{s}(P), \tau_P, \subseteq, \mathcal{B}_P \rangle,$$

- for every morphism $h\colon P \to Q$ in $\mathbb{BMODP}$,

$$\Phi(h) := R_h \subseteq Q_* \times P_*.$$

Clearly, for $\mathrm{id}_P\colon P \to P$, the identity morphism for P in $\mathbb{BMODP}$, we have that $\Phi(\mathrm{id}_P) = R_{\mathrm{id}_P} = \subseteq$ and this is the identity morphism for $\Phi(P) = P_*$ in $\mathbb{PPS}$. Hence, by Lemmas 4.1.13, 4.2.11 and 4.2.16, we obtain the the functor Φ is well-defined. On the other hand, we define the functor $\Psi\colon \mathbb{PPS} \to \mathbb{BMODP}$ as follows:

- for every poset Priestley space $X = \langle X, \tau, \leq, \mathcal{B} \rangle$,

$$\Psi(X) := P_{\mathcal{B}} = \langle \mathcal{B}, \subseteq \rangle,$$

- for every morphism $R \subseteq X \times Y$ in $\mathbb{PPS}$,

$$\Psi(R) := h_R\colon P_{\mathcal{B}_Y} \to P_{\mathcal{B}_X}.$$

Clearly, for $\leq_X$, the identity morphism for X in $\mathbb{PPS}$, we have $\Psi(\leq_X) = h_{\leq_X} = \mathrm{id}_{P_{\mathcal{B}_X}}$, that is, $\Psi(\leq_X)$ is the identity morphism for $\Psi(X)$ in $\mathbb{BMODP}$. Hence, by Lemmas 4.1.18, 4.2.12 and 4.2.17, we obtain that the functor Ψ is well-defined. Therefore, we are able to establish the main aim of this section.

THEOREM 4.2.21. *The categories $\mathbb{BMODP}$ and $\mathbb{PPS}$ are dually equivalent via the functors Φ and Ψ.*

PROOF. First we need to define the natural equivalences $\widetilde{\varphi}\colon \mathrm{Id}_{\mathbb{BMODP}} \cong \Psi \circ \Phi$ and $\widetilde{\varepsilon}\colon \mathrm{Id}_{\mathbb{PPS}} \cong \Phi \circ \Psi$. So we consider the following definition:

- for every bounded mo-distributive poset P,

$$\widetilde{\varphi}(P) := \varphi_P\colon P \to P_{\mathcal{B}_P};$$

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d, "\varphi_P"'] & Q \arrow[d, "\varphi_Q"] \\ P_{\mathcal{B}_P} \arrow[r, "h_{R_h}"'] & P_{\mathcal{B}_Q} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "R"] \arrow[d, "R_{\varepsilon_X}"'] & Y \arrow[d, "R_{\varepsilon_Y}"] \\ (P_{\mathcal{B}_X})_* \arrow[r, "R_{h_R}"'] & (P_{\mathcal{B}_Y})_* \end{tikzcd}$$

FIGURE 4.1. Commutative diagrams of morphisms in the categories $\mathbb{BMODP}$ and $\mathbb{PPS}$.

- for every poset Priestley space $X = \langle X, \tau, \leq, \mathcal{B}\rangle$,

$$\widetilde{\varepsilon}(X) := R_{\varepsilon_X} \subseteq X \times (P_{\mathcal{B}})_*$$

where the binary relation R_{ε_X} is defined as follows:

$$xR_{\varepsilon_X}\varepsilon_X(x') \iff \varepsilon_X(x) \subseteq \varepsilon_X(x')$$

for all $x, x' \in X$.

We have, by Lemma 4.1.4, that for every bounded mo-distributive poset P, $\widetilde{\varphi}(P)$ is an isomorphism in $\mathbb{BMODP}$. Using Theorem 4.2.5, it is not hard to show that for every poset Priestley space X the relation $R_{\varepsilon_X} \subseteq X \times (P_{\mathcal{B}})_*$ is a poset Priestley morphism. If we define the relation $\widehat{R_{\varepsilon_X}} \subseteq (P_{\mathcal{B}_X})_* \times X$ as $\varepsilon_X(x')\widehat{R_{\varepsilon_X}}x \iff \varepsilon_X(x') \subseteq \varepsilon_X(x)$ for every $x, x' \in X$, then by Theorem 4.2.5 again it follows that $\widehat{R_{\varepsilon_X}}$ is a poset Priestley morphism and moreover $\widehat{R_{\varepsilon_X}} * R_{\varepsilon_X} = R_{\varepsilon_X} \circ \widehat{R_{\varepsilon_X}} = \leq_X$ and $R_{\varepsilon_X} * \widehat{R_{\varepsilon_X}} = \widehat{R_{\varepsilon_X}} \circ R_{\varepsilon_X} = \leq_{(P_{\mathcal{B}_X})_*}$. Hence, R_{ε_X} is an isomorphism in the category $\mathbb{PPS}$ whose inverse morphism is $\widehat{R_{\varepsilon_X}}$.

Let now $h\colon P \to Q$ be a morphism in $\mathbb{BMODP}$ and let $R \subseteq X \times Y$ be a morphism in $\mathbb{PPS}$. By Lemma 4.2.8, we obtain that $h_{R_h} \circ \varphi_P = \varphi_Q \circ h$. And by Lemma 4.2.13, it is not hard to check that $R_{h_R} * R_{\varepsilon_X} = R_{\varepsilon_Y} * R$, although it is a little tedious and thus we leave the details to the reader. Then, we have proved that the diagrams in Figure 4.1 commute. Hence $\widetilde{\varphi}\colon \mathrm{Id}_{\mathbb{BMODP}} \cong \Psi \circ \Phi$ and $\widetilde{\varepsilon}\colon \mathrm{Id}_{\mathbb{PPS}} \cong \Phi \circ \Psi$ are natural equivalences. Therefore we have obtained that the functors Φ and Ψ establish a dual equivalence between $\mathbb{BMODP}$ and $\mathbb{PPS}$.

$\square$

4.3. Applying the Priestley-style duality

One of the two purposes in this section is to apply the Priestley-style duality for bounded mo-distributive posets given in Theorem 4.2.21 to two subcategories of $\mathbb{BMODP}$. And the second purpose is to show that our Priestley-style duality is a generalization of the classical Priestley duality for bounded distributive lattices [**52**].

4.3.1. Strong Priestley morphisms on poset Priestley spaces. The aim of this subsection is to obtain two dualities for two particular subcategories of $\mathbb{BMODP}$ through of the Priestley-style duality developed for $\mathbb{BMODP}$. Consider the category of all bounded mo-distributive posets and all inf-homomorphisms preserving bottom. We denote this category by $\mathbb{BMODP}^{\perp}$. Let us denote by $\mathbb{BMODP}^{\mathrm{s}}$ the category of all bounded mo-distributive posets and all strong inf-homomorphisms. It should be noted that the composition in $\mathbb{BMODP}^{\mathrm{s}}$ is the usual set-theoretic composition between functions and the identity morphism is the identity function.

Let P be a bounded mo-distributive poset. Recall that the dual poset Priestley space of P is $P_* = \langle \mathsf{Opt_s}(P), \tau_P, \subseteq, \mathcal{B}_P \rangle$. Moreover, recall also that if Q is a bounded mo-distributive poset and $h\colon P \to Q$ is an inf-homomorphism, then the dual poset Priestley morphism $R_h \subseteq Q_* \times P_*$ of h is defined as:

$$V R_h U \iff h^{-1}[V] \subseteq U$$

for every $V \in Q_*$ and $U \in P_*$.

LEMMA 4.3.1. *Let P and Q be bounded mo-distributive posets and let $h\colon P \to Q$ be an inf-homomorphism. Then:*

(1) *h preserves bottom if and only if $R_h^{-1}[P_*] = Q_*$;*
(2) *h is a strong inf-homomorphism if and only if for each $V \in Q_*$ there is $U \in P_*$ such that $R_h[V] = {\uparrow} U$.*

PROOF.

(1) Assume that h preserves bottom, i.e., $h(\perp_P) = \perp_Q$. It is clear that $R_h^{-1}[P_*] \subseteq Q_*$. Let $V \in Q_*$. Since V is proper, it follows that $h(\perp_P) = \perp_Q \notin V$ and then $\perp_P \notin h^{-1}[V]$. Thus, there exists $U \in \mathsf{Opt_s}(P) = P_*$ such that $h^{-1}[V] \subseteq U$ and $\perp_P \notin U$. Then $V R_h U$ and hence $V \in R_h^{-1}[P_*]$. Hence $Q_* = R_h^{-1}[P_*]$. Conversely, assume that $R_h^{-1}[P_*] = Q_*$. Suppose towards a contradiction that $h(\perp_P) \nleq \perp_Q$. So, there is $V \in \mathsf{Opt_s}(Q) = Q_*$ such that $h(\perp_P) \in V$. Thus $\perp_P \in h^{-1}[V]$. As $V \in Q_*$ and $R_h^{-1}[P_*] = Q_*$, there is $U \in P_*$ such that $V R_h U$. So, $h^{-1}[V] \subseteq U$ and this implies that $\perp_P \in U$, a contradiction because U is proper. Hence $h(\perp_P) = \perp_Q$.
(2) Assume that h is a strong inf-homomorphism. By Lemma 2.4.32, we have that $h^{-1}[V] \in P_*$ for all $V \in Q_*$. Then, by definition of R_h, it is clear that $R_h[V] = {\uparrow} h^{-1}[V]$ for every $V \in Q_*$. Conversely, assume that for each $V \in Q_*$, $R_h[V] = {\uparrow} U$ for some $U \in P_*$. To prove that h is a strong inf-homomorphism we will use the characterization of Lemma 2.4.32. Let $V \in Q_*$. So, there is $U \in P_*$ such that $R_h^{-1}[V] = {\uparrow} U$. Then $U \in R_h^{-1}[V]$ and thus $h^{-1}[V] \subseteq U$. Suppose towards a contradiction that $U \nsubseteq h^{-1}[V]$. So, there exists $a \in U$ such that $a \notin h^{-1}[V]$. Then there is $U' \in P_*$ such

that $h^{-1}[V] \subseteq U'$ and $a \notin U'$. Hence, we obtain that $U' \in R_h[V]$ and $U' \notin \uparrow U$, which is a contradiction. Then $h^{-1}[V] = U \in P_*$. We thus obtain that $h^{-1}[V] \in P_*$ for all $V \in Q_*$. Therefore, by Lemma 2.4.32, we have that h is a strong inf-homomorphism.

□

Now we introduce the following definition that is motivated by the previous lemma.

Definition 4.3.2. Let X and Y be poset Priestley spaces and let $R \subseteq X \times Y$ be a poset Priestley morphism.

(1) R is called *total* if $R^{-1}[Y] = X$;
(2) R is called *strong* if for each $x \in X$ there is $y \in Y$ such that $R[x] = \uparrow y$

We will refer to a strong poset Priestley morphism by *strong Priestley morphism* for short.

It should be noted that every strong Priestley morphism is total. By Lemma 4.3.1, we obtain directly the following lemma.

Lemma 4.3.3. *Let P and Q be bounded mo-distributive posets and let $h\colon P \to Q$ be an inf-homomorphism. Then:*

(1) *h preserves bottom if and only if R_h is total;*
(2) *h is a strong inf-homomorphism if and only if R_h is strong.*

Let X and Y be poset Priestley spaces and let $R \subseteq X \times Y$ be a poset Priestley morphism. Recall that $P_{\mathcal{B}_X} = \langle \mathcal{B}_X, \subseteq \rangle$ is the dual bounded mo-distributive poset of X and $h_R\colon P_{\mathcal{B}_Y} \to P_{\mathcal{B}_X}$ defined as in (4.3) on page 135 is the dual inf-homomorphism of R. By Lemma 4.3.3 and Lemma 4.2.13, it is straightforward to show directly that the next lemma holds.

Lemma 4.3.4. *Let X and Y be poset Priestley spaces and let $R \subseteq X \times Y$ be a poset Priestley morphism. Then:*

(1) *R is total if and only if h_R preserves bottom.;*
(2) *R is strong if and only if h_R is a strong inf-homomorphism.*

Recall that the composition $*$ of the category $\mathbb{PPS}$ between two poset Priestley morphisms $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is defined as: for every $x \in X$ and $z \in Z$,

$$x(S * R)z \iff (\forall W \in \mathcal{B}_Z)(x \in (h_R \circ h_S)(W) \implies z \in W).$$

Next we show that the composition $*$ between two strong Priestley morphisms becomes more simple, in fact, we show that $*$ coincide with the usual set-theoretic composition $\circ$ between relations. First we need to show the following property of poset Priestley morphisms.

Lemma 4.3.5. *Let X and Y be poset Priestley spaces and let $R \subseteq X \times Y$ be a poset Priestley morphism. Then:*

(1) *for each $x, x' \in X$,*

$$x \leq x' \implies R[x'] \subseteq R[x];$$

(2) *for each $y, y' \in Y$,*

$$y \leq y' \implies R^{-1}[y] \subseteq R^{-1}[y'].$$

Proof. (1) and (2) follow from the definition of poset Priestley morphism (Definition 4.2.10), (4.3) and Condition (P4.3) in Definition 4.1.11. $\square$

Lemma 4.3.6. *Let X, Y and Z be poset Priestley spaces and let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be strong Priestley morphisms. Then $S * R = R \circ S$.*

Proof. Let $x \in X$ and $z \in Z$. First assume that $x(R \circ S)z$. So there is $y \in Y$ such that xRy and ySz. Thus $y \in R[x]$ and $z \in S[y]$. Let $W \in \mathcal{B}_Z$ be such that $x \in (h_R \circ h_S)(W)$. Then $R[x] \subseteq h_S(W)$ and thus $y \in h_S(W)$. Then $S[y] \subseteq W$ and this implies that $z \in W$. Hence, by definition of $*$, we obtain that $x(S * R)z$. Now, conversely, assume that $x(S * R)z$. Since R and S are strong Priestley morphisms, it follows that there is $y \in Y$ such that $R[x] = \uparrow y$ and then there exists $z' \in Z$ such that $S[y] = \uparrow z'$. Let us show that $z' \leq z$ using Condition (P4.3) in Definition 4.1.11. To this, let $W \in \mathcal{B}_Z$ be such that $z' \in W$. Since W is an up-set of Z, it follows that $S[y] = \uparrow z' \subseteq W$ and then $y \in h_S(W)$. Since $h_S(W) \in \mathcal{B}_Y$, it follows that $h_S(W)$ is an up-set of Y and thus $R[x] = \uparrow y \subseteq h_S(W)$. Then $x \in (h_R \circ h_S)(W)$ and, since $x(S * R)z$, we obtain that $z \in W$. So we have proved that $(\forall W \in \mathcal{B}_Z)(z' \in W \implies z \in W)$, which implies that $z' \leq z$. Now, by Lemma 4.3.5, we have $S^{-1}[z'] \subseteq S^{-1}[z]$. As $z' \in S[y]$, $y \in S^{-1}[z']$ and thus $y \in S^{-1}[z]$. So ySz. Then, we have xRy and ySz and hence $x(R \circ S)z$. This finishes the proof. $\square$

Let X, Y and Z be poset Priestley spaces and let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be poset Priestley morphisms. First notice that is straightforward to check directly that the identity morphism $\leq_X$ of X in $\mathbb{PPS}$ is a total strong Priestley morphism. By Lemma 4.3.4 and Lemma 4.2.17, it is straightforward to prove that

(1) if R and S are total, then $S * R$ is total;
(2) if R and S are strong, then $S * R = R \circ S$ is strong.

Hence, we now can define the following subcategories of the category $\mathbb{PPS}$:

- the category of all poset Priestley spaces and all total poset Priestley morphisms, that we denote by $\mathbb{PPS}^{\mathrm{T}}$;
- the category of all poset Priestley spaces and all strong Priestley morphisms, that we denote by $\mathbb{PPS}^{\mathrm{s}}$.

Let $X = \langle X, \tau, \leq, \mathcal{B}\rangle$ be a poset Priestley space. We recall from Theorem 4.2.5 that $\varepsilon_X \colon X \to (P_{\mathcal{B}})_*$ defined as $\varepsilon_X(x) = \{U \in P_{\mathcal{B}} : x \in U\}$ is an order-isomorphism and a homeomorphism. We also recall that the poset Priestley morphism $R_{\varepsilon_X} \subseteq X \times (P_{\mathcal{B}})_*$ defined as: $xR_{\varepsilon_X}\varepsilon_X(x') \iff \varepsilon_X(x) \subseteq \varepsilon_X(x')$ is an isomorphism of the category $\mathbb{PPS}$. It is not hard to check that $R_{\varepsilon_X}^{-1}[(P_{\mathcal{B}})_*] = X$ and for each $x \in X$, $R_{\varepsilon_X}[x] = {\uparrow}\varepsilon_X(x)$. Hence, R_{ε_X} is a total strong Priestley morphism. That is, R_{ε_X} is a morphism of the categories $\mathbb{PPS}^{\mathrm{T}}$ and $\mathbb{PPS}^{\mathrm{s}}$. Moreover, it can be also proved that the inverse morphism of R_{ε_X}, $\widehat{R_{\varepsilon_X}}$ (see on page 142), is a total strong Priestley morphism. Thus R_{ε_X} is an isomorphism in $\mathbb{PPS}^{\mathrm{T}}$ and $\mathbb{PPS}^{\mathrm{s}}$.

Hence, putting all the previous results together and applying Theorem 4.2.21, it follows the next theorem.

Theorem 4.3.7. *The categories $\mathbb{BMODP}^{\perp}$ and $\mathbb{PPS}^{\mathrm{T}}$ are dually equivalent via the appropriate restrictions of the functors Φ and Ψ and the categories $\mathbb{BMODP}^{\mathrm{s}}$ and $\mathbb{PPS}^{\mathrm{s}}$ are also dually equivalent via the appropriate restrictions of the functors Φ and Ψ.*

Now we will show that the strong Priestley morphisms between poset Priestley spaces can be characterized by certain functions between poset Priestley spaces. We start with the following definition.

Definition 4.3.8. Let X and Y be poset Priestley spaces. A map $f \colon X \to Y$ is called *strong-continuous* if f is order-preserving and $f^{-1}[V] \in \mathcal{B}_X$ for all $V \in \mathcal{B}_Y$.

It should be noted that every strong-continuous map $f \colon X \to Y$ is a continuous map because $\mathcal{B}_X \cup \{U^c : U \in \mathcal{B}_X\}$ and $\mathcal{B}_Y \cup \{U^c : U \in \mathcal{B}_Y\}$ are subbases of the poset Priestley spaces X and Y, respectively.

Let X and Y be poset Priestley spaces and let $R \subseteq X \times Y$ be a strong Priestley morphism. We define the map $f^R \colon X \to Y$ as follows: for every $x \in X$,

$$f^R(x) = y \quad \text{if and only if} \quad R[x] = {\uparrow}y. \tag{4.5}$$

The next lemma is a property of the composition between two strong Priestley morphisms that we will need for what follows.

Lemma 4.3.9. *Let X, Y and Z be poset Priestley spaces and let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be strong Priestley morphisms. Then, for every $x \in X$ and $z \in Z$,*

$$(R \circ S)[x] = {\uparrow}z \iff (\exists y \in Y)(R[x] = {\uparrow}y \text{ and } S[y] = {\uparrow}z).$$

Proof. Let $x \in X$ and $z \in Z$. First assume that $(R \circ S)[x] = {\uparrow}z$. As R is a strong Priestley morphism, there exists $y \in Y$ such that $R[x] = {\uparrow}y$. We show that $S[y] = {\uparrow}z$. Let $z' \in S[y]$. So, we have xRy and ySz' and then $z' \in (R \circ S)[x] = {\uparrow}z$. Thus $S[y] \subseteq {\uparrow}z$. Let now $z' \in {\uparrow}z$. So $z' \in (R \circ S)[x]$ and then there is $y' \in Y$ such that xRy' and $y'Sz'$. As $y' \in R[x] = {\uparrow}y$, we have $y \leq y'$. Then, by Lemma 4.3.5, we obtain $S[y'] \subseteq S[y]$ and thus $z' \in S[y]$. Hence ${\uparrow}z \subseteq S[y]$. Therefore $S[y] = {\uparrow}z$. Now, conversely, assume that there exists $y \in Y$ such that $R[x] = {\uparrow}y$ and $S[y] = {\uparrow}z$. Let $z' \in (R \circ S)[x]$. So there is $y' \in Y$ such that xRy' and $y'Sz'$. Then $y' \in R[x] = {\uparrow}y$ and thus $y \leq y'$. By Lemma 4.3.5, we have $S[y'] \subseteq S[y]$ and this implies that $z' \in S[y] = {\uparrow}z$. Hence $(R \circ S)[x] \subseteq {\uparrow}z$. Let now $z' \in {\uparrow}z$. So $z' \in S[y]$. We thus have xRy and ySz'. Then $z' \in (R \circ S)[x]$ and hence ${\uparrow}z \subseteq (R \circ S)[x]$. Therefore, $(R \circ S)[x] = {\uparrow}z$. This completes the proof. $\square$

Lemma 4.3.10. *Let X and Y be poset Priestley spaces and let $R \subseteq X \times Y$ be a strong Priestley morphism. Then $f^R \colon X \to Y$ is a strong-continuous map. Moreover, if Z is a poset Priestley space and $S \subseteq Y \times Z$ is a strong Priestley morphism then $f^S \circ f^R = f^{R \circ S}$.*

Proof. Let $x, x' \in X$ be such that $x \leq x'$. So $f^R(x) = y$ and $f^R(x') = y'$ for some $y, y' \in Y$ such that $R[x] = {\uparrow}y$ and $R[x'] = {\uparrow}y'$. Since $x \leq x'$ and by Lemma 4.3.5, it follows that $R[x'] \subseteq R[x]$. Then ${\uparrow}y' \subseteq {\uparrow}y$ and thus $y \leq y'$. We thus obtain $f^R(x) \leq f^R(x')$ and hence f^R is order-preserving. Let now $V \in \mathcal{B}_Y$ and let $x \in X$. Then, we have

$$\begin{aligned} x \in {f^R}^{-1}[V] &\iff f^R(x) \in V \\ &\iff {\uparrow}f^R(x) \subseteq V \\ &\iff R[x] \subseteq V \\ &\iff x \in h_R(V). \end{aligned}$$

Thus ${f^R}^{-1}[V] = h_R(V) \in \mathcal{B}_X$. Then, we have proved that ${f^R}^{-1}[V] \in \mathcal{B}_X$ for all $V \in \mathcal{B}_Y$. Therefore f^R is a strong-continuous map. Now we show that $f^S \circ f^R = f^{R \circ S}$. Let $x \in X$ and $z \in Z$. Then, by Lemma 4.3.9, we have

$$\begin{aligned} (f^S \circ f^R)(x) = z &\iff f^S(f^R(x)) = z \\ &\iff S[f^R(x)] = {\uparrow}z \\ &\iff (\exists y \in Y)(f^R(x) = y \text{ and } S[y] = {\uparrow}z) \\ &\iff (\exists y \in Y)(R[x] = {\uparrow}y \text{ and } S[y] = {\uparrow}z) \\ &\iff (R \circ S)[x] = {\uparrow}z \\ &\iff f^{R \circ S}(x) = z. \end{aligned}$$

Hence $(f^S \circ f^R)(x) = f^{R \circ S}(x)$ for all $x \in X$. Therefore $f^S \circ f^R = f^{R \circ S}$. $\square$

Let X and Y be poset Priestley spaces and let $f \colon X \to Y$ be a strong-continuous map. We define the relation $R^f \subseteq X \times Y$ as follows: for every $x \in X$ and $y \in Y$,

$$xR^f y \quad \text{if and only if} \quad f(x) \leq y.$$

Notice that clearly for each $x \in X$, $xR^f f(x)$ and $R^f[x] = {\uparrow} f(x)$.

Lemma 4.3.11. *Let X and Y be poset Priestley spaces and let $f\colon X \to Y$ be a strong-continuous map. Then $R^f \subseteq X \times Y$ is a strong Priestley morphism. Moreover, if Z is a poset Priestley space and $g\colon Y \to Z$ is a strong-continuous map, then $R^f \circ R^g = R^{g \circ f}$.*

Proof. First we show that R^f is a poset Priestley morphism. Let $V \in \mathcal{B}_Y$ and $x \in X$. Then

$$\begin{aligned} x \in h_{R^f}(V) &\iff R^f[x] \subseteq V \\ &\iff {\uparrow} f(x) \subseteq V \\ &\iff f(x) \in V \\ &\iff x \in f^{-1}[V]. \end{aligned}$$

Thus $h_{R^f}(V) = f^{-1}[V] \in \mathcal{B}_X$. Hence Condition (PM1) holds. Now let $y \in Y$ be such that $y \in V$ for all $V \in B_Y$ such that $R^f[x] \subseteq V$. We want to show that $y \in R^f[x]$, i.e., we show that $f(x) \leq y$. Let $V \in B_Y$ be such that $f(x) \in V$. So $R^f[x] = {\uparrow} f(x) \subseteq V$ and then $y \in V$. Thus $f(x) \leq y$ and hence $y \in R^f[x]$. Then, we have proved that $R^f[x] = \bigcap\{V \in B_Y : \ R^f[x] \subseteq V\}$. Hence R^f satisfies Condition (PM2). Therefore R^f is a poset Priestley morphism. Since $R^f[x] = {\uparrow} f(x)$ for all $x \in X$, it follows that R^f is a strong Priestley morphism. Now, assume moreover that Z is a poset Priestley space and $g\colon Y \to Z$ is a strong-continuous map. Let $x \in X$ and $z \in Z$. Then,

$$\begin{aligned} x(R^f \circ R^g)z &\iff (\exists y \in Y)(xR^f y \text{ and } yR^g z) \\ &\iff (\exists y \in Y)(f(x) \leq y \text{ and } g(y) \leq z) \\ &\iff g(f(x)) \leq z \\ &\iff (g \circ f)(x) \leq z \\ &\iff xR^{g\circ f} z. \end{aligned}$$

Hence $R^f \circ R^g = R^{g\circ f}$. □

It is straightforward to check directly that the composition of two strong continuous-maps between poset Priestley spaces is a strong-continuous map and moreover the identity map is clearly a strong-continuous map. Then, we can consider the category of all poset Priestley spaces and all strong-continuous maps. We denote this category by $\mathbb{PPS}^{\mathrm{sc}}$.

