UNIVERSITAT DE BARCELONA

Investigations into the role of translations in abstract algebraic logic

Tommaso Moraschini

Investigations into the role of translations in abstract algebraic logic

Tommaso Moraschini

University of Barcelona
Ph.D. Program in Pure and Applied Logic
2016

Supervisors:
Josep Maria Font i Llovet and Ramon Jansana i Ferrer
Tutor:
Ramon Jansana i Ferrer

A Amanda

Contents

Acknowledgements

I wish to express my deepest gratitude to my supervisors Ramon Jansana and Josep Maria Font for their advice and guidance in this journey along abstract algebraic logic and its neighbourhood. Thanks are due to Ramon for introducing me to the study of ordered algebraic structures and for his patience and confidence while I was trying to understand McKenzie's work on category equivalence. On the other hand, Josep Maria is responsible for my interest in abstract algebraic logic, which I learned from his master courses. I thank him for being a great teacher and for his patience with the constant lack of focus in my interests.

There is no word to acknowledge the influence of James Raftery on this memoir: the first part is inspired by his work on truth-equational logics, while the second one is motivated by a conversation we had on the notion of equivalence between deductive systems. I hope that this memoir has taken some advantage from his insightful vision of mathematics. Thanks are due also to the staff of Pretoria University and, in particular, to Jamie Wannenburg for making my visits at Pretoria University very enjoyable.

Also in Prague I received a very warm welcome from the research group on non-classical logics. Especially, I wish to thank Carles Noguera and the catalan community for a very nice trip to the highest mountain (hill?) of the Czech Republic and Petr Cintula for sharing the knowledge of a secret Whiskey Tower. In Prague I met also Jiří Velebil and Matěj Dostál, who made me some very interesting remarks on the second part of this memoir. All of them made my stay there very nice and amusing.

My gratitude goes to the logic group of Barcelona, in particular to the members of the non-classical logics seminar. Among them, Antoni Torrens introduced me to the basic constructions of universal algebra, which always surprises me for its beauty, and Joan Gispert to the algebraic study of non-classical logics. Sometimes it is more difficult to ask the right questions than to solve them: I am very grateful to Félix Bou for posing the problems addressed in Section 4. I wish to thank also Silvio Bozzi and Maurizio Negri for their friendship and for inspiring my early interest in logic. Moreover, Stefano Aguzzoli, Tommaso Flaminio and Vincenzo Marra welcomed me in my first attempt to become a Ph.D. student in Milan some years ago. Even if I eventually decided to move to Barcelona, I am very grateful to them.

Several people made this work possible with their friendship and love. I wish to thank Luciano Gonzalez for a trip to Italy and one to Buenos Aires, Darllan Pinto with his categorical monsters' zoo and Hugo Albuquerque for the hours spent brewing together. Pedro Esteban and Eduardo Reyes have been my chaotic family in Barcelona: with them I spent marvellous days and nights between parallel universe conversations, terrible noise, random walks, enlightenment and poisoning. In other words, they made this world a good place to stay for a while. Many thanks also to the friends from Italy: Alessandro Cimino and Stefano Fiorentini for their visits to Barcelona and Francesco Genderini for a trip to Naples: it is a big luck to have them around. My parents and my brother supported me with all their love, which is something invaluable. It is hard to imagine how my life would have been without Amanda Vidal by my side. Even if I never fully understood the tradition of dedicating a mathematical piece to a lover, I will conform to this tradition with the promise to remedy in the future.

Abstract

This memoir is divided into two parts, devoted to two different but related topics in (abstract) algebraic logic. In the first part we develop a hierarchy in which propositional logics $\mathcal{L}$ are classified according to the definability conditions enjoyed by the truth sets of the matrix semantics $\mathsf{Mod}^*\mathcal{L}$. More precisely, we focus on conditions belonging to the proper conceptual framework of the Leibniz hierarchy, in the sense that they can be characterized by means of the order-theoretic behaviour of the Leibniz operator (restricted to deductive filters). Thus, the hierarchy we present is an extension of the standard Leibniz hierarchy. The starting point of the discussion is the observation that the methods of [83] can be applied to capture the fact that truth is definable in $\mathsf{Mod}^*\mathcal{L}$ by means of universally quantified equations leaving one variable free. We study the logics that satisfy this condition and investigate their relation with the Frege hierarchy. Subsequently we move our attention to logics for which truth is implicitly definable in $\mathsf{Mod}^*\mathcal{L}$ and show that the injectivity of the Leibniz operator does not transfer in general from theories to deductive filters over arbitrary algebras, answering the open question [83, Problem 1]. Nevertheless, we show that injectivity transfers for logics expressed in a countable language. Finally we consider an intermediate condition on the truth sets in $\mathsf{Mod}^*\mathcal{L}$ that corresponds to the order-reflection of the Leibniz operator. We conclude the first part of this memoir by taking a computational glimpse to the two hierarchies typical of abstract algebraic logic. More precisely, we show that the problems of classifying the logic of a given Hilbert calculus inside the Leibniz or Frege hierarchies is undecidable. On the other hand, we show that the problem of classifying the logic of a given finite set of finite matrices of finite type in most classes of the Leibniz hierarchy is decidable.

In the second part of this memoir we present an algebraic and combinatorial description of right adjoint functors between generalized quasi-varieties, inspired by [73]. This result is achieved by developing a correspondence between the concept of adjunction and a new notion of translation between relative equational consequences. More precisely, we introduce a notion of translation that satisfy the following condition: given two generalized quasi-varieties K and K', every translation of the equational consequence relative to K into the one relative to K' corresponds to a right adjoint functor from K' to K and vice-versa. This correspondence between adjunctions and translations provides a general explanation of the correspondence that appears in some well-known translations between logics, e.g., Gödel's translation of intuitionistic logic into the gobal modal logic $\mathcal{S}4$ corresponds to the functor that takes an interior algebra to the Heyting algebra of its open elements and Kolmogorov's translation of classical logic into intuitionistic logic corresponds to the functor that takes a Heyting algebra to the Boolean algebra of its regular elements. Then, we investigate the preservation of some logico-algebraic properties, such as the (contextual) deduction theorem, the inconsistency lemma and the fact of having a generalized disjunction, in presence of an adjunction. We conclude this second part by showing that every prevariety is categorically equivalent to the equivalent algebraic semantics of an algebraizable logic.

Resum

Aquesta memòria es divideix en dues parts, dedicades respectivament a dos temes diferents i alhora relacionats de lògica algebraica abstracta. En la primera part descrivim una jerarquia on es classifiquen les lògiques proposicionals $\mathcal{L}$ segons les condicions de definibilitat que compleixen els conjunts de veritat de la semàntica de matrius $\mathsf{Mod}^*\mathcal{L}$. Més concretament, ens centrem en l'estudi de condicions de definibilitat que pertanyen al marc conceptual propi de la jerarquia de Leibniz, en el sentit que es poden caracteritzar gràcies al comportament de l'operador de Leibniz (restringit als filtres deductius). Per tant, la jerarquia que presentem és una extensió de la jerarquia de Leibniz estàndard. El nostre punt de partida és l'observació que les eines introduïdes en [83] es poden fer servir per capturar el fet que la veritat sigui definible en $\mathsf{Mod}^*\mathcal{L}$ a través d'equacions quantificades universalment deixant lliure una variable. Estudiem la classe de les lògiques que compleixen aquesta condició i les seves relacions amb la jerarquia de Frege. Després considerem la família de les lògiques que tenen la veritat implícitament definible en la semàntica $\mathsf{Mod}^*\mathcal{L}$ i provem que la injectivitat de l'operador de Leibniz no es transfereix en general de les teories als filtres deductius sobre àlgebres arbitràries, resolent un problema obert de [83, Problem 1]. No obstant això, mostrem que la injectivitat de l'operador de Leibniz sí es transfereix per a lògiques formulades en un llenguatge numerable. Finalment, considerem una condició de definibilitat intermèdia sobre els conjunts de veritat de $\mathsf{Mod}^*\mathcal{L}$, que correspon al fet que l'operador de Leibniz reflecteixi l'ordre. Concloem la primera part d'aquesta memòria amb una aproximació computacional a les dues jerarquies de la lògica algebraica abstracta. Concretament, provem que els problemes de classificar la lògica d'un càlcul de Hilbert en la jeraquia de Leibniz o en la de Frege són en general indecidibles. D'altra banda, mostrem que el problema de classificar la lògica determinada per un conjunt finit de matrius finites en un llenguatge finit en la majoria dels nivells de la jerarquia de Leibniz és decidible.

En la segona part d'aquesta memòria presentem una descripció algebraica i combinatòria dels functors adjunts a la dreta entre quasi-varietats generalitzades, inspirada pel treball [73]. Aquest resultat s'obté desenvolupant una correspondència entre el concepte d'adjunció i el d'una nova noció de traducció entre conseqüències equacionals relatives. Concretament, in-

troduïm una noció de traducció que compleix la següent condició: donades dues quasi-varietats generalitzades K i K', cada traducció de la conseqüència equacional relativa a K en la relativa a K' correspon a un functor adjunt a la dreta de K' a K i viceversa. Aquesta correspondència entre adjuncions i traduccions proporciona una explicació general de la correspondència que apareix en algunes traduccions conegudes entre lògiques. Per exemple, la traducció de Gödel de la lógica intuïcionista en la lògica modal global $\mathcal{S}4$ correspon al functor que envia una àlgebra d'interior a l'àlgebra de Heyting dels seus elements oberts; i la traducció de Kolmogorov de la lògica clàssica en la lògica intuïcionista correspon al functor que envia una àlgebra de Heyting a l'àlgebra de Boole dels seus elements regulars. A continuació estudiem la preservació, en presència d'una adjunció, d'algunes propietats lògico-algebraiques, tals com el teorema (contextual) de la deducció, el lema d'inconsistència o el fet de tenir una disjunció generalitzada. Concloem aquesta segona part mostrant que cada prevarietat és categorialment equivalent a la semàntica algebraica equivalent d'una lògica algebritzable.

Introduction

The goal of this dissertation is to make a contribution to abstract algebraic logic in two quite different directions, both related to the fundamental intuitions that give the field its unity and distinctive character. One of the basic ideas that motivate the algebraic approach to the study of propositional logics is that the deducibility relation of every logic can be mimicked by a semantic consequence relation defined in terms of algebraic structures. Historically the first algebraic *completeness* theorems had the following structure (here described in modern terms). A propositional logic $\mathcal{L}$ was related to a class of algebras K by means of a structural translation $\boldsymbol{\tau}$ of formulas into equations. Thus the completeness theorem stated that a deduction holds in the logic $\mathcal{L}$ if and only if its translation holds in the equational consequence relative to the class of algebras K, in symbols:

$$\Gamma \vdash_{\mathcal{L}} \varphi \iff \boldsymbol{\tau}(\Gamma) \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi) \tag{$*$}$$

for every set of formulas $\Gamma \cup \{\varphi\}$. This phenomenon is reflected for example in the completeness theorems of superintuitionistic logics with respect to varieties of Heyting algebras. In these cases $\boldsymbol{\tau}$ is the translation that converts every formula φ into the equation $\varphi \approx 1$.

The situation expressed in $(*)$ can be generalized in at least two different ways, which correspond to the two parts in which this memoir is structured. First observe that condition $(*)$ means that the logic $\mathcal{L}$ is complete with respect to the class of logical matrices whose algebraic reducts are the algebras in K and whose designated elements are the solutions of the equations $\boldsymbol{\tau}(x)$, i.e., with respect to matrices of the form

$$\langle \boldsymbol{A}, \{a \in A : \boldsymbol{A} \vDash \boldsymbol{\tau}(a)\}\rangle \text{ with } \boldsymbol{A} \in \mathsf{K}.$$

Keeping this in mind, we can rephrase $(*)$ in terms of a *definability* condition on the set of designated elements of a matrix semantics. More precisely, $(*)$ amounts to the fact that the logic $\mathcal{L}$ is complete with respect to the class of logical matrices based on K, whose set of designated elements are *equationally definable* by means of the equations $\boldsymbol{\tau}(x)$. It is therefore natural to ask the following question (addressed in Part I of this memoir):

- Can we conceive completeness results with respect to classes of logical matrices, whose designated elements enjoy some definability condition (possibly) weaker than the equational one? And, in particular, can we classify logics according to the definability conditions enjoyed by the designated elements of their matrix semantics?

A different direction in which we can generalize the situation expressed by ($*$) comes from a slightly more abstract reading of the concept of completeness. More precisely, the completeness phenomenon occurring in ($*$) amounts to the fact that the deducibility relation of the logic $\mathcal{L}$ can be *interpreted* into the equational consequence relative to the class of algebras K by means of the structural translation $\boldsymbol{\tau}$ (of formulas in equations). We can abstract this idea and say that a completeness theorem is in general any result that relates two consequence relations $\vdash$ and $\vdash'$ between possibly different syntactic objects (e.g., formulas, equations or whatever) through a suitable translation $\boldsymbol{\tau}$ in such a way that

$$A \vdash a \iff \boldsymbol{\tau}(A) \vdash' \boldsymbol{\tau}(a)$$

for every set $A \cup \{a\}$ of syntactic objects of $\vdash$. It is therefore natural to ask the following question (addressed in Part II of this memoir):

- Can we conceive a general notion of translation between equational consequences relative to classes of algebras in (possibly) different languages? And, in particular, can we do this in such a way that the equational consequences relative to two classes of algebras K and K' are related by a completeness theorem exactly when there is some strong mathematical relation between K and K'?

In the following we discuss how this memoir addresses the two questions separately.

Part I: Truth Predicates in Matrix Semantics

The investigation of this part is developed in the framework of classical abstract algebraic logic [27, 38, 40, 41], which is a theory that aims to provide general tools for the algebraic study of arbitrary propositional logics. One of the most striking achievements of this theory is the discovery of the importance of the so-called *Leibniz operator*. This is the map $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathcal{P}(A) \to \mathrm{Con}\boldsymbol{A}$, defined for every algebra $\boldsymbol{A}$, that sends every subset $F \subseteq A$ to the largest congruence θ of $\boldsymbol{A}$ such that F is a union of blocks of θ. The importance of the Leibniz operator is two-fold. On the one hand it allows to associate a special class of matrices $\mathsf{Mod}^*\mathcal{L}$ with every logic $\mathcal{L}$, obtaining a completeness result for $\mathcal{L}$. Remarkably, in the best-known cases the class of matrices $\mathsf{Mod}^*\mathcal{L}$ coincides with the intended algebraic semantics of $\mathcal{L}$, e.g., in the case of superintuitionistic logics, $\mathsf{Mod}^*\mathcal{L}$ is the class of matrices based on a variety of Heyting algebras with the top element as designated element. In order to explain the other fundamental usage of the Leibniz operator, we take a very short detour. It is well known that a logical matrix $\langle \boldsymbol{A}, F\rangle$ can be regarded as a first-order structure, namely, as an algebra equipped with the interpretation of a predicate symbol $P(x)$. The intuitive reading of logical matrices suggests that the set of designated elements F represents truth inside the set of truth-values A. Accordingly $P(x)$ can be understood as a *truth predicate* and F as a *truth set*. Keeping this in mind, it makes sense to refer to the truth sets of a class of matrices. Now, the importance of the Leibniz operator comes from the fact that its behaviour on the deductive filters of a given logic $\mathcal{L}$ determines interesting facts about *logical equivalence* and about the *truth sets* of the matrix semantics $\mathsf{Mod}^*\mathcal{L}$. This discovery led to the development of the so-called *Leibniz hierarchy*, where logics are classified according to properties related to the behaviour of the Leibniz operator. Keeping this in mind, we can reformulate in a more precise fashion the objective of this part of the dissertation:

- We develop a hierarchy in which propositional logics $\mathcal{L}$ are classified according to the definability conditions enjoyed by the truth sets of $\mathsf{Mod}^*\mathcal{L}$. In particular, we will do this in such a way that these conditions fit inside the Leibniz hierarchy, i.e., they are characterized by the behaviour of the Leibniz operator (when restricted to deductive filters).

The program described above has already been considered in the abstract algebraic logic literature, especially in [22, 30, 56, 83]. Thus in this memoir we will both propose some original contributions and organize the previous work under a general perspective (credits will be given along the way).

The definability condition reflected in completeness theorems of the form ($*$) is the following. We say that truth is *equationally definable* in $\mathsf{Mod}^*\mathcal{L}$ if there is a set of equations $\boldsymbol{\tau}(x)$ such that the truth sets of $\mathsf{Mod}^*\mathcal{L}$ are exactly the solutions of the equations $\boldsymbol{\tau}$. This idea can be generalized by considering equational completeness theorems in which we admit the presence of universally quantified equations. More precisely, we say that truth is *almost universally definable* if there is a set of equations $\boldsymbol{\tau}(x, \vec{y})$ such that the *non-empty* truth sets in $\mathsf{Mod}^*\mathcal{L}$ are exactly the sets of solutions of the equations $\boldsymbol{\tau}(x, \vec{y})$ once we bound the parameters $\vec{y}$ by a universal quantifier. The reader may wonder why we restrict this last definition to non-empty truth sets. This is because we prove that, when applied to all the truth sets of $\mathsf{Mod}^*\mathcal{L}$, the notion of equational and universal definability coincide (Corollary 2.10). In particular, this implies that these two definability conditions are equivalent for logics with theorems. However, theorem-less logics have always a reduced model of the form $\langle \mathbf{1}, \emptyset\rangle$ (where $\mathbf{1}$ is the trivial algebra), whose truth set is never definable by equations (with or without parameters), and hence the restriction was unavoidable. A family of logics $\mathcal{L}$ whose truth sets are almost universally, but not equationally, definable in $\mathsf{Mod}^*\mathcal{L}$ comes from the consideration of lattice-based examples. Among them, we find the conjunctive and disjunctive fragments of classical logic (Examples 2.15 and 2.18), a logic associated with normal Kleene algebras (Example 2.20) and the logic of distributive bilattices (Example 2.21).

Logics whose truth sets are equationally and almost universally definable can be characterized by means of the behaviour of the Leibniz operator. More precisely, it turns out that truth is equationally (resp. almost universally) definable in $\mathsf{Mod}^*\mathcal{L}$ if and only if $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is completely order-reflecting over (resp. non-empty) deductive filters for every algebra $\boldsymbol{A}$. This condition on the Leibniz operator can be equivalently restricted to the theories of the logic $\mathcal{L}$ (Theorems 2.8 and 2.9). These results were first discovered by Raftery for equational definability in [83]. Then we move our attention to the relation between these logics and the *Frege hierarchy*, a hierarchy in which logics are classified by means of general replacement properties [40]. We show that if a Fregean logic $\mathcal{L}$ has a very weak kind of disjunction or conjuction, then truth is almost universally definable in $\mathsf{Mod}^*\mathcal{L}$. We show that for logics whose truth sets are almost universally definable, the Frege hierarchy reduces to three distinct classes (Theorem 2.29). In particular, we describe a characterization of equational and universal definability in terms of the behaviour of full generalized models (Lemma 2.27 and Corollary 2.28). Some of these results have an antecedent in [6], where only equational definability is taken into account.

Until now we focused on logics whose truth sets can be defined by means of some linguistic translation of formulas into equations. This idea presents some analogy with the one of *explicit definability* in first-order logic, in the sense that it requires that the definition of the truth sets is witnessed by some linguistic construction, i.e., by sets of equations. Now, Beth's definability theorem states that in first-order logic explicit definability and implicit definability coincide. Building on this analogy, it is natural to consider some suitable version of the notion of *implicit definability* in the framework of truth sets and to ask under which conditions these two kinds of definability coincide. Accordingly, given a logic $\mathcal{L}$, we say that truth is *implicitly definable* in $\mathsf{Mod}^*\mathcal{L}$ if the matrices in $\mathsf{Mod}^*\mathcal{L}$ are determined by their algebraic reduct. The analogy with Beth's definability theorem culminates in the discovery [30, 56, 58] that the notions of implicit and equational definability coincide when $\mathcal{L}$ is a protoalgebraic logic (here obtained as Corollary 3.4).

Now, implicit definability can be characterized in terms of the behaviour of the Leibniz operator. More precisely, it has long been known that truth is implicitly definable in $\mathsf{Mod}^*\mathcal{L}$ if and only if the Leibniz operator is injective over the deductive filters of *every* algebra (Lemma 3.2). This fact posed the problem of whether the injectivity of the Leibniz operator transfers from the theories, i.e., the filters over the term algebra with countably many free generators, to filters over arbitrary algebras [83, Problem 1]. The feeling that this question could have a positive answer is motivated by the fact that the main conditions on the Leibniz operator considered in the literature transfer from theories to filters over arbitrary algebras. In fact Czelakowski and Jansana provided in [30] a positive answer under the assumption of protoalgebraicity (Theorem 3.6). We solve this problem by showing that its answer depends on the cardinality of the language in which the logic is formulated. More precisely, if the language is countable, then the injectivity of the Leibniz operator transfers from theories to arbitrary filters (Theorem 3.8). On the other hand, we show that it is possible to construct counterexamples for logics expressed in uncountable languages (Example 3.9).

An intermediate definability condition that we take into account is the following: we say that truth is *small* in $\mathsf{Mod}^*\mathcal{L}$ when the truth sets in this class are the smallest deductive filters of the logic $\mathcal{L}$. We prove that this condition is equivalent to the fact that the Leibniz operator is order-reflecting over deductive filters of every algebra (Lemma 3.12). As it was the case for injectivity, the order-reflection of the Leibniz operator transfers from theories to filters over arbitrary algebras for logics expressed in a countable language (Theorem 3.13), while there are counterexamples among logics whose language is uncountable.

The work described until now originates an expansion of the Leibniz hierarchy with additional, weaker classes of logics corresponding to the definability conditions on the truth sets considered here. The expanded hierarchy is depicted in Figure 1 on page xviii.

Finally we consider the computational aspects of the problem of classifying logics according to the way their truth sets can be defined. While doing this, we adopt a wider perspective and consider, from a computational point of view, the problem of classifying logics within the Leibniz and Frege hierarchies in general. We show that the problem of classifying the logic determined by a finite consistent Hilbert calculus in the Leibniz (or Frege) hierarchy is undecidable (Theorems 4.10 and 4.23). This is achieved by reducing Hilbert's tenth problem on Diophantine equations to the one of classifying logics of Hilbert calculi in the Leibniz hierarchy, and by reducing the problem of determining the equational theory in one variable of relation algebras to the one of classifying logics of Hilbert cacluli in the Frege hierarchy. Remarkably, our proof shows that the problem of classifying logics in the Frege hierarchy remains undecidable even if we restrict our attention to Hilbert calculi that determine a finitary and finitely algebraizable logic. On the other hand, the situation changes if we consider a semantic version of the same question. We show that the problem of classifying the logic determined by a finite set of finite matrices (of finite type) in the main classes of the Leibniz hierarchy is decidable (Theorem 4.17), and that the same happens for some classes in the Frege hierarchy (Theorem 4.24 and Corollary 4.25). However, the classification problem of logics semantically presented is still open in other cases (Problems 3 and 4).

Part II: Adjunctions as Translations

As we mentioned, the aim of this part of the dissertation is to provide a notion of *translation* between equational consequences relative to classes K and K' of algebras possibly in different languages. Moreover, we aim to do this in such a way that the existence of a translation of this kind corresponds to the existence some strong *mathematical relation* between the classes K and K'. It is evident that this project has to be made more precise, since it can lead to completely different outcomes depending on the mathematical relation between K and K'. In fact we restrict our attention to the case where K and K', viewed as categories, are related by an *adjunction*. There are at least two reasons that motivate this choice. On the one hand, category theory [2, 8, 69] provides a formalism to describe the relations between different collections of mathematical objects and, within this formalism, adjunctions are probably the most prominent example of this kind of relations. On the other hand, it turns out that adjunctions can be associated with a certain kind of translations, that encompass some well-known examples in the logical literature. Thus the objective of this part of the dissertation can be made more precise as follows:

- We aim to develop a notion of translation between equational consequences relative to classes of algebras K and K' in such a way that each of these translations corresponds to an adjunction between K and K' and vice-versa.

Our approach to this problem is inspired by the work of McKenzie on category equivalences [73]. Roughly speaking, McKenzie discovered an algebraic and combinatorial characterization of category equivalence between prevarieties of algebras. In particular, he showed that if two prevarieties K and K' are categorically equivalent, then we can transform K into K' by applying two kinds of deformations to K. The first of these deformations is the *matrix power* construction. Very roughly speaking, the matrix power with exponent n of an algebra $\boldsymbol{A}$ is a new algebra $\boldsymbol{A}^{[n]}$ with universe A^n and whose basic m-ary operations are all n-sequences of $(m \times n)$-ary term functions of $\boldsymbol{A}$, which are applied component-wise. Again roughly speaking, the other basic deformation is defined as follows. Suppose that $\sigma(x)$ is a unary term. Then, given an algebra $\boldsymbol{A}$, we let $\boldsymbol{A}(\sigma)$ be the algebra whose universe is the range of the term-function $\sigma^{\boldsymbol{A}} \colon A \to A$ and whose n-ary operations are the restrictions to $\sigma[A]$ of the term functions of $\boldsymbol{A}$ of the form $\sigma t(x_1, \ldots, x_n)$, where t is an n-ary term. McKenzie's work shows that the prevarieties categorically equivalent to K are exactly the ones obtained deforming K by means of the matrix power and $\sigma(x)$ constructions, where σ is a unary term satisfying some additional condition (Theorem 5.18). This algebraic approach to the study of category equivalence has been reformulated in categorical terms for example in [81, 82] and has an antecedent in [36].

Building on McKenzie's work, we show that every right adjoint functor between generalized quasi-varieties (which are particular kinds of prevarieties) can be decomposed into a combination of two deformations that generalize the ones devised by McKenzie in the special case of category equivalence. These deformations are matrix powers with (possibly) *infinite* exponent and the following generalization of the $\sigma(x)$ construction. Given an algebra $\boldsymbol{A}$, we say that a set of equations in one variable θ is *compatible* with a sublanguage $\mathscr{L}$ of the language of $\boldsymbol{A}$ if the set of solutions of θ in $\boldsymbol{A}$ is closed under the restriction of the operations in $\mathscr{L}$. In this case we let $\boldsymbol{A}(\theta, \mathscr{L})$ be the algebra obtained by equipping the set of solutions of θ in $\boldsymbol{A}$ with the restriction of the operations in $\mathscr{L}$. We show (Theorem 5.29) that every right adjoint functor between generalized quasi-varieties is, up to a natural isomorphism, a composition of the matrix power construction and the generalized $\sigma(x)$ construction. Moreover, every functor obtained as a composition of these deformations is a right adjoint.

Now observe that the two deformations above have a clear syntactic flavour, as they are defined by means of term-functions, equations and sublanguages. It turns out that we can exploit the syntactic nature of these deformations to introduce a new notion of *translation* between relative equational consequences in a way that our decomposition of right adjoints can be rephrased as follows. Given two generalized quasi-varieties K and K', every translation of the equational consequence relative to K into the one relative to K' corresponds to a right adjoint functor from K' to K and vice-versa (Theorems 5.23 and 5.27). In other words, right adjoints reverse the direction of translations. It should be mentioned that the notion of translation that we obtain in this way is not especially simple. Roughly speaking the idea is that a translation of this kind is a pair $\langle \boldsymbol{\tau}, \Theta \rangle$, where $\boldsymbol{\tau}$ is a map that converts every equation of K into κ-many equations of K' (where κ is a fixed cardinal) and Θ is a set of equations of K' in κ-many variables. The translation is required to satisfy the following condition:

$$\Phi \vDash_{\mathsf{K}} \varepsilon \approx \delta \Longrightarrow \boldsymbol{\tau}(\Phi) \cup \bigcup_{j<\lambda} \Theta(\vec{x}_j) \vDash_{\mathsf{K}'} \boldsymbol{\tau}(\varepsilon \approx \delta) \tag{$\circledast$}$$

for every set of equations $\Phi \cup \{\varepsilon \approx \delta\}$ in λ variables. There are some remarkable points about the condition expressed in ($\circledast$). First observe that in general we do not require that the right-to-left direction in ($\circledast$) holds. In fact it turns out that it holds exactly when the left adjoint functor in the adjunction induced by the translation is faithful (Lemma 6.4). Second, observe that in the antecedent of the right-hand side of ($\circledast$) the equations Θ appear. This represents an asymmetry with respect to the situation expressed in ($*$), where the *context* Θ is not present.

Nevertheless it should be remarked that the context Θ appears, although not explicitly, in some of the best-known translations between propositional logics, when reformulated (using algebraizability) as translations between the the equational consequences relative to the corresponding quasi-varieties. For example, Gödel's translation induces an interpretation of the equational consequence relative to the variety of Heyting algebra into the one relative to the variety of interior algebras, where the role of Θ is played by the equation $\Box x \approx x$. This is related to the fact that this translation is associated with the functor that extracts the Heyting algebra of open elements, i.e., the solutions of $\Box x \approx x$, from an interior algebra (Examples 5.21 and 5.24). Similarly Kolmogorov's translation induces an interpretation of the equational consequence relative to the variety of Boolean algebras into the one relative to the variety of Heyting algebras, where the role of Θ is played by the equation $\neg\neg x \approx x$. Again, this corresponds to the fact that this translation is associated with the functor that extracts the Boolean algebra of regular elements, i.e., the solutions of $\neg\neg x \approx x$, out of a Heyting algebra (Examples 5.22 and 5.24).

In the last chapter we apply the machinery, constructed to describe the correspondence between adjunctions and translations, to develop some applications to the preservation of logico-algebraic properties related in some way to congruences. It is easy to see that every left adjoint functor $\mathcal{F} \colon \mathsf{X} \to \mathsf{Y}$ between generalized quasi-varieties induces a residuated map $\gamma_{\boldsymbol{A}} \colon \mathrm{Con}_{\mathsf{X}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Y}}\mathcal{F}(\boldsymbol{A})$ for every $\boldsymbol{A} \in \mathsf{X}$. Then we find sufficient and necessary conditions under which the global map γ is a complete lattice embedding (Theorem 6.5). In this spirit, we find sufficient and necessary conditions under which γ preserves compact congruences and $\mathcal{F}$ preserves finitely generated algebras (resp. Lemmas 6.9 and 6.15), and we apply them to the study of the preservation of EDPRC and its generalization known as ESRPC (resp. Theorems 6.11 and 6.17). Finally, we provide a logical interpretation of these results. In particular, we study the preservation of the (contextual) deduction theorem [20], of generalized disjunctions [27] and of the inconsistency lemma [84] between algebraizable logics whose equivalent algebraic semantics are related by an adjunction (Theorems 6.22, 6.21 and 6.23). Moreover, we show how the matrix power construction can be applied to prove that every prevariety is categorically equivalent to the equivalent algebraic semantics of an algebraizable logic expressed in enough variables (Theorem 6.26). This contrasts with the well-known fact that there are varieties that are not the equivalent algebraic semantics of any algebraizable logic (e.g., all non-trivial varieties of lattices). The memoir closes with a section (Section 6.5) of a somehow collateral interest in which we relate our logical interpretation of adjunctions to the general theory of equivalence between structural closure operators initiated in [16].

Conclusions

This memoir presents two contributions to the field of abstract algebraic logic. On the one hand we study some definability conditions on the truth sets of classes of matrices of the form $\mathsf{Mod}^*\mathcal{L}$. We characterize these conditions by means of the behaviour of the Leibniz operator and investigate their interaction with the Frege hierarchy. Finally, we consider the problem of classifying logics in the Leibniz and Frege hierarchies from a computational point of view. On the other hand, we describe a correspondence that relate adjunctions between generalized quasi-varieties and translations between their relative equational consequences. Then we study the preservation of logico-algebraic properties between algebraizable logics whose equivalent algebraic semantics are related by an adjunction. These investigations are both related to the fundamental idea, which is found everywhere in abstract algebraic logic, of interpreting a consequence relation into another one.

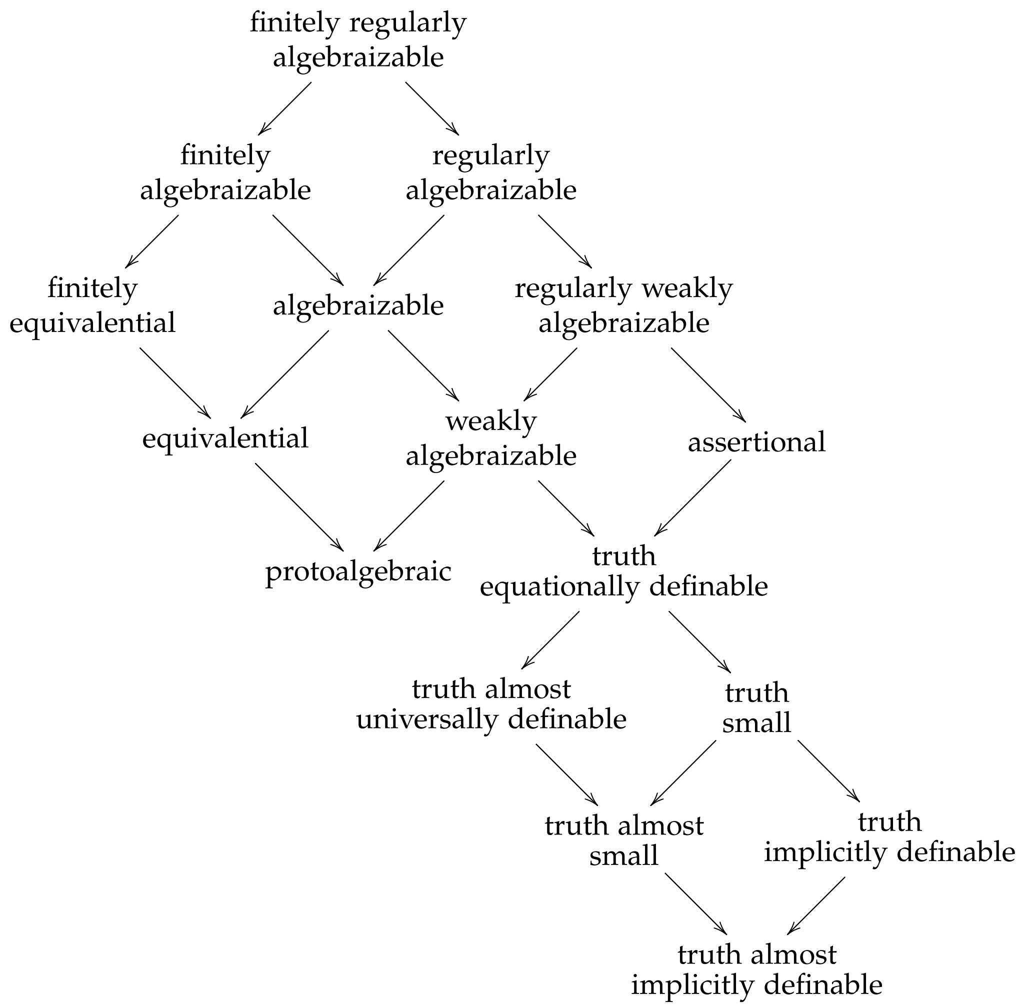


Figure 1: The extended Leibniz hierarchy

CHAPTER 1

Preliminaries

1.1 Sets and functions

For a general background on residuation theory we refer the reader to [48]. Consider two posets $\langle A, \leqslant \rangle$ and $\langle B, \leqslant \rangle$. A map $f \colon A \to B$ is *residuated* if there exists another map $f^+ \colon B \to A$ such that

$$f(a) \leqslant b \iff a \leqslant f^+(b)$$

for every $a \in A$ and $b \in B$. In this case f^+ is unique and is called the *residuum* of f. For example, consider an arbitrary function $f \colon X \to Y$ between two sets X and Y. The map f can be lifted to the power-sets as $f \colon \mathcal{P}(X) \to \mathcal{P}(Y)$ setting

$$f(A) := \{f(a) : a \in X\}$$

for every $A \subseteq X$. It is easy to see that f is a residuated map between the posets $\langle \mathcal{P}(X), \subseteq \rangle$ and $\langle \mathcal{P}(Y), \subseteq \rangle$. The residuum of f is the map $f^{-1} \colon \mathcal{P}(Y) \to \mathcal{P}(X)$ defined as

$$f^{-1}(B) := \{a \in A : f(a) \in B\}$$

for every $B \subseteq Y$. The map f^{-1} is the *inverse image* of f. Residuated maps between complete lattices enjoy a very useful characterization. More precisely, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are complete lattices, then a map $f \colon A \to B$ is residuated if and only if it commutes with arbitrary joins.

We denote cardinal numbers by λ, κ, μ etc. Given two cardinals κ and λ, we denote their product as cardinals by $\kappa \cdot \lambda$ and their cartesian product by $\kappa \times \lambda$. Keep this in mind, since sometimes it will be useful to distinguish between the two operations. We denote the set of natural numbers by ω.

1.2 Closure operators

A *closure operator* on a set A is a monotone function $C\colon \mathcal{P}(A) \to \mathcal{P}(A)$ such that $X \subseteq C(X) = C\big(C(X)\big)$ for every $X \in \mathcal{P}(A)$. A *closure system* on A is a family $\mathcal{C} \subseteq \mathcal{P}(A)$ closed under arbitrary intersections and such that $A \in \mathcal{C}$. It is well known that the closed sets (fixed points) of a closure operator on A form a closure system and that, given a closure system $\mathcal{C}$, one can construct a closure operator C by letting $C(X) = \bigcap\{Y \in \mathcal{C} \mid X \subseteq Y\}$ for every $X \in \mathcal{P}(A)$. These transformations are indeed inverse to one another. Therefore definitions and results established for closure operators transfer naturally to closure systems and vice-versa. Given a closure system $\mathcal{C} \subseteq \mathcal{P}(A)$ and $F \in \mathcal{C}$, we let $\mathcal{C}^F := \{G \in \mathcal{C} : F \subseteq G\}$. It is easy to prove that $\mathcal{C}^F$ is still a closure system on A.

A closure operator $C\colon \mathcal{P}(A) \to \mathcal{P}(A)$ is *finitary* if for every $X \cup \{a\} \subseteq A$,

$$a \in C(X) \text{ if and only if there is a finite } Y \subseteq A \text{ such that } a \in C(Y).$$

A closure operator is finitary if and only if the corresponding closure system is inductive, i.e., it is closed under unions of upward directed non-empty subfamilies. Let $\mathcal{C} \subseteq \mathcal{P}(A)$ be a closure system on A. We say that an element $a \in A$ is a *theorem* of $\mathcal{C}$ if $a \in \bigcap \mathcal{C}$. Accordingly, we say that $\mathcal{C}$ has theorems if $\bigcap \mathcal{C} \neq \emptyset$. If this is not the case, we say that $\mathcal{C}$ is *purely inferential*. There are at least two other extreme cases of closure systems that deserve a special name: $\mathcal{C}$ is *inconsistent* if $\mathcal{C} = \{A\}$ and *almost inconsistent* if $\mathcal{C} = \{\emptyset, A\}$. We say that $\mathcal{C}$ is *trivial* if it is either inconsistent or almost inconsistent.

1.3 Logics and relative equational consequences

Given an algebraic language $\mathscr{L}$ and a set X, we denote by $Fm(\mathscr{L}, X)$ the set of formulas over $\mathscr{L}$ built up with the variables in X, and by $\boldsymbol{Fm}(\mathscr{L}, X)$ the corresponding absolutely free algebra. We also denote by $Eq(\mathscr{L}, X)$ the set of equations built up from X. Formally speaking, equations are pairs of formulas, i.e., $Eq(\mathscr{L}, X) := Fm(\mathscr{L}, X) \times Fm(\mathscr{L}, X)$. Moreover, $End(\mathscr{L}, X)$ denotes the monoid of endomorphisms of $\boldsymbol{Fm}(\mathscr{L}, X)$. When the language $\mathscr{L}$ is clear from the context, we simply write $Fm(X)$, $Eq(X)$, $\boldsymbol{Fm}(X)$ and $End(X)$.* Since every cardinal κ is a set, sometimes we write $Fm(\mathscr{L}, \kappa)$ to stress the cardinality of the set of variables. The same convention applies to equations, term algebras and endomorphisms monoids.

While working with a fixed algebraic language $\mathscr{L}$, we will denote by Fm the set of formulas over $\mathscr{L}$ built up with denumerably many variables Var denoted by x, y, z, etc., and by $\boldsymbol{Fm}$ the corresponding algebra. A *logic* $\mathcal{L}$ is a closure operator $C_{\mathcal{L}}\colon \mathcal{P}(Fm) \to \mathcal{P}(Fm)$ which is *structural* in the sense

*In particular, in Part I we will work with a fixed but arbitrary algebraic language, while in Part II we will compare classes of algebras with different languages.

that $\sigma C_{\mathcal{L}}(\Gamma) \subseteq C_{\mathcal{L}}\sigma(\Gamma)$ for every $\Gamma \subseteq Fm$ and every endomorphism σ of $\boldsymbol{Fm}$ (also called *substitution*). Given $\Gamma \cup \{\varphi\} \subseteq Fm$ we write $\Gamma \vdash_{\mathcal{L}} \varphi$ to denote the fact that $\varphi \in C_{\mathcal{L}}(\Gamma)$. Sometimes expressions of the form $\Gamma \vdash_{\mathcal{L}} \varphi$ will be called informally *deductions* of $\mathcal{L}$. In practice, we often define a logic $\mathcal{L}$ by describing the relation $\vdash_{\mathcal{L}}$. We denote the closure system associated with $C_{\mathcal{L}}$ by $\mathcal{T}h\mathcal{L}$. A set $\Gamma \subseteq Fm$ is a *theory* of $\mathcal{L}$ if $\Gamma \in \mathcal{T}h\mathcal{L}$.

Given two logics $\mathcal{L}$ and $\mathcal{L}'$, we will write $\mathcal{L} \leqslant \mathcal{L}'$ if $C_{\mathcal{L}}(\Gamma) \subseteq C_{\mathcal{L}'}(\Gamma)$ for every $\Gamma \subseteq Fm$; in this case we say that $\mathcal{L}'$ is an *extension* of $\mathcal{L}$. A logic $\mathcal{L}$ is a *conversative expansion* of a logic $\mathcal{L}'$ if the language of $\mathcal{L}$ includes the language of $\mathcal{L}'$ and for every set of formulas $\Gamma \cup \{\varphi\}$ in the language of $\mathcal{L}'$, we have $\Gamma \vdash_{\mathcal{L}} \varphi$ if and only if $\Gamma \vdash_{\mathcal{L}'} \varphi$.

Given a class of algebras K and $\Phi \cup \{\varphi \approx \psi\} \subseteq Eq(X)$, we define

$$\Phi \vDash_{\mathsf{K}} \varphi \approx \psi \iff \text{for every } \boldsymbol{A} \in \mathsf{K} \text{ and every } h\colon \boldsymbol{Fm}(X) \to \boldsymbol{A} \\ \text{if } h\varepsilon = h\delta \text{ for every } \varepsilon \approx \delta \in \Phi, \text{ then } h\alpha = h\beta.$$

The relation $\vDash_{\mathsf{K}}$ is called the *equational consequence relative to* K. The map $C_{\mathsf{K}}\colon \mathcal{P}(Eq(X)) \to \mathcal{P}(Eq(X))$ defined by the rule

$$C_{\mathsf{K}}(\Phi) := \{\varphi \approx \psi : \Phi \vDash_{\mathsf{K}} \varphi \approx \psi\}, \text{ for every } \Phi \subseteq Eq(X)$$

is a closure operator over $Eq(X)$.

1.4 Classes of algebras

Our main references for universal algebra are [13, 24, 74]. We denote by $\mathbb{I}$, $\mathbb{H}$, $\mathbb{S}$, $\mathbb{P}$, $\mathbb{P}_{\text{SD}}$ and $\mathbb{P}_{\text{U}}$ respectively the class operators of isomorphism, homomorphic images, subalgebras, direct products, (isomorphic copies of) subdirect products and ultraproducts. We assume that product-style class operators admit empty set of indexes. We denote algebras by bold capital letters $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, etc. (with universes A, B, C, etc.). Given a class of algebras K, we denote its language by $\mathscr{L}_{\mathsf{K}}$. A *prevariety* is a class of algebras axiomatized by arbitrary generalized quasi-equations or, equivalently, a class closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}$. A *generalized quasi-variety* is a class of algebras axiomatized by generalized quasi-equations whose number of variables is bounded by some infinite cardinal. These can be equivalently characterized [16] as the classes of algebras closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{U}_{\kappa}$ (for some infinite cardinal κ), where for every class of algebras K:

$$\mathbb{U}_{\kappa}(\mathsf{K}) := \{\boldsymbol{A} : \boldsymbol{B} \in \mathsf{K} \text{ for every } \kappa\text{-generated subalgebra } \boldsymbol{B} \leqslant \boldsymbol{A}\}.$$

A *quasi-variety* is a class of algebras axiomatized by quasi-equations or, equivalently, a class closed under $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$ and $\mathbb{P}_{\text{U}}$. A *variety* is a class of algebras axiomatized by equations or, equivalently, closed under $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$. Given a class of algebras K, we will denote by $\mathbb{GQ}_{\kappa}(\mathsf{K})$ the models of the generalized quasi-equations in κ-many variables that hold in K and respectively by $\mathbb{Q}(\mathsf{K})$ and $\mathbb{V}(\mathsf{K})$ the quasi-variety and the variety generated by K. It is well known that

$$\mathbb{GQ}_\kappa(\mathsf{K}) = \mathbb{U}_\kappa\mathbb{ISP}(\mathsf{K}) \quad \mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K}) \quad \mathbb{V}(\mathsf{K}) = \mathbb{HSP}(\mathsf{K}).$$

It is worth to remark that both the existence and the non-existence of a pre-variety that is not a generalized quasi-variety are consistent with Neumann-Bernays-Gödel (NBG) class theory with the axiom of choice. In fact in NBG the assumption that every prevariety is a generalized quasi-variety is equivalent to the *Vopěnka Principle*, according to which every class of pair-wise non-embeddable models of a first-order theory is a set [1] (see also [55, Proposition 2.3.18]).

Given a class of algebras K and a set $X \neq \emptyset$, we denote by $\boldsymbol{Fm}_{\mathsf{K}}(X)$ the free algebra over K with free generators X. In general the free algebra $\boldsymbol{Fm}_{\mathsf{K}}(X)$ is constructed as a quotient of the term algebra $\boldsymbol{Fm}(X)$ and its elements are congruence classes of formulas equivalent in K. Sometimes we identify the universe of $\boldsymbol{Fm}_{\mathsf{K}}(X)$ with a set of its representatives, i.e., with a set of formulas in variables X. It is well known that $\boldsymbol{Fm}_{\mathsf{K}}(X) \in \mathbb{ISP}(\mathsf{K})$. Thus prevarieties contain free algebras with arbitrary large sets of free generators. Prevarieties contain also trivial algebras, which we denote generically by $\mathbf{1}$.

Given a class of algebras K and an algebra $\boldsymbol{A}$, we say that a congruence θ of $\boldsymbol{A}$ is a K-*congruence* if $\boldsymbol{A}/\theta \in \mathsf{K}$, and denote the collection of K-congruences by $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$. In particular, we will denote by $\pi_\theta \colon \boldsymbol{A} \to \boldsymbol{A}/\theta$ the canonical projection on the quotient and by $0_{\boldsymbol{A}}$ and $1_{\boldsymbol{A}}$ the identity and total congruence of $\boldsymbol{A}$. If K is a prevariety, we have that $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ forms a closure system when ordered under the inclusion relation: this is due to the fact that prevarieties are closed under the formation of subdirect products and contain trivial algebras. Accordingly, we denote by $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}$ the closure operator of generation of K-congruences. While speaking of congruence generation in the absolute sense, we will simply write $\mathrm{Cg}^{\boldsymbol{A}}$. It is easy to see that for every class of algebras K, the validity of generalized quasi-equations in K corresponds to the validity of deductions in the equational consequence relative to K in the sense that

$$\mathsf{K} \vDash \bigwedge_{i\in I} \varphi_i \approx \psi_i \to \varepsilon \approx \delta \Longleftrightarrow \{\varphi_i \approx \psi_i : i \in I\} \vDash_{\mathsf{K}} \varepsilon \approx \delta.$$

This is reflected in the fact that if K is a prevariety, then the set of fixed points of $C_{\mathsf{K}} \colon : \mathcal{P}(Eq(X)) \to \mathcal{P}(Eq(X))$ coincides with $\mathrm{Con}_{\mathsf{K}}\boldsymbol{Fm}(X)$. Now let K be a quasi-variety and $\boldsymbol{A}$ an arbitrary algebra. The lattice $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ is algebraic and its compact elements $\mathrm{Comp}_{\mathsf{K}}\boldsymbol{A}$ are the finitely generated K-congruences. In particular, the closure operator $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}$ is finitary. An algebra $\boldsymbol{A} \in \mathsf{K}$ is K-*finitely presentable* if there is some $n \in \omega$ and some finitely generated K-congruence θ of $\boldsymbol{Fm}_{\mathsf{K}}(n)$ such that $\boldsymbol{A}$ is isomorphic to $\boldsymbol{Fm}_{\mathsf{K}}(n)/\theta$.

A non-trivial algebra $\boldsymbol{A}$ is *subdirectly irreducible* if for every class of algebras K, if $\boldsymbol{A} \in \mathbb{P}_{\mathrm{SD}}(\mathsf{K})$, then $\boldsymbol{A} \in \mathbb{I}(\mathsf{K})$. Equivalently, $\boldsymbol{A}$ is subdirectly irreducible when the poset $\langle \mathrm{Con}\boldsymbol{A} \smallsetminus \{0_A\}, \subseteq\rangle$ has a minimum element (called the *monolith* of $\boldsymbol{A}$). Given a class of algebras K, we denote by K_{si} the collection of its subdirectly irreducible members. It is well known that if K is a variety, then $\mathsf{K} = \mathbb{P}_{\mathrm{SD}}(\mathsf{K}_{si})$. A variety K is *congruence distributive* when $\mathrm{Con}\boldsymbol{A}$ is a distributive lattice for every $\boldsymbol{A} \in \mathsf{K}$. Every variety of algebras with a lattice-reduct is congruence distributive. Jónsson's lemma states that if K is a class of algebras that generates a congruence distributive variety, then $\mathbb{V}(\mathsf{K})_{si} \subseteq \mathbb{HSP}_{\mathrm{U}}(\mathsf{K})$. When K is a finite set of finite algebras, this specializes to $\mathbb{V}(\mathsf{K})_{si} \subseteq \mathbb{HS}(\mathsf{K})$.

1.5 Logical congruences

From now on we will review well-known concepts and results of abstract algebraic logic. For further informations we refer the reader to [18, 19, 27, 38, 40, 41, 85]. Given an algebra $\boldsymbol{A}$ and a set $F \subseteq A$, a congruence $\theta \in \mathrm{Con}\boldsymbol{A}$ is *compatible* with F when

$$\text{if } a \in F \text{ and } \langle a, b\rangle \in \theta, \text{ then } b \in F$$

for every $a, b \in A$. There exists the largest congruence of $\boldsymbol{A}$ compatible with F. We denote this congruence by $\boldsymbol{\Omega}^{\boldsymbol{A}}F$ and refer to it as the *Leibniz congruence* of F (over $\boldsymbol{A}$). In the case of the term algebra $\boldsymbol{Fm}$ we will omit the superscript in the Leibniz congruence.

Lemma 1.1. *Let $h\colon \boldsymbol{A} \to \boldsymbol{B}$ be an homomorphism and $F \subseteq B$.*

1. $h^{-1}\boldsymbol{\Omega}^{\boldsymbol{B}}F \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}h^{-1}[F]$.
2. *If h is surjective, then $\boldsymbol{\Omega}^{\boldsymbol{A}}h^{-1}[F] = h^{-1}\boldsymbol{\Omega}^{\boldsymbol{B}}F$.*

Given an algebra $\boldsymbol{A}$, a closure system $\mathcal{C}$ on A and a closed set $F \in \mathcal{C}$, the *Suzko congruence* (relative to $\mathcal{C}$) of F is

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{C}}F := \bigcap\{\boldsymbol{\Omega}^{\boldsymbol{A}}G : G \in \mathcal{C} \text{ and } F \subseteq G\}.$$

In particular, we have that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{C}}F \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}F$. A useful property of the Suszko congruence is the following:

$$\text{if } F \subseteq G, \text{ then } \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{C}}F \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{C}}G$$

for every $F, G \in \mathcal{C}$. Given an algebra $\boldsymbol{A}$ and a closure system $\mathcal{C}$ on A, the *Tarski congruence* of $\mathcal{C}$ (over $\boldsymbol{A}$) is

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\mathcal{C} := \bigcap\{\boldsymbol{\Omega}^{\boldsymbol{A}}F : F \in \mathcal{C}\}.$$

The Leibniz, Suszko and Tarski congruences can be characterized in terms of the indiscernibility of elements with respect to filters and closure systems in the following way. Given an algebra $\boldsymbol{A}$, a function $p\colon A^n \to A$ is a *polynomial function* if there are a natural number m, a term $\varphi(x_1, \dots, x_{n+m})$ and elements $b_1, \dots, b_m \in A$ such that

$$p(a_1, \dots, a_n) = \varphi^{\boldsymbol{A}}(a_1, \dots, a_n, b_1, \dots, b_m)$$

for every $a_1, \dots, a_n \in A$. Observe that the notation $\varphi(x_1, \dots, x_{n+m})$ means just that the variables really occurring in φ are among, but not necessarily all, $\{x_1, \dots, x_{n+m}\}$.

