UNIVERSIDAD DE MURCIA

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Going further in the L_p-Brunn-Minkowski theory:
a p-difference of convex bodies

(Ampliando la teoría L_p de Brunn-Minkowski:
una p-diferencia de cuerpos convexos)

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Llegado este punto es inevitable mirar atrás y acordarse de todas aquellas personas que de un modo u otro han contribuido a la realización de este trabajo.

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Las palabras mágicas son: amor, humor y libertad.

(Juan Tamariz)

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Introducción/Preface

La teoría de Brunn-Minkowski puede considerarse como el núcleo de la Geometría de Cuerpos Convexos. Sus orígenes se remontan a finales del siglo XIX y comienzos del siglo XX, principalmente con los trabajos de Cauchy, Steiner, Brunn y, en particular, Minkowski.

La teoría de Brunn-Minkowski es el resultado de combinar dos conceptos elementales para conjuntos en el espacio euclídeo: la *suma de Minkowski* $+$ (suma vectorial), y el *volumen* $\mathrm{vol}(\cdot)$ (medida de Lebesgue). Esto conduce a la noción de volúmenes mixtos y a la famosa y fundamental desigualdad de Brunn-Minkowski. Un hecho crucial, y en cierto modo sorprendente, en este contexto, fue descubierto por Steiner en 1840 en el caso particular de la bola euclídea: el volumen de la suma de Minkowski de un cuerpo convexo K y un dilatado positivo λE de un cuerpo *gauge* E (lo que se conoce como *cuerpo paralelo exterior relativo de* K *a distancia* λ), viene dado por un polinomio de grado a lo sumo n en la variable λ, la *fórmula de Steiner relativa.* De forma precisa,

$$\mathrm{vol}(K+\lambda E)=\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)\lambda^i. \tag{$\dagger$}$$

Sus coeficientes son, salvo constantes, medidas particulares asociadas a K y a E, las llamadas *quermassintegrales relativas.* El volumen $\mathrm{W}_0(K;E)=\mathrm{vol}(K)$ es un caso particular de tales medidas.

Resulta natural considerar una operación "opuesta" a la suma de Minkowski de cuerpos convexos, la denominada *diferencia de Minkowski*, $\sim$:

$$K\sim E=\{x\in\mathbb{R}^n : x+E\subseteq K\}.$$

Así se pueden definir los *cuerpos paralelos interiores* de K respecto a E, que, junto con los cuerpos paralelos exteriores considerados anteriormente, conducen a la definición de *sistema completo de cuerpos paralelos*:

$$K_\lambda=\begin{cases} K\sim|\lambda|E & \text{si} \quad -\mathrm{r}(K;E)\le\lambda\le 0,\\ K+\lambda E & \text{si} \quad 0\le\lambda<\infty,\end{cases}$$

donde $\mathrm{r}(K;E)=\max\{r\ge 0 : \exists\, x\in\mathbb{R}^n \text{ con } x+rE\subseteq K\}$ es el *inradio relativo* de K respecto a E. La fórmula de Steiner ($\dagger$) permite obtener fácilmente la *fórmula de Minkowski*, la cual establece que la derivada por la derecha en 0 del funcional volumen $\mathrm{vol}(\lambda)=\mathrm{vol}(K_\lambda)$ es n veces la primera quermassintegral relativa:

$$n\mathrm{W}_1(K;E)=\lim_{\lambda\to 0^+}\frac{\mathrm{vol}(K_\lambda)-\mathrm{vol}(K)}{\lambda}.$$

A mediados del siglo XX, Bol, Hadwiger y Matheron demostraron la derivabilidad de $\mathrm{vol}(\lambda)$ en todo el intervalo $\bigl(-\mathrm{r}(K;E),\infty\bigr)$, proporcionando la fórmula explícita

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{vol}(\lambda) = n\mathrm{W}_1(K_\lambda;E). \tag{$\spadesuit$}$$

Recientemente, Hernández Cifre y Saorín Gómez han estudiado la derivabilidad de las quermassintegrales relativas $\mathrm{W}_i(K_\lambda;E)$ como funciones del parámetro $\lambda\in\bigl(-\mathrm{r}(K;E),\infty\bigr)$. En particular, gracias a la conocida expresión polinómica en $\lambda\geq 0$ para $\mathrm{W}_i(K_\lambda;E)$, se obtienen la derivabilidad y el valor explícito de la derivada, cuando $\lambda\geq 0$: para todo $i=0,\dots,n-1$,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{W}_i(K_\lambda;E) = (n-i)\mathrm{W}_{i+1}(K_\lambda;E). \tag{$\clubsuit$}$$

Además, dado un cuerpo convexo K, su *función soporte* $h(K,\cdot):\mathbb{S}^{n-1}\longrightarrow\mathbb{R}$, definida sobre la esfera unidad $\mathbb{S}^{n-1}$, es $h(K,u)=\sup\bigl\{\langle x,u\rangle : x\in K\bigr\}$, donde $\langle\cdot,\cdot\rangle$ denota el producto escalar usual en $\mathbb{R}^n$. Así, tiene sentido estudiar la derivabilidad de la función soporte $h(K_\lambda,u)$ con respecto al parámetro $\lambda\in\bigl(-\mathrm{r}(K;E),\infty\bigr)$. *Heurísticamente*, esta derivada juega el papel de la *velocidad de crecimiento* de los cuerpos paralelos K_λ con respecto a λ. En 1979, Chakerian y Sangwine-Yager demostraron que una cota inferior para la derivada de $h(K_\lambda,u)$ es siempre la función soporte de E,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}h(K_\lambda,u)\geq h(E,u), \tag{$\heartsuit$}$$

siempre que tal derivada exista, probando que la igualdad se da para todo $u\in\mathbb{S}^{n-1}$, y para casi todo punto en $\bigl[-\mathrm{r}(K;E),0\bigr]$, si, y sólo si, $K=K_{-\mathrm{r}(K;E)}+\mathrm{r}(K;E)E$.

Por otro lado, el hecho de que un cuerpo convexo está determinado de forma única por su función soporte, permitió a Firey introducir, en 1962, la p-suma de dos cuerpos convexos K,E conteniendo el origen, para $1\leq p\leq\infty$, como el cuerpo convexo cuya función soporte es

$$h(K+_p E,u)=\bigl(h(K,u)^p+h(E,u)^p\bigr)^{1/p}.$$

El caso $p=1$ es la suma de Minkowski, mientras que si $p=\infty$ se obtiene la envoltura convexa de los conjuntos, $K+_\infty E=\mathrm{conv}(K\cup E)$. Esta p-suma de cuerpos convexos fue el punto de partida de la hoy en día conocida como teoría L_p de Brunn-Minkowski (o teoría de Firey-Brunn-Minkowski).

Considerando el reciente y profundo desarrollo de la teoría L_p de Brunn-Minkowski, es en cierto modo sorprendente que no se haya definido hasta ahora un p-concepto equivalente de diferencia de cuerpos convexos. Esta cuestión fue la motivación inicial de nuestro trabajo, y se podría decir que la memoria está dedicada a introducir una nueva operación "opuesta" a la p-suma, que llamaremos *p-diferencia*, de cuerpos convexos, y a investigar sus propiedades. Además, estudiamos la derivabilidad de las quermassintegrales relativas del correspondiente sistema completo de cuerpos p-paralelos, respecto al parámetro de definición. En particular, obtenemos que, como en el caso clásico, el volumen es siempre derivable, dando la expresión explícita de su derivada. A continuación vamos a describir el contenido específico de cada uno de los cuatro capítulos en los que se ha estructurado este trabajo.

La memoria comienza con un *primer capítulo* introductorio, en el que se establece la notación a seguir y se presentan brevemente los conceptos y resultados que serán fundamentales en el posterior desarrollo de los contenidos, tanto de convexidad, en general, como de teoría (L_p) de Brunn Minkowski, en particular. Así, en una primera sección, se recuerdan las nociones básicas, como la de suma de Minkowski, cuerpo convexo, función soporte, hiperplano soporte, etc. A continuación, se introduce la diferencia de Minkowski, estableciendo las relaciones elementales que tiene con la suma de Minkowski. En este contexto, consideramos también la noción de *Wulff-shape* asociada a un subconjunto cerrado $\Omega \subseteq \mathbb{S}^{n-1}$ de la esfera y a una función no-negativa $\psi : \Omega \longrightarrow [0, \infty)$, señalando algunas propiedades importantes que se usarán más adelante. La suma y la diferencia de Minkowski conducen a los conceptos de inradio relativo y sistema completo de cuerpos paralelos. Una propiedad clave de esta familia de conjuntos es la concavidad. También recordamos la noción de cuerpo tangencial y el importante resultado de Schneider que caracteriza estos conjuntos como los únicos cuerpos convexos cuyos cuerpos paralelos interiores son copias homotéticas de ellos. Proseguimos el capítulo con la teoría L_p de Brunn-Minkowski, estudiando la noción y principales propiedades de la p-suma de cuerpos convexos (que contienen al origen). Finalmente, tras introducir los volúmenes mixtos y las quermassintegrales, establecemos varias desigualdades conocidas de la teoría (L_p) de Brunn-Minkowski, las cuales jugarán un papel importante en el desarrollo de esta memoria.

El *segundo capítulo* está dedicado a estudiar el concepto principal de nuestro trabajo, la p-diferencia $\sim_p$ de cuerpos convexos, que desempeña el papel de operación "opuesta" a la p-suma. En la primera sección, establecemos la definición de p-diferencia de dos maneras distintas, probando que, de hecho, son equivalentes, y estudiamos las primeras propiedades de esta nueva operación. En particular, obtenemos que la p-diferencia de dos cuerpos convexos puede verse como una Wulff-shape asociada a las funciones soporte de ambos conjuntos. Tal y como ocurre cuando se trabaja con la suma y la diferencia de Minkowski clásicas, se obtienen las correspondientes relaciones naturales entre la p-suma y la p-diferencia, así como la no-continuidad de la p-diferencia con respecto a la métrica de Hausdorff. En la segunda sección discutimos la necesidad de trabajar con una subfamilia particular de cuerpos convexos $\mathcal{K}^n_{00}(E)$ para un cuerpo gauge fijo $E \subset \mathbb{R}^n$, con el fin de desarrollar un estudio estructurado y sistemático de la p-diferencia, que también sea válido cuando $p = 1$. También introducimos el concepto de *p-núcleo* de K respecto a E, $\ker_p(K; E)$, y probamos que, como en el caso clásico, éste nunca tiene dimensión n. Finalmente, la última sección está dedicada a definir y estudiar el llamado *sistema completo de cuerpos p-paralelos*, esto es, la colección de todos los *cuerpos p-paralelos exteriores e interiores*:

$$K^p_\lambda = \begin{cases} K \sim_p |\lambda| E & \text{si} \quad -\mathrm{r}(K; E) \leq \lambda \leq 0, \\ K +_p \lambda E & \text{si} \quad 0 \leq \lambda < \infty. \end{cases}$$

De forma análoga al caso clásico, probamos que el sistema completo de cuerpos p-paralelos satisface una propiedad de tipo concavidad respecto a la inclusión, así como la continuidad en el parámetro

λ respecto a la métrica de Hausdorff. Llegados a este punto, introducimos una *p-suma* $+_p$ *en* $\mathbb{R}$, con el fin de establecer los resultados de forma precisa. Esta definición extiende la p-media clásica de números reales positivos (salvo una constante). También estudiamos, en la última parte de esta sección, el comportamiento de los cuerpos p-paralelos interiores para dos familias especiales de cuerpos convexos: los cuerpos tangenciales, y aquellos conjuntos obtenidos como p-paralelos exteriores de conjuntos de menor dimensión. En particular, caracterizamos los cuerpos tangenciales como los únicos cuerpos convexos cuyos p-paralelos interiores son copias homotéticas de ellos, generalizando de esta forma al caso $p > 1$ el ya mencionado resultado de Schneider.

En el *tercer capítulo* abordamos el estudio de la derivabilidad de las quermassintegrales relativas $\mathrm{W}_i(K_\lambda^p; E)$ respecto a la familia uniparamétrica de los cuerpos p-paralelos. En la primera sección, recordamos la definición de la *quermassintegral mixta* $\mathrm{W}_{p,i}(K, L; E)$, $0 \leq i \leq n-1$, un concepto clave en el desarrollo de la teoría L_p de Brunn-Minkowski, definidas por Lutwak en 1993 mediante un argumento variacional que involucra p-sumas:

$$\mathrm{W}_{p,i}(K, L; E) = \frac{p}{n-i} \lim_{\varepsilon \to 0^+} \frac{\mathrm{W}_i\big(K +_p \varepsilon^{1/p} L; E\big) - \mathrm{W}_i(K; E)}{\varepsilon}.$$

También recordamos que estas quermassintegrales mixtas admiten una representación integral muy útil a través de funciones soporte y medidas de área de superficie mixtas.

Con el fin de estudiar la posible derivabilidad de $\mathrm{W}_i(K_\lambda^p; E)$, consideramos una propiedad de $+_p$-*concavidad* para funciones reales definidas en un intervalo $I \subseteq \mathbb{R}$, y probamos que esta propiedad está relacionada con la concavidad usual y, en particular, con la quasi-concavidad. A continuación, en la segunda sección del capítulo, nos centramos en el estudio de la derivabilidad de las quermassintegrales relativas. En particular, probamos que la función $\mathrm{W}_i(\lambda) = \mathrm{W}_i(K_\lambda^p; E)$ es derivable en $(0, \infty)$, siendo su derivada

$$\mathrm{W}_i'(\lambda) = \lambda^{p-1}(n-i)\mathrm{W}_{p,i}(K_\lambda^p, E; E).$$

Observamos que este resultado extiende (♣) cuando $p = 1$, ya que $\mathrm{W}_{1,i}(K_\lambda, E; E) = \mathrm{W}_{i+1}(K_\lambda; E)$. El argumento para obtenerlo es, sin embargo, completamente diferente, ya que no existe una expresión polinómica para las quermassintegrales de una p-suma. Seguidamente, se demuestra que hay derivabilidad también en $\lambda = 0$ si $p > 1$, siendo el valor de la derivada siempre 0. La sección, y el capítulo, finalizan con el caso más especial del volumen: profundos resultados de Lutwak nos permiten probar que, como en el caso clásico (cf. (♠)), el funcional volumen $\mathrm{vol}(K_\lambda^p)$ es siempre derivable en $\lambda \in \big(-\mathrm{r}(K; E), \infty\big)$, proporcionando la expresión explícita para su derivada. Como consecuencia de ello, obtenemos una fórmula integral para el volumen de K en función de sus cuerpos p-paralelos interiores.

En el *cuarto* y último *capítulo* de esta memoria nos ocupamos de la estructura de la frontera de los cuerpos p-paralelos interiores: estudiamos sus *vectores extremos* y su relación con los del cuerpo convexo original. Estos vectores extremos permiten definir un nuevo cuerpo convexo, el *cuerpo* *p-forma*, que será útil para obtener, en la última sección, (nuevas) desigualdades que proporcionan cotas para las quermassintegrales de los cuerpos p-paralelos interiores. Antes de esto, estudiamos la derivabilidad de la función soporte $h(K_\lambda^p,\cdot)$ respecto a la familia uniparamétrica de cuerpos p-paralelos interiores. En primer lugar probamos que, para cuerpos convexos adecuados K, E,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}h(K_\lambda^p,u)\geq\frac{|\lambda|^{p-1}h(E,u)^p}{h(K_\lambda^p,u)^{p-1}}$$

para casi todo punto en $\bigl(-\mathrm{r}(K;E),0\bigr]$, con igualdad, para todo $u\in\mathbb{S}^{n-1}$ y para casi todo punto en $\bigl[-\mathrm{r}(K;E),0\bigr]$ si, y sólo si, $K=\ker_p(K;E)+_p\mathrm{r}(K;E)E$. Este resultado es el L_p-análogo de (♡), y nos permitirá caracterizar los cuerpos convexos para los cuales la quermassintegral $(n-1)$-ésima $\mathrm{W}_{n-1}(K_\lambda^p;E)$ es derivable. En particular, obtenemos que $\mathrm{W}'_{n-1}(\lambda)=|\lambda|^{p-1}\mathrm{W}_{p,n-1}(K_\lambda^p,E;E)$ para $\lambda\in\bigl(-\mathrm{r}(K;E),0\bigr)$ si, y sólo si, $K=\ker_p(K;E)+_p\mathrm{r}(K;E)E$. Además, y de forma análoga al caso clásico, la anterior cota inferior para la derivada de la función soporte puede mejorarse usando el cuerpo forma de los cuerpos p-paralelos interiores.

Los resultados originales que se encuentran recogidos en esta memoria pueden encontrarse en nuestros trabajos [13, 14, 21].

Brunn-Minkowski's theory is the classical center of the geometry of convex bodies. Its origin goes back to the end of the nineteenth century and the beginning of the twentieth century, mainly due to the works of Cauchy, Steiner, Brunn and, essentially, of Minkowski.

One could say that Brunn-Minkowski's theory is the result of combining two elementary notions for sets in the Euclidean space: the *Minkowski sum* $+$ (i.e., the vectorial addition), and the *volume* $\mathrm{vol}(\cdot)$ (the Lebesgue measure). It leads to the notion of mixed volumes and to the famous and fundamental Brunn-Minkowski inequality. A very relevant and somehow amazing look like fact in this context was discovered by Steiner (1840) in the particular case of the Euclidean ball: the volume of the Minkowski addition of a convex body K and a positive dilation λE of a *gauge* convex body E (the so-called *relative outer parallel body of K at distance λ*), is a polynomial of degree at most n in the variable λ, the *relative Steiner formula.* More precisely,

$$\mathrm{vol}(K+\lambda E)=\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)\lambda^i. \qquad (\star)$$

The coefficients are, up to constants, particular measures associated to K and E, the so-called *relative quermassintegrals.* The volume $\mathrm{W}_0(K;E)=\mathrm{vol}(K)$ is a particular case of these measures.

It is natural to consider an operation somehow "opposite" to the Minkowski addition of convex bodies, the so-called *Minkowski difference*, $\sim$:

$$K\sim E=\{x\in\mathbb{R}^n : x+E\subseteq K\}.$$

Then, *inner parallel bodies* of K with respect to E can be defined, which, together with the previously considered outer parallel sets, lead to the definition of *full system of parallel bodies*:

$$K_\lambda=\begin{cases} K\sim|\lambda|E & \text{if} \quad -\mathrm{r}(K;E)\leq\lambda\leq 0,\\ K+\lambda E & \text{if} \quad 0\leq\lambda<\infty,\end{cases}$$

where $\mathrm{r}(K;E)=\max\{r\geq 0 : \text{there exists } x\in\mathbb{R}^n \text{ with } x+rE\subseteq K\}$ is the *relative inradius* of K with respect to E. Relative Steiner's formula $(\star)$ allows us to easily obtain the so-called *Minkowski formula*, which states that the right-hand derivative at 0 of the volume functional $\mathrm{vol}(\lambda)=\mathrm{vol}(K_\lambda)$ is n times the first (relative) quermassintegral:

$$n\mathrm{W}_1(K;E)=\lim_{\lambda\to 0^+}\frac{\mathrm{vol}(K_\lambda)-\mathrm{vol}(K)}{\lambda}.$$

Later, in the middle of the twentieth century, Bol, Hadwiger and Matheron proved the differentiability of $\mathrm{vol}(\lambda)$ on the full interval $\bigl(-\mathrm{r}(K;E),\infty\bigr)$, providing the explicit formula

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{vol}(\lambda)=n\mathrm{W}_1(K_\lambda;E). \tag{$\natural$}$$

Recently, Hernández Cifre and Saorín Gómez studied the differentiability of the relative quermassintegrals $\mathrm{W}_i(K_\lambda;E)$ regarded as functions in the parameter $\lambda\in\bigl(-\mathrm{r}(K;E),\infty\bigr)$. In particular, due to the known polynomial expression in $\lambda\geq 0$ for $\mathrm{W}_i(K_\lambda;E)$, both, the differentiability and the explicit value of its derivative, can be obtained when $\lambda\geq 0$: for all $i=0,\dots,n-1$,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{W}_i(K_\lambda;E)=(n-i)\mathrm{W}_{i+1}(K_\lambda;E). \tag{$\sharp$}$$

Moreover, given a convex body K, its *support function* $h(K,\cdot):\mathbb{S}^{n-1}\longrightarrow\mathbb{R}$, defined on the unit sphere $\mathbb{S}^{n-1}$, is $h(K,u)=\sup\bigl\{\langle x,u\rangle : x\in K\bigr\}$, where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{R}^n$. Thus, it makes sense to study the differentiability of the support function $h(K_\lambda,u)$ with respect to the parameter $\lambda\in\bigl(-\mathrm{r}(K;E),\infty\bigr)$. *Heuristically*, this derivative plays the role of the *growth velocity* of the parallel bodies K_λ with respect to λ. In 1979, Chakerian and Sangwine-Yager proved that the support function of E is always a lower bound for the derivative of $h(K_\lambda,u)$,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}h(K_\lambda,u)\geq h(E,u), \tag{$\bowtie$}$$

wherever it exists, showing that equality holds for all $u\in\mathbb{S}^{n-1}$, almost everywhere on $\bigl[-\mathrm{r}(K;E),0\bigr]$, if and only if $K=K_{-\mathrm{r}(K;E)}+\mathrm{r}(K;E)E$.

On the other hand, the fact that a convex body is uniquely determined by its support function allowed Firey to introduce in 1962 the p-sum of two convex bodies K,E containing the origin, $1\leq p\leq\infty$, as the convex body whose support function is given by the expression

$$h(K+_p E,u)=\bigl(h(K,u)^p+h(E,u)^p\bigr)^{1/p}.$$

Minkowski's addition is obtained when $p=1$, whereas for $p=\infty$ we get the convex hull of the sets, $K+_\infty E=\mathrm{conv}(K\cup E)$. The p-sum of convex bodies was the starting point of the nowadays known as the L_p-Brunn-Minkowski (or Firey-Brunn-Minkowski) theory.

Considering however the recent and deep development of the L_p-Brunn-Minkowski theory, it is somehow surprising that no equivalent p-concept of difference of convex bodies has been defined up to now. This question was the starting motivation of our work, and we could say that this dissertation is devoted to introduce a new "opposite" operation to the p-sum of sets, that we will call *p-difference* of convex bodies, and to investigate its properties. Moreover, we study the differentiability of the relative quermassintegrals of the corresponding full system of p-parallel bodies with respect to its definition parameter. In particular, we obtain that, as in the classical case, the volume functional is always differentiable, providing the explicit expression for its derivative. Next we describe the specific contents of each chapter in which this dissertation has been organized.

The work starts with an introductory *first chapter* in which we establish the notation and introduce the concepts and results that will be needed further on, both, about general Convexity and, in particular, about (L_p-)Brunn-Minkowski theory. Thus, in a first section, we recall the basic notions, such as Minkowski's addition, convex body, support function, support hyperplane, etc. Next, Minkowski's difference is introduced, stating the elementary relations that it has with the Minkowski addition. In this context, we also consider the notion of *Wulff-shape* associated with a closed subset $\Omega \subseteq \mathbb{S}^{n-1}$ of the sphere and a non-negative continuous function $\psi : \Omega \longrightarrow [0, \infty)$, pointing out some important properties which will be used later on. Minkowski's addition and subtraction lead to the concepts of relative inradius and of full system of parallel bodies. A key property satisfied by this family of sets is the concavity. We also recall the notion of tangential body and the important result of Schneider characterizing these sets as the only convex bodies whose inner parallel bodies are homothetic copies of them. Next we move to the L_p-Brunn-Minkowski theory, studying the notion and main properties of the p-sum of convex bodies (containing the origin). Finally, after introducing the mixed volumes and quermassintegrals, we state several known inequalities in the (L_p-)Brunn-Minkowski theory, which will play an important role in the development of our work.

The *second chapter* is devoted to the main concept of this work, the p-difference $\sim_p$ of convex bodies, $1 \leq p \leq \infty$, which plays a role of "opposite" operation to the p-sum of convex bodies. In the first section, we define the p-difference in two possible ways, we prove that they are in fact equivalent, and we study the first properties of this new operation. In particular, we obtain that the p-difference of two convex bodies can be regarded as a Wulff-shape associated with the support functions of both sets. As it occurs when dealing with the classical Minkowski sum and difference, we show the corresponding natural relations between the p-sum and the p-difference, as well as the non-continuity of the p-difference with respect to the Hausdorff metric. In the second section we discuss the necessity of working on a restricted subfamily of convex bodies $\mathcal{K}^n_{00}(E)$ for a fixed gauge convex body $E \subset \mathbb{R}^n$, in order to develop a structured and systematic study of the p-difference, valid for $p = 1$ too. We also introduce the concept of *p-kernel* of K with respect to E, $\ker_p(K; E)$, and we prove that, as in the classical setting, it can never have dimension n. Finally, the last section is devoted to define and study the so-called *full system of p-parallel bodies*, i.e., the collection of all *p-outer and inner parallel bodies*:

$$K^p_\lambda = \begin{cases} K \sim_p |\lambda| E & \text{if} \quad -\mathrm{r}(K; E) \leq \lambda \leq 0, \\ K +_p \lambda E & \text{if} \quad 0 \leq \lambda < \infty. \end{cases}$$

In analogy to the classical case, we prove that the full system of p-parallel bodies satisfies a kind of concavity property with respect to the inclusion, as well as that it is continuous in the parameter λ with respect to the Hausdorff metric. At this point we introduce a *p-sum $+_p$ in $\mathbb{R}$*, in order to state the results in a precise way. This definition extends (up to a constant) the classical p-mean of positive real numbers. We also study, in the last part of this section, the behavior of the p-inner parallel bodies for two special families of convex bodies: tangential bodies, and those sets obtained as p-outer parallel bodies of a lower dimensional set. In particular, we characterize tangential bodies as the only convex bodies such that their p-inner parallel bodies are homothetic copies of them, generalizing in this way to $p > 1$ the already mentioned result by Schneider.

In the *third chapter* we focus on the differentiability of the relative quermassintegrals $\mathrm{W}_i(K_\lambda^p; E)$ with respect to the one-parameter family of the p-parallel bodies. In the first section, we recall the definition of *mixed quermassintegral* $\mathrm{W}_{p,i}(K, L; E)$, $0 \leq i \leq n-1$, a key concept in the development of the L_p-Brunn-Minkowski theory, which was defined by Lutwak in 1993 via a variational argument involving p-sums:

$$\mathrm{W}_{p,i}(K, L; E) = \frac{p}{n-i} \lim_{\varepsilon \to 0^+} \frac{\mathrm{W}_i(K +_p \varepsilon^{1/p} L; E) - \mathrm{W}_i(K; E)}{\varepsilon}.$$

We also recall that these mixed quermassintegrals do admit a very useful integral representation by means of support functions and mixed surface area measures.

In order to study the possible differentiability of $\mathrm{W}_i(K_\lambda^p; E)$, we consider a $+_p$-*concavity* property for real-valued functions defined on an interval $I \subseteq \mathbb{R}$, and prove that it is related to the usual concavity. More particulary, it is related to quasi-concavity. Then, in the second section of the chapter, we focus on studying the differentiability of the relative quermassintegrals. In particular, we prove that the function $\mathrm{W}_i(\lambda) = \mathrm{W}_i(K_\lambda^p; E)$ is differentiable on $(0, \infty)$, being its derivative

$$\mathrm{W}_i'(\lambda) = \lambda^{p-1}(n-i)\mathrm{W}_{p,i}(K_\lambda^p, E; E).$$

We notice that this result extends ($\sharp$) when $p = 1$, because $\mathrm{W}_{1,i}(K_\lambda, E; E) = \mathrm{W}_{i+1}(K_\lambda; E)$. The approach is, however, completely different, since it is known that there is no polynomial expression for the quermassintegrals of a p-sum. Next, we prove that the there is also differentiability at $\lambda = 0$ if $p > 1$, being the value of the derivative always 0. The section, and the chapter, finish with the most special case of the volume: deep results by Lutwak allow us to prove that, as in the classical case (cf. ($\natural$)), the volume functional $\mathrm{vol}(K_\lambda^p)$ is always differentiable at $\lambda \in \big(-\mathrm{r}(K; E), \infty\big)$, providing the explicit expression for its derivative. As a consequence, we obtain an integral expression for the volume of K in terms of its p-inner parallel bodies.

In the *fourth* and last *chapter* of this dissertation we deal with the structure of the boundary of the p-inner parallel bodies of a convex body: we study their *extreme vectors*, and relate them to the ones of the original set. They will allow us to define a new convex body, the *p-form body*, which will be useful to obtain, in the last section, (new) inequalities providing bounds for the quermassintegrals of the p-inner parallel bodies. Previous to this, we also study the differentiability of the support function $h(K_\lambda^p, \cdot)$ with respect to the one-parameter family of p-inner parallel bodies. First we prove that for suitable convex bodies K, E,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} h(K_\lambda^p, u) \geq \frac{|\lambda|^{p-1} h(E, u)^p}{h(K_\lambda^p, u)^{p-1}}$$

almost everywhere on $\bigl(-\mathrm{r}(K;E),0\bigr]$, with equality for all $u \in \mathbb{S}^{n-1}$ and almost everywhere on $\bigl[-\mathrm{r}(K;E),0\bigr]$, if and only if $K = \ker_p(K;E) +_p \mathrm{r}(K;E)E$. This result is the L_p-analogue of ($\bowtie$), and will allow us to characterize the convex bodies for which the $(n-1)$-st quermassintegral $\mathrm{W}_{n-1}(K_\lambda^p;E)$ is differentiable. In particular, we get that $\mathrm{W}'_{n-1}(\lambda) = |\lambda|^{p-1}\mathrm{W}_{p,n-1}(K_\lambda^p, E;E)$ for $\lambda \in \bigl(-\mathrm{r}(K;E),0\bigr)$ if and only if $K = \ker_p(K;E) +_p \mathrm{r}(K;E)E$. Moreover, and analogously to the classical case, the above lower bound for the derivative of the support function can be improved using the form body of the p-inner parallel bodies.

The original work contained in this dissertation can be found in the papers [13, 14, 21].

Chapter 1

Preliminaries

In this first chapter we make a brief survey of the main definitions, properties and results of convex bodies and Brunn-Minkowski theory which will be needed throughout this dissertation.

1.1 Convex bodies and their properties

We will use the following standard notation. We write $\mathbb{R}^n$ to denote the n-dimensional Euclidean space, endowed with the standard inner product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $|\cdot|$. We denote by e_i the i-th canonical unit vector in $\mathbb{R}^n$, $i = 1, \ldots, n$. Let B_n be the n-dimensional unit ball and $\mathbb{S}^{n-1}$ the $(n-1)$-dimensional unit sphere of $\mathbb{R}^n$. We will use $C_n = [-1, 1]^n$ to denote the n-dimensional cube of edge-length 2 centered at the origin.

The closure of a set $M \subseteq \mathbb{R}^n$ is denoted by $\operatorname{cl} M$, its boundary by $\operatorname{bd} M$ and its interior by $\operatorname{int} M$. The *dimension* of M, i.e., the dimension of the smallest affine subspace containing M (its *affine hull*, $\operatorname{aff} M$) is denoted by $\dim M$. Regarding the dimension of M, we write $\operatorname{relint} M$ to denote the relative interior of M, i.e., the interior of the set M relative to its affine hull. Given $x, y \in \mathbb{R}^n$, the notation $[x, y]$ stands for the line segment with end points x and y

The following definitions and properties are well known and can be found in any book on Convexity, for instance [2, 10, 27].

Definition 1.1.1. *A (non-empty) subset $M \subseteq \mathbb{R}^n$ is said to be* convex *if, whenever two points $x, y \in M$, then the segment $[x, y]$ is contained in M, i.e., the convex combination $(1-\lambda)x+\lambda y \in M$, for $0 \leq \lambda \leq 1$.*

Definition 1.1.2. *A* convex body $K \subsetneq \mathbb{R}^n$ *is a non-empty compact convex set. Moreover, a convex body is called* strictly convex *if its boundary does not contain a segment.*

From now on $\mathcal{K}^n$ will denote the set of all convex bodies in $\mathbb{R}^n$, and $\mathcal{K}^n_0$ will denote the subset of $\mathcal{K}^n$ consisting of all convex bodies containing the origin 0. We also denote by $\mathcal{K}^n_n$ (respectively, $\mathcal{K}^n_{(0)}$) the subset of $\mathcal{K}^n$ having interior points (0 as an interior point).

The *Minkowski sum* of two non-empty sets $A, B \subseteq \mathbb{R}^n$ is their (vectorial) addition, i.e.,

$$A + B = \{a + b : a \in A,\, b \in B\}\,,$$

and we write $A + x := A + \{x\}$, for $x \in \mathbb{R}^n$. Moreover, $\lambda A = \{\lambda x : x \in A\}$, for $\lambda \in \mathbb{R}$, and we shorten $A - B = A + (-B) = A + (-1)B$.

If $K, L \in \mathcal{K}^n$, then clearly $K + L \in \mathcal{K}^n$ (see Figure 1.1). Two convex bodies $K, L \in \mathcal{K}^n$ are called *homothetic* if $K = \lambda K + t$ for some $t \in \mathbb{R}^n$ and $\lambda > 0$. Moreover, we say that L is a *summand* of K if there exists a convex body $M \in \mathcal{K}^n$ such that $K = L + M$.

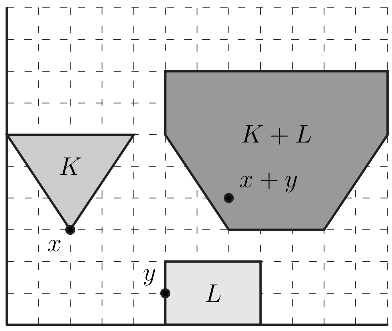

Figure 1.1: The Minkowski (vectorial) addition.

The intersection of all convex sets containing $M \subsetneq \mathbb{R}^n$ is the *convex hull* of M, and it will be denoted by $\operatorname{conv} M$; thus $\operatorname{conv} M$ is the smallest convex set containing M. We denote by $\operatorname{pos} M$ the *positive hull* of M, namely, $\operatorname{pos} M = \{\lambda x : \lambda \geq 0,\, x \in M\}$. Finally, the *linear hull* of M, $\operatorname{lin} M$, is the intersection of all linear subspaces in $\mathbb{R}^n$ containing M. The convex hull of a compact set is always a convex body. In particular, the convex hull of a finite number of points is so and the family of all of them defines a very important class of convex bodies:

Definition 1.1.3. *A* polytope *is the convex hull of finitely many points in* $\mathbb{R}^n$ *(its* vertices*).*

In spite of the fact that many of the following properties and definitions are valid for closed convex sets, in order to simplify the exposition we will restrict them to compact ones, since later we will always work under the hypothesis of compactness.

Definition 1.1.4. *A hyperplane H is a* supporting hyperplane *of $K \in \mathcal{K}^n$ if $H \cap K \neq \emptyset$ and K is contained in one of the two halfspaces determined by H, which is called its* supporting halfspace*.*

The following classical results concerning supporting hyperplanes will be needed in the following.

Theorem 1.1.5. *At every boundary point of $K \in \mathcal{K}^n$ there exists a supporting hyperplane to K. Furthermore, for all $u \in \mathbb{S}^{n-1}$ there is a supporting hyperplane of K with outer normal vector u.*

Supporting hyperplanes are used to characterize convexity: if $K \subsetneq \mathbb{R}^n$ is compact, $\operatorname{int} K \neq \emptyset$, then K is convex if and only if for every $x \in \operatorname{bd} K$ there is a supporting hyperplane to K. As a consequence, we get that any convex body is the intersection of its supporting halfspaces. Moreover:

Theorem 1.1.6. *Any $K \in \mathcal{K}^n$ with $\operatorname{int} K \neq \emptyset$ is the intersection of its supporting halfspaces at regular points, i.e., those boundary points x such that the supporting hyperplane to K at x is unique.*

Definition 1.1.7. *A convex body $K \in \mathcal{K}^n$ is said to be* regular *if every $x \in \operatorname{bd} K$ is a* regular point.

We observe that if $K \in \mathcal{K}^n$ is regular, then necessarily $\operatorname{int} K \neq \emptyset$.

There is no doubt that convex functions play an important role in the theory of convex bodies.

Definition 1.1.8. *A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is* convex *if for any $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$,*

$$f\big((1-\lambda)x + \lambda y\big) \leq (1-\lambda)f(x) + \lambda f(y).$$

A function f is concave *if $-f$ is convex.*

The following properties of convex functions will be needed later. For references and further study we refer for instance to [24, 27].

Proposition 1.1.9. *A convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous in $\operatorname{int}\operatorname{dom} f$ and, if $n = 1$:*

i) *The left and right derivatives, denoted respectively by $(\mathrm{d}^-/\mathrm{d}x)f(x)$ and $(\mathrm{d}^+/\mathrm{d}x)f(x)$, do exist at every point $x \in \mathbb{R}$ and they are increasing functions. Moreover,*

$$\frac{\mathrm{d}^-}{\mathrm{d}x} f(x) \leq \frac{\mathrm{d}^+}{\mathrm{d}x} f(x).$$

ii) *If $f : [a, b] \longrightarrow \mathbb{R}$, the left (respectively, right) derivative exists at $x = b$ (respectively, $x = a$).*

Lemma 1.1.10 ([23, Problem/Remark B, p. 13]). *If $f : I \longrightarrow \mathbb{R}$ is a convex (or concave) function, where $I \subseteq \mathbb{R}$ is an interval, then f is absolutely continuous on I.*

Definition 1.1.11. *The* support function *of a convex body $K \in \mathcal{K}^n$ in the direction $u \in \mathbb{S}^{n-1}$, denoted by $h(K, u)$, is the real valued function defined on the sphere by*

$$h(K, u) = \max\big\{\langle x, u\rangle : x \in K\big\}.$$

The support function may be extended to $\mathbb{R}^n$, but for our purposes we consider it defined on $\mathbb{S}^{n-1}$. It has many useful properties; here we detail just the ones we will need further on.

Proposition 1.1.12. *Let $K, L \in \mathcal{K}^n$ and $u, v \in \mathbb{S}^{n-1}$.*

i) *$h(K+L, u) = h(K, u) + h(L, u)$ and $h(\lambda K, u) = \lambda h(K, u)$ for all $\lambda \geq 0$.*

ii) *If $K \subseteq L$ then $h(K, u) \leq h(L, u)$.*

iii) *$h(K, \lambda u) = \lambda h(K, u)$ for all $\lambda \geq 0$, i.e., $h(K, \cdot)$ is* positively homogeneous.

iv) *$h(K, u+v) \leq h(K, u) + h(K, v)$, i.e., $h(K, \cdot)$ is* subadditive.

In particular, $h(K, \cdot)$ is convex. The above last two properties are usually expressed by saying that $h(K, \cdot)$ is *sublinear*. In fact, it turns out that they characterize support functions:

Theorem 1.1.13. *Let h be a sublinear (real-valued) function defined on $\mathbb{S}^{n-1}$. Then there is a unique $K \in \mathcal{K}^n$ such that $h = h(K, \cdot)$. Moreover, any $K \in \mathcal{K}^n$ is determined by its support function.*

We observe that if $K \in \mathcal{K}^n$, then

$$K = \left\{x \in \mathbb{R}^n : \langle x, u\rangle \leq h(K, u) \text{ for every } u \in \mathbb{S}^{n-1}\right\},$$

or equivalently,

$$K = \bigcap_{u \in \mathbb{S}^{n-1}} \left\{x \in \mathbb{R}^n : \langle x, u\rangle \leq h(K, u)\right\}. \tag{1.1}$$

Property i) in Proposition 1.1.12 allows to prove the *cancellation law* for $(\mathcal{K}^n, +)$, this is, if $K + M = L + M$ for $K, L, M \in \mathcal{K}^n$, then $K = L$.

The space of convex bodies $\mathcal{K}^n$ is endowed with the *Hausdorff metric* $\delta_{\mathcal{H}}$, namely

$$\delta_{\mathcal{H}}(K, L) = \min\left\{\lambda \geq 0 : K \subseteq L + \lambda B_n,\ L \subseteq K + \lambda B_n\right\} \quad \text{for } K, L \in \mathcal{K}^n. \tag{1.2}$$

From now on we will consider the continuity of functionals defined on $\mathcal{K}^n$, as well as the convergence of convex bodies, with respect to the topology determined by the Hausdorff metric. We finish this section by formulating the famous *Blaschke selection theorem*, which provides a very useful tool in providing the existence of convex bodies with specific properties. The proof of this theorem can be found, for instance, in [27, Theorem 1.8.6].

Theorem 1.1.14 (Blaschke's selection theorem). *Every bounded sequence of convex bodies in $\mathbb{R}^n$ has a convergent subsequence (in the Hausdorff metric $\delta_{\mathcal{H}}$) to a convex body.*

1.2 Minkowski's difference

Minkowski's difference (though it was not introduced by Minkowski) can be regarded as the subtraction counterpart of the Minkowski sum: for two non-empty sets $A, B \subseteq \mathbb{R}^n$, the Minkowski difference of A and B is defined by

$$A \sim B = \{x \in \mathbb{R}^n : B + x \subseteq A\}, \tag{1.3}$$

this is, $A \sim B$ is the largest set such that $(A \sim B) + B \subseteq A$.

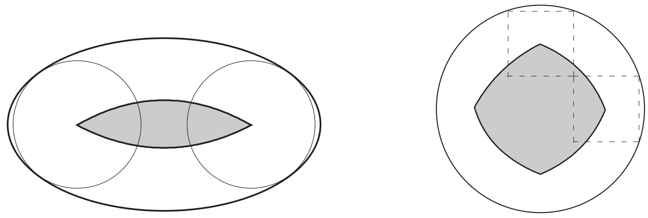

Figure 1.2: Inner parallel body of an ellipse (relative to B_2) and a circle (relative to the square).

Next lemma collects some useful rules which relate Minkowski's addition and subtraction.

Lemma 1.2.1. *Let $A, B, C \subseteq \mathbb{R}^n$ be non-empty sets. Then*

i) $(A + B) \sim B \supseteq A$. *If $A, B \in \mathcal{K}^n$, then there is equality.*

ii) $(A \sim B) + B \subseteq A$. *If $A, B \in \mathcal{K}^n$, equality holds if and only if B is a summand of A.*

iii) $(A \sim B) + C \subseteq (A + C) \sim B$.

iv) $(A \sim B) \sim C = A \sim (B + C)$.

v) $A + B \subseteq C$ *if and only if* $A \subseteq C \sim B$.

As we shall see later, Minkowski's difference gives rise to the notion of inner parallel bodies, a notion which has many applications in the geometry of convex bodies: according to [27, Section 7.5] "some of the deeper investigations of inequalities for mixed volumes make essential use of the method of inner parallel bodies"; we refer the reader to [27, Note 2 for Section 7.5] for further applications of inner parallel bodies.

For convex bodies $K, E \in \mathcal{K}^n$, the Minkowski difference $K \sim E$ can be also defined as

$$K \sim E = \bigcap_{u \in \mathbb{S}^{n-1}} \bigl\{x \in \mathbb{R}^n : \langle x, u \rangle \leq h(K, u) - h(E, u)\bigr\}. \tag{1.4}$$

Unlike what happens with the Minkowski sum (Proposition 1.1.12 i)), the support function of the Minkowski difference $K \sim E$ cannot be given, in general, by an easy combination of the support functions of K and E. Nevertheless, one can easily get the following bound:

$$h(K \sim E, u) \leq h(K, u) - h(E, u) \quad \text{ for all } u \in \mathbb{S}^{n-1}.$$

We would like to point out that, in general, there is no equality in the above inequality relating support functions (cf. Lemma 1.2.1 ii)).

Definition 1.2.2. *Let $\psi : \mathbb{S}^{n-1} \longrightarrow [0, \infty)$ be a non-negative continuous function and $\Omega \subseteq \mathbb{S}^{n-1}$ be a closed subset, not lying in a closed hemisphere. The* Wulff-shape *associated with (Ω, ψ) is the set*

$$\mathrm{WS}(\Omega, \psi) = \bigcap_{u \in \Omega} \bigl\{x \in \mathbb{R}^n : \langle x, u \rangle \leq \psi(u)\bigr\}.$$

In the particular case $\Omega = \mathbb{S}^{n-1}$, we will just write $\mathrm{WS}(\psi) = \mathrm{WS}(\mathbb{S}^{n-1}, \psi)$.

For further details about Wulff-shapes we refer to [27, Section 7.5] and the references therein.

Remark 1.2.3. $\mathrm{WS}(\Omega, \psi)$ *is a convex body containing the origin ([27, Section 7.5]), and moreover,* $h\big(\mathrm{WS}(\Omega, \psi), u\big) \leq \psi(u)$ *for all* $u \in \Omega$*. Equality holds for particular vectors (see Remark 4.1.4).*

We observe that the difference body $K \sim E$ of two convex bodies $K, E \in \mathcal{K}^n$ can be seen as the Wulff-shape associated with the function $\psi(u) = h(K, u) - h(E, u)$.

The following lemma can be found in [27, Lemma 7.5.2]:

Lemma 1.2.4. *If* $K_j = \mathrm{WS}(\Omega, \psi_j)$ *for* $j \in \mathbb{N} \cup \{0\}$ *and if* $(\psi_j)_{j \in \mathbb{N}}$ *converges uniformly to* ψ_0*, then* $(K_j)_{j \in \mathbb{N}}$ *converges to* K_0 *with respect to the Hausdorff metric.*

We have seen two equivalent geometric constructions giving rise to the Minkowski difference, namely, equations (1.3) and (1.4). The first definition makes use of the natural connection of the subtraction with the Minkowski sum: $K \sim E$ *should be maximal among all convex bodies which (Minkowski) added to* E *keep the result within* K. The second definition takes advantage of the connection of the Minkowski sum with the support function via a Wulff-shape, since differences of support functions need not be support functions.

1.3 The full system of parallel bodies

When dealing with the Minkowski difference of convex bodies, the notions of inradius and kernel play a prominent role (see e.g. [25, 27] and the references therein).

Definition 1.3.1. *For two convex bodies* $K, E \in \mathcal{K}^n$*, the* relative inradius $\mathrm{r}(K; E)$ *of* K *with respect to* E *is defined by*

$$\mathrm{r}(K; E) = \max\big\{r \geq 0 : x + rE \subseteq K \text{ for some } x \in \mathbb{R}^n\big\}, \tag{1.5}$$

whereas

$$\ker(K; E) = K \sim \mathrm{r}(K; E)E$$

is the set of relative incenters of K*, usually called* kernel *of* K *with respect to* E*.*

It is well-known (see [2, p. 59]) that the dimension of $\ker(K; E)$ satisfies

$$\dim \ker(K; E) \leq n - 1. \tag{1.6}$$

For two convex bodies $K, E \in \mathcal{K}^n$, and a non-negative real number λ, the *outer parallel body of* K *(relative to* E*) at distance* λ is the Minkowski sum $K + \lambda E$. For $-\mathrm{r}(K; E) \leq \lambda \leq 0$ the *inner parallel body of* K *(relative to* E*) at distance* $|\lambda|$ is the Minkowski difference $K \sim |\lambda| E$. Inner parallel bodies and their properties have been studied in [1, 11, 25], among others.

For $E \in \mathcal{K}^n$, joining outer and inner parallel bodies, the full system of relative parallel bodies of K is defined as

$$K_\lambda = \begin{cases} K \sim |\lambda| E & \text{if} \quad -\mathrm{r}(K;E) \leq \lambda \leq 0, \\ K + \lambda E & \text{if} \quad 0 \leq \lambda < \infty. \end{cases}$$

Obviously we have $K_0 = K$ and $K_{-\mathrm{r}(K;E)} = \ker(K;E)$. Moreover, for $K, L \in \mathcal{K}^n$ and arbitrary $\mu \geq -\mathrm{r}(K;E)$, $\sigma \geq -\mathrm{r}(L;E)$, the rule

$$K_\mu + L_\sigma \subseteq (K+L)_{\mu+\sigma} \tag{1.7}$$

is valid (see [27, (3.20)]). As a consequence, a very useful property of the full system of relative parallel bodies of a convex set is obtained: it is concave with respect to inclusion and Minkowski's addition (see [27, Lemma 3.1.13]), i.e.,

$$(1-\lambda)K_\mu + \lambda K_\sigma \subseteq K_{(1-\lambda)\mu+\lambda\sigma}. \tag{1.8}$$

A special family of convex bodies for which their inner parallel bodies can be easily characterized, are the so-called tangential bodies. Tangential bodies can be defined in several equivalent ways; here we will use the following one:

Definition 1.3.2. *A convex body $K \in \mathcal{K}^n$ containing $E \in \mathcal{K}^n$, is called a* tangential body *of E, if through each boundary point of K there exists a support hyperplane to K that also supports E.*

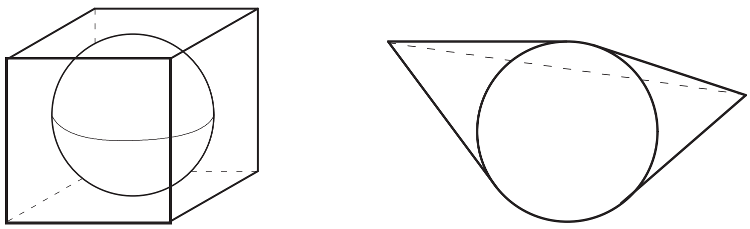

Figure 1.3: Examples of tangential bodies of B_n.

We notice that if K is a tangential body of E, then $\mathrm{r}(K;E) = 1$. The n-dimensional cube C_n is an example of this type of bodies for $E = B_n$ (see Figure 1.3). For an exhaustive study of the more general defined p-tangential bodies we refer to [27, Section 2.2 and p. 149].

The following result was proved by Schneider and characterizes tangential bodies as the only convex bodies whose inner parallel bodies are homothetic copies of them.

Theorem 1.3.3 ([27, Lemma 3.1.14]). *Let $K, E \in \mathcal{K}^n_n$ and let $\lambda \in \big(-\mathrm{r}(K;E), 0\big)$. Then K_λ is homothetic to K if and only if K is homothetic to a tangential body of E.*

1.4 p-sum of convex bodies

In 1962 Firey [6] introduced the following generalization of the classical Minkowski addition.

Definition 1.4.1. *For $p \geq 1$ and $K, E \in \mathcal{K}_0^n$, the p*-sum *(or L_p*-sum*) of K and E is the convex body $K +_p E \in \mathcal{K}_0^n$ whose support function is given by*

$$h(K +_p E, u) = \Big(h(K,u)^p + h(E,u)^p\Big)^{1/p}, \quad u \in \mathbb{S}^{n-1}. \tag{1.9}$$

When $p = 1$, the latter defines the usual Minkowski addition, whereas for $p = \infty$ we have that

$$h(K +_\infty E, u) = \max\big\{h(K,u), h(E,u)\big\},$$

this is,

$$K +_\infty E = \mathrm{conv}(K \cup E) \tag{1.10}$$

(see Figure 1.4). We notice also that when combining the p-sum with the scalar multiplication μK, the following facts hold: for all $K, E \in \mathcal{K}_0^n$, $\mu, \lambda > 0$ and $p \geq 1$,

$$\mu K +_p \mu E = \mu(K +_p E)$$

and

$$\mu K +_p \lambda K = \big(\mu^p + \lambda^p\big)^{1/p} K. \tag{1.11}$$

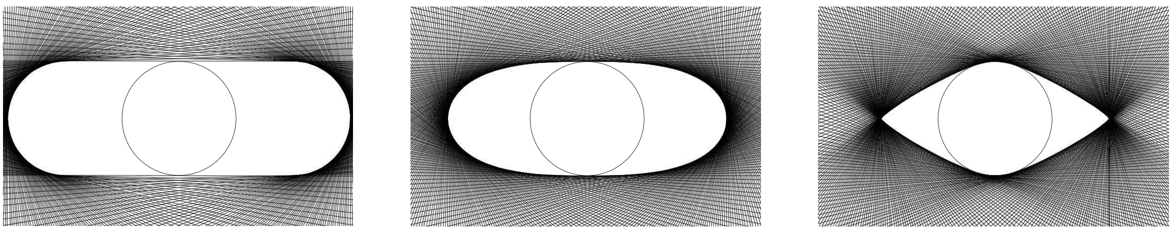

Figure 1.4: $[-2e_1, 2e_1] +_p B_2$, where $p = 1,\ 1.5,\ 10$.

Moreover, in [6, Theorem 1] it is shown that for all $1 \leq p \leq q \leq \infty$,

$$K +_q E \subseteq K +_p E \tag{1.12}$$

and

$$\frac{1}{2^{(p-1)/p}}(K + E) \subseteq K +_p E. \tag{1.13}$$

We notice that the cancellation law for convex bodies $K, L, M \in \mathcal{K}_0^n$ also works for the p-sum $+_p$ for all $1 \leq p < \infty$, i.e., if $K +_p M = L +_p M$, then $K = L$.

We also would like to observe a straightforward *geometrically important* difference between the Minkowski sum and the p-sum: the loss of the translation invariance.

The works [19, 20] of Lutwak, where a systematic study of means of convex bodies is taken up, constitute the outstanding rising of the nowadays known as L_p-Brunn-Minkowski theory. In the last years many important developments of this theory have come out. For further details, as well as detailed bibliography on the topic we refer to [27, Chapter 9] and the references therein.

1.5 Mixed volumes and quermassintegrals

Given a convex body $K \in \mathcal{K}^n$, its *volume* is defined as its Lebesgue measure and it will be denoted by $\mathrm{vol}(K)$. Therefore, $\mathrm{vol}(\cdot)$ satisfies the following known properties:

Lemma 1.5.1. *Let $K, L \in \mathcal{K}^n$.*

i) *If* $\dim K = n$, *then* $\mathrm{vol}(K) > 0$. *If* $\dim K \leq n-1$, *then* $\mathrm{vol}(K) = 0$.

ii) $\mathrm{vol}(\lambda K) = \lambda^n \mathrm{vol}(K)$ *for* $\lambda \geq 0$.

iii) *The volume* $\mathrm{vol} : \mathcal{K}^n \longrightarrow [0, \infty)$ *is a continuous function on the space of convex bodies.*

iv) *If* $L \subseteq K$ *then* $\mathrm{vol}(L) \leq \mathrm{vol}(K)$ *and equality holds, for* $\dim L = n$, *if and only if* $L = K$.

Combining the notions of volume and Minkowski sum, the concept of mixed volume appears. For a detailed study of mixed volumes we refer to Section 5.1 in [27].

Theorem 1.5.2. *Let $K_1, \dots, K_m \in \mathcal{K}^n$ and $\lambda_1, \dots, \lambda_m \geq 0$. There are coefficients* $\mathrm{V}(K_{i_1}, \dots, K_{i_n})$, $1 \leq i_1, \dots, i_n \leq m$, *called* mixed volumes, *which are symmetric in the indices and such that*

$$\mathrm{vol}\big(\lambda_1 K_1 + \cdots + \lambda_m K_m\big) = \sum_{i_1, \dots, i_n = 1}^{m} \mathrm{V}(K_{i_1}, \dots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}. \tag{1.14}$$

We notice that the polynomial expression given in (1.14) does not exist for the p-sum $+_p$ when $p > 1$ (see e.g. [7]).

Further, for $K_1, \dots, K_n \in \mathcal{K}^n$ it is known that there exist finite Borel measures on $\mathbb{S}^{n-1}$, the *mixed surface area measures* $\mathrm{S}(K_2, \dots, K_n, \cdot)$, such that

$$\mathrm{V}(K_1, \dots, K_n) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(K_1, u) \, \mathrm{dS}(K_2, \dots, K_n, u).$$

From now on, for the sake of brevity we will use the abbreviation

$$\big(K_1[r_1], \dots, K_m[r_m]\big) := \Big(K_1, \overset{(r_1)}{\dots}, K_1, \dots, K_m, \overset{(r_m)}{\dots}, K_m\Big).$$

If only two convex bodies $K, E \in \mathcal{K}^n$ are involved, the mixed volume $\mathrm{V}\big(K[n-i], E[i]\big) = \mathrm{W}_i(K; E)$ is called the *i-th quermassintegrals* of K (relative to E), $i = 0, \dots, n$. In particular, we have $\mathrm{W}_0(K; E) = \mathrm{vol}(K)$ and $\mathrm{W}_n(K; E) = \mathrm{vol}(E)$. We notice that

$$\mathrm{W}_i(K; E) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(K, u) \, \mathrm{dS}\big(K[n-i-1], E[i], u\big). \tag{1.15}$$

Some useful properties of the mixed volumes and mixed surface area measures are listed in the following proposition; they will be needed throughout this work.

Proposition 1.5.3. *Let $K, L, K_1, \dots, K_n \in \mathcal{K}^n$. The following properties hold:*

i) $\mathrm{V}(K, \dots, K) = \mathrm{vol}(K)$.

ii) $\mathrm{S}(K_2, \dots, K_n, \cdot)$ *is symmetric in the indices for any permutation.*

iii) $\mathrm{V}(rK + sL, K_2, \dots, K_n) = r\mathrm{V}(K, K_2, \dots, K_n) + s\mathrm{V}(L, K_2, \dots, K_n)$ *for every* $r, s \geq 0$, *and* $\mathrm{S}(rK + sL, K_2, \dots, K_{n-1}, \cdot) = r\mathrm{S}(K, K_2, \dots, K_{n-1}, \cdot) + s\mathrm{S}(L, K_2, \dots, K_{n-1}, \cdot)$, *i.e., mixed volumes and surface area measures are linear in each argument.*

iv) *Mixed volumes are continuous functions on* $(\mathcal{K}^n)^n$, *and rigid motion invariant. Mixed surface area measures are weakly continuous on* $(\mathcal{K}^n)^{n-1}$.

v) *If* $K \subseteq L$ *then* $0 \leq \mathrm{V}(K, K_2, \dots, K_n) \leq \mathrm{V}(L, K_2, \dots, K_n)$.

If only two convex bodies are considered, Theorem 1.5.2 yields the so-called *relative Steiner formula* or *Minkowski-Steiner formula*.

Theorem 1.5.4 (The (relative) Steiner formula. Steiner, [28]). *Let $K, E \in \mathcal{K}^n$. The volume of the outer parallel body of K with respect to E at distance $\lambda \geq 0$ is expressed as*

$$\mathrm{vol}(K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K; E) \lambda^i. \tag{1.16}$$

We notice that if $E \in \mathcal{K}^n_n$, then the polynomial on the right-hand side of (1.16), the so-called *relative Steiner polynomial*, has degree n, i.e., the dimension of the space.

Taking into account that quermassintegrals are particular cases of mixed volumes, the following *Steiner formula* for the relative quermassintegrals can be also obtained.

Theorem 1.5.5 (Steiner formula for relative quermassintegrals). *Let $K, E \in \mathcal{K}^n$ and $\lambda \geq 0$. The relative i-th quermassintegral, $i = 0, \dots, n$, of the outer parallel body of K (relative to E), $K + \lambda E$, can be expressed as a polynomial in the parameter λ,*

$$\mathrm{W}_i(K + \lambda E; E) = \sum_{k=0}^{n-i} \binom{n-i}{k} \mathrm{W}_{i+k}(K; E) \lambda^k. \tag{1.17}$$

1.6 Inequalities for mixed volumes and other related results

In this section we collect some of the most relevant inequalities regarding mixed volumes, which will be needed throughout this work. Relating the volume to the Minkowski addition of convex bodies, one is led to the famous Brunn-Minkowski inequality: it ensures the concavity of the n-th root of the volume functional, $\mathrm{vol}^{1/n} : \mathcal{K}^n \longrightarrow [0, \infty)$:

Theorem 1.6.1 (Brunn-Minkowski's inequality). *For convex bodies $K, L \in \mathcal{K}^n$ and $\lambda \in [0,1]$,*

$$\mathrm{vol}\big((1-\lambda)K + \lambda L\big)^{1/n} \geq (1-\lambda)\mathrm{vol}(K)^{1/n} + \lambda \mathrm{vol}(L)^{1/n}. \tag{1.18}$$

Equality for some $\lambda \in (0,1)$ holds if and only if K and L either lie in parallel hyperplanes or are homothetic.

Using the monotonicity of the volume (Lemma 1.5.1) and properties relating Minkowski sum and difference (Lemma 1.2.1), one gets a Brunn-Minkowski inequality for the Minkowski difference:

$$\mathrm{vol}(K \sim L)^{1/n} \leq \mathrm{vol}(K)^{1/n} - \mathrm{vol}(L)^{1/n}. \tag{1.19}$$

In the setting of the L_p-Brunn-Minkowski theory, the L_p-Brunn-Minkowski inequality, i.e., a Brunn-Minkowski type inequality for the p-sum, establishes that if $K, L \in \mathcal{K}^n_{(0)}$, $1 \leq p < \infty$, then

$$\mathrm{vol}(K +_p L)^{p/n} \geq \mathrm{vol}(K)^{p/n} + \mathrm{vol}(L)^{p/n}, \tag{1.20}$$

with equality if and only if K and L are homothetic (see e.g. [27, Theorem 9.1.3]).

There exists also a general Brunn-Minkowski theorem stating an analogous inequality for mixed volumes (see [27, Theorem 7.4.5]), and in particular, for every (relative) quermassintegral, which we will need later: if $K, L, E \in \mathcal{K}^n$ and $\lambda \in [0,1]$, then, for all $i = 0, \dots, n-2$,

$$\mathrm{W}_i\big((1-\lambda)K + \lambda L; E\big)^{1/(n-i)} \geq (1-\lambda)\mathrm{W}_i(K;E)^{1/(n-i)} + \lambda \mathrm{W}_i(L;E)^{1/(n-i)},$$

whereas

$$\mathrm{W}_{n-1}\big((1-\lambda)K + \lambda L; E\big) = (1-\lambda)\mathrm{W}_{n-1}(K;E) + \lambda \mathrm{W}_{n-1}(L;E).$$

In the setting of the L_p-Brunn-Minkowski theory, the corresponding inequality takes the following form (see [27, Corollary 9.1.5]):

Theorem 1.6.2. *For convex bodies $E \in \mathcal{K}^n_n$, $K, L \in \mathcal{K}^n_{(0)}$, and for all $p \geq 1$, $i \in \{0, \dots, n-1\}$,*

$$\mathrm{W}_i\big(K +_p L; E\big)^{p/(n-i)} \geq \mathrm{W}_i(K;E)^{p/(n-i)} + \mathrm{W}_i(L;E)^{p/(n-i)}. \tag{1.21}$$

An additional inequality that we will need throughout the thesis is Minkowski's inequality for sums (see e.g. [10, Corollary 1.6]). It states that if $x_1, y_1, \dots, x_n, y_n \geq 0$ and $p \geq 1$, then

$$\Big((x_1+y_1)^p + \cdots + (x_n+y_n)^p\Big)^{1/p} \leq \big(x_1^p + \cdots + x_n^p\big)^{1/p} + \big(y_1^p + \cdots + y_n^p\big)^{1/p} \tag{1.22}$$

1.7 On hypergeometric functions

In this final section, we collect some well-known functions, as Euler's Beta and Gamma functions, as well as their relation with the so-called *hypergeometric function* ${}_2F_1$. This will allow us to compute explicitly some Beta-type integrals that will be used in the last chapter of this thesis.

The *Beta function of Euler*, $\mathrm{B} : (0,\infty) \times (0,\infty) \longrightarrow \mathbb{R}$, is defined by

$$\mathrm{B}(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,\mathrm{d}t,$$

and the *Gamma function of Euler*, $\Gamma : (0,\infty) \longrightarrow \mathbb{R}$, is given by

$$\Gamma(p) = \int_0^\infty t^{p-1}e^{-t}\,\mathrm{d}t.$$

Both functions are related by means of

$$\mathrm{B}(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

The so-called *hypergeometric function* ${}_2F_1(a,b;c;x)$, given by the integral representation

$$ {}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-xt)^{-a}t^{b-1}(1-t)^{c-b-1}\,\mathrm{d}t, \tag{1.23}$$

with $|x| < 1$, $a \in \mathbb{R}$ and $c > b > 0$ (see e.g. [8, (1.11.10)]), happens to be related with the Beta and Gamma functions and will appear in some inequalities at the end of the work.

In the last chapter the following integral expression, closed related with the above defined hypergeometric function, will be needed: for $\alpha \in \mathbb{R}$, $\mathrm{r} > 0$ and $p \geq 1$, let

$$\mathrm{I}_{\alpha,p}(\lambda) := \int_{-\mathrm{r}}^{\lambda} |t|^p\big(\mathrm{r}^p - |t|^p\big)^{\alpha/p}\,\mathrm{d}t, \quad -\mathrm{r} < \lambda \leq 0. \tag{1.24}$$

Lemma 1.7.1. *Let $p \geq 1$, $\alpha > -p$ and $\mathrm{r} > 0$. Then, for all $-\mathrm{r} < \lambda \leq 0$,*

$$\mathrm{I}_{\alpha,p}(\lambda) = \frac{\mathrm{r}^{\alpha+p+1}}{p}\left[\mathrm{B}\left(\frac{p+1}{p}, \frac{p+\alpha}{p}\right) - \frac{p}{p+1}\frac{|\lambda|^{p+1}}{\mathrm{r}^{p+1}}\,{}_2F_1\left(-\frac{\alpha}{p}, \frac{p+1}{p}; \frac{2p+1}{p}; \frac{|\lambda|^p}{\mathrm{r}^p}\right)\right].$$

Proof. With the change of variable $u = \big(|t|/\mathrm{r}\big)^p$, we have

$$\begin{aligned}
\mathrm{I}_{\alpha,p}(\lambda) &= \int_{-\mathrm{r}}^{\lambda} |t|^p\big(\mathrm{r}^p - |t|^p\big)^{\alpha/p}\,\mathrm{d}t = \mathrm{r}^\alpha \int_{(|\lambda|/\mathrm{r})^p}^1 \frac{\mathrm{r}^{p+1}}{p}\, u^{1/p}(1-u)^{\alpha/p}\,\mathrm{d}u \\
&= \frac{\mathrm{r}^{\alpha+p+1}}{p}\left(\int_0^1 u^{1/p}(1-u)^{\alpha/p}\,\mathrm{d}u - \int_0^{(|\lambda|/\mathrm{r})^p} u^{1/p}(1-u)^{\alpha/p}\,\mathrm{d}u\right) \\
&= \frac{\mathrm{r}^{\alpha+p+1}}{p}\left(\mathrm{B}\left(1+\frac{1}{p}, 1+\frac{\alpha}{p}\right) - \int_0^{(|\lambda|/\mathrm{r})^p} u^{1/p}(1-u)^{\alpha/p}\,\mathrm{d}u\right).
\end{aligned}$$

Finally, taking into account (1.23), we get the value of the integral in the above expression:

$$\begin{aligned}
\int_0^{(|\lambda|/\mathrm{r})^p} u^{1/p}(1-u)^{\alpha/p}\,\mathrm{d}u &= \frac{|\lambda|^{p+1}}{\mathrm{r}^{p+1}} \int_0^1 s^{1/p}\left(1 - \frac{|\lambda|^p}{\mathrm{r}^p}s\right)^{\alpha/p}\mathrm{d}s \\
&= \frac{|\lambda|^{p+1}}{\mathrm{r}^{p+1}} \frac{\Gamma\left(1+\frac{1}{p}\right)\Gamma(1)}{\Gamma\left(2+\frac{1}{p}\right)}\,{}_2F_1\left(-\frac{\alpha}{p}, 1+\frac{1}{p}; 2+\frac{1}{p}; \frac{|\lambda|^p}{\mathrm{r}^p}\right) \\
&= \frac{p}{p+1}\frac{|\lambda|^{p+1}}{\mathrm{r}^{p+1}}\,{}_2F_1\left(-\frac{\alpha}{p}, 1+\frac{1}{p}; 2+\frac{1}{p}; \frac{|\lambda|^p}{\mathrm{r}^p}\right).
\end{aligned}$$

□

Chapter 2

p-difference of convex bodies: First properties

As it was stated in the previous chapter, Minkowski's difference is the subtraction counterpart of the Minkowski sum. In this chapter we introduce the notion of p-difference, which is an extension of the Minkowski difference in the setting of the L_p-Brunn-Minkowski theory. We prove several properties of this new operation, introducing also the notion of p-(inner) parallel bodies. We prove an analog of the concavity of the family of classical parallel bodies for the p-parallel ones, as well as the continuity of this new family, in its definition parameter. Further results on inner parallel bodies are extended to p-inner ones; for instance, we show that tangential bodies are characterized as the only convex bodies whose p-inner parallel bodies are homothetic copies of them. The original work that we collect in this chapter can be found in [21].

2.1 p-difference of convex bodies. Definition and first properties

There are several definitions of Minkowski's difference, all of which turn out to be equivalent (see [27, p. 146]). On one hand, as mentioned already in Section 1.2, the Minkowski difference of two non-empty sets $A, B \subseteq \mathbb{R}^n$ can be defined by (1.3). On the other hand, for convex bodies $K, E \in \mathcal{K}^n$, the Minkowski difference $K \sim E$ can be obtained as the Wulff-shape of the function $\psi(u) = h(K, u) - h(E, u)$, as in (1.4).

These two constructions settle down the basis for describing the p-difference of convex bodies. We define this operation in the following way.

Definition 2.1.1. *Let $K, E \in \mathcal{K}_0^n$, $E \subseteq K$, and let $p \geq 1$. The* p-difference *of K and E is the largest convex body $K \sim_p E \in \mathcal{K}_0^n$ such that*

$$(K \sim_p E) +_p E \subseteq K. \tag{2.1}$$

On the one hand, it is clear from the above definition that

$$K \sim_p E = \bigcup_{\substack{M \in \mathcal{K}_0^n \\ M +_p E \subseteq K}} M, \tag{2.2}$$

because the above union is a convex body. Indeed, writing $\mathcal{F} = \{M \in \mathcal{K}_0^n : M +_p E \subseteq K\}$, if $K_1, K_2 \in \mathcal{F}$ then also $\operatorname{conv}(K_1 \cup K_2) \in \mathcal{F}$, which implies that the above standard union is a convex set. Now given a sequence of points $(x_n)_n \subseteq \bigcup_{M \in \mathcal{F}} M$ with $\lim_{n \to \infty} x_n = x$, there exists a sequence $(M_n)_n \subseteq \mathcal{F}$ with $x_n \in M_n$ for each $n \in \mathbb{N}$. By Blaschke's Selection Theorem 1.1.14 we can choose $(M_n)_n$ to be convergent to a convex body $\overline{M}$, and it is clear that $\overline{M} \in \mathcal{F}$. Therefore, $x \in \overline{M} \subseteq \bigcup_{M \in \mathcal{F}} M$.

Taking (1.10) into account, it is easy to check that $K \sim_\infty E = K$, and for $p = 1$ we obviously obtain the classical Minkowski difference of K and E.

On the other hand, and looking back at (1.4), it would be desirable that such a kind of expression also works for the p-difference, in order to use the powerful Wulff-shape structure and its connection with the support function. The following theorem shows that this is the case. First we will assume that $1 \leq p < \infty$. The case $p = \infty$ will be treated later.

Theorem 2.1.2. *Let $1 \leq p < \infty$ and let $K, E \in \mathcal{K}_0^n$ with $E \subseteq K$. Then,*

$$K \sim_p E = \bigcap_{u \in \mathbb{S}^{n-1}} \left\{ x \in \mathbb{R}^n : \langle x, u \rangle \leq \big(h(K,u)^p - h(E,u)^p \big)^{1/p} \right\}. \tag{2.3}$$

Proof. We show (2.3) using the already known expression (2.2) for $K \sim_p E$. Let

$$L = \bigcap_{u \in \mathbb{S}^{n-1}} \left\{ x \in \mathbb{R}^n : \langle x, u \rangle \leq \big(h(K,u)^p - h(E,u)^p \big)^{1/p} \right\}.$$

Remark 1.2.3 ensures that

$$h(L,u) \leq \big(h(K,u)^p - h(E,u)^p \big)^{1/p},$$

and so we have $h(L,u)^p + h(E,u)^p \leq h(K,u)^p$ for all $u \in \mathbb{S}^{n-1}$. It yields $L \subseteq K \sim_p E$.

Conversely, if $x \in K \sim_p E$, then there exists $M \in \mathcal{K}_0^n$ satisfying $M +_p E \subseteq K$, such that $x \in M$, and from this condition we obtain that

$$h(M,u)^p + h(E,u)^p \leq h(K,u)^p \quad \text{for all } u \in \mathbb{S}^{n-1}.$$

It implies that $\langle x, u \rangle \leq \big(h(K,u)^p - h(E,u)^p \big)^{1/p}$ for all $u \in \mathbb{S}^{n-1}$, i.e., $x \in L$, which shows the reverse inclusion and concludes the proof. $\square$

We observe that by Remark 1.2.3, $K \sim_p E$ is a convex body whose support function satisfies

$$h(K \sim_p E, u) \leq \bigl(h(K,u)^p - h(E,u)^p\bigr)^{1/p}. \tag{2.4}$$

For $p = \infty$, the right-hand side in the defining inequality in (2.3) shall be seen as the limit when $p \to \infty$. Then, the case $p = \infty$ is not achieved in the above result as the following example shows.

Example 2.1.3. *Clearly for the cube C_n we have*

$$\bigcap_{u \in \mathbb{S}^{n-1}} \left\{ x \in \mathbb{R}^n : \langle x, u \rangle \leq \lim_{p \to \infty} \bigl(h(C_n,u)^p - h(B_n,u)^p\bigr)^{1/p} \right\} = \{0\},$$

whereas $C_n \sim_\infty B_n = C_n$ using (2.2).

The problem relies on the fact that $h(K,u) = h(E,u)$ for some $u \in \mathbb{S}^{n-1}$ provokes a *devastating geometrical* effect on the intersection expression in (2.3), whereas it is almost unseen by the union used in (2.2). Indeed, if $h(K,u) = h(E,u)$ holds for some $u \in \mathbb{S}^{n-1}$ then

$$\lim_{p \to \infty} \bigl(h(K,u)^p - h(E,u)^p\bigr)^{1/p} = 0.$$

However, if $\operatorname{bd} K \cap \operatorname{bd} E = \emptyset$, as $E \subseteq \operatorname{int} K$, we have

$$\lim_{p \to \infty} \bigl(h(K,u)^p - h(E,u)^p\bigr)^{1/p} = h(K,u)$$

obtaining that

$$\bigcap_{u \in \mathbb{S}^{n-1}} \left\{ x \in \mathbb{R}^n : \langle x, u \rangle \leq \lim_{p \to \infty} \bigl(h(K,u)^p - h(E,u)^p\bigr)^{1/p} \right\} = K.$$

Remark 2.1.4. *From now on, we set $K \sim_\infty E = K$, i.e.,*

$$\bigl(h(K,u)^p - h(E,u)^p\bigr)^{1/p} = h(K,u) \quad \text{for } p = \infty \text{ and all } u \in \mathbb{S}^{n-1},$$

which is the limit when $p \to \infty$ except if $h(K,u) = h(E,u) \neq 0$ for some $u \in \mathbb{S}^{n-1}$. With this convention, Theorem 2.1.2 remains true for $p = \infty$ too.

Next we state the p-analogue of Lemma 1.2.1 and further properties of the p-difference.

Lemma 2.1.5. *Let $K, E, M \in \mathcal{K}_0^n$ and $p \geq 1$. Then, assuming the suitable inclusions among the sets, the following properties hold:*

i) $(K \sim_p E) +_p E \subseteq K$. *Equality holds if and only if E is a p-summand of K, i.e., if there exists $L \in \mathcal{K}_0^n$ with $K = L +_p E$.*

ii) $(K \sim_p E) +_p M \subseteq (K +_p M) \sim_p E$.

iii) $(K \sim_p E) \sim_p M = K \sim_p (E +_p M)$.

iv) $K +_p E \subseteq M$ *if and only if* $K \subseteq M \sim_p E$.

v) $(K +_p E) \sim_p E = K$, *for* $p \neq \infty$.

vi) $(\lambda K) \sim_p (\mu K) = \big(\lambda^p - \mu^p\big)^{1/p} K$, *for* $0 \leq \mu \leq \lambda$.

vii) $\lambda(K \sim_p E) = (\lambda K) \sim_p (\lambda E)$, *for all* $\lambda > 0$.

Proof. The proofs of i), ii) and iii) are direct applications of (2.3) and (1.9), whereas iv) follows directly from Definition 2.1.1 (cf. (2.2)).

For v), we observe that since $h\big((K +_p E) \sim_p E, u\big)^p \leq h(K,u)^p$ for all $u \in \mathbb{S}^{n-1}$ (cf. (2.4)), we obtain that $(K +_p E) \sim_p E \subseteq K$. Now, iv) for $M = K +_p E$ yields $K \subseteq (K +_p E) \sim_p E$, and thus, $K = (K +_p E) \sim_p E$.

In order to prove vi), we first notice that from (1.9) we get $\big(\lambda^p - \mu^p\big)^{1/p} K +_p \mu K = \lambda K$. Then, by v) we obtain the result.

Finally we show vii). Taking support functions and using (2.4), it is immediate to see that $\big(\lambda(K \sim_p E)\big) +_p \lambda E \subseteq \lambda K$, which yields the inclusion $\lambda\big(K \sim_p E\big) \subseteq (\lambda K) \sim_p \big(\lambda E\big)$. Then, applying this relation to λK, λE and $1/\lambda$, we finally get

$$K \sim_p E \subseteq \frac{1}{\lambda}\Big[(\lambda K) \sim_p \big(\lambda E\big)\Big] \subseteq K \sim_p E. \qquad \square$$

The following lemma is an easy consequence of (1.12) and (2.2).

Lemma 2.1.6. *Let* $K, E \in \mathcal{K}^n_0$, $E \subseteq K$, *and let* $1 \leq p \leq q \leq \infty$. *Then*

$$K \sim_p E \subseteq K \sim_q E. \tag{2.5}$$

Remark 2.1.7. *We observe that the inclusion* (2.5) *may be strict, as relation vi) of Lemma 2.1.5 shows, because the map* $t \mapsto (1 - \varepsilon^t)^{1/t}$, $0 \leq \varepsilon \leq 1$, *is strictly increasing.*

Finally we deal with the continuity of this new operation in $\mathcal{K}^n_0$. It is known (see [27, Remark 3.1.12]) that Minkowski's subtraction is not continuous with respect to the Hausdorff metric $\delta_{\mathcal{H}}$. Next we prove that the same holds for the p-difference of convex bodies, for any $1 < p < \infty$. For $p = \infty$, the continuity holds trivially, since $K \sim_\infty E = K$ (cf. Remark 2.1.4).

Proposition 2.1.8. *Let* $1 < p < \infty$. *The p-difference is not continuous with respect to the Hausdorff metric on* $\mathcal{K}^n_0$.

Proof. We consider the convex bodies

$$K = \operatorname{conv}\Big(B_2 \cup \big\{(2,1)^\intercal, (2,-1)^\intercal\big\}\Big),$$
$$K_i = \operatorname{conv}\Big(B_2 \cup \big\{(2,1)^\intercal, (2,-1+1/i)^\intercal\big\}\Big), \quad i \in \mathbb{N}.$$

Clearly, K_i converges to K with respect to the Hausdorff metric in $\mathcal{K}^2_0$. Indeed, it can be seen that $\delta_{\mathcal{H}}(K_i, K) \leq 1/i$.

On the one hand, we have (see (2.5)) $K \sim_p B_2 \supseteq K \sim B_2 = [0, e_1]$ for all $p > 1$. On the other hand, we claim that $K_i \sim_p B_2 = \{0\}$ for every $i \in \mathbb{N}$ and all $p > 1$, and hence we could conclude that $K_i \sim_p B_2$ does not converge to $K \sim_p B_2$, as required.

In order to prove the claim, let $i \in \mathbb{N}$ and we suppose, by contradiction, that there exists $u = (a, b)^\intercal \in K_i \sim_p B_2$, $u \neq 0$, which yields $[0, u] +_p B_2 \subseteq K_i$. If $b \neq 0$ then

$$h\big([0, u] +_p B_2, \operatorname{sgn}(b)e_2\big) = \big(1 + |b|^p\big)^{1/p} > 1 = h(K_i, \operatorname{sgn}(b)e_2),$$

where, as usual, sgn denotes the sign function. Clearly it is not possible, and therefore $b = 0$, i.e., $u = ae_1$. Now, if $a < 0$ then

$$h\big([0, ae_1] +_p B_2, -e_1\big) = \big(1 + |a|^p\big)^{1/p} > 1 = h\big(K_i, -e_1\big),$$

again a contradiction. Hence, $a > 0$.

Let $u_i = (\cos\theta_i, \sin\theta_i)^\intercal \in \mathbb{S}^1$ be the unit outer normal vector to K_i at the "inclined bottom edge", i.e., the unique vector on $\mathbb{S}^1$ with coordinates $\cos\theta_i > 0$, $\sin\theta_i < 0$ (see Figure 2.1).

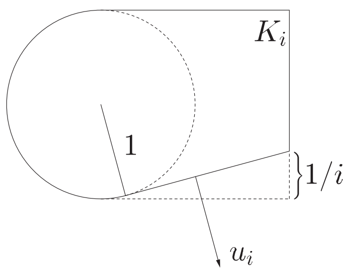

Figure 2.1: The p-difference is not continuous.

Then we have

$$h\big([0, ae_1] +_p B_2, u_i\big) = (1 + a^p \cos^p\theta_i)^{1/p} > 1 = h(B_2, u_i) = h(K_i, u_i),$$

which is impossible. Therefore, $K_i \sim_p B_2 = \{0\}$. $\square$

We conclude the section with a brief observation on a Brunn-Minkowski type inequality. Taking into account (1.20), the inclusion (2.1) provides, in a straightforward manner, a Brunn-Minkowski type inequality for the p-difference of two convex bodies (cf. (1.19)):

Proposition 2.1.9. *Let $K, E \in \mathcal{K}^n_{(0)}$ with $E \subseteq \operatorname{int} K$, and let $1 \leq p < \infty$. Then*

$$\operatorname{vol}(K \sim_p E)^{p/n} \leq \operatorname{vol}(K)^{p/n} - \operatorname{vol}(E)^{p/n}.$$

Equality holds if and only if K and E are homothetic convex bodies.

Proof. Combining (2.1) with the monotonicity of the volume (Lemma 1.5.1), we obtain that $\mathrm{vol}\big((K \sim_p E) +_p E\big)^{p/n} \leq \mathrm{vol}(K)^{p/n}$. Now, since $E \subseteq \operatorname{int} K$ then $0 \in \operatorname{int}(K \sim_p E)$, and hence we can apply (1.20) to get $\mathrm{vol}\big((K \sim_p E) +_p E\big)^{p/n} \geq \mathrm{vol}(K \sim_p E)^{p/n} + \mathrm{vol}(E)^{p/n}$. Joining both inequalities we get the result.

Next we deal with the equality case. If K and L are homothetic, then the equality is a direct consequence of the homogeneity of the volume (Lemma 1.5.1) and item vi) of Lemma 2.1.5.

Conversely, if $\mathrm{vol}(K \sim_p E)^{p/n} = \mathrm{vol}(K)^{p/n} - \mathrm{vol}(E)^{p/n}$, then we have equality in the two above inequalities, which implies, on the one hand, that $K = (K \sim_p E) +_p E$; and, on the other hand, that there exist $\mu > 0$ such that $K \sim_p E = \mu E$ (see the equality case in (1.20)). Therefore, we get $K = (K \sim_p E) +_p E = \mu E +_p E = (1 + \mu^p)^{1/p} E$ (cf. (1.11)). $\square$

Analogously, a Brunn-Minkowski type inequality for the (relative) quermassintegrals of the p-difference of two convex bodies can be obtained, but without characterizing the equality case.

2.2 p-inradius and p-kernel

From now on, taking Example 2.1.3 and Remark 2.1.4 into account, we will assume $p \neq \infty$.

When dealing with the Minkowski difference, the notions of inradius and kernel play a prominent role (see e.g. [25, 27] and the references therein). In addition to the classical (relative) inradius (see (1.5)), there exists another type of inradius:

Definition 2.2.1. *For two convex bodies $K, E \in \mathcal{K}^n_0$, the* (relative) inradius at the origin *of K with respect to E is given by $\rho(K; E) = \max\{\rho \geq 0 : \rho E \subseteq K\}$.*

Regarding (any of) the definitions of p-difference, one would be tempted to introduce, for $K, E \in \mathcal{K}^n_0$, $E \subseteq K$, an analogue of the relative inradius, i.e., a *p-inradius of K relative to E* as

$$\max\big\{r \geq 0 : M +_p rE \subseteq K \text{ for some } M \in \mathcal{K}^n_0\big\}.$$

However, it is immediate to see that the above number, for $p \geq 1$, coincides with the (relative) inradius at the origin $\rho(K; E)$. Indeed, if there exists $M \in \mathcal{K}^n_0$ such that $M +_p \rho E \subseteq K$, then

$$\rho E = \{0\} +_p \rho E \subseteq M +_p \rho E \subseteq K.$$

We observe that since the "naturally defined" p-inradius does not depend on p, and since, in general, $\mathrm{r}(K; E) \neq \rho(K; E)$, in order to develop a structured and systematic study of the p-difference, also valid for $p = 1$, we have the *heuristic necessity* of introducing a subfamily of $\mathcal{K}^n_0$ where also the trivial cases are avoided. Thus, for $E \in \mathcal{K}^n_0$, we define the subfamily, strongly depending on the *geometry* of the body $E \in \mathcal{K}^n_0$, given by

$$\mathcal{K}^n_{00}(E) = \big\{K \in \mathcal{K}^n_0 : \mathrm{r}(K; E) = \rho(K; E)\big\} = \big\{K \in \mathcal{K}^n_0 : 0 \in \ker(K; E)\big\}. \tag{2.6}$$

The last equality of sets follows easily: if $0 \in \ker(K;E)$ then $\mathrm{r}(K;E)E \subseteq K$, and thus we have $\mathrm{r}(K;E) \leq \rho(K;E)$, being the reverse inequality a direct consequence of the definition of inradius. Conversely, if $\mathrm{r}(K;E) = \rho(K;E)$ then $\mathrm{r}(K;E)E \subseteq K$, which implies that $0 \in \ker(K;E)$.

Definition 2.2.2. *For $E \in \mathcal{K}^n_0$, $K \in \mathcal{K}^n_{00}(E)$ and $1 \leq p < \infty$, we define the* p-kernel of K with respect to E *as*

$$\ker_p(K;E) = K \sim_p \mathrm{r}(K;E)E.$$

Then, using (2.5) it follows that, for $1 \leq p \leq q < \infty$,

$$\ker_p(K;E) \subseteq \ker_q(K;E). \tag{2.7}$$

As in the case of the kernel for $p = 1$ (see (1.6)), the following result shows that the p-kernel of $K \in \mathcal{K}^n_{00}(E)$ with respect to E, for any $1 \leq p < \infty$, has always dimension strictly less than n.

Proposition 2.2.3. *For $E \in \mathcal{K}^n_0$, let $K \in \mathcal{K}^n_{00}(E)$. Then, for any $1 \leq p < \infty$,*

$$\dim\bigl(\ker_p(K;E)\bigr) \leq n - 1.$$

Proof. Without loss of generality, we may assume that $\mathrm{r}(K;E) = 1$. Then the set of vectors $U = \bigl\{u \in \mathbb{S}^{n-1} : h(K,u) = h(E,u)\bigr\} \neq \emptyset$. We observe that if we show that

$$\dim \bigcap_{u \in U} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leq 0\bigr\} \leq n-1, \tag{2.8}$$

then, using (2.3) we would get that

$$K \sim_p E \subseteq \bigcap_{u \in U} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leq 0\bigr\},$$

which would finish the proof. Therefore, we have to prove (2.8).

Thus we assume, by contradiction, that

$$\dim \bigcap_{u \in U} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leq 0\bigr\} = n.$$

Let $v_1, \dots, v_n \in \mathbb{R}^n$ be n linearly independent vectors so that

$$A = \mathrm{pos}\{v_1, \dots, v_n\} \subseteq \mathrm{int} \bigcap_{u \in U} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leq 0\bigr\},$$

and let $u_1, \dots, u_n \in \mathbb{S}^{n-1}$ be n unit vectors such that

$$\bigcap_{u \in \{u_1,\dots,u_n\}} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leq 0\bigr\} = \bigcap_{u \in \mathrm{pos}\{u_1,\dots,u_n\}} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \leq 0\bigr\} = A.$$

Denoting by $\widetilde{U} = \operatorname{pos}\{u_1, \dots, u_n\} \cap \mathbb{S}^{n-1}$, we clearly have that $U \subseteq \operatorname{relint} \widetilde{U}$. Thus,

$$\varepsilon = \min\Big\{h(K,u) - h(E,u) : \, u \in \operatorname{cl}\big(\mathbb{S}^{n-1} \backslash \widetilde{U}\big)\Big\}$$

is a positive real number and hence,

$$\begin{aligned} A \cap \varepsilon B_n &= \left(\bigcap_{u \in \widetilde{U}} \{x \in \mathbb{R}^n : \langle x, u\rangle \le 0\}\right) \cap \left(\bigcap_{u \in \mathbb{S}^{n-1}} \{x \in \mathbb{R}^n : \langle x, u\rangle \le \varepsilon\}\right) \\ &= \left(\bigcap_{u \in \widetilde{U}} \{x \in \mathbb{R}^n : \langle x, u\rangle \le 0\}\right) \cap \left(\bigcap_{u \in \mathbb{S}^{n-1} \backslash \widetilde{U}} \{x \in \mathbb{R}^n : \langle x, u\rangle \le \varepsilon\}\right) \\ &\subseteq \bigcap_{u \in \mathbb{S}^{n-1}} \Big\{x \in \mathbb{R}^n : \langle x, u\rangle \le h(K,u) - h(E,u)\Big\} = K \sim E. \end{aligned}$$

This implies that $K \sim E$ has interior points (cf. (1.6)), a contradiction. □

Moreover, for a given $K \in \mathcal{K}^n_{00}(E)$, the dimension of the p-kernel may depend on the parameter p. Before stating in a precise way this property, we need the following result, which allows to determine directly the inradius and the p-kernel in a special situation.

Lemma 2.2.4. *For $E \in \mathcal{K}^n_0$, let $K \in \mathcal{K}^n_{00}(E)$ and $1 \le p < \infty$. If $K = L +_p E$ with $L \in \mathcal{K}^n_0$ such that $\dim L < \dim(L + E)$, then $\mathrm{r}(K;E) = 1$ and $\ker_p(K;E) = L$.*

Proof. Since $E \subseteq L +_p E = K$, then $\mathrm{r}(K;E) \ge 1$. Moreover, by (1.12) we have that $L +_p E \subseteq L + E$, and since $\dim L < \dim(L+E)$, we get $1 \le \mathrm{r}(K;E) \le \mathrm{r}(L+E;E) = 1$, i.e., $\mathrm{r}(K;E) = 1$. Finally, by Lemma 2.1.5 v),

$$\ker_p(K;E) = K \sim_p \mathrm{r}(K;E)E = K \sim_p E = (L +_p E) \sim_p E = L.$$ □

Proposition 2.2.5. *For $E \in \mathcal{K}^n_0$, let $K \in \mathcal{K}^n_{00}(E)$ and $1 \le p \le q < \infty$. Then*

$$\dim\big(\ker_p(K;E)\big) \le \dim\big(\ker_q(K;E)\big).$$

The inequality may be strict.

Proof. The statement is an immediate consequence of (2.7). The following example shows that the inequality may be strict.

Let $1 \le p < q < \infty$ and let $K = [-e_1, e_1] +_q B_n$. Then, by Lemma 2.2.4, $\ker_q(K;B_n) = [-e_1, e_1]$, and we claim that $\ker_p(K;B_n) = \{0\}$, which would show the statement.

Since $K \sim_p B_n \subseteq K \sim_q B_n = [-e_1, e_1]$ (see Lemma 2.1.6), we suppose, by contradiction, that there exists $\lambda e_1 \in K \sim_p B_n$ with $0 < \lambda \le 1$. It implies that $[0, \lambda e_1] \subseteq K \sim_p B_n$, i.e., $[0, \lambda e_1] +_p B_n \subseteq K$, and then

$$h\big([0,\lambda e_1] +_p B_n, u\big)^p \le h(K,u)^p = h\big([-e_1, e_1] +_q B_n, u\big)^p$$

for all $u \in \mathbb{S}^{n-1}$. In particular, taking

$$u = \left(\lambda^{p/(q-p)}, \left(1 - \lambda^{2p/(q-p)}\right)^{1/2}, 0, \dots, 0\right)^{\intercal} \in \mathbb{S}^{n-1},$$

the above inequality becomes $\lambda^{pq/(q-p)} + 1 \leq \left(\lambda^{pq/(q-p)} + 1\right)^{p/q}$, which is a contradiction because $p < q$ and $\lambda > 0$.

Due to the symmetry, the same argument shows that for all $-1 \leq \lambda < 0$, $\lambda e_1 \notin K \sim_p B_n$. Therefore, $K \sim_p B_n = \{0\}$, as claimed. □

2.3 p-inner parallel bodies

In this section we define a *full system of p-parallel bodies* of K for $1 < p < \infty$ and prove, in the spirit of the classical case, several properties of such a system. Since we will work with convex bodies lying in $\mathcal{K}^n_{00}(E)$, the lower bound for the parameters will be always (minus) the classical relative inradius (cf. (2.6)).

Definition 2.3.1. *Let $E \in \mathcal{K}^n_0$ and $K \in \mathcal{K}^n_{00}(E)$. Then, for any $1 \leq p < \infty$,*

$$K^p_\lambda = \begin{cases} K \sim_p |\lambda| E & \text{if } \; -\mathrm{r}(K;E) \leq \lambda \leq 0, \\ K +_p \lambda E & \text{if } \qquad\qquad 0 \leq \lambda < \infty. \end{cases}$$

*We refer to K^p_λ as the p-*inner *(respectively, p-*outer*) parallel body of K at distance $|\lambda|$ relative to E.*

2.3.1 On the continuity and the concavity of the family of p-parallel bodies.

Next we show that, similarly as in the case $p = 1$, the full system $\mu \mapsto K^p_\mu$ is $+_p$-concave with respect to set inclusion. First we introduce some notation for the p-sum of two real numbers. This sum will perfectly fit when operating with p-parallel bodies, and thus will play an important role in the following. Since negative real numbers are allowed, this definition extends (up to a constant) the classical p-mean of positive real numbers (see [12]).

Definition 2.3.2. *Let $+_p : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ denote the binary operation defined by*

$$a +_p b = \begin{cases} \mathrm{sgn}_2(a,b) \left(|a|^p + |b|^p\right)^{1/p} & \text{if } ab \geq 0, \\ \mathrm{sgn}_2(a,b) \left(\max\{|a|,|b|\}^p - \min\{|a|,|b|\}^p\right)^{1/p} & \text{if } ab < 0, \end{cases} \tag{2.9}$$

being $\mathrm{sgn}_2 : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ the function given by

$$\mathrm{sgn}_2(a,b) = \begin{cases} \mathrm{sgn}(a) = \mathrm{sgn}(b) & \text{if } ab > 0, \\ \mathrm{sgn}(a) & \text{if } ab \leq 0 \text{ and } |a| \geq |b|, \\ \mathrm{sgn}(b) & \text{if } ab \leq 0 \text{ and } |a| < |b|. \end{cases}$$

For $ab \geq 0$ this notion corresponds, up to maybe a signed constant, to the classical p-mean ([12, Chapter II]) and does not correspond to any of the more general ϕ-means considered in [12, Chapter III]. Commutativity, associativity and distributivity of $+_p$ can be easily proved distinguishing the sign of the involved real numbers. We collect these properties in the following lemma.

Lemma 2.3.3. *Let $a, b, c \in \mathbb{R}$. Then:*

i) $a +_p b = b +_p a$,

ii) $(a +_p b) +_p c = a +_p (b +_p c) = (a +_p c) +_p b$,

iii) $a(b +_p c) = (ab) +_p (ac)$.

In the setting of the L_p-Brunn-Minkowski theory, given $K \in \mathcal{K}_0^n$, a *p-scalar multiplication* is usually defined by

$$\lambda \cdot K = \lambda^{1/p} K \qquad \text{for } \lambda \geq 0$$

(see e.g. [27, p. 490]). We use the analogous notation to the above one in order to define, for $\lambda \geq 0$ and $a \in \mathbb{R}$, the product

$$\lambda \cdot a = \lambda^{1/p} a.$$

With the above notation, the following result on the *p-sum* of (arbitrary) real numbers shows that its behavior fits in this context.

Lemma 2.3.4. *Let $a, b \in \mathbb{R}$, $a \leq b$ and $\lambda \in [0, 1]$. Then, for all $p \geq 1$,*

$$(1 - \lambda) \cdot a +_p \lambda \cdot b \in [a, b].$$

Proof. First, if $ab \geq 0$ then

$$\begin{aligned}
(1 - \lambda) \cdot a +_p \lambda \cdot b &= \left[(1 - \lambda)^{1/p} a\right] +_p \left[\lambda^{1/p} b\right] \\
&= \operatorname{sgn}_2\left((1 - \lambda)^{1/p} a, \lambda^{1/p} b\right) \left((1 - \lambda)|a|^p + \lambda |b|^p\right)^{1/p} \\
&= \operatorname{sgn}(a)\big((1 - \lambda)|a|^p + \lambda |b|^p\big)^{1/p},
\end{aligned}$$

and thus, in both cases $a \geq 0$ and $a \leq 0$, we get, from the above identity,

$$a \leq (1 - \lambda) \cdot a +_p \lambda \cdot b \leq b.$$

So, we assume $ab \leq 0$, i.e., $a \leq 0 \leq b$. If $(1 - \lambda)^{1/p}|a| \geq \lambda^{1/p}|b|$, then

$$\operatorname{sgn}_2\left((1 - \lambda)^{1/p} a, \lambda^{1/p} b\right) = \operatorname{sgn}\left((1 - \lambda)^{1/p} a\right) = -1.$$

Therefore (see (2.9)),

$$\begin{aligned}
(1 - \lambda) \cdot a +_p \lambda \cdot b &= \operatorname{sgn}_2\left((1 - \lambda)^{1/p} a, \lambda^{1/p} b\right) \left((1 - \lambda)|a|^p - \lambda |b|^p\right)^{1/p} \\
&= -\big((1 - \lambda)|a|^p - \lambda |b|^p\big)^{1/p} \leq 0 \leq b
\end{aligned}$$

and

$$(1-\lambda)\cdot a +_p \lambda\cdot b \geq -\big((1-\lambda)|a|^p\big)^{1/p} = -(1-\lambda)^{1/p}|a| \geq -|a| = a.$$

The proof in the case $(1-\lambda)^{1/p}|a| \leq \lambda^{1/p}|b|$ is analogous to the previous one. $\square$

The defined p-sum of real numbers (2.9) turns out to be the right operation in order to describe the behavior of the system of p-parallel bodies, as the following proposition shows. The proof follows the one of (1.7) (see [27, pp. 148–149]), just interchanging the Minkowski sum and difference of convex bodies by the p-sum and p-difference, and the usual sum of real numbers by the p-sum defined in (2.9). We include the proof here for completeness.

Proposition 2.3.5. *For $E \in \mathcal{K}_0^n$, let $K, L \in \mathcal{K}_{00}^n(E)$, $-\mathrm{r}(K;E) \leq \mu < \infty$ and $-\mathrm{r}(L;E) \leq \sigma < \infty$. Then, for all $1 \leq p < \infty$, we have*

$$K_\mu^p +_p L_\sigma^p \subseteq (K +_p L)_{\mu +_p \sigma}^p. \tag{2.10}$$

Proof. Let $\mu, \sigma \geq 0$. From (1.11) we have $\mu E +_p \sigma E = \big(\mu^p + \sigma^p\big)^{1/p} E = (\mu +_p \sigma)E$, and then

$$K_\mu^p +_p L_\sigma^p = (K +_p \mu E) +_p (L +_p \sigma E) = (K +_p L) +_p (\mu +_p \sigma)E = (K +_p L)_{\mu +_p \sigma}^p.$$

Next, item i) of Lemma 2.1.5 yields

$$K_{-\mu}^p +_p L_{-\sigma}^p +_p (\mu +_p \sigma)E = (K \sim_p \mu E) +_p \mu E +_p (L \sim_p \sigma E) +_p \sigma E \subseteq K +_p L,$$

and hence, by Lemma 2.3.3, we get $K_{-\mu}^p +_p L_{-\sigma}^p \subseteq (K +_p L)_{-(\mu +_p \sigma)}^p = (K +_p L)_{(-\mu)+_p(-\sigma)}^p$.

If $\mu \geq \sigma$, using again item i) of Lemma 2.1.5 and Lemma 2.3.3 we obtain

$$\begin{aligned} K_\mu^p +_p L_{-\sigma}^p &= (K +_p \mu E) +_p (L \sim_p \sigma E) = K +_p (L \sim_p \sigma E) +_p \sigma E +_p \big(\mu +_p (-\sigma)\big)E \\ &\subseteq K +_p L +_p \big(\mu +_p (-\sigma)\big)E = (K +_p L)_{\mu +_p (-\sigma)}^p. \end{aligned}$$

Finally, if $\mu \leq \sigma$, items v) and i) of Lema 2.1.5 yield

$$\begin{aligned} K_\mu^p +_p L_{-\sigma}^p +_p \big(\sigma +_p (-\mu)\big)E &= (K +_p \mu E) +_p (L \sim_p \sigma E) +_p \big(\sigma +_p (-\mu)\big)E \\ &= K +_p (L \sim_p \sigma E) +_p \sigma E \subseteq K +_p L, \end{aligned}$$

which, together with Lemma 2.3.3, implies that

$$K_\mu^p +_p L_{-\sigma}^p \subseteq (K +_p L) \sim_p \big(\sigma +_p (-\mu)\big)E = (K +_p L)_{-(\sigma +_p(-\mu))}^p = (K +_p L)_{\mu +_p (-\sigma)}^p. \quad \square$$

As we already noticed when dealing with the p-difference, its combination with the p-sum is not necessarily commutative if the difference is taken first (cf. Lemma 2.1.5). Next result shows how this fact is translated into the setting of p-parallel bodies. For $p = 1$, i.e., for the usual relative parallel bodies, it can be found in [11].

Proposition 2.3.6. *For $E \in \mathcal{K}^n_0$, let $K \in \mathcal{K}^n_{00}(E)$, and let $\lambda, \mu \geq 0$. The following relations hold for any $1 \leq p < \infty$:*

i) $\bigl(K^p_\lambda\bigr)^p_\mu = K^p_{\lambda +_p \mu}$.

ii) $\bigl(K^p_{-\lambda}\bigr)^p_\mu \subseteq K^p_{(-\lambda)+_p\mu}$ *if* $\lambda \leq \mathrm{r}(K;E)$.

iii) $\bigl(K^p_{-\lambda}\bigr)^p_{-\mu} = K^p_{(-\lambda)+_p(-\mu)}$ *if* $\lambda +_p \mu \leq \mathrm{r}(K;E)$.

iv) $\bigl(K^p_\lambda\bigr)^p_{-\mu} = K^p_{\lambda+_p(-\mu)}$ *if* $\mu \leq \mathrm{r}(K;E) +_p \lambda$.

v) $\lambda K^p_\sigma = \bigl(\lambda K\bigr)^p_{\lambda\sigma}$ *for all* $-\mathrm{r}(K;E) \leq \sigma < \infty$.

Proof. Items i), ii) and iii) follow directly from the definition of p-sum, relation (2.10) with $L = \{0\}$ and Lemma 2.1.5 iii), respectively, taking into account (1.11).

In order to prove iv) we notice first that if $\lambda \geq \mu$ then, by i),

$$K^p_{\lambda+_p(-\mu)} +_p \mu E = K^p_{[\lambda+_p(-\mu)]+_p\mu} = K^p_\lambda,$$

and using Lemma 2.1.5 v) we obtain $K^p_{\lambda+_p(-\mu)} = (K^p_\lambda)^p_{-\mu}$.

Now if $\lambda < \mu$, item ii) yields

$$K^p_{\lambda+_p(-\mu)} +_p \mu E \subseteq K^p_{[\lambda+_p(-\mu)]+_p\mu} = K^p_\lambda,$$

and again from Lemma 2.1.5 v) we deduce that $K^p_{\lambda+_p(-\mu)} \subseteq (K^p_\lambda)^p_{-\mu}$. Moreover, using Lemma 2.1.5 ii) and v) , we obtain

$$\begin{aligned}(K^p_\lambda)^p_{-\mu} +_p |\lambda +_p (-\mu)|E &= (K^p_\lambda \sim_p \mu E) +_p |\lambda +_p (-\mu)|E \subseteq (K^p_\lambda +_p |\lambda +_p (-\mu)|E) \sim_p \mu E\\ &= K^p_{\lambda+_p|\lambda+_p(-\mu)|} \sim_p \mu E = K^p_\mu \sim_p \mu E = K,\end{aligned}$$

which shows the opposite inclusion $(K^p_\lambda)^p_{-\mu} \subseteq K^p_{\lambda+_p(-\mu)}$.

Finally, v) is straightforward from the definition of p-sum if $\sigma \geq 0$, and a direct consequence of Lemma 2.1.5 vii) if $\sigma \leq 0$. $\square$

From (2.10) and Proposition 2.3.6 v) we obtain the following result (cf. (1.8)).

Theorem 2.3.7. *For $E \in \mathcal{K}^n_0$, let $K \in \mathcal{K}^n_{00}(E)$. The full system of p-parallel sets of K relative to E, $1 \leq p < \infty$, is $+_p$-concave with respect to inclusion, i.e., for $\lambda \in [0,1]$ and $\mu, \sigma \in \bigl[-\mathrm{r}(K;E), \infty\bigr)$,*

$$(1-\lambda)\cdot K^p_\mu +_p \lambda \cdot K^p_\sigma \subseteq K^p_{(1-\lambda)\cdot\mu+_p\lambda\cdot\sigma}.$$

Proof. We notice that, by Lemma 2.3.4, $(1-\lambda)\cdot\mu +_p \lambda\cdot\sigma \geq -\mathrm{r}(K;E)$. Then

$$\begin{aligned}(1-\lambda)\cdot K^p_\mu +_p \lambda\cdot K^p_\sigma &= \bigl((1-\lambda)^{1/p}K^p_\mu\bigr) +_p \bigl(\lambda^{1/p}K^p_\sigma\bigr)\\ &= \bigl[(1-\lambda)^{1/p}K\bigr]^p_{(1-\lambda)^{1/p}\mu} +_p \bigl[\lambda^{1/p}K\bigr]^p_{\lambda^{1/p}\sigma} \subseteq K^p_{(1-\lambda)\cdot\mu +_p \lambda\cdot\sigma}.\end{aligned}$$ $\square$

Before showing the continuity of the full system of p-parallel bodies in the parameter λ with respect to the Hausdorff metric (cf. Proposition 2.1.8), we state a couple of simple but useful facts about p-parallel bodies:

Lemma 2.3.8. *Let $E \in \mathcal{K}^n_0$, $K \in \mathcal{K}^n_{00}(E)$ and $1 \leq p < \infty$. Then, for all $-\mathrm{r}(K;E) \leq \lambda < \infty$,*

i) $\mathrm{r}(K^p_\lambda;E) = \mathrm{r}(K;E) +_p \lambda$,

ii) $K^p_\lambda \in \mathcal{K}^n_{00}(E)$ *and*

iii) $\ker_p(K^p_\lambda;E) = \ker_p(K;E)$.

Proof. For the sake of brevity, we write $\mathrm{r} = \mathrm{r}(K;E)$. We prove i). Since $+_p$ is associative, if $\lambda \leq 0$ we have that $(\mathrm{r} +_p \lambda)E +_p |\lambda|E = \mathrm{r}E \subseteq K$, whereas for $\lambda > 0$ we get $(\mathrm{r} +_p \lambda)E \subseteq K +_p \lambda E = K^p_\lambda$. Then, in both cases, we can conclude that $(\mathrm{r} +_p \lambda)E \subseteq K^p_\lambda$ and thus $\mathrm{r}(K^p_\lambda;E) \geq \mathrm{r} +_p \lambda$. We suppose by contradiction that $\mathrm{r}(K^p_\lambda;E) > \mathrm{r} +_p \lambda$. Then, there exists $\delta > 0$ such that $\mathrm{r}(K^p_\lambda;E) = \delta +_p (\mathrm{r} +_p \lambda)$, which yields $\big(\delta +_p (\mathrm{r} +_p \lambda)\big)E \subseteq K^p_\lambda$. Now, if $\lambda \leq 0$ then

$$(\delta +_p \mathrm{r})E = \big(\delta +_p (\mathrm{r} +_p \lambda)\big)E +_p |\lambda|E \subseteq K,$$

whereas for $\lambda > 0$ we have $\big((\delta +_p \mathrm{r}) +_p \lambda\big)E \subseteq K +_p \lambda E$, which implies, by the cancellation law, that $(\delta +_p \mathrm{r})E \subseteq K$. Hence, in both cases, $\mathrm{r} \geq \delta +_p \mathrm{r} > \mathrm{r}$, a contradiction.

In order to prove ii) we observe that, by i) we have $(\mathrm{r} +_p \lambda)E \subseteq K$, for all $-\mathrm{r} \leq \lambda < \infty$. Then, $0 \in \ker(K^p_\lambda;E)$ and thus $K^p_\lambda \in \mathcal{K}^n_{00}(E)$.

Finally, iii) is a direct consequence of item i) and Proposition 2.3.6. □

Next we show the already mentioned continuity of the full system of p-parallel bodies in the parameter λ with respect to the Hausdorff metric:

Proposition 2.3.9. *Let $E \in \mathcal{K}^n_0$ and $K \in \mathcal{K}^n_{00}(E)$. The function $\Phi : \big[-\mathrm{r}(K;E), \infty\big) \longrightarrow \mathcal{K}^n_{00}(E)$ given by $\Phi(\lambda) = K^p_\lambda$ is continuous with respect to the Hausdorff metric in $\mathcal{K}^n_0$, for $1 \leq p < \infty$.*

Proof. From Lemma 2.3.8 ii) we have that the image of Φ is well-defined. We consider a sequence $\{\lambda_i\}_{i=1}^\infty \subseteq \big[-\mathrm{r}(K;E), \infty\big)$ such that $\lim_{i\to\infty} \lambda_i = \lambda$. We have to prove that $\lim_{i\to\infty} \Phi(\lambda_i) = \Phi(\lambda)$. We notice first that

$$\Phi(\lambda_i) = K^p_{\lambda_i} = \mathrm{WS}(\varphi_{\lambda_i}),$$
$$\Phi(\lambda) = K^p_\lambda = \mathrm{WS}(\varphi_\lambda),$$

where $\varphi_\mu : \mathbb{S}^{n-1} \longrightarrow [0, \infty)$ is the (continuous) function given by

$$\varphi_\mu(u) = \Big(h(K,u)^p + \mathrm{sgn}(\mu)|\mu|^p h(E,u)^p\Big)^{1/p}.$$

From the continuity of the functions $\varphi_{\lambda_i}, \varphi_\lambda$ and the compactness of $\mathbb{S}^{n-1}$ we deduce that there exist constants $M_{\lambda_i}, M_\lambda \geq 0$, $i \in \mathbb{N}$, such that

$$\varphi_{\lambda_i}\big(\mathbb{S}^{n-1}\big) \subseteq [0, M_{\lambda_i}], \quad \varphi_\lambda\big(\mathbb{S}^{n-1}\big) \subseteq [0, M_\lambda].$$

Since $\lim_{i\to\infty} \lambda_i = \lambda$, the sequence $\{\lambda_i\}_{i=1}^\infty$ is bounded, and then there exists a constant $M > 0$ such that $M \geq M_\lambda$ and $M \geq M_{\lambda_i}$, $i \in \mathbb{N}$. If $\lambda \neq 0$, then $\operatorname{sgn}(\lambda_i) = \operatorname{sgn}(\lambda)$ for i large enough, whereas if $\lambda = 0$, then

$$\varphi_{\lambda_i}^p - \varphi_0^p = \operatorname{sgn}(\lambda_i)|\lambda_i|^p h(E,\cdot)^p.$$

Therefore we have, in both cases, that

$$\big\|\varphi_{\lambda_i}^p - \varphi_\lambda^p\big\|_\infty = \Big\|\operatorname{sgn}(\lambda_i)\big(|\lambda_i|^p - |\lambda|^p\big)h(E,\cdot)^p\Big\|_\infty = \Big||\lambda_i|^p - |\lambda|^p\Big|\big\|h(E,\cdot)^p\big\|_\infty$$

for i large enough, and thus, $\lim_{i\to\infty}\big\|\varphi_{\lambda_i}^p - \varphi_\lambda^p\big\|_\infty = 0$.

Since the function $[0, M] \longrightarrow \mathbb{R}$ given by $t \mapsto t^{1/p}$, is uniformly continuous, then

$$\lim_{i\to\infty} \|\varphi_{\lambda_i} - \varphi_\lambda\|_\infty = 0.$$

Now, Lemma 1.2.4 implies that $\lim_{i\to\infty} \mathrm{WS}(\varphi_{\lambda_i}) = \mathrm{WS}(\varphi_\lambda)$, as desired. □

2.3.2 p-inner parallel bodies of special families of sets.

As it occurs when dealing with the p- and the Minkowski sums, for which the first one happens to be more difficult to *visualize*, the p-difference is, in general, also more difficult to deal with than the Minkowski difference. However, there are particular cases in which the p-difference is easy to determine. In this subsection we deal with special families of sets, for which p-parallel bodies can be explicitly determined. One of them are the tangential bodies (see Definition 1.3.2), whose p-inner parallel bodies can be easily obtained (see Figure 2.2, cf. Lemma 2.1.5 vi)).

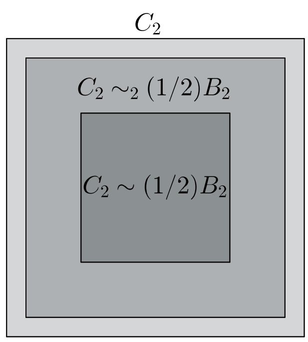

Figure 2.2: 1 and 2-difference of the square C_2 and the ball $(1/2)B_2$.

Proposition 2.3.10. *Let $E \in \mathcal{K}_0^n$ and let $K \in \mathcal{K}_0^n$ be a tangential body of E. Then, for all $1 \le p < \infty$ and any $\lambda \in [0,1]$,*

$$K^p_{-\lambda} = (1-\lambda^p)^{1/p} K. \tag{2.11}$$

Proof. We recall that $\mathrm{r}(K;E)=1$. Let $\mathcal{U} \subseteq \mathbb{S}^{n-1}$ be the set of those outer normal vectors for which the support hyperplane to K also supports E, i.e., such that $h(K,u)=h(E,u)$. Since any outer normal vector at a regular point of K is of this type, Theorem 1.1.6 ensures that

$$K = \bigcap_{u\in\mathcal{U}} \bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \le h(K,u)\bigr\}.$$

Therefore we get, on the one hand,

$$\begin{aligned} K^p_{-\lambda} = K \sim_p \lambda E &= \bigcap_{u\in\mathbb{S}^{n-1}} \Bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \le \bigl(h(K,u)^p - \lambda^p h(E,u)^p\bigr)^{1/p}\Bigr\} \\ &\subseteq \bigcap_{u\in\mathcal{U}} \Bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \le (1-\lambda^p)^{1/p} h(K,u)\Bigr\} = (1-\lambda^p)^{1/p} K. \end{aligned}$$

On the other hand, since $E \subseteq K$, then

$$h\bigl((1-\lambda^p)^{1/p}K,u\bigr)^p = (1-\lambda^p)h(K,u)^p \le h(K,u)^p - \lambda^p h(E,u)^p$$

for all $u \in \mathbb{S}^{n-1}$. Hence

$$\begin{aligned} K^p_{-\lambda} &= \bigcap_{u\in\mathbb{S}^{n-1}} \Bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \le \bigl(h(K,u)^p - \lambda^p h(E,u)^p\bigr)^{1/p}\Bigr\} \\ &\supseteq \bigcap_{u\in\mathbb{S}^{n-1}} \Bigl\{x \in \mathbb{R}^n : \langle x,u\rangle \le h\bigl((1-\lambda^p)^{1/p}K,u\bigr)\Bigr\} = (1-\lambda^p)^{1/p} K. \end{aligned}$$ □

Moreover, tangential bodies can be characterized by (2.11), i.e., as the only convex bodies such that their p-inner parallel bodies are homothetic copies of them (see Figure 2.2). The case $p=1$ was proved by Schneider (see Theorem 1.3.3). In order to prove it we need the following auxiliary result, which shows that p-inner parallel bodies of tangential bodies are strongly related to the classical inner ones when dealing with dilations. For the sake of brevity we will assume that $\mathrm{r}(K;E)=1$.

Proposition 2.3.11. *Let $K,E \in \mathcal{K}_0^n$ with $E \subseteq K$ and $\mathrm{r}(K;E)=1$. Let $1 \le p < \infty$ and $\lambda \in [0,1]$. If $K^p_{-\lambda} = \theta K$ for some $\theta \in [0,1]$, then $\theta = (1-\lambda^p)^{1/p}$ and $K_{-(1-\theta)} = \theta K$.*

Proof. First we prove that

$$\text{if } \; K^p_{-\lambda} = \theta K \; \text{ for } \; 0 \le \lambda \le 1, \; \text{ then } \; \theta = (1-\lambda^p)^{1/p}. \tag{2.12}$$

Indeed, since $(1-\lambda^p)^{1/p}K +_p \lambda E \subseteq (1-\lambda^p)^{1/p}K +_p \lambda K = K$, then we get

$$(1-\lambda^p)^{1/p}K \subseteq K \sim_p \lambda E = \theta K,$$

which yields $\theta \geq (1-\lambda^p)^{1/p}$. Moreover, since $\mathrm{r}(K;E)=1$, there exists $u\in\mathbb{S}^{n-1}$ such that $h(K,u)=h(E,u)>0$ (see [2, p. 59]). Therefore,

$$\theta h(K,u)=h(\theta K,u)=h(K\sim_p\lambda E,u)\leq\big(h(K,u)^p-\lambda^p h(E,u)^p\big)^{1/p}=(1-\lambda^p)^{1/p}h(K,u),$$

and since $h(K,u)>0$, we get $\theta\leq(1-\lambda^p)^{1/p}$, which shows (2.12).

Now we prove the second statement of the proposition.

First we observe that $\theta K+(1-\theta)E\subseteq\theta K+(1-\theta)K=K$, which yields

$$\theta K\subseteq K\sim(1-\theta)E=K_{-(1-\theta)},$$

and we assume, by contradiction, that there exists $x\in\big(K\sim(1-\theta)E\big)\backslash\theta K$. In particular, we have $x\notin\theta K=K^p_{-\lambda}$, and so (cf. (2.3)) there is $u_x\in\mathbb{S}^{n-1}$ such that

$$\langle x,u_x\rangle>\big(h(K,u_x)^p-\lambda^p h(E,u_x)^p\big)^{1/p}. \tag{2.13}$$

Moreover, since $x+(1-\theta)E\subseteq K$, taking support functions we get

$$\langle x,u_x\rangle+(1-\theta)h(E,u_x)\leq h(K,u_x), \tag{2.14}$$

and joining both inequalities (2.13) and (2.14), we obtain

$$\big(h(K,u_x)^p-\lambda^p h(E,u_x)^p\big)^{1/p}<h(K,u_x)-(1-\theta)h(E,u_x). \tag{2.15}$$

We notice that $h(K,u_x)>0$ since $h(K,u_x)\geq\langle x,u_x\rangle>0$ (cf. (2.13)). Thus, writing

$$\alpha=\frac{h(E,u_x)}{h(K,u_x)}\in[0,1],$$

inequality (2.15) becomes

$$(1-\lambda^p\alpha^p)^{1/p}<1-(1-\theta)\alpha. \tag{2.16}$$

In order to get the contradiction, we define $f(\alpha)=(1-\lambda^p\alpha^p)^{1/p}$ on $[0,1]$. Direct calculations yield

$$f''(\alpha)=-(p-1)\lambda^p\alpha^{p-2}(1-\lambda^p\alpha^p)^{(1-2p)/p}\leq 0 \quad \text{for all } \alpha\in(0,1),$$

i.e., f is a concave function, with $f(0)=1$ and $f(1)=(1-\lambda^p)^{1/p}=\theta$ (cf. (2.12)), which implies that $f(\alpha)\geq 1-(1-\theta)\alpha$ for all $\alpha\in[0,1]$. It contradicts (2.16), and shows the result. $\square$

Remark 2.3.12. *Proposition 2.3.11 states that there is a bijection between p-inner parallel bodies and the inner parallel bodies of K, when they all are homothetic to K, given by*

$$K^p_{-\lambda}\longleftrightarrow K_{-1+(1-\lambda^p)^{1/p}}.$$

Theorem 2.3.13. *Let $K,E\in\mathcal{K}^n_0$, $\operatorname{int}E\neq\emptyset$, with $E\subseteq K$ and $\mathrm{r}(K;E)=1$. Let $1\leq p<\infty$ and $\lambda\in(0,1)$. Then K is a tangential body of E if and only if $K^p_{-\lambda}$ is homothetic to K.*

Proof. If K is a tangential body of E, then $K^p_{-\lambda} = (1-\lambda^p)^{1/p}K$ (Proposition 2.3.10). Conversely, if $K^p_{-\lambda} = \theta K$ for some $\theta \in (0,1)$, then by Proposition 2.3.11 we get $K_{-(1-\theta)} = \theta K$ with $\theta = (1-\lambda^p)^{1/p}$. Finally, Theorem 1.3.3 shows that K is a tangential body of E. □

Other convex bodies for which their p-inner parallel bodies can be easily determined are those which are obtained as p-outer parallel bodies of a lower dimensional set.

Proposition 2.3.14. *For $E \in \mathcal{K}^n_0$, let $K \in \mathcal{K}^n_{00}(E)$ be given by $K = L +_p \mu E$, with $L \in \mathcal{K}^n_0$, $\dim L < \dim(L+E)$, and $\mu \geq 0$. Then, for all $\lambda \in \bigl[-\mu, \infty\bigr)$,*

$$K^p_\lambda = L +_p (\mu +_p \lambda)E.$$

Proof. For $\lambda \geq 0$ the result follows directly from the definitions of p-sums of convex bodies and real numbers. If $-\mu \leq \lambda \leq 0$ and since $\mathrm{r}(L;E) = 0$, we can use Proposition 2.3.6 iv) in order to obtain

$$K^p_\lambda = K \sim_p |\lambda| E = (L +_p \mu E) \sim_p |\lambda| E = L +_p (\mu +_p \lambda)E. \qquad \square$$

We notice moreover that in this case $\ker_p(K;E) = L$ and $\mathrm{r}(K;E) = \mu$ (see Lemma 2.2.4). Besides, if we remove the assumption $\dim L < \dim(L+E)$, then the result also holds in the appropriate range of λ.

Proposition 2.3.14 indicates that for some convex bodies, the whole family of p-parallel bodies is made of only *p-outer parallel bodies.* We wonder whether it is possible to characterize the convex bodies satisfying such a property, i.e., whether a converse for the proposition is also true. This question will be answered in Theorem 2.3.16.

When $p = 1$, this fact was studied by Sangwine-Yager, who introduced, for a given convex body K and the particular case $E = B_n$, the set

$$S = \Bigl\{\tau \in \bigl[-\mathrm{r}(K;B_n), \infty\bigr) : K_\tau + (\lambda - \tau)B_n = K_\lambda, \text{ for all } \lambda \geq \tau\Bigr\},$$

proving in [25, Lemma 1.5] that it is always a left-hand closed interval, namely, that there exists $\sigma \in \bigl[-\mathrm{r}(K;B_n), 0\bigr]$ such that $S = [\sigma, \infty)$.

The natural extension of S to the case $p \geq 1$ is considered in the next. For $E \in \mathcal{K}^n_0$, let $K \in \mathcal{K}^n_{00}(E)$ and let $1 \leq p < \infty$. We define the set

$$S_p = \Bigl\{\tau \in \bigl[-\mathrm{r}(K;E), \infty\bigr) : K^p_\tau +_p \bigl(\lambda +_p (-\tau)\bigr)E = K^p_\lambda, \text{ for all } \lambda \geq \tau\Bigr\}.$$

Although S_p strongly depends on the convex bodies K, E, we shall write just S_p for short. We observe that when $E = B_n$ and $p = 1$ we recover S. We also notice that, in general, $S_p \neq S_q$ if $p \neq q$ and thus, in particular, $S_p \neq S$ for all $p > 1$. Indeed, for $K = [-e_1, e_1] +_q B_n$, Proposition 2.3.14 yields $S_q = [-1, \infty)$; however, for $p < q$, since $\ker_p(K;B_n) = \{0\}$ (cf. proof of Proposition 2.2.5), if $S_p = S_q$ it would be $K = \ker_p(K;B_n) +_p B_n = B_n$, which is not possible.

The following lemma shows, as it happened in Sangwine-Yager's result ([25, Lemma 1.5]), that S_p is also a left-hand closed interval for all $p \geq 1$. The proof is based mostly on the proof of [25, Lemma 1.5].

Lemma 2.3.15. *Let $E \in \mathcal{K}_0^n$, $K \in \mathcal{K}_{00}^n(E)$ and $1 \leq p < \infty$. Then there exists $\sigma \in \big[-\mathrm{r}(K;E), 0\big]$ such that $S_p = [\sigma, \infty)$.*

Proof. First, we prove that $[0,\infty) \subseteq S_p$. Clearly $K_0^p +_p \big(\lambda +_p (-0)\big)E = K +_p \lambda E = K_\lambda^p$ for all $\lambda \geq 0$, and so $0 \in S_p$. Now let $\tau > 0$ and $\lambda \geq \tau$. Since $\lambda +_p (-\tau) = (\lambda^p - \tau^p)^{1/p} > 0$, Proposition 2.3.6 i) ensures that

$$K_\tau^p +_p \big(\lambda +_p (-\tau)\big)E = K^p_{\tau +_p (\lambda +_p (-\tau))} = K_\lambda^p.$$

Hence, $\tau \in S_p$ for all $\tau \geq 0$ and thus, $[0,\infty) \subseteq S_p$, as required.

Next we see that if $\tau \in S_p$, $\tau < 0$, then $[\tau, 0) \subseteq S_p$. Indeed, for such a value $\tau \in S_p$, let $\mu \in (\tau, 0)$ and $\lambda \geq \mu$. Clearly, both numbers satisfy

$$\mu +_p (-\tau) = \big(|\tau|^p - |\mu|^p\big)^{1/p} > 0 \quad \text{and}$$
$$\lambda +_p (-\mu) = \begin{cases} \big(\lambda^p + |\mu|^p\big)^{1/p} & \geq 0 \quad \text{if } \lambda \geq 0, \\ \big(|\mu|^p - |\lambda|^p\big)^{1/p} & \geq 0 \quad \text{if } \lambda < 0, \end{cases}$$

and thus, since $\tau \in S_p$ and $\lambda \geq \mu > \tau$, we get, from Proposition 2.3.6 i) and Lemma 2.3.3, that

$$\begin{aligned} K_\mu^p +_p \big(\lambda +_p (-\mu)\big)E &= \Big[K_\tau^p +_p \big(\mu +_p (-\tau)\big)E\Big] +_p \big(\lambda +_p (-\mu)\big)E \\ &= K_\tau^p +_p \Big[\big(\mu +_p (-\tau)\big) +_p \big(\lambda +_p (-\mu)\big)\Big]E \\ &= K_\tau^p +_p \big(\lambda +_p (-\tau)\big)E = K_\lambda^p \end{aligned}$$

for all $\lambda \geq \mu$. Hence, $\mu \in S_p$.

At this point, we have shown that if $\tau \in S_p$ then $[\tau, \infty) \subseteq S_p$. Finally, let $\sigma = \inf S_p$, which clearly satisfies $-\mathrm{r}(K;E) \leq \sigma \leq 0$. We have to prove that $\sigma \in S_p$. For $\lambda > \sigma$ let $\{\tau_i\}_{i=1}^\infty \subseteq S_p$ be a decreasing sequence with $\lim_{i\to\infty} \tau_i = \sigma$ and $\tau_1 \leq \lambda$. Since $\tau_i \in S_p$ for all $i \in \mathbb{N}$ we have

$$K_{\tau_i}^p +_p \big(\lambda +_p (-\tau_i)\big)E = K_\lambda^p,$$

and taking limits as $i \to \infty$, the continuity of the full system of p-parallel bodies (Proposition 2.3.9) ensures that

$$K_\sigma^p +_p \big(\lambda +_p (-\sigma)\big)E = K_\lambda^p$$

for all $\lambda \geq \sigma$, i.e., $\sigma \in S_p$. It concludes the proof. $\square$

Lemma 2.3.15 and Proposition 2.3.14 allow us to determine those convex bodies for which the set S_p is maximal.

Theorem 2.3.16. *Let $E \in \mathcal{K}_0^n$, $K \in \mathcal{K}_{00}^n(E)$, and $1 \leq p < \infty$. Then $K = \ker_p(K;E) +_p \mathrm{r}(K;E)E$ if and only if $S_p = \big[-\mathrm{r}(K;E), \infty\big)$.*

Chapter 3

Differentiability properties of the family of p-parallel bodies

In this chapter we investigate the differentiability of the quermassintegrals with respect to the one-parameter family of the p-parallel bodies. As in the classical case, we obtain that the volume is always differentiable in the full range $\bigl(-\mathrm{r}(K;E),\infty\bigr)$. Moreover, although there is no polynomial expression for the quermassintegrals of a p-sum, we prove that all quermassintegrals are also differentiable on positive values of the parameter, as well as at $\lambda=0$. The original work that we collect in this chapter can be found in [13].

3.1 Introduction and preliminary results

Differentiability properties of functions that depend on one-parameter families of convex bodies play an important role in some proofs in Convex Geometry (see e.g. [27, Theorem 7.6.19 and Notes to Section 7.6]). In particular, for $E\in\mathcal{K}^n_n$ and $K\in\mathcal{K}^n$, the differentiability of functions depending on the full-system of 1-parallel bodies was already addressed by Bol [1], Hadwiger [11] and Chakerian and Sangwine-Yager [5]. In this case, i.e., when $p=1$, the considered functions are the quermassintegrals $\mathrm{W}_i(K_\lambda;E)$, $i=0,\dots,n-1$, and the support function $h(K_\lambda;E)$.

One of the most useful classical tools in this context is the differentiability of the function $\mathrm{vol}(K_\lambda)$ on $-\mathrm{r}(K;E)\le\lambda\le 0$, and the following consequence of its explicit computation:

$$\mathrm{vol}(K)=n\int_{-\mathrm{r}(K;E)}^{0}\mathrm{W}_1(K_\lambda;E)\,\mathrm{d}\lambda. \tag{3.1}$$

A relevant consequence of (3.1) is the possibility of obtaining important inequalities (see e.g. [27, Section 7.2]). Further applications of the differentiability of quermassintegrals with respect to the one-parameter family of parallel bodies can be found in [18] and the references therein.

One of the aims of this chapter is to approach the differentiability of the quermassintegrals $\mathrm{W}_i(K_\lambda^p;E)$ as functions of the parameter $\lambda \in \bigl(-\mathrm{r}(K;E),\infty\bigr)$. We prove that they are always differentiable on $[0,\infty)$, providing an explicit expression for the derivative, while, in general, we only have differentiability almost everywhere on $\bigl(-\mathrm{r}(K;E),0\bigr)$.

If $K,E \in \mathcal{K}_0^n$, using a variational argument involving the p-sum and the quermassintegrals, other functionals can be introduced. This is the case, for example, of the so-called mixed quermassintegrals defined by Lutwak in [19]; for other functionals defined in such a variational way, we refer to [27, Section 9.1] and the references therein. Next theorem gathers fundamental results on L_p-Brunn-Minkowski theory, some of the proofs of this paper are based on. We notice that we need the stronger assumption $K,L \in \mathcal{K}_{(0)}^n$ and $E \in \mathcal{K}_n^n$ in order the integral expression to make sense.

Theorem 3.1.1 ([19], see also [27, Theorems 9.1.1 and 9.1.2]). *Let $K,L \in \mathcal{K}_{(0)}^n$ and $E \in \mathcal{K}_n^n$. Let $1 \leq p < \infty$ and $0 \leq i \leq n-1$. Then*

$$\begin{aligned}\frac{n-i}{p}\mathrm{W}_{p,i}(K,L;E) &:= \lim_{\varepsilon\to 0^+}\frac{\mathrm{W}_i(K+_p\varepsilon\cdot L;E)-\mathrm{W}_i(K;E)}{\varepsilon}\\ &= \frac{n-i}{p}\frac{1}{n}\int_{\mathbb{S}^{n-1}} h(L,u)^p h(K,u)^{1-p}\,\mathrm{dS}\bigl(K[n-i-1],E[i],u\bigr)\end{aligned}\tag{3.2}$$

and

$$\mathrm{W}_{p,i}(K,L;E)^{n-i} \geq \mathrm{W}_i(K;E)^{n-i-p}\mathrm{W}_i(L;E)^p.\tag{3.3}$$

The following inequality between real numbers can be easily obtained as a consequence of the mean value theorem applied to the function t^p. It will be useful in the next.

Lemma 3.1.2. *Let $0 \leq a \leq b$ and $1 \leq p < \infty$. Then,*

$$p\,(b-a)a^{p-1} \leq b^p - a^p \leq p\,(b-a)b^{p-1}.\tag{3.4}$$

3.1.1 $+_p$-concavity

Along this chapter we will be dealing with functions concerning p-parallel bodies, which instead of being concave, satisfy an analogue inequality involving $+_p$. In order to address this property we will name it $+_p$-concavity in the following definition. We notice that given an interval $I \subseteq \mathbb{R}$, $x,y \in I$ and $\lambda \in [0,1]$, it follows from Lemma 2.3.4 that $(1-\lambda)\cdot x +_p \lambda\cdot y \in I$.

Definition 3.1.3. *Let $f : I \longrightarrow \mathbb{R}$, with $I \subseteq \mathbb{R}$ an interval, and let $1 \leq p < \infty$. We say that f is* $+_p$-concave *if for all $x,y \in I$ and $\lambda \in [0,1]$,*

$$f\bigl((1-\lambda)\cdot x +_p \lambda\cdot y\bigr) \geq (1-\lambda)f(x) + \lambda f(y).$$

We say that f is $+_p$-convex *if $-f$ is $+_p$-concave.*

If $p = 1$, it is the usual definition of concavity. $+_p$-concave functions are not as *nice* as concave functions, but sometimes they share their good properties. Next we prove the existence of derivatives almost everywhere (cf. Proposition 1.1.9), as well as absolute continuity (cf. Lemma 1.1.10) for monotone $+_p$-concave functions in appropriate intervals, since they are indeed concave.

Lemma 3.1.4. *Let $f : I \longrightarrow \mathbb{R}$, with $I \subseteq (-\infty, 0]$ an interval, be an increasing $+_p$-concave function, $1 \leq p < \infty$. Then f is a concave function.*

Proof. Let $x, y \in I$ and $\lambda \in [0, 1]$. Using the concavity of t^p for $t \geq 0$ we get

$$(1-\lambda) \cdot x +_p \lambda \cdot y = -\big((1-\lambda)(-x)^p + \lambda(-y)^p\big)^{1/p} \leq (1-\lambda)x + \lambda y,$$

and since f is increasing and $+_p$-concave, we get that f is concave on I. □

Definition 3.1.5. *Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \longrightarrow \mathbb{R}$ is said to be* quasi-concave *if for all $x, y \in I$ and $\lambda \in [0, 1]$ it holds*

$$f\big((1-\lambda)x + \lambda y\big) \geq \min\big\{f(x), f(y)\big\}.$$

Quasi-concave functions are characterized as those functions such that every upper level set $\big\{x \in I : f(x) \geq t\big\}$ is convex. It is clear that every concave function is quasi-concave, but the converse is not true, as the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} -x & \text{if } x \leq 0, \\ -x^2 & \text{if } x > 0, \end{cases}$$

shows. For further details and properties on quasi-concave functions we refer to [27, p. 520] and [3].

Next we prove that $+_p$-concave functions are quasi-concave, although there is no direct relation between $+_p$-concave functions and concave ones.

Lemma 3.1.6. *Let $I \subseteq \mathbb{R}$ be an interval and let $1 \leq p < \infty$. If $f : I \longrightarrow \mathbb{R}$ is $+_p$-concave, then f is quasi-concave.*

Proof. We observe that the intermediate value theorem ensures the existence of $\mu_\lambda \in [0, 1]$ such that $(1-\lambda)x + \lambda y = (1-\mu_\lambda) \cdot x +_p \mu_\lambda \cdot y$. Therefore,

$$f\big((1-\lambda)x + \lambda y\big) = f\big((1-\mu_\lambda) \cdot x +_p \mu_\lambda \cdot y\big) \geq (1-\mu_\lambda)f(x) + \mu_\lambda f(y) \geq \min\big\{f(x), f(y)\big\}.$$ □

Remark 3.1.7. *In general, there is no relation between $+_p$-concavity and concavity. Indeed, let $f(x) = x^p$, $p > 1$, which is a convex function on $[0, \infty)$. Then:*

(i) f is $+_q$-convex (and not $+_q$-concave) if $1 \leq q < p$.

(ii) f is $+_q$-concave (and not $+_q$-convex) if $p < q < \infty$.

*(iii) f is $+_p$-*linear*, i.e., $f\big((1-\lambda) \cdot x +_p \lambda \cdot y\big) = (1-\lambda)f(x) + \lambda f(y)$, for all $x, y \in [0, \infty)$ and any $\lambda \in [0, 1]$.*

3.2 Quermassintegrals of K_λ^p as functions of λ

The problem of studying the differentiability of the quermassintegrals $\mathrm{W}_i(K_\lambda; B_3)$ of a convex body K with respect to the parameter λ of definition of the full system of parallel bodies of K, in the 3-dimensional case and with respect to the Euclidean unit ball B_3, goes back to Bol, [1]. In [11], Hadwiger addressed a close related question, providing some partial solutions to it. This last question was posed and studied for a general gauge body E and arbitrary dimension n in [18], where the original problem was solved. In this section we study differentiability properties of the functions $\mathrm{W}_i(K_\lambda^p; E)$. For the sake of brevity, we write $\mathrm{W}_i(\lambda) = \mathrm{W}_i(K_\lambda^p; E)$; if the distinction of p is necessary we write $\mathrm{W}_i(\lambda; p)$. Moreover, we shorten $\mathrm{W}_{p,i}(\lambda, L; E) = \mathrm{W}_{p,i}(K_\lambda^p, L; E)$ and $h(\lambda, u) = h(K_\lambda^p, u)$, $u \in \mathbb{S}^{n-1}$. Finally, we will write, from now on, $\mathrm{r} = \mathrm{r}(K; E)$.

Given $a \in \mathbb{R}$ and $b \geq 0$, we denote by $\mu(a, b)$ the real number satisfying

$$\begin{aligned} \text{either} \quad & a + b = a +_p \mu(a, b) \quad \text{– in this case } \mu(a, b) = (a + b) +_p (-a), \\ \text{or} \quad & a - b = a +_p \big(-\mu(a, b)\big) \quad \text{– now } \mu(a, b) = a +_p \big(-(a - b)\big). \end{aligned} \tag{3.5}$$

Of course $\mu(a, b)$ will strongly depend on the "size" of a and b and their signs.

First we prove a lower bound for the right derivative of $\mathrm{W}_i(\lambda)$ with respect to λ, for the whole range of definition $[-\mathrm{r}, \infty)$.

Proposition 3.2.1. *Let $E \in \mathcal{K}^n_{(0)}$, $K \in \mathcal{K}^n_{00}(E)$, $1 \leq p < \infty$ and $0 \leq i \leq n - 1$. Then, wherever the right derivative exists,*

$$\frac{\mathrm{d}^+}{\mathrm{d}\lambda} \mathrm{W}_i(\lambda) \geq |\lambda|^{p-1}(n - i)\mathrm{W}_{p,i}(\lambda, E; E) \quad \textit{on } [-\mathrm{r}, \infty), \tag{3.6}$$

and equality holds if $\lambda \in [0, \infty)$.

For the proof of this result we need the following property.

Lemma 3.2.2. *Let $E \in \mathcal{K}^n_{(0)}$, $K \in \mathcal{K}^n_{00}(E)$, $1 \leq p < \infty$ and $0 \leq i \leq n - 1$, and let $\lambda \in [-\mathrm{r}, \infty)$ and $\varepsilon > 0$. If there exist suitable positive constants C, c not depending on ε, such that:*

i) $K^p_{\lambda+\varepsilon} \supseteq K^p_\lambda +_p (\varepsilon C)^{1/p} E$ *for all* $\varepsilon \leq c$, *then*

$$\frac{\mathrm{d}^+}{\mathrm{d}\lambda} \mathrm{W}_i(\lambda) \geq C \, \frac{n - i}{p} \mathrm{W}_{p,i}(\lambda, E; E);$$

ii) $K^p_{\lambda+\varepsilon} \subseteq K^p_\lambda +_p (\varepsilon C)^{1/p} E$ *for all* $\varepsilon \leq c$, *then*

$$\frac{\mathrm{d}^+}{\mathrm{d}\lambda} \mathrm{W}_i(\lambda) \leq C \, \frac{n - i}{p} \mathrm{W}_{p,i}(\lambda, E; E).$$

Proof. We prove i), and thus we assume that $K^p_{\lambda+\varepsilon} \supseteq K^p_\lambda +_p (\varepsilon C)^{1/p}E$. Then, the monotonicity of the mixed volumes (see Proposition 1.5.3) yields

$$\frac{\mathrm{W}_i(\lambda+\varepsilon)-\mathrm{W}_i(\lambda)}{\varepsilon} \geq C\frac{\mathrm{W}_i\left(K^p_\lambda +_p (\varepsilon C)^{1/p}E;E\right)-\mathrm{W}_i(\lambda)}{\varepsilon C}$$

for $0<\varepsilon\leq c$, and thus, computing the limit as ε approaches 0 to the right and taking into account (3.2), we get

$$\frac{\mathrm{d}^+}{\mathrm{d}\lambda}\mathrm{W}_i(\lambda) \geq C\lim_{\eta\to 0^+}\frac{\mathrm{W}_i\big(K^p_\lambda +_p \eta^{1/p}E;E\big)-\mathrm{W}_i(\lambda)}{\eta} = C\frac{n-i}{p}\mathrm{W}_{p,i}(\lambda,E;E).$$

Item ii) is analogous. □

Proof of Proposition 3.2.1. Let $\varepsilon>0$, $\alpha\in(0,1)$ and let $\mu(\lambda,\varepsilon)$ satisfy $\lambda+\varepsilon=\lambda+_p\mu(\lambda,\varepsilon)$ (cf. (3.5)).

First, we assume $\lambda\in[-\mathrm{r},0)$ and we observe that, since we aim to take limits as $\varepsilon\to 0$, we may suppose that $-\mathrm{r}\leq\lambda<\lambda+\varepsilon<0$. In this case, $\mu(\lambda,\varepsilon)=\left(|\lambda|^p-|\lambda+\varepsilon|^p\right)^{1/p}$, and we are going to prove that

$$\mu(\lambda,\varepsilon)\geq(\varepsilon\, C_{p,\alpha,\lambda})^{1/p} \quad \text{for all } \ 0<\varepsilon\leq c(p,\alpha,\lambda), \tag{3.7}$$

with $C_{p,\alpha,\lambda}=p(1-\alpha)|\lambda|^{p-1}$, and

$$c(p,\alpha,\lambda)=\begin{cases}\left[1-(1-\alpha)^{1/(p-1)}\right]|\lambda| & \text{if } p>1,\\ |\lambda| & \text{if } p=1.\end{cases}$$

If $p=1$, then $\mu(\lambda,\varepsilon)=\varepsilon>(1-\alpha)\varepsilon=\varepsilon\, C_{1,\alpha,\lambda}$ for all $\varepsilon\leq|\lambda|=c(1,\alpha,\lambda)$, which establishes (3.7) in this case. So, let $p>1$ and $\varepsilon\leq c(p,\alpha,\lambda)$. Then

$$(1-\alpha)^{1/(p-1)}|\lambda|\leq|\lambda|-\varepsilon=|\lambda+\varepsilon|,$$

i.e., $(1-\alpha)|\lambda|^{p-1}\leq|\lambda+\varepsilon|^{p-1}$, and with Lemma 3.1.2 for $a=|\lambda+\varepsilon|$ and $b=|\lambda|$ we get that

$$\mu(\lambda,\varepsilon)^p=|\lambda|^p-|\lambda+\varepsilon|^p\geq p\,\varepsilon|\lambda+\varepsilon|^{p-1}\geq\varepsilon\, C_{p,\alpha,\lambda}$$

for all $\varepsilon\leq c(p,\alpha,\lambda)$, which concludes the proof of (3.7).

Using Proposition 2.3.6 ii) and (3.7), we immediately get

$$K^p_{\lambda+\varepsilon}=K^p_{\lambda+_p\mu(\lambda,\varepsilon)}\supseteq(K^p_\lambda)^p_{\mu(\lambda,\varepsilon)}=K^p_\lambda+_p\mu(\lambda,\varepsilon)E\supseteq K^p_\lambda+_p(\varepsilon\, C_{p,\alpha,\lambda})^{1/p}E.$$

Thus, Lemma 3.2.2 ensures that

$$\frac{\mathrm{d}^+}{\mathrm{d}\lambda}\mathrm{W}_i(\lambda)\geq C_{p,\alpha,\lambda}\frac{n-i}{p}\mathrm{W}_{p,i}(\lambda,E;E)=(1-\alpha)|\lambda|^{p-1}(n-i)\mathrm{W}_{p,i}(\lambda,E;E)$$

for all $\alpha\in(0,1)$. It proves (3.6) when $\lambda<0$.

If $\lambda = 0$ then, writing $\eta = \varepsilon^p$ and using (3.2),

$$\frac{\mathrm{d}^+}{\mathrm{d}\lambda}\bigg|_{\lambda=0} \mathrm{W}_i(\lambda) = \lim_{\varepsilon\to 0^+} \varepsilon^{p-1} \lim_{\eta\to 0^+} \frac{\mathrm{W}_i\big(0 +_p \eta^{1/p}\big) - \mathrm{W}_i(0)}{\eta}$$
$$= \begin{cases} 0 & \text{if } p > 1, \\ (n-i)\mathrm{W}_{1,i}(0, E; E) & \text{if } p = 1. \end{cases}$$

Therefore (3.6) holds with equality.

Next, we assume $\lambda > 0$. Now $\mu(\lambda, \varepsilon) = \big((\lambda+\varepsilon)^p - \lambda^p\big)^{1/p}$, and therefore, Lemma 3.1.2 yields

$$\big(p\varepsilon\lambda^{p-1}\big)^{1/p} \leq \mu(\lambda, \varepsilon) \leq \big(p\varepsilon(\lambda+\varepsilon)^{p-1}\big)^{1/p}. \tag{3.8}$$

Using Proposition 2.3.6 i), the left inequality in (3.8) implies

$$K^p_{\lambda+\varepsilon} = K^p_{\lambda +_p \mu(\lambda,\varepsilon)} = (K^p_\lambda)^p_{\mu(\lambda,\varepsilon)} \supseteq K^p_\lambda +_p \left(\varepsilon p \lambda^{p-1}\right)^{1/p} E \supseteq K^p_\lambda +_p \left(\varepsilon(1-\alpha)p\lambda^{p-1}\right)^{1/p} E$$

for all $\varepsilon > 0$, and Lemma 3.2.2 yields

$$\frac{\mathrm{d}^+}{\mathrm{d}\lambda}\mathrm{W}_i(\lambda) \geq (1-\alpha)\lambda^{p-1}(n-i)\mathrm{W}_{p,i}(\lambda, E; E)$$

for any $\alpha \in (0,1)$. It shows (3.6) on $(0,\infty)$.

Now we deal with the equality case when $\lambda > 0$. Noticing that $(\lambda+\varepsilon)^{p-1} \leq (1+\alpha)\lambda^{p-1}$ if and only if $\varepsilon \leq \lambda\left[(1+\alpha)^{1/(p-1)} - 1\right]$, we get from the right inequality in (3.8) that

$$\mu(\lambda, \varepsilon) \leq \left(\varepsilon p(1+\alpha)\lambda^{p-1}\right)^{1/p},$$

and hence, by Proposition 2.3.6 i), that

$$K^p_{\lambda+\varepsilon} = K^p_\lambda +_p \mu(\lambda, \varepsilon)E \subseteq K^p_\lambda +_p \left(\varepsilon p(1+\alpha)\lambda^{p-1}\right)^{1/p} E \tag{3.9}$$

for $\varepsilon \leq \lambda\left[(1+\alpha)^{1/(p-1)} - 1\right]$. Now, applying Lemma 3.2.2 we obtain

$$\frac{\mathrm{d}^+}{\mathrm{d}\lambda}\mathrm{W}_i(\lambda) \leq (1+\alpha)\lambda^{p-1}(n-i)\mathrm{W}_{p,i}(\lambda, E; E)$$

for any $\alpha \in (0,1)$ which, together with (3.6), proves the equality case and concludes the proof. □

The above proposition shows that the right derivative always exists on the range $[0,\infty)$. We notice also that if we work on the range $(-\mathrm{r}, 0]$, the inclusion in (3.9) would be reversed, and we cannot expect to get equality in (3.6). Therefore, in this case we may only have differentiability almost everywhere on $(-\mathrm{r}, 0)$ and thus we obtain the following result:

Proposition 3.2.3. *Let $E \in \mathcal{K}^n_0$, $K \in \mathcal{K}^n_{00}(E)$ and $1 \leq p < \infty$. Then $\mathrm{W}_i(\lambda)$ is differentiable with the exception of at most countably many points on $(-\mathrm{r}, 0)$, $0 \leq i \leq n-1$, and*

$$\frac{\mathrm{d}^-}{\mathrm{d}\lambda}\mathrm{W}_i(\lambda) \geq \frac{\mathrm{d}^+}{\mathrm{d}\lambda}\mathrm{W}_i(\lambda) \geq |\lambda|^{p-1}(n-i)\mathrm{W}_{p,i}(\lambda, E; E).$$

Proof. Inequality (1.21) and Theorem 2.3.7 imply that the function $\mathrm{W}_i(\lambda)^{p/(n-i)}$ is $+_p$-concave and increasing on $(-\mathrm{r}, 0)$. Then, Lemma 3.1.4 ensures that it is concave on this range. Hence there exist left and right derivatives of $\mathrm{W}_i(\lambda)$ and they satisfy the required inequality on $(-\mathrm{r}, 0)$. Finally, (3.6) concludes the proof. □

In order to get similar properties for the left derivative on $[0, \infty)$, we need a different approach. Next result cannot be obtained as a consequence of the $+_p$-concavity of the full system of p-parallel bodies (Theorem 2.3.7), since there is no analogue of Lemma 3.1.4 for $+_p$-concave increasing functions defined on $[0, \infty)$ (see Remark 3.1.7).

Proposition 3.2.4. *Let $E \in \mathcal{K}_0^n$, $K \in \mathcal{K}_{00}^n(E)$, $1 \le p < \infty$ and $0 \le i \le n-1$. Then, wherever the left derivative exists for $\lambda \ge 0$,*

$$\frac{\mathrm{d}^-}{\mathrm{d}\lambda}\mathrm{W}_i(\lambda) \ge \frac{\mathrm{d}^+}{\mathrm{d}\lambda}\mathrm{W}_i(\lambda).$$

Proof. By Theorem 2.3.7 and Lemma 2.3.3, it is easy to check that

$$K^p_{\lambda +_p (-t)} +_p K^p_{\lambda +_p t} \subseteq 2^{1/p} K^p_\lambda \tag{3.10}$$

for all $t > 0$ such that $\lambda +_p (-t) > -\mathrm{r}$. Then, (1.21) yields

$$\mathrm{W}_i\big(2^{1/p}K^p_\lambda; E\big)^{p/(n-i)} \ge \mathrm{W}_i\big(\lambda +_p (-t)\big)^{p/(n-i)} + \mathrm{W}_i(\lambda +_p t)^{p/(n-i)},$$

which, by the homogeneity of W_i (Proposition 1.5.3) amounts to

$$\mathrm{W}_i(\lambda)^{p/(n-i)} - \mathrm{W}_i\big(\lambda +_p (-t)\big)^{p/(n-i)} \ge \mathrm{W}_i(\lambda +_p t)^{p/(n-i)} - \mathrm{W}_i(\lambda)^{p/(n-i)}. \tag{3.11}$$

Let $\varepsilon > 0$ with $-\mathrm{r} < \lambda - \varepsilon$. By (3.5) we write $\lambda - \varepsilon = \lambda +_p \big(-\mu(\lambda, \varepsilon)\big) > -\mathrm{r}$, and with

$$m(a, b) := \frac{\mathrm{W}_i(b)^{p/(n-i)} - \mathrm{W}_i(a)^{p/(n-i)}}{\mathrm{W}_i(b) - \mathrm{W}_i(a)},$$

inequality (3.11) implies that

$$\begin{aligned}
\mathrm{W}_i(\lambda) - \mathrm{W}_i(\lambda - \varepsilon) &= \frac{\mathrm{W}_i(\lambda)^{p/(n-i)} - \mathrm{W}_i(\lambda - \varepsilon)^{p/(n-i)}}{m(\lambda - \varepsilon, \lambda)} \\
&\ge \frac{\mathrm{W}_i\big(\lambda +_p \mu(\lambda, \varepsilon)\big)^{p/(n-i)} - \mathrm{W}_i(\lambda)^{p/(n-i)}}{m(\lambda - \varepsilon, \lambda)} \\
&= \Big(\mathrm{W}_i\big(\lambda +_p \mu(\lambda, \varepsilon)\big) - \mathrm{W}_i(\lambda)\Big)\frac{m\big(\lambda, \lambda +_p \mu(\lambda, \varepsilon)\big)}{m(\lambda - \varepsilon, \lambda)}.
\end{aligned} \tag{3.12}$$

We notice that $m(a, b)$ is the slope in $\mathbb{R}^2$ of the straight line joining the points $\big(\mathrm{W}_i(a), \mathrm{W}_i(a)^{p/(n-i)}\big)^\intercal$ and $\big(\mathrm{W}_i(b), \mathrm{W}_i(b)^{p/(n-i)}\big)^\intercal$, which yields

$$\lim_{a \to b^-} m(a, b) = \lim_{c \to b^+} m(b, c) = \frac{p}{n-i}\mathrm{W}_i(b)^{(p/(n-i))-1}. \tag{3.13}$$

In order to compute the limit in (3.12) we need to *control* the size of the right-hand side in the latter inequality. Since $\mu(\lambda,\varepsilon)=\bigl(\lambda^p-(\lambda-\varepsilon)^p\bigr)^{1/p}$, given $\alpha\in(0,1)$, an easy computation proves that, for ε small enough,

$$\lambda+_p\mu(\lambda,\varepsilon)=\bigl(2\lambda^p-(\lambda-\varepsilon)^p\bigr)^{1/p}\geq\lambda+(1-\alpha)\,\varepsilon: \tag{3.14}$$

indeed, if $\lambda=0$ (3.14) is valid for all $\varepsilon>0$, whereas if $\lambda>0$ it suffices to consider

$$\varepsilon\in\left(0,\lambda\,\frac{1-(1-\alpha)^{1/(p-1)}}{1+(1-\alpha)^{p/(p-1)}}\right].$$

Thus, for $\varepsilon>0$ small enough we get

$$\frac{\mathrm{W}_i(\lambda)-\mathrm{W}_i(\lambda-\varepsilon)}{\varepsilon}\geq\frac{\mathrm{W}_i\bigl(\lambda+(1-\alpha)\,\varepsilon\bigr)-\mathrm{W}_i(\lambda)}{\varepsilon}\,\frac{m\bigl(\lambda,\lambda+_p\mu(\lambda,\varepsilon)\bigr)}{m(\lambda-\varepsilon,\lambda)}.$$

Then, taking limits as $\varepsilon\to0$ to the right in the above inequality, since, by (3.13),

$$\lim_{\varepsilon\to0^+}\frac{m\bigl(\lambda,\lambda+_p\mu(\lambda,\varepsilon)\bigr)}{m(\lambda-\varepsilon,\lambda)}=1,$$

we obtain that, for all $\alpha\in(0,1)$,

$$\begin{aligned}\frac{\mathrm{d}^-}{\mathrm{d}\lambda}\mathrm{W}_i(\lambda)&\geq(1-\alpha)\lim_{\varepsilon\to0^+}\frac{\mathrm{W}_i\bigl(\lambda+(1-\alpha)\,\varepsilon\bigr)-\mathrm{W}_i(\lambda)}{(1-\alpha)\varepsilon}\\&=(1-\alpha)\lim_{\eta\to0^+}\frac{\mathrm{W}_i(\lambda+\eta)-\mathrm{W}_i(\lambda)}{\eta}=(1-\alpha)\frac{\mathrm{d}^+}{\mathrm{d}\lambda}\mathrm{W}_i(\lambda).\end{aligned}$$

We notice that the above expression can be written because the right derivative always exists on $[0,\infty)$ (Proposition 3.2.1). □

We would like to observe that, for $\lambda<0$, (3.10) does not hold in general.

Remark 3.2.5. *At this point we observe that, in the classical case $p=1$, the differentiability of the quermassintegral $\mathrm{W}_i(\lambda;1)$ on $(0,\infty)$, $0\leq i\leq n-1$, follows immediately from the fact that $\mathrm{W}_i(K+\lambda E;E)$ can be written as a polynomial in $\lambda\geq0$ (see Theorem 1.5.5).*

Our main aim is to establish the differentiability of $\mathrm{W}_i(\lambda)$ on $(0,\infty)$. In order to do it, and taking into account Proposition 3.2.4, we will prove that the expression for the right derivative given in (3.6) provides also an upper bound for the left derivative. As usual, when we write f' for a function f, we mean that the left and right derivatives do exist and coincide.

Theorem 3.2.6. *Let $E\in\mathcal{K}^n_{(0)}$, $K\in\mathcal{K}^n_{00}(E)$ and let $1\leq p<\infty$. Then $\mathrm{W}_i(\lambda)$ is differentiable on $(0,\infty)$, $0\leq i\leq n-1$, and*

$$\mathrm{W}_i'(\lambda)=\lambda^{p-1}(n-i)\mathrm{W}_{p,i}(\lambda,E;E).$$

Proof. We are going to prove that

$$\frac{\mathrm{d}^-}{\mathrm{d}\lambda}\mathrm{W}_i(\lambda) \le \lambda^{p-1}(n-i)\mathrm{W}_{p,i}(\lambda, E; E) \tag{3.15}$$

which, together with Propositions 3.2.1 (equality case) and 3.2.4, will conclude the proof.

Let $\lambda > 0$ and $\varepsilon > 0$ with $\lambda - \varepsilon > 0$, and let $\mu(\lambda,\varepsilon) = \big(\lambda^p - (\lambda-\varepsilon)^p\big)^{1/p}$, which satisfies $\lambda - \varepsilon = \lambda +_p \big(-\mu(\lambda,\varepsilon)\big)$ (cf. (3.5)). From Lemma 3.1.2 we get $\mu(\lambda,\varepsilon) \le \big(p\varepsilon\lambda^{p-1}\big)^{1/p}$, and hence

$$\lambda - \varepsilon \ge \lambda +_p \left[-\big(p\varepsilon\lambda^{p-1}\big)^{1/p}\right].$$

It implies, by Proposition 2.3.6 iv) and the monotonicity of the mixed volumes (Proposition 1.5.3), that for all $0 < \varepsilon < \lambda$

$$\frac{\mathrm{W}_i(\lambda) - \mathrm{W}_i(\lambda-\varepsilon)}{\varepsilon} \le \frac{\mathrm{W}_i(\lambda) - \mathrm{W}_i\left(\lambda +_p \left[-\big(p\varepsilon\lambda^{p-1}\big)^{1/p}\right]\right)}{\varepsilon}. \tag{3.16}$$

We need some properties of the latter quermassintegral, for which we argue, where it applies, as in the proof of [19, Theorem (1.1)]. We show the argument for completeness. For the sake of brevity we write $\mathrm{W}_{1,i}(\mu,\tau) := \mathrm{W}_{1,i}\big(K_\mu^p, K_\tau^p; E\big)$, $\tau,\mu \ge 0$, and let $\lambda(\varepsilon) := \lambda +_p \left[-\big(p\varepsilon\lambda^{p-1}\big)^{1/p}\right]$ and

$$g(\varepsilon) := \mathrm{W}_i\left(\lambda +_p \left[-(p\varepsilon\lambda^{p-1})^{1/p}\right]\right)^{1/(n-i)} = \mathrm{W}_i\big(\lambda(\varepsilon)\big)^{1/(n-i)}.$$

We also define

$$\ell_i := \liminf_{\varepsilon\to 0^+} \frac{\mathrm{W}_i(\lambda) - \mathrm{W}_{1,i}\big(\lambda,\lambda(\varepsilon)\big)}{\varepsilon}, \quad \ell_s := \limsup_{\varepsilon\to 0^+} \frac{\mathrm{W}_{1,i}\big(\lambda(\varepsilon),\lambda\big) - \mathrm{W}_i\big(\lambda(\varepsilon)\big)}{\varepsilon}.$$

Since $K_{\lambda(\varepsilon)}^p \subseteq K_\lambda^p$ for $\varepsilon < \lambda$, the monotonicity of the mixed volumes (Proposition 1.5.3, cf. (3.2)) yields that ℓ_i and ℓ_s are the $\liminf$ and $\limsup$, respectively, of nonnegative functions for $0 < \varepsilon < \lambda$. Using inequality (3.3) we obtain

$$\begin{aligned}\ell_i &\le \liminf_{\varepsilon\to 0^+} \frac{\mathrm{W}_i(\lambda) - \mathrm{W}_i(\lambda)^{(n-i-1)/(n-i)}\mathrm{W}_i\big(\lambda(\varepsilon)\big)^{1/(n-i)}}{\varepsilon}\\ &= \mathrm{W}_i(\lambda)^{\frac{n-i-1}{n-i}} \liminf_{\varepsilon\to 0^+} \frac{\mathrm{W}_i(\lambda)^{1/(n-i)} - \mathrm{W}_i\big(\lambda(\varepsilon)\big)^{1/(n-i)}}{\varepsilon},\end{aligned}$$

and analogously,

$$\ell_s \ge \limsup_{\varepsilon\to 0^+} \mathrm{W}_i\big(\lambda(\varepsilon)\big)^{\frac{n-i-1}{n-i}} \frac{\mathrm{W}_i(\lambda)^{1/(n-i)} - \mathrm{W}_i\big(\lambda(\varepsilon)\big)^{1/(n-i)}}{\varepsilon}.$$

The continuity of the full system of p-parallel bodies with respect to the Hausdorff metric (Proposition 2.3.9) and of the quermassintegrals W_i on $\mathcal{K}^n$ (Proposition 1.5.3) prove that g is continuous at 0. Hence we may write

$$\begin{aligned}\ell_i &\le \mathrm{W}_i(\lambda)^{\frac{n-i-1}{n-i}} \liminf_{\varepsilon\to 0^+} \frac{\mathrm{W}_i(\lambda)^{1/(n-i)} - \mathrm{W}_i\big(\lambda(\varepsilon)\big)^{1/(n-i)}}{\varepsilon}\\ &\le \mathrm{W}_i(\lambda)^{\frac{n-i-1}{n-i}} \limsup_{\varepsilon\to 0^+} \frac{\mathrm{W}_i(\lambda)^{1/(n-i)} - \mathrm{W}_i\big(\lambda(\varepsilon)\big)^{1/(n-i)}}{\varepsilon} \le \ell_s.\end{aligned} \tag{3.17}$$

Moreover, using the integral expressions of W_i and $\mathrm{W}_{1,i}$ given in (1.15) and (3.2), respectively, we can write

$$\ell_i = \liminf_{\varepsilon\to 0^+} \frac{1}{n}\int_{\mathbb{S}^{n-1}} \frac{h(\lambda,u)-h\big(\lambda(\varepsilon),u\big)}{\varepsilon}\,\mathrm{dS}\left(K_\lambda^p[n-i-1],E[i],u\right)$$

and

$$\ell_s = \limsup_{\varepsilon\to 0^+} \frac{1}{n}\int_{\mathbb{S}^{n-1}} \frac{h(\lambda,u)-h\big(\lambda(\varepsilon),u\big)}{\varepsilon}\,\mathrm{dS}\left(K_{\lambda(\varepsilon)}^p[n-i-1],E[i],u\right).$$

Since

$$\lim_{\varepsilon\to 0^+} \frac{h(\lambda,u)-h\big(\lambda(\varepsilon),u\big)}{\varepsilon} = \lambda^{p-1}h(\lambda,u)^{1-p}h(E,u)^p$$

uniformly on $\mathbb{S}^{n-1}$, the continuity of $\big(h(\lambda,u)-h(\lambda(\varepsilon),u)\big)/\varepsilon$ on $\varepsilon\in(0,\lambda)$ and the weak convergence $\mathrm{S}\big(K_{\lambda(\varepsilon)}^p[n-i-1],E[i],\cdot\big)\to\mathrm{S}\big(K_\lambda^p[n-i-1],E[i],\cdot\big)$ (Propositions 1.5.3 and 2.3.9) when $\varepsilon\to 0^+$ prove that

$$\ell_i=\ell_s=\frac{\lambda^{p-1}}{n}\int_{\mathbb{S}^{n-1}} h(\lambda,u)^{1-p}h(E,u)^p\,\mathrm{dS}\big(K_\lambda^p[n-i-1],E[i],u\big). \tag{3.18}$$

Now, since $\ell_i=\ell_s$, we get from (3.17) that the right derivative of g^{n-i} at 0 does exist and satisfies

$$\lim_{\varepsilon\to 0^+}\frac{g(\varepsilon)^{n-i}-g(0)^{n-i}}{\varepsilon} = (n-i)g(0)^{n-i-1}\left.\frac{\mathrm{d}^+}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} g(\varepsilon).$$

It implies (cf. (3.17))

$$\lim_{\varepsilon\to 0^+}\frac{\mathrm{W}_i(\lambda)-\mathrm{W}_i\big(\lambda(\varepsilon)\big)}{\varepsilon} = (n-i)\ell_i = (n-i)\ell_s. \tag{3.19}$$

Thus, (3.16), (3.19), (3.18), and (3.2) yield

$$\begin{aligned}
\frac{\mathrm{d}^-}{\mathrm{d}\lambda}\mathrm{W}_i(\lambda) &= \lim_{\varepsilon\to 0^+}\frac{\mathrm{W}_i(\lambda)-\mathrm{W}_i(\lambda-\varepsilon)}{\varepsilon} \le \lim_{\varepsilon\to 0^+}\frac{\mathrm{W}_i(\lambda)-\mathrm{W}_i\big(\lambda(\varepsilon)\big)}{\varepsilon} = (n-i)\,\ell_i \\
&= \frac{n-i}{n}\lambda^{p-1}\int_{\mathbb{S}^{n-1}} h(E,u)^p h(\lambda,u)^{1-p}\,\mathrm{dS}\big(K_\lambda^p[n-i-1],E[i],u\big) \\
&= (n-i)\lambda^{p-1}\mathrm{W}_{p,i}(\lambda,E;E)
\end{aligned}$$

for $\lambda>0$, which proves (3.15) and concludes the proof. □

Remark 3.2.7. *We would like to point out that none of the results proved so far provides a proof of the differentiability of* W_i *at* $\lambda=0$*. In order to deal with this we will need a slightly different approach. This will be treated in Corollary 3.2.12.*

There exist families of convex bodies for which the functions $\mathrm{W}_i(\lambda)$ are differentiable on $(-\mathrm{r},0)$, $0\le i\le n-1$. This is, for instance, the case of the tangential bodies (see Definition 1.3.2).

In Theorem 2.3.13 it was proven that K is a tangential body of E if and only if K_λ^p is homothetic to K for all $\lambda\in(-\mathrm{r},0)$, with factor $(1-|\lambda|^p)^{1/p}$. This property, the homogeneity of quermassintegrals (Proposition 1.5.3) and the differentiability of $(1-|\lambda|^p)^{1/p}$ on $(-1,0)$ immediately prove the following result. We notice that E is always assumed to lie on $\mathcal{K}_0^n$, and any other assumption complements this one.

Lemma 3.2.8. *Let $E \in \mathcal{K}_n^n$ and $K \in \mathcal{K}_0^n$ be a tangential body of E, and let $1 \le p < \infty$. Then $\mathrm{W}_i(\lambda)$ is differentiable on $(-1,0)$, $0 \le i \le n-1$, and*

$$\mathrm{W}_i'(\lambda) = (n-i)|\lambda|^{p-1}\big(1-|\lambda|^p\big)^{((n-i)/p)-1}\mathrm{W}_i(0).$$

Remark 3.2.9. *We notice that for $\lambda \in (0,\infty)$, Theorem 3.2.6 ensures that*

$$\mathrm{W}_i'(\lambda) = \lambda^{p-1}(n-i)\mathrm{W}_{p,i}(\lambda, E; E)$$

which, in general, is not the same function provided by Lemma 3.2.8.

3.2.1 Differentiability of quermassintegrals at $\lambda = 0$

Next, we deal with the differentiability of $\mathrm{W}_i(\lambda)$ at $\lambda = 0$. Indeed we will prove that all quermassintegrals are differentiable at 0 for $p > 1$, being the value of the derivative always 0. First we prove a lemma that will be used to provide an upper bound for the left derivative of $\mathrm{W}_i(\lambda)$, involving $\mathrm{W}_i(\lambda)$ itself. The case $p = 1$ was obtained in [25, Lemma 4.7].

Lemma 3.2.10. *Let $E \in \mathcal{K}_n^n$, $K \in \mathcal{K}_{00}^n(E)$ and $1 \le p < \infty$. For all $-\mathrm{r} \le \lambda \le 0$,*

$$\frac{\mathrm{r} +_p \lambda}{\mathrm{r}} K \subseteq K_\lambda^p. \tag{3.20}$$

Equality holds for some $\lambda \in (-\mathrm{r}, 0)$ if and only if K is homothetic to a tangential body of E.

Proof. Since $K \in \mathcal{K}_{00}^n(E)$ we have $\mathrm{r}E \subseteq K$, which yields $\mathrm{r}h(E,u) \le h(K,u)$ for all $u \in \mathbb{S}^{n-1}$. Thus, $h(K,u)^p/\mathrm{r}^p - h(E,u)^p \ge 0$ for all $u \in \mathbb{S}^{n-1}$, and so

$$\frac{\mathrm{r}^p - |\lambda|^p}{\mathrm{r}^p} h(K,u)^p + |\lambda|^p h(E,u)^p \le h(K,u)^p, \qquad \text{for all } \ u \in \mathbb{S}^{n-1}.$$

It implies, as required, that

$$h\left(\frac{\mathrm{r} +_p \lambda}{\mathrm{r}} K +_p |\lambda| E, u\right) \le h(K,u), \qquad \text{for all } \ u \in \mathbb{S}^{n-1}.$$

The equality case is provided by Theorem 2.3.13, which ensures that (3.20) holds with equality for some $\lambda \in (-\mathrm{r}, 0)$ if and only if K is homothetic to a tangential body of E. □

Now we are ready to prove the mentioned upper bound for the left derivative of $\mathrm{W}_i(\lambda)$. The case $p = 1$ of this lemma was obtained in [17, Lemma 2.2].

Proposition 3.2.11. *Let $E \in \mathcal{K}_n^n$, $K \in \mathcal{K}_{00}^n(E)$, $1 \le p < \infty$ and $0 \le i \le n-1$. Then, the left derivative exists on $(-\mathrm{r}, 0]$ and*

$$\frac{\mathrm{d}^-}{\mathrm{d}\lambda}\mathrm{W}_i(\lambda) \le (n-i)\frac{|\lambda|^{p-1}}{\mathrm{r}^p - |\lambda|^p}\mathrm{W}_i(\lambda). \tag{3.21}$$

For $0 \le i \le n-2$, equality holds almost everywhere on $(-\mathrm{r}, 0)$ if and only if K is homothetic to a tangential body of E.

Proof. The existence of the left derivative is guaranteed by the concavity of W_i (Proposition 1.1.9). Let $\lambda\in(-\mathrm{r},0]$ and $\varepsilon\geq 0$ be such that $-\mathrm{r}<\lambda-\varepsilon\leq\lambda$. By (3.5) and Proposition 2.3.6 iii) we get

$$K^p_{\lambda-\varepsilon}=K^p_{\lambda+_p(-\mu(\lambda,\varepsilon))}=(K^p_\lambda)^p_{-\mu(\lambda,\varepsilon)}.$$

Then, Lemma 3.2.10 and the monotonicity and the homogeneity of the mixed volumes (Proposition 1.5.3) yield

$$\left(\frac{\mathrm{r}+_p\lambda+_p\left(-\mu(\lambda,\varepsilon)\right)}{\mathrm{r}+_p\lambda}\right)^{n-i}\mathrm{W}_i(\lambda)\leq\mathrm{W}_i(\lambda-\varepsilon),$$

and thus,

$$\frac{\mathrm{d}^-}{\mathrm{d}\lambda}\mathrm{W}_i(\lambda)=\lim_{\varepsilon\to0^+}\frac{\mathrm{W}_i(\lambda)-\mathrm{W}_i(\lambda-\varepsilon)}{\varepsilon}\leq\lim_{\varepsilon\to0^+}\frac{1-\left(\frac{\mathrm{r}^p-|\lambda-\varepsilon|^p}{\mathrm{r}^p-|\lambda|^p}\right)^{(n-i)/p}}{\varepsilon}\mathrm{W}_i(\lambda)=(n-i)\frac{|\lambda|^{p-1}}{\mathrm{r}^p-|\lambda|^p}\mathrm{W}_i(\lambda).$$

Next we deal with the equality case. From Proposition 3.2.3 we know that, with the exception of at most countably many points, the function $\mathrm{W}_i(\lambda)$ is differentiable on $(-\mathrm{r},0)$. Hence, assuming equality in (3.21) we have

$$\mathrm{W}_i'(\lambda)=(n-i)\frac{|\lambda|^{p-1}}{\mathrm{r}^p-|\lambda|^p}\mathrm{W}_i(\lambda)$$

almost everywhere on $(-\mathrm{r},0)$. Then, for $\mu\in(-\mathrm{r},0)$,

$$\int_\mu^0\frac{\mathrm{W}_i'(\lambda)}{\mathrm{W}_i(\lambda)}\,\mathrm{d}\lambda=(n-i)\int_\mu^0\frac{|\lambda|^{p-1}}{\mathrm{r}^p-|\lambda|^p}\,\mathrm{d}\lambda,$$

and thus we obtain that

$$\mathrm{W}_i(\mu)=\left(\frac{\mathrm{r}+_p\mu}{\mathrm{r}}\right)^{n-i}\mathrm{W}_i(0)=\mathrm{W}_i\left(\frac{\mathrm{r}+_p\mu}{\mathrm{r}}K;E\right). \tag{3.22}$$

Therefore, because of the inclusion provided by Lemma 3.2.10, we can conclude that

$$\frac{\mathrm{r}+_p\mu}{\mathrm{r}}K=K^p_\mu\quad\text{ for }0\leq i\leq n-2.$$

Now, Theorem 2.3.13 implies that K is homothetic to a tangential body of E.

Conversely, if K is homothetic to a tangential body of E then Lemma 3.2.8 yields

$$\mathrm{W}_i'(\lambda)=(n-i)|\lambda|^{p-1}\frac{\left(\mathrm{r}^p-|\lambda|^p\right)^{\frac{n-i}{p}-1}}{\mathrm{r}^{n-i}}\mathrm{W}_i(0)=(n-i)\frac{|\lambda|^{p-1}}{\mathrm{r}^p-|\lambda|^p}\mathrm{W}_i(\lambda). \qquad\square$$

We observe that the equality case in (3.21) when $i=n-1$ cannot be deduced from (3.22). The differentiability of W_{n-1} will be treated in a different way in Theorem 4.3.4 of Chapter 4.

As a direct consequence we get the expected result.

Corollary 3.2.12. *Let $E\in\mathcal{K}^n_n$, $K\in\mathcal{K}^n_{00}(E)$, $1<p<\infty$ and $0\leq i\leq n-1$. Then $\mathrm{W}_i(\lambda)$ is differentiable at 0 and $\mathrm{W}_i'(0)=0$.*

Proof. Proposition 1.1.9 and Proposition 3.2.11 yield that the left derivative exists at $\lambda = 0$ and $(\mathrm{d}^-/\mathrm{d}\lambda)\big|_{\lambda=0}\mathrm{W}_i(\lambda) \leq 0$. Moreover, using Proposition 3.2.1, we can assure that the right derivative of $\mathrm{W}_i(\lambda)$ at $\lambda = 0$ does exist. Finally, the equality case for (3.6) and Proposition 3.2.4 allow us to conclude the result:

$$0 = \frac{\mathrm{d}^+}{\mathrm{d}\lambda}\bigg|_{\lambda=0} \mathrm{W}_i(\lambda) \leq \frac{\mathrm{d}^-}{\mathrm{d}\lambda}\bigg|_{\lambda=0} \mathrm{W}_i(\lambda) \leq 0. \qquad \square$$

We observe that the above result is not true in the classical case $p = 1$. In fact, the argument can be reproduced in order to get, on the one hand, the value

$$\frac{\mathrm{d}^+}{\mathrm{d}\lambda}\bigg|_{\lambda=0} \mathrm{W}_i(\lambda) = (n-i)\mathrm{W}_{1,i}(0, E; E) = (n-i)\mathrm{W}_{i+1}(K; E)$$

(Proposition 3.2.1) and, on the other hand, the bound

$$\frac{\mathrm{d}^-}{\mathrm{d}\lambda}\bigg|_{\lambda=0} \mathrm{W}_i(\lambda) \leq (n-i)\frac{1}{\mathrm{r}}\mathrm{W}_i(K; E)$$

(Proposition 3.2.11). However, they are equal only when $K = E$.

3.2.2 The differentiability of the volume functional

In the following lemma we provide an *alternative* expression for the left derivative of $\mathrm{W}_i(\lambda)$ involving the p-sum in the computing limit.

Lemma 3.2.13. *Let $E \in \mathcal{K}_0^n$, $K \in \mathcal{K}_{00}^n(E)$, $1 \leq p < \infty$ and $0 \leq i \leq n-1$. Then, for all $\lambda \in (-\mathrm{r}, 0)$,*

$$\frac{\mathrm{d}^-}{\mathrm{d}\lambda}\mathrm{W}_i(\lambda) = p|\lambda|^{p-1} \lim_{\varepsilon\to 0^+} \frac{\mathrm{W}_i(\lambda) - \mathrm{W}_i\big(\lambda +_p (-\varepsilon^{1/p})\big)}{\varepsilon}.$$

Proof. Let $\varepsilon > 0$ be such that $-\mathrm{r} < \lambda - \varepsilon$ and $\mu(\lambda, \varepsilon) = \big(|\lambda - \varepsilon|^p - |\lambda|^p\big)^{1/p}$, which satisfies (cf. (3.5)) $\lambda - \varepsilon = \lambda +_p \big(-\mu(\lambda, \varepsilon)\big)$. From Lemma 3.1.2 we get $p\varepsilon|\lambda|^{p-1} \leq \mu(\lambda, \varepsilon)^p \leq p\varepsilon|\lambda - \varepsilon|^{p-1}$, and hence

$$K_\lambda^p \sim_p \big(p\varepsilon|\lambda|^{p-1}\big)^{1/p} E \supseteq K_{\lambda-\varepsilon}^p \supseteq K_\lambda^p \sim_p \big(p\varepsilon|\lambda-\varepsilon|^{p-1}\big)^{1/p} E.$$

Then, using the monotonicity of the mixed volumes (Proposition 1.5.3) we can write

$$\mathrm{W}_i\left(\lambda +_p \big(-p\varepsilon|\lambda|^{p-1}\big)^{1/p}\right) \geq \mathrm{W}_i(\lambda - \varepsilon) \geq \mathrm{W}_i\left(\lambda +_p \big(-p\varepsilon|\lambda - \varepsilon|^{p-1}\big)^{1/p}\right).$$

Therefore, since the left derivative does exist (see the proof of Proposition 3.2.3),

$$\begin{aligned} p|\lambda|^{p-1} \lim_{\varepsilon\to 0^+} \frac{\mathrm{W}_i(\lambda) - \mathrm{W}_i\left(\lambda +_p \big(-p|\lambda|^{p-1}\varepsilon\big)^{1/p}\right)}{p|\lambda|^{p-1}\varepsilon} &\leq \frac{\mathrm{d}^-}{\mathrm{d}\lambda}\mathrm{W}_i(\lambda) \\ &\leq \lim_{\varepsilon\to 0^+} p|\lambda-\varepsilon|^{p-1} \frac{\mathrm{W}_i(\lambda) - \mathrm{W}_i\left(\lambda +_p \big(-p|\lambda-\varepsilon|^{p-1}\varepsilon\big)^{1/p}\right)}{p|\lambda-\varepsilon|^{p-1}\varepsilon}, \end{aligned}$$

which proves the result. $\square$

The case $i = 0$ can be already found in the literature, directly related to p-sums, though not in the context of p-inner parallel bodies. In [19], Lutwak proved the following integral expression for *a p-variation of the volume functional.*

Theorem 3.2.14 ([19, Lemma (3.2)]). *Let $K, E \in \mathcal{K}^n_{(0)}$ and $1 \leq p < \infty$. Then,*

$$\begin{aligned}\frac{n}{p}\,\mathrm{W}_{p,0}(K, E; E) &= \lim_{\varepsilon \to 0} \frac{\mathrm{vol}(K +_p \varepsilon \cdot E) - \mathrm{vol}(K)}{\varepsilon} \\ &= \frac{1}{p}\int_{\mathbb{S}^{n-1}} h(E, u)^p h(K, u)^{1-p} \mathrm{dS}\big(K[n-1], u\big).\end{aligned}$$

We observe that the above formula is not a particular case of (3.2) when $i = 0$, since here the limit is taking both, left and right from 0. In the case of the left limit, the result was established using a variation of the support function, which turns out to be equivalent to the p-difference considered in this work. Using Lutwak's proof for an arbitrary $-\mathrm{r} \leq \lambda \leq 0$, we prove in Theorem 3.2.15 that the volume function of the system of parallel bodies, $\mathrm{vol}(\lambda) = \mathrm{vol}(K^p_\lambda)$, is differentiable on its whole range of definition $(-\mathrm{r}, \infty)$. We would like to notice that the limit appearing in Theorem 3.2.14 does not coincide with the usual limit defining the derivative, since in the first one the special product $\varepsilon \cdot E$ plays a prominent role.

Theorem 3.2.15. *Let $E \in \mathcal{K}^n_{(0)}$, $K \in \mathcal{K}^n_{00}(E)$ and $1 \leq p < \infty$. Then, for all $\lambda \in (-\mathrm{r}, \infty)$,*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{vol}(\lambda) &= n|\lambda|^{p-1}\mathrm{W}_{p,0}\big(\lambda, E; E\big) \\ &= |\lambda|^{p-1}\int_{\mathbb{S}^{n-1}} h(E, u)^p h(\lambda, u)^{1-p}\, \mathrm{dS}\big(K^p_\lambda[n-1], u\big).\end{aligned} \tag{3.23}$$

Proof. Theorems 3.2.6 and 3.2.14 ensure that $\mathrm{vol}(\lambda)$ is differentiable on $[0, \infty)$, having the desired derivative. Thus, let $\lambda \in (-\mathrm{r}, 0)$. Since $K^p_\lambda \in \mathcal{K}^n_{00}(E)$ (see Lemma 2.3.8), using Proposition 3.2.3, Lemma 3.2.13 for $i = 0$ and Theorem 3.2.14, we get

$$\begin{aligned}n|\lambda|^{p-1}\mathrm{W}_{p,0}\big(\lambda, E; E\big) \leq \frac{\mathrm{d}^+}{\mathrm{d}\lambda}\mathrm{vol}(\lambda) &\leq \frac{\mathrm{d}^-}{\mathrm{d}\lambda}\mathrm{vol}(\lambda) \\ &= |\lambda|^{p-1}\int_{\mathbb{S}^{n-1}} h(E, u)^p h(\lambda, u)^{1-p}\, \mathrm{dS}\big(K^p_\lambda[n-1], u\big) \\ &= n|\lambda|^{p-1}\mathrm{W}_{p,0}\big(\lambda, E; E\big),\end{aligned}$$

i.e., the volume function is differentiable and satisfies (3.23). □

Remark 3.2.16. *Since* $\dim K^p_{-\mathrm{r}} \leq n - 1$ *(Proposition 2.2.3), the latter result provides an integral formula for* $\mathrm{vol}(K)$ *in terms of functionals evaluated on the p-inner parallel bodies of K (cf.* (3.1)*):*

$$\begin{aligned}\mathrm{vol}(K) &= n\int_{-\mathrm{r}}^{0} |\lambda|^{p-1}\mathrm{W}_{p,0}(\lambda, E; E)\,\mathrm{d}\lambda \\ &= \int_{-\mathrm{r}}^{0} |\lambda|^{p-1}\left(\int_{\mathbb{S}^{n-1}} h(E, u)^p h(\lambda, u)^{1-p}\, \mathrm{dS}\big(K^p_\lambda[n-1], u\big)\right) \mathrm{d}\lambda.\end{aligned}$$

The case $p=1$ in Theorem 3.2.14 is connected to the theory of Wulff-shapes. We refer to [27, Section 7.5] and the references therein for detailed information, in particular, to Lemma 7.5.3. It provides, in the same way we have just done, the proof of the differentiability of $\mathrm{W}_0(\lambda;1)$.

We would like to conclude this section with a final remark. In the classical setting, the differentiability of the volume functional $\mathrm{vol}(\lambda)$ was proved by Matheron [22] using a clever and nice argument, which strongly relies on the following result, nowadays known as the *convexity lemma*:

Lemma 3.2.17 (Convexity lemma, [22]). *Let $K,E\in\mathcal{K}^n$, $E\subseteq K$. Then,*

$$\mathrm{vol}(K)-\mathrm{vol}(K\sim\varepsilon E)\leq\mathrm{vol}(K+\varepsilon E)-\mathrm{vol}(K)\quad \textit{for all}\ \ 0\leq\varepsilon\leq\mathrm{r}(K;E).$$

It can be proved that if $p>n$, for $K\in\mathcal{K}^n_{(0)}$ and $0\leq\varepsilon\leq1$, noticing that $K+_p\varepsilon K=(1+\varepsilon^p)^{1/p}K$ (cf. (1.11)) and $K\sim_p\varepsilon K=(1-\varepsilon^p)^{1/p}K$ (cf. relation vi) of Lemma 2.1.5), one gets

$$\mathrm{vol}(K+_p\varepsilon K)-\mathrm{vol}(K)\leq\mathrm{vol}(K)-\mathrm{vol}(K\sim_p\varepsilon K).$$

Therefore, there is no p-version of the convexity lemma for $p>n$, and hence the differentiability of the volume (in the above sense) cannot be obtained as in [22]. Thus, a different technique has been needed. We would like to notice that for $p\leq n$ this argument does not apply. Indeed, $\mathrm{vol}(K)-\mathrm{vol}(K\sim_p\varepsilon K)\leq\mathrm{vol}(K+_p\varepsilon K)-\mathrm{vol}(K)$, and hence, following Matheron's arguments, some differentiability properties for the volume can be proved in this range.

Chapter 4

Bounding quermassintegrals of p-inner parallel bodies

In this chapter we deal with the structure of the boundary of the p-inner parallel bodies of a convex body, studying their so-called extreme normal vectors and relating them with the ones of the original set. Then we will define a new convex body, the p-form body, which will allow to obtain new inequalities that will provide bounds for the quermassintegrals of the p-inner parallel bodies. As in the previous chapter, we will write, for $K, E \in \mathcal{K}^n$, $\mathrm{r} = \mathrm{r}(K; E)$.

The original work that we collect in this chapter can be found mainly in [14]. The differentiability properties of W_{n-1} that we will treat in Section 4.3 is contained in [13].

4.1 Extreme vectors and related notions

In order to establish most of the results contained in this chapter, we need some previous definitions and known facts about extreme normal vectors, as well as other related notions associated to a convex body.

Definition 4.1.1. *A vector $u \in \mathbb{S}^{n-1}$ is an* extreme normal vector *of a convex body $K \in \mathcal{K}^n$ if we cannot write $u = u_1 + u_2$, with u_1, u_2 linearly independent normal vectors at one and the same boundary point of K.*

The set of extreme normal vectors of K (also called 0-extreme normal vectors) will be denoted by $\mathcal{U}_0(K)$. For a detailed study of the more general defined r-extreme normal vectors we refer to [27, Section 2.2].

Remark 4.1.2. *The set of extreme normal vectors turns out to be the smallest subset of vectors which is needed in order to determine the convex body. More precisely, in [25, (2.9)] it is proved that given $K \in \mathcal{K}_n^n$, then (cf.* (1.1) *and Theorem 1.1.6; see also [27, p. 386])*

$$K = \bigcap_{u \in \mathcal{U}_0(K)} \bigl\{x \in \mathbb{R}^n : \langle x, u\rangle \le h(K, u)\bigr\}.$$

The following definition is standard, and it will allow us to relate the set of extreme normal vectors to the mixed surface area measures.

Definition 4.1.3. *If μ is a Borel measure on $\mathbb{S}^{n-1}$, its* support*, denoted by* $\operatorname{supp}\mu$*, is the closure of the set of all vectors $u \in \mathbb{S}^{n-1}$ such that every open neighborhood of u has positive μ-measure.*

Then, it can be proved (see e.g. [27, Theorem 4.5.3]) that for a convex body $K \in \mathcal{K}_n^n$,

$$\operatorname{cl}\mathcal{U}_0(K) = \operatorname{supp}\mathrm{S}\bigl(K[n-1], \cdot\bigr). \tag{4.1}$$

Remark 4.1.4. *We notice that for $K \in \mathcal{K}_n^n$, if $x \in \operatorname{bd} K$ is a regular point, then the (only) unit outer normal vector u at x is an extreme normal vector of K. Thus, extreme normal vectors characterize the regularity of convex bodies:*

$$K \in \mathcal{K}^n \text{ is regular if and only if } \mathcal{U}_0(K) = \mathbb{S}^{n-1} = \operatorname{supp}\mathrm{S}(K[n-1], \cdot).$$

Moreover, given a Wulff-shape $\mathrm{WS}(\psi)$*, for a continuous non-negative function $\psi : \mathbb{S}^{n-1} \longrightarrow [0, \infty)$, we have (see Remark 1.2.3)*

$$h\bigl(\mathrm{WS}(\psi), u\bigr) = \psi(u) \quad \text{for all } \ u \in \operatorname{cl}\mathcal{U}_0\bigl(\mathrm{WS}(\psi)\bigr).$$

The following characterization of extreme normal vectors in terms of the support function of K will be needed later on. It can be found in [25, Lemma 2.3].

Lemma 4.1.5. *Let $K \in \mathcal{K}^n$ and $u \in \mathbb{S}^{n-1}$ be an outer normal vector to K. Then $u \in \mathcal{U}_0(K)$ if and only if for any distinct $u_1, u_2 \in \mathbb{S}^{n-1}$ and $\alpha, \beta > 0$ such that $u = \alpha u_1 + \beta u_2$,*

$$h(K, u) < \alpha h(K, u_1) + \beta h(K, u_2).$$

Extreme normal vectors allow us to define the following remarkable convex body, which can be crucially used, among others, in order to obtain improvements of classical inequalities as the isoperimetric one. For further details we refer, for instance, to [27, Section 7.2]

Definition 4.1.6. *Let $K, E \in \mathcal{K}_n^n$. The relative* form body *K^* of K with respect to E (for short,* form body of *K) is the intersection of the supporting half-spaces to E with outer normals vectors in $\mathcal{U}_0(K)$, i.e.,*

$$K^* = \bigcap_{u \in \mathcal{U}_0(K)} \bigl\{x \in \mathbb{R}^n : \langle x, u\rangle \le h(E, u)\bigr\}. \tag{4.2}$$

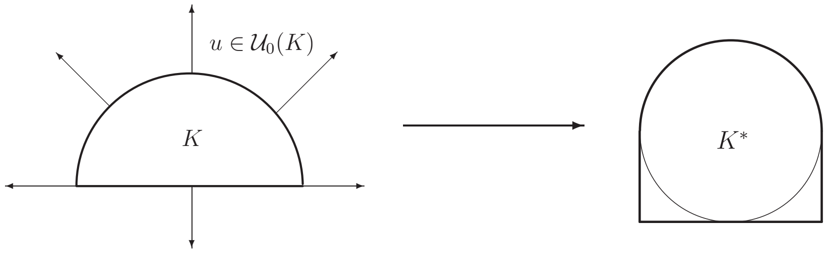

Figure 4.1: The form body of a half-circle with respect to the circle.

For the sake of brevity, we will use the notation K^*, without any reference to the convex body E. In the case of a polytope P (with interior points), the extreme normal vectors of P are the outer normal vectors to its facets, and hence, the form body of a polytope is always a polytope.

In view of its definition, one might think that the extreme normal vectors of the form body should be related to the ones of the original body K. Indeed, Sangwine-Yager proved in [25, Lemma 4.6] that for $K, E \in \mathcal{K}_n^n$,

$$\mathcal{U}_0(K^*) \subseteq \operatorname{cl} \mathcal{U}_0(K). \tag{4.3}$$

The equality case was treated in [16, Lemma 2.1]: if $E \in \mathcal{K}_n^n$ is regular, then equality holds in (4.3) for all $K \in \mathcal{K}_n^n$.

Remark 4.1.7. *We notice that if K is a tangential body of the convex body E, then $K^* = K$. Indeed, any form body is a tangential body (of the gauge body E).*

4.1.1 Extreme normal vectors of p-inner parallel bodies

We notice that there is no possible connection between the extreme normal vectors of two sets K and L when $K \subseteq L$. However, if we deal with the p-sum of two convex bodies, its extreme normal vectors are somehow related to the ones of both convex bodies. For the classical Minkowski addition of two convex bodies $K, L \in \mathcal{K}^n$, it was proved in [25, Lemma 2.4] that

$$\mathcal{U}_0(K) \cup \mathcal{U}_0(L) \subseteq \mathcal{U}_0(K + L). \tag{4.4}$$

It is not difficult to see that (4.4) also holds in the case $1 < p < \infty$:

Proposition 4.1.8. *Let $K, L \in \mathcal{K}_0^n$ and $1 \leq p < \infty$. Then*

$$\mathcal{U}_0(K) \cup \mathcal{U}_0(L) \subseteq \mathcal{U}_0(K +_p L).$$

Proof. Let $u \in \mathcal{U}_0(K)$ and let $u_1, u_2 \in \mathbb{S}^{n-1}$, $u_1 \neq u_2$, be such that $u = \alpha u_1 + \beta u_2$, with $\alpha, \beta > 0$. Then, by Lemma 4.1.5 we have

$$h(K, u) < \alpha h(K, u_1) + \beta h(K, u_2).$$

For L, the subadditivity of the support function gives also $h(L,u)\le \alpha h(L,u_1)+\beta h(L,u_2)$. These inequalities, together with Minkowski's inequality for sums (1.22), yield

$$\begin{aligned} h(K+_p L,u) &= \big(h(K,u)^p+h(L,u)^p\big)^{1/p}\\ &< \Big(\big[\alpha h(K,u_1)+\beta h(K,u_2)\big]^p+\big[\alpha h(L,u_1)+\beta h(L,u_2)\big]^p\Big)^{1/p}\\ &\le \Big(\big(\alpha h(K,u_1)\big)^p+\big(\alpha h(L,u_1)\big)^p\Big)^{1/p}+\Big(\big(\beta h(K,u_2)\big)^p+\big(\beta h(L,u_2)\big)^p\Big)^{1/p}\\ &= \alpha\Big(h(K,u_1)^p+h(L,u_1)^p\Big)^{1/p}+\beta\Big(h(K,u_2)^p+h(L,u_2)^p\Big)^{1/p}\\ &= \alpha\, h(K+_p L,u_1)+\beta\, h(K+_p L,u_2) \end{aligned}$$

Then, Lemma 4.1.5 implies that $u\in\mathcal{U}_0(K+_p L)$, and thus $\mathcal{U}_0(K)\subseteq\mathcal{U}_0(K+_p L)$. Analogously, we get $\mathcal{U}_0(L)\subseteq\mathcal{U}_0(K+_p L)$, which concludes the proof. □

Remark 4.1.9. *In view of* (4.4)*, and since* $K+_p L\subseteq K+L$ *for* $p\ge 1$ *(see* (1.12)*), one could consider the possible validity of the inclusion* $\mathcal{U}_0(K+L)\subseteq\mathcal{U}_0(K+_p L)$*, which would show Proposition 4.1.8. However, there is no possible relation of inclusion between the extreme normal vectors of* $K+L$ *and* $K+_p L$*, as the following example shows. It is known that the unit* q*-ball associated to the* q*-norm* $|\cdot|_q$ *is a* p*-sum of segments for* $p\ge 1$ *with* $1/p+1/q=1$ *(see e.g. [9]), namely,*

$$B_2^q=\left\{(x_1,x_2)^\intercal\in\mathbb{R}^2 : |x|_q=\Big(|x_1|^q+|x_2|^q\Big)^{1/q}\le 1\right\}=[-e_1,e_1]+_p[-e_2,e_2]$$

(see Figure 4.2). For all $1<p\le 2$*, we have the strict inclusion*

$$\{\pm e_1,\pm e_2\}=\mathcal{U}_0\big([-e_1,e_1]+[-e_2,e_2]\big)\subsetneq\mathcal{U}_0\big([-e_1,e_1]+_p[-e_2,e_2]\big).$$

However, if $p>2$*, then*

$$\mathbb{S}^1=\mathcal{U}_0\big([-e_1,e_1]+_2[-e_2,e_2]\big)\supsetneq\mathcal{U}_0\big([-e_1,e_1]+_p[-e_2,e_2]\big).$$

We can even find no inclusion at all if we consider, e.g., $p=1$ *and* $p>2$*.*

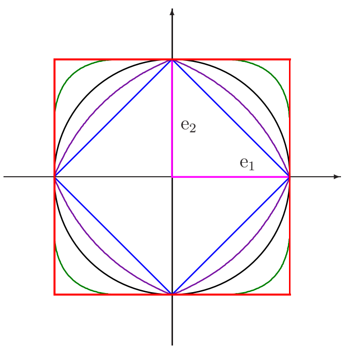

Figure 4.2: q-balls for $q\ge 1$.

Regarding inner parallel bodies, the following result states a relation between the extreme normal vectors of $K \in \mathcal{K}^n$ and the ones of its inner parallel bodies. The inclusion can be found in [25, Lemma 4.5], whereas the equality case was proved in [15, Lemma 3.2].

Lemma 4.1.10 ([25, 15]). *Let $K, E \in \mathcal{K}^n$ and $-\mathrm{r} < \lambda \leq 0$. Then*

$$\mathcal{U}_0(K_\lambda) \subseteq \mathcal{U}_0(K). \tag{4.5}$$

Equality holds if K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$.

Next proposition extends (4.5) to the case $1 < p < \infty$.

Proposition 4.1.11. *Let $E \in \mathcal{K}^n_0$, $K \in \mathcal{K}^n_{00}(E)$ and $1 \leq p < \infty$. Then, for all $-\mathrm{r} < \lambda < 0$,*

$$\mathcal{U}_0(K^p_\lambda) \subseteq \mathcal{U}_0(K).$$

Proof. Let $-\mathrm{r} < \lambda < 0$ and let $u \in \mathcal{U}_0(K^p_\lambda)$. Using (2.3) and Remark 4.1.4 we get

$$h(K^p_\lambda, u) = \big(h(K, u)^p - |\lambda|^p h(E, u)^p\big)^{1/p}. \tag{4.6}$$

Now let $u_1, u_2 \in \mathbb{S}^{n-1}$ and $\alpha, \beta > 0$ be such that $u = \alpha u_1 + \beta u_2$. Then Lemma 4.1.5 yields

$$h(K^p_\lambda, u) < \alpha h(K^p_\lambda, u_1) + \beta h(K^p_\lambda, u_2).$$

Thus, the above two relations together with the subadditivity of the support function $h(E, \cdot)$ and Minkowski's inequality for sums (1.22), yield

$$\begin{aligned}
h(K, u) &= h(K^p_\lambda, u) +_p |\lambda| h(E, u) \\
&< \Big(\alpha h(K^p_\lambda, u_1) + \beta h(K^p_\lambda, u_2)\Big) +_p \Big(\alpha|\lambda| h(E, u_1) + \beta|\lambda| h(E, u_2)\Big) \\
&\leq \Big(\alpha h(K^p_\lambda, u_1) +_p \alpha|\lambda| h(E, u_1)\Big) + \Big(\beta h(K^p_\lambda, u_2) +_p \beta|\lambda| h(E, u_2)\Big) \\
&= \alpha\Big(h(K^p_\lambda, u_1) +_p |\lambda| h(E, u_1)\Big) + \beta\Big(h(K^p_\lambda, u_2) +_p |\lambda| h(E, u_2)\Big) \\
&\leq \alpha h(K, u_1) + \beta h(K, u_2).
\end{aligned}$$

By Lemma 4.1.5 we can conclude that $u \in \mathcal{U}_0(K)$. □

Remark 4.1.12. *The above proposition implies that, for any $1 \leq p < \infty$, every p-inner parallel body of a polytope is also a polytope.*

4.2 The p-form body

Given $K, E \in \mathcal{K}^n_n$, the following relation between the (classical) inner parallel bodies of K and its form body is well-known (see [25, Lemma 4.8]): for all $-\mathrm{r} \leq \lambda \leq 0$,

$$K_\lambda + |\lambda| K^* \subseteq K. \tag{4.7}$$

When $K, E \in \mathcal{K}^n_n$ and E is regular, the equality case was characterized in [16, Theorem 2.2].

Since no much is known in this context when the gauge body E is not regular and strictly convex, from now on we will assume these conditions. Moreover, it is easy to see that

$$K_\lambda = K \sim |\lambda| E = K \sim |\lambda| K^*.$$

This inequality says in a way that the geometry of K^* "describes" the geometry of the (classical) inner parallel bodies. If we move to p-inner parallel bodies, we already know that for $E \in \mathcal{K}_0^n$, $K \in \mathcal{K}_{00}^n(E)$ and $1 \leq p < \infty$, the p-inner parallel body K_λ^p, $-\mathrm{r} \leq \lambda \leq 0$, is the largest convex body such that $K_\lambda^p +_p |\lambda| E \subseteq K$ (Definition 2.1.1). Therefore it is a natural question whether we can find a convex body L containing E which, on the one hand, can replace E in the above inclusion, i.e., such that $K_\lambda^p +_p |\lambda| L \subseteq K$, for all $-\mathrm{r} \leq \lambda \leq 0$; and, on the other hand, generalizes the usual form body K^*. Keeping these ideas in mind, we introduce the following definition.

Definition 4.2.1. *Let $E \in \mathcal{K}_0^n$ be regular and strictly convex, $K \in \mathcal{K}_{00}^n(E)$ and $1 \leq p < \infty$. The (relative)* p-form body *of K* with respect to E, *denoted by K_p^*, is defined as*

$$K_p^* := \bigcap_{u \in \mathcal{U}_0(K)} \bigl\{x \in \mathbb{R}^n : \langle x, u\rangle \leq f_{p,K}(u)\bigr\},$$

where $f_{p,K} : \mathbb{S}^{n-1} \longrightarrow [0, \infty)$ is the function given by

$$f_{p,K}(u) = \inf_{-\mathrm{r} \leq \lambda < 0} \frac{1}{|\lambda|}\Bigl(h(K,u)^p - h\left(K_\lambda^p, u\right)^p\Bigr)^{1/p}.$$

As in the case of the (classical) form body, K_p^* depends on the gauge body E, but we will not write this dependence explicitly for the sake of brevity.

Remark 4.2.2. *We observe that $f_{p,K}(u) \geq h(E,u)$ for all $u \in \mathbb{S}^{n-1}$, and therefore, $E \subseteq K_p^*$ (see Remark 4.1.2). In particular, since* $\dim E = n$ *then also* $\dim K_p^* = n$.

It can be proved that in some particular cases, the p-form body does generalize the usual form body. Unfortunately, we do not have a proof of this fact in the full generality yet, but we continue working on it. We have included this construction here since it will allow us to get inequalities for the quermassintegrals of p-inner parallel bodies. Next proposition collect several properties that the p-form body shares with the classical one.

Proposition 4.2.3. *Let $E \in \mathcal{K}_0^n$ be regular and strictly convex, $K \in \mathcal{K}_{00}^n(E)$ and $1 \leq p < \infty$. Then:*

i) $K_p^* \in \mathcal{K}_0^n$ *and* $K^* \subseteq K_p^*$.

ii) $K_\lambda^p +_p |\lambda| K_p^* \subseteq K$, *for all* $-\mathrm{r} \leq \lambda \leq 0$.

iii) $K_\lambda^p = K \sim_p |\lambda| K_p^*$

iv) *For any* $\mu > 0$, $(\mu K)_p^* = K_p^*$.

Proof. i) By definition, K_p^* is an intersection of closed halfspaces, and hence it is closed and convex. Moreover, from $f_{p,K}(u)\leq (1/\mathrm{r})\bigl(h(K,u)^p-h(K_{-\mathrm{r}}^p,u)^p\bigr)^{1/p}<\infty$ for all $u\in\mathbb{S}^{n-1}$, we get

$$K_p^*\subseteq\bigcap_{u\in\mathcal{U}_0(K)}\left\{x\in\mathbb{R}^n:\langle x,u\rangle\leq\frac{1}{\mathrm{r}}\bigl(h(K,u)^p-h(K_{-\mathrm{r}}^p,u)^p\bigr)^{1/p}\right\},$$

and therefore, since $\mathcal{U}_0(K)$ are the extreme normal vectors of a convex body K with interior points, it is bounded. Finally, $0\in K_p^*$ because $0\in E\subseteq K_p^*$ (Remark 4.2.2). Now, let $x\in K^*$. Then, $\langle x,u\rangle\leq h(E,u)\leq f_{p,K}(u)$ for all $u\in\mathcal{U}_0(K)$ (see Remark 4.2.2), and hence we get $x\in K_p^*$.

ii) Let $-\mathrm{r}\leq\lambda\leq 0$. For all $u\in\mathcal{U}_0(K)$

$$h\bigl(K_\lambda^p+_p|\lambda|K_p^*,u\bigr)\leq\Bigl(h(K_\lambda^p,u)^p+|\lambda|^pf_{p,K}(u)^p\Bigr)^{1/p}\leq h(K,u).$$

Then $K_\lambda^p+_p|\lambda|K_p^*\subseteq K$ (see Remark 4.1.2).

iii) Item ii) implies that $K_\lambda^p\subseteq K\sim_p|\lambda|K_p^*$. The reverse inclusion follows from the fact that $E\subseteq K^*\subseteq K_p^*$ (item i)).

iv) Let $u\in\mathcal{U}_0(\mu K)$. Then, by Proposition 2.3.6 v), and since $\mathrm{r}(\mu K;E)=\mu\mathrm{r}(K;E)$, we get $f_{p,\mu K}(u)=f_{p,K}(u)$. Moreover, since $\mathcal{U}_0(\mu K)=\mathcal{U}_0(K)$, we obtain the result. □

We observe that for $E\in\mathcal{K}_0^n$ regular and strictly convex, $K\in\mathcal{K}_{00}^n(E)$ and $1\leq p<\infty$, since $E\subseteq K_p^*$ (Remark 4.2.2), item ii) in Proposition 4.2.3 yields

$$\Bigl(K_{-\mathrm{r}}^p+_p(\mathrm{r}+_p\lambda)K_p^*\Bigr)+_p|\lambda|E\subseteq K_{-\mathrm{r}}^p+_p\mathrm{r}K_p^*\subseteq K,$$

and therefore,

$$K_{-\mathrm{r}}^p+_p(\mathrm{r}+_p\lambda)K_p^*\subseteq K_\lambda^p\quad\text{for all}\quad-\mathrm{r}\leq\lambda\leq 0.\tag{4.8}$$

As it happens for the classical form body (see Remark 4.1.7), we are going to prove that, in the particular case of tangential bodies (see Definition 1.3.2), the p-form body $K_p^*=K$.

Lemma 4.2.4. *Let $E\in\mathcal{K}_0^n$ be regular and strictly convex and let $K\in\mathcal{K}_0^n$ be a tangential body of E. Then $K_p^*=K$ for all $1\leq p<\infty$. Moreover, $(K_{-\lambda}^p)_p^*=K$ for all $\lambda\in[0,1]$.*

Proof. From Proposition 2.3.10 we know that if K is a tangential body of E then, for all $\lambda\in[0,1]$, $K_{-\lambda}^p=(1-\lambda^p)^{1/p}K$. Thus,

$$f_{p,K}(u)=\inf_{0<\lambda\leq 1}\frac{1}{\lambda}\Bigl(h(K,u)^p-h(K_{-\lambda}^p,u)^p\Bigr)^{1/p}=h(K,u)$$

for all $u\in\mathcal{U}_0(K)$, and then $K_p^*=K$ (see Remark 4.1.2). The last assertion directly follows from Proposition 4.2.3 iv). □

4.3 Differentiability properties of the support function

Throughout this section, the convex body E does not need to be regular and strictly convex.

For $K, E \in \mathcal{K}^n$, the concavity of the family of parallel bodies of K in $-\mathrm{r} \leq \lambda < \infty$ is translated into concavity of the support function, as a function in $\lambda \in (-\mathrm{r}, \infty)$, which implies the existence of derivatives almost everywhere. Even more, in [5] it was proved that wherever the derivative exists, it satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} h(K_\lambda^p, u) \geq h(E, u), \tag{4.9}$$

and equality holds for all $u \in \mathbb{S}^{n-1}$, all $\lambda \in (0, \infty)$ and almost everywhere on $(-\mathrm{r}, 0)$, if and only if $K = K_{-\mathrm{r}} + \mathrm{r}E$. A slightly better bound for the above derivative was shown in [25, Lemma 4.9]: wherever the derivative exists, it satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} h(K_\lambda^p, u) \geq h\big((K_\lambda)^*, u\big). \tag{4.10}$$

In the case $p \geq 1$, Lemma 3.1.4 ensures the existence of derivatives of $h(\lambda, u) = h(K_\lambda^p, u)$ almost everywhere, and it makes sense to ask for an analogue of (4.9) when $1 \leq p < \infty$.

Theorem 4.3.1. *Let $E \in \mathcal{K}^n_{(0)}$, $K \in \mathcal{K}^n_{00}(E)$ and let $1 \leq p < \infty$. Then, for all $u \in \mathbb{S}^{n-1}$,*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} h(\lambda, u) \geq \frac{|\lambda|^{p-1} h(E, u)^p}{h(\lambda, u)^{p-1}} \tag{4.11}$$

almost everywhere on $(-\mathrm{r}, 0]$. Equality holds for all $u \in \mathbb{S}^{n-1}$, almost everywhere on $[-\mathrm{r}, 0]$, if and only if $K = K_{-\mathrm{r}}^p +_p \mathrm{r}E$.

Proof. The existence of the derivative of $h(\lambda, u)$ almost everywhere on $(-\mathrm{r}, 0)$ is ensured by Lemma 3.1.4. Writing $\lambda + \varepsilon = \lambda +_p \mu(\lambda, \varepsilon)$ (cf. (3.5)) and using Proposition 2.3.6 ii), we have

$$\begin{aligned} h(\lambda + \varepsilon, u) - h(\lambda, u) &\geq h\big(K_\lambda^p +_p \mu(\lambda, \varepsilon)E, u\big) - h(\lambda, u) \\ &= \big[h(\lambda, u)^p + \mu(\lambda, \varepsilon)^p h(E, u)^p\big]^{1/p} - h(\lambda, u) \\ &\geq \frac{\mu(\lambda, \varepsilon)^p h(E, u)^p}{p\big[h(\lambda, u)^p + \mu(\lambda, \varepsilon)^p h(E, u)^p\big]^{(p-1)/p}}, \end{aligned}$$

where the last inequality follows from the right-hand side of (3.4). Since

$$\lim_{\varepsilon \to 0^+} \big[h(\lambda, u)^p + \mu(\lambda, \varepsilon)^p h(E, u)^p\big]^{(p-1)/p} = h(\lambda, u)^{p-1}$$

and $\lim_{\varepsilon \to 0^+} \mu(\lambda, \varepsilon)^p/\varepsilon = p|\lambda|^{p-1}$, we may conclude that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} h(\lambda, u) = \lim_{\varepsilon \to 0^+} \frac{h(\lambda + \varepsilon, u) - h(\lambda, u)}{\varepsilon} \geq \frac{|\lambda|^{p-1} h(E, u)^p}{h(\lambda, u)^{p-1}}.$$

Now we deal with the equality case in (4.11). If $K = K^p_{-\mathrm{r}} +_p \mathrm{r}E$, it is not difficult to check that, for all $u \in \mathbb{S}^{n-1}$,

$$h(\lambda, u)^p = h(-\mathrm{r}, u)^p + (\mathrm{r} +_p \lambda)^p h(E, u)^p,$$

and a direct computation proves that, for all $\lambda \in [-\mathrm{r}, 0]$ and $u \in \mathbb{S}^{n-1}$,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} h(\lambda, u) = \frac{|\lambda|^{p-1} h(E, u)^p}{h(\lambda, u)^{p-1}}.$$

Conversely, we assume that, for all $u \in \mathbb{S}^{n-1}$ and almost everywhere on $[-\mathrm{r}, 0]$, equality holds in (4.11). For $u \in \mathbb{S}^{n-1}$, we consider the function

$$\psi(\lambda) := h(\lambda, u)^p - h(-\mathrm{r}, u)^p - (\mathrm{r} +_p \lambda)^p h(E, u)^p.$$

Since $h(\lambda, u)^p$ is increasing and $+_p$-concave on $(-\mathrm{r}, 0)$, Lemmas 1.1.10 and 3.1.4 yield that it is absolutely continuous. Therefore ψ is absolutely continuous on $[-\mathrm{r}, 0]$, and since $\psi(-\mathrm{r}) = 0$ and $\psi'(\lambda) = 0$ almost everywhere on $[-\mathrm{r}, 0]$, we get that $\psi \equiv 0$ for any $u \in \mathbb{S}^{n-1}$. In particular, $\psi(0) = 0$ for any $u \in \mathbb{S}^{n-1}$, which yields $K = K^p_{-\mathrm{r}} +_p \mathrm{r}E$. □

We notice that if $\lambda \geq 0$, the existence of the derivative, as well as its explicit expression, follow from the fact that $h(\lambda, u)^p = h(0, u)^p + \lambda^p h(E, u)^p$, i.e., equality holds in (4.11).

Next we will slightly relax the equality conditions in Theorem 4.3.1, for which we will impose regularity on E:

Proposition 4.3.2. *If $E \in \mathcal{K}^n_{(0)}$ is regular, then equality holds in* (4.11) *almost everywhere on* $[-\mathrm{r}, 0]$ *and* $\mathrm{S}\big(E[n-1], \cdot\big)$*-almost everywhere on $\mathbb{S}^{n-1}$ if and only if $K = K^p_{-\mathrm{r}} +_p \mathrm{r}E$.*

Proof. In order to prove the proposition, it suffices to see that if $K, L, E \in \mathcal{K}^n$, $K \subseteq L$ with E regular, such that $\mathrm{S}\big(E[n-1], \cdot\big)$-almost everywhere on $\mathbb{S}^{n-1}$, $h(K, u) = h(L, u)$, then $K = L$. Indeed, under these assumptions, by (1.15) we get $\mathrm{W}_{n-1}(K; E) = \mathrm{W}_{n-1}(L; E)$, and hence

$$\int_{\mathbb{S}^{n-1}} \big[h(L, u) - h(K, u)\big] \mathrm{d}\mathrm{S}\big(E[n-1], u\big) = 0.$$

Then $h(L, u) = h(K, u)$ for all $u \in \operatorname{supp} \mathrm{S}\big(E[n-1], \cdot\big) = \mathbb{S}^{n-1}$ (cf. Remark 4.1.4), and so $K = L$. Doing $L = K^p_{-\mathrm{r}} +_p \mathrm{r}E$ we get the result. □

We point out that Proposition 4.3.2 fails for an arbitrary E. Let $M := \operatorname{supp} \mathrm{S}\big(E[n-1], \cdot\big) \subsetneq \mathbb{S}^{n-1}$ and let $u_0 \in \mathbb{S}^{n-1} \backslash M$. Since $\mathbb{S}^{n-1} \backslash M$ is open on the sphere, there exists an open neighborhood $\Omega \subseteq \mathbb{S}^{n-1} \backslash M$ of u_0. Then taking $L = \operatorname{conv}\big\{B_n, (1 + \varepsilon) u_0\big\}$, with $\varepsilon > 0$ small enough such that $\operatorname{cl}(L \backslash B_n) \cap \mathbb{S}^{n-1} \subseteq \Omega$, we have $h(B_n, u) = h(L, u)$ for all $u \in M$, but $L \neq B_n$.

Analogously to the classical case, where we have the two bounds (4.9) and (4.10), an improvement of Theorem 4.3.1, using the form body of the p-inner parallel bodies, can be obtained, but without a characterization of the equality case.

Theorem 4.3.3. *Let $E \in \mathcal{K}^n_{(0)}$ be regular and strictly convex, $K \in \mathcal{K}^n_{00}(E)$ and $1 \le p < \infty$. Then, for all $u \in \mathbb{S}^{n-1}$,*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} h(\lambda, u) \ge \frac{|\lambda|^{p-1} h\big((K^p_\lambda)^*, u\big)^p}{h(\lambda, u)^{p-1}} \tag{4.12}$$

almost everywhere on $(-\mathrm{r}, 0]$.

Proof. Let $-\mathrm{r} \le \lambda < 0$. First we notice that for $\varepsilon > 0$ such that $-\mathrm{r} \le \lambda < \lambda + \varepsilon < 0$, Proposition 4.2.3 ii) applied to $K^p_{\lambda+\varepsilon} \in \mathcal{K}^n_{00}(E)$ yields

$$K^p_\lambda +_p \mu(\lambda, \varepsilon)(K^p_{\lambda+\varepsilon})^*_p = (K^p_{\lambda+\varepsilon})^p_{-\mu(\lambda,\varepsilon)} +_p \mu(\lambda, \varepsilon)(K^p_{\lambda+\varepsilon})^*_p \subseteq K^p_{\lambda+\varepsilon},$$

where $\mu(\lambda, \varepsilon) = \big(|\lambda|^p - \big(|\lambda| - \varepsilon\big)^p\big)^{1/p}$ satisfies $\lambda + \varepsilon = \lambda +_p \mu(\lambda, \varepsilon)$ (cf. (3.5)). Then, item i) in Proposition 4.2.3 implies that

$$K^p_\lambda +_p \mu(\lambda, \varepsilon)(K^p_{\lambda+\varepsilon})^* \subseteq K^p_{\lambda+\varepsilon}. \tag{4.13}$$

Following the proof of Theorem 4.3.1, and due to the continuity of the full system of p-parallel bodies (see Proposition 2.3.9), we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} h(\lambda, u) = \lim_{\varepsilon \to 0^+} \frac{h(\lambda + \varepsilon, u) - h(\lambda, u)}{\varepsilon} \ge \frac{|\lambda|^{p-1}}{h(\lambda, u)^{p-1}} \lim_{\varepsilon \to 0^+} h\left((K^p_{\lambda+\varepsilon})^*, u\right)^p.$$

Finally, following the argument in [25, Lemma 3.1], we can conclude the result: since $\lambda < \lambda + \varepsilon$, then $(K^p_\lambda)^* \supseteq (K^p_{\lambda+\varepsilon})^* \supseteq E$, which implies that $h\big((K^p_\lambda)^*, u\big)$ is a monotone decreasing function in λ and continuous almost everywhere on $[-\mathrm{r}, 0]$. Therefore the above limit can be computed, $\lim_{\varepsilon \to 0^+} h\left((K^p_{\lambda+\varepsilon})^*, u\right)^p = \big((K^p_\lambda)^*, u\big)^p$, and we obtain (4.12). □

The key idea of the proof goes back to Sangwine-Yager [25]. However, we notice that without using the p-form body, further technical results would be necessary. We expect the same lower bound to hold also when $(K^p_\lambda)^*$ is replaced by $(K^p_\lambda)^*_p$, which would improve the bound.

As mentioned at the beginning of Section 3.2, Hadwiger proposed to determine the convex bodies for which $\mathrm{W}_i(\lambda, 1)$ is differentiable, $1 \le i \le n-1$, with $\mathrm{W}'_i(\lambda, 1) = (n-i)\mathrm{W}_{i+1}(\lambda, 1)$. In [16, 18] the cases $i = n-1, n-2$ were solved, respectively. We conclude the paper using the previous discussion in order to solve a similar p-problem for $i = n-1$. It will provide also the characterization of the equality case in (3.6) when $i = n-1$.

Theorem 4.3.4. *Let $E \in \mathcal{K}^n_0$ be regular, $K \in \mathcal{K}^n_{00}(E)$ and $1 \le p < \infty$. Then $\mathrm{W}_{n-1}(\lambda)$ is differentiable on $(-\mathrm{r}, 0)$ with $\mathrm{W}'_{n-1}(\lambda) = |\lambda|^{p-1}\mathrm{W}_{p,n-1}(\lambda, E; E)$, if and only if $K = K^p_{-\mathrm{r}} +_p \mathrm{r}E$.*

Proof. First we assume that $\mathrm{W}'_{n-1}(\lambda) = |\lambda|^{p-1}\mathrm{W}_{p,n-1}(\lambda, E; E)$. Then, integration and (3.2), together with Fubini and Theorem 4.3.1 yield

$$\begin{aligned}\mathrm{W}_{n-1}(K) - \mathrm{W}_{n-1}(K^p_{-\mathrm{r}}) &= \frac{1}{n}\int_{-\mathrm{r}}^{0}\left(\int_{\mathbb{S}^{n-1}} \frac{|\lambda|^{p-1}h(E,u)^p}{h(\lambda,u)^{p-1}}\,\mathrm{dS}\big(E[n-1],u\big)\right)\mathrm{d}\lambda\\ &\leq \frac{1}{n}\int_{\mathbb{S}^{n-1}}\left(\int_{-\mathrm{r}}^{0}\frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_{\mu=\lambda} h(\mu,u)\,\mathrm{d}\lambda\right)\mathrm{dS}(E[n-1],u)\\ &= \mathrm{W}_{n-1}(K) - \mathrm{W}_{n-1}(K^p_{-\mathrm{r}}).\end{aligned}$$

Hence, we have equality all over the above expression, and thus

$$\int_{-\mathrm{r}}^{0}\frac{|\lambda|^{p-1}h(E,u)^p}{h(\lambda,u)^{p-1}}\,\mathrm{d}\lambda = \int_{-\mathrm{r}}^{0}\frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_{\mu=\lambda}h(\mu,u)\,\mathrm{d}\lambda$$

$\mathrm{S}\big(E[n-1],\cdot\big)$-almost everywhere on $\operatorname{supp}\mathrm{S}(E[n-1],\cdot) = \mathbb{S}^{n-1}$, because E is regular. From Proposition 4.3.2 we get $K = K^p_{-\mathrm{r}} +_p \mathrm{r}E$.

Conversely, if $K = K^p_{-\mathrm{r}} +_p \mathrm{r}E$ then, for all $\lambda \in (-\mathrm{r}, 0)$, by (1.15), Theorem 4.3.1 and (3.2),

$$\begin{aligned}\mathrm{W}'_{n-1}(\lambda) &= \frac{1}{n}\int_{\mathbb{S}^{n-1}}\frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_{\mu=\lambda}h(\mu,u)\,\mathrm{dS}\big(E[n-1],u\big)\\ &= \frac{1}{n}\int_{\mathbb{S}^{n-1}}\frac{|\lambda|^{p-1}h(E,u)^p}{h(\lambda,u)^{p-1}}\,\mathrm{dS}\big(E[n-1],u\big) = |\lambda|^{p-1}\mathrm{W}_{p,n-1}(\lambda,E;E).\end{aligned}$$ □

4.4 Inequalities for quermassintegrals of p-inner parallel bodies

This final section is devoted to obtain new inequalities that provide bounds for the quermassintegrals of p-inner parallel bodies. They will extend the known results for $p = 1$ that can be found in [4, 15, 26]. First we prove an inequality relating the mixed quermassintegral $\mathrm{W}_{p,i}$ and the relative quermassintegral W_{i+1} of the p-inner parallel bodies of a convex set. It will be needed later.

Proposition 4.4.1. *Let $E \in \mathcal{K}^n_{(0)}$, $K \in \mathcal{K}^n_{00}(E)$ and $1 \leq p < \infty$. Then, for all $-\mathrm{r} \leq \lambda < \infty$ and any $i = 0, \ldots, n-1$, we have*

$$(\mathrm{r} +_p \lambda)^{p-1}\mathrm{W}_{p,i}(K^p_\lambda, E; E) \leq \mathrm{W}_{i+1}(K^p_\lambda; E).$$

Proof. We notice that the case $p = 1$ leads to the trivial equality $\mathrm{W}_{1,i}(K_\lambda, E; E) = \mathrm{W}_{i+1}(K_\lambda; E)$. By Lemma 2.3.8 we have $(\mathrm{r} +_p \lambda)E \subseteq K^p_\lambda$, and thus $h(K^p_\lambda, u)^{1-p} \leq (\mathrm{r} +_p \lambda)^{1-p}h(E,u)^{1-p}$. Then,

$$\begin{aligned}\mathrm{W}_{p,i}(K^p_\lambda, E; E) &= \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(E,u)^p h(K^p_\lambda,u)^{1-p}\,\mathrm{dS}\big(K^p_\lambda[n-i-1], E[i], u\big)\\ &\leq \frac{1}{n}(\mathrm{r}+_p\lambda)^{1-p}\int_{\mathbb{S}^{n-1}} h(E,u)\,\mathrm{dS}\big(K^p_\lambda[n-i-1], E[i], u\big) = (\mathrm{r} +_p \lambda)^{1-p}\,\mathrm{W}_{i+1}(K^p_\lambda; E).\end{aligned}$$ □

First we consider upper bounds for the i-th quermassintegral of the p-inner parallel body in terms of the magnitudes of the original body. The proof of the next result follows the arguments of [15, Theorem 2.3], where the case $p = 1$ was proved. From now on, and for the sake of brevity, we will write $\alpha_p := 2^{(1/p)-1}$.

Theorem 4.4.2. *Let $E \in \mathcal{K}_0^n$, $K \in \mathcal{K}_{00}^n(E)$ and $1 \le p < \infty$. Then, for all $-\mathrm{r} \le \lambda \le 0$ and any $i = 0, \dots, n-1$ we have*

$$\begin{aligned}\alpha_p^{n-i}\mathrm{W}_i(K_\lambda^p; E) \le \mathrm{W}_i(K; E) &- |\lambda|\alpha_p \mathrm{V}\bigl(K_p^*, K[n-i-1], E[i]\bigr)\\ &- |\lambda| \sum_{j=1}^{n-i-1} \alpha_p^j \mathrm{V}\bigl(K_\lambda^p[j], K[n-i-j-1], K_p^*, E[i]\bigr).\end{aligned}$$

Proof. Using Proposition (4.2.3) ii), the monotonicity and the linearity of the mixed volumes (Proposition 1.5.3), and the inclusion (1.13), we get

$$\begin{aligned}\mathrm{W}_i(K; E) &= \mathrm{V}\bigl(K[n-i], E[i]\bigr) \ge \mathrm{V}\bigl(K_\lambda^p +_p |\lambda| K_p^*, K[n-i-1], E[i]\bigr)\\ &\ge \alpha_p \mathrm{V}\bigl(K_\lambda^p + |\lambda| K_p^*, K[n-i-1], E[i]\bigr)\\ &\ge \alpha_p \Bigl(\mathrm{V}\bigl(K_\lambda^p, K[n-i-1], E[i]\bigr) + |\lambda| \mathrm{V}\bigl(K_p^*, K[n-i-1], E[i]\bigr)\Bigr)\\ &\ge \alpha_p |\lambda| \mathrm{V}\bigl(K_p^*, K[n-i-1], E[i]\bigr) + \alpha_p \mathrm{V}\bigl(K_\lambda^p, K_\lambda^p +_p |\lambda| K_p^*, K[n-i-2], E[i]\bigr)\\ &\ge \alpha_p |\lambda| \mathrm{V}\bigl(K_p^*, K[n-i-1], E[i]\bigr) + \alpha_p^2 \mathrm{V}\bigl(K_\lambda^p[2], K[n-i-2], E[i]\bigr)\\ &\quad + |\lambda| \alpha_p \mathrm{V}\bigl(K_p^*, K_\lambda^p, K[n-i-2], E[i]\bigr)\\ &\ge \dots\\ &\ge \alpha_p^{n-i}\mathrm{W}_i(K_\lambda^p; E) + \alpha_p |\lambda| \mathrm{V}\bigl(K_p^*, K[n-i-1], E[i]\bigr)\\ &\quad + |\lambda| \sum_{j=1}^{n-i-1} \alpha_p^j \mathrm{V}\bigl(K_\lambda^p[j], K[n-i-j-1], K_p^*, E[i]\bigr).\end{aligned}$$ □

Since $E \subseteq K_p^*$ (Remark 4.2.2), next result is an immediate consequence of the above theorem.

Corollary 4.4.3. *Let $E \in \mathcal{K}_0^n$, $K \in \mathcal{K}_{00}^n(E)$ and $1 \le p < \infty$. Then, for all $-\mathrm{r} \le \lambda \le 0$ and any $i = 0, \dots, n-1$ we have*

$$\begin{aligned}\alpha_p^{n-i}\mathrm{W}_i(K_\lambda^p; E) \le \mathrm{W}_i(K; E) &- |\lambda|\alpha_p \mathrm{W}_{i+1}(K; E)\\ &- |\lambda| \sum_{j=1}^{n-i-1} \alpha_p^j \mathrm{V}\bigl(K_\lambda^p[j], K[n-i-j-1], E[i+1]\bigr).\end{aligned}$$

In particular, when $i = 0$, the following upper bound for the volume is obtained.

Corollary 4.4.4. *Let $E \in \mathcal{K}_0^n$, $K \in \mathcal{K}_{00}^n(E)$ and $1 \le p < \infty$. Then, for all $-\mathrm{r} \le \lambda \le 0$, we have*

$$\alpha_p^n \mathrm{vol}(K_\lambda^p) \le \mathrm{vol}(K) - |\lambda| \left[\alpha_p \mathrm{W}_1(K; E) - \sum_{j=1}^{n-1} \alpha_p^j \,\mathrm{V}\bigl(K_\lambda^p[j], K[n-j-1], E\bigr)\right].$$

A different (not comparable) upper bound for the volume can be obtained by integration. But in order to do it, we need to "remove" the parameter λ, appearing in the p-inner parallel body, from the mixed volumes. In this bound, the particular integrals $\mathrm{I}_{\alpha,p}(\lambda)$, which take finite and known values in terms of hypergeometric functions, will appear (see Section 1.7).

Theorem 4.4.5. *Let $E \in \mathcal{K}^n_{(0)}$, $K \in \mathcal{K}^n_{00}(E)$ and $1 \leq p < \infty$. Then, for all $-\mathrm{r} < \lambda \leq 0$ we have*

$$\begin{aligned}\mathrm{vol}(K^p_\lambda) \leq& \frac{n}{\alpha_p^{n-1}}(\mathrm{r} +_p \lambda)\mathrm{W}_1(K;E) - \frac{n}{\alpha_p^{n-2}}\mathrm{I}_{1-p,p}(\lambda)\mathrm{V}\big(K[n-2], K^*_p, E\big)\\ &- n\sum_{j=1}^{n-2}\sum_{k=0}^{j}\alpha_p^{2j-n+1}\binom{j}{k}\mathrm{I}_{1-p+j-k,p}(\lambda)\mathrm{V}\big(K^p_{-\mathrm{r}}[k], K^*_p[j-k+1], K[n-j-2], E\big).\end{aligned}$$

Proof. For the sake of brevity we write, for $1 \leq j \leq n-2$ and $0 \leq k \leq j$,

$$\big(K^p_{-\mathrm{r}}[k], K^*_p[j-k+1], K[n-j-2], E\big) \equiv (\mathcal{C}_{jk})$$

Using (4.8) and (1.13) we can bound the mixed volume

$$\begin{aligned}\mathrm{V}\big(K^p_\lambda[j], K[n-j-2], K^*_p, E\big) &\geq \mathrm{V}\big(K^p_{-\mathrm{r}} +_p (\mathrm{r} +_p \lambda)K^*_p[j], K[n-j-2], K^*_p, E\big)\\ &\geq \alpha_p^j\,\mathrm{V}\big(K^p_{-\mathrm{r}} + (\mathrm{r} +_p \lambda)K^*_p[j], K[n-j-2], K^*_p, E\big)\\ &= \alpha_p^j\sum_{k=0}^{j}\binom{j}{k}(\mathrm{r} +_p \lambda)^{j-k}\mathrm{V}(\mathcal{C}_{jk}).\end{aligned}$$

This, together with Theorem 4.4.2 for $i = 1$ yield

$$\begin{aligned}\alpha_p^{n-1}\mathrm{W}_1(K^p_\lambda;E) &\leq \mathrm{W}_1(K;E) - |\lambda|\alpha_p\mathrm{V}\big(K[n-2], K^*_p, E\big) - |\lambda|\sum_{j=1}^{n-2}\alpha_p^j\mathrm{V}\big(K^p_\lambda[j], K[n-j-2], K^*_p, E\big)\\ &\leq \mathrm{W}_1(K;E) - |\lambda|\alpha_p\mathrm{V}\big(K[n-2], K^*_p, E\big) - |\lambda|\sum_{j=1}^{n-2}\alpha_p^{2j}\sum_{k=0}^{j}\binom{j}{k}(\mathrm{r} +_p \lambda)^{j-k}\mathrm{V}(\mathcal{C}_{jk}).\end{aligned}$$

Now, Remark 3.2.16 and Proposition 4.4.1 for $i = 0$ give

$$\frac{1}{n}\mathrm{vol}(K^p_\lambda) = \int_{-\mathrm{r}}^{\lambda}|t|^{p-1}\mathrm{W}_{p,0}(K^p_t, E; E)\,\mathrm{d}t \leq \int_{-\mathrm{r}}^{\lambda}\frac{|t|^{p-1}}{(\mathrm{r} +_p t)^{p-1}}\mathrm{W}_1(K^p_t; E)\,\mathrm{d}t,$$

and thus,

$$\begin{aligned}\frac{1}{n}\mathrm{vol}(K^p_\lambda) \leq& \frac{\mathrm{W}_1(K;E)}{\alpha_p^{n-1}}\int_{-\mathrm{r}}^{\lambda}\frac{|t|^{p-1}}{\big(\mathrm{r}^p - |t|^p\big)^{(p-1)/p}}\mathrm{d}t - \frac{\mathrm{V}\big(K[n-2], K^*_p, E\big)}{\alpha_p^{n-2}}\int_{-\mathrm{r}}^{\lambda}\frac{|t|^p}{\big(\mathrm{r}^p - |t|^p\big)^{(p-1)/p}}\mathrm{d}t\\ &-\sum_{j=1}^{n-2}\alpha_p^{2j-n+1}\sum_{k=0}^{j}\binom{j}{k}\mathrm{V}(\mathcal{C}_{jk})\int_{-\mathrm{r}}^{\lambda}\frac{|t|^p}{\big(\mathrm{r}^p - |t|^p\big)^{(p-1-j+k)/p}}\mathrm{d}t.\\ =& \frac{\big(\mathrm{r}^p - |\lambda|^p\big)^{1/p}}{\alpha_p^{n-1}}\mathrm{W}_1(K;E) - \frac{\mathrm{I}_{1-p,p}(\lambda)}{\alpha_p^{n-2}}\mathrm{V}\big(K[n-2], K^*_p, E\big)\\ &-\sum_{j=1}^{n-2}\sum_{k=0}^{j}\alpha_p^{2j-n+1}\binom{j}{k}\mathrm{I}_{1-p+j-k,p}(\lambda)\mathrm{V}(\mathcal{C}_{jk}).\end{aligned}$$

□

Next we will make it our concern to get a lower bound for the volume of the p-inner parallel bodies. For the sake of brevity, we write $h_p(\lambda)=\mathrm{I}_{1-p,p}(0)-\mathrm{I}_{1-p,p}(\lambda)$, $-\mathrm{r}<\lambda\leq 0$, and it is easy to check (cf. Lemma 1.7.1) that

$$h_p(\lambda)=\mathrm{I}_{1-p,p}(0)-\mathrm{I}_{1-p,p}(\lambda)=\frac{1}{p+1}\frac{|\lambda|^{p+1}}{\mathrm{r}^{p-1}}\;{}_2F_1\left(\frac{p-1}{p},\frac{p+1}{p};\frac{2p+1}{p};\left(\frac{|\lambda|}{\mathrm{r}}\right)^p\right).$$

We observe that Theorem 3.2.15 provides us with the following expression relating the volume of a convex body with the volume of any p-inner parallel body:

$$\mathrm{vol}(K)-\mathrm{vol}(K_\lambda^p)=n\int_\lambda^0|t|^{p-1}\mathrm{W}_{p,0}(t,E;E)\,\mathrm{d}t. \tag{4.14}$$

This relation will allow to get the announced lower bound for the volume of p-inner parallel bodies.

Theorem 4.4.6. *Let $E\in\mathcal{K}^n_{(0)}$, $K\in\mathcal{K}^n_{00}(E)$ and $1\leq p<\infty$. Then, for all $-\mathrm{r}<\lambda\leq 0$,*

$$\mathrm{vol}(K_\lambda^p)\geq\mathrm{vol}(K)+\frac{n}{\alpha_p^{n-2}}h_p(\lambda)\mathrm{W}_2(K;E)-n\frac{\mathrm{r}-\left(\mathrm{r}^p-|\lambda|^p\right)^{1/p}}{\alpha_p^{n-1}}\mathrm{W}_1(K;E).$$

Proof. Corollary 4.4.3 for $i=1$ implies that

$$\alpha_p^{n-1}\mathrm{W}_1(K_\lambda^p;E)\leq\mathrm{W}_1(K;E)-|\lambda|\alpha_p\mathrm{W}_2(K;E).$$

This, together with Proposition 4.4.1 for $i=0$ yield

$$n|\lambda|^{p-1}\mathrm{W}_{p,0}(K_\lambda^p,E;E)\leq\frac{n}{\alpha_p^{n-1}}\frac{|\lambda|^{p-1}}{(\mathrm{r}+_p\lambda)^{p-1}}\mathrm{W}_1(K;E)-\frac{n}{\alpha_p^{n-2}}\frac{|\lambda|^p}{(\mathrm{r}+_p\lambda)^{p-1}}\mathrm{W}_2(K;E).$$

Then, integrating this expression in $(\lambda,0)$ and using (4.14) (see also (1.24)), we get the result:

$$\begin{aligned}\mathrm{vol}(K)-\mathrm{vol}(K_\lambda^p)&=n\int_\lambda^0|t|^{p-1}\mathrm{W}_{p,0}(K_t^p,E;E)\,\mathrm{d}t\\&\leq\frac{n\mathrm{W}_1(K;E)}{\alpha_p^{n-1}}\int_\lambda^0\frac{|t|^{p-1}}{\left(\mathrm{r}^p-|t|^p\right)^{(p-1)/p}}\,\mathrm{d}t-\frac{n\mathrm{W}_2(K;E)}{\alpha_p^{n-2}}\int_\lambda^0\frac{|t|^p}{\left(\mathrm{r}^p-|t|^p\right)^{(p-1)/p}}\,\mathrm{d}t\\&=\frac{n\mathrm{W}_1(K;E)}{\alpha_p^{n-1}}\left(\mathrm{r}-\left(\mathrm{r}^p-|\lambda|^p\right)^{1/p}\right)-\frac{n\mathrm{W}_2(K;E)}{\alpha_p^{n-2}}h_p(\lambda).\end{aligned}$$

□

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