Lemma 4.3.12. *Let X and Y be poset Priestley spaces and let $R \subseteq X \times Y$ be a strong Priestley morphism and $f\colon X \to Y$ a strong-continuous map. Then, $R^{f^R} = R$ and $f^{R^f} = f$.*

PROOF. Let $x \in X$ and $y \in Y$. Then,

$$\begin{aligned} xR^{f^R}y &\iff f^R(x) \leq y \\ &\iff y \in {\uparrow} f^R(x) = R[x] \\ &\iff xRy \end{aligned}$$

and we also have

$$\begin{aligned} f^{R^f}(x) = y &\iff R^f[x] = {\uparrow} y \\ &\iff f(x) = y. \end{aligned}$$

Hence, $R^{f^R} = R$ and $f^{R^f} = f$. □

It is easy to check directly that for every poset Priestley space X, $f^{\leq} = \mathrm{id}_X$ and $R^{\mathrm{id}_X} = \leq$. Therefore by the previous results together we obtain the following theorem.

THEOREM 4.3.13. *The categories $\mathbb{PPS}^{\mathrm{s}}$ and $\mathbb{PPS}^{\mathrm{sc}}$ are isomorphic via the functors:*

(1) $\Phi^*\colon \mathbb{PPS}^{\mathrm{s}} \to \mathbb{PPS}^{\mathrm{sc}}$ *is defined as follows:*
- *for every poset Priestley space X,*
$$\Phi^*(X) := X;$$
- *for every morphism $R \subseteq X \times Y$ in $\mathbb{PPS}^{\mathrm{s}}$,*
$$\Phi^*(R) := f^R\colon X \to Y;$$

(2) $\Psi^*\colon \mathbb{PPS}^{\mathrm{sc}} \to \mathbb{PPS}^{\mathrm{s}}$ *is defined as follows:*
- *for every poset Priestley space X,*
$$\Psi^*(X) := X;$$
- *for every morphism $f\colon X \to Y$ in $\mathbb{PPS}^{\mathrm{sc}}$,*
$$\Psi^*(f) := R^f \subseteq X \times Y.$$

The following corollary is an immediate consequence by Theorems 4.3.7 and 4.3.13.

COROLLARY 4.3.14. *The categories $\mathbb{BMODP}^{\mathrm{s}}$ and $\mathbb{PPS}^{\mathrm{sc}}$ are dually equivalent.*

The following lemma allow us describe explicitly the functors that leading to the dual equivalence between the categories $\mathbb{BMODP}^{\mathrm{s}}$ and $\mathbb{PPS}^{\mathrm{sc}}$.

LEMMA 4.3.15. *Let X and Y be poset Priestley spaces and let P and Q be bounded mo-distributive posets.*

(1) *Let* $f\colon X \to Y$ *be a strong-continuous map. Then for every* $V \in \mathcal{B}_Y$*,* $h_{R^f}(V) = f^{-1}[V]$*.*
(2) *Let* $h\colon P \to Q$ *be a strong inf-homomorphism. Then for every* $V \in Q_* = \mathsf{Opt_s}(Q)$*,* $f^{R_h}(V) = h^{-1}[V]$*.*

PROOF.
(1) Let $V \in \mathcal{B}_Y$ and $x \in X$. Then

$$\begin{aligned} x \in h_{R^f}(V) &\iff R^f[x] \subseteq V \\ &\iff {\uparrow} f(x) \subseteq V \\ &\iff f(x) \in V \\ &\iff x \in f^{-1}[V]. \end{aligned}$$

Hence $h_{R^f}(V) = f^{-1}[V]$ for all $V \in \mathcal{B}_Y$.
(2) Let $V \in Q_* = \mathsf{Opt_s}(Q)$. Since $R_h[V] = {\uparrow} h^{-1}[V]$ and moreover by definition of f^{R_h}, it follows that $f^{R_h}(V) = h^{-1}[V]$.

□

Therefore, by Theorems 4.3.7 and 4.3.13 and by Lemma 4.3.15, we obtain the following theorem. We leave the details to the reader.

THEOREM 4.3.16. *The categories* $\mathbb{BMODP}^{\mathrm{s}}$ *and* $\mathbb{PPS}^{\mathrm{sc}}$ *are dually equivalent via the functors:*

(1) $\Phi^{\mathrm{s}}\colon \mathbb{BMODP}^{\mathrm{s}} \to \mathbb{PPS}^{\mathrm{sc}}$ *is defined as follows:*
- *for every bounded mo-distributive poset* P*,*

$$\Phi^{\mathrm{s}}(P) := P_* = \langle \mathsf{Opt_s}(P), \tau_P, \subseteq, \mathcal{B}_P \rangle;$$

- *for every morphism* $h\colon P \to Q$ *in* $\mathbb{BMODP}^{\mathrm{s}}$*,*

$$\Phi^{\mathrm{s}}(h) := h^{-1}\colon Q_* \to P_*;$$

(2) $\Psi^{\mathrm{s}}\colon \mathbb{PPS}^{\mathrm{sc}} \to \mathbb{BMODP}^{\mathrm{s}}$ *is defined as follows:*
- *for every poset Priestley space* X*,*

$$\Psi^{\mathrm{s}}(X) := P_{\mathcal{B}_X} = \langle \mathcal{B}_X, \subseteq \rangle;$$

- *for every morphism* $f\colon X \to Y$ *in* $\mathbb{PPS}^{\mathrm{sc}}$*,*

$$\Psi^{\mathrm{s}}(f) := f^{-1}\colon P_{\mathcal{B}_Y} \to P_{\mathcal{B}_X}.$$

In Table 4.1 we summarize all dual equivalences that we have obtained so far in this chapter.

Categories of posets			Categories of topological spaces	
bounded mo-distributive posets and inf-homomorphisms	$\mathbb{BMODP}$	dually equivalent to	$\mathbb{PPS}$	poset Priestley spaces and poset Priestley morphisms
bounded mo-distributive posets and inf-homomorphisms preserving bottom	$\mathbb{BMODP}^{\perp}$	dually equivalent to	$\mathbb{PPS}^{\mathrm{T}}$	poset Priestley spaces and total poset Priestley morphisms
bounded mo-distributive posets and strong inf-homomorphisms	$\mathbb{BMODP}^{\mathrm{s}}$	dually equivalent to	$\mathbb{PPS}^{\mathrm{s}}$	poset Priestley spaces and strong Priestley morphisms
	$\mathbb{BMODP}^{\mathrm{s}}$	dually equivalent to	$\mathbb{PPS}^{\mathrm{sc}}$	poset Priestley spaces and strong-continuous maps

TABLE 4.1. Dual equivalences between categories of bounded mo-distributive posets and poset Priestley spaces.

4.3.2. The Priestley duality as particular case. The main aim of this subsection is to show that the classical Priestley duality for bounded distributive lattices [**52**] is a particular case of the Priestley-style duality that we have obtained for bounded mo-distributive posets in Theorem 4.2.21.

In [**4**] Bezhanishvili and Jansana showed that the Priestley duality for bounded distributive lattices is a particular case of their Priestley-style duality for bounded distributive meet-semilattices. So, to show that the Priestley duality [**52**] can be obtained of our Priestley-style duality, first we will show that the Priestley-style duality for bounded distributive meet-semilattices due to Bezhanishvili and Jansana is a particular case of our Priestley-style duality for bounded mo-distributive posets. Then we show explicitly how can be obtained the Priestley duality by means our Priestley-style duality.

Here recall the notations and the definition of generalized Priestley space given in Definition 4.1.15 on page 130.

Let M be a bounded distributive meet-semilattice. Then, by Lemma 2.4.24, we have that

$$M_* = \mathsf{Opt_s}(M) = \mathsf{Opt}(M).$$

Then, the dual poset Priestley space of M is $M_* = \langle \mathsf{Opt}(M), \tau_M, \subseteq, \mathcal{B}_M \rangle$. By Lemma 4.1.17, we know that $\langle \mathsf{Opt}(M), \tau_M, \subseteq, X_0^{\mathcal{B}_M} \rangle$ is a generalized Priestley space. Recall that $X_0^{\mathcal{B}_M} = \mathsf{Fi^{pr}}(M)$. Since M is a meet-semilattice, it follows that $\varphi_M \colon M \to P_{\mathcal{B}_M}$ is a meet-isomorphism. Then, by Lemma 4.1.9, we have

$$\begin{aligned} X_*^{\mathcal{B}_M} &= \{\mathcal{U} \in \mathsf{CLUp}(M_*) : \ \max(\mathcal{U}^c) \subseteq \mathsf{Fi^{pr}}(M)\} \\ &= \{\varphi_M(a) : \ a \in M\} \\ &= \mathcal{B}_M. \end{aligned}$$

Hence $X^* = X_*^{\mathcal{B}_M} = \mathcal{B}_M$. We thus have $\varphi_M \colon M \to \langle X^*, \cap \rangle$ is a meet-isomorphism and therefore this implies that $\langle \mathsf{Opt}(M), \tau_M, \subseteq, \mathsf{Fi^{pr}}(M) \rangle$ is the dual generalized Priestley space of M.

LEMMA 4.3.17. *Let $X = \langle X, \tau, \leq, X_0\rangle$ be a generalized Priestley space. Then $\langle X, \tau, \leq, X^*\rangle$ is a poset Priestley space.*

PROOF. We need to prove Conditions (P1)-(P5) in Definition 4.1.11. First we denote $\mathcal{B} := X^*$. Notice that for each $x \in X$, we have $I_x^{\mathcal{B}} = I_x$ and then $X_0^{\mathcal{B}} = X_0$. Hence, by Conditions (1)-(3) in Definition 4.1.15, we obtain that Conditions (P1)-(P3) hold. By [**5**, Remark 5.6, Proposition 5.10] and by Condition (5) of Definition 4.1.15, we obtain that Conditions (P4.1)-(P4.3) hold. Given that $\mathcal{B} = X^*$ and $\langle X^*, \cap\rangle$ is a distributive meet-semilattice (see Corollary 2.2.13 on page 36), then Condition (P4.4) holds. Lastly, Condition (P5) is straightforward because $X_*^{\mathcal{B}} = X^* = \mathcal{B}$. □

Let $X = \langle X, \tau, \leq, X_0\rangle$ be a generalized Priestley space. By Lemma 4.3.17, we have that $\langle X, \tau, \leq, \mathcal{B}\rangle$, with $\mathcal{B} := X^*$, is a poset Priestley space. Therefore we obtain that $P_{\mathcal{B}} = \langle \mathcal{B}, \subseteq\rangle = \langle X^*, \subseteq\rangle$ is a bounded distributive meet-semilattice.

Let M_1 and M_2 be bounded distributive meet-semilattices and let $h\colon M_1 \to M_2$ be a map. Then, by Lemma 2.3.5, h is a meet-homomorphism preserving top if and only if h is an inf-homomorphism. Now let X and Y be generalized Priestley spaces and let $R \subseteq X \times Y$ be a binary relation. By Lemma 4.3.17, it is straightforward to show that R is a generalized Priestley morphism (see [**5**, Definition 6.2]) if and only if R is a poset Priestley morphism.

Consider the category of all bounded distributive meet-semilattices and all meet-homomorphisms preserving top element. We denote this category by $\mathbb{BDML}^{\top}$. Let $\mathbb{GPS}$ denote the category of all generalized Priestley spaces and all generalized Priestley morphisms (see [**5**, pp.107]). Hence, putting all the previous results together and by Theorem 4.2.21, we obtain the following theorem.

THEOREM 4.3.18. *The categories $\mathbb{BDML}^{\top}$ and $\mathbb{GPS}$ are dually equivalent.*

Therefore we have obtained the dual equivalence between the categories $\mathbb{BDML}^{\top}$ and $\mathbb{GPS}$ due to Bezhanishvili and Jansana [**5, 4**] by means of the dual equivalence between the categories $\mathbb{BMDOP}$ and $\mathbb{PPS}$ (Theorem 4.2.21). And since the classical Priestley duality is a particular case of the Priestley-style duality due to Bezhanishvili and Jansana [**4**, Section 10], it follows that the Priestley duality is a particular case of our Priestley-style duality.

Now we show a direct proof of the fact that the Priestley duality for bounded distributive lattices can be obtained of our Priestley-style duality.

Let L be a bounded distributive lattice. Then

$$\mathsf{Opt_s}(L) = \mathsf{Opt}(L) = \mathsf{Fi^{pr}}(L).$$

So for every $a \in L$, we have $\varphi_L(a) = \{F \in \mathsf{Fi^{pr}}(L) :\ a \in F\}$. Hence the dual poset Priestley space $L_* = \langle \mathsf{Fi^{pr}}(L), \tau_L, \subseteq, \mathcal{B}_L\rangle$ is such that $\langle \mathsf{Fi^{pr}}(L), \tau_L, \subseteq\rangle$ is the dual

Priestley space of L and $\mathcal{B}_L = \mathsf{CLUp}(L_*)$. Reciprocally, let $\langle X, \tau, \leq \rangle$ be a Priestley space. Let $\mathcal{B} := \mathsf{CLUp}(X)$. Then, it is straightforward check that $\langle X, \tau, \leq, \mathcal{B} \rangle$ is a poset Priestley space. Now we need the following lemma.

LEMMA 4.3.19. *Let L_1 and L_2 be bounded distributive lattices and let $h\colon L_1 \to L_2$ be a map. Then, h is a strong inf-homomorphism if and only if h is a bounded lattice homomorphism.*

PROOF. First assume that h is a strong inf-homomorphism. By Lemma 2.4.29, we have that h is an inf-sup-homomorphism and by Lemma 2.3.5, we obtain that h is a bounded lattice homomorphism. Now, conversely assume that h is a bounded lattice homomorphism. Let $A, B \subseteq_\omega L_1$ be such that $A^{\mathrm{u}} \subseteq B^{\mathrm{lu}}$. Without loss of generality we can suppose that $A \neq \emptyset$ and $B \neq \emptyset$. Because if $A = \emptyset$, then $A^{\mathrm{u}} = L_1 = {\uparrow}\bot_1$ and if $B = \emptyset$, then $B^{\mathrm{lu}} = \{\top_1\} = {\uparrow}\top_1$. Now, since $A^{\mathrm{u}} \subseteq B^{\mathrm{lu}}$ and since L_1 is a lattice, it follows that $\bigwedge B \leq \bigvee A$. As h is a bounded lattice homomorphism, we have $\bigwedge h[B] \leq \bigvee h[A]$. Then, $h[A]^{\mathrm{u}} \subseteq h[B]^{\mathrm{lu}}$. This completes the proof. $\square$

Let $\langle X, \tau_X, \leq_X \rangle$ and $\langle Y, \tau_Y, \leq_Y \rangle$ be Priestley spaces. Consider the poset Priestley spaces $\langle X, \tau_X, \leq_X, \mathcal{B}_X \rangle$ and $\langle Y, \tau_Y, \leq_Y, \mathcal{B}_Y \rangle$ where $\mathcal{B}_X = \mathsf{CLUp}(X)$ and $\mathcal{B}_Y = \mathsf{CLUp}(Y)$. Let $f\colon X \to Y$ be a map. Then, f is a Priestley morphism (continuous order-preserving map) if and only if f is a strong-continuous map (Definition 4.3.8). Let us denote by $\mathbb{PRI}$ the category of all Priestley spaces and all Priestley morphisms and the category of all bounded distributive lattices and all bounded lattice homomorphisms is denoted by $\mathbb{BDL}$. Hence, by Theorem 4.3.16 we obtain the following theorem. We leave the details to the reader.

THEOREM 4.3.20. *The categories $\mathbb{BDL}$ and $\mathbb{PRI}$ are dually equivalent.*

4.4. The duality running

In this last section of this chapter we use the Priestley-style duality for bounded mo-distributive posets obtained in Section 4.2 in first place to obtain the distributive meet-semilattice envelope (see Section 2.4) and make clear the relations between a bounded mo-distributive poset, its distributive meet-semilattice envelope and distributive lattice envelope, and their corresponding Priestley-style dual spaces. Secondly, for a bounded mo-distributive poset we characterize topologically several classes of Frink-filters. Finally, we use the Priestley-style duality for obtain the Frink completion (see Section 3.5) of a bounded mo-distributive poset.

4.4.1. A topological viewpoint of the distributive meet-semilattice envelope. The purpose of this subsection is to obtain the distributive meet-semilattice envelope of a bounded mo-distributive poset using the Priestley-style duality presented in the previous section. Moreover we establish the connection between the distributive meet-semilattice envelope and distributive lattice envelope of a bounded mo-distributive poset and their Priestley-style dual spaces.

Let $X = \langle X, \tau, \leq, \mathcal{B}\rangle$ be a poset Priestley space. By Lemma 4.1.17, recall that $\langle X, \tau, \leq, X_0^{\mathcal{B}}\rangle$ is a generalized Priestley space and by Lemma 4.1.16, we have that $\langle X^*, \cap, \emptyset, X\rangle$ is a bounded distributive meet-semilattice where $X^* = X_*^{\mathcal{B}} = \{U \in \mathsf{CLUp}(X) : \max(U^c) \subseteq X_0^{\mathcal{B}}\}$.

Lemma 4.4.1. *Let $\langle X, \tau, \leq, \mathcal{B}\rangle$ be a poset Priestley space. Then, $\langle X_*^{\mathcal{B}}, \cap, \emptyset, X\rangle$ is the distributive meet-semilattice envelope of the bounded mo-distributive poset $P_{\mathcal{B}} = \langle \mathcal{B}, \subseteq\rangle$.*

Proof. Let $X = \langle X, \tau, \leq, \mathcal{B}\rangle$ be a poset Priestley space. We need to prove Conditions (DE1) and (DE2) of Definition 2.4.1 (see on page 49) for $P_{\mathcal{B}}$ and $\langle X_*^{\mathcal{B}}, \cap, \emptyset, X\rangle$ instead of P and M. First, by Condition (P5) of Definition 4.1.11, we have that $P_{\mathcal{B}} \subseteq X_*^{\mathcal{B}}$ and then we consider the inclusion map $i\colon P_{\mathcal{B}} \to X_*^{\mathcal{B}}$, which is an order-embedding. By Condition (P5) again, we obtain that $P_{\mathcal{B}}$ is finitely meet-dense on $X_*^{\mathcal{B}}$. Then (DE1) holds. Notice that the top and bottom elements of the poset $P_{\mathcal{B}}$ are, respectively, the top and bottom elements of the meet-semilattice $X_*^{\mathcal{B}}$. To prove Condition (DE2), first we recall that the meet of two elements in $P_{\mathcal{B}}$ is the intersection, if it exists. So, we have that the map $i\colon P_{\mathcal{B}} \to X_*^{\mathcal{B}}$ is a $\wedge$-homomorphism preserving top and then, by Lemma 2.3.16, we obtain that i is an inf-homomorphism. Now, using Condition (P5) is straightforward to show that i is a sup-homomorphism. Then, since i is an order-embedding, we obtain that i is an inf-sup-embedding and hence Condition (DE2) holds. Therefore $\langle X_*^{\mathcal{B}}, \cap, \emptyset, X\rangle$ is the distributive meet-semilattice envelope of $P_{\mathcal{B}} = \langle \mathcal{B}, \subseteq\rangle$. $\square$

Let P be a bounded mo-distributive poset and $P_* = \langle \mathsf{Opt_s}(P), \tau_P, \subseteq, \mathcal{B}_P\rangle$ its dual poset Priestley space. Recall from Subsection 2.4.1 that $M(P)$ denotes the distributive meet-semilattice envelope of P. By the previous lemma, we have that $X_*^{\mathcal{B}_P} = \langle X_*^{\mathcal{B}_P}, \cap, \emptyset, \mathsf{Opt_s}(P)\rangle$ is the distributive meet-semilattice envelope of $P_{\mathcal{B}_P}$ and by Lemma 4.1.4, we have that $P \cong P_{\mathcal{B}_P}$. Then, by the uniqueness of the distributive meet-semmilattice envelope we have that

$$X_*^{\mathcal{B}_P} \cong M(P).$$

Hence, by the Priestley-style duality for distributive meet-semilattices presented in [**5**] (see also [**4**]), we obtain that $\langle \mathsf{Opt_s}(P), \tau_P, \subseteq, \mathsf{Fi}_{\mathsf{F}}^{\mathsf{pr}}(P)\rangle$ is the dual generalized Priestley space of $M(P)$. Moreover, by Lemma 4.1.1 we know that $\langle \mathsf{Opt_s}(P), \tau_P, \subseteq\rangle$ is the dual Priestley space of $D(P)$. The diagram of Figure 4.2 intend summarize the previous statements.

$$\begin{tikzcd} P \arrow[r] \arrow[d] & M(P) \arrow[r] \arrow[d, dashed] & D(P) \arrow[d, dashed] \\ P_* \arrow[r] & \langle \mathsf{Opt_s}(P), \tau_P, \subseteq, \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)\rangle \arrow[r] & \langle \mathsf{Opt_s}(P), \tau_P, \subseteq\rangle \end{tikzcd}$$

FIGURE 4.2. Relations between the distributive meet-semilattice and lattice envelope of a bounded mo-distributive poset and their Priestley-style dual spaces.

4.4.2. A topological characterization of Frink-filters. In this subsection we intend to obtain by means of poset Priestley spaces topological characterizations for several classes of Frink-filters of bounded mo-distributive posets. To this end, first we recall the topological characterizations of filters of a distributive lattice by means of its dual Priestley space.

For every ordered topological space X, let us denote by $\mathsf{CUp}(X)$ the collection of all closed up-sets of X. Now let L be a bounded distributive lattice and X its dual Priestley space. The maps $\Upsilon\colon \mathsf{Fi}(L) \to \mathsf{CUp}(X)$ and $\Xi\colon \mathsf{CUp}(X) \to \mathsf{Fi}(L)$ are defined as follows: for every $F \in \mathsf{Fi}(L)$ and $C \in \mathsf{CUp}(X)$,

$$\Upsilon(F) = \bigcap_{a\in F} \varphi(a) \quad \text{and} \quad \Xi(C) = \{a \in L :\ C \subseteq \varphi(a)\}$$

THEOREM 4.4.2. *Let L be a bounded distributive lattice and X its dual Priestley space. Then, the maps Υ and Ξ are dual lattice isomorphisms between the lattices $\mathsf{Fi}(L)$ and $\mathsf{CUp}(X)$, one inverse of the other.*

In [**5**] Bezhanishvili and Jansana extended the definitions of the previous two maps to the distributive meet-semilattice setting to obtain a topological characterization of the filters of a bounded distributive meet-semilattice and thus they obtained the following theorem:

THEOREM 4.4.3. ([**5**, Theorem 8.5]). *Let L be a bounded distributive meet-semilattice and let X be its generalized Priestley space. Then the maps $\Upsilon\colon \mathsf{Fi}(L) \to \{C \in \mathsf{CUp}(X) :\ X \setminus C = {\downarrow}(X_0 \setminus C)\}$ and $\Xi\colon \{C \in \mathsf{CUp}(X) :\ X \setminus C = {\downarrow}(X_0 \setminus C)\} \to \mathsf{Fi}(L)$ defined as:*

$$\Upsilon(F) = \bigcap_{a\in F} \varphi(a) \quad \textit{and} \quad \Xi(C) = \{a \in L :\ C \subseteq \varphi(a)\}$$

for every $F \in \mathsf{Fi}(L)$ and $C \in \{C \in \mathsf{CUp}(X) :\ X \setminus C = {\downarrow}(X_0 \setminus C)\}$, set an dual lattice isomorphisms.

Thus, motivated by Theorem 4.4.2, we extend the definitions of the maps Υ and Ξ from the distributive lattice setting to the mo-distributive poset setting.

Let P be a fixed but arbitrary bounded mo-distributive poset and P_* its dual poset Priestley space. We define $\Upsilon\colon \mathsf{Fi}_{\mathsf{F}}(P) \to \mathsf{CUp}(P_*)$ and $\Xi\colon \mathsf{CUp}(P_*) \to \mathsf{Fi}_{\mathsf{F}}(P)$ as follows: for every $F \in \mathsf{Fi}_{\mathsf{F}}(P)$ and $C \in \mathsf{CUp}(P_*)$,

$$\Upsilon(F) = \bigcap_{a\in F} \varphi_P(a) \quad \text{and} \quad \Xi(C) = \{a \in P:\ C \subseteq \varphi_P(a)\}. \tag{4.6}$$

It is clear that Υ is a well-defined map and by Lemma 4.1.6, it is straightforward to show directly that Ξ is well-defined.

Recall that the distributive lattice envelope $D(P)$ of P is the distributive envelope ([**5, 4**]) of $M(P)$. By Theorem 2.4.16, we have that $\mathsf{Fi}_{\mathsf{F}}(P) \cong \mathsf{Fi}(M(P))$. If $\mathsf{Fi}_{\mathsf{F}}(P) \cong \mathsf{Fi}(D(P))$, then $\mathsf{Fi}(M(P)) \cong \mathsf{Fi}(D(P))$, which is not true in general. Hence, the lattices $\mathsf{Fi}_{\mathsf{F}}(P)$ and $\mathsf{Fi}(D(P))$ are not necessarily isomorphic. Then, by Theorem 4.4.2 (also see Figure 4.2) we obtain that the lattices $\mathsf{Fi}_{\mathsf{F}}(P)$ and $\mathsf{CUp}(P_*)$ are not necessarily isomorphic.