Lemma 1.2. *Let $\boldsymbol{A}$ be an algebra, $\mathcal{C} \subseteq \mathcal{P}(A)$ be a closure system, $F \subseteq A$ and $a, b \in A$.*

1. $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F \iff (p(a) \in F$ *if and only if* $p(b) \in F)$ *for every unary polynomial function p on $\boldsymbol{A}$.*
2. $\langle a, b\rangle \in \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{C}} F \iff C(F \cup \{p(a)\}) = C(F \cup \{p(b)\})$ *for every unary polynomial function p on $\boldsymbol{A}$.*
3. $\langle a, b\rangle \in \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}} \mathcal{C} \iff C(p(a)) = C(p(b))$ *for every unary polynomial function p on $\boldsymbol{A}$.*

Given an algebra $\boldsymbol{A}$, the *Leibniz operator* (relative to $\boldsymbol{A}$) is the function $\boldsymbol{\Omega}^{\boldsymbol{A}}\colon \mathcal{P}(A) \to \mathrm{Con}\boldsymbol{A}$ that associates $\boldsymbol{\Omega}^{\boldsymbol{A}} F$ with a set $F \subseteq A$. Given a closure system $\mathcal{C}$ over A, the *Suszko operator* (relative to $\boldsymbol{A}$ and $\mathcal{C}$) is the map $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Con}\boldsymbol{A}$ that associates $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{C}} F$ with a set $F \in \mathcal{C}$.

1.6 Matrix semantics

A pair $\langle \boldsymbol{A}, F\rangle$ is a *matrix* when $\boldsymbol{A}$ is an algebra and $F \subseteq A$. A matrix $\langle \boldsymbol{A}, F\rangle$ is *trivial* if $\boldsymbol{A}$ the trivial algebra and $F = A$. Every class of matrices M determines a logic in the following way. Given $\Gamma \cup \{\varphi\} \subseteq Fm$ we let

$$\Gamma \vdash_{\mathsf{M}} \varphi \iff \text{for every } \langle \boldsymbol{A}, F\rangle \in \mathsf{M} \text{ and every homomorphism } h\colon \boldsymbol{Fm} \to \boldsymbol{A}, \text{ if } h[\Gamma] \subseteq F, \text{ then } h(\varphi) \in F.$$

A logic $\mathcal{L}$ is *complete* w.r.t. a class of matrices M if $\mathcal{L}$ coincides with the logic determined by M.

Given a logic $\mathcal{L}$ and an algebra $\boldsymbol{A}$, a set $F \subseteq A$ is a *deductive filter* (or simply a *filter*) of $\mathcal{L}$ over $\boldsymbol{A}$ when

$$\text{if } \Gamma \vdash_{\mathcal{L}} \varphi, \text{ then for every homomorphism } h\colon \boldsymbol{Fm} \to \boldsymbol{A}, \text{ if } h[\Gamma] \subseteq F, \text{ then } h(\varphi) \in F$$

for every $\Gamma \cup \{\varphi\} \subseteq Fm$. We denote by $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ the set of deductive filters of $\mathcal{L}$ over $\boldsymbol{A}$, which turns out to be a closure system. Observe that the filters of $\mathcal{L}$ over $\boldsymbol{Fm}$ coincide with the theories of $\mathcal{L}$. A matrix $\langle \boldsymbol{A}, F\rangle$ is a *model* of a logic $\mathcal{L}$ when $F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$. The class of models of $\mathcal{L}$ is closed under the natural extension to matrices of the operators $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{P}_{\text{SD}}$. The *reduction* of a matrix $\langle \boldsymbol{A}, F\rangle$ is $\langle \boldsymbol{A}, F\rangle^* := \langle \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F, F/\boldsymbol{\Omega}^{\boldsymbol{A}}F\rangle$. A matrix $\langle \boldsymbol{A}, F\rangle$ is *reduced* when $\boldsymbol{\Omega}^{\boldsymbol{A}}F = 0_{A}$. A matrix and its reduction determine the same logic. The Leibniz congruence allows to associate two special classes of models and a special class of algebras with a logic $\mathcal{L}$:

$$\begin{aligned} \mathsf{LMod}^*\mathcal{L} &:= \mathbb{I}\{\langle \boldsymbol{Fm}, \Gamma\rangle : \Gamma \in \mathcal{T}h\mathcal{L} \text{ and } \boldsymbol{\Omega}\Gamma = 0_{\boldsymbol{Fm}}\} \\ \mathsf{Mod}^*\mathcal{L} &:= \{\langle \boldsymbol{A}, F\rangle : F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \text{ and } \boldsymbol{\Omega}^{\boldsymbol{A}}F = 0_{A}\} \\ \mathsf{Alg}^*\mathcal{L} &:= \{\boldsymbol{A} : \text{there is } F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \text{ such that } \boldsymbol{\Omega}^{\boldsymbol{A}}F = 0_{A}\}. \end{aligned}$$

We will refer to $\mathsf{LMod}^*\mathcal{L}$ and $\mathsf{Mod}^*\mathcal{L}$ respectively as to the classes of *Lindenbaum-Tarski models* and *reduced models* of $\mathcal{L}$. It is well known that $\mathcal{L}$ is complete w.r.t. both $\mathsf{LMod}^*\mathcal{L}$ and $\mathsf{Mod}^*\mathcal{L}$.

When dealing with the closure system $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ associated with a fixed logic $\mathcal{L}$, we shall write $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}}F$ instead of $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}}F$. It is often useful to restrict the Leibniz and Suszko operators to $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$. For example we will say that the Leibniz (resp. Suszko) operator *is injective over* $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ if the restriction $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \to \mathrm{Con}\boldsymbol{A}$ (resp. $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}} \colon \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \to \mathrm{Con}\boldsymbol{A}$) is injective. The same convention applies to order-theoretic properties such as monotonicity, order-reflection, etc.

The *Suszko-reduction* of a model $\langle \boldsymbol{A}, F\rangle$ of $\mathcal{L}$ is the matrix $\langle \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}}F, F/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}}F\rangle$. A matrix $\langle \boldsymbol{A}, F\rangle$ is *Suszko-reduced*, when $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}}F = 0_{A}$. The Suszko congruence allows to associate a class of models and a class of algebras with a logic $\mathcal{L}$:

$$\begin{aligned} \mathsf{Mod}^{\mathsf{Su}}\mathcal{L} &:= \{\langle \boldsymbol{A}, F\rangle : F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \text{ and } \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}}F = 0_{A}\} \\ \mathsf{Alg}\mathcal{L} &:= \{\boldsymbol{A} : \text{there is } F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \text{ such that } \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}}F = 0_{A}\}. \end{aligned}$$

$\mathsf{Mod}^{\mathsf{Su}}\mathcal{L}$ is the class of *Suszko-reduced models* of $\mathcal{L}$ and $\mathsf{Alg}\mathcal{L}$ is the *algebraic counterpart* of $\mathcal{L}$. It turns out that $\mathsf{Mod}^{\mathsf{Su}}\mathcal{L} = \mathbb{P}_{\text{SD}}\mathsf{Mod}^*\mathcal{L}$ and $\mathsf{Alg}\mathcal{L} = \mathbb{P}_{\text{SD}}\mathsf{Alg}^*\mathcal{L}$. Observe that $\mathcal{L}$ is complete w.r.t. $\mathsf{Mod}^{\mathsf{Su}}\mathcal{L}$.

Lemma 1.3. *If $\mathcal{L}$ is complete w.r.t. a class of matrices* M, *then* $\mathsf{Alg}\mathcal{L}$ *is included into the variety generated by the algebraic reducts of* M.

A finite set of finite matrices of finite type is called *strongly finite*. A logic is *strongly finite* if it is complete w.r.t. a strongly finite set of matrices.[†] Given a class of matrices M and a set X, we denote by $\boldsymbol{Fm}_{\mathsf{M}}(X)$ the free algebra with free generators X over the variety generated by the algebraic reducts of M.

[†]Observe that the finiteness of the type is not required in the usual definition of a strongly finite logic [96].

1.7 G-matrix semantics

A pair $\langle \boldsymbol{A}, \mathcal{C} \rangle$ is a *g-matrix* when $\boldsymbol{A}$ is an algebra and $\mathcal{C}$ is a closure system over A. Every logic $\mathcal{L}$ can be seen as the g-matrix $\langle \boldsymbol{Fm}, \mathcal{T}h\mathcal{L} \rangle$. Given two g-matrices $\langle \boldsymbol{A}, \mathcal{C} \rangle$ and $\langle \boldsymbol{B}, \mathcal{D} \rangle$, a *strict homomorphism* $h \colon \langle \boldsymbol{A}, \mathcal{C} \rangle \to \langle \boldsymbol{B}, \mathcal{D} \rangle$ is an homomorphism $h \colon \boldsymbol{A} \to \boldsymbol{B}$ such that $h^{-1}\mathcal{D} = \mathcal{C}$.

Lemma 1.4. *Let $\langle \boldsymbol{A}, \mathcal{C} \rangle$ and $\langle \boldsymbol{B}, \mathcal{D} \rangle$ be g-matrices and $h \colon \boldsymbol{A} \to \boldsymbol{B}$ a surjective homomorphism. $h \colon \langle \boldsymbol{A}, \mathcal{C} \rangle \to \langle \boldsymbol{B}, \mathcal{D} \rangle$ is strict if and only if the extended function $h \colon \mathcal{P}(A) \to \mathcal{P}(B)$ restricts to an order isomorphism between $\mathcal{C}$ and $\mathcal{D}$ (both ordered under set-theoretical inclusion) and its inverse is the residual function h^{-1}.*

Let M be a class of g-matrices. The logic determined by M is the logic determined by the class of matrices

$$\{\langle \boldsymbol{A}, F \rangle : F \in \mathcal{C} \text{ for some } \langle \boldsymbol{A}, \mathcal{C} \rangle \in \mathsf{M}\}.$$

A logic $\mathcal{L}$ is complete w.r.t. a class of g-matrices M, when $\mathcal{L}$ coincides with the logic determined by M. A g-matrix $\langle \boldsymbol{A}, \mathcal{C} \rangle$ is a *g-model* of a logic $\mathcal{L}$ if $\mathcal{C} \subseteq \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$. The *reduction* of a g-matrix $\langle \boldsymbol{A}, \mathcal{C} \rangle$ is the g-matrix $\langle \boldsymbol{A}, \mathcal{C} \rangle^* := \langle \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\mathcal{C}, \mathcal{C}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\mathcal{C} \rangle$. Given a logic $\mathcal{L}$, we will denote by $\widetilde{\boldsymbol{\Omega}}\mathcal{L}$ the Tarski congruence associated with the g-matrix $\langle \boldsymbol{Fm}, \mathcal{T}h\mathcal{L} \rangle$. The congruence $\widetilde{\boldsymbol{\Omega}}\mathcal{L}$ is fully invariant. Given a logic $\mathcal{L}$, we have that

$$\mathsf{Alg}\mathcal{L} = \{\boldsymbol{A} : \text{ there is a g-model } \langle \boldsymbol{A}, \mathcal{C} \rangle \text{ of } \mathcal{L} \text{ such that } \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\mathcal{C} = 0_{A}\}.$$

1.8 The Leibniz hierarchy

The study of the Leibniz hierarchy is one of the main topics of abstract algebraic logic. Since the reader is expected to have some working knowledge of the topic, we will not recall here all the relevant standard definitions and results (that can be found for example in [27]). In particular, we will state explicitly only the results that will be used later on. As a consequence, some of the most famous theorems of the general theory are omitted.

A logic $\mathcal{L}$ is *protoalgebraic* if there is a set of formulas $\boldsymbol{\rho}(x, y, \vec{z})$ in two variables x and y (possibly) with parameters $\vec{z}$ such that for every algebra $\boldsymbol{A}$, every $F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ and every $a, b \in A$

$$\langle a, b \rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}F \Longleftrightarrow \boldsymbol{\rho}^{\boldsymbol{A}}(a, b, \vec{c}) \subseteq F \text{ for every } \vec{c} \in A. \tag{1.1}$$

In this case $\boldsymbol{\rho}(x, y, \vec{z})$ is a set of *congruence formulas with parameters* for $\mathcal{L}$. Analogously, a logic is called (*finitely*) *equivalential* if there is a (finite) set of formulas $\boldsymbol{\rho}(x, y)$ in two variables and without parameters that satisfies (1.1). In this case $\boldsymbol{\rho}(x, y)$ is called a set of *congruence formulas* for $\mathcal{L}$. Clearly equivalential logics are protoalgebraic, but the converse is not true in general. Protoalgebraic and equivalential logics can be characterized syntactically. Let us explain briefly how. A set of formulas in two variables $\boldsymbol{\rho}(x,y)$ is a set of *protoimplication formulas* for a logic $\mathcal{L}$ if the following conditions hold:

$$\emptyset \vdash_{\mathcal{L}} \boldsymbol{\rho}(x,x) \tag{R}$$

$$x, \boldsymbol{\rho}(x,y) \vdash_{\mathcal{L}} y. \tag{MP}$$

Then it is possible to show that the existence of a set of protoimplication formulas characterizes protoalgebraic logics.

Theorem 1.5.

1. *$\mathcal{L}$ is protoalgebraic $\iff$ it has a set of protoimplication formulas.*
2. *$\mathcal{L}$ is equivalential $\iff$ it has a set of protoimplication formulas $\boldsymbol{\rho}(x,y)$ such that*

$$\bigcup_{i \leqslant n} \boldsymbol{\rho}(x_i, y_i) \vdash_{\mathcal{L}} \boldsymbol{\rho}(\lambda(x_1 \dots x_n), \lambda(y_1 \dots y_n)) \tag{Rep}$$

for every n-ary function symbol λ.

It is natural to ask how to construct a set of congruence formulas with parameters from a given set of protoimplication formulas. The next lemma provides the answer.

Lemma 1.6. *If $\boldsymbol{\rho}(x,y)$ is a set of protoimplication formulas for $\mathcal{L}$, then*

$$\boldsymbol{\mu}(x,y,\vec{z}) := \{\alpha(\varphi(x,\vec{z}), \varphi(y,\vec{z})) : \alpha(x,y) \in \boldsymbol{\rho}(x,y) \text{ and } \varphi(x,\vec{z}) \in Fm\}$$

is a set of congruence formulas with parameters for $\mathcal{L}$.

Given a set of formulas $\Delta(x,y,\vec{z})$ in two variables x and y and parameters $\vec{z}$, we define

$$\Delta\langle x,y\rangle := \bigcup\{\Delta(x,y,\vec{\delta}) : \vec{\delta} \in Fm\}.$$

Lemma 1.7. *Let $\Delta(x,y,\vec{z})$ and $\Delta'(x,y,\vec{z})$ be sets of formulas and $\mathcal{L}$ a logic. If $\Delta(x,y,\vec{z})$ is a set of congruence formulas with parameters for $\mathcal{L}$, then also $\Delta'(x,y,\vec{z})$ is so if and only if $\Delta\langle x,y\rangle \dashv\vdash_{\mathcal{L}} \Delta'\langle x,y\rangle$.*

Protoalgebraic logics can be characterized by means of the behaviour of the Leibniz and Suszko operators over deductive filters.

Theorem 1.8. *The following conditions are equivalent:*

(i) *$\mathcal{L}$ is protoalgebraic.*

(ii) *$\boldsymbol{\Omega}^{\boldsymbol{A}}$ is monotone over $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ for every algebra $\boldsymbol{A}$.*

(iii) *$\boldsymbol{\Omega}$ is monotone over $\mathcal{T}h\mathcal{L}$.*

(iv) *$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}} F = \boldsymbol{\Omega}^{\boldsymbol{A}} F$ for every $F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ and every algebra $\boldsymbol{A}$.*

In this case $\mathsf{Mod}^{*}\mathcal{L} = \mathsf{Mod}^{\mathsf{Su}}\mathcal{L}$.

Now we move our attention to the theory of algebrizable logics. A *structural transformer* $\boldsymbol{\tau}\colon \mathcal{P}(Fm) \to \mathcal{P}(Eq)$ (from formulas into equations) is a residuated map that commutes with substitutions. It is easy to see that a structural transformer can be identified with a set of equations $\boldsymbol{\tau}(x)$ in just one variable x by requiring that

$$\boldsymbol{\tau}(\Gamma) \coloneqq \bigcup\{\boldsymbol{\tau}(\gamma) : \gamma \in \Gamma\}$$

for every $\Gamma \subseteq Fm$. Structural transformers $\boldsymbol{\rho}\colon \mathcal{P}(Eq) \to \mathcal{P}(Fm)$ (from equations into formulas) are defined analogously and can be identified with sets of formulas $\boldsymbol{\rho}(x, y)$ in two variables x and y. A logic $\mathcal{L}$ is *algebraizable* with *equivalent algebraic semantics* the generalized quasi-variety K (axiomatized by generalized quasi-equations in countably many variables) if there are structural transformers $\boldsymbol{\tau}\colon \mathcal{P}(Eq) \longleftrightarrow \mathcal{P}(Eq)\colon \boldsymbol{\rho}$ for which the following conditions hold:

A1. $\Gamma \vdash_{\mathcal{L}} \varphi$ if and only if $\boldsymbol{\tau}\Gamma \vDash_{\mathsf{K}} \boldsymbol{\tau}\varphi$;

A2. $x \approx y \mathrel{=\!\mid\!\models}_{\mathsf{K}} \boldsymbol{\tau}\boldsymbol{\rho}\,(x \approx y)$

for every $\Gamma \cup \{\varphi\} \subseteq Fm$. Condition A2 is usually referred to as stating that $\boldsymbol{\tau}(x)$ and $\boldsymbol{\rho}(x, y)$ are mutually inverse (modulo K). Algebraizable logics enjoy a syntactic characterization:

Theorem 1.9. *A logic $\mathcal{L}$ is algebraizable if and only if there are a set of equations $\boldsymbol{\tau}(x)$ and a set of formulas $\boldsymbol{\rho}(x, y)$ such that:*

$$\emptyset \vdash_{\mathcal{L}} \boldsymbol{\rho}(x, x) \tag{R}$$

$$x, \boldsymbol{\rho}(x, y) \vdash_{\mathcal{L}} y \tag{MP}$$

$$\bigcup_{i \leqslant n} \boldsymbol{\rho}(x_i, y_i) \vdash_{\mathcal{L}} \boldsymbol{\rho}(\lambda x_1 \dots x_n, \lambda y_1 \dots y_n) \tag{Rep}$$

$$x \dashv\vdash_{\mathcal{L}} \boldsymbol{\rho}\boldsymbol{\tau}(x) \tag{A3}$$

for every n-ary function symbol λ.

In particular, this implies that every algebraizable logic has a set of congruence formulas $\boldsymbol{\rho}(x, y)$ and, therefore, is equivalential.

There are some prominent strengthenings and weakenings of the concept of algebraizability. A logic $\mathcal{L}$ is *finitely algebraizable* if it is algebraizable through some finite structural transformer $\boldsymbol{\rho}(x, y)$. A logic $\mathcal{L}$ is *weakly algebraizable* if it is protoalgebraic and truth is equationally definable in $\mathsf{Mod}^*\mathcal{L}$.‡

The last family of logics that we consider embraces many of the best-known examples, such as superintuitionistic logics. A logic $\mathcal{L}$ is *assertional* if

‡The reader may have noticed that we have not explained yet what does it mean that truth is equationally definable in $\mathsf{Mod}^*\mathcal{L}$. Since this will not cause any confusion, we chose to postpone this explanation to Definition 2.2.

there is a class K of algebras with a constant term 1 such that $\mathcal{L}$ is complete with respect to the class of matrices

$$\{\langle \boldsymbol{A}, \{1^{\boldsymbol{A}}\}\rangle : \boldsymbol{A} \in \mathsf{K}\}.$$

Assertionality can be strengthened as follows. A logic $\mathcal{L}$ is *regularly* (resp. *weakly*) *algebraizable* if it is assertional and (resp. weakly) algebraizable. A logic $\mathcal{L}$ is *finitely regularly algebraizable* if it is regularly and finitely algebraizable. In order to characterize assertional logics, we need to introduce some concepts. Let M be a class of matrices. M is *almost unital* if F is either empty or a singleton for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{M}$. M is *unital* if F is a singleton for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{M}$.

Theorem 1.10.

1. *If $\mathcal{L}$ is determined by an almost unital class of matrices M, then $\mathsf{Mod}^*\mathcal{L}$ is also almost unital.*
2. *$\mathcal{L}$ is assertional if and only if $\mathsf{Mod}^*\mathcal{L}$ is unital.*

Lemma 1.11. *Let $\mathcal{L}$ be an algebraizable logic. $\mathcal{L}$ is regularly algebraizable if and only if*

$$x, y \vdash_{\mathcal{L}} \boldsymbol{\rho}(x, y) \tag{G}$$

for some (or, equivalently, every) of its sets of congruence formulas $\boldsymbol{\rho}(x, y)$.

The classes of logics considered so far form the *Leibniz hierarchy* depicted in Figure 1.1 (where arrows denote the inclusion relation).[§]

1.9 The Frege hierarchy

The study of the Frege hierarchy is another pillar of abstract algebraic logic. Again, we don't pursue here a systematic exposition of the topic (that can be found for example in [40]) and state only the results that will be used later on. The central concepts of the Frege hierarchy are built on the notion of a particular kind of g-models. A g-matrix $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a *full g-model* of a logic $\mathcal{L}$ if $\mathcal{C}/\widetilde{\boldsymbol{\Omega}}^{A}\mathcal{C} = \mathcal{F}i_{\mathcal{L}}(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{A}\mathcal{C})$. A useful characterization of full g-models is the following one:

Lemma 1.12. *Let $\langle \boldsymbol{A}, \mathcal{C}\rangle$ be a g-matrix, $\mathcal{L}$ a logic and $\boldsymbol{A}$ an algebra. $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a full g-model of $\mathcal{L}$ if and only if $\mathcal{C} = \{F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}^{A}\mathcal{C} \subseteq \boldsymbol{\Omega}^{A}F\}$.*

Corollary 1.13. *If $\mathcal{L}$ is purely inferential, then $\emptyset \in \mathcal{C}$ for every full g-model $\langle \boldsymbol{A}, \mathcal{C}\rangle$ of $\mathcal{L}$.*

[§]Looking at Figure 1.1, the reader may notice that the class of truth-equational logic has not been defined yet. We chose to postpone its discussion, but the interested reader may consult the footnote at pag. 20.

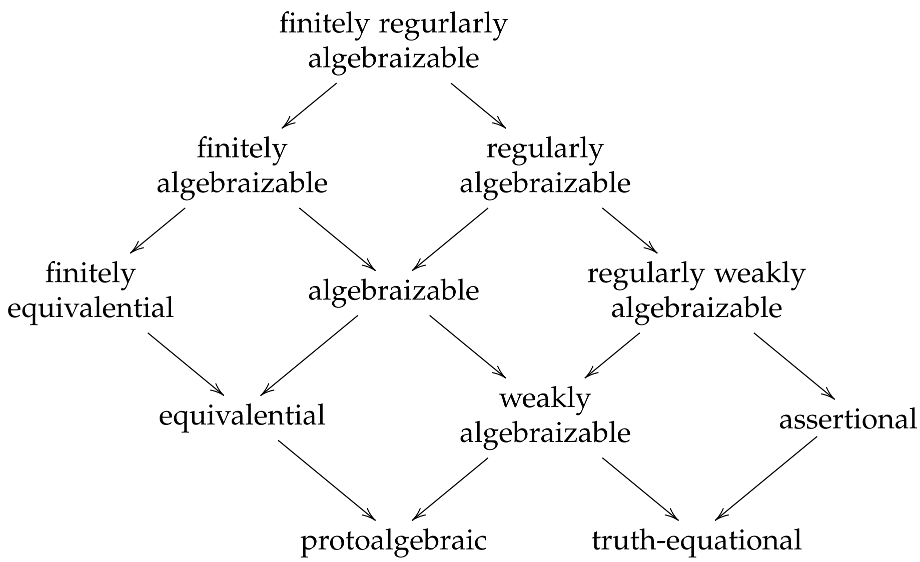


Figure 1.1: The Leibniz hierarchy

The *Frege relation* of a closure system $\mathcal{C}$ on A is the following relation:

$$\boldsymbol{\Lambda}\mathcal{C} := \{\langle a,b\rangle \in A \times A : C\{a\} = C\{b\}\}.$$

This construction applies to logics: we write $\boldsymbol{\Lambda}\mathcal{L}$ or $\dashv\vdash_{\mathcal{L}}$ for the Frege relation of $\mathcal{T}h\mathcal{L}$. If A is the universe of an algebra $\boldsymbol{A}$, there is a strong connection between the Frege relation and the Tarski congruence of $\mathcal{C}$, namely that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\mathcal{C}$ is the largest congruence of $\boldsymbol{A}$ below $\boldsymbol{\Lambda}\mathcal{C}$.

We say that a g-matrix $\langle \boldsymbol{A}, \mathcal{C}\rangle$ has the (PCONG), i.e., the *property of congruence*, if the relation $\boldsymbol{\Lambda}\mathcal{C}$ is a congruence on $\boldsymbol{A}$. Then $\langle \boldsymbol{A}, \mathcal{C}\rangle$ has the (SPCONG), i.e., the *strong property of congruence*, if $\langle \boldsymbol{A}, \mathcal{C}^F\rangle$ has the (PCONG) for every $F \in \mathcal{C}$. This allows us to classify logics according to the sort of replacement properties they satisfy. Let $\mathcal{L}$ be a logic, then:

1. $\mathcal{L}$ is *selfextensional* when $\langle \boldsymbol{Fm}, \mathcal{T}h\mathcal{L}\rangle$ has the (PCONG).
2. $\mathcal{L}$ is *Fregean* when $\langle \boldsymbol{Fm}, \mathcal{T}h\mathcal{L}\rangle$ has the (SPCONG).
3. $\mathcal{L}$ is *fully selfextensional* when all its full g-models have the (PCONG).
4. $\mathcal{L}$ is *fully Fregean* when all its full g-models have the (SPCONG).

These classes of logics form the so-called *Frege hierarchy*, depicted in Figure 1.2.

Lemma 1.14. *If $\mathcal{L}$ is selfextensional and $\varphi \dashv\vdash_{\mathcal{L}} \psi$, then $\mathsf{Alg}\mathcal{L} \vDash \varphi \approx \psi$.*

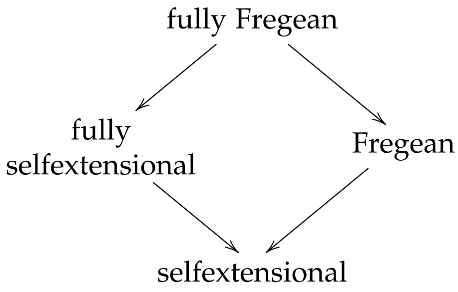


Figure 1.2: The Frege hierarchy

1.10 Adjoint functors

For a systematic treatment of category theory we refer the reader to [2, 8, 69]. In particular, we assume that the reader is familiar with the basic notions of category, functor, natural transformation, limit and colimit. In fact the usage of category theory in this dissertation is very limited and will be centred on the notion of adjoint functors. For this reason we chose to recall some basic facts and definitions related to adjunctions. To this end, we will limit our discussion to *locally small categories*, i.e., categories whose hom-sets are ordinary sets.

An *adjunction* between two categories X and Y is a tuple $\langle \mathcal{F}, \mathcal{G}, \varepsilon, \eta \rangle$ where $\mathcal{F} \colon \mathsf{X} \to \mathsf{Y}$ and $\mathcal{G} \colon \mathsf{Y} \to \mathsf{X}$ are functors and $\eta \colon 1_{\mathsf{X}} \to \mathcal{G}\mathcal{F}$ and $\varepsilon \colon \mathcal{F}\mathcal{G} \to 1_{\mathsf{Y}}$ are natural transformations such that

$$1_{\mathcal{F}(\boldsymbol{A})} = \varepsilon_{\mathcal{F}(\boldsymbol{A})} \circ \mathcal{F}(\eta_{\boldsymbol{A}}) \text{ and } 1_{\mathcal{G}(\boldsymbol{B})} = \mathcal{G}(\varepsilon_{\boldsymbol{B}}) \circ \eta_{\mathcal{G}(\boldsymbol{B})}$$

for every $\boldsymbol{A} \in \mathsf{X}$ and $\boldsymbol{B} \in \mathsf{Y}$. In this case we say that $\mathcal{F}$ is *left adjoint* to $\mathcal{G}$ and that $\mathcal{G}$ is *right adjoint* to $\mathcal{F}$, in symbols $\mathcal{F} \dashv \mathcal{G}$. Moreover, η and ε are respectively the *unit* and *counit* of the adjunction. We say that a functor is *right adjoint* (resp *left adjoint*) if it is right (resp. left) adjoint to some functor. It is worth to remark that if a functor has two right (left) adjoints, they are naturally isomorphic. Right adjoint functors preserve limits and left adjoint functors preserve colimits. A *category equivalence* between two categories X and Y is an adjunction $\langle \mathcal{F}, \mathcal{G}, \varepsilon, \eta \rangle$ where ε and η are natural isomorphisms. In this case the functors $\mathcal{F}$ and $\mathcal{G}$ preserve all categorical constructions. We say that two categories are *categorically equivalent* when they are related by a category equivalence.

A *hom-set adjunction* between two categories X and Y is a triple $\langle \mathcal{F}, \mathcal{G}, \mu \rangle$ where $\mathcal{F} \colon \mathsf{X} \to \mathsf{Y}$ and $\mathcal{G} \colon \mathsf{Y} \to \mathsf{X}$ are functors and μ is a natural isomorphism between the functors:

$$\hom_{\mathsf{Y}}(\mathcal{F}(\cdot), \cdot) \colon \mathsf{X}^{op} \times \mathsf{Y} \to \mathsf{Set} \text{ and } \hom_{\mathsf{X}}(\cdot, \mathcal{G}(\cdot)) \colon \mathsf{X}^{op} \times \mathsf{Y} \to \mathsf{Set}.$$

Here $\hom_{\mathsf{Y}}(\mathcal{F}(\cdot), \cdot)$ is the functor that takes a pair $\langle \boldsymbol{A}, \boldsymbol{B} \rangle$, where $\boldsymbol{A} \in \mathsf{X}$ and $\boldsymbol{B} \in \mathsf{Y}$, to the set of arrows $\hom_{\mathsf{Y}}(\mathcal{F}(\boldsymbol{A}), \boldsymbol{B})$ and that takes a pair $\langle f, g \rangle \colon \langle \boldsymbol{A}_1, \boldsymbol{B}_1 \rangle \to \langle \boldsymbol{A}_2, \boldsymbol{B}_2 \rangle$, where $f \colon \boldsymbol{A}_2 \to \boldsymbol{A}_1$ and $g \colon \boldsymbol{B}_1 \to \boldsymbol{B}_2$ are arrows in X and Y respectively, to the set-theoretic function

$$g \circ (\cdot) \circ \mathcal{F}(f) \colon \hom_{\mathsf{Y}}(\mathcal{F}(\boldsymbol{A}_1), \boldsymbol{B}_1) \to \hom_{\mathsf{Y}}(\mathcal{F}(\boldsymbol{A}_2), \boldsymbol{B}_2).$$

The functor $\hom_{\mathsf{X}}(\cdot, \mathcal{G}(\cdot))$ is defined in a similar fashion. When $\langle \mathcal{F}, \mathcal{G}, \mu \rangle$ is a hom-set adjunction as above, we say that $\mathcal{F}$ is *left adjoint* to $\mathcal{G}$ and that $\mathcal{G}$ is *right adjoint* to $\mathcal{F}$, in symbols $\mathcal{F} \dashv \mathcal{G}$.

Adjunctions and hom-set adjunctions are two sides of the same coin. To explain why, consider an adjunction $\langle \mathcal{F}, \mathcal{G}, \varepsilon, \eta \rangle$ between X and Y with $\mathcal{F} \dashv \mathcal{G}$. Then for every $\langle \boldsymbol{A}, \boldsymbol{B} \rangle \in \mathsf{X}^{op} \times \mathsf{Y}$ we let

$$\gamma_{\langle \boldsymbol{A}, \boldsymbol{B} \rangle} \colon \hom_{\mathsf{Y}}(\mathcal{F}(\boldsymbol{A}), \boldsymbol{B}) \to \hom_{\mathsf{X}}(\boldsymbol{A}, \mathcal{G}(\boldsymbol{B}))$$

be the map that sends an arrow f to $\mathcal{G}(f) \circ \eta_{\boldsymbol{A}}$. It turns out that the global map

$$\gamma \colon \hom_{\mathsf{Y}}(\mathcal{F}(\cdot), \cdot) \to \hom_{\mathsf{X}}(\cdot, \mathcal{G}(\cdot))$$

is a natural isomorphism. Thus the triple $\langle \mathcal{F}, \mathcal{G}, \gamma \rangle$ is a hom-set adjunction between X and Y with $\mathcal{F} \dashv \mathcal{G}$. Vice-versa consider a hom-set adjunction $\langle \mathcal{F}, \mathcal{G}, \gamma \rangle$ between X and Y with $\mathcal{F} \dashv \mathcal{G}$. For every $\boldsymbol{A} \in \mathsf{X}$ and $\boldsymbol{B} \in \mathsf{Y}$ we define $\eta_{\boldsymbol{A}} := \gamma_{\langle \boldsymbol{A}, \mathcal{F}(\boldsymbol{A}) \rangle}(1_{\mathcal{F}(\boldsymbol{A})})$ and $\varepsilon_{\boldsymbol{B}} := \gamma^{-1}_{\langle \mathcal{G}(\boldsymbol{B}), \boldsymbol{B} \rangle}(1_{\mathcal{G}(\boldsymbol{B})})$. It turns out that the global maps $\eta \colon 1_{\mathsf{X}} \to \mathcal{G}\mathcal{F}$ and $\varepsilon \colon \mathcal{F}\mathcal{G} \to 1_{\mathsf{Y}}$ are natural transformations and that $\langle \mathcal{F}, \mathcal{G}, \varepsilon, \eta \rangle$ is an adjunction with $\mathcal{F} \dashv \mathcal{G}$. Keeping this in mind we can speak of the hom-set adjunction associated with an adjunction and vice-versa.

Part I

Truth predicates in matrix semantics

CHAPTER 2

Equational definability

In the late 80's the theory of algebraizable logics was introduced by Blok and Pigozzi [19] as a general mathematical framework to express the relation between a propositional logic $\mathcal{L}$ and its natural algebra-based semantics K, called the equivalent algebraic semantics of $\mathcal{L}$. Its key point is the use of two structural transformers $\boldsymbol{\tau}\colon \mathcal{P}(Fm) \longleftrightarrow \mathcal{P}(Eq)\colon \boldsymbol{\rho}$ of sets formulas into sets of equations and vice-versa to establish a deductive equivalence between the consequence of $\mathcal{L}$ and the equational consequence relative to K. Blok and Pigozzi considered that also one half of the relation between $\mathcal{L}$ and K (induced by $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$) was interesting on its own. More precisely, they say that a logic $\mathcal{L}$ has an *algebraic semantics* if there is a class of algebras K and a structural transformer $\boldsymbol{\tau}\colon \mathcal{P}(Fm) \to \mathcal{P}(Eq)$ of sets of formulas into sets of equations that allows to interpret $\mathcal{L}$ into the equational consequence relative to K in the sense of (A1) on page 10. Of course, the main difference between this and the notion of algebraizability is that in this case we do not require $\boldsymbol{\tau}$ to be invertible in the sense of (A2) on page 10, with respect to another structural transformer $\boldsymbol{\rho}$.

The study of the notion of algebraic semantics was further developed by Blok and Rebagliato in [22]. Among other results, they proved that every logic that is complete with respect to a class of matrices with an idempotent basic n-ary operation has an algebraic semantics [22, Theorem 3.6]. In particular, every logic that is defined by means of a class of matrices with a semilattice reduct has an algebraic semantics. Moreover, it turned out that the same logic may have different algebraic semantics, which, in addition, generate different prevarieties. This happens already in the classical case, since by Glivenko's theorem the varieties of Boolean and of Heyting algebras are both algebraic semantics for classical propositional logic through the structural transformer $\boldsymbol{\tau}(x) := \{\neg\neg x \approx 1\}$. These facts inspire the feeling that it is unlikely to discover a general method of highlighting an algebraic semantics for a given logic among the ones that it may have, and that the notion of an algebraic semantics does not fit naturally inside the study of the Leibniz hierarchy in the same sense that the other classes do.

Nevertheless, building on the works of Hermann [57, 58] and Jansana and Czelakowski [30], Raftery characterized *truth-equational* logics, i.e., logics $\mathcal{L}$ for which there is a structural transformer $\boldsymbol{\tau}\colon \mathcal{P}(Fm) \to \mathcal{P}(Eq)$ that defines the truth sets of $\mathsf{Mod}^{*}\mathcal{L}$ [83]. In this case both $\mathsf{Alg}^{*}\mathcal{L}$ and $\mathsf{Alg}\mathcal{L}$ are algebraic semantics for $\mathcal{L}$ through the structural transformer $\boldsymbol{\tau}(x)$. Raftery's characterization (Theorems 2.9 and 2.13) is made in terms of concepts typical of the Leibniz hierarchy and can therefore be viewed as providing the best approximation to the notion of algebraic semantics that fits within the framework of classical abstract algebraic logic.

Both the concept of an algebraic semantics and that of a truth-equational logic rely on the idea of translating formulas into equations in a way that yields an equational completeness theorem. In this chapter we generalize this idea by considering equational completeness theorems in which we admit the presence of universally quantified equations. When this kind of completeness theorem holds w.r.t. the class $\mathsf{Mod}^{*}\mathcal{L}$ we say that the truth sets of $\mathcal{L}$ are *universally definable*. The theory of logics whose truth sets are universally definable retains several features of the original framework of truth-equationality. In particular, these logics can be characterized by means of the behaviour of the Leibniz operator (Theorem 2.8). Then we motivate the study of this new class with some concrete examples, mainly related to the variety of lattices (e.g., Lemma 2.16 and Theorem 2.17). We conclude by investigating the relations that hold between logics whose truth sets are universally definable and the Frege hierarchy (Theorem 2.29).

2.1 Definability with parameters

Before beginning our trip through the study of truth predicates, let us spend a few words on some terminological convention, which will considerably simplify the formulation of the main results. We say that the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}}\colon \mathcal{P}(A) \to \mathrm{Con}\boldsymbol{A}$ enjoys a certain set- or order-theoretic property over $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$, if its restriction to $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ enjoys it. Moreover, we say that $\boldsymbol{\Omega}^{\boldsymbol{A}}$ *almost* enjoys that property over $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ if its restriction to $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \smallsetminus \{\emptyset\}$ enjoys it. For example we will say that $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is *almost injective* over $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ if $\boldsymbol{\Omega}^{\boldsymbol{A}}\colon \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \smallsetminus \{\emptyset\} \to \mathrm{Con}\boldsymbol{A}$ is injective. The reader may wonder why do we care so much about the empty filter. This is because we will be concerned with several examples of *purely inferential* logics, i.e., logics without theorems and it is easy to prove that $\emptyset \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ if and only if $\mathcal{L}$ is purely inferential. Therefore it will be often the case that the collection of deductive filters of our logics contains the empty-set, which represents a limit case and shall be eliminated in the formulation of the main results (that would be false otherwise). An analogous expedient will apply to matrices as follows. We say that a matrix $\langle \boldsymbol{A}, F\rangle$ is *almost trivial* if $F = \emptyset$. Observe that the unique almost trivial reduced matrix is $\langle \mathbf{1}, \emptyset\rangle$. A class of matrices M *almost* enjoys a certain property, if every non-almost trivial member of M enjoys it.

Definition 2.1. A *universal translation* is a set $\boldsymbol{\tau}(x, \vec{y}) \subseteq Eq$ of equations in a distinguished variable x with parameters $\vec{y}$. An *equational translation* is a universal translation without parameters.

Universal and equational translations witness the definability of truth sets in classes of matrices by bounding parameters (if any) by an universal quantifier and considering the solutions of the resulting universally quantified equations. This motivates the name for the two kind of translations, since only in the presence of parameters the universal quantification plays a meaningful role in the determination of the corresponding solution set. More precisely, given an universal translation $\boldsymbol{\tau}(x, \vec{y})$ and an algebra $\boldsymbol{A}$, we let

$$\mathrm{Sl}^{\boldsymbol{A}}(\boldsymbol{\tau}) := \{a \in A : \boldsymbol{A} \vDash \boldsymbol{\tau}(a, \vec{c}) \text{ for every } \vec{c} \in A\}. \tag{2.1}$$

When $\boldsymbol{\tau}(x, \vec{y}) = \{\varepsilon \approx \delta\}$, we shall simply write $\mathrm{Sl}^{\boldsymbol{A}}(\varepsilon \approx \delta)$ instead of $\mathrm{Sl}^{\boldsymbol{A}}(\{\varepsilon \approx \delta\})$. Observe that if $\boldsymbol{\tau}(x, \vec{y}) = \boldsymbol{\tau}(x)$ is an equational translation, then (2.1) simplifies to the following:

$$\mathrm{Sl}^{\boldsymbol{A}}(\boldsymbol{\tau}) = \{a \in A : \boldsymbol{A} \vDash \boldsymbol{\tau}(a)\}.$$

Definition 2.2. A universal (resp. equational) translation $\boldsymbol{\tau}$ *defines truth* in M, if $\mathrm{Sl}^{\boldsymbol{A}}(\boldsymbol{\tau}) = F$ for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{M}$. Truth is *universally* (resp. *equationally*) *definable* in M if there is a universal (resp. equational) translation that defines truth in M.

Example 2.3 (Lattices)**.** Let $\boldsymbol{A}$ be a lattice with a maximum element a. Then consider the matrix $\langle \boldsymbol{A}, \{a\}\rangle$. For every $b \in A$ we have that

$$\begin{aligned} b = a \quad &\Longleftrightarrow \quad c \leqslant b \text{ for every } c \in A \\ &\Longleftrightarrow \quad c \wedge b = c \text{ for every } c \in A \\ &\Longleftrightarrow \quad \boldsymbol{A} \vDash \boldsymbol{\tau}(b, c) \text{ for every } c \in A \end{aligned}$$

where $\boldsymbol{\tau}$ is the universal translation $\{x \wedge y \approx y\}$. This shows that $\boldsymbol{\tau}$ defines truth in $\langle \boldsymbol{A}, \{a\}\rangle$. On the other hand if $\boldsymbol{A}$ is non-trivial, there is no equational translation that defines truth in $\langle \boldsymbol{A}, \{a\}\rangle$. This is due to the fact that (up to equivalence) the unique lattice equation in variable x is $x \approx x$. The situation changes if we add a constant 1 to the type of $\boldsymbol{A}$. In particular, let $\boldsymbol{A}^{+}$ be the expansion of $\boldsymbol{A}$ where 1 is interpreted as a. Then truth is equationally definable in $\langle \boldsymbol{A}^{+}, \{a\}\rangle$ by the equational translation $x \approx 1$. ⊠

Observe that when truth is almost universally definable in M, the almost non-trivial matrices in M are determined by their algebraic reduct. More precisely, if $\langle \boldsymbol{A}, F\rangle, \langle \boldsymbol{A}, G\rangle \in \mathsf{M}$ are almost non-trivial, then $F = G$. This observation will be used in several proofs. It is clear that if truth is equationally definable in M, then it is universally definable too.

We will be interested in logics $\mathcal{L}$ for which truth is universally or equationally definable in $\mathsf{Mod}^{*}\mathcal{L}$. For this reason it will be convenient to introduce some terminological convention. We say that the truth sets of a logic $\mathcal{L}$ are universally (resp. equationally) definable, as an abbreviation for the fact that truth is universally (resp. equationally) definable in $\mathsf{Mod}^{*}\mathcal{L}$. In the case of equational definability, logics that satisfy this property have been studied in depth by Raftery [83].*

Formally speaking an equation equation $\varepsilon \approx \delta$ is just a pair $\langle \varepsilon, \delta\rangle$. Thus, given an algebra $\boldsymbol{A}$ and a tuple $\vec{a} \in A$, the expression $\varepsilon^{\boldsymbol{A}}(\vec{a}) \approx \delta^{\boldsymbol{A}}(\vec{a})$ will denote the pair $\langle \varepsilon^{\boldsymbol{A}}(\vec{a}), \delta^{\boldsymbol{A}}(\vec{a})\rangle \in A \times A$. Keeping this in mind, we have the following:

Lemma 2.4. *Let $\boldsymbol{\tau}(x, \vec{y})$ be a universal translation and $\langle \boldsymbol{A}, F\rangle$ a non-almost trivial matrix. $\boldsymbol{\tau}(x, \vec{y})$ defines truth in $\langle \boldsymbol{A}, F\rangle^{*}$ if and only if for every $a \in A$,*

$$a \in F \Longleftrightarrow \boldsymbol{\tau}^{\boldsymbol{A}}(a, \vec{c}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} F \text{ for every } \vec{c} \in A.$$

Proof. Apply the fact that $\boldsymbol{\Omega}^{\boldsymbol{A}} F$ is compatible with F. ⊠

The main goal of this section will be to characterize whose truth sets are almost universally definable in terms of the behaviour of the Leibniz operator over deductive filters. The first step in this direction consists in the following transfer result.

Lemma 2.5. *A universal translation almost defines truth in $\mathsf{Mod}^{*}\mathcal{L}$ if and only if it almost defines truth in $\mathsf{LMod}^{*}\mathcal{L}$.*

Proof. It will be enough to check the "if" part. Let $\boldsymbol{\tau}$ be a universal translation which almost defines truth in $\mathsf{LMod}^{*}\mathcal{L}$. We want to show that $\boldsymbol{\tau}$ almost defines truth in $\mathsf{Mod}^{*}\mathcal{L}$ too. By Lemma 2.4 this amounts to proving that for every algebra $\boldsymbol{A}$, $F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \smallsetminus \{\emptyset\}$ and $a \in A$:

$$a \in F \Longleftrightarrow \boldsymbol{\tau}^{\boldsymbol{A}}(a, \vec{c}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} F \text{ for every } \vec{c} \in A. \tag{2.2}$$

First recall that $\mathsf{LMod}^{*}\mathcal{L}$ is (up to isomorphism) the class of countably generated reduced models of $\mathcal{L}$. Together with Lemma 2.4, this implies that (2.2) holds in case $\boldsymbol{A}$ is countably generated.

*Raftery calls *truth-equational* the logics $\mathcal{L}$ for which truth is equationally definable in $\mathsf{Mod}^{*}\mathcal{L}$. Here we prefer not to give them any particular name, in order to obtain a more uniform naming scheme when dealing with different kinds of definability conditions.

Then consider the case where $\boldsymbol{A}$ is not countably generated. We begin by proving the "if" part of (2.2). Let $a \in A$ and $F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \smallsetminus \{\emptyset\}$ be such that $\boldsymbol{\tau}^{\boldsymbol{A}}(a, \vec{c}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}F$ for every $\vec{c} \in A$. Then choose any $b \in F$ and consider the subalgebra of $\boldsymbol{B}$ of $\boldsymbol{A}$ generated by $\{a, b\}$. Let $G := F \cap B$. Observe that $G \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{B} \smallsetminus \{\emptyset\}$, since $b \in G$. Moreover, we have $\boldsymbol{\tau}^{\boldsymbol{B}}(a, \vec{c}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}F \cap (B \times B) \subseteq \boldsymbol{\Omega}^{\boldsymbol{B}}G$ for every $\vec{c} \in B$. Since $\boldsymbol{B}$ is countably generated, we conclude that $a \in G \subseteq F$.

Now we prove the "only if" part of (2.2). Suppose that $a \in F$. Then consider $\varepsilon \approx \delta \in \boldsymbol{\tau}$ and $\vec{c} \in A$. Let also $p(x) \colon \boldsymbol{A} \to \boldsymbol{A}$ be a unary polynomial function. By definition there is an $(n+1)$-ary term φ and a sequence $\vec{e}$ of n elements of A such that $\varphi^{\boldsymbol{A}}(b, \vec{e}) = p(b)$ for every $b \in A$. We will prove that

$$p(\varepsilon^{\boldsymbol{A}}(a, \vec{c})) \in F \Longleftrightarrow p(\delta^{\boldsymbol{A}}(a, \vec{c})) \in F. \tag{2.3}$$

First suppose that $p(\varepsilon^{\boldsymbol{A}}(a, \vec{c})) \in F$. Then consider the subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$ generated by $\{a, e_1, \dots, e_n, c_1, \dots, c_k\}$, where $c_1, \dots c_k$ are the elements of $\vec{c}$ corresponding to the variables in $\vec{y}$ occurring in ε and δ. Then let $G := F \cap B$. We know that $G \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{B} \smallsetminus \{\emptyset\}$, because $a \in G$. Since $\boldsymbol{B}$ is countably generated and $a \in G$, we have that

$$\langle \varepsilon^{\boldsymbol{B}}(a, c_1, \dots, c_k), \delta^{\boldsymbol{B}}(a, c_1, \dots, c_k) \rangle \in \boldsymbol{\Omega}^{\boldsymbol{B}}G.$$

Since $p(\varepsilon^{\boldsymbol{B}}(a, \vec{c})) \in G$, by compatibility we obtain that $p(\delta^{\boldsymbol{B}}(a, \vec{c})) \in G \subseteq F$. This establishes condition (2.3). By point 1 of Lemma 1.2, we conclude that $\langle \varepsilon^{\boldsymbol{A}}(a, \vec{c}), \delta^{\boldsymbol{A}}(a, \vec{c}) \rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}F$. ⊠

One may wonder why in the above lemma we were interested in logics $\mathcal{L}$ for which truth is *almost* universally definable, and not simply universally definable. This is because logics $\mathcal{L}$ for which truth is universally definable in $\mathsf{Mod}^*\mathcal{L}$ coincide with logics $\mathcal{L}$ for which truth is equationally definable in $\mathsf{Mod}^*\mathcal{L}$ (Corollary 2.10).

The following technical result is stated without a detailed proof in [28, Proposition 1.5(8)] and will be needed in the sequel.

Lemma 2.6 (Czelakowski). *$\sigma\widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}C_{\mathcal{L}}\{x\} \subseteq \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}C_{\mathcal{L}}\{\sigma x\}$ for every substitution σ.*

Proof. Consider a pair $\langle \varphi, \psi \rangle \in \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}C_{\mathcal{L}}\{x\}$. We have to prove that $\langle \sigma\varphi, \sigma\psi \rangle \in \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}C_{\mathcal{L}}\{\sigma x\}$. By point 2 of Lemma 1.2 it will be enough to check that

$$\gamma(\sigma(\varphi), \vec{z}), \sigma x \dashv\vdash_{\mathcal{L}} \sigma x, \gamma(\sigma(\psi), \vec{z}), \text{ for every } \gamma(x, \vec{z}) \in Fm.$$

To this end, consider $\gamma(x, \vec{z}) \in Fm$ and a new substitution σ such that:

1. σ and σ' coincide on the variables actually occurring in φ and ψ.
2. For every variable $v \neq x$ actually occurring in γ, there is a variable u such that $\sigma'(u) = v$.

Now, consider the formula δ obtained by replacing in γ each variable $v \neq x$ by the corresponding u. Applying point 2 of Lemma 1.2 to the fact that $\langle \varphi, \psi \rangle \in \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}} C_{\mathcal{L}}\{x\}$, we obtain that

$$x, \delta(\varphi, \vec{u}) \dashv\vdash_{\mathcal{L}} \delta(\psi, \vec{u}), x.$$

By structurality we obtain that

$$\sigma' x, \sigma'\delta(\varphi, \vec{u}) \dashv\vdash_{\mathcal{L}} \sigma'\delta(\psi, \vec{u}), \sigma' x.$$

But this is exactly $\gamma(\sigma(\varphi), \vec{z}), \sigma x \dashv\vdash_{\mathcal{L}} \sigma x, \gamma(\sigma(\psi), \vec{z})$. ⊠

Definition 2.7. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be complete lattices and $f \colon X \to Y$ be a map.

1. f is *order-reflecting* if for every $a, b \in X$, if $f(a) \leqslant f(b)$, then $a \leqslant b$.
2. f is *completely order-reflecting* if for every $A \cup \{b\} \subseteq X$,

$$\text{if } \bigwedge_{a \in A} f(a) \leqslant f(b), \text{ then } \bigwedge_{a \in A} a \leqslant b.$$

Observe that every completely order-reflecting map is order-reflecting and that every order-reflecting map is injective. We are now ready to state our desired characterization result of logics $\mathcal{L}$ whose truth sets are almost universally definable. It is worth to remark that it opens the door of the Leibniz hierarchy to non-trivial purely inferential logics.†

Theorem 2.8. *The following conditions are equivalent:*

(i) *Truth is almost universally definable in* $\mathsf{Mod}^*\mathcal{L}$.

(ii) $\boldsymbol{\Omega}^{\boldsymbol{A}}$ *is almost completely order-reflecting over* $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ *for every algebra* $\boldsymbol{A}$.

(iii) $\boldsymbol{\Omega}$ *is almost completely order-reflecting over* $\mathcal{T}h\mathcal{L}$.

In this case $\boldsymbol{\tau}(x, \vec{y}) := \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}} C_{\mathcal{L}}\{x\}$ *almost defines truth in* $\mathsf{Mod}^*\mathcal{L}$.

Proof. (i)⇒(ii): Consider an arbitrary algebra $\boldsymbol{A}$ and let $\mathcal{F} \cup \{G\} \subseteq \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \smallsetminus \{\emptyset\}$ such that $\bigcap\{\boldsymbol{\Omega}^{\boldsymbol{A}}F : F \in \mathcal{F}\} \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}G$. Then consider $a \in \bigcap \mathcal{F}$. From Lemma 2.4 and the assumptions it follows that

$$\boldsymbol{\tau}^{\boldsymbol{A}}(a, \vec{c}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}F \text{ for every } \vec{c} \in A \text{ and } F \in \mathcal{F}.$$

This implies that $\boldsymbol{\tau}^{\boldsymbol{A}}(a, \vec{c}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}G$ for every $\vec{c} \in A$. With another application of Lemma 2.4 we conclude that $a \in G$.

(ii)⇒(iii): Straightforward. (iii)⇒(i): Choose a variable x and define $\boldsymbol{\tau}(x, \vec{y}) := \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}} C_{\mathcal{L}}\{x\}$ where $\vec{y}$ is the list of all variables different from x.

† Up to now the weakest conditions considered in the study of the Leibniz hierarchy were protoalgebraicity and having truth sets equationally definable. It is well known that every logic $\mathcal{L}$ whose truth sets are equationally definable in $\mathsf{Mod}^*\mathcal{L}$ has theorems (see Corollary 2.10) and that the unique purely inferential protoalgebraic logic (in a given language) is the almost inconsistent one.

Thanks to Lemma 2.5 it will be enough to prove that $\boldsymbol{\tau}$ almost defines truth in $\mathsf{LMod}^{*}\mathcal{L}$. By Lemma 2.4 this reduces to proving the following:

$$\Gamma \vdash_{\mathcal{L}} \varphi \text{ if and only if } \boldsymbol{\tau}(\varphi, \vec{\gamma}) \subseteq \boldsymbol{\Omega}\Gamma \text{ for every } \vec{\gamma} \in Fm \tag{2.4}$$

for every $\varphi \in Fm$ and $\Gamma \in \mathcal{T}h\mathcal{L} \smallsetminus \{\emptyset\}$. For the "only if" part of (2.4) suppose that $\Gamma \vdash_{\mathcal{L}} \varphi$. Then consider any sequence $\vec{\gamma} \in Fm$ and let σ be a substitution sending x to φ and $\vec{y}$ to $\vec{\gamma}$. Applying Lemma 2.6, we obtain that

$$\begin{aligned}\boldsymbol{\tau}(\varphi, \vec{\gamma}) &= \sigma\boldsymbol{\tau}(x, \vec{y}) = \sigma\widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}C_{\mathcal{L}}\{x\} \subseteq \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}C_{\mathcal{L}}\{\sigma x\} \\ &= \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}C_{\mathcal{L}}\{\varphi\} \subseteq \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}\Gamma \subseteq \boldsymbol{\Omega}\Gamma.\end{aligned}$$

Then we turn to prove the "if" part of (2.4). Suppose that $\boldsymbol{\tau}(\varphi, \vec{\gamma}) \subseteq \boldsymbol{\Omega}\Gamma$ for every sequence $\vec{\gamma} \in Fm$. Recall that $\Gamma \neq \emptyset$. Then we can choose a formula $\psi \in \Gamma$ and consider the substitution σ defined as

$$\sigma(z) = \begin{cases} \varphi & \text{if } z = x \\ \psi & \text{otherwise} \end{cases}$$

for every variable z. From point 1 of Lemma 1.1 and the assumption it follows that

$$\widetilde{\boldsymbol{\Omega}}C_{\mathcal{L}}\{x\} = \boldsymbol{\tau}(x, \vec{y}) \subseteq \sigma^{-1}\boldsymbol{\Omega}\Gamma \subseteq \boldsymbol{\Omega}\sigma^{-1}\Gamma.$$

Since inverse images of theories under substitutions are theories, we know that $\sigma^{-1}\Gamma \in \mathcal{T}h\mathcal{L}$. Moreover, observe that $y \in \sigma^{-1}\Gamma$ for every variable different from x. Thus $\sigma^{-1}\Gamma \neq \emptyset$. Therefore we can apply the fact that $\boldsymbol{\Omega}$ is completely order-reflecting over $\mathcal{T}h\mathcal{L} \smallsetminus \{\emptyset\}$ and get $C_{\mathcal{L}}\{x\} \subseteq \sigma^{-1}\Gamma$. This yields that $\varphi \in \Gamma$ and concludes the proof of condition (2.4). ⊠

Now we provide a proof of the characterization of logics $\mathcal{L}$ whose truth sets are equationally definable that first appeared in [83, Theorem 28].

Theorem 2.9 (Raftery). *The following conditions are equivalent:*

(i) *Truth is equationally definable in* $\mathsf{Mod}^{*}\mathcal{L}$.

(ii) $\boldsymbol{\Omega}^{\boldsymbol{A}}$ *is completely order-reflecting on* $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ *for every algebra* $\boldsymbol{A}$.

(iii) $\boldsymbol{\Omega}$ *is completely order-reflecting on* $\mathcal{T}h\mathcal{L}$.

In this case $\boldsymbol{\tau}(x) := \sigma_x\widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}C_{\mathcal{L}}\{x\}$ *defines truth in* $\mathsf{Mod}^{*}\mathcal{L}$, *where* σ_x *is the substitutions sending every variable to* x.

Proof. (i)⇒(ii): By the assumption truth is also almost universally definable in $\mathsf{Mod}^{*}\mathcal{L}$. Then consider an arbitrary algebra $\boldsymbol{A}$. From Theorem 2.8 it follows that $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is almost completely order-reflecting over $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$. Recall that the matrix $\langle \mathbf{1}, \{1\}\rangle$ is a reduced model of every logic. Since truth is equationally definable in $\mathsf{Mod}^{*}\mathcal{L}$, this implies that $\langle \mathbf{1}, \emptyset\rangle \notin \mathsf{Mod}^{*}\mathcal{L}$. Hence $\mathcal{L}$ has theorems and $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A} = \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \smallsetminus \{\emptyset\}$, which means that $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is completely order-reflecting over $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$.

(ii)⇒(iii): Straightforward. (iii)⇒(i): Observe that $\mathcal{L}$ has theorems. Suppose the contrary towards a contradiction, i.e., that $\emptyset \in \mathcal{T}h\mathcal{L}$. We would have that $\boldsymbol{\Omega} Fm = Fm \times Fm = \boldsymbol{\Omega}\emptyset$ and $Fm \nsubseteq \emptyset$, contradicting the assumption that $\boldsymbol{\Omega}$ is (completely) order reflecting. Then define $\boldsymbol{\tau}(x) := \sigma_x \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}} C_{\mathcal{L}}\{x\}$ where σ_x is the substitution sending each variable to x. Together with the fact that $\mathcal{L}$ has theorems, Lemmas 2.4 and 2.5 imply that it will be enough to show that

$$\Gamma \vdash_{\mathcal{L}} \varphi \text{ if and only if } \boldsymbol{\tau}(\varphi) \subseteq \boldsymbol{\Omega}\Gamma \tag{2.5}$$

for every $\varphi \in Fm$ and $\Gamma \in \mathcal{T}h\mathcal{L}$. To prove the "only if" part of (2.5) suppose that $\varphi \in \Gamma$. In the proof of part (iii)⇒(i) of Theorem 2.8 we showed that $\sigma \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}} C_{\mathcal{L}}\{x\} \subseteq \boldsymbol{\Omega}\Gamma$ for every substitution σ such that $\sigma(x) = \varphi$. Together with Lemma 2.6 this implies that

$$\boldsymbol{\tau}(\varphi) = \sigma\sigma_x \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}\{x\} \subseteq \sigma \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}} C_{\mathcal{L}}\{x\} \subseteq \boldsymbol{\Omega}\Gamma.$$

Then we turn to check the "if" part of (2.5). Suppose that $\boldsymbol{\tau}(\varphi) \subseteq \boldsymbol{\Omega}\Gamma$. Then let σ be any substitution such that $\sigma(x) = \varphi$. From point 1 of Lemma 1.1 and the assumption it follows that

$$\widetilde{\boldsymbol{\Omega}} C_{\mathcal{L}}\{x\} \subseteq \sigma_x^{-1}\sigma^{-1}\boldsymbol{\Omega}\Gamma \subseteq \boldsymbol{\Omega}\sigma_x^{-1}\sigma^{-1}\Gamma.$$

Since $\boldsymbol{\Omega}$ is completely order reflecting, we obtain that $x \in \sigma_x^{-1}\sigma^{-1}\Gamma$ and therefore $\varphi \in \Gamma$. ⊠

Combining Theorems 2.8 and 2.9 we can prove a surprising result, namely that truth is universally definable in the whole class of Leibniz-reduced models of a logic if and only if it is equationally definable in it. In other words, it turns out that the notion of universal definability makes sense only for purely inferential logics since, in the presence of theorems, it collapses into that of equational definability.

Corollary 2.10. *The following conditions are equivalent:*

(i) *Truth is equationally definable in* $\mathsf{Mod}^*\mathcal{L}$.

(ii) *Truth is universally definable in* $\mathsf{Mod}^*\mathcal{L}$.

(iii) *Truth is almost universally definable in* $\mathsf{Mod}^*\mathcal{L}$ *and* $\mathcal{L}$ *has theorems.*

In particular, if truth is equationally definable in $\mathsf{Mod}^*\mathcal{L}$*, then* $\mathcal{L}$ *has theorems.*

Proof. (i)⇒(ii): Straightforward. (ii)⇒(iii): Suppose towards a contradiction that $\mathcal{L}$ is purely inferential, i.e., that $\emptyset \in \mathcal{T}h\mathcal{L}$. Then both $\langle \mathbf{1}, \{1\}\rangle$ and $\langle \mathbf{1}, \emptyset\rangle$ are reduced models of $\mathcal{L}$. But this contradicts the fact that truth is universally definable in $\mathsf{Mod}^*\mathcal{L}$. (iii)⇒(i): Together with Theorem 2.8, the assumption implies that $\boldsymbol{\Omega}$ is completely order-reflecting over $\mathcal{T}h\mathcal{L}$. Thus with an application of Theorem 2.9 we are done. ⊠

Corollary 2.11. *Let $\mathcal{L}$ be a logic for which truth is almost universally definable in* $\mathsf{Mod}^*\mathcal{L}$*. There is a conservative expansion $\mathcal{L}'$ of $\mathcal{L}$, where the expansion consists in adding a new constant symbol* 1*, such that truth is equationally definable in* $\mathsf{Mod}^*\mathcal{L}'$*.*

Proof. Let $\boldsymbol{\tau}(x, \vec{y})$ be the universal translation that almost defines truth in $\mathsf{Mod}^*\mathcal{L}$. Then let K be the class of algebras obtained as follows. We expand the algebras in $\mathsf{Alg}^*\mathcal{L}$ by adding to them a new constant 1 that is interpreted arbitrarily in the set of solutions of $\boldsymbol{\tau}$. Observe that an algebra $\boldsymbol{A} \in \mathsf{K}$ can be expanded in different ways if $\mathrm{Sl}^{\boldsymbol{A}}(\boldsymbol{\tau})$ has more than one element. Then let $\mathcal{L}'$ be the logic determined by the following class of matrices:

$$\{\langle \boldsymbol{A}, \mathrm{Sl}^{\boldsymbol{A}}(\boldsymbol{\tau})\rangle : \boldsymbol{A} \in \mathsf{K}\}.$$

It is easy to see that $\mathcal{L}'$ is a conservative expansion of $\mathcal{L}$. In particular, this implies that if $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*\mathcal{L}'$, then $\langle \boldsymbol{A}', F\rangle \in \mathsf{Mod}^*\mathcal{L}$ where $\boldsymbol{A}'$ is the 1-free reduct of $\boldsymbol{A}$. As a consequence we obtain that truth is almost universally definable in $\mathsf{Mod}^*\mathcal{L}'$. Now observe that $\mathcal{L}'$ has theorems, since $\emptyset \vdash_{\mathcal{L}'} 1$. Thus, with an application of Corollary 2.10 we conclude that truth is equationally definable in $\mathsf{Mod}^*\mathcal{L}'$. ⊠

Remarkably, equational definability can be characterized by means of the behaviour of the Suszko operator. In order to prove this, we will make use of the following technical result.

Lemma 2.12. *The following conditions are equivalent:*

(i) *$\boldsymbol{\Omega}^{\boldsymbol{A}}$ is almost completely order-reflecting over $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$.*

(ii) *For every $F, G \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ such that $G \neq \emptyset$, if $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}}F \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}G$, then $F \subseteq G$.*

Proof. (i)⇒(ii): Straightforward. (ii)⇒(i): Let $\mathcal{F} \cup \{G\} \subseteq \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \smallsetminus \{\emptyset\}$ such that $\bigcap_{F\in\mathcal{F}} \boldsymbol{\Omega}^{\boldsymbol{A}}F \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}G$. Observe that

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}} \bigcap \mathcal{F} \subseteq \bigcap_{F\in\mathcal{F}} \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}}F \subseteq \bigcap_{F\in\mathcal{F}} \boldsymbol{\Omega}^{\boldsymbol{A}}F.$$

From the assumption it follows that $\bigcap \mathcal{F} \subseteq G$. ⊠

The equational definability of truth sets can be characterized by means of the behaviour of the Suszko operator [83, Theorem 11]. Since we will make use of this fact, we chose to include a proof for the sake of completeness.

Theorem 2.13 (Raftery). *Truth is equationally definable in* $\mathsf{Mod}^*\mathcal{L}$ *if and only if $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}}$ is injective over $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ for every algebra $\boldsymbol{A}$.*

Proof. We begin by the "only if" part. Suppose that truth is equationally definable in $\mathsf{Mod}^*\mathcal{L}$. Then consider $F, G \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}}F = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}}G$. We have that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}}F \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}}G \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}G$. By Theorem 2.8 and Lemma 2.12 we obtain that $F \subseteq G$. The fact that $G \subseteq F$ is proved analogously.

Then we turn to prove the "if" part. We will make use of the following observation [28, Theorem 7.8]: For every $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^{\mathrm{Su}}\mathcal{L}$, we have that $\min \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} = F$. To prove this, let $G := \min \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$. Since the Suszko operator is monotone, we have that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}} G = 0_A = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}} F$. Together with the assumption, this implies that $F = G$ as desired.

Now, observe that $\mathcal{L}$ has theorems, otherwise $\widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}$ would not be injective over $\mathcal{T}h\mathcal{L}$. Thus by Corollary 2.10 it will be enough to prove that truth is almost universally definable in $\mathsf{Mod}^{*}\mathcal{L}$. To this end, we will make use of Lemma 2.12. Consider an algebra $\boldsymbol{A}$ and and $F, G \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}} F \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} G$. Let $h \colon \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}} F \to \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}} G$ be the natural epimorphism. We have that $h^{-1}[G/\boldsymbol{\Omega}^{\boldsymbol{A}} G] \in \mathcal{F}i_{\mathcal{L}}(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}} F)$. Therefore we can apply the assumption, yielding that $F/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}} F \subseteq h^{-1}[G/\boldsymbol{\Omega}^{\boldsymbol{A}} G]$. Consider $a \in F$. We have that $a/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}} F \in F/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}} F \subseteq h^{-1}[G/\boldsymbol{\Omega}^{\boldsymbol{A}} G]$ and, therefore, $a/\boldsymbol{\Omega}^{\boldsymbol{A}} G \in G/\boldsymbol{\Omega}^{\boldsymbol{A}} G$. By compatibility, this implies that $a \in G$ as desired. ⊠

Problem 1. It is natural to ask whether the following generalization of the theorem above holds: Truth is almost universally definable in $\mathsf{Mod}^{*}\mathcal{L}$ if and only if $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}}$ is almost injective over $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ for every algebra $\boldsymbol{A}$.

If the reader is interested in solving this problem, she may find useful the following remarks. The proof of Theorem 2.13 relies on the fact that for every algebra $\boldsymbol{A}$, if $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}}$ is injective over $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ and $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^{\mathrm{Su}}\mathcal{L}$, then $F = \min \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$. However, the following generalization is *false*: for every algebra $\boldsymbol{A}$, if $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}}$ is almost injective over $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ and $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^{\mathrm{Su}}\mathcal{L}$ is almost non-trivial, then $F = \min(\mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \smallsetminus \{\emptyset\})$. This is shown in the following counterexample.

Example 2.14. Let $\boldsymbol{A} = \langle \{a, b, c\}, \Box, a, b, c\rangle$ be the algebra where $\Box$ is the unary operation defined as follows: $\Box a = \Box b = a$ and $\Box c = c$. Then consider the logic $\mathcal{L}$ determined by the matrices $\langle \boldsymbol{A}, \{a\}\rangle$ and $\langle \boldsymbol{A}, \{c\}\rangle$. From the definition of $\mathcal{L}$ it follows that $a, c \vdash_{\mathcal{L}} x$ and $b \vdash_{\mathcal{L}} x$. In particular, this implies that $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A} = \{\emptyset, \{a\}, \{c\}, A\}$. With this information at hand, one can check that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{L}}$ is almost injective over $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$. Finally observe that the model $\langle \boldsymbol{A}, \{a\}\rangle$ is Suszko reduced and that $\{a\} \neq \emptyset = \min(\mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \smallsetminus \{\emptyset\})$. ⊠

2.2 Lattice-based examples

In general it is not easy to find logics $\mathcal{L}$ for which truth is almost universally definable, but not equationally definable in $\mathsf{Mod}^{*}\mathcal{L}$. In this section we will review a family of natural examples of this kind, that comes from logics whose algebra-based semantics has an unbounded lattice (or semilattice) reduct. The easiest one is probably the following:

Example 2.15 (Distributive Lattices)**.** Let $\mathcal{CPC}_{\wedge\vee}$ be the $\langle \wedge, \vee \rangle$-fragment of classical propositional logic. For every non-almost trivial $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^{*}\mathcal{CPC}_{\wedge\vee}$ the following conditions hold:

(i) $\boldsymbol{A}$ is a distributive lattice with a maximum 1.

(ii) $F = \{1\}$.

(iii) For every $a, b \in A$, if $a < b$, then there is $c \in A$ such that $a \vee c \neq 1$ and $a \vee c = 1$.

This was proved in [39, Pag. 127], but see [45] for further information on the logic $\mathcal{CPC}_{\wedge\vee}$. In particular, this result implies that the truth sets of $\mathcal{CPC}_{\wedge\vee}$ are almost universally definable through the universal translation $\boldsymbol{\tau}(x, \vec{y}) = \{x \wedge y \approx y\}$, as shown in Example 2.3. It is worth to remark that $\mathsf{Alg}\mathcal{CPC}_{\wedge\vee}$ is the variety of distributive lattices DL [45, Corollary 4.5], while $\mathsf{Alg}^{*}\mathcal{CPC}_{\wedge\vee}$ does not contain the three element chain and, therefore, is strictly included into DL. ⊠

It is well known that for every $\boldsymbol{A} \in \mathsf{DL}$ the collection $\mathcal{F}i_{\mathcal{CPC}_{\wedge\vee}}\boldsymbol{A}$ coincides with the set of (possibly empty) lattice filters. Next results generalize this situation to logics related to (possibly non-distributive) classes of lattices. Keep in mind that these logics are expressed in the language of lattices $\langle \wedge, \vee \rangle$.

Lemma 2.16. *Let $\mathcal{L}$ be a logic whose truth sets are almost universally definable. If $\mathsf{Alg}\mathcal{L}$ is a class of lattices, then one of the following conditions holds:*

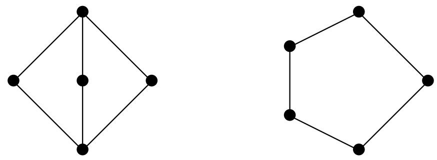
1. *$\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ is a set of (possibly empty) lattice filters, for every $\boldsymbol{A} \in \mathsf{Alg}\mathcal{L}$.*

2. *$\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ is a set of (possibly empty) lattice ideals, for every $\boldsymbol{A} \in \mathsf{Alg}\mathcal{L}$.*

Proof. We claim that F is closed under $\wedge$ and $\vee$ for every $F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ with $\boldsymbol{A} \in \mathsf{Alg}\mathcal{L}$. Suppose towards a contradiction that F is not closed under $\wedge$. Then there are $a, b \in F$ such that $a \wedge b \notin F$. Clearly a and b are incomparable. Then consider the submodel of $\langle \boldsymbol{A}, F \rangle$ with universe $\{a \wedge b, a\}$. It is isomorphic to $\langle \mathbf{2}, \{1\} \rangle$, where $\mathbf{2}$ is the two-element lattice. We will prove that F is closed under $\vee$. Suppose towards a contradiction that $c \vee d \notin F$ for some $c, d \in A$. Similarly this implies that $\langle \mathbf{2}, \{0\} \rangle$ is a model of $\mathcal{L}$. But then we would have two different non-almost trivial reduced models of $\mathcal{L}$ with the same algebraic reduct, contradicting the assumption. Thus F is closed under $\vee$. In particular, this implies that $a \vee b \in F$. Then consider the submodel of $\langle \boldsymbol{A}, F \rangle$ with universe $\{a, b, a \wedge b, a \vee b\}$. It is the four-element diamond, where everything except the bottom is designated. It is easy to check that it is reduced. But observe that also the direct square of $\langle \mathbf{2}, \{1\} \rangle$ is a model of $\mathcal{L}$. This is the four-element diamond, where only the top is designated. Also this model is reduced. Thus we constructed two different non-almost trivial reduced models with the same algebraic reduct, contradicting the assumptions. This shows that F is closed under $\wedge$. A dual argument shows that F is closed under $\vee$, establishing the claim.

Now suppose towards a contradiction that conditions 1 and 2 are false. Then there are two models $\langle \boldsymbol{A}, F\rangle$ and $\langle \boldsymbol{B}, G\rangle$ of $\mathcal{L}$ with $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{Alg}\mathcal{L}$ such that F is not a filter and G is not an ideal. By the claim we know that F is closed under the lattice operations. Then F is not an up-set. Then there are elements $a < b$ such that $a \in F$ and $b \notin F$. The submodel of $\langle \boldsymbol{A}, F\rangle$ with universe $\{a, b\}$ is isomorphic to $\langle \mathbf{2}, \{0\}\rangle$. Applying a dual argument to G, it is easy to show that also $\langle \mathbf{2}, \{1\}\rangle$ is a model of $\mathcal{L}$. This contradicts the fact that truth is almost universally definable in $\mathsf{Mod}^*\mathcal{L}$. ⊠

It is well known that a lattice is distributive if and only if it has no subalgebra isomorphic to one of the two lattices depicted below (called respectively $\boldsymbol{M}_3$ and $\boldsymbol{N}_5$). Moreover a lattice is modular if and only if it contains no subalgebra isomorphic to $\boldsymbol{N}_5$.



Theorem 2.17. *Let* K *be a non-trivial variety of lattices.*

1. K *is not the algebraic counterpart of a logic whose truth sets are equationally definable.*
2. $\mathbb{V}(\boldsymbol{M}_3)$ *is the algebraic counterpart of a logic whose truth sets are almost universally definable.*
3. *If* K *is non-modular or contains the variety of modular lattices, then it is not the algebraic counterpart of any logic whose truth sets are almost universally definable.*

Proof. 1. Suppose towards a contradiction that K is the algebraic counterpart of a logic $\mathcal{L}$ for which truth is equationally definable in $\mathsf{Mod}^*\mathcal{L}$. Consider a non-trivial subdirectly irreducible algebra $\boldsymbol{A} \in \mathsf{K}$. Since $\mathbb{P}_{\text{SD}}\mathsf{Alg}^*\mathcal{L} = \mathsf{Alg}\mathcal{L} = \mathsf{K}$, we have that $\boldsymbol{A} \in \mathsf{Alg}^*\mathcal{L}$. Then there must be a reduced model of the form $\langle \boldsymbol{A}, F\rangle$, where F is equationally definable. Up to equivalence the unique lattice equation in one variable x is $x \approx x$. This is a consequence of the fact that the free one-generated lattice is trivial. Thus we conclude that $F = A$ and, therefore, that $\boldsymbol{A}$ is the trivial algebra, against the assumption.

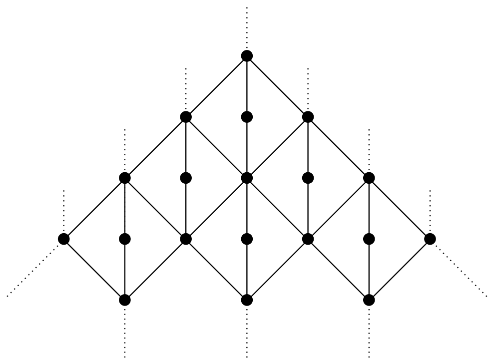
2. From Jónsson's lemma it follows that the subdirectly irreducible members of $\mathbb{V}(\boldsymbol{M}_3)$ are $\boldsymbol{M}_3$ and $\mathbf{2}$. Then consider the logic $\mathcal{L}$ determined by the matrices $\langle \boldsymbol{M}_3, \{1\}\rangle$ and $\langle \mathbf{2}, \{1\}\rangle$. Since these matrices are reduced, we obtain that

$$\mathbb{V}(\boldsymbol{M}_3) = \mathbb{P}_{\text{SD}}(\boldsymbol{M}_3, \mathbf{2}) \subseteq \mathbb{P}_{\text{SD}}\mathsf{Alg}^*\mathcal{L} = \mathsf{Alg}\mathcal{L}.$$

On the other hand $\mathsf{Alg}\mathcal{L} \subseteq \mathbb{V}(\boldsymbol{M}_3)$ by Lemma 1.3. Hence $\mathbb{V}(\boldsymbol{M}_3)$ is the algebraic counterpart of $\mathcal{L}$. Observe that $\mathcal{L}$ has been defined by a class of almost unital matrices. By Theorem 1.10 this implies that F is a singleton for every non-almost trivial $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*\mathcal{L}$. Together with Lemma 2.16 this implies that if $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*\mathcal{L}$ is non-almost trivial, then $\boldsymbol{A}$ has a maximum 1 and $F = \{1\}$. As a consequence, truth is almost universally definable in $\mathsf{Mod}^*\mathcal{L}$ via $\boldsymbol{\tau}(x, \vec{y}) := \{x \wedge y \approx y\}$.

3. Suppose that K is non-modular. Then $\boldsymbol{N}_5 \in \mathsf{K}$. Suppose towards a contradiction that K is the algebraic counterpart of a logic $\mathcal{L}$ such that truth is almost universally definable in $\mathsf{Mod}^*\mathcal{L}$. We denote by $\{a, b, c, 0, 1\}$ the universe of $\boldsymbol{N}_5$, where $a < b$ are incomparable with c. Since $\mathbb{P}_{\text{SD}}\mathsf{Alg}^*\mathcal{L} = \mathsf{Alg}\mathcal{L} = \mathsf{K}$ and $\boldsymbol{N}_5 \in \mathsf{K}$ is subdirectly irreducible, we have that $\boldsymbol{N}_5 \in \mathsf{Alg}^*\mathcal{L}$. Thus there must be $F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{N}_5$ that separates a and b. From Lemma 2.16 we know that the deductive filters of $\mathcal{L}$ are either lattice filters or lattice ideals. Since $\boldsymbol{N}_5$ is isomorphic to its dual, we can assume w.l.o.g. that they are lattice filters. Hence $\{b, 1\} \subseteq F$ and $a \notin F$. Now let $f \colon \boldsymbol{N}_5 \to \boldsymbol{N}_5$ be the homomorphism that sends b to a and is the identity on $N_5 \smallsetminus \{b\}$. Clearly $G := f^{-1}(F)$ is a deductive filter. Moreover, it is easy to check that $G \subsetneq F$ and $a, b \notin G$. Clearly $\boldsymbol{\Omega}^{\boldsymbol{N}_5}F = 0_{N_5} \subseteq \boldsymbol{\Omega}^{\boldsymbol{N}_5}G$. Together with Theorem 2.8, this implies that $F \subseteq G$, which is false. Hence we conclude that K is not the algebraic counterpart of any logic $\mathcal{L}$ such that truth is almost universally definable in $\mathsf{Mod}^*\mathcal{L}$.

Then consider the case where K contains the variety of modular lattices. Let $\boldsymbol{A}$ be the infinite modular lattice depicted below:



We claim that $\boldsymbol{A}$ is simple. To prove this, consider two different $a, b \in A$ and let $\theta := \mathrm{Cg}(a, b)$. We can assume w.l.o.g. that $a \wedge b < a$. Clearly θ identifies the interval $[a \wedge b, a]$. Observe that $[a \wedge b, a]$ contains two elements that belong to one of the small copies $\boldsymbol{B}$ of $\boldsymbol{M}_3$. Thus θ identifies the whole $\boldsymbol{B}$. Then θ identifies also the small copies of $\boldsymbol{M}_3$ that share a side with $\boldsymbol{B}$. Repeating this argument, we obtain that θ is the total relation. Hence $\boldsymbol{A}$ is simple.

In particular this implies that $\boldsymbol{A} \in \mathsf{Alg}^*\mathcal{L}$, since $\boldsymbol{A} \in \mathsf{K} = \mathsf{Alg}\mathcal{L} = \mathbb{P}_{\text{SD}}\mathsf{Alg}^*\mathcal{L}$. Then there is a reduced model $\langle \boldsymbol{A}, F\rangle$ of $\mathcal{L}$ with $F \neq \emptyset$. From Lemma 2.16 we know that F is either a lattice filter or a lattice ideal. Suppose w.l.o.g. that F is a lattice filter. Since $F \neq A$, there are elements $a < b$ with $a \notin F$ and $b \in F$.

Clearly the interval $[a, b]$ has finite height. Thus there are elements $c, d \in A$ such that d covers c and $c \notin F$ and $d \in F$. Tacking a closer look at $\boldsymbol{A}$, this implies that there is a small copy $\boldsymbol{B}$ of $\boldsymbol{M}_3$ inside $\boldsymbol{A}$ such that $G := B \cap F$ is a proper filter of $\boldsymbol{B}$ that contains at least two elements. Let $\{0, 1, a, b, c\}$ be the universe of $\boldsymbol{B}$. We can assume w.l.o.g. that $G = \{a, 1\}$. Since there is an automorphism of $\boldsymbol{B}$ that interchanges a and b, we obtain that also $\langle \boldsymbol{B}, \{b, 1\} \rangle$ is a model of $\mathcal{L}$. Together with the fact that $\boldsymbol{B}$ is simple, this implies that there are two different reduced models of $\mathcal{L}$ with the same algebraic reduct. But this contradicts the fact that the truth is almost universally definable in $\mathsf{Mod}^*\mathcal{L}$. ⊠

Problem 2. Characterize the varieties of lattices that are the algebraic counterpart of a logic $\mathcal{L}$ such that truth is almost universally definable in $\mathsf{Mod}^*\mathcal{L}$. We know that these varieties must be properly contained into the variety of modular lattices. Observe that the argument applied in point 2 of Lemma 2.16 can be used to produce several examples of varieties that are the algebraic counterpart of a logic whose truth sets are almost universally definable, e.g., all semisimple finitely generated varieties.

Another family of examples comes from the study of semilattices, that are the algebraic counterpart of the conjunctive and of the disjunctive fragment of classical logic.

Example 2.18 (Semilattices)**.** Let $\mathcal{CPC}_\wedge$ be the $\langle \wedge \rangle$-fragment of classical propositional logic. Let also $\mathbf{2} = \langle \{0, 1\}, \wedge \rangle$ be the two-element meet semilattice with $0 < 1$. Every almost non-trivial member of $\mathsf{Mod}^*\mathcal{CPC}_\wedge$ is an isomorphic copy either of $\langle \mathbf{2}, \{1\} \rangle$ or of $\langle \mathbf{1}, \{1\} \rangle$. This was first claimed in [89, Pag. 68-69], but see [42, Corollary 6.3] for an explicit proof. It follows that the truth sets of $\mathcal{CPC}_\wedge$ are almost universally definable by the universal translation $\boldsymbol{\tau}(x, \vec{y}) = \{x \wedge y \approx y\}$.

Let $\mathcal{CPC}_\vee$ be the $\langle \vee \rangle$-fragment of classical propositional logic. The almost non-trivial members of $\mathsf{Mod}^*\mathcal{CPC}_\vee$ can be characterized exactly as those of $\mathcal{CPC}_{\wedge\vee}$ in Example 2.15, but replacing *distributive lattice* by *semilattice* [89, Pag. 68-69]. It follows that the truth sets of $\mathcal{CPC}_\vee$ are almost universally definable by the universal translation $\boldsymbol{\tau}(x, \vec{y}) = \{x \vee y \approx x\}$.

Finally, observe that the algebraic counterpart of both $\mathcal{CPC}_\wedge$ and $\mathcal{CPC}_\vee$ is the variety of semilattices SL [42, Example 4.4]. An argument, similar to the one used in the proof of point 1 of Theorem 2.17, shows that SL is not the algebraic counterpart of any logic $\mathcal{L}$ whose truth sets are equationally definable in $\mathsf{Mod}^*\mathcal{L}$. ⊠

Until now we met four examples of logics whose truth sets are almost universally (but not equationally) definable, namely the fragments $\mathcal{CPC}_{\wedge\vee}, \mathcal{CPC}_\wedge$ and $\mathcal{CPC}_\vee$, and the logic associated with the variety $\mathbb{V}(\boldsymbol{M}_3)$ in Theorem 2.17. All these logics are exemplifications of the following general phenomenon:

Lemma 2.19. *Let* K *be a class of algebras with meet (join) semilattice reduct. Truth is almost universally definable in the class of reduced models of the logic determined by the class of matrices*

$$\{\langle \boldsymbol{A}, \{1\}\rangle : \boldsymbol{A} \in \mathsf{K} \text{ has a top element } 1\}.$$

Proof. First observe that $\mathcal{L}$ is determined by matrices whose filters are singletons. By Theorem 1.10 F is a singleton for every non-almost trivial $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*\mathcal{L}$. Then consider one of these models $\langle \boldsymbol{A}, F\rangle$. From Lemma 1.3 it follows that $\boldsymbol{A} \in \mathbb{V}(\mathsf{K})$, thus $\boldsymbol{A}$ has a meet-semilattice reduct $\langle A, \wedge\rangle$. Observe that by definition of $\mathcal{L}$

$$x, y \vdash_{\mathcal{L}} x \wedge y \qquad x \wedge y \vdash_{\mathcal{L}} x \qquad x \wedge y \vdash_{\mathcal{L}} y.$$

Thus F is a filter of $\langle A, \wedge\rangle$ which, moreover, is a singleton. Hence $\langle A, \wedge\rangle$ has a top element 1 such that $F = \{1\}$. We conclude that truth is almost universally definable in $\mathsf{Mod}^*\mathcal{L}$ via $\boldsymbol{\tau}(x, \vec{y}) := \{x \wedge y \approx y\}$. ⊠

Example 2.20 (Normal Kleene Algebras)**.** Let $\boldsymbol{C}_3 = \langle C_3, \wedge, \vee, \neg, n\rangle$ be the three-element lattice $0 < n < 1$ with an involution $\neg$ whose unique fixed point is n. The class of *normal Kleene algebras* is the variety generated by $\boldsymbol{C}_3$. In [76] the logic $\mathcal{L}$ determined by the matrix $\langle \boldsymbol{C}_3, \{1\}\rangle$ is studied. From Lemma 2.19 it follows that the truth sets of $\mathcal{L}$ are almost universally definable. In particular, we have that for every non-almost trivial $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*\mathcal{L}$ the following conditions hold:

(i) $\boldsymbol{A}$ is a normal Kleene algebra with maximum 1.

(ii) $F = \{1\}$.

(iii) For every $a, b \in A$ such that $a, b \geqslant n$, if $a < b$, then there is $c \in A$ such that $a \vee c < b \vee c = 1$.

This was proved in [76, Theorem 3.5]. From Jónsson's lemma it follows that $\boldsymbol{C}_3$ is the unique subdirectly irreducible normal Kleene algebras. Together with the above characterization of $\mathsf{Mod}^*\mathcal{L}$ and the fact that $\mathsf{Alg}\mathcal{L} = \mathbb{P}_{\text{SD}}\mathsf{Alg}^*\mathcal{L}$, this implies that $\mathsf{Alg}\mathcal{L}$ is the variety of normal Kleene algebras [76, Theorem 3.2]. It is not difficult to see that this variety is also the algebraic counterpart of a logic $\mathcal{L}'$ whose truth sets are equationally definable, namely the one determined by the matrix $\langle \boldsymbol{C}_3, \{n, 1\}\rangle$. ⊠

It makes sense to wonder whether there are examples of meaningful logics $\mathcal{L}$ for which truth is almost universally (but not equationally) definable in $\mathsf{Mod}^*\mathcal{L}$, except those that fall under the scope of Lemma 2.19. One of them comes from the study of bilattices, i.e., algebras which have two lattice-theoretic order relations.

Example 2.21 (Distributive Bilattices)**.** An algebra $\boldsymbol{L} = \langle L, \wedge, \vee, \otimes, \oplus\rangle$ is a *pre-bilattice* if $\langle L, \wedge, \vee\rangle$ and $\langle L, \otimes, \oplus\rangle$ are lattices. In this case we will denote by $\leqslant$

the lattice order associated with $\langle L, \wedge, \vee \rangle$ and by $\sqsubseteq$ the one associated with $\langle L, \otimes, \oplus \rangle$. An algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \otimes, \oplus, \neg \rangle$ is a *bilattice* if $\langle A, \wedge, \vee, \otimes, \oplus \rangle$ is a pre-bilattice such that $\neg\neg a = a$ and

$$a \leqslant b \Longrightarrow (\neg b \leqslant \neg a \text{ and } \neg a \sqsubseteq \neg b)$$

for every $a, b \in A$ (see [53]). A bilattice is *distributive* if the four lattice operations satisfy all the combined distributive axioms. Distributive bilattices form a variety which we denote by DBL. This variety is generated by the bilattice $\boldsymbol{B}_4$ with universe $\{0, 1, a, b\}$, where $\langle B_4, \sqsubseteq \rangle$ is the four-element diamond bounded by $0 < 1$, while $\langle B_4, \leqslant \rangle$ is the four-element diamond bounded by $a < b$, and $\neg$ interchanges a and b and is the identity on $\{0, 1\}$.

The so-called *logic of distributive bilattices* $\mathcal{LB}$ is defined through the matrix $\langle \boldsymbol{B}_4, \{b, 1\} \rangle$, see [23, 90]. By Corollary 2.10 truth is not equationally definable in $\mathsf{Mod}^*\mathcal{LB}$, since $\mathcal{LB}$ is purely inferential. This is a consequence of the fact that $\{0\}$ is the universe of a subalgebra of $\boldsymbol{B}_4$. Our goal will be to prove that truth is almost universally definable in $\mathsf{Mod}^*\mathcal{LB}$. To this end, we need to recall the following construction from [23]. Given a distributive lattice $\boldsymbol{L} = \langle L, \sqcap, \sqcup \rangle$, let $\boldsymbol{L} \odot \boldsymbol{L} = \langle L \times L, \wedge, \vee, \otimes, \oplus, \neg \rangle$ be the twist structure defined as

$$\begin{aligned} \langle a_1, a_2 \rangle \wedge \langle b_1, b_2 \rangle &\coloneqq \langle a_1 \sqcap b_1, a_2 \sqcup b_2 \rangle \\ \langle a_1, a_2 \rangle \vee \langle b_1, b_2 \rangle &\coloneqq \langle a_1 \sqcup b_1, a_2 \sqcap b_2 \rangle \\ \langle a_1, a_2 \rangle \otimes \langle b_1, b_2 \rangle &\coloneqq \langle a_1 \sqcap b_1, a_2 \sqcap b_2 \rangle \\ \langle a_1, a_2 \rangle \oplus \langle b_1, b_2 \rangle &\coloneqq \langle a_1 \sqcup b_1, a_2 \sqcup b_2 \rangle \\ \neg \langle a_1, a_2 \rangle &\coloneqq \langle a_2, a_1 \rangle \end{aligned}$$

for every $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in L \times L$. It turns out that $\boldsymbol{L} \odot \boldsymbol{L} \in \mathsf{DBL}$.

We claim that the universal translation $\boldsymbol{\tau}(x, \vec{y}) \coloneqq \{(x \oplus y) \wedge x \approx x \oplus y\}$ almost defines truth in $\mathsf{Mod}^*\mathcal{LB}$. If $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^*\mathcal{LB}$ is non-almost trivial, then $\boldsymbol{A} \cong \boldsymbol{L} \odot \boldsymbol{L}$ for some lattice $\boldsymbol{L}$ such that the following conditions hold:

(i) $\boldsymbol{L}$ is a distributive lattice with maximum 1.

(ii) $F \cong \{1\} \times L$.

(iii) For every $a, b \in L$, if $a < b$, then there is $c \in L$ such that $a \sqcup c < b \sqcup c = 1$.

This was proved in [23, Theorem 4.13]. For the sake of simplicity, we assume that $\boldsymbol{A} = \boldsymbol{L} \odot \boldsymbol{L}$. For every $\langle a_1, a_2, \rangle, \langle b_1, b_2 \rangle \in A$ we have that

$$\boldsymbol{A} \vDash \boldsymbol{\tau}(\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle) \Longleftrightarrow b_1 \leqslant a_1.$$

This in particular yields that $\mathsf{Sl}^{\boldsymbol{A}}(\boldsymbol{\tau}) = \{1\} \times L$, matching condition (ii) above. ⊠

All the logics considered so far were equipped either with a disjunction or with a conjunction, which was interpreted as a semilattice operation. Remarkably, within the landscape of Fregean logics, the presence of a weak disjunction (conjunction) forces the truth sets to be almost universally definable. To explain how, let us recall the following concept, originating in [80, Convention 3.1]:‡

Definition 2.22. A logic $\mathcal{L}$ has a *protodisjunction* if there is an (at most) binary term $\vee$ such that $x \vdash_{\mathcal{L}} x \vee y$ and $y \vdash_{\mathcal{L}} x \vee y$.

Observe that every logic with theorems has a protodisjunction. This can be easily proved by showing that each theorem can be converted into a protodisjunction, by replacing each variable occurring in it by the variable x.

Theorem 2.23. *If $\mathcal{L}$ is Fregean and has a protodisjunction, then the universal translation $\boldsymbol{\tau}(x, \vec{y}) = \{x \vee y \approx x\}$ almost defines truth in* $\mathsf{Mod}^{*}\mathcal{L}$.

Proof. Thanks to Lemma 2.5 it will be enough to prove that $\boldsymbol{\tau}$ almost universally defines truth in $\mathsf{LMod}^{*}\mathcal{L}$. By Lemma 2.4 this amount to checking that

$$\Gamma \vdash_{\mathcal{L}} \varphi \text{ if and only if } \langle \varphi, \varphi \vee \psi \rangle \in \boldsymbol{\Omega}\Gamma \text{ for every } \psi \in Fm$$

for every $\Gamma \in \mathcal{T}h\mathcal{L} \smallsetminus \{\emptyset\}$ and $\varphi \in Fm$. For the "only if" part assume that $\Gamma \vdash_{\mathcal{L}} \varphi$ and observe that $\varphi, \Gamma \dashv\vdash_{\mathcal{L}} \Gamma, \varphi \vee \psi$, because $\vee$ is a protodisjunction. Since $\mathcal{L}$ is Fregean, this implies that $\langle \varphi, \varphi \vee \psi \rangle \in \boldsymbol{\Lambda}(\mathcal{T}h\mathcal{L})^{\Gamma} = \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}\Gamma \subseteq \boldsymbol{\Omega}\Gamma$. For the "if" part choose $\psi \in \Gamma$. This can be done since $\Gamma \neq \emptyset$. Since $\vee$ is a protodisjunction, we have that $\Gamma \vdash_{\mathcal{L}} \varphi \vee \psi$. By compatibility we conclude that $\Gamma \vdash_{\mathcal{L}} \varphi$. ⊠

Drawing consequences from this result, we obtain an essentially different proof of the following known result [6, Theorem 14]:

Corollary 2.24. *A Fregean logic $\mathcal{L}$ has theorems if and only if truth is equationally definable in* $\mathsf{Mod}^{*}\mathcal{L}$.

Proof. The "if" part was proven in Corollary 2.10. For the "only if" part, let $\mathcal{L}$ be a Fregean logic with theorems. In particular, $\mathcal{L}$ has a protodisjunction. Thus from Theorem 2.23 it follows that truth is almost universally definable in $\mathsf{Mod}^{*}\mathcal{L}$. Together with Corollary 2.10 and the fact that $\mathcal{L}$ has theorems, this implies that truth is equationally definable in $\mathsf{Mod}^{*}\mathcal{L}$. ⊠

Now we consider Fregean logics with a binary connective which behaves like a weak conjunction.

Definition 2.25. A logic $\mathcal{L}$ has a *protoconjunction* if there is an (at most) binary term $\wedge$ such that $\{x, y\} \vdash_{\mathcal{L}} x \wedge y$, $\{x, x \wedge y\} \vdash_{\mathcal{L}} y$ and $\{y, x \wedge y\} \vdash_{\mathcal{L}} x$.

‡It is worth to remark that in [80, Convention 3.1] protodisjunctions are defined as *sets of formulas*. Here we will assume they are just *formulas* and our results depend on this assumption.

Theorem 2.26. *If $\mathcal{L}$ is Fregean and has a protoconjunction, then the universal translation $\boldsymbol{\tau}(x,\vec{y}) = \{x \wedge y \approx y\}$ almost defines truth in* $\mathsf{Mod}^*\mathcal{L}$.

Proof. Again it will be enough to prove that

$$\Gamma \vdash_{\mathcal{L}} \varphi \text{ if and only if } \langle \psi, \varphi \wedge \psi \rangle \in \boldsymbol{\Omega}\Gamma \text{ for every } \psi \in Fm$$

for every $\Gamma \in \mathcal{T}h\mathcal{L} \smallsetminus \{\emptyset\}$ and $\varphi \in Fm$. For the "only if" part observe that if $\Gamma \vdash_{\mathcal{L}} \varphi$, then $\psi, \Gamma \dashv\vdash_{\mathcal{L}} \Gamma, \varphi \wedge \psi$ since $\wedge$ is a protodisjunction. Since $\mathcal{L}$ is Fregean, this implies that $\langle \psi, \varphi \wedge \psi \rangle \in \boldsymbol{\Lambda}(\mathcal{T}h\mathcal{L})^{\Gamma} = \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}\Gamma \subseteq \boldsymbol{\Omega}\Gamma$. For the "if" part choose $\psi \in \Gamma$. This can be done since $\Gamma \neq \emptyset$. By compatibility we have that $\Gamma \vdash_{\mathcal{L}} \varphi \wedge \psi$. Since $\wedge$ is a protoconjunction, we conclude that $\Gamma \vdash_{\mathcal{L}} \varphi$. ⊠

2.3 The Frege hierarchy

We saw that logics $\mathcal{L}$ whose truth sets are almost universally definable in $\mathsf{Mod}^*\mathcal{L}$ fit inside the framework of the Leibniz hierarchy (Theorem 2.8). Thus it is natural investigate which is their behaviour with respect to the Frege hierarchy, the other hierarchy typical of abstract algebraic logic. In particular, we will show that for these logics the Frege hierarchy reduces to three classes that are mutually different. Even if we don't provide precise quotations, all the results presented in this section have an antecedent in [6], where only equational definability is taken into account.

Since the definition of some levels of the Frege hierarchy refers to the notion of a full g-model, it will be useful to obtain a characterization of logics whose truth sets are almost universally definable in terms of the behaviour of full g-models. To this end, given a logic $\mathcal{L}$, an algebra $\boldsymbol{A}$ and a deductive filter $F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$, we define

$$\mathcal{F}i^{\circ}_{\mathcal{L}}\boldsymbol{A}^{F} := \begin{cases} \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}^{F} & \text{if } \mathcal{L} \text{ has theorems} \\ \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}^{F} \cup \{\emptyset\} & \text{otherwise.} \end{cases}$$

Given a theory $\Gamma \in \mathcal{T}h\mathcal{L}$, we will write simply $\mathcal{T}h^{\circ}\mathcal{L}^{\Gamma}$ instead of $\mathcal{F}i^{\circ}_{\mathcal{L}}\boldsymbol{Fm}^{\Gamma}$.

Lemma 2.27. *The following conditions are equivalent:*

(i) *Truth is almost universally definable in* $\mathsf{Mod}^*\mathcal{L}$.

(ii) *The g-matrix $\langle \boldsymbol{A}, \mathcal{F}i^{\circ}_{\mathcal{L}}\boldsymbol{A}^{F} \rangle$ is a full g-model of $\mathcal{L}$, for every $F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ and every algebra $\boldsymbol{A}$.*

(iii) *The g-matrix $\langle \boldsymbol{Fm}, \mathcal{T}h^{\circ}\mathcal{L}^{\Gamma} \rangle$ is a full g-model of $\mathcal{L}$, for every $\Gamma \in \mathcal{T}h\mathcal{L}$.*

Proof. (i)⇒(ii): Consider an algebra $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$. Observe that $\widetilde{\boldsymbol{\Omega}}^{A}\mathcal{F}i^{\circ}_{\mathcal{L}}\boldsymbol{A}^{F} = \widetilde{\boldsymbol{\Omega}}^{A}_{\mathcal{L}}F$, since $\boldsymbol{\Omega}^{A}\emptyset = A \times A$. By Lemma 1.12 it will be enough to show that

$$\mathcal{F}i^{\circ}_{\mathcal{L}}\boldsymbol{A}^{F} = \{G \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}^{A}_{\mathcal{L}}F \subseteq \boldsymbol{\Omega}^{A}G\}.$$

The inclusion from left to right follows from the definition of Suszko congruence, together with the fact that $\boldsymbol{\Omega}^{A}\emptyset = A \times A$. To prove the other inclusion, consider $G \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}^{A}_{\mathcal{L}}F \subseteq \boldsymbol{\Omega}^{A}G$. If $G = \emptyset$, then $\mathcal{L}$ is purely inferential and we are done. Then suppose that $G \neq \emptyset$. With an application of Theorem 2.8 and Lemma 2.12 we conclude that $F \subseteq G$.

(ii)⇒(iii): Straightforward. (iii)⇒(i): Again by Theorem 2.8 and Lemma 2.12 it will be enough to prove that for every $\Gamma, \Gamma' \in \mathcal{T}h\mathcal{L}$ such that $\Gamma' \neq \emptyset$ if $\widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}\Gamma \subseteq \boldsymbol{\Omega}\Gamma'$, then $\Gamma \subseteq \Gamma'$. Since $\boldsymbol{\Omega}\emptyset = Fm \times Fm$ we have that

$$\widetilde{\boldsymbol{\Omega}}\mathcal{T}h^{\circ}\mathcal{L}^{\Gamma} = \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}\Gamma \subseteq \boldsymbol{\Omega}\Gamma'.$$

By Lemma 1.12 we conclude that $\Gamma' \in (\mathcal{T}h^{\circ}\mathcal{L})^{\Gamma}$. Since $\Gamma' \neq \emptyset$, this yields that $\Gamma \subseteq \Gamma'$. ⊠

Corollary 2.28. *The following conditions are equivalent:*

(i) *Truth is equationally definable in* $\mathsf{Mod}^{*}\mathcal{L}$.

(ii) *The g-matrix* $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}^{F} \rangle$ *is a full g-model of* $\mathcal{L}$*, for every* $F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ *and every algebra* $\boldsymbol{A}$.

(iii) *The g-matrix* $\langle \boldsymbol{Fm}, \mathcal{T}h\mathcal{L}^{\Gamma} \rangle$ *is a full g-model of* $\mathcal{L}$*, for every* $\Gamma \in \mathcal{T}h\mathcal{L}$.

Proof. (i)⇒(ii): By Corollary 2.10 the logic $\mathcal{L}$ has theorems. In particular, this implies that $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A} = \mathcal{F}i^{\circ}_{\mathcal{L}}\boldsymbol{A}^{F}$ for every algebra $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$. Thus we can apply Lemma 2.27. (ii)⇒(iii): Straightforward. (iii)⇒(i): By assumption the g-matrix $\langle \boldsymbol{Fm}, \{Fm\} \rangle$ is a full g-model of $\mathcal{L}$. Together with Corollary 1.13, this implies that $\mathcal{L}$ has theorems. Thus $\mathcal{T}h\mathcal{L}^{\Gamma} = \mathcal{T}h^{\circ}\mathcal{L}^{\Gamma}$ for every $\Gamma \in \mathcal{T}h\mathcal{L}$. Then we can apply Lemma 2.27, obtaining that truth is almost universally definable in $\mathsf{Mod}^{*}\mathcal{L}$. Keeping in mind that $\mathcal{L}$ has theorems, with an application of Corollary 2.10 we are done. ⊠

Building on this characterization, we obtain the following:

Theorem 2.29. *A logic whose truth sets are almost universally definable is fully selfextensional if and only if it is fully Fregean. Moreover, inside the class of logics whose truth sets are almost universally definable, the classes of selfextensional, Fregean and fully Fregean logics are different.*

Proof. Suppose that $\mathcal{L}$ is fully selfextensional and that truth is almost universally definable in $\mathsf{Mod}^{*}\mathcal{L}$. Then consider a full g-model $\langle \boldsymbol{A}, \mathcal{C} \rangle$ of $\mathcal{L}$ and $F \in \mathcal{C}$. We have to prove that $\langle \boldsymbol{A}, \mathcal{C}^{F} \rangle$ has the (PCONG). We define

$$\mathcal{C}^{\circ F} := \begin{cases} \mathcal{C}^{F} & \text{if } \mathcal{L} \text{ has theorems} \\ \mathcal{C}^{F} \cup \{\emptyset\} & \text{otherwise.} \end{cases}$$

Observe that $\langle \boldsymbol{A}, \mathcal{C}^{F} \rangle$ has the (PCONG) if and only if $\langle \boldsymbol{A}, \mathcal{C}^{\circ F} \rangle$ has it. Since $\mathcal{L}$ is fully selfextensional, it will be enough to check that $\langle \boldsymbol{A}, \mathcal{C}^{\circ F} \rangle$ is a full g-model of $\mathcal{L}$. Since $\langle \boldsymbol{A}, \mathcal{C} \rangle$ if a full g-model of $\mathcal{L}$, we know that

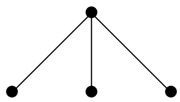
$\mathcal{C}/\widetilde{\Omega}^{A}\mathcal{C} = \mathcal{F}i_{\mathcal{L}}(A/\widetilde{\Omega}^{A}\mathcal{C})$. From Lemma 1.4 it follows that the natural surjection from A to $A/\widetilde{\Omega}^{A}\mathcal{C}$ extends to an isomorphism between $\mathcal{C}^{\circ F}$ and $(\mathcal{F}i^{\circ}_{\mathcal{L}}A/\widetilde{\Omega}^{A}\mathcal{C})^{F/\widetilde{\Omega}^{A}\mathcal{C}}$ and, therefore, to a strict surjective homomorphism between the corresponding g-matrices. This yields in particular that

$$\langle A, \mathcal{C}^{\circ F}\rangle^{*} \cong \langle A/\widetilde{\Omega}^{A}\mathcal{C}, (\mathcal{F}i^{\circ}_{\mathcal{L}}A/\widetilde{\Omega}^{A}\mathcal{C})^{F/\widetilde{\Omega}^{A}\mathcal{C}}\rangle^{*}. \tag{2.6}$$

From Lemma 2.27 we know that $\langle A/\widetilde{\Omega}^{A}\mathcal{C}, (\mathcal{F}i^{\circ}_{\mathcal{L}}A/\widetilde{\Omega}^{A}\mathcal{C})^{F/\widetilde{\Omega}^{A}\mathcal{C}}\rangle$ is a full g-model of $\mathcal{L}$. Together with (2.6), this implies that $\langle A, \mathcal{C}^{\circ F}\rangle$ is also a full g-model of $\mathcal{L}$. This establishes the first statement of the theorem.

To justify the second statement of the theorem, we reason as follows. In [9] a Fregean logic, which is not fully Fregean, is presented. Since this logic has theorems, by Corollary 2.24 we know that truth is equationally definable in the class of its reduced models. Therefore it will be enough to construct a selfextensional, but not Fregean logic $\mathcal{L}$ for which truth is almost universally definable in $\mathsf{Mod}^{*}\mathcal{L}$. This is done in the following example. ⊠

Example 2.30 (Non-Fregean Logic)**.** Consider the language $\langle \to, \Box, a, b, c, 1\rangle$ of type $\langle 2, 1, 0, 0, 0, 0\rangle$, and the set $A := \{a, b, c, 1\}$ with the order structure given by the following graph, whose top element is 1:



We equip it with the structure of an algebra $A = \langle A, \to, \Box, a, b, c, 1\rangle$ of the above similarity type, where the four constants are interpreted in the obvious way, and for every $p, q \in A$,

$$p \to q := \begin{cases} 1 & \text{if } p \leqslant q, \\ q & \text{otherwise,} \end{cases} \qquad \Box p := \begin{cases} b & \text{if } p \in \{1, a, c\}, \\ 1 & \text{otherwise.} \end{cases}$$

Observe that the implicative reduct of A is a Hilbert algebra. Let $\mathcal{L}$ be the logic determined by the g-matrix $\langle A, \mathcal{C}\rangle$, where

$$\mathcal{C} := \{\{1\}, \{a, 1\}, \{c, 1\}, A\}.$$

Observe that all the members of $\mathcal{C}$ are implicative filters.

Fact 2.30.1. *$\mathcal{L}$ is a finitary algebraizable logic.*

$\mathcal{L}$ is finitary, since it is determined by a finite set of finite g-matrices. Then we turn to check that $\mathcal{L}$ is algebraizable. The implicative fragment of $\mathcal{L}$ is a logic defined by a family of implicative filters of a Hilbert algebra, and is therefore an implicative logic in the sense of [88]. Moreover, it is easy to check that

$$x \to y, y \to x \vdash_{\mathcal{L}} \Box x \to \Box y.$$

As a consequence, $\mathcal{L}$ itself is implicative and, therefore, algebraizable [19, § 5.2].

Fact 2.30.2. *$\mathcal{L}$ is selfextensional.*

Observe that the closure system $\mathcal{C}$ separates points in A, therefore $\boldsymbol{\Lambda}\mathcal{C} = 0_A$, and hence $\widetilde{\boldsymbol{\Omega}}^A\mathcal{C} = 0_A$, that is, the g-matrix $\langle \boldsymbol{A}, \mathcal{C}\rangle$ has the property of congruence (and is reduced). This easily implies [31, Theorem 82] that $\langle \boldsymbol{Fm}, \mathcal{T}h\mathcal{L}\rangle$ has the property of congruence, that is, the logic $\mathcal{L}$ is selfextensional.

Fact 2.30.3. *$\mathcal{L}$ is not fully Fregean.*

It is easy to see that the following deductions hold

$$\emptyset \vdash_{\mathcal{L}} 1 \qquad a, c \vdash_{\mathcal{L}} x \qquad b \vdash_{\mathcal{L}} x,$$

and that this implies that $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A} = \mathcal{C}$. Therefore $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a full g-model of $\mathcal{L}$. Now, consider the closure system $\mathcal{C}^{\{a,1\}} = \{\{a,1\}, A\}$. It is clear that $\langle c, b\rangle \in \boldsymbol{\Lambda}\mathcal{C}^{\{a,1\}}$, because c and b belong to the same members of $\mathcal{C}^{\{a,1\}}$, and that $\langle \Box c, \Box b\rangle \notin \boldsymbol{\Lambda}\mathcal{C}^{\{a,1\}}$, because $\Box c = b \notin \{a,1\}$ while $\Box b = 1 \in \{a,1\}$. Hence the g-model $\langle \boldsymbol{A}, \mathcal{C}^{\{a,1\}}\rangle$ does not have the property of congruence, which is to say that the full g-model $\langle \boldsymbol{A}, \mathcal{C}\rangle$ has not the strong property of congruence.

Fact 2.30.4. *$\mathcal{L}$ is not Fregean.*

A finitary protoalgebraic logic is Fregean if and only if it is fully Fregean [31, Corollary 80]. Since $\mathcal{L}$ is a non-Fregean finitary algebraizable logic, we conclude that it is also non-Fregean. ⊠

Corollary 2.31. *A logic with theorems is fully Fregean if and only if it is both Fregean and fully selfextensional.*

Proof. Let $\mathcal{L}$ be a Fregean and fully selfextensional logic with theorems. From Corollary 2.24 it follows that the truth is equationally definable in $\mathsf{Mod}^*\mathcal{L}$. Thus we can apply Theorem 2.29, obtaining that $\mathcal{L}$ is fully Fregean. ⊠

CHAPTER 3

Definability without equations

In the previous chapter we focused on logics whose truth predicates can be defined by means of some linguistic translation of formulas into equations in one variable and, possibly, with parameters. In this chapter we consider some definability conditions that make no refrences to equations or to any other syntactic object. More precisely, we say that truth is *implicitly definable* in a class of matrices M if the matrices in M are determined by their algebraic reduct. It is well known that, when applied to the Leibniz reduced models $\mathsf{Mod}^{*}\mathcal{L}$ of a given logic $\mathcal{L}$, the notion of implicit definability corresponds to the injectivity of the Leibniz operator over $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ for every algebra $\boldsymbol{A}$ (Lemma 3.2). Czelakowski and Jansana showed in [30] that for protoalgebraic logics the injectivity of the Leibniz operator transfers from theories to filters over arbitrary algebras (Theorem 3.6). This fact posed the problem of whether the injectivity of the Leibniz operator transfers for arbitrary logics [83, Problem 1]. We solve this problem by showing that its answer depends on the cardinality of the language in which the logic is expressed. More precisely, if the language is countable, then the injectivity of the Leibniz operator transfers from theories to arbitrary filters (Theorem 3.8). On the other hand, we construct a counterexample with an uncountable language (Example 3.9).

When applied to classes of matrices, the concept of implicit definability has inspired an analogy with Beth's definability theorem which states that in first-order logic explicit definability and implicit definability coincide. In particular, Hermann [56, 58] and Czelakowski and Jansana [30] proved a series of results, collectively called *Beth's definability theorems*, whose main outcome is that the notion of equational and implicit definability coincide, when referred to the class of reduced models of a protoalgebraic logic (Corollary 3.4). Remarkably this result is an achievement of matrix theory which cannot be inferred from the original Beth definability theorem, since the class of reduced models of a logic is not an elementary class in general.

As we mentioned, a natural way to introduce new conditions on the truth sets of a matrix semantics is to move the attention to conditions that make no reference to linguistic objects (such as structural transformers), as in the case of implicit definability. In this spirit we introduce the concept of a class of matrices M whose truth sets are *small*, in the sense that the they are the smallest non-empty deductive filters of the logic determined by M. It turns out that this condition, when applied to the class of reduced models $\mathsf{Mod}^*\mathcal{L}$, corresponds to the fact that the Leibniz operator is order-reflecting over $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ for every algebra $\boldsymbol{A}$ (Lemma 3.12). Again, the order-reflection of the Leibniz operator transfers from theories to filters over arbitrary algebras for logics expressed in a countable language (Theorem 3.13), while there are counterexamples among logics whose language is uncountable.

3.1 Implicit definability

The next step we make in the analysis of truth sets in matrix semantics, is that of considering classes of matrices whose truth sets are implicitly definable.

Definition 3.1. Truth is *implicitly definable* in M if matrices in M are determined by their algebraic reducts, i.e., if $\langle \boldsymbol{A}, F\rangle, \langle \boldsymbol{A}, G\rangle \in \mathsf{M}$, then $F = G$.

It is part of the folklore of abstract algebraic logic that the implicit definability of truth sets in the class of Leibniz-reduced models of a logic, corresponds to the fact that the Leibniz operator is injective on deductive filters over arbitrary algebras. For the sake of completenes, we sketch a proof of this fact.

Lemma 3.2 (Folklore). *Truth is (almost) implicitly definable in* $\mathsf{Mod}^*\mathcal{L}$ *if and only if* $\boldsymbol{\Omega}^{\boldsymbol{A}}$ *is (almost) injective on* $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ *for every algebra* $\boldsymbol{A}$.

Proof. Suppose that truth is implicitly definable in $\mathsf{Mod}^*\mathcal{L}$. Then let $F, G \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ such that $\boldsymbol{\Omega}^{\boldsymbol{A}}F = \boldsymbol{\Omega}^{\boldsymbol{A}}G$. Thus the reduced models $\langle \boldsymbol{A}, F\rangle^*$ and $\langle \boldsymbol{A}, G\rangle^*$ have the same algebraic reduct. Since the truth sets of $\mathcal{L}$ are implicitly definable, we conclude that $F/\boldsymbol{\Omega}^{\boldsymbol{A}}F = G/\boldsymbol{\Omega}^{\boldsymbol{A}}G$, and hence by compatibility $F = G$. For the converse suppose that $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is injective on $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ for every algebra $\boldsymbol{A}$, and consider two reduced models $\langle \boldsymbol{A}, F\rangle$ and $\langle \boldsymbol{A}, G\rangle$ of $\mathcal{L}$. Clearly we have that $\boldsymbol{\Omega}^{\boldsymbol{A}}F = 0_A = \boldsymbol{\Omega}^{\boldsymbol{A}}G$. From the assumption it follows that $F = G$. The *almost* case is handled just restricting this argument to non-empty filters. ⊠

Clearly implicit definability generalizes equational and universal definability:

Lemma 3.3. *Let* M *be a class of matrices.*

1. *If truth is equationally definable in* M*, then it is also implicitly definable in* M*.*

2. *If truth is almost universally definable in* M*, then it is also almost implicitly definable in* M*.*

Under certain assumptions implicit and equational definability may coincide. This phenomenon has been interpreted as providing a version of *Beth's definability theorem* of first-order (classical) logic, namely one that is intrinsic to propositional logics and logical matrices. More precisely, we have [83, Theorem 28 and Corollary 29]:

Corollary 3.4 (Hermann and Raftery)**.** *The following conditions are equivalent:*

(i) *Truth is equationally definable in* $\mathsf{Mod}^*\mathcal{L}$*.*

(ii) *Truth is equationally definable in* $\mathsf{Mod}^{\mathsf{Su}}\mathcal{L}$*.*

(iii) *Truth is implicitly definable in* $\mathsf{Mod}^{\mathsf{Su}}\mathcal{L}$*.*

In particular, if $\mathcal{L}$ *is protoalgebraic, then truth is equationally definable in* $\mathsf{Mod}^*\mathcal{L}$ *if and only if it is implicitly definable in* $\mathsf{Mod}^*\mathcal{L}$*.*

Proof. (i)⇒(ii): This is a consequence of the fact that $\mathsf{Mod}^{\mathsf{Su}}\mathcal{L}$ is the closure under subdirect products of $\mathsf{Mod}^*\mathcal{L}$. (ii)⇒(iii): Straightforward. (iii)⇒(i): Suppose that truth is implicitly definable in $\mathsf{Mod}^{\mathsf{Su}}\mathcal{L}$. An argument analogous to the one in the proof of Lemma 3.2 shows that $\widetilde{\Omega}^{\boldsymbol{A}}_{\mathcal{L}}$ is injective over $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ for every algebra $\boldsymbol{A}$. Thus, with an application of Theorem 2.13, we are done. The last observation follows from the fact that the class of Leibniz and Suszko reduced models coincide for protoalgebraic logics, by Theorem 1.8. ⊠

The next example shows that there are fragments $\mathcal{L}$ of modal logics for which truth is implicitly but not equationally definable in $\mathsf{Mod}^*\mathcal{L}$. Two *ad hoc* logics of this kind have already been constructed in [83, Examples 2 and 3].

Example 3.5 (Modal Fragment)**.** Let $\mathcal{L}$ be the $\{\Box, 1\}$-fragment of the local modal system $\mathcal{S}4$. Working with Kripke semantics, it is easy to show that $\mathcal{L}$ is axiomatized by the following set of Hilbert-style rules:

$$\emptyset \vdash \Box 1 \qquad \Box x \vdash x \qquad \Box x \vdash \Box\Box x.$$

Let $\boldsymbol{A}_3 = \langle \{a, b, 1\}, \Box, 1\rangle$ be the algebra where $\Box a = \Box b = b$ and $\Box 1 = 1$. Moreover, let $\boldsymbol{A}_2$ be the subalgebra of $\boldsymbol{A}_3$ with universe $\{1, b\}$.

Fact 3.5.1. $\mathsf{Mod}^*\mathcal{L}$ *is the closure under isomorphism of* $\langle \boldsymbol{A}_3, \{1, a\}\rangle$*,* $\langle \boldsymbol{A}_2, \{1\}\rangle$ *and the trivial matrix.*

The inclusion from right to left is a straightforward application of the Hilbert-style characterization of $\mathcal{L}$. We turn to prove the other one. Consider $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*\mathcal{L}$. If $\boldsymbol{A}$ is trivial, then $\langle \boldsymbol{A}, F\rangle$ is the trivial matrix since $\mathcal{L}$ has theorems. Then suppose that $\boldsymbol{A}$ is non-trivial. From point 1 of Lemma 1.2 it follows that for every $a, b \in A$:

$$a = b \text{ if and only if } (a \in F \Leftrightarrow b \in F \text{ and } \Box a \in F \Leftrightarrow \Box b \in F). \tag{3.1}$$

We know that $1 \in F$. Moreover, since $\boldsymbol{A}$ is non-trivial, there is $b \notin F$. We claim that $A \smallsetminus F = \{b\}$ and that $\Box b = b$. To prove this, let $c \in A \smallsetminus F$. From $b, c \notin F$ and $\Box x \vdash_{\mathcal{L}} x$ it follows that $\Box b, \Box c \notin F$. By (3.1) we conclude that $b = c$. Hence $A \smallsetminus F = \{b\}$. Analogously, the fact that $\Box b \notin F$ together with $\Box x \vdash_{\mathcal{L}} x$, implies that $\Box\Box b \notin F$. Therefore, with an application of (3.1), we conclude that $\Box b = b$. This establishes our claim.

It is easy to show that $1, \Box 1$ and $\Box\Box 1$ are theorems of $\mathcal{L}$. Together with (3.1) this implies that $\Box 1 = 1$. Hence if $F = \{1\}$, we conclude that $\langle \boldsymbol{A}, F \rangle \cong \langle \boldsymbol{A}_2, \{1\} \rangle$. Then suppose that there is $a \in F$ different from 1. Together with (3.1), this implies that $\Box a \notin F$. Observe that $\Box\Box a \notin F$, since $\Box x \vdash_{\mathcal{L}} x$. From (3.1) we obtain $\Box a = b$. Now we turn to prove that $A = \{a, b, 1\}$. Suppose the contrary towards a contradiction. Since $A \smallsetminus F = \{b\}$, there is $c \in F \smallsetminus \{a, 1\}$. Clearly either $\Box c \in F$ or $\Box c \notin F$. In both cases we have that $c \in \{1, a\}$ by (3.1), contradicting the assumptions. Hence $A = \{a, b, 1\}$ and therefore $\langle \boldsymbol{A}, F \rangle \cong \langle \boldsymbol{A}_3, \{1, a\} \rangle$.

Fact 3.5.2. *$\boldsymbol{\Omega}^{\boldsymbol{A}_3}$ is not order-reflecting over $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}_3$.*

Observe that $\{1\}, \{1, a\} \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}_3$ and that $\boldsymbol{\Omega}^{\boldsymbol{A}_3}\{1, a\} \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}_3}\{1\}$.

Fact 3.5.3. *Truth is implicitly, but not equationally, definable in* $\mathsf{Mod}^*\mathcal{L}$.

The fact that truth is implicitly definable in $\mathsf{Mod}^*\mathcal{L}$ is a direct consequence of the characterization of $\mathsf{Mod}^*\mathcal{L}$. Moreover, Fact 3.5.2 and Theorem 2.9 imply that truth is not equationally definable in $\mathsf{Mod}^*\mathcal{L}$. ⊠

Up to now it was an open problem whether the injectivity of the Leibniz operator transfers, in general, from the theories of a given logic to its deductive filters over arbitrary algebras [83, Problem 1] (see also [35]). The main goal of this section is two-fold. On the one hand we will prove that this is the case for protoalgebraic logics (Theorem 3.6) and for logics expressed in a countable language (Theorem 3.8). On the other hand, we will show that the injectivity of the Leibniz operator does not transfer in general (Example 3.9). The transfer result for protoalgebraic logics was first proved in [30, Proposition 3.7], but the following proof is considerably easier.

Theorem 3.6 (Czelakowski and Jansana)**.** *For protoalgebraic logics the injectivity of the Leibniz operator transfers from theories to filters over arbitrary algebras.*

Proof. Suppose that $\mathcal{L}$ is protoalgebraic and that $\boldsymbol{\Omega}$ is injective over $\mathcal{T}h\mathcal{L}$. We will show that $\boldsymbol{\Omega}$ is also completely order-reflecting over $\mathcal{T}h\mathcal{L}$. By Lemma 2.12 it will be enough to show that

$$\text{if } \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}\Gamma \subseteq \boldsymbol{\Omega}\Gamma', \text{ then } \Gamma \subseteq \Gamma' \tag{3.2}$$

for every $\Gamma, \Gamma' \in \mathcal{T}h\mathcal{L}$. Suppose that $\widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}\Gamma \subseteq \boldsymbol{\Omega}\Gamma'$. By Theorem 1.8 the Leibniz operator (when restricted to $\mathcal{T}h\mathcal{L}$) commutes with intersections and coincides with the Suszko operator. Therefore we obtain that

$$\boldsymbol{\Omega}\Gamma = \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}\Gamma = \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}\Gamma \cap \boldsymbol{\Omega}\Gamma' = \boldsymbol{\Omega}\Gamma \cap \boldsymbol{\Omega}\Gamma' = \boldsymbol{\Omega}(\Gamma \cap \Gamma').$$

Since $\boldsymbol{\Omega}$ is injective over $\mathcal{T}h\mathcal{L}$ we conclude that $\Gamma \subseteq \Gamma'$, establishing (3.2). With an application of Theorem 2.9 we are done. ⊠

Now we turn to prove the transfer result for logics expressed in a countable language. To this end, we will make use of the following technical result which generalizes [27, Proposition 0.7.6].

Lemma 3.7. *Let κ be an infinite cardinal larger or equal to $|\mathscr{L}|$ and let $\langle \boldsymbol{A}, F\rangle$ and $\langle \boldsymbol{A}, G\rangle$ be a pair of reduced matrices. Every κ-generated subalgebra $\boldsymbol{C}$ of $\boldsymbol{A}$ can be extended to another κ-generated subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$ such that the matrices $\langle \boldsymbol{B}, F \cap B\rangle$ and $\langle \boldsymbol{B}, G \cap B\rangle$ are reduced.*

Proof. First observe that κ-generated algebras are of cardinality $\leqslant \kappa$, since κ is infinite and larger or equal to $|\mathscr{L}|$. Then we define recursively an infinite family of subsets of A. We begin with $X_0 := C$. To define X_{n+1}, we go through the following construction: For every pair of different elements $a, b \in \mathrm{Sg}^{\boldsymbol{A}}(X_n)$ we pick two finite sequences $\vec{c}$ and $\vec{d}$ of elements of A for which there is a pair of formulas $\varphi(x, \vec{y})$ and $\psi(x, \vec{z})$ such that

$$\varphi^{\boldsymbol{A}}(a, \vec{c}) \in F \Longleftrightarrow \varphi^{\boldsymbol{A}}(b, \vec{c}) \notin F$$
$$\psi^{\boldsymbol{A}}(a, \vec{d}) \in G \Longleftrightarrow \psi^{\boldsymbol{A}}(b, \vec{d}) \notin G.$$

The existence of the sequences $\vec{c}$ and $\vec{d}$ is ensured by point 1 of Lemma 1.2 together with the fact that $\langle a, b\rangle \notin 0_{\boldsymbol{A}} = \boldsymbol{\Omega}^{\boldsymbol{A}}F = \boldsymbol{\Omega}^{\boldsymbol{A}}G$. We then let Y_n be the set of all elements in the sequences constructed in this way. Finally we set

$$X_{n+1} := X_n \cup Y_n.$$

Now consider the union $\bigcup_{n\in\omega} \mathrm{Sg}^{\boldsymbol{A}}(X_n)$. It is easy to prove that it is the universe of a subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$. Clearly $\boldsymbol{B}$ extends $\boldsymbol{C}$, since $X_0 = C$. We claim that $\langle \boldsymbol{B}, F \cap B\rangle$ is reduced. To prove this, consider two different $a, b \in B$. There is $n \in \omega$ such that $a, b \in \mathrm{Sg}^{\boldsymbol{A}}(X_n)$. By definition of X_{n+1}, we know that there is a finite sequence $\vec{c}$ of elements of X_{n+1} and a formula $\varphi(x, \vec{y})$ such that $\varphi^{\boldsymbol{A}}(a, \vec{c}) \in F \Longleftrightarrow \varphi^{\boldsymbol{A}}(b, \vec{c}) \notin F$. Since $a, b, \vec{c} \in B$ and $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$, we conclude that $\varphi^{\boldsymbol{B}}(a, \vec{c}), \varphi^{\boldsymbol{B}}(b, \vec{c}) \in B$ and, finally, that

$$\varphi^{\boldsymbol{B}}(a, \vec{c}) \in F \cap B \Longleftrightarrow \varphi^{\boldsymbol{B}}(b, \vec{c}) \notin F \cap B.$$

By point 1 of Lemma 1.2 we conclude that $\langle a, b\rangle \notin \boldsymbol{\Omega}^{\boldsymbol{B}}(F \cap B)$ and therefore that $\boldsymbol{\Omega}^{\boldsymbol{B}}(F \cap B) = 0_{\boldsymbol{B}}$. This concludes the proof of our claim. An analogous argument yields that the the matrix $\langle \boldsymbol{B}, G \cap B\rangle$ is reduced too.

It only remains to show that $\boldsymbol{B}$ is κ-generated. We begin by showing inductively that $|X_n| \leqslant \kappa$ for every $n \in \omega$. For $n = 0$ we have that $X_0 = C$. Recall from the assumption that $\boldsymbol{C}$ is κ-generated and, therefore, of cardinality $\leqslant \kappa$. For the $n+1$ case observe that, by the inductive hypothesis, $|X_n| \leqslant \kappa$. Then $\mathrm{Sg}^{\boldsymbol{A}}(X_n)$ is κ-generated and again of cardinality $\leqslant \kappa$. Now observe that, while constructing Y_n, we added to X_n at most a finite number of elements for every pair of elements of $\mathrm{Sg}^{\boldsymbol{A}}(X_n)$. Therefore the cardinality of Y_n can be bounded above by $\aleph_0 \cdot |\mathrm{Sg}^{\boldsymbol{A}}(X_n)| \cdot |\mathrm{Sg}^{\boldsymbol{A}}(X_n)|$. In particular, this yields that

$$|Y_n| \leqslant \aleph_0 \cdot |\mathrm{Sg}^{\boldsymbol{A}}(X_n)| \cdot |\mathrm{Sg}^{\boldsymbol{A}}(X_n)| \leqslant \aleph_0 \cdot (\kappa \cdot \kappa) \leqslant \kappa.$$

Since $X_{n+1} = X_n \cup Y_n$ is the union of two sets of cardinality smaller or equal to κ, we conclude that $|X_{n+1}| \leqslant \kappa$. This concludes our proof by induction. Then let $n \in \omega$. The fact that $|X_n| \leqslant \kappa$ implies that $\mathrm{Sg}^{\boldsymbol{A}}(X_n)$is of cardinality $\leqslant \kappa$. Thus $B = \bigcup_{n\in\omega} \mathrm{Sg}^{\boldsymbol{A}}(X_n)$ is the union of countably many sets of cardinality smaller of equal to κ. Since κ is infinite, we conclude that $\boldsymbol{B}$ has cardinality $\leqslant \kappa$, hence a fortiori it is κ-generated. ⊠

We are now ready to prove our transfer result:

Theorem 3.8. *For logics expressed in a countable language the (almost) injectivity of the Leibniz operator tansfers from theories to filters over arbitrary algebras.*

Proof. We apply Lemma 3.2. Consider two reduced models $\langle \boldsymbol{A}, F\rangle$ and $\langle \boldsymbol{A}, G\rangle$ of $\mathcal{L}$. We have to prove that $F = G$. By symmetry it is enough to prove that $F \subseteq G$. Consider an element $a \in F$ and let $\boldsymbol{C}$ be the subalgebra of $\boldsymbol{A}$ generated by $\{a\}$. Clearly $\boldsymbol{C}$ is countably generated. Since $|\mathscr{L}| \leqslant \aleph_0$, we can apply Lemma 3.7 and extend $\boldsymbol{C}$ to a countably generated subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$ such that both $\langle \boldsymbol{B}, F\cap B\rangle$ and $\langle \boldsymbol{B}, G\cap B\rangle$ are reduced matrices. Clearly they are both models of $\mathcal{L}$. Since $\boldsymbol{B}$ is countably generated, we can choose a surjective homomorphism $h\colon \boldsymbol{Fm} \to \boldsymbol{B}$. Then we define $\Gamma := h^{-1}[F\cap B]$ and $\Gamma' := h^{-1}[G\cap B]$. We have $\Gamma, \Gamma \in \mathcal{T}h\mathcal{L}$. With an application of point 2 of Lemma 1.1 we obtain

$$\begin{aligned} \boldsymbol{\Omega}\Gamma &= \boldsymbol{\Omega} h^{-1}[F\cap B] = h^{-1}\boldsymbol{\Omega}^{\boldsymbol{B}}(F\cap B) = h^{-1}0_{\boldsymbol{B}} \\ &= h^{-1}\boldsymbol{\Omega}^{\boldsymbol{B}}(G\cap B) = \boldsymbol{\Omega} h^{-1}[G\cap B] = \boldsymbol{\Omega}\Gamma'. \end{aligned}$$

Together with the assumption, this implies that $\Gamma = \Gamma'$. In particular, this implies that $a \in F\cap B = G\cap B \subseteq G$. This concludes the proof that $F \subseteq G$ and therefore we are done. The *almost* case follows by restricting the proof to non-empty filters. ⊠

As we mentioned, if we move our attention to logics expressed in uncountable languages, it is possible to construct examples where the injectivity of the Leibniz operator does not transfer from theories to deductive filters over arbitrary algebras.

Example 3.9 (Transfer Problem)**.** Let $\mathbb{R}$ be the set of real numbers. We consider the algebraic type that consists of a set of binary connectives $\{\multimap_i : i \in \mathbb{R} \smallsetminus \{1,2\}\}$, a set of constants $\{c_i : i \in \mathbb{R} \smallsetminus \{2\}\}$ and a unary connective $\Box$. Then let $\boldsymbol{A}$ be the algebra with universe

$$A := ((\mathbb{R} \smallsetminus \{2\}) \times \{0\}) \cup (\mathbb{R} \times \{1\})$$

and operations defined as follows:

$$\begin{aligned} \langle a_1, a_2 \rangle \multimap_i \langle b_1, b_2 \rangle &:= \begin{cases} \langle 1,1 \rangle & \text{if } a_1 = b_1 = i,\, a_2 = 0 \text{ and } b_2 = 1 \\ \langle 1,0 \rangle & \text{otherwise} \end{cases} \\ \Box \langle a_1, a_2 \rangle &:= \begin{cases} \langle 1,1 \rangle & \text{if either } a_2 = 0 \text{ or } (a_2 = 1 \text{ and } a_1 < 2) \\ \langle 1,0 \rangle & \text{otherwise} \end{cases} \\ c_i &:= \langle i, 1 \rangle \end{aligned}$$

for every $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in A$. To simplify the notation put

$$F := \mathbb{R} \times \{1\} \text{ and } G := (\mathbb{R} \times \{1\}) \smallsetminus \{\langle 2,1 \rangle\}.$$

We consider the logic $\mathcal{L}$ determined by the pair of matrices $\langle \boldsymbol{A}, F \rangle$ and $\langle \boldsymbol{A}, G \rangle$.

Fact 3.9.1. $\boldsymbol{\Omega}^{\boldsymbol{A}}$ *is not injective over* $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$.

We claim that for every pair of different $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in A$, there is a polynomial function $p(z)$ of $\boldsymbol{A}$ that satisfies one of the following conditions:

$$p\langle a_1, a_2 \rangle \in F \cap G \text{ and } p\langle b_1, b_2 \rangle \notin F \cup G$$
$$p\langle b_1, b_2 \rangle \in F \cap G \text{ and } p\langle a_1, a_2 \rangle \notin F \cup G.$$

We split the proof of the claim in three main cases:

1. $a_2 \neq b_2$.
2. $a_2 = b_2 = 0$.
3. $a_2 = b_2 = 1$.

1. Assume w.l.o.g. that $a_2 = 0$ and $b_2 = 1$. If $b_1 \neq 2$, then $\langle a_1, a_2 \rangle \notin F \cup G$ and $\langle b_1, b_2 \rangle \in F \cap G$. Then suppose that $b_1 = 2$. We have that

$$\Box \langle a_1, a_2 \rangle = \langle 1,1 \rangle \in F \cap G \text{ and } \Box \langle b_1, b_2 \rangle = \langle 1,0 \rangle \notin F \cup G.$$

2. Since $\langle a_1, a_2 \rangle \neq \langle b_1, b_2 \rangle$, we have that either $a_1 \neq 1$ or $b_1 \neq 1$. Assume w.l.o.g. that $a_1 \neq 1$. We consider the unary polynomial function

$$p(z) := z \multimap_{a_1} \langle a_1, 1 \rangle.$$

Observe that the operation $\multimap_{a_1}$ exists, since $a_1 \notin \{1,2\}$. Clearly we have that $p\langle a_1, a_2 \rangle \in F \cap G$, while $p\langle b_1, b_2 \rangle \notin F \cup G$.

3. We have two subcases: either $\{a_1, b_1\} \neq \{1,2\}$ or $\{a_1, b_1\} = \{1,2\}$. Suppose that $\{a_1, b_1\} \neq \{1,2\}$. Since $\langle a_1, a_2\rangle \neq \langle b_1, b_2\rangle$ and $a_2 = b_2$, we know that $a_1 \neq b_2$. Together with the fact that $\{a_1, b_1\} \neq \{1,2\}$, this implies that either $a_1 \notin \{1,2\}$ or $b_1 \notin \{1,2\}$. Assume w.l.o.g. that $a_1 \notin \{1,2\}$. Then we can safely consider the polynomial function

$$p(z) := \langle a_1, 0\rangle \multimap_{a_1} z.$$

It is easy to see that $p\langle a_1, a_2\rangle \in F \cap G$, while $p\langle b_1, b_2\rangle \notin F \cup G$. Then we consider the case where $\{a_1, b_1\} = \{1,2\}$. Assume w.l.o.g. that $a_1 = 1$ and $b_1 = 2$. We have that

$$\Box\langle a_1, a_2\rangle = \langle 1,1\rangle \in F \cap G \text{ and } \Box\langle b_1, b_2\rangle = \langle 1,0\rangle \notin G \cup F.$$

This establishes our claim. From Lemma 1.2 it follows that $\boldsymbol{\Omega}^{\boldsymbol{A}}F = \boldsymbol{\Omega}^{\boldsymbol{A}}G$. Since $F \neq G$ we are done.

Fact 3.9.2. *Consider $\Gamma \in \mathcal{T}h\mathcal{L}$, $\varphi \in \Gamma$ and a formula of the form $\alpha(z) \multimap_i \beta(z)$ in which z actually occurs. If $\Gamma \vdash_{\mathcal{L}} \alpha(\varphi) \multimap_i \beta(\varphi)$, then $\langle \varphi, c_i\rangle \in \boldsymbol{\Omega}\Gamma$.*

Suppose that $\Gamma \vdash_{\mathcal{L}} \alpha(\varphi) \multimap_i \beta(\varphi)$. By Lemma 1.2 it will be enough to prove that $h(\varphi) = h(c_i)$ for every homomorphism $h \colon \boldsymbol{Fm} \to \boldsymbol{A}$ such that $h[\Gamma] \subseteq \mathbb{R} \times \{1\}$. Then consider an homomorphism h of this kind. Since $h[\Gamma] \subseteq \mathbb{R} \times \{1\}$ we have that $h(\alpha(\varphi) \multimap_i \beta(\varphi)) \in \mathbb{R} \times \{1\}$. Looking at the definition of $\multimap_i$, it is easy to see that this happens only if $h(\alpha(\varphi) \multimap_i \beta(\varphi)) = \langle 1,1\rangle$. In particular, this is to say that

$$h\alpha(\varphi) = \langle i, 0\rangle \text{ and } h\beta(\varphi) = \langle i, 1\rangle.$$

Looking at the definition of the basic operations of $\boldsymbol{A}$ and keeping in mind that $i \in \mathbb{R} \smallsetminus \{1,2\}$, it is possible to see that $\alpha(\varphi)$ must be a variable and that $\beta(\varphi)$ must be either a variable or c_i. Then we have cases:

$$\text{either } \alpha(\varphi) \multimap_i \beta(\varphi) = x \multimap_i y \text{ or } \alpha(\varphi) \multimap_i \beta(\varphi) = x \multimap_i c_i \tag{3.3}$$

for some variables x and y. Now, from the assumption we know that z occurs in $\alpha(z) \multimap_i \beta(z)$. We claim that z does not appear really in $\alpha(z)$. To prove this, suppose the contrary towards a contradiction. By (3.3) we would have that $\varphi = x$. Then

$$h\varphi = h\alpha(\varphi) = \langle i, 0\rangle \notin \mathbb{R} \times \{1\}.$$

But this contradicts the fact that $\Gamma \vdash_{\mathcal{L}} \varphi$, establishing the claim. In particular, the claim implies that z occurs in $\beta(z)$. Together with (3.3), this means that $\beta(\varphi) = \varphi$. This easily implies that

$$h(\varphi) = h\beta(\varphi) = \langle i, 1\rangle = h(c_i).$$

Fact 3.9.3. *If φ is neither a variable nor a constant, then $\emptyset \vdash_{\mathcal{L}} \Box\varphi$.*

Observe that $\Box\Box x$ and $\Box(x \multimap_i y)$ are theorems of $\mathcal{L}$, for every $i \notin \{1,2\}$. This follows directly from the definition of $\mathcal{L}$. In particular, this implies that also $\Box\varphi$ is a theorem.

Fact 3.9.4. *For every $\Gamma \cup \{\varphi\} \subseteq Fm$ one of the following conditions hold:*

1. *For every $i < 2$ and formula $\alpha(z)$ in which z actually occurs: $\Gamma \vdash_{\mathcal{L}} \Box\alpha(\varphi) \iff \Gamma \vdash_{\mathcal{L}} \Box\alpha(c_i)$.*
2. *For every $i > 2$ and formula $\alpha(z)$ in which z actually occurs: $\Gamma \vdash_{\mathcal{L}} \Box\alpha(\varphi) \iff \Gamma \vdash_{\mathcal{L}} \Box\alpha(c_i)$.*

First consider the case in which $\Gamma \vdash_{\mathcal{L}} \Box\alpha(\varphi)$ for every formula $\alpha(z)$ in which z really occurs. We want to prove that condition 1 is satisfied. Then consider c_i with $i < 2$ and a formula $\alpha(z)$ in which z really occurs. If $\alpha(z)$ is not a variable, then $\Box\alpha(c_i)$ is a theorem by Fact 3.9.3. Then consider the case where $\alpha(z)$ is a variable. Clearly $\alpha = z$. Also in this case $\Box\alpha(c_i)$ is a theorem. Thus we conclude that condition 1 is satisfied.

Then consider the case where there is at least one formula $\alpha(z)$, in which z really occurs, such that $\Gamma \nvdash_{\mathcal{L}} \Box\alpha(\varphi)$. From Fact 3.9.3 it follows that $\alpha(z) = z$ and, therefore, that $\Gamma \nvdash_{\mathcal{L}} \varphi$. Together with Fact 3.9.3 this implies that for every formula $\beta(z)$, in which z really occurs, we have that

$$\Gamma \vdash_{\mathcal{L}} \Box\beta(\varphi) \iff \beta \neq z.$$

Then consider a constant c_i with $i > 2$ and a formula $\beta(z)$ in which z really occurs. From Fact 3.9.3 and from the definition of $\Box$ it follows that $\Gamma \vdash_{\mathcal{L}} \Box\beta(c_i)$ if and only if $\beta \neq z$. Thus condition 2 is satisfied.

Fact 3.9.5. *For every $\Gamma \in \mathcal{T}h\mathcal{L}$ and $\varphi \in \Gamma$, there is $i \in \mathbb{R} \smallsetminus \{2\}$ such that $\langle \varphi, c_i \rangle \in \boldsymbol{\Omega}\Gamma$.*

If Γ is the inconsistent theory, then $\boldsymbol{\Omega}\Gamma = Fm \times Fm$ and, therefore, we are done. Then consider the case where Γ is consistent. Suppose towards a contradiction that $\langle \varphi, c_i \rangle \notin \boldsymbol{\Omega}\Gamma$ for every $i \in \mathbb{R} \smallsetminus \{2\}$. From Lemma 1.2 it follows that for every $i \in \mathbb{R} \smallsetminus \{2\}$ there is a formula $p(z)$ such that

$$\Gamma \vdash_{\mathcal{L}} p(\varphi) \iff \Gamma \nvdash_{\mathcal{L}} p(c_i). \tag{3.4}$$

By Fact 3.9.4 one of the following conditions hold:

1. For every $i < 2$ and formula $\alpha(z)$ in which z actually occurs: $\Gamma \vdash_{\mathcal{L}} \Box\alpha(\varphi) \iff \Gamma \vdash_{\mathcal{L}} \Box\alpha(c_i)$.
2. For every $i > 2$ and formula $\alpha(z)$ in which z actually occurs: $\Gamma \vdash_{\mathcal{L}} \Box\alpha(\varphi) \iff \Gamma \vdash_{\mathcal{L}} \Box\alpha(c_i)$.

Assume that condition 1 holds (the proof for case 2 is analogous). Then consider $1 \neq i < 2$. There is a polynomial function $p(z)$ that satisfies (3.4). Thus z actually occurs in $p(z)$. By condition 1, we know that the main connective of $p(z)$ cannot be $\Box$. Therefore $p(z) = \alpha(z) \multimap_j \beta(z)$ for some $j \in \mathbb{R} \smallsetminus \{1, 2\}$ and formulas α and β. Together with Fact 3.9.2 and $\langle \varphi, c_i \rangle \notin \boldsymbol{\Omega}\varGamma$, this implies that

$$\varGamma \vdash_{\mathcal{L}} \alpha(c_i) \multimap_j \beta(c_i).$$

Since $\varGamma$ is consistent, there is a homomorphism $h \colon \boldsymbol{Fm} \to \boldsymbol{A}$ such that $h[\varGamma] \subseteq \mathbb{R} \times \{1\}$. We have that $h(\alpha(c_i) \multimap_j \beta(c_i)) \in \mathbb{R} \times \{1\}$. But this is to say that

$$h\alpha(c_i) = \langle j, 0 \rangle \text{ and } h\beta(c_i) = \langle j, 1 \rangle.$$

From the definition of $\multimap_j$ it follows that $\alpha(\varphi)$ must be a variable y_i and that $\beta(\varphi)$ must be either a variable or c_j (this is because $i \neq 1$). Since z actually occurs in $\alpha(z) \multimap_j \beta(z)$ we conclude that $\alpha(c_i) \multimap_j \beta(c_i) = y_i \multimap_j c_i$. Keeping in mind that $h(y_i \multimap_j c_i) \in \mathbb{R} \times \{1\}$, we obtain that $j = i$. Therefore we conclude that $\varGamma \vdash_{\mathcal{L}} y_i \multimap_i c_i$. Now, we proved that for every $1 \neq i < 2$ there is a variable y_i such that

$$\varGamma \vdash_{\mathcal{L}} y_i \multimap_i c_i. \tag{3.5}$$

Consider the homomorphism h above. From (3.5) it follows that $h(y_i) = \langle i, 0 \rangle$ for every $i < 2$. But this contradicts the fact that there are uncountably many reals smaller than 2 and different from 1 and only countably many variables in Fm.

Fact 3.9.6. *$\boldsymbol{\Omega}$ is order-reflecting (and therefore injective) over $\mathcal{T}h\mathcal{L}$.*

Consider two theories $\varGamma, \varGamma' \in \mathcal{T}h\mathcal{L}$ such that $\boldsymbol{\Omega}\varGamma \subseteq \boldsymbol{\Omega}\varGamma'$. Then pick $\varphi \in \varGamma$ (we can always do this, since $\mathcal{L}$ has theorems). By Fact 3.9.5 there is a constant c_i such that $\langle \varphi, c_i \rangle \in \boldsymbol{\Omega}\varGamma \subseteq \boldsymbol{\Omega}\varGamma'$. Since $\emptyset \vdash_{\mathcal{L}} c_i$, by compatibility we obtain that $\varphi \in \varGamma'$. Hence $\boldsymbol{\Omega}\varGamma \subseteq \boldsymbol{\Omega}\varGamma'$ as desired. ⊠

3.2 Small truth predicates

The last condition on the truth sets of a matrix semantics that we consider is the following:

Definition 3.10. Let M be a class of matrices and $\mathcal{L}$ the logic it defines. Truth is *small* in M, if $\min(\mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \smallsetminus \{\emptyset\})$ exists and coincides with F, for every $\langle \boldsymbol{A}, F \rangle \in$ M.

In general there are classes of matrices is which truth is small but not equationally definable (Example 3.14) and vice-versa (Example 3.15). The notion of *smallness* becomes better behaved if we restrict the attention to classes of matrices of the form $\mathsf{Mod}^*\mathcal{L}$ for some logic $\mathcal{L}$. In particular, we say that *the truth sets of logic $\mathcal{L}$ are small*, if truth is small in $\mathsf{Mod}^*\mathcal{L}$. Observe that the truth sets of $\mathcal{L}$ are small if and only if they are almost small and $\mathcal{L}$ has theorems. This means that, in order to prove that the truth sets of a logic with theorems are small, it is enough to prove that they are almost small. We have the following:

Lemma 3.11.

1. *If truth is (almost) small in* $\mathsf{Mod}^*\mathcal{L}$*, then it is (almost) implicitly definable in* $\mathsf{Mod}^*\mathcal{L}$*.*
2. *If truth is (almost universally) equationally definable in* $\mathsf{Mod}^*\mathcal{L}$*, then it is also (almost) small in* $\mathsf{Mod}^*\mathcal{L}$*.*

Proof. 1. Suppose that truth is almost small in $\mathsf{Mod}^*\mathcal{L}$. Then consider two non-almost trivial reduced models $\langle \boldsymbol{A}, F\rangle$ and $\langle \boldsymbol{A}, G\rangle$ of $\mathcal{L}$. Since truth is almost small in $\mathsf{Mod}^*\mathcal{L}$, we have that $F = \min(\mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \smallsetminus \{\emptyset\}) = G$.

2. Suppose that truth is almost universally definable in $\mathsf{Mod}^*\mathcal{L}$. Then consider a non-almost trivial reduced model $\langle \boldsymbol{A}, F\rangle$ of $\mathcal{L}$ and any $G \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \smallsetminus \{\emptyset\}$. We have that $\boldsymbol{\Omega}^{\boldsymbol{A}}F = 0_A \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}G$. With an application of Theorem 2.8 we obtain that $F \subseteq G$. Since $F \neq \emptyset$, we conclude that $F = \min(\mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \smallsetminus \{\emptyset\})$. ⊠

The smallness of truth sets can be characterized by means of the behaviour of the Leibniz operator as follows:

Theorem 3.12. *Truth is (almost) small in* $\mathsf{Mod}^*\mathcal{L}$ *if and only if* $\boldsymbol{\Omega}^{\boldsymbol{A}}$ *is (almost) order reflecting over* $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ *for every algebra* $\boldsymbol{A}$*.*

Proof. We begin by the "only if" part. Consider $F, G \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \smallsetminus \{\emptyset\}$ such that $\boldsymbol{\Omega}^{\boldsymbol{A}}F \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}G$. Let $h\colon \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F \to \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}G$ be the natural surjection. We have that $h^{-1}[G/\boldsymbol{\Omega}^{\boldsymbol{A}}G] \in \mathcal{F}i_{\mathcal{L}}(\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F)$. By the assumption we know that $F/\boldsymbol{\Omega}^{\boldsymbol{A}}F = \min(\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F \smallsetminus \{\emptyset\})$. Since $h^{-1}[G/\boldsymbol{\Omega}^{\boldsymbol{A}}G] \neq \emptyset$, we conclude that $F/\boldsymbol{\Omega}^{\boldsymbol{A}}F \subseteq h^{-1}[G/\boldsymbol{\Omega}^{\boldsymbol{A}}G]$. Then let $a \in F$. We have that $a/\boldsymbol{\Omega}^{\boldsymbol{A}}F \subseteq h^{-1}[G/\boldsymbol{\Omega}^{\boldsymbol{A}}G]$ and, therefore, $a \in G$.

Then we turn to check the "if" part. Consider a non-almost trivial reduced model $\langle \boldsymbol{A}, F\rangle$ of $\mathcal{L}$ and a filter $G \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \smallsetminus \{\emptyset\}$. We have that $\boldsymbol{\Omega}^{\boldsymbol{A}}F = 0_A \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}G$. By the assumption we obtain $F \subseteq G$. Since $F \neq \emptyset$, we conclude that $F = \min(\mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \smallsetminus \{\emptyset\})$. ⊠

It is natural to ask whether the order-reflection of the Leibniz operator transfers from theories to filters over arbitrary algebras. In Example 3.9 we already showed that this is not the case in general. Nevertheless, a natural adaptation of the proof of Theorem 3.8 yields the following result:

Theorem 3.13. *For logics expressed in a countable language the (almost) order-reflection of the Leibniz operator transfers from theories to filters over arbitrary algebras.*

Proof. We apply Theorem 3.12. Suppose that $\boldsymbol{\Omega}$ is order-reflecting over $\mathcal{T}h\mathcal{L}$. This easily implies that $\mathcal{L}$ has theorems and, therefore, that its deductive filters are non-empty. Consider a reduced model $\langle \boldsymbol{A}, F\rangle$ of $\mathcal{L}$. We have to prove that $F = \min \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \smallsetminus \{\emptyset\}$. Then consider $G \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \smallsetminus \{\emptyset\}$ and $a \in F$. We know that $G \neq \emptyset$. Thus we can choose an element $b \in G$. We let $\boldsymbol{C}$ be the subalgebra of $\boldsymbol{A}$, generated by $\{a, b\}$. Since $\boldsymbol{C}$ is finitely generated, we can apply Lemma 3.7 and extend it to a countably generated subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$ such that $\langle \boldsymbol{B}, F \cap B\rangle$ is a reduced model of $\mathcal{L}$.

Since $\boldsymbol{B}$ is countable, there is a surjective homomorphism $h\colon \boldsymbol{Fm} \to \boldsymbol{B}$. Then let $\Gamma := h^{-1}[F \cap B]$ and $\Gamma' := h^{-1}[G \cap B]$. Notice that $F \cap B$ and $G \cap B$ are non-empty, and hence Γ and Γ' are non-empty as well. Clearly $\Gamma, \Gamma' \in \mathcal{T}h\mathcal{L}$. From point 2 of Lemma 1.1 we obtain that

$$\boldsymbol{\Omega}\Gamma = \boldsymbol{\Omega}h^{-1}[F \cap B] = h^{-1}\boldsymbol{\Omega}^{\boldsymbol{B}}(F \cap B) = h^{-1}0_{\boldsymbol{B}} = \mathrm{Ker}(h).$$

Moreover, since $\mathrm{Ker}(h)$ is compatible with Γ', we know that $\boldsymbol{\Omega}\Gamma = \mathrm{Ker}(h) \subseteq \boldsymbol{\Omega}\Gamma'$. Hence we can apply the assumption and conclude that $\Gamma \subseteq \Gamma'$. Together with the fact that h is surjective, this implies that

$$F \cap B = hh^{-1}[F \cap B] = h[\Gamma] \subseteq h[\Gamma'] = hh^{-1}[G \cap B] \subseteq G \cap B.$$

Therefore we obtain that $a \in F \cap B \subseteq G \cap B \subseteq G$. This shows that $F \subseteq G$. ⊠

We already met tacitly the example of logic whose truth sets are implicitly definable but not small. More precisely, let $\mathcal{L}$ be the logic defined in Example 3.5. In Fact 3.5.3 we proved that the truth sets of $\mathcal{L}$ are implicitly definable. Moreover from Fact 3.5.2 and Theorem 3.12 it follows that the truth sets of $\mathcal{L}$ are not small. The next example present a logic whose truth sets as small but not equationally definable.

Example 3.14 (Small Truth)**.** Let $\mathcal{L}$ be the logic, expressed in the language $\langle \Box, 1\rangle$ of type $\langle 1, 0\rangle$, axiomatized by the following Hilbert-style rules:

$$\emptyset \vdash 1 \quad \emptyset \vdash \Box 1 \quad \Box\Box x \vdash y.$$

We will prove that truth is small, but not equationally definable, in $\mathsf{Mod}^*\mathcal{L}$. To this end, let $\boldsymbol{A}_4 = \langle \{a, b, c, 1\}, \Box, 1\rangle$ be the algebra where $\Box$ is defined for every $x \in A_4$ as follows:

$$\Box p = \begin{cases} a & \text{if } p \in \{1, c\} \\ b & \text{otherwise.} \end{cases}$$

Let $\boldsymbol{A}_3$ be the subalgebra of $\boldsymbol{A}_4$ with universe $\{1, a, b\}$.

Fact 3.14.1. *$\mathsf{Mod}^*\mathcal{L}$ is the closure under isomorphism of $\langle \boldsymbol{A}_4, \{1, a\}\rangle$, $\langle \boldsymbol{A}_3, \{1, a\}\rangle$ and the trivial matrix.*

The inclusion from right to left follows from the definition of $\mathcal{L}$. Then we turn to prove the other inclusion. Consider $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*\mathcal{L}$. If $\boldsymbol{A}$ is trivial, then also $\langle \boldsymbol{A}, F\rangle$ is trivial since $\mathcal{L}$ has theorems. The suppose that $\boldsymbol{A}$ is non-trivial. The fact that $\Box\Box x \vdash_{\mathcal{L}} y$ implies that that characterization of the Leibniz congruence given in point 1 of Lemma 1.2 can be finitized, yielding the following result: for every $p, q \in A$

$$p = q \text{ if and only if } (p \in F \Leftrightarrow q \in F \text{ and } \Box p \in F \Leftrightarrow \Box q \in F). \qquad (3.6)$$

Since $\emptyset \vdash_{\mathcal{L}} 1$ and $\emptyset \vdash_{\mathcal{L}} \Box 1$, we know that $1, \Box 1 \in F$. The facts that $\Box\Box x \vdash_{\mathcal{L}} y$ and that $\langle \boldsymbol{A}, F\rangle$ is non-trivial and reduced, imply that $\Box^n 1 \notin F$ for every $n \geqslant 2$. In particular, this implies that $\Box 1 \neq 1$ and $\Box^2 1 = \Box^3 1$ by (3.6). Now let $p \in F$. We have cases: either $\Box p \in F$ or $\Box p \notin F$. By (3.6) in both cases $p \in \{1, \Box 1\}$. Hence $F = \{1, \Box 1\}$.

If $A = \{1, \Box 1, \Box\Box 1\}$, then $\langle \boldsymbol{A}, F\rangle \cong \langle \boldsymbol{A}_3, \{1, a\}\rangle$. Then consider the case where $A \neq \{1, \Box 1, \Box\Box 1\}$. There is $p \in A \smallsetminus \{1, \Box 1, \Box\Box 1\}$. In particular, this yields that $p \notin F$. By (3.6) and the fact that $p \neq \Box\Box 1$, we know that $\Box p \in F$. Again, since $\Box\Box x \vdash_{\mathcal{L}} y$ and $\boldsymbol{A}$ is non-trivial, we obtain that $\Box\Box p \notin F$. Thus from (3.6) we conclude that $\Box p = \Box 1$.

If $A = \{1, \Box 1, \Box\Box 1, p\}$, then $\langle \boldsymbol{A}, F\rangle \cong \langle \boldsymbol{A}_4, \{1, a\}\rangle$. Suppose the contrary towards a contradiction. Then there is $q \in A \smallsetminus \{1, \Box 1, \Box\Box 1, c\}$. Since $F = \{1, \Box 1\}$, we know that $q \notin F$. But from (3.6) it follows that either $q = \Box\Box 1$ or $q = p$, against the assumption.

Fact 3.14.2. *Truth is small in* $\mathsf{Mod}^*\mathcal{L}$.

This is a direct application of the definition of $\mathcal{L}$ to the characterization of the class $\mathsf{Mod}^*\mathcal{L}$ given in Fact 3.14.1.

Fact 3.14.3. *Truth is not equationally definable in* $\mathsf{Mod}^*\mathcal{L}$.

Observe that the terms in one variable x up to equivalence in $\mathbb{V}(\boldsymbol{A}_4)$ are $\{x, \Box x, \Box\Box x, 1, \Box 1\}$. It is easy to check that $x \approx x$ is the only equation, built up with these terms, that is satisfied by the designated elements of $\langle \boldsymbol{A}_4, \{1, a\}\rangle$. Since $\langle \boldsymbol{A}_4, \{1, a\}\rangle$ is a reduced model of $\mathcal{L}$, we conclude that truth is not equationally definable in $\mathsf{Mod}^*\mathcal{L}$. ⊠

Example 3.15 (Non-Small Truth)**.** We will construct a matrix in which truth is equationally definable, but not small. Let $\boldsymbol{A} = \langle \{a, b, c, 1\}, \Box, 1\rangle$ be the algebra equipped with the unary operation $\Box$ defined for every $p \in A$ as

$$\Box p = \begin{cases} 1 & \text{if } p \in \{a, b\} \\ c & \text{otherwise.} \end{cases}$$

Let $\mathcal{L}$ be the logic determined by the matrix $\langle \boldsymbol{A}, \{a, b\}\rangle$.

Fact 3.15.1. *The equation* $\Box x \approx 1$ *defines truth in* $\langle \boldsymbol{A}, \{a, b\}\rangle$.

Fact 3.15.2. *$\mathcal{L}$ is axiomatized by the rules $\Box x \vdash y$ and $1 \vdash y$.*

Clearly these rules are sound for $\mathcal{L}$. Then we turn to prove completeness. Let $\mathcal{L}'$ be the logic determined by the two rules above. Consider $\Gamma \cup \{\varphi\} \subseteq Fm$ such that $\Gamma \vdash_{\mathcal{L}} \varphi$. Observe that $\Gamma \neq \emptyset$, since $\mathcal{L}$ is purely inferential. We have two cases: either Γ is a set of variables or not. In the first case, taking a look at the matrix $\langle \boldsymbol{A}, \{a, b\} \rangle$, it is easy to see that $\varphi \in \Gamma$. Thus $\Gamma \vdash_{\mathcal{L}'} \varphi$. In the second case, there is $\gamma \in \Gamma$ that is not a variable. Hence either $\gamma = 1$ or $\gamma = \Box^n x$ for some variable x and some $n \geqslant 1$. In both cases the rules axiomatizing $\mathcal{L}'$ yield that $\gamma \vdash_{\mathcal{L}'} \varphi$, therefore $\Gamma \vdash_{\mathcal{L}'} \varphi$ and we are done.

Fact 3.15.3. *Truth is not small in $\langle \boldsymbol{A}, \{a, b\} \rangle$.*

By means of the Hilbert-style axiomatization of $\mathcal{L}$, it is easy to prove that $\{a\} \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$. Thus we conclude that truth is not small in $\langle \boldsymbol{A}, \{a, b\} \rangle$. ⊠

In Figure 3.1 the reader can find a diagram (where arrows represent inclusions) that subsume the definability conditions considered so far in the framework of the Leibniz hierarchy. It is worth to remark that all classes in that diagram are different.

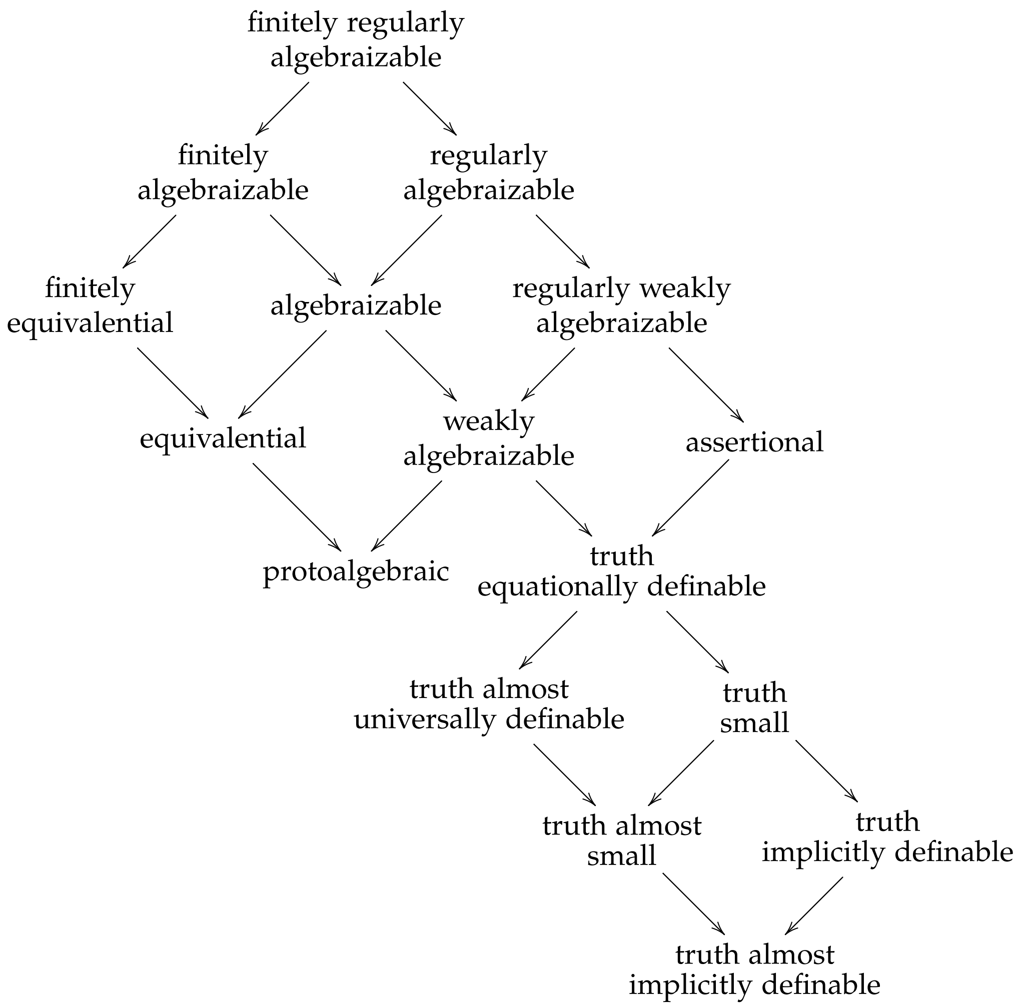


Figure 3.1: The Leibniz hierarchy

CHAPTER 4

Computational aspects

This chapter studies several computational aspects of the problem of classifying logics within the Leibniz and Frege hierarchies. Since logics can be presented *syntactically* and *semantically*, we have studied the problem for these two kind of logics separately. Accordingly, on the one hand we will consider the following problems:

- Let K be a level of the Leibniz (resp. Frege) hierarchy. Is it possible to decide whether the logic of a given finite consistent Hilbert calculus in a finite language belongs to K?

It turns out that in general the answer is negative both for the Leibniz and the Frege hierarchies. To show the first case, we reduce Hilbert's tenth problem on Diophantine equations to the problem of classifying the logic of a (finite consistent) Hilbert calculus in the Leibniz hierarchy, thus obtaining that also the last one is undecidable (Theorem 4.10). In the process we will also describe and axiomatize a new logic, whose deductions mimic the equational theory of commutative rings with unit (Theorem 4.6). In order to prove that also the problem of classifying the logic of a (finite and consistent) Hilbert calculus in the Frege hierarchy is undecidable, we rely on the undecidability of the equational theory of relation algebras in a single variable. Remarkably, our proof shows that this classification problem remains undecidable even if we restrict our attention to Hilbert calculi that determine a finitary algebraizable logic (Theorem 4.23).

On the other hand, we consider the semantic version of the same problem:

- Let K be a level of the Leibniz (resp. Frege) hierarchy. Is it possible to decide whether the logic of a given strongly finite set of matrices belongs to K?

In this case the situation is different and the change is due to the fact that most levels of the Leibniz hierarchy admit a characterization in terms of the existence of sets of formulas (or equations) with certain properties. Building on the observation that the finitely generated free algebras over a given finite set of finite algebras of finite type can de described mechanically, we will be able to prove or disprove algorithmically the existence of those sets of formulas (or equations). As a consequence we obtain that the problem of classifying logics semantically presented inside the main classes of the Leibniz hierarchy is decidable (Theorem 4.17). The situation becomes more delicate if we shift our attention to the Frege hierarchy. This is because not all the replacement principles typical of the Frege hierarchy can be expressed as a requirement on formulas: some of them involve separation properties that refer to the behaviour of deductive filters on arbitrary algebras. An argument analogous to the one devised for the Leibniz hierarchy allows to prove that it is possible to determine whether a semantically presented logic enjoys the replacement principles of the Frege hierarchy that refer only to the behaviour of formulas (Theorem 4.24 and Corollary 4.25). But it is still an open question (Problem 4) to solve the same problem for the remaining classes of the Frege hierarchy (see also Problem 3). The majority of the results of this chapter are contained in [78].

4.1 The classification problem in the Leibniz hierarchy

We will begin our work on some computational aspects of abstract algebraic logic by considering the problem of classifying logics in the Leibniz hierarchy. In this section the expression *Leibniz hierarchy* will refer to the hierarchy depicted in Figure 3.1. In particular, we will focus on two problems. Let K be a level of the Leibniz hierarchy. The first one asks, given a finite and consistent Hilbert-style calculus H in a finite language, to decide whether the logic determined by H belongs to K (Theorem 4.10). The second one asks, given a strongly finite set of matrices M, to decide whether the logic determined by M belongs to K (Theorem 4.17). Accordingly, we chose to divide the rest of the section into two parts that correspond respectively to the analysis of the first and of the second problem.

The syntactic case

The main outcome of this part is that the problem of classifying logics determined by a finite consistent Hilbert calculus in a finite language in the Leibniz hierarchy is in general undecidable (Theorem 4.10). Our strategy is the following: for every Diaphantine equation $p \approx 0$ we define a finite consistent Hilber calculus $\mathcal{L}(p)$ such that $\mathcal{L}(p)$ belongs to a given level of the Leibniz hierarchy if and only if $p \approx 0$ has an integer solution. For the sake of completeness we recall some concepts. An algebra $\boldsymbol{A} = \langle A, +, \cdot, -, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ is a *commutative ring* if $\langle A, +, -, 0\rangle$ is an Abelian group,

$\langle A, \cdot, 1\rangle$ is a commutative monoid, and

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

for every $a, b, c \in A$. We denote by CR the variety of commutative rings. The algebra $\mathbb{Z} := \langle Z, +, \cdot, -, 0, 1\rangle$, where Z is the collection of integer numbers, is a commutative ring. A *Diophantine equation* is an equation of the form $p(z_1, \ldots, z_1) \approx 0$, where $p(z_1, \ldots, z_1)$ is a term in the language of commutative rings. Hilbert's tenth problem asked for an algorithm that, given a Diophantine equation, tells us whether it has a solution in $\mathbb{Z}$ or not. It is well known that such an algorithm does not exist (see for example [12]). In other words, it turned out that the problem of determining whether a given Diophantine equation has an integer solution is undecidable.

In order to relate this problem to the that of the classification of logics into the Leibniz hierarchy, we will construct a logic that mimics the behaviour of (Diophantine) equations in commutative rings. In [42, 43] a way of doing this for arbitrary varieties is described. More precisely, given a non-trivial variety V, we let $\mathcal{L}_{\mathsf{V}}$ be the logic determined by the following class of matrices:

$$\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{V} \text{ and } F \subseteq A\}.$$

Given $\Gamma \cup \{\varphi\} \subseteq Fm$, we will write $\Gamma \vdash_{\mathsf{V}} \varphi$ as a shortening for $\Gamma \vdash_{\mathcal{L}_{\mathsf{V}}} \varphi$. The following result will be used later on:

Lemma 4.1. *Let V be a non-trivial variety and $\Gamma \cup \{\varphi\} \subseteq Fm$.*

1. $\mathsf{Alg}^*\mathcal{L}_{\mathsf{V}} \subseteq \mathsf{Alg}\mathcal{L}_{\mathsf{V}} = \mathsf{V}$.
2. *$\mathcal{L}_{\mathsf{V}}$ is fully selfextensional.*
3. *$\Gamma \vdash_{\mathsf{V}} \varphi$ if and only if there is $\gamma \in \Gamma$ such that $\mathsf{V} \vDash \gamma \approx \varphi$.*

Proof. 1. It is a general fact that $\mathsf{Alg}^*\mathcal{L}_{\mathsf{V}} \subseteq \mathsf{Alg}\mathcal{L}_{\mathsf{V}}$, while the inclusion $\mathsf{Alg}\mathcal{L}_{\mathsf{V}} \subseteq \mathsf{V}$ is a consequence of Lemma 1.3. Therefore it only remains to prove that $\mathsf{V} \subseteq \mathsf{Alg}\mathcal{L}_{\mathsf{V}}$. Since $\mathsf{Alg}\mathcal{L}_{\mathsf{V}} = \mathbb{P}_{\text{SD}}\mathsf{Alg}^*\mathcal{L}_{\mathsf{V}}$, it will be enough to prove that $\mathsf{V}_{si} \subseteq \mathsf{Alg}^*\mathcal{L}_{\mathsf{V}}$. To this end, consider a non-trivial $\boldsymbol{A} \in \mathsf{V}_{si}$. We know that there are two different elements $a, b \in A$ such that $\mathrm{Cg}^{\boldsymbol{A}}(a, b)$ is the monolith of $\boldsymbol{A}$. It is easy to see that the matrix $\langle \boldsymbol{A}, \{a\}\rangle$ is reduced. Since $\langle \boldsymbol{A}, \{a\}\rangle$ is a model of $\mathcal{L}_{\mathsf{V}}$, we conclude that $\boldsymbol{A} \in \mathsf{Alg}^*\mathcal{L}_{\mathsf{V}}$ as desired.

2. Recall that a logic $\mathcal{L}$ is fully selfextensional if and only if for every $\boldsymbol{A} \in \mathsf{Alg}\mathcal{L}$ and $a, b \in A$:

$$Fi^{\boldsymbol{A}}_{\mathcal{L}}\{a\} = Fi^{\boldsymbol{A}}_{\mathcal{L}}\{b\} \Longleftrightarrow a = b.$$

By point 1 we know that $\mathsf{Alg}\mathcal{L}_{\mathsf{V}} = \mathsf{V}$. Moreover, for every $\boldsymbol{A} \in \mathsf{V}$ we have that $\mathcal{F}i_{\mathcal{L}_{\mathsf{V}}}\boldsymbol{A} = \mathcal{P}(A)$. In particular, this implies that $Fi^{\boldsymbol{A}}_{\mathcal{L}_{\mathsf{V}}}\{a\} = \{a\}$ for every $a \in A$. Therefore the characterization of full selfextensionality given above is trivially satisfied.

3. The "if" part follows from the fact that $\mathcal{L}_{\mathsf{V}}$ is defined through a class of matrices whose algebraic reducts live in V. Now we turn to prove the "only if" part. Suppose that $\Gamma \vdash_{\mathsf{V}} \varphi$. Then consider the natural surjection $\pi \colon \boldsymbol{Fm} \to \boldsymbol{Fm}_{\mathsf{V}}(Var)$. We denote the elements of the free algebra $\boldsymbol{Fm}_{\mathsf{V}}(Var)$ by $[\![\gamma]\!]$, since they are congruence classes of equivalent formulas. The matrix $\mathcal{M} = \langle \boldsymbol{Fm}_{\mathsf{V}}(Var), \{[\![\gamma]\!] : \gamma \in \Gamma\}\rangle$ is a model of $\mathcal{L}_{\mathsf{V}}$ by definition. Clearly h maps Γ into the filter of $\mathcal{M}$. Thus the same happens for φ. But this means that $[\![\gamma]\!] = [\![\varphi]\!]$ for some $\gamma \in \Gamma$. Since $\boldsymbol{Fm}_{\mathsf{V}}(Var)$ is the free algebra of $\mathcal{L}_{\mathsf{V}}$, we conclude that $\mathsf{V} \vDash \gamma \approx \varphi$. ⊠

Digression. Next we are going to axiomatize the logic $\mathcal{L}_{\mathsf{CR}}$, but first let us consider the general problem of axiomatizing the logic $\mathcal{L}_{\mathsf{V}}$ for an arbitrary non-trivial variety V. Suppose that we are given an equational basis $\mathcal{V}$ for V. It would be nice to have a natural way of transforming $\mathcal{V}$ into an axiomatization of the logic $\mathcal{L}_{\mathsf{V}}$. This can be done easily, adapting Birkhoff's Completeness Theorem of equational logic [24, Theorem 14.19, Section II], if what we are looking for is a Gentzen system adequate to $\mathcal{L}_{\mathsf{V}}$ [43, Theorem 1.2].* On the contrary, there is no natural way of building a nice Hilbert calculus for $\mathcal{L}_{\mathsf{V}}$ out of $\mathcal{V}$. In particular, the obvious idea of considering the logic axiomatized by the rules $\alpha \dashv\vdash \beta$ for all $\alpha \approx \beta \in \mathcal{V}$ does not work in general. For example, it is possible to see that the logic $\mathcal{S}$ determined by the rules

$$x \dashv\vdash x \wedge x \qquad x \wedge y \dashv\vdash y \wedge x \qquad x \wedge (y \wedge z) \dashv\vdash (x \wedge y) \wedge z \tag{4.1}$$

is not the logic $\mathcal{L}_{\mathsf{SL}}$ associated with the variety of semilattices SL (cfr. [42, Example 3.8]). This is because the matrix $\langle \mathbb{Z}_3, \{1, 2\}\rangle$, where $\mathbb{Z}_3$ is the additive semigroup of integers modulo 3, is easily proved to be a reduced model of $\mathcal{S}$. In particular this implies that $\mathbb{Z}_3 \in \mathsf{Alg}\mathcal{S}$ and, therefore, that $\mathsf{Alg}\mathcal{S} \neq \mathsf{SL}$. Hence $\mathcal{S} \neq \mathcal{L}_{\mathsf{SL}}$, by point 1 of Lemma 4.1. In order to obtain a complete Hilbert-style axiomatization of $\mathcal{L}_{\mathsf{SL}}$ one has to add to the rules in (4.1) the following ones:

$$\begin{gathered} u \wedge x \dashv\vdash u \wedge (x \wedge x) \\ u \wedge (x \wedge y) \dashv\vdash u \wedge (y \wedge x) \\ u \wedge (x \wedge (y \wedge z)) \dashv\vdash u \wedge ((x \wedge y) \wedge z) \end{gathered}$$

This is mainly due to the fact that selfextensionality, which is not easily expressible by means of Hilbert-style rules, fails for $\mathcal{S}$, while $\mathcal{L}_{\mathsf{SL}}$ is selfextensional by point 2 of Lemma 4.1. Even if it is not straightforward to present an explicit Hilbert-style axiomatization of $\mathcal{L}_{\mathsf{V}}$, given a base $\mathcal{V}$ for V, one may wonder whether there exists (no matter which one) a finite Hilbert-style axiomatization of $\mathcal{L}_{\mathsf{V}}$ when $\mathcal{V}$ is finite. In the next example we show that in general this is not the case.

*Even if we won't pursue this here, it is possible to show that there is no Gentzen system fully adequate to $\mathcal{L}_{\mathsf{V}}$.

Example 4.2 (Finite Axiomatizability)**.** An algebra $\boldsymbol{A} = \langle A, \cdot \rangle$ is a *commutative magma* if $\cdot$ is a binary commutative operation. Clearly the class of commutative magmas forms a finitely based variety, which we denote by CM. We will prove that the logic $\mathcal{L}_{\mathsf{CM}}$ is not axiomatizable by means of a finite set of Hilbert-style rules. In order to do this, let $\mathcal{CM}$ be a finite set of rules holding in $\mathcal{L}_{\mathsf{CM}}$. We will show that there is a model of $\mathcal{CM}$ that is not a model of $\mathcal{L}_{\mathsf{CM}}$. First observe that there is a natural number n that bounds the number of occurrences of (possibly equal) variables in terms appearing in the rules of $\mathcal{CM}$. We can assume, without loss of generality, that $n \geqslant 2$. Then we consider the algebra $\boldsymbol{A} = \langle \{0, 1, 2, \dots, n\}, \cdot \rangle$ equipped with a binary operation such that $1 \cdot 2 := 2$ and $2 \cdot 1 := 1$ and

$$a \cdot b = b \cdot a := \begin{cases} a & \text{if } a \neq n \text{ and } b = 0 \\ 0 & \text{if } a = n \text{ and } b = 0 \\ a & \text{if } b = a - 1 \text{ and } a \geqslant 3 \\ a - 1 & \text{if } b = a - 2 \text{ and } a \geqslant 3 \\ 1 & \text{otherwise} \end{cases}$$

for every $a, b \in A$ such that $\{a, b\} \neq \{1, 2\}$.

We first show that $\langle \boldsymbol{A}, \{0\} \rangle$ is not a model of $\mathcal{L}_{\mathsf{CM}}$. Observe that $\boldsymbol{A} \notin \mathsf{CM}$, since $1 \cdot 2 \neq 2 \cdot 1$. By point 1 of Lemma 4.1 we know that it will be enough to prove that $\langle \boldsymbol{A}, \{0\} \rangle$ is a reduced matrix. By point 1 of Lemma 1.2 this amounts to checking whether for every different $a, b \in A \smallsetminus \{0\}$ there is a polynomial function $p \colon \boldsymbol{A} \to \boldsymbol{A}$ such that $p(a) = 0$ if and only if $p(b) \neq 0$. This is what we do now: consider a pair of different $a, b \in A \smallsetminus \{0\}$. Assume, without loss of generality, that $a < b$. Then we consider the polynomial function

$$p(x) := (\dots(((\dots((\dots((1 \cdot 2) \cdot 3) \cdot \dots) \cdot a) \cdot \dots \cdot b - 1) \cdot x) \cdot b + 1) \cdot \dots \cdot n) \cdot 0.$$

It is easy to see that $p(b) = 0$. Then we turn to show that $p(a) \neq 0$. We consider two cases, whether $b - 1 < 3$ or not. First consider the case in which $b - 1 < 3$. We have that either ($a = 1$ and $b = 2$) or ($a = 1$ and $b = 3$) or ($a = 2$ and $b = 3$). It is easy to prove that

$$\begin{aligned} &\text{if } a = 1 \text{ and } b = 2, \text{ then } p(a) = n - 1 \\ &\text{if } a = 1 \text{ and } b = 3, \text{ then } p(a) = 1 \\ &\text{if } a = 2 \text{ and } b = 3, \text{ then } p(a) = 1. \end{aligned}$$

Then we turn to the case in which $3 \leqslant b - 1$. We have that:

$$p(a) = (\dots((b - 1 \cdot a) \cdot b + 1) \dots \cdot n) \cdot 0 = \begin{cases} n & \text{if } a = b - 2 \\ 1 & \text{otherwise.} \end{cases}$$

Therefore we obtain that $p(a) \neq 0$. This concludes the proof that $\langle \boldsymbol{A}, \{0\} \rangle$ is not a model of $\mathcal{L}_{\mathsf{CM}}$.

Now we turn to prove that $\langle \boldsymbol{A}, \{0\}\rangle$ is a model of $\mathcal{CM}$. Consider a rule $\Gamma \vdash \varphi$ in $\mathcal{CM}$. Then pick a homomorphism $h\colon \boldsymbol{Fm} \to \boldsymbol{A}$ such that $h[\Gamma] \subseteq \{0\}$. From point 3 of Lemma 4.1 we know that there is $\gamma \in \Gamma$ such that $\mathsf{CM} \vDash \gamma \approx \varphi$. In particular, we have that $h(\gamma) = 0$. We claim that $\{h(\varepsilon), h(\delta)\} \neq \{1, 2\}$ for every subformula $\varepsilon \cdot \delta$ of γ. Suppose the contrary towards a contradiction. Then there is a subformula $\varepsilon \cdot \delta$ of γ such that $\{h(\varepsilon), h(\delta)\} = \{1, 2\}$. Since at most n (possibly equal) variables occur in γ, and $\varepsilon \cdot \delta$ contains at least two of them, we know that if we draw the subformulas tree of γ there are at most $n - 2$ nodes $\gamma_1 < \gamma_2 < \cdots < \gamma_{n-2}$ (with $\gamma_{n-2} = \gamma$) strictly above $\varepsilon \cdot \delta$. Looking at the definition of $\cdot$ it is possible to see that $1 \leqslant h(\gamma_m) \leqslant m + 2$ for every $m \leqslant n - 2$. Therefore we obtain that $1 \leqslant h(\gamma) \leqslant n$, against the assumption that $h(\gamma) = 0$. This concludes the proof of our claim.

Now, recall that $\mathsf{CM} \vDash \gamma \approx \varphi$. It is easy to prove by induction on the length of the proofs of Birkhoff's equational logic that φ is obtained from γ in the following way. We replace a subformula $\gamma_1 \cdot \gamma_2$ of γ by $\gamma_2 \cdot \gamma_1$ and denote by γ' the formula obtained in this way. Then we repeat this process on γ'. Iterating this construction a finite number of times we reach φ. Now, observe that the operation $\cdot$ in $\boldsymbol{A}$ commutes always except for the case in which its arguments exhaust the set $\{1, 2\}$. This fact, together with our claim and the observation on equational logic, implies that $h(\gamma) = h(\varphi)$. In particular, this means that $h(\varphi) = 0$. Hence $\langle \boldsymbol{A}, \{0\}\rangle$ is a model of the rules in $\mathcal{CM}$. ⊠

End of digression

The digression shows that the quest for a finite Hilbert-style axiomatization of the logic $\mathcal{L}_{\mathsf{CR}}$ is not in principle a trivial one: such an axiomatization may even fail to exist, as in the case of commutative magmas of Example 4.2. Nevertheless we will provide an explicit and finite Hilbert calculus for the logic $\mathcal{L}_{\mathsf{CR}}$ (Theorem 4.6). To this end we introduce the following

Definition 4.3. Let $\mathcal{CR}$ be the following Hilbert calculus and the logic it determines in the language of commutative rings:

$$w + (u \cdot ((x \cdot y) \cdot z)) \dashv\vdash w + (u \cdot (x \cdot (y \cdot z))) \tag{A}$$
$$w + (u \cdot (x \cdot y)) \dashv\vdash w + (u \cdot (y \cdot x)) \tag{B}$$
$$w + (u \cdot (x \cdot 1)) \dashv\vdash w + (u \cdot x) \tag{C}$$
$$w + (u \cdot ((x + y) + z)) \dashv\vdash w + (u \cdot (x + (y + z))) \tag{D}$$
$$w + (u \cdot (x + y)) \dashv\vdash w + (u \cdot (y + x)) \tag{E}$$
$$w + (u \cdot (x + 0)) \dashv\vdash w + (u \cdot x) \tag{F}$$
$$w + (u \cdot (x + -x)) \dashv\vdash w + (u \cdot 0) \tag{G}$$
$$w + (u \cdot (x \cdot (y + z))) \dashv\vdash w + (u \cdot ((x \cdot y) + (x \cdot z))) \tag{H}$$
$$w + (u \cdot -(x + y)) \dashv\vdash w + (u \cdot (-x + -y)) \tag{I}$$

$$w + (u \cdot -(x \cdot y)) \dashv\vdash w + (u \cdot (-x \cdot y)) \quad \text{(L)}$$
$$w + (u \cdot -(x \cdot y)) \dashv\vdash w + (u \cdot (x \cdot -y)) \quad \text{(M)}$$
$$0 + x \dashv\vdash x \quad \text{(N)}$$
$$x + (1 \cdot y) \dashv\vdash x + y \quad \text{(O)}$$

The following technical result on the special nature of deductions of $\mathcal{CR}$, will be useful in the subsequent proofs.

Lemma 4.4. *Let $\Gamma \cup \{\varphi\} \subseteq Fm$. If $\Gamma \vdash_{\mathcal{CR}} \varphi$, then there is $\gamma \in \Gamma$ and a finite sequence of formulas $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ such that $\alpha_1 = \gamma$, $\alpha_n = \varphi$ and for every $m < n$ there is a rule $\varepsilon \dashv\vdash \delta$ of the calculus $\mathcal{CR}$ and a substitution σ such that $\{\sigma\varepsilon, \sigma\delta\} = \{\alpha_m, \alpha_{m+1}\}$.*

Proof. The result follows from two observations. First, that the calculus $\mathcal{CR}$ consists of rules with only one premise. Second, that if $\alpha \vdash \beta \in \mathcal{CR}$, then $\beta \vdash \alpha \in \mathcal{CR}$ too. ⊠

Our main goal will be to prove that $\mathcal{CR}$ is in fact a finite axiomatization of $\mathcal{L}_{\mathsf{CR}}$. To do this we will make use of the following lemma, whose easy proof involves some tedious calculations. The reader may safely choose to skip it in order keep track of the proof of the main result of the section.

Lemma 4.5. *The logic $\mathcal{CR}$ is selfextensional.*

Proof. First we check that $\dashv\vdash_{\mathcal{CR}}$ preserves the connective $-$. To do this, consider $\psi, \varphi \in Fm$ such that $\psi \dashv\vdash_{\mathcal{CR}} \varphi$: we have to prove that $-\psi \dashv\vdash_{\mathcal{CR}} -\varphi$. By Lemma 4.4 it will be enough to check that $-\alpha \dashv\vdash_{\mathcal{CR}} -\beta$ for every $\alpha \dashv\vdash \beta \in \mathcal{CR}$. We shall make use of some deductions. In particular, by suitable substitutions and applying (N) and (O) to (B), (D), (E), (H), (I), (L) and (M), we obtain that

$$x \cdot y \dashv\vdash_{\mathcal{CR}} y \cdot x \quad \text{(B')}$$
$$(x + y) + z \dashv\vdash_{\mathcal{CR}} x + (y + z) \quad \text{(D')}$$
$$x + y \dashv\vdash_{\mathcal{CR}} y + x \quad \text{(E')}$$
$$x \cdot (y + z) \dashv\vdash_{\mathcal{CR}} (x \cdot y) + (x \cdot z) \quad \text{(H')}$$
$$-(x + y) \dashv\vdash_{\mathcal{CR}} -x + -y \quad \text{(I')}$$
$$w + -(x \cdot y) \dashv\vdash_{\mathcal{CR}} w + (-x \cdot y) \quad \text{(L')}$$
$$w + -(x \cdot y) \dashv\vdash_{\mathcal{CR}} w + (x \cdot -y) \quad \text{(M')}$$

Then suppose that we are given a rule of our calculus (X), different from (N) and (O). Then (X) is of the form $w + (u \cdot \varepsilon) \dashv\vdash w + (u \cdot \delta)$ for some formulas

ε and δ. But we have that

$$
\begin{aligned}
-(w+(u\cdot\varepsilon)) &\dashv\vdash_{\mathcal{CR}} -w+-(u\cdot\varepsilon) && \text{(I}'\text{)}\\
&\dashv\vdash_{\mathcal{CR}} -w+(-u\cdot\varepsilon) && \text{(L}'\text{)}\\
&\dashv\vdash_{\mathcal{CR}} -w+(-u\cdot\delta) && \text{(X)}\\
&\dashv\vdash_{\mathcal{CR}} -w+-(u\cdot\delta) && \text{(L}'\text{)}\\
&\dashv\vdash_{\mathcal{CR}} -(w+(u\cdot\delta)) && \text{(I}'\text{)}
\end{aligned}
$$

Therefore it only remains to prove the cases of (N) and (O). This is what we do now:

$$
\begin{aligned}
-(0+x) &\dashv\vdash_{\mathcal{CR}} 0+-(0+x) && \text{(N)}\\
&\dashv\vdash_{\mathcal{CR}} 0+(1\cdot-(0+x)) && \text{(O)}\\
&\dashv\vdash_{\mathcal{CR}} 0+-(1\cdot(0+x)) && \text{(M}'\text{)}\\
&\dashv\vdash_{\mathcal{CR}} 0+(-1\cdot(0+x)) && \text{(L}'\text{)}\\
&\dashv\vdash_{\mathcal{CR}} 0+(-1\cdot x) && \text{(F)}\\
&\dashv\vdash_{\mathcal{CR}} 0+-(1\cdot x) && \text{(L}'\text{)}\\
&\dashv\vdash_{\mathcal{CR}} 0+(1\cdot -x) && \text{(M}'\text{)}\\
&\dashv\vdash_{\mathcal{CR}} 0+-x && \text{(O)}\\
&\dashv\vdash_{\mathcal{CR}} -x && \text{(N)}
\end{aligned}
$$

and

$$
\begin{aligned}
-(x+(1\cdot y)) &\dashv\vdash_{\mathcal{CR}} -x+-(1\cdot y) && \text{(I}'\text{)}\\
&\dashv\vdash_{\mathcal{CR}} -x+(1\cdot -y) && \text{(M}'\text{)}\\
&\dashv\vdash_{\mathcal{CR}} -x+-y && \text{(o)}\\
&\dashv\vdash_{\mathcal{CR}} -(x+y) && \text{(I}'\text{)}
\end{aligned}
$$

Therefore we conclude that $\dashv\vdash_{\mathcal{CR}}$ preserves $-$.

Then we turn to prove that $\dashv\vdash_{\mathcal{CR}}$ preserves $+$ and $\cdot$ too. In order to do this, it will be enough to show that if $\psi \dashv\vdash_{\mathcal{CR}} \varphi$, then $\chi+\psi \dashv\vdash_{\mathcal{CR}} \chi+\varphi$ and $\chi\cdot\psi \dashv\vdash_{\mathcal{CR}} \chi\cdot\varphi$ for every formula χ. Let us explain briefly why. Suppose that this condition, call it (Y), holds. Then consider $\psi_1, \psi_2, \varphi_1, \varphi_2 \in Fm$ such that $\psi_1 \dashv\vdash_{\mathcal{CR}} \varphi_1$ and $\psi_2 \dashv\vdash_{\mathcal{CR}} \varphi_2$. We would have that

$$
\begin{aligned}
\psi_1+\psi_2 &\dashv\vdash_{\mathcal{CR}} \psi_1+\varphi_2 && \text{(Y)}\\
&\dashv\vdash_{\mathcal{CR}} \varphi_2+\psi_1 && \text{(E}'\text{)}\\
&\dashv\vdash_{\mathcal{CR}} \varphi_2+\varphi_1 && \text{(Y)}\\
&\dashv\vdash_{\mathcal{CR}} \varphi_1+\varphi_2 && \text{(E}'\text{)}
\end{aligned}
$$

and

$$\begin{aligned}
\psi_1 \cdot \psi_2 &\dashv\vdash_{\mathcal{CR}} \psi_1 \cdot \varphi_2 && \text{(Y)}\\
&\dashv\vdash_{\mathcal{CR}} \varphi_2 \cdot \psi_1 && \text{(B}'\text{)}\\
&\dashv\vdash_{\mathcal{CR}} \varphi_2 \cdot \varphi_1 && \text{(Y)}\\
&\dashv\vdash_{\mathcal{CR}} \varphi_1 \cdot \varphi_2 && \text{(B}'\text{)}
\end{aligned}$$

concluding the proof. Therefore we turn to prove that if $\psi \dashv\vdash_{\mathcal{CR}} \varphi$, then $\chi + \psi \dashv\vdash_{\mathcal{CR}} \chi + \varphi$ and $\chi \cdot \psi \dashv\vdash_{\mathcal{CR}} \chi \cdot \varphi$ for every formula χ. Suppose that $\psi \dashv\vdash_{\mathcal{CR}} \varphi$ and consider an arbitrary formula χ. By Lemma 4.4, to prove that $\chi + \psi \dashv\vdash_{\mathcal{CR}} \chi + \varphi$ and $\chi \cdot \psi \dashv\vdash_{\mathcal{CR}} \chi \cdot \varphi$, it will be enough to check that $\chi + \alpha \dashv\vdash_{\mathcal{CR}} \chi + \beta$ and $\chi \cdot \alpha \dashv\vdash_{\mathcal{CR}} \chi \cdot \beta$ for every rule $\alpha \dashv\vdash \beta$ in $\mathcal{CR}$. Then suppose that we are given a rule (X) of $\mathcal{CR}$, different from (N) and (O). Then (X) is of the form $w + (u \cdot \varepsilon) \dashv\vdash w + (u \cdot \delta)$ for some formulas ε and δ. We have that

$$\begin{aligned}
\chi + (w + (u \cdot \varepsilon)) &\dashv\vdash_{\mathcal{CR}} (\chi + w) + (u \cdot \varepsilon) && \text{(D}'\text{)}\\
&\dashv\vdash_{\mathcal{CR}} (\chi + w) + (u \cdot \delta) && \text{(X)}\\
&\dashv\vdash_{\mathcal{CR}} \chi + (w + (u \cdot \delta)) && \text{(D}'\text{)}
\end{aligned}$$

and

$$\begin{aligned}
\chi \cdot (w + (u \cdot \varepsilon)) &\dashv\vdash_{\mathcal{CR}} (\chi \cdot w) + (\chi \cdot (u \cdot \varepsilon)) && \text{(H}'\text{)}\\
&\dashv\vdash_{\mathcal{CR}} (\chi \cdot w) + (1 \cdot (\chi \cdot (u \cdot \varepsilon))) && \text{(O)}\\
&\dashv\vdash_{\mathcal{CR}} (\chi \cdot w) + (1 \cdot ((\chi \cdot u) \cdot \varepsilon)) && \text{(A)}\\
&\dashv\vdash_{\mathcal{CR}} (\chi \cdot w) + ((\chi \cdot u) \cdot \varepsilon) && \text{(O)}\\
&\dashv\vdash_{\mathcal{CR}} (\chi \cdot w) + ((\chi \cdot u) \cdot \delta) && \text{(X)}\\
&\dashv\vdash_{\mathcal{CR}} (\chi \cdot w) + (1 \cdot ((\chi \cdot u) \cdot \delta)) && \text{(O)}\\
&\dashv\vdash_{\mathcal{CR}} (\chi \cdot w) + (1 \cdot (\chi \cdot (u \cdot \delta))) && \text{(A)}\\
&\dashv\vdash_{\mathcal{CR}} (\chi \cdot w) + (\chi \cdot (u \cdot \delta)) && \text{(O)}\\
&\dashv\vdash_{\mathcal{CR}} \chi \cdot (w + (u \cdot \delta)) && \text{(H}'\text{)}
\end{aligned}$$

Therefore it only remains to check cases (N) and (O). For what concerns (N) we have that:

$$\begin{aligned}
\chi + (0 + x) &\dashv\vdash_{\mathcal{CR}} (0 + x) + \chi && \text{(E}'\text{)}\\
&\dashv\vdash_{\mathcal{CR}} 0 + (x + \chi) && \text{(D}'\text{)}\\
&\dashv\vdash_{\mathcal{CR}} x + \chi && \text{(N)}\\
&\dashv\vdash_{\mathcal{CR}} \chi + x && \text{(E}'\text{)}
\end{aligned}$$

and

$$\begin{aligned}
\chi \cdot (0 + x) &\dashv\vdash_{\mathcal{CR}} 0 + (\chi \cdot (0 + x)) && \text{(N)}\\
&\dashv\vdash_{\mathcal{CR}} 0 + (\chi \cdot (x + 0)) && \text{(E)}\\
&\dashv\vdash_{\mathcal{CR}} 0 + (\chi \cdot x) && \text{(F)}\\
&\dashv\vdash_{\mathcal{CR}} \chi \cdot x && \text{(N)}
\end{aligned}$$

Then we turn to prove the case of (O). We have that

$$\begin{aligned}
\chi + (x + (1 \cdot y)) &\dashv\vdash_{\mathcal{CR}} (\chi + x) + (1 \cdot y) && \text{(D}'\text{)}\\
&\dashv\vdash_{\mathcal{CR}} (\chi + x) + y && \text{(O)}\\
&\dashv\vdash_{\mathcal{CR}} \chi + (x + y) && \text{(D}'\text{)}
\end{aligned}$$

and

$$\begin{aligned}
\chi \cdot (x + (1 \cdot y)) &\dashv\vdash_{\mathcal{CR}} (\chi \cdot x) + (\chi \cdot (1 \cdot y)) && \text{(H}'\text{)}\\
&\dashv\vdash_{\mathcal{CR}} (\chi \cdot x) + (\chi \cdot (y \cdot 1)) && \text{(B)}\\
&\dashv\vdash_{\mathcal{CR}} (\chi \cdot x) + (\chi \cdot y) && \text{(C)}\\
&\dashv\vdash_{\mathcal{CR}} \chi \cdot (x + y) && \text{(H}'\text{)}
\end{aligned}$$

This concludes the proof that $\mathcal{CR}$ is selfextensional. ⊠

Drawing consequences from the fact that $\mathcal{CR}$ is selfextensional, the following result is easy to obtain.

Theorem 4.6. *The rules $\mathcal{CR}$ provide a finite axiomatization of $\mathcal{L}_{\mathsf{CR}}$.*

Proof. First observe that each of the rules in $\mathcal{CR}$ corresponds to an equation, which holds in CR. Together with point 3 of Lemma 4.1, this implies that $\mathcal{CR} \leqslant \mathcal{L}_{\mathsf{CR}}$. Now, applying (N) and (O), it is easy to prove that

$$\begin{aligned}
(x \cdot y) \cdot z &\dashv\vdash_{\mathcal{CR}} x \cdot (y \cdot z)\\
x \cdot y &\dashv\vdash_{\mathcal{CR}} y \cdot x\\
x \cdot 1 &\dashv\vdash_{\mathcal{CR}} x\\
(x + y) + z &\dashv\vdash_{\mathcal{CR}} x + (y + z)\\
x + y &\dashv\vdash_{\mathcal{CR}} y + x\\
x + 0 &\dashv\vdash_{\mathcal{CR}} x\\
x + -x &\dashv\vdash_{\mathcal{CR}} 0\\
x \cdot (y + z) &\dashv\vdash_{\mathcal{CR}} (x \cdot y) + (x \cdot z)
\end{aligned}$$

Since $\mathcal{CR}$ is selfextensional by Lemma 4.5, we can apply Lemma 1.14 obtaining $\mathsf{Alg}\mathcal{CR} \subseteq \mathsf{CR}$. On the other hand, from point 1 of Lemma 4.1 and the fact that $\mathcal{CR} \leqslant \mathcal{L}_{\mathsf{CR}}$, it follows that $\mathsf{CR} = \mathsf{Alg}\mathcal{L}_{\mathsf{CR}} \subseteq \mathsf{Alg}\mathcal{CR}$. Therefore we conclude that $\mathsf{Alg}\mathcal{CR} = \mathsf{CR}$. Since for every $\boldsymbol{A} \in \mathsf{CR}$ and $F \subseteq A$ the matrix $\langle \boldsymbol{A}, F \rangle$ is a model of $\mathcal{L}_{\mathsf{CR}}$, this implies that $\mathcal{L}_{\mathsf{CR}} \leqslant \mathcal{CR}$. Thus we conclude that $\mathcal{CR} = \mathcal{L}_{\mathsf{CR}}$ as desired. ⊠

Once we have built the machinery necessary to speak about commutative rings by means of the propositional logic $\mathcal{L}_{\mathsf{CR}}$, we turn back to the classification of logics determined by finite Hilbert calculi in the Leibniz hierarchy. We begin by describing a general way of associating a logic with each Diophantine equation.

Definition 4.7. Let $p(z_1, \ldots, z_n) \approx 0$ be a Diophantine equation and x and y two new different variables. Let $\mathcal{L}(p)$ be the logic, in the language of commutative rings expanded with a new binary operation symbol $\leftrightarrow$, determined by the following Hilbert calculus:

$$\emptyset \vdash x \leftrightarrow x \quad \text{(R)}$$
$$x \leftrightarrow y \vdash y \leftrightarrow x \quad \text{(S)}$$
$$x \leftrightarrow y, y \leftrightarrow z \vdash x \leftrightarrow z \quad \text{(T)}$$
$$x \leftrightarrow y \vdash -x \leftrightarrow -y \quad \text{(Rep1)}$$
$$x \leftrightarrow y, z \leftrightarrow u \vdash (x+z) \leftrightarrow (y+u) \quad \text{(Rep2)}$$
$$x \leftrightarrow y, z \leftrightarrow u \vdash (x \cdot z) \leftrightarrow (y \cdot u) \quad \text{(Rep3)}$$
$$x \leftrightarrow y, z \leftrightarrow u \vdash (x \leftrightarrow z) \leftrightarrow (y \leftrightarrow u) \quad \text{(Rep4)}$$
$$p(z_1, \ldots, z_n) \leftrightarrow 0, x, x \leftrightarrow y \vdash y \quad \text{(MP')}$$
$$p(z_1, \ldots, z_n) \leftrightarrow 0, x \dashv\vdash x \leftrightarrow (x \leftrightarrow x), p(z_1, \ldots, z_n) \leftrightarrow 0 \quad \text{(A3')}$$
$$p(z_1, \ldots, z_n) \leftrightarrow 0, x, y \vdash x \leftrightarrow y \quad \text{(G')}$$
$$\emptyset \vdash \alpha \leftrightarrow \beta \quad \text{(CR)}$$

for every $\alpha \dashv\vdash \beta \in \mathcal{CR}$.

Observe that $\mathcal{L}(p)$ is determined by an explicit finite Hilbert calculus in a finite language. It turns out that there is a strong relation between the existence of an integer solution to $p(z_1, \ldots, z_n) \approx 0$ and the location of $\mathcal{L}(p)$ in the Leibniz hierarchy.

Theorem 4.8. *Let $p(z_1, \ldots, z_n) \approx 0$ be a Diophantine equation. The following conditions are equivalent:*

(i) *$\mathcal{L}(p)$ is finitely regularly algebraizable.*

(ii) *Truth is almost implicitly definable in $\mathsf{Mod}^*\mathcal{L}(p)$.*

(iii) *$\mathcal{L}(p)$ is protoalgebraic.*

(iv) *The equation $p(z_1, \ldots, z_n) \approx 0$ has an integer solution.*

Proof. Clearly (i) implies (ii) and (iii). Now we turn to prove (ii)⇒(iv). We reason by contraposition. Suppose that $p(z_1, \ldots, z_n) \approx 0$ has no integer solution. Now choose two different integers n and m. Then let $\boldsymbol{Z}$ be the expansion of $\mathbb{Z}$ with a new binary function $\leftrightarrow$ defined as follows:

$$a \leftrightarrow b := \begin{cases} n & \text{if } a = b \\ m & \text{otherwise} \end{cases}$$

for every $a, b \in Z$. Pick $k \notin \{n, m\}$. It is easy to check that $\langle \boldsymbol{Z}, \{n\}\rangle$ and $\langle \boldsymbol{Z}, \{n, k\}\rangle$ are models of $\mathcal{L}(p)$, since $p(z_1, \ldots, z_n) \approx 0$ has no integer solution. Moreover they are reduced. In order to see this, pick two different $a, b \in Z$ and consider the polynomial function $q(z) := a \leftrightarrow z$. We have that $q(a) = n$ and that $q(b) = m \notin \{n, k\}$. By point 1 of Lemma 1.2 we conclude both that $\langle a, b\rangle \notin \boldsymbol{\Omega}^{\boldsymbol{Z}}\{n\}$ and that $\langle a, b\rangle \notin \boldsymbol{\Omega}^{\boldsymbol{Z}}\{n, k\}$. But this means that there are two different reduced and non-almost trivial models of $\mathcal{L}(p)$ with the same algebraic reduct and, therefore, that truth is not almost implicitly definable in $\mathsf{Mod}^{*}\mathcal{L}(p)$.

Now we prove (iii)⇒(iv). Suppose that $\mathcal{L}(p)$ is protoalgebraic. Then there is a set of protoimplication formulas $\boldsymbol{\rho}(x, y)$. In particular, we have that $x, \boldsymbol{\rho}(x, y) \vdash_{\mathcal{L}(p)} y$. Then there is a finite proof π of y from the premises in $\{x\} \cup \boldsymbol{\rho}(x, y)$. Taking a closer look at the axiomatization of $\mathcal{L}(p)$, it is easy to see that either an application of (MP′) or of (A3′) must occur in π. This is because the other rules yield complex conclusions. In particular, this implies that there is a substitution σ such that $x, \boldsymbol{\rho}(x, y) \vdash_{\mathcal{L}(p)} \sigma p(z_1, \ldots, z_n) \leftrightarrow \sigma 0$. From Theorem 1.5 we know that $\emptyset \vdash_{\mathcal{L}(p)} \boldsymbol{\rho}(x, x)$. In particular, this means that $x \vdash_{\mathcal{L}(p)} \sigma_x \sigma p(z_1, \ldots, z_n) \leftrightarrow \sigma_x \sigma 0$, where σ_x is the substitution which sends all variables to x. But observe that every substitution leaves 0 fixed. Therefore we conclude that

$$x \vdash_{\mathcal{L}(p)} \sigma_x \sigma p(z_1, \ldots, z_n) \leftrightarrow 0. \tag{4.2}$$

Now we consider the algebra $\boldsymbol{Z}$ built in the proof of part (ii)⇒(iv). It is easy to check that $\langle \boldsymbol{Z}, \{n\}\rangle$ is a model of $\mathcal{L}(p)$. Therefore, pick an homomorphism $h \colon \boldsymbol{Fm} \to \boldsymbol{Z}$ that sends x to n. From (4.2) it follows that $h(\sigma_x \sigma p(z_1, \ldots, z_n) \leftrightarrow 0) = n$. But observe that

$$\begin{aligned} h(\sigma_x \sigma p(z_1, \ldots, z_n) \leftrightarrow 0) = n &\iff h(p(\sigma_x \sigma(z_1), \ldots, \sigma_x \sigma(z_n)) \leftrightarrow 0) = n \\ &\iff p^{\boldsymbol{Z}}(h\sigma_x \sigma(z_1), \ldots, h\sigma_x \sigma(z_n)) \leftrightarrow h0 = n \\ &\iff p^{\boldsymbol{Z}}(h\sigma_x \sigma(z_1), \ldots, h\sigma_x \sigma(z_n)) = h0 \\ &\iff p^{\boldsymbol{Z}}(h\sigma_x \sigma(z_1), \ldots, h\sigma_x \sigma(z_n)) = 0. \end{aligned}$$

Therefore we conclude that $h\sigma_x \sigma(z_1), \ldots, h\sigma_x \sigma(z_n)$ is an integer solution to the equation $p(z_1, \ldots, z_n) \approx 0$.

It only remains to prove (iv)⇒(i). Suppose that the equation $p(z_1, \ldots, z_n) \approx 0$ admits an integer solution. Recall that the free commutative ring with free generators $\{z_1, \ldots, z_n\}$ is the polynomial ring $\mathbb{Z}[z_1, \ldots, z_n]$. Since $\mathbb{Z}$ is a subalgebra of $\mathbb{Z}[z_1, \ldots, z_n]$, this implies that there are constant terms (in the language of commutative rings) $\alpha_1, \ldots, \alpha_n$ such that $\mathbb{Z}[z_1, \ldots, z_n] \vDash p(\alpha_1, \ldots, \alpha_n) \approx 0$ and, therefore, that $\mathsf{CR} \vDash p(\alpha_1, \ldots, \alpha_n) \approx 0$. From point 3 of Lemma 4.1 it follows that $p(\alpha_1, \ldots, \alpha_n) \dashv\vdash_{\mathsf{CR}} 0$. By Theorem 4.6 this is equivalent to the fact that $p(\alpha_1, \ldots, \alpha_n) \dashv\vdash_{\mathcal{CR}} 0$. Therefore we can apply Lemma 4.4 obtaining a finite sequence $\langle \gamma_1, \gamma_2, \ldots, \gamma_m\rangle$, where $\gamma_1 = p(\alpha_1, \ldots, \alpha_n)$,

$\gamma_m = 0$ and for every $k < m$ there is a rule $\alpha \dashv\vdash \beta \in \mathcal{CR}$ and a substitution σ (in the language of commutative rings) such that $\{\sigma\alpha, \sigma\beta\} = \{\gamma_k, \gamma_{k+1}\}$. For each such $k < m$, α, β and σ consider the substitution σ in the language of $\mathcal{L}(p)$. Recall that $\alpha \leftrightarrow \beta$ is an axiom of $\mathcal{L}(p)$ by (CR). Therefore, by structurality, we obtain $\emptyset \vdash_{\mathcal{L}(p)} \sigma\alpha \leftrightarrow \sigma\beta$. Applying (S) if necessary, this yields that $\emptyset \vdash_{\mathcal{L}(p)} \gamma_k \leftrightarrow \gamma_{k+1}$. Hence we proved that $\emptyset \vdash_{\mathcal{L}(p)} \gamma_k \leftrightarrow \gamma_{k+1}$ for every $k \leqslant m - 1$. Applying $m - 1$ times (T) we conclude that

$$\emptyset \vdash_{\mathcal{L}(p)} p(\alpha_1, \ldots, \alpha_n) \leftrightarrow 0. \tag{4.3}$$

Recall that the variables x and y do not appear in the equation $p(z_1, \ldots, z_n) \approx 0$ therefore, we can safely consider the substitution σ that sends z_i to α_i for every $i \leqslant n$ and leaves the other variables untouched. By (MP'), (A3'), (G') and (4.3) we obtain that

$$x, x \leftrightarrow y \vdash_{\mathcal{L}(p)} y \quad x \dashv\vdash_{\mathcal{L}(p)} x \leftrightarrow (x \leftrightarrow x) \text{ and } x, y \vdash_{\mathcal{L}(p)} x \leftrightarrow y.$$

Now it is easy to see that what we have proved is just the syntactic characterization of algebraizability of Theorem 1.9 for $\boldsymbol{\tau}(x) := \{x \approx x \leftrightarrow x\}$ and $\boldsymbol{\rho}(x, y) := \{x \leftrightarrow y\}$. Therefore, with an application of Lemma 1.11, we conclude that $\mathcal{L}(p)$ is finitely regularly algebraizable. ⊠

Corollary 4.9. *The logic $\mathcal{L}(p)$ is consistent for every Diophantine equation $p \approx 0$.*

Proof. It is easy to see that the matrix $\langle \mathbf{Z}, \{n\} \rangle$ defined in the above proof is a model of $\mathcal{L}(p)$. Since $\{n\} \neq Z$, we conclude that $\mathcal{L}(p)$ is consistent. ⊠

Observe that every class of the Leibniz hierarchy is contained either into the class of protoalgebraic logic or into the one of logics whose truth sets are almost implicitly definable. Moreover every class of the Leibniz hierarchy is contained into the one of finitely regularly algebraizable logics. Keeping this is mind, Theorem 4.8 shows that the problem of determining whether a logic of the form $\mathcal{L}(p)$ belong to a *given* level of the Leibniz hierarchy is equivalent to the one of determining whether it belong to any level of the Leibniz hierarchy. This does not contradicts the fact that *in general* these two problems are different. By means of this peculiar feature of logics of the form $\mathcal{L}(p)$, we are able to establish at once the undecidability of the various problems (one for each level of the Leibniz hierarchy) of determining whether a logic belong to a given level of the Leibniz hierarchy.

Theorem 4.10. *Let K be a level of the Leibniz hierarchy in Figure 3.1. The problem of determining whether the logic of a given consistent finite Hilbert calculus in a finite language belongs to K is undecidable.*

Proof. Suppose towards a contradiction that there is an algorithm A_1 that, given a consistent finite Hilbert calculus in a finite language, determines whether its logic belongs to K. Then we define a new algorithm A_2 as follows:

given a Diophantine equation $p \approx 0$, we construct the logic $\mathcal{L}(p)$ and check with A_1 whether it belongs to K. Observe that we can do this, since $\mathcal{L}(p)$ is consistent by Corollary 4.9. Then in the positive case A_2 returns *yes*, while *no* otherwise. Observe that K contains the class of finitely regularly algebraizable logics and is included either in the class of protoalgebraic logics or in the class of logics whose truth sets are almost implicitly definable. Therefore we can apply Theorem 4.8, yielding that:

$$\mathcal{L}(p) \in \mathsf{K} \Longleftrightarrow p \approx 0 \text{ has an integer solution.}$$

Therefore A_2 would provide a decision procedure for Hilbert's tenth problem. Since we know that such a procedure does not exist, we obtain a contradiction as desired. ⊠

The semantic case

We saw that the problem of classifying syntactically presented logics in the Leibniz hierarchy is in general undecidable. If we move the attention from syntax to semantics, we obtain a completely opposite situation. More precisely, we will prove that for most levels K of the Leibniz hierarchy there is an algorithm that determines whether the logic defined by a strongly finite set of matrices belongs to K (Theorem 4.17). While doing this, we will provide also some algorithmic constructions of syntactic objects related to each level of the Leibniz hierarchy: for example if the logic of a strongly finite set of matrices is protoalgebraic, then we will describe a mechanical way of constructing a set of protoimplication and a set of congruence formulas with parameters for it. The proofs of these results rely on the well-known fact that there is an algorithm that, given a finite set of finite algebras $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n$ of finite type and a natural $k \in \omega$, calculates a set of representatives for the elements of $\boldsymbol{Fm}_{\mathsf{V}}(k)$, where V is the variety generated by $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n$. The Universal Algebra Calculator [46] contains an implementation of this algorithm. Building on it, we implemented the algorithms described in this section and developed a freely available software application [79]. In what follows we will identify systematically the universe of $\boldsymbol{Fm}_{\mathsf{V}}(k)$ with the set of representatives.

We begin our analysis by the classes of protoalgebraic and equivalential logics, which admit a syntactic characterization (Theorem 1.5).

Lemma 4.11. *There is an algorithm that determines whether the logic of a given strongly finite set of matrices is protoalgebraic and in the positive case returns a set of protoimplication formulas $\boldsymbol{\rho}(x, y)$ and a set of congruence formulas with parameters $\boldsymbol{\mu}(x, y, \vec{z})$ for it.*

Proof. We split our algorithm into two parts. The first one is intended to answer the question of whether the logic of a strongly finite set of matrices is protoalgebraic or not and to provide a set of protoimplication formulas for it in the positive case. Let `protoimplication` be the algorithm defined as follows. Given a strongly finite set of matrices M, we construct the finite set of terms $Fm_{\mathsf{M}}(x,y)$. Then we compute the set

$$\boldsymbol{\rho}(x,y) := \{\alpha \in Fm_{\mathsf{M}}(x,y) : \emptyset \vdash_{\mathcal{L}} \alpha(x,x)\}$$

where $\mathcal{L}$ is the logic of M. Finally we check whether the deduction $x, \boldsymbol{\rho}(x,y) \vdash y$ holds in $\mathcal{L}$. In the positive case our algorithm returns *yes* and $\boldsymbol{\rho}(x,y)$, while *no* otherwise. Now, recall from Theorem 1.5 that a logic is protoalgebraic if and only if it has a set of protoimplication formulas. Keeping this in mind it is an easy exercise to check that `protoimplication` works as expected.

In order to complete the proof, it will be enough to build a new algorithm that provides a set of congruence formulas with parameters for $\mathcal{L}$ in case `protoimplication` outputs *yes*. We define this new algorithm `p-congruence` as follows. Given a strongly finite set of matrices M, we run `protoimplication`. If it outputs *no*, then our new algorithm outputs *no* too. Then suppose `protoimplication` outputs *yes* and provides $\boldsymbol{\rho}(x,y)$ as set of protoimplication formulas. We define

$$t := \max\{|A| : \langle \boldsymbol{A}, F\rangle \in \mathsf{M}\}$$

and construct the set of terms $Fm_{\mathsf{M}}(z_1,\dots,z_{t+1})$. Then we construct the set

$$\begin{aligned}\boldsymbol{\mu}(x,y,\vec{z}) :=\{&\alpha(\varphi(\vec{z},x),\varphi(\vec{z},y)) : \alpha(x,y) \in \boldsymbol{\rho}(x,y)\\ &\text{and } \varphi(\vec{z},z_{t+1}) \in Fm_{\mathsf{M}}(z_1,\dots,z_{t+1})\}.\end{aligned}$$

Then our algorithm outputs *yes* and $\boldsymbol{\mu}(x,y,\vec{z})$.

It only remains to prove that `p-congruence` works as expected. If $\mathcal{L}$ is not protoalgebraic, then `p-congruence` returns *no*. Then consider the case in which $\mathcal{L}$ is protoalgebraic. In this case `protoimplication` returns *yes* and a set of protoimplication formulas $\boldsymbol{\rho}(x,y)$ for $\mathcal{L}$. In this case `p-congruence` returns a set of formulas $\boldsymbol{\mu}(x,y,z_1,\dots,z_t)$. Therefore it will be enough to prove that this is a set of congruence formulas with parameters for $\mathcal{L}$. To do this, recall from Lemma 1.6 that

$$\Delta(x,y,\vec{z}) := \{\alpha(\varphi(\vec{z},x),\varphi(\vec{z},y)) : \alpha(x,y) \in \boldsymbol{\rho}(x,y) \text{ and } \varphi(\vec{z},x) \in Fm\}$$

is a set of congruence formulas with parameters for $\mathcal{L}$, since $\boldsymbol{\rho}(x,y)$ is a set of protoimplication formulas for $\mathcal{L}$. Then we claim that $\boldsymbol{\mu}\langle x,y\rangle \dashv\vdash_{\mathcal{L}} \Delta\langle x,y\rangle$. Obviously, $\boldsymbol{\mu}(x,y,z_1,\dots,z_t) \subseteq \Delta(x,y,\vec{z})$ and, therefore, $\Delta\langle x,y\rangle \vdash_{\mathcal{L}} \boldsymbol{\mu}\langle x,y\rangle$. Then we turn to prove the other direction. Let $\beta \in \Delta\langle x,y\rangle$. We want to prove that $\boldsymbol{\mu}\langle x,y\rangle \vdash_{\mathcal{L}} \beta$. In order to do this, observe that by definition of $\Delta(x,y,\vec{z})$ there are $\alpha(x,y) \in \boldsymbol{\rho}(x,y)$, $\varphi(v_1,\dots,v_k,x) \in Fm$ and $\delta_1,\dots,\delta_k \in Fm$ such that $\beta = \alpha(\varphi(\delta_1,\dots,\delta_k,x),\varphi(\delta_1,\dots,\delta_k,y))$. Then consider $\langle \boldsymbol{A}, F\rangle \in \mathsf{M}$ and a homomorphism $h\colon \boldsymbol{Fm} \to \boldsymbol{A}$ such that $h[\boldsymbol{\mu}\langle x,y\rangle] \subseteq F$. We have that

$$h(\beta) = \alpha^{\boldsymbol{A}}(\varphi^{\boldsymbol{A}}(h(\delta_1),\dots,h(\delta_k),h(x)),\varphi^{\boldsymbol{A}}(h(\delta_1),\dots,h(\delta_k),h(y))). \tag{4.4}$$

Since $|A| \leqslant t$, there are $\gamma_1, \dots, \gamma_n \in \{\delta_1, \dots, \delta_k\}$ with $n \leqslant t$ and a surjective function $f \colon \{1, \dots, k\} \to \{1, \dots, n\}$ such that $h(\delta_m) = h(\gamma_{f(m)})$ for every $1 \leqslant m \leqslant k$. Pick n variables $z_1, \dots, z_n$ different from x and y. We consider the formula $\psi(z_1, \dots, z_n, x)$ obtained by replacing in $\varphi(v_1, \dots, v_k, x)$ the variable v_m by $z_{f(m)}$ for every $1 \leqslant m \leqslant k$. Now observe that $\alpha(\psi(\vec{z}, x), \psi(\vec{z}, y)) \in \boldsymbol{\mu}(x, y, z_1, \dots, z_t)$ and therefore that

$$\alpha(\psi(\gamma_1, \dots, \gamma_n, x), \psi(\gamma_1, \dots, \gamma_n, y)) \in \boldsymbol{\mu}\langle x, y\rangle.$$

Together with (4.4), this implies that $h(\beta) \in F$. But this means that $\boldsymbol{\mu}\langle x, y\rangle \vdash_{\mathcal{L}} \beta$, thus concluding the proof of our claim.

Applying Lemma 1.7 to our claim and to the fact that $\Delta(x, y, \vec{z})$ is a set of congruence formulas with parameters for $\mathcal{L}$, we conclude that $\boldsymbol{\mu}(x, y, z_1, \dots, z_t)$ is a set of congruence formulas with parameters for $\mathcal{L}$ too. ⊠

Lemma 4.12. *There is an algorithm that determines whether the logic of a given strongly finite set of matrices is equivalential and in the positive case returns a set of congruence formulas* $\boldsymbol{\rho}(x, y)$ *for it.*

Proof. Consider the algorithm defined as follows. Given a strongly finite set of matrices M, we construct the set of formulas $Fm_{\mathsf{M}}(x, y)$. Then we check whether there is a subset $\boldsymbol{\rho}(x, y) \subseteq Fm_{\mathsf{M}}(x, y)$ that satisfies the conditions of point 2 of Theorem 1.5 for the logic of M. If this is the case, then our algorithm returns *yes* and $\boldsymbol{\rho}(x, y)$. Otherwise it outputs *no*. ⊠

Now we move our attention to logics whose truth sets are almost universally definable. We will make use of the following technical result:

Lemma 4.13. *Let* $\mathcal{L}$ *be complete w.r.t. a finite set* M *of finite matrices,* $n = |\boldsymbol{Fm}_{\mathsf{M}}(x, y)|$ *and* $\boldsymbol{\tau}(x, y_1, \dots, y_n) := Fm_{\mathsf{M}}(x, y_1, \dots, y_n)^2 \cap \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}} C_{\mathcal{L}}\{x\}$. *Truth is almost universally definable in* $\mathsf{Mod}^*\mathcal{L}$ *if and only if for every* $\Gamma \in \mathcal{F}i_{\mathcal{L}} \boldsymbol{Fm}_{\mathsf{M}}(x, y) \smallsetminus \{\emptyset\}$ *and* $\varphi \in Fm_{\mathsf{M}}(x, y)$*:*

$$\varphi \in \Gamma \iff \boldsymbol{\tau}(\varphi, \vec{\gamma}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}_{\mathsf{M}}(x,y)} \Gamma \text{ for every } \vec{\gamma} \in Fm_{\mathsf{M}}(x, y).$$

In this case $\boldsymbol{\tau}$ *almost defines truth in* $\mathsf{Mod}^*\mathcal{L}$.

Proof. We begin by the "only if" part. Suppose that truth is almost universally definable in $\mathsf{Mod}^*\mathcal{L}$. By Theorem 2.8 this fact is witnessed by the universal translation $\boldsymbol{\tau}'(x, \vec{z}) := \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}} C_{\mathcal{L}}\{x\}$ where $\vec{z} = \langle z_k : k \in \omega \rangle$ is the infinite tuple of all variables different from x. Then consider $\Gamma \in \mathcal{F}i_{\mathcal{L}} \boldsymbol{Fm}_{\mathsf{M}}(x, y) \smallsetminus \{\emptyset\}$ and $\varphi \in Fm_{\mathsf{M}}(x, y)$. We have that

$$\varphi \in \Gamma \iff \boldsymbol{\tau}'(\varphi, \vec{\gamma}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}_{\mathsf{M}}(x,y)} \Gamma \text{ for every } \vec{\gamma} \in Fm_{\mathsf{M}}(x, y). \tag{4.5}$$

First suppose that $\varphi \in \Gamma$. By (4.5) we obtain that for every $\vec{\gamma} \in Fm_{\mathsf{M}}(x, y)$:

$$\boldsymbol{\tau}(\varphi, \vec{\gamma}) \subseteq \boldsymbol{\tau}'(\varphi, \vec{\gamma}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}_{\mathsf{M}}(x,y)} \Gamma.$$

Now suppose that $\boldsymbol{\tau}(\varphi, \vec{\gamma}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}_{\mathsf{M}}(x,y)}\Gamma$ for every $\vec{\gamma} \in Fm_{\mathsf{M}}(x,y)$. We want to prove that $\varphi \in \Gamma$. By (4.5) it will be enough to show that $\boldsymbol{\tau}'(\varphi, \vec{\gamma}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}_{\mathsf{M}}(x,y)}\Gamma$ for every $\vec{\gamma} \in Fm_{\mathsf{M}}(x,y)$. To this end, consider $\varepsilon \approx \delta \in \boldsymbol{\tau}'$ and $\vec{\gamma} \in Fm_{\mathsf{M}}(x,y)$. Observe that the sequence $\vec{\gamma}$ contains at most n distinct formulas $\delta_1, \ldots, \delta_n$. Then consider the substitution σ such that

$$\sigma(v) := \begin{cases} x & \text{if } v = x \\ y_i & \text{if } v = z_k \text{ and } \gamma_k = \delta_i \end{cases}$$

for every variable v. Define $\varepsilon'(x, y_1, \ldots, y_n) := \sigma\varepsilon$ and $\delta'(x, y_1, \ldots, y_n) := \sigma\delta$. We have that

$$\varepsilon'(\varphi, \gamma_1, \ldots, \gamma_n) = \varepsilon(\varphi, \vec{\gamma}) \text{ and } \delta'(\varphi, \gamma_1, \ldots, \gamma_n) = \delta(\varphi, \vec{\gamma}).$$

From Lemma 2.6 it follows that:

$$\varepsilon' \approx \delta' \in \sigma\boldsymbol{\tau}'(x, \vec{z}) = \sigma\widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}C_{\mathcal{L}}\{x\} \subseteq \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}C_{\mathcal{L}}\{\sigma x\} = \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}C_{\mathcal{L}}\{x\} = \boldsymbol{\tau}'(x, \vec{z}).$$

Since ε' and δ' are in variables $x, y_1, \ldots, y_n$ there are formulas $\varepsilon'', \delta'' \in Fm_{\mathsf{M}}(x, y_1, \ldots, y_n)$ such that $\boldsymbol{A} \vDash \{\varepsilon' \approx \varepsilon'', \delta' \approx \delta''\}$ for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{M}$. In particular, this implies that $\varepsilon'' \approx \delta'' \in \boldsymbol{\tau}'$ and, therefore, $\varepsilon'' \approx \delta'' \in \boldsymbol{\tau}$. By the assumption we have that

$$\langle \varepsilon''(\varphi, \gamma_1, \ldots, \gamma_n), \delta''(\varphi, \gamma_1, \ldots, \gamma_n)\rangle \in \boldsymbol{\Omega}^{\boldsymbol{Fm}_{\mathsf{M}}(x,y)}\Gamma.$$

Since $\boldsymbol{A} \vDash \{\varepsilon''(\varphi, \gamma_1, \ldots, \gamma_n) \approx \varepsilon(\varphi, \vec{\gamma}), \delta''(\varphi, \gamma_1, \ldots, \gamma_n) \approx \delta(\varphi, \vec{\gamma})\}$ for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{M}$, this implies that $\langle \varepsilon(\varphi, \vec{\gamma}), \delta(\varphi, \vec{\gamma})\rangle \in \boldsymbol{\Omega}^{\boldsymbol{Fm}_{\mathsf{M}}(x,y)}\Gamma$.

Now we turn to prove the "if" part. From Lemma 2.5 it will be enough to prove that for every $\Gamma \in \mathcal{T}h\mathcal{L} \smallsetminus \{\emptyset\}$ and $\varphi \in Fm$:

$$\Gamma \vdash_{\mathcal{L}} \varphi \Longleftrightarrow \boldsymbol{\tau}(\varphi, \vec{\gamma}) \subseteq \boldsymbol{\Omega}\Gamma \text{ for every } \vec{\gamma} \in Fm.$$

First suppose that $\Gamma \vdash_{\mathcal{L}} \varphi$. Consider $\vec{\gamma} \in Fm$ and a substitution σ such that $\sigma\boldsymbol{\tau}(x, \vec{y}) = \boldsymbol{\tau}(\varphi, \vec{\gamma})$. By Lemma 2.6 we have

$$\boldsymbol{\tau}(\varphi, \vec{\gamma}) = \sigma\boldsymbol{\tau}(x, \vec{y}) \subseteq \sigma\widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}C_{\mathcal{L}}\{x\} \subseteq \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}C_{\mathcal{L}}\{\varphi\} \subseteq \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}\Gamma \subseteq \boldsymbol{\Omega}\Gamma.$$

Then suppose that $\boldsymbol{\tau}(\varphi, \vec{\gamma}) \subseteq \boldsymbol{\Omega}\Gamma$ for every $\vec{\gamma} \in Fm$. Consider $\vec{\gamma} \in Fm_{\mathsf{M}}(x,y)$. Then choose a formula $\psi \in \Gamma$ and let σ be a substitution such that $\sigma(y) = \psi$ and $\sigma(x) = \varphi$. We have that $\sigma\boldsymbol{\tau}(x, \vec{\gamma}) \subseteq \boldsymbol{\Omega}\Gamma$. Together with Lemma 1.1, this implies that

$$\boldsymbol{\tau}(x, \vec{\gamma}) \subseteq \sigma^{-1}\boldsymbol{\Omega}\Gamma \subseteq \boldsymbol{\Omega}\sigma^{-1}\Gamma$$

Now let $\pi\colon \boldsymbol{Fm}(x,y) \to \boldsymbol{Fm}_{\mathsf{M}}(x,y)$ be the canonical projection. Observe that $\mathrm{Ker}(\pi)$ is compatible with $Fm(x,y) \cap \sigma^{-1}(\Gamma)$ since $\mathcal{L}$ is complete w.r.t. M. Moreover, observe that $Fm(x,y) \cap \sigma^{-1}(\Gamma) \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{Fm}(x,y) \smallsetminus \{\emptyset\}$ since

$y \in \sigma^{-1}(\Gamma)$. Thus there is $\Gamma' \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{Fm}_{\mathsf{M}}(x,y) \smallsetminus \{\emptyset\}$ such that $\pi^{-1}(\Gamma') = Fm(x,y) \cap \sigma^{-1}(\Gamma)$. We have that

$$\begin{aligned}\boldsymbol{\tau}(x,\vec{\gamma}) &\subseteq Fm(x,y)^2 \cap \boldsymbol{\Omega}\sigma^{-1}\Gamma \subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}(x,y)}(Fm(x,y) \cap \sigma^{-1}(\Gamma)) \\ &= \boldsymbol{\Omega}^{\boldsymbol{Fm}(x,y)}\pi^{-1}\Gamma' = \pi^{-1}\boldsymbol{\Omega}^{\boldsymbol{Fm}_{\mathsf{M}}(x,y)}\Gamma'.\end{aligned}$$

In particular, this implies that $\boldsymbol{\tau}(x,\vec{\gamma}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}_{\mathsf{M}}(x,y)}\Gamma'$ for every $\vec{\gamma} \in Fm_{\mathsf{M}}(x,y)$. Since $\Gamma' \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{Fm}_{\mathsf{M}}(x,y) \smallsetminus \{\emptyset\}$, we can apply the assumptions yielding $x \in \Gamma'$. In particular, this means that $x \in \sigma^{-1}(\Gamma)$ and, therefore, that $\varphi = \sigma(x) \in \Gamma$. ⊠

The result above contains the key to describe a decision procedure for the universal definability of truth sets. More precisely, we have the following:

Lemma 4.14. *There are algorithms that determine whether the logic $\mathcal{L}$ of a given strongly finite set of matrices has truth almost universally definable (resp. equationally definable) in* $\mathsf{Mod}^*\mathcal{L}$*, and in the positive case return a set of defining equations (resp. without parameters)* $\boldsymbol{\tau}(x,\vec{y})$ *for it.*

Proof. Let `universal` be the algorithm defined as follows. Given a strongly finite set of matrices M, we compute the cardinality of $Fm_{\mathsf{M}}(x,y)$. Suppose that it is $n \in \omega$. Then we construct the terms $Fm_{\mathsf{M}}(x,y_1,\dots,y_n)$. We define $m := \max\{|A| : \langle \boldsymbol{A}, F\rangle \in \mathsf{M}\}$. Let $\mathcal{L}$ be the logic determined by M. Choose variables $z_1,\dots,z_m$ such that $\{z_1,\dots,z_m\} \cap \{x,y_1,\dots,y_n\}) = \emptyset$. For every equation $\varepsilon \approx \delta$ with $\varepsilon,\delta \in Fm_{\mathsf{M}}(x,y_1,\dots,y_n)$ we check whether

$$\varphi(\varepsilon,z_1,\dots,z_m), x \dashv\vdash_{\mathcal{L}} x, \varphi(\delta,z_1,\dots,z_m)$$

for every $\varphi \in Fm_{\mathsf{M}}(x,z_1,\dots,z_m)$. Then let $\boldsymbol{\tau}(x,\vec{y})$ be the set of equations $\varepsilon \approx \delta$ that satisfy this condition. Finally we check whether for every $\Gamma \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{Fm}_{\mathsf{M}}(x,y) \smallsetminus \{\emptyset\}$ and $\varphi \in Fm_{\mathsf{M}}(x,y)$:

$$\varphi \in \Gamma \iff \boldsymbol{\tau}(\varphi,\vec{\gamma}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}_{\mathsf{M}}(x,y)}\Gamma \text{ for every } \vec{\gamma} \in Fm_{\mathsf{M}}(x,y).$$

Observe that this can be done mechanically, since the set $\mathcal{F}i_{\mathcal{L}}\boldsymbol{Fm}_{\mathsf{M}}(x,y)$ can be constructed by means of the completeness of $\mathcal{L}$ w.r.t. M. In the positive case our algorithm returns *yes* and $\boldsymbol{\tau}(x,\vec{y})$, while in the negative case it returns *no*.

Now we show that `universal` determines whether truth is almost universally definable in $\mathsf{Mod}^*\mathcal{L}$. First observe that the set $\boldsymbol{\tau}$ described above coincides with $Fm_{\mathsf{M}}(x,y_1,\dots,y_n)^2 \cap \widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}C_{\mathcal{L}}\{x\}$. This follows from point 2 of Lemma 1.2 and the fact that m bounds the cardinality of the matrices in M. Keeping this in mind, the fact that `universal` works well follows from 4.13.

Now we describe a decision procedure for the equational definability of truth sets. Let `equational` be the algorithm defined as follows. Given a strongly finite set of matrices M, we first run `universal`. If it outputs *no*, then also our new algorithm returns *no*. Otherwise `universal` returns *yes* and a set $\boldsymbol{\tau}$. In this case we check whether the logic $\mathcal{L}$ of M has theorems. This can be done mechanically, since $\mathcal{L}$ has theorems if and only if there is a theorem among the formulas $Fm_{\mathsf{M}}(x)$. In case $\mathcal{L}$ is purely inferential, `equational` returns *no*. Otherwise it returns *yes* and $\sigma_x\boldsymbol{\tau}(x,\vec{y})$.

Now observe that `equational` returns *yes* exactly when truth is equationally definable in $\mathsf{Mod}^*\mathcal{L}$. This is due to the fact that truth is equationally definable in $\mathsf{Mod}^*\mathcal{L}$ if and only if truth is almost universally definable in $\mathsf{Mod}^*\mathcal{L}$ and $\mathcal{L}$ has theorems (Corollary 2.10). Then it only remains to prove that if `equational` returns *yes* and $\sigma_x\boldsymbol{\tau}(x,\vec{y})$, then $\sigma_x\boldsymbol{\tau}(x,\vec{y})$ defines truth in $\mathsf{Mod}^*\mathcal{L}$. From Lemma 2.6 it follows that

$$\sigma_x\boldsymbol{\tau}(x,\vec{y}) = Fm_{\mathsf{M}}(x) \cap \sigma_x\widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}C_{\mathcal{L}}\{x\}. \tag{4.6}$$

Moreover, by Lemma 1.3 we know that $\mathsf{Alg}^*\mathcal{L} \subseteq \mathbb{V}\{\boldsymbol{A} : \langle \boldsymbol{A}, F\rangle\}$. Together with (4.6), this implies that $\sigma_x\boldsymbol{\tau}(x,\vec{y})$ defines truth in $\mathsf{Mod}^*\mathcal{L}$ if and only if $\sigma_x\widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}C_{\mathcal{L}}\{x\}$ does so. Hence, since $\sigma_x\widetilde{\boldsymbol{\Omega}}_{\mathcal{L}}C_{\mathcal{L}}\{x\}$ defines truth in $\mathsf{Mod}^*\mathcal{L}$ by Theorem 2.9, we are done. ⊠

One may wonder why the algorithm used in the proof of Lemma 4.14 needs to compute filters over free algebras, instead of looking just at the behaviour of matrices in M. This is because, even under very strong assumptions, there need not be any direct relation between the truth sets of M and the ones of $\mathsf{Mod}^*\mathcal{L}$, where $\mathcal{L}$ is the logic determined by M. The next example presents an instance of this phenomenon.

Example 4.15. Let $\boldsymbol{A} = \langle\{a,b,1\},\Box,\Diamond,1\rangle$ be the algebra with a constant 1 and unary operations $\Box$ and $\Diamond$ defined as follows:

$$\Box a = \Box b = b \quad \Box 1 = 1$$

$$\Diamond b = \Diamond 1 = 1 \quad \Diamond a = b.$$

Then we consider the matrix $\langle \boldsymbol{A}, \{a,1\}\rangle$ and let $\mathcal{L}$ be its logic. The first facts are easy to check:

Fact 4.15.1. *The matrix $\langle \boldsymbol{A}, \{a,1\}\rangle$ is reduced.*

Fact 4.15.2. *Truth is equationally definable in $\langle \boldsymbol{A}, \{a,1\}\rangle$ by $\Box x \approx \Diamond x$.*

Fact 4.15.3. *$\mathcal{L}$ has theorems, namely $\emptyset \vdash_{\mathcal{L}} 1$.*

Fact 4.15.4. *For every $\Gamma \subseteq Fm$ and $x \in Var$, if $\Gamma \vdash_{\mathcal{L}} x$, then there is $n \in \omega$ such that $\Box^n x \in \Gamma$.*

We reason by contraposition. Suppose that $\Box^n x \notin \Gamma$ for every $n \in \omega$. Consider the evaluation $h\colon \boldsymbol{Fm} \to \boldsymbol{A}$ defined as

$$h(z) := \begin{cases} b & \text{if } z = x \\ 1 & \text{if } z \neq x \end{cases}$$

for every variable $z \in Var$. Observe that $h(\gamma) = 1$ for every $\gamma \in \Gamma$ with $Var(\gamma) \neq \{x\}$. Then consider $\gamma \in \Gamma$ such that $Var(\gamma) = \{x\}$. By assumption we know that $\gamma = \Delta\Diamond\nabla x$ for some (possibly empty) finite sequences Δ and ∇ of connectives in $\{\Box, \Diamond\}$. But this implies that $h(\gamma) = h(\Delta\Diamond\nabla x) = \Delta\Diamond\nabla(b) = \Delta 1 = 1 \in \{a, 1\}$. Hence we conclude that $h[\Gamma] \subseteq \{a, 1\}$ and $h(x) = b \notin \{a, 1\}$. This shows that $\Gamma \nvdash_{\mathcal{L}} x$.

Fact 4.15.5. *Truth is not implicitly definable in* $\mathsf{Mod}^*\mathcal{L}$.

First observe that $\{1\} \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$. To prove this, let $\Gamma \cup \{\varphi\} \subseteq Fm$ such that $\Gamma \vdash_{\mathcal{L}} \varphi$. Then consider a homomorphism $h\colon \boldsymbol{Fm} \to \boldsymbol{A}$ such that $h[\Gamma] \subseteq \{1\}$. Since $\langle \boldsymbol{A}, \{a, 1\}\rangle$ is a model of $\mathcal{L}$, we know that $h(\varphi) \in \{a, 1\}$. Now, we have cases: either $h(\varphi) = a$ or $h(\varphi) = 1$. Suppose that $h(\varphi) = a$. Then, by definition of $\boldsymbol{A}$, we would have that $\varphi = x$ for some $x \in Var$. By Fact 4.15.4 this implies that there is $n \in \omega$ such that $\Box^n x \in \Gamma$. Therefore we would have that $h(\Box^n x) = 1$ against the assumption that $h(\Box^n x) = \Box^n h(x) = \Box^n a \in \{a, b\}$. Therefore $h(\varphi) \neq a$, and hence $h(\varphi) = 1$. We conclude that $\{1\} \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$. It is easy to prove that the matrix $\langle \boldsymbol{A}, \{1\}\rangle$ is reduced. Therefore both $\langle \boldsymbol{A}, \{a, 1\}\rangle$ and $\langle \boldsymbol{A}, \{1\}\rangle$ are reduced models of $\mathcal{L}$. ⊠

Finally, we consider the problem of determining whether the logic of a strongly finite set of matrices is assertional.

Lemma 4.16. *There is an algorithm that determines whether the logic of a given strongly finite set of matrices is assertional.*

Proof. Let `assertional` be the algorithm defined as follows. Given a strongly finite set of matrices M, we check whether the set

$$\{F/\boldsymbol{\Omega}^{\boldsymbol{A}}F : \langle \boldsymbol{A}, F\rangle \in \mathsf{M}\}$$

is a collection of singletons. In case it is not, then our algorithm outputs *no*. Otherwise we construct the terms $Fm_{\mathsf{M}}(x)$ and check whether there is at least one formula $\varphi \in Fm_{\mathsf{V}}(x)$ such that $\emptyset \vdash_{\mathcal{L}} \varphi$, where $\mathcal{L}$ is the logic of M. In the positive case our algorithm outputs *yes*, while *no* otherwise.

To see that `assertional` works as expected, we reason as follows. First suppose that $\mathcal{L}$ is assertional. From Theorem 1.10 we know that $\mathsf{Mod}^*\mathcal{L}$ is a unital class of matrices. Therefore the first step succeeds and our algorithm goes on checking whether there is a theorem of $\mathcal{L}$ inside $Fm_{\mathsf{M}}(x)$. Since assertional logics have theorems, by structurality we know that $\mathcal{L}$ has a theorem with at most the variable x. Therefore there is a formula $\varphi \in Fm_{\mathsf{V}}(x)$ such that $\emptyset \vdash_{\mathcal{L}} \varphi$ and our algorithm outputs *yes*.

Now suppose that $\mathcal{L}$ is not assertional. If $\{F/\boldsymbol{\Omega}^{A}F : \langle \boldsymbol{A}, F\rangle \in \mathsf{M}\}$ is not a collection of singletons, the algorithm outputs *no*. Now consider the case where it is so. Recall that the classes M and M^{*} determine the same logic. Then by Theorem 1.10 we know that $\mathsf{Mod}^{*}\mathcal{L}$ is a almost unital class of matrices that is not unital. In particular, this implies that $\mathcal{L}$ has no theorems. Therefore our algorithm outputs *no*. ⊠

Now observe that equivalential logics determined by a strongly finite set of matrices are always finitely equivalential. Therefore combining the results obtained so far we obtain the following:

Theorem 4.17. *Let K be a level of the Leibniz hierarchy contained either in the class of protoalgebraic logics or in the that of logics $\mathcal{L}$ whose truth sets are almost universally definable in $\mathsf{Mod}^{*}\mathcal{L}$. The problem of determining whether the logic of a given strongly finite set of matrices belongs to K is decidable.*

Our analysis leaves open the following:

Problem 3. Is there a decision procedure that, given a strongly finite set of matrices, determines whether the truth sets of its logic $\mathcal{L}$ are implicitly definable (small) in $\mathsf{Mod}^{*}\mathcal{L}$?

4.2 The classification problem in the Frege hierarchy

Now we move our attention to the problem of classifying logics in the Frege hierarchy, which deals with several kinds of replacement properties. Also in this case we will focus on two problems. Let K be a level of the Frege hierarchy. The first one is, given a consistent finite Hilbert-style calculus H in a finite language, to decide whether the logic determined by H belongs to K (Theorem 4.23). The second one is, given a strongly finite set of matrices M, to decide whether the logic determined by M belongs to K (Theorem 4.24 and Corollary 4.25). Again, we divide the rest of the section into two parts that correspond respectively to the analysis of the first and of the second problem.

The syntactic case

The aim of this section is to prove a result analogous to the one obtained for the Leibniz hierarchy. More precisely, we will show that the problem of classifying the logic of a consistent finite Hilbert calculus in a finite language in the Frege hierarchy is in general undecidable (Theorem 4.23). Remarkably, our argument shows that this classification problem remains undecidable even if we restrict our attention to Hilbert calculi that determine a finitary algebraizable logic. Our strategy consists reducing an undecidable problem related to the equational theory of relation algebras to the problem of classifying logics of finite consistent Hilbert calculi in the Frege hierarchy. For this reason we will recall some basic definitions. A *relation algebra* is an algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \neg, \cdot, ^{\smile}, 1\rangle$ of type $\langle 2, 2, 1, 2, 1, 0\rangle$ such that:

1. $\langle A, \wedge, \vee, \neg\rangle$ is a Boolean algebra.
2. $\langle A, \cdot, 1\rangle$ is a monoid.
3. The operations $\cdot$ and $^{\smile}$ distribute over $\vee$, i.e.

$$x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z) \text{ and } (x \vee y)^{\smile} = x^{\smile} \vee y^{\smile}.$$

4. The operation $^{\smile}$ is an involution, i.e.

$$x^{\smile\smile} = x \text{ and } (x \cdot y)^{\smile} = y^{\smile} \cdot x^{\smile}.$$

5. The inequality $x^{\smile} \cdot (x \cdot y)^{\smile} \leqslant y^{\smile}$ holds.

We denote by RA the variety of relation algebras and by Φ its finite basis obtained from the definition above.

Example 4.18 (Relation Algebras)**.** Let X be an arbitrary set and $\mathcal{R}(X)$ the collection of binary relations on it, i.e. $\mathcal{P}(X \times X)$. Let $\cap, \cup$ and $\neg$ be respectively the set-theoretic intersection, union and complement operations. Moreover, for every pair of relations $R, S \in \mathcal{R}(X)$ we define

$$\begin{aligned} R \cdot S &:= \{\langle x, y\rangle : \langle x, z\rangle \in R \text{ and } \langle z, y\rangle \in S \text{ for some } z \in X\} \\ R^{\smile} &:= \{\langle x, y\rangle : \langle y, x\rangle \in R\} \\ 1 &:= \{\langle x, x\rangle : x \in X\}. \end{aligned}$$

Then the structure $\langle \mathcal{R}(X), \cap, \cup, \neg, \cdot, ^{\smile}, 1\rangle$ is a relation algebra. ⊠

The following result can be deduced from [94, Section 8.5(viii)]:[†]

Theorem 4.19 (Tarski and Givant)**.** *The set of all equations in a single variable that hold in* RA *is undecidable.*

We will denote by $Eq(x)$ the set of equations in the language of relation algebras in the variable x. Now we introduce a logic associated with every equation of relation algebras in a given variable x. Observe that this logic is expressed in the language of relation algebras expanded with two new connectives $\Box$ and $\to$ that witness the algebraizability of the logic.

[†]See for example pag. 398 of [70] for an explicit statement of this fact.

Definition 4.20. Let $\alpha \approx \beta \in Eq(x)$. Then $\mathcal{L}(\alpha, \beta)$ is the logic in the language $\langle \wedge, \vee, \neg, \cdot, ^{\smile}, 1, \Box, \to \rangle$ of type $\langle 2, 2, 1, 2, 1, 0, 1, 2 \rangle$ axiomatized by the following Hilbert calculus:

$$\emptyset \vdash x \to x \tag{R}$$

$$x, x \to y \vdash y \tag{MP}$$

$$x \to y, y \to x \vdash \Diamond x \to \Diamond y \tag{Rep1}$$

$$x \to y, y \to x, z \to u, u \to z \vdash (x * z) \to (y * u) \tag{Rep2}$$

$$x \dashv\vdash \Box x \to x, x \to \Box x \tag{A3}$$

$$\emptyset \vdash \varepsilon \to \delta, \delta \to \varepsilon \tag{RA}$$

$$\alpha(\varphi) \to \beta(\varphi) \vdash \varphi \tag{DDT}$$

for every $\varepsilon \approx \delta \in \Phi$, every $* \in \{\wedge, \vee, \cdot, \to\}$, $\Diamond \in \{\neg, ^{\smile}, \Box\}$ and every formula φ of the following list:

$\varphi_1 := x \to (y \to x)$
$\varphi_2 := (x \to (y \to z)) \to ((x \to y) \to (x \to z))$
$\varphi_3 := (x \to y) \to ((y \to x) \to (\Diamond x \to \Diamond y))$
$\varphi_4 := (x \to y) \to ((y \to x) \to ((z \to u) \to ((u \to z) \to ((x * z) \to (y * u)))))$
$\varphi_5 := x \to (x \to \Box x)$
$\varphi_6 := x \to (\Box x \to x)$
$\varphi_7 := (\Box x \to x) \to ((x \to \Box x) \to x)$

again with $*$ and $\Diamond$ ranging over the unary and binary connectives respectively.

The name of the set of rules (DDT) stands for *deduction-detachment theorem*. The motivation of this choice will become evident in the proof of part (iii)⇒(i) of Theorem 4.23. Before going on, observe that the syntactic characterization of algebraizability (Theorem 1.9) implies that $\mathcal{L}(\alpha, \beta)$ is finitely algebraizable through the structural transformers

$$\boldsymbol{\tau}(x) = \{x \approx \Box x\} \text{ and } \boldsymbol{\rho}(x, y) = \{x \to y, y \to x\}.$$

The next result expresses the relation between the validity of the equation $\alpha \approx \beta$ in RA and the location of the logic $\mathcal{L}(\alpha, \beta)$ in the Frege hierarchy.

Theorem 4.21. *Let $\alpha \approx \beta \in Eq(x)$. The following conditions are equivalent:*

(i) *$\mathcal{L}(\alpha, \beta)$ is fully Fregean.*

(ii) *$\mathcal{L}(\alpha, \beta)$ is selfextensional.*

(iii) *$\mathsf{RA} \vDash \alpha \approx \beta$.*

Proof. (i)⇒(ii) is trivial. Then we turn to prove (ii)⇒(iii). Suppose towards a contradiction that $\mathcal{L}(\alpha, \beta)$ is selfextensional and that $\mathsf{RA} \nvDash \alpha \approx \beta$. Then let $\boldsymbol{A}$ be the free relation algebra with countably many free generators $\{[\![z]\!] : z \in Var\}$. Now we expand $\boldsymbol{A}$ with two new unary and binary operations $\Box$ and $\to$ defined as

$$\Box a := \begin{cases} [\![1]\!] & \text{if } a = [\![x]\!] \\ a & \text{otherwise} \end{cases} \qquad a \to b := \begin{cases} [\![x]\!] & \text{if } a \neq b \\ [\![1]\!] & \text{if } a = b \neq [\![x]\!] \\ [\![y]\!] & \text{if } a = b = [\![x]\!] \end{cases}$$

for every $a, b \in A$. Let $\boldsymbol{B}$ be the result of the expansion and put $F := B \smallsetminus \{[\![x]\!]\}$.

Claim 4.21.1. $\langle \boldsymbol{B}, F\rangle \in \mathsf{Mod}^*\mathcal{L}(\alpha, \beta)$.

First observe that $\langle \boldsymbol{B}, F\rangle$ is a reduced matrix. To prove this, consider two different elements $a, b \in B$ and the polynomial function $p(x) := x \to a$. It is easy to see that $p(a) \in F$ and $p(b) \notin F$. By point 1 of Lemma 1.2 we conclude that $\langle a, b\rangle \notin \boldsymbol{\Omega}^{\boldsymbol{B}} F$ and, therefore, that $\langle \boldsymbol{B}, F\rangle$ is reduced. Then we turn to prove that it is a model of $\mathcal{L}(\alpha, \beta)$. The closure of F under the rules axiomatizing $\mathcal{L}(\alpha, \beta)$ is easily proved, except perhaps for the set of rules (DDT). Consider a rule $\alpha(\varphi) \to \beta(\varphi) \vdash \varphi$ in (DDT) and a homomorphism $h\colon \boldsymbol{Fm} \to \boldsymbol{B}$ such that $h\alpha(\varphi) \to h\beta(\varphi) \in F$. Clearly we have that $h\alpha(\varphi) = h\beta(\varphi)$. Since $\mathsf{RA} \nvDash \alpha \approx \beta$ and $\boldsymbol{A}$ is the free relation algebra, we know that $h(\varphi) \notin \{[\![z]\!] : z \in Var\}$. Together with the fact that the principal connective of φ is $\to$, this implies that $h(\varphi) = [\![1]\!] \in F$, as desired. This establishes the claim.

Now observe that $x \leftrightarrow x$ and $1 \leftrightarrow 1$ are instances of (R) and, therefore, they are theorems of $\mathcal{L}(\alpha, \beta)$. In particular this means that $x \leftrightarrow x \dashv\vdash_{\mathcal{L}(\alpha,\beta)} 1 \leftrightarrow 1$. Since $\mathcal{L}(\alpha, \beta)$ is selfextensional, we obtain that $\mathsf{Alg}\mathcal{L}(\alpha, \beta) \vDash x \leftrightarrow x \approx 1 \leftrightarrow 1$ by Lemma 1.14. Let $h\colon \boldsymbol{Fm} \to \boldsymbol{B}$ the the natural surjection, which sends z to $[\![z]\!]$ for every $z \in Var$. It is easy to see that

$$h(x \leftrightarrow x) = [\![y]\!] \neq [\![1]\!] = h(1 \leftrightarrow 1).$$

But from Claim 4.21.1 it follows that $\boldsymbol{B} \in \mathsf{Alg}\mathcal{L}$. Therefore we reach a contradiction as desired.

It only remains to prove part (iii)⇒(i). Suppose that $\mathsf{RA} \vDash \alpha \approx \beta$. Recall that $\mathcal{L}(\alpha, \beta)$ is algebraizable with set of congruence formulas $\boldsymbol{\rho}(x, y) = \{x \to y, y \to x\}$. In particular, this implies that

$$\emptyset \vdash_{\mathcal{L}(\alpha,\beta)} \boldsymbol{\rho}(\gamma, \eta) \iff \mathsf{Alg}\mathcal{L}(\alpha, \beta) \vDash \gamma \approx \eta$$

for every $\gamma, \eta \in Fm$. Applying this observation to the axiom (RA), we conclude that $\mathsf{Alg}\mathcal{L}(\alpha, \beta)$ is a class of expanded relation algebras. Together with the fact that $\mathsf{RA} \vDash \alpha \approx \beta$, this implies that $\alpha \to \beta$ is a theorem of $\mathcal{L}(\alpha, \beta)$. Then consider any $i \leqslant 7$. By structurality $\alpha(\varphi_i) \to \beta(\varphi_i)$ is a theorem of

$\mathcal{L}(\alpha, \beta)$. Therefore we can apply the rules (DDT), obtaining that also φ_i is a theorem of $\mathcal{L}(\alpha, \beta)$.

Now, let $\mathcal{L}$ be the logic axiomatized only by $\{\varphi_i : i \leqslant 7\}$, (RA) and (MP).

Claim 4.21.2. $\mathcal{L} = \mathcal{L}(\alpha, \beta)$.

Clearly we have that $\mathcal{L} \leqslant \mathcal{L}(\alpha, \beta)$. To show the other direction, observe that $\mathcal{L}$ satisfies the rules (DDT). Moreover, with an application of (MP), it is easy to see that $\mathcal{L}$ satisfies (Rep1), (Rep2) and (A3) too. It only remains to show that $\mathcal{L}$ satisfies (R). But observe that $\mathcal{L}$ is an expansion of the implication fragment $\mathcal{IPC}_{\rightarrow}$ of propositional intuitionistic logic, which is axiomatized by φ_1, φ_2 and (MP). Since $x \rightarrow x$ is a theorem of $\mathcal{IPC}_{\rightarrow}$, we conclude that $\mathcal{L}$ satisfies (R) and, therefore, that $\mathcal{L}(\alpha, \beta) \leqslant \mathcal{L}$. Thus we established the claim.

It is well known that a finitary logic in a language containing $\rightarrow$ satisfies the classical version of the deduction-detachment theorem if and only if it is an axiomatic extension of the logic defined in its language by the axioms φ_1 and φ_2 and the rule (MP) [96, Theorem 2.4.2]. Together with Claim 4.21.2, this implies that

$$\Gamma \vdash_{\mathcal{L}(\alpha,\beta)} \gamma \rightarrow \psi \Longleftrightarrow \Gamma, \gamma \vdash_{\mathcal{L}(\alpha,\beta)} \psi$$

for every $\Gamma \cup \{\gamma, \psi\} \subseteq Fm$. In particular, this means that for every $\gamma, \psi \in Fm$:

$$\gamma \dashv\vdash_{\mathcal{L}(\alpha,\beta)} \psi \Longleftrightarrow \emptyset \vdash_{\mathcal{L}(\alpha,\beta)} \boldsymbol{\rho}(\gamma, \psi) \Longleftrightarrow \mathsf{Alg}\mathcal{L} \vDash \gamma \approx \psi.$$

Keeping this in mind, it is easy to see that $\dashv\vdash_{\mathcal{L}(\alpha,\beta)}$ is a congruence of $\boldsymbol{Fm}$. Hence we conclude that $\mathcal{L}(\alpha, \beta)$ is selfextensional. The fact that $\mathcal{L}(\alpha, \beta)$ is also fully Fregean, is a consequence of two general results of abstract algebraic logic. First, every finitary selfextensional logic with a classical deduction-detachment theorem is fully selfextensional [40, Theorem 4.46] (see also [61]).‡ Secondly, every truth-equational fully selfextensional logic is also fully Fregean [6, Theorem 22]. ⊠

Corollary 4.22. *The logic* $\mathcal{L}(\alpha, \beta)$ *is consistent for every* $\alpha \approx \beta \in Eq(x)$.

Proof. The fact that $\mathcal{L}(\alpha, \beta)$ is consistent when $\mathsf{RA} \nvDash \alpha \approx \beta$ has been proved in part (i)⇒(ii) of Theorem 4.23. Then consider the case where $\mathsf{RA} \vDash \alpha \approx \beta$. In part (iii)⇒(i) of Theorem 4.23, we showed that $\mathcal{L}(\alpha, \beta)$ is the logic axiomatized by only by $\{\varphi_i : i \leqslant 7\}$, (RA) and (MP). Then consider any non-trivial Boolean algebra $\boldsymbol{A}$ and expand it to a relation algebra by interpreting $\cdot$ as $\wedge$, $^{\smile}$ as the identity map and 1 as the top element. Moreover let $\rightarrow$ be the usual Boolean implication and $\Box$ be the constant function with value 1. Let $\boldsymbol{B}$ be the result of the expansion. It is very easy to see that $\langle \boldsymbol{B}, \{1\} \rangle$ is a model of $\{\varphi_i : i \leqslant 7\}$, (RA) and (MP). ⊠

‡In these references the classical deduction-detachment theorem is called *uniterm* deduction-detachment theorem. The expression *uniterm* refers to the fact that this DDT is witnessed by a single formula in two variables in contrast to the cases where the DDT is witnesses by a set of formulas as in [20, 32].

Now it is easy to complete the proof of our main result on the classification of syntactically presented logics in the Frege hierarchy.

Theorem 4.23. *Let* K *be a level of the Frege hierarchy. The problem of determining whether the logic of a given finite consistent Hilbert calculus in a finite language belongs to* K *is undecidable. Moreover, the problem remains undecidable when restricted to the classification of finite consistent Hilbert calculi that determine a finitely algebraizable logic.*

Proof. Let K be a level of the Frege hierarchy. Suppose towards a contradiction that there is an algorithm A_1 which, given a finite consistent Hilbert calculus in a finite language that moreover determines a finitely algebraizable logic, decides whether its logic belongs to K. Then we define a new algorithm A_2 as follows: given an equation $\alpha \approx \beta \in Eq(x)$, we construct the logic $\mathcal{L}(\alpha, \beta)$ and check with A_1 if it belongs to K. Observe that we can do this, since $\mathcal{L}(\alpha, \beta)$ is finitely algebraizable and consistent by Corollary 4.22. In the positive case A_2 returns *yes*, while *no* otherwise. Since K contains the class of selfextensional logics and is included in the of class of fully Fregean ones, we can apply Theorem 4.23 obtaining that

$$\mathcal{L}(\alpha, \beta) \in \mathsf{K} \iff \mathsf{RA} \vDash \alpha \approx \beta.$$

Therefore A_2 would provide a decision procedure for the validity in RA of equations in one variable. From Theorem 4.19 we know that such a procedure does not exist. Hence we obtain a contradiction as desired. ⊠

The semantic case

Now we briefly consider the problem of classifying logics determined by a strongly finite set of matrices in the Frege hierarchy. The main result in this direction is the following:

Theorem 4.24. *There is an algorithm that determines whether the logic of a given strongly finite set of matrices is selfextensional.*

Proof. We define the algorithm `selfextensional` as follows. Given a finite set M of finite matrices of finite type, we compute $n := \max\{|A| : \langle \boldsymbol{A}, F\rangle \in \mathsf{M}\}$ and, subsequently, the terms $Fm_{\mathsf{M}}(x_1, \ldots, x_n)$. Then we check whether for every $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \in Fm_{\mathsf{M}}(x_1, \ldots, x_n)$ and every k-ary function symbol λ the following condition holds:

$$\text{if } \alpha_i \dashv\vdash_{\mathcal{L}} \beta_i \text{ for every } i \leqslant k, \text{ then } \lambda(\alpha_1, \ldots, \alpha_k) \dashv\vdash_{\mathcal{L}} \lambda(\alpha_1, \ldots, \alpha_k), \tag{4.7}$$

where $\mathcal{L}$ is the logic of M. In the positive case our algorithm returns *yes*, while *no* otherwise.

We check that this algorithm works as intended. It is easy to see that if $\mathcal{L}$ is selfextensional, then it returns *yes*. Conversely, suppose that `selfextensional` returns *yes*. Then consider a k-ary function symbol λ and formulas $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_k$ such that $\alpha_i \dashv\vdash_{\mathcal{L}} \beta_i$ for every $i \leqslant k$. Then consider a matrix $\langle \boldsymbol{A}, F \rangle \in \mathsf{M}$ and a homomorphism $h\colon \boldsymbol{Fm} \to \boldsymbol{A}$ such that $h\lambda(\alpha_1, \ldots, \alpha_k) \in F$. Then for every $a \in h[Var]$ we choose a variable $v_a \in h^{-1}(a)$ and define a substitution $\sigma\colon \boldsymbol{Fm} \to \boldsymbol{Fm}$ as $\sigma(x) = v_{h(x)}$ for every $x \in Var$. Observe that $h\sigma(\varphi) = h(\varphi)$ for every formula φ. By structurality we know that $\sigma(\alpha_i) \dashv\vdash_{\mathcal{L}} \sigma(\beta_i)$ for every $i \leqslant k$. Moreover, we have

$$|Var\{\sigma(\alpha_i) : i \leqslant k\} \cup Var\{\sigma(\beta_i) : i \leqslant k\}| \leqslant |A| \leqslant n.$$

This means that the formulas $\sigma(\alpha_1), \ldots, \sigma(\alpha_k), \sigma(\beta_1), \ldots, \sigma(\beta_k)$ can be viewed as elements of $Fm_{\mathsf{M}}(x_1, \ldots, x_n)$. Therefore we can safely apply the assumption, obtaining that $\sigma\lambda(\alpha_1, \ldots, \alpha_k) \dashv\vdash_{\mathcal{L}} \sigma\lambda(\beta_1, \ldots, \beta_k)$. Since $h\sigma\lambda(\alpha_1, \ldots, \alpha_k) = h\lambda(\alpha_1, \ldots, \alpha_k) \in F$, this means that $h\sigma\lambda(\beta_1, \ldots, \beta_k) \in F$. But this is to say that $h\lambda(\beta_1, \ldots, \beta_k) = h\sigma\lambda(\beta_1, \ldots, \beta_k) \in F$. This shows that $\lambda(\alpha_1, \ldots, \alpha_k) \vdash_{\mathcal{L}} \lambda(\beta_1, \ldots, \beta_k)$. Hence we conclude that $\mathcal{L}$ is selfextensional. ⊠

Corollary 4.25. *There is an algorithm that determines whether the logic of a strongly finite set of matrices is Fregean.*

Proof. The same strategy used in the proof of Theorem 4.24 allows to construct a decision procedure for Fregeanity. The only difference is that the notion of Fregean logic involves some premises Γ on both sides of the interderivability relation. More precisely, let `Fregean` be the algorithm defined as `selfextensional` except for the fact that, instead of checking condition (4.7), it check the following: if $\Gamma, \alpha_i \dashv\vdash_{\mathcal{L}} \beta_i, \Gamma$ for every $i \leqslant k$, then $\Gamma, \lambda(\alpha_1, \ldots, \alpha_k) \dashv\vdash_{\mathcal{L}} \lambda(\beta_1, \ldots, \beta_k), \Gamma$ for every $\Gamma \cup \{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k\} \subseteq Fm_{\mathsf{M}}(x_1, \ldots, x_n)$. If this condition holds `Fregean` returns *yes*, while it returns *no* otherwise. It is easy to show that `Fregean` works as expected. ⊠

The fact that it is possible to decide whether a logic of a strongly finite set M of finite matrices is selfextensional or Fregean was to be expected. This is because selfextensionality and Fregeanity are purely syntactic conditions and, therefore, can be checked on the finitely generated free algebras over M, which are finite. Full selfextensionality and full Fregeanity, on the contrary, make reference to semantic concepts such as that of generated $\mathcal{L}$-filter. Therefore in principle it may be complicated to find a decision procedure for them, even in the context of strongly finite logics. This poses the following open question:

Problem 4. Are there decision procedures that, given a strongly finite set of matrices, determines whether its logic is fully selfextensional (resp. fully Fregean)?

Part II

A logical and algebraic characterization of adjunctions between generalized quasi-varieties

CHAPTER 5

Canonical decomposition

In the very influential paper [73] McKenzie discovered that category equivalences between prevarieties can be described in purely algebraic terms. Namely, he showed that if two prevarieties X and Y are categorically equivalent, then we can transform X into Y by applying two kind of deformations to X. In this chapter we generalize this approach to the study of adjunctions between generalized quasi-varieties. Our main result will be that every right adjoint functor between generalized quasi-varieties can be decomposed into a combination of deformations that generalize the ones introduced by McKenzie (Theorem 5.29). This result is achieved by developing a *correspondence* between the concept of *adjunction* and a new notion of *translation* between relative equational consequences (Theorems 5.23 and 5.27). This provides a general explanation of the correspondence that appears in some well-known translations between logics:

1. Gödel's translation of $\mathcal{IPC}$ into $\mathcal{S}4$ *corresponds* to the functor that takes an interior algebra to the Heyting algebra of its open elements (Examples 5.21 and 5.24).
2. Kolmogorov's translation of $\mathcal{CPC}$ into $\mathcal{IPC}$ *corresponds* to the functor that takes a Heyting algebra to the Boolean algebra of its regular elements (Examples 5.22 and 5.24).

5.1 Categorical universal algebra

We already met classes of algebras in our road. However we implicitly chose to overlook their categorical structure and to delay discussing it until the moment where it becomes fundamental. Now that we want to give a logical interpretation of the categorical notion of an adjunction, this moment has come. In particular, we will spend a few words on two distinct but related topics. First we discuss some universal constructions in the framework of prevarieties, and show how these constructions can be applied to characterize right adjoint functors between generalized quasi-varieties. Accordingly, we divide the rest of the section into two parts.

Universal constructions

All the constructions that we will review can be found in [5]. In particular, our aim is to show that prevarieties seen as categories (where algebras are objects and homomorphisms are arrows) are *bicomplete*, i.e., they have small limits and small colimits. To this end, we will conform to the following convention:* An algebraic language $\mathscr{L}$ admits *empty* models if and only if $\mathscr{L}$ does not contain constant symbols. In particular, a prevariety K contains the empty algebra if and only if its language does not contain a constant symbol. This convention ensures the existence of the 0-generated free algebra over K.

We are now ready to describe the structure of limits and colimits in prevarieties. Let K be a prevariety. Categorical *products* in K coincide with direct products. Moreover, given a parallel pair of arrows $f, g \colon \boldsymbol{A} \rightrightarrows \boldsymbol{B}$ in K, we have that $h \colon \boldsymbol{C} \to \boldsymbol{A}$ is an *equalizer* of f and g if and only if h is an embedding and $h[C] = \{a \in A : f(a) = g(a)\}$. Observe that if the language of K does not contain constant symbols, then $\boldsymbol{C}$ may happen to be empty. It is well known that all other limits can be obtained as a combination of these two constructions. The description of colimits is slightly more complicated. Consider a family of algebras $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{K}$ and assume w.l.o.g. that their universes are pair-wise disjoint. For every $i \in I$ we let $\pi_i \colon \boldsymbol{Fm}_{\mathsf{K}}(A_i) \to \boldsymbol{A}_i$ be the unique surjective homomorphism that maps identically A_i onto A_i. Then consider the set $X := \bigcup_{i \in I} A_i$ and define

$$\theta := \mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{Fm}_{\mathsf{K}}(X)} \bigcup_{i \in I} \mathrm{Ker}(\pi_i).$$

The algebra $\boldsymbol{Fm}_{\mathsf{K}}(X)/\theta$, together with the maps $p_i \colon \boldsymbol{A}_i \to \boldsymbol{Fm}_{\mathsf{K}}(X)/\theta$ that send $a \in A_i$ to a/θ, is a *coproduct* of $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{K}$. Observe that the maps p_i need not to be injective. Moreover, it is worth to remark that the free κ-generated algebra is a κ-th copower of the free 1-generated algebra. In general, if $\boldsymbol{A}$ is a coproduct in K of the family $\{\boldsymbol{A}_i : i \in I\}$ and $f_i \colon \boldsymbol{A}_i \to \boldsymbol{B}$ with $i \in I$ are arrows in K, then we will denote by $\langle f_i : i \in I \rangle \colon \boldsymbol{A} \to \boldsymbol{B}$ the map induced by the universal property of the coproduct. In case $\boldsymbol{A} = \boldsymbol{Fm}_{\mathsf{K}}(X)/\theta$ as above, the arrow $\langle f_i : i \in I \rangle$ is defined by the rule

$$\varphi(a_1, \ldots, a_n)/\theta \longmapsto \varphi^{\boldsymbol{B}}(f_{k_1}(a_1), \ldots, f_{k_n}(a_n))$$

*This convention was tacitly *not* applied in Part I, where algebras were assumed to have non-empty universes.

for every $\varphi(a_1, \ldots, a_n)/\theta \in Fm_{\mathsf{K}}(X)/\theta$ with $a_1 \in A_{k_1}, \ldots, a_n \in A_{k_n}$.

Now, we move our attention to the other basic kind of colimit. Given a parallel pair of arrows $f, g\colon \boldsymbol{A} \rightrightarrows \boldsymbol{B}$ in K, we have that $h\colon \boldsymbol{B} \to \boldsymbol{C}$ is a *coequalizer* of f and g if and only if it is surjective and

$$\mathrm{Ker}(h) = \mathrm{Cg}^{\boldsymbol{B}}_{\mathsf{K}}\{\langle f(a), g(a)\rangle : a \in A\}.$$

It is worth to remark that every surjective homomorphism in K arises as the coequalizer of a pair of arrows. In particular, observe that every congruence $\theta \in \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ of $\boldsymbol{A} \in \mathsf{K}$ can be seen as a subalgebra of the direct product $\boldsymbol{A} \times \boldsymbol{A}$. Keeping this in mind, θ can be associated with two homomorphisms $l, r\colon \theta \rightrightarrows \boldsymbol{A}$ that sends a pair $\langle a, b\rangle \in \theta$ respectively to its left and right component. It is easy to prove that π_θ is a coequalizer of l and r. Finally, it is well known that all other colimits can be constructed as a combination of coproducts and coequalizers.

Observe that the *terminal object* of K is the trivial algebra, while its *initial object* is $\boldsymbol{Fm}_{\mathsf{K}}(0)$. Therefore the initial object of K is empty if and only if the language of K contains no constant symbols. Given two prevarieties X and Y, the functors $\mathcal{F}\colon \mathsf{X} \longleftrightarrow \mathsf{Y} \colon \mathcal{G}$, where $\mathcal{F}$ sends everything to the initial object and $\mathcal{G}$ sends everything to the terminal object, always form an adjunction $\mathcal{F} \dashv \mathcal{G}$. We call *trivial* the adjunctions of this kind. In particular, we say that a left (right) adjoint functor between prevarieties is *trivial* if it sends everything to the initial (terminal) object.

It is worth to spend some more time on a special kind of colimit constructions. These are *κ-directed colimits*, i.e., colimits of diagrams indexed by posets in which every subset of cardinality $< \kappa$ has an upper bound, for a regular cardinal κ. In varieties they are constructed as usual, by taking the disjoint union of the factors and identifying the elements that become eventually equal. In the case of prevarieties K the only difference is that we have to factor out the resulting algebra by its smallest K-congruence. Remarkably, this last step can be avoided when K is a generalized quasi-variety that can be axiomatized by generalized quasi-equations, whose number of variables is less than κ. Then the κ-directed colimits of families of algebras in K are obtained by just identifying elements that become eventually equal. In particular, in quasi-varieties this is the case for usual $\aleph_0$-directed colimits.

Right adjoint functors

All the definitions reviewed here can be found in [4]. Let κ be a regular cardinal and K be a locally small category, i.e., a category whose hom-sets are sets (as opposed to proper classes). An object $\boldsymbol{A}$ in K is *κ-presentable* if the functor $\hom(\boldsymbol{A}, \cdot)$ preserves κ-directed colimits. More explicitly, this means that for every κ-directed diagram $\{\boldsymbol{B}_i : i \in I\}$ with colimit $g_i\colon \boldsymbol{B}_i \to \boldsymbol{B}$ and for every arrow $h\colon \boldsymbol{A} \to \boldsymbol{B}$ the following conditions hold:

1. There is $i \in I$ and an arrow $p\colon \boldsymbol{A} \to \boldsymbol{B}_i$ such that $g_i \circ p = h$.

2. The map p is essentially unique, in the sense that for every other arrow $q\colon \boldsymbol{A} \to \boldsymbol{B}_i$ such that $g_i \circ q = h$ there is $j \geqslant i$ such that $f_{ij} \circ p = f_{ij} \circ q$, where $f_{ij}\colon \boldsymbol{B}_i \to \boldsymbol{B}_j$ is an arrow of the κ-directed diagram.

Then K is *locally κ-presentable* if it is cocomplete, and has a *set* J of κ-presentable objects such that every object in K is a κ-directed colimit of objects in J. Finally, K is *locally presentable* if it is locally κ-presentable for some regular cardinal κ. The following result is [4, Theorem 1.66]:

Theorem 5.1 (Adámek and Rosicky)**.** *A functor between locally presentable categories is right adjoint if and only if it preserves limits and κ-directed colimits for some regular cardinal κ.*

The aim of this section is to show that generalized quasi-varieties are locally presentable categories and, therefore, that the above theorem characterizes right adjoints between generalized quasi-varieties. The following is essentially [4, Theorem 3.12]:

Lemma 5.2. *Let κ be a regular cardinal and K be a generalized quasi-variety axiomatized by generalized quasi-equations in less than κ variables. An algebra $\boldsymbol{A} \in \mathsf{K}$ is κ-presentable if and only if it is (isomorphic to) a quotient of $\boldsymbol{Fm}_{\mathsf{K}}(\lambda)$ under a μ-generated K-congruence for some $\lambda, \mu < \kappa$.*

Proof. Observe that generalized quasi-varieties are locally small categories and, therefore, that the definition of κ-presentable object makes sense in K. First suppose that $\boldsymbol{A}$ is κ-presentable. Then consider the diagram whose objects are the λ-generated subalgebras of $\boldsymbol{A}$ for every $\lambda < \kappa$ and inclusion maps between them. Since κ is a regular cardinal, this diagram is κ-directed. It is easy to see that $\boldsymbol{A}$ is the colimit of the diagram. We consider the situation where the role of the map h in the definition of κ-presentable object is played by the identity map $1_{\boldsymbol{A}}\colon \boldsymbol{A} \to \boldsymbol{A}$. Thus we obtain a λ-generated subalgebra $\boldsymbol{B} \leqslant \boldsymbol{A}$ for some $\lambda < \kappa$ and a homomorphism $p\colon \boldsymbol{A} \to \boldsymbol{B}$ such that $p = 1_{\boldsymbol{A}}$. In particular, this implies that $\boldsymbol{A}$ is λ-generated. Then there is a K-congruence θ and an isomorphism $h\colon \boldsymbol{A} \to \boldsymbol{Fm}_{\mathsf{K}}(\lambda)/\theta$.

Consider a new diagram whose objects are algebras of the form $\boldsymbol{Fm}_{\mathsf{K}}(\lambda)/\phi$, where $\phi \subseteq \theta$ is a μ-generated K-congruence for some $\mu < \kappa$, and whose arrows are the canonical projections $f_{\phi\eta}\colon \boldsymbol{Fm}_{\mathsf{K}}(\lambda)/\phi \to \boldsymbol{Fm}_{\mathsf{K}}(\lambda)/\eta$ when $\phi \subseteq \eta$. Since κ is a regular cardinal, this diagram is κ-directed. Moreover, its colimit is given by $\boldsymbol{Fm}_{\mathsf{K}}(\lambda)/\theta$ and by the canonical projections $g_{\phi}\colon : \boldsymbol{Fm}_{\mathsf{K}}(\lambda)/\phi \to \boldsymbol{Fm}_{\mathsf{K}}(\lambda)/\theta$ for every object $\boldsymbol{Fm}_{\mathsf{K}}(\lambda)/\phi$ in the diagram. Thus, applying the fact that $\boldsymbol{A}$ is κ-presentable, we obtain a homomorphism $p\colon \boldsymbol{A} \to \boldsymbol{Fm}_{\mathsf{K}}(\lambda)/\phi$ (where ϕ is μ-generated for some $\mu < \kappa$) such that $g_{\phi} \circ p = h$. Now let $\{x_j/\theta : j < \lambda\}$ be the natural generators of $\boldsymbol{Fm}_{\mathsf{K}}(\lambda)/\theta$. We know that there are $\{a_j : j < \lambda\} \subseteq A$ such that $h(a_j) = x_j/\theta$. Then for every $j < \lambda$ we choose a term $t_j \in Fm(\lambda)$ such that t_j belongs to the congruence class $p(a_j)$. We have that

$$x_j/\theta = h(a_j) = g_\phi \circ p(a_j) = g_\phi(t_j/\phi) = t_j/\theta.$$

Hence we obtain that $\langle x_j, t_j \rangle \in \theta$ for every $j < \lambda$. Then consider the K-congruence η of $\boldsymbol{Fm}_{\mathsf{K}}(\lambda)$ generated by the union $X \cup \{\langle x_j, t_j \rangle : j < \lambda\}$, where X is the set of generators of ϕ. The algebra $\boldsymbol{Fm}_{\mathsf{K}}(\lambda)/\eta$ belongs to the diagram. Then we consider the map $f_{\phi\eta} \circ p \colon \boldsymbol{A} \to \boldsymbol{Fm}_{\mathsf{K}}(\lambda)/\eta$. It is very easy to check that $f_{\phi\eta} \circ p$ is injective. The fact that it is surjective follows from the fact that its image covers the generators of $\boldsymbol{Fm}_{\mathsf{K}}(\lambda)/\eta$. We conclude that $\boldsymbol{A}$ is isomorphic to $\boldsymbol{Fm}_{\mathsf{K}}(\lambda)/\eta$, where η is a μ-generated K-congruence and $\lambda, \mu < \kappa$.

For the other direction, let $\boldsymbol{A}$ be the quotient of $\boldsymbol{Fm}_{\mathsf{K}}(\lambda)$ under a μ-generated K-congruence θ for some $\lambda, \mu < \kappa$. Then consider a κ-directed diagram $\{\boldsymbol{B}_i : i \in I\}$ with arrows $f_{ij} \colon \boldsymbol{B}_i \to \boldsymbol{B}_j$ when $i \leqslant j$. Let $\boldsymbol{C}$ with arrows $g_i \colon \boldsymbol{B}_i \to \boldsymbol{C}$ be the colimit of the diagram. Suppose we have a homomorphism $h \colon \boldsymbol{A} \to \boldsymbol{C}$. Let $\{x_j/\theta : j < \lambda\}$ be the natural generators of $\boldsymbol{Fm}_{\mathsf{K}}(\lambda)/\theta$. Since $\lambda < \kappa$ and the diagram is κ-directed, there is $i \in I$ and elements $\{c_j : j < \lambda\} \subseteq B_i$ such that $g_i(c_j) = h(x_j/\theta)$ for every $j < \lambda$.

Recall that θ is μ-generated. For each of its generators $\langle \varphi, \psi \rangle$, we have that $g_i(\varphi(\vec{c})) = g_i(\psi(\vec{c}))$. From the fact that K is axiomatized by generalized quasi-equations in less than κ variables, it follows that the colimit $\boldsymbol{C}$ is obtained by considering the disjoint union of the objects in the diagram and identifying elements that become eventually equal along the arrows of the diagram. Thus there is $j \geqslant i$ such that $f_{ij}(\varphi(\vec{c})) = f_{ij}(\psi(\vec{c}))$. Since the generators of θ are less than κ and the diagram is κ-directed, there is a new $i \in I$ and new elements $\{c_j : j < \lambda\} \subseteq B_i$ such that

$$g_i(c_j) = h(x_j/\theta) \text{ and } g_i(\varphi(\vec{c})) = g_i(\psi(\vec{c}))$$

for every $j < \lambda$ and every pair $\langle \varphi, \psi \rangle$ in the generators of θ. In particular, this means the map $p \colon \boldsymbol{A} \to \boldsymbol{B}_i$, defined as $p(\varphi/\theta) := \varphi^{\boldsymbol{B}_i}(\bar{c})$ for every $\varphi/\theta \in A$, is a well-defined homomorphism such that $g_i \circ p = h$.

It only remains to prove that p is essentially unique. To this end, consider any other arrow $q \colon \boldsymbol{A} \to \boldsymbol{B}_i$ such that $g_i \circ q = h$. In particular, we have that $g_i \circ p(x_j/\theta) = g_i \circ q(x_j/\theta)$ for every $j < \lambda$. Again, recall that the colimit $\boldsymbol{C}$ is obtained by considering the disjoint union of the objects in the diagram and identifying elements that become eventually equal along the arrows of the diagram. Then for every $j < \lambda$ there is $r \geqslant i$ such that $f_{ir} \circ p(x_j/\theta) = f_{ir} \circ q(x_j/\theta)$. Since the diagram is κ-directed, there is a new $r \geqslant i$ such that $f_{ir} \circ p(x_j/\theta) = f_{ir} \circ q(x_j/\theta)$ for every $j < \lambda$. Thus the arrows $f_{ir} \circ p$ and $f_{ir} \circ q$ coincide on the generators of $\boldsymbol{A}$ and, therefore, are equal. ⊠

As a consequence we obtain the following:

Lemma 5.3. *Generalized quasi-varieties are locally presentable.*

Proof. Let K be a generalized quasi-variety. Clearly K is a locally small category. Moreover, we know that K is cocomplete. K is axiomatized by generalized quasi-equations in less than κ variables for some regular cardinal κ. We want to prove that K is a locally κ-presentable category.

From Lemma 5.2 it follows that κ-presetable objects (up to isomorphism) form a set. Then consider an algebra $\boldsymbol{A} \in \mathsf{K}$. Let $\pi\colon \boldsymbol{Fm}_{\mathsf{K}}(A) \to \boldsymbol{A}$ be the natural surjection and define $\theta := \mathrm{Ker}(\pi)$. We consider the set of algebras $\boldsymbol{Fm}_{\mathsf{K}}(B)/\phi$ where $B \subseteq A$ has cardinality $< \kappa$ and $\phi \subseteq \theta$ is a K-congruence generated by a set of cardinality $< \kappa$. We equip this set with maps $f_{\phi\eta}\colon \boldsymbol{Fm}_{\mathsf{K}}(B)/\phi \to \boldsymbol{Fm}_{\mathsf{K}}(C)/\eta$ such that $f_{\phi\eta}(b/\phi) := b/\eta$ for every $b \in B$, when $B \subseteq C$ and $\phi \subseteq \eta$. By Lemma 5.2 this is a diagram of κ-presentable objects that, moreover, is κ-directed. Since κ is strictly larger than the number of variables occurring in the generalized quasi-equations that axiomatize K, we know that the colimit of this diagram is obtained constructing the disjoint union of the factors and then identifying the elements that become eventually equal. Keeping this in mind, it is easy to see that this colimit is isomorphic to $\boldsymbol{A}$. ⊠

Thanks to Lemma 5.3 a particular instance of Theorem 5.1 is the following:

Theorem 5.4 (Adámek and Rosicky)**.** *A functor between generalized quasi-varieties is right adjoint if and only if it preserves limits and κ-directed colimits for some regular cardinal κ.*

5.2 Two deformations

In this section we will describe two general methods to deform a given generalized quasi-variety, obtaining a new generalized quasi-variety that is related to the first one by an adjunction. In particular, it turns out that every right adjoint between generalized quasi-variety arises (up to natural isomorphism) as a combination of these deformations (Theorem 5.29). Remarkably, in the particular case of category equivalence, these deformations coincide with the ones identified by McKenzie in [73] (see Examples 5.10 and 5.15).

The first deformation that we consider is just an infinite version of the usual *finite matrix power* construction. Let X be a class of similar algebras and κ be a cardinal. Then observe that every term $\varphi \in Fm(\kappa)$ induces a map $\varphi\colon A^{\kappa} \to A$ for every $\boldsymbol{A} \in \mathsf{X}$. Obviously this does not mean that κ-many variables actually occur in φ, as κ may be infinite.

Definition 5.5. Let $\kappa > 0$ be a cardinal and X a class of similar algebras. Then $\mathscr{L}_{\mathsf{X}}^{\kappa}$ is the algebraic language whose n-ary operations (for every $n \in \omega$) are all κ-sequences $\langle t_i : i < \kappa \rangle$ of terms t_i of the language of X built up with variables

$$\{x_m^j : 1 \leqslant m \leqslant n \text{ and } j < \kappa\}.$$

Observe that each t_i has a finite number of variables, possibly none, of each sequence $\vec{x}_m := \langle x_m^j : j < \kappa \rangle$ with $1 \leqslant m \leqslant n$. We will write $t_i = t_i(\vec{x}_1, \dots, \vec{x}_n)$ to denote this fact.

Example 5.6. Consider the variety of distributive lattices DL. Examples of basic binary operations of $\mathscr{L}^2_{\mathsf{DL}}$ are:

$$\langle x_1^1, x_1^2 \rangle \sqcap \langle x_2^1, x_2^2 \rangle := \langle x_1^1 \wedge x_2^1, x_1^2 \vee x_2^2 \rangle$$
$$\langle x_1^1, x_1^2 \rangle \sqcup \langle x_2^1, x_2^2 \rangle := \langle x_1^1 \vee x_2^1, x_1^2 \wedge x_2^2 \rangle.$$

Moreover, $\neg \langle x_1^1, x_1^2 \rangle := \langle x_1^2, x_1^1 \rangle$ is a basic unary operation of $\mathscr{L}^2_{\mathsf{DL}}$. ⊠

Definition 5.7. Consider an algebra $\boldsymbol{A} \in \mathsf{X}$ and a cardinal $\kappa > 0$. We let $\boldsymbol{A}^{[\kappa]}$ be the algebra of type $\mathscr{L}^\kappa_{\mathsf{X}}$ with universe A^κ where a n-ary operation $\langle t_i : i < \kappa \rangle$ is interpreted as

$$\langle t_i : i < \kappa \rangle (a_1, \dots, a_n) = \langle t_i^{\boldsymbol{A}}(a_1 / \vec{x}_1, \dots, a_n / \vec{x}_n) : i < \kappa \rangle$$

for every $a_1, \dots, a_n \in A^\kappa$. In other words $\langle t_i : i < \kappa \rangle (a_1, \dots, a_n)$ is the κ-sequence of elements of A defined as follows. Consider $i < \kappa$. Observe that only a finite number of variables occurs in t_i, say

$$t_i = t_i(x_1^{\alpha_1^1}, \dots, x_1^{\alpha_{m_1}^1}, \dots, x_n^{\alpha_1^n}, \dots, x_n^{\alpha_{m_n}^n})$$

where $\alpha_1^1, \dots, \alpha_{m_1}^1, \dots, \alpha_1^n, \dots, \alpha_{m_n}^n < \kappa$. Then the i-th component of the sequence $\langle t_i : i < \kappa \rangle (a_1, \dots, a_n)$ is

$$t_i^{\boldsymbol{A}}(a_1(\alpha_1^1), \dots, a_1(\alpha_{m_1}^1), \dots, a_n(\alpha_1^n), \dots, a_1(\alpha_{m_n}^n)).$$

If X is a class of similar algebras, we set

$$\mathsf{X}^{[\kappa]} := \mathbb{I}\{\boldsymbol{A}^{[\kappa]} : \boldsymbol{A} \in \mathsf{X}\}$$

and call it the κ-th *matrix power* of X.

Now, let $[\kappa]$ be the map defined as follows:

$$\boldsymbol{A} \longmapsto \boldsymbol{A}^{[\kappa]}$$
$$f \colon \boldsymbol{A} \to \boldsymbol{B} \longmapsto f^{[\kappa]} \colon \boldsymbol{A}^{[\kappa]} \to \boldsymbol{B}^{[\kappa]}$$

where $f^{[\kappa]} \langle a_i : i < \kappa \rangle := \langle f(a_i) : i < \kappa \rangle$, for every $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{X}$ and every homomorphism f.

Example 5.8. In Example 5.6 we described two binary operations $\sqcap$ and $\sqcup$ and a unary operation $\neg$ in $\mathscr{L}^2_{\mathsf{DL}}$. Let us explain how they are interpreted in the matrix power construction. Consider $\boldsymbol{A} \in \mathsf{DL}$. The universe of $\boldsymbol{A}^{[2]}$ is just the cartesian product $A \times A$. We have that:

$$\begin{aligned} \langle a, b\rangle \sqcap \langle c, d\rangle &= \langle a \wedge c, b \vee d\rangle \\ \langle a, b\rangle \sqcup \langle c, d\rangle &= \langle a \vee c, b \wedge d\rangle \\ \neg\langle a, b\rangle &= \langle b, a\rangle \end{aligned}$$

for every $\langle a, b\rangle, \langle c, d\rangle \in A \times A$.

Examples of matrix powers with infinite exponent are technically, but not conceptually, more involved. We review two of them in Examples 6.10 and 6.14. ⊠

Theorem 5.9. *Let X be a generalized quasi-variety and $\kappa > 0$ a cardinal. If Y is a generalized quasi-variety such that $\mathsf{X}^{[\kappa]} \subseteq \mathsf{Y}$, then $[\kappa] \colon \mathsf{X} \to \mathsf{Y}$ is a right adjoint functor.*

Proof. It is not difficult to see that the map $[\kappa]$ is a functor that preserves direct products and equalizers. Since all limits can be constructed as combination of products and equalizers, we conclude that $[\kappa]$ preserves limits. In view of Theorem 5.4 it only remains to show that it preserves λ-directed colimits for some regular cardinal λ. To this end, let λ be a regular cardinal that is larger than the number of variables occurring in the generalized quasi-equations axiomatizing X and Y. This makes sense, since X and Y are generalized quasi-varieties. Moreover, assume that λ is larger than κ. Then consider a λ-directed diagram $\{\boldsymbol{A}_i : i \in I\}$ with arrows $f_{ij} \colon \boldsymbol{A}_i \to \boldsymbol{A}_j$ when $i \leqslant j$ in X. Since λ is larger than the number of variables occurring in the generalized quasi-equations axiomatizing X, the directed colimit of this diagram is the algebra $\boldsymbol{A}$ obtained as follows. First we consider the disjoint union $\{\langle a, i\rangle : a \in A_i \text{ and } i \in I\}$. Then we factor out by the quotient with respect to the following equivalence relation

$$\theta := \{\langle\langle a, i\rangle, \langle b, j\rangle\rangle : \text{ there is } k \geqslant i, j \text{ such that } f_{ik}(a) = f_{jk}(b)\}$$

and define operations in the natural way. Analogously, the colimit in Y of the λ-directed diagram $\{\boldsymbol{A}_i^{[\kappa]} : i \in I\}$ with arrows $f_{ij}^{[\kappa]} \colon \boldsymbol{A}_i^{[\kappa]} \to \boldsymbol{A}_j^{[\kappa]}$ when $i \leqslant j$ is the algebra $\boldsymbol{B}$ obtained as follows. First we consider the disjoint union $\{\langle \vec{a}, i\rangle : \vec{a} \in A_i^{\kappa} \text{ and } i \in I\}$, then we factor it out by the equivalence relation

$$\phi := \{\langle\langle \vec{a}, i\rangle, \langle \vec{b}, j\rangle\rangle : \text{ there is } k \geqslant i, j \text{ such that } f_{ik}^{[\kappa]}(\vec{a}) = f_{jk}^{[\kappa]}(\vec{b})\}$$

and finally we define operations in the natural way.

We claim that the map $g \colon \boldsymbol{B} \to \boldsymbol{A}^{[\kappa]}$ defined as

$$g(\langle \vec{a}, i\rangle / \phi) := \langle\langle \vec{a}(r), i\rangle / \theta : r < \kappa\rangle$$

for every $\langle \vec{a}, i\rangle/\phi \in B$ is an isomorphism. It is very easy to see that g is well defined. To see that it is injective, we reason as follows. Suppose that $g(\langle \vec{a}, i\rangle/\phi) = g(\langle \vec{b}, j\rangle/\phi)$. This means that for every $r < \kappa$ there is $k_r \geqslant i, j$ such that $f_{ik_r}(\vec{a}(r)) = f_{jk_r}(\vec{b}(r))$. But since our diagram is λ-directed and $\kappa < \lambda$, there is $k \in I$ such that $k_r \leqslant k$ for every $r < \kappa$. In particular, this implies that $f_{ik}(\vec{a}(r)) = f_{jk}(\vec{b}(r))$ for every $r < \kappa$ and, therefore, that $f_{ik}^{[\kappa]}(\vec{a}) = f_{jk}^{[\kappa]}(\vec{b})$. This means that $\langle \vec{a}, i\rangle/\phi = \langle \vec{b}, j\rangle/\phi$, as desired.

Then we turn to show that g is surjective. Consider an element $\langle\langle a_r, i_r\rangle/\theta : r < \kappa\rangle \in \boldsymbol{A}^{[\kappa]}$. Again, since our diagram is λ-directed and $\kappa < \lambda$, there is $k \in I$ such that $k \geqslant i_r$ for every $r < \kappa$. In particular, this implies that $\langle\langle a_r, i_r\rangle/\theta : r < \kappa\rangle = \langle\langle f_{i_r k}(a_r), k\rangle/\theta : r < \kappa\rangle$. Now observe that the element $\vec{b} := \langle f_{i_r k}(a_r) : r < \kappa\rangle$ belongs to $A_k^{[\kappa]}$. Moreover, we have that $g(\langle \vec{b}, k\rangle/\phi) = \langle\langle f_{i_r k}(a_r), k\rangle/\theta : r < \kappa\rangle$, as desired.

It only remains to show that g is a homomorphism. To this end, let φ be a basic n-ary operation of Y and consider $\langle \vec{a}_1, i_1\rangle/\phi, \dots, \langle \vec{a}_n, i_n\rangle/\phi \in B$. Consider an index $j \geqslant i_1, \dots, i_n$. Then for every $s < \kappa$, we have the following (where φ_s is the s-th component of φ):

$$
\begin{aligned}
& \varphi^{\boldsymbol{A}^{[\kappa]}}(g\langle \vec{a}_1, i_1\rangle/\phi, \dots, g\langle \vec{a}_n, i_n\rangle/\phi)(s) \\
= \; & \varphi^{\boldsymbol{A}^{[\kappa]}}(\langle\langle \vec{a}_1(r), i_1\rangle/\theta : r < \kappa\rangle, \dots, \langle\langle \vec{a}_n(r), i_n\rangle/\theta : r < \kappa\rangle)(s) \\
= \; & \varphi^{\boldsymbol{A}^{[\kappa]}}(\langle\langle f_{i_1 j}(\vec{a}_1(r)), j\rangle/\theta : r < \kappa\rangle, \dots, \langle\langle f_{i_n j}(\vec{a}_n(r)), j\rangle/\theta : r < \kappa\rangle)(s) \\
= \; & \varphi_s^{\boldsymbol{A}}(\vec{x}_1/\langle\langle f_{i_1 j}(\vec{a}_1(r)), j\rangle/\theta : r < \kappa\rangle, \dots, \vec{x}_n/\langle\langle f_{i_n j}(\vec{a}_n(r)), j\rangle/\theta : r < \kappa\rangle) \\
= \; & \langle \varphi_s^{\boldsymbol{A}_j}(\vec{x}_1/f_{i_1 j}(\vec{a}_1), \dots, \vec{x}_n/f_{i_n j}(\vec{a}_n)), j\rangle/\theta \\
= \; & \langle \varphi^{\boldsymbol{A}_j^{[\kappa]}}(f_{i_1 j}^{[\kappa]}(\vec{a}_1), \dots, f_{i_n j}^{[\kappa]}(\vec{a}_n))(s), j\rangle/\theta \\
= \; & g(\langle \varphi^{\boldsymbol{A}_j^{[\kappa]}}(f_{i_1 j}^{[\kappa]}(\vec{a}_1), \dots, f_{i_n j}^{[\kappa]}(\vec{a}_n)), j\rangle/\phi)(s) \\
= \; & g(\varphi^{\boldsymbol{B}}(\langle f_{i_1 j}^{[\kappa]}(\vec{a}_1), j\rangle/\phi, \dots, \langle f_{i_n j}^{[\kappa]}(\vec{a}_n), j\rangle/\phi))(s) \\
= \; & g(\varphi^{\boldsymbol{B}}(\langle \vec{a}_1, i_1\rangle/\phi, \dots, \langle \vec{a}_n, i_n\rangle/\phi))(s).
\end{aligned}
$$

This concludes the proof of our claim.

To prove that $[\kappa]$ preserves λ-directed colimits, it only remains to show that $g \circ q_i = p_i^{[\kappa]}$ for every $i \in I$, where $q_i \colon \boldsymbol{A}_i^{[\kappa]} \to \boldsymbol{B}$ and $p_i \colon \boldsymbol{A}_i \to \boldsymbol{A}$ are the maps associated with the colimits $\boldsymbol{B}$ and $\boldsymbol{A}$ respectively. But this is an easy consequence of the fact that

$$q_i(\vec{a}) := \langle \vec{a}, i\rangle/\phi \text{ and } p_i(a) := \langle a, i\rangle/\theta$$

for every $\vec{a} \in A_i^{\kappa}$ and $a \in A_i$. Therefore we can apply Theorem 5.4, concluding that $[\kappa]$ is a right adjoint functor. ⊠

Example 5.10 (Finite Exponent)**.** It is worth to remark that, when κ is finite, the functor $[\kappa]\colon \mathsf{X} \to \mathsf{X}^{[\kappa]}$ is a category equivalence [73, Theorem 2.3.(i)]. In this case if X is a prevariety (or a generalized quasi-variety, a quasi-variety, a variety), then so is $\mathsf{X}^{[\kappa]}$. This is not the case in general: when κ is infinite it may happen that $[\kappa]\colon \mathsf{X} \to \mathsf{Y}$ fails to be a category equivalence for every prevariety Y containing $\mathsf{X}^{[\kappa]}$.

To construct the necessary counterexample, we reason as follows. First observe that, given a prevariety K, an *infinite* algebra $\boldsymbol{A} \in \mathsf{K}$ has cardinality λ if and only if the following conditions hold:

1. The set $\hom(\boldsymbol{B}, \boldsymbol{A})$ has cardinality $\leqslant \lambda$ for every finitely generated algebra $\boldsymbol{B} \in \mathsf{K}$.
2. There is a finitely generated algebra $\boldsymbol{B} \in \mathsf{K}$ such that $\hom(\boldsymbol{B}, \boldsymbol{A})$ has exactly cardinality λ.

To see this, observe if $\boldsymbol{A}$ has infinite cardinality λ and $\boldsymbol{B}$ is n-generated, then the cardinality of $\hom(\boldsymbol{B}, \boldsymbol{A})$ is $\leqslant \lambda^n = \lambda$. Moreover there is a finitely generated algebra $\boldsymbol{B}$, e.g., the one-generated free algebra, such that $\hom(\boldsymbol{B}, \boldsymbol{A})$ has cardinality λ. This justifies the equivalence between having cardinality λ and conditions 1 and 2.

Together with the fact that the notion of a finitely generated algebra is categorical in prevarieties [73, Theorem 3.1.(5)] and that category equivalences preserve the cardinality of hom-sets, this implies that category equivalences preserve also infinite cardinalities.[†] We will use this fact to construct the desired counterexample. Consider a generalized quasi-variety X of finite type and an infinite cardinal κ. We know that the free algebra $\boldsymbol{Fm}_{\mathsf{X}}(\kappa)$ has cardinality κ and that its matrix power $\boldsymbol{Fm}_{\mathsf{X}}(\kappa)^{[\kappa]}$ has cardinality κ^κ. Since $\kappa < \kappa^\kappa$, we conclude that the functor $[\kappa]$ does not preserve infinite cardinalities. Thus $[\kappa]\colon \mathsf{X} \to \mathsf{Y}$ is not a category equivalence, for every prevariety Y containing $\mathsf{X}^{[\kappa]}$. ⊠

In order to describe the second kind of deformation, we need to introduce a new concept:

Definition 5.11. Let X be a class of similar algebras and $\mathscr{L} \subseteq \mathscr{L}_{\mathsf{X}}$. A set of equations $\theta \subseteq Eq(\mathscr{L}_{\mathsf{X}}, 1)$ is *compatible* with $\mathscr{L}$ in X if for every n-ary operation $\varphi \in \mathscr{L}$ we have that:

$$\theta(x_1) \cup \cdots \cup \theta(x_n) \vDash_{\mathsf{X}} \theta(\varphi(x_1, \ldots, x_n)).$$

In other words θ is compatible with $\mathscr{L}$ in X when the solution sets of θ in X are closed under the interpretation of the operations and constants in $\mathscr{L}$.

[†]This contrast with the fact that category equivalences between pravarieties do not preserve the cardinality of *finite* algebras. Nevertheless, they preserve the fact of being *finite* [73, Theorem 3.1.(7)].

Now we will explain how it is possible to build a functor out of a set of equations θ compatible with $\mathscr{L} \subseteq \mathscr{L}_{\mathsf{X}}$. For every $\boldsymbol{A} \in \mathsf{X}$, we let $\boldsymbol{A}(\theta, \mathscr{L})$ be the algebra of type $\mathscr{L}$ whose universe is

$$A(\theta, \mathscr{L}) := \{a \in A : \boldsymbol{A} \vDash \theta(a)\}$$

equipped with the restriction of the operations in $\mathscr{L}$. We know that $\boldsymbol{A}(\theta, \mathscr{L})$ is well-defined, since its universe is closed under the interpretation of the operations in $\mathscr{L}$ and contains the interpretation of the constants in $\mathscr{L}$. Observe that by definition of compatibility $\boldsymbol{A}(\theta, \mathscr{L})$ can be empty if and only if $\mathscr{L}$ contains no constant symbol. Given a homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ in X, we denote its restriction to $A(\theta, \mathscr{L})$ by

$$\theta_{\mathscr{L}}(f)\colon \boldsymbol{A}(\theta, \mathscr{L}) \to \boldsymbol{B}(\theta, \mathscr{L}).$$

It is easy to see that $\theta_{\mathscr{L}}(f)$ is a well-defined homomorphism. Now, consider the following class of algebras:

$$\mathsf{X}(\theta, \mathscr{L}) := \mathbb{I}\{\boldsymbol{A}(\theta, \mathscr{L}) : \boldsymbol{A} \in \mathsf{X}\}.$$

Let $\theta_{\mathscr{L}}\colon \mathsf{X} \to \mathsf{X}(\theta, \mathscr{L})$ be the map defined by the following rule:

$$\begin{aligned} \boldsymbol{A} &\longmapsto \boldsymbol{A}(\theta, \mathscr{L}) \\ f\colon \boldsymbol{A} \to \boldsymbol{B} &\longmapsto \theta_{\mathscr{L}}(f)\colon \boldsymbol{A}(\theta, \mathscr{L}) \to \boldsymbol{B}(\theta, \mathscr{L}). \end{aligned}$$

It is easy to check that $\theta_{\mathscr{L}}$ is a functor.

Theorem 5.12. *Let X be a generalized quasi-variety and $\theta \subseteq Eq(\mathscr{L}_{\mathsf{X}}, 1)$ a set of equations compatible with $\mathscr{L} \subseteq \mathscr{L}_{\mathsf{X}}$. If Y is a generalized quasi-variety such that $\mathsf{X}(\theta, \mathscr{L}) \subseteq \mathsf{Y}$, then $\theta_{\mathscr{L}}\colon \mathsf{X} \to \mathsf{Y}$ is a right adjoint functor.*

Proof. By Theorem 5.4 we know that the functor $\theta_{\mathscr{L}}$ is a right adjoint if and only if it preserves limits and κ-directed colimits for some regular cardinal κ. We begin by proving that $\theta_{\mathscr{L}}$ preserves limits. It will be enough to show that it preserves direct products and equalizers. To do this, consider a family $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{X}$. It is easy to see that

$$(\prod_{i \in I} \boldsymbol{A}_i)(\theta, \mathscr{L}) = \prod_{i \in I} \boldsymbol{A}_i(\theta, \mathscr{L})$$

and that projections are sent to projections. As to equalizers, the situation is analogous. Consider two homomorphisms $f, g\colon \boldsymbol{A} \rightrightarrows \boldsymbol{B}$ in X. Their equalizer is the inclusion map $e\colon \boldsymbol{C} \to \boldsymbol{A}$, where $\boldsymbol{C}$ is the subalgebra of $\boldsymbol{A}$ with universe $\{a \in A : f(a) = g(a)\}$. Keeping this in mind, it is clear that $\theta_{\mathscr{L}}(e)\colon \boldsymbol{C}(\theta, \mathscr{L}) \to \boldsymbol{A}(\theta, \mathscr{L})$ is an inclusion map, whose range consists of objects on which $\theta_{\mathscr{L}}(f)$ and $\theta_{\mathscr{L}}(g)$ are identical. Therefore it only remains to prove that $\theta_{\mathscr{L}}(e)$ covers all the elements on which $\theta_{\mathscr{L}}(f)$ and $\theta_{\mathscr{L}}(g)$

coincide. Pick $a \in A(\theta, \mathscr{L})$ such that $\theta_{\mathscr{L}}(f)(a) = \theta_{\mathscr{L}}(g)(a)$. This means that $f(a) = g(a)$ and, therefore, that $a \in C$. Moreover a is a solution to all the equations in θ, therefore we obtain that $a \in C(\theta, \mathscr{L})$. This concludes the proof that $\theta_{\mathscr{L}}$ preserves limits.

It only remains to prove that $\theta_{\mathscr{L}}$ preserves κ-directed colimits for some regular cardinal κ. Let κ be regular cardinal larger than the number of variables occurring in the generalized quasi-equations axiomatizing X. This makes sense, since X is a generalized quasi-variety. Consider a κ-directed diagram $\{\boldsymbol{A}_i : i \in I\}$ with arrows $f_{ij} \colon \boldsymbol{A}_i \to \boldsymbol{A}_j$ when $i \leqslant j$ in X. Its directed colimit is the algebra $\boldsymbol{A}$ obtained as follows. First we consider the disjoint union $\{\langle a, i\rangle : a \in A_i \text{ and } i \in I\}$. Then we pass to the quotient with respect to the following equivalence relation

$$\phi := \{\langle\langle a, i\rangle, \langle b, j\rangle\rangle : \text{ there is } k \geqslant i, j \text{ such that } f_{ik}(a) = f_{jk}(b)\}$$

and define operations in the natural way. It is now clear that the algebra $\boldsymbol{A}(\theta, \mathscr{L})$ is obtained analogously out of the κ-directed diagram $\{\boldsymbol{A}_i(\theta, \mathscr{L}) : i \in I\}$ and $\theta_{\mathscr{L}}(f_{ij})$ for $i \leqslant j$. Therefore the directed colimit of this diagram is the quotient of $\boldsymbol{A}(\theta, \mathscr{L})$ with respect to its smallest Y-congruence. But this congruence is the identity, because we assume that $\boldsymbol{A}(\theta, \mathscr{L}) \in$ Y. Therefore we conclude $\boldsymbol{A}(\theta, \mathscr{L})$ is the directed colimit of the diagram as desired. ⊠

A familiar instance of the above construction is the following:

Example 5.13 (Subreducts)**.** Let X be a (generalized) quasi-variety and $\mathscr{L} \subseteq \mathscr{L}_{\mathsf{X}}$. A *$\mathscr{L}$-subreduct* of an algebra $\boldsymbol{A} \in$ X is a subalgebra of the $\mathscr{L}$-reduct of $\boldsymbol{A}$. From [55, Proposition 2.3.19] it is easy to infer that the class Y of $\mathscr{L}$-subreducts of algebras in X is a (generalized) quasi-variety. For quasi-varieties this fact was proved by Maltsev [72]. Consider the forgetful functor $\mathcal{U} \colon \mathsf{X} \to \mathsf{Y}$. It is easy to see that $\mathcal{U} = \theta_{\mathscr{L}}$ where $\theta = \emptyset$. From Theorem 5.12 it follows that $\mathcal{U}$ has a left adjoint. ⊠

In the next example we will illustrate how the two deformations introduced so far can be combined to describe right adjoint functors.

Example 5.14 (Kleene Algebras)**.** A *Kleene algebra* $\boldsymbol{A} = \langle A, \sqcap, \sqcup, \neg, 0, 1\rangle$ is a De Morgan algebra in which the equation $x \sqcap \neg x \leqslant y \sqcup \neg y$ holds. We denote by KA the variety of Kleene algebras and by DL_{01} the variety of bounded distributive lattices. In [25] (but see also [62]) a way of constructing Kleene algebras out of bounded distributive lattices is described. More precisely, given $\boldsymbol{A} \in \mathsf{DL}_{01}$, the Kleene algebra $\mathcal{G}(\boldsymbol{A})$ has universe

$$G(A) := \{\langle a, b\rangle \in A^2 : a \wedge b = 0\}$$

and operations defined as

$$\begin{aligned}\langle a,b\rangle \sqcap \langle c,d\rangle &:= \langle a\wedge c, b\vee d\rangle\\ \langle a,b\rangle \sqcup \langle c,d\rangle &:= \langle a\vee c, b\wedge d\rangle\\ \neg\langle a,b\rangle &:= \langle b,a\rangle\\ 1 &:= \langle 1,0\rangle\\ 0 &:= \langle 0,1\rangle\end{aligned}$$

for every $\langle a,b\rangle, \langle c,d\rangle \in G(A)$. Moreover, given a homomorphism $f\colon \boldsymbol{A}\to\boldsymbol{B}$ in DL_{01}, the map $\mathcal{G}(f)\colon \mathcal{G}(\boldsymbol{A})\to\mathcal{G}(\boldsymbol{B})$ is defined by replicating f componentwise. It turns out that $\mathcal{G}\colon \mathsf{DL}_{01}\to\mathsf{KA}$ is a right adjoint functor [25, Theorem 1.7].

It is worth to remark that DL_{01} and KA are *not* categorically equivalent and, therefore, that $\mathcal{G}$ is not a category equivalence. This follows from the following observations:

1. Category equivalences between prevarieties preserve the fact of being a non-trivial subdirectly irreducible algebra.
2. DL_{01} has up to isomorphism only one non-trivial subdirectly irreducible member (the two-element chain), while KA has two (the two and the three-element chains).

In order to decompose $\mathcal{G}$ into a combination of our two deformations, we reason as follows. First consider the matrix power functor $[2]\colon \mathsf{DL}_{01}\to\mathsf{DL}_{01}^{[2]}$. Recall from Example 5.10 that it is a category equivalence and that $\mathsf{DL}_{01}^{[2]}$ is a variety. Now consider the following sublanguage $\mathscr{L}$ of the language of $\mathsf{DL}_{01}^{[2]}$ (cfr. Example 5.6):

$$\begin{aligned}\langle x^1,x^2\rangle \sqcap \langle y^1,y^2\rangle &:= \langle x^1\wedge y^1, x^2\vee y^2\rangle\\ \langle x^1,x^2\rangle \sqcup \langle y^1,y^2\rangle &:= \langle x^1\vee y^1, x^2\wedge y^2\rangle\\ \neg\langle x^1,x^2\rangle &:= \langle x^2,x^1\rangle\\ 1 &:= \langle 1,0\rangle\\ 0 &:= \langle 0,1\rangle.\end{aligned}$$

Then consider the set of equations

$$\theta := \{\langle x^1\wedge x^2, x^1\wedge x^2\rangle \approx \langle 0,0\rangle\} \subseteq Eq(\mathscr{L}_{\mathsf{DL}_{01}^{[2]}},1).$$

The set of equations θ is compatible with $\mathscr{L}$. For example the compatibility of θ w.r.t. $\sqcap$ amounts to the following condition: For every $\boldsymbol{A}\in\mathsf{DL}_{01}$ and $\langle a,b\rangle,\langle c,d\rangle \in A\times A$ if

$$\langle x^1\wedge x^2, x^1\wedge x^2\rangle^{\boldsymbol{A}^{[2]}}(\langle a,b\rangle) = \langle 0,0\rangle \text{ and } \langle x^1\wedge x^2, x^1\wedge x^2\rangle^{\boldsymbol{A}^{[2]}}(\langle c,d\rangle) = \langle 0,0\rangle$$

then

$$\langle x^1 \wedge x^2, x^1 \wedge x^2 \rangle^{A^{[2]}} (\langle a, b \rangle \sqcap^{A^{[2]}} \langle c, d \rangle) = \langle 0, 0 \rangle.$$

The condition above is equivalent to the following elementary fact: For every $\boldsymbol{A} \in \mathsf{DL}_{01}$ and $\langle a, b \rangle, \langle c, d \rangle \in A \times A$

$$\text{if } a \wedge b = 0 \text{ and } c \wedge d = 0, \text{ then } (a \wedge c) \wedge (b \vee d) = 0.$$

This shows that θ is compatible with $\sqcap$. A similar argument shows that θ is compatible with the whole $\mathscr{L}$.

Moreover, for every $\boldsymbol{A} \in \mathsf{DL}_{01}$ and $a, b \in A$ we have that:

$$\begin{aligned} \langle a, b \rangle \in \mathcal{G}(\boldsymbol{A}) &\iff a \wedge b = 0 \\ &\iff \langle a \wedge b, a \wedge b \rangle = \langle 0, 0 \rangle \\ &\iff \boldsymbol{A}^{[2]} \vDash \langle x^1 \wedge x^2, x^1 \wedge x^2 \rangle \approx \langle 0, 0 \rangle [\![a, b]\!] \\ &\iff \langle a, b \rangle \in A(\theta, \mathscr{L}). \end{aligned}$$

Hence we conclude $\boldsymbol{A}^{[2]}(\theta, \mathscr{L}) = \mathcal{G}(\boldsymbol{A}) \in \mathsf{KA}$ for every $\boldsymbol{A} \in \mathsf{DL}_{01}^{[2]}$. But this implies that $\theta_{\mathscr{L}} \colon \mathsf{DL}_{01}^{[2]} \to \mathsf{KA}$ is a right adjoint functor by Theorem 5.12. Finally, the functor $\mathcal{G}$ coincides with the composition $\theta_{\mathscr{L}} \circ [2]$ as desired. Observe that we showed that $\mathcal{G}$ is the composition of two right adjoint functors. Thus we obtained a new and purely combinatorial proof of the fact that $\mathcal{G}$ is a right adjoint functor. ⊠

Before concluding this section, it is worth to remark that the deformations described until now can be applied to decompose equivalence functors between prevarieties. In order to explain how, let us recall the definition of a special version of the $\theta_{\mathscr{L}}$ construction.

Example 5.15 (Idempotent and Invertible Terms)**.** Suppose that X is a prevariety and $\sigma(x)$ a unary term. We say that $\sigma(x)$ is *idempotent* if $\mathsf{X} \vDash \sigma\sigma(x) \approx \sigma(x)$ and that $\sigma(x)$ is *invertible* if there are an n-ary term t and unary terms $t_1, \ldots, t_n$ such that

$$\mathsf{X} \vDash t(\sigma t_1(x), \ldots, \sigma t_n(x)) \approx x.$$

Given a unary and idempotent term $\sigma(x)$ of X, we define

$$\mathscr{L} := \{\sigma t : t \text{ is a basic symbol of } \mathsf{X}^{[1]}\}$$

and $\theta := \{x \approx \sigma(x)\}$. Moreover, we define

$$\mathsf{X}(\sigma) := \mathsf{X}^{[1]}(\theta, \mathscr{L}).$$

McKenzie proved that the functor $\sigma \colon \mathsf{X} \to \mathsf{X}(\sigma)$ defined as the composition $\theta_{\mathscr{L}} \circ [1]$ is a category equivalence [73, Theorem 2.2.(ii)]. Moreover, if X is a prevariety (or a generalized quasi-variety, a quasi-variety, a variety), then so is $\mathsf{X}(\sigma)$. Following the literature, we will write $\boldsymbol{A}(\sigma)$ instead of $\sigma(\boldsymbol{A})$ for every $\boldsymbol{A} \in \mathsf{X}$. ⊠

To introduce McKenzie's characterization of category equivalence we need to recall some basic concepts [13, Definitions 4.76 and 4.77]:

Definition 5.16. Let X and Y be prevarieties. An *interpretation* of X in Y is a map $\boldsymbol{\tau}\colon \mathscr{L}_{\mathsf{X}} \to Fm(\mathscr{L}_{\mathsf{Y}}, \omega)$ such that:

1. $\boldsymbol{\tau}$ sends n-ary basic symbols to at most n-ary terms for every $n \geqslant 1$.
2. $\boldsymbol{\tau}$ sends constants symbols to at most unary terms.
3. $\boldsymbol{A}^{\boldsymbol{\tau}} := \langle A, \{\boldsymbol{\tau}(\lambda) : \lambda \in \mathscr{L}_{\mathsf{X}}\}\rangle \in \mathsf{X}$.

Definition 5.17. Two prevarieties X and Y are *term-equivalent* if there are interpretations $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ of X' in Y' and of Y' in X' respectively such that for every $\boldsymbol{A} \in \mathsf{X}'$ and $\boldsymbol{B} \in \mathsf{Y}'$

$$(\boldsymbol{A}^{\boldsymbol{\rho}})^{\boldsymbol{\tau}} = \boldsymbol{A} \text{ and } (\boldsymbol{B}^{\boldsymbol{\tau}})^{\boldsymbol{\rho}} = \boldsymbol{B}$$

where X' and Y' are the prevarieties obtained from replacing the constant symbols by new constant unary terms in X and Y respectively.

When two prevarieties X and Y without constant symbols are term-equivalent as in condition 4 of the above definition, the map that sends $\boldsymbol{A} \in \mathsf{X}$ to $\boldsymbol{A}^{\boldsymbol{\rho}}$ and is the identity on arrows is a category equivalence $\mathcal{F}^{\boldsymbol{\rho}}\colon \mathsf{X} \to \mathsf{Y}$. Then we have the following [73, Theorem 6.1]:

Theorem 5.18 (McKenzie)**.** *If $\mathcal{G}\colon \mathsf{X} \to \mathsf{Y}$ is a category equivalence between prevarieties without constant symbols, then there is a natural $n > 0$, a unary idempotent and invertible term $\sigma(x)$ of $\mathsf{X}^{[n]}$ such that*

1. *Y is term-equivalent to $\mathsf{X}^{[n]}(\sigma)$ under some interpretation $\boldsymbol{\rho}$ of Y in $\mathsf{X}^{[n]}(\sigma)$.*
2. *The functors $\mathcal{G}$ and $\mathcal{F}^{\boldsymbol{\rho}} \circ (\sigma \circ [n])$ are naturally isomorphic.*

Observe the restriction to prevarieties without constant symbols in Theorem 5.18 is not very important. To see this, observe that, given a prevarety K, we can always replace the constant symbols of K by constant unary operations obtaining a new prevareity K' whose only difference with K is the presence of the empty algebra.

5.3 From translations to right adjoints

As we mentioned, our aim is to develop a correspondence between the *adjunctions* between two generalized quasi-varieties X and Y and the *translations* between the equational consequences relative to X and Y. The first step we make in this direction is to introduce a precise notion of translation between relative equational consequences. Subsequently, we use these translations to construct right adjoint functors (Theorem 5.23). To simplify the notation, we will assume all along the section that X and Y are two fixed generalized quasi-varieties (possibly in different languages).

Definition 5.19. Consider a cardinal $\kappa > 0$. A *κ-translation* $\boldsymbol{\tau}$ of $\mathscr{L}_{\mathsf{X}}$ into $\mathscr{L}_{\mathsf{Y}}$ is a map from $\mathscr{L}_{\mathsf{X}}$ to $\mathscr{L}_{\mathsf{Y}}^{\kappa}$ that preserves the arities of function symbols.

In other words, if a basic symbol $\varphi \in \mathscr{L}_{\mathsf{X}}$ is n-ary, we have that $\boldsymbol{\tau}(\varphi) = \langle t_i : i < \kappa \rangle$ for some terms $t_i = t_i(\vec{x}_1, \ldots, \vec{x}_n)$ of language of Y, where $\vec{x}_m = \langle x_m^j : j < \kappa \rangle$. It is worth to remark that $\boldsymbol{\tau}$ sends constant symbols to sequences of constant symbols. Thus if $\mathscr{L}_{\mathsf{X}}$ contains a constant symbol, then also $\mathscr{L}_{\mathsf{Y}}$ must contain one for a translation to exists.

A κ-translation $\boldsymbol{\tau}$ extends naturally to arbitrary terms. Let us explain briefly how. Given a cardinal λ, let $Fm(\mathscr{L}_{\mathsf{X}}, \lambda)$ be the set of terms of X written with variables in $\{x_j : j < \lambda\}$ and let $Fm(\mathscr{L}_{\mathsf{Y}}, \kappa \times \lambda)$ be the set of terms of Y written with variables in $\{x_j^i : j < \lambda, i < \kappa\}$. We define recursively a map

$$\boldsymbol{\tau}_* \colon Fm(\mathscr{L}_{\mathsf{X}}, \lambda) \to Fm(\mathscr{L}_{\mathsf{Y}}, \kappa \times \lambda)^{\kappa}.$$

For variables and constants we set

$$\boldsymbol{\tau}_*(x_j) := \langle x_j^i : i < \kappa \rangle, \text{ for every } j < \lambda$$
$$\boldsymbol{\tau}_*(c) := \boldsymbol{\tau}(c).$$

For complex terms we reason as follows. Suppose that we are given an n-ary symbol $\psi \in \mathscr{L}_{\mathsf{X}}$ and $\varphi_1, \ldots, \varphi_n \in Fm(\mathscr{L}_{\mathsf{X}}, \lambda)$. We have that $\boldsymbol{\tau}(\psi) = \langle t_i : i < \kappa \rangle$ where $t_i = t_i(\vec{x}_1, \ldots, \vec{x}_n)$. Keeping this in mind, we set

$$\boldsymbol{\tau}_*(\psi(\varphi_1, \ldots, \varphi_n))(i) := t_i(\boldsymbol{\tau}_*(\varphi_1)/\vec{x}_1, \ldots, \boldsymbol{\tau}_*(\varphi_n)/\vec{x}_n) \text{ for every } i < \kappa.$$

The map $\boldsymbol{\tau}_*$ can be lifted to sets of equations yielding a new function

$$\boldsymbol{\tau}^* \colon \mathcal{P}(Eq(\mathscr{L}_{\mathsf{X}}, \lambda)) \to \mathcal{P}(Eq(\mathscr{L}_{\mathsf{Y}}, \kappa \times \lambda))$$

defined by the following rule:

$$\Phi \longmapsto \{\boldsymbol{\tau}_*(\varepsilon)(i) \approx \boldsymbol{\tau}_*(\delta)(i) : i < \kappa \text{ and } \varepsilon \approx \delta \in \Phi\}.$$

Observe that there is a qualitative difference between $\boldsymbol{\tau}^*$ and $\boldsymbol{\tau}_*$: the map $\boldsymbol{\tau}^*$ translates sets of equations of X into sets of equations of Y, while $\boldsymbol{\tau}_*$ translates terms of X into κ-sequences of terms (and not simply terms) of Y.

Definition 5.20. Consider a cardinal $\kappa > 0$. A *κ-translation* of $\vDash_{\mathsf{X}}$ into $\vDash_{\mathsf{Y}}$ is a pair $\langle \boldsymbol{\tau}, \Theta \rangle$ where $\boldsymbol{\tau}$ is a κ-translation of $\mathscr{L}_{\mathsf{X}}$ into $\mathscr{L}_{\mathsf{Y}}$ and $\Theta(\vec{x}) \subseteq Eq(\mathscr{L}_{\mathsf{Y}}, \kappa)$ is a set of equations written with variables among $\{x^i : i < \kappa\}$ that satisfies the following conditions:

1. For every cardinal λ and equations $\Phi \cup \{\varepsilon \approx \delta\} \subseteq Eq(\mathscr{L}_{\mathsf{X}}, \lambda)$ written with variables among $\{x_j : j < \lambda\}$:

$$\text{if } \Phi \vDash_{\mathsf{X}} \varepsilon \approx \delta, \text{ then } \boldsymbol{\tau}^*(\Phi) \cup \bigcup_{j<\lambda} \Theta(\vec{x}_j) \vDash_{\mathsf{Y}} \boldsymbol{\tau}^*(\varepsilon \approx \delta).$$

2. For every n-ary operation $\psi \in \mathscr{L}_{\mathsf{X}}$:

$$\Theta(\vec{x}_1) \cup \cdots \cup \Theta(\vec{x}_n) \vDash_{\mathsf{Y}} \Theta(\boldsymbol{\tau}_* \psi(x_1, \ldots, x_n)).$$

In 1 and 2 it is intended that $\vec{x}_j = \langle x_j^i : i < \kappa \rangle$.

A κ-translation $\langle \boldsymbol{\tau}, \Theta \rangle$ of $\vDash_{\mathsf{X}}$ into $\vDash_{\mathsf{Y}}$ is *non-trivial*‡ provided that: If there is a (non-empty) sequence $\vec{\varphi} \in Fm(\mathscr{L}_{\mathsf{Y}}, 0)^{\kappa}$ of constant symbols such that $\mathsf{Y} \vDash \Theta(\vec{\varphi})$, then there is $i < \kappa$ and sequences of variables

$$\vec{x} = \langle x^i : i < \kappa \rangle \text{ and } \vec{y} = \langle y^i : i < \kappa \rangle$$

such that

$$\Theta(\vec{x}) \cup \Theta(\vec{y}) \nvDash_{\mathsf{Y}} x^i \approx y^i.$$

Several translations between logics in the literature provide examples of this general definition of translation between relative equational consequences. Let us recall some of them:

Example 5.21 (Heyting and Interior Algebras)**.** As shown by Gödel in [54], it is possible to interpret the intuitionistic propositional calculus $\mathcal{IPC}$ into the consequence relation associated with the global modal system $\mathcal{S}4$ [67, 68]. Since these two logics are algebraizable with equivalent algebraic semantics the variety of Heyting algebras HA and of interior algebras IA respectively, this interpretation can be lifted from terms to equations. More precisely, let $\boldsymbol{\tau}$ be the 1-translation of $\mathscr{L}_{\mathsf{HA}}$ into $\mathscr{L}_{\mathsf{IA}}$ defined as follows:

$$x \star y \longmapsto x \star y \quad \neg x \longmapsto \Box \neg x \quad x \to y \longmapsto \Box(x \to y)$$

for $\star \in \{\wedge, \vee\}$. The original version of Gödel's translation reads as follows:

$$\Gamma \vdash_{\mathcal{IPC}} \varphi \Longleftrightarrow \sigma \boldsymbol{\tau}_*(\Gamma) \vdash_{\mathcal{S}4} \sigma \boldsymbol{\tau}_*(\varphi) \tag{5.1}$$

for every $\Gamma \cup \{\varphi\} \subseteq Fm(\mathscr{L}_{\mathsf{HA}}, \lambda)$, where σ is the substitution sending every variable x to its necessitation $\Box x$. In fact Gödel proved the direction from left to right of (5.1), while the other one was supplied in [75] (cfr. also [37, 71] for some generalizations). In order to present this translation in our framework, we have to deal with the fact that we allow only translations that send variables to variables. This problem is overcome by introducing a context in the premises. To explain how, we recall that the terms $Fm(\mathscr{L}_{\mathsf{HA}}, \lambda)$ are written with variables among $\{x_j : j < \lambda\}$. Then we have that:

$$\sigma \boldsymbol{\tau}_*(\Gamma) \vdash_{\mathcal{S}4} \sigma \boldsymbol{\tau}_*(\varphi) \Longleftrightarrow \boldsymbol{\tau}_*(\Gamma) \cup \{x_j \leftrightarrow \Box x_j : j < \lambda\} \vdash_{\mathcal{S}4} \boldsymbol{\tau}_*(\varphi). \tag{5.2}$$

The left-to-right direction of (5.2) follows from the fact that the algebraic meaning of $x_j \leftrightarrow \Box x_j$ is $x_j \approx \Box x_j$. To prove the other direction, suppose that

‡This condition of *non-triviality* is designed in order to identify translations that correspond to non-trivial adjunctions. This will become clear in the proof of Theorem 5.23.

the right-hand deduction holds. Then by structurality we can apply to it the substitution σ. This fact, together with $\emptyset \vdash_{\mathcal{S}4} \Box x \leftrightarrow \Box\Box x$, yields the desired conclusion. Now, using the completeness of $\mathcal{IPC}$ and $\mathcal{S}4$ with respect to the corresponding equivalent algebraic semantics, we obtain that:

$$\Phi \vDash_{\mathsf{HA}} \varepsilon \approx \delta \Longleftrightarrow \boldsymbol{\tau}^*(\Phi) \cup \bigcup_{j<\lambda} \Theta(x_j) \vDash_{\mathsf{IA}} \boldsymbol{\tau}^*(\varepsilon \approx \delta) \tag{5.3}$$

for every $\Phi \cup \{\varepsilon \approx \delta\} \subseteq Eq(\mathscr{L}_{\mathsf{HA}}, \lambda)$, where $\Theta(x) = \{x \approx \Box x\}$. Observe that (5.3) implies condition 1 of Definition 5.20. Moreover, observe that in this case condition 2 of the same definition amounts to the following deductions, which are all easy to check:

$$x \approx \Box x \vDash_{\mathsf{IA}} \Box\neg x \approx \Box\Box\neg x$$
$$x \approx \Box x, y \approx \Box y \vDash_{\mathsf{IA}} x \star y \approx \Box(x \star y)$$
$$x \approx \Box x, y \approx \Box y \vDash_{\mathsf{IA}} \Box(x \to y) \approx \Box\Box(x \to y)$$

for each $\star \in \{\wedge, \vee\}$. Therefore we conclude that $\langle \boldsymbol{\tau}, \Theta \rangle$ is a translation of $\vDash_{\mathsf{HA}}$ into $\vDash_{\mathsf{IA}}$. ⊠

Example 5.22 (Heyting and Boolean Algebra)**.** The same trick can be applied to subsume in our framework Kolmogorov's interpretation of classical propositional calculus $\mathcal{CPC}$ into $\mathcal{IPC}$ [66]. Let $\boldsymbol{\tau}$ be the 1-translation defined as follows:

$$0 \longmapsto 0 \quad 1 \longmapsto 1 \quad \neg x \longmapsto \neg x \quad x \star y \longmapsto \neg\neg(x \star y)$$

for every $\star \in \{\wedge, \vee, \to\}$. The original translation of Kolmogorov states that

$$\Gamma \vdash_{\mathcal{CPC}} \varphi \Longleftrightarrow \sigma\boldsymbol{\tau}_*(\Gamma) \vdash_{\mathcal{IPC}} \sigma\boldsymbol{\tau}_*(\varphi)$$

for every $\Gamma \cup \{\varphi\} \subseteq Fm(\mathscr{L}, \lambda)$, where σ is the substitution sending every variable x to its double negation $\neg\neg x$. Combining it with the observation that $\emptyset \vdash_{\mathcal{IPC}} \neg x \leftrightarrow \neg\neg\neg x$, it is easy to see that $\langle \boldsymbol{\tau}, \Theta \rangle$ with $\Theta = \{x \approx \neg\neg x\}$ is a translation of $\vDash_{\mathsf{BA}}$ into $\vDash_{\mathsf{HA}}$, where BA is the variety of Boolean algebras. ⊠

The importance of non-trivial κ-translations of $\vDash_{\mathsf{X}}$ into $\vDash_{\mathsf{Y}}$ is that they correspond to non-trivial right adjoint functors from Y to X. In other words, right adjoints reverse the direction of translations and vice-versa. To explain how, consider a non-trivial κ-translation $\langle \boldsymbol{\tau}, \Theta \rangle$ of $\vDash_{\mathsf{X}}$ into $\vDash_{\mathsf{Y}}$. Then consider the set:

$$\mathscr{L} := \{\boldsymbol{\tau}(\psi) : \psi \in \mathscr{L}_{\mathsf{X}}\} \subseteq \mathscr{L}_{\mathsf{Y}}^{\kappa}. \tag{5.4}$$

Observe that $\mathscr{L}$ is a sublanguage of the language of the matrix power $\mathsf{Y}^{[\kappa]}$. Then consider the set

$$\theta := \{\vec{\varepsilon} \approx \vec{\delta} : \varepsilon \approx \delta \in \Theta\}$$

where $\vec{\varepsilon}$ and $\vec{\delta}$ are the κ-sequences constant on ε and δ respectively. Observe that θ is a set of identities between κ-sequences of terms of Y in κ-many variables. Now, κ-sequences of terms of Y in κ-many variables can be viewed as *unary* terms of the matrix power $\mathsf{Y}^{[\kappa]}$. Thus θ can be viewed as a set of equations in one variable in the language of $\mathsf{Y}^{[\kappa]}$. Hence we have the three basic ingredients of our construction: a matrix power $\mathsf{Y}^{[\kappa]}$, a sublanguage $\mathscr{L} \subseteq \mathscr{L}_{\mathsf{Y}}^{\kappa}$, and a set of equations $\theta \subseteq Eq(\mathscr{L}_{\mathsf{Y}}^{\kappa}, 1)$.

There is still a technicality we must take into account: when κ is infinite the matrix power $\mathsf{Y}^{[\kappa]}$ may fail to be a generalized quasi-variety. Let K be the class of algebras defined as follows:

$$\mathsf{K} := \begin{cases} \mathbb{Q}(\mathsf{Y}^{[\kappa]}) & \text{if } \mathsf{X} \text{ and } \mathsf{Y} \text{ are quasi-varieties} \\ & \text{and } \mathrm{Cg}_{\mathsf{Y}}^{\boldsymbol{Fm}_{\mathsf{Y}}(\kappa)}(\Theta) \text{ is finitely generated} \\ \mathbb{GQ}_{\lambda}(\mathsf{Y}^{[\kappa]}) & \text{otherwise, where } \lambda \geqslant \kappa \text{ is infinite and } \mathbb{U}_{\lambda}(\mathsf{X}) = \mathsf{X}. \end{cases}$$

Observe that in the above definition λ is not uniquely determined, but any choice would be equivalent for our purposes.

Theorem 5.23. *Let* X *and* Y *be generalized quasi-varieties and* $\langle \boldsymbol{\tau}, \Theta \rangle$ *be a non-trivial* κ*-translation of* $\vDash_{\mathsf{X}}$ *into* $\vDash_{\mathsf{Y}}$ *and let* K *be the class just introduced. The maps* $[\kappa]\colon \mathsf{Y} \to \mathsf{K}$ *and* $\theta_{\mathscr{L}}\colon \mathsf{K} \to \mathsf{X}$ *defined above are right adjoint functors. In particular, the composition* $\theta_{\mathscr{L}} \circ [\kappa]\colon \mathsf{Y} \to \mathsf{X}$ *is a non-trivial right adjoint.*

Proof. Observe that K is a generalized quasi-variety. Therefore we can apply Theorem 5.9, yielding that $[\kappa]\colon \mathsf{Y} \to \mathsf{K}$ is a right adjoint functor. Now we turn to prove the same for $\theta_{\mathscr{L}}$. We will detail the case where X and Y are quasi-varieties and $\mathrm{Cg}_{\mathsf{Y}}^{\boldsymbol{Fm}_{\mathsf{Y}}(\kappa)}(\Theta)$ finitely generated, since the other one is analogous. Since Y is a quasi-variety and $\mathrm{Cg}_{\mathsf{Y}}^{\boldsymbol{Fm}_{\mathsf{Y}}(\kappa)}(\Theta)$ is finitely generated, there is a finite set $\{\langle \alpha_i, \beta_i \rangle : i < n\} \subseteq \Theta$ such that $\{\langle \alpha_i, \beta_i \rangle : i < n\} \mathrel{=\!\!|\!\vDash_{\mathsf{Y}}} \Theta$. It is easy to see that

$$\{\vec{\alpha}_i \approx \vec{\beta}_i : i < n\} \mathrel{=\!\!|\!\vDash_{\mathsf{Y}^{[\kappa]}}} \theta \tag{5.5}$$

where $\vec{\alpha}_i$ and $\vec{\beta}_i$ are the κ-sequences constant on α_i and β_i respectively.

Now from condition 2 of Definition 5.20 it follows that θ is compatible with $\mathscr{L}$ in $\mathsf{Y}^{[\kappa]}$, where $\mathscr{L}$ is the language defined in (5.4). From (5.5) we know that this compatibility condition can be expressed by a set of deductions, whose antecedent is finite, of the equational consequence relative to $\mathsf{Y}^{[\kappa]}$, i.e.,

$$\bigcup_{j \leqslant m} \{\vec{\alpha}_i \approx \vec{\beta}_i : i < n\}(\vec{x}_j) \vDash_{\mathsf{Y}^{[\kappa]}} \theta(\boldsymbol{\tau}(\psi)(\vec{x}_1, \dots, \vec{x}_n))$$

for every m-ary $\psi \in \mathscr{L}$. In particular, this implies that θ is still compatible with $\mathscr{L}$ in K (recall that K is the *quasi-variety* generated by $\mathsf{Y}^{[\kappa]}$).

We claim that $\boldsymbol{A}(\theta, \mathscr{L}) \in \mathsf{X}$ for every $\boldsymbol{A} \in \mathsf{K}$. To prove this, consider any finite deduction

$$\varphi_1 \approx \psi_1, \dots, \varphi_m \approx \psi_m \vDash_{\mathsf{X}} \varepsilon \approx \delta.$$

Let $x_1, \ldots, x_p$ be the variables that occur in it. From condition 1 of Definition 5.20 it follows that

$$\{\boldsymbol{\tau}_*(\varphi_t) \approx \boldsymbol{\tau}_*(\psi_t) : t \leqslant m\} \cup \bigcup_{j \leqslant p} \theta(\vec{x}_j) \vDash_{\mathsf{Y}^{[\kappa]}} \boldsymbol{\tau}_*(\varepsilon) \approx \boldsymbol{\tau}_*(\delta)$$

where $\vec{x}_j = \langle x_j^i : i < \kappa \rangle$. Thanks to (5.5) the above deduction can be expressed by a collection of deductions, whose antecedent is finite, of the equational consequence relative to $\mathsf{Y}^{[\kappa]}$, i.e.,

$$\{\boldsymbol{\tau}_*(\varphi_t) \approx \boldsymbol{\tau}_*(\psi_t) : t \leqslant m\} \cup \bigcup_{j \leqslant p} \{\vec{\alpha}_i \approx \vec{\beta}_i : i < n\}(\vec{x}_j) \vDash_{\mathsf{Y}^{[\kappa]}} \boldsymbol{\tau}_*(\varepsilon) \approx \boldsymbol{\tau}_*(\delta).$$

Since K is the *quasi-variety* generated by $\mathsf{Y}^{[\kappa]}$, we know that the above deduction persists in K. Together with the fact that $\{\vec{\alpha}_i \approx \vec{\beta}_i : i < n\} \subseteq \theta$, this implies that for every $\boldsymbol{A} \in \mathsf{K}$ and every $a_1, \ldots, a_p \in A(\theta, \mathscr{L})$, we have that:

$$\begin{aligned} &\text{if } \boldsymbol{A} \vDash \boldsymbol{\tau}_*(\varphi_1) \approx \boldsymbol{\tau}_*(\psi_1), \ldots, \boldsymbol{\tau}_*(\varphi_m) \approx \boldsymbol{\tau}_*(\psi_m)[\![a_1, \ldots, a_p]\!], \\ &\text{then } \boldsymbol{A} \vDash \boldsymbol{\tau}_*(\varepsilon) \approx \boldsymbol{\tau}_*(\delta)[\![a_1, \ldots, a_p]\!]. \end{aligned}$$

But this means exactly that

$$\begin{aligned} &\text{if } \boldsymbol{A}(\theta, \mathscr{L}) \vDash \varphi_1 \approx \psi_1, \ldots, \varphi_m \approx \psi_m[\![a_1, \ldots, a_p]\!], \\ &\text{then } \boldsymbol{A}(\theta, \mathscr{L}) \vDash \varepsilon \approx \delta[\![a_1, \ldots, a_p]\!]. \end{aligned}$$

Thus we showed that $\boldsymbol{A}(\theta, \mathscr{L})$ satisfies every quasi-equation that holds in X. Since X is a quasi-variety, we conclude that $\boldsymbol{A}(\theta, \mathscr{L}) \in \mathsf{X}$. This establishes our claim. Hence we can apply Theorem 5.12 yielding that $\theta_{\mathscr{L}} \colon \mathsf{K} \to \mathsf{X}$ is a right adjoint functor. We conclude that $\theta_{\mathscr{L}} \circ [\kappa] \colon \mathsf{Y} \to \mathsf{X}$ is a right adjoint functor.

It only remains to prove that $\theta_{\mathscr{L}} \circ [\kappa]$ is non-trivial, i.e., that it does not send every algebra to the trivial one. First consider the case where there is no sequence $\vec{\varphi} \in Fm(\mathscr{L}_{\mathsf{Y}}, 0)^\kappa$ of constant symbols such that $\mathsf{Y} \vDash \Theta(\vec{\varphi})$. Then consider the free algebra $\boldsymbol{Fm}_{\mathsf{Y}}(0)$. We have that $\boldsymbol{Fm}_{\mathsf{Y}}(0)^{[\kappa]}(\theta, \mathscr{L}) = \emptyset$, otherwise the equations Θ would have a constant solution (which is not the case). Thus in this case the functor $\theta_{\mathscr{L}} \circ [\kappa]$ is non-trivial. Then consider the case where there is a non-empty sequence $\vec{\varphi} \in Fm(\mathscr{L}_{\mathsf{Y}}, 0)^\kappa$ such that $\mathsf{Y} \vDash \Theta(\vec{\varphi})$. Since $\langle \boldsymbol{\tau}, \Theta \rangle$ is non-trivial, we have that

$$\Theta(\vec{x}) \cup \Theta(\vec{y}) \nvDash_{\mathsf{Y}} x^i \approx y^i.$$

This means that there is an algebra $\boldsymbol{A} \in \mathsf{Y}$ and sequences $\vec{a}, \vec{c} \in A^\kappa$ such that that $\vec{a}, \vec{c} \in A^{[\kappa]}(\theta, \mathscr{L})$ and $\vec{a} \neq \vec{c}$. Thus the algebra $\boldsymbol{A}^{[\kappa]}(\theta, \mathscr{L})$ has at least two elements and, therefore, is non-trivial as desired. ⊠

If we apply the above construction to Gödel and Kolmogorov's translations, we obtain some well-known transformations:

Example 5.24 (Open and Regular Elements)**.** Given $\boldsymbol{A} \in \mathsf{IA}$, an element $a \in A$ is *open* if $\Box a = a$. The set of open elements $\mathrm{Op}(\boldsymbol{A})$ of $\boldsymbol{A}$ is closed under the lattice operations and contains the bounds. Moreover we can equip it with an implication $\multimap$ and with a negation $\sim$ defined for every $a, b \in \mathrm{Op}(\boldsymbol{A})$ as follows:

$$a \multimap b := \Box^{\boldsymbol{A}}(a \to^{\boldsymbol{A}} b) \text{ and } \sim a := \Box^{\boldsymbol{A}} \neg^{\boldsymbol{A}} a.$$

It is well known that

$$\mathbf{Op}(\boldsymbol{A}) := \langle \mathrm{Op}(\boldsymbol{A}), \wedge, \vee, \multimap, \sim, 0, 1 \rangle$$

is a Heyting algebra. Now, every homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ between interior algebras restricts to a homomorphism $f\colon \mathbf{Op}(\boldsymbol{A}) \to \mathbf{Op}(\boldsymbol{B})$. Therefore the application $\mathrm{Op}\colon \mathsf{IA} \to \mathsf{HA}$ can be regarded as a functor. As the reader may have guessed, it is in fact the right adjoint functor induced by Gödel's translation of $\mathcal{IPC}$ into $\mathcal{S}4$ (Example 5.21).

A similar correspondence arises from Kolmogorov's translation of $\mathcal{CPC}$ into $\mathcal{IPC}$. More precisely, given $\boldsymbol{A} \in \mathsf{HA}$, an element $a \in A$ is *regular* if $\neg\neg a = a$. It is well known that the set of regular elements $\mathrm{Reg}(\boldsymbol{A})$ of $\boldsymbol{A}$ is closed under $\wedge, \neg$ and $\to$ and contains the bounds. Moreover we can equip it with a new join $\sqcup$ defined for every $a, b \in \mathrm{Reg}(\boldsymbol{A})$ as follows:

$$a \sqcup b := \neg^{\boldsymbol{A}} \neg^{\boldsymbol{A}} (a \vee b).$$

It is well known that

$$\mathbf{Reg}(\boldsymbol{A}) := \langle \mathrm{Reg}(\boldsymbol{A}), \wedge, \sqcup, \to, \neg, 0, 1 \rangle$$

is a Boolean algebra. Now, every homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ between Heyting algebras restricts to a homomorphism $f\colon \mathbf{Reg}(\boldsymbol{A}) \to \mathbf{Reg}(\boldsymbol{B})$. Therefore the application $\mathrm{Reg}\colon \mathsf{HA} \to \mathsf{BA}$ can be regarded as a functor, which is exactly the right adjoint functor induced by Kolmogorov's translation (Example 5.22). ⊠

5.4 From right adjoints to translations

In the previous section we described one half of the correspondence between translations and adjunctions, namely how to build an adjunction out of a translation. Now we provide the other half, showing how to construct a translation (between relative equational consequences) out of an adjunction (between generalized quasi-varieties). To this end, in this section we will work with a fixed (but arbitrary) non-trivial left adjoint functor $\mathcal{F}\colon \mathsf{X} \to \mathsf{Y}$ between generalized quasi-varieties. Our goal is to construct a translation of $\vDash_{\mathsf{X}}$ into $\vDash_{\mathsf{Y}}$. We will rely on the following observation:

Lemma 5.25. *The universe of $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1))$ is non-empty.*

Proof. Suppose towards a contradiction that $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1))$ has an empty universe. Then it is the initial object of Y. Now consider any other algebra $\boldsymbol{A} \in \mathsf{X}$. We know that there is a surjective homomorphism $f \colon \boldsymbol{B} \to \boldsymbol{A}$, where $\boldsymbol{B}$ is a suitable free algebra of X. Categorically speaking we have that:

1. The arrow f is a coequalizer.[§]
2. $\boldsymbol{B}$ is a copower of $\boldsymbol{Fm}_{\mathsf{X}}(1)$.

Now recall that left adjoint functors preserve colimits, e.g., coequalizers and copowers. Therefore $\mathcal{F}(\boldsymbol{B})$ is a copower of $\emptyset$. Since copowers of the empty algebras are empty, we obtain that $\mathcal{F}(\boldsymbol{B}) = \emptyset$. In particular, this implies that $\mathcal{F}(f) \colon \emptyset \to \mathcal{F}(\boldsymbol{A})$ is a coequalizer. As $\mathcal{F}(f)$ has empty domain, it must be the coequalizer of a parallel pair $\emptyset \rightrightarrows \emptyset$. It is easy to check that the coequalizer of $\emptyset \rightrightarrows \emptyset$ is the unique arrow $\emptyset \to \emptyset$. Thus we conclude that $\mathcal{F}(\boldsymbol{A}) = \emptyset$. This shows that $\mathcal{F}$ sends every algebra to $\emptyset$, that is the initial object of Y. But this contradicts the assumption that the left adjoint $\mathcal{F} \colon \mathsf{X} \to \mathsf{Y}$ is non-trivial. ⊠

Now we construct the announced translation $\langle \boldsymbol{\tau}, \Theta \rangle$ out of $\mathcal{F} \colon \mathsf{X} \to \mathsf{Y}$. By Lemma 5.25 we know that $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1)) \neq \emptyset$. Then we can choose a cardinal $\kappa > 0$ and a surjective homomorphism $\pi \colon \boldsymbol{Fm}_{\mathsf{Y}}(\kappa) \to \mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1))$. Let Θ be the kernel of π and observe that it can be viewed as a set of equations in $Eq(\mathscr{L}_{\mathsf{Y}}, \kappa)$. This is the set of equations of our translation.

In order to construct the κ-translation $\boldsymbol{\tau}$ of $\mathscr{L}_{\mathsf{X}}$ into $\mathscr{L}_{\mathsf{Y}}$, we do the following. Consider a cardinal $\lambda > 0$. Since $\mathcal{F}$ preserves copowers and the algebra $\boldsymbol{Fm}_{\mathsf{X}}(\lambda)$ is the λ-th copower of $\boldsymbol{Fm}_{\mathsf{X}}(1)$, we know that $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(\lambda))$ is the λ-th copower of $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1))$. Keeping in mind how coproducts look like in prevarieties (see Section 5.1 if necessary), we can identify $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(\lambda))$ with the quotient of the free algebra $\boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times \lambda)$ with free generators $\{x^i_j : i < \kappa, j < \lambda\}$ under the Y-congruence generated by

$$\bigcup_{j<\lambda} \Theta(\vec{x}_j) \text{ where } \vec{x}_j = \langle x^i_j : i < \kappa \rangle.$$

The above construction can be carried out also for the case $\lambda = 0$ as follows. Recall that $\mathcal{F}$ preserves initial objects, since these are special colimits. Thus we can assume that $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(0)) = \boldsymbol{Fm}_{\mathsf{Y}}(0)$. Now we have that $\boldsymbol{Fm}_{\mathsf{Y}}(0)$ is exactly the quotient of $\boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times 0)$ under the Y-congruence generated by the union of zero-many copies of Θ, i.e., under the identity relation. Thus we identified $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(\lambda))$ with a quotient of $\boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times \lambda)$ for every cardinal λ. Accordingly, we denote by $\pi_\lambda \colon \boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times \lambda) \to \mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(\lambda))$ the corresponding canonical projection.

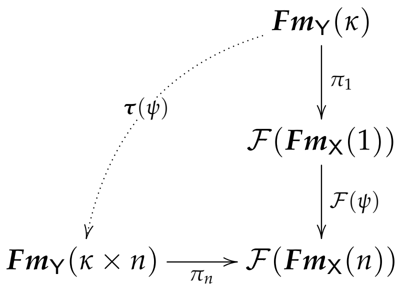
[§] In Section 5.1 we showed that in prevarieties every surjective homomorphism is a coequalizer.

Definition 5.26. Let λ be a cardinal and $\varphi \in Fm(\mathscr{L}_{\mathsf{X}}, \lambda)$. We denote also by $\varphi\colon \boldsymbol{Fm}_{\mathsf{X}}(1) \to \boldsymbol{Fm}_{\mathsf{X}}(\lambda)$ the unique homomorphism that sends x to φ, where x is the free generator of $\boldsymbol{Fm}_{\mathsf{X}}(1)$.

We are finally ready to construct the κ-translation $\boldsymbol{\tau}$ of $\mathscr{L}_{\mathsf{X}}$ into $\mathscr{L}_{\mathsf{Y}}$. Consider an n-ary basic operation $\psi \in \mathscr{L}_{\mathsf{X}}$. By the above definition it can be viewed as an arrow $\psi\colon \boldsymbol{Fm}_{\mathsf{X}}(1) \to \boldsymbol{Fm}_{\mathsf{X}}(n)$. Since π_n is surjective and $\boldsymbol{Fm}_{\mathsf{Y}}(\kappa)$ is projective in Y (as it is a free algebra), there is a homomorphism

$$\boldsymbol{\tau}(\psi)\colon \boldsymbol{Fm}_{\mathsf{Y}}(\kappa) \to \boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times n)$$

that makes the following diagram commute.



(5.6)

The map $\boldsymbol{\tau}(\psi)$ can be identified with its values on the generators $\{x^i : i < \kappa\}$ of $\boldsymbol{Fm}_{\mathsf{Y}}(\kappa)$. In this way it become a κ-sequence

$$\langle \boldsymbol{\tau}(\psi)(x^i) : i < \kappa \rangle \text{ of terms in variables } \{x^i_j : i < \kappa, 1 \leqslant j \leqslant n\}.$$

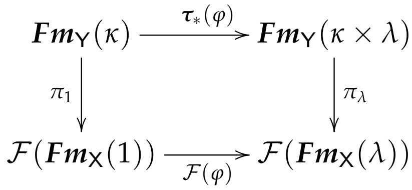
Let $\boldsymbol{\tau}$ be the κ-translation of $\mathscr{L}_{\mathsf{X}}$ into $\mathscr{L}_{\mathsf{Y}}$ obtained by applying this construction to every $\psi \in \mathscr{L}_{\mathsf{X}}$. Hence we constructed a pair $\langle \boldsymbol{\tau}, \Theta \rangle$, where $\boldsymbol{\tau}$ is a κ-translation of $\mathscr{L}_{\mathsf{X}}$ into $\mathscr{L}_{\mathsf{Y}}$ and $\Theta \subseteq Eq(\mathscr{L}_{\mathsf{Y}}, \kappa)$.

Theorem 5.27. *Let $\mathcal{F}\colon \mathsf{X} \to \mathsf{Y}$ be a non-trivial left adjoint functor between generalized quasi-varieties. The pair $\langle \boldsymbol{\tau}, \Theta \rangle$ defined above is a non-trivial translation of $\vDash_{\mathsf{X}}$ into $\vDash_{\mathsf{Y}}$.*

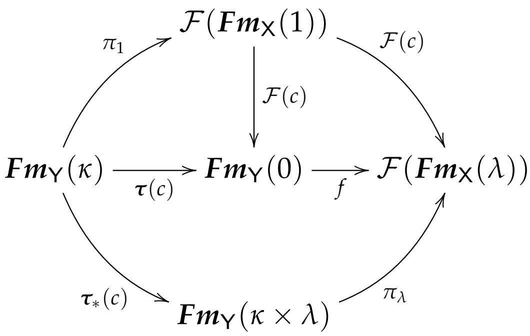
Proof. Consider a cardinal λ. We know that $\boldsymbol{\tau}$ can be extended to a function $\boldsymbol{\tau}_*\colon Fm(\mathscr{L}_{\mathsf{X}}, \lambda) \to Fm(\mathscr{L}_{\mathsf{Y}}, \kappa \times \lambda)^\kappa$, where the terms $Fm(\mathscr{L}_{\mathsf{X}}, \lambda)$ and $Fm(\mathscr{L}_{\mathsf{Y}}, \kappa \times \lambda)$ are built respectively with variables among $\{x_j : j < \lambda\}$ and $\{x^i_j : i < \kappa, j < \lambda\}$. Then consider $\varphi \in Fm(\mathscr{L}_{\mathsf{X}}, \lambda)$. Observe that $\boldsymbol{\tau}_*(\varphi)$ is a κ-sequence of terms of Y in variables $\{x^i_j : i < \kappa, j < \lambda\}$. Thus $\boldsymbol{\tau}_*(\varphi)$ can be regarded as a map from the free generators of $\boldsymbol{Fm}_{\mathsf{Y}}(\kappa)$ to $\boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times \lambda)$. Since $\boldsymbol{Fm}_{\mathsf{Y}}(\kappa)$ is a free algebra, this assignment extends uniquely to a homomorphism

$$\boldsymbol{\tau}_*(\varphi)\colon \boldsymbol{Fm}_{\mathsf{Y}}(\kappa) \to \boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times \lambda).$$

Claim 5.27.1. *For every cardinal λ and $\varphi \in Fm(\mathscr{L}_{\mathsf{X}}, \lambda)$, the following diagram commutes.*

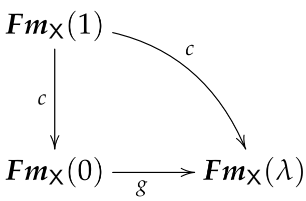


The proof works by induction on φ. We begin by the base case: φ is either a variable or a constant. We can assume w.l.o.g. that the identification of $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(\lambda))$ with a quotient of $\boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times \lambda)$ described above is done in a way that the claim holds for variables. Then we consider the case where φ is a constant c. Then consider the following diagram.



(5.7)

Recall that we identified $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(0))$ with $\boldsymbol{Fm}_{\mathsf{Y}}(0)$ and that, under this identification, the map π_0 becomes the identity map $1 \colon \boldsymbol{Fm}_{\mathsf{Y}}(0) \to \boldsymbol{Fm}_{\mathsf{Y}}(0)$. Keeping this in mind, we look at the left upper quadrant of diagram (5.7). It is an instance of diagram (5.6), where we deleted the identity map π_0 since it plays no significant role. Therefore this quadrant commutes by construction of $\boldsymbol{\tau}$. Then we consider the right upper quadrant of diagram (5.7), where f is the unique homomorphism given by the universal property of the initial object, i.e., the map that sends each constant term to its interpretation in $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(\lambda))$. Then let $g \colon \boldsymbol{Fm}_{\mathsf{X}}(0) \to \boldsymbol{Fm}_{\mathsf{X}}(\lambda)$ be the inclusion map. It is clear that the following diagram commutes.



In particular, this implies that the image under $\mathcal{F}$ of the above diagram commutes too. But observe that $\mathcal{F}(g) = f$, since $\boldsymbol{Fm}_{\mathsf{Y}}(0)$ is the initial object of Y. This shows that the right upper quadrant of diagram (5.7) commutes.

We are now ready to prove the claim for $\varphi = c$. Let $\{x^i : i < \kappa\}$ be the free generators of $\boldsymbol{Fm}_{\mathsf{Y}}(\kappa)$. Then consider $i < \kappa$ and let $c_i \in \mathscr{L}_{\mathsf{Y}}$ be the constant symbol that is the i-th component of the κ-sequence $\boldsymbol{\tau}(c)$. Since the upper part of diagram (5.7) commutes, we have that:

$$\mathcal{F}(c) \circ \pi_1(x^i) = f \circ \boldsymbol{\tau}(c)(x^i) = c_i^{\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(\lambda))}.$$

Moreover, observe that $\boldsymbol{\tau}_*(c) = \boldsymbol{\tau}(c)$ by definition of $\boldsymbol{\tau}_*$. Together with the fact that c_i is a constant, this implies that

$$\pi_\lambda \circ \boldsymbol{\tau}_*(c)(x^i) = \pi_\lambda(c_i^{\boldsymbol{Fm}_{\mathsf{Y}}(\kappa\times\lambda)}) = c_i^{\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(\lambda))}.$$

We conclude that $\pi_\lambda \circ \boldsymbol{\tau}_*(c) = \mathcal{F}(c) \circ \pi_1$. This establishes the base case.

Then we turn to prove the inductive case. Consider a basic n-ary operation $\psi \in \mathscr{L}_{\mathsf{X}}$ and $\varphi_1, \ldots, \varphi_n \in Fm(\mathscr{L}_{\mathsf{Y}}, \lambda)$. Recall that the angle-bracket notation was introduced in Section 5.1 to denote arrows induced by the universal property of the coproduct. Applying in succession the inductive hypothesis and the fact that $\mathcal{F}$ preserves coproducts, we obtain that

$$\begin{aligned} \pi_\lambda \circ \langle \boldsymbol{\tau}_*(\varphi_1), \ldots, \boldsymbol{\tau}_*(\varphi_n)\rangle &= \langle \mathcal{F}(\varphi_1), \ldots, \mathcal{F}(\varphi_n)\rangle \circ \pi_n \\ &= \mathcal{F}\langle \varphi_1, \ldots, \varphi_n\rangle \circ \pi_n \end{aligned}$$

where $\varphi_j \colon \boldsymbol{Fm}_{\mathsf{X}}(1) \to \boldsymbol{Fm}_{\mathsf{X}}(\lambda)$ and $\boldsymbol{\tau}_*(\varphi_j) \colon \boldsymbol{Fm}_{\mathsf{Y}}(\kappa) \to \boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times \lambda)$ for every $j \leqslant n$. Recall from the definition of $\boldsymbol{\tau}$ that $\pi_n \circ \boldsymbol{\tau}(\psi) = \mathcal{F}(\psi) \circ \pi_1$, where $\psi \colon \boldsymbol{Fm}_{\mathsf{X}}(1) \to \boldsymbol{Fm}_{\mathsf{X}}(n)$. Hence we conclude that

$$\begin{aligned} \mathcal{F}(\psi(\varphi_1, \ldots, \varphi_n)) \circ \pi_1 &= \mathcal{F}(\langle \varphi_1, \ldots, \varphi_n\rangle \circ \psi) \circ \pi_1 \\ &= \mathcal{F}\langle \varphi_1, \ldots, \varphi_n\rangle \circ \mathcal{F}(\psi) \circ \pi_1 \\ &= \mathcal{F}\langle \varphi_1, \ldots, \varphi_n\rangle \circ \pi_n \circ \boldsymbol{\tau}(\psi) \\ &= \pi_\lambda \circ \langle \boldsymbol{\tau}_*(\varphi_1), \ldots, \boldsymbol{\tau}_*(\varphi_n)\rangle \circ \boldsymbol{\tau}(\psi) \\ &= \pi_\lambda \circ \boldsymbol{\tau}_*(\psi(\varphi_1, \ldots, \varphi_n)). \end{aligned}$$

This establishes the claim.

Claim 5.27.2. *$\langle \boldsymbol{\tau}, \Theta\rangle$ satisfies condition 1 of Definition 5.20.*

Consider a cardinal λ and equations $\Phi \cup \{\varepsilon \approx \delta\} \subseteq Eq(\mathscr{L}_{\mathsf{X}}, \lambda)$ such that $\Phi \vDash_{\mathsf{X}} \varepsilon \approx \delta$. Define $\mu := |\Phi|$. For the sake of simplicity we identify μ with the set Φ. Then consider the map $\boldsymbol{\tau}_* \colon Fm(\mathscr{L}_{\mathsf{X}}, \lambda) \to Fm(\mathscr{L}_{\mathsf{Y}}, \kappa \times \lambda)^\kappa$. Consider also the free algebras $\boldsymbol{Fm}_{\mathsf{X}}(\mu)$ and $\boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times \mu)$ with free generators $\{x_{\alpha\approx\beta} : \alpha \approx \beta \in \Phi\}$ and $\{x^i_{\alpha\approx\beta} : i < \kappa, \alpha \approx \beta \in \Phi\}$ respectively. Then let

$$p_l \colon \boldsymbol{Fm}_{\mathsf{X}}(\mu) \to \boldsymbol{Fm}_{\mathsf{X}}(\lambda) \quad \text{and} \quad q_l \colon \boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times \mu) \to \boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times \lambda)$$

be the homomorphisms defined respectively by the following rules:

$$x_{\alpha\approx\beta} \longmapsto \alpha \text{ and } x^i_{\alpha\approx\beta} \longmapsto \boldsymbol{\tau}_*(\alpha)(i).$$

Observe that the following diagram commutes.

$$\begin{array}{ccc} \boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times \mu) & \xrightarrow{q_l} & \boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times \lambda) \\ {\scriptstyle \pi_\mu}\downarrow & & \downarrow{\scriptstyle \pi_\lambda} \\ \mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(\mu)) & \xrightarrow[\mathcal{F}(p_l)]{} & \mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(\lambda)) \end{array} \tag{5.8}$$

To prove this, it will be enough to show that $\pi_\lambda \circ q_l(x^i_{\alpha\approx\beta}) = \mathcal{F}(p_l) \circ \pi_\mu(x^i_{\alpha\approx\beta})$ for every $i < \kappa$ and $\alpha \approx \beta \in \Phi$. Consider the maps

$$\begin{aligned} \boldsymbol{\tau}_*(x_{\alpha\approx\beta})&\colon \boldsymbol{Fm}_{\mathsf{Y}}(\kappa) