UNIVERSITAT DE BARCELONA

Ads/CFT Correspondence and Superconductivity: Various Approaches and Magnetic Phenomena

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A dissertation submitted to the University of Barcelona
for the degree of Doctor of Philosophy
Programa de Doctorado en Física August 2015

To Alma and Daniela

"*For there are these three things that endure: Faith, Hope and Love,*
but the greatest of these is Love."

Acknowledgements

First of all, I wish to express my deepest and heartfelt gratitude to my advisor, Jorge G. Russo. Having the incredible privilege to work under his guidance has been the single most profoundly changing experience of my life, both professionally and personally. I will forever strive to live up to his standards of intellectual rigor and scientific excellence. Whatever the future may hold for me, I know I will always be able to look back at these extraordinary years and say to myself with a deep sense of pride: "I worked under Jorge Russo." My eternal gratitude to you, Jorge.

I would like to thank Domènec Espriu and Alexander A. Andrianov for giving me the tremendous opportunity to pursuit my doctoral studies at the University of Barcelona. Also, my gratitude to Alberto Güijosa, David J. Vergara and Hugo Morales-Tecotl, for their unconditional support for me to move abroad. Special thanks to Joan Soto for his constant assistance during these years.

I wish to express my sincere gratitude to Francesco Aprile for his invaluable help and advise during my doctoral studies. I am indebted to you, my dear Francesco. I also wish to thank Diederik Roest and Andrea Borghese for our fruitful collaboration.

My thanks go to Alejandro Barranco and Daniel Fernandez, comrades-in-PhD on whose support and companionship I could always rely. I also thank Miguel Escobedo, Albert Puig and Cedric Potterat, the best friends I could ever hope for. I would like to express my deepest appreciation to Elias Lopez, Antonio Perez-Calero, Dani Puigdomenech, Blai Garolera, Markus Fröb, Ivan Latella and Mariano Chernicoff. Very special thanks to my very special friends Liliana Vazquez and Julian Barragan.

More importantly, I want to thank my mother and sister, whose love and care gave me strength throughout these very intense years. I also thank my father for his unwavering support. Finally, I would like to thank my very dear uncles Carmen and Roberto, who where always there for me whenever I needed them. I know for sure my uncle would have liked to hold in his hands a copy of this work.

I was supported financially during my doctoral studies by CONACyT grant, No.306769. I also wish to acknowledge additional support provided by the Departament d'Estructura i Constituents de la Matèria.

Introduction

> *"The history of science is rich in the example of the fruitfulness of bringing two sets of techniques, two sets of ideas, developed in separate contexts for the pursuit of new truth, into touch with one another."*
>
> – J. Robert Oppenheimer

The AdS/CFT correspondence [1, 2] is one of the most important developments in the history of theoretical physics. Using as a binding bridge superstring theory, or, more concretely, some theoretical aspects in the interaction between superstrings and D-branes, the Maldacena conjecture establishes that the physics of a strongly-coupled, perturbatively-inaccessible quantum field theory in d-dimensions can be described equivalently in terms of the dynamics of a dual classical gravitational theory in $(d+1)$-dimensional AdS space. Two particular aspects of the duality are of great importance. First, the duality ascertains that the quantum field theory lives in the boundary of the AdS space in which the dual gravitational system exists, and that the two-point functions of the dual field theory are computed in terms boundary-to-bulk propagators [3, 4]. This difference between the dimensions of the theories makes the duality *holographic*, so giving evidence to the idea that a quantum theory of gravity should be indeed a holographic in nature [5, 6].

The second important aspect we must mention is that the AdS/CFT correspondence is a strong/weak-coupling duality: it allows one to formulate a strongly-coupled quantum problem in terms of the classical Einstein equations of the dual higher-dimensional gravitational system. Because of this particular nature of the duality, it provides a promising new way of studying quantum gauge theories in the strongly-coupled regime, where the usual perturbative methods fail to apply. The gauge/gravity duality has thus been used to gain insight in a wide variety of physical systems where a satisfactory description in terms of standard methods is lacking, such as the quark-gluon plasma or in condensed matter theory.

This thesis will be devoted to the study through holographic techniques of one of such "problematic" systems, namely high-T_c superconductors, or *cuprates*. As will be seen in Chapter 1, the main problem with a standard theoretical description of the cuprates can be tracked down to their temperature vs. doping phase diagram. In it one finds that the normal phase of the material above optimum doping is described by Non-Fermi Liquid physics. In particular, this will mean that the strong interactions between the components of the system will make the usual quasi-particle description of electrons near the Fermi surface to break down, and therefore make the description of the mechanism behind Cooper-pairing intractable by the usual field theory methods. As we will see throughout this thesis, the AdS/CFT correspondence, because of its strong/weak-coupling duality, can be very fruitfully applied to show us that strongly-coupled field theories indeed present a superconducting phase and, even more importantly, it will allow us to study such phases in a non perturbative fashion with the aid of holographic techniques. In this sense, the AdS/CFT correspondence can be said to provide a natural theoretical definition of superconductivity in the strong-coupling regime. The holographic models of superconductivity we will study are called *holographic superconductors* [7] and currently represent a very exciting and active area of research.

All the systems studied in these thesis are s-wave holographic superconductors. Even though high-T_c superconductors have condensates with d-wave symmetry, it is nevertheless expected that the main results obtained in the s-wave case remain valid in a d-wave case. Similarly, it is also believed that the technology developed through the study of s-wave holographic superconductors can be equally applied to d-wave ones. The construction of a d-wave holographic superconductor is a currently open challenge in the area [8, 9, 10].

When constructing phenomenological bulk models in the bottom-up approach, we will focus our attention on the $D = 5$ case. The reason for this choice of dimension is that, as noted in [11, 12], dimensionality may play an important role in the way external magnetic fields act in the dual superconducting system. The standard argument is that in a $2 + 1$ ($D = 4$) dimensional superconductor an external 3+1 dimensional magnetic field will always penetrate the material because the energy needed to expel the field scales as the volume, while the energy that the system gains from being in a superconducting state scales as the area. This results in the system being a Type II superconductor. In the case of a $3 + 1$ ($D = 5$) dimensional system such as the ones we study, both energies scale as the volume and one has therefore a direct competition that does not exclude the possibility of obtaining a Type I superconductor. Also, while high-T_c samples are typically composed of 2-dimensional CuO_2 layers (*cuprate superconductors*), it is important to examine the effect of thickness when the system is probed by external magnetic fields.

This thesis is organized in three parts.

- **Chapters 1, 2 and 3. Introductory Concepts.**

These chapters provide an introduction to the topics relevant for the rest of the thesis. Chapter 1 is a general overview of superconductivity. We detail the phenomenological description of superconductors provided by Ginzburg-Landau theory. We then review the general theoretical aspects of Fermi Liquid theory and use this to introduce the most successful microscopic theory of standard superconductivity, BCS theory. We then continue with an introduction to high-temperature superconductors showing their basic phenomenology and, finally, we briefly summarize some of the most challenging problems in the construction of a satisfactory theoretical description of the system using standard field theory techniques.

Chapter 2 provides an introduction to the AdS/CFT correspondence. We review the general theoretical aspects of Type IIB supergravity, $\mathcal{N} = 4$ Super

Yang-Mill theory and D-brane theory. We then present the Maldacena conjecture and its various limits. Finally, because of its importance to the holographic superconducting models of the remaining chapters, we present and account of scalar fields in AdS and their holographic description

Chapter 3 attempts to merge the preceding two chapter by providing an introduction to the proper subject of this thesis, holographic superconductivity. We begin by introducing the main ingredients needed for a consistent holographic superconducting model. We then continue to describe the general details, both theoretical and computational, of an holographic superconductor using as an example the phenomenological HHH model of holographic superconductivity. We finish by describing how magnetic phenomena are introduced in the subject.

- **Chapters 4 and 5. Bottom-Up Approach to Holographic superconductivity.**

These chapters describe *bottom-up* models of holographic superconductivity, that is, bulk models that do not arise from a particular truncation of string theory, but are rather constructed by hand to probe the phenomenology of the superconducting physics of the dual field theory. In Chapter 4, we construct a family of minimal phenomenological models for holographic superconductors in $d = 4 + 1$ AdS spacetime and study the effect of scalar and gauge field fluctuations. By making a Ginzburg-Landau interpretation of the dual field theory, we determine through holographic techniques a phenomenological Ginzburg-Landau Lagrangian and the temperature dependence of physical quantities in the superconducting phase. We obtain insight on the behaviour of the Ginzburg-Landau parameter and whether the systems behave as a Type I or Type II superconductor. Finally, we apply a constant external magnetic field in a perturbative approach following previous work by D'Hoker and Kraus, and obtain droplet solutions which signal the appearance of the Meissner effect.

In Chapter 5 we continue our bottom-up research in the Ginzburg-Landau approach to holographic superconductivity. We investigate the effects of Lifshitz dynamical critical exponent z on a family of minimal $D = 4 + 1$ holographic superconducting models, with a particular focus on magnetic phenomena. We see that it is possible to have a consistent Ginzburg-Landau approach to holographic superconductivity in a Lifshitz background. By following this phenomenological approach we are able to compute a wide array of physical quantities. We also calculate the Ginzburg-Landau parameter for different condensates, and conclude that in systems with higher dynamical critical exponent, vortex formation is more strongly unfavored energetically and exhibit a stronger Type I behavior. Finally, following the perturbative approach proposed by Maeda, Natsuume and Okamura, we calculate the critical magnetic field of our models for different values of z. These two chapters are based on the original research done in [12, 13].

- **Chapter 6. Top-Down Approach to Holographic Superconductivity.**

This chapter is meant to provide a working example of *top-down* holographic superconducting models, which are bulk-models that arise naturally as smaller sector of consistent truncation of Type IIB supergravity, and whose holographic dual field theories present superconducting behaviour. We construct a one-parameter family of five-dimensional $\mathcal{N} = 2$ supergravity Lagrangians with an $SU(2,1)/U(2)$ hypermultiplet. For certain values of the parameter, these are argued to describe the dynamics of scalar modes of superstrings on $AdS_5 \times T^{1,1}$, and therefore to be dual to specific chiral primary operators of Klebanov-Witten superconformal field theory. We demonstrate that, below a critical temperature, the thermodynamics is dominated by charged black holes with hair for the scalars that are dual to the operator of lowest conformal dimension $\Delta = 3/2$. The system thus enters into a superconducting phase where $\langle Tr[A_k B_l] \rangle$ condenses. This chapter is based on the original research presented in [14].

Compendi de la Tesi i Resultats Obtinguts

Compendi de la Tesi

Aquesta tesi s'organitza en tres parts.

- **Capítols 1, 2 i 3. Conceptes Introductoris**

Aquests capítols ofereixen una introducció als temes d'interès per a la resta de la tesi. El Capìtol 1 és una revisió general de la superconductivitat. Detallem la descripció fenomenològica dels superconductors proporcionada per la teoria de Ginzburg-Landau. A continuació, repassem els aspectes teòrics generals de la teoria del Líquid de Fermi i fem servir això per introduir la descripció microscòpica de més èxit de la superconductivitat estàndard, la teoria BCS. Continuem amb una introducció als superconductors d'alta temperatura mostrant la seva fenomenologia bàsica i, finalment, es resumeixen breument alguns dels problemes més difícils que apareixen a la construcció d'una descripció teòrica satisfactòria d'aquests sistemes quan es ultilizan tècniques de teoria de camps estàndard .

El Capítol 2 ofereix una introducció a la correspondència AdS/CFT. Es revisen els aspectes teòrics generals de supergravetat Tipus IIB, la teoria de Super Yang-Mill $\mathcal{N} = 4$ i teoria de D-branas. A continuació, presentem la conjectura de Maldacena i els seus diversos límits. Finalment, per la seva importància per als models de la superconductivitat hologràfiques dels capítols restants, es presenta una descripció de camps escalars en espai AdS i la seva interpretació holográfica.

El Capítol 3 intenta combinar els dos capítols anteriors en proporcionar una introducció a la matèria pròpia d'aquesta tesi, la superconductivitat holográfica.

Comencem amb la introducció dels principals ingredients necessaris per a un model hologràfic superconductor consistent. Continuem descrivint els detalls generals, tant teòrics i computacionals, d'un superconductor hologràfic usant com a exemple el model fenomenològic HHH de la superconductivitat hologràfica. Acabem amb una descripció de com s'introdueixen els fenòmens magnètics en el tema.

- **Capítols 4 i 5. Aproximació Bottom-Up a la superconductivitat hologràfica**

Aquests capítols descriuen models *bottom-up* de la superconductivitat holográfica, és a dir, models en el bulk que no es deriven d'un truncament particular de la teoria de cordes, sinó que estan construïts a mà per sondejar la fenomenologia de la fase superconductora en la teoria de camps dual. En el Capítol 4 es construeix una família de models fenomenològics mínims per als superconductors hologràfiques en espai-temps $d = 4+1$ AdS i estudiem l'efecte de fluctuacions en els camps escalars i gauge. En fer una interpretació Ginzburg-Landau de la teoria de camps dual, determinem a través de tècniques hologràfiques un Lagrangià fenomenològic tipus Ginzburg-Landau, així com la dependència en la temperatura de certes quantitats físiques en la fase superconductora. Obtenim informació sobre el comportament del paràmetre de Ginzburg-Landau i si el sistema dual es comporta com superconductor Tipus I o Tipus II. Finalment, s'aplica un camp magnètic extern constant en un enfocament perturbatiu seguint a D'Hoker i Kraus, i obtenim solucions que assenyalen l'aparició de l'efecte Meissner.

En el Capítol 6 investiguem els efectes de l'exponent dinàmic crític de Lifshitz z en una família de models mínims de la superconductivitat hologràfica en $d = 4+1$, amb un enfocament particular en els fenòmens magnètics. Veiem que és possible tenir un interpretació Ginzburg-Landau consistent per superconductivitat hologràfica en un fons Lifshitz. Seguint aquest enfocament fenomenològic som capaços de calcular una àmplia gamma de quantitats físiques. També calculem el paràmetre de Ginzburg-Landau per diferents condensats i vam concloure que en els sistemes amb major exponent crític dinàmic la formació de vòrtex està més fortament desfavorecidad energèticament i exhibeixen un comportament de Tipus I més fort. Finalment, seguint l'enfocament perturbatiu proposat per Maeda, Natsuume i Okamura, calculem el camp magnètic crític dels nostres models per a diferents valors de z. Aquests dos capítols estan basats en la investigació original realitzada en [12, 13].

- **Capítol 6. Aproximació Top-Down a la Superconductividad Hologràfica**

Aquest capítol té per objecte proporcionar un exemple d'un model superconductor hologràfic *Top-Down*, que són models en el bulk que sorgeixen naturalment com un sector petit d'un truncament consistent de supergravetat Tipus IIB, i on la teoria de camps dual presenta comportament superconductor. Construïm una família d'un sol paràmetre de Lagrangianes de supergravetat $\mathcal{N} = 2$ en cinc dimensions amb un hypermultiplet $SU(2,1)/U(2)$. Per certs valors del paràmetre, vam argumentar que aquests descriuen la dinàmica dels modes escalars de supercordes en $AdS_5 \times T^{1,1}$, i per tant han de ser duals a certs operadors primaris quirals específics de la teoria de camps superconforme Klebanov-Witten. Es demostra que, per sota d'una temperatura crítica, la termodinàmica està dominada per forats negres carregats amb cabells escalars duals a l'operador de menor dimensió conforme $\Delta = 3/2$. El sistema entra així en una fase superconductora on l'operador $\langle Tr[A_k B_l]\rangle$ condensa. Aquest capítol està basat en investigacion original presentada en [14].

Resultats Obtinguts en Aquesta Tesi

En aquesta tesi s'ha demostrat que la correspondència AdS/CFT ofereix una nova manera d'estudiar la fase superconductora de les teories large-N al règim fortament acoblat. En discutir els cuprats al final del Capítol 1, veiem ia algunes de les deficiències que els acercaminentos usuals basats en teories de camps tenen quan es tracta d'abordar sistemes de molts cossos fortament acoblats.

Potser que la més greu d'aquestes deficiències és el col·lapse del concepte de quasi-partícula a causa de les fortes interaccions involucrades. Com hem vist, la dualitat gauge/gravity ens permet plantejar problemes gairebé intractables en sistemes quàntics de molts cossos en termes de la dinàmica clàssica d'un sistema dual de gravetat en l'espai AdS. Usant aquest nou punt de vista hologràfic, la condensació de parells de Cooper al costat de la teoria del camps es tradueix en la creació espontània de solucions amb pèl carregat al costat gravitatori de la dualitat. Això dóna lloc a una fase en la teoria del camps dual, on es recuperen els aspectes fenomenològics fonamentals de la superconductivitat. Crida l'atenció que amb només observar al problema des d'un punt de vista hologràfic, es pot demostrar q'aquests sistemes completament intractables en el règim d'acoblament fort presenten una fase superconductora. a causa de l'èxit del mètode hologràfic i les dificultats ja esmentades sobre els enfocaments estàndard basats en teories de camps, pot ser que no sigui massa agosarat imaginar que, efectivament, la definició teòrica natural de la superconductivitat en el règim fort acoblament està donada pel sistema dual de gravetat.

Amb aquestes consideracions generals en ment, en aquesta tesi ens hem esforçat a presentar una imatge el més completa possible dels diferents enfocaments seguits en la superconductivitat hologràfica. Així, hem presentat exemples, tant en la de apropaments bottom-up (Capítols 4 i 5) i top-down (Capítol 6). Vegem ara algunes conclusions de cada un d'aquests capítols.

En el Capítol 4 hem pres com a punt de partida una família de models superconductors hologràfics mínims en espai-temps $d = 4 + 1$ AdS, caracteritzats per càrrega q del seu camp escalar (o, equivalentment, per la seva temperatura crítica T_c). Hem introduït primer una petita pertorbació magnètica en la component x_1 del camp gauge, així com una petita pertorbació del camp escalar al voltant de la solució condensada. En fer una interpretació fenomenològica tipus Ginzburg-Landau de la teoria de camps dual, es van calcular els paràmetres de Ginzburg-Landau i longituds característiques en funció de la temperatura. Hem trobat que tenen un comportament consistent amb el dels sistemes superconductors habituals descrits per la teoria de camp mig. També es va calcular el paràmetre de Ginzburg-Landau κ per a diferents valors de la càrrega del camp escalar q. A partir d'aquest càlcul trobem que, en augmentar el valor de q, el paràmetre de Ginzburg-Landau s'acosta asimptòticament al valor $\kappa \sim 0.55 < 1/\sqrt{2}$. D'això podem concloure que el sistema es comportarà com un superconductor de Tipus I per a tots els valors de q considerats. També hem calculat la densitat d'energia lliure de Helmholtz del sistema utilitzant l'enfocament de Ginzburg-Landau proposat, i l'hem comparat amb l'energia lliure calculada amb les tècniques hologràfiques estàndard. Es va trobar que tots dos enfocaments són consistents entre si prop de T_c. També, a través de càlculs de l'energia lliure del sistema, l'enfocament Ginzburg-Landau es va comparar amb el mètode desenvolupat en [15] per al càlcul dels paràmetres α i β. Tots dos mètodes van demostrar estar en excel·lent acord.

A continuació, hem apaguat la fluctuació magnètica i vam sondejar el nostre sistema amb un camp magnètic constant B. Això es va fer mitjançant l'ús de la solució de brana negra de [16] a $d = 4 + 1$ AdS fins a ordre B^2. Amb aquesta solució perturbativa com a fons fix mostrem la formació de solucions amb condensat tipus gota i calculem el camp magnètic crític per sobre del qual la fase superconductora es trenca. El camp obtingut d'aquesta manera es va comparar amb el camp magnètic crític obtingut per mitjà del nostre enfocament Ginzburg-Landau. Encara que tots dos camps mesuren diferents aspectes de la resposta del sistema a un camp magnètic, es va trobar que prop de T_c tots dos camps es comporten com $B_c \sim B_0(1 - T/T_c)$ i que els seus corresponents factors B_0 es comporten com $\sim 1/q^{1/3}$ (o, equivalentment, com $\sim 1/T_c$) per a valors grans q. Un dels principals resultats d'aquest treball és mostrar que a partir d'un model fenomenològic molt simple en espai-temps $d = 4+1$ AdS podem construir una descripció consistent tipus Ginzburg-Landau de la teoria de camps a la frontera, on tots els paràmetres de Ginzburg-Landau i longituds característiques es poden calcular utilitzant mètodes hologràfics, i el comportament s'ajusta al predit per la teoria de camp mitjà tradicional. D'altra banda, també s'observa que, en augmentar el valor de la càrrega del camp escalar q, el paràmetre de Ginzburg-Landau del model tendeix asimptòticament a un valor ben definit que caracteritza el sistema superconductor dual com Tipus I.

El Capítol 5 és una continuació natural de l'anterior. En aquest capítol hem optat per estudiar un model mínim en $D = 5$ de superconductivitat hologràfica en el probe limit, amb un fons de forat negre Lifshitz. Dins d'aquest marc, hem estudiat diferents casos de condensació, variant dins de cada un d'ells l'exponent crític dinàmic a fi d'obtenir una visió sobre com el sistema es veu afectat per z respecte al seu comportament isotròpic usual. Igual que en el capítol anterior, hem afegit petites flucutaciones escalars i de camp gauge als camps components originals, per tal de calcular holográficamente les longituds de penetració i la coherència del sistema superconductor. Observem que les dues longituds característiques prop de T_c tenen la depencia funcional estàndard respecte a la temperatura, per a tots els casos de condensat i tots els valors de z. No obstant això, l'exponent crític z si afecta la magnitud de les longituds característiques, com es fa evident en el canvi del valor del seu ràtio, donat pel paràmetre de Ginzburg-Landau κ. També hem vist que és possible construir una interpretació fenomenològica Ginzburg-Landau consistent fins i tot en una teoria dual amb escalament de Lifshitz. Hem calculat a través de tècniques hologràfiques dels coeficients de Ginzburg-Landau α i β i, igual que en el cas de les longituds característiques, la conclusió és que, prop de T_c, tenen una dependència funcional estàndard respecte a la temperatura, per a tots els casos de condensat i tots els valors de z. No obstant això, la presència de z té un efecte no trivial en aquests paràmetres fenomenològics, disminuint el valor dels seus coeficients numèrics mentre el valor de z augmenta.

També hem calculat amb tècniques hologràfiques el paràmetre de Ginzburg-Landau κ del sistema. Per tot cas de condensació i tots els valors de z, observem que $\kappa < 1/\sqrt{2}$. Això vol dir que per a tots els casos el sistema dual es comportarà com un superconductor Tipus I. D'altra banda, també es va observar que, per a cada cas de condensació considerat, el valor de κ disminueix a mesura que el valor de z augmenta. Això significa que en els sistemes amb major anisotropia, la formació de vòrtex és més fortament desfavorida energèticament i aquests exhibeixen un comportament de Tipus I més fort.

Finalment, es va calcular el camp magnètic crític B_c necessari per trencar la fase superconductora del sistema, seguint el procediment perturbatiu desenvolupat primer en [17]. Hem observat que el camp crític prop de T_c té la dependència funcional amb la temperatura que prediu la teoria de Ginzburg-Landau. No obstant això, també observem que el valor del camp magnètic crític és cada vegada menor mentre el valor de z augmenta. A més, dins d'aquest enfocament perturbatiu, hem confirmat holográficamente la conjectura plantejada en [18], que diu que el camp magnètic crític és inversament proporcional al quadrat de la longitud de correlació, en acord amb la teoria de Ginzburg-Landau.

El càlcul hologràfic del paràmetre de Ginzburg-Landau κ presentat en aquests dos capítols pot servir com una sonda útil per posar a prova la viabilitat d'un model superconductor hologràfic com una possible descripció d'un superconductor d'alta T_c del món real. De fet, tots els cuprats fins ara descoberts presenten un comportament de Tipus II. Per tant, seria una propietat molt desitjable en un superconductor hologràfic que tingués valor de κ a la regió de Tipus II. Una cosa similar es pot dir dels sistemes estudiats en el Capítol 5, on es va concloure que els sistemes amb major anisotropia tenen un comportament de Tipus I més fort. En aquest sentit, és natural preguntar-se com el paràmetre de Ginzburg-Landau obtingut en aquests capítols podria canviar amb l'elecció d'altres models, com ara, per exemple, superconductors hologràfics d'ona-d [8, 9, 10], superconductors hologràfics d'ona-p [19], models amb correccions d'ordre mayors en el potencial del camp escalar, com ara els que apareixen en els enfocaments top-down [20, 21, 14] o models menys convencionals, com ara els que tenen termes de Chern-Simons, acoblaments amb derivades d'ordre superior, o aquells dins el context de New Massive Gravity [22, 23, 24]. Això requereix d'una major investigació.

Passem ara al sistema top-down estudiat al Capítol 6. En resum, hem construït explícitament una Lagrangiana per supergravetat $\mathcal{N} = 2$ galgada, acoblada a un hypermultiplete escalar $SU(2,1)/U(2)$. El model resultant està determinat únicament per un sol paràmetre β, que representa la barreja entre els generadors $U(1)$ de $SU(2)$ amb $U(1)$. Quan $\beta = 1$, el sistema descriu dos escalars complexos ζ_1, ζ_2 amb masses $m_1^2 = -3$ i $m_2^2 = 0$. En aquest cas, la Lagrangiana resultant coincideix exactament amb la Lagrangiana de [20], amb l'extensió que incorpora el dilatón complex que es troba en [25, 26, 27, 28]. Aquest aparellament implica un potencial escalar no trivial i acoblaments no trivials, i no ens hauria de sorprendre que no hi hagi un altre model possible per a un hypermultiplete $SU(2,1)/U(2)$ amb aquestes masses

De la mateixa manera, la pròpia naturalesa única de la Lagrangiana indica fortament que el model amb $\beta = 0$ certament ha de descriure els dos camps escalars complexos de masses $m^2 = -15/4$ que són duals a l'operador de dimensió més baixa $\Delta = 3/2$ en la teoria superconformal Klebanov-Witten. Hem demostrat explícitament que aquest mode domina la termodinàmica a baixes temperatures. Seria molt interessant veure si la el model $\beta = 0$ representa un truncament consistent de supergravetat Tipus IIB. Tot i que els camps escalars tenen números quàntics de Kaluza-Klein no trivials $(1/2, 1/2)$, són tanmateix els estats més baixos en l'espectre KK, el que suggereix que el truncament pot ser consistent. Demostrar aixó últim pot requerir una construcció explícita d'un ansatz Tipus IIB que reprodueixi les mateixes equacions de moviment.

Contents

1

Superconductivity

One of the main objectives in the research on **Holographic Superconductivity** is to provide an holographic theoretical description of the strongly-coupled phenomenon of high-temperature superconductivity. In order to get a general picture of the former subject and why holographic methods may be applicable in its study, it is reasonable to start by explaining the basic features of conventional superconductors. We do this by introducing two very successful theoretical descriptions of superconductivity: *Ginzburg-Landau theory* and *BCS theory*. Ginzburg-Landau theory provides an effective, phenomenological description of superconductors near the critical temperature in terms of very simple degrees of freedom. We review it because it introduces some important concepts like spontaneous symmetry breaking in the context of condensed matter and the Meissner effect, and because it also explains in very simple fashion some important superconducting phenomena that will become relevant in the remaining chapters.

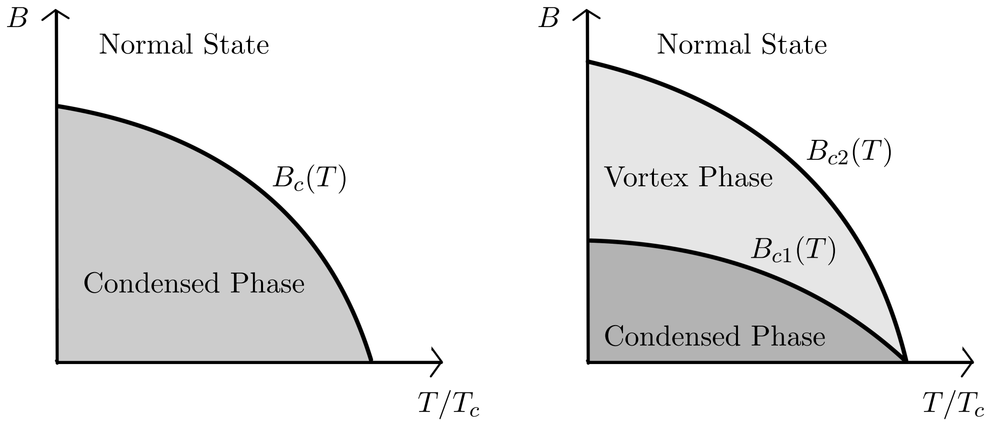
BCS theory, on the other hand, is a microscopical theory of superconductivity, based on the concept of quasi-particle fermionic interaction. We review it because it is the most successful description of conventional superconductivity and provides a starting point to understand the problems related to having a similar theory for high-T_c superconductivity. Before introducing BCS theory, we briefly review some aspects of Fermi liquid theory which will give us important physical insights on both conventional and high-T_c superconductors. Finally, we briefly review the most important aspects of high-T_c superconductivity, focusing on the particular case of cuprate superconductors and briefly reviewing some of its more important physical properties. We comment on one of the most promising attempts to provide a theoretical description of the phenomenon by means of standard methods, and its limitations. By doing this, it will become clear why holographic methods can be of importance when studying these systems.

1.1 Ginzburg-Landau Theory

In presenting this topic we will follow the exposition made by [29, 30, 31]. The Ginzburg-Landau theory [33] is a phenomenological description of superconductivity. One of its strongest points is that it is an universal theory in the sense that it describes superconducting phenomena regardless of the microscopic details of the material. Its weakness relies on the fact that it is only valid near the critical temperature T_c, more precisely in the *transition region*

$$\frac{|T - T_c|}{T_c} \ll 1\,. \tag{1.1.1}$$

Ginzburg-Landau theory is exceptionally well suited to provide a phenomenological description of the physics of superconductors in presence of external electromagnetic fields. A superconducting material's reaction to external magnetic fields provide one of the two main phenomenological definitions of superconductivity, the first one being the loss of resistivity. The second is perfect diamagnetism. Indeed, a superconductor has the physical property of expelling external magnetic fields from its volume, a phenomenon called the *Meissner effect* [32]. Furthermore, one of the most important ways of classifying superconducting materials is related also to magnetic phenomena. Briefly stated, a superconductor



(a) Type I

(b) Type II

Figure 1.1: Schematic magnetic phase diagram of Type I and Type II superconductors

can be classified in two classes. In a *Type I* superconductor the system goes from the superconducting to the normal phase in a first order transition as the value of the external magnetic is increased beyond a critical value B_c. On the other hand, a *Type II* superconductor has two critical values: below the first critical value B_{c_1} the system is in a superconducting phase, but as the value of the field is increased, a stable vortex lattice (*Abrikosov vortices*) begins to form inside the material where the magnetic field can penetrate until a second critical value B_{c_2} is reached and the system enters fully in the normal phase. In this case the phase transitions are second order in B. In any case, it is experimentally observed that the value of the critical field near the critical temperature behaves as

$$B_c \sim (1 - T/T_c) \ . \tag{1.1.2}$$

In figures (1.1a) and (1.1b) we show an schematic magnetic phase diagram for Type I and Type II superconductors, respectively.

The first attempt to provide a description of the diamagnetic currents that expel magnetic fields from a material in the superconducting phase was the phenomenological London theory [34], which used the number density of superconducting carriers n_s as an order parameter for the system, and related the current with the applied electromagnetic potential in the *London equation*

$$\mathbf{J} = -\frac{e^2}{m} n_s \mathbf{A} \,, \tag{1.1.3}$$

were e and m are the charge and mass of the superconducting carriers, respectively. However, the London theory relied on the approximation that n_s is a constant, an assumption that cannot hold with increasing magnetic fields. Landau therefore constructed a phenomenological theory that reproduced and extended the London theory results in the case of a non-homogenous number density. Therefore he proposed the complex Ginzburg-Landau order parameter as

$$\Psi(\mathbf{r}) = \sqrt{n_s(\mathbf{r})} e^{i\varphi(\mathbf{r})} \,, \tag{1.1.4}$$

so that

$$|\Psi|^2 = n_s \,, \tag{1.1.5}$$

and where $\varphi(\mathbf{r})$ is a phase that we will set to zero in the following, for simplicity.

Then, using as a starting point Landau's own theory of second order transitions, Ginzburg-Landau theory proposes that the system's free energy density difference can be expanded near the critical temperature as

$$\Delta f = \alpha(T) \left|\Psi\right|^2 + \frac{1}{2}\beta(T) \left|\Psi\right|^4 + \frac{\hbar^2}{2m} \left| \left(\nabla - \frac{ie}{\hbar}\mathbf{A} \right) \Psi \right|^2 + \frac{B^2}{2\mu_0} \,, \tag{1.1.6}$$

where $\Delta f = f_{\mathrm{sc}} - f_{\mathrm{n}}$, with f_{sc} and f_{n} being the free energy densities in the superconducting and normal phases of the system, respectively. Also, α and β are phenomenological parameters that have a temperature dependence in general. We note the addition of the gauge field A_i and the corresponding magnetic energy in order to describe a charged system. We will adopt the usual convention $\alpha < 0$, $\beta > 0$.

When the external field and gradients are negligible, the free energy density difference (1.1.6) can be approximated by

$$\Delta f = \alpha \left|\Psi\right|^2 + \frac{1}{2}\beta \left|\Psi\right|^4 , \tag{1.1.7}$$

which is minimized at

$$|\Psi_\infty| = \sqrt{\frac{|\alpha|}{\beta}} . \tag{1.1.8}$$

Since deep inside the superconductor the external fields and gradients can be neglected, the critical parameter Ψ will approach the value Ψ_∞ as it goes deeper into the volume of the system. Inserting this value back in (1.1.6), we get inside the material

$$\Delta f = -\frac{\alpha^2}{2\beta} . \tag{1.1.9}$$

This last equation can be related to the critical magnetic field H_c, which is the value of the magnetic field needed to be applied to the system in a condensed phase in order to break superconductivity. Indeed, this field is determined by the specific magnetic energy density that needs to be added to the condensation energy to take the system into the normal phase, that is

$$f_{\rm sc} + \frac{\mu_0}{2} H_c^2 = f_{\rm n} , \tag{1.1.10}$$

or, equivalently

$$\Delta f = -\frac{\mu_0}{2} H_c^2 . \tag{1.1.11}$$

Equating (1.1.9) and (1.1.11), we obtain

$$H_c^2 = \frac{\alpha^2}{\mu_0 \beta} . \tag{1.1.12}$$

which corresponds to the value where the magnetic field destroys superconductivity in the system, since for values $H > H_c$ it will be energetically more favorable for the system to be in the normal phase.

A few words about the functional dependence on temperature of the coefficients α and β. Since at $T = T_c$ we must have $\left|\Psi\right|^2 = 0$ and a finite value for

$T < T_c$, then from (1.1.8) we must have $\alpha = 0$ at $T = T_c$ and $\alpha < 0$ for $T < T_c$. A simple assumption is therefore

$$\alpha \sim (T/T_c - 1)\,. \tag{1.1.13}$$

Comparing the near-T_c behaviour (1.1.2) of the critical magnetic field and relation (1.1.12), we see that such a functional dependence for α is only possible if β behaves as a positive constant near the critical temperature. This finally let us conclude from (1.1.8) that the order parameter behaves near-T_c as

$$\Psi \sim (1 - T/T_c)^{1/2}\,, \qquad T < T_c\,, \tag{1.1.14}$$

which is a result that is confirmed experimentally and in BCS theory.

Minimizing (1.1.6) with respect to $\mathbf{A}$, and using $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, we arrive at

$$\mathbf{J} = -\frac{e^2}{m}\,|\Psi|^2\,\mathbf{A}\,, \tag{1.1.15}$$

from which, when substituting $|\Psi|^2 = n_s$, one recovers the original expression for the London current (1.1.3).

Continuing with the gauge field equations, one arrives at the following equation

$$\nabla^2 \mathbf{B} = \frac{1}{\lambda^2}\mathbf{B}\,, \tag{1.1.16}$$

which has magnetic field solutions that decay exponentially inside the superconductor, with decay length λ, called the *penetration length*, and given by

$$\lambda^2 = \frac{m}{\mu_0 e^2 n_s}. \tag{1.1.17}$$

This length corresponds to the inverse mass of the gauge field after symmetry breaking. Combining (1.1.8), (1.1.12) and (1.1.17), we arrive at the following expressions for α and β

$$\alpha = -\frac{e^2 \mu_0^2}{m} H_c^2 \lambda^2\,, \tag{1.1.18}$$

$$\beta = \frac{e^4 \mu_0^3}{m^2} H_c^2 \lambda^4\,. \tag{1.1.19}$$

Now, minimizing (1.1.6) with respect to the order parameter Ψ^*, one obtains

$$\alpha\Psi + \beta\,|\Psi|^2\,\Psi - \frac{\hbar^2}{2m}\left(\nabla - \frac{ie}{\hbar}\mathbf{A}\right)^2\Psi = 0\,. \tag{1.1.20}$$

Then, if we consider the case without fields present $\mathbf{A} = 0$, we have

$$\alpha\Psi + \beta\Psi^3 - \frac{\hbar^2}{2m}\Psi'' = 0\,, \tag{1.1.21}$$

where for simplicity we assumed that Ψ is real and only depends on the dimension x. Expanding around the minimum as

$$\Psi(x) = \sqrt{\frac{|\alpha|}{\beta}} + \eta(x)\,, \qquad |\eta| \ll 1\,, \tag{1.1.22}$$

and inserting in (1.1.21), we have, up to second order the equation

$$2\,|\alpha|\,\eta - \frac{\hbar^2}{2m}\eta'' = 0\,, \tag{1.1.23}$$

which has the physical solution

$$\eta(x) \sim e^{-\frac{|x|}{\xi_0}}\,, \tag{1.1.24}$$

where ξ_0, defined as

$$\xi_0^2 = \frac{\hbar^2}{4m\,|\alpha|}\,, \tag{1.1.25}$$

is the superconductor *correlation length*, and it is a measure of the spatial decay of a small perturbation of Ψ from its equilibrium value. It is customary, however, to work with the *Ginzburg-Landau correlation length* ξ, given by $\xi^2 = 2\,\xi_0^2$, that is

$$\xi^2 = \frac{\hbar^2}{2m\,|\alpha|}\,. \tag{1.1.26}$$

Finally, from the characteristic lengths λ and ξ one can construct the *Ginzburg-Landau parameter* κ, defined as:

$$\kappa = \frac{\lambda}{\xi}\,, \tag{1.1.27}$$

whose value, based on surface energy calculations made by Abrikosov [35], characterizes the behaviour of the system in a superconducting phase as:

$$\kappa < \frac{1}{\sqrt{2}} \qquad \text{Type I Superconductor} \tag{1.1.28}$$

$$\kappa > \frac{1}{\sqrt{2}} \qquad \text{Type II Superconductor} \tag{1.1.29}$$

where, as said before, a Type II superconductor is one which allows partial penetration of a magnetic field, while a Type I superconductor is one where the magnetic field is fully expelled from its volume by the Meissner effect.

We conclude by saying that, even though Ginzburg-Landau theory can be consider a triumph in physical intuition and that it correctly describes many superconducting phenomena, it is nevertheless a phenomenological theory that gives us little information about the microscopic mechanism behind superconductivity.

1.2 Fermi Liquid

Before presenting BCS theory, it is useful to have a general knowledge of *Fermi Liquid Theory*. For a more detailed treatment, see [36, 37, 38]. Briefly stated, Fermi Liquid theory is a general microscopical description of electrons in a metal, and is constructed as a quantum theory of interacting many-fermions. It also introduces some very important concepts that will become very relevant in our discussion of both conventional and high-T_c superconductivity.

We begin by writing the generic microscopic Hamiltonian

$$\mathcal{H}_{\mathrm{FL}} = \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \mathcal{H}_{\mathrm{int.}}\,, \tag{1.2.1}$$

where $c_{\mathbf{k}\sigma}$ and $c^{\dagger}_{\mathbf{k},\sigma}$ are fermionic creation and annihilation operators for one-particle states with momentum $\mathbf{k}$ and spin σ. Also, $\varepsilon_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$, with kinetic energy $\epsilon_{\mathbf{k}} = k^2/(2m)$. The interaction term can be giving in very general terms as

$$\mathcal{H}_{\text{int.}} = -\sum_{\mathbf{k}\mathbf{k}'\mathbf{q}\mathbf{q}'} c^\dagger_{\mathbf{k}} c_{\mathbf{q}} \hat{V} c^\dagger_{\mathbf{k}'} c_{\mathbf{q}} \delta(\mathbf{k} + \mathbf{k}' - \mathbf{q} - \mathbf{q}') \,. \tag{1.2.2}$$

For the purposes of our present discussion, we will not be concerned with its particular details.

We now recall that, according to the Pauli exclusion principle in a many-body fermionic system at zero temperature, the ground state is obtained by filling all energy levels up to μ. This defines a sphere in momentum space with radio given by $k_F = \sqrt{2m\mu}$, called the *Fermi surface* of the system. This is a very important concept that will continue to appear in the remaining of this chapter.

We proceed our description of Fermi liquid theory from view point of its Green's function, which is defined in general as

$$G(\mathbf{k}, t) = -i\langle\Psi_0|T\{c_{\mathbf{k}} c^\dagger_{\mathbf{k}}\}|\Psi_0\rangle \tag{1.2.3}$$

where $|\Psi_0\rangle$ is the ground state of the system and $c_{\mathbf{k}}$ is an operator of the theory. For a system of free fermions, one obtains by direct calculation in frequency space that

$$G_0(\mathbf{k}, \omega) = \frac{1}{\omega - \varepsilon_{\mathbf{k}} + i\delta_{\mathbf{k}}} \,, \tag{1.2.4}$$

where $\delta_{\mathbf{k}}$ is a real infinitesimal quantity defined to go around the pole at $\omega - \varepsilon_{\mathbf{k}} = \omega - (\epsilon_{\mathbf{k}} - \mu)$ as

$$\delta_{\mathbf{k}} = \begin{cases} +\delta \text{ If } \epsilon_{\mathbf{k}} - \mu > 0 \\ -\delta \text{ If } \epsilon_{\mathbf{k}} - \mu < 0 \end{cases} \tag{1.2.5}$$

Before we deal with interactions between fermions, we bring out an important conceptual development first put forward by Landau. In general terms, Landau proposed that when interaction couplings are slowly turned on by an adiabatic process, the states of the free theory evolve smoothly into states in the interacting theory. More precisely, this means that during this process the quantum numbers of the free states, namely charge, momentum and spin, remain unchanged, and therefore continue to label the interacting states. Such states are called *quasi-particles*, since they can almost be treated as non-interacting states. However, as will be seen below, the quasi-particle concept is well-defined only in the vicinity of the Fermi surface.

Taking into account interactions, we must instead consider the *dressed Green's function*

$$G(\mathbf{k},\omega) = \frac{1}{\omega - \varepsilon_{\mathbf{k}} - \Sigma(\mathbf{k},\omega) + i\delta_{\mathbf{k}}} \tag{1.2.6}$$

were $\Sigma(\mathbf{k},\omega)$ is the irreducible self-energy calculated through perturbation theory, and where again $\varepsilon_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$. Its real part $\mathrm{Re}\Sigma(\mathbf{k},\omega)$ represents a shift in the quasi-particles kinetic energy, while its imaginary part $\mathrm{Im}\Sigma(\mathbf{k},\omega)$ is related to the quasi-particles lifetime τ. Additionally, the imaginary part of the Green function is always non-positive[1].

At this point, we introduce two conditions that serve as a definition of a Fermi Liquid from the self-energy point of view. First, for a Fermi liquid the imaginary part of the self-energy always has the following specific form

$$\mathrm{Im}\Sigma(\mathbf{k},\omega) = -C_{\mathbf{k}}\,\omega^2\,, \tag{1.2.9}$$

where ω is close to zero and $C_{\mathbf{k}}$ is a positive constant. Because of this functional dependence of $\mathrm{Im}\Sigma$ on ω, the denominator of $G(\mathbf{k},0)$ is real. The second condition is that there always exists a momentum vector $\mathbf{k}_F$ where the denominator vanishes. That is, for a Fermi liquid there always exists a value $\mathbf{k}_F$ such that

$$\mu - \epsilon_{\mathbf{k}_F} - \mathrm{Re}\Sigma_1(\mathbf{k}_F,0) = 0\,. \tag{1.2.10}$$

[1]This can be seen from the spectral density, defined as

$$A(\mathbf{k},\omega) = -\frac{1}{\pi}\mathrm{Im}G(\mathbf{k},\omega)\,, \tag{1.2.7}$$

which can be written in term of the self energy as

$$A(\mathbf{k},\omega) = -\frac{1}{\pi}\frac{\mathrm{Im}\Sigma(\mathbf{k},\omega)}{(\omega-\mu-\epsilon_{\mathbf{k}}-\mathrm{Re}\Sigma(\mathbf{k},\omega))^2+\mathrm{Im}\Sigma(\mathbf{k},\omega)^2}\,. \tag{1.2.8}$$

Since the spectral density is positive, then $\mathrm{Im}\Sigma(\mathbf{k},\omega) \leq 0$.

Basically, this condition tells us that in a Fermi Liquid there is always a well-defined Fermi surface even in the presence of interactions.

Since we are interested in the low-energy physics, we perform a Taylor expansion of $\mathrm{Re}\Sigma(\mathbf{k},\omega)$ around $k = k_F$ (where we are denoting $k = |\mathbf{k}|$) and $\omega = 0$

$$\mathrm{Re}\Sigma(\mathbf{k},\omega) = \mathrm{Re}\Sigma(\mathbf{k}_F,0) + (k-k_F)\partial_k \mathrm{Re}\Sigma(\mathbf{k}_F,0) + \omega\partial_\omega \mathrm{Re}\Sigma(\mathbf{k},0) + \cdots \quad (1.2.11)$$

and we also expand the kinetic energy $\epsilon_{\mathbf{k}}$ around $k = k_F$

$$\epsilon_{\mathbf{k}} = \epsilon_{\mathbf{k}_F} + (k-k_F)\frac{k_F}{m} + \cdots \quad (1.2.12)$$

In this approximation the dressed Green's function is

$$G(\mathbf{k},\omega) = \frac{Z}{\omega - (k-k_F)\,\frac{k_F}{m^*} + iZC_{\mathbf{k}_F}\omega^2}\,, \quad (1.2.13)$$

where we have defined

$$\frac{1}{Z} = 1 - \omega\partial_\omega \mathrm{Re}\Sigma(k_F,0)\,, \quad (1.2.14)$$

$$\frac{1}{m^*} = Z\left(\frac{1}{m} + \frac{1}{k_F}\partial_k \mathrm{Re}\Sigma(k_F,0)\right). \quad (1.2.15)$$

The quantities Z and m^* are called the *quasi-particle residue* and the *effective mass*, respectively. When $\omega \to 0$, the ω^2 term becomes negligible and one has a pole at

$$\omega = \mathcal{E}_k - \mu\,, \quad (1.2.16)$$

where we have defined

$$\mathcal{E}_k = \mu + \frac{k_F}{m^*}(k-k_F)\,. \quad (1.2.17)$$

With this value of ω, the dressed Green's function can be written as

$$G(k,\omega) = \frac{Z}{\omega - (k-k_F)\frac{k_F}{m^*} + i/(2\tau)} \quad (1.2.18)$$

where the quasi-particle lifetime τ is defined as

$$\frac{1}{\tau} \equiv 2ZC_k(\mathcal{E}_k - \mu)^2 \sim \omega^2\,. \quad (1.2.19)$$

In this form, (1.2.18) represents the dressed Green's function of a quasi-particle of mass m^* and energy $\mathcal{E}_k$. Its lifetime τ near the Fermi surface is

$$\frac{1}{\tau} \sim (k - k_F)^2 \,. \tag{1.2.20}$$

We then conclude that the quasi-particle's lifetime goes to infinity at $k = k_F$ and its states are stable and well defined in that limit. This is a very important result of Fermi-Liquid theory.

In relation to the quasi-particle residue Z, we note the following [39]. The value of the particle number operator $\langle n_{\mathbf{k}} \rangle$ is related to the Green function by

$$\langle n_{\mathbf{k}} \rangle = -i \lim_{t \to 0} G(\mathbf{k}, \omega) \,, \tag{1.2.21}$$

which in the case of the dressed function (1.2.18) takes the form

$$\langle n_{\mathbf{k}} \rangle = Z\, \theta(\mu - \mathcal{E}_{\mathbf{k}}) \,, \tag{1.2.22}$$

where θ is the Heaviside step function. This result has an important implication, namely that in an interacting system the Fermi surface exists, provided $Z \neq 0$ and that perturbation theory is applicable under the interaction considered. The existence and stability of the Fermi surface in turn guarantees that the quasi-particles are well defined and that the low-energy physics of the system can be determined by these quasi-particles excitations near the Fermi surface as in the free case.

1.3 BCS Theory

In 1957, Bardeen, Cooper and Schrieffer (BCS) published one of the most successful theories in the history of physics [40]. Starting from very general physical assumptions and sensible simplifications, BCS theory provides a microscopic description of conventional superconductors that accounts for a wide array of their physical phenomena. One of the most important clues necessary for the construction of BCS theory can be found in the experimental observation of the *isotope effect* [41]

$$T_c \propto M^{-1/2}\,, \tag{1.3.1}$$

where T_c is the superconductor critical temperature and M is the mass of the crystal lattice ions. This dependence indicates that electron-phonon interactions play an important role in conventional superconductivity.

Indeed, BCS theory is ultimately a theory of electron-phonon interaction. The first physical intuition of how this interaction can be realized came from Fröhlich [42], who made the observation that conduction electrons could attract each other due to interaction with the material's ion cores. The physical picture would be that, on passing through the metallic grid, a first electron conduction attracts a positive ions in its vicinity, while this excess of positive ions in turns attracts a second electron, creating an effective attractive interaction between both electrons. Another way to look at it is that the lattice deformation creates phonons, which mediate attractively between electrons. Starting from a very general model of electron-phonon interaction, Fröhlich arrived at the following Hamiltonian [43]

$$\begin{aligned}\mathcal{H}_{\text{Frölich}} &= \sum_{\mathbf{k},\sigma} \varepsilon_{\mathbf{k}} c^\dagger_{\mathbf{k},\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(B^\dagger_{\mathbf{k}} B_{\mathbf{k}} + \frac{1}{2} \right) \\ &\quad + \sum_{\mathbf{k}\mathbf{k}'\mathbf{q},\sigma\sigma'} V_{\mathbf{k}q} c^\dagger_{\mathbf{k}'-\mathbf{q},\sigma'} c^\dagger_{\mathbf{k}+\mathbf{q},\sigma} c_{\mathbf{k},\sigma} c_{\mathbf{k}',\sigma'} + \cdots \end{aligned} \tag{1.3.2}$$

where the first term represents free quasi-electrons, the second term represents free phonons, and the third term represents electron-phonon interaction. Here, $\varepsilon_{\mathbf{k}}$ is the quasi-electron energy measured with respect to the Fermi energy, and $\omega_{\mathbf{k}}$ is the phonon frequency. The interaction $V_{\mathbf{k}\mathbf{k}'}$ is given as

$$V_{\mathbf{k}\mathbf{k}'} = \frac{4\pi e^2}{k'^2 + \lambda^2} + \frac{2\omega_{\mathbf{k}'} \left| M_{\mathbf{k}'} \right|^2}{(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}+\mathbf{k}'})^2 - \omega^2_{\mathbf{k}'}}\,. \tag{1.3.3}$$

The first term corresponds to the shielded Coulomb interaction in the Fermi-Thomas approximation, λ is related to the screening length, and $|M_{\mathbf{q}}|$ is proportional to the shielded electron-phonon coupling. We now make some observations about the second term in (1.3.3). First, we substitute $\omega_{\mathbf{k}} \to \omega_D$, where ω_D is the Debye frequency. The reason for this substitution is that ω_D is the typical phonon frequency and acts as a cutoff scale in the electron-phonon interaction. The second thing we note is the experimental observation that superconductivity comes from electrons near the Fermi surface. This observation is in agreement with the Fermi Liquid theory result, that quasi-particles have well defined meaning near-k_F. Therefore we arrive at the original BCS assumption, based on Fermi Liquid theory, that the relevant quasi-electrons have energies in a thin shell of width $\pm\omega_D$ near the Fermi surface.

$$|\varepsilon_{\mathbf{k}}| < \omega_D \,. \tag{1.3.4}$$

With these assumptions, the second term is negative and represents an attractive interaction. Moreover, in a superconductor the electron-phonon coupling $|M_{\mathbf{k}}|$ is large, with the result that the second term is dominant and the effective total interaction between quasi-particles is attractive.

The next building block in BCS theory was set by Cooper in 1956 [44], when he showed the surprising result that two electrons outside the Fermi sea subjected to an attractive interaction between them would form a bound state, regardless of how weak the interaction is. More precisely stated, the Fermi sea is unstable against the "pairing" of an electron in a state $\mathbf{k},\uparrow$ with an electron with $-\mathbf{k},\downarrow$, forming a *Cooper Pair*.

A simple way to see this in looking at the two-particle wave function for such electrons[2]

$$\Psi(\mathbf{r}_1,\sigma_1;\mathbf{r}_2,\sigma_2) = \phi(\mathbf{r}_1-\mathbf{r}_2)\varphi_{\sigma_1\sigma_2} \,, \tag{1.3.5}$$

where $\varphi_{\sigma_1\sigma_2}$ represents the spin part of the wave function, which can be a spin singlet or triplet. The function $\phi(\mathbf{r}_1-\mathbf{r}_2)$ can be written as

$$\phi(\mathbf{r}_1-\mathbf{r}_2) = \sum_{\mathbf{k}} g(\mathbf{k})\, e^{i\mathbf{k}(\mathbf{r}_1-\mathbf{r}_2)} \,, \tag{1.3.6}$$

where $|g(\mathbf{k})|^2$ is the probability of finding a electron with momentum $\mathbf{k}$ and the other with momentum $-\mathbf{k}$. Since according to the Pauli exclusion principle all

[2]We are working in the center of mass frame.

electronic states with $|\mathbf{k}| < |\mathbf{k}_F|$ are completely filled and cannot be occupied by other electrons, then

$$g(\mathbf{k}) = 0, \quad \text{If} \quad |\mathbf{k}| < |\mathbf{k}_F| \,. \tag{1.3.7}$$

The Schrödinger equation for the two electrons system is given by

$$\frac{1}{2m}\left(\nabla_1^2 + \nabla_2^2\right)\Psi + V(\mathbf{r}_1, \mathbf{r}_2)\Psi = (E + 2\epsilon_F)\Psi\,, \tag{1.3.8}$$

where ϵ_F is the Fermi energy and $V(\mathbf{r}_1, \mathbf{r}_2)$ is the attractive interaction potential. Substituting (1.3.6) in (1.3.8) we get the following equation

$$\frac{k^2}{m}g(\mathbf{k}) + \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'}g(\mathbf{k}') = (E + 2\epsilon_F)g(\mathbf{k})\,. \tag{1.3.9}$$

Since we know that only electrons that fulfill condition (1.3.4) are relevant to superconductivity, we follow the *BCS approximation* that the interaction potential has the form

$$V_{\mathbf{k}\mathbf{k}'} = -V\,, \quad \text{If} \quad |\varepsilon_{\mathbf{k}}|\,, |\varepsilon_{\mathbf{k}'}| < \omega_D\,, \tag{1.3.10}$$

and equal to zero otherwise. Then the equation (1.3.9) is

$$g(\mathbf{k})\left(E + 2\epsilon_F - \frac{k^2}{m}\right) = -V\sum_{\mathbf{k}'} g(\mathbf{k}') = \lambda\,, \tag{1.3.11}$$

where λ is a separation constant. We then obtain the consistency equation

$$1 = V\sum_{\mathbf{k}} \frac{1}{\frac{k^2}{m} - E - 2\epsilon_F}\,. \tag{1.3.12}$$

We now introduce the density of electron states per spin direction

$$N(\xi) = \frac{4\pi\, k^2}{(2\pi)^2}\frac{dk}{d\xi}\,, \tag{1.3.13}$$

and the summation can be substituted by the integral

$$1 = V\int_0^{\omega_D} N(\xi)\frac{1}{2\xi - E}d\xi\,. \tag{1.3.14}$$

We can now replace $N(\epsilon) \approx N(0)$, with $N(0) = mk_F^2/2\pi^2$ (the density of electron states at the Fermi surface), since in metals $\omega_D \ll \epsilon_F$. Then we have

$$1 = \frac{N(0)V}{2}\ln\frac{E - 2\omega_D}{E}\,, \tag{1.3.15}$$

from where we finally obtain

$$E_{\text{crit.}} \approx -2\omega_D e^{-1/N(0)V}\,, \tag{1.3.16}$$

where we have taken the limit $N(0)V \ll 1$. This approximation is justified by the fact that most standard superconductors have $N(0)V < 0.3$. Indeed, this dimensionless quantity $N(0)V$ characterizes the strength of the interaction and therefore we conclude that *BCS is a weakly coupled theory.* Since $E_{\text{crit.}} < 1$, the two electrons form a bound state, regardless of how small the value of V is. The same pairing process can be realized for the case of many electrons. This leads us to take only in consideration interaction terms in (1.3.2) that occur in Cooper pairs. The result is the effective *pairing* or *reduced* Hamiltonian

$$\mathcal{H}_{\text{pair.}} = \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}} c_{\mathbf{k}\sigma} c^{\dagger}_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}\,. \tag{1.3.17}$$

The Hamiltonian written above represents a fully interacting system, which is difficult to solve it exactly. We can further simplify it by nothing that the ground state will be composed of coherent Cooper pairs that can have non-zero expectation values

$$b_{\mathbf{k}} = \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle\,. \tag{1.3.18}$$

and that any fluctuation around these expectation values $b_{\mathbf{k}}$ can be neglected because of the large number of particles. We can therefore write

$$c_{-\mathbf{k}\downarrow} c_{-\mathbf{k}\uparrow} = b_{\mathbf{k}} + \left(c_{-\mathbf{k}\downarrow} c_{-\mathbf{k}\uparrow} - b_{\mathbf{k}}\right), \tag{1.3.19}$$

and neglect any bilinear terms in the fluctuation term in parenthesis. We then finally obtain the *model Hamiltonian*

$$\mathcal{H}_{\text{mod.}} = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k},\sigma} + \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \left(b_{\mathbf{k}'} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} + b^{\dagger}_{\mathbf{k}} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} - b^{*}_{\mathbf{k}} b_{\mathbf{k}'} \right). \tag{1.3.20}$$

We now define

$$\begin{aligned} \Delta_{\mathbf{k}} &= -\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} b_{\mathbf{k}'} \\ &= -\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \rangle\,. \end{aligned} \tag{1.3.21}$$

Substituting in (1.3.20) we obtain

$$\mathcal{H} = \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} \left(\Delta_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\downarrow} c_{-\mathbf{k}\downarrow} + \Delta^{*}_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} - \Delta_{\mathbf{k}} b^{*}_{\mathbf{k}} \right) , \tag{1.3.22}$$

which is now put in an bilinear form which can be diagonalized. Following Bogoliubov and Valatin [45], we propose the canonical transformation

$$c_{\mathbf{k}\uparrow} = u_{\mathbf{k}} \alpha_{\mathbf{k}} + v_{\mathbf{k}} \beta^{\dagger}_{\mathbf{k}} , \tag{1.3.23}$$

$$c_{-\mathbf{k}\downarrow} = u_{\mathbf{k}} \beta_{\mathbf{k}} - v_{\mathbf{k}} \alpha^{\dagger}_{\mathbf{k}} . \tag{1.3.24}$$

with the unitarity condition

$$u^2_{\mathbf{k}} + v^2_{\mathbf{k}} = 1 . \tag{1.3.25}$$

These transformations simplifies our system to that of two different types of quasi-fermions $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$, each in a ideal Fermi gas model, if we set

$$u^2_{\mathbf{k}} = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{\sqrt{\Delta^2_{\mathbf{k}} + \xi^2_{\mathbf{k}}}}\right) , \tag{1.3.26}$$

$$v^2_{\mathbf{k}} = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{\sqrt{\Delta^2_{\mathbf{k}} + \xi^2_{\mathbf{k}}}}\right) , \tag{1.3.27}$$

$$u_{\mathbf{k}} v_{\mathbf{k}} = -\frac{\Delta_{\mathbf{k}}}{\sqrt{\xi^2_{\mathbf{k}} + |\Delta_{\mathbf{k}}|^2}} . \tag{1.3.28}$$

These very specific values for the coefficients $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ can alternatively be obtained by making a variational analisis in order to find a minimal BCS ground-state wave-function for the pairing Hamiltonian. (See, for instance [29]). Then the operators $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ obey the algebra

$$\left\{\alpha_{\mathbf{k}}, \alpha^{\dagger}_{\mathbf{k}'}\right\} = \left\{\beta_{\mathbf{k}}, \beta^{\dagger}_{\mathbf{k}'}\right\} = \delta_{\mathbf{k}\mathbf{k}'} , \tag{1.3.29}$$

and other commutation combinations are equal to zero. The resulting diagonal Hamiltonian is

$$\mathcal{H} = E_0 + \sum_{\mathbf{k}} \sqrt{\varepsilon^2_{\mathbf{k}} + |\Delta_{\mathbf{k}}|^2} \left(\alpha^{\dagger}_{\mathbf{k}} \alpha_{\mathbf{k}} + \beta^{\dagger}_{\mathbf{k}} \beta_{\mathbf{k}} \right) , \tag{1.3.30}$$

where

$$E_0 = \sum_{\mathbf{k}} \left(\varepsilon_{\mathbf{k}} - \sqrt{\varepsilon_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2} + \Delta_{\mathbf{k}} b_{\mathbf{k}} \right) \tag{1.3.31}$$

is a constant term and represents the ground state energy of the interacting superconductor. The second term in (1.3.30) represent the increase of energy above the ground state. In this sense, the operators $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ represent the quasi-particle excitations of the superconductor. These quasi-particles are called *Bogoliubons*, and have energy

$$E_{\mathbf{k}} = \sqrt{\varepsilon_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}\,. \tag{1.3.32}$$

We see that $\Delta_{\mathbf{k}}$ represents an energy gap between the ground state energy and that of the first excited state. The quantity $\Delta_{\mathbf{k}}$ is known as the *order parameter* of the superconductor, and is of great physical significance. As proved by Gorkov [46], it can be directly related to the Ginzburg-Landau order parameter Ψ introduced in Section 1.1.

We now will try to determine the order parameter $\Delta_{\mathbf{k}}$. We can do it by going back to its original definition (1.3.21), which can be rewritten in terms of the operators $\alpha_{\mathbf{k}}$, $\beta_{\mathbf{k}}$ as

$$\begin{aligned} \Delta_{\mathbf{k}} &= -\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \rangle \\ &= -\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} u_{\mathbf{k}'} v_{\mathbf{k}'} \langle 1 - \alpha_{\mathbf{k}'}^\dagger \alpha_{\mathbf{k}'} - \beta_{\mathbf{k}'}^\dagger \beta_{\mathbf{k}'} \rangle\,. \end{aligned} \tag{1.3.33}$$

Since the total average number of quasi-particles $\alpha_{\mathbf{k}}$, $\beta_{\mathbf{k}}$ is not fixed as in the electron case, their chemical potential is zero in thermal equilibrium. Also, as noted in the Hamiltonian (1.3.30), the quasiparticles do not interact, thanks to our choice of coefficients $u_{\mathbf{k}}$, $v_{\mathbf{k}}$. Therefore, the quasi-particles will have occupation given by the Fermi-Dirac distribution

$$f(E_{\mathbf{k}}) = \frac{1}{e^{\beta E_{\mathbf{k}}} + 1}\,, \tag{1.3.34}$$

with $\beta = 1/(kT)$. Then (1.3.33) becomes

$$\Delta_{\mathbf{k}} = -\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} u_{\mathbf{k}'} v_{\mathbf{k}'} \left(1 - f(E_{\mathbf{k}})\right) = \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \frac{\Delta_{\mathbf{k}'}}{2E_{\mathbf{k}'}} \tanh \frac{\beta E_{\mathbf{k}'}}{2}\,. \tag{1.3.35}$$

Equation (1.3.35) is called the *gap equation* and is very important in BCS theory since it predicts the temperature dependence of $\Delta_{\mathbf{k}}$ and the critical temperature T_c of the system. We now make the BCS approximation (1.3.10) and assume that $V_{\mathbf{kk'}}$ is equal to a constant value $V_{\mathbf{kk'}} = -V$ in a thin shell around the Fermi surface. We also assume a similar behavior for the order parameter and set $\Delta_{\mathbf{k}} = \Delta$. Then the gap equation is

$$1 = \frac{V}{2} \sum_{\mathbf{k}} \frac{\tanh(\beta E_{\mathbf{k}}/2)}{E_{\mathbf{k}}} . \tag{1.3.36}$$

Since we are working within the thin shell of states around the Fermi energy, with $|\xi_{\mathbf{k}}| < \hbar\omega_D$, we can replace the summation (1.3.36) with the integral

$$1 = \frac{N(0)V}{2} \int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{\tanh(\beta E/2)}{E} d\xi \, , \tag{1.3.37}$$

where $E = \sqrt{\xi^2 + \Delta^2}$. From this equation in the limit $\Delta \approx 0$ one can estimate the value of the critical temperature T_c

$$T_c \approx 1.13\, \omega_D e^{-1/(N(0)V)} \, , \tag{1.3.38}$$

and similarly, in the limit $T = 0$ on finds

$$\Delta(0) \approx 2\, \omega_D e^{-1/(N(0)V)} \, , \tag{1.3.39}$$

where we used the small coupling limit. From these equation we can deduce the famous BCS result

$$T_c = 0.56\, \Delta(0) \, , \tag{1.3.40}$$

which holds in numerous superconductors. Additionally, from (1.3.37) one finds that the order parameter behaves near the critical temperature as

$$\Delta(T \approx T_c) \sim (1 - T/T_c)^{1/2} \, , \tag{1.3.41}$$

which is a robust functional dependence and confirms the Ginzburg-Landau theory result (1.1.14).

To finish this section, lets consider BCS theory from the point of view of the Green's function, just as we did with the Fermi Liquid theory. We follow closely [37]. We consider the Bogoliubon's Hamiltonian (1.3.30). The Green's function should be of the form

$$G(\mathbf{k},\omega) \sim \frac{1}{\omega - \sqrt{\varepsilon_{\mathbf{k}}^2 + |\Delta|_{\mathbf{k}}^2} + i\delta} \,. \tag{1.3.42}$$

One can expand the Green's function (1.3.42) in term of Feynman diagrams in the usual fashion, where the interaction will be given by originally by (1.3.3) that can be set to a constant $-V$ in the BCS approximation. However, when one calculates the Green function perturbatively, one finds that there are some classes of one-loop diagrams that render the expansion unstable. These diagrams in fact represent physical processes where particles with momenta of equal magnitude but opposite direction (that is, Cooper pairs) scatter against each other. (See [37] for details on the calculation.) Calculating the contribution of these diagrams to the self energy up to one-loop level one can obtain the result that the dressed vertex is

$$\tilde{V}(\omega) = -\frac{V}{1 - VN(0)\left\{\frac{1}{2}\ln\left|\frac{(2\omega_D - \omega^2)}{\omega^2}\right| + i\frac{\pi}{2}\theta_{2\omega_D-\omega}\theta_{\omega+2\omega_D}\right\}} \tag{1.3.43}$$

where ω_D has been taken as a cutoff. We observe that there is a pole in the denominator of (1.3.43) which in the $\omega \ll \omega_D$ limit is given by

$$\omega_{\text{pole}} = i\, 2\omega_D\, e^{1/(N(0)V)}. \tag{1.3.44}$$

The pole is purely imaginary and is located on the half upper side of complex frequency space. Moreover, it is also to be found in the retarded Green's function, which must be analytic in the upper-half plane. This leads to the conclusion that the perturbation series leading to (1.3.43) is invalid and indicates that the normal system is unstable and will undergo a phase transition into a superconducting phase. This breakdown of perturbation theory, according to our Fermi Liquid theory discussion, means that the states in the normal phase and those in the superconducting phase are qualitatively different and that an adiabatic process is invalid when there is a phase transition involved in the middle. This observation will bear relevance in the following section.

1.4 High-Temperature Superconductivity

The area of *High-Temperature Superconductivity* was inaugurated by the discovery by Bednorz and Müller in 1986 of the onset of superconductivity around $T_c = 35K$ [47]. In general, a material is considered a high-T_c superconductor if its critical temperature is $T_c \sim 30K$ or higher. In their original discovery, Bednorz and Müller used a kind of material called *cuprate*. Given that many high-T_c superconductors belong to this class of materials, one should provide a general description of them. For more details on the physical properties and phenomenology of the cuprates, see [48]. Cuprates are originally antiferromagnetic Mott Insulators that, after being slightly doped, become superconducting on cooling. Regarding their specific microscopic structure, they are a variation of the crystal type known as *Perovskite*. These are minerals with tetragonal structure whose chemical formula is given in general by ABX_3 or AB_2X_3, where the element X is usually oxygen.

As a very important property, we note that they are structurally composed of 2-dimensional CuO_2 layers, and superconductivity in fact occurs in these copper-oxide planes. The CuO_2 layers are mediated by layers of other elements, called charge reservoirs, which provide the charge carriers necessary for superconductivity. It is found experimentally that the distance between the charge reservoirs and the CuO_2 planes has a high impact on the value of the critical temperature, and that higher-T_c is correlated with shorter layer distances.

Apart from the inter-layer distance mentioned above, cuprate superconductors have many other parameters that affect the value of their critical temperature. Because of its importance, we will focus on the doping p of the material. In conventional superconductors, one finds a linear relation between doping an

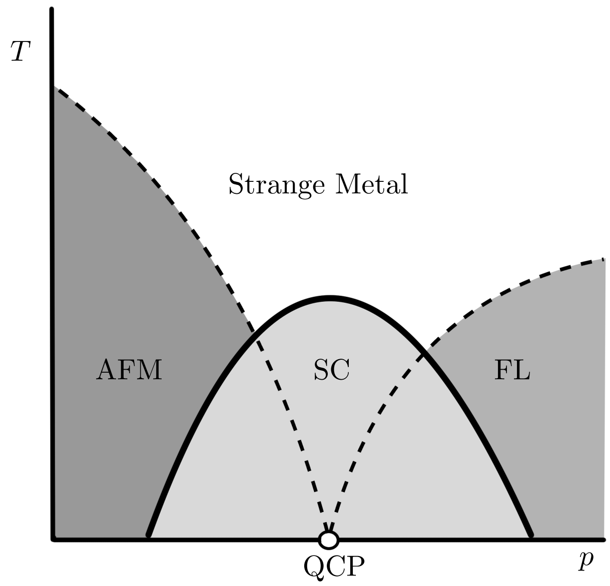


Figure 1.2: Generic phase diagram of a high-T_c superconductor. We show the regions corresponding to the antiferromagnetic phase, superconducting phase, Fermi liquid phase and strange metal phase. Inside the *superconducting dome* (dashed line) we find a quantum critical point.

critical temperature, $T_c(p) \sim p$. This is very much changed in the case of high-T_c superconductivity, where the relation is non-monotonic. Indeed, one finds in the case of hole-doped cuprates a general dependency

$$T_c(p) \sim T_{c,\max}\left(1 - \alpha(p - \beta)^2\right) , \tag{1.4.1}$$

where α, β are fitting parameters. In either hole or electron-doped cuprates, one finds a bell-like profile, with a particular value of p, called the *optimal doping*, where the critical temperature attains its highest value $T_{c,\max}$. This optimum value for T_c is taken as the critical temperature for the system.

High-T_c superconductors have a very rich phase diagram in terms of the doping parameter. In figure (1.2) we show an schematic phase diagram for cuprate superconductors. The same basic structure is found in a wide variety of cuprates. Some observations regarding this diagram are in order. We first observe that at low doping the system finds itself in the anti-ferromagnetic phase of its parent compound. Then, after more doping the system enters the superconducting phase, in a region which has the bell profile mentioned above and that known as the *superconducting dome*. In the high doping region, the system enters a region of Fermi-Liquid behaviour. Above the superconducting dome, in the normal phase of the superconductor, the system finds itself in a *strange metal region*, meaning that the material has Non Fermi Liquid behaviour. For instance, experimental evidence reveals that near optimal doping and in the normal phase, the cuprates quasi-particles have lifetime behaviour

$$\frac{1}{\tau} \sim \omega \,, \tag{1.4.2}$$

as opposed to the quadratic Fermi Liquid behaviour described in (1.2.19). Another different behavior can be found in the temperature dependence of the resistivity in cuprates above T_c, which is of the form $\rho \sim T$, whereas Fermi Liquid theory predicts a dependency of the form $\rho \sim T^2$ for 3-dimensional systems and $\rho \sim T^2 \log(1/T)$ for 2-dimensional systems. One of the proposed explanations for this Non Fermi Liquid behaviour is that the normal state is close to a quantum phase transition for some value of doping, where strong quantum fluctuations would cause the deviation from standard Fermi Liquid theory. Thus, removing the superconducting dome, the conjecture is that there is a quantum critical point separating the antiferromagnetic and the Fermi Liquid phases. This is shown in figure (1.2).

Let us pause for a second and go back to the superconducting phase of the cuprates. By means of ARPES[3] measurements on the copper-oxide layers, the

[3]In Angle Resolved Photo Emission (ARPES), an X-ray photon of know energy and momentum excites an electron out of the surface of the superconductor. By measuring the emitted electron's energy and momentum one can determine its original energy and crystal momentum. Finally, by comparing the spectra as a function of temperature, one can map the order

order parameter of the superconductor was found to be of the form

$$\Delta_{\mathbf{k}} \sim (\cos(k_x) - \cos(k_y)) \ . \tag{1.4.3}$$

This means that a cuprate behaves as an unconventional superconductor, where the latter type is defined in general terms as superconductivity where the ground state has different symmetry from that of the BCS ground state. More precisely, given a symmetry transformation $\hat{T}$ in momentum space, a superconductor is unconventional if

$$\Delta_{\mathbf{k}} \neq \Delta_{\hat{T}\mathbf{k}} \, , \tag{1.4.4}$$

while the equality holds in the conventional case. For an unconventional superconductor the order parameter can be written as

$$\Delta_{\mathbf{k}} = \sum_{\Gamma} \eta_{\Gamma} f_{\Gamma}(\mathbf{k}) \, , \tag{1.4.5}$$

where $f_{\Gamma}(\mathbf{k})$ are a set of functions defined in terms of the irreducible representations Γ of the symmetry group, and η_{Γ} are expansion coefficients.

The cuprate order parameter Δ_k has $d_{x^2-y^2}$ symmetry and describes a superconductor with a spin-singlet, $l = 1$ pairing state. It is called a *d-wave* order parameter, since it has the same symmetry as an atomic d spherical harmonic function. The particular symmetry of $\Delta_{\mathbf{k}}$ suggest that the kind of pairing mechanism relies on strong electron-electron repulsion at short range. The most likely type of pairing mechanism that leads to $d_{x^2-y^2}$ Cooper pairs are based on systems with strong electron-electron repulsion. In materials with strong repulsive energy it is favorable to form wave functions that are zero at $\mathbf{r}_1 = \mathbf{r}_2$, which can be accomplished by paired states with $l \neq 0$. This is further supported by the fact that the parent compound of the cuprates are antiferromagnetic Mott insulators, which have this kind of strong interaction.

This symmetry therefore makes it reasonable to introduce spin degrees in a theory describing the strange-metal phase of the cuprates. These spin fluctua-

parameter $|\Delta_{\mathbf{k}}|$ at any point of the Fermi surface.

tions would be similar to those found in the well known Fermi Liquid description of ^{3}He, where in that case ferromagnetic spin fluctuations take the place of phonons in the electronic interactions[4]. One of the proposed models trying to describe the strange metal physics of the cuprates while incorporating the physical clues mentioned above is the *Spin-Fermion model*. We review it very briefly, but knowledge of it may provide us with the general flavor of some relevant difficulties that appear when applying standard theoretical methods in this problem.

1.5 A Field Theoretical Model.

The theoretical model we will now consider, the Spin-Fermion model [49, 50], is a low energy theory which attempts to provide a generic description of the quantum phase transition between the Fermi-Liquid and the antiferromagnetic phases. In this sense, it is an attempt to explain the anomalous normal state properties of high T_c materials. It has the natural fermionic degrees of freedom $c_{\mathbf{k}}$, given that electrons have arbitrary low energy near the Fermi surface. It however introduces spin degrees of freedom, based on the proximity of the antiferromagnetic phase. These spin fluctuations are given by the Spin Density Wave $\mathbf{S}_{\mathbf{k}}$, and are bosonic collective modes of the fermions. The main idea is that at low energies there is a dominant channel in the fermion-fermion interaction which introduces as spin collective mode that mediates between them. In this sense, as mentioned above, they are spinful analogs of phonons in BCS theory. The Hamiltonian of the theory can be written as [49]

$$\mathcal{H}_{\text{S-F}} = \sum_{\mathbf{k}\alpha} G_0(\mathbf{k},\omega) c^\dagger_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha} + \sum_{\mathbf{k}} \chi_0^{-1}(\mathbf{k},\omega) \mathbf{S}_{\mathbf{k}} \mathbf{S}_{-\mathbf{k}} + g \sum_{\mathbf{k}\mathbf{k}'\alpha\beta} c^\dagger_{\mathbf{k}+\mathbf{k}',\alpha} \sigma_{\alpha,\beta} c_{\mathbf{k}\beta} \cdot \mathbf{S}_{-\mathbf{k}'}\,, \tag{1.5.1}$$

[4]Phonon interactions are believed not to play an important role in high-T_c superconductivity. Indeed, the isotope effect (1.3.1) is experimentally found to be extremely small in the cuprates. This is usually taken as evidence that phonons should play a negligible part in a theory of cuprate superconductivity.

where $G_0(\mathbf{k},\omega)$ is the fermions bare green's function, $\chi_0(\mathbf{k},\omega)$ is the bare spin propagator, σ_i are the Pauli matrices and α, β are spin projection indexes. The first term in (1.5.1) represents free fermions; the second represents free spin degrees of freedom, and the third represents the spin-fermion interaction. The first result comes from the bare spin propagator. It is given by the Ornstein-Zernike form

$$\chi_0(\mathbf{k},\omega) = \frac{\chi_0}{\xi^{-2} + (\mathbf{k}-\mathbf{K})^2 - (\omega/v_s)^2}\,, \tag{1.5.2}$$

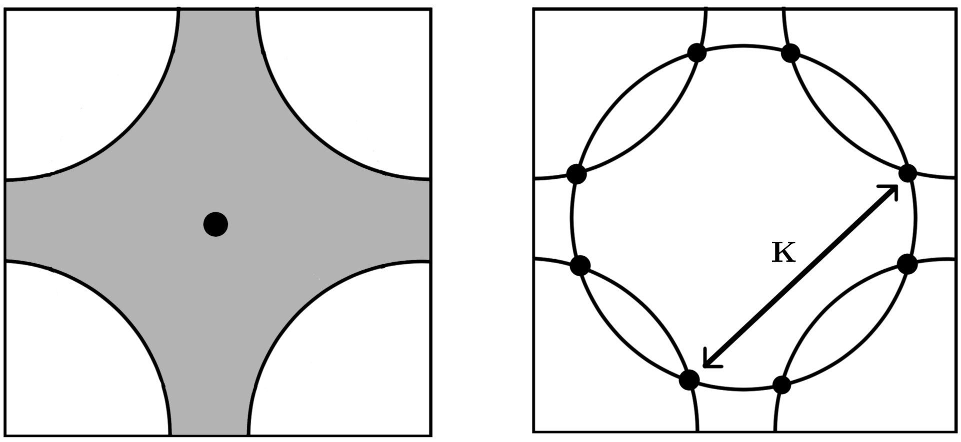
where ξ is the spin correlation length and v_s is the spin velocity of the order v_F, since the spin degrees of freedom are made of fermions. We have also introduced the *ordering spin wave vector* $\mathbf{K}$. In this case it was found that the spin-singlet gap equation is

$$\Delta_{\mathbf{k}} = -\frac{3}{2}g^2 \int \chi_0(\mathbf{k}-\mathbf{q})\Delta_{\mathbf{k}} \frac{\tanh\left(\beta\sqrt{E_{\mathbf{q}}^2+\Delta_{\mathbf{q}}^2}\Big/2\right)}{2\sqrt{E_{\mathbf{q}}^2+\Delta_{\mathbf{q}}^2}} d\mathbf{q}\,, \tag{1.5.3}$$

where the minus sign comes from projection to the spin singlet channel. Comparing with (1.3.35), we note that a BCS-like s-wave solution is not possible because of this negative sign. However, since $\chi(\mathbf{k},\omega)$ is peaked near $\mathbf{K}$, the pairing interaction relates the gap at momentum $\mathbf{k}$ and $\mathbf{k}+\mathbf{K}$. Thus we can eliminate the minus sign by proposing the ansatz $\Delta_{\mathbf{k}} = -\Delta_{\mathbf{k}+\mathbf{K}}$. For the cuprates one has $\mathbf{K} = (\pi,\pi)$ and this implies $d_{x^2-y^2}$ symmetry for $\Delta_{\mathbf{k}}$. Thus, the spin fluctuation proposed gives rise to d-wave order parameter $\Delta_{\mathbf{k}} \sim (\cos(k_x) - \cos(k_y))$.

However, although the spin-fermion model correctly predicts the d-wave behavior of the order parameter, it also predicts that the system is strongly coupled. Indeed, so far the theory relies on the parameters g, χ_0, ξ and v_F. From the first two, we can construct the effective coupling $\overline{g} = g^2\chi_0$, which is a combination that appears naturally in perturbation theory. From the renamed parameter one can construct the energy scale v_F/ξ. From these two energies one can construct the dimensionless constant

$$\lambda = \frac{3\overline{g}}{4\pi v_F \xi^{-1}}\,. \tag{1.5.4}$$



(a) Fermi Liquid Phase (b) Antiferromanetic Order

Figure 1.3: The transformation of the cuprate Fermi surface by antiferromagnetism. In figure (a) we show the Fermi surface in the Fermi Liquid phase. The central point represents momentum $\mathbf{k} = (0, 0)$. In figure (b) we show the original Fermi surface along with the Fermi surface shifted by the wave vector $\mathbf{K} = (\pi, \pi)$. These surfaces intersect at the *hot spots*, represented by the filled circles. Electrons near the hot spots, separated by $\mathbf{K}$, have opposite sign in the pairing amplitude $\Delta_{\mathbf{k}} = -\Delta_{\mathbf{k}+\mathbf{K}}$, leading to unconventional superconductivity.

As it turns out, the dimensionless constant λ is the natural coupling in the perturbative calculations for the fermionic and bosonic self energies, and is thus the effective coupling for the system. In the case of cuprates, experimental observations estimate a value of $\lambda \sim 2$ near optimal doping, making thus the system strongly-coupled. This sets a limit to the applicability of weak-coupling perturbative methods.

More characteristics of the spin fermion model can be obtained from the structure of the Fermi surface for the cuprates. In general, the addition of antiferromagnetic spin fluctuations alter the dispersion relation $\varepsilon_{\mathbf{k}}$ for the fermions, which in turn also changes the structure of the Fermi surface. More concretely, the antiferromagnetic order mixes electrons states with $\mathbf{k}$ and $\mathbf{k} + \mathbf{K}$, shifting the Fermi surface by $\mathbf{K}$, where, as said before, for the cuprates takes the value $\mathbf{K} = (\pi, \pi)$. This is shown in figures (1.3a) and (1.3b), where we see how the Fermi surface is shifted, intersecting at eight points, usually called *hot spots*. The hot spots are of great importance, as can be seen in the following.

We very generally present the results of [51, 52], where the authors made an study of the spin-fermion model (1.5.1) near the hot spots. In the proximity of one hot spot there are two Fermi lines, and the relevant fermionic quasi-particles along these lines are referred as ψ_{1a}, ψ_{2a}, with $a = \uparrow, \downarrow$, with their momentum measured with respect to the hot spot momentum $\mathbf{k}_h$. Near the hot spots the fermions are described by the Lagrangian [52]

$$\mathcal{L} = \psi_{1a}^{\dagger} \left(\partial_t - i\mathbf{v}_1 \cdot \nabla\right) \psi_{1a} + \psi_{2a}^{\dagger} \left(\partial_t - i\mathbf{v}_2 \cdot \nabla\right) \psi_{2a} \, , \tag{1.5.5}$$

where $\mathbf{v}_i$ is the Fermi velocity at $\mathbf{k}_h$. These fermions are to be coupled to the spin density wave represented by the ferromagnetic parameter ϕ_a, with $a = x, y, z$ spin components. These are described by the Lagrangian term [52]

$$\mathcal{L}_{\psi\phi} = \frac{1}{2} \left(\nabla \phi_a\right)^2 + \frac{r}{2} \phi_a^2 + \frac{u}{4} \left(\phi_a^2\right)^2 + U \phi_\alpha \sigma_{ab}^{\alpha} \left(\psi_{1a}^{\dagger} \psi_{2b} + \psi_{2a}^{\dagger} \psi_{1b}\right) \, , \tag{1.5.6}$$

where σ are the Pauli matrices. We see in (1.5.5) and (1.5.6) the same structure as in (1.5.1). Very interestingly, the authors find a vertex instability similar to the one encountered in the BCS case (1.3.43) that could lead to a superconducting phase. However, they also find that, in the vicinity of the hot spot, the ψ_1 Green's function has the general structure

$$G_{\text{hot spot}} \sim \frac{1}{\sqrt{i\omega} - \mathbf{v}_1 \cdot \mathbf{k}} \, , \tag{1.5.7}$$

so there is no quasi-particle pole and therefore quasi-particles are not well defined at the hot spots of the cuprates Fermi surface.

We thus see in the spin fermion model some of the virtues and defects of the usual field-theoretical approaches to high-temperature superconductivity. We find that the theory's effective coupling is strong, which seriously limits the extent in which perturbation theory is applicable. Furthermore, we observe the breakdown of the quasi-particle picture, a fact that is more profound and could point to the unsuitability of standard physical assumptions when trying to describe the cuprates. It is therefore necessary to look for different theoretical approaches to the problem, and holographic methods are suited to achieve just that.

2

AdS/CFT. An Introduction

The AdS/CFT correspondence is the most important recent development in theoretical physics. In its strongest form, it suggests that every non-Abelian gauge theory is equivalent to a consistent theory of gravity. The bridge connecting these apparently disconnected areas of physics is string theory, and its construction involves some of the fundamental actors of modern physics: non-abelian gauge symmetries, quantum field theory, general relativity, black hole physics, supersymmetry, physics in higher dimensions, etc. The AdS/CFT duality is a profound advancement in the understanding of fundamental physics both because of its conceptual depth and the wide range of its implications and applications.

The most studied example of AdS/CFT correspondence, or Maldacena duality, is the duality between Type IIB superstring and $\mathcal{N} = 4$ $SU(N)$ Super Yang-Mills theory. It is therefore sensible to start this chapter with a brief summary of these two theories. In particular, the correspondence relates the quantum physics of the strongly coupled gauge quantum field theory with the classical dynamics of gravity in higher dimensions. Therefore, in sections 1.3 to 1.5 of this chapter, this is the particular path we will follow when introducing the duality, first by looking at how the duality can be motivated by different string theory perspectives, by analyzing its different limits and by looking at evidence in favor of the conjecture. Finally, in section 1.6 we will review the basic aspects of scalar field holography, since this topic will be of particular importance for the holographic superconducting models we will be considering in the remaining of this thesis. Having thus set this basic theoretical background, in the next chapter we will merge the present discussion on the AdS/CFT duality with the one in preceding chapter on superconductivity, thereby introducing the subject of holographic superconductors, and explaining how these models can be realized in their simplest setup and illustrating their fundamental properties. This chapter follows closely the expositions of the duality presented in [53, 54, 55, 56].

2.1 Type IIB Supergravity

2.1.1 Field Content and Symmetries.

Ten-dimensional Type IIB supergravity [57, 58, 59, 60] is one of the two halves of the AdS/CFT correspondence, and we will briefly describe its basic characteristics. We start by its Lagrangian. Type IIB supergravity is the low-energy limit of Type IIB string theory, and the particle content of the former is given by the massless spectrum of the latter. We will only focus on the bosonic field content of Type IIB supergravity, which in the NS-NS sector is composed of the metric (*zehnbein*), the dilaton Φ and the two form B_2 with field strength $H_3 = dB_2$, while in the R-R sector one has the form fields C_0, C_2, C_4. The latter has a self-dual strength given by $\tilde{F}_5 = dC_4$. The existence of this self dual strength in the theory is a very particular property of Type IIB supergravity.

In opposition to the case of Type IIA gravity, the corresponding effective Lagrangian for Type IIB supergravity cannot be obtained from dimensional reduction of eleven-dimensional supergravity, which would corresponding to the low-energy limit of M theory. Furthermore, the presence self-dual five form becomes an impediment to formulating in a manifestly covariant form the action of the theory. Instead, the way to construct the Type IIB supergravity Lagrangian is to start from the covariant equations of motion, consistent with gauge invariance and supersymmetry, and then construct an action that yields these particular equations. The self duality of $\tilde{F}_5$ is then implemented as an additional field equation.

Constructed in this fashion, the bosonic part of the Type IIB supergravity action is given by

$$S_{\rm IIB} = S_{\rm NS} + S_{\rm R} + S_{\rm CS}\,, \tag{2.1.1}$$

with each term being

$$\begin{aligned} S_{\rm NS} &= \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\right)\,, & (2.1.2)\\ S_{\rm R} &= -\frac{1}{4\kappa^2}\int d^{10}x\sqrt{-g}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right)\,, & (2.1.3)\\ S_{\rm CS} &= -\frac{1}{4\kappa^2}\int C_4\wedge H_3\wedge F_3\,, & (2.1.4)\end{aligned}$$

and where the field strengths are written as $F_{n+1} = dC_n$, $H_3 = dB_2$ and we defined the gauge-invariant combinations

$$\begin{aligned} \tilde{F}_3 &= F_3 - C_0 H_3\,, & (2.1.5)\\ \tilde{F}_5 &= F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3\,. & (2.1.6)\end{aligned}$$

The self duality condition for the five-form

$$\tilde{F}_5 = *\tilde{F}_5\,, \tag{2.1.7}$$

has to be imposed as an additional constraint to the equations of motion that arise from (2.1.1).

Type IIB gravity has a hidden non-compact global symmetry $\mathrm{SL}(2,\mathbb{R})$. In particular, the two-form fields B_2 and C_2 transform as a doublet under this symmetry group. Then to make the global symmetry apparent, we introduce the following transformation Λ

$$\Lambda = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})\,, \tag{2.1.8}$$

with a, b, c, d real numbers such that $ad-bc=1$. Now we rename the two-form fields as $B_2 = B_2^{(1)}$, $C_2 = B_2^{(2)}$ and define the column vector

$$\mathcal{B}_2 = \begin{pmatrix} B_2^{(1)} \\ B_2^{(2)} \end{pmatrix}\,, \tag{2.1.9}$$

which in turns gives us the vector $\mathcal{H}_3 = d\mathcal{B}_2$, whose entries are given by the two-forms field strengths. This vector transforms under Λ as

$$\mathcal{B}_2 \to \Lambda \mathcal{B}_2\,, \tag{2.1.10}$$

and similarly for $\mathcal{H}_3$. We also introduce the complex scalar field

$$\tau = C_0 + i e^{-\Phi}\,, \tag{2.1.11}$$

which is called the *axion-dilaton field* because of its component fields and transforms under $\mathrm{SL}(2,\mathbb{R})$ as

$$\tau \to \frac{a\tau + b}{c\tau + c}\,. \tag{2.1.12}$$

We now rewrite the action (2.1.1) using the matrix

$$\mathcal{M} = e^{\Phi} \begin{pmatrix} |\tau|^2 & -C_0 \\ -C_0 & 1 \end{pmatrix}\,, \tag{2.1.13}$$

which transforms under Λ as

$$\mathcal{M}' = (\Lambda^{-1})^T \mathcal{M} \Lambda^{-1}\,. \tag{2.1.14}$$

With these definitions, the action (2.1.1) is finally rewritten as

$$\begin{aligned} S &= \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g^E} \left(R^E - \frac{1}{12} \mathcal{H}^T_{\mu\nu\rho} \mathcal{M} \mathcal{H}^{\mu\nu\rho} + \frac{1}{4} \mathrm{tr}\left(\partial^\mu \mathcal{M} \partial_\mu \mathcal{M}^{-1}\right)\right) \\ &\quad - \frac{1}{8\kappa^2} \left(\int d^{10}x \sqrt{-g^E} \left|\tilde{F}_5\right|^2 + \int \varepsilon_{IJ} C_4 \wedge H_3^I \wedge H_3^J \right)\,, \end{aligned} \tag{2.1.15}$$

which is manifestly invariant under global the $SL(2,\mathbb{R})$ symmetry, and where we have used the *Einstein-frame metric* $g^E_{\mu\nu}$, which is defined in terms of the usual string-frame metric $g_{\mu\nu}$ as

$$g^E_{\mu\nu} = e^{-\Phi/2} g_{\mu\nu} \,. \tag{2.1.16}$$

2.1.2 Brane Solutions in Type IIB Supergravity.

Brane supergravity solutions were historically the starting point to formulating the AdS/CFT correspondence. In this section we briefly review some of its general aspects. We start by recalling the definition of a $(p+1)$-form

$$A_{p+1} = \frac{1}{(p+1)!} A_{\mu_1 \cdots \mu_{p+1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}} \,, \tag{2.1.17}$$

Similarly to the manner in which a charged point particle couples to a gauge field, a $(p+1)$-form can be coupled naturally to a geometrical object Σ_{p+1} of space-time dimension $(p+1)$ by means of an action term

$$S_{p+1} = Q_p \int_{\Sigma_{p+1}} A_{p+1} \,, \tag{2.1.18}$$

where there is a pullback from the bulk space to Σ_{p+1}

$$\int_{\Sigma_{p+1}} A_{p+1} = \frac{1}{(p+1)!} \int_{\Sigma_{p+1}} A_{\mu_1 \cdots \mu_{p+1}} \frac{\partial x^{\mu_1}}{\partial \sigma^0} \cdots \frac{\partial x^{\mu_{p+1}}}{\partial \sigma^p} d^{p+1}\sigma \,. \tag{2.1.19}$$

Supergravity solutions with non-trivial A_{p+1} charge are called *p-branes*, and they represent geometrical objects with space dimension p and well defined charge Q_p. From (2.1.17) we can construct a gauge invariant $(p+2)$-form field strength $F_{p+2} = dA_{p+1}$, with

$$F_{p+2} = \frac{1}{(p+2)!} F_{\mu_1 \mu_2 \cdots \mu_{p+2}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+2}} \,, \tag{2.1.20}$$

and which is invariant under gauge transformations $\delta A_{p+1} = d\Lambda_p$, since the exterior derivative is closed, $d^2 = 0$. In complete analogy to the case of a charged point-particle, the charge of a p-brane is computed by encircling it by a S^{D-p-2} sphere and calculating its flux. Then, by Gauss's law in general D-dimensions

$$Q_p = \int_{S^{D-p-2}} *F_{p+2} \,, \tag{2.1.21}$$

where $*F_{p+2}$ is the Hodge-dual to the field strength. In general, given a charged p-brane, there is a dual *magnetic* $(D-p-4)$-*brane*, whose magnetic charge Q_{D-p-4} is computed from

$$Q_{D-p-4} = \int_{S^{p+2}} F_{p+2}\,. \tag{2.1.22}$$

In the particular case when $F_{p+2} = *F_{p+2}$, one says that the field strength is self-dual.

In the case of $D = 10$ Type IIA/B supergravity, p-branes are referred to as *Dp-branes* when the charge they carry come from a $(p+1)$-form in the R-R sector. In the context of string perturbation theory, a Dp-brane may be described as as $(p+1)$-dimensional hypersurface in flat 10-dimensional space-time on which open string can end [62]. In the particular case of Type IIB supergravity, these forms are given by C_0, C_2 and C_4 introduced above. Then, if we wish to construct Dp-branes solutions in Type IIB theory, we must consider the action

$$S^{(p)} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\,\left(e^{-2\Phi}(R+4/(\partial\Phi)^2) - \frac{1}{2}|F_{p+2}|^2\right)\,, \tag{2.1.23}$$

where we have already taken into considerations that in Dp-brane solutions, the NS-NS two-form B_2 vanishes. In the special case $p = 3$, the self-duality constraint $F_5 = \star F_5$ has to be imposed by hand and an extra 1/2 factor should be added in the term $|F_{p+2}|^2$.

Before presenting the general Dp-brane solution to (2.1.23), it is useful to make some general considerations about its geometry. We consider the case of general dimension D, for generality. As said above, a Dp-brane is a $(p+1)$-dimensional flat hypersurface, with Poincaré invariance $\mathbb{R}^{p+1}\times \mathrm{SO}(1,p)$. Therefore, the transverse space is $(D-p-1)$-dimensional and one can always find solutions with maximal rotational symmetry SO(D-p-1). Thus, the total symmetry of the Dp-brane solutions in $D = 10$ Type IIB supergravity is given by $\mathbb{R}^{p+1}\times \mathrm{SO}(1,p)\times \mathrm{SO}(9-p)$. The symmetry of a D$p$-brane just described can tell us a great deal of the general form of the solution: the Poincaré invariance in the space parallel to the Dp-brane tells us that the metric solution in those directions has to be a rescaling of the Minkowski metric, while the rotational invariance in the transverse directions tells us that the metric in those directions has to be a rescaling of the Euclidean metric. With this in consideration , one finds that the extremal Dp-brane solution is given for every p by [61]

$$ds^2 = H_p(r)^{-1/2}\eta_{ij}dx^i dx^j + H_p(r)^{1/2}\left(dr^2 + r^2 d\Omega_{8-p}\right) , \tag{2.1.24}$$

where r is a "radial" coordinate transverse to the brane. Assuming maximal SO(9-p) symmetry in the transverse directions, and using the fact that the metric should tend to flat-space time at $r \to \infty$, the most general solution for $H_p(r)$ is given by the harmonic function

$$H_p(r) = 1 + \left(\frac{L_p}{r}\right)^{7-p} , \tag{2.1.25}$$

where L_p is a numerical constant to be determined latter. The first part of the solution (2.1.24) correspond to the $(p+1)$-dimensional Lorentz metric along the brane, while the second term corresponds to the $(9-p)$-dimensional euclidean metric in the transversal directions. The solution to the R-R field is given by

$$F_{p+2} = dH_p^{-1} \wedge dx^0 \wedge dx^1 \wedge \cdots \wedge dx^p , \tag{2.1.26}$$

while the dilaton solution is

$$e^{\Phi} = g_s H_p(r)^{(3-p)/4} . \tag{2.1.27}$$

We note that in the limit $r \to \infty$, then $H_p \to 1$ and the dilaton is equal to g_s. Then, the parameter g_s is the string coupling constant at infinity.

If we consider the important case of N-coincident Dp-branes, the above solutions remain unchanged. The flux from the N Dp-branes can be written as

$$N = \int_{S^{8-p}} *F_{p+2} , \tag{2.1.28}$$

where we are stating the fact that the N coincident Dp-branes carry a total amount of N units of charge. The constant L_p can be deduced to be [63]

$$L_p^{7-p} = (2\sqrt{\pi})^{5-p}\Gamma\left(\frac{7-p}{2}\right) g_s N\, \alpha'^{(7-p)/2} . \tag{2.1.29}$$

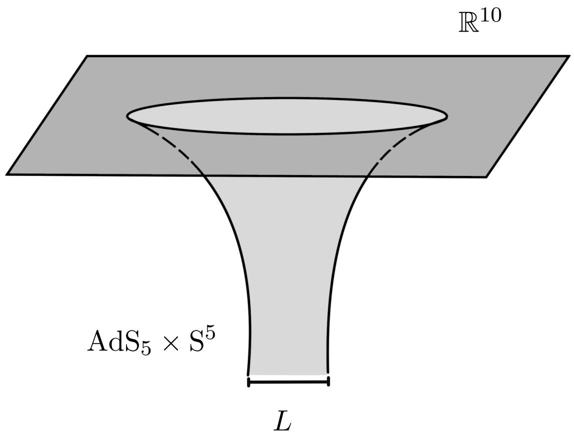


Figure 2.1: Schematic representation of the throat-like geometry of D3-branes solutions.

One relevant particularity about extremal Dp-branes solutions in Type IIA/B supergravity is that they preserve 16 out of the original 32 supersymmetries of the theory. This will mean that the associated open-string spectrum will have this much supersymmetry and will result to be tachyon free. Thus, these solutions are also referred to as *half-BPS Dp-branes*.

2.1.3 D3-Brane Solutions.

We now consider the particular case of D3-branes. D3-branes solutions in Type IIB supergravity are associated to the four-form C_4. We note that its field strength F_5 and its corresponding Hodge-dual $*F_5$ are self-dual five-forms. Therefore, C_4 couples both to an electric and a magnetic D3-branes. Moreover, since the field strength is self-dual, the D3-branes carry then a self-dual charge, and the two branes are in fact the same.

The D3-brane solution has a total $\mathbb{R}^4 \times \mathrm{SO}(1,2) \times \mathrm{SO}(8)$ symmetry, and is obviously given by the $p = 3$ case of the general solution presented above

$$ds^2 = H(r)^{-1/2}\eta_{ij}dx^i dx^j + H(r)^{1/2}\left(dr^2 + r^2 d\Omega_5\right) , \tag{2.1.30}$$

with

$$H(r) = 1 + \frac{L^4}{r^4} , \tag{2.1.31}$$

where we have dropped the subindexes in the harmonic function and the constant L, which is given by

$$L^4 = 4\pi g_s N \alpha'^2 . \tag{2.1.32}$$

The five-form is given by

$$F_5 = dH^{-1} \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 . \tag{2.1.33}$$

with flux

$$N = \int_{S^5} F_5 . \tag{2.1.34}$$

Meanwhile, the dilaton field has solution

$$e^{\Phi} = g_s , \tag{2.1.35}$$

while for the axion field C_0 and the two forms C_2, B_2 we have

$$C_0 = \text{constant.} , \tag{2.1.36}$$

$$B_2 = 0 , \qquad C_2 = 0 . \tag{2.1.37}$$

We note the important fact that the D3-brane solution for the dilaton (2.1.35) is a constant.

Regarding the geometry of the metric solution (2.1.30), we first note in (2.1.30) that in the regime $r \gg L$ we recover flat space-time $\mathbb{R}^{10}$. On the other hand, in the region $r < R$ we find a curved geometry referred customarily as the *throat*. (See figure (2.1).) The radius of the throat approaches the value L asymptotically at $r \to 0$. Another very important fact about the solution

(2.1.30) comes from the way it can be decomposed. Indeed, if we define the inverse coordinate

$$z = \frac{L^2}{r}\,, \tag{2.1.38}$$

then the metric (2.1.30) in the *near-throat region* ($z \gg L$) becomes

$$ds^2 = L^2\left[\frac{1}{z^2}\left(\eta_{ij}dx^i dy^j + dz^2\right) + d\Omega_5^2\right]\,, \tag{2.1.39}$$

which is manifestly regular and describes the product geometry $\mathrm{AdS}_5\times\mathrm{S}^5$, where both factors have radius L. Again, we note that this characteristic about the metric solution is only attainable when $p = 3$.

An important general property of Dp-branes is that they are objects that carry mass, so they can backreact to the surrounding geometry. Clearly then, in order to see how strong the deformation of space-time in the presence of a Dp-branes, we need to calculate its mass. This can be realized by dimensional reduction of the Dp-brane on its spatial directions and by then reading the large r behaviour of the g^E_{00} metric component in the Einstein frame. The result of this calculation for the case of N D3-branes yields the result

$$\frac{M}{V_3} = \frac{N}{(2\pi)^3\alpha'^2}\frac{1}{g_s}\,. \tag{2.1.40}$$

The gravitational field produced by an object is proportional to its mass time the Newton constant, which goes as $G_N \sim g_s^2$. (See equation (2.3.6).) Then we conclude that the gravitational field goes as $\sim g_s$, and that in the limit $g_s \to 0$ (which means a vanishing throat radius L) the metric reduces to Minkowski. Therefore, Dp-branes in the small string coupling regime admit a flat theory description.

The D3-brane solution is of essential importance to formulating the AdS/CFT correspondence from the gravity side. To finish this section, we summarize some of its unique properties that will prove relevant when explaining the duality: it is a half-BPS solution, it is asymptotically flat, it has a $\mathrm{AdS}_5 \times \mathrm{S}^5$ near-throat geometry and it has a constant dilaton.

2.2 $\mathcal{N}=4$ Super Yang-Mills

2.2.1 Field Content and Symmetries.

In the preceding section we have outlined the "gravity" side of the AdS/CFT correspondence. Here we will describe the other half of the duality, corresponding to the full quantum $\mathcal{N}=4$ Super Yang-Mills theory. We begin by writing the theory's Lagrangian [64]

$$\begin{aligned}\mathcal{L}_{\rm SYM} &= \mathrm{tr}\Big\{-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu}+\frac{\theta_I}{8\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}-\sum_a i\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda_a\\ &\quad -\sum_i D_\mu X^i D^\mu X^i+\sum_{a,b,i}\Big(g\,C_i^{ab}\lambda_a\left[X^i,\lambda_b\right]+g\,\bar{C}_{iab}\bar{\lambda}^a\left[X^i,\bar{\lambda}^b\right]\Big)\\ &\quad +\frac{g^2}{2}\sum_{i,j}\left[X^i,X^j\right]\Big\}\,,\end{aligned} \tag{2.2.1}$$

where the constants C_i^{ab} and C_{iab} are related to the Clifford Dirac matrices for the internal R-symmetry group $\mathrm{SO(6)_R}\sim\mathrm{SU(4)_R}$. The Lagrangian (2.2.1) is invariant under $\mathcal{N}=4$ Poincaré symmetry, whose transformation laws are

$$\begin{aligned}\delta X^i &= \left[Q^a_\alpha,X^i\right]=C^{iab}\lambda_{\alpha b}\,,\\ \delta\lambda_b &= \{Q^a_\alpha,\lambda_{\beta b}\}=F^+_{\mu\nu}\left(\sigma^{\mu\nu}\right)^\alpha{}_\beta\,\delta^a_b+\left[X^i,X^j\right]\varepsilon_{\alpha\beta}\left(C_{ij}\right)^a{}_b\,,\\ \delta\bar{\lambda}^b_{\dot\beta} &= \left\{Q^a_\alpha,\bar{\lambda}^b_{\dot\beta}\right\}=C_i^{ab}\bar{\sigma}^\mu_{\alpha\dot\beta}D_\mu X^i\,,\\ \delta A_\mu &= \left[Q^a_\alpha,A_\mu\right]=\left(\sigma_\mu\right)_\alpha{}^{\dot\beta}\,\bar{\lambda}^a_{\dot\beta}\,,\end{aligned} \tag{2.2.2}$$

where $F^\pm_{\mu\nu}=\frac{1}{2}\left(F_{\mu\nu}\pm\frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\right)$ and the constants $\left(C_{ij}\right)^a_b$ are related to bilinears in Clifford Dirac matrices for $\mathrm{SO(6)_R}$.

An important property of $\mathcal{L}_{\rm SYM}$ is that it is classically *scale invariant*. Indeed, the standard mass dimension for the theory's fields and couplings is

$$[A_\mu]=\left[X^i\right]=1\,,\qquad [\lambda_a]=\frac{3}{2}\,,\qquad [g]=[\theta_I]=0\,, \tag{2.2.3}$$

so all terms in the Lagrangian have dimension 4. Furthermore, upon perturbative quantization it is found that the β function of the theory vanishes, and therefore the scale invariance is preserved at quantum level. In this manner, scale invariance and Poincaré invariance combine to form a larger conformal symmetry $SO(2,4) \sim SU(2,2)$. In turn, this symmetry combined with the $\mathcal{N}=4$ supersymmetry produce a *superconformal symmetry*, given by $SU(2,2|4)$. In more detail, the components of $SU(2,2|4)$ are [53]

- The *R-symmetry group* $SO(6)_R \sim SU(4)_R$, with generators T^A, $A = 1,\ldots,15$.
- The $\mathcal{N}=4$ *Poincaré supersymmetry*, generated by the supercharges Q^a_α and their complex conjugates $\bar{Q}_{\dot{\alpha}a}$, $a=1,\ldots,4$.
- The *conformal group* $SO(2,4) \sim SU(2,2)$, generated by translations P^μ, Lorentz transformations $M_{\mu\nu}$, dilations D and special conformal transformations K^μ.
- The *conformal supersymmetries*, generated by the supercharges $S_{\alpha a}$ and their complex conjugates $\bar{S}^a_{\dot{\alpha}}$. These symmetries arise from the fact that the special conformal transformations K_μ and the Poincaré supercharges Q^a_α do not commute and, since both are symmetries, their commutator is also a symmetry. These commutators are precisely the generators $S_{\alpha a}$.

The dimension of these various generators is

$$\begin{aligned} &[D] = [M_{\mu\nu}] = \left[T^A\right] = 0\,, \\ &[P^\mu] = +1\,, \qquad [K_\mu] = -1\,, \\ &[Q] = +1/2\,, \qquad [S] = -1/2\,. \end{aligned} \tag{2.2.4}$$

2.2.2 Local Operators and Multiplets.

We are now interested in the construction and classification of local gauge invariant operators. These class of operators are to be built from the gauge covariant fields X^i, λ_a, $F_{\mu\nu}$ and the covariant derivative D_μ, whose dimension is $[D_\mu]=1$, and the operators are to be polynomials in the mentioned components. This restriction will result in gauge invariant operators of definite positive dimension and that the number of operators whose dimension is less than a given number is finite. In particular, the only operator with dimension 0 will be the unity operator.

Since the conformal supercharges S have dimension $-1/2$, then the successive application of S to any of the mentioned combinations will eventually yield 0. Otherwise, one would start generating operators of negative dimension, which is impossible in a unitary representation. We then define a *superconformal primary operator* $\mathcal{O}$ to be a non-vanishing operator such that

$$[S, \mathcal{O}] = 0\,, \tag{2.2.5}$$

and the same for the anticommutator in the fermionic case. An alternate, equivalent definition of a superconformal primary operator is as the lowest dimensional operator in a given superconformal representation.

An operator $\mathcal{O}'$ is called an *superconformal descendant operator* of a local polynomial gauge invariant operator $\mathcal{O}$ if it is obtained as

$$\mathcal{O}' = [Q, \mathcal{O}]\ , \tag{2.2.6}$$

and the same for the anticommutator in the fermionic case. The dimensions of both operators are related by $\Delta_{\mathcal{O}'} = \Delta_{\mathcal{O}} + 1/2$, and therefore $\mathcal{O}'$ cannot be a primary operator. Furthermore, if an operator $\mathcal{O}'$ is a descendant from $\mathcal{O}$, then both operators belong to the same superconformal representation. Then, in a given irreducible superconformal representation there is always a single superconformal primary operator and a tower of superconformal descendants arising from this primary.

In the case of $\mathcal{N} = 4$ super Yang-Mills theory, it is useful to begin by looking for which operators are *not* superconformal primary operators. These would be operators that arise from the commutation with the supercharge Q, for in that case, we would instead have a descendant, according to (2.2.6). We need to see then how the gauge covariant operators of $\mathcal{N} = 4$ SYM commute with Q. These commutation relations are given schematically as

$$\begin{aligned} \{Q, \lambda\} &= F^+ + [X, X] \,, & [Q, X] &= \lambda \,, \\ \left\{Q, \bar{\lambda}\right\} &= DX \,, & [Q, F] &= D\lambda \,. \end{aligned} \tag{2.2.7}$$

Therefore any local polynomial operator containing any of the elements in the right-hand side of the relations above cannot be primary. In particular, we note that they cannot be composed of the gauginos λ, the field strength F, or the derivative and commutators of X. This means that superconformal primary operators are gauge invariant scalars involving just X. The simplest one of these are *single trace operators*, which are of the form

$$\operatorname{tr}\left(X^{\{I_1} X^{I_2} \cdots X^{I_n\}}\right) \,, \tag{2.2.8}$$

where the indexes I_k, $k = 1, \dots, n$ are symmetrized and belong to the fundamental $SO(6)_R$ representation. Since $\operatorname{tr}X^i = 0$, then the simplest primary operators are

$$\begin{aligned} \sum_I \operatorname{tr}X^I X^I &\sim \text{Konishi multiplet} \,, \\ \operatorname{tr}X^{\{I} X^{J\}} &\sim \text{Supergravity multiplet} \,. \end{aligned} \tag{2.2.9}$$

By contrast, one defines *multitrace operators* as those operators built from the product of single trace operators.

The unitary representation of the superconformal algebra can be labeled by the quantum numbers of the bosonic group

$$SO(1,3) \times SO(1,1) \times SU(4)_R \,. \tag{2.2.10}$$

The first factor corresponds to the Lorentz group, and has (s_+, s_-) spin quantum numbers. The second group has quantum number Δ, which corresponds to the positive or zero dimension of the operator, while the third group corresponds to the R-symmetry, whose representation is determined by Dynkin label given by the triplets $[r_1, r_2, r_3]$ [65]. Each representation of $SU(4)_R$ can be labeled in terms of its dimension, given by

$$\dim[r_1, r_2, r_3] = \frac{1}{12}\bar{r}_1\bar{r}_2\bar{r}_3(\bar{r}_1 + \bar{r}_2)(\bar{r}_2 + \bar{r}_3)(\bar{r}_1 + \bar{r}_2 + \bar{r}_3)\,, \qquad (2.2.11)$$

where $\bar{r}_n = r_n + 1$. In unitary representations, the number Δ is bounded from below by the other quantum numbers. In the case of primary operators, which as we know have the lowest dimension in a given multiplet, the spin number vanish since the operator is a scalar. In those cases, one can find primary operators that commute with at least one of the supercharges Q. Such representations are shortened and are called *BPS multiplets*. These have the particularity that their dimension is protected from having quantum corrections, and play a special role in the AdS/CFT correspondence.

2.3 Type IIB Strings: Two Perspectives

In the preceding sections we have described two very different theories: Type IIB supergravity in $D = 10$ dimensions on one hand, and $\mathcal{N} = 4$ super Yang-Mills theory on $D = 4$ on the other. Furthermore, we have stated that the AdS/CFT correspondence proposes an equivalence between both theories, and that the bridge to constructing the duality comes from string theory. More concretely, the starting point is Type IIB string theory in the presence of N coinciding D3-branes. Since we will be working in the limit of very low energies, the effective action for the system will be given by the Type IIB supergravity solutions already described. In order to provide evidence for the conjecture, we will follow the following steps

- Identify $SU(N)$ super Yang-Mills theory as a sector of the low-energy limit of Type IIB strings in the presence of N parallel D3-branes in the weak coupling limit [66].
- Identify the space-time geometry arising from the D3-branes as an $\mathrm{AdS}_5 \times$

S^5 throat embedded in an asymptotically flat region whose gravitational modes decouple from the string modes on the throat.

- Compare the two low-energy descriptions to identify the super Yang-Mills sector with the gravitational $AdS_5 \times S^5$ sector, with a very specific mapping of each theory's parameters.

The first two points are achieved by analyzing Type IIB string theory in the presence of N parallel D3-branes from two different perspectives. The first one, which we will call the *D-brane perspective*, will be related to $\mathcal{N} = 4$ super Yang-Mills theory. The second one, which we shall call the *black-brane perspective*, will be related to Type IIB supergravity in a $AdS_5 \times S^5$ background. When studying both perspectives, we will recur to some very particular limits of the theory in order to make it more tractable. The most important of such limits is called the *Maldacena limit*, which will be explained in detail in the remaining of this section. Taking this limit will result in a particular decoupling of the theory in distinct sectors as seen from the two different point of views. Finally, since both point of views are in fact different equivalent descriptions of the same theory, the decoupled sectors can be identified, leading to the AdS/CFT conjecture.

2.3.1 The D-brane perspective.

Let us begin by consider again $D = 10$ Type IIB string theory with N coinciding D3-branes. This theory will have two kind of excitations. These will consist of closed strings living in the bulk of the theory, and of open strings with its ends attached to the D3-branes.

An open string with both its ends attached to one of these branes can have arbitrarily short length, and therefore must be massless. (See figure (2.2).) In general, the massless modes coming from open strings with both ends on the same D3-brane generate a $U(1)$ gauge theory living in the $D = 4$ brane world volume. Naively one would have $U(1)^N$ for the case of the whole N D3-brane

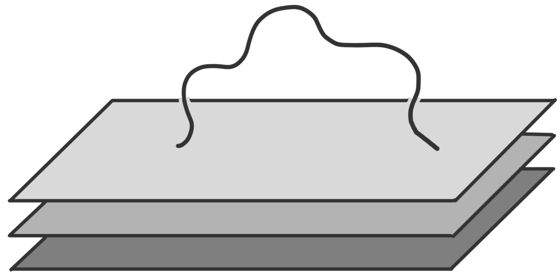


Figure 2.2: Schematic representation of a stack of N coincident D3-branes, with open string attached.

system, but one must also consider strings whose ends are attached to different branes. Since the N branes are coinciding, the string modes remain massless, so that $U(1)^N$ gauge symmetry is enhanced to a $U(N)$ gauge symmetry. We ignore the diagonal $U(1)$ factor, which corresponds to the overall center-of-mass position of the D3-branes. These translational degrees of freedom will decouple in the low-energy limit and will pay no role in the AdS/CFT correspondence. We are then left with a theory with $SU(N)$ gauge symmetry living in an effectively flat space-time in $D = 4$ [67]. Furthermore, since as we know the brane solutions break half of the total number of supersymmetries, the gauge theory must then have $\mathcal{N} = 4$ Poincaré supersymmetry. In the low-energy limit, as we shall see, the open-string degrees of freedom are described by $\mathcal{N} = 4$ SYM, with $SU(N)$ gauge group.

For energies $E \ll 1/\ell_s$, the massive states of the spectrum of theory become inaccessible and one is left with massless excitations. We will be mainly interested in the massless spectrum, and therefore it is sensible to be more accurate about the regime where we wish to work. So far in the discussion, we have introduced the number of D3-branes N. Furthermore, Type IIB string theory contains the string coupling g_s, and the Regge slope $\alpha' = \ell_s^2$. Then, the low-energy limit can be achieved by keeping all energies fixed and taking $\alpha' \to 0$. We define the *Maldacena low-energy limit* [2] as

- **Maldacena Limit.** g_s and N are kept fixed, as well as all dimensionless parameters, while taking $\alpha' \to 0$.

Then, in the Maldacena limit the effective action for the massless fields can then be written as

$$S_1 = S_{\text{brane}} + S_{\text{bulk}} + S_{\text{interaction}}\,, \tag{2.3.1}$$

where $\mathrm{S_{brane}}$ describes the open string modes on the 4-dimensional brane world-volume, $\mathrm{S_{bulk}}$ describes the closed string modes in the 10-dimensional bulk of the theory, and $\mathrm{S_{interaction}}$ describes open-closed string interactions.

In the $\alpha' \to 0$ limit, the D3-brane theory reduces to $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory. This can be seen from the DBI action of the D3-brane

$$S_{\text{brane}} = -T_{\text{D3}} \int d^4x\; e^{-\Phi} \sqrt{G_{\alpha\beta} + \mathcal{F}_{\alpha\beta}} + \cdots \tag{2.3.2}$$

with $\mathcal{F}_{\alpha\beta} = B_{\alpha\beta} + (2\pi\alpha')F_{\alpha\beta}$ and where $F_{\alpha\beta}$ is the usual Maxwell field strength and $G_{\alpha\beta}$ and $B_{\alpha\beta}$ incorporate supersymmetry explicitly. Furthermore, the D3-brane tension T_{D3} is given by

$$T_{D3} = \frac{1}{(2\pi)^3 g_s \alpha'^2}\,. \tag{2.3.3}$$

For a flat target space-time, the D3-brane action (2.3.2) can be expanded as

$$\begin{aligned} S_{\text{brane}} &= \frac{1}{2g_{\text{YM}}^2} \int d^4x\; F_{\alpha\beta}F^{\alpha\beta} + \cdots + \mathcal{O}(\alpha') \\ &= S_{\mathcal{N}=4} + \mathcal{O}(\alpha')\,, \end{aligned} \tag{2.3.4}$$

where the $\mathcal{N} = 4$ super Yang-Mills theory coupling is given in terms of the string coupling as

$$g_{\text{YM}}^2 = 4\pi g_s\,. \tag{2.3.5}$$

Regarding the S_{bulk} term, in the Maldacena limit we may do an expansion for small powers of κ, which is given in terms of g_s and α' as

$$\kappa^2 = 64\pi^7 g_s^2 \alpha'^2\,. \tag{2.3.6}$$

Then, expanding the metric as $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ we obtain schematically

$$\begin{aligned} S_{\text{bulk}} &= \frac{1}{2\kappa^2}\int d^{10}x\,\sqrt{-g}\,R + \mathcal{O}(R^2) \\ &\sim \int d^{10}x\left\{(\partial h)^2 + \kappa(\partial h)^2 h + \cdots\right\}, \end{aligned} \tag{2.3.7}$$

where other massless bulk fields have not been explicitly indicated, for simplicity. Likewise, the interaction terms between open-closed strings in $S_{\text{int.}}$ are proportional to positive powers of κ. Therefore, taking the Maldacena limit $\alpha' \to 0$, the dynamics in the bulk decouple from the brane dynamics and gravity becomes *IR-free*.

We see then that, within the D-brane perspective and in the Maldacena limit, Type IIB string theory decouples in two distinct systems

$$S_{\text{I}} = \mathcal{A}_I + \mathcal{B}_I\,, \tag{2.3.8}$$

where

- **System** $\mathcal{A}_I$: $\mathcal{N} = 4$ $SU(N)$ Super Yang-Mills theory in $\mathbb{R}^4$.
- **System** $\mathcal{B}_I$: Free supergravity in $\mathbb{R}^{10}$.

2.3.2 The Black-Brane Perspective

Let us now study the same system from a different perspective, by taking the Maldacena limit in the non-linear sigma model for string theory on a D3-brane background. We rewrite the solution (2.1.30) using the inverse coordinate $z = L^2/r$. The result is

$$\begin{aligned} ds^2 &= \frac{L^2}{z^2}\left[\tilde{H}(z)^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + \tilde{H}(z)^{1/2}\left(dz^2 + z^2 d\Omega_5^2\right)\right] \\ &= G_{MN}dx^M dx^N\,, \end{aligned} \tag{2.3.9}$$

with

$$\tilde{H}(u) = 1 + \frac{L^4}{z^4}\,. \tag{2.3.10}$$

Substituting this metric in the Polyakov action, one has

$$\begin{aligned} S_{\mathrm{p}} &= \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}h^{ab}G_{MN}(x)\partial_a x^M\partial_b x^N \\ &= \frac{L^2}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}h^{ab}\tilde{G}_{MN}(x)\partial_a x^M\partial_b x^N\,, \end{aligned} \tag{2.3.11}$$

where we have made a rescaling of the metric as $\tilde{G}_{MN}(x) = G_{MN}(x)/L^2$. We notice that the prefactor in (2.3.11) can be rewritten as

$$\frac{L^2}{4\pi\alpha'} = \sqrt{\frac{\lambda}{4\pi}}\,, \tag{2.3.12}$$

where we have introduced the *t'Hooft coupling*

$$\lambda = g_s N\,. \tag{2.3.13}$$

In the Maldacena limit, g_s and N (and consequently λ) are kept fixed while $\alpha' \to 0$. This means

$$L^4 = 4\pi\lambda\,\alpha'^2 \to 0\,, \tag{2.3.14}$$

from which we have $\hat{H}(z) \to 1$ and the metric $\tilde{G}_{MN}$ becomes

$$\tilde{G}_{MN}dx^M dx^N = \frac{1}{z^2}\left(\eta_{\mu\nu}d^\mu dx^\nu + dz^2 + z^2 d\Omega_5^2\right)\,. \tag{2.3.15}$$

Then, in this regime the metric is $\mathrm{AdS}_5 \times \mathrm{S}^5$, with unit radius. Therefore, the Maldacena limit "zooms" into the near-throat region of the D3-brane solution and the Polyakov action reduces to a string sigma-model on $\mathrm{AdS}_5 \times \mathrm{S}^5$, with string tension proportional to $T \sim 1/\sqrt{\lambda}$. Physically, this means that in the Maldacena limit only the the near-horizon $\mathrm{AdS}_5 \times \mathrm{S}^5$ region contributes dynamically to the physical description of the system, while the dynamics of the asymptotically flat region decouples from the theory. A simple schematic way to how this can be so, is by proposing a series expansion on α' of the effective action in a background with Riemann tensor R

$$\mathcal{L}_{\text{eff.}} = c_1\alpha' R + c_2\alpha'^2 R^2 + c_3\alpha'^3 R^3 + \cdots\,. \tag{2.3.16}$$

The asymptotically flat region is characterized by scales $r \gg R$, where we note that we are using the original radial coordinate for the metric, r. Then, one proposes that in this region the Riemann tensor scales as $R \sim 1/r^2$. We then obtain

$$\mathcal{L}_{\text{eff.}} = c_1\alpha' r^{-2} + c_2\alpha'^2 r^{-4} + c_3\alpha'^3 r^{-6} + \cdots\,, \tag{2.3.17}$$

so, if one keeps the scale r fixed, then in the limit $\alpha' \to 0$ the contribution from the asymptotic modes vanish.

In this manner then, the system is decoupled in low energy massless supergravity modes in the asymptotically flat region of the bulk on the one hand, and in arbitrary energy excitations near the AdS throat on the other. Let the energy in string-length units of these near-throat excitations as measured from the radial position r be $\sqrt{\alpha'}E_r =$ const. These near-throat energies can in principle be arbitrarily large. However, because of the gravitational red-shift, from the point of view of an observer at $r \to \infty$ these excitations look like

$$E_\infty = \sqrt{-g_{00}}E_r = \left(1 + \frac{L^4}{r^4}\right)^{-1/4} E_r\,. \tag{2.3.18}$$

Since we want to consider modes that are in the near-throat region, then, while taking the Maldacena limit the radial distance of these modes must satisfy

$$\frac{r}{\alpha'} \quad \text{Fixed}\,. \tag{2.3.19}$$

With this condition in mind, the red-shift factor is

$$\begin{aligned}\left(1 + \frac{L^4}{r^4}\right)^{-1/4} = \left(1 + \frac{4\pi\lambda\,\alpha'^2}{r^4}\right)^{-1/4} &= \left(1 + 4\pi\lambda\left(\frac{\alpha'}{r}\right)^2 \frac{1}{r^2}\right)^{-1/4} \\ &\approx \left(4\pi\lambda\left(\frac{\alpha'}{r}\right)^2 \frac{1}{r^2}\right)^{-1/4}\end{aligned} \tag{2.3.20}$$

where in the second line we have used the fact that λ and r/α' remain fixed while $r \to \infty$. Then

$$E_\infty \sim \frac{r}{\sqrt{\alpha'}}\,E_r \sim \frac{r}{\alpha} = \text{const.} \tag{2.3.21}$$

The conclusion is that one can have any kind of string excitations close to $r = 0$, since their energy measured at the asymptotic flat region is finite. These modes are then decoupled from the massless modes in the bulk region.

From these discussions we conclude that, from the black-brane perspective, in the Maldacena limit our original Type IIB string theory decouples in two systems

$$S_{II} = \mathcal{A}_{II} + \mathcal{B}_{II}\,, \tag{2.3.22}$$

where

- **System $\mathcal{A}_{II}$**: Full quantum superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$.
- **System $\mathcal{B}_{II}$**: Free supergravity on $\mathbb{R}^{10}$.

2.4 The Maldacena Conjecture

The main conclusion from the discussion in the previous section is that, both in the black-brane perspective and the D-brane perspective, the Maldacena limit realizes a decoupling of the original Type IIB theory in two well-defined and distinct systems, $\mathcal{A}$ and $\mathcal{B}$. Moreover, one finds that system $\mathcal{B}$, which is supergravity in flat space-time, is the same in both perspectives: $\mathcal{B}_I = \mathcal{B}_{II}$. Since the actions S_I and S_{II} provide equivalent descriptions to the same system, then one is led to identify $\mathcal{A}_I = \mathcal{A}_{II}$. This conjecture would implies in fact to identify $\mathcal{N} = 4$ $SU(N)$ Super-Yang-Mills theory in $3+1$ flat space-time with Type IIB superstring theory on a $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. This is the main essence of the Maldacena conjecture, which we now state in its *strong form.* The AdS/CFT correspondence conjectures the equivalence of two theories:

- $\mathcal{N} = 4$ $SU(N)$ Super Yang-Mills theory in $\mathbb{R}^4$, generated by massless *open* string modes.
- Type IIB Superstring the theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$, with integer flux of the five-form Ramond-Ramond field strength $N = \int_{S^5} F_5$, generated by the massless *closed* string modes.

The parameters of both theories are related by

$$g_s = 4\pi g_{\mathrm{YM}}^2\,, \qquad L^4 = 4\pi g_s N \alpha'^{\,2}\,, \tag{2.4.1}$$

and the parameter N, which labels the the gauge group of the field theory, corresponds to the five-form flux in the dual Type IIB string theory description.

As said, this is the strong statement of the conjecture. We have already seen that in order to motivate the conjecture, one has to take the Maldacena low-energy limit. However, for us to gain a better understanding, we must investigate additional limits. One of them is given by

- **t'Hooft Limit.** Take $N \to \infty$, while keeping the t'Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N = g_s N$ fixed.

The consequences of taking the t'Hooft limit are well known in both sides of the duality. In the $\mathcal{N} = 4$ super Yang-Mills theory, this limit corresponds to the planar sector in perturbation theory. Indeed, in large N theory, the theory has a convenient topological expansion. Starting from double-line notation for adjoint $U(N)$ fields [68], it is found that each Feynman diagram can be mapped into a two-dimensional surface, which can be assigned an Euler characteristic $\chi = 2 - 2g$, where g is the genus of the surface. It is found that vacuum-to-vacuum diagrams are proportional to

$$N^{2-2g} \lambda^{E-V} , \tag{2.4.2}$$

where E and V are the number of propagators and vertices, respectively. Then, since $g \geq 0$, in the large N limit diagrams whose associated surface have $g \neq 0$ are suppressed by $1/N^{2g}$, leaving only the *planar* (genus $g = 0$) diagrams contribution in the perturbative expansion.

Meanwhile, in the gravity side of the duality, the string coupling can be expressed as $g_s = \lambda/N$, so the regime of fixed λ and $N \to \infty$ means that $g_s \to 0$. This corresponds to weak-coupling string perturbation theory. The perturbation expansion of string theory is a genus expansion of the world-sheet. Correlators on a surface of genus g scale as g_s^{2g-2}. Then, since in this regime $g_s \to 0$, contributions from higher-genus surfaces are dropped. From this we conclude that the t'Hooft limit corresponds to classical string theory.

We see then that this weaker form of the original conjecture nevertheless still proposes a very non-trivial equivalence between the large N, planar limit of $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$ and classical Type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. However, as opposed the flat space-time case, classical string theory in curved backgrounds with R-R fluxes is still poorly understood. It is then convenient to look for a more tractable regime. Given that in the t'Hooft limit one is left with λ as the only free parameter, one can take the additional limit

- **Large λ Limit.** Take both $N \to \infty$ with fixed $\lambda = g_{YM}^2 N$, and then consider $\lambda \gg 1$.

In this limit, the $\mathcal{N} = 4$ super Yang-Mills theory enters the strongly-coupled, non-perturbative sector. However, in the gravity side of the duality, we find that the Type IIB string theory reduces to classical supergravity. To see this, we repeat the expansion on α' performed in (2.3.16), only that in this case we focus on the near-throat region, where the relevant scale is given by the radius L. Then the Riemann tensor scales as $R \sim 1/L^2 \sim \lambda^{-1/2}/\alpha'$, so the effective Lagrangian has a power series expansion in $\lambda^{-1/2}$

$$\begin{aligned} \mathcal{L}_{\text{eff.}} &= c_1 \alpha' R + c_2 \alpha'^2 R^2 + c_3 \alpha'^3 R^3 + \cdots \\ &= c_1 \lambda^{-1/2} + c_2 \lambda^{-1} + c_3 \lambda^{-3/2} + \cdots . \end{aligned} \tag{2.4.3}$$

The substitution of α' by λ as the effective expansion parameter in the near-throat region is in agreement with the sigma model action (2.3.12). Therefore, by taking the large λ limit, any higher-curvature derivatives drop out from the effective Lagrangian and the superstring theory reduces to classical supergravity.

From this form of the AdS/CFT conjecture, we observe an equivalence between classic Type IIB supergravity on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and $\mathcal{N} = 4$ super Yang-Mills in flat *4D* at strong coupling. We then see clearly that when the field theory side of the duality is strongly coupled, the dual string theory finds itself on the classical supergravity regime. This strong/weak coupling nature of the duality will be a very important property when studying holographic models of superconductivity.

2.5 Evidence for the Conjecture.

2.5.1 Mapping of Global Symmetries.

If the AdS/CFT conjecture is to be truth, then both dual theories should have the same global symmetries. Since these symmetries do not depend on parameters λ or N, it does not matter if any side of the duality is in the strongly-coupled regime. The superconformal group of $\mathcal{N} = 4$ super Yang-Mills theory is $SU(2, 2|4)$, which has a maximal bosonic subgroup given by

$$SU(2, 2|4) \supset SO(2, 4) \times SU(4)_R \,, \tag{2.5.1}$$

where the first factor corresponds to the conformal group and the second corresponds to the internal R-symmetry group. These two groups are matched by the isometry group of $\mathrm{AdS}_5 \times \mathrm{S}^5$, given by $SO(2, 4)$ and $SO(6) \cong SU(4)_R$, respectively. The completion into the full $SU(2, 2|4)$, which has 32 supersymmetries is less straightforward. The N D3-branes present in the Type IIB string theory are half-BPS, which preserve only half of the theory's original Poincaré supersymmetries $32 \to 16$. The remaining 16 supersymmetries needed to a complete match are supplemented by 16 conformal supersymmetries in the AdS limit [69].

2.5.2 Mapping Bulk Fields to Boundary States.

In a previous section, we have described how irreducible representations of $\mathcal{N} = 4$ SYM theory can be described by the spectrum of superconformal local operators, and the special importance of primary superconformal operators in the construction of such a given irreducible representation. We now need to describe how such representation of the gauge theory can be mapped to the bulk theory. To do this, one focus on the massive and massless Type IIB string degrees of freedom living in AdS$_5 \times$ S^5. Let one such stringy generic degree of freedom be referred as $\phi(z,y)$, where z^μ, $\mu = 0,1,\ldots,4$ are AdS$_5$ coordinates and y^m, $m = 1,\ldots,5$ are S^5 coordinates. Such a generic field can then be decomposed as

$$\phi(z,x) = \sum_{\Delta=0}^{\infty} Y_\Delta(y)\phi_\Delta(z)\,, \tag{2.5.2}$$

where $\phi_\Delta(z)$ lives on AdS$_5$, $Y_\Delta(y)$ is a complete basis of spherical harmonics on S^5 and Δ labels the $SO(6)_R$ representations. Because of this compactification, the fields receive a mass contribution. Computing the eigenvalues of the Laplacian on S^5 for different spins, one finds the following relations between bulk-field masses and boundary operator's scaling dimensions [70, 71]

$$\text{Scalar} \qquad m^2L^2 = \Delta(\Delta-4) \tag{2.5.3}$$

$$\text{Spin } 1/2,\, 3/2 \qquad |m|\,L = \Delta - 2 \tag{2.5.4}$$

$$p\text{-form} \qquad m^2L^2 = (\Delta-p)(\Delta+p-4) \tag{2.5.5}$$

$$\text{Spin } 2 \qquad m^2L^2 = \Delta(\Delta-4) \tag{2.5.6}$$

Therefore, the stringy degrees of freedom are compactified in a Kaluza-Klein reduction and organized in terms of the quantum numbers of the shared global symmetries of both theories. Then they are matched to the boundary-theory's superconformal operators in a given representation labeled by those same numbers.

2.6 Scalar Fields in AdS$_5$ and their Holographic description.

We now focus on the study of scalar fields on the bulk side of the duality. We consider the Klein-Gordon Lagrangian for such field in the general case AdS$_{d+1}$, which we write as

$$ds^2 = \frac{1}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right)\,, \tag{2.6.1}$$

where we have set $L = 1$ for simplicity. Then the Klein-Gordon Lagrangian for the scalar field is

$$\begin{aligned} S &= -\frac{1}{2}\int dz\, d^d x \sqrt{-g}\left(g^{MN}\partial_M\phi\partial_N\phi + m^2\phi^2\right) \\ &= -\frac{1}{2}\int dz\, d^d x \frac{1}{z^{d+1}}\left(z^2(\partial_z\phi)^2 + z^2\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right) . \end{aligned} \tag{2.6.2}$$

We rescale ϕ as $\phi = z^{d/2}\psi$ and define the coordinate $y = -\ln z$ such that the kinetic term for ψ becomes canonical

$$S = -\frac{1}{2}\int dy\, d^d x\left\{(\partial_y\psi)^2 + e^{-2y}\eta^{\mu\nu}\partial_\mu\psi\partial_\nu\psi + \left(m^2 + \frac{d^2}{4}\right)\psi^2\right\} + \frac{d}{4}\int d^d x\, \psi^2\big|_{y=-\infty}^{y=\infty} . \tag{2.6.3}$$

Focusing in the ψ field's mass, we will have a positive Hamiltonian if m^2 satisfies

$$m^2 \geq -\frac{d^2}{4} , \tag{2.6.4}$$

which is called the *Breitenlohner-Freedman (BF) bound* [72]. Therefore the system is stable for scalars with mass-squared above the BF bound. We note that the BF-bound gives a window of possibility for the existence of *tachyonic scalar fields*. This is a consequence of the AdS geometry.

In addition to the bound on the mass, we can obtain similar bounds on the dimension Δ. These bounds come from the requirement of having normalizable scalar field solutions. In general terms, any scalar field solution $\phi(z, x)$ such that the action remains finite $S[\phi] < \infty$ is called a *normalizable solution*. One can have in fact two ways to define the scalar field norm, which differ to one another by boundary terms [73]. The first of these norms comes directly from the scalar field action. Indeed, starting from (2.6.2) and assuming a general near-boundary behavior $\phi \sim F(x)z^\Delta$ one can easily find that the z-integral is finite near $z = 0$ only if

$$\Delta > \frac{d}{2} \tag{2.6.5}$$

On the other hand, one can define a second norm by taking the original scalar field action (2.6.2) and by integrating it by parts to obtain

$$\begin{aligned} S &= -\frac{1}{2}\int dz\, d^d x \sqrt{-g}\,\phi\left(-\nabla^2+m^2\right)\phi \\ &= -\frac{1}{2}\int dz\, d^d x \frac{1}{z^{d+1}}\phi\left\{\left(-\partial_z^2\phi+\frac{d-1}{z}\partial_z\phi-\frac{m^2}{z}\phi\right)-d\,\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right\}, \end{aligned} \tag{2.6.6}$$

where in the second line we expanded the operator ∇^2. If we again propose a near-horizon decomposition $\phi \sim z^{\Delta}F(x)$, then the term inside the parenthesis in the second line is proportional to $(\Delta(d-\Delta)+m^2)F(x)$ and vanishes on-shell. We are then left with the contribution from the kinetic energy in the transverse x-coordinate space, and in order to have an finite action near $z=0$ the mass must satisfy

$$\Delta > \frac{d}{2}-1\,. \tag{2.6.7}$$

The different normalization bounds (2.6.5) and (2.6.7) also set additional bounds on the scalar field mass. As a starting point to see this, one must first look at the asymptotic $z \to 0$ fallout of the scalar field, which can be found from its equation of motion

$$\frac{1}{\sqrt{-g}}\partial_M\left(\sqrt{-g}g^{MN}\partial_N\phi(z,x)\right)-m^2\phi(z,x)=0\,. \tag{2.6.8}$$

Proposing a plane wave ansatz $\phi(z,x)=e^{ik\cdot x}\phi(z)$, one has

$$z^{d+1}\partial_z\left(z^{1-d}\partial_z\phi\right)-\left(m^2+k^2z^2\right)\phi=0\,. \tag{2.6.9}$$

In the asymptotic AdS boundary $z\to 0$, the k^2 term can be neglected and the field solution behaves as

$$\phi(z,x)\approx\phi_1(x)z^{\Delta_+}+\phi_0(x)z^{\Delta_-}+\cdots\,, \tag{2.6.10}$$

where we define

$$\Delta_\pm=\frac{d}{2}\pm\sqrt{\frac{d^2}{4}+m^2}\,. \tag{2.6.11}$$

We note that (2.6.11) can be obtained from the equation

$$\Delta(\Delta-d)=m^2\,, \tag{2.6.12}$$

which is a d-dimensional generalization of the relation between mass and scaling dimension already found in (2.5.3).

We now return to the normalization conditions and compare them with (2.6.11). In the case of the first condition (2.6.5) we find that, for instance in the $m^2 > 0$ case, only the Δ_+ is normalizable, while the Δ_- is divergent. If one lowers the mass in the range $-d^2/4 < m^2 < 0$ one finds that both modes vanish but only one is normalizable. On the other hand, comparing the alternative normalization condition (2.6.7) one finds that in the range $-d^2/4 < m^2 < -d^2/4 + 1$ both $\Delta_\pm$ modes are normalizable. We note that the lower limit coincides with the BF bound (2.6.4). In the limit case $m^2 = -d^2/4$ one finds that $\Delta_+ = \Delta_-$ and there is the appearance of a non-normalizable logarithmic term. For lower values, one has complex dimensions Δ, which reflects the fact that the theory is unstable as mentioned before.

We will now show how the bulk field modes $\Delta_\pm$ can be matched holographically to expectation values of operators on the boundary field theory. For simplicity, the normalization status of each mode is referred with respect to the first of the possible norms explained above. We then accordingly refer to the Δ_+ and Δ_- modes as the *normalizable* and *non-normalizable* modes, respectively. The normalizable solution describes bulk excitations and decay at the AdS boundary. On the other hand, the non-normalizable modes define boundary fields given by

$$\phi_0(x) = \lim_{z \to 0} z^{-\Delta_-} \phi(z, x) \,. \qquad (2.6.13)$$

An important fact to see how the non-normalizable mode can be interpreted holographically is that the boundary data ϕ_0 entirely determines the bulk field $\phi(z, x)$ and in consequence the regular mode $\phi_1(x)$ follows from ϕ_0 and the equations of motion. The freedom with which one can specify the boundary field ϕ_0 corresponds to the freedom of adding an arbitrary source in the boundary field theory. Therefore ϕ_0 should be seen as the source for an operator $\mathcal{O}_\Delta(x)$ living on the boundary. From the field theory side then one has a generating functional $\Gamma[\phi_0]$ given by

$$e^{-\Gamma[\phi_0]} \equiv \left\langle \exp\left(- \int d^d x \phi_0(x) \mathcal{O}_\Delta(x) \right) \right\rangle . \tag{2.6.14}$$

One now relates holographically the field theory generating functional with the partition functional on the string theory side $Z_{\text{string}}(\phi_0)$ by [3, 4]

$$e^{-\Gamma[\phi_0]} \equiv \left\langle \exp\left(- \int d^d x \phi_0(x) \mathcal{O}_\Delta(x) \right) \right\rangle = Z_{\text{string}}(\phi_0) . \tag{2.6.15}$$

where $Z_{\text{string}}(\phi_0)$ is evaluated on-shell with boundary value ϕ_0. Equation (2.6.15) is sometimes referred to as the *master equation* of the AdS/CFT duality. The generating functional can be expanded as

$$\begin{aligned} \Gamma[\phi_0] &= \Gamma[0] + \int d^d x \phi_0(x) \left\langle \mathcal{O}_\Delta(x) \right\rangle_c \\ &\quad -\frac{1}{2} \int d^d x_1 d^d x_2 \phi_0(x_1) \phi_0(x_2) \left\langle \mathcal{O}_\Delta(x_1) \mathcal{O}_\Delta(x_2) \right\rangle_c \\ &\quad + \cdots , \end{aligned} \tag{2.6.16}$$

which clearly satisfies the usual quantum field theory relation for the generating functional

$$\frac{\delta \Gamma[\phi_0]}{\delta \phi_0(x)} = \left\langle \mathcal{O}_\Delta(x) \right\rangle_c , \tag{2.6.17}$$

$$\frac{\delta^2 \Gamma[\phi_0]}{\delta \phi_0(x_1) \delta \phi_0(x_2)} = - \left\langle \mathcal{O}_\Delta(x_1) \mathcal{O}_\Delta(x_2) \right\rangle_c , \tag{2.6.18}$$

$$\vdots$$

$$\frac{\delta^{(n)} \Gamma[\phi_0]}{\delta \phi_0(x_1) \cdots \delta \phi_0(x_n)} = (-1)^n \left\langle \mathcal{O}_\Delta(x_1) \cdots \mathcal{O}_\Delta(x_n) \right\rangle_c , \tag{2.6.19}$$

and where in the right hand side of these equations we always refer to the QFT connected correlation functions $\langle \cdots \rangle_c$.

Meanwhile, in the supergravity limit, the action in the string partition function $Z_{\text{string}}(\phi_0)$ is the classical action (2.6.2), and one obtains on-shell

$$Z_{\text{string}}(\phi_0) = - \lim_{z \to 0} z^{d+1-2\Delta} \int dx^d \left(z^{\Delta-d} \phi(z,x) \right) \partial_z \left(z^{\Delta-d} \phi(z,x) \right) , \tag{2.6.20}$$

But near the boundary one can substitute

$$z^{\Delta-d}\phi(z,x) \;\to\; \phi_0(x)\,, \tag{2.6.21}$$

$$\partial_z\left(z^{\Delta-d}\phi(z,x)\right) \;\to\; (2\Delta-d)z^{2\Delta-d-1}\phi_1(x)\,. \tag{2.6.22}$$

So comparing (2.6.16) and (2.6.21) one identifies the non-normalizable mode with the vacuum expectation value of the field theory operator $\mathcal{O}_\Delta$

$$\langle\mathcal{O}_\Delta(x)\rangle \sim \phi_1(x)\,. \tag{2.6.23}$$

We then see that the holographic relation (2.6.15) identifies the scalar field asymptotic coefficient ϕ_1 with the field theory operator $\mathcal{O}_\Delta(x)$, of scaling dimension Δ_+ as given by (2.6.11), while it interprets ϕ_0 with an external source to that same operator.

Regarding the secondary definition of scalar field norm, as we already pointed out, there is an interval of masses in which both $\Delta_\pm$ are normalizable, which leads to the matter of how these modes are to be interpreted holographically. It turns out that when both modes are normalizable, it is possible to choose from either one of them which one is to be dual to a boundary operator of scaling dimension given by the mode of choice. The choice of the Δ_+ mode is called the *standard quantization*, while that of the Δ_- mode is called the *alternative quantization*. Thus, by choosing the alternative quantization, one then has to interpret the asymptotic coefficient $\phi_1(x)$ as the external field theory source to an operator of dimension Δ_-. Then one concludes that in this case it is the choice of mode as much as the particular value of the mass that defines the field theory on the boundary in the sense that, for masses in the interval $-d^2/4 < m^2 < -d^2/4+1$, one can have two different AdS theories that correspond to different CFT's, one with an operator of dimension Δ_+ and the other with an operator of dimension Δ_-. Furthermore, the generating functionals $\Gamma[\phi_1]$ and $\Gamma[\phi_0]$ are related by a Legendre transformation [73].

3

Holographic Superconductivity

In this chapter we will intend to merge our previous discussions on high temperature superconductivity and on the AdS/CFT duality. Our principal aim will be to realize a consistent holographic description of superconducting phenomena, to which we will refer to as *holographic superconductivity*. For additional reviews on applications of the AdS/CFT duality in condensed matter system in general, and in holographic superconductivity in particular, see e.g. [74, 75, 76, 77, 78, 79, 80].

As we have already seen from the spin-fermion model at the end of Chapter 1, there is strong indication that any microscopic theory attempting to describe the high-T_c superconductivity will be in the strong-coupling regime, so that the usual perturbative techniques will no longer be applicable. Furthermore, one also finds that the standard quasi-particle picture of interactions may not longer be suited to this kind of systems, and that there is need for a different kind of fundamental approach in order to describe high-T_c superconductors. Now, as we have seen in the previous chapter, the AdS/CFT correspondence is a strong/weak coupling duality, meaning that one can describe a strongly coupled boundary quantum field theory in terms of the dynamics of a dual classical supergravity theory living on the bulk. It should be noted that in the holographic setup the dual field theory will be a $SU(N)$ gauge theory in the large-N regime. This class of theories can in principle have a different microscopic Lagrangian, with different degrees of freedom and mechanisms for fermion-fermion coupling than that of usual high-T_c superconductors, like the cuprates. However, they are nevertheless theories in a strong-coupling regime which, as will be shown, indeed exhibit superconducting phenomenology. There is also some indication that these holographic models of strongly-coupled superconducting theories do exhibit some universal phenomena that are shared by the cuprates [81, 82]. In any case, being able to solve toy-models supplied by holographic superconductivity may help us to gain important physical insight on how real world systems work.

Concerning holographic superconducting models, there are two possible approaches one can follow in their construction. The first one of these is the *bottom-up* approach. This approach was introduced and is best exemplified in [83, 84, 7]. In this approach, one constructs reasonably simple bulk models which are intended to generate specific superconducting phenomena in the dual field theory. In this sense, it is an effective approach in the bulk-theory side, whose main purpose is to give a phenomenological description of the physics of the dual field theory. The drawback to the wide array of physical phenomena that can be modeled in this approach is that there may not be a clear way to embedded these simple bulk models in the context of a full blown supergravity or string theory, with a well understood dual field theory. In order to address this question, one can instead follow the opposite way in a *top-down* approach, and start with a well defined, higher-dimensional gravitational theory, like string theory, M-theory or supergravity and then compactify and make a consistent truncation to a particular, definite sector of this "parent" theory. This truncated sector would in turn be dual through the AdS/CFT correspondence to a gauge theory with superconducting behavior. For some excellent examples of research in this approach, see [20, 85, 86, 21]. However, although by following this approach one could have obtained a consistent holographic description of superconductivity which is traceable to a full supergravity theory, this approach also has a drawback. These truncations usually constrain the parameters of the resulting relevant bulk sector so much that there is not much room for accommodating the phenomenology of real systems. In any case, both approaches are complementary to each other and needed in equal share if one wishes to have a complete, consistent understanding of holographic superconductivity.

3.1 Minimal Superconductivity.

When confronted to determine the particular minimal components and form of a holographic superconductor, one can gain valuable insight from general phenomenological considerations. In particular, one can look for the very basic symmetries and components of *any* given superconducting theory, so one can get an idea of what sort of bulk-theory characteristics are needed to generate them holographically. In order to do this, we will follow the arguments posed by Weinberg in [87]. In very general terms, one starts by assuming that any superconducting system allows a theoretical description as a quantum gauge theory. More concretely, this theory will posses the usual electromagnetic gauge invariance, which means the presence of a $U(1)$ field. One then must demand invariance under the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x) \,, \tag{3.1.1}$$

where the transformation acts in the usual manner in the theory's fermionic or bosonic degrees of freedoms. The next step is to make the phenomenological assumption that the superconducting phase transition is caused by spontaneous symmetry breaking of the $U(1)$. This generates a massless Goldstone boson which behaves as a phase and therefore transforms under $U(1)$ as a shift

$$G(x) \to G(x) + \alpha(x) \,. \tag{3.1.2}$$

Gauge invariance of the theory means that the Lagrangian describing the gauge and Goldstone fields must have the general structure

$$\mathcal{L} = \int d^4x \, \mathcal{F}\left[A_\mu - \partial_\mu G\right] , \tag{3.1.3}$$

where the function $\mathcal{F}$ depends on the combination $A_\mu - \partial_\mu G$, which is invariant under the gauge transformation described above. The electric and charge density are then given by

$$J^i = \frac{\delta \mathcal{F}}{\delta A_i} , \tag{3.1.4}$$

$$J^0 = \rho = \frac{\delta \mathcal{F}}{\delta A^0} = -\frac{\delta \mathcal{F}}{\delta(\partial_t G)} , \tag{3.1.5}$$

where in the last equation one uses that fact that $\mathcal{F}$ depends on the gauge invariant combination $A - dG$. From (3.1.5) one sees that $-\rho$ is the conjugate momentum to G. Therefore, in the Hamiltonian description the energy density $\mathcal{H}$ is a function of ρ and G, and the Hamilton equation to G is then

$$\partial_t G = -\frac{\delta \mathcal{H}}{\delta \rho} . \tag{3.1.6}$$

This equation represents the change of energy density due to a variation of the charge density ρ, i.e. the electric potential V. Then, one can relate the time derivative of the Goldstone boson with the potential as

$$\partial_t G = -V . \tag{3.1.7}$$

One can now consider the stationary case, which physically means a steady current flow. Since in that case there is no explicit time dependence, then $\partial_t G = 0$ and the electric potential is therefore zero. One has then obtained a system with a steady flow of current through it and without any electric potential to sustain it. This is then a system with infinite conductivity, i.e. a system in a superconducting state.

We have thus found that a minimal theory describing basic, defining superconducting physics can be obtained from very simple phenomenological assumptions, namely, the presence of $U(1)$ gauge symmetry and its simultaneous breaking. Furthermore, the particular microscopic details of the fermion-pairing or of the symmetry breaking mechanism are not essential in this minimal description. This explains the fact that an effective description such as Ginzburg-Landau theory can provide such a good phenomenological description of superconductivity starting from rough physical intuition.

3.2 Minimal Bulk Field Content.

From the analysis in the previous section we have obtained a very general idea of the indispensable physical characteristics that any bulk gravitational theory should be able to generate holographically in the dual field theory at the boundary. In the most basic setup, the $U(1)$ symmetry in the field theory requires a $U(1)$ gauge field in the bulk theory, while the requirement for symmetry breaking calls for the introduction of a scalar field, charged under the $U(1)$. In this line of thought, we can the list the very minimal ingredients any bulk model of holographic superconductivity (in its simplest setting) must contain:

- $U(1)$ **local symmetry.** According to the AdS/CFT dictionary, a local $U(1)$ symmetry in the bulk theory will correspond to a global $U(1)$ symmetry in the boundary. It is important to note, however, that the theoretical description of superconducting phenomena require a dynamical photon and that a global $U(1)$ symmetry in the boundary theory is actually more suited to the description of a superfluid. Nonetheless, when studying magnetic phenomena in holographic superconductors, one observes that the bulk gravitational models do actually give rise to diamagnetic currents in the boundary field theory that account for the Meissner effect, which is an eminently dynamical phenomena. As we shall argue latter, one can always assume that the global $U(1)$ symmetry can be weakly gauged.

- $U(1)$ **gauge field** A_μ**.** This field is required by $U(1)$ symmetry invariance and is holographically dual to a global $U(1)$ current in the boundary. Additionally, its temporal component will introduce the presence of a charge density in the boundary theory. This charge density will in turn add an energy scale to the system.

- **Complex Scalar field Ψ.** This bulk degree of freedom represents the charged condensate as it will be dual to a s-wave order parameter in the superconducting theory. It can be physically interpreted as an effective holographic description of the bosonic condensate of some multi-fermion bound state in the dual field theory. Furthermore, when following a phenomenological bottom-up approach, the scalar field's mass and charge introduce the most basic set of input parameters in the bulk model.

- **Gravity.** In the minimal setup, one considers Einstein-Hilbert gravity with a negative cosmological constant in order to have AdS vacuum solutions. Since superconductivity is a thermal theory, one introduces temperature by considering black-hole solutions in the bulk. This way, the Hawking temperature of the black hole is translated to the temperature of the superconducting system, according to the AdS/CFT dictionary [88]. Since we want our dual field theory to have a chemical potential or charge density, we must look for charged black hole solutions. Indeed, as we shall see, the presence of a charged black hole in the system will allow us to have gauge field solutions that will add a chemical potential or a charge density needed to describe superconductivity in the dual field theory.

3.3 Minimal Bulk Theory.

Having proposed the minimal ingredients of the bulk theory, one now determines their dynamics by writing the simplest action, which in the general dimensional case is

$$S = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{g} \left\{ R - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{d(d-1)}{L^2} + (D_\mu \Psi)^* D^\mu \Psi - m^2 |\Psi|^2 \right\} \tag{3.3.1}$$

where R is the scalar curvature and the Einstein-Hilbert action is coupled to the complex scalar field Ψ and the gauge field A_μ. Also, the field strength is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the gauge-covariant derivative is

$$D_\mu \Psi = \partial_\mu \Psi - iqA_\mu \Psi \,. \tag{3.3.2}$$

The Lagrangian (3.3.1) contains the parameters m^2 and q, which are the scalar field's mass and charge, respectively. In a phenomenological approach, these are considered input parameters which can be varied within the acceptable ranges in order to probe some physical properties of the dual field theory, like its temperature. In more refined approaches one can modify the action (3.3.1) with different structure functions and potentials [89, 90].

The choice of dimension of the bulk model (3.3.1) is phenomenological. As we saw in Chapter 1, in the case of the cuprates superconductivity is realized in the copper-oxide planes, so it can be regarded as a quasi-two dimensional system. Therefore, for most models of holographic superconductivity, the dual bulk theory lives in AdS_4 or AdS_5. We call attention in particular to the $D = 3+1$ case, by which the general subject of holographic superconductivity was introduced in [7], and to which we will refer to as the *HHH model*, that we review in what follows.

The equations of motion arising from (3.3.1) are given in general by

$$D_\mu D^\mu \Psi = m^2 \Psi \,, \tag{3.3.3}$$

$$\nabla^\mu F_{\mu\nu} = iq\left(\Psi^* D_\nu \Psi - \Psi (D_\nu \Psi)^*\right) \,, \tag{3.3.4}$$

$$\begin{aligned} R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} - \frac{d(d-1)}{2L^2} g_{\mu\nu} &= \frac{1}{2} F_{\mu\lambda} F_\nu{}^\lambda - \frac{F_{\alpha\beta}F^{\alpha\beta}}{8} g_{\mu\nu} - \frac{m^2\left|\Psi\right|^2}{2} g_{\mu\nu} - \frac{\left|D\Psi\right|^2}{2} g_{\mu\nu} \\ &\quad + \frac{1}{2}\left(D_\mu \Psi D_\nu^* \Psi^* + D_\nu \Psi D_\mu^* \Psi^*\right) \,, \end{aligned} \tag{3.3.5}$$

where we have set $2\kappa^2 = 1$, for simplicity. We check that, in the absence of scalar and gauge fields, the theory does contain the AdS_{d+1} solution

$$\begin{aligned} ds^2 &= \frac{r^2}{L^2}\left(-dt^2 + dx_{d-1}^2\right) + L^2 \frac{dr^2}{r^2} \,, \\ \Psi(t,r,\vec{x}) &= 0 \,, \qquad A_\mu(t,r,\vec{x}) = 0 \,. \end{aligned} \tag{3.3.6}$$

In order to solve the equations of motion (3.3.3)-(3.3.5), one proposes the following ansatz for the metric and fields

$$ds^2 = -g(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{g(r)} + \frac{r^2}{L^2}dx_{d-1}^2\,, \tag{3.3.7}$$

$$A = \Phi(r)dt\,, \qquad \Psi = \Psi(r)\,. \tag{3.3.8}$$

In taking the gauge field ansatz (3.3.8), we have implicitly chosen the *radial gauge* $A_r(r,t,\vec{x}) = 0$. This choice has the particular advantage that it results in a constant phase for the complex scalar field, which can then be set to zero. We then redefine the scalar field in term of a real function $\psi(r)$ as

$$\Psi(r) = \psi(r)\,. \tag{3.3.9}$$

For the case of $D = 3+1$ bulk-dimensions, the equations of motion under ansatz (3.3.7)-(3.3.8) are

$$\psi'' + \left(\frac{g'}{g} - \frac{\chi'}{2} + \frac{2}{r}\right)\psi' + \frac{q^2\phi^2e^{\chi}}{g^2}\psi - \frac{m^2}{2g}\psi = 0\,, \tag{3.3.10}$$

$$\phi'' + \left(\frac{\chi'}{2} + \frac{2}{r}\right)\phi' - \frac{2q^2\psi^2}{g}\phi = 0\,, \tag{3.3.11}$$

$$\chi' + r\psi'^2 + \frac{rq^2\phi^2\psi^2e^{\chi}}{g^2} = 0\,, \tag{3.3.12}$$

$$\frac{1}{2}\psi'^2 + \frac{\phi'^2e^{\chi}}{4g} + \frac{g'}{gr} + \frac{1}{r^2} - \frac{3}{gL^2} + \frac{m^2\psi^2}{2g} + \frac{q^2\psi^2\phi^2e^{\chi}}{2g^2} = 0\,, \tag{3.3.13}$$

where we have set $L = 1$, for simplicity. We see from the equation for the scalar field ψ the presence of an effective mass, given by

$$m^2_{\text{eff.}} = m^2 - \frac{q^2\phi^2e^{\chi}}{g}\,, \tag{3.3.14}$$

and where we note a negative sign in the gauge contribution. The presence of this relative sign is a very important fact since, as will be explained below, it will ultimately result in making the gravitational system unstable and produce condensation in the dual field theory.

In looking for solutions for the system, one can make use of the following scaling symmetries

$$r \to ar\,, \qquad (t,x_i) \to (t,x_i)/a\,, \qquad g \to a^2g\,, \qquad \phi \to a\phi\,, \tag{3.3.15}$$

and

$$e^{\chi} \to a^2 e^{\chi}\,, \qquad t \to at\,, \qquad \phi \to a\phi\,, \tag{3.3.16}$$

which leave the equations of motion and the metric unchanged. By means of these symmetries one can set $r_h = 1$ and $\chi(r_h) = 1$ at the moment of solving the equations of motion, thus simplifying the numerical computations in a considerable degree.

In order to solve the equations of motion (3.3.10)-(3.3.13), one has to impose boundary conditions. Regarding the metric ansatz (3.3.7), one must demand that it behaves asymptotically AdS. This means that the functions $f(r)$ and $\chi(r)$ must behave at $r \to \infty$ as

$$g(r) \approx r^2 + \cdots\,, \qquad \chi(r) \approx 0 + \cdots\,, \tag{3.3.17}$$

where the dots denote subleading corrections in r. In addition to this, one must also impose the condition $g(r_h) = 0$ in order to obtain a black hole solution with horizon r_h. As said before, this introduces a finite temperature at the dual field theory through the Hawking temperature of the black hole, which can be obtained by analyzing the near-horizon behavior of $g(r)$. Indeed, making the change to euclidean time $t = i\tau$ and the near-horizon coordinate $z^2 = (r - r_h)$, the metric ansatz (3.3.7) becomes

$$ds^2 \approx dz^2 + \left(\frac{g'(r_h)e^{-\chi(r_h)/2}}{2}\right)^2 z^2 d\tau^2 + \cdots\,. \tag{3.3.18}$$

Asking for the absence of a conical singularity at the origin, one then requires the euclidean time to have the periodicity

$$\tau \sim \tau + \frac{4\pi}{g'(r_h)e^{-\chi(r_h)/2}}\,, \tag{3.3.19}$$

and recalling that the inverse period of the Euclidean times is equal to the temperature one has

$$T = \frac{g'(r_h)e^{-\chi(r_h)/2}}{4\pi}\,. \tag{3.3.20}$$

Regarding the gauge field, one notes that, because of the presence of a horizon in the metric, the norm of the gauge field's time component $A_t = \phi(r)$ diverges near the horizon

$$g^{tt}(r_h)\phi(r_h)^2 \to \infty \,. \tag{3.3.21}$$

Therefore, in order to obtain finite physical quantities, one imposes the following horizon regularity condition

$$\phi(r_h) = 0 \,. \tag{3.3.22}$$

Regarding its asymptotic, near-boundary ($r \to \infty$) behaviour, from the equations of motion one obtains in the general dimensional case

$$\phi(r) = \mu + \frac{\rho}{r^{d-2}} + \cdots \,. \tag{3.3.23}$$

The asymptotic coefficients ρ and μ are identified holographically with the charge density and the chemical potential, respectively. To see this, one remembers that, according to the AdS/CFT dictionary, the leading, non-normalizable bulk mode μ is associated with a source of the temporal component of a $U(1)$ current J_μ in the boundary field theory, while the subleading term is proportional to the vacuum expectation value of J_t. Generally speaking, the source $A_\mu(\infty)$ is coupled to a vector current J_μ as

$$\int d^d x \; A^{\mu}_{(0)} J_\mu \,, \tag{3.3.24}$$

where $A^{\mu}_{(0)} \equiv A^{\mu}(\infty)$ is the boundary value of the bulk gauge field. Since in our particular ansatz only the gauge field's time component is different from zero, we have

$$\int d^d x \; \phi_{(0)} J_t = \int d^d x \; \phi_{(0)} \rho \,, \tag{3.3.25}$$

where we used the fact that the temporal component of the current density is the charge density of the system. Since from (3.3.23) $\phi_{(0)} = \mu$, with μ depending only on the radial coordinate, then

$$\mu \int d^d x \; \rho = \mu \, Q \,, \tag{3.3.26}$$

where we have now used the fact that the integral on the left-hand-side of (3.3.26) represents the system's total charge Q. In this manner we see that the holographic identification of the bulk gauge field's asymptotic modes is physically consistent and that the charged black hole solutions in the bulk represent a dual field theory with finite charge density. We choose to fix the value of the charge density of the system to unity $\rho = 1$, for simplicity. By fixing this value, we are effectively choosing to work in the canonical ensemble. On the other hand one could choose to set a fixed chemical potential μ, thus working in the grand canonical ensemble. As it turns out, both ensembles are connected by a Legendre transformation [15]. For more on boundary conditions for vector fields, see [91].

Regarding the asymptotic $r \to \infty$ behaviour of the bulk scalar field ψ, one finds through the equations of motion

$$\psi \approx \frac{\psi_-}{r^{\Delta_-}} + \frac{\psi_+}{r^{\Delta_-}} + \cdots \,, \tag{3.3.27}$$

where, as has already been seen in the previous chapter, $\Delta_\pm$ come from solving the equation $m^2 = \Delta(\Delta - d)$. As we have discussed in the previous chapter, one of the modes ψ_- or ψ_+ (depending on whether we are using standard or alternative quantization) will be fixed and will correspond to a source in the dual theory. It will be fixed to 0 as a boundary condition, while the remaining mode will be associated with the vacuum expectation value of a conjugate operator. In this manner, one has two different quantization schemes that correspond to different boundary field theories.

In choosing a particular quantization scheme, we are interested in obtaining spontaneous condensation of the scalar field. Indeed, a non-zero profile for the bulk scalar field will correspond to a non-trivial expectation value for the dual field theory operator. Since this operator is charged under a global $U(1)$ symmetry, then this symmetry will be spontaneously broken. This spontaneous symmetry breaking will lead the dual boundary system to a superconducting phase. Therefore, we will look for bulk solutions that translate into a scalar operator that acquires an unsourced, non-trivial vacuum expectation value. This means setting one of the modes to zero, and the quantization schemes are summarized as

$$\psi_- = 0\,, \qquad \psi_+ = \langle \mathcal{O}_+ \rangle\,, \tag{3.3.28}$$

or, for the alternative quantization (when possible)

$$\psi_+ = 0\,, \qquad \psi_- = \langle \mathcal{O}_- \rangle\,. \tag{3.3.29}$$

From the point of view of the boundary field theory, the operators $\langle \mathcal{O}_\pm \rangle$ are interpreted physically as the superconducting order parameters of the system.

In the concrete case of the HHH model, and setting the value of the mass as $m^2 = -2$, then the asymptotic behaviour of the scalar field will be given by

$$\psi \approx \frac{\mathcal{O}_1}{r} + \frac{\mathcal{O}_2}{r^2} + \cdots\,, \tag{3.3.30}$$

so that the quantization schemes (3.3.28) or (3.3.29) we can consider will result in condensates $\mathcal{O}_1$, $\mathcal{O}_2$ of dimension $\Delta = 1$ or $\Delta = 2$, respectively.

3.4 The Normal Phase.

In a superconductor in the normal phase, the order parameter vanishes. This fact translates holographic to a gravitational solution with a null bulk scalar field. The normal phase of the superconductor is then equal to a hairless, asymptotically AdS, Reissner-Nordström black hole solution. This is given in the general dimensional case by

$$g(r) = r^2 - \frac{1}{r^{d-2}}\left(r_h^d + \frac{Q^2}{r_h^{d-2}}\right) + \frac{d-2}{d-1}\frac{\rho^2}{2r^{2d-4}}\,, \qquad \chi(r) = 0\,, \tag{3.4.1}$$

with

$$Q^2 = \frac{d-2}{2(d-1)}\rho^2\,, \tag{3.4.2}$$

while the gauge field solution is

$$\phi(r) = \rho\left(\frac{1}{r_h^{d-2}} - \frac{1}{r^{d-2}}\right)\,. \tag{3.4.3}$$

We note that all solutions are expressed in terms of the charge density, thus effectively representing the normal phase solutions in the canonical ensemble. They are also expressed in terms of r_h, which is then traded by temperature. (See equation (3.5.1) below.)

3.5 The Superconducting Instability.

We now study the zero-temperature limit of the Reissner-Nordström black hole solution. This limit describes a finite charge density state analog to a Fermi Liquid. As we have seen, in Fermi Liquid theory the superconducting instability comes from calculating the scattering amplitude in the Cooper channel. In an analogous line of reasoning, we now investigate whether scalar field perturbations are able to destabilize the background gravitational solution in this limit. If the Reissner-Nordström background is unstable under such scalar perturbations, then the full backreacted solution will give rise to a non-trivial scalar field profile in an holographic analogous to the Cooper instability. For interesting research on scalar instabilities under different AdS setups, see for example [92, 93, 94, 95, 96].

We will follow the arguments of [7, 73, 83, 97]. The Hawking temperature of the Reissner-Nordström black hole (3.4.1) is given in the general dimensional case as

$$T = \frac{2d(d-1)r_h - (d-2)^2 r_h^{3-2d}\rho^2}{8(d-1)\pi} \,. \tag{3.5.1}$$

We can make use of the scaling symmetries (3.3.15)-(3.3.16) in order to set $r_h = 1$. Then, in order to have zero-temperature solution we require

$$\rho = \frac{(d-2)}{\sqrt{2d(d-1)}} \,. \tag{3.5.2}$$

Now, we can make an expansion of the function $g(r)$ around the horizon

$$g(r) = g(1) + g'(1)(r-1) + \frac{1}{2}g''(1)(r-1)^2 + \cdots \,, \tag{3.5.3}$$

and since

$$g(1) = 0 \,, \qquad g'(1) \sim T = 0 \,, \qquad g''(1) = 2d(d-1) \,, \tag{3.5.4}$$

then we obtain

$$g(r) = d(d-1)(r-1)^2 + \cdots \,. \tag{3.5.5}$$

Then, defining the coordinate $\tilde{r} = r - 1$, the metric near the horizon can be written as

$$ds^2 \approx -d(d-1)\tilde{r}^2 dt^2 + \frac{1}{d(d-1)\tilde{r}^2} d\tilde{r}^2 + dx_{d-1}^2 \,, \tag{3.5.6}$$

which can be recognized as $\mathrm{AdS}_2 \times \mathbb{R}^{d-1}$. If we were to recover the original AdS radius L which we have previously set to one, one would find that the radius squared of the AdS_2 part is

$$L_{(2)}^2 = \frac{L^2}{d(d-1)} \,. \tag{3.5.7}$$

By plugging this near-horizon, zero-temperature limit of the metric into the equation of motion for the scalar field, one obtains

$$\psi'' + \frac{2}{\tilde{r}}\psi' + \frac{2q^2 - m^2}{d(d-1)\tilde{r}^2}\psi = 0 \,, \tag{3.5.8}$$

where we notice that the effective mass of the scalar field is now given by

$$m_{\text{eff.}}^2 = \frac{m^2 - 2q^2}{d(d-1)} \,. \tag{3.5.9}$$

As we have already seen before, an AdS_{d+1} background is unstable under scalar field perturbations if the field's mass in below the Breitenlohner-Friedman bound. In the case of AdS_2, this bound is given by $m_{\mathrm{BF}}^2 = -1/4$, so in the present case the system is unstable if

$$m^2 - 2q^2 < -\frac{d(d-1)}{4} \,. \tag{3.5.10}$$

Therefore, remembering that the mass already satisfies the AdS_{d+1} bound, we obtain the following range of values for m^2

$$-\frac{d^2}{4} < m^2 < 2q^2 - \frac{d(d-1)}{4} \,. \tag{3.5.11}$$

Then, by complying to these bounds, one obtains an asymptotic AdS_{d+1} geometry and an instability in the near-horizon AdS_2.

3.6 The Helmholtz Free Energy

In general terms, the grand-canonical potential is given by

$$\Omega = -T \log Z \,, \tag{3.6.1}$$

where Z is the field theory's thermal partition function. According to the AdS/CFT prescription, one can compute Z in terms of the gravitational action, evaluated on-shell. In the semi-classical approximation, one will have

$$Z \approx \exp\left[-S^{\text{gravity}}_{\text{on-shell}}\right] \,, \tag{3.6.2}$$

so one obtains

$$\Omega = -T S^{\text{gravity}}_{\text{on-shell}} \,. \tag{3.6.3}$$

Therefore, in order to calculate the grand-canonical potential of the field theory, one must compute the on-shell action of the gravitational system. For definiteness, let us now consider the HHH-model. We start from the Euclidean version of the original bulk action (3.3.1)

$$S_E = -\frac{1}{2\kappa^2} \int d^4x \sqrt{-g}\, \mathcal{L} \,, \tag{3.6.4}$$

and where we will set $2\kappa^2 = 1$ in the following. In order to have a simple expression for the on-shell Lagrangian, we consider the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} \,, \tag{3.6.5}$$

whose trace is given by

$$G^{\mu}_{\ \mu} = -R \,. \tag{3.6.6}$$

We now note that, since our ansatz (3.3.7)-(3.3.8) has null spatial gauge field components and has only dependence on the coordinate r, then the xx and yy components of the Einsteins equations are

$$G_{xx} = G_{yy} = \frac{1}{2} r^2 \left(\mathcal{L} - R\right) \,. \tag{3.6.7}$$

Therefore, substituting (3.6.7) in (3.6.6), one obtains

$$-R = G^t_{\ t} + G^r_{\ r} + \mathcal{L} - R\,, \tag{3.6.8}$$

which yields the simple expression

$$\mathcal{L} = -G^t_{\ t} - G^r_{\ r} = -\frac{1}{r^2}\left((rg)' + (rge^{-\chi})'e^{\chi}\right)\,. \tag{3.6.9}$$

Substituting back in (3.6.4) and simplifying, we obtain

$$S_E = \int d^3x \int_{r_h}^{r_\infty} dr \left(2rge^{-\chi/2}\right)' = \int d^3x \,\left(2rge^{-\chi/2}\right)\Big|_{r_\infty}\,. \tag{3.6.10}$$

In the case of an unsourced theory such as ours, where one of the asymptotic coefficients in (3.3.27) is set to zero, the asymptotic analysis of the Einstein equations reveals

$$e^{-\chi}g = r^2 - \frac{\epsilon}{2r} + \cdots\,, \tag{3.6.11}$$

where we have denoted the black hole mass as ϵ, which will be interpreted holographically as the dual theory energy density [7, 98]. Substituting this asymptotic expansion in (3.6.10), we clearly see that the action is divergent at $r \to \infty$ because of the first term in (3.6.11). In order to make the integral finite, we follow [7, 99] and regulate the action by adding as counter-terms the Gibbons-Hawking-York term plus a boundary cosmological constant term

$$S_{GH} = \int d^3x\sqrt{-h}\left(2K + \frac{4}{L}\right)\,, \tag{3.6.12}$$

where h is the induced boundary metric and K is the trace of the extrinsic curvature

$$K^{\mu\nu} = -\frac{1}{2}\left(\nabla^\mu n^\nu + \nabla^\nu n^\mu\right)\,, \tag{3.6.13}$$

with n^μ a unit vector normal to the boundary, pointing outwards. With the addition of this counter-term, the renormalized action is

$$S_{\text{ren.}} = S_E + S_{GH}\,, \tag{3.6.14}$$

which is written explicitly as

$$S_{\text{ren}} = -\frac{1}{2}\int d^3x\,\epsilon\,. \tag{3.6.15}$$

Then, the grand canonical potential is

$$\begin{aligned}\Omega = T\,S_{\text{ren}} &= -\frac{T}{2}\int d^3x\,\epsilon \\ &= -\frac{V_2}{2}\,\epsilon = -\frac{E}{2}\,,\end{aligned} \tag{3.6.16}$$

where V_2 is the volume obtained by integrating over the spatial dimensions of the boundary and where E is the total energy of the dual system. We have also performed an integration over the compact time dimension, giving the inverse temperature and canceling T from the last expression.

In order to study the thermodynamical properties of the system in the canonical ensemble (ρ fixed), we must look into the Helmholtz free energy F. It is related to the grand canonical potential Ω by a Legendre transformation

$$F = \Omega + \mu\,Q = -PV + \mu\,Q\,. \tag{3.6.17}$$

Substituting the result (3.6.16) for Ω in the second equality, we have

$$PV = \frac{E}{2}\,. \tag{3.6.18}$$

We now consider the *Euler equation*

$$E = -P\,V + S\,T + \mu\,Q\,, \tag{3.6.19}$$

where S is the total entropy of the system. Then, substituting (3.6.18) in (3.6.19) we get

$$\mu\,Q = \frac{3}{2}E - ST\,, \tag{3.6.20}$$

and substituting this result and (3.6.16) back in the first equation (3.6.17), we finally have

$$F = E - TS\,, \tag{3.6.21}$$

which is the definition for F, showing thus that our framework is consistent. It is usually more useful to rewrite (3.6.21) in terms of energy densities

$$f = \epsilon - s\,T\,, \tag{3.6.22}$$

where we have rescaled each thermodynamical potential with the total boundary spatial volume V_2 and where s is the entropy density.

We call attention to the fact that the relation (3.6.18) can alternatively be obtained by considering that the boundary stress-energy tensor is traceless. Indeed, from this fact we have

$$0 = \int d^2x\, T = -\int d^2x\, T^{tt} + 2\int d^2x\, T^{ii} = -E + 2P\,V\,, \tag{3.6.23}$$

leading to (3.6.18).

A much more convenient expression for F, from the computational point of view, can be obtained by rewriting (3.6.20) as

$$E = \frac{2}{3}\left(S\,T + \mu\,Q\right)\,, \tag{3.6.24}$$

and substituting this result in (3.6.21), we obtain

$$F = \frac{1}{3}\left(2\mu\,Q - S\,T\right)\,, \tag{3.6.25}$$

or, in terms of energy, entropy and charge densities

$$f = \frac{1}{3}\left(2\mu\rho - s\,T\right)\,, \tag{3.6.26}$$

We see that, in this form, the Helmholtz free energy can be computed holographically in terms of the asymptotic bulk-gauge field modes and on the black hole area and temperature. In the following, the relevant quantity we will want to compute is given by

$$\Delta f = f_{\text{Supercond.}} - f_{\text{norm.}}\,, \tag{3.6.27}$$

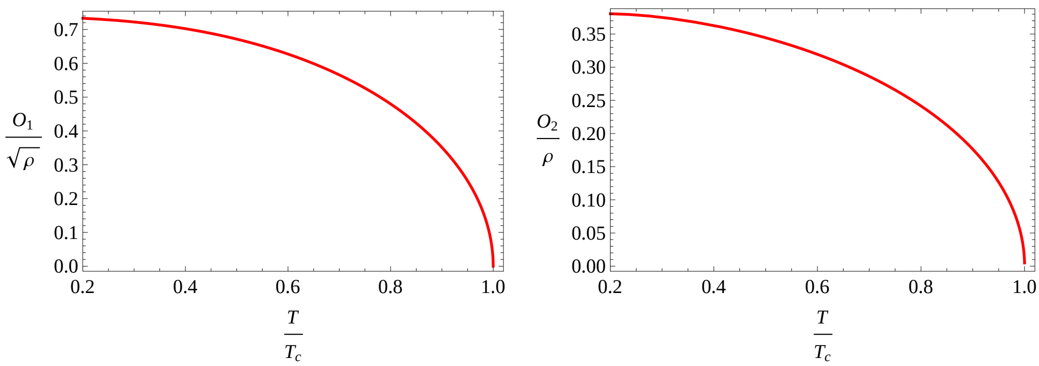
which is the difference between the free energy in the superconducting and normal phases. Thus, $\Delta f < 0$ will mean that the superconducting phase has a lower free energy than the normal phase, implying that the hairy solution is thermodynamically favorable.

3.7 Condensation.

To summarize, in the previous sections we have set a consistent general theoretical frame for a model of holographic superconductivity. To do this, we have proposed a very simple bulk theory, consistent with the minimal features required for superconducting behavior in the dual boundary theory. Through the AdS/CFT dictionary, we have given a physical interpretation to each of the components of the gravity theory, and explained how every allowed bulk theory solution translates to a different superconducting phase in the dual field theory. In particular, the holographic interpretation asserts that hairy black hole solutions with well-defined boundary conditions correspond to a spontaneous symmetry breaking phase in the dual theory. We have also investigated the general conditions on the bulk side for instability against hair creation, and claimed that this bulk instability can be interpreted as an holographic dual of the Cooper instability in the boundary side of the duality.

Referring to the details of the bulk computations, by making use of the scaling symmetries (3.3.15)-(3.3.16) and by setting the input parameters m^2 and q at fixed values from the beginning, we have engineered our system to depend only on the temperature T. By solving the equations of motion for each field numerically with the prescribed boundary conditions through the *shooting method*, one can find the desired solutions for the bulk scalar field which, as said before, will be dual do the superconducting order parameter in the boundary field theory. These solutions are found to undergo a phase transition at some definite value $T = T_c$.[1] Below this critical temperature, the system admits two different solutions, one being the usual Reissner-Nordström solution with null scalar field, and the other one being a hairy black hole solution with non-trivial scalar profile. Both below-T_c solutions are characterized by a particular

[1]In the present HHH model, the phase transition is second-order. Other models of holographic superconductivity, however, can show a first-order phase transition behaviour. One of such models is presented in Chapter 6.



(a) (b)

Figure 3.1: Value of the condensate as a function of temperature, for value of the scalar field charge $q = 1$. Figure (a) shows the $\mathcal{O}_1$ condensate, while figure (b) shows the $\mathcal{O}_2$ condensate.

behaviour of the free energy, and the solution with a lower free energy is to be thermodynamically favored over the other. In figures (3.1a) and (3.1b) we show the value of the condensates of the HHH model as a function of temperature, for each quantization scheme as described by the asymptotic expansion (3.3.30), and where we have set the scalar mass and charge as $m^2 = -2$ and $q = 1$.

A very non-trivial result can be observed in the near-T_c behaviour of the condensates. Indeed, in that region the condensate has a functional dependence on temperature

$$\langle \mathcal{O} \rangle \sim (1 - T/T_c)^{1/2} , \tag{3.7.1}$$

which, as we have seen in Chapter 1, is the typical mean-field behaviour for the superconducting order parameter. As we have already commented, the proposed bulk theory model is very phenomenological in nature, in the sense that is intended to describe holographically through a simple scalar field the physical behaviour of the dual field theory's order parameter. This approach can therefore be interpreted as a holographic mean-field description of the superconducting system. Non-mean field behaviour should then be found moving away from the large-N supergravity approximation, and therefore deviations from the standard behaviour (3.7.1) should be controlled by $1/N$ corrections.

3.8 Magnetic Phenomena.

Since in the following chapters we will be very interested in studying magnetic phenomena in holographic superconductivity, it could prove of great usefulness to give a general overview of how magnetic phenomenology is implemented in the holographic context. We will follow the standard treatment, as first introduced in [77, 100, 101, 102, 103].

3.8.1 The Meissner Effect

The main focus of our interest in magnetic phenomena will be given by the *Meissner effect* and the holographic determination of the critical magnetic field B_c. As already said in Chapter 1, the behaviour of a superconducting system under the presence of an external magnetic field provides the means for the phenomenological classification of the material as a Type I or Type II superconductor. Whether an holographic superconductor can present Type I or II behaviour is a subject we will treat in detail in the following chapters. For the present account, let is suffice to say that the critical magnetic field B_c we will be computing holographically will be the greater critical magnetic field, that is, the value of the magnetic field mediating within the normal phase and any of the superconducting (Type I) or quasi-superconducting (Type II) phases.

The usual way to define the critical magnetic field B_c is to start in the superconducting phase and no magnetic field present and, for each value of temperature $T < T_c$, increase the magnitude of the magnetic field and look for the particular value above which the system returns to the normal phase. An equivalent but more useful point of view, however, could be to start in the normal phase with a large value of the magnetic field and, for each value of temperature

$T < T_c$, lower the magnitude of the magnetic field until one finds a particular value B_c under which condensation occurs. Holographically, this would mean starting from a normal, magnetic solution and look for values of the magnetic field that develop an instability for a hairy black hole solution in the bulk theory scalar. In the particular $D = 3 + 1$ case of the HHH model, such a vacuum, magnetic solution is given exactly by the dyonic metric [104]

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2\left(du^2 + u^2 d\varphi^2\right)\,, \tag{3.8.1}$$

with

$$g(r) = r^2 - \frac{1}{4rr_h}\left(4r_h^4 + \rho^2 + B^2\right) + \frac{1}{4r^2}\left(\rho^2 + B^2\right)\,, \tag{3.8.2}$$

and where we are working in polar coordinates for the spatial dimensions of the boundary theory, $dx^2 + dy^2 = du^2 + u^2 d\varphi^2$. The gauge field now includes a magnetic term

$$A = \rho\left(\frac{1}{r_h} - \frac{1}{r}\right)dt + \frac{1}{2}Bu^2 d\varphi\,, \tag{3.8.3}$$

from where a boundary magnetic field perpendicular to the x-y plane can be read

$$F_{xy}\big|_{r\to\infty} = \frac{1}{u}F_{u\varphi}\big|_{r\to\infty} = B\,. \tag{3.8.4}$$

The Hawking temperature of this black hole solution is then given by

$$T = \frac{12r_h^4 - \rho^2 - B^2}{16\pi r_h^3}\,. \tag{3.8.5}$$

In order to look for hairy solutions, we will fix the dyonic solution described above and treat the scalar field ψ as a perturbation under this background, with boundary conditions at the boundary given by any quantization scheme we choose from the asymptotic fall-off (3.3.30). We also choose the scalar field to depend only on the bulk-radial coordinate r and on the boundary-radial coordinate u, that is, $\psi = \psi(r, u)$. As always with the HHH model, we set the scalar mass as $m^2 = -2$. In this manner, the equation of motion for ψ is given by

$$\frac{1}{u}\partial_u\left(u\partial_u\psi\right) + \partial_r\left(r^2 g\partial_r\psi\right) + \left(\frac{q^2\rho^2}{gr_h^2}(r - r_h)^2 - \frac{q^2u^2}{4}B^2 + 2r^2\right)\psi = 0\,, \tag{3.8.6}$$

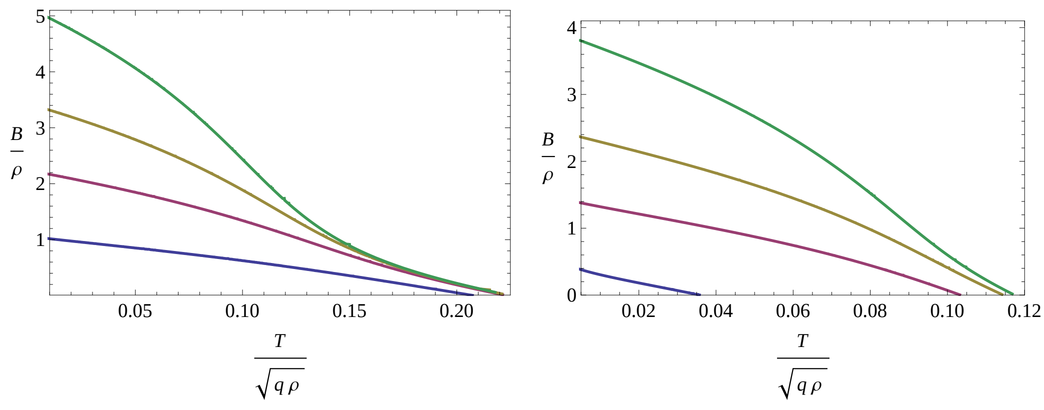


Figure 3.2: Value of the critical magnetic field B_c as a function of temperature, for various values of the scalar field charge q. Figure (a) shows the case where $\mathcal{O}_1$ condenses, while figure (b) shows the case where $\mathcal{O}_2$ condenses. In both figures, going from the top to the bottom figure, the value of the charge is $q = 12, 6, 3, 1$.

which can be solved by separation of variables

$$\psi(r,u) = R(r)U(u)\,, \tag{3.8.7}$$

so that the equation we obtain are

$$\begin{aligned} U'' + \frac{1}{u}U' + \left(\frac{quB}{2}\right)^2 U &= -\lambda U\,, & (3.8.8)\\ \left(r^2 g R'\right)' + \left(\frac{q^2\rho^2(r-r_h)^2}{gr^2} + 2r^2\right) R &= \lambda R\,, & (3.8.9) \end{aligned}$$

with the separation constant $\lambda = qnB$. The equation (3.8.8) for the function $U(u)$ is that of a harmonic oscillator with frequency determined by B. We will expect the lowest mode $n = 1$ of this oscillating boundary profile to be the most stable solution after condensation. Therefore, the solution for $U(u)$ is given by the simple Gaussian profile

$$U(u) = \exp\left(-qBu^2/4\right)\,. \tag{3.8.10}$$

In this manner, the superconducting solution will be localized as a droplet condensate in the boundary field theory. This is customarily called the *droplet* *solution*. Meanwhile, the asymptotic behaviour of the scalar field (3.3.30) is now carried by $R(r)$, so that one sets the boundary conditions for the equation (3.8.9) following the quantization scheme for ψ that one wishes to follow. Solving the system in this manner, the result will be a curve of solutions (T, B_c). In figures (3.2a) and (3.2b) we show the behaviour of the critical field as a function of temperature, for both quantizations schemes and for various values of the scalar charge q. It is very important to note that the critical magnetic field has a near-T_c behaviour

$$B_c \sim (1 - T/T_c) \ , \tag{3.8.11}$$

which is in agreement with the behaviour predicted by mean-field theory.

3.8.2 London Currents and Dynamical Photons

In standard superconductivity, the physical picture behind the Meissner effect is the generation, by the external magnetic field, of diamagnetic currents in the superconductor. Such currents will in turn generate a magnetic field of equal magnitude and opposite direction that will cancel the external field inside the sample. This generation of currents clearly requires the presence of a dynamic electromagnetic field. In fact, the introduction of such a dynamical vector field was the main cause behind the success of the phenomenological description of superconductivity provided by Ginzburg-Landau theory, as we have seen in Chapter 1. In this description, superconductivity is generated by the breaking of a local $U(1)$ symmetry, a fact that provides the photon with mass and explains, for instance, the superconductor penetration length λ, among other phenomenological features. However, when we match this field theory expectation with our holographic setup that we have built in order to model superconductivity, we inmediately face some issues. Indeed, according to the AdS/CFT dictionary, the *local* $U(1)$ symmetry breaking in the bulk theory that we have been using to generate hairy black hole solutions, translates to a *global* $U(1)$ symmetry breaking in the dual field theory, as mentioned earlier. This means that the dual boundary theory does not contain a dynamical photon, that the material cannot produce its own opposite magnetic fields and that, strictly speaking, the Meissner effect should not exist in a holographic superconductor. As first noted in [7] the solution to this comes from the fact that holographic superconductors do generate the currents required to expel external magnetic fields (London currents), and that the dual field theory can be consistently gauged.

To see how the London currents arise in holographic superconductivity, let us first discuss the London equation (1.1.3) in frequency-momentum space

$$J_i\left(\omega,k\right) = -n_s A_i\left(\omega,k\right)\ , \tag{3.8.12}$$

which, as we have already seen in Chapter 1, was proposed to explain both the infinite conductivity and the Meissner effect. In the limit $k = 0$ and $\omega \to 0$ we can take a time derivative of both sides of the London equation and find

$$J_i\left(\omega,0\right) = \frac{i n_s}{\omega} E_i\left(\omega,0\right)\ . \tag{3.8.13}$$

This relation explains infinite DC conductivity. On the other hand, by considering the curl of the London equation and the limit $k \to 0$ and $\omega = 0$ we obtain

$$i\epsilon_{ijl} k^j J^l\left(0,k\right) = -n_s B_i\left(0,k\right)\ . \tag{3.8.14}$$

This relation together with the Maxwell equation $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ results in the magnetic field equation $\nabla^2 \mathbf{B} = \mathbf{B}/\lambda^2$ (1.1.16) which has as solution an exponentially decaying field inside the superconductor. (See section 1.1 in Chapter 1.) It is important that both the superconductivity and the Meissner effect follow from the London equation.

To see how the London equation (3.8.12) can be consistently accommodated in the holographic setup, let us allow for the presence of bulk gauge field perturbations in our holographic superconducting model, with a momentum and frequency dependence of the kind

$$\delta A = e^{-i\omega t + iky} A_x(r) dx\,. \tag{3.8.15}$$

In order to simplify things, let us work in the *probe limit* [84], which amounts to taking the limit $q \to \infty$ and in which the metric can be safely kept fixed to be simply the Schwarzschild AdS black hole

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2\left(dx^2 + dy^2\right) , \qquad (3.8.16)$$

where $g(r) = r^2 - r_h^3/r$. Assuming then that we have consistently solved for the gauge field ϕ and scalar field ψ, the differential equation for the perturbation A_x will be given by

$$\left(\frac{\omega^2}{g} - \frac{k^2}{r^2}\right) A_x + \left(gA_x'\right)' = 2q^2\psi^2 A_x \,. \qquad (3.8.17)$$

Ignoring the radial dependence at infinity, this equation describes a vector field with mass proportional to $q^2\psi^2$. This mass, which implies an underlying Higgs mechanism in the bulk theory, should give rise to the usual effects of superconductivity and therefore equation (3.8.17) can be thought of as the holographic dual of the London equation (3.8.12).

Finally, in order to expel any external magnetic field, the London currents should be able to couple to dynamical photons [7]. These are introduced by gauging the dual field theory. The $2 + 1$ dimensional boundary theory can be coupled to a photon through the standard $J_\mu A^\mu$ interaction. To make the photon dynamical, we add an F^2 term to the action, with $F = dA$. Electromagnetic phenomena are described by the effective action for the photon. We can obtain this effective action by integrating out all the other degrees of freedom. Starting from the Euclidean partition function, one has

$$\mathcal{Z} = \int \mathcal{D}A\mathcal{D}\mathcal{Y} \, \exp\left\{-S[\mathcal{Y}] - \frac{1}{4}\int d^3x F^2 - \int d^3x J_\mu A^\mu\right\} \qquad (3.8.18)$$

$$= \int \mathcal{D}A \, \exp\left\{-S_{\text{eff.}}[A]\right\} , \qquad (3.8.19)$$

where $\mathcal{Y}$ denotes the remaining degrees of freedom of the boundary field theory. The effective action $S_{\text{eff.}}$ can be obtained by expanding the exponent in (3.8.18), integrating over $\mathcal{Y}$ and finally exponentiating back. The result is

$$S_{\text{eff.}}[A] = \frac{1}{4}\int d^3x\, F^2 + \frac{1}{2}\int d^3x d^3y \,\langle J^\mu J^\nu\rangle_c\,(x-y)\, A_\mu(x)A_\nu(y) + \cdots , \qquad (3.8.20)$$

where $\langle J^\mu J^\nu \rangle_c$ is the connected 2-point function of the current, and where the expectation value is evaluated in the theory without the dynamical photons. In writing (3.8.20) we have omitted additional terms that do not contribute to the dynamics and neglected higher order terms in A because they are suppressed in the large-N limit [7].

Examining (3.8.20), we see that it is possible to obtain the photon mass from $\langle J^\mu J^\nu \rangle_c$. Since we are working in a field theory where Lorentz invariance is broken by the presence of a background charge density, we therefore choose to define the photon mass as the energy of photons in a frame where they are at rest relative to such charge density. Therefore, to find the photon mass we need to exhibit an on-shell photon mode with $k = 0$ and the energy ω of this mode will be photon mass m_γ.

In order to obtain the photon spectrum, we go back to action (3.8.20) and derive the corresponding equations of motion. We restrict to a mode in which A_x is the only nonvanishing component and is of the magnetic form

$$A_x(\omega) = B\, y\, e^{-i\omega t} \,, \tag{3.8.21}$$

where the y dependence has been included so that the mode is not pure gauge at $\omega = 0$. Rotating back to Lorentzian signature, the equation of motion for this mode is

$$\left(\omega^2 + G^R_{J^x J^x}(\omega)\right) A_x(\omega) = 0 \,, \tag{3.8.22}$$

where we have used the fact that the retarded Green's function in momentum space is the Fourier transform of the Euclidean Green's function in momentum space. For a more detailed account, see [105]. Using the result from linear response theory, we have $G^R_{J^x J^x}(\omega) = i\omega\sigma(\omega)$, where $\sigma(\omega)$ is the conductivity. Thus, equation (3.8.22) becomes

$$\omega \left(\omega + i\sigma(\omega)\right) = 0 \,. \tag{3.8.23}$$

The solution to this equation gives the photon mass $m_\gamma = \omega$. From (3.8.23) we can see that $\omega = 0$ (a massless photon) is a solution, provided that the conductivity does not have a pole at $\omega \to 0$. The intuitive expectation we have is to have a massive photon in phases where the electromagnetic $U(1)$ is broken and, conversely, to have a massless photon where the electromagnetic symmetry remains unbroken. Holographic studies in the conductivity, such as [84, 7, 89, 90], show that this intuition turns in fact to be right. When studying the superconducting phase of holographic superconductors, the imaginary part of $\sigma(\omega)$ has a pole as $\omega \to 0$, so that the photon will have a nonzero mass. Meanwhile, in the normal phase one finds that the conductivity σ is a constant, meaning that there is no pole as $\omega \to 0$ and the photon is therefore massless.

4

Bottom-Up Approach, Part I: Ginzburg-Landau Approach to Holographic Superconductivity

In this and in the following chapter, we will be working using an holographic superconducting model in a phenomenological bottom-up approach. The following models are characterized by a very minimal bulk action that, through tractable computations, gives rise holographically to superconducting behaviour in the dual field theory. Furthermore, the bulk model contains input parameters, such as the mass of the scalar field charge, that can be varied in order to gain insight on the phenomenological behaviour of these dual superconducting physical quantities.

Having set the basic phenomenology of this simple model, we will then implement an effective description of the dual field theory on terms of Ginzburg-Landau theory. Using the gauge/gravity duality and some basic physical assumptions, we will be able to compute in a self-consistent fashion the main input parameters of an effective Ginzburg-Landau action to the dual field theory near the critical temperature. Using this phenomenological description, we will the compute the characteristic lengths of the system and the Ginzburg-Landau parameter κ of the holographic superconductor. This parameter is of particular importance in the general theory of superconductivity, since it value is related to the basic classification of superconducting materials as Type I or Type II.(See Chapter 1.) In the present model, it will be shown that the system always shows a Type I behaviour, irrespective of the values of the input parameters considered. Additionally, since this classification is directly related to magnetic phenomena in superconductivity, we will also submit our system to an external magnetic field, using the perturbative gravitational solution constructed by D'Hoker and Kraus [16], for the first time in the context of holographic superconductivity.

As we have already said in Chapter 3, the area of holographic superconductivity in its present form is very phenomenological in nature. It can, for example, give us insight on the physical properties of, say, the dual condensing operators, although without describing the microscopic details behind the pairing mechanism leading to condensation. In this sense, a Ginzburg-Landau approach to the dual theory is very well suited: Ginzburg-Landau theory is itself a very successful "coarse-grained" description of superconductors, which focuses on the phenomenology of physical quantities and which does not require an underlying microscopical theory for these phenomena. This is principal the reason of why Ginzburg-Landau theory can be used to describe the phenomenology, within its reaches of validity, of a very wide array of different physical systems, including high-T_c superconductors. This universality of Ginzburg-Landau theory is also in tune with one of the main motivations behind holographic superconductivity, where one aims to construct models of strongly-coupled superconducting theories that could exhibit some universal phenomena shared by the cuprates.

In this and the following chapter, we will choose to work with a bulk model in $4 + 1$ dimensions, meaning that the dual field theory will live in $3 + 1$ dimensions. In a $3 + 1$ dimensional system subjected to a $3 + 1$ electromagnetic field, both free energies scale with the volume (see Chapter 1), and hence there is a direct thermodynamical competition that can drive the system to a Type I superconducting state. In contrast, there is some evidence that holographic superconductors describing $2+1$ dimensional field theories mostly exhibit Type II behaviour [7, 106]. The standard argument [7] is that, when applying an external $3+1$ dimensional magnetic field to a $2+1$ dimensional system, the free energy needed to expel it scales as the volume, while the free energy that the system gains from being in a superconducting state scales as the area. This indicates that in most of these $2+1$ dimensional systems, B_{c_1} must be zero. However, recently $2+1$ dimensional models exhibiting Type I behaviour in some region of parameter space have been constructed in the interesting paper [11].

4.1 A Minimal Holographic Superconductor in $d = 4 + 1$ AdS

4.1.1 The Model

We will work using a minimal phenomenological model in $d = 4 + 1$ AdS space-time, in the same spirit as in [7], containing a scalar field Ψ and a $U(1)$ gauge field A_μ

$$\mathcal{L} = R + \frac{12}{L^2} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - |D\Psi|^2 - M^2|\Psi|^2 , \tag{4.1.1}$$

where, $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, and $D_\mu\Psi = \nabla_\mu\Psi - iqA_\mu\Psi$. The parameter q corresponds to the charge of the scalar field and, as it will be shown below, different values of q will correspond to superconducting systems with different critical temperature. The general equations of motion for this system are

$$\begin{aligned} D^2\Psi &= M^2\Psi , & (4.1.2)\\ \nabla_\mu F^{\mu\nu} &= qJ^\nu + q^2|\Psi|^2 A^\nu , & (4.1.3)\\ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left(R + \frac{12}{L^2}\right) &= \frac{1}{2}g_{\mu\nu}\left(-\frac{1}{4}F^2 - |D\Psi|^2 - M^2|\Psi|^2\right) & \\ &\quad + \frac{1}{2}F^\lambda{}_\mu F_{\lambda\nu} + D_{(\mu}\Psi D^*_{\nu)}\Psi^* , & (4.1.4) \end{aligned}$$

where

$$J_\mu = i\left(\Psi^*\nabla_\mu\Psi - \Psi\nabla_\mu\Psi^*\right) . \qquad (4.1.5)$$

We will set $L = 1$ for the rest of this chapter.

4.1.2 The Normal and Superconducting Phases

In this section we will briefly review the normal and superconductor regimes of our model, with no external magnetic field to begin with, and with full backreaction included. As is usual, we use the following ansatz for the metric

$$ds^2 = -g(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{g(r)} + r^2\left(dx_1^2 + dx_2^2 + dx_3^2\right) , \qquad (4.1.6)$$

which is the most general ansatz with space-rotation and time-translation symmetry. We will demand that solutions for this ansatz are asymptotically AdS and that they have a black hole geometry, with an event horizon at some $r = r_h$. For the scalar and gauge field we use the ansatz

$$A = \phi(r)dt\,, \qquad \Psi(r) = \frac{1}{\sqrt{2}}\psi(r)\,, \qquad (4.1.7)$$

where ψ is a real function. Introducing the new coordinate $z = r_h/r$, equations (4.1.2-4.1.4) under this ansatz turn to be

$$\begin{aligned}
\psi'' + \left(-\frac{\chi'}{2} - \frac{1}{z} + \frac{g'}{g}\right)\psi' + \frac{r_h^2}{z^4}\left(\frac{e^\chi q^2\phi^2}{g^2} - \frac{M^2}{g}\right)\psi &= 0\,, &(4.1.8)\\
\phi'' + \left(\frac{\chi'}{2} - \frac{1}{z}\right)\phi' - \frac{r_h^2 q^2\psi^2}{z^4 g}\phi &= 0\,, &(4.1.9)\\
3\chi' - z\psi'^2 - \frac{e^\chi q^2\phi^2\psi^2}{z^3 g^2} &= 0 &(4.1.10)\\
\frac{1}{2}\psi'^2 + \frac{e^\chi\phi'^2}{2g} - \frac{3g'}{zg} + \frac{6}{z^2} - \frac{12r_h^2}{z^4 g} + \frac{r_h^2 M^2\psi^2}{2z^4 g} + \frac{e^\chi r_h^2 q^2\phi^2\psi^2}{2z^4 g^2} &= 0 &(4.1.11)
\end{aligned}$$

This system of equations admit a $\psi(z) = 0$ solution. This no-hair solution is given by

$$\begin{aligned}
g(r) &= \frac{r_h^2}{z^2} + \frac{z^4\rho^2}{3r_h^4} - \frac{z^2\left(3r_h^6 + \rho^2\right)}{3r_h^4}\,, &(4.1.12)\\
\chi(r) &= 0\,, &(4.1.13)\\
\phi(r) &= \frac{\rho}{r_h^2}\left(1 - z^2\right)\,, &(4.1.14)
\end{aligned}$$

which is the usual Reissner-Nordström-AdS solution, and corresponds to the normal phase of the superconductor.

We will now consider solutions with scalar hair $\psi \neq 0$. We will set $M^2L^2 = -3$ for the scalar field mass, which is above the Breitenlohner-Freedman bound $M^2_{\rm BF}L^2 = -4$. This choice of mass appears naturally in top-down models of holographic superconductors coming from consistent truncations of supergravity [20, 21][1]. With this choice, ψ behaves at $z \to \infty$ as

$$\psi \approx \mathcal{O}_1 \frac{z}{r_h} + \mathcal{O}_3 \frac{z^3}{r_h^3} + \dots \tag{4.1.15}$$

while for the gauge field the near-boundary behaviour is

$$\phi \approx \mu - \rho \frac{z^2}{r_h^2} + \dots \tag{4.1.16}$$

According to the gauge-gravity correspondence, $\mathcal{O}_3$ corresponds to the vacuum expectation value of an operator of dimension 3 in the dual field theory, while $\mathcal{O}_1$ corresponds to a source to that same operator. Also, μ and ρ will correspond to the chemical potential and charge density of the dual field theory, respectively. To solve our equations of motion, we will impose the boundary condition $\mathcal{O}_1 = 0$ in (4.1.15) and take $\mathcal{O}_3$ as the superconductor order parameter. Setting the source to zero will result in spontaneous breaking of the global $U(1)$ symmetry in the dual field theory and the system enters then in a superconducting phase [7, 83].

We will choose to work in the canonical ensemble, fixing $\rho = 1$. As mentioned above, we will also impose $g(z = 1) = 0$ for some non-zero value of r_h in order to have black hole solutions to our ansatz and introduce temperature to the dual field theory. The Hawking temperature of the system will be given by

$$T_H = -\frac{e^{\chi} g'}{4\pi r_h}\bigg|_{z=1} . \tag{4.1.17}$$

[1] These models have a different potential from ours, arising from higher order terms in ψ. However, they have the same critical temperature, since this only depends on the values of m and q.

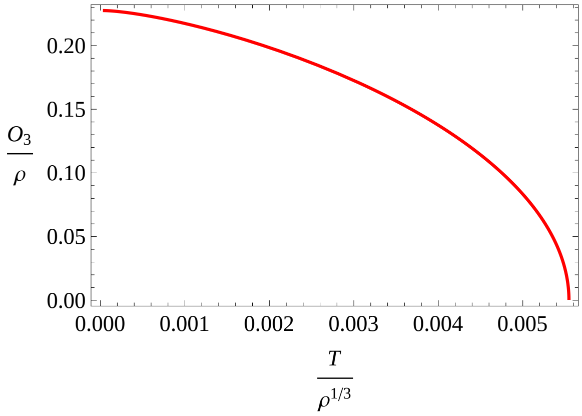


Figure 4.1: The value of the condensate as a function of temperature, for $q = 1$. In this case, $T_c = 0.0055$ approximately.

From equation (4.1.8) for ψ we see that regularity of the solutions at the horizon $z = 1$ requires that

$$\psi'(1) = \frac{r_h^2 M^2 \psi(1)}{g'(1)} \,. \tag{4.1.18}$$

Regularity at the horizon also requires $\phi(1) = 0$. The model has the following scaling symmetries

$$e^{\chi} \to \; a^2 e^{\chi} \,, \;\; t \to at \,, \;\; \phi \to \phi/a \,, \tag{4.1.19}$$

$$r \to ar \,, \;\; (t, x_i) \to (t, x_i)\,/a \;\;\; g \to a^2 g \,, \;\; \phi \to a\phi \,. \tag{4.1.20}$$

This scale invariance helps us to further reduce the number of independent parameters in our model to only one, which we will take to be the temperature of the black hole. Solutions to equations (4.1.8)-(4.1.11) are found via the shooting method, enforcing the no-source condition mentioned above for ψ.

In figure (4.1) we show the behaviour of the order parameter $\mathcal{O}_3$ as a function of temperature for the case $q = 1$, signaling condensation below some critical temperature T_c. One can find by a numerical analysis for different values of q that near T_c the condensate behaves as

$$\mathcal{O}_3 \sim \mathcal{O}_0 \left(1 - T/T_c\right)^{1/2} \,. \tag{4.1.21}$$

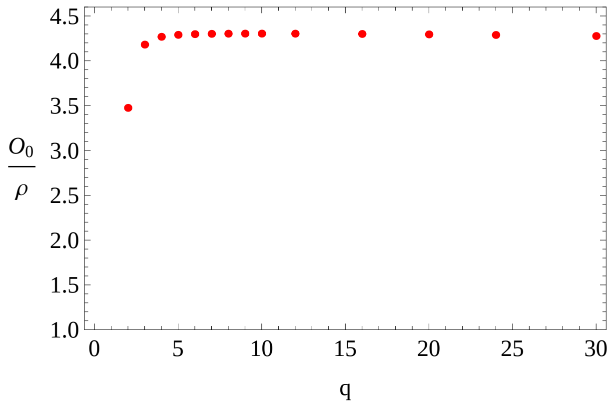


Figure 4.2: The value of the near-T_c coefficient $\mathcal{O}_0$ (see eq. (4.1.21)) as a function of the scalar field charge q.

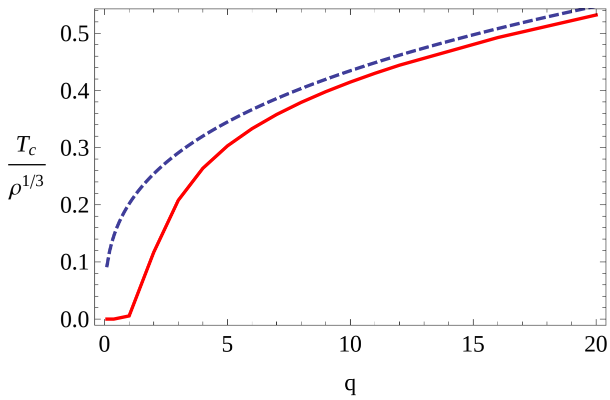


Figure 4.3: The solid line represents the value of the critical temperature T_c as a function of the charge q. The dashed line represents the analytical approximation (4.1.22).

The behaviour of the coefficient $\mathcal{O}_0$ as a function of the scalar field charge q is shown in figure (4.2). For large values of q, we find that $\mathcal{O}_0 \sim$ const.

In the bold line of figure (4.3) we show how the critical temperature T_c behaves for different values of the charge q. As in the $2 + 1$ dimensional case of [7], the behaviour of T_c near zero q is caused because the charged scalar field backreacts to the metric more strongly in that region, decreasing the temperature. Since we have a one-to-one relation between T_c and q, we will use q to vary the critical temperature of our model. Therefore, we will have a set of different superconducting systems characterized by different q.

For large values of q (the probe limit) one can obtain a fair analytical approximation for T_c using the matching method introduced in [107], getting

$$T_c^{\text{large } q} = \frac{1}{\pi}\left(\sqrt{\frac{5}{309}}2\rho\, q\right)^{\frac{1}{3}} . \tag{4.1.22}$$

This is shown as a dashed line in figure (4.3).[2]

4.2 Ginzburg-Landau Description.

We introduce our Ginzburg-Landau interpretation of the dual field theory by first studying the system under a small perturbation of the gauge field on the bulk.

4.2.1 A Magnetic Perturbation

We now add a small magnetic perturbation of the gauge field, in the specific form

$$A = \phi(r)\, dt + \delta A_x(r,t,y)\, dx; \tag{4.2.1}$$

with

$$\delta A_x(t,r,y) = e^{-i\,\omega\, t + i\, k\, y} A_x(r)\,, \qquad |A_x| \ll 1 \tag{4.2.2}$$

This perturbation has an harmonic dependence on time and carries momentum along the y-direction. To linearized level, the equation of motion for A_x in the z coordinate is given by

$$A_x'' + \left(\frac{g'}{g} + \frac{1}{z} - \frac{\chi'}{2}\right) A_x' + \frac{r_h^2}{z^2 g}\left(\frac{e^{\chi}\omega^2}{z^2 g} - \frac{k^2}{r_h^2} - \frac{q^2\psi^2}{z^2}\right) A_x = 0\,. \tag{4.2.3}$$

[2] For applications of the matching method on the study of magnetic effects in holographic superconductors, see. e.g. [108, 109].

We will work in the low-frequency/small-momentum regime, where $k\,,\omega$ are much smaller than the scale of the condensate, so that quadratic terms in k, ω can be neglected in (4.2.3). To solve this equation, we use the following boundary conditions

$$A_x(1) = A_0\,, \qquad A_x'(1) = -\frac{6q^2 r_h^2 \psi_0^2}{e^{\chi_0}\phi_0^2 + r_h^2\left(M^2\psi_0^2 - 24\right)} A_0\,, \tag{4.2.4}$$

where we use the notation $\psi_0 = \psi(1)$, $\phi_0' = \phi(1)$, and where the second condition is needed for regularity at the horizon. As before, $M^2 = -3$. Since the equation (4.2.3) is linear, with no loss of generality we set $A_0 = 1$.

From equation (4.2.3) we can read the behaviour of A_x at $z \to 1$

$$A_x = A_x^{(0)} + J_x \frac{z^2}{r_h^2} + \ldots\,. \tag{4.2.5}$$

According to the AdS/CFT dictionary, $A_x^{(0)}$ and J_x correspond to a vector potential and the conjugated current on the dual field theory, respectively. We can identify these asymptotic values with the London current on the dual superconducting field theory (see eq. (1.1.3))

$$J_x = -\frac{q^2}{m} n_s A_x^{(0)}\,, \tag{4.2.6}$$

were n_s is the number density of superconducting carriers and q and m are the charge and mass of the superconducting carriers, respectively. At this point, it is worth mentioning that, as stated in [7], the London equation is valid only when k and ω are small compared to the scale of the condensate, in consistency with our low-frequency/small-momentum regime. From (4.2.6) we can read the value of the quantity $q^2 n_s/m$ holographically as

$$\frac{q^2}{m} n_s = -\frac{J_x}{A_x^{(0)}}\,. \tag{4.2.7}$$

For simplicity, we define the quantity

$$\tilde{n}_s \equiv \frac{q^2}{m} n_s\,, \tag{4.2.8}$$

which is a rescaling of the carrier number density. Numerically one finds that $\tilde{n}_s$ behaves near T_c as $\tilde{n}_s \sim (1 - T/T_c)$. The value of $\tilde{n}_s$ as a function of temperature for charge $q = 4$ is shown in figure (4.4).

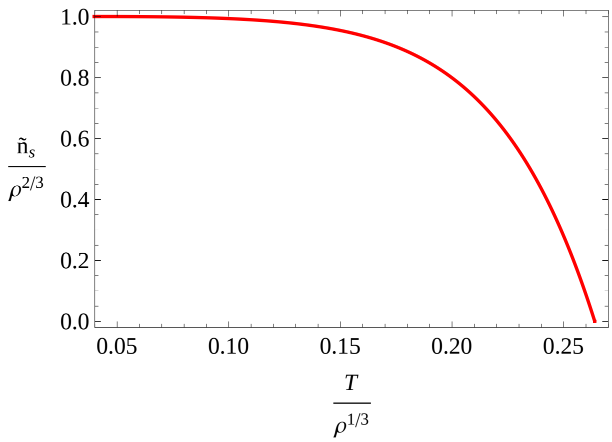


Figure 4.4: Value of $\tilde{n} = \frac{q^2}{m} n_s$ as a function of temperature, for the $q = 4$ case.

4.2.2 Ginzburg-Landau Interpretation of the Dual Field Theory

In this section we will implement a phenomenological Ginzburg-Landau description of our superconducting system by assuming that the dual $d = 3 + 1$ field theory at non-zero temperature can be described phenomenologically by an effective Ginzburg-Landau field theory. This will be given by a vector field $\mathcal{A}_\mu$, $\mu = 0, \dots, 3$, and a scalar field $\Psi_{\rm GL}$ which acts as an order parameter for the theory and effectively represents the operator that condenses in the underlying dual field theory, which in principle could have very different degrees of freedom. This Ginzburg-Landau description is only valid near the critical temperature, where the order parameter $\Psi_{\rm GL}$ is small, and where the effective action for the dual field theory can be written as

$$S_{\rm eff} \approx \frac{1}{T} \int d^3x \left\{ \alpha \, |\Psi_{\rm GL}|^2 + \frac{\beta}{2} \, |\Psi_{\rm GL}|^4 + \frac{1}{2m} \, |D_i \Psi_{\rm GL}|^2 + \dots \right\} , \qquad (4.2.9)$$

where $D_i = \partial_i - iq\,\mathcal{A}_i$, and α and β are phenomenological parameters with a temperature dependence[3]. According to the AdS/CFT dictionary, the vector

[3]For a discussion about effective field approximations in the dual field theory, see [106]. For other works on aspects of Ginzburg-Landau theory in the context of holography, see, e.g. [15, 110, 111, 22].

components $\mathcal{A}_0$ and $\mathcal{A}_x$ correspond respectively to the chemical potential μ in (4.1.16) and to $A_x^{(0)}$ in (4.2.5). We have consistently identified the charge of the superconducting carrier of the phenomenological Ginzburg-Landau Lagrangian with the charge of the bulk scalar field q. We will be mainly interested in electromagnetic phenomena present in superconductivity, which require a dynamical gauge field in the boundary theory. However, we know that the $U(1)$ local symmetry in the bulk translates to a global $U(1)$ symmetry in the boundary according to the gauge/gravity dictionary. In order to overcome this, we will assume that the $U(1)$ global symmetry in the boundary can be promoted to local, by adding a F^2 term using the procedure described in [7]. Indeed, this is the underlying procedure behind most studies of magnetic phenomena in holographic superconductivity. In terms of our current effective field theory description of the boundary theory, this will mean that the Ginzburg-Landau theory approach to electromagnetic phenomena can be applied in our case, especially concerning its determination of the critical magnetic field and of the Ginzburg-Landau parameter κ, which requires a balance between the superconducting and the purely magnetic parts of the free energy of the system (see Chapter 1).

The VEV of the scalar operator that condenses in the underlying dual field theory will be proportional to $\mathcal{O}_3$ to the required power to match dimensions. The Ginzburg-Landau order parameter $\Psi_{\rm GL}$ has mean field critical exponent 1/2. Then, in order to match this critical exponent with the critical exponent of $\mathcal{O}_3$ we must identify

$$|\Psi_{\rm GL}|^2 = N_q \mathcal{O}_3^2\,, \tag{4.2.10}$$

where N_q is a proportionality constant that depends on the value of the charge q of the scalar bulk field.

Regarding the parameters α, β shown in (4.2.9), one sets $\beta > 0$ in order for the lowest free energy to be at finite $|\Psi_{\rm GL}|^2$. Also, in order to have a superconducting phase, one requires that $\alpha < 0$. All definitions and conventions that will be used regarding the Ginzburg-Landau theory can be found in the Chapter 1,

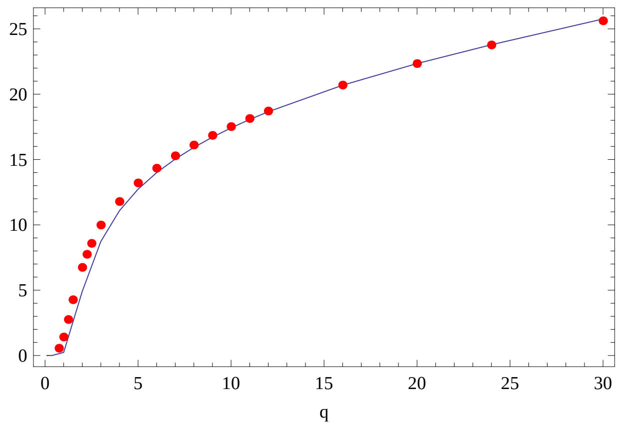


Figure 4.5: Comparison between $q\,\mathcal{O}_3^2/\tilde{n}_s$, corresponding to red points, and $C_0T_c(q)$, corresponding to continuous line.

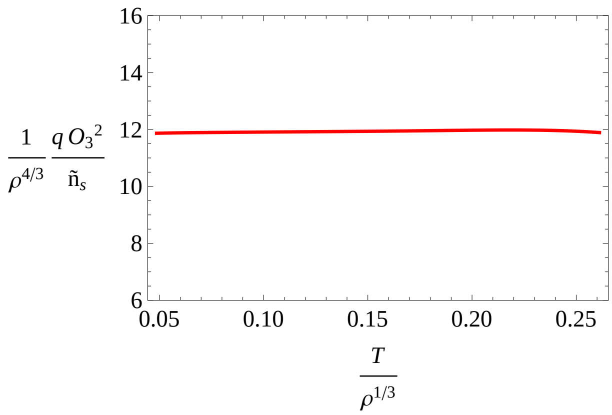


Figure 4.6: Value of the ratio $q\frac{\mathcal{O}_3^2}{\tilde{n}_s}$ as a function of temperature, for the case $q = 4$.

where we have set the physical constants $\hbar = 1$ and $\mu_0 = 4\pi$ (their values in natural units), while preserving numerical factors. The superconducting carrier mass m can be absorbed into a redefinition of the other parameters, so, with no loss of generality, we will set $m = 1$.

At this point, we have two phenomenological Ginzburg-Landau parameters

α and β, and introduced the proportionality constant N_q. We should be able to fully determine them in order for our Ginzburg-Landau description to be as complete and consistent as possible. In order to do it, we will make use of the numeric identity

$$q\frac{\mathcal{O}_3^2}{\tilde{n}_s} = C_0 T_c(q)\,, \tag{4.2.11}$$

where the ratio at the left hand side is evaluated at the critical temperature, and C_0 is a proportionality constant, approximately equal to $C_0 \approx 41.99$. In figure (4.5) we show how this equality holds for various values of q. To have a better understanding of this equality, one can see through the matching method in the large q limit that $\mathcal{O}_3 \sim \frac{T_c^3}{q}(1-T/T_c)^{1/2}$ and $\tilde{n}_s \sim T_c^2(1-T/T_c)$, so the left hand side of the equality goes as $q\,\mathcal{O}_3^2/\tilde{n}_s \sim T_c^4/q$, and because $q \sim T_c^3$ (see (4.1.22)), we indeed have $q\,\mathcal{O}_3^2/\tilde{n}_s \sim T_c$. Another point worth mentioning is that the left hand side of equation (4.2.11) is constant as a function of temperature, for most values of q. This is shown in figure (4.6), where we plot $q\,\mathcal{O}_3^2/\tilde{n}_s$ versus temperature, for the $q = 4$ case.

Rewriting (4.2.11) in terms of n_s instead of $\tilde{n}_s$, we have

$$\frac{\mathcal{O}_3^2}{q\,n_s} = C_0 T_c\,. \tag{4.2.12}$$

According to the Ginzburg-Landau theory, the relation between the order parameter $|\Psi_{\rm GL}|$ and the charge carrier density n_s is given by (see (1.1.5))

$$|\Psi_{\rm GL}|^2 = n_s\,. \tag{4.2.13}$$

Substituting our identification (4.2.10) in (4.2.13), and matching with (4.2.12) we obtain

$$N_q = \frac{1}{q\,C_0 T_c(q)}\,. \tag{4.2.14}$$

The behaviour of N_q as a function of q is shown in figure (4.7). For large q we find $N_q \sim q^{-4/3}$.

In order to determine the remaining parameters, we must calculate first the Ginzburg-Landau coherence length ξ. To do this, we consider small fluctuations

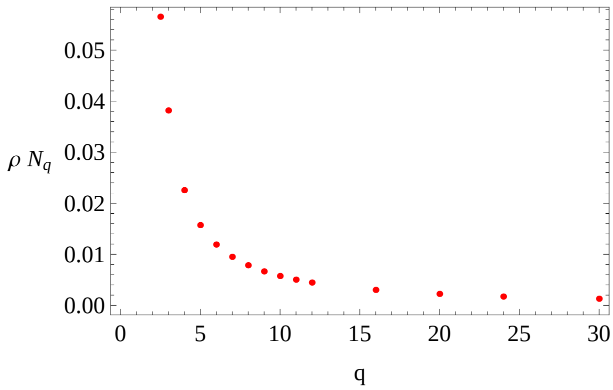


Figure 4.7: Value of the proportionality factor N_q as a function of the scalar field charge q.

around the condensed phase of our system in the bulk. More concretely, we write the original complex scalar field Ψ in our model (4.1.1) as

$$\Psi(r,y) = \frac{1}{\sqrt{2}}\left(\psi(r) + e^{i\,k\,y}\eta(r)\right)\,, \tag{4.2.15}$$

where ψ is the full back-reacted solution associated with the order parameter $\mathcal{O}_3$ described in section 2, and the term $e^{i\,k\,y}\eta(r)$ is a small fluctuation ($|\eta| \ll 1$) around this condensed solution. The equation of motion for η to linearized level is

$$\eta'' + \left(\frac{g'}{g} - \frac{\chi'}{2} - \frac{1}{z}\right)\eta' + \frac{1}{z^2 g}\left(\frac{e^{\chi} q^2 r_h^2 \Phi^2}{z^2 g} - \frac{M^2 r_h^2}{z^2} - k^2\right)\eta = 0\,, \tag{4.2.16}$$

which can be put as in the form of an eigenvalue equation

$$\mathcal{L}\left\{\eta\right\} = k^2 \eta\,, \tag{4.2.17}$$

with $\mathcal{L}$ the same linear operator that acts on ψ. The boundary conditions at the horizon $z = 1$ are:

$$\eta(1) = \eta_0\,, \qquad \eta'(1) = -\frac{6\left(k^2 + M^2 r_h^2\right)}{e^{\chi_0}\Phi_0^2 + r_h^2\left(M^2\psi_0^2 - 24\right)}\eta_0\,, \tag{4.2.18}$$

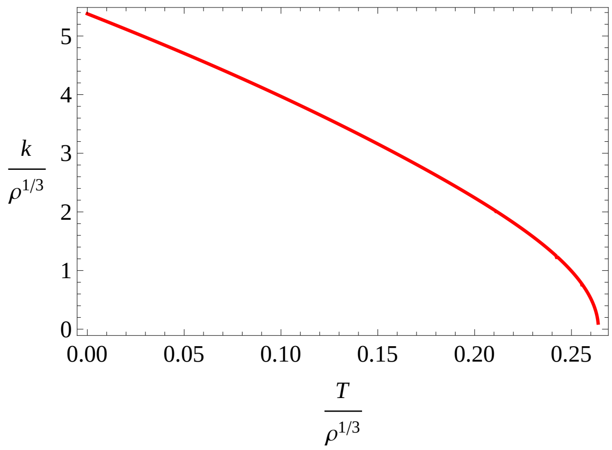


Figure 4.8: Value of the wave number k as a function of temperature, for the case $q = 4$.

while near $z = 0$ we will have the asymptotic behaviour

$$\eta(z) \approx (\delta\mathcal{O}_1)\,\frac{z}{r_h} + (\delta\mathcal{O}_3)\,\frac{z^3}{r_{h^3}} + \cdots\,, \tag{4.2.19}$$

and will demand the same conditions as for ψ, namely $(\delta\mathcal{O}_1) = 0$. Since, as will be seen below, we will not be concerned with the absolute normalization of η, we will take advantage of the linearity of (4.2.16) and set $\eta_0 = 1$.

As noted in [110], the coherence length of the superconducting system is equal to the correlation length ξ_0 of the order parameter. In turn, the correlation length is the inverse of the pole of the correlation function of the order parameter written in Fourier space

$$\langle\mathcal{O}(k)\mathcal{O}(-k)\rangle \sim \frac{1}{|k|^2 + 1/\xi_0^2}\,. \tag{4.2.20}$$

This pole will be given by the eigenvalue of (4.2.17). Therefore, we must solve equation (4.2.16) and calculate the value of the wave number k consistent with the desired boundary conditions for η. This was done near the critical temperature. The behaviour of the wave number k as a function of temperature is shown in figure (4.8), for $q = 4$. From the wave number k we obtain the

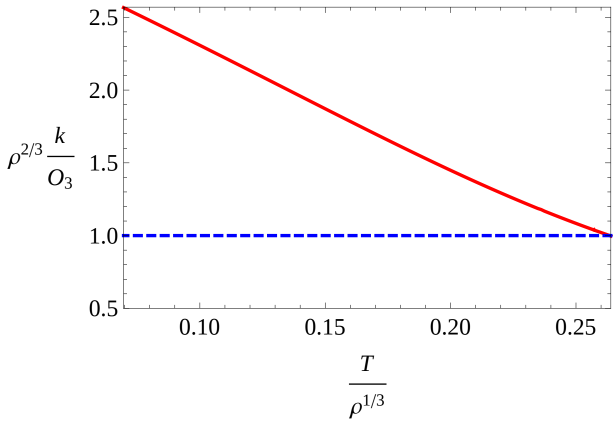


Figure 4.9: Value of the ratio $k/\mathcal{O}_3$ as a function of temperature, for the case $q = 4$. The dashed line corresponds to the respective value of A_q, which in this case is close to one.

coherence length ξ_0 simply as

$$|\xi_0| = \frac{1}{|k|}\,. \tag{4.2.21}$$

whose behaviour as a function of temperature is shown in figure (4.12a), also for the value $q = 4$.

It should be pointed out that the wave number k near the critical temperature becomes equal to the order parameter $\mathcal{O}_3$ times a proportionality constant A_q, which depend on the value of the charge q considered. The value of A_q is given by the ratio between k and $\mathcal{O}_3$ evaluated at T_c

$$A_q = \left.\frac{k}{\mathcal{O}_3}\right|_{T=T_c}, \tag{4.2.22}$$

which, for every case considered, was a finite number. The value of the ratio $k/\mathcal{O}_3$ as a function of temperature can be seen in figure (4.9), for $q = 4$. The value of A_q as a function of the charge q is shown in figure (4.10), and is found numerically to behave as $q^{1/3}$ for large values of q. From (4.2.21) and (4.2.22), one has near the critical temperature

$$\frac{1}{\xi_0} \approx A_q\mathcal{O}_3\,, \qquad (T \approx T_c)\,. \tag{4.2.23}$$

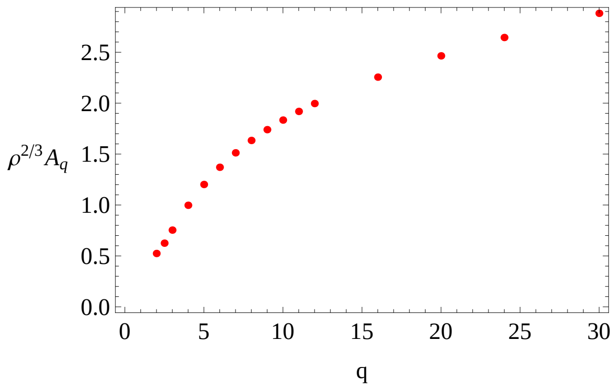


Figure 4.10: Value of the proportionality constant A_q as a function of the scalar field charge q.

With the calculation of the correlation length of the order parameter, and its identification as the superconductor coherence length, we now resort to the Ginzburg-Landau theory relation (1.1.25), which gives us the parameter $|\alpha|$ as

$$|\alpha| = \frac{1}{4\,\xi_0^2}\,. \tag{4.2.24}$$

Since, as we mentioned above, near the critical temperature $\xi_0 \approx A_q/\mathcal{O}_3$, then

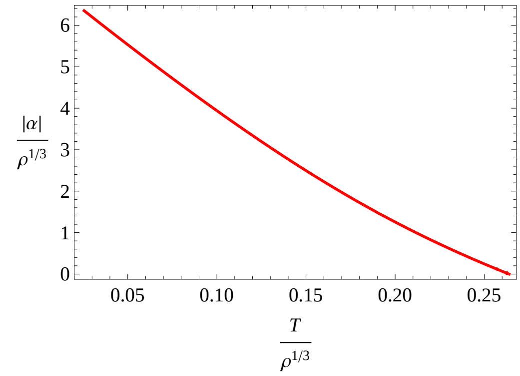
$$|\alpha| \approx \frac{A_q^2}{4}\,\mathcal{O}_3^2 \sim (1 - T/T_c)\;, \qquad (T \approx T_c) \tag{4.2.25}$$

which is the correct near-critical temperature behaviour for $|\alpha|$ according to Ginzburg-Landau theory. In figure (4.11a), we show the behaviour of α as a function of temperature, for the case $q = 4$.

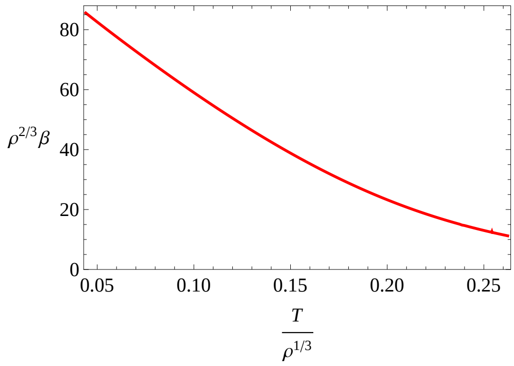
In order to calculate the remaining Ginzburg-Landau parameter β, we will assume that the superconducting order parameter $|\Psi_{\rm GL}|$ does not differ significantly from (see (1.1.8))

$$|\Psi_\infty|^2 = \frac{|\alpha|}{\beta}\,, \tag{4.2.26}$$

which is the value of the order parameter that minimizes the Ginzburg-Landau free energy and physically is the value of $|\Psi_{\rm GL}|$ deep inside the volume of the



(a) α



(b) β

Figure 4.11: Value of the Ginzburg-Landau parameters α and β as a function of temperature, for the case $q = 4$.

superconductor. As stated in the Chapter 1, this can only be so in the case where the external fields and gradients are negligible. This is indeed the case for our gauge perturbation (4.2.1). Substituting our identification (4.2.10) in (4.2.26) we get

$$N_q \mathcal{O}_3^2 = \frac{|\alpha|}{\beta}\,, \tag{4.2.27}$$

from where we obtain, making use of (4.2.14) and (4.2.24)

$$\beta = \frac{q\, C_0 T_c(q)}{4} \frac{1}{\xi_0^2 \mathcal{O}_3^2}\,. \tag{4.2.28}$$

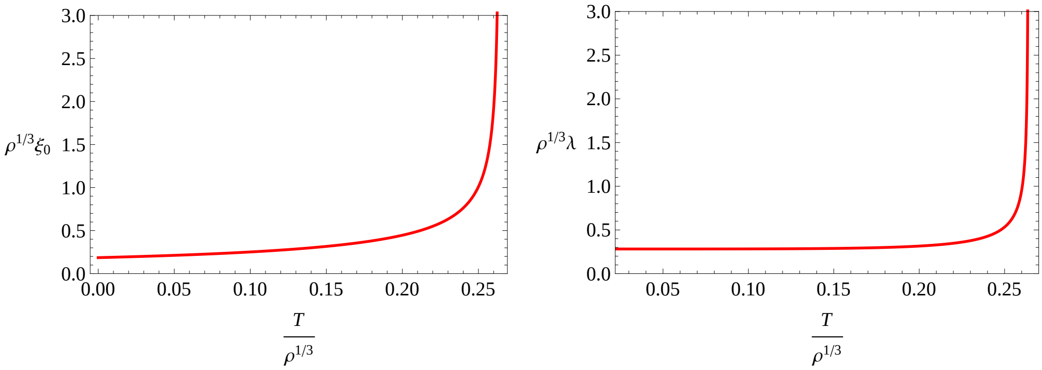
In figure (4.11b) we show the behaviour of β as a function of temperature, for the $q = 4$ case.

Having determined the correlation length ξ_0, we can also calculate the remaining characteristic length of the superconductor, namely the Ginzburg-Landau penetration length λ. This can be done directly from its definition as (1.1.17)

$$\lambda^2 = \frac{1}{4\pi\, q^2 n_s}\,, \tag{4.2.29}$$

or, in terms of $\tilde{n}_s$

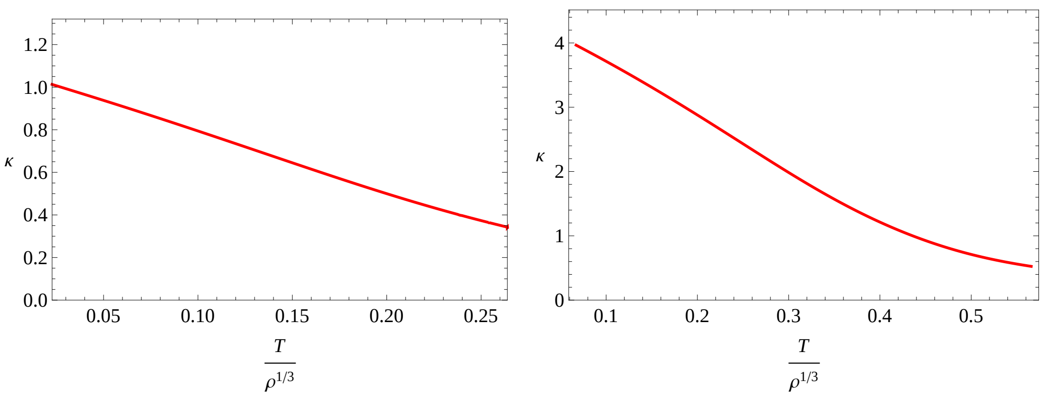
$$\lambda^2 = \frac{1}{4\pi \tilde{n}_s}\,, \tag{4.2.30}$$



(a) Coherence length ξ_0, $q = 4$

(b) Penetration length λ, $q = 4$

Figure 4.12: Value of the characteristic lengths ξ_0 and λ as a function of temperature, for the case $q = 4$.



(a) $q = 4$

(b) $q = 24$

Figure 4.13: Value of the Ginzburg-Landau parameter κ as a function of temperature, for the cases $q = 4$, and $q = 24$.

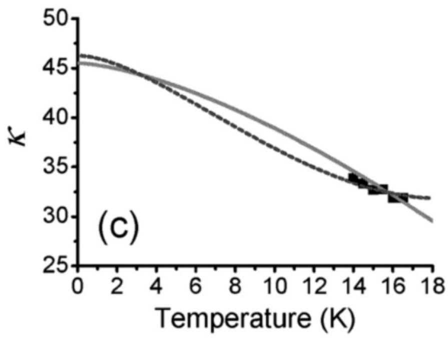


Figure 4.14: Temperature dependence of the Ginzburg-Landau parameter κ. The dashed line corresponds to the empirical curve of the form (4.2.32) for the high-T_c material Nb_3Sn. Figure taken from [114].

where, as we have seen, $\tilde{n}_s$ is given holographically by (4.2.8). In figure (4.12b) we show its behaviour as a function of temperature, for the $q = 4$ case. With both characteristic lengths, we can consequently obtain numerical values for the Ginzburg-Landau parameter, defined as $\kappa = \lambda/\xi$ (see (1.1.27)).[4] We note that the definition of κ uses the Ginzburg-Landau coherence length ξ, which is related to the superconducting coherence length calculated above by $\xi^2 = 2\xi_0^2$. We obtain

$$\kappa = \sqrt{\frac{1}{8\pi\, \tilde{n}_s\, \xi_0^2}}\,. \tag{4.2.31}$$

The behaviour of κ as a function of temperature is shown in figures (4.13a) and (4.13b) for the cases $q = 4$ and $q = 24$, respectively. A striking feature concerning the large-q Ginzburg-Landau parameter, like the $q = 24$ case presented in (4.13b), is that its qualitative behaviour can be modeled using the same kind of empirical fitting already used for high-T_c superconducting material Nb_3Sn in [113], where the authors determined the temperature dependence for κ to be

[4]For previous research on the κ parameter in a holographic context, see [11, 110]. See also [112].

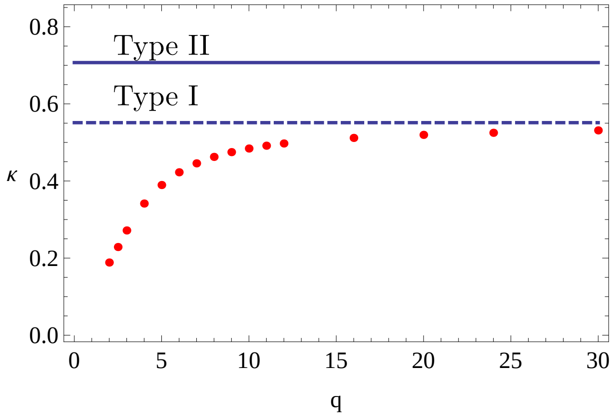


Figure 4.15: Evolution of the Ginzburg-Landau parameter κ as a function of the scalar field charge q.

given by

$$\kappa(T) = \kappa(0)\left(a_0 - b_0(T/T_c)^2\,(1 - c_0\,\log(T/T_c))\right)\,, \tag{4.2.32}$$

with a_0, b_0 and c_0 given empirically. This is shown in figure (4.14). This curve has the same shape of figure (4.13b). Indeed, the same formula can be used to fit our results to very good approximation, giving rise to the essentially same plot shown in figure (4.13b). The same can be done with the other large-q cases.

In figure (4.15) we show the evolution of κ as the value of q increases. The plot was made by taking the value of κ closest to the critical temperature for each charge. We also show the line $\kappa = 1/\sqrt{2}$ (bold line) corresponding to the value where, according to Ginzburg-Landau theory, the system turns from a Type I to a Type II superconductor. Since numerical factors have been maintained in our Ginzburg-Landau interpretation, this exact value still holds. What can be seen is that the system behaves as a Type I superconductor, with the value of κ increasing monotonically and approaching the asymptotic value $\kappa \approx 0.55$, shown as a dashed line in figure (4.15), which is below $\kappa = 1/\sqrt{2}$. [5]

[5]We note that the asymptotic constant behaviour of κ as the value of q grows can be

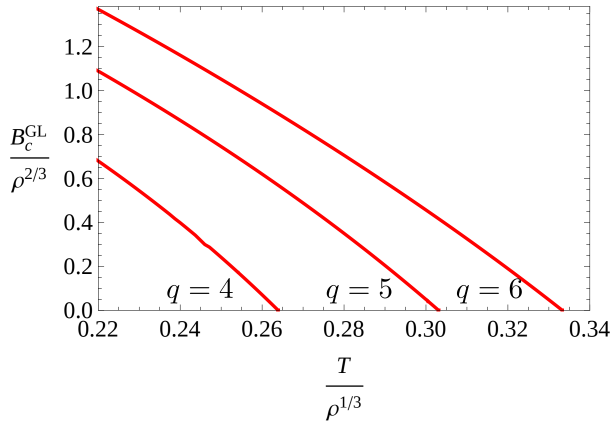


Figure 4.16: Value of the Ginzburg-Landau critical magnetic field B_c^{GL} as a function of temperature, for $q = 4\ ,5\ ,6$.

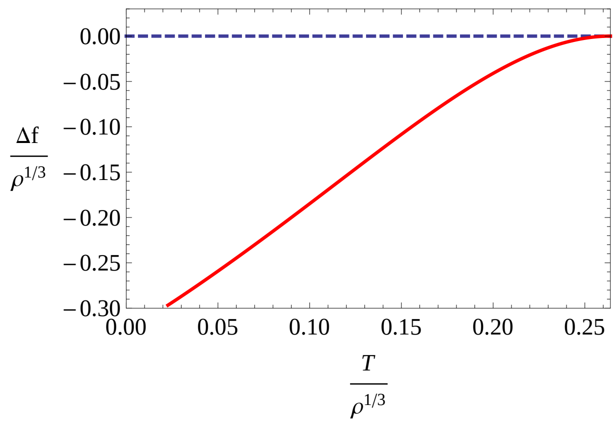


Figure 4.17: Value of the Helmholtz free energy difference computed through standard holographic techniques Δf as a function of temperature, for $q = 4$.

seen directly from (4.2.31), where, making use of the fact that at the critical temperature $\xi_0 = 1/A_q\mathcal{O}_3$, we can write κ as

$$\kappa = \sqrt{\frac{A_q^2 \mathcal{O}_3^2}{8\pi\,\tilde{n}_s}}\,, \tag{4.2.33}$$

and, using (4.2.11)

$$\kappa = \sqrt{\frac{C_0 A_q^2\, T_c(q)}{8\pi\, q}}\,. \tag{4.2.34}$$

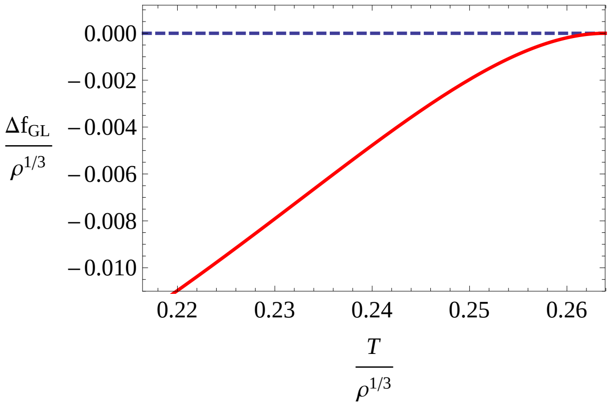


Figure 4.18: Value of the Helmholtz free energy difference computed by the Ginzburg-Landau approach $\Delta f_{\rm GL}$ as a function of temperature, for $q = 4$.

An interesting fact about the phenomenological Ginzburg-Landau description is that, according to it, we can calculate the value of the critical magnetic field that breaks the superconducting phase of the theory. According to the Ginzburg-Landau theory, this critical field, which we will refer to as $B_c^{\rm GL}$, is given by (1.1.12)

$$B_c^{\rm GL} = \sqrt{4\pi}\frac{|\alpha|}{\sqrt{\beta}}\,, \tag{4.2.35}$$

where we used the fact that for holographic superconductors $H = B/\mu_0$. It is important to notice that this critical field arises in Ginzburg-Landau theory from balancing the condensate part of the free energy against its purely magnetic part (see Chapter 1). This field points in the x_3-direction, and should be related to the real part of

$$F_{x_1,x_2} = i\,k\,A_x^{(0)}\,. \tag{4.2.36}$$

After substitution of (4.2.24) and (4.2.28) in (4.2.35) we have

$$B_c^{\rm GL} = \sqrt{\frac{\pi}{q\,C_0 T_c}\frac{\mathcal{O}_3}{\xi_0}}\,. \tag{4.2.37}$$

Since for large q we know that both A_q and T_c behave as $q^{1/3}$, then in that limit we will have $\kappa \sim$ const.

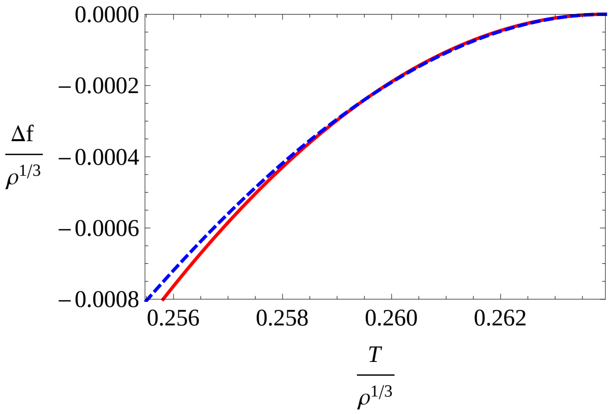


Figure 4.19: Comparison between free energy densities as a function of temperature, for $q = 4$. The bold line corresponds to Δf, while the dashed line corresponds to $\Delta f_{\rm GL}$.

In figure (4.16) we show how this critical field behaves as a function of temperature for the cases $q = 4, 5, 6$. Near T_c, using (4.2.23), the last expression becomes

$$B_c^{\rm GL} \approx \sqrt{\frac{\pi}{q\, C_0 T_c}} A_q \mathcal{O}_3^2 \,, \tag{4.2.38}$$

where we see that $B_c^{\rm GL}$ has a near-T_c behaviour $B_c^{\rm GL} \sim (1 - T/T_c)$, consistent with mean field theory.

Finally, we want to see how our current Ginzburg-Landau approach holds up with regard to the Helmholtz free energy density of the system.[6] The Helmholtz free energy density f is given in general by

$$f = \epsilon - T s \,, \tag{4.2.39}$$

where ϵ and s are the total energy and entropy density, respectively. In order to calculate the Helmholtz free energy, we follow [7] and make use of the fact that the stress-energy tensor must be traceless. For our particular case, this implies

[6] The Helmholtz free energy density is the appropriate thermodynamic potential in our case, given our choice to work with fixed charge density ρ, i.e. in the canonical ensemble.

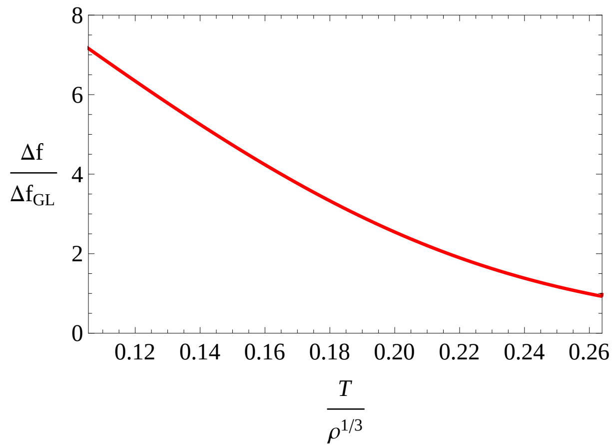


Figure 4.20: Value of the ratio $\Delta f/\Delta f_{\rm GL}$ as a function of temperature, for $q = 4$.

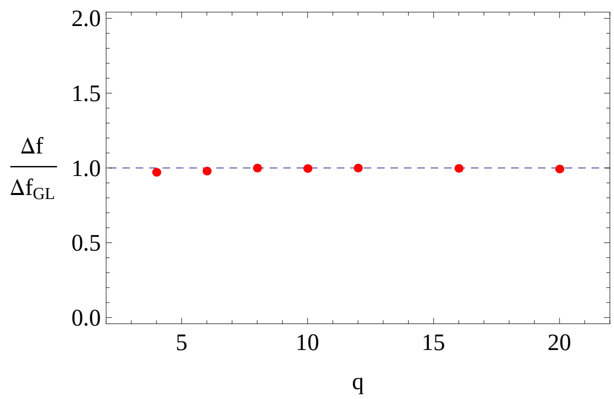


Figure 4.21: Value of the ratio $\Delta f/\Delta f_{\rm GL}$ evaluated at $T = T_c$, for different values of q.

that $\epsilon = 3P$, where P is the pressure. Substituting in the thermodynamic identity $\epsilon = sT + \mu\rho - P$, and in the formal definition (4.2.39) we obtain the expression

$$f = \frac{1}{4}\left(3\mu\rho - sT\right) , \tag{4.2.40}$$

which is used to compute f in both the condensed and normal phases, as a function of T and for different values of q. We focus on the free energy difference $\Delta f = f_{\rm sc} - f_{\rm n}$, where $f_{\rm sc}$ corresponds to the free energy in the superconducting phase, while $f_{\rm n}$ corresponds to the free energy in the normal phase. The free energy difference Δf of the system is shown in figure (4.17) as a function of temperature, for the particular case $q = 4$.

Meanwhile, according to Ginzburg-Landau theory, the free energy difference is given by equation (1.1.6) in the Chapter 1. Since we are working in the approximation where the order parameter $|\Psi| \approx |\Psi_\infty|$, near T_c we can safely focus on the first two terms

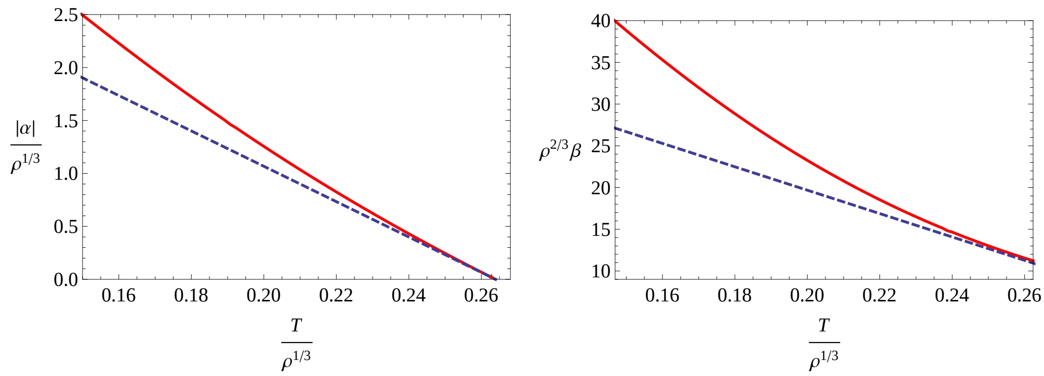
$$\Delta f_{\rm GL} \approx \alpha\,|\Psi|^2 + \frac{1}{2}\beta\,|\Psi|^4 \,, \qquad (T \approx T_c)\,. \tag{4.2.41}$$

Substituting in (4.2.41) the values obtained holographically earlier in this section for $|\Psi|$, α and β, we have

$$\Delta f_{\rm GL} = -\frac{1}{8\,q\,C_0 T_c}\frac{\mathcal{O}_3^2}{\xi_0^2}\,. \tag{4.2.42}$$

In figure (4.18) we show the behaviour of $\Delta f_{\rm GL}$ as a function of temperature, for the $q = 4$ case. We then compare both free energy differences Δf and $\Delta f_{\rm GL}$. Figure (4.19) compares the free energies computed by the two different methods. We see that there is an excellent agreement, showing that both descriptions should be more accurate near the critical temperature. In figure (4.20) we show the ratio $\Delta f/\Delta f_{\rm GL}$ as a function of temperature, for the $q = 4$. We find that the ratio reaches the constant value ~ 0.99 at $T = T_c$. Moreover, this value of the ratio at $T = T_c$ is found to be the same for all values of q considered. This is shown in figure (4.21), where the value of the ratio $\Delta f/\Delta f_{\rm GL}$ evaluated at T_c is shown for different values of q.

It is interesting to compare the present results with the results of [15] for a $d = 3 + 1$-bulk system. The authors in this paper, using a rather different method, performed a fit of the free energy using the Ginzburg-Landau form (4.2.41) with the corresponding order parameter $\mathcal{O}_i$. By doing this, they obtain near-T_c expressions for α and β as functions of temperature which agree with



(a) α (b) β

Figure 4.22: Near-T_c comparison of the Ginzburg-Landau parameters α and β obtained through the method developed in [15] (dashed line) and through our current Ginzburg-Landau approach (bold line), for the case $q = 4$. The parameters computed through [15] are presented only to linear level in T.

the results of standard Ginzburg-Landau theory. Applying the same procedure to fit the free energy in our $d = 4+1$ bulk dimensional system[7], we find that, in the particular $q = 4$ case, the Ginzburg-Landau parameters α and β behave near-T_c and up to lineal level in T as

$$|\alpha| = 4.41\,(1 - T/T_c)\ , \qquad \beta = 10.95 + 36.75\,(1 - T/T_c)\ , \tag{4.2.43}$$

Meanwhile, in our current Ginzburg-Landau approach, the parameters α and β, which are computed through equations (4.2.24) and (4.2.28) respectively, can be expressed near-T_c and up to linear level in T as

$$|\alpha_{\rm GL}| = 4.45\,(1 - T/T_c)\ , \qquad \beta_{\rm GL} = 11.23 + 35.2\,(1 - T/T_c)\ . \tag{4.2.44}$$

Comparing (4.2.43) and (4.2.44) we see that near T_c both results are quantitatively very similar. In figures (4.22a) and (4.22b) we show how the expressions

[7] I order to apply the methods developed in [15], we note that our system is a $d = 4+1$ dimensional version of the model they work with, with no spatial component of the gauge field (superfluid velocity $\xi = 0$, in the authors notation), and that we are working in the canonical ensemble while in [15] the authors consider the grand canonical ensemble.

(4.2.43) for α and β obtained through the methods used in [15] compare near-T_c with the parameters computed by our Ginzburg-Landau approach. Observing this good agreement between both results, we conclude that the methods developed in [15] and in this chapter can be viewed as complementary. We notice that, in the Ginzburg-Landau approach, the whole functional dependency of α, β and the free energy on T is contained entirely on simple combinations of $\mathcal{O}_3^2$ and ξ_0^2, which arise naturally when looking for consistency.

4.3 Constant External Magnetic Field

4.3.1 A Constant Magnetic Field Background

We will now introduce a uniform external magnetic field into our model. To do this, we use the procedure described in [16] to build perturbatively an asymptotically-AdS fixed magnetic background. The starting point is a $d = 4 + 1$ Einstein-Maxwell action with a negative cosmological constant

$$S = \int d^5x\sqrt{-g}\left(R + \frac{12}{L^2} - \frac{1}{4}F^2\right). \tag{4.3.1}$$

We consider a magnetic ansatz for the gauge field

$$A = \phi(r)dt + \frac{B}{2}\ (-x_2 dx_1 + x_1 dx_2)\ , \tag{4.3.2}$$

which means that we will have a constant external magnetic field pointing in the x_3-direction of the dual field theory, given by $F_{x_1,x_2} = B$. For the metric, we propose the ansatz

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + e^{2V(r)}\left(dx_1^2 + dx_2^2\right) + e^{2W(r)}dx_3^2\,. \tag{4.3.3}$$

Such an ansatz has a $SO(2)$ isometry in the $x_1 - x_2$ plane, and is invariant under translations in the x_3 direction, due to the fact that the magnetic field will define a preferred direction in the $(x_1,\ x_2,\ x_3)$ space. We will look for asymptotically AdS black hole solutions for the metric. The Einstein equations for this system are

$$R_{\mu\nu} + g_{\mu\nu}\left(\frac{1}{12}F^2 + \frac{4}{L^2}\right) + \frac{1}{2}F_{\mu}{}^{\lambda}F_{\nu\lambda} = 0\,. \qquad (4.3.4)$$

Substituting the ansatz (4.3.2) and (4.3.3) into these equations, we get

$$\begin{aligned} 2V'^2 + W'^2 + 2V'' + W'' &= 0\,, & (4.3.5)\\ \frac{B^2}{2}e^{-4V} + \left(g\,(V-W)'\right)' + g\,(2V-W)'\,(V-W)' &= 0\,, & (4.3.6)\\ -\frac{B^2}{3}e^{-4V} - \frac{2}{3}\phi'^2 - \frac{8}{L^2} + g'\,(2V+W)' + g'' &= 0\,, & (4.3.7) \end{aligned}$$

while the gauge field equation is given by

$$(2V+W)'\,\phi' + \phi'' = 0\,. \qquad (4.3.8)$$

One then considers the following expansion in powers of B around $B = 0$, up to second order:

$$\begin{aligned} g(r) &= g_0(r) + B^2 g_2(r) + \ldots & (4.3.9)\\ V(r) &= V_0(r) + B^2 V_2(r) + \ldots & (4.3.10)\\ W(r) &= W_0(r) + B^2 W_2(r) + \ldots & (4.3.11)\\ \phi(r) &= \phi_0(r) + B^2 \phi_2(r) + \ldots\,. & (4.3.12) \end{aligned}$$

As described in [16], this expansion is reliable for $B \ll T^2$. The B^0-order equations are solved by the usual AdS Reissner-Nordström solution:

$$\begin{aligned} \phi_0(r) &= \frac{1}{2} - \frac{\rho}{r^2}\,, & (4.3.13)\\ g_0(r) &= \frac{r^2}{L^2} + \frac{\rho^2}{3r^4} - \frac{3r_h^6 + L^2\rho^2}{3L^2 r_h^2 r^2}\,, & (4.3.14)\\ V_0(r) &= W_0(r) = \log r\,. & (4.3.15) \end{aligned}$$

From now on, we will set $L = 1$, following our previous convention. The B^2-order equations are:

$$\begin{aligned} \left(r^2\,(2V_2 + W_2)'\right)' &= 0\,, & (4.3.16)\\ \frac{1}{2r} + \left(r^3 g_0\,(V_2 - W_2)'\right)' &= 0\,, & (4.3.17)\\ -\frac{1}{3r} + \left(r^3 g_2'\right)' + r^3 g_0'\,(2V_2 + W_2)' &= 0\,, & (4.3.18)\\ 2\rho\,(2V_2 + W_2)' + \left(r^3\phi_2'\right)' &= 0\,. & (4.3.19) \end{aligned}$$

From (4.3.16), demanding that V_2 and W_2 vanish at infinity and be regular at the horizon, we obtain

$$2V_2 + W_2 = 0. \tag{4.3.20}$$

Substituting this result in (4.3.19), and demanding that ϕ_2 vanishes at both the horizon and infinity, we have $\phi_2 = 0$. Also, from (4.3.18) and demanding that g_2 vanishes also at the horizon and infinity, the solution for g_2 is

$$g_2(r) = -\frac{1}{6r^2}\log\left(\frac{r}{r_h}\right). \tag{4.3.21}$$

Finally, from equation (4.3.17) we get

$$V_2(r) = -\frac{1}{6}\int_\infty^r dr' \frac{\log(r'/r_h)}{r'^3 g_0(r')}. \tag{4.3.22}$$

and W_2 given by (4.3.20). From the solution up to second order in B for $g(r)$

$$g(r) = r^2 + \frac{\rho^2}{3\,r^4} - \frac{3\,r_h^6 + \rho^2}{3r^2 r_h^2} - \frac{B^2 \log(r/r_h)}{6\,r^2}, \tag{4.3.23}$$

we can obtain the Hawking temperature of the system

$$T_H = \frac{24\,r_h^6 - 4\rho^2 - B^2\,r_h^2}{24\pi\,r_h^5}. \tag{4.3.24}$$

Since we will continue to work in the canonical ensemble, we will set $\rho = 1$ for the remainder of this section.

4.3.2 Droplet solution and critical magnetic field

We will now turn on a small scalar field in the fixed background given by the solutions constructed in the previous subsection. This will be analogous to the analysis made by [7, 101] in a $d = 3 + 1$ AdS. (For other, less conventional models, see e.g. [23].) We propose an ansatz for the scalar field

$$\Psi(r,u) = \frac{1}{\sqrt{2}} R(r)\,U(u), \tag{4.3.25}$$

where we have made the change to cylindrical coordinates $dx_1^2 + dx_2^2 = du^2 + u^2 d\theta^2$. The equation (4.1.2) turns to be separable in this case, resulting in the equations

$$U'' + \frac{1}{u}U' + \left(\lambda - B^2 q^2 u^2\right) U = 0\,, \tag{4.3.26}$$

$$R'' + \left(\frac{g'}{g} + \frac{3}{r}\right) R' + \frac{1}{g}\left(\frac{q^2\phi^2}{g} - e^{-2\,V}\lambda - M^2\right) R = 0\,, \tag{4.3.27}$$

where λ is the separation constant, and must be equal to $\lambda_n = n\,q\,B$ in order for $U(u)$ to be finite as $u \to \infty$. We choose the $n = 1$ mode, since this corresponds to the most stable solution [7, 101]. In this case, the solution for (4.3.26) is a gaussian function

$$U(u) = \exp\left(-\frac{q\,B}{4}\,u^2\right)\,, \tag{4.3.28}$$

which is the same result obtained in [7] for a $d = 3 + 1$-dimensional bulk.

Substituting λ_1 in (4.3.27) and changing to the $z = r_h/r$ coordinate, we get

$$R'' + \left(\frac{g'}{g} - \frac{1}{z}\right) R' + \frac{r_h^2}{g\,z^4}\left(\frac{q^2\phi^2}{g} - 2\,q\,B\,e^{-2\,V} - M^2\right) R = 0\,, \tag{4.3.29}$$

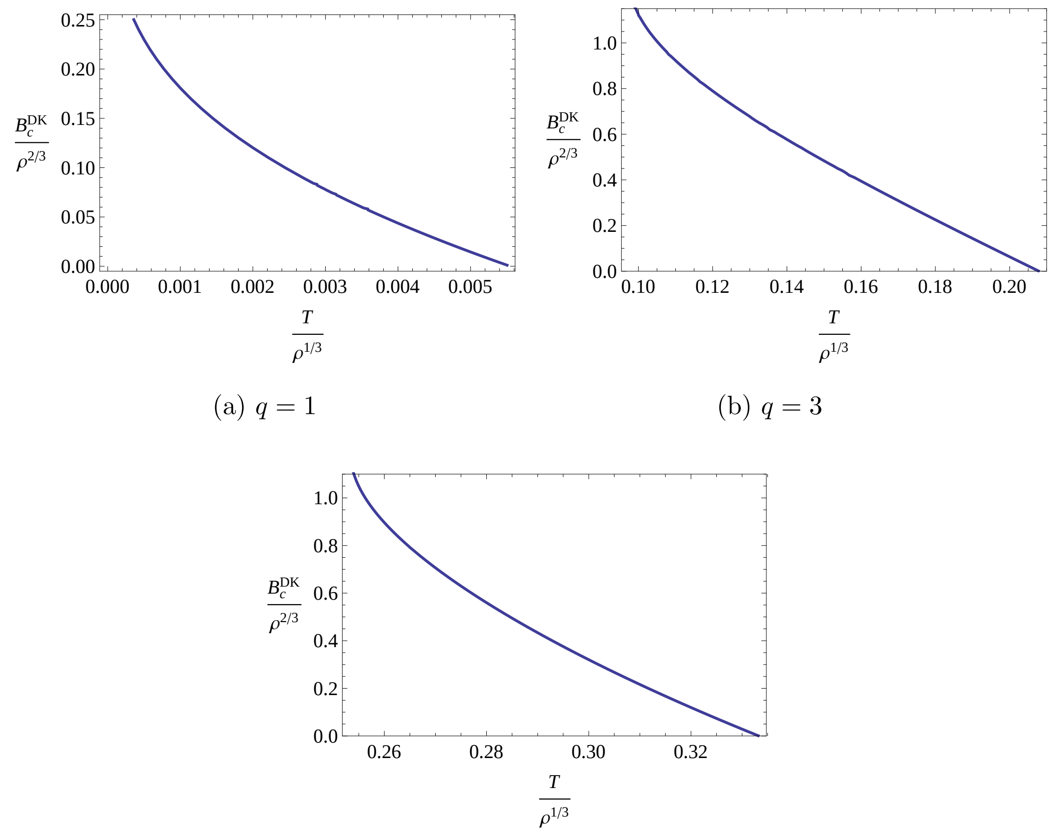
from where we derive the boundary regularity condition

$$R'(1) = \frac{r_h^2}{g'(1)}\left(2\,q\,B\,e^{-2V(1)} + M^2\right) R_0\,, \tag{4.3.30}$$

where $R_0 = R(1)$. Again, we choose $M^2 = -3$, which gives the asymptotic behaviour

$$R = \mathcal{O}_1\frac{z}{r_h} + \mathcal{O}_3\frac{z^3}{r_h^3} + \dots\,. \tag{4.3.31}$$

Since we will not be concerned about the absolute normalization of $\mathcal{O}_3$, we will take advantage of the linearity of (4.3.29) and set $R_0 = 1$. This will leave B and r_h as the only input parameters in the equation. As in the previous section, we will choose to set $\mathcal{O}_1 = 0$ and solve the differential equation (4.3.29) enforcing this choice through the shooting method. This leaves r_h, and therefore T_H in (4.3.24), as the only free parameter of the system and will allow us to determine the value of B as a function of temperature. This magnetic field B_c will correspond to the value above which superconductivity is broken. From the holographic point of view, the critical magnetic field obtained above measures an instability of the



(a) $q = 1$

(b) $q = 3$

(c) $q = 6$

Figure 4.23: Value of the critical magnetic field B_c^{DK} as a function of temperature, for different values of q.

bulk scalar field ψ. Indeed, from the effective mass of the scalar field

$$M_{\text{eff}}^2 = M^2 - \frac{q^2}{g}\Phi^2 + \frac{q^2}{4}e^{-2V}u^2B^2\,, \tag{4.3.32}$$

we see that the magnetic term has an opposite sign to the electric term, which is responsible for lowering the effective mass below the Breitenlohner-Freedman bound and making the field tachyonic. The sign difference means then that the magnetic term lowers the critical temperature under which the scalar field becomes unstable [102]. We will refer to the critical magnetic field obtained in this section as B_c^{DK}, in order to distinguish it from the critical magnetic field as given by Ginzburg-Landau theory, B_c^{GL}, which was introduced in the preceding section.

In figures (4.23a)-(4.23c) we show the value of the critical magnetic field B_c^{DK} for the cases $q = 1\,,3\,,6$. We only show the region near the critical temperature where our approximation is valid. The divergence of B_c^{DK} as the temperature moves away from T_c is typical of the no-backreaction approach we are using, as observed in [102].

Finally, we find numerically that near-T_c the critical magnetic field B_c^{DK} behaves as

$$B_c^{\text{DK}} \sim B_0^{\text{DK}}\,(1 - T/T_c)\,, \tag{4.3.33}$$

in accordance to mean field theory. The behaviour of the factor B_0^{DK} as a function of the scalar field charge q is shown in figure (4.24). For large q, one finds numerically that $B_0^{\text{DK}} \sim q^{-1/3}$.

It is interesting to note that the critical magnetic fields B_c^{GL} and B_c^{DK} measure different aspects of the response of the system to a magnetic field: with B_c^{DK} measuring an instability in the scalar bulk field, and B_c^{GL} arising from a balancing between the condensate part and the purely magnetic part of the free energy according to Ginzburg-Landau theory. We found that near T_c both critical magnetic fields behave as $\sim (1 - T/T_c)$. Explicitly

$$B_c^{\text{DK}} \sim B_0^{\text{DK}}\,(1 - T/T_c)\,, \qquad B_c^{\text{GL}} \sim B_0^{\text{GL}}\,(1 - T/T_c)\,, \tag{4.3.34}$$

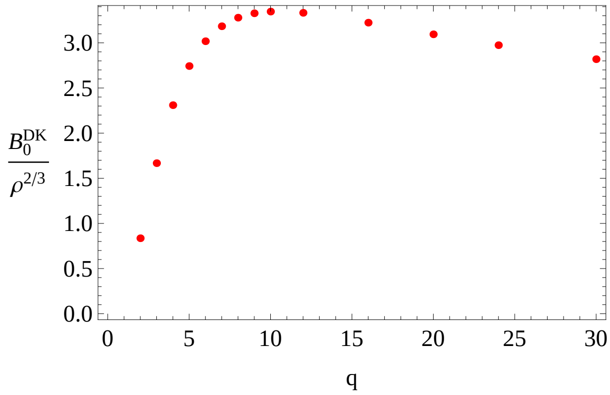


Figure 4.24: Behaviour of the near-T_c coefficient $B_0^{\rm DK}$ as a function of the scalar field charge q.

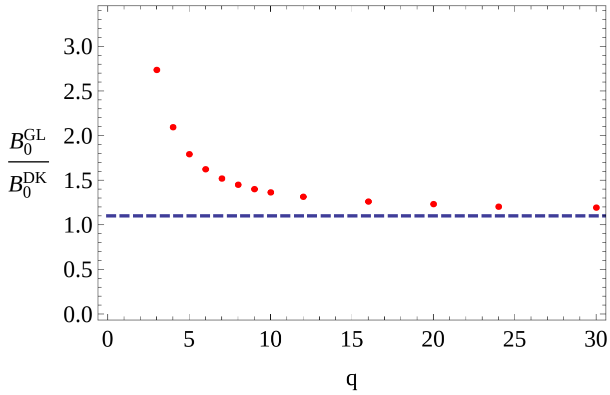


Figure 4.25: Behaviour of the ratio $B_0^{\rm GL}/B_0^{\rm DK}$ as a function of the scalar field charge q. The dashed line corresponds to the asymptotic limit ~ 1.1

with

$$B_0^{\rm GL} \equiv \sqrt{\frac{\pi}{qC_0T_c}} A_q \mathcal{O}_0^2 \,. \tag{4.3.35}$$

(See equation (4.2.38).) Since we know that for large q we have $A_q \sim q^{1/3}$, $T_c \sim q^{1/3}$ and $\mathcal{O}_0 \sim q^0$, then we conclude that, in this limit, $B_0^{\rm GL} \sim 1/q^{1/3}$ (or equivalently, $B_0^{\rm GL} \sim 1/T_c$) and thus, we find that both $B_0^{\rm DK}$ and $B_0^{\rm GL}$ have the same large-q behaviour. Indeed, this can be seen in figure (4.25), where we show the ratio $B_0^{\rm GL}/B_0^{\rm DK}$ as a function of the scalar field charge q, and where we find numerically that it tends asymptotically to the constant value ~ 1.1.

5

Bottom-Up Approach, Part II: Magnetic Phenomena in Holographic Superconductivity with Lifshitz Scaling

Having developed in the previous chapter the main framework for a Ginzburg-Landau approach to holographic superconductivity, in this chapter we will be applying this effective approach to holographic superconducting models with Lifshitz scaling. In very general terms, the usual models of holographic superconductivity are built around a local gauge group symmetry breaking by one of the component fields in the gravity side, where the gravitational solution is an asymptotically AdS charged black hole. This symmetry breaking in the gravity side signals the beginning of a superconducting phase in the dual field theory. (See, e.g. [83, 7].) It has been found, however, that in some condensed matter systems phase transitions are governed by *Lifshitz-like fixed points*. These exhibit the particular anisotropic spacetime scaling symmetry

$$t \to \lambda^z t\,, \qquad x \to \lambda x\,, \tag{5.0.1}$$

where z is the *Lifshitz dynamical critical exponent* governing the degree of anisotropy. This anisotropy breaks Lorentz invariance and the systems are non-relativistic in nature. Therefore, in order to study such field theories holographically, the dual gravitational description has to be modified. Indeed, it was found in [115] that these Lifshitz-like fixed points can be described by the gravitational dual

$$ds^2 = L^2 \left(-r^{2z}dt^2 + r^2 dr^2 + r^2 \sum_{i=1}^{d} d\vec{x}^2 \right)\,, \tag{5.0.2}$$

which, for $z = 1$ reduces back to the usual AdS_{d+2} metric, but for $z \neq 1$ satisfies the anisotropic scaling (5.0.1). A black hole generalization of this metric was found in [116].

The purpose of this chapter then is to explore various aspects of holographic superconductivity with Lifshitz-like fixed points, with a particular focus on magnetic phenomena. We do this by starting from a minimal bulk model and by studying various choices of condensates. More concretely, we want to investigate how the dynamical critical exponent z affects our system with respect to its behavior in the isotropic $z = 1$ case. Most of the existing research on the subject was realized in $D = 4$. See, for instance [117, 118, 119, 120, 121]. In [122], the authors do make an interesting treatment of the $D = 5$ case, but have their interest put mainly on studying different kinds of superconductors (s-wave, p-wave, soliton) and on the computation of condensation and conductivity. Regarding the study of magnetic effects in a Lifshitz background, we note in particular [123, 18]. In the first reference the authors also treat the $D = 5$ case, but using a different condensate as the ones we will propose, and with a focus on the applicability of the matching method.

In this respect, in this chapter we see that it is possible to have a consistent Ginzburg-Landau phenomenological approach to holographic superconductivity

[12] in a Lifshitz background. We then apply this Ginzburg-Landau approach to compute, among other physical quantities, the Ginzburg-Landau parameter of the system, and to see how it is affected by the dynamical critical exponent z. We will also study the effect of an external magnetic field acting directly on the system, using the approach proposed in [17]. In order to have a more complete study of the system's properties, we managed to study a wide array of condensation cases, always within the $D = 5$ framework, so that the general tendencies in the behavior of physical quantities become more clear.

5.1 Minimal Holographic Superconductor in Lifshitz Background

5.1.1 General Setup

As mentioned in the Introduction, the D=d+2 gravitational dual (5.0.2) can be generalized to a black hole solution [116]

$$ds^2 = L^2 \left(-r^{2z} f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 \sum_{i=1}^{d} dx_i^2 \right) , \tag{5.1.1}$$

where

$$f(r) = 1 - \frac{r_h^{z+d}}{r^{z+d}} , \tag{5.1.2}$$

and where r_h is the horizon of the black hole. The Lifshitz dynamical critical exponent can take values $1 \leq z \leq d$. The gravitational solution (5.1.1) can be obtained from the action [124]

$$S = \frac{1}{16\pi\, G_{d+1}} \int d^{d+2}x \sqrt{g} \left(R + \Lambda - \frac{1}{2} (\partial\varphi)^2 - \frac{1}{4} e^{\lambda\varphi} \mathcal{F}^2 \right) , \tag{5.1.3}$$

with the action-extremizing solution for the fields

$$\begin{aligned} \mathcal{F}_{rt} &= q e^{-\lambda\varphi} , \qquad e^{\lambda\varphi} = r^{\lambda\sqrt{2(z-1)d}} , \\ \lambda^2 &= \frac{2d}{z-1} , \qquad q^2 = 2L^2 (z-1)(z+d) , \\ \Lambda &= -\frac{(z+d-1)(z+d)}{2L^2} , \end{aligned} \tag{5.1.4}$$

For the remaining of the chapter we will set $L^2 = 1$. We will also prefer to work with the coordinate $u = r_h/r$. This change of coordinates gives

$$ds^2 = -\frac{r_h^{2z} f(u)}{u^{2z}} dt^2 + \frac{1}{u^2 f(u)} du^2 + \frac{r_h^2}{u^2} \sum_{i=1}^{d} dx_i^2 \,, \tag{5.1.5}$$

where

$$f(u) = 1 - u^{z+d} \,, \tag{5.1.6}$$

and the Hawking temperature is

$$T_H = \frac{(z+d)}{4\pi} r_h^z \,. \tag{5.1.7}$$

It is therefore the action (5.1.3) that will provide us the gravitational Lifshitz background (5.1.5)-(5.1.6). We will now construct our minimal phenomenological model of holographic superconductivity by adding to (5.1.3) the action term

$$S_m = \int d^{d+2}x \sqrt{-g} \left(-\frac{1}{4} F^2 - |D\Psi|^2 - m^2 |\Psi|^2 \right) \,, \tag{5.1.8}$$

where we have introduced a charged scalar field Ψ and a $U(1)$ gauge field A_μ, following [7], and where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $D_\mu = \nabla_\mu - iA_\mu$. We will assume that there is negligible interaction with the gravitational background and therefore it remains fixed and given by the Lifshitz black hole solution (5.1.5)-(5.1.6). This lack of back reaction means we are effectively working in the probe limit (very large scalar field charge). As we will explain below, the scalar field mass will be chosen so as to get particular dimensions for the condensate under study.

The general equations of motion for these fields are

$$D^2\Psi = m^2\Psi \,, \tag{5.1.9}$$

$$\nabla_\mu F^{\mu\nu} = J^\nu + |\Psi|^2 A^\nu \,, \tag{5.1.10}$$

where

$$J_\mu = i \left(\Psi^* \nabla_\mu \Psi - \Psi \nabla_\mu \Psi^* \right) \,. \tag{5.1.11}$$

We propose the following ansatz for the component fields

$$\Psi(u) = \frac{1}{\sqrt{2}}\psi(u)\,, \qquad A = \phi(u)dt\,, \tag{5.1.12}$$

where $\psi(u)$ is a real function. Under this ansatz the equations of motion (5.1.9)-(5.1.10) become

$$\begin{aligned} \psi'' + \left(\frac{f'}{f} - \frac{d+z-1}{u}\right)\psi' - \frac{1}{u^2 f}\left(m^2 - \frac{u^{2z}\phi^2}{r_h^{2z} f}\right)\psi &= 0\,, &(5.1.13)\\ \phi'' - \frac{d-z-1}{u}\phi' - \frac{\psi^2}{u^2 f}\phi &= 0\,. &(5.1.14)\end{aligned}$$

This system of equations admits the no-hair solution $\psi(r) = 0$. In this case the gauge field has solutions

$$\begin{aligned} \phi(u) &= \mu - \rho\,\frac{u^{d-z}}{r_h^{d-z}}\,, \qquad (z \neq d)\,, &(5.1.15)\\ \phi(u) &= \mu - \rho\log\left(\frac{\xi r_h}{u}\right)\,, \qquad (z = d)\,, &(5.1.16)\end{aligned}$$

where ξ is a constant. This no-hair solution will correspond to the normal phase of the superconductor. The superconducting phase will be given by solutions with $\psi(u) \neq 0$. From the equations of motion (5.1.13)-(5.1.14) we see that the asymptotic $u \to 0$ behavior of the fields is

$$\psi(u) \approx \mathcal{O}_- \frac{u^{\Delta_-}}{r_h^{\Delta_-}} + \mathcal{O}_+ \frac{u^{\Delta_+}}{r_h^{\Delta_+}} + \cdots\,, \tag{5.1.17}$$

and

$$\begin{aligned} \phi(u) &\approx \mu - \rho\frac{u^{d-z}}{r_h^{d-z}} + \cdots\,, \qquad (z \neq d)\,, &(5.1.18)\\ \phi(u) &\approx \mu - \rho\log\left(\frac{\xi r_h}{u}\right) + \cdots\,, \qquad (z = d)\,, &(5.1.19)\end{aligned}$$

with

$$\Delta_\pm = \frac{1}{2}\left((z+d) \pm \sqrt{(z+d)^2 + 4m^2}\right)\,, \tag{5.1.20}$$

from where we get the BF-bound on the mass

$$m^2 \geq -\frac{(z+d)^2}{4}\,. \tag{5.1.21}$$

According to the AdS/CFT dictionary, the asymptotic coefficient $\mathcal{O}_+$ corresponds to the vacuum expectation value of an operator of dimension Δ_+, while $\mathcal{O}_-$ corresponds to a source in the boundary theory. Meanwhile, μ and ρ correspond to the chemical potential and the charge density of the dual field theory, respectively. In order to solve Eqs. (5.1.13)-(5.1.14) we will impose the regularity condition at the horizon $\phi(u=1)=0$. Also, from Eq. (5.1.13) we obtain the additional condition at $u=1$

$$\psi'(1) = \frac{m^2}{f'(1)}\psi(1)\,. \tag{5.1.22}$$

Additionally, in order to simplify the numerical calculations, we will make use of the scaling symmetries

$$r \to ar\,,\;\; t \to \frac{1}{a^z}t\,,\;\; x_i \to \frac{1}{a}x_i\,,\;\; g \to a^2 g\,,\;\; \phi \to a^z\phi\,. \tag{5.1.23}$$

As explained in the Introduction, we will focus on the particular case $D=5$. This means we will take $d=3$. When looking for hairy solutions to the equations of motion one has two possible boundary conditions at $u \to 0$ (5.1.17). Having set either one of these boundary conditions, one can proceed to solve the equations of motion through the shooting method.

5.1.2 Different Cases of Condensation

Going back to the allowed values for the dynamical critical exponent, we see that for the $D=5$ case we can have $1 \leq z \leq 3$. Throughout this chapter, for both brevity and simplicity, we will choose to work with the integer values $z=1,2$. This suits perfectly our primary objective, stated in the Introduction, which is to have a general idea of how the dynamical critical exponent z affects our holographic superconductor with respect to its behavior in the usual $(z=1)$ isotropic realization of the gauge/gravity duality. As will be seen in the following, the general tendency in the behavior of the physical quantities of the system will be very clear when treating these values.

In order to have a more comprehensive study of the effect of the dynamical critical exponent z on our holographic superconductor, we will choose to work in the following cases:

- **Case I**. We set the value of the scalar field mass as

$$m^2 = -3\,z\,. \tag{5.1.24}$$

In this way, we have

$$\Delta_- = z\,, \qquad \Delta_+ = 3\,, \tag{5.1.25}$$

so that the asymptotic behavior of the scalar field at $u \to 0$ is

$$\psi(u) \approx \mathcal{O}_z \frac{u^z}{r_h^z} + \mathcal{O}_3 \frac{u^3}{r_h^3} + \cdots\,. \tag{5.1.26}$$

In this case, we will set $\mathcal{O}_z = 0$ for all values of z considered, so that the superconducting order parameter of the system will be given by $\mathcal{O}_3$ of dimension 3.[1]

- **Case II**. We set the scalar mass as

$$m^2 = -(z+2)\,. \tag{5.1.27}$$

This choice of mass results in

$$\Delta_- = 1\,, \qquad \Delta_+ = z+2\,, \tag{5.1.28}$$

so that near $u \to 0$ we have

$$\psi(u) \approx \mathcal{O}_1 \frac{u}{r_h} + \mathcal{O}_{z+2} \frac{u^{z+2}}{r_h^{z+2}} + \cdots\,. \tag{5.1.29}$$

Here we will choose to set $\mathcal{O}_{z+2} = 0$ and the order parameter of the superconductor will be given by $\mathcal{O}_1$ of dimension 1.

We must point out that, even though the $\mathcal{O}_1$ mode is non-normalizable (it has the lowest dimension), it is useful to also study the case of the alternative

[1] The same condensate was used in [122], but magnetic properties were not studied in that paper.

boundary condition where $\mathcal{O}_{z+2} = 0$. Indeed, our main focus is to have a general insight on the way the critical dynamical parameter z alters the system with respect to its isotropic behavior, and, as will be seen in the following, the study of this mode will allow us to do just that, confirming all the results and phenomenology obtained in **Case I**.

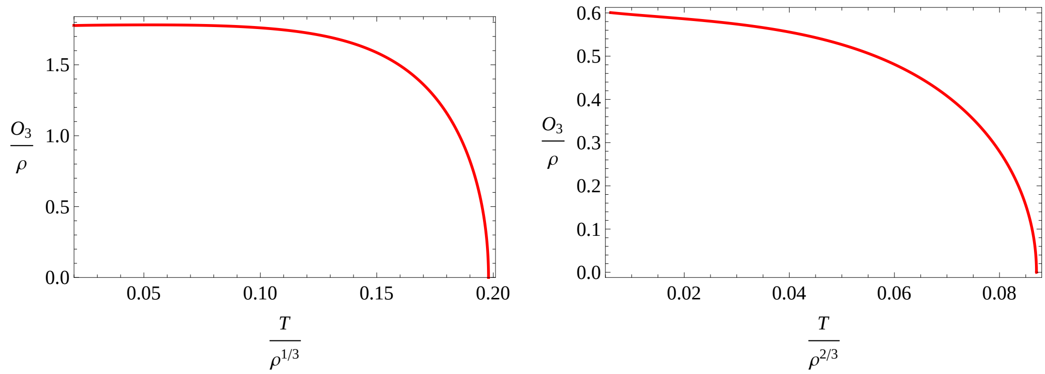
In figures (5.1)-(5.2) we show the value of the condensate $\mathcal{O}_\Delta$ as a function of temperature for each one of the cases described above. We notice that the near-T_c the condensate behaves as

$$\mathcal{O}_\Delta \sim (1 - T/T_c)^{1/2} \;, \tag{5.1.30}$$

for all values of z. Therefore, the dynamical critical exponent does not alter the mean-field theory behavior of the order parameter. In Table (5.1) we show the value of the critical temperature T_c for our different cases. We notice that the value of the critical temperature decreases with z for all cases, and therefore a large dynamical critical exponent inhibits superconduction.

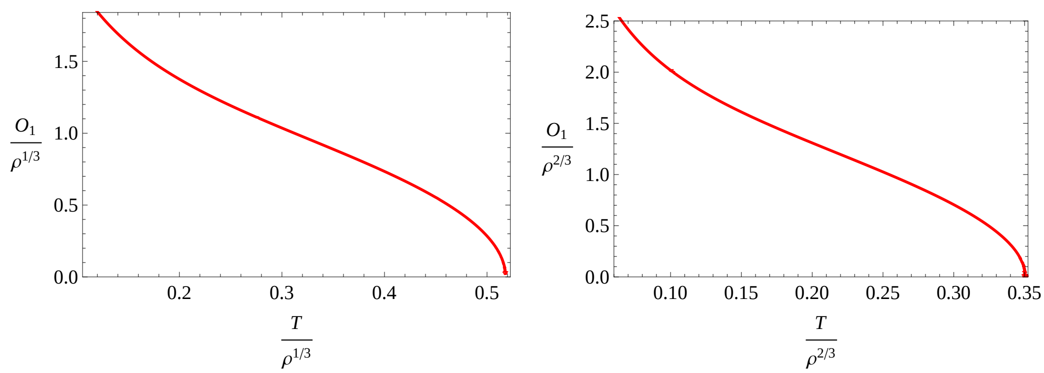
Comparing figs. (5.1a) and (4.1), we can notice a difference in the value of T_c, even though for **Case I**, $z = 1$, we are dealing with the same system and studying the same condensate $\mathcal{O}_3$. The reason for this difference is that in the present case we are dealing with a fixed background (the probe limit), while in Chapter 4 we considered full backreaction. Indeed, backreaction has the property to lower the value of T_c [7]. This can be observed in figure (4.3), where the dashed line corresponds closely to the probe limit. According to the analytical approximation to the critical temperature given by (4.1.22) obtained through the matching method, the probe limit value of T_c at $q = 1$ is ~ 0.201, while here we have obtained numerically ~ 0.198, showing good agreement within numerical limits.

Value of the condensate $\mathcal{O}_\Delta$ as a function of temperature, for different cases.



(a) $z = 1$ (b) $z = 2$

Figure 5.1: **Case I**.



(a) $z = 1$ (b) $z = 2$

Figure 5.2: **Case II**.

Table 5.1: Value of critical temperature T_c, for different cases.

$T_c/\rho^{z/3}$	$z=1$	$z=2$
Case I	0.198	0.087
Case II	0.517	0.351

5.2 Field Fluctuations

5.2.1 Gauge Field Fluctuation

In this section we will add small fluctuations to the component fields of our model. As explained before, we will set $d = 3$. We begin by adding the following gauge field fluctuation

$$A = \phi(u)dt + \delta A_x(t, u, y)dx \,, \tag{5.2.1}$$

where

$$\delta A_x(t, u, y) = e^{-i\omega t + iky} A_x(u) \,. \tag{5.2.2}$$

The corresponding equation of motion for $A_x(u)$ is, to linear order

$$A_x'' + \left(\frac{f'}{f} - \frac{d-z-3}{z}\right) A_x' + \left(\frac{u^{2z-2}\omega^2}{r_h^{2z} f^2} - \frac{k^2}{r_h^2 f} - \frac{\psi^2}{z^2 f}\right) A_x = 0 \,. \tag{5.2.3}$$

We will consider the case where ω and k are much smaller than the scale of the condensate (low-frequency/small-momentum regime), so that quadratic terms in k, ω in (5.2.3) can be neglected. Demanding regularity at the horizon $u = 1$, from (5.2.3) we have the following conditions

$$A_x(1) = A_{x0} \,, \qquad A_x'(1) = \frac{\psi_0^2}{f'(1)} A_{x0} \,. \tag{5.2.4}$$

where $\phi'(1) \equiv \psi_0$. Since Eq. (5.2.3) is linear, we will set $A_{x0} = 1$ without loss of generality.

From (5.2.3) we see that the asymptotic behavior of A_x at $u \to 0$ is

$$A_x(u) \approx A_x^{(0)} + J_x \frac{u^{d+z-2}}{r_h^{d+z-2}} + \cdots \tag{5.2.5}$$

According to the AdS/CFT dictionary, $A_x^{(0)}$ corresponds to a vector potential in the dual field theory, while J_x corresponds to its conjugate current [7]. We can relate both physical quantities through the London equation

$$J_x = -\frac{1}{m_s} n_s A_x^{(0)} , \tag{5.2.6}$$

where n_s is the superconducting carrier density number and m_s is the superconductor carrier mass. For simplicity we define the quantity

$$\tilde{n}_s \equiv \frac{1}{m_s} n_s , \tag{5.2.7}$$

which can be computed holographically from (5.2.5) and (5.2.6) as $\tilde{n}_s = -J_x/A_x^{(0)}$. In figures (5.3)-(5.4) we show the value of $\tilde{n}_s$ as a function of temperature, for different cases. We find that $\tilde{n}_s$ behaves near-T_c as

$$\tilde{n}_s \sim (1 - T/T_c) . \tag{5.2.8}$$

It is found numerically that the ratio of $\mathcal{O}_\Delta^2/\tilde{n}_s$ as a function of temperature behaves almost constantly and has a definite value at $T = T_c$ that varies according to the value of z within a specific case of condensation. We define this ratio at T_c as

$$\frac{\mathcal{O}_\Delta^2}{\tilde{n}_s} = C_z . \tag{5.2.9}$$

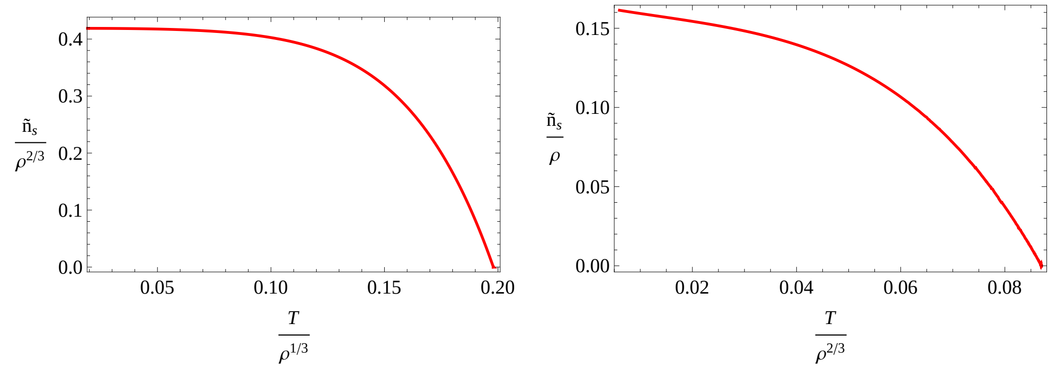
We show in Table (5.2) how the constant C_z varies for different cases. In figure (5.5) we show this ratio as a function of temperature, for **Case I**, $z = 1$, and **Case II**, $z = 1$. We can observe that the ratio behaves almost like a constant with respect to T. The fact that $\mathcal{O}_\Delta^2$ and $\tilde{n}_s$ have the same temperature behaviour can be shown in a similar way as we did in the previous chapter, using the matching method. The ratio (5.2.9) will be important in the next section, when we apply the Ginzburg-Landau interpretation to our system.

5.2.2 Scalar Field Fluctuation

We now consider a small fluctuation to the scalar field of the form

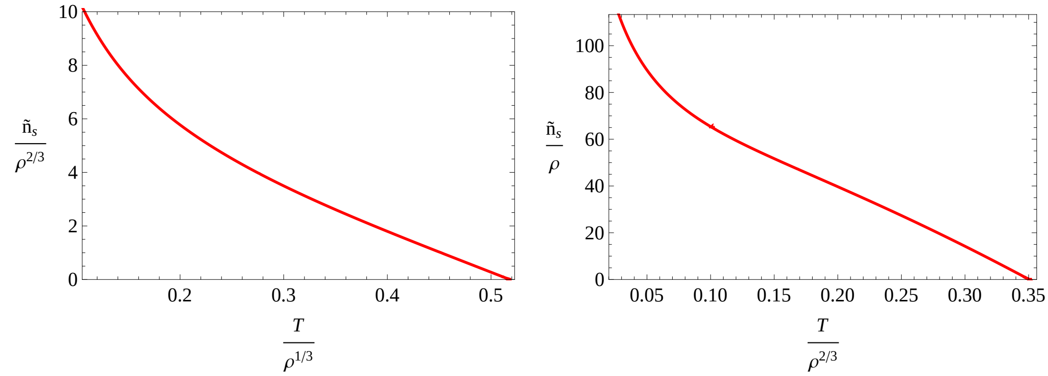
$$\Psi = \frac{1}{\sqrt{2}} \left(\psi(u) + \delta\psi(u, y) \right) , \tag{5.2.10}$$

Value of $\tilde{n}_s$ as a function of temperature, for different cases.



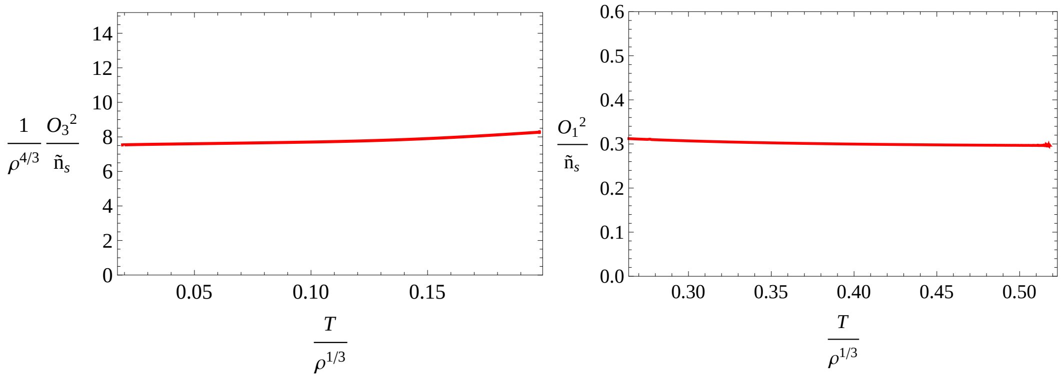
(a) $z = 1$ (b) $z = 2$

Figure 5.3: **Case I**.



(a) $z = 1$ (b) $z = 2$

Figure 5.4: **Case II**.



(a) **Case I**. $z = 1$ (b) **Case II**. $z = 1$

Figure 5.5: Value of the ratio $\mathcal{O}_\Delta^2/\tilde{n}_s$ as a function of temperature, for different cases.

Table 5.2: Value of C_z for different cases.

$C_z/\rho^{(2\Delta-z-1)/3}$	$z=1$	$z=2$
Case I	8.272	2.068
Case II	0.297	0.032

with

$$\delta\psi(u,y) = e^{iky}\eta(u)\,. \tag{5.2.11}$$

The corresponding equation of motion is

$$\eta'' + \left(\frac{f'}{f} - \frac{(d+z-1)}{u}\right)\eta' - \frac{1}{u^2 f}\left(m^2 - \frac{u^{2z}\phi^2}{r_h^{2z} f} + \frac{u^2}{r_h^2}k^2\right)\eta = 0\,, \tag{5.2.12}$$

where we set $d = 3$. Demanding regularity at the horizon $u = 1$, from (5.2.12) we have the following conditions

$$\eta(1) \equiv \eta_0\,, \qquad \eta'(1) = \frac{1}{f'(1)}\left(m^2 + \frac{k^2}{r_h^2}\right)\eta_0\,, \tag{5.2.13}$$

while at $u \to 0$ we have the asymptotic behavior

$$\eta(u) \approx (\delta\mathcal{O}_-)\,\frac{u^{\Delta_-}}{r_h^{\Delta_-}} + (\delta\mathcal{O}_+)\,\frac{u^{\Delta_+}}{r_h^{\Delta_+}} + \cdots \tag{5.2.14}$$

When solving equation (5.2.12) we set the same boundary conditions at $u \to 0$ as for the field ψ. Since we will not be concerned about the normalization of η, we set $\eta_0 = 1$.

Following [110], we can compute holographically the correlation length of the boundary operator by calculating the wave number k. Indeed, the correlation length ξ_0 is the inverse of the pole of the correlation function of the order parameter written in Fourier space

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle \sim \frac{1}{|k|^2 + 1/\xi_0^2} \,. \tag{5.2.15}$$

Following the same method as in Chapter 4, we obtain the wave number k by solving Eq. (5.2.12) as an eigenvalue problem consistent with the boundary conditions. This is done near the critical temperature. Once having computed k, one obtains the correlation length simply as

$$|\xi_0| = \frac{1}{|k|} \,. \tag{5.2.16}$$

In figures (5.6)-(5.7) we show the value of k as a value of temperature for our different cases. Also, in figures (5.8)-(5.9) we show the value of ξ_0 as a function of temperature, for our cases. We find that near the critical temperature, $k \sim (1 - T/T_c)^{1/2}$, and equivalently

$$\xi_0 \sim \frac{1}{(1 - T/T_c)^{1/2}} \,, \tag{5.2.17}$$

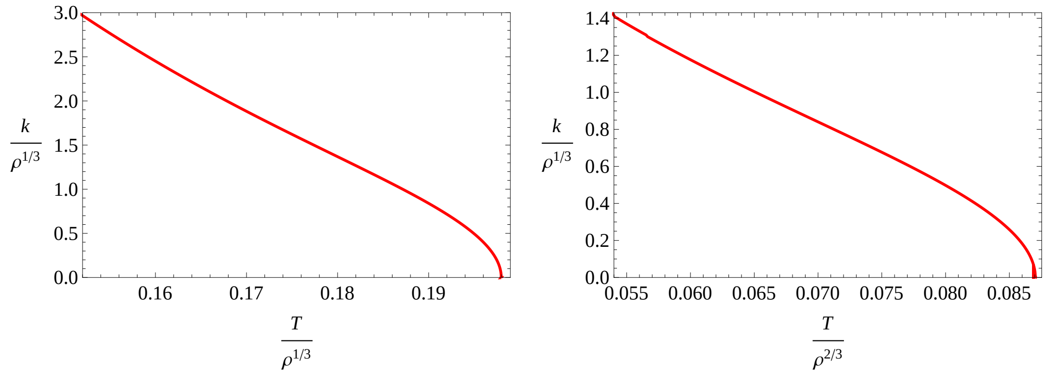
for all values of z, which is in agreement with mean field theory.

5.3 Ginzburg-Landau Approach

At this point we implement a phenomenological Ginzburg-Landau approach to our holographic superconductor, following [12]. The main assumption of this approach is that the dual field theory can be described near the critical temperature by the effective action

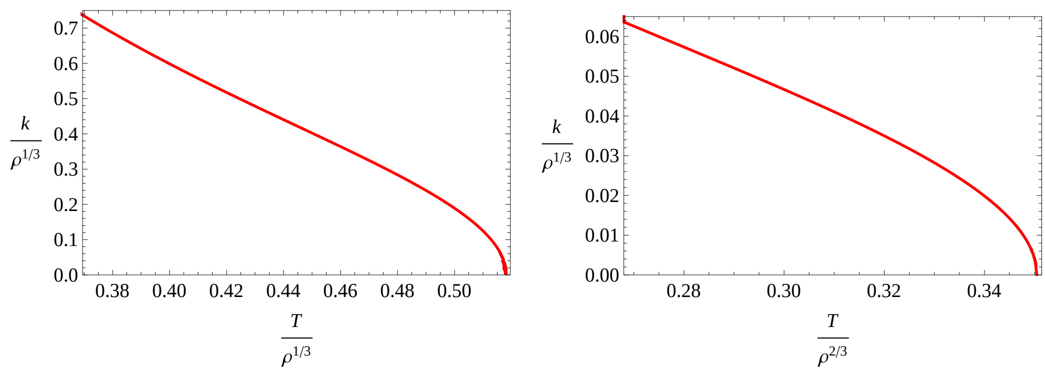
$$S_{\text{eff}} \approx \int d^4x \left\{ \alpha \left|\Psi_{\text{GL}}\right|^2 + \frac{\beta}{2}\left|\Psi_{\text{GL}}\right|^4 + \frac{1}{2m_s}\left|D\Psi_{\text{GL}}\right|^2 + \cdots \right\} \,. \tag{5.3.1}$$

Value of the wave number k as a function of temperature, for different cases.



(a) $z = 1$ (b) $z = 2$

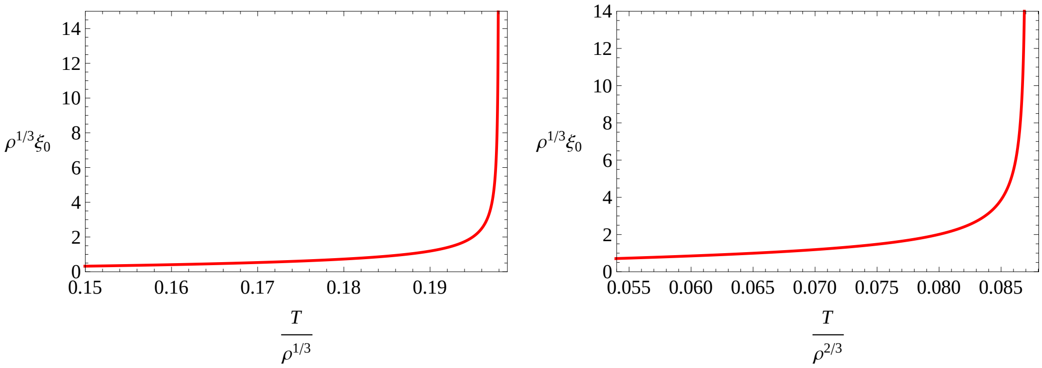
Figure 5.6: **Case I**.



(a) $z = 1$ (b) $z = 2$

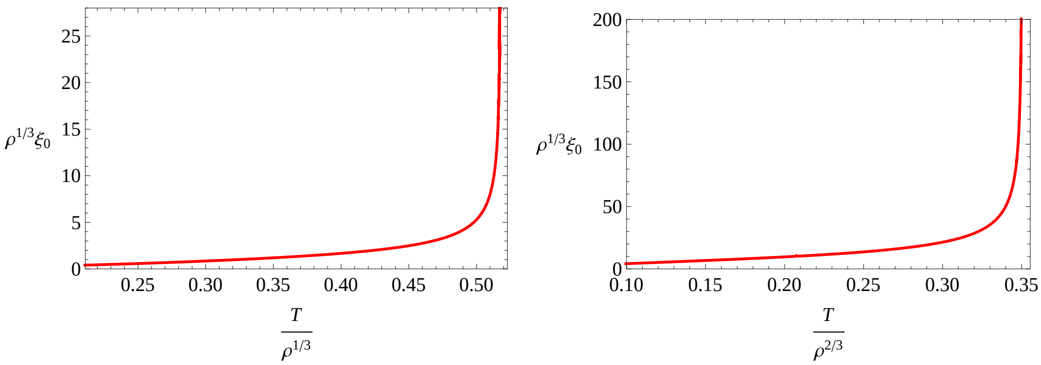
Figure 5.7: **Case II**.

Value of the correlation length ξ_0 as a function of temperature, for different cases.



(a) $z = 1$ (b) $z = 2$

Figure 5.8: **Case I**.



(a) $z = 1$ (b) $z = 2$

Figure 5.9: **Case II**.

where the component fields are a scalar field $\Psi_{\rm GL}$ representing the order parameter of the theory, and a vector field $\mathcal{A}_\mu$, with $\mu = 0, ..., 3$ and where $D_i = \partial_i - i\mathcal{A}_i$. Also, m_s is the superconductor carrier mass and α, β are phenomenological parameters with temperature dependence. According to the AdS/CFT dictionary, the vector field components $\mathcal{A}_0$ and $\mathcal{A}_x$ correspond to the chemical potential μ in (5.1.18) and to vector potential $A_x^{(0)}$ in (5.2.5) respectively. According to mean field theory, the order parameter $|\Psi_{\rm GL}|$ has critical exponent 1/2. In order to match this critical exponent with that of $\mathcal{O}_\Delta$ we propose the identification

$$|\Psi_{\rm GL}|^2 = N_z \mathcal{O}_\Delta^2 \,, \tag{5.3.2}$$

where N_z is a proportionality constant that depends on z and changes according to every model we consider.

In the remaining of this chapter, we adopt the same notation and conventions of [12]. In particular, the superconducting carrier mass m_s can be absorbed in definitions of the other parameters in Ginzburg-Landau theory, and we can therefore safely set $m_s = 1$. Going back to (5.2.7), this means in particular that $\tilde{n}_s = n_s$ and the numerical formula (5.2.9) can be written as

$$\frac{\mathcal{O}_\Delta^2}{n_s} = C_z \,. \tag{5.3.3}$$

However, according to Ginzburg-Landau theory, one has the following relation between the order parameter $|\Psi_{\rm GL}|$ and the charge carrier density n_s

$$|\Psi_{\rm GL}|^2 = n_s \,. \tag{5.3.4}$$

Then, substituting (5.3.4) and our identification (5.3.2) in (5.3.3) we obtain

$$N_z = \frac{1}{C_z} \,. \tag{5.3.5}$$

In Table 5.3 we show the value of the proportionality constant N_z for various cases.

We can also calculate the penetration length λ of the superconductor, defined as

$$\lambda = \frac{1}{\sqrt{4\pi n_s}} \,. \tag{5.3.6}$$

Table 5.3: Value of N_z for different cases.

$\rho^{(2\Delta-z-2)/3}N_z$	$z=1$	$z=2$
Case I	0.121	0.484
Case II	3.367	30.986

In figures 5.10-5.11 we show the value of λ as a function of temperature, for our different cases. As in the case of ξ_0, we have that the behavior of λ at $T \approx T_c$ is

$$\lambda \sim \frac{1}{(1-T/T_c)^{1/2}}\,, \tag{5.3.7}$$

for all z. This result is in agreement with mean field theory.

In order to have a consistent Ginzburg-Landau description of the dual field theory, we must be able to determine by holographic methods the parameters $|\alpha|$ and β. Regarding $|\alpha|$, we can determine it directly from the Ginzburg-Landau theory relation[2]

$$|\alpha| = \frac{1}{4\xi_0^2}\,. \tag{5.3.9}$$

In figures (5.12)-(5.13) we show the value of $|\alpha|$ as a function of temperature, for our various cases. We see that the near-T_c behavior of $|\alpha|$ is

$$|\alpha| \sim \alpha_0\,(1-T/T_c)\;, \tag{5.3.10}$$

which is in agreement with usual Ginzburg-Landau theory, for all z. We find numerically that the value of the coefficient α_0 decreases as the value of z raises.

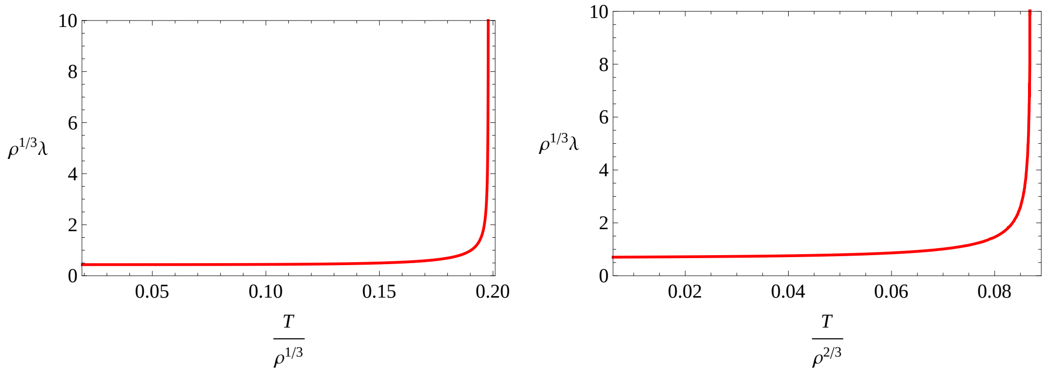
The remaining phenomenological parameter β can be computed through the

[2]The actual Ginzburg-Landau theory relation is

$$|\alpha| = \frac{\hbar^2}{2m_s\xi^2}\,, \tag{5.3.8}$$

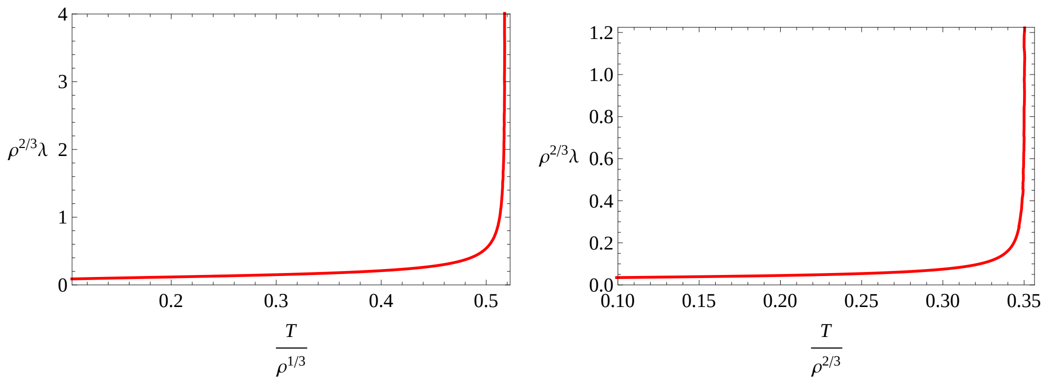
where ξ is the Ginzburg-Landau coherence length. The coherence length is in turn related to the correlation length ξ_0 as $\xi^2 = 2\xi_0^2$. See (1.1.25) and below.

Value of the penetration length λ as a function of temperature, for different cases.



(a) $z = 1$ (b) $z = 2$

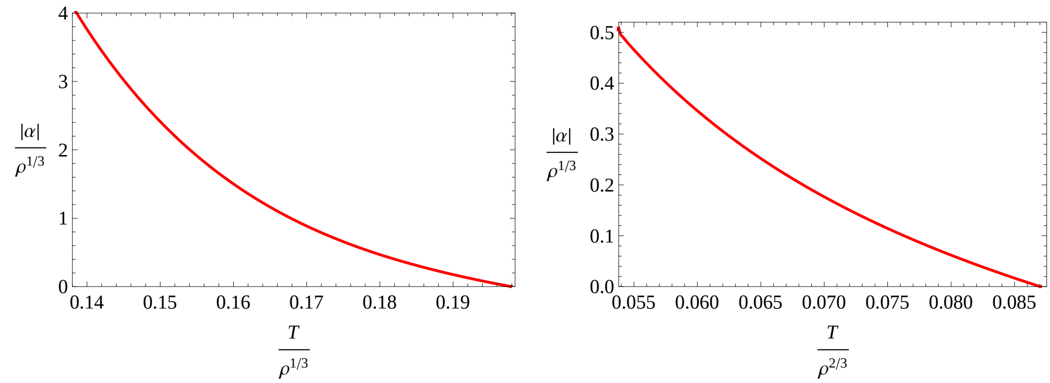
Figure 5.10: **Case I**.



(a) $z = 1$ (b) $z = 2$

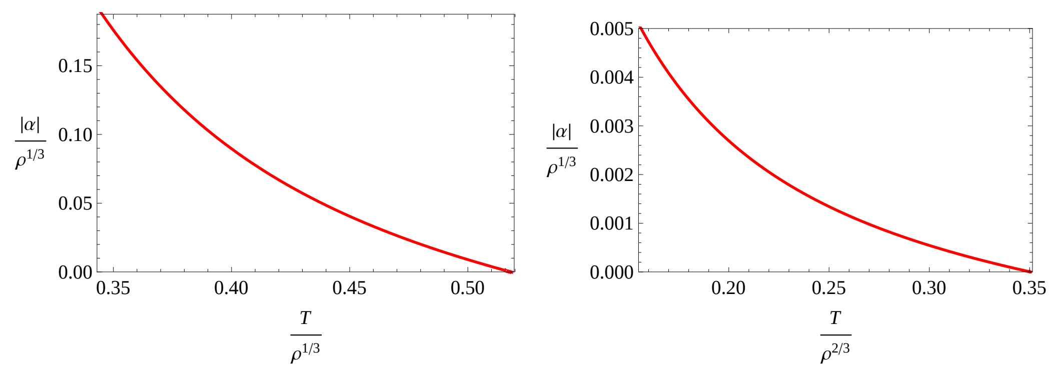
Figure 5.11: **Case II**.

Value of the parameter α as a function of temperature, for different cases.



(a) $z = 1$ (b) $z = 2$

Figure 5.12: **Case I**.



(a) $z = 1$ (b) $z = 2$

Figure 5.13: **Case II**.

Ginzburg-Landau theory relation

$$|\Psi_\infty|^2 = \frac{|\alpha|}{\beta}\,. \tag{5.3.11}$$

where $|\Psi_\infty|$ is the value of the condensate deep inside the superconductor, where external fields and gradients are negligible. Since we are in the limit of small field perturbations, we indeed find ourselves in that approximation. Substituting (5.3.2) and (5.3.9) in (5.3.11) we obtain the following expression

$$\beta = \frac{1}{4N_z}\frac{1}{\mathcal{O}_\Delta^2 \xi_0^2}\,. \tag{5.3.12}$$

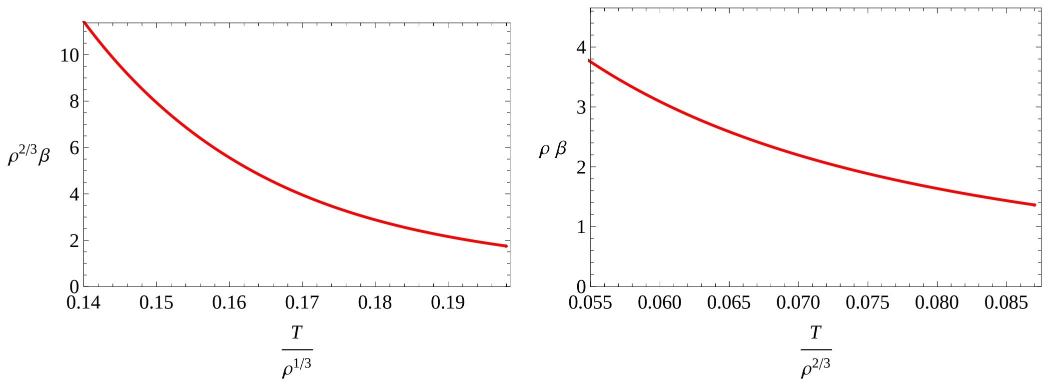
In figures (5.14)-(5.15) we show the behavior of β as a function of temperature, for our different condensation cases. We observe that, near-T_c, β behaves in agreement with Ginzburg-Landau theory, having a definite value at $T = T_c$. We also observe that this value decreases as the value of z raises.

Having calculated the characteristic lengths of the system ξ_0 and λ, we can compute the Ginzburg-Landau parameter κ, defined as

$$\kappa = \frac{\lambda}{\xi}\,, \tag{5.3.13}$$

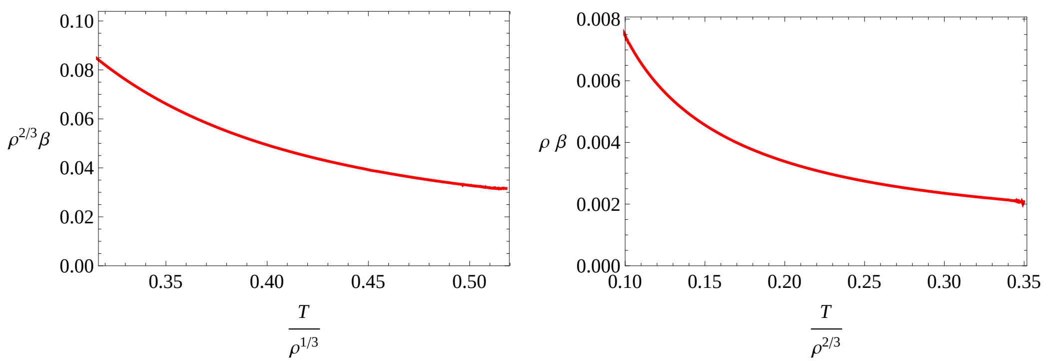
where ξ is the Ginzburg-Landau coherence length, which is related to our correlation length ξ_0 as $\xi^2 = 2\xi_0^2$. (See [12].) In figures (5.16)-(5.17) we show how the Ginzburg-Landau parameter κ behaves as a function of temperature, for our different cases. We notice that all plots have a definite value at $T = T_c$. We will take this to be the value of κ of our holographic superconductor for each case considered. The value of κ for different cases is shown in Table 5.4. We note that all values of κ are lower than $1/\sqrt{2} \sim 0.707$ for all cases of z considered, which means that our system behaves always as a Type I superconductor. Also, we notice that the value of κ is always lower for $z = 2$, which means that in holographic superconductors with higher dynamical critical exponent, vortex formation is more strongly unfavored energetically and has a stronger Type I behavior.

Value of the parameter β as a function of temperature, for different cases.



(a) $z = 1$ (b) $z = 2$

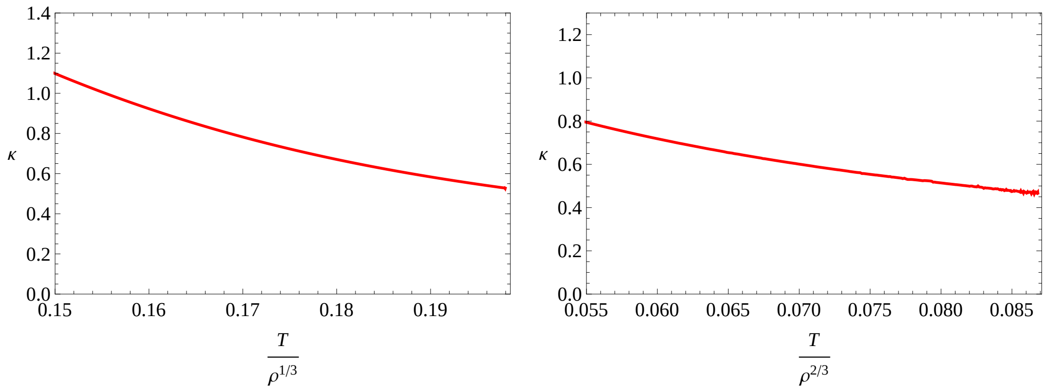
Figure 5.14: **Case I**.



(a) $z = 1$ (b) $z = 2$

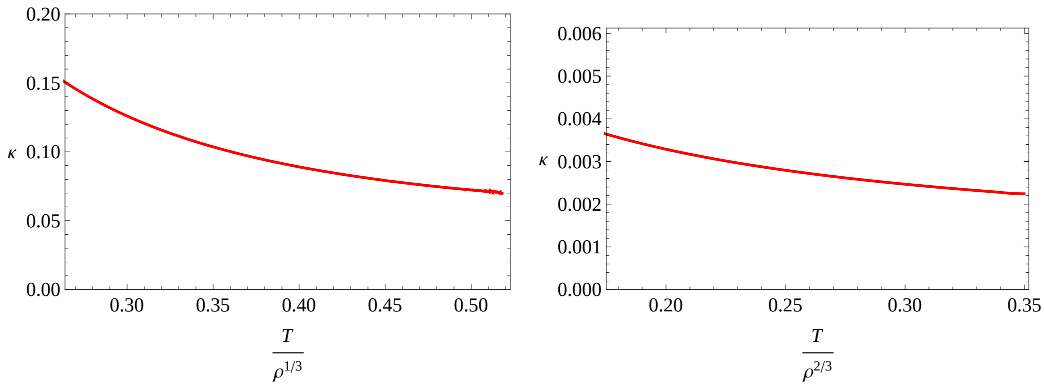
Figure 5.15: **Case II**.

Value of the Ginzburg-Landau parameter κ as a function of temperature, for different cases.



(a) $z = 1$ (b) $z = 2$

Figure 5.16: **Case I**.



(a) $z = 1$ (b) $z = 2$

Figure 5.17: **Case II**.

Table 5.4: Value of the Ginzburg-Landau parameter κ, for different cases.

κ	$z = 1$	$z = 2$
Case I	0.527	0.467
Case II	0.070	0.002

To finalize this section, let us change our point of view on our bulk model (5.1.8) and give it a different physical interpretation in the dual field theory. Indeed, instead of studying our system as a model for a dual superconducting system, let us consider the case when we keep the $U(1)$ symmetry ungauged in the boundary field theory, keeping it thus global. By proceeding in this fashion, our same gravity system will instead be considered a model of *holographic superfluidity* [125, 15]. Since standard Ginzburg-Landau theory makes some precise predictions about superfluid phenomenology, we can therefore use this holographic superfluid interpretation to test the consistency of our current Ginzburg-Landau approach and to see how this phenomenology is altered by the presence of the dynamical exponent z. As an example, we will compute the the *critical supercurrent* J_c and will verify that its near-T_c functional dependency on the temperature is in precise agreement with the usual formulas derived from GL theory of the superfluid.

To introduce the critical supercurrent in this new holographic superfluid context, we must first go back to equation (5.2.3), which, having neglected the terms in ω and k, is just a homogeneous (only u-dependent) equation for A_x. In holographic superfluidity, switching on a nonzero $A_x(u)$ corresponds, as described first in [125, 15], to turning on a supercurrent in the system . In that context, the asymptotic coefficient $A_x^{(0)}$ in (5.2.5) corresponds to the source, or superfluid velocity v_x, while J_x corresponds to the supercurrent. We note that, since we are considering A_x as a perturbation where the backreaction on the fields ψ and ϕ is neglected, then one is effectively switching a perturbative supercurrent J_x.

The relation between the supercurrent J_x and the superfluid velocity is very well known in the context of Ginzburg-Landau theory, where it is studied in the limit where the modulus of the order parameter $|\Psi_{\rm GL}|$ has a constant value. This limit is associated with the physical situation where the charged superfluid is confined to a thin film. (See, e.g. [29, 125].) Using holographic methods, the relation between J_x and v_x has been previously studied in [126], where the authors started from a minimal 3+1 dimensional bulk-model in the same spirit as ours, and considered three fields equivalent to our ψ, ϕ and A_x, and considered them to be all of the same order and to fully backreact between them[3]. In the study of the relation between J_x and v_x, one the most important conclusions they reached is that for temperatures close to T_c, the system has the same behavior predicted by Ginzburg-Landau theory. Building on this result and on the essential similarities between our bulk models, we can compute the critical current J_c, that corresponds to the value of the supercurrent above which the system passes to the normal phase [29]. According to Ginzburg-Landau theory, it is given by the general expression

$$J_c = q_s\,|\Psi_\infty|^2\left(\frac{2}{3}\right)^{3/2}\sqrt{\frac{|\alpha|}{m_s}}\,, \qquad (5.3.14)$$

which we can rewrite in terms of our holographically-computed quantities $\mathcal{O}_\Delta$ and ξ_0 as

$$J_c = \frac{1}{2C_z}\left(\frac{2}{3}\right)^{3/2}\frac{\mathcal{O}_\Delta^2}{\xi_0}\,. \qquad (5.3.15)$$

In figures (5.18)-(5.19) we show the value of J_c as a function of temperature, for our various cases of condensation. We note that the value of the supercurrent decreases as the value of z rises. More importantly, we see numerically that the

[3]For additional studies of the supercurrent density in the presence of a superfluid velocity, see [127, 128].

near-T_c behavior of J_c is

$$J_c \sim (1 - T/T_c)^{3/2} , \tag{5.3.16}$$

which is in accordance with the predictions of usual Ginzburg-Landau theory.

5.4 Constant Magnetic Field

We will now study the effect of an external magnetic field to the superconducting phase of our models. As done before, we begin with in the general dimensional case and then focus on $D = 5$. We follow the procedure developed by [17] and will proceed in a perturbative fashion by proposing a series expansion for the component fields

$$\Psi(\vec{x}, u) = \epsilon^{1/2}\Psi^{(1)}(\vec{x}, u) + \epsilon^{3/2}\Psi^{(2)}(\vec{x}, u) + \cdots \tag{5.4.1}$$

$$A_\mu(\vec{x}, u) = A_\mu^{(0)}(\vec{x}, u) + \epsilon A_\mu^{(1)}(\vec{x}, u) + \cdots \tag{5.4.2}$$

where $\vec{x} = (x, y)$, and the expansion parameter is given by

$$\epsilon = \frac{B_c - B}{B_c} , \qquad \epsilon \ll 1 , \tag{5.4.3}$$

were B_c is the value of the magnetic field that breaks the superconducting phase (*critical magnetic field*). Since this expansion is done near the value $B = B_c$, this means that we find ourselves near the point where the condensate vanishes. We substitute expansions (5.4.1)-(5.4.2) in the general equations on motion (5.1.9)-(5.1.10). The zero order equation for the gauge field is

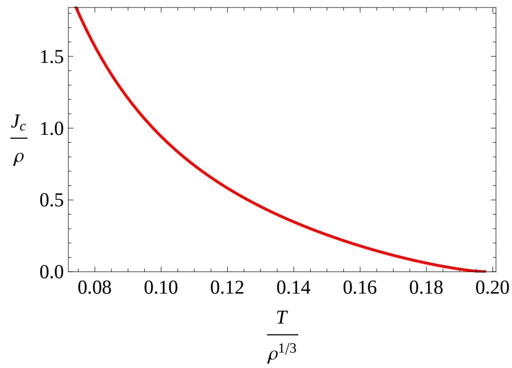
$$\frac{1}{\sqrt{g}}\partial_\mu \left(\sqrt{g}F^{\mu\nu}_{(0)}\right) = 0 , \tag{5.4.4}$$

and has solutions

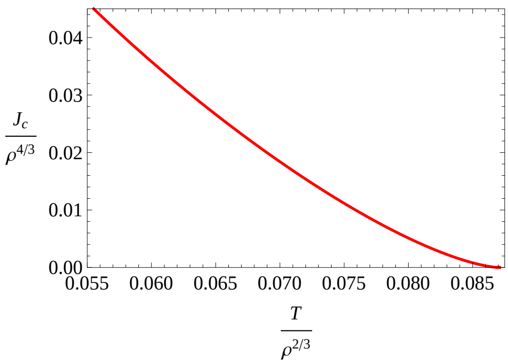
$$A_t^{(0)}(u) = \mu - \rho\,\frac{u^{d-z}}{r_h^{d-z}} , \tag{5.4.5}$$

$$A_y^{(0)}(x) = B_c\, x , \tag{5.4.6}$$

Value of the critical current J_c as a function of temperature, for different cases.

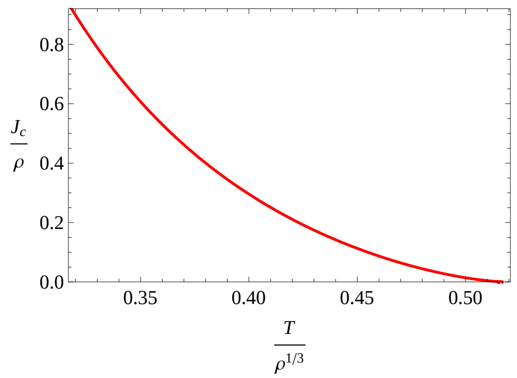


(a) $z = 1$

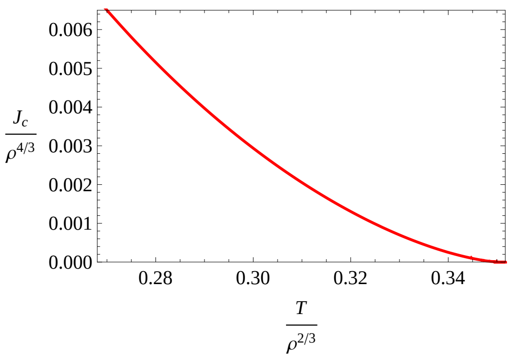


(b) $z = 2$

Figure 5.18: **Case I**.



(a) $z = 1$



(b) $z = 2$

Figure 5.19: **Case II**.

and the rest of spatial components equal to zero: $A_i^{(0)} = 0$, $i \neq y$. Since the solution for $A_t^{(0)}$ is equal to solution (5.1.15), we set the notation $A_t^{(0)} = \phi$, for simplicity. Meanwhile, the general scalar field equation is

$$u^{d+1-z}\partial_u\left(\frac{f}{u^{z+d-1}}\partial_u\Psi^{(1)}\right) - \left(\frac{m^2}{u^{2z}} - \frac{\phi^2}{r_h^{2z}f}\right)\Psi^{(1)} = -\frac{1}{r_h^2u^{2z-2}}\delta^{IJ}D_ID_J\Psi^{(1)}\,. \tag{5.4.7}$$

where $I, J = x, y$. Eq. (5.4.7) is clearly separable. We follow the standard treatment and propose

$$\Psi^{(1)}(\vec{x}, u) = e^{ipy}\varphi^{(p)}(x, u)\,, \tag{5.4.8}$$

so on the right hand side of (5.4.7) we have

$$\delta^{IJ}D_ID_J\Psi^{(1)} = \left(\partial_x^2 + (\partial_y - iB_c\,x)^2\right)\Psi^{(1)} = e^{ipy}\left(\partial_x^2 - (p - B_c\,x)^2\right)\varphi^{(p)}\,, \tag{5.4.9}$$

and we get the following equation

$$u^{d+1-z}\partial_u\left(\frac{f}{u^{z+d-1}}\partial_u\varphi^{(p)}\right) - \left(\frac{m^2}{u^{2z}} - \frac{\phi^2}{r_h^{2z}f}\right)\varphi^{(p)} = \frac{1}{r_h^2u^{2z-2}}\left(-\partial_x^2 + (p - B_c\,x)^2\right)\varphi^{(p)}\,. \tag{5.4.10}$$

Now, we make the separation

$$\varphi_n^{(p)}(x, u) = \rho_n(u)\gamma_n^{(p)}(x)\,, \tag{5.4.11}$$

and define the variable

$$X = \sqrt{2B_c}\left(x - \frac{p}{B_c}\right)\,, \tag{5.4.12}$$

so that the operator on the right hand side of (5.4.10) becomes

$$\left[-\partial_x^2 + (p - B_cx)^2\right] = (2B_c)\left[-\partial_X^2 + \frac{1}{4}X^2\right]\,, \tag{5.4.13}$$

and acting on $\gamma_n^{(p)}$ we have the eigenvalue equation

$$\left(-\partial_X^2 + \frac{1}{4}X^2\right)\gamma_n^{(p)} = \frac{\lambda_n}{2}\gamma_n^{(p)}\,, \tag{5.4.14}$$

that has as a solution the eigenfunctions

$$\gamma_n^{(p)}(x) = e^{-X^2/4}H_n(X)\,, \tag{5.4.15}$$

where H_n are the Hermite polynomials and

$$\lambda_n = 2n+1\,, \qquad n = 0, 1 \ldots \tag{5.4.16}$$

We choose the $n = 0$ mode, which corresponds to the most stable solution [7, 17, 101, 129]. As described originally in [17], the more general solution to the scalar field is given by linear superposition of the solution obtained above, with different values of p. (We adopt the authors notation in the following). Going back to (5.4.8), (5.4.11) and (5.4.15), we write our solution explicitly as

$$\Psi^{(1)}\left(u, \vec{x}\right) = \rho_0(u) \sum_{l=-\infty}^{\infty} C_l e^{ip_l y} \gamma_0\left(x; p_l\right)\,, \tag{5.4.17}$$

where

$$\gamma_0\left(x; p_l\right) = \exp\left\{-\frac{B_c}{2}\left(x - \frac{p_l}{B_c}\right)^2\right\}\,, \tag{5.4.18}$$

and where C_l and p_l are chosen explicitly as

$$C_l = \exp\left(-i\frac{\pi a_2}{a_1^2} l^2\right)\,, \qquad p_l = \frac{2\pi\sqrt{B_c}}{a_1} l\,, \tag{5.4.19}$$

with a_1, a_2 real parameters. Solution (5.4.17) can be rewritten as

$$\Psi^{(1)}\left(u, \vec{x}\right) = \frac{1}{L}\rho_0(u) e^{-\frac{B_c x^2}{2}} \vartheta_3(\upsilon, \tau)\,, \tag{5.4.20}$$

where $\vartheta_3(\upsilon, \tau)$ is the *elliptic theta function*, defined as

$$\vartheta_3(\upsilon, \tau) = \sum_{l=-\infty}^{\infty} e^{i\pi\tau l^2} e^{2i\pi\upsilon l}\,, \tag{5.4.21}$$

and where the variables υ and τ are defined as

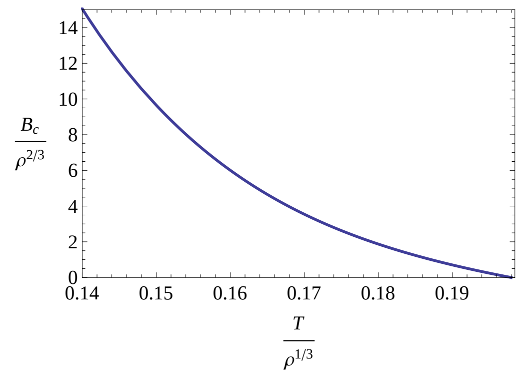
$$\upsilon \equiv \frac{\sqrt{B_c}}{a_1}\left(-ix + y\right)\,, \qquad \tau \equiv \frac{1}{a_1^2}\left(2i\pi - a_2\right)\,. \tag{5.4.22}$$

Owing to the elliptical theta function ϑ_3, the scalar field solution $\Psi^{(1)}$ has the following pseudo-periodicity in the $x - y$ plane

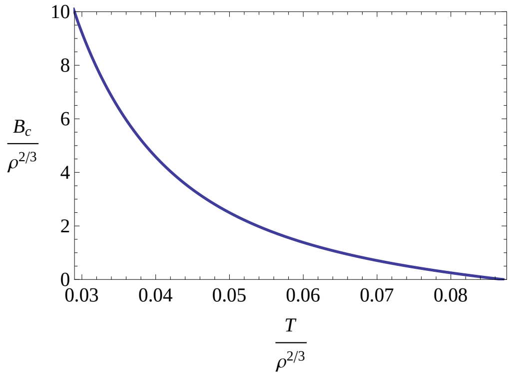
$$\Psi^{(1)}\left(u, x, y\right) = \Psi^{(1)}(u, x, y + a_1)\,, \tag{5.4.23}$$

$$\Psi^{(1)}\left(u, x + \frac{2\pi}{\sqrt{B_c} a_1}, y + \frac{a_2}{\sqrt{B_c} a_1}\right) = \exp\left[\frac{2\pi i}{a_1}\left(\sqrt{B_c} y + \frac{a_2}{2a_1}\right)\right] \Psi^{(1)}\left(u, x, y\right)\,. \tag{5.4.24}$$

Value of the critical magnetic field B_c as a function of temperature, for different cases.

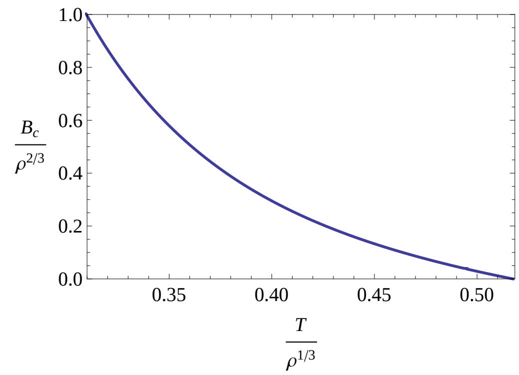


(a) $z = 1$

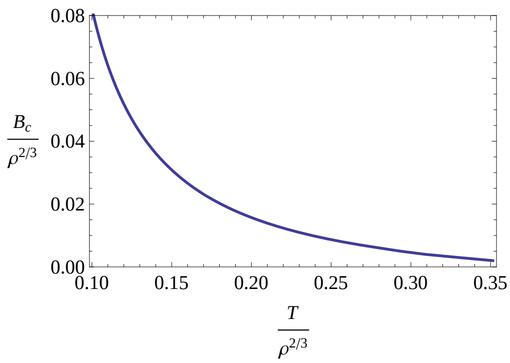


(b) $z = 2$

Figure 5.20: **Case I**.



(a) $z = 1$



(b) $z = 2$

Figure 5.21: **Case II**.

In addition to this, the ϑ_3 function has zeros located periodically at

$$\vec{V} = \left(m + \frac{1}{2}\right)\vec{v}_1 + \left(n + \frac{1}{2}\right)\vec{v}_2\,, \tag{5.4.25}$$

where the $\vec{v}_i$ vectors are given by

$$\vec{v}_1 = \frac{a_1}{\sqrt{B_c}}\frac{\partial}{\partial y}\,, \qquad \vec{v}_2 = \frac{2\pi}{\sqrt{B_c}a_1}\frac{\partial}{\partial x} + \frac{a_2}{\sqrt{B_c}a_2}\frac{\partial}{\partial y}\,. \tag{5.4.26}$$

Thus, the $\Psi^{(1)}$ solution has a lattice profile in the $(x - y)$ plane, spanned by the vectors $\vec{v}_i$. We note that, in our given approximation, we will get a 2-dimensional plane, orthogonal to the remaining $(d-2)$-dimensional boundary space, where the vortices live. We should note that the presence of the vortex solutions given above does not contradict the fact that our system was found in the previous section to be Type I[4]. Indeed, the computation of κ presented above comes from an energetic analysis, conducted directly from the dual system's Ginzburg-Landau action. (See [12].) This shows that, according to Ginzburg-Landau theory, the formation of the above vortex solutions costs more energy to the system than the energy needed for the system staying in a superconducting state. (See, e.g. [29].) To be more specific, the case where $B > B_c$ corresponds to a physical situation where the energy of the superconducting state is bigger than that of the normal state, while for $B < B_c$ the inverse situation holds true, with a phase transition occurring at $B = B_c$. Furthermore, since our system was found to be Type I, the phase transition in the magnetic field is first order and, also as a consequence of being Type I, vortex formation is not energetically favored. Therefore the system undergoes a phase transition from a homogeneous superconducting phase to the normal phase as the magnetic field is increased.

Returning to the scalar field equation (5.4.10) and substituting the results given above, we obtain the following equation for the radial function ρ

$$u^{d+1-z}\partial_u\left(\frac{f}{u^{z+d-1}}\partial_u\rho(u)\right) - \left(\frac{m^2}{u^{2z}} - \frac{\phi^2}{r_h^{2z}f} + \frac{B_c}{r_h^2u^{2z-2}}\right)\rho(u) = 0\,, \tag{5.4.27}$$

[4]A dynamical approach to vortex solutions in $D = 4$ can be found in [11], where it was concluded that, for some values of the system's parameters, the dual superconducting system was Type I. For other dynamical approaches, see [106, 130].

which can be written as

$$\rho'' + \left(\frac{f'}{f} - \frac{d+z-1}{u}\right)\rho' - \frac{1}{u^2 f}\left(m^2 - \frac{u^{2z}\phi^2}{r_h^{2z} f} + \frac{u^2}{r_h^2}B_c\right)\rho = 0\,. \tag{5.4.28}$$

This equation of course has the same behavior at $u \to 0$ as (5.1.17)

$$\rho \sim C_- u^{\Delta_-} + C_+ u^{\Delta_+}\,, \tag{5.4.29}$$

with $\Delta_\pm$ given by (5.1.20). We set the same boundary conditions at $u \to 0$ as for the field ψ in (5.1.12). By applying the shooting method to Eq. (5.4.27) we find the value of the critical magnetic field that breaks the superconducting phase of the system. In figures (5.20)-(5.21) we show the value of the critical magnetic field B_c as a function of temperature, for our different cases. We see that near-T_c the critical magnetic field B_c behaves as

$$B_c \sim (1 - T/T_c)\,, \tag{5.4.30}$$

which is in agreement with mean field theory, for all values of z. We also note by comparing Eqs. (5.4.28) and (5.2.12), that the procedure to obtain the near-T_c values of the square of the wave number k and the critical field B_c is the same. This in turn confirms the relation between the correlation length and the critical magnetic field put forward in [18]

$$B_c \approx \frac{1}{\xi_0^2}\,, \qquad (T \approx T_c)\,. \tag{5.4.31}$$

6

Top-Down Approach: Superconductors from Superstrings on $\mathbf{AdS}_5 \times \mathbf{T}^{1,1}$.

Having seen in the previous two chapters an example of the phenomenological-focused bottom-up approach to holographic superconductivity, we now turn our attention to an example of a top-down model. To summarize, we will consider holographic superconductors arising from a family of $\mathcal{N} = 2$ supergravity bulk theories. As it will be shown, there is strong evidence that these theories can in fact be embedded in 10 dimensional Type IIB superstring theory on $\mathrm{AdS}_5 \times T^{1,1}$.

Many of the most important properties and motivations of the top-down approach will become evident in this chapter. The first thing one should notice is that, in building holographic superconducting models from consistent truncations, we are taking advantage of the fact that superstring theory can be framed within a relatively wide landscape of gravitational backgrounds. The most relevant of these is of course given by $AdS_d \times S^{10-d}$ type backgrounds. However, as we will show, a different choice of the internal manifold can have a dual superconducting interpretation. Moreover, one can speculate that certain universal properties of holographic superconductors could be intimately related to a certain class of internal manifolds chosen in the Kaluza-Klein compactifications. Another commonly appearing property in top-down models is the fact that one usually has a very good description of the dual field theory operators. Indeed, this is the case for the system studied in the present chapter, where we argue that the dual field theory operators are well defined operators belonging to *Klebanov-Witten superconformal field theory* [131]. Generally speaking, detailed knowledge of the dual operators is a very desirable property from the condensed matter theory point of view, since it may help to shed some light on the microscopic details of the condensing operators.

In the case of $\mathrm{AdS}_5 \times \mathrm{S}^5$, the dual field theory in the boundary is given by $\mathcal{N} = 4$ Super Yang Mills in four dimensions. Meanwhile, at large t'Hooft coupling the bulk dual is given by five dimensional $\mathcal{N} = 8$ supergravity. In this case, it is the main result of the gauge/gravity duality to be able to identify the fields living in the bulk with quantum operators in $\mathcal{N} = 4$ SYM. The basic idea in this chapter is to study consistent truncations of $\mathcal{N} = 8$ supergravity to smaller, more tractable sectors with gauge and scalar field content. Since various of these truncations will lead to $\mathcal{N} = 2$ supergravity theories, we will then choose these as our starting point.

6.1 The Bosonic Sector of $\mathcal{N} = 2$ Supergravity.

We will start by considering the bosonic sector of $\mathcal{N} = 2$ supergravity in $D = 4{+}1$ bulk-dimensions. In general terms, the theory contains the following multiplets

- **The graviton multiplet.** It consists of the metric and a single gauge field referred to as the *graviphoton*.

- **The gauge multiplet $\mathcal{A}$.** It consists of a vector field A_μ and a real scalar ϕ.

- **The hypermultiplet $\mathcal{H}$.** It consists of two complex scalars, or equivalently, by four real scalars q^u, with $u = 1, \ldots, 4$.

In addition to this bosonic content, each multiplet has also fermion degrees of freedom, but they vanish in a classical background.

We will be now be mainly interested with a collection of the described multiplets. More concretely, we will consider

- n_V gauge multiplets $\mathcal{A}_I$, labeled by $I = 1, \ldots, n_V$.
- n_H hypermultiplets $\mathcal{H}_J$, labeled by $J = 1, \ldots, n_H$.

We will refer to the real scalar fields in the n_V gauge multiplets as ϕ^x, where $x = 1, \ldots, n_V$. Meanwhile, the real hyperscalars will be referred to as q^u, with $u = 1, \ldots, n_H$. The whole set of scalars coming from both multiplets $\mathcal{A}$ and $\mathcal{H}$ will be denoted by the vector $\vec{v}$, with components v^i, $i = 1, \ldots, n_V + n_H$. On the other hand, the totality of gauge fields will be denoted as A^I_μ, with $I = 0, \ldots, n_V$. We observe that the total number of vector fields is $n_V + 1$, since we have taken into account the graviphoton.

Pure $\mathcal{N} = 2$ supergravity consists only of the graviton multiplet, in which case the Lagrangian is uniquely determined and its bosonic part is given simply by Einstein-Maxwell theory with a negative cosmological constant [132]. Additional field content and matter fields are introduced to the pure theory by means of the gauge multiplets and hypermultiplets. The couplings among matter the additional fields and the original pure $\mathcal{N} = 2$ supergravity are constrained by the requirement that the complete Lagrangian is invariant under local $\mathcal{N} = 2$ transformations. Although these supersymmetry transformations can be quite complicated, we can mention three geometrical facts that are enough to determine this Lagrangian. For a more detailed account of the construction of the Lagrangian, see [133, 134, 135].

6.2 The Scalar Manifold

In order to be able to construct $\mathcal{N}=2$ supergravity, we first need to focus on the geometry describing the scalar manifold. More formally, this scalar manifold will be a *Kähler special manifold.* Let us begin then by defining a Kähler manifold: a Kähler manifold is a Riemannian manifold $\mathbf{M}$ of real dimension $2n$, endowed with the almost complex structure J and the hermitian metric g, such that J is covariantly constant with respect to the Levi-Civita connection. If we consider a local chart of $2n$ real coordinates $\vec{\varphi}$, an almost complex structure is then given by a real valued tensor $J_i^j(\vec{\varphi})$ living on the tangent space of the manifold, and defined by the property $J^2=-1$. In particular, the hermitian metric satisfies the relation $JgJ^T=g$. As a consequence of the definition, $\mathbf{M}$ will also be a complex manifold which can then be covered by local holomorphic charts with n complex coordinates $\{z^\alpha, \bar{z}^\alpha\}$. In these complex coordinates the metric can be written with indexes $g_{\alpha\overline{\beta}}$.

Going back to $\mathcal{N}=2$ supergravity, the scalar manifold $\mathcal{M}$ of the theory has the following geometric properties:

- The scalars belonging to the hypermultiplets are coordinates for a quaternionic Kähler manifold $\mathcal{Q}$, whose choice fixes the self-interaction of the hypermultiplets.

- The scalar belonging to the vector multiplets are coordinates for a very special real manifold $\mathcal{V}$, whose choice fixes the self-interaction of the scalars and their couplings to the gauge vectors.

The scalar manifold is then given by the direct product $\mathcal{M}=\mathcal{V}\otimes\mathcal{Q}$, and is equipped with a smooth metric g_{ij} of euclidean signature. This metric defines a non-linear sigma-model kinetic term for the scalars $\vec{v}$ as

$$\mathcal{L}^{\text{Kin.}}_{\mathcal{N}=2} \sim g_{ij}\partial_\mu v^i \partial^\mu v^j \,, \tag{6.2.1}$$

where the metric g_{ij} has block-diagonal form

$$g_{ij} = \begin{pmatrix} G_{xy} & 0 \\ 0 & H_{uv} \end{pmatrix} , \tag{6.2.2}$$

and where G_{xy}, H_{uv} correspond to the product spaces $\mathcal{V}$, $\mathcal{Q}$, respectively. Additionally, we simply state that the hyperscalar manifold $\mathcal{Q}$ is further geometrically constrained by the R-symmetry group of the theory, $SU(2)$.

Invariance under $\mathcal{N} = 2$ supersymmetry also places constrains on the non-linear sigma-model terms for the field strengths $F^I = dA^I$. Concretely, these constraints result in a kinetic matrix N_{IJ}, with indexes $I, J = 0, \ldots, n_V$, therefore including the graviphoton. The kinetic matrix couples the field strengths with the scalars of the vector multiplets as $N_{IJ} = N_{IJ}(\phi^x)$ and its specific form is determined by the geometry of the scalar manifold.

Determined in this manner by the geometry of the scalar manifold $\mathcal{M}$, the kinetic term of the bosonic part of $\mathcal{N} = 2$ supergravity coupled to matter will be given by

$$\mathcal{L}_{\mathcal{N}=2} \sim R + g_{ij}\partial_\mu v^i \partial^\mu v^j + N_{IJ} F^I_{\mu\nu} F^{\mu\nu J} . \tag{6.2.3}$$

In order to complete the construction of the theory, one must now introduce a gauge group. This gauging procedure will result in a replacement of the partial derivatives with gauge-covariant ones and in the appearance of a scalar potential term $V(\mathcal{M})$.

In order to understand the gauging procedure, we have to first introduce some general background. As we have seen above, the kinetic terms of the bosonic sector of the theory are given by a non-linear sigma-model on target space $\mathcal{M}$. In particular, the metric is a non-trivial function of the scalars $g_{ij} = g_{ij}(\vec{v})$. An isometry of this metric will be a field reparametrization $\vec{v} \to F(\vec{v})$ which leaves unchanged the functional form of g_{ij}. A Killing vector is defined as the infinitesimal generator of this isometry. The Killing symmetry associated with a Killing vector is an infinitesimal symmetry δ_θ, where θ is the infinitesimal transformation parameter. Then, a Killing vector $K^i_\Lambda(\vec{v})$ can be defined by

$$\delta_\theta v^i = \theta^\Lambda K^i_\Lambda(\vec{v}) \,, \tag{6.2.4}$$

where the index Λ counts the number of isometries. Meanwhile, the isometry itself is reflected in the relation

$$\delta_\theta g_{ij}(\vec{v}) = 0 \,. \tag{6.2.5}$$

We now consider the case where the hypermultiplet scalar manifold $\mathcal{Q}$ is a *homogeneous space*. Formally speaking, a manifold $\mathbf{M}$ is said to be a homogenous space for a group $\mathcal{G}$ if the map

$$\mathcal{G} \times \mathbf{M} \ni (\mathbf{a}, x) \mapsto T_{\mathbf{a}}(x) \in \mathcal{M} \tag{6.2.6}$$

is a diffeomorphism and it acts transitively on $\mathbf{M}$, and where the $T_{\mathbf{a}}(x)$ denotes the action of $\mathcal{G}$ on $\mathbf{M}$. The isotropy group $\mathcal{I}_x$ is defined as the set of elements of $\mathcal{G}$ that leaves $x \in \mathbf{M}$ unchanged. Furthermore, there turns out to be a one-to-one mapping between the manifold $\mathbf{M}$ and the coset space $\mathcal{G}/\mathcal{I}$ [136]. The important result that we want to consider is that, in the case when our hyperscalar manifold $\mathcal{Q}$ is an homogeneous space of the type $\mathcal{G}/\mathcal{I}$, with $\mathcal{G}$ non-compact and $\mathcal{I}$ its maximally compact isotropy group, then the R-symmetry of the theory will be embedded in $\mathcal{I}$. The particular homogeneous space we will study is given by the coset

$$\frac{SU(2,1)}{U(2)} \,. \tag{6.2.7}$$

For the gauging of $\mathcal{N}=2$ supergravity we will choose to realize the Yang-Mills gauge group G_{YM} as a subgroup of the isometries of the scalar manifold $\mathcal{M}$. This gauging procedure then involves the introduction of at most n_V+1 Killing vectors acting infinitesimally on $\mathcal{M}$ as

$$\phi^x \;\to\; \phi^x + \theta^\Lambda K^x_\Lambda(\phi) \,, \tag{6.2.8}$$

$$q^u \;\to\; q^u + \theta^\Lambda K^u_\Lambda(q) \,. \tag{6.2.9}$$

Then, the gauging procedure modifies the covariant derivatives for the hyper-scalars and the vector multiplet scalars as

$$\begin{aligned} \partial_\mu q^u &\rightarrow D_\mu q^u = \partial_\mu q^u - c_{\rm YM} A_\mu^\Lambda K_\Lambda^u(q)\,, \quad (6.2.10)\\ \partial_\mu \phi^x &\rightarrow D_\mu \phi^x = \partial_\mu \phi^x - c_{\rm YM} A_\mu^\Lambda K_\Lambda^x(\phi)\,, \quad (6.2.11)\end{aligned}$$

while the gauge field strengths will be given by

$$F_{\mu\nu}^\Lambda = \partial_\mu A_\nu^\Lambda - \partial_\nu A_\mu^\Lambda + c_{\rm YM} f_{MN}^\Lambda A_\mu^M A_\nu^N\,, \quad (6.2.12)$$

where f_{MN}^λ are the gauge group structure constants. We must also mention that the gauging procedure also brings the introduction of certain "prepotentials" P_Λ, which are functionals acting on the scalar manifold. Briefly stated, the prepotentials are introduced because the special and quaternionic Kähler nature of the scalar manifolds allows the Killing equations to be solved in terms of them. They bear importance to our discussion because the scalar potential $V(q^u, \phi^x)$ is expressed mostly in terms of the prepotentials, as will be seen below.

The resulting bosonic sector of the $\mathcal{N}=2$ supergravity Lagrangian after the gauging procedure is given by [133]

$$\mathcal{L}_{\mathcal{N}=2} = R + \mathcal{L}_{\rm vectors} + \mathcal{L}_{\rm scalars} + \mathcal{L}_{\rm CS}\,, \quad (6.2.13)$$

where

$$\begin{aligned} \mathcal{L}_{\rm vectors} &= -\frac{1}{2} N_{IJ} F_{\mu\nu}^I F^{\mu\nu I}\,, \quad (6.2.14)\\ \mathcal{L}_{\rm scalars} &= -G_{xy} D_\mu \phi^x D^\mu \phi^y - H_{uv} D_\mu q^u D^\mu q^v - V(q,\phi)\,, \quad (6.2.15)\\ \mathcal{L}_{\rm CS} &= \frac{1}{3\sqrt{6}} C_{IJK} \epsilon^{\mu\nu\rho\sigma\tau} F_{\mu\nu}^I F_{\rho\sigma}^J A_\tau^K\,. \quad (6.2.16)\end{aligned}$$

The scalar potential term is given by

$$V(q,\phi) = -2c_R^2 P^r P^r + c_R^2 P_x^r P_y^r G^{xy} + \frac{3}{2} c_{\rm YM}^2 K^u K^v H_{uv}\,, \quad (6.2.17)$$

where the factors P^r, P_x^r are related directly to the prepotentials acting on the scalar manifold $\mathcal{M}$ and which here we only mention for the sake of exposition. For further details, see [133].

6.3 The “Universal Multiplet” $SU(2,1)/U(2)$.

Before we move further on, we must briefly consider the geometry of the coset space $\mathcal{H} = SU(2,1)/U(2)$, known as the *universal multiplet* [137]. Formally, it is a quaternionic manifold of real dimension four with the topology of the open ball $\mathbb{C}^2$ [138], whose metric can be written in a simple fashion by introducing the complex coordinates ζ_1, ζ_2

$$\frac{1}{2}ds^2 = \frac{d\zeta_1\overline{d\zeta_1} + d\zeta_2\overline{d\zeta_2}}{\left(1 - |\zeta_1|^2 - |\zeta_2|^2\right)} + \frac{(\bar{\zeta}_1 d\zeta_1 + \bar{\zeta}_2 d\zeta_2)(\zeta_1\overline{d\zeta_1} + \zeta_2\overline{d\zeta_2})}{\left(1 - |\zeta_1|^2 - |\zeta_2|^2\right)^2}\,, \tag{6.3.1}$$

where $|\zeta_1|^2 + |\zeta_2|^2 < 1$. The metric (6.3.1) is a Kähler metric with an associated Kähler potential

$$K = -\log\left(1 - |\zeta_1|^2 - |\zeta_2|^2\right)\,. \tag{6.3.2}$$

The isotropy group of the manifold $\mathcal{H}$ is given by $U(2) = SU(2)_R \times U(1)$, which has four isometries and four Killing vectors associated to them. In order to complete the gauging procedure, we must know these Killing vectors. From the Kähler metric and potential we see that the combination $|\zeta_1|^2 + |\zeta_2|^2$ is invariant under two independent $U(1)$ symmetries

$$\zeta_1 \to e^{i\theta_1}\zeta_1\,, \qquad \zeta_2 \to e^{i\theta_2}\zeta_2\,, \tag{6.3.3}$$

where both transformations represent independent $U(1) \cong SO(2)$ rotations of the complex planes ζ_1 and ζ_2. The remaining isometries involve rotations of the coordinates $\{\zeta_1, \zeta_2\}$ as a doublet of $SU(2)$. This is becomes apparent after looking at the functional form in the metric terms

$$d\zeta_1\overline{d\zeta_1} + d\zeta_2\overline{d\zeta_2}\,, \qquad \overline{\zeta_1}d\zeta_1 + \overline{\zeta_2}d\zeta_2\,, \qquad \zeta_1\overline{d\zeta_1} + \zeta_2\overline{d\zeta_2}\,. \tag{6.3.4}$$

A convenient choice for this third isometry is given by

$$\begin{pmatrix} \zeta_1 \\ \overline{\zeta_2} \end{pmatrix} \to \mathcal{R}(\theta_3) \begin{pmatrix} \zeta_1 \\ \overline{\zeta_2} \end{pmatrix}\,, \tag{6.3.5}$$

where the rotation matrix $\mathcal{R}(\theta_3)$ is given by

$$\mathcal{R}(\theta_3) = \begin{pmatrix} \cos\theta_3 & \sin\theta_3 \\ -\sin\theta_3 & \cos\theta_3 \end{pmatrix} . \tag{6.3.6}$$

Finally, the last isometry of $U(2)$ is the complex conjugate of (6.3.5), which is independent on $\mathbb{C}^2$.

The four isometries described above are generated by the Killing vectors

$$\begin{aligned} H_1 &= \zeta_1\partial_{\zeta_1} - \overline{\zeta}_1\overline{\partial}_{\zeta_1} \quad & H_2 &= \zeta_2\partial_{\zeta_2} - \overline{\zeta}_2\overline{\partial}_{\zeta_2} \\ L_1 &= \zeta_2\partial_{\zeta_1} - \overline{\zeta}_1\overline{\partial}_{\zeta_2} \quad & \overline{L}_1 &= \overline{\zeta}_2\overline{\partial}_{\zeta_1} - \zeta_1\partial_{\zeta_2} \,, \end{aligned} \tag{6.3.7}$$

which can be rearranged in terms of the two subalgebras $U(2) = SU(2) \times U(1)$

$$SU(2) \times U(1) = \begin{cases} F_1 = \frac{1}{2}\left(L_1 - \overline{L}_1\right) \,, \\ F_2 = \frac{1}{2i}\left(L_1 + \overline{L}_1\right) \,, \\ F_3 = \frac{1}{2}\left(H_2 - H_1\right) \,, \\ F_8 = \frac{\sqrt{3}}{2}\left(H_1 + H_2\right) \,. \end{cases} \tag{6.3.8}$$

The $SU(2)$ algebra is generated by (F_1, F_2, F_3) and satisfies the standard algebra $[F_i, F_j] = i\,\epsilon_{ijk}F_k$, with indexes running over $\{1, 2, 3\}$. Meanwhile, the $U(1)$ algebra is generated by F_8, which is orthogonal to the $SU(2)$ subgroup, $[F_8, F_i] = 0$. There are also additional non-linear and non-compact Killing vectors, which we will omit. For more details, see [138, 139]. We will associate the transformation parameters $\{\alpha_1, \alpha_2, \alpha_3\}$ with the $SU(2)$ subgroup and an α_8 parameter with the $U(1)$ transformation.

6.4 Obtaining a One-Parameter Family of Theories.

We can now finish the construction of $\mathcal{N} = 2$ gauged supergravity coupled to the universal multiplet. Since the gauge group we consider is abelian, then no other vectors but the graviphoton will be present. As said before, the gauging procedure requires the abelian gauge group to be embedded in the isotropy group $U(2)$ of the scalar manifold. This embedding is not unique, and can be parametrized by the Killing vector [139]

$$K(\zeta_1, \zeta_2) = -i\sqrt{6}\left(\alpha_3 F_3 + \frac{1}{\sqrt{3}}\alpha_8 F_8\right), \tag{6.4.1}$$

where we have used the $SU(2)$ invariance to set the parameters $\alpha_1 = \alpha_2 = 0$ without any loss of generality. The ratio α_8/α_3 specifies the direction of the Killing vector K inside the $U(2)$ isotropy group. We refer to this ratio as β and will use it to parametrize a continuous family of abelian $\mathcal{N} = 2$ supergravity coupled to the universal multiplet $\mathcal{H}$.

The Lagrangian of gauge $\mathcal{N} = 2$ supergravity has many contributions, given by (6.2.13). Looking at the gauge kinetic term (6.2.14), we note that the kinetic matrix in our case will be one-dimensional $N_{IJ} = N_{00}$, corresponding to the case where there is only the graviphoton present and no vector multiplets. This matrix is fixed by supersymmetry as $N_{00} = 1$. The standard kinetic term for the gauge boson can be realized by rescaling $A_0 \to A_0/\sqrt{2}$. Regarding the scalar fields, we will use the complex coordinates ζ_1, ζ_2, since it is useful to express the metric of manifold $\mathcal{H}$ in terms of holomorphic and anti-holomorphic indexes, as in (6.3.1), which we rewrite

$$h_{i\bar{j}} d\zeta_i d\bar{\zeta}_{\bar{j}} = \frac{d\zeta_1 d\bar{\zeta}_1 + d\zeta_2 d\bar{\zeta}_2}{\left(1 - |\zeta_1|^2 - |\zeta_2|^2\right)} + \frac{(\bar{\zeta}_1 d\zeta_1 + \bar{\zeta}_2 d\zeta_2)(\zeta_1 d\bar{\zeta}_1 + \zeta_2 d\bar{\zeta}_2)}{\left(1 - |\zeta_1|^2 - |\zeta_2|^2\right)^2}. \tag{6.4.2}$$

The gauge-covariant derivatives are obtained according to (6.2.8)-(6.2.9) and using the Killing vector (6.4.1), from where we obtain

$$D_\mu \zeta_1 = \partial_\mu \zeta_1 - i A_\mu \frac{\sqrt{3}}{2}(\beta + 1)\zeta_1, \tag{6.4.3}$$

$$D_\mu \zeta_2 = \partial_\mu \zeta_2 - i A_\mu \frac{\sqrt{3}}{2}(\beta - 1)\zeta_2, \tag{6.4.4}$$

Proceeding in this manner, the Lagrangian of the $SU(2,1)/U(2)$ model we are constructing is given by

$$\mathcal{L}_{\mathcal{H}} = R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - 2h_{i\bar{j}} D^\mu \zeta_i \overline{D_\mu \zeta_j} - P(\zeta_1, \zeta_2), \tag{6.4.5}$$

with a potential term

$$P(\zeta_1, \zeta_2) = -\frac{3}{2}\frac{8 - \mathbf{V}}{\left(1 - |\zeta_1|^2 - |\zeta_2|^2\right)^2}\,, \tag{6.4.6}$$

$$\begin{aligned} \mathbf{V} &= \left(11 - 2\beta + 3\beta^2\right)|\zeta_1|^2 + \left(11 + 2\beta + 3\beta^2\right)|\zeta_2|^2 \\ &\quad -2(\beta - 1)^2|\zeta_1|^4 - 2(\beta + 1)^2|\zeta_2|^4 - 4(\beta^2 + 2)|\zeta_1|^2|\zeta_2|^2\,. \end{aligned} \tag{6.4.7}$$

6.5 Holographic Superconductivity from the Hyperscalars

In the previous sections we have constructed family of $\mathcal{N} = 2$ abelian supergravities parametrized by a real number β that specifies the gauging direction in the $U(2)$ isotropy group of the scalar manifold $SU(2,1)/U(2)$. An important thing to notice is that the resulting gauged Lagrangian (6.4.5) has the minimal required setting needed to construct a $D = 5$ holographic superconductor as described in previous chapters, with field content consisting of a gauge field A_μ and two complex scalar fields (ζ_1, ζ_2) coming from the graviton multiplet and the hypermultiplet, respectively. In the special case when both scalar fields vanish $\zeta_1 = \zeta_2 = 0$, one obtains for the scalar potential $P(0,0) = -12$, and the whole system has an asymptotically unit-radius-AdS_5 Reissner-Nordström black hole solution. We will in the following study holographic superconducting behaviour arising from the dynamics of the two charged superscalars (ζ_1, ζ_2). To do it, we will consider particular values of β and study how the resulting $\mathcal{N} = 2$ supergravity theory at those values can be embedded in the context of Type IIB superstring. Also, for any value of β considered, the Reissner-Nordström solution mentioned above corresponds to the normal phase of the dual superconducting system.

The masses of the hyperscalars depend on the value of the parameter β. To see this, one can go to the AdS_5 vacuum and read the masses directly from the scalar potential by evaluating the Hessian matrix. The masses obtained in this

Table 6.1: Value of scalar masses m_i^2, charges r_i and dual-operator dimensions Δ, for different integer values of the parameter β. The charges are defined as $q_i = \sqrt{3}\, r_i/2$, with $r_i = \beta - (-1)^i$.

β	(m_1^2, m_2^2)	(r_1, r_2)	(Δ_1, Δ_2)
0	$\left(-\frac{15}{4}, -\frac{15}{4}\right)$	$(1, -1)$	$\left(\frac{3}{2}, \frac{5}{2}\right)$
1	$(-3, 0)$	$(2, 0)$	$(3, 4)$
2	$\left(\frac{9}{4}, \frac{33}{4}\right)$	$(3, 1)$	$\left(\frac{9}{2}, \frac{11}{2}\right)$
3	$(12, 21)$	$(4, 2)$	$(6, 7)$

manner are given by

$$m_1^2 = -\frac{3}{4}\,(1+\beta)\,(5-3\beta)\;, \qquad m_2^2 = -\frac{3}{4}\,(1-\beta)\,(5+3\beta)\;. \tag{6.5.1}$$

The Lagrangian (6.4.5) is symmetric under the transformation $\zeta_1 \leftrightarrow \zeta_2$ with $\beta \leftrightarrow -\beta$. Therefore, without any loss of generality we can take $\beta \geq 0$. The parameter is related holographically to the value of the dimension Δ of the dual field theory operator. In Table (6.1) we show the values of the scalar masses, charges and dual-operator dimensions for different integer values of the parameter β.

The equations of motions arising from our $SU(2,1)/U(2)$ Lagrangian (6.4.5) can be written in terms of the complex variables ζ_1, ζ_2 as

$$(\nabla_\mu - iq_1 A_\mu)\,(\nabla^\mu - iq_1 A^\mu)\,\zeta_1 + (\partial_\mu - iq_1 A_\mu \zeta_1)\,\chi^\mu - DV_1\zeta_1 = 0\,, \tag{6.5.2}$$

$$(\nabla_\mu - iq_2 A_\mu)\,(\nabla^\mu - iq_2 A^\mu)\,\zeta_2 + (\partial_\mu - iq_2 A_\mu \zeta_2)\,\chi^\mu - DV_2\zeta_2 = 0\,, \tag{6.5.3}$$

where q_1 and q_2 can be read directly from the gauge derivative definitions (6.4.3)-(6.4.3). The function χ_μ is defined as

$$\chi_\mu = 2\left(\frac{\bar{\zeta}_1 D_\mu \zeta_1 + \bar{\zeta}_2 D_\mu \zeta_2}{1 - |\zeta_1|^2 - |\zeta_2|^2}\right)\,, \tag{6.5.4}$$

and we define the quantities

$$DV_1 = -\frac{3}{4}(1+\gamma)\left(\frac{-5+3\beta+(7-\beta)|\zeta_1|^2+(3-\beta)|\zeta_2|^2}{1-|\zeta_1|^2-|\zeta_2|^2}\right), \qquad (6.5.5)$$

$$DV_2 = -\frac{3}{4}(1-\gamma)\left(\frac{-5-3\beta+(7+\beta)|\zeta_1|^2+(3+\beta)|\zeta_2|^2}{1-|\zeta_1|^2-|\zeta_2|^2}\right), \qquad (6.5.6)$$

where the symmetry under $\zeta_1 \leftrightarrow \zeta_2$ with $\beta \leftrightarrow -\beta$ is apparent. From (6.5.5) and (6.5.6) we can recover the masses m_1^2, m_2^2 given in (6.5.1).

In the following sections we shall study the superconducting phase of the system for different values of β. in order to solve the equations of motion we shall propose the standard metric and gauge ansatz

$$ds^2 = -e^{\chi(r)}f(r)dt^2 + \frac{dr^2}{f(r)} + d\vec{x}^2\,, \qquad A = \Phi(r)dt\,. \qquad (6.5.7)$$

The superconducting solutions for the equations of motion have the following asymptotic $r \to \infty$ behaviour

$$e^{\chi} f(r) = e^{-\chi_\infty}\left(r^2 - \frac{M}{r^2} + \cdots\right), \qquad (6.5.8)$$

$$\Phi(r) = \mu - \frac{\rho}{r^2} + \cdots \qquad (6.5.9)$$

$$\zeta_i = \frac{\mathcal{O}_i}{r^{\Delta_i}} + \cdots, \qquad (6.5.10)$$

where $i = 1, 2$ and Δ_1, Δ_2 are the dimensions of the dual condensing operators. We will choose to work in the canonical ensemble, so that the charge density of the dual theory is fixed as $\rho = 1$. The energy and entropy density of the field theory are given by

$$E = \frac{3M}{16\pi G}\,, \qquad s = \frac{r_h^3}{4G}\,, \qquad (6.5.11)$$

and the Helmholtz free energy of the system is given by

$$f = E - T\,s = \frac{1}{8\pi G}\left(\frac{3}{2}M - 2\pi r_h^3 T\right). \qquad (6.5.12)$$

In order to know which hairy solutions are thermodynamically preferred, we will in the following compute the free energy relative to the Reissner-Nordström solutions $\Delta f = f_{\text{Hairy}} - f_{\text{RN}}$.

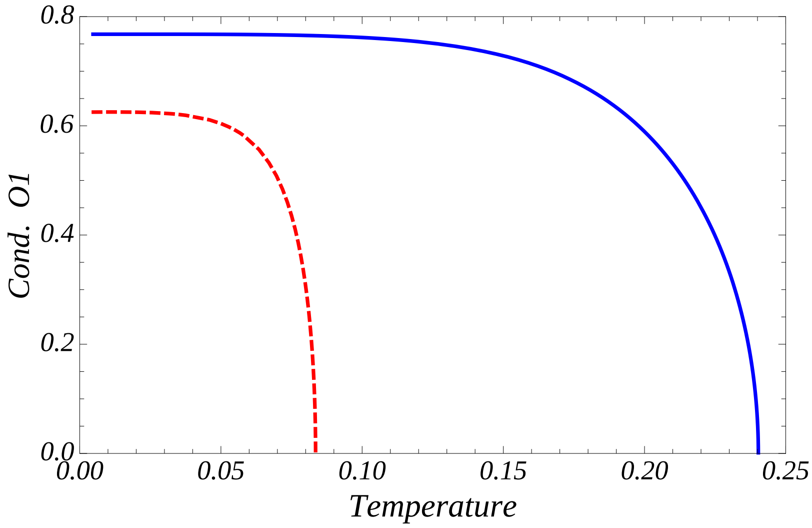


Figure 6.1: The condensates $\mathcal{O}$ for the operators of conformal dimensions $\Delta = 3/2$ ($\beta = 0$ case, solid blue line) and $\Delta = 3$ ($\beta = 1$ case, Truncation I, dashed red line), as a function of temperature.

6.6 The $\beta = 1$ Condensate

When one sets the free parameter as $\beta = 1$, the system is then characterized by a scalar potential term

$$P(\zeta_1, \zeta_2)\big|_{\beta=1} = -6\left(1 - |\zeta_2|^2\right) , \tag{6.6.1}$$

so that the DV_1, DV_2 terms in the general equations of motion become

$$DV_1 = -3\left(\frac{1 - 3|\zeta_1|^2 - 2|\zeta_2|^2}{\left(1 - |\zeta_1|^2 - |\zeta_2|^2\right)^2}\right) , \tag{6.6.2}$$

$$DV_2 = 0 . \tag{6.6.3}$$

Since $DV_2 = 0$, the field ζ_2 has trivial dynamics and we can focus on solutions for ζ_1. We define the real fields η and θ by

$$\zeta_1 \equiv e^{i\theta} \tanh\frac{\eta}{2} , \tag{6.6.4}$$

so that the Lagrangian (6.4.5) can be written as

$$\mathcal{L}_{\mathcal{N}=2}^{\beta=1} = -\frac{1}{2}\left[(\partial\eta)^2 + \sinh^2\eta\left(\partial\theta - \sqrt{3}A\right)^2\right] - 3\cosh^2\frac{\eta}{2}\,(5 - \cosh\eta) . \tag{6.6.5}$$

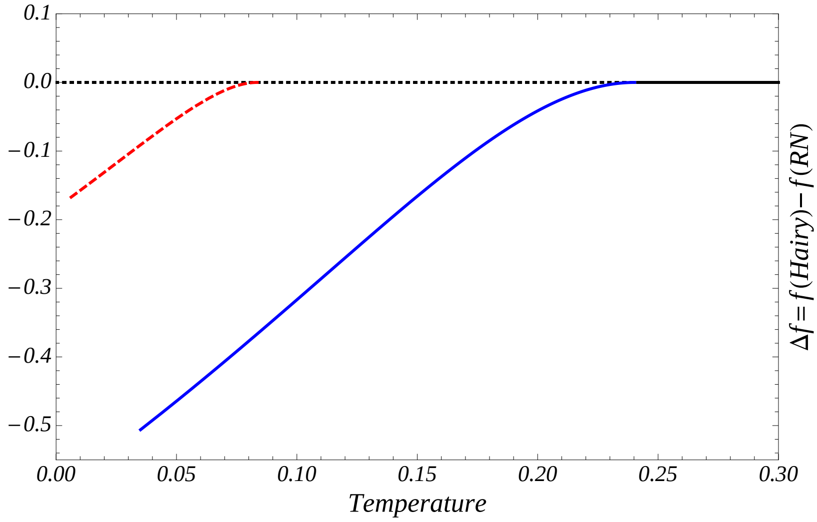


Figure 6.2: The free energy relative to the Reissner-Nordström solution for the operators of conformal dimensions $\Delta = 3/2$ ($\beta = 0$ case, solid blue line) and $\Delta = 3$ ($\beta = 1$ case, Truncation I, dashed red line), as a function of temperature.

From (6.6.5) we can read the asymptotic $r \to \infty$ behaviour of η

$$\eta(r) \approx \frac{\mathcal{O}_1}{r} + \frac{\mathcal{O}_3}{r^3} + \cdots . \tag{6.6.6}$$

In order to have spontaneous symmetry breaking, we will set $\mathcal{O}_1 = 0$, so that the dual condensate will have dimension $\Delta = 3$. The behaviour of the condensate as a function of temperature can be seen in figure (6.1) (red dashed line), where we find that system undergoes a second-order phase transition at $T_c \approx 0.083$. This can be seen in figure (6.2) (red dashed line), where we plot the free energy as function of temperature and we find no discontinuity at the critical temperature.

6.7 The $\beta = 0$ Condensate

The $\beta = 0$ case is characterized by the scalar potential

$$P(\zeta_1, \zeta_2) = -\frac{3}{2}\left(\frac{8 - 11(|\zeta_1|^2 + |\zeta_2|^2) + 2(|\zeta_1|^2 + |\zeta_2|^2) + 6\,|\zeta_1|^2\,|\zeta_2|^2}{\left(1 - |\zeta_1|^2 - |\zeta_2|^2\right)^2}\right), \tag{6.7.1}$$

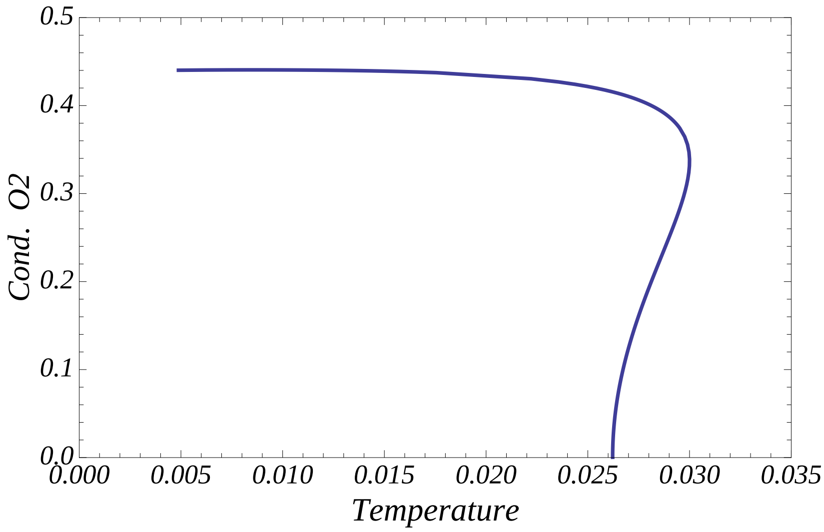


Figure 6.3: The condensates $\mathcal{O}$ for the operator of conformal dimension $\Delta = 5/2$ ($\beta = 0$ case, Truncation I) as a function of temperature.

which leads to the terms

$$DV_1 = -\frac{3}{4}\left(\frac{5 - 7\,|\zeta_1|^2 - 3\,|\zeta_2|^2}{1 - |\zeta_1|^2 - |\zeta_2|^2}\right), \tag{6.7.2}$$

$$DV_2 = DV_1\,. \tag{6.7.3}$$

As one can see, the scalar fields in the $\beta = 0$ case have the same masses $m_1^2 = m_2^2 = -15/4$ and also the same interaction terms $DV_1 = DV_2$. In order to have a single charge scalar field, one can consider the following distinct truncations

$$\textbf{Truncation I} \qquad \zeta_1 = e^{i\theta}\tanh\frac{\eta}{2}\,, \qquad \zeta_2 = 0\,, \tag{6.7.4}$$

$$\textbf{Truncation II} \qquad \zeta_1 = \zeta_2 \equiv \zeta\,, \qquad \zeta \equiv \frac{1}{\sqrt{2}}e^{i\theta}\tanh\frac{\eta}{2}\,. \tag{6.7.5}$$

Each truncation results in a different form for the Lagrangian, which can be written as

$$\mathcal{L}^{\beta=0}_{\mathcal{N}=4} = -\frac{1}{2}(\partial\eta)^2 - \frac{1}{2}J^i(\eta)A^2 - V^i(\eta)\,, \tag{6.7.6}$$

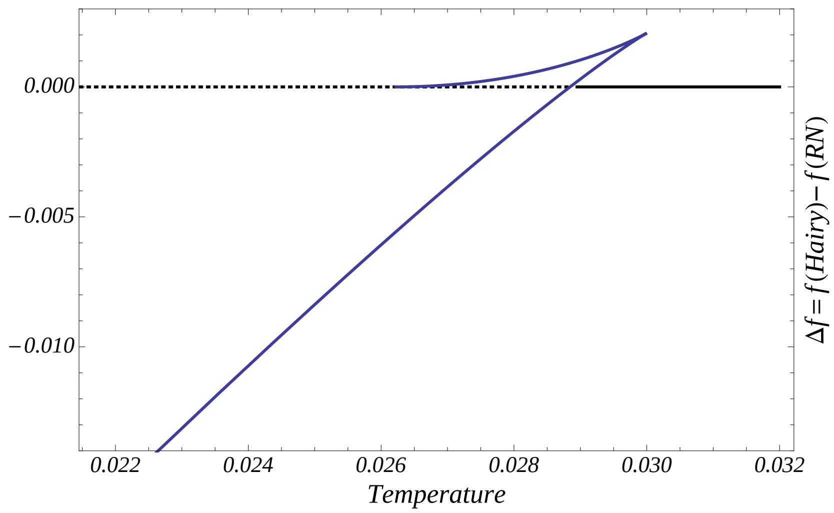


Figure 6.4: The free energy relative to the Reissner-Nordström solution for the operator of conformal dimension $\Delta = 5/2$ ($\beta = 0$ case, Truncation I) as a function of temperature.

where the structure function J^i and potential V^i characterize each different truncation. In the case of truncation (6.7.4) one has

$$\textbf{Truncation I} \qquad \begin{cases} J^{\rm I} = \frac{3}{4}\sinh^2\eta\,, \\ V^{\rm I} = \frac{3}{8}\left(\cosh^2\eta - 12\cosh\eta - 21\right)\,, \end{cases} \tag{6.7.7}$$

while for the second truncation (6.7.5) one obtains

$$\textbf{Truncation II} \qquad \begin{cases} J^{\rm II} = 3\sinh^2\frac{\eta}{2}\,, \\ V^{\rm II} = \frac{3}{2}\left(3 + 5\cosh^2\frac{\eta}{2}\right)\,. \end{cases} \tag{6.7.8}$$

In each of these cases the asymptotic $r \to \infty$ behaviour of the field η is the same

$$\eta(r) \approx \frac{\mathcal{O}_1}{r^{3/2}} + \frac{\mathcal{O}_2}{r^{5/2}} + \cdots\,. \tag{6.7.9}$$

The value of the mass $m^2 = -15/4$ allows us to consider both the standard and alternative quantization schemes. Therefore, we have the possibility to consider two distinct condensation cases within each truncation: one by setting $\mathcal{O}_1 = 0$, so that the condensate has dimension $\Delta = 5/2$, and the other by setting $\mathcal{O}_2$ in which case one has a condensate of dimension $\Delta = 3/2$ [14]. However, as it turns out after performing numerical computations, in both quantization cases of Truncation II (6.7.8) one obtains a *retrograde condensate* [21, 140]. This corresponds to a black hole solution where a non-trivial profile for the scalar field appears for temperatures *above* a critical temperature, instead of below as in the case of usual condensation. Furthermore, in our particular case, this solution has a free energy than is higher that that of the Reissner-Nordström black hole solution. Therefore this case is of not thermodynamically favored and is of no physical interest.

Therefore, we can focus only on the condensates corresponding to the Truncation I (6.7.7). The condensate of dimension $\Delta = 3/2$ as a function of temperature can be seen in figure (6.1) (solid blue line). Just as in the case of the $\beta = 1$ condensate, the system undergoes a second-order phase transition at $T_c \approx 0.24$ as can be seen in figure (6.2) (solid blue line), where we plot the free energy as function of temperature and we find no discontinuity at the critical temperature. In figure (6.3) we can see the $\Delta = 5/2$ condensate as a function of temperature. In contrast with all of the preceding cases, the system here undergoes a first-order phase transition at $T_c \approx 0.029$. In this case the critical temperature T_c is defined by the temperature at which the free energy becomes lower than that of the Reissner-Nordström black hole solution. This can be seen in figure (6.4), where the free energy of the system is plotted as a function of temperature. The phase transition is discontinuous because the condensate has a jump at T_c from zero to a non-zero value. The general picture of the $\beta = 0$ model is strikingly similar to the analogous $4D$ model of [141].

Comparing both $\beta = 0$ and $\beta = 1$ models, we note that the $\Delta = 3/2$ ($\beta = 0$) operator condenses at higher temperatures than the dimension $\Delta = 3$ operator which is dual to the $m^2 = -3$ scalar in the $\beta = 1$ model. Additionally, from making a comparison between the free energies of these particular operators, we conclude that the thermodynamics is dominated by the phase in which the $\Delta = 3/2$ operator condenses for $T < T_c$, as this phase has the lowest free energy.

6.8 Embedding of the Theories

In this section we outline how both the $\beta = 0, 1$ $\mathcal{N} = 2$ supergravity theories studied here can be embedded in Type IIB superstring theory. The mathematical details behind these embeddings are quite involved, so for simplicity we will limit ourselves to present only the general aspects of the construction. For a more detailed account, see [20, 25, 28].

6.8.1 The $\beta = 1$ Embedding: Sasaki-Einstein Compactification

This $\mathcal{N} = 2$ supergravity theory can be directly embedded in Type IIB theory. We first begin by generally describing the embedding of the field ζ_1 [20]. In order to do this, we start by decomposing Type IIB theory fields according to possible deformations of a Sasaki-Einstein manifold. Briefly stated, a Sasaki-Einstein manifold Y can be seen as a $U(1)$ fibration over a base Kähler-Einstein manifold B

$$ds_Y^2 = ds_B^2 + \xi \otimes \xi \,, \tag{6.8.1}$$

where ξ is a globally defined 1-form. An Einstein-Sasaki manifold is characterized by three globally defined 2-forms $\{J^1, J^2, J^3\}$ that satisfy the following conditions

$$J^a \wedge J^b = 2\delta^{ab}\,\mathrm{Vol}(B) \,, \qquad i_\xi(J^a) = 0 \,, \tag{6.8.2}$$

$$d\Omega_2 = 3i\,\xi \wedge \Omega_2 \,, \qquad d\xi = 2\omega \,, \tag{6.8.3}$$

where we have defined $J^1 = \omega$ and $\Omega_2 = J^2 + iJ^3$, and where Vol(B) is the volume of the base manifold. Thus, a Sasaki-Einstein deformation in the ten-dimensional metric in the Einstein frame can be written as

$$ds^2 = G_{MN}dx^M dx^M = e^{\frac{4}{3}(4U+V)}ds_5^2 + e^{2U}ds_B^2 + e^{2V}\left(\xi + A\right) \otimes \left(\xi + A\right) \,, \tag{6.8.4}$$

where $U(x)$, $V(x)$ are scalars fields and $A(x)$ is a 1-form. The coordinates x^μ, $\mu = 0, \dots 4$ are coordinates for ds_5^2, while y^m, $m = 5, \dots, 9$ are coordinates for the Sasaki-Einstein manifold.

The next step is to realize a Kaluza-Klein compactification of the Type IIB theory 2-forms B_2, C_2 and the self-dual 4-form C_4. This reduction is achieved by performing an expansion around the structure forms ξ, ω and Ω_2 [131]. The KK compactification will in fact introduce a five-dimensional gauge transformation, which is induced by reparametrizations of the fiber coordinate. A simple example of how this happens can be seen in a toroidal compactification, where the metric can be written as

$$ds^2 = G_{MN} dx^M dx^N = G_{\mu\nu} dx^\mu dx^\nu + e^{2V} \left(d\xi + A_\sigma dx^\sigma \right)^2 \,, \tag{6.8.5}$$

and where the compact dimension is $x^9 = \xi$. As usual, the Kaluza-Klein compactification will introduce a vector $A_\sigma(x) \sim G_{\sigma 9}$ and a scalar $V(x) \sim G_{99}$. Then, reparametrizations of the form $\xi \to \xi + \Lambda(x)$ will induce a transformation $A_\sigma \to A_\sigma + \partial_\sigma \Lambda(x)$, signifying that A_σ is indeed a gauge field.

Having this in mind, in the case of Sasaki-Einstein compactifications, since we want our reduction ansatz to be gauge invariant we must then look at how the fields transform under reparametrizations. To do so, we consider the Lie derivative along the Killing vector of the fiber isometry. We denote such Killing vector as $K = k(x)\partial/\partial\xi$, so that the Lie derivative will be given by $L_K = i_K \cdot d + d \cdot i_K$. Acting on the structure 2-forms one has

$$L_K \omega = 0 \,, \qquad L_K \Omega_2 = 3ik(x)\Omega_2 \,, \tag{6.8.6}$$

so that we conclude that Ω_2 will not be gauge invariant. Thus, the five-dimensional harmonics of Ω_2 that will appear in the expansion of the Type IIB forms B_2, C_2 and C_4 will be charged under the $U(1)$ field. The expansion of the ten-dimensional fields has to be realized in terms of $\xi + A$, instead of only ξ [25]. This way, the ansatz for the Type IIB fields needed for the ζ_1 truncation is given by

$$e^{2V} = \cosh\frac{\eta}{2}\,, \qquad e^{-2U} = \cosh\frac{\eta}{2}\,, \tag{6.8.7}$$

$$B_2 = \mathrm{Re}\left(b^\Omega \Omega_2\right)\,, \quad C_2 = \mathrm{Im}\left(c^\Omega \Omega_2\right)\,, \tag{6.8.8}$$

where we define

$$b^\Omega = e^{i\theta}\tanh^2\frac{\eta}{2}\,, \qquad c^\Omega = i\,b^\Omega\,, \tag{6.8.9}$$

and where we have chosen for the time being to set the axion field C_0 to zero, and the dilaton field ϕ to a constant. The self-dual five form F_5 is given by

$$\begin{aligned}
\mathcal{F} &= \cosh^2\frac{\eta}{2}\,(5-\cosh\eta)\,\mathrm{Vol}(ds_5^2) - (\star_5 dA)\wedge\omega \\
&\quad +\frac{1}{4}e^{8U}\sinh^2\eta\,(d\theta-3A)\wedge\omega\wedge\omega\,, && (6.8.10)\\
\star\mathcal{F} &= \frac{1}{2}e^{4U}\,(\cosh\eta-5)\,(\xi+A)\wedge\omega\wedge\omega + dA\wedge(\xi+A)\wedge A \\
&\quad +\frac{1}{2}\sinh^2\eta\,(\star\,(d\theta-3A))\wedge(\xi+A)\,, && (6.8.11)\\
F_5 &= \mathcal{F}+\star\mathcal{F}\,. && (6.8.12)
\end{aligned}$$

Finally, the resulting truncated action is the given by

$$\begin{aligned}
\mathcal{L}^{\zeta_1}_{IIB} &= R - \frac{3}{2}\,(\star_5 dA)\wedge dA + A\wedge dA\wedge dA \\
&\quad -\frac{1}{2}\left(d\eta^2 + \sinh^2\eta\,(d\theta-3A)^2 - 6\cosh^2\frac{\eta}{2}\,(5-\cosh\eta)\right)\,.
\end{aligned} \tag{6.8.13}$$

The truncated Type IIB Lagrangian (6.8.13) clearly coincides with the $\mathcal{N}=2$ Lagrangian (6.6.5). However, by setting the axion and dilaton fields to zero and constant values respectively in the ansatz (6.8.7)-(6.8.8), we have in fact retained only half of the hypermultiplet, i.e. the ζ_1 field. Indeed, the remaining half of the hypermultiplet, given by the field ζ_2, is a complex chargeless scalar with trivial dynamics $DV_2 = 0$ and dimension $\Delta = 4$. There is one candidate in Type IIB theory with these same characteristics, namely the axion-dilaton field $\tau = C_0 + i\,e^{-\phi}$. Therefore, in order to regain the complete $\beta = 1$ supergravity theory, we must give dynamics to C_0 and ϕ. To do this, one proposes the ansatz

$$c^\Omega = b^\Omega \tau = e^{\phi/2} e^{i\theta} \tanh \frac{\eta}{2} \,, \tag{6.8.14}$$

which results in

$$\begin{aligned} \mathcal{L}_{\mathcal{H}} - \mathcal{L}_{\text{IIB}}^{\zeta_1} &= -\frac{1}{2} \cosh^2 \frac{\eta}{2} \left(d\phi^2 + e^{2\phi} \cosh^2 \frac{\eta}{2} dC_0^2 \right) \\ & \quad -\frac{1}{2} e^{\phi} \sinh^2 \eta \, dC_0 \left(d\theta - 3A \right) . \end{aligned} \tag{6.8.15}$$

The matching of this Lagrangian to the full $\beta = 1$, $\mathcal{N} = 2$ supergravity Lagrangian (6.4.5) can be realized by making the following field definitions

$$\zeta_1 = e^{i\theta} \sqrt{1 - |\zeta_2|^2} \, \tanh \frac{\eta}{2} \sqrt{\frac{1 + i\,\overline{\tau}}{1 - i\,\tau}} \,, \qquad \zeta_2 = \frac{1 + i\,\tau}{1 - i\,\tau} \,. \tag{6.8.16}$$

We thus see how our $\mathcal{N} = 2$ supergravity describes the universal hypermultiplet belonging to the class (6.8.4) of Sasaki-Einstein compactifications. Furthermore, since we have not specified the base space of the Sasaki-Einstein manifold, the compactification we have performed contains in fact a large class of theories. This fact will be of importance when discussing the possible $\beta = 0$ embedding in the following subsection.

6.8.2 The $\beta = 0$ Embedding: Type IIB on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$

Let us now see how $\mathcal{N} = 2$ supergravity with the $\beta = 0$ gauging can be connected to Type IIB superstring theory. An important fact is that the spectrum of five-dimensional $\mathcal{N} = 8$ supergravity arising from Type IIB theory compactified on $\mathrm{AdS}_5 \times \mathrm{S}^5$ does not support Kaluza-Klein modes with the value of scalar field mass $m^2 = -15/4$ that the $\beta = 0$ truncations have. A possible solution to this is to consider Type IIB theory compactified on spaces with an internal manifold different from S^5 and whose KK spectrum matches our $\beta = 0$ model. As we will show, there is strong evidence that this can indeed be achieved. Following work by Klebanov and Witten [131] we review a gauge/gravity duality where gravitational theory has an internal manifold given by the homogeneous space

$$T^{1,1} = \frac{SU(2) \times SU(2)}{U(1)} \,. \tag{6.8.17}$$

Let us then very generally describe this space. Formally speaking, the space $T^{1,1}$ is an Einstein manifold with positive curvature and can be geometrically identified as the transverse space of the Calabi-Yau three-fold Y_6. The manifold Y_6 can be realized by the surface

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\,, \tag{6.8.18}$$

or equivalently by

$$z_1 z_2 - z_3 z_4 = 0\,, \tag{6.8.19}$$

where $\{z_1, z_2, z_3, z_4\}$ are coordinates on $\mathbb{CP}^4$. The space Y_6 has a conical singularity at the origin where the metric can be written as a cone over $T^{1,1}$

$$ds_Y^2 = dr^2 + r^2 ds_T^2\,, \tag{6.8.20}$$

where ds_T^2 is the metric of $T^{1,1}$. The gauge/duality duality is then established by studying a stack of N D3-branes located at the conical singularity, following a similar path to the original Maldacena derivation of the standard AdS/CFT duality. In this particular setup, supersymmetry is reduced from $\mathcal{N} = 4$ to $\mathcal{N} = 2$ and the dual field theory is given by a 4-dimensional $\mathcal{N} = 1$ superconformal field theory with gauge group $SU(N) \times SU(N)$. This field theory contains two chiral gauge superfields $W_{1\alpha}$, $W_{2\alpha}$ and the chiral superfields (A_1, A_2) and (B_1, B_2). The A and B fields can be seen as $N \times N$ matrices whose eigenvalues parametrize the position of the D3-branes. This field theory has a global symmetry group $SU(2)_A \times SU(2)_B$, under which the superfields transform as a doublet. This can be seen from equation (6.8.19), which is solved by making the substitution

$$z_1 \to A_1 B_1\,, \quad z_2 \to A_2 B_2\,, \quad z_3 \to A_1 B_2\,, \quad z_4 \to A_2 B_1\,. \tag{6.8.21}$$

In addition to all this, the $\mathcal{N} = 1$ field theory has a unique superpotential

$$W \sim \epsilon^{ij} \epsilon^{kl} \mathrm{Tr}\, A_i B_k A_j B_l\,. \tag{6.8.22}$$

In this way, there is a conjectured duality between a Type IIB superstring theory on $AdS_5 \times T^{1,1}$ with N units of Ramond-Ramond flux through $T^{1,1}$, and a four-dimensional $SU(N) \times SU(N)$, $\mathcal{N} = 1$ superconformal gauge theory perturbed by a superpotential (6.8.22). It is customary to refer to the dual field theory as *Klebanov-Witten superconformal theory.*

To see how our $\beta = 0$ $\mathcal{N} = 2$ theory can be related to Type IIB theory on $AdS_5 \times T^{1,1}$, we go back to Table (6.1) and point out the fact that, for integer values of β, the masses and charges of the $SU(2,1)/U(1)$ hypermultiplet are in *precise* correspondence with those of the chiral AdS multiplets of the Klebanov-Witten field theory, with specific global $SU(2) \times SU(2)$ quantum numbers (j, l). In particular, one has the following two Kaluza-Klein towers [142, 143]

- A Kaluza-Klein tower coming from the complex IIB zero and two-forms with quantum numbers satisfying $2j = 2l = \beta - 1$ with $\beta \geq 1$. This is dual to field theory operators

$$\mathcal{A}_{\beta-1} = \mathrm{Tr}\left[\left(W_1^2 + W_2^2\right)\left(A_k B_l\right)^{\beta-1}\right] + \cdots . \tag{6.8.23}$$

- A Kaluza-Klein tower originating from the IIB metric, four-form and complex four-form and has quantum number satisfying $2j = 2l = \beta + 1$ with $\beta \geq 0$. The corresponding dual field theory operators are of the form

$$\mathcal{B}_{\beta+1} = \mathrm{Tr}\left[\left(A_k B_l\right)^{\beta+1}\right] . \tag{6.8.24}$$

The resulting mass spectrum as a function of $2j = 2l$ is shown in figure (6.5). Regarding the dimension Δ of the dual operators of each KK tower $\mathcal{A}_{\beta-1}$ and $\mathcal{B}_{\beta+1}$, we note that the chiral superfields A and B have dimension $\Delta = 3/4$, whereas the chiral gauge superfields $W_{I\alpha}$ have also dimension $\Delta = 3/2$. This means that the $\mathcal{A}$ operators have dimension

$$\Delta\left(\mathcal{A}\right) = 3 + \frac{3}{2}\left(\beta - 1\right) , \tag{6.8.25}$$

so that the lowest state in this Kaluza-Klein tower has $\beta = 1$ and $\Delta = 3$. Likewise, the $\mathcal{B}$ operators have

$$\Delta\left(\mathcal{B}\right) = \frac{3}{2}\left(\beta + 1\right) , \tag{6.8.26}$$

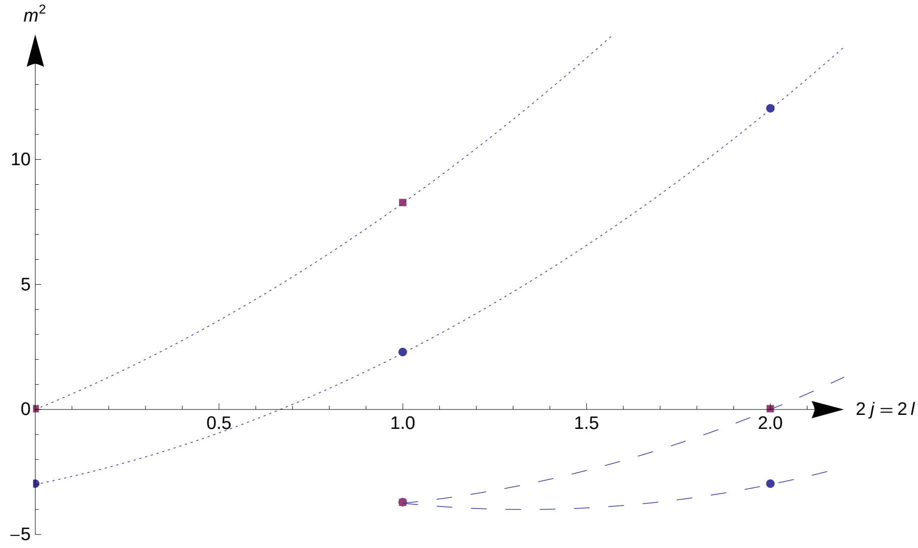


Figure 6.5: The two Kaluza-Klein towers of scalar fields on $T^{1,1}$. The dotted (dashed) line indicates hypers with $2j = \beta \mp 1$.

so that the lowest state in this Kaluza-Klein tower has $\beta = 0$ and $\Delta = 3/2$. This is the operator that dominates the thermodynamics in the superconducting phase of the system.

Comparing the mass spectrum and the KK tower dimensions of Klebanov-Witten theory described above with the values in Table (6.1) of our family of $\mathcal{N} = 2$ supergravities, we can observe the same relation between mass and dimension when taking integer values of β. To this exact correspondence we add the fact that the Type IIB Sasaki-Einstein embedding described for the $\beta = 1$ includes the Calabi-Yau three-fold Y_6. These facts leads one to claim that the lowest Kaluza-Klein state in the $\mathcal{A}$-tower corresponds to $SU(2,1)/U(2)$ hypermultiplet with $\beta = 1$. The $SU(2) \times SU(2)$ quantum numbers in this case are $j = l = 0$, which means that the hypermultiplet transforms as a singlet. On the other hand, the lowest Kaluza-Klein state in the $\mathcal{B}$-tower has $\beta = 0$ and quantum numbers $j = l = 1/2$. It is only natural to ask whether the $\beta = 0$ $\mathcal{N} = 2$ supergravity theory can be considered a consistent truncation within Type IIB theory compactified on $T^{1,1}$. A major obstruction to this is that, because of the non-trivial quantum numbers mentioned above, the hypermultiplet cannot be considered a singlet under $SU(2) \times SU(2)$ and the identification is not straightforward as in the $\beta = 1$ case.

7

Conclusions

In this thesis we have shown that the AdS/CFT correspondence provides a new way to study the superconducting phase of large-N theories in the strongly-coupled regime. When discussing the cuprates at the end of Chapter 1, we have already seen some of the shortcomings of the usual field theory approaches to strongly-coupled many body systems, perhaps the most serious of which being the break-down of the quasi-particle picture because of the strong interactions involved. As we have seen, the gauge/gravity duality allows to pose intractable many-body quantum problems in terms of the classical dynamics of a dual gravitational system in AdS space. Using this novel holographic point of view, the condensation of Cooper-pairs in the field theory side is translated to the spontaneous creation of charged hairy black hole solutions in the gravity side. This gives rise to a phase in the dual field theory where the core phenomenological aspects of superconductivity are recovered. It is striking that by only looking to the problem from an holographic point of view, these utterly intractable systems in the strong-coupling regime can be shown to present a superconducting phase. Given the success of the holographic method and the difficulties already mentioned about standard field theoretical approaches, it may not be too bold to imagine that indeed the natural theoretical definition of superconductivity in the strong coupling regime is given by the gravitational dual system.

With these general considerations in mind, in this thesis we have endeavored to present a complete picture of the different approaches followed in holographic superconductivity. We have thus presented working examples in both the bottom-up (Chapters 4 and 5) and top-down (Chapter 6) approaches. Let us now draw some conclusions from each of these chapters.

In Chapter 4, using as a starting point a family of minimal holographic superconducting models in $d = 4+1$ AdS spacetime, we have constructed a consistent Ginzburg-Landau phenomenological interpretation of the corresponding dual field theory. This was realized by making use of some non-trivial numerical identities related to the bulk-side fields, by identifying the scalar field relevant asymptotic mode with the Ginzburg-Landau theory order parameter and by making some sensible and simple physical assumptions. We have checked that our Ginzburg-Landau description consistently reproduces all expected properties in great detail.

By making a study of small fluctuations of the gauge and scalar fields in the bulk theory around the condensed solution, we were then able to compute the penetration length λ and coherence length ξ of the dual superconducting system. We found through these holographic computations that the characteristic lengths have the expected non-trivial functional dependency on temperature near T_c, in accord with the one observed in real-world superconductors. Next, we computed holographically the parameters α and β of the Ginzburg-Landau Lagrangian for the dual field theory, and found in both cases that their near-T_c functional dependence on temperature is in complete agreement with the behaviour predicted by standard Ginzburg-Landau theory. We then proceeded to compute the Ginzburg-Landau parameter κ for different values of the scalar field charge q. From this calculation we find that, as the value of q increases, the Ginzburg-Landau parameters approaches asymptotically the value

$\kappa \sim 0.55 < 1/\sqrt{2}$. From this we can conclude that the system will behave as a Type I superconductor for all values of q considered. Strikingly, we found that the temperature dependence of the Ginzburg-Landau parameter κ found in this paper can be modeled using the same kind of empirical fitting already used for the high-T_c superconducting material Nb_3Sn in [113]. This can be observed in the striking qualitative similarity between figures (4.13b) and (4.14).

We have also calculated the Helmholtz free energy density of the system using our Ginzburg-Landau approach, and compared it with the free energy computed with the standard holographic techniques. It was found that both approaches give mutually consistent results in the near-T_c region. Additionally, through calculations of the free energy of the system, the Ginzburg-Landau approach was compared with the method developed in [15] for calculating the parameters α and β. Both methods were shown to be in excellent agreement.

Next, we probed our system with a constant magnetic field B. This was done by using the black brane solution of [16] in $d = 4+1$ AdS up to order B^2. This is the first use of this solution in the context of holographic superconductivity. With this perturbative solution, we showed the formation of droplet condensate solutions in this fixed background and calculated the critical magnetic field above which the superconducting phase is broken. The field obtained in this fashion was compared with the critical magnetic field obtained in the Ginzburg-Landau approach. While both fields measure different aspects of the response of the system to a magnetic field, we found that near T_c both fields behave as $B_c \sim B_0(1 - T/T_c)$ and that their corresponding factors B_0 behave as $\sim 1/q^{1/3}$ (or equivalently as $\sim 1/T_c$) for large q. In conclusion, one of the main results of this chapter is to show that a very simple phenomenological model in $d = 4+1$ AdS spacetime allows for a consistent Ginzburg-Landau description of the boundary theory, where all the Ginzburg-Landau parameters and characteristic lengths can be calculated using holographic methods, and whose behaviour is in accordance to the one predicted by traditional mean field theory. Moreover, we also observe that, as the value of the scalar field charge q increases, the Ginzburg-Landau parameter of the model tends asymptotically to a well defined value that characterizes the dual superconducting system as Type I.

Chapter 5 is a natural continuation of the previous one. In this chapter we have chosen to study a $D = 5$ minimal model of holographic superconductivity in the probe limit, with a Lifshitz black hole background. Within this framework, we have studied different cases of condensation, varying within each of them the dynamical critical exponent in order to gain insight on how the system is affected by z with respect to its usual isotropic behavior. As in the previous chapter, we have added small scalar and gauge field fluctuations to the original component fields in order to compute holographically the penetration and coherence length of the superconducting system. We saw that both characteristic lengths have the standard near-T_c functional dependency on temperature for all condensate cases and all values of z. However, the dynamical critical exponent z does affect the value of the characteristic lengths, as it becomes evident in the change of the value of their ratio as given by the Ginzburg-Landau parameter κ. We also saw that it is possible to construct a consistent Ginzburg-Landau phenomenological interpretation of the dual theory with Lifshitz scaling. We computed through holographic techniques the Ginzburg-Landau Lagrangian parameters α and β and, as with the characteristic lengths, concluded that they have the standard near-T_c functional dependency on temperature for all condensate cases and all values of z. However, the presence of z does have a non-trivial effect on this phenomenological parameters, diminishing the value of their numerical coefficients as z raises.

We have also computed with holographic techniques the Ginzburg-Landau parameter κ of the system. For all case of condensation and all values of z, we saw that $\kappa < 1/\sqrt{2}$. This means that for all cases the dual system will behave as a Type I superconductor. Moreover, we also observed that, for each case of condensation considered, the value of κ became lower for higher values of z. This means that in systems with higher anisotropy, vortex formation is more strongly unfavored energetically and exhibit a stronger Type I behavior.

In addition to this, by making an holographic superfluid interpretation of our bulk system and using as a starting point previous research on the relation between the supercurrent density J_c and superfluid velocity v_x of the system [126], we used the Ginzburg-landau quantities obtained previously and computed the critical supercurrent J_c, which is the value of the supercurrent above which the system passes to its normal phase. The results obtained have a near-T_c temperature dependence which is in complete agreement with the one predicted by standard Ginzburg-Landau theory.

Finally, we computed the critical magnetic field B_c needed to break the superconducting phase of the system, following the perturbative procedure first developed in [17]. We observed that the critical field near-T_c functional dependence on temperature is the one predicted by Ginzburg-Landau theory. However, we also note that the value of the critical magnetic field is smaller for higher values of z. Additionally, within this perturbative approach, we have confirmed holographically the conjecture posed in [18] that the critical magnetic field is inversely proportional to the square of the correlation length, in accordance to Ginzburg-Landau theory.

The holographic computation of the Ginzburg-Landau parameter κ presented in these two chapters can serve as an useful probe to test the viability of an holographic superconducting model as a possible description of real world high-T_c superconductors. Indeed, all the cuprates so far discovered present a very Type II behaviour. Therefore, it would be a very desirable property of an holographic superconductor to have a value of κ in the Type II region. A similar thing can be said about the systems studied in Chapter 5, where we concluded that the systems with greater anisotropy will be have a stronger Type I behaviour. In this respect it is natural to ask how the Ginzburg-Landau parameter obtained in these chapters could change by the choice of other models such as, for instance, d-wave holographic superconductors [8, 9, 10], p-wave holographic superconductors [19], models with higher corrections to the scalar field potential such as the ones that appear in top-down approaches [20, 21, 14] or less conventional models such as ones with Chern-Simons terms, higher-derivative couplings or in the context of New Massive Gravity [22, 23, 24]. This calls for further research.

Let us now turn to the top-down system studied in Chapter 6. To summarize, we have explicitly constructed a Lagrangian for the five-dimensional $\mathcal{N} = 2$ gauged supergravity coupled to an $SU(2,1)/U(2)$ scalar hypermultiplet. The resulting model is uniquely determined by a single parameter β representing the mixing between the $U(1)$ generators of $SU(2)$ and $U(1)$. When $\beta = 1$, the system describes two complex scalars ζ_1, ζ_2, with masses $m_1^2 = -3$ and $m_2^2 = 0$. In this case, the resulting Lagrangian exactly coincides with the Lagrangian of [20], with the extension that incorporates the complex dilaton found in [25, 26, 27, 28]. This match involves a non-trivial scalar potential and non-trivial couplings, and should not come as a surprise as there is no other possible model for an $SU(2,1)/U(2)$ hypermultiplet with such masses.

Similarly, the same uniqueness property of the Lagrangian strongly indicates that the model with $\beta = 0$ indeed must describe the two complex scalar fields of masses $m^2 = -15/4$ which are dual to the operator of lowest dimension $\Delta = 3/2$ in the Klebanov-Witten superconformal theory. We have explicitly demonstrated that this mode dominates the thermodynamics at low temperatures. It would be extremely interesting to see if the $\beta = 0$ model represents a consistent truncation of Type IIB supergravity. While the scalar fields have non-trivial Kaluza-Klein quantum numbers $(1/2, 1/2)$, they are however the lowest states in the KK spectrum, which suggests that the truncation might nevertheless be consistent. Proving the latter may require an explicit construction of a Type IIB ansatz that reproduces the same equations of motion.

In presenting both approaches in contrast, we can compare their strong and weak spots more clearly. For instance, by having the complete freedom to define our bulk model and the freedom to choose the value of the input parameters (scalar charge q in Chapter 5 and dynamical exponent z and scalar field mass M^2 in Chapter 6), we have been able to probe the general phenomenological behaviour of the relevant physical quantities in the condensed phase of the dual field theory. In particular, in Chapter 5 we have seen that the Ginzburg-Landau parameter κ reaches an asymptotic upper constant value as the value of the scalar charge increases. Therefore, in looking for possible holographic candidates to model real world high-T_c superconductors, we could expect that a system with a large scalar charge will be more likely to show Type II behaviour. In a similar fashion, from Chapter 5 we can conclude that a larger degree of anisotropy will inhibit Type II behaviour. The main downside to this bottom-up approach is that it is not clear the manner in which any of these phenomenological simple models could be obtained from a consistent truncation of Type IIB theory, and how they could provide insight on the microscopical details behind the superconducting phase of the dual theory. Of course, one can always rightfully say that these systems are not designed to have these properties, but rather to give us practical knowledge on the phenomenology of the dual superconducting systems and its working relation with the bulk models from where they arise.

On the other hand, in the top-down holographic superconductor described in Chapter 6, by considering a bulk system with a different internal manifold, we were able to construct a gravity model whose holographic dual is nothing but the Klebanov-Witten superconformal field theory. Moreover, we found through holographic techniques that the very specific K-W operator $\mathrm{Tr}\,(A_k B_l)$ (the one with the lowest dimension) condenses and dominates the thermodynamics of the system, thus showing that Klebanov-Witten has a well defined superconducting phase.

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