Lemma 4.4.4. *For every $F \in \mathsf{Fi}_{\mathsf{F}}(P)$, we have $(\Xi \circ \Upsilon)(F) = F$.*

Proof. By (4.6), we obtain that $(\Xi \circ \Upsilon)(F) = \{a \in P:\ \bigcap_{b\in F} \varphi_P(b) \subseteq \varphi_P(a)\}$. So, it is clear that $F \subseteq (\Xi \circ \Upsilon)(F)$. Now let $a \in (\Xi \circ \Upsilon)(F)$. Suppose towards a contradiction that $a \notin F$. Then, by Theorem 2.4.27, there exists $G \in \mathsf{Opt}_{\mathsf{s}}(P) = P_*$ such that $F \subseteq G$ and $a \notin G$. We thus have $G \in \bigcap_{b\in F} \varphi_P(b)$ and $G \notin \varphi_P(a)$, which is a contradiction. Hence $(\Xi \circ \Upsilon)(F) \subseteq F$. □

An immediate consequence of the previous lemma is that Ξ is onto. So, to characterize the Frink-filters of P, we need find an adequate subfamily of $\mathsf{CUp}(P_*)$. To this, we consider Theorem 4.4.3 and we obtain a similar result in the setting of mo-distributive posets. Here, for a poset P, we denote $P_0 := \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$ for short.

Theorem 4.4.5. *Let P be a bounded mo-distributive poset and P_* its dual poset Priestley space. Then, the maps $\Upsilon\colon \mathsf{Fi}_{\mathsf{F}}(P) \to \{C \in \mathsf{CUp}(P_*):\ P_* \setminus C = {\downarrow}(P_0 \setminus C)\}$ and $\Xi\colon \{C \in \mathsf{CUp}(P_*):\ P_* \setminus C = {\downarrow}(P_0 \setminus C)\} \to \mathsf{Fi}_{\mathsf{F}}(P)$ defined as in* (4.6) *are dual lattice isomorphisms, one inverse of the other.*

Proof. As we know, Ξ is well-defined. To show that Υ is well-defined, let $F \in \mathsf{Fi}_{\mathsf{F}}(P)$. We know that $\Upsilon(F) \in \mathsf{CUp}(P_*)$ and moreover, since $\Upsilon(F)$ is an up-set of P_*, it is clear that ${\downarrow}(P_0 \setminus \Upsilon(F)) \subseteq P_* \setminus \Upsilon(F)$. Now let $G \in P_* \setminus \Upsilon(F)$. So, there is $a \in F$ such that $a \notin G$. Then, by Corollary 2.2.18, there exists $H \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P) = P_0$ such that $G \subseteq H$ and $a \notin H$. That is, $H \in P_0 \setminus \Upsilon(F)$ and $G \subseteq H$. Thus, $G \in {\downarrow}(P_0 \setminus \Upsilon(F))$. Hence $P_* \setminus \Upsilon(F) = {\downarrow}(P_0 \setminus \Upsilon(F))$ and therefore Υ is well-defined. It is clear that Υ and Ξ are order-reversing and by Lemma 4.4.4, we have $\Xi \circ \Upsilon = \mathrm{id}_{\mathsf{Fi}_{\mathsf{F}}(P)}$. So, it only remains to prove that for every $C \in \mathsf{CUp}(P_*)$

satisfying $P_* \setminus C = {\downarrow}(P_0 \setminus C)$, $(\Upsilon \circ \Xi)(C) = C$ holds. Let $C \in \mathsf{CUp}(P_*)$ such that $P_* \setminus C = {\downarrow}(P_0 \setminus C)$. By definition, $(\Upsilon \circ \Xi)(C) = \bigcap\{\varphi_P(a) : a \in P \text{ and } C \subseteq \varphi_P(a)\}$ and thus it is clear that $C \subseteq (\Upsilon \circ \Xi)(C)$. Let $F \in (\Upsilon \circ \Xi)(C)$. So,

$$F \in \varphi_P(a) \quad \text{for all } a \in P \text{ such that } C \subseteq \varphi_P(a). \tag{4.7}$$

Suppose towards a contradiction that $F \notin C$. So $F \in P_* \setminus C = {\downarrow}(P_0 \setminus C)$, which implies that there is $G \in P_0$ such that $F \subseteq G$ and $G \notin C$. Since C is a closed up-set of the Priestley space P_*, it follows that C is the intersection of all clopen up-sets of P_* containing C. Then, there is a clopen up-set $\mathcal{U}$ of P_* such that $C \subseteq \mathcal{U}$ and $G \notin \mathcal{U}$. Using Lemma 4.1.5, we have $\mathcal{U} = \bigcup_{i=1}^{n} \bigcap \varphi_P[A_i]$ for some non-empty $A_1, \ldots, A_n \subseteq_\omega P$. Then, $G \notin \varphi_P[A_i]$ for all $i \in \{1, \ldots, n\}$. So, for every $i \in \{1, \ldots, n\}$ there exists $a_i \in A_i$ such that $a_i \notin G$. We thus obtain that $\{a_1, \ldots, a_n\} \subseteq G^c$. As G is a prime Frink-filter, G^c is an order-ideal. Then there is $a \in G^c$ such that $a_1, \ldots, a_n \leq a$. Hence $G \notin \varphi_P(a)$. Now we show that $C \subseteq \varphi_P(a)$. Let $F' \in C$. So $F' \in \mathcal{U}$ and this implies that there exists $i_0 \in \{1, \ldots, n\}$ such that $A_{i_0} \subseteq F'$. Then, $a_{i_0} \in F'$ and thus $a \in F'$. So $F' \in \varphi_P(a)$. Hence, we have proved that $C \subseteq \varphi_P(a)$. Then, since $F \subseteq G$ and by (4.7), we obtain that $G \in \varphi_P(a)$, a contradiction. Then, $F \in C$ and hence $(\Upsilon \circ \Xi)(C) = C$. This completes the proof. □

Now we want to obtain a characterization of the prime and s-optimal Frink-filters. To this, we note that for every $U \in \mathsf{Opt_s}(P)$, the closed up-set $C = {\uparrow}U$ of P_* holds $P_* \setminus C = {\downarrow}(P_0 \setminus C)$. So, we obtain that $\{C \in \mathsf{CUp}(P_*) : C = {\uparrow}F \text{ for some } F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)\} \subseteq \{C \in \mathsf{CUp}(P_*) : C = {\uparrow}U \text{ for some } U \in \mathsf{Opt_s}(P)\} \subseteq \{C \in \mathsf{CUp}(P_*) : P_* \setminus C = {\downarrow}(P_0 \setminus C)\}$. It is clear that for every $U \in \mathsf{Opt_s}(P)$, we have that $\Upsilon(U) = {\uparrow}U$ and it is also straightforward to prove, using Theorem 2.4.27, that for every $U \in \mathsf{Opt_s}(P)$, $\Xi({\uparrow}U) = \{a \in P : {\uparrow}U \subseteq \varphi_P(a)\} = U$. Hence, we obtain the following theorem.

Theorem 4.4.6. *Let P be a bounded mo-distributive poset and P_* its dual poset Priestley space.*

(1) *The maps $\Upsilon\colon \mathsf{Opt_s}(P) \to \{C \in \mathsf{CUp}(P_*) : C = {\uparrow}U \text{ for some } U \in \mathsf{Opt_s}(P)\}$ and $\Xi\colon \{C \in \mathsf{CUp}(P_*) : C = {\uparrow}U \text{ for some } U \in \mathsf{Opt_s}(P)\} \to \mathsf{Opt_s}(P)$ establish dual order-isomorphisms, one inverse of the other.*
(2) *The maps $\Upsilon\colon \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P) \to \{C \in \mathsf{CUp}(P_*) : C = {\uparrow}F \text{ for some } F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)\}$ and $\Xi\colon \{C \in \mathsf{CUp}(P_*) : C = {\uparrow}F \text{ for some } F \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)\} \to \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$ establish dual order-isomorphisms, one inverse of the other.*

4.4.3. The Frink completion by means of the Priestley-style duality. In this subsection we derive the Frink completion of a bounded mo-distributive poset by means of the Priestley-style duality developed in Section 4.2. We refer the reader to Section 3.5 for the definition of the Frink completion and its properties.

Let $X = \langle X, \tau, \leq, \mathcal{B}\rangle$ be a poset Priestley space. By (P4.3) of Definition 4.1.11, we know that every $U \in \mathcal{B}$ is an up-set of X. For our purpose we consider the sub-poset $\langle X_0^{\mathcal{B}}, \leq\rangle$ of $\langle X, \leq\rangle$. So, for every $U \in \mathcal{B}$ we have that $U \cap X_0^{\mathcal{B}}$ is an up-set of $X_0^{\mathcal{B}}$. Let us denote by $\mathsf{Up}(X_0^{\mathcal{B}})$ the family of all up-sets of $X_0^{\mathcal{B}}$. Hence, we know that $\langle \mathsf{Up}(X_0^{\mathcal{B}}), \cap, \cup\rangle$ is a completely distributive algebraic lattice and moreover $\{U \cap X_0^{\mathcal{B}} : U \in \mathcal{B}\} \subseteq \mathsf{Up}(X_0^{\mathcal{B}})$. Recall that $P_{\mathcal{B}} = \langle \mathcal{B}, \subseteq\rangle$. We define the map $e\colon P_{\mathcal{B}} \to \mathsf{Up}(X_0^{\mathcal{B}})$ as follows: for every $U \in P_{\mathcal{B}}$,

$$e(U) = U \cap X_0^{\mathcal{B}}.$$

It is clear that e is order-preserving. To show that e is an order-embedding we note the following fact. Let $U \in P_{\mathcal{B}}$. So U is a clopen subset of X. Since U is an open subset of X and $X_0^{\mathcal{B}}$ is a dense subset of X, it follows that $U \cap X_0^{\mathcal{B}}$ is dense in U. Then, since U is a closed subset of X, we obtain that $\mathrm{cl}(U \cap X_0^{\mathcal{B}}) = U$. Now, let $U_1, U_2 \in P_{\mathcal{B}}$. Then,

$$e(U_1) \subseteq e(U_2) \iff U_1 \cap X_0^{\mathcal{B}} \subseteq U_2 \cap X_0^{\mathcal{B}} \iff U_1 \subseteq U_2.$$

Hence e is an order-embedding. Therefore, we have proved that $\langle \mathsf{Up}(X_0^{\mathcal{B}}), e\rangle$ is a completion of $P_{\mathcal{B}}$.

Lemma 4.4.7. *Let $X = \langle X, \tau, \leq, \mathcal{B}\rangle$ be a poset Priestley space. Then, $\langle \mathsf{Up}(X_0^{\mathcal{B}}), e\rangle$ is the Frink completion of $P_{\mathcal{B}}$.*

Proof. We need to prove that the completion $\langle \mathsf{Up}(X_0^{\mathcal{B}}), e\rangle$ of $P_{\mathcal{B}}$ holds conditions (C) and (D) (see Definition 3.5.2 on page 113). To show Condition (C), let $\mathcal{F} \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathcal{B}})$ and $\mathcal{I} \in \mathsf{Id}_{\mathsf{or}}(P_{\mathcal{B}})$ and assume that $\bigcap e[\mathcal{F}] \subseteq \bigcup e[\mathcal{I}]$. We need to prove that $\mathcal{F} \cap \mathcal{I} \neq \emptyset$. Suppose towards a contradiction that $\mathcal{F} \cap \mathcal{I} = \emptyset$. Since $P_{\mathcal{B}}$ is mo-distributive, it follows that there is $\mathcal{H} \in \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathcal{B}})$ such that $\mathcal{F} \subseteq \mathcal{H}$ and $\mathcal{H} \cap \mathcal{I} = \emptyset$. By Lemma 4.2.6, we have that $\varepsilon^{-1}(\mathcal{H}) \in X_0^{\mathcal{B}}$ and, by (P4.3) of Definition 4.1.11 and by Theorem 4.2.5 , it is not hard to show that $\bigcap \mathcal{H} = {\uparrow}\varepsilon^{-1}(\mathcal{H})$ (the principal up-set is considered in X). Then $\varepsilon^{-1}(\mathcal{H}) \in \bigcap e[\mathcal{H}]$. Now, since $\mathcal{F} \subseteq \mathcal{H}$, it follows that $\bigcap e[\mathcal{H}] \subseteq \bigcap e[\mathcal{F}]$ and thus $\bigcap e[\mathcal{H}] \subseteq \bigcup e[\mathcal{I}]$. Then $\varepsilon^{-1}(\mathcal{H}) \in \bigcup e[\mathcal{I}]$ and hence there exists $U \in \mathcal{I}$ such that $\varepsilon^{-1}(\mathcal{H}) \in U$. Thus $U \in \varepsilon(\varepsilon^{-1}(\mathcal{H})) = \mathcal{H}$. Then $\mathcal{H} \cap \mathcal{I} \neq \emptyset$, which is a contradiction. Therefore, we conclude that $\mathcal{F} \cap \mathcal{I} \neq \emptyset$. Hence $\langle \mathsf{Up}(X_0^{\mathcal{B}}), e\rangle$ satisfies Condition (C).

To prove Condition (D), let $A \in \mathsf{Up}(X_0^{\mathcal{B}})$. First we show that

$$A = \bigcup\{\bigcap e[\mathcal{F}] : \mathcal{F} \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathcal{B}}) \text{ and } \bigcap e[\mathcal{F}] \subseteq A\}. \tag{4.8}$$

Clearly $\bigcup\{\bigcap e[\mathcal{F}] : \mathcal{F} \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathcal{B}}) \text{ and } \bigcap e[\mathcal{F}] \subseteq A\} \subseteq A$. Now let $x_0 \in A$. Since $x_0 \in X_0^{\mathcal{B}}$, by Lemma 4.2.3 it follows that $\mathcal{F} := \varepsilon(x_0) = \{U \in P_{\mathcal{B}} : x_0 \in U\} \in$

$\mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P_{\mathcal{B}})$. Notice that $x_0 \in \bigcap e[\mathcal{F}]$. Now we show that $\bigcap e[\mathcal{F}] \subseteq A$. Let $y_0 \in \bigcap e[\mathcal{F}]$. So $y_0 \in U \cap X_0^{\mathcal{B}}$ for all $U \in \mathcal{F}$ and then, by (P4.3), we obtain that $x_0 \leq y_0$. Since A is an up-set of $X_0^{\mathcal{B}}$, it follows that $y_0 \in A$. Hence $\bigcap e[\mathcal{F}] \subseteq A$. Then we have proved that $x_0 \in \bigcup\{\bigcap e[\mathcal{F}] : \mathcal{F} \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathcal{B}})$ and $\bigcap e[\mathcal{F}] \subseteq A\}$ and thus $A \subseteq \bigcup\{\bigcap e[\mathcal{F}] : \mathcal{F} \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathcal{B}})$ and $\bigcap e[\mathcal{F}] \subseteq A\}$. Hence (4.8) holds. Now, for the second part of Condition (D), we need to prove that

$$A = \bigcap\{\bigcup e[\mathcal{I}] : \mathcal{I} \in \mathsf{Id}_{\mathsf{or}}(P_{\mathcal{B}}) \text{ and } A \subseteq \bigcup e[\mathcal{I}]\}. \tag{4.9}$$

First, it is clear that $A \subseteq \bigcap\{\bigcup e[\mathcal{I}] : \mathcal{I} \in \mathsf{Id}_{\mathsf{or}}(P_{\mathcal{B}})$ and $A \subseteq \bigcup e[\mathcal{I}]\}$. Now let $x_0 \in \bigcap\{\bigcup e[\mathcal{I}] : \mathcal{I} \in \mathsf{Id}_{\mathsf{or}}(P_{\mathcal{B}})$ and $A \subseteq \bigcup e[\mathcal{I}]\}$. Since $x_0 \in X_0^{\mathcal{B}}$, it follows that $\mathcal{I} := I_{x_0}^{\mathcal{B}} = \{U \in P_{\mathcal{B}} : x_0 \notin U\}$ is an up-directed subset of $P_{\mathcal{B}}$ and then $\mathcal{I} = I_{x_0}^{\mathcal{B}} \in \mathsf{Id}_{\mathsf{or}}(P_{\mathcal{B}})$. Suppose towards a contradiction that $x_0 \notin A$. We claim that $A \subseteq \bigcup e[\mathcal{I}]$. In fact, if $y_0 \in A$, then $y_0 \nleq x_0$, because A is an up-set of $X_0^{\mathcal{B}}$ and $x_0 \notin A$. Thus, by (P4.3), we have that there is $U \in P_{\mathcal{B}}$ such that $y_0 \in U$ and $x_0 \notin U$. Then $y_0 \in U \cap X_0^{\mathcal{B}} = e(U)$ and $U \in I_{x_0}^{\mathcal{B}} = \mathcal{I}$, which implies that $y_0 \in \bigcup e[\mathcal{I}]$. Hence, since $A \subseteq \bigcup e[\mathcal{I}]$, we obtain that $x_0 \in \bigcup e[\mathcal{I}]$, a contradiction. Then $x_0 \in A$ and hence $\bigcap\{\bigcup e[\mathcal{I}] : \mathcal{I} \in \mathsf{Id}_{\mathsf{or}}(P_{\mathcal{B}})$ and $A \subseteq \bigcup e[\mathcal{I}]\} \subseteq A$. Thus (4.9) holds. Hence the completion $\langle \mathsf{Up}(X_0^{\mathcal{B}}), e\rangle$ of $P_{\mathcal{B}}$ satisfies Condition (D). This completes the proof. $\square$

Therefore, the following corollary is an immediate consequence from the previous lemma and Theorem 4.2.21.

COROLLARY 4.4.8. *Let P be a bounded mo-distributive poset and P_* its dual poset Priestley space. Then the lattice $\langle \mathsf{Up}(P_0), \cap, \cup\rangle$, where $P_0 = X_0^{\mathcal{B}_P} = \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P)$, is the Frink completion of P.*

4.5. The strong Frink completion

This section is devoted to study another Δ_1-completion of a bounded mo-distributive poset. To this end, as we did in Section 3.5 for the Frink completion, we apply the theory developed in [**27**].

Recall that $\mathsf{Id}_{\mathsf{sF}}(P)$ denotes the collection of all strong Frink-ideals of a poset P. Let P be a poset and let $\langle L, e\rangle$ be a completion of P. Recall that an element $x \in L$ is called *Frink-closed* if there exists $F \in \mathsf{Fi}_{\mathsf{F}}(P)$ such that $x = \bigwedge_L e[F]$ and the collection of all Frink-closed elements of L is denoted by $\mathsf{K}_{\mathsf{F}}(L)$. An element $y \in L$ is called *strong Frink-open* if there exists $I \in \mathsf{Id}_{\mathsf{sF}}(P)$ such that $y = \bigvee_L e[I]$. Let us denote by $\mathsf{O}_{\mathsf{sF}}(L)$ the set of all strong Frink-open elements of L. In the sequel, we omit the subscript L when denoting joins and meets in the lattice L and only use it when we need to indicate which lattice is under consideration.

THEOREM 4.5.1. ([**27**, Theorem 5.10]). *Let P be a poset. Then, there exists a unique up to isomorphism completion $\langle L, e\rangle$ of P such that the following conditions are satisfied:*

(sC) *for every $F \in \mathsf{Fi_F}(P)$ and $I \in \mathsf{Id_{sF}}(P)$ if $\bigwedge e[F] \leq \bigvee e[I]$, then $F \cap I \neq \emptyset$.*
(sD) *each element of L is both the join of all the Frink-closed elements below it and the meet of all the strong Frink-open elements above it. That is, for all $a \in L$, we have*

$$a = \bigvee\{x \in \mathsf{K_F}(L) : \ x \leq a\} \quad \textit{and} \quad a = \bigwedge\{y \in \mathsf{O_{sF}}(L) : \ a \leq y\}.$$

DEFINITION 4.5.2. Let P be a poset. The *strong Frink completion* of P is the unique up to isomorphism completion of P such that Conditions (sC) and (sD) hold.

For every poset P, we denote the strong Frink completion of P by $\langle P^{\mathrm{sF}}, e\rangle$ or simply by P^{sF}.

It is straightforward to prove directly that $\mathsf{Id_{sF}}(P)$ is closed under unions of up-directed families. Then, analogously as we did in Section 3.5, we can introduce some results that are a consequence of the fact that $\mathsf{Fi_F}(P)$ is an algebraic closure system and $\mathsf{Id_{sF}}(P)$ is closed under unions of up-directed families; they are obtained by a direct application of the results in [**27**] and thus we omit their proofs leaving the details to the reader.

LEMMA 4.5.3. ([**27**, Proposition 6.4]). *Let P be a poset and P^{sF} its strong Frink completion. Then:*

(1) $\mathcal{J}^\infty(P^{\mathrm{sF}}) \subseteq \mathsf{K_F}(P^{\mathrm{sF}})$*;*
(2) $\mathcal{M}^\infty(P^{\mathrm{sF}}) \subseteq \mathsf{O_{sF}}(P^{\mathrm{sF}})$*.*

LEMMA 4.5.4. ([**27**, Proposition 6.5]). *Let P be a poset and P^{sF} its strong Frink completion. Then, $\mathcal{J}^\infty(P^{\mathrm{sF}})$ is join-dense in P^{sF} and $\mathcal{M}^\infty(P^{\mathrm{sF}})$ is meet-dense in P^{sF}.*

Let P be a poset and let $F \in \mathsf{Fi_F}(P)$ and $I \in \mathsf{Id_{sF}}(P)$. We say that $\langle F, I\rangle$ is a *strong maximal pair* of P provided F is maximal in the set $\{G \in \mathsf{Fi_F}(P) : \ G \cap I = \emptyset\}$ and I is maximal in the set $\{J \in \mathsf{Id_{sF}}(P) : \ J \cap F = \emptyset\}$. Given $F \in \mathsf{Fi_F}(P)$, we will say that F is in a maximal pair if there is a strong Frink-ideal I such that $\langle F, I\rangle$ is a maximal pair.

LEMMA 4.5.5. *Let P be a mo-distributive poset and let $F \in \mathsf{Fi_F}(P)$. Then, F is in a maximal pair if and only if F is a s-optimal Frink-filter.*

PROOF. The proof is similar to that of Lemma 3.5.7. □

Corollary 4.5.6. *Let P be a mo-distributive poset and let $F \in \mathsf{Fi}_{\mathsf{F}}(P)$ and $I \in \mathsf{Id}_{\mathsf{sF}}(P)$. Then, $\langle F, I\rangle$ is a maximal pair of P if and only if $I = F^c$.*

Proof. The proof is similar to that of Lemma 3.5.8. $\square$

Lemma 4.5.7. ([**27**, Propositions 5.4 and 6.9]). *Let P be a mo-distributive poset and $\langle P^{\mathrm{sF}}, e\rangle$ its strong Frink completion. Let $\Phi_s \colon \mathsf{Fi}_{\mathsf{F}}(P) \to \mathsf{K}_{\mathsf{F}}(P^{\mathrm{sF}})$ and $\Psi_s \colon \mathsf{Id}_{\mathsf{sF}}(P) \to \mathsf{O}_{\mathsf{sF}}(P^{\mathrm{sF}})$ be the maps defined by*

$$\Phi_s(F) = \bigwedge e[F] \quad \text{and} \quad \Psi_s(I) = \bigvee e[I]$$

for every $F \in \mathsf{Fi}_{\mathsf{F}}(P)$ and every $I \in \mathsf{Id}_{\mathsf{sF}}(P)$, respectively. Then, Φ_s is a dual order-isomorphism and Ψ_s is an order-isomorphism. Moreover, Φ_s restricts to a dual order-isomorphism from $\mathsf{Opt}_{\mathsf{s}}(P)$ onto $\mathcal{J}^{\infty}(P^{\mathrm{sF}})$ and Ψ_s restricts to an order-isomorphism from $\{F^c : F \in \mathsf{Opt}_{\mathsf{s}}(P)\}$ onto $\mathcal{M}^{\infty}(P^{\mathrm{sF}})$.

Lemma 4.5.8. ([**27**, Proposition 6.10]). *Let P be a poset and P^{sF} its strong Frink completion. Then, the finite meets and joins existing in P are preserved in P^{sF}.*

Lemma 4.5.9. *Let P_1 and P_2 be bounded posets. Then*

$$\mathsf{Id}_{\mathsf{sF}}(P_1 \times P_2) = \mathsf{Id}_{\mathsf{sF}}(P_1) \times \mathsf{Id}_{\mathsf{sF}}(P_2).$$

Proof. Let $I_1 \in \mathsf{Id}_{\mathsf{sF}}(P_1)$ and $I_2 \in \mathsf{Id}_{\mathsf{sF}}(P_2)$. We prove that $I_1 \times I_2$ is a strong Frink-ideal of $P_1 \times P_2$. Since I_1 and I_2 are, respectively, non-empty down-sets of P_1 and P_2, it follows that $I_1 \times I_2$ is a non-empty down-set of $P_1 \times P_2$. Now let $X \subseteq_\omega I_1 \times I_2$ and $Y \subseteq_\omega P_1 \times P_2$ be non-empty. Assume that $X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}}$. We consider the sets $X_i := \pi_i[X]$ with $i = 1, 2$, where π_1 and π_2 are the corresponding projections. Then, for every $i = 1, 2$, we have that $X_i \subseteq_\omega I_i$, $Y_i \subseteq_\omega P_i$ and $X_i^{\mathrm{u}} \subseteq Y_i^{\mathrm{lu}}$. Thus $Y_i^{\mathrm{lu}} \cap I_i \neq \emptyset$ for all $i = 1, 2$. Let $a \in Y_1^{\mathrm{lu}} \cap I_1$ and $b \in Y_2^{\mathrm{lu}} \cap I_2$. Hence $(a, b) \in (I_1 \times I_2) \cap Y^{\mathrm{lu}}$ and thus we obtain that $(I_1 \times I_2) \cap Y^{\mathrm{lu}} \neq \emptyset$. Then, $I_1 \times I_2 \in \mathsf{Id}_{\mathsf{sF}}(P_1 \times P_2)$ and therefore $\mathsf{Id}_{\mathsf{sF}}(P_1) \times \mathsf{Id}_{\mathsf{sF}}(P_2) \subseteq \mathsf{Id}_{\mathsf{sF}}(P_1 \times P_2)$. Now let $I \in \mathsf{Id}_{\mathsf{sF}}(P_1 \times P_2)$. Consider the sets

$$I_1 := \{a \in P_1 : (a, \bot_2) \in I\} \quad \text{and} \quad I_2 := \{b \in P_2 : (\bot_1, b) \in I\}.$$

Since I is a down-set and it is closed under existing finite joins, it follows that $I = I_1 \times I_2$. We prove that I_1 and I_2 are strong Frink-ideals of P_1 and P_2, respectively. Since I is a non-empty down-set of $P_1 \times P_2$, it follows easily that I_1 and I_2 are non-empty down-sets of P_1 and P_2, respectively. Now, let $X_1 \subseteq_\omega I_1$ and $Y_1 \subseteq_\omega P_1$ be non-empty and such that $X_1^{\mathrm{u}} \subseteq Y_1^{\mathrm{lu}}$. We consider the sets $X := \{(x, \bot_2) : x \in X_1\}$ and $Y := \{(y, \bot_2) : y \in Y_1\}$. It is clear that $X \subseteq_\omega I$. Let us show that $X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}}$. Let $(a, b) \in X^{\mathrm{u}}$. So $x \leq a$ for all $x \in X_1$ and then $a \in X_1^{\mathrm{u}}$. If $(a', b') \in Y^{\mathrm{l}}$, then $a' \leq y$ for all $y \in Y_1$ and $b' = \bot_2$. So $a' \in Y_1^{\mathrm{l}}$ and then $a' \leq a$. As $b' = \bot_2$, it follows that $(a', b') \leq (a, b)$ and hence $(a, b) \in Y^{\mathrm{lu}}$. Now, since $X^{\mathrm{u}} \subseteq Y^{\mathrm{lu}}$ and I is a strong Frink-ideal of $P_1 \times P_2$, we obtain that $Y^{\mathrm{lu}} \cap I \neq \emptyset$. Let $(a, b) \in Y^{\mathrm{lu}} \cap I$. As I is a down-set, $(a, \bot_2) \in I$ and moreover $a \in Y_1^{\mathrm{lu}}$. Then $a \in I_1$ and thus we obtain that $a \in Y_1^{\mathrm{lu}} \cap I_1$. Hence $Y_1^{\mathrm{lu}} \cap I_1 \neq \emptyset$. Therefore I_1 is a strong Frink-ideal of P_1. With a similar argument we can prove that I_2 is a strong Frink-ideal of P_2. Hence, $I = I_1 \times I_2 \in \mathsf{Id}_{\mathsf{sF}}(P_1) \times \mathsf{Id}_{\mathsf{sF}}(P_2)$. Then $\mathsf{Id}_{\mathsf{sF}}(P_1 \times P_2) \subseteq \mathsf{Id}_{\mathsf{sF}}(P_1) \times \mathsf{Id}_{\mathsf{sF}}(P_2)$. This completes the proof. □

Lemma 4.5.10. *Let P_1 and P_2 be bounded posets. Then, $(P_1 \times P_2)^{\mathrm{sF}} = P_1^{\mathrm{sF}} \times P_2^{\mathrm{sF}}$.*

Proof. It is a consequence from Lemmas 4.5.9 and 3.5.11 and by [**27**, Proposition 6.12]. □

Now we use the Priestley-style duality for bounded mo-distributive posets to prove the existence of the strong Frink completion of a bounded mo-distributive poset. This allows us moreover to show, similarly as in the case of the Frink completion, that the strong Frink completion is a completely distributive algebraic lattice.

Let $X = \langle X, \tau, \leq, \mathcal{B}\rangle$ be a poset Priestley space and $P_{\mathcal{B}} = \langle \mathcal{B}, \subseteq\rangle$ its dual bounded mo-distributive poset. Let us denote by $\mathsf{Up}(X)$ the collection of all up-sets of the poset $\langle X, \leq\rangle$. By Condition (P4.3) we have that $\mathcal{B} \subseteq \mathsf{Up}(X)$ and hence $P_{\mathcal{B}}$ is a subposet of the lattice $\langle \mathsf{Up}(X), \cap, \cup\rangle$. That is, $\langle \mathsf{Up}(X), i\rangle$, with $i\colon P_{\mathcal{B}} \to \mathsf{Up}(X)$ the inclusion map, is a completion of $P_{\mathcal{B}}$.

Theorem 4.5.11. *Let $X = \langle X, \tau, \leq, \mathcal{B}\rangle$ be a poset Priestley space and $P_{\mathcal{B}}$ its dual bounded mo-distributive poset. Then, $\langle \mathsf{Up}(X), \cap, \cup\rangle$ is the strong Frink completion of $P_{\mathcal{B}}$.*

Proof. By Theorem 4.5.1, we need to prove that Conditions (sC) and (sD) hold. To show (sD), let $A \in \mathsf{Up}(X)$. We need to prove that

(i) $A = \bigcup\{\bigcap \mathcal{F} : \mathcal{F} \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathcal{B}}) \text{ and } \bigcap \mathcal{F} \subseteq A\}$ and
(ii) $A = \bigcap\{\bigcup \mathcal{I} : \mathcal{I} \in \mathsf{Id}_{\mathsf{sF}}(P_{\mathcal{B}}) \text{ and } A \subseteq \bigcup \mathcal{I}\}$.

It is clear that $\bigcup\{\bigcap \mathcal{F} : \mathcal{F} \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathcal{B}}) \text{ and } \bigcap \mathcal{F} \subseteq A\} \subseteq A$ and $A \subseteq \bigcap\{\bigcup \mathcal{I} : \mathcal{I} \in \mathsf{Id}_{\mathsf{sF}}(P_{\mathcal{B}}) \text{ and } A \subseteq \bigcup \mathcal{I}\}$. Now let $x \in A$. By Lemma 4.2.2, we have that $\varepsilon(x) = \{U \in P_{\mathcal{B}} : x \in U\}$ is an s-optimal Frink-filter of $P_{\mathcal{B}}$. Thus $\mathcal{F} := \varepsilon(x) \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathcal{B}})$. It is clear that $x \in \bigcap \mathcal{F}$. We show that $\bigcap \mathcal{F} \subseteq A$. Let $y \in \bigcap \mathcal{F}$. Using Condition (P4.4) in Definition 4.1.11, we obtain that $x \leq y$. Since A is an up-set of X, we have that $y \in A$. Then $\bigcap \mathcal{F} \subseteq A$ and hence $x \in \bigcup\{\bigcap \mathcal{F} : \mathcal{F} \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathcal{B}}) \text{ and } \bigcap \mathcal{F} \subseteq A\}$. Therefore (i) holds. Now let $x \in \bigcup \mathcal{I}$ for all $\mathcal{I} \in \mathsf{Id}_{\mathsf{sF}}(P_{\mathcal{B}})$ such that $A \subseteq \bigcup \mathcal{I}$. Since $\varepsilon(x) \in \mathsf{Opt}_{\mathsf{s}}(P_{\mathcal{B}})$, we have that $\mathcal{I} := \varepsilon(x)^c \in \mathsf{Id}_{\mathsf{sF}}(P_{\mathcal{B}})$. Suppose towards a contradiction that $x \notin A$. Let us show that $A \subseteq \bigcup \mathcal{I}$. Let $y \in A$. Since A is an up-set and since $x \notin A$, it follows that $y \nleq x$. By Condition (P4.3) in Definition 4.1.11, there exists $U \in P_{\mathcal{B}}$ such that $y \in U$ and $x \notin U$. Then $U \in \mathcal{I}$ and thus $y \in \bigcup \mathcal{I}$. Hence $A \subseteq \bigcup \mathcal{I}$. Notice that clearly $x \notin \bigcup \mathcal{I}$, which is a contradiction. Then $x \in A$ and hence $\bigcap \{\bigcup \mathcal{I} : \mathcal{I} \in \mathsf{Id}_{\mathsf{sF}}(P_{\mathcal{B}}) \text{ and } A \subseteq \bigcup \mathcal{I}\} \subseteq A$. Therefore (ii) holds.

Now we show that Condition (sC) holds. Let $\mathcal{F} \in \mathsf{Fi}_{\mathsf{F}}(P_{\mathcal{B}})$ and $\mathcal{I} \in \mathsf{Id}_{\mathsf{sF}}(P_{\mathcal{B}})$ be such that $\bigcap \mathcal{F} \subseteq \bigcup \mathcal{I}$. Suppose towards a contradiction that $\mathcal{F} \cap \mathcal{I} = \emptyset$. Since $P_{\mathcal{B}}$ is a mo-distributive poset and by Theorem 2.4.27, there exists $\mathcal{U} \in \mathsf{Opt}_{\mathsf{s}}(P_{\mathcal{B}})$ such that $\mathcal{F} \subseteq \mathcal{U}$ and $\mathcal{U} \cap \mathcal{I} = \emptyset$. By Lemma 4.2.4, there exists $x \in X$ such that $\varepsilon(x) = \mathcal{U}$. Then $x \in \bigcap \mathcal{U} \subseteq \bigcap \mathcal{F}$ and thus $x \in \bigcup \mathcal{I}$. So, there is $U \in \mathcal{I}$ such that $x \in U$. Hence $U \in \mathcal{U} \cap \mathcal{I}$. Then $\mathcal{U} \cap \mathcal{I} \neq \emptyset$, a contradiction. Hence $\mathcal{F} \cap \mathcal{I} \neq \emptyset$. Therefore Condition (sC) holds. $\square$

Let P be a bounded mo-distributive poset and $P_* = \langle \mathsf{Opt}_{\mathsf{s}}(P), \tau, \subseteq, \mathcal{B} \rangle$ its dual poset Priestley space. Recall that $P_{\mathcal{B}_P} = \langle \mathcal{B}_P, \subseteq \rangle$ where $\mathcal{B}_P = \{\varphi_P(a) : a \in P\}$ and for each $a \in P$, $\varphi_P(a) = \{U \in \mathsf{Opt}_{\mathsf{s}}(P) : a \in U\}$. By Lemma 4.1.4, we know that $\varphi_P \colon P \to P_{\mathcal{B}_P}$ is an order-isomorphism. Hence, we obtain the following corollary.

Corollary 4.5.12. *Let P be a bounded mo-distributive poset and P_* its dual poset Priestley space. Then, $\langle \mathsf{Up}(P_*), \varphi_P \rangle$ is the strong Frink completion of P.*

Now the following corollary is clear.

Corollary 4.5.13. *Let P be a bounded mo-distributive poset and P^{sF} its strong Frink completion. Then,*

(1) *P^{sF} is a completely distributive algebraic lattice;*
(2) *$\mathcal{J}^{\infty}(P^{\mathrm{sF}})$ and $\mathcal{M}^{\infty}(P^{\mathrm{sF}})$ are isomorphic posets.*

Remark 4.5.14. Let P be a bounded mo-distributive poset and P^{F} its Frink completion and P^{sF} its strong Frink completion. If P^{F} and P^{sF} are isomorphic, then $\mathcal{J}^{\infty}(P^{\mathrm{F}})$ and $\mathcal{J}^{\infty}(P^{\mathrm{sF}})$ are order-isomorphic (both respectively ordered with respect to the restriction of the ordered of P^{F} and P^{sF}).

Example 4.5.15. In this example we show that the strong Frink completion and the Frink completion of a bounded mo-distributive poset can be different. Let P be the poset depicted in Figure 2.3 (see on page 30). In Figure 4.3 are depicted the posets $\langle \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P), \subseteq \rangle$ and $\langle \mathsf{Opt}_{\mathsf{s}}(P), \subseteq \rangle$. Hence, it is clear that $\mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P) \ncong \mathsf{Opt}_{\mathsf{s}}(P)$. Then, by Lemmas 3.5.9 and 4.5.7 we have that $\mathcal{J}^{\infty}(P^{\mathrm{F}}) \ncong \mathcal{J}^{\infty}(P^{\mathrm{sF}})$. Therefore, by Remark 4.5.14 we obtain that $P^{\mathrm{F}} \ncong P^{\mathrm{sF}}$.

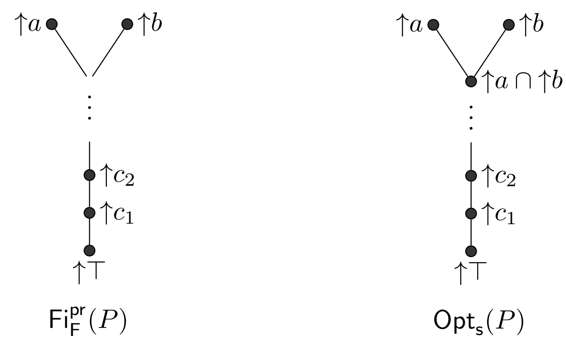


FIGURE 4.3. The posets $\langle \mathsf{Fi}^{\mathsf{pr}}_{\mathsf{F}}(P), \subseteq \rangle$ and $\langle \mathsf{Opt}_{\mathsf{s}}(P), \subseteq \rangle$ for the poset P given in Figure 2.3.

CHAPTER 5

A general topological duality for posets

As we mentioned in the introduction of this dissertation, the theory of topological duality arose with Stone's works [**54, 55**] and followed by Priestley's work [**52**]. In both cases the ordered algebraic structures, Boolean algebras and bounded distributive lattices, have an important property: distributivity. The dualities developed by Stone and Priestley are very useful to find and develop possible topological dualities for those ordered algebraic structures that have a distributivity condition, see for instance [**8, 5, 22, 23**]. But there are other important ordered algebras which do not have a distributivity condition, for instance lattices, expansions of lattices, meet-semilattices. So, it is important to study another way to find topological dualities for these ordered algebraic structures.

In the literature there are several topological dualities for bounded lattices, for instance [**56, 36, 35, 48**]. In [**48**] Moshier and Jipsen give a topological duality for bounded lattices and a topological duality for meet-semilattices with last element in a way that the corresponding dual categories are subcategories of the category of topological spaces. Thus, we can say that their dualities follow the line of Stone's duality. In [**49**], the second part of [**48**], Moshier and Jipsen use the duality developed in [**48**] to give, in a topological framework, a characterization of lattice expansions.

In this final chapter we develop a topological duality for arbitrary posets. The fundamental concept to build our duality is the notion of order-filter. We intend that the dual category of the posets with their order-preserving maps that in addition satisfy that the inverse image of an order-filter is an order-filter form a subcategory of the category of topological spaces, and that our duality generalizes the duality given by Moshier and Jipsen for bounded lattices.

The dual spaces of posets will be the sober spaces $\langle X, \tau \rangle$ with the property that the compact open order-filters of X with respect to the specialization order form a base for the topology τ. We call these spaces P-spaces. The duals of the morphisms between posets of our category will be the continuous functions with the property that the inverse image of a compact open order-filter is a compact open order-filter, which we call F-continuous maps.

5.1. Topological representation of posets

In this first section we present a topological representation theorem for posets through a certain kind of topological spaces. These topological spaces are Scott spaces built by means of posets. Our main purpose in this part is to generalize the topological representation for lattices and meet-semilattices given in [**48**] to posets. For this, we apply the underlying idea in [**48**] in a more general context.

We start introducing the notion of *Scott space*. The Scott topology arises in a natural way by means of posets. Here we choose to give an abstract definition of Scott spaces and we show the intrinsic connection with posets. The following concepts and results are well known, and so we leave the details to the reader. References for Scott spaces are [**40**] and [**57**]. Recall that given a T_0-space $\langle X, \tau\rangle$, the specialization order of X is denoted by $\preceq$ (see page 1.6).

Definition 5.1.1. A topological space $\langle X, \tau\rangle$ is said to be *Scott* if:

(1) X is T_0;
(2) for every subset U of X, U is open if and only if U is an up-set and it is inaccessible by up-directed joins (w.r.t. $\preceq$). That is, for each up-directed $D \subseteq X$, if $\bigvee^{\uparrow} D \in U$, then $U \cap D \neq \emptyset$.

In the previous definition, with $\bigvee^{\uparrow} D \in U$ we mean that D is an up-directed subset and the join of D in the poset $\langle X, \preceq\rangle$ exists and belongs to U. We keep in mind this convention throughout the chapter.

Example 5.1.2. Let $\langle P, \leq\rangle$ be a poset. The *Scott topology* on P determined by the order $\leq$ is the collection τ_P of all subsets U of P which are up-sets and inaccessible by up-directed joins with respect to $\leq$. So, it is clear that $\langle P, \tau_P\rangle$ is a Scott space. Moreover, $\leq$ is its specialization order $\preceq$.

Lemma 5.1.3. *Let X and Y be Scott spaces and $f\colon X \to Y$ a function. Then, f is continuous if and only if f preservers up-directed joins, i.e., $f(\bigvee^{\uparrow} D) = \bigvee^{\uparrow} f[D]$.*

We denote by $\mathbb{P}^{\Uparrow}$ the category whose objects are all posets and whose morphisms are all the functions between posets that preserve up-directed joins and by $\mathbb{TOP}(S)$ we denote the category of all Scott spaces and all continuous functions between them.

Lemma 5.1.4. *The categories $\mathbb{P}^{\Uparrow}$ and $\mathbb{TOP}(S)$ are isomorphic via the functors:*

(1) $\Gamma\colon \mathbb{P}^{\Uparrow} \to \mathbb{TOP}(S)$ *where*
  - $\Gamma(P) := \langle P, \tau_P\rangle$ *for every poset P;*
  - *for every morphism $f\colon P \to Q$ in $\mathbb{P}^{\Uparrow}$, $\Gamma(f)\colon \Gamma(P) \to \Gamma(Q)$ is given by $\Gamma(f) = f$.*

(2) $\Delta\colon \mathbb{TOP}(S) \to \mathbb{P}^{\Uparrow}$ *where*
   - $\Delta(X) := \langle X, \preceq \rangle$ *for every Scott space* X*;*
   - *for every morphism* $f\colon X \to Y$ *in* $\mathbb{TOP}(S)$*,* $\Delta(f)\colon \Delta(X) \to \Delta(Y)$ *is defined by* $\Delta(f) = f$*.*

Let P be a poset. Recall that $\mathsf{Fi}_{\mathsf{or}}(P)$ denotes the collection of all order-filters of P. Let us consider the poset $\langle \mathsf{Fi}_{\mathsf{or}}(P), \subseteq \rangle$. Notice that in the poset $\mathsf{Fi}_{\mathsf{or}}(P)$ every up-directed family of order-filters has join and equals its union. We define the topological space $\langle \mathsf{Fi}_{\mathsf{or}}(P), \tau_{\mathsf{Fi}_{\mathsf{or}}(P)} \rangle$ where $\tau_{\mathsf{Fi}_{\mathsf{or}}(P)}$ is the Scott topology of the poset $\mathsf{Fi}_{\mathsf{or}}(P)$ (see Example 5.1.2). For short we write $X_P := \langle \mathsf{Fi}_{\mathsf{or}}(P), \tau_{\mathsf{Fi}_{\mathsf{or}}(P)} \rangle$. It should be noted that the specialization order $\preceq$ of the space X_P is the order of inclusion $\subseteq$. That is, for all $F, G \in \mathsf{Fi}_{\mathsf{or}}(P)$, $F \preceq G$ if and only if $F \subseteq G$.

For every $a \in P$ we define the set

$$\varphi_a := \{F \in \mathsf{Fi}_{\mathsf{or}}(P) : a \in F\}.$$

The proofs of the following two lemmas are similar to the ones given for the analogous facts in the case of meet-semilattice with top element in [**48**]. We give more details here than in [**48**], because we work in the more general setting of posets.

Lemma 5.1.5. *Let* P *be a poset. Then the family* $\{\varphi_a : a \in P\}$ *is a base for the space* X_P*.*

Proof. We prove this lemma in two steps.

- Let $a \in P$. It is clear that φ_a is an up-set (of the poset $\langle \mathsf{Fi}_{\mathsf{or}}(P), \subseteq \rangle$). Let $\mathcal{A}$ be an up-directed collection of order-filters of P and suppose that $\bigvee^{\uparrow} \mathcal{A} \in \varphi_a$. Since $\mathcal{A}$ is an up-directed family of order-filters of P with respect to the inclusion order, we have that $\bigvee^{\uparrow} \mathcal{A} = \bigcup \mathcal{A}$. So, $a \in \bigcup \mathcal{A}$. This implies that there exists $F \in \mathcal{A}$ such that $a \in F$ and thus $F \in \varphi_a$. Then, $\varphi_a \cap \mathcal{A} \neq \emptyset$. Hence, φ_a is a Scott open of the space X_P.
- Now we will prove that the family $\{\varphi_a : a \in P\}$ is a base for the space X_P. Let $U \subseteq \mathsf{Fi}_{\mathsf{or}}(P)$ be a Scott open set of X_P and let $F \in U$. Let us take the set $\mathcal{D} := \{\uparrow a : a \in F\}$. $\mathcal{D}$ is an up-directed subset of $\mathsf{Fi}_{\mathsf{or}}(P)$ because F is an order-filter of P. So, $F = \bigcup \mathcal{D} = \bigvee^{\uparrow} \mathcal{D} \in U$ and, since U is Scott open, we obtain $U \cap \mathcal{D} \neq \emptyset$. Then, there is $a \in F$ such that $\uparrow a \in U$. This implies that $F \in \varphi_a \subseteq U$.

$\square$

Lemma 5.1.6. *Let* P *be a poset. For every* $a \in P$*,* φ_a *is a compact open order-filter of* X_P*.*

Proof. Let $a \in P$. From the previous lemma we know that φ_a is open. Moreover, clearly, φ_a is an order-filter of X_P. Now, let us prove that φ_a is compact. Let $\{U_i : i \in I\}$ be a family of open subsets of X_P and suppose that $\varphi_a \subseteq \bigcup_{i \in I} U_i$. Since $\uparrow a \in \varphi_a$, it follows that $\uparrow a \in \bigcup_{i \in I} U_i$. So, for some $i \in I$, $\uparrow a \in U_i$. As U_i is open and the specialization order in X_P is $\subseteq$, we have $\varphi_a \subseteq U_i$. $\square$

We provide a characterization of the posets P whose space X_P is compact. In particular, it turns out that if P has a last element, then the space X_P is compact.

Lemma 5.1.7. *Let P be a poset. The space X_P is compact if and only if the set of maximal elements of P is finite and for every $a \in P$ there exists a maximal element $b \in P$ such that $a \leq b$.*

Proof. Suppose that the set $\max(P)$ of maximal elements of P is finite and for every $a \in P$ there exists a maximal element $b \in P$ such that $a \leq b$. Let us consider a cover $\{\varphi_a : a \in Z\}$ of X_P by basic open sets. Let $b \in \max(P)$. Then $\{b\}$ is an order-filter of P. Therefore $\{b\} \in \varphi_a$ for some $a \in Z$, and then $a = b$. It follows that $\max(P) \subseteq Z$. Now, since by assumption for every $a \in P$ there exists $b \in \max(P)$ such that $a \leq b$, we obtain that $\{\varphi_b : b \in \max(P)\}$ is a finite subcover of $\{\varphi_a : a \in Z\}$. We conclude that X_P is compact. Conversely, assume that X_P is compact and the set of maximal elements of P is infinite or there exists $a \in P$ such that for no $b \in \max(P)$, $a \leq b$. If $\max(P)$ is infinite, then $\{\varphi_b : b \in \max(P)\} \cup \{\varphi_a : (\forall b \in \max(P))(a \not\leq b)\}$ is a cover of X_P without any finite subcover. If there exists $a \in P$ such that for no $b \in \max(P)$, $a \leq b$, let a_0 be such an element. Then there exists a strictly increasing infinite chain $a_0 < a_1 < \cdots < a_n < a_{n+1} < \ldots$. Let $\{\varphi_{a_n} : n \in \omega\} \cup \{\varphi_a : (\forall n \in \omega)(a \not\leq a_n)\}$. This family is a cover of X_P and has no finite subcover. $\square$

Let us denote by $\mathsf{KOF}(X)$ the collection of all compact open order-filters (with respect to the specialization order of X) of a T_0-space X. So, Lemma 5.1.6 tells us that $\{\varphi_a : a \in P\} \subseteq \mathsf{KOF}(X_P)$. We next prove that all compact open order-filters of the space X_P are the form φ_a for some $a \in P$. Then applying Lemma 5.1.5, we have that $\mathsf{KOF}(X_P)$ is a base for X_P. Before we need a technical lemma and a corollary.

Let $\langle X, \tau \rangle$ be a T_0-space. An element $a \in X$ is called *finite* if $\uparrow a$ is an open subset of X and we let $\mathsf{Fin}(X) := \{a \in X : \uparrow a \text{ is an open subset of } X\}$.

Lemma 5.1.8. ([**48**, Lemma 2.4]). *Let $\langle X, \tau \rangle$ be a T_0-space and let $F_1, \ldots, F_n$ be pairwise incomparable order-filters of X. Then $F_1 \cup \cdots \cup F_n$ is compact if and only if each F_i is a principal order-filter.*

Proof. It is clear that each principal order-filter $\uparrow x$ is compact and thus every finite union of principal order-filters is compact. Now let $F_1, \dots, F_n$ be pairwise incomparable order-filters of X and assume that $F_1 \cup \dots \cup F_n$ is compact. Suppose that F_n is not principal. Let $\mathcal{D}$ be the family of all opens U such that $F_n \setminus U \neq \emptyset$. For $x \in F_n$, there is an element $y \in F_n$ such that $x \npreceq y$. So there is an open U of X such that $x \in U$ and $y \notin U$. Since the order-filters $F_1, \dots, F_n$ are pairwise incomparable, it follows that for $x \in F_i$ with $i < n$ there is $y \in F_n$ such that $x \npreceq y$. Again there is an open U such that $x \in U$ and $y \notin U$. Thus, D is an open cover of $F_1 \cup \dots \cup F_n$. Let $U, V \in \mathcal{D}$. So there are elements $x \in F_n \setminus U$ and $y \in F_n \setminus U$. Since F_n is an order-filter, we have that there exists $z \in F_n$ such that $z \preceq x$ and $z \preceq y$. Then $z \in F_n \setminus (U \cup V)$. Thus, we have proved that $\mathcal{D}$ is up-directed. By construction, no $U \in \mathcal{D}$ covers $F_1 \cup \dots \cup F_n$, which is a contradiction. $\square$

Notice that for every T_0-space X we can restrict the specialization order to the subset $\mathsf{Fin}(X)$. Then $\langle \mathsf{Fin}(X), \preceq \rangle$ is a subposet of $\langle X, \preceq \rangle$.

Corollary 5.1.9. *Let $\langle X, \tau \rangle$ be a T_0-space. Then, we have that $\mathsf{KOF}(X) = \{\uparrow a : a \in \mathsf{Fin}(X)\}$. Therefore, the posets $\langle \mathsf{KOF}(X), \subseteq \rangle$ and $\langle \mathsf{Fin}(X), \preceq \rangle$ are dual order-isomorphic.*

Lemma 5.1.10. *Let P be a poset. For every compact open order-filter U of X_P there is $a \in P$ such that $U = \varphi_a$.*

Proof. Let $U \in \mathsf{KOF}(X_P)$. By Corollary 5.1.9, we have that $U = \{G \in \mathsf{Fi}_{\mathsf{or}}(P) : F \subseteq G\}$ for some $F \in \mathsf{Fi}_{\mathsf{or}}(P)$. Let $\mathcal{D} := \{\uparrow a : a \in F\}$ and so, $\mathcal{D}$ is an up-directed family of order-filters of P. Then, $\bigvee \mathcal{D} = \bigcup \mathcal{D} = F \in U$. As U is a Scott open, $U \cap \mathcal{D} \neq \emptyset$. Thus, there exists $a \in F$ such that $\uparrow a \in U$. Then, we obtain that $F = \uparrow a$ and hence,

$$\begin{aligned} U &= \{G \in \mathsf{Fi}_{\mathsf{or}}(P) : \uparrow a \subseteq G\} \\ &= \{G \in \mathsf{Fi}_{\mathsf{or}}(P) : a \in G\} \\ &= \varphi_a. \end{aligned}$$

$\square$

Therefore, bringing together the above results we have shown that $\mathsf{KOF}(X_P) = \{\varphi_a : a \in P\}$. Let us consider on $\mathsf{KOF}(X_P)$ the inclusion order $\subseteq$. We can now present the main result of this section, namely the representation theorem for posets.

Theorem 5.1.11. *Let P be a poset. Then, the map $\varphi_P \colon P \to \mathsf{KOF}(X_P)$ defined by $\varphi_P(a) = \varphi_a$ for all $a \in P$ is an order-isomorphism.*

Proof. By Lemma 5.1.10, it is clear that the map φ is onto. Let $a, b \in P$. Then, we have

$$\begin{aligned} a \leq b &\iff (\forall F \in \mathsf{Fi}_{\mathsf{or}}(P))(a \in F \Longrightarrow b \in F) \\ &\iff \varphi_a \subseteq \varphi_b \\ &\iff \varphi_P(a) \subseteq \varphi_P(b). \end{aligned}$$

Therefore, φ is an order-isomorphism from P onto $\mathsf{KOF}(X_P)$. □

Remark 5.1.12. Let P be a poset and $a, b, c \in P$. Note that $\varphi_c = \varphi_a \cap \varphi_b$ if and only if the greatest lower bound of a, b exists and is c. Thus in P the greatest lower bound of any two elements exists if and only if $U \cap V \in \mathsf{KOF}(X_P)$ for every $U, V \in \mathsf{KOF}(X_P)$. Also note that P has a top element (i.e., a greatest upper bound) if and only if $X_P \in \mathsf{KOF}(X_P)$.

5.2. Topological duality

In this section we define the topological spaces that will be the duals of the posets in the categorical duality that we want to establish. These spaces should be an abstract characterization of the spaces X_P constructed by means of posets P as in the previous section. Then, a topological space X dual to a poset should be such that $\mathsf{KOF}(X)$ is a base for the space. Moreover, we observe that the spaces X_P have very nice properties with respect to the specialization order. So, because our duality is a kind of Stone duality, a suitable property to consider can be to be sober. We begin by giving the following definition.

Definition 5.2.1. A topological space $\langle X, \tau \rangle$ is a P-*space* if it satisfies the following conditions:

(P1) X is a sober space;
(P2) $\mathsf{KOF}(X)$ is a base for τ.

The notion of P-space is a direct generalization of the notion of HMS-space introduced in [**48**]. HMS-spaces are duals of meet-semilattices with a top element. We will discuss HMS-spaces in Section 5.7. The following lemma, which is a characterization of P-spaces, can be useful to show that certain topological spaces are P-spaces. For a topological space $\langle X, \tau \rangle$, let us denote by $\mathrm{N}^{\mathrm{o}}(x)$ the collection of all neighbourhoods of an element $x \in X$, i.e., $\mathrm{N}^{\mathrm{o}}(x) = \{U \in \tau : x \in U\}$.

Lemma 5.2.2. *Let X be a topological space. Then, X is a P-space if and only if the following conditions are satisfied:*

(1) *X is a Scott space;*
(2) *$\mathsf{KOF}(X)$ is a base for X;*
(3) *every up-directed subset of X (w.r.t. $\preceq$) has a join.*

PROOF. First, we assume that X is a P-space and we prove that the three conditions above hold.

(1) Since X is sober, it is a T_0-space. Let U be an open set. Then, U is an up-set of X and by Lemma 1.6.8, U is inaccessible by up-directed joins. Now, let U be an up-set of X which is inaccessible by up-directed joins. Let $x \in U$. The following set

$$D := \{a \in \mathsf{Fin}(X) : a \preceq x\}$$

is up-directed and non-empty, because $\mathsf{KOF}(X)$ is a base for X. Then, since X is sober, there exists $\bigvee^{\uparrow} D$. Let us see that $x = \bigvee^{\uparrow} D$. It is clear that $\bigvee^{\uparrow} D \preceq x$. To prove the reverse inequality we use the fact that $\mathsf{KOF}(X)$ is a base. Let ${\uparrow}b \in \mathsf{KOF}(X)$ be such that $x \in {\uparrow}b$. So, $b \preceq x$ and thus, $b \in D$. Then, $b \preceq \bigvee^{\uparrow} D$, which implies that $\bigvee^{\uparrow} D \in {\uparrow}b$. Then, $x \preceq \bigvee^{\uparrow} D$. Thus, we have that $\bigvee^{\uparrow} D = x \in U$ and, since U is inaccessible by up-directed joins, we obtain that $D \cap U \neq \emptyset$. Then, there is $a \in D$ such that $a \in U$. Consequently, $a \in \mathsf{Fin}(X)$ and $a \preceq x$ and hence, $x \in {\uparrow}a$. So, we obtain that $x \in {\uparrow}a \subseteq U$, which implies that U is an open set of X. Therefore, X is a Scott space.

(2) By hypothesis, $\mathsf{KOF}(X)$ is a base for X.

(3) By Lemma 1.6.8, every up-directed subset of X has a join.

Now, we assume that X satisfies the three conditions of the lemma. We need only to prove that X is sober. Since X is a Scott space, we have that X is a T_0-space. Let $\mathcal{A}$ be a completely prime filter of the lattice of open sets. Let $D := \{a \in X : {\uparrow}a \in \mathcal{A}\}$. Since $\mathcal{A}$ is a filter, there is $U \in \mathcal{A}$. So, $U = \bigcup\limits_{i \in I} {\uparrow}a_i$ for some family $\{a_i : i \in I\} \subseteq \mathsf{Fin}(X)$. As $\bigcup\limits_{i \in I} {\uparrow}a_i \in \mathcal{A}$ and $\mathcal{A}$ is completely prime, there exists $i_0 \in I$ such that ${\uparrow}a_{i_0} \in \mathcal{A}$. So, $a_{i_0} \in D$. Thus, D is non-empty. Let us see that D is an up-directed subset of X. Let $a, b \in D$. So, since $\mathcal{A}$ is a filter, it follows that ${\uparrow}a \cap {\uparrow}b \in \mathcal{A}$. As ${\uparrow}a \cap {\uparrow}b$ is open, ${\uparrow}a \cap {\uparrow}b = \bigcup\limits_{i \in I} {\uparrow}c_i$ for some family $\{c_i : i \in I\} \subseteq \mathsf{Fin}(X)$. Then, ${\uparrow}c_i \in \mathcal{A}$ for some $i \in I$. Thus $c_i \in D$ and $a \preceq c_i$ and $b \preceq c_i$ and hence, D is up-directed. By Condition (3), let $x := \bigvee^{\uparrow} D$. We show that $\mathrm{N}^{\mathrm{o}}(x) = \mathcal{A}$. Let $U \in \mathrm{N}^{\mathrm{o}}(x)$. So, $x \in U$ and this implies that $\bigvee^{\uparrow} D \in U$. By Condition (1), we have that $D \cap U \neq \emptyset$. Then, there is $a \in D \cap U$. Since U is an up-set of X and $a \in U$, it follows that ${\uparrow}a \subseteq U$. As $a \in D$, ${\uparrow}a \in \mathcal{A}$ and then, $U \in \mathcal{A}$. Hence, $\mathrm{N}^{\mathrm{o}}(x) \subseteq \mathcal{A}$. Conversely, let $U \in \mathcal{A}$. So, $U = \bigcup\limits_{i \in I} {\uparrow}a_i$ for some ${\uparrow}a_i \in \mathsf{KOF}(X)$. Since $\mathcal{A}$ is completely prime, there exists $i \in I$ such that ${\uparrow}a_i \in \mathcal{A}$. Thus, $a_i \in D$. This implies that $a_i \preceq x$. Then, $x \in U$ because $a_i \in U$. Hence, $U \in \mathrm{N}^{\mathrm{o}}(x)$. Therefore, $\mathcal{A} \subseteq \mathrm{N}^{\mathrm{o}}(x)$. $\square$

THEOREM 5.2.3. *Let P be a poset. Then, X_P is a* P*-space.*

PROOF. By definition, the space X_P is a Scott space. From Lemmas 5.1.5 and 5.1.6, $\mathsf{KOF}(X_P)$ is a base for X_P. Lastly, since the specialization order $\preceq$ of X_P is the inclusion order, it is clear that the joins of all up-directed subsets of X_P exist. Then, the three conditions (1)–(3) of Lemma 5.2.2 are satisfied and therefore, X_P is a P-space. □

Let X be a topological space. We denote the poset $\langle \mathsf{KOF}(X), \subseteq \rangle$ by P_X. Using the construction of the previous section, we can obtain the topological space $X_{P_X} := \langle \mathsf{Fi}_{\mathsf{or}}(P_X), \tau_{\mathsf{Fi}_{\mathsf{or}}(P_X)} \rangle$.

THEOREM 5.2.4. *Let X be a* P*-space. Then, X is homeomorphic to X_{P_X}.*

PROOF. We define the map $\theta_X : X \to X_{P_X}$ as follows

$$\theta_X(x) := \{U \in \mathsf{KOF}(X) : x \in U\}$$

for each $x \in X$. We show that θ_X is a homeomorphism in several steps.

- θ_X is well defined. Let $x \in X$. It is clear that $\theta_X(x)$ is an up-set of $P_X = \langle \mathsf{KOF}(X), \subseteq \rangle$. Let $U_1, U_2 \in \theta_X(x)$. So, $x \in U_1 \cap U_2$ and, since $U_1 \cap U_2$ is open, there exists $U_3 \in \mathsf{KOF}(X)$ such that $x \in U_3 \subseteq U_1 \cap U_2$. Then, $U_3 \in \theta_X(x)$ and $U_3 \subseteq U_1, U_2$. Hence, $\theta_X(x)$ is an order-filter of the poset P_X.
- θ_X is injective. Let $x, y \in X$ and suppose that $x \npreceq y$. So, since $\mathsf{KOF}(X)$ is a base for X, there exists $U \in \mathsf{KOF}(X)$ such that $x \in U$ and $y \notin U$. Then, $\theta_X(x) \nsubseteq \theta_X(y)$.
- θ_X is onto. Let $F \in X_{P_X} = \mathsf{Fi}_{\mathsf{or}}(P_X)$. Let $D := \{a \in X : \uparrow a \in F\}$. As F is an order-filter of P_X, D is an up-directed subset of X. Then, since X is a P-space, there exists $x := \bigvee^{\uparrow} D$. We want to show that $\theta_X(x) = F$. Let $\uparrow a \in F$. So, $a \in D$ and then $x \in \uparrow a$. Which implies that $\uparrow a \in \theta_X(x)$. Now, let $\uparrow a \in \theta_X(x)$. By definition of θ_X and since $x = \bigvee^{\uparrow} D$, it follows that $\bigvee^{\uparrow} D \in \uparrow a$. As X is a P-space, the open subsets of X are inaccessible by up-directed joins and so, $D \cap \uparrow a \neq \emptyset$. Then, there exists $d \in D \cap \uparrow a$. Since F is an order-filter, $\uparrow d \subseteq \uparrow a$ and $\uparrow d \in F$, it follows that $\uparrow a \in F$. Therefore, $F = \theta_X(x)$.
- θ_X is continuous. Let φ_U be a basic open set of the space X_{P_X}. Recall that for $U \in P_X = \mathsf{KOF}(X)$

$$\varphi_U = \{F \in \mathsf{Fi}_{\mathsf{or}}(P_X) : U \in F\}.$$

For every $x \in X$ we have

$$x \in \theta_X^{-1}[\varphi_U] \iff \theta_X(x) \in \varphi_U$$

$$\iff U \in \theta_X(x)$$
$$\iff x \in U.$$

Then, $\theta_X^{-1}[\varphi_U] = U$ is an open set of X and therefore, θ_X is continuous.

- θ_X is an open map. Let $U \in \mathsf{KOF}(X)$. We show that $\theta_X[U] = \varphi_U$. Let $F \in \theta_X[U]$. So, there is $x \in U$ such that $\theta_X(x) = F$. Since $x \in U$, it follows that $F \in \varphi_U$. Now, let $F \in \varphi_U$. So, $U \in F$. Since θ_X is onto, there exists $x \in X$ such that $\theta_X(x) = F$. As, $U \in F = \theta_X(x)$, $x \in U$. Then, $F \in \theta_X[U]$.

Therefore, from all these points, we can conclude that θ_X is a homeomorphism. □

Let us denote by $\mathbb{PO}$ the category whose objects are posets and whose morphisms are the order-preserving maps between posets and such that the inverse image of an order-filter is an order-filter. That is, $j\colon P \to Q$ is a morphism of $\mathbb{PO}$ if it is an order-preserving map and for all $G \in \mathsf{Fi}_{\mathsf{or}}(Q)$, $j^{-1}[G] \in \mathsf{Fi}_{\mathsf{or}}(P)$. By $\mathbb{PS}$ we denote the category of P-spaces and F-continuous maps. A map $f\colon X \to Y$ from the P-space X to the P-space Y is called *F-continuous* if for all $U \in \mathsf{KOF}(Y)$ we have that $f^{-1}[U] \in \mathsf{KOF}(X)$. When this condition holds we say that f^{-1} preserves compact open order-filters. Note that every F-continuous map between P-spaces is continuous.

Now, we extend the representation theorem for posets to a duality between the categories $\mathbb{PO}$ and $\mathbb{PS}$.

THEOREM 5.2.5. *The categories $\mathbb{PO}$ and $\mathbb{PS}$ are dually equivalent via the functors:*

(1) $\Gamma\colon \mathbb{PO} \to \mathbb{PS}$ *defined by*
  - $\Gamma(P) := X_P$, *for each poset P;*
  - *for every morphism $j\colon P \to Q$ in the category $\mathbb{PO}$, $\Gamma(j)\colon X_Q \to X_P$ is given by $\Gamma(j) := j^{-1}$.*

(2) $\Delta\colon \mathbb{PS} \to \mathbb{PO}$ *defined by*
  - $\Delta(X) := P_X$, *for each P-space X;*
  - *for every morphism $f\colon X \to Y$ in the category $\mathbb{PS}$, $\Delta(f)\colon P_Y \to P_X$ is given by $\Delta(f) := f^{-1}$.*

PROOF.

(1) Let $j\colon P \to Q$ be a morphism in $\mathbb{PO}$. We show that $\Gamma(j) = j^{-1}\colon X_Q \to X_P$ is F-continuous. Let $U \in \mathsf{KOF}(X_P)$. By Lemma 5.1.10, there is $a \in P$ such that $U = \varphi_a$. Then, we have

$$G \in \Gamma(j)^{-1}[\varphi_a] \iff \Gamma(j)(G) \in \varphi_a$$

$$
\begin{aligned}
&\iff j^{-1}[G] \in \varphi_a \\
&\iff a \in j^{-1}[G] \\
&\iff j(a) \in G \\
&\iff G \in \varphi_{j(a)}.
\end{aligned}
$$

Hence, $\Gamma(j)^{-1}[\varphi_a] = \varphi_{j(a)} \in \mathsf{KOF}(X_Q)$.

(2) Let $f\colon X \to Y$ be a morphism of the category $\mathbb{PS}$. Since $\Delta(f) = f^{-1}$, it is clear that $\Delta(f)$ is an order-preserving map from P_Y to P_X. Now, let $F \in \mathsf{Fi}_{\mathsf{or}}(P_X)$. From Theorem 5.2.4 we know that $F = \theta_X(x)$ for some $x \in X$. Let $U \in \mathsf{KOF}(Y)$. Then,

$$
\begin{aligned}
U \in \Delta(f)^{-1}[F] &\iff \Delta(f)(U) \in F = \theta_X(x) \\
&\iff f^{-1}[U] \in \theta_X(x) \\
&\iff f(x) \in U \\
&\iff U \in \theta_Y(f(x)).
\end{aligned}
$$

Hence, we obtain that $\Delta(f)^{-1}[F] = \theta_Y(f(x)) \in \mathsf{Fi}_{\mathsf{or}}(P_Y)$. Thus, $\Delta(f)$ is a morphism of $\mathbb{PO}$.

To conclude the proof, we need to show that for every morphism $j\colon P \to Q$ in $\mathbb{PO}$ and every morphism $f\colon X \to Y$ in $\mathbb{PS}$ the diagrams in Figure 5.1 commute. Let $a \in P$ and $G \in \mathsf{Fi}_{\mathsf{or}}(Q)$. Then, we have

$$
\begin{aligned}
G \in (\Delta(\Gamma(j)) \circ \varphi_P)(a) &\iff G \in \Gamma(j)^{-1}[\varphi_P(a)] \\
&\iff \Gamma(j)(G) \in \varphi_P(a) \\
&\iff j^{-1}[G] \in \varphi_P(a) \\
&\iff a \in j^{-1}[G] \\
&\iff j(a) \in G \\
&\iff G \in \varphi(j(a)).
\end{aligned}
$$

Hence,

$$\Delta(\Gamma(j)) \circ \varphi_P = \varphi_Q \circ j.$$

Finally, let $x \in X$ and $U \in \mathsf{KOF}(Y)$. Then

$$
\begin{aligned}
U \in (\Gamma(\Delta(f)) \circ \theta_X)(x) &\iff U \in \Delta(f)^{-1}[\theta_X(x)] \\
&\iff \Delta(f)(U) \in \theta_X(x) \\
&\iff f^{-1}[U] \in \theta_X(x) \\
&\iff x \in f^{-1}[U] \\
&\iff f(x) \in U
\end{aligned}
$$

$$\begin{tikzcd} P \arrow[r, "j"] \arrow[d, "\varphi_P"'] & Q \arrow[d, "\varphi_Q"] \\ \mathsf{KOF}(X_P) \arrow[r, "\Delta(\Gamma(j))"'] & \mathsf{KOF}(X_Q) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\theta_X"'] & Y \arrow[d, "\theta_X"] \\ X_{P_X} \arrow[r, "\Gamma(\Delta(f))"'] & X_{P_Y} \end{tikzcd}$$

FIGURE 5.1. Commutative diagrams of morphisms in the categories $\mathbb{PO}$ and $\mathbb{PS}$.

$$\iff U \in \theta_Y(f(x)).$$

Hence,

$$\Gamma(\Delta(f)) \circ \theta_X = \theta_Y \circ f.$$

□

5.3. Canonical extension for posets

In this section we use the duality between the categories $\mathbb{PO}$ and $\mathbb{PS}$ of the previous section to show the existence of the canonical extension of a poset from a topological viewpoint. Here, the canonical extension of a poset is taken as in [**17**]. Our proof is a topological alternative proof to the algebraic proofs in [**17**], in a similar way than the proof of the existence of a canonical extension for lattices given in [**48**] is a topological alternative proof to the purely algebraic proof supplied in [**26**].

Let X be a T_0-space. Let us denote by $\mathsf{OF}(X)$ the family of all open order-filters of X. We take the closure system $\mathsf{Fsat}(X)$ on X generated by the family $\mathsf{OF}(X)$. That is, $\mathsf{Fsat}(X)$ is the collection of all subsets of X that are intersections of open order-filters. The elements of $\mathsf{Fsat}(X)$ are called *F-saturated* sets. We denote the associated closure operator of $\mathsf{Fsat}(X)$ by $\mathrm{Fsat}(.)$. So, for every $A \subseteq X$,

$$\mathrm{Fsat}(A) = \bigcap\{F \in \mathsf{OF}(X) : \ A \subseteq F\}.$$

Then, we have the complete lattice $\langle \mathsf{Fsat}(X), \bigcap, \bigvee \rangle$ where $\bigvee \mathcal{A} = \mathrm{Fsat}(\bigcup \mathcal{A})$ for each $\mathcal{A} \subseteq \mathsf{Fsat}(X)$ and, moreover, $\mathsf{KOF}(X) \subseteq \mathsf{OF}(X) \subseteq \mathsf{Fsat}(X)$. So, it is clear that the lattice $\mathsf{Fsat}(X)$ is a completion of the poset $P_X = \langle \mathsf{KOF}(X), \subseteq \rangle$.

Let P be a poset. We will prove that $\mathsf{Fsat}(X_P) = \langle \mathsf{Fsat}(X_P), \bigcap, \bigvee \rangle$ is the canonical extension of P with the embedding $\varphi_P : P \to \mathsf{Fsat}(X_P)$.

According to the terminology in [**17**], an element of $\mathsf{Fsat}(X_P)$ is a *closed element* if it is the infimum in $\mathsf{Fsat}(X_P)$ of $\varphi_P[F]$ for some order-filter F of P. And an element of $\mathsf{Fsat}(X_P)$ is *open element* if it is the supremum in $\mathsf{Fsat}(X_P)$ of $\varphi_P[I]$ for some order-ideal I of P.

Lemma 5.3.1. *Let P be a poset. An element $U \in \mathsf{Fsat}(X_P)$ is a closed element if there is an order-filter F of P such that $U = {\uparrow}F$ in $\langle \mathsf{Fi}_{\mathsf{or}}(P), \subseteq\rangle$. Similarly U is an open element if there exists an order-ideal I of P such that $U = \{G \in \mathsf{Fi}_{\mathsf{or}}(P) : G \cap I \neq \emptyset\}$.*

Proof. First note that if $F \in \mathsf{Fi}_{\mathsf{or}}(P)$, then ${\uparrow}F = \{G \in \mathsf{Fi}_{\mathsf{or}}(P) : F \subseteq G\} = \bigcap\{\varphi_P(a) : a \in F\} = \bigcap \varphi_P[F]$. Thus, ${\uparrow}F \in \mathsf{Fsat}(X_P)$ and is a closed element. Now if $U \in \mathsf{Fsat}(X_P)$ is a closed element, then there exists $F \in \mathsf{Fi}_{\mathsf{or}}(P)$ such that $U = \bigcap \varphi_P[F]$. Then, $U = {\uparrow}F$.

Let I be an order-ideal of P. Then $\{G \in \mathsf{Fi}_{\mathsf{or}}(P) : G \cap I \neq \emptyset\} = \bigcup \varphi_P[I]$. Note that since I is an order-ideal, $\varphi_P[I]$ is up-directed. Thus, $\bigvee \varphi_P[I] = \mathrm{Fsat}(\bigcup \varphi_P[I]) = \bigcup \varphi_P[I]$. Hence, $\{G \in \mathsf{Fi}_{\mathsf{or}}(P) : G \cap I \neq \emptyset\} \in \mathsf{Fsat}(X_P)$ and is an open element. Now if $U \in \mathsf{Fsat}(X_P)$ is an open element, let I be an order-ideal of P such that $U = \bigvee \varphi_P[I]$. Then, $U = \{G \in \mathsf{Fi}_{\mathsf{or}}(P) : G \cap I \neq \emptyset\}$. $\square$

Lemma 5.3.2. *Let P be a poset. If $\mathcal{F}$ is an open order-filter of X_P, then there exists an order-ideal I of P such that $\mathcal{F} = \bigvee \varphi_P[I]$.*

Proof. Let $\mathcal{F}$ be an open order-filter of X_P, thus it is an up-set which is down-directed and being an open set is inaccessible by up-directed joins. Let

$$I := \{a \in P : {\uparrow}a \in \mathcal{F}\}$$

We claim that I is an order-ideal of P. If $a \in I$ and $b \leq a \in P$, then ${\uparrow}a \in \mathcal{F}$ and ${\uparrow}a \subseteq {\uparrow}b$. Hence, ${\uparrow}b \in \mathcal{F}$ and so $b \in I$. Suppose now that $a, b \in I$, so that ${\uparrow}a, {\uparrow}b \in \mathcal{F}$. There exists $F \in \mathcal{F}$ such that $F \subseteq {\uparrow}a, {\uparrow}b$. Note that since F is an order-filter of P, the set $\{{\uparrow}c : c \in F\}$ is up-directed and its join is F. Using that $\mathcal{F}$ is inaccessible by up-directed joins, there exists $c \in F$ such that ${\uparrow}c \in \mathcal{F}$. It follows that $a, b \leq c \in I$.

To conclude the proof we show that $\mathcal{F} = \bigvee \varphi_P[I]$. First note that $\varphi_P[I]$ is up-directed because I is an order-ideal. Thus $\bigvee \varphi_P[I] = \bigcup \varphi_P[I]$. Let $G \in \bigcup \varphi_P[I]$. So, there exists $a \in I$ such that $a \in G$. Hence, since ${\uparrow}a \in \mathcal{F}$ and ${\uparrow}a \subseteq G$, we have $G \in \mathcal{F}$. To prove the other inclusion suppose that $G \in \mathcal{F}$. Since $G = \bigvee\{{\uparrow}c : c \in G\} \in \mathcal{F}$ and the set $\{{\uparrow}c : c \in G\}$ is up-directed, it follows that there is $c \in G$ such that ${\uparrow}c \in \mathcal{F}$. Therefore, $c \in I$ and $G \in \varphi_P(c)$, so that $G \in \bigcup \varphi_P[I]$. $\square$

Lemma 5.3.3. *Let P be a poset. Then, the complete lattice $\mathsf{Fsat}(X_P)$ is the canonical extension of the poset P (with the embedding φ_P).*

Proof. Density: Let $U \in \mathsf{Fsat}(X_P)$. First note that U is an up-set of the poset $\langle \mathsf{Fi}_{\mathsf{or}}(P), \subseteq\rangle$ because it is an intersection of open order-filters of X_P and these are up-sets. Thus, $U = \bigcup\{{\uparrow}F : F \in U\}$. Hence, using Lemma 5.3.1, we have

$$U = \mathrm{Fsat}(U) = \bigvee\{V \in \mathsf{Fsat}(X_P) : V \text{ is a closed element and } V \subseteq U\}.$$

Now we prove that

$$U = \bigcap\{\bigvee \varphi_P[I] : \ I \text{ is an order-ideal of } P \text{ and } U \subseteq \bigvee \varphi_P[I]\},$$

that is, using Lemma 5.3.1, we show that

$$U = \bigcap\{V \in \mathsf{Fsat}(X_P) : \ V \text{ is an open element and } U \subseteq V\}.$$

One inclusion is obvious, to prove the other inclusion let $G \in \mathsf{Fi}_{\mathsf{or}}(P)$ be such that $G \notin U$. We find an order-ideal I of P such that $U \subseteq \bigvee \varphi_P[I]$ and $G \notin \bigvee \varphi_P[I]$. Since $U \in \mathsf{Fsat}(X_P)$ there is a family $\{\mathcal{F}_k : \ k \in K\}$ of open order-filters of X_P such that $U = \bigcap_{k\in K} \mathcal{F}_k$. Then there exists $\mathcal{F}_k$ such that $G \notin \mathcal{F}_k$. We consider the set

$$I := \{a \in P : \ \uparrow a \in \mathcal{F}_k\}$$

By Lemma 5.3.2 we have $\mathcal{F}_k = \bigvee \varphi_P[I]$. Thus $U \subseteq \bigvee \varphi_P[I]$ and $G \notin \bigvee \varphi_P[I]$.

Compactness: Let D be a non-empty down-directed subset of P and E a non-empty up-directed subset of P such that $\bigwedge \varphi_P[D] \subseteq \bigvee \varphi_P[E]$. Then $\varphi_P[E]$ is up-directed and therefore $\bigvee \varphi_P[E] = \bigcup \varphi_P[E]$. So, $\bigcap \varphi_P[D] \subseteq \bigcup \varphi_P[E]$. Let $F := \{d \in P : \ \exists a \in D \ a \leq d\}$. This set is an order-filter of P and $F \in \bigcap \varphi_P[D]$. Thus $F \in \bigcup \varphi_P[E]$. Then there exists $d \in E$ such that $d \in F$. Hence there is $a \in D$ such that $a \leq d$. This proves the compactness condition. □

5.4. The dual space of P^∂

In this section we will see how to obtain the dual P-space of the dual poset P^∂ of a poset P. This characterization of the dual P-space of P^∂ can be useful when trying to obtain topological representations of n-ary operations on posets that are order-preserving or order-reversing in each coordinate.

Given a P-space X, we consider the Scott topology of the poset $\langle \mathsf{OF}(X), \subseteq\rangle$. We refer to the resulting space simply by $\mathsf{OF}(X)$. For every $x \in X$ we define the sets

$$\psi_x := \{F \in \mathsf{OF}(X) : \ x \in F\}.$$

LEMMA 5.4.1. *Let X be a P-space. Then, for every $x \in X$, ψ_x is an order-filter of $\langle \mathsf{OF}(X), \subseteq\rangle$.*

PROOF. Let $x \in X$. Since $\mathsf{KOF}(X)$ is a base of X, there is $a \in \mathsf{Fin}(X)$ such that $x \in \uparrow a$. Then $\uparrow a \in \psi_x$ and thus $\psi_x \neq \emptyset$. It is clear that ψ_x is an up-set. Now let $F, G \in \psi_x$. So $x \in F \cap G$. Since $F \cap G$ is an open subset of X, it follows that there exists $\uparrow a \in \mathsf{KOF}(X)$ such that $x \in \uparrow a \subseteq F \cap G$. Then $\uparrow a \in \psi_x$ and $\uparrow a \subseteq F$ and $\uparrow a \subseteq G$. Hence, ψ_x is an order-filter of $\mathsf{OF}(X)$. □

LEMMA 5.4.2. *Let X be a P-space. Then the family $\{\psi_x : \ x \in X\}$ is a base for the space $\mathsf{OF}(X)$.*

PROOF. Let $x \in X$. First, we prove that ψ_x is a Scott open subset of $\mathsf{OF}(X)$. It is clear that ψ_x is an up-set of $\mathsf{OF}(X)$. Now, let $\{F_k : k \in K\}$ be an up-directed family of open order-filters of X and suppose that $\bigvee^{\uparrow} F_k \in \psi_x$. Since the family $\{F_k : k \in K\}$ is up-directed, $\bigvee^{\uparrow} F_k = \bigcup F_k$. So, $x \in F_k$ for some $k \in K$. Then, $\{F_k : k \in K\} \cap \psi_x \neq \emptyset$. Hence, ψ_x is a Scott open set of the space $\mathsf{OF}(X)$.

To prove that the family $\{\psi_x : x \in X\}$ is a base for $\mathsf{OF}(X)$, let U be a Scott open set of the space $\mathsf{OF}(X)$ and let $F \in U$. Since X is a P-space,

$$F = \bigcup \{\uparrow a : a \in F \cap \mathsf{Fin}(X)\} = \bigvee^{\uparrow} \{\uparrow a : a \in F \cap \mathsf{Fin}(X)\}.$$

As U is inaccessible by up-directed joins, there exists $a \in F \cap \mathsf{Fin}(X)$ such that $\uparrow a \in U$. Since U is an up-set, we have $F \in \psi_a \subseteq U$. □

In [**49**] Moshier and Jipsen consider for a HMS-space X the topology on $\mathsf{OF}(X)$ generated by the family $\{\psi_x : x \in X\}$ and then they show that this family is a base. In our case, in the setting of posets, the proof that the family $\{\psi_x : x \in X\}$ is a base is completely different from theirs. Moreover we are seeing that this family is a base for the Scott topology on $\mathsf{OF}(X)$.

In the next lemma we show a relation between a P-space X and the space $\mathsf{OF}(X)$. Consider the poset $\mathsf{Fin}(X) := \langle \mathsf{Fin}(X), \preceq \rangle$, which is a sub-poset of the space X with respect to the specialization order. So, we can consider the dual P-space $X_{\mathsf{Fin}(X)} = \mathsf{Fi}_{\mathsf{or}}(\mathsf{Fin}(X))$ of the poset $\mathsf{Fin}(X)$.

From the previous lemma, we know that the space $\mathsf{OF}(X)$ has the family $\{\psi_x : x \in X\}$ as a base, but since $\mathsf{KOF}(X)$ is a base for the space X we can take a smaller family as a base for the space $\mathsf{OF}(X)$ and this is $\{\psi_a : a \in \mathsf{Fin}(X)\}$. To show this, let U be an open set of the space $\mathsf{OF}(X)$ and let $F \in U$. So, there is $x \in X$ such that $F \in \psi_x \subseteq U$. Since $x \in F$ and F is an open set of X, there exists $a \in \mathsf{Fin}(X)$ such that $x \in \uparrow a \subseteq F$. Then, we obtain $F \in \psi_a \subseteq U$. Now we are ready to prove the following lemma.

LEMMA 5.4.3. *Let X be a P-space. Then, the spaces $X_{\mathsf{Fin}(X)}$ and $\mathsf{OF}(X)$ are homeomorphic.*

PROOF. We define the map $\alpha \colon X_{\mathsf{Fin}(X)} \to \mathsf{OF}(X)$ as follows

$$\alpha(F) := \bigcup \{\uparrow a : a \in F\}$$

for each $F \in X_{\mathsf{Fin}(X)}$.

- α is well defined. Let $F \in X_{\mathsf{Fin}(X)}$. Since $F \subseteq \mathsf{Fin}(X)$, $\alpha(F)$ is an open subset of X and moreover, it is an up-set. Let $x, y \in \alpha(F)$. So, there are $a, b \in F$ such that $x \in \uparrow a$ and $y \in \uparrow b$. Given that F is an order-filter of the poset $\mathsf{Fin}(X)$, there is $c \in F$ such that $c \preceq a, b$. Then, $c \preceq x, y$ and $c \in \alpha(F)$. Hence, $\alpha(F) \in \mathsf{OF}(X)$.

- α is injective. Let $F_1, F_2 \in X_{\mathsf{Fin}(X)}$ and assume that $\alpha(F_1) = \alpha(F_2)$. Since $F_1 \subseteq \alpha(F_1)$, we have $F_1 \subseteq \alpha(F_2)$. Let $a \in F_1$. So, $a \in \alpha(F_2)$ and this implies that there exists $b \in F_2$ such that $a \in {\uparrow} b$. Then, $a \in F_2$. Thus, $F_1 \subseteq F_2$. Similarly, we can show that $F_2 \subseteq F_1$. Hence, $F_1 = F_2$.
- α is onto. Let $G \in \mathsf{OF}(X)$. We take $F = G \cap \mathsf{Fin}(X)$. Let $a, b \in \mathsf{Fin}(X)$ and suppose that $a \preceq b$ and $a \in F$. Since G is an order-filter of X, $b \in G$ and then, $b \in F$. Let $a, b \in F$. Since G is an order-filter, there is $x \in G$ such that $x \preceq a, b$. Given that G is an open set of X, there exists $c \in \mathsf{Fin}(X)$ such that $x \in {\uparrow} c \subseteq G$. So, we have $c \in G$ and $c \preceq x$. Then, $c \in F$ and $c \preceq a, b$. Which implies that $F \in X_{\mathsf{Fin}(X)}$. Finally, we need to show that $\alpha(F) = G$. Let $x \in \alpha(F)$. So, there is $a \in F$ such that $a \preceq x$. Then, $a \in G$ and thus $x \in G$. Hence $\alpha(F) \subseteq G$. Let $x \in G$. So, there exists $a \in \mathsf{Fin}(X)$ such that $x \in {\uparrow} a \subseteq G$. Then, $a \in G \cap \mathsf{Fin}(X) = F$ and consequently, $x \in \alpha(F)$. Thus, $G \subseteq \alpha(F)$. Therefore, $\alpha(F) = G$.
- α is continuous. Let $a \in \mathsf{Fin}(X)$. We have that

$$\begin{aligned} F \in \alpha^{-1}[\psi_a] &\iff \alpha(F) \in \psi_a \\ &\iff a \in \alpha(F) \\ &\iff F \in \varphi_a. \end{aligned}$$

  Then, $\alpha^{-1}[\psi_a]$ is an open subset of the space $X_{\mathsf{Fin}(X)}$ and hence, α is continuous. Notice that here φ_a is restricted to the poset $\langle \mathsf{Fin}(X), \preceq \rangle$, that is, $\varphi_a = \{F \in \mathsf{Fi_{or}}(\mathsf{Fin}(X)) : \ a \in F\}$.
- α is an open map. By the previous point, for every $a \in \mathsf{Fin}(X)$, $\alpha^{-1}[\psi_a] = \varphi_a$. So, since we already know that α is a bijection we obtain

$$\psi_a = \alpha[\alpha^{-1}[\psi_a]] = \alpha[\varphi_a].$$

  Hence, α is open. This completes the proof.

$\square$

It should be noted that, by the previous lemma, for every P-space X the space $\mathsf{OF}(X)$ is a P-space.

Corollary 5.4.4. *Let P be a poset. If X is the dual P-space of P, then $\mathsf{OF}(X)$ is the dual P-space of the poset P^∂.*

Proof. Let X be a P-space. It is clear that $\mathsf{KOF}(X)^\partial = \langle \mathsf{KOF}(X), \supseteq \rangle$ and $\langle \mathsf{Fin}(X), \preceq \rangle$ are order-isomorphic. Thus, by Theorem 5.2.5, we obtain that the spaces $X_{\mathsf{KOF}(X)^\partial}$ and $X_{\mathsf{Fin}(X)}$ are homeomorphic. Then, by Lemma 5.4.3, we have $X_{\mathsf{KOF}(X)^\partial}$ and $\mathsf{OF}(X)$ are homeomorphic and therefore the P-space $\mathsf{OF}(X)$ is the dual of the poset $\mathsf{KOF}(X)^\partial$. $\square$

Lemma 5.4.5. *Let X be a* P*-space and $x \in X$. Then,*

$$x \in \mathsf{Fin}(X) \iff \psi_x \text{ is compact.}$$

Proof. Let $x \in \mathsf{Fin}(X)$. Notice that $\psi_x = \{F \in \mathsf{OF}(X) : {\uparrow}x \subseteq F\}$. Since ${\uparrow}x \in \mathsf{OF}(X)$, it follows that ψ_x is the compact open principal order-filter of the space $\mathsf{OF}(X)$ generated by ${\uparrow}x$. Reciprocally, let $x \in X$ and assume that ψ_x is compact. So, by Lemma 5.4.1 $\psi_x \in \mathsf{KOF}(\mathsf{OF}(X))$. Then, there exists $G \in \mathsf{OF}(X)$ such that

$$\psi_x = \{F \in \mathsf{OF}(X) : G \subseteq F\}.$$

Since $G \in \psi_x$, it follows that $x \in G$. So, we have ${\uparrow}x \subseteq G$. Now, let $a \in G$ and suppose that $x \not\preceq a$. Thus, there is $U \in \mathsf{KOF}(X)$ such that $x \in U$ and $a \notin U$. Then, we have $U \in \psi_x$ and hence, $G \nsubseteq U$, which is a contradiction. Then $x \preceq a$. So, we obtain that $G \subseteq {\uparrow}x$. Therefore, $G = {\uparrow}x$ and $x \in \mathsf{Fin}(X)$. □

Lemma 5.4.6. *If X is a* P*-space, then the map $\eta\colon X \to \mathsf{OF}(\mathsf{OF}(X))$ defined by*

$$\eta(x) := \psi_x = \{F \in \mathsf{OF}(X) : x \in F\},$$

for every $x \in X$, is a homeomorphism.

5.5. Topological representation of quasi-monotone maps

The main aim of this section is to characterize topologically the maps $j\colon P_1 \times \cdots \times P_n \to P_{n+1}$ where $P_1, \dots, P_{n+1}$ are posets and that in each coordinate either preserve or reverse the order. We will call such maps *quasi-monotone maps*. If P is a poset and $j\colon P^n \to P$ is a quasi-monotone map, then we say that j is an *n-ary quasi-monotone map*. In [**17**] a structure $\langle P, (j_i)_{i\in I}\rangle$ where P is a poset and every j_i is an n_i-ary quasi-monotone map on P is called a *monotone poset expansion*.

For every quasi-monotone map $j\colon P_1 \times \cdots \times P_n \to P_{n+1}$ there is a *monotonicity type* $\epsilon = \langle \epsilon_1, \dots, \epsilon_n\rangle$ associated with j where for every $i = 1, \dots, n$, $\epsilon_i = 1$ or $\epsilon_i = \partial$ depending on whether j preserves or reverses the order in the coordinate i. If we let $P_i^{\epsilon_i} = P_i$ or $P_i^{\epsilon_i} = P_i^{\partial}$, depending on whether $\epsilon_i = 1$ or $\epsilon_i = \partial$, then the map $j\colon P_1^{\epsilon_1} \times \cdots \times P_n^{\epsilon_n} \to P_{n+1}$ is order-preserving.

To represent topologically quasi-monotone maps $j\colon P_1 \times \cdots \times P_n \to P_{n+1}$ for arbitrary posets $P_1, \dots, P_{n+1}$, it is then enough to represent order-preserving maps in each coordinate. Indeed, if $\epsilon = \langle \epsilon_1, \dots, \epsilon_n\rangle$ is the monotonicity type associated with j, letting $X_k^{\epsilon_n}$ to be the dual space of $P_k^{\epsilon_n}$ for $1 \leq k \leq n$, we will take as a representation of $j\colon P_1 \times \cdots \times P_n \to P_{n+1}$ the representation $f\colon X_1^{\epsilon_1} \times \cdots \times X_n^{\epsilon_n} \to X_{n+1}$ of j taken as the order-preserving map $j\colon P_1^{\epsilon_1} \times \cdots \times P_n^{\epsilon_n} \to P_{n+1}$.

5.5.1. Product of P-spaces. Here we prove that the topological product $X_1 \times \cdots \times X_n$ of a finite number of P-spaces is a P-space. Let $\{X_i\}_{i\in I}$ be an arbitrary family of topological spaces. Let us denote by $\prod_{i\in I} X_i$ the product topological space, i.e., we consider on $\prod_{i\in I} X_i$ the product topology. The fact in the following lemma is well-known and we omit its proof.

LEMMA 5.5.1. *Let $\{X_i\}_{i\in I}$ be a non-empty arbitrary family of T_0-spaces. Then $\prod_{i\in I} X_i$ is a T_0-space.*

Let $\{X_i\}_{i\in I}$ be a family of T_0-spaces. It is straightforward to show directly that the specialization order of the product space $\prod_{i\in I} X_i$ is the order of the product poset corresponding to $\{\langle X_i, \preceq_i\rangle\}_{i\in I}$. That is, for every $\overline{x}, \overline{y} \in \prod_{i\in I} X_i$,

$$\overline{x} \preceq \overline{y} \iff x_i \preceq y_i \text{ for all } i \in I.$$

The following lemma is also probably well known, but we do not have a reference. So, here we give a proof of it.

LEMMA 5.5.2. *If $\{X_i\}_{i\in I}$ is a family of sober topological spaces, then the product space $X = \prod_{i\in I} X_i$ is sober.*

PROOF. Since for each $i \in I$ the space X_i is T_0, it follows that the product topological space X is T_0. Let F be an irreducible closed subset of X. We need to prove that there exists $\overline{x} \in X$ such that $F = {\downarrow}\overline{x}$.

For every $i \in I$ we consider the following closed subset $F_i := \mathrm{cl}(\pi_i[F])$ of X_i, where $\pi_i \colon \prod_{i\in I} X_i \to X_i$ is the projection map. Now let us see that for each $i \in I$, the closed subset F_i is irreducible. Let $i \in I$. Assume that $F_i \subseteq F_1 \cup F_2$ with F_1 and F_2 closed subsets of X_i. So $\pi_i^{-1}[F_i] \subseteq \pi_i^{-1}[F_1] \cup \pi_i^{-1}[F_2]$, that is, $\pi_i^{-1}[\mathrm{cl}(\pi_i[F])] \subseteq \pi_i^{-1}[F_1] \cup \pi_i^{-1}[F_2]$. We note that $F \subseteq \pi_i^{-1}[\pi_i[F]] \subseteq \pi_i^{-1}[\mathrm{cl}(\pi_i[F])]$ and then $F \subseteq \pi_i^{-1}[F_1] \cup \pi_i^{-1}[F_2]$. Also we note that $\pi_i^{-1}[F_1]$ and $\pi_i^{-1}[F_2]$ are closed subsets of X, because the projection maps are all continuous. Hence, since F is an irreducible closed, it follows that $F \subseteq \pi_i^{-1}[F_1]$ or $F \subseteq \pi_i^{-1}[F_2]$. Suppose that $F \subseteq \pi_i^{-1}[F_1]$ (similarly if $F \subseteq \pi_i^{-1}[F_2]$). Thus $\pi_i[F] \subseteq F_1$ and then $F_i = \mathrm{cl}(\pi_i[F]) \subseteq \mathrm{cl}(F_1) = F_1$. Hence each F_i is irreducible. Thus, since for every $i \in I$ X_i is sober, we have that for every $i \in I$ there exists $x_i \in X_i$ such that $F_i = {\downarrow}x_i$. Let $\overline{x} := (x_i)_{i\in I}$. Now we prove that $F = {\downarrow}\overline{x}$. First, let $\overline{y} \in F$ and $i \in I$. So $y_i = \pi_i(\overline{y}) \in \pi_i[F] \subseteq F_i = {\downarrow}x_i$ and then $y_i \preceq x_i$. Thus, we have obtained that $y_i \preceq x_i$ for all $i \in I$ and hence $\overline{y} \preceq \overline{x}$. Then $\overline{y} \in {\downarrow}\overline{x}$. Hence $F \subseteq {\downarrow}\overline{x}$. Now let $\overline{y} \in {\downarrow}\overline{x}$. So $y_i \preceq x_i$ for all $i \in I$. This implies that $y_i \in {\downarrow}x_i = F_i$ for all $i \in I$. Then $y_i \in \mathrm{cl}(\pi_i[F])$ for all $i \in I$. Suppose towards a contradiction that $\overline{y} \notin F$. Then

$\overline{y} \in F^c$. Since F^c is an open subset, it follows that there are $i_1, \ldots, i_n \in I$ and $U_1 \in \mathsf{O}(X_{i_1}), \ldots, U_n \in \mathsf{O}(X_{i_n})$ such that

$$\overline{y} \in \pi_{i_1}^{-1}[U_1] \cap \cdots \cap \pi_{i_n}^{-1}[U_n] \subseteq F^c. \tag{5.1}$$

From (5.1) we note that $y_{i_1} \in U_1, \ldots, y_{i_n} \in U_n$ and $F \subseteq \pi_{i_1}^{-1}[U_1]^c \cup \cdots \cup \pi_{i_n}^{-1}[U_n]^c$. As F is irreducible, there is $k \in \{1, \ldots, n\}$ such that $F \subseteq \pi_{i_k}^{-1}[U_k]^c$. Then $\pi_{i_k}[F] \subseteq \pi_{i_k}[\pi_{i_k}^{-1}[U_k]^c] = \pi_{i_k}[\pi_{i_k}^{-1}[U_k^c]]$ and, since π_{i_k} is an onto map, it follows that $\pi_{i_k}[F] \subseteq U_k^c$. We thus obtain $F_{i_k} = \mathrm{cl}(\pi_{i_k}[F]) \subseteq U_k^c$. As $y_{i_k} \notin U_k^c$, we have that $y_{i_k} \notin F_{i_k}$ and this is a contradiction. Hence $\overline{y} \in F$ and then ${\downarrow}\overline{x} \subseteq F$. Thus $F = {\downarrow}\overline{x}$. Therefore, we can conclude that the product topological space X is sober. $\square$

Next we proceed to characterize the compact open order-filters of finite products of P-spaces.

LEMMA 5.5.3. *Let $X_1, \ldots X_n$ be P-spaces and let $X = X_1 \times \cdots \times X_n$. Then*

$$\mathsf{KOF}(X) = \{{\uparrow}x_1 \times \cdots \times {\uparrow}x_n : x_1 \in \mathsf{Fin}(X_1), \ldots, x_n \in \mathsf{Fin}(X_n)\}.$$

PROOF. Let $x_1 \in \mathsf{Fin}(X_1), \ldots, x_n \in \mathsf{Fin}(X_n)$ and let $\overline{x} = \langle x_1, \ldots, x_n \rangle$. It is easy to see that ${\uparrow}\overline{x} = {\uparrow}x_1 \times \cdots \times {\uparrow}x_n$ and since this last set is an open set of X, $\overline{x} \in \mathsf{Fin}(X)$. Thus, ${\uparrow}x_1 \times \cdots \times {\uparrow}x_n \in \mathsf{KOF}(X)$. Now let $\overline{y} = \langle y_1, \ldots, y_n \rangle \in \mathsf{Fin}(X)$. It is clear that ${\uparrow}\overline{y} = {\uparrow}y_1 \times \cdots \times {\uparrow}y_n$. Moreover, for every $i \in \{1, \ldots, n\}$, $\pi_i[{\uparrow}\overline{y}] = {\uparrow}y_i$; therefore ${\uparrow}y_i$ is open and then $y_i \in \mathsf{Fin}(X_i)$. $\square$

From the proof of the previous lemma we have that for a finite number of P-spaces $X_1, \ldots, X_n$, if $X = X_1 \times \cdots \times X_n$ is their product space, then an element $\overline{x} = \langle x_1, \ldots, x_n \rangle$ of X is finite if and only if each x_i is a finite element in the space X_i.

THEOREM 5.5.4. *Let $X_1, \ldots, X_n$ be P-spaces and let $X = X_1 \times \cdots \times X_n$. Then X is a P-space.*

PROOF. For every $i \in \{1, \ldots, n\}$, $\mathsf{KOF}(X_i) = \{{\uparrow}x : x \in \mathsf{Fin}(X_i)\}$ is a base for X_i. Thus, from previous lemma follows that $\mathsf{KOF}(X)$ is a base for X. Moreover, since the product of sober spaces is sober, it follows that X is sober. Hence X is a P-space. $\square$

The open order-filters of a finite product of P-spaces are characterized as follows:

LEMMA 5.5.5. *Let $X_1, \ldots, X_n$ be P-spaces and let $X = X_1 \times \cdots \times X_n$. Then*

$$\mathsf{OF}(X) = \{F_1 \times \cdots \times F_n : F_1 \in \mathsf{OF}(X_1), \ldots, F_n \in \mathsf{OF}(X_n)\}.$$

Proof. Let $F_1 \in \mathsf{OF}(X_1), \dots, F_n \in \mathsf{OF}(X_n)$. Then, $F_1 \times \cdots \times F_n$ is an open set of X and it is clear that it is an order-filter (see Lemma 3.5.12). Assume now that $F \in \mathsf{OF}(X)$. Let $F_i := \pi_i[F]$ for every $i \in \{1, \dots, n\}$. Then F_i is an open order-filter of X_i. Moreover, using that F is an order-filter we have that $F = F_1 \times \cdots \times F_n$ (see also Lemma 3.5.12). $\square$

Now we move to some considerations on the P-space dual of the direct product of a finite number of posets. Let $P_1, \dots, P_n$ be posets and consider their direct product $P = P_1 \times \cdots \times P_n$, whose order is given coordinatewise. Note that by Lemma 3.5.12 the order-filters of P are the sets of the form $F_1 \times \cdots \times F_n$ where F_i is an order-filter of P_i for every $i \in \{1, \dots, n\}$.

Lemma 5.5.6. *Let $P_1, \dots, P_n$ be posets and let $P = P_1 \times \cdots \times P_n$. Then, the P-spaces X_P and $X_{P_1} \times \cdots \times X_{P_n}$ are homeomorphic.*

Proof. We define the map $f \colon X_{P_1} \times \cdots \times X_{P_n} \to X_P$ as follows:

$$f(\langle F_1, \dots F_n \rangle) = F_1 \times \cdots \times F_n$$

for all $\langle F_1, \dots F_n \rangle \in X_{P_1} \times \cdots \times X_{P_n}$. This map is clearly a bijection. The compact open order-filters of X_P are the sets of the form $\varphi_{\langle a_1, \dots, a_n \rangle}$ with $\langle a_1, \dots, a_n \rangle \in P$ and the compact open order-filters of $X_{P_1} \times \cdots \times X_{P_n}$ are sets of the form $\varphi_{a_1} \times \cdots \times \varphi_{a_n}$ with $\langle a_1, \dots, a_n \rangle \in P$. Let $\langle a_1, \dots, a_n \rangle \in P$. Then it is easy to check that $f^{-1}[\varphi_{\langle a_1, \dots, a_n \rangle}] = \{\langle F_1, \dots F_n \rangle : \langle a_1, \dots, a_n \rangle \in F_1 \times \cdots \times F_n\} = \varphi_{a_1} \times \cdots \times \varphi_{a_n}$, and hence a compact open order-filter of $X_{P_1} \times \cdots \times X_{P_n}$. Similarly, we have $f[\varphi_{a_1} \times \cdots \times \varphi_{a_n}] = \varphi_{\langle a_1, \dots, a_n \rangle}$. Therefore, f is a continuous and open map. We conclude that f is a homeomorphism. $\square$

5.5.2. Quasi-monotone maps. In [**49**] Moshier and Jipsen present a topological representation of n-ary quasioperators. From the definition of n-ary quasioperator clearly follows that they are n-ary quasi-monotone maps. We apply the ideas developed in [**49**] to the poset setting to obtain a topological representation of quasi-monotone maps as maps between P-spaces. Hence, the topological representation of n-ary quasi-monotone maps in the setting of posets that we develop in this section can be considered a generalization of the topological representation for n-ary quasioperators in the setting of lattices due to Moshier and Jipsen.

As we mentioned at the beginning of the section, to represent topologically quasi-monotone maps it is enough to represent order-preserving maps. Let us consider arbitrary posets $P_1, \dots, P_{n+1}$. Any map $j \colon P_1 \times \cdots \times P_n \to P_{n+1}$ that is order-preserving in each coordinate is an order-preserving map from the direct product $P_1 \times \cdots \times P_n$ of the posets $P_1, \dots, P_n$ to the poset P_{n+1}. So, considering also Lemma 5.5.6, it will be enough to represent order-preserving maps between posets.

Let P and Q be posets and let $j\colon P \to Q$ be an order-preserving map. We define the map $f_j\colon X_P \to X_Q$ as follows:

$$f_j(F) = \{b \in Q : (\exists a \in F)(j(a) \leq b)\} = {\uparrow} j[F] \tag{5.2}$$

for every $F \in X_P$. Let us see that f is well defined in the sense that its range is included in X_Q. Clearly $f_j(F)$ is a non-empty up-set. Let us see that it is down-directed. Let $b_1, b_2 \in f_j(F)$. Fix $a_1, a_2 \in F$ such that $j(a_1) \leq b_1$ and $j(a_2) \leq b_2$. Let $c \in F$ be such that $c \leq a_1, a_2$. Then $j(c) \leq j(a_1), j(a_2)$ and thus, $j(c) \leq b_1, b_2$ and $j(c) \in f_j(F)$. Hence, $f_j(F)$ is down-directed and therefore it is an order-filter of Q.

Lemma 5.5.7. *Let P and Q be posets and let $j\colon P \to Q$ be an order-preserving map. Then, the map f_j is continuous.*

Proof. Let U be a basic open subset of the P-space X_Q. We know that, by Lemma 5.1.10, $U = \varphi_b$ for some $b \in Q$. Notice that, by definition of f_j,

$$\begin{aligned} F \in f_j^{-1}[\varphi_b] &\iff f_j(F) \in \varphi_b \\ &\iff b \in f_j(F) \\ &\iff \exists a \in F(j(a) \leq b). \end{aligned}$$

Let $F \in f_j^{-1}[\varphi_b]$. So, there exist $a \in F$ such that $j(a) \leq b$. Clearly, φ_a is an open subset of the P-space X_P and $F \in \varphi_a$. Next, we show that $\varphi_a \subseteq f_j^{-1}[\varphi_b]$. Let $G \in \varphi_a$. So $a \in G$ and, since $j(a) \leq b$, it follows that $G \in f_j^{-1}[\varphi_b]$. Thus, $f_j^{-1}[\varphi_b]$ is an open subset of the P-space X_P and therefore f_j is continuous. $\square$

Let X be a P-space. We define the binary relation $\ll$ on X as follows: for every $x_0, x_1 \in X$,

$$\begin{aligned} x_0 \ll x_1 \iff &\exists F \in \mathsf{OF}(X) \text{ such that } x_1 \in F \text{ and} \\ &(\forall G \in \mathsf{OF}(X))(x_0 \in G \implies F \subseteq G). \end{aligned}$$

We say that a map $f\colon X \to Y$ between P-spaces is *strongly-continuous* if it is continuous and preserves the relation $\ll$, that is,

$$x_0 \ll x_1 \implies f(x_0) \ll f(x_1).$$

The next two lemmas are easy consequences of the definition of the relation $\ll$ and thus we leave the details to the reader.

Lemma 5.5.8. *Let X be a P-space and let $x, y \in X$. Then,*

$$\begin{aligned} x \ll y \iff &\exists a \in \mathsf{Fin}(X) \textit{ such that } y \in {\uparrow} a \textit{ and} \\ &(\forall b \in \mathsf{Fin}(X))(x \in {\uparrow} b \implies {\uparrow} a \subseteq {\uparrow} b). \end{aligned}$$

LEMMA 5.5.9. *Let X be a* P*-space. Then, for every $x \in X$ we have, $x \ll x$ if and only if $x \in \mathsf{Fin}(X)$.*

The following lemma is a useful characterization of the relation $\ll$ in a product of a finite number of P-spaces.

LEMMA 5.5.10. *Let $X_1, \ldots, X_n$ be* P*-spaces and let $X = X_1 \times \cdots \times X_n$ be the space with the product topology. Let $\overline{x}, \overline{y} \in X$. Then,*

$$\overline{x} \ll \overline{y} \iff x_i \ll y_i \text{ for all } i = 1, \ldots, n.$$

PROOF. First, we assume that $\overline{x} \ll \overline{y}$. So, there exists $F \in \mathsf{OF}(X)$ such that $\overline{y} \in F$. Then for every $i = 1, \ldots, n$ there exists $F_i \in \mathsf{OF}(X_i)$ such that $F = F_1 \times \cdots \times F_n$. Let $i \in \{1, \ldots, n\}$. Then $y_i \in F_i$. Let now $G_i \in \mathsf{OF}(X_i)$ be such that $x_i \in G_i$. Fix $G_j \in \mathsf{OF}(X_j)$ such that $x_j \in G_j$ for every $j \in \{1, \ldots n\}$ different from i. Then $\overline{x} \in G = G_1 \times \cdots \times G_n \in \mathsf{OF}(X)$. Therefore, $F \subseteq G$. This implies that $F_i \subseteq G_i$. Hence, $x_i \ll y_i$.

Conversely, we assume that $x_i \ll y_i$ for all $i = 1, \ldots, n$. So, for each $i = 1, \ldots, n$, there exists $a_i \in \mathsf{Fin}(X_i)$ such that $y_i \in {\uparrow} a_i$ and it holds $(\forall b_i \in \mathsf{Fin}(X_i))\big(x_i \in {\uparrow} b_i \Longrightarrow {\uparrow} a_i \subseteq {\uparrow} b_i\big)$. We define $\overline{a} := \langle a_1, \ldots, a_n \rangle \in \mathsf{Fin}(X)$. Notice that $\overline{y} \in {\uparrow}\overline{a}$. Let $\overline{b} \in \mathsf{Fin}(X)$ be such that $\overline{x} \in {\uparrow}\overline{b}$. Then, for every $i = 1, \ldots, n$, $b_i \in \mathsf{Fin}(X_i)$ and $x_i \in {\uparrow} b_i$. Thus, ${\uparrow} a_i \subseteq {\uparrow} b_i$ for all $i = 1, \ldots, n$ and, consequently, ${\uparrow}\overline{a} \subseteq {\uparrow}\overline{b}$. Therefore, $\overline{x} \ll \overline{y}$. □

REMARK 5.5.11. Let P be a poset and consider the dual P-space X_P of P. Then, the relation $\ll$ on X_P is reduced to

$$F \ll G \iff (\exists a \in P)(F \subseteq {\uparrow} a \subseteq G).$$

LEMMA 5.5.12. *Let P and Q be posets and let $j \colon P \to Q$ be an order-preserving map. Then, the map $f_j \colon X_P \to X_Q$ is strongly-continuous.*

PROOF. By Lemma 5.5.7, it only remains to prove that f_j preserves the relation $\ll$. Let $F, G \in X_P$ be such that $F \ll G$. By Remark 5.5.11, there is $a \in P$ such that $F \subseteq {\uparrow} a \subseteq G$. We take $b := j(a) \in Q$. Then, by definition of f_j and since the map j is order-preserving, it follows that

$$f_j(F) \subseteq {\uparrow} b \subseteq f_j(G).$$

Hence, $f_j(\mathcal{F}) \ll f_j(G)$. □

Now, we want to obtain the reverse construction. Let X and Y be P-spaces and let $f \colon X \to Y$ be a strongly-continuous map. The map $j_f \colon \mathsf{KOF}(X) \to \mathsf{KOF}(Y)$ is defined as follows:

$$j_f({\uparrow} a) = {\uparrow} f(a) \tag{5.3}$$

for every $a \in \mathsf{Fin}(X)$. Notice that if $a \in \mathsf{Fin}(X)$, $a \ll a$. Then, given that f preserves the relation $\ll$, we have that $f(a) \ll f(a)$, which implies that $f(a) \in \mathsf{Fin}(Y)$. So, we have that j_f is well defined.

Lemma 5.5.13. *Let X and Y be P-spaces and let $f\colon X \to Y$ be a strongly-continuous map. Then, the map j_f is order-preserving.*

Proof. Let $a_1, a_2 \in \mathsf{Fin}(X)$ be such that $\uparrow a_1 \subseteq \uparrow a_2$. Then, $a_2 \preceq a_1$. Since f is a continuous map, it follows that f is order-preserving (with respect to the specialization order). Then $f(a_2) \preceq f(a_1)$. We thus obtain $\uparrow f(a_1) \subseteq \uparrow f(a_2)$. Hence, $j_f(\uparrow a_1) \subseteq j_f(\uparrow a_2)$. Therefore, j_f is order-preserving. $\square$

We are in a position to show that the application that sends order-preserving maps between posets to strongly-continuous maps between P-spaces $j \mapsto f_j$ and the application that sends strongly-continuous maps between P-spaces to order-preserving maps between posets $f \mapsto f_j$ are, essentially, one inverse of the other.

Let P and Q be posets and let $j\colon P \to Q$ be an order-preserving map and consider the map $f_j\colon X_P \to X_Q$ defined as in (5.2). Then, we have the map $j_{f_j}\colon \mathsf{KOF}(X_P) \to \mathsf{KOF}(X_Q)$ defined as in (5.3). We want to show that the maps j and j_{f_j} are, essentially, the same. Recall from Theorem 5.1.11 that $\varphi_P\colon P \to \mathsf{KOF}(X_P)$ is an order-isomorphism. So, we should prove that $j_{f_j}(\varphi_P(a)) = \varphi_Q(j(a))$ for all $a \in P$. That is, that the the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, "j"] \arrow[d, "\varphi_P"'] & Q \arrow[d, "\varphi_Q"] \\ \mathsf{KOF}(X_P) \arrow[r, "j_{f_j}"] & \mathsf{KOF}(X_Q) \end{tikzcd}$$

Let $a \in P$. First we observe, by definition of f_j, that $f_j(a) = \uparrow j(a)$ and, moreover, $j_{f_j}(\varphi_P(a)) = \uparrow f_j(\uparrow a)$. Then, for every $F \in \mathsf{Fi}_{\mathsf{or}}(P)$ we have

$$\begin{aligned} F \in j_{f_j}(\varphi_P(a)) &\iff f_j(\uparrow a) \subseteq F \\ &\iff j(a) \in F \\ &\iff F \in \varphi_Q(j(a)). \end{aligned}$$

Reciprocally, we now consider a strongly-continuous map $f\colon X \to Y$ from a P-spaces X to a P-space Y. So, we have the map $j_f\colon P_X \to P_Y$ given by (5.3). Then, we consider the strongly-continuous map

$$f_{j_f}\colon \mathsf{Fi}_{\mathsf{or}}(P_X) \to \mathsf{Fi}_{\mathsf{or}}(P_Y)$$

and we prove that f and f_{j_f} are, essentially, the same maps. That is, we prove that $f_{j_f}(\theta_X(x)) = \theta_Y(f(x))$ for all $x \in X$, where $\theta_X : X \to \mathsf{Fi_{or}}(P_X)$ is the homeomorphism given by Theorem 5.2.4 and similarly for θ_Y. In other words, we prove that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\theta_X"'] & Y \arrow[d, "\theta_Y"] \\ \mathsf{Fi_{or}}(P_X) \arrow[r, "f_{j_f}"] & \mathsf{Fi_{or}}(P_Y) \end{tikzcd}$$

Let $x \in X$ and let $\uparrow y \in \mathsf{KOF}(Y)$. First, we assume $\uparrow y \in f_{j_f}(\theta_X(x))$. So, there exists $\uparrow z \in \theta_X(x)$ such that $j_f(\uparrow z) \subseteq \uparrow y$. Then, $\uparrow f(z) \subseteq \uparrow y$, which implies that $f(z) \in \uparrow y$. Since $x \in \uparrow z$ and f is order-preserving, we have that $f(z) \preceq f(x)$. Thus, $f(x) \in \uparrow y$. Hence $\uparrow y \in \theta_Y(f(x))$. Now we assume that $\uparrow y \in \theta_Y(f(x))$. So, $f(x) \in \uparrow y$. Then $x \in f^{-1}[\uparrow y]$ and thus there exists $z \in \mathsf{Fin}(X)$ such that $x \in \uparrow z \subseteq f^{-1}[\uparrow y]$. So, $f[\uparrow z] \subseteq \uparrow y$. Then, $\uparrow z \in \theta_X(x)$ and $f(z) \in \uparrow y$. Therefore $\uparrow y \in f_{j_f}(\theta_X(x))$.

Let $P_1, \dots, P_{n+1}$ be posets and let $j : P_1 \times \cdots \times P_n \to P_{n+1}$ be a map such that is order-preserving in each coordinate. Let P be the direct product of $P_1, \dots, P_n$. Recall that the order-filters of P are the sets of the form $F_1 \times \cdots \times F_n$ where for every $i = 1, \dots, n$, $F_i \in \mathsf{Fi_{or}}(P_i)$. So, for every $F_1 \times \cdots \times F_n \in \mathsf{Fi_{or}}(P)$

$$f_j(F_1 \times \cdots \times F_n) = \{a \in P_{n+1} : \exists b_1 \in F_1, \dots, \exists b_n \in F_n \\ \text{such that } j(\langle b_1, \dots, b_n \rangle) \leq a\}.$$

Thus we can obtain a map $\overline{f_j} : X_{P_1} \times \cdots \times X_{P_n} \to X_{P_{n+1}}$ defined by

$$\overline{f_j}(\langle F_1, \dots, F_n \rangle) = \{a \in P_{n+1} : \exists b_1 \in F_1, \dots, \exists b_n \in F_n \\ \text{such that } j(\langle b_1, \dots, b_n \rangle) \leq a\}.$$

This map is, thanks to the homeomorphism between X_P and the product space $X_{P_1} \times \cdots \times X_{P_n}$, strongly-continuous.

Let now $X_1, \dots, X_{n+1}$ be P-spaces and let $f : X_1 \times \cdots \times X_n \to X_{n+1}$ be a strongly-continuous map. Let P be the poset of the compact open order-filters of the product space $X_1 \times \cdots \times X_n$. This poset is isomorphic to the direct product $\mathsf{KOF}(X_1) \times \cdots \times \mathsf{KOF}(X_n)$. We have the order-preserving map $j_f : P \to \mathsf{KOF}(X_{n+1})$. Using the isomorphism we obtain the order-preserving in each coordinate map $\overline{j_f} : \mathsf{KOF}(X_1) \times \cdots \times \mathsf{KOF}(X_n) \to \mathsf{KOF}(X_{n+1})$ given by

$$\overline{j_f}(\langle \uparrow a_1, \dots, \uparrow a_n \rangle) = \uparrow f(\langle a_1, \dots, a_n \rangle)$$

for every $a_i \in \mathsf{Fin}(X_i)$ with $i = 1, \dots, n$.

5.6. The extension of a strongly-continuous map between P-spaces to their lattices of F-saturated sets

Let X and Y be P-spaces. We show in this short section how to extend a strongly-continuous map f from X to Y to a map from $\mathsf{Fsat}(X)$ to $\mathsf{Fsat}(Y)$ in such a way that the image of $\uparrow x$, with $x \in X$, is $\uparrow f(x)$.

We will exploit the fact that according to the results on Section 5.3 for every P-space X, $\mathsf{Fsat}(X)$ is (up to isomorphism) the canonical extension of $\mathsf{KOF}(X)$, with the identity as the embedding map, and the theory developed in [**17**] of the extension of maps between posets to their canonical extensions.

Let $f\colon X \to Y$ be a strongly-continuous map. We know that $j_f\colon \mathsf{KOF}(X) \to \mathsf{KOF}(Y)$ is order-preserving. Thus, using the results in [**17**], it has two extensions $(j_f)^\sigma$ and $(j_f)^\pi$ from the canonical extension $\mathsf{Fsat}(X)$ of $\mathsf{KOF}(X)$ to the canonical extension $\mathsf{Fsat}(Y)$ of $\mathsf{KOF}(Y)$. We provide a description in our setting of the maps $(j_f)^\sigma$ and $(j_f)^\pi$.

First, let us characterize for a given P-space X the open and closed elements of $\mathsf{Fsat}(X)$ taken as the canonical extension of $\mathsf{KOF}(X)$. According to [**17**] a set $U \in \mathsf{Fsat}(X)$ is a closed element if there is an order-filter $\mathcal{F}$ of the poset $\langle \mathsf{KOF}(X), \subseteq\rangle$ such that $U = \bigcap \mathcal{F}$. And it is an open element if there is an order-ideal $\mathcal{I}$ of the poset $\langle \mathsf{KOF}(X), \subseteq\rangle$ such that $U = \bigvee \mathcal{I}$.

Note that $\mathcal{I}$ is an order-ideal of $\langle \mathsf{KOF}(X), \subseteq\rangle$ if and only if there is $F_{\mathcal{I}} \in \mathsf{OF}(X)$ such that $\mathcal{I} = \{\uparrow a : a \in F_{\mathcal{I}} \cap \mathsf{Fin}(X)\}$ and in $\mathsf{Fsat}(X)$, $\bigvee \mathcal{I} = F_{\mathcal{I}}$. Thus the open elements of $\mathsf{Fsat}(X)$ are the open order-filters of X. Now note that $\mathcal{F}$ is an order-filter of $\langle \mathsf{KOF}(X), \subseteq\rangle$ if and only if there is an order-ideal I of $\mathsf{Fin}(X)$ such that $\mathcal{F} = \{\uparrow a : a \in I\}$.

Thus, $U \in \mathsf{Fsat}(X)$ is an open element if and only if there exists $F \in \mathsf{OF}(X)$ such that $U = \bigvee\{\uparrow a : a \in F \cap \mathsf{Fin}(X)\}$ and it is a closed element if and only if there exists an order-ideal I of $\mathsf{Fin}(X)$ such that $U = \bigcap\{\uparrow a : a \in I\}$.

LEMMA 5.6.1. *Let X, Y be P-spaces and $f\colon X \to Y$ a strongly-continuous map. Then for every $U \in \mathsf{Fsat}(X)$,*

(1) $(j_f)^\pi(U) = \bigcap\{\bigvee\{\uparrow f(a) :\ a \in F \cap \mathsf{Fin}(X)\} :\ F \in \mathsf{OF}(X)$ *and* $U \subseteq F\}$.
(2) $(j_f)^\sigma(U) = \bigvee\{\bigcap\{\uparrow f(a) :\ a \in I\} :\ I \in \mathsf{Id_{or}}(\mathsf{Fin}(X))$ *and* $\bigcap\{\uparrow a :\ a \in I\} \subseteq U\}$.

PROOF. (1) By definition of the map $(j_f)^\pi\colon \mathsf{Fsat}(X) \to \mathsf{Fsat}(Y)$ ([**17**, Definition 3.2]), we have

$$(j_f)^\pi(U) = \bigcap\{\bigvee\{j_f(\uparrow a) :\ \uparrow a \in \mathcal{I}\} :\ \mathcal{I} \in \mathsf{Id_{or}}(\mathsf{KOF}(X)) \text{ and } U \subseteq \bigvee \mathcal{I}\}.$$

Hence,

$$(j_f)^\pi(U) = \bigcap\{\bigvee\{\uparrow f(a) :\ a \in F \cap \mathsf{Fin}(X)\} :\ F \in \mathsf{OF}(X) \text{ and } U \subseteq F\}.$$

Similarly, by the definition of the map $(j_f)^\sigma\colon \mathsf{Fsat}(X) \to \mathsf{Fsat}(Y)$ ([**17**, Definition 3.2]), we have

$$(j_f)^\sigma(U) = \bigvee\{\bigcap\{j_f(\uparrow a) : \ \uparrow a \in \mathcal{F}\} : \ \mathcal{F} \in \mathsf{Fi_{or}}(\mathsf{KOF}(X)) \text{ and } \bigcap \mathcal{F} \subseteq U\}.$$

Thus,

$$(j_f)^\sigma(U) = \bigvee\{\bigcap\{\uparrow f(a) : \ a \in I\} : I \in \mathsf{Id_{or}}(\mathsf{Fin}(X)) \text{ and } \bigcap\{\uparrow a : \ a \in I\} \subseteq U\}.$$

$\square$

5.7. Meet-semilattices and maps that preserve meet

In [**48**] Moshier and Jipsen develop a topological duality for meet-semilattices with top element of which our duality for posets is a generalization. But our duality also provides a duality for meet-semilattices in general. We proceed to expound this duality and see how it specifies to Moshier and Jipsen's.

We consider the category of meet-semilattices (as posets) and meet-homomorphisms. We denote this category by $\mathbb{MSL}$. Recall that $\mathbb{PO}$ denotes the category of all posets and all order-preserving maps such that the inverse image of order-filters are order-filters. It is not hard to check that the category $\mathbb{MSL}$ is a full subcategory of $\mathbb{PO}$.

We say that a topological space X is an *almost HMS-space*, *AHMS-space* for short, if it satisfies the following conditions:

(1) X is sober;
(2) $\mathsf{KOF}(X)$ forms a base;
(3) $\mathsf{KOF}(X)$ is closed under intersection (i.e., if $U, V \in \mathsf{KOF}(X)$, then $U \cap V \in \mathsf{KOF}(X)$).

This notion of almost HMS-space is essentially due to Moshier and Jipsen [**48**]. Since they work with meet-semilattices with top element, they require in addition $\mathsf{KOF}(X)$ to be closed under intersections of arbitrary finite subsets of $\mathsf{KOF}(X)$ or, equivalently, that X has a least element with respect to specialization order. Moshier and Jipsen call their spaces *HMS-spaces* in honor to Hoffman, Mislove and Stralka.

It is clear that every almost HMS-space is a P-space. Thus we may consider the full subcategory $\mathbb{AHMS}$ of $\mathbb{PS}$ with objects the almost HMS-spaces (and hence with morphisms the F-continuous maps between them). The full subcategory $\mathbb{HMS}$ of $\mathbb{AHMS}$ of the HMS-spaces is the category that Moshier and Jipsen prove in [**48**] to be dually equivalent to the category of meet-semilattices with top element and meet-preserving maps that also preserve the top element.

If we apply the duality for posets given in Theorem 5.2.5 to the full subcategory of meet-semilattices we obtain, taking into account Remark 5.1.12, that this category is dual to the category $\mathbb{AHMS}$ and if we apply that theorem to the category of meet-semilattices with top element we obtain, taking again into account Remark 5.1.12, the duality given by Moshier and Jipsen between that category and the category of HMS-spaces.

Now, we restrict our attention to those maps $j\colon M_1 \times \cdots \times M_n \to M_{n+1}$, where $M_1, \ldots, M_{n+1}$ are meet-semilattices, that are meet-preserving in each coordinate. We apply the topological representation presented in Subsection 5.5.2 to the map j. Firstly, observe that if M is a meet-semilattice and F_1, F_2 are filters of M, then the filter $F_1 \vee F_2 := \{a \in M : \ b_1 \wedge b_1 \leq a$ for some $b_1 \in F_1$ and $b_2 \in F_2\}$ is the least upper bound of F_1 and F_2 in $\mathsf{Fi}(M)$ with respect to inclusion order. Hence, using the duality between meet-semilattices and almost HMS-spaces, we have that every almost HMS-space X is a join-semilattice with respect to specialization order.

Lemma 5.7.1. *Let $M_1, \ldots, M_{n+1}$ be meet-semilattices. The maps $j\colon M_1 \times \cdots \times M_n \to M_{n+1}$ that preserve meets in each coordinate are topologically represented by the maps $f\colon X_{M_1} \times \cdots \times X_{M_n} \to X_{M_{n+1}}$ that are strongly-continuous and preserve joins in each coordinate (w.r.t. the specialization order).*

Proof. Let $M_1, \ldots, M_{n+1}$ be meet-semilattices and let $j\colon M_1 \times \cdots \times M_n \to M_{n+1}$ be a map that preserves meets in each coordinate. It is clear that j is order-preserving. We thus define the map $\overline{f_j}\colon X_{M_1} \times \cdots \times X_{M_n} \to X_{M_{n+1}}$ (where X_{M_i} is the dual almost HMS-space of the meet-semilattice M_i) as in Subsection 5.5.2. It only remains to prove that $\overline{f_j}$ preserves joins in each coordinate. Let $H, G \in \mathsf{Fi}(M_1)$ and $F_i \in \mathsf{Fi}(M_i)$ for every $i = 2, \ldots, n$. We need to prove that (with an analogous argument can be proved for the other coordinates)

$$\overline{f_j}(\langle H \vee G, F_2, \ldots, F_n\rangle) = \overline{f_j}(\langle H, F_2, \ldots, F_n\rangle) \vee f_j(\langle G, F_2, \ldots, F_n\rangle).$$

Let $a \in \overline{f_j}(\langle H \vee G, F_2, \ldots, F_n\rangle)$. So, $j(\langle a_1, b_2, \ldots, b_n\rangle) \leq a$ for some $a_1 \in H \vee G$, $b_2 \in F_2, \ldots,\ b_n \in F_n$. Then, there exist $h \in H$ and $g \in G$ such that $h \wedge g \leq a_1$. Thus, we have

$$j(\langle h \wedge g, b_2, \ldots, b_n\rangle) \leq j(\langle a_1, b_2, \ldots, b_n\rangle) \leq a,$$

and since j preserve meets in each coordinate

$$j(\langle h, b_2, \ldots, b_n\rangle) \wedge j(\langle g, b_2, \ldots, b_n\rangle) \leq a.$$

Moreover, it is clear that

$$j(\langle h, b_2, \ldots, b_n\rangle) \in \overline{f_j}(\langle H, F_2, \ldots, F_n\rangle)$$

and

$$j(\langle g, b_2, \ldots, b_n\rangle) \in \overline{f_j}(\langle G, F_2, \ldots, F_n\rangle).$$

Hence,

$$a \in \overline{f_j}(\langle H, F_2, \dots, F_n\rangle) \vee \overline{f_j}(\langle G, F_2, \dots, F_n\rangle).$$

On the other hand, if $a \in \overline{f_j}(\langle H, F_2, \dots, F_n\rangle) \vee \overline{f_j}(\langle G, F_2, \dots, F_n\rangle)$ then, there exist $h \in \overline{f_j}(\langle H, F_2, \dots, F_n\rangle)$ and $g \in \overline{f_j}(\langle G, F_2, \dots, F_n\rangle)$ such that $h \wedge g \leq a$. Thus, by definition of $\overline{f_j}$, we obtain $j(\langle h_1, b_2, \dots, b_n\rangle) \leq h$ for some $h_1 \in H$ and $b_i \in F_i$ con $i = 2, \dots, n$ and $j(\langle g_1, b'_2, \dots, b'_n\rangle) \leq g$ for some $g_1 \in G$ and $b'_i \in F_i$ for $i = 2, \dots, n$. Now, for each $2 \leq i \leq n$, we put $c_i := b_i \wedge b'_i$. We note that $c_i \in F_i$ for all $i = 2, \dots, n$. Then,

$$j(\langle h_1, c_2, \dots, c_n\rangle) \leq h \text{ and } j(\langle g_1, c_2, \dots, c_n\rangle) \leq g.$$

So, from the previous inequalities and since the map j preserves meet in each argument we have

$$j(\langle h_1 \wedge g_1, c_2, \dots, c_n\rangle) \leq h \wedge g \leq a.$$

This implies

$$a \in \overline{f_j}(\langle H \vee G, F_2, \dots, F_n\rangle).$$

Now, let $X_1, \dots, X_{n+1}$ be almost HMS-spaces and let $f \colon X_1 \times \cdots \times X_n \to X_{n+1}$ be a strongly-continuous map such that preserve joins in each coordinate (w.r.t. specialization order). We consider the map $\overline{j_f} \colon \mathsf{KOF}(X_1) \times \cdots \times \mathsf{KOF}(X_n) \to \mathsf{KOF}(X_{n+1})$. Let $\uparrow a, \uparrow b \in \mathsf{KOF}(X_1)$, $\uparrow a_2 \in \mathsf{KOF}(X_2)$, $\dots$, $\uparrow a_n \in \mathsf{KOF}(X_n)$. Then,

$$\begin{aligned}
\overline{f_j}(\langle \uparrow a \cap \uparrow b, \uparrow a_2, \dots, \uparrow a_n\rangle) &= \uparrow f(\langle a \vee b, a_2, \dots, a_n\rangle)\\
&= \uparrow (f(\langle a, a_2, \dots, a_n\rangle) \vee f(\langle b, a_2, \dots, a_n\rangle))\\
&= \uparrow f(\langle a, a_2, \dots, a_n\rangle) \cap \uparrow f(\langle b, a_2, \dots, a_n\rangle).
\end{aligned}$$

Similarly for the rest of the coordinates. It follows that $\overline{f_j}$ preserve meets in each coordinate. $\square$

Summary and conclusions

In this dissertation we aimed to show that the classical topological dualities for bounded distributive lattices by Stone and Priestley can be generalized to partially ordered sets having a distributivity condition. Thus we developed two topological dualities for meet-order distributive posets: spectral-style and Priestley-style. We have used these dualities to obtain two new completions of mo-distributive partially ordered sets in an analogous way that the canonical extension for distributive lattices was obtained. Moreover we aimed to show a topological duality for the class of all partially ordered sets.

We have studied the notions of order-filter (order-ideal), Frink-filter (Frink-ideal) and meet-filter (join-ideal) proving that the collection of Frink-filters (Frink-ideals) and the collection of meet-filters (join-ideals) are both algebraic closure systems and the collection of order-filters (order-ideals) is not necessarily a closure system. We have considered the notion of meet-order distributivity on posets due David and Erné [**16**] and we saw that this condition (as was also proved in [**16**]) is characterized by the condition that the lattice of Frink-filters is distributive. We saw also that the notions of Frink-filter (Frink-ideal) and meet-filter (join-ideal) coincide on mo-distributive posets. We have shown that a poset is mo-distributive if and only if each pair of different points can be separated by means of prime Frink-filters. We have defined the notions of inf-homomorphism and sup-homomorphism and we studied their properties. We proved that a map between posets is an inf-homomorphism (sup-homomorphism) if and only if the inverse image of Frink-filters (Frink-ideals) are Frink-filters (Frink-ideals). Moreover we have proved that on mo-distributive (jo-distributive) posets the inf-homomorphisms (sup-homomorphisms) coincide with the maps that preserve existing finite meets (joins) and preserving top (bottom), if it exists. We have studied two extensions for mo-distributive posets: the distributive meet-semilattice envelope and the distributive lattice envelope. In both extensions the poset is embedded in a very nice way. For instance, we proved that the Frink-filters, prime Frink-filters and s-optimal Frink-filters of a mo-distributive poset respectively correspond to the filters, prime filters and optimal filters of its distributive meet-semilattice envelope; and the s-optimal Frink-filters correspond to the prime filters of its distributive lattice envelope.

Once we have obtained the algebraic tools, we focused on developing two topological dualities for mo-distributive posets and one topological duality for the class of all posets. The notion of prime Frink-filter on mo-distributive posets is used to build up the spectral-style duality and the notion of s-optimal Frink-filter on mo-distributive posets is the one used to build up the Priestley-style duality. Both dualities allowed us to obtain two different completions of mo-distributive posets: the Frink completion and the strong Frink completion. Moreover they are also different from the canonical extension (taken as in [**17**]). The Frink completion and the strong Frink completion of a mo-distributive poset are both completely distributive algebraic lattices. These properties on the completions are very interesting and may be useful to study the extensions of certain order-preserving maps between mo-distributive posets. An analogous situation was considered in the setting of distributive lattice [**28, 29, 30**]. A future work, in this line, would be to try to apply the theory of the σ- and π-extensions (as in [**17**]) for order-preserving maps between mo-distributive posets to their Frink completion and/or strong Frink completion and to study their properties. Then it would be possible to study what equations and/or inequalities that hold in certain monotone poset expansions are preserved on its completion. A more ambitious project would use these new completions to find possible complete relational semantics following the line of work developed in [**17**].

The order-filters are used to build up the topological duality for the class of all posets. The dual topological spaces to the posets are those characterized as the sober spaces in which the family of all compact open order-filters form a base for the space. We have used this duality to prove the existence of the canonical extension ([**17**]) of a poset. We have characterized topologically the quasi-monotone maps.

Resumen y conclusiones

En esta tesis doctoral nos propusimos mostrar que las dualidades topológicas clásicas para retículos distributivos acotados debidas a Stone y a Priestley se pueden generalizar a conjuntos parcialmente ordenados que tienen una condición de distributividad. Así, hemos desarrollado dos dualidades topológicas para posets meet-orden distributivos: una estilo espectral y una estilo Priestley. Hemos utilizado estas dualidades para obtener dos nuevas compleciones de conjuntos parcialmente ordenados mo-distributivos de una manera análoga en la que se obtuvo la extensión canónica para retículos distributivos. Además nos propusimos mostrar una dualidad topológica para la clase de todos los conjuntos parcialmente ordenados.

Hemos estudiado las nociones de filtro de orden (ideal de orden), filtro de Frink (ideal de Frink) y filtro inferior (ideal superior) probando que la colección de filtros de Frink (ideales de Frink) y la colección de filtros inferiores (ideales superiores) son las dos sistemas clausuras algebraicos y que la colección de filtros de orden (ideales de orden) no es necesariamente un sistema clausura. Hemos considerado la noción de meet-orden distributividad para posets debida a David y Erné [**16**] y vimos que esta condición (como también se demostró en [**16**]) es caracterizada por la condición de que el retículo de filtros de Frink es distributivo. Vimos también que las nociones de filtro de Frink (ideal de Frink) y filtro inferior (ideal superior) coinciden sobre los posets mo-distributivos. Hemos demostrado que un conjunto parcialmente ordenado es mo-distributivo si y sólo si cada par de puntos diferentes se pueden separar por medio de filtros de Frink primos. Hemos definido las nociones de inf-homomorfismo y sup-homomorfismo y estudiado sus propiedades. Hemos demostrado que una función entre posets es un inf-homomorfismo (sup-homomorfismo) si y sólo si la imagen inversa de filtros de Frink (ideales de Frink) son filtros de Frink (ideales de Frink). Por otra parte, hemos demostrado que sobre posets mo-distributivos (jo-distributivos) los inf-homomorfismos (sup-homomorfismos) coinciden con las aplicaciones que preservan ínfimos (supremos) finitos existentes y preservan último elemento (primer elemento), si existe. Hemos estudiado dos extensiones para conjunto parcialmente ordenado mo-distributivos: la envoltura meet-semiretículo distributiva y la envoltura retículo distributiva. En ambos extensiones el poset es incrustado en una forma muy buena. Por ejemplo, hemos podido comprobar que los filtros de Frink, los filtros de Frink primos y los filtros de Frink s-óptimos de un conjunto parcialmente ordenado mo-distributivo corresponden respectivamente a los filtros, los filtros primos y filtros óptimos de su envoltura meet-semiretículo distributiva; y los filtros de Frink s-óptimos corresponden a los filtros primos de su envoltura retículo distributiva.

Una vez que hemos obtenido las herramientas algebraicas, nos centramos en desarrollar dos dualidades topológicas para conjuntos parcialmente ordenados mo-distributivos y una dualidad topológica para la clase de todos los conjuntos parcialmente ordenados. La noción de filtro de Frink primo sobre posets mo-distributivos se utiliza para construir la dualidad estilo espectral y la noción de filtro de Frink s-óptimo sobre posets mo-distributivos es la que se utiliza para construir la dualidad estilo Priestley. Ambas dualidades nos permitieron obtener dos compleciones diferentes de posets mo-distributivos: la compleción Frink y la compleción Frink fuerte. Además, ellas son también diferentes de la extensión canónica (tomada como en [**17**]). La compleción Frink y la compleción Frink fuerte de un conjunto parcialmente ordenado mo-distributivo son ambas retículos algebraicos completamente distributivos. Estas propiedades sobre las compleciones son interesantes y pueden ser útiles para estudiar las extensiones de ciertas aplicaciones que preservan orden entre posets mo-distributivos. Una situación análoga se consideró en el marco de retículo distributivo [**28, 29, 30**]. Un trabajo futuro, en esta línea, sería tratar de aplicar la teoría de las σ y π-extensiones (como en [**17**]) para aplicaciones que preservan orden entre posets mo-distributivos a su compleción Frink y/o su compleción Frink fuerte y estudiar sus propiedades. Entonces sería posible estudiar qué ecuaciones y/o desigualdades que se cumplen en ciertas expansiones monótonas de posets se conservan en su compleción. Un proyecto más ambicioso sería utilizar estas nuevas compleciones para encontrar posibles semánticas relacionales completas siguiendo la línea de Dunn et al. [**17**].

Los filtros de orden se utilizan para construir la dualidad topológica para la clase de todos los conjuntos parcialmente ordenados. Los espacios topológicos duales a los posets son aquellos caracterizados como los espacios sober en los cuales la familia de todos los filtros de orden abiertos y compactos forman una base para el espacio. Hemos utilizado esta dualidad para probar la existencia de la extensión canónica ([**17**]) de un conjunto parcialmente ordenado. También hemos caracterizado topológicamente las aplicaciones cuasi-monótonas.

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$\mathsf{Id}_{\mathsf{F}}^{\mathsf{pr}}(P)$	prime Frink-ideals of a poset P	27
$\mathsf{Opt}(P)$	optimal Frink-filters of a poset P	29
$\mathsf{Fi}_{\mathsf{m}}(P)$	meet-filters of a poset P	31
$\mathsf{Id}_{\mathsf{j}}(P)$	join-ideals of a poset P	31
$M(P)$	the distributive meet-semilattice envelope of a mo-distributive poset P	52
$\mathsf{Id}_{\mathsf{sF}}(P)$	strong Frink-ideals of a poset P	59
$\mathsf{Opt}_{\mathsf{s}}(P)$	s-optimal Frink-filters of a poset P	61
$D(P)$	the distributive lattice envelope of a mo-distributive poset P	76
$\mathbf{X}(P)$	dual DP-space of a mo-distributive poset P	85
$P_{\mathbf{X}}$	dual mo-distributive poset of a DP-space $\mathbf{X}$	86
$\mathcal{B}(\mathbf{X})$	base associated to the DP-space $\mathbf{X}$	85
P^{F}	Frink completion of a poset P	113
P^{σ}	canonical extension of a poset P	113
$\mathsf{CLUp}(X)$	clopen up-sets of an ordered topological space X	125
P_{*}	dual poset Priestley space of a bounded mo-distributive poset P	129
$P_{\mathcal{B}}$	dual bounded mo-distributive poset of a poset Priestley space $X = \langle X, \tau, \leq, \mathcal{B}\rangle$	131
P^{sF}	strong Frink completion of a poset P	160
$\mathsf{KOF}(X)$	compact open order-filters of a T_0-space X	168
$\mathsf{Fin}(X)$	finite elements of a T_0-space X	168
X_P	dual P-space of a poset P	171
$\mathsf{OF}(X)$	open order-filters of a T_0-space X	175
$\mathsf{Fsat}(X)$	F-saturated subsets of a T_0-space X	175
$\mathrm{Fsat}(A)$	F-saturated subset generated by A	175

Acronyms (categories)

Bibliography

[1] Allwein, G., Dunn, J.M.: *Kripke models for linear logic*. The Journal of Symbolic Logic **58**, 514–545 (1993).

[2] Balbes, R.: *A representation theory for prime and implicative semilattices*. Trans. Amer. Math. Soc. **136**, 261–267 (1969).

[3] Balbes, R., Dwinger, P.: Distributive Lattices. University of Missouri Press, Columbia (1975).

[4] Bezhanishvili, G., Jansana, R.: *Duality for distributive and implicative semilattices*. Manuscript (2008).

[5] Bezhanishvili, G., Jansana, R.: *Priestley style duality for distributive meet-semilattices*. Studia Logica **98**, 83–123 (2011).

[6] Birkhoff, G.: Lattice Theory, 3rd ed. American Mathematical Society Colloquium Publications **25**, Providence (1967).

[7] Burris, S., Sankappanavar, H. P.: A course in Universal Algebra. The Millennium edition (2000).

[8] Celani, S. A.: *Topological representation of distributive semilattices*. Scientiae Mathematicae Japonicae **8**, 561–572 (2003).

[9] Celani, S. A., Calomino, I.: *Some remarks on distributive semilattices*. Comment.Math.Univ.Carolin. **54**, 407–428 (2013).

[10] Celani, S. A., Jansana, R.: *On the free implicative semilattice extension of a Hilbert algebra*. Math. Log. Quart **58**, 188–207 (2012).

[11] Chajda, I., Halaš, R., Kühr, J.: Semilattice structures. Heldermann Velarg, Germany (2007).

[12] Cornish, W. H.: *On H. Priestley's dual of the category of bounded distributive lattices*. Matematički Vesnik **12**, 329–332 (1975).

[13] Cornish, W. H., Hickman, R. C.: *Weakly distributive semilattices*. Acta Math. Acad. Sci. Hungar. **32**, 5–16 (1978).

[14] Davey, B. A., Haviar, M., Priestley, H. A.: *Boolean topological distributive lattices and canonical extensions*. Applied Categorical Structures **15**, 225–241 (2007).

[15] Davey, B. A., Priestley, H. A.: Introduction to lattices and order, 2nd edn. Cambridge University Press, New York (2002).

[16] David, E., Erné, M.: *Ideal completion and Stone representation of ideal-distributive ordered sets*. Topology and its Application **44**, 95–113 (1992).

[17] Dunn, J.M., Gehrke, M., Palmigiano, A.: *Canonical extensions and relational completeness of some substructural logics*. The Journal of Symbolic Logic **70**, 713–740 (2005).

[18] Engelking, R.: General Topology, Sigma series in Pure Mathematics vol. 6. Heldermann Verlag, Berlin (1989).

[19] Erné, M.: Adjunctions and standard constructions for posets. In: Contributions to General Algebra (Klagenfurt, 1982), **2**, pp. 77–106. Hölder-Pichler-Tempsky, Vienna (1983).

[20] Erné, M.: *The Dedekind-MacNeille completion as a reflector*. Order **8**, 159–173 (1991).

[21] Erné, M.: Adjunctions and Galois connections: Origins, history and development. In: Denecke, K. et al. (eds.) Galois connections and Applications, pp. 1–138. Kluwer (2004).

[22] Esakia, L.: *Topological Kripke models*. Soviet Mathematics Doklady **15**, 147–151 (1974).

[23] Esteban, M.: Duality Theory and Abstract Algebraic Logic. PhD Tesis, University of Barcelona (2013).

[24] Frink, O.: *Ideals in partially ordered sets*. Amer. Math. Monthly **61**, 223–234 (1954).

[25] Gehrke, M.: *Generalized Kripke frames*. Studia Logica **84**, 241–275 (2006).

[26] Gehrke, M., Harding, J.: *Bounded lattice expansions*. Journal of Algebra **238**, 345–371 (2001).

[27] Gehrke, M., Jansana, R., Palmigiano, A.: Δ_1-*completions of a poset*. Order **30**, 39–64 (2013).

[28] Gehrke, M., Jónsson, B.: *Bounded distributive lattices with operators*. Math. Japon. **40**, 207–215 (1994).

[29] Gehrke, M., Jónsson, B.: *Monotone bounded distributive lattice expansions*. Math. Japon. **52**, 197–213 (2000).

[30] Gehrke, M., Jónsson, B.: *Bounded distributive lattice expansions*. Math. Scand. **94**, 13–45 (2004).

[31] Gehrke, M., Priestley, H. A.: *Dualities for double quasioperator algebras via their canonical extensions*. Studia Logica **86**, 31–68 (2007).

[32] Gehrke, M., Priestley, H.A.: *Canonical extensions and completions of posets and lattices*. Report on Mathematical Logic **43**, 133–152 (2008).

[33] Goldblatt, R.: *Varieties of complex algebras*. Ann. Pure App. Logic **44**, 173–242 (1989).

[34] Grätzer, G.: General Lattice Theory, 2nd edn. Birkhäuser Verlag (1998).

[35] Hartonas, C., Dunn, J.M.: *Stone duality for lattices*. Algebra Universalis **37**, 391–401 (1997).

[36] Hartung, G.: *A topological representation of lattices*. Algebra Universalis **29**, 273–299 (1992).

[37] Hickman, R. C.: *Mildly distributive semilattices*. J. Austral. Math. Soc. **36**, 287–315 (1984).

[38] Hickman, R. C., Monro, G. P.: *Distributive partially ordered sets*. Fundamenta Mathematicae **120**, 151–166 (1984).

[39] Hofmann, K.H.: The low separation axioms T_0 and T_1. In: Hart, K.P., Nagata, J.I., Vaughan, J.E. (eds.) Encyclopedia of General Topology, pp. 155–157. Elsevier/North-Holland (2004).

[40] Johnstone, P. T.: Stone spaces. Cambridge University Press, Cambridge (1982).

[41] Jónsson, B., Tarski, A.: *Boolean algebras with operators, I*. Amer. J. Math. **73**, 891–939 (1951).

[42] Jónsson, B., Tarski, A.: *Boolean algebras with operators, II*. Amer. J. Math. **74**, 127–162 (1952).

[43] Kamide, N.: *Kripke semantics for modal substructural logics*. Journal of Logic, Language and Information **11**, 453–470 (2002).

[44] MacCaull, W.: *Relational semantics and a relational proof system for full Lambek calculus*. The Journal of Symbolic Logic **63**, 623–637 (1998).

[45] Mac Lane, S.: Categories for the working mathematician. Springer, New York (1998).

[46] MacNeille, H.M.: *Partially ordered sets*. Trans. Amer. Math. Soc. **42**, 416–460 (1937).

[47] Morton, W.: *Canonical extensions of posets*. Algebra Universalis **72**, 167–200 (2014).

[48] Moshier, M. A., Jipsen, P.: *Topological duality and lattice expansions, I: A topological construction of canonical extensions*. Algebra Universalis **71**, 109–126 (2014).

[49] Moshier, M. A., Jipsen, P.: *Topological duality and lattice expansions, II: Lattice expansions with quasioperators*. Algebra Universalis **71**, 221–234 (2014).

[50] Pierce, B. C.: Basic category theory for computer scientist. The MIT Press, Cambridge (1991).

[51] Pigozzi, D.: *Partially ordered varieties and quasivarieties*. Preprint available at http://www. math. iastate. edu/dpigozzi (2004).

[52] Priestley, H. A.: *Representation of distributive lattices by means of ordered Stone spaces*. Bulletin of the London Mathematical Society **2**, 186–190 (1970).

[53] Priestley, H. A.: *Ordered sets and duality for distributive lattices*. North-Holland Mathematics Studies **99**, 39–60 (1984).

[54] Stone, M. H.: *The theory of representations for Boolean algebras*. Trans. American Math Soc. **40**, 37–111 (1936).

[55] Stone, M. H.: *Topological representations of distributive lattices and Brouwerian logics*. Časopis pro pěstování matematiky a fysiky **67**, 1–25 (1938).

[56] Urquhart, A.: *A topological representation theorem for lattices*. Algebra Universalis **8**, 45–58 (1978).

[57] Vickers, S.: Topology via Logic. Cambridge University Press, Cambridge (1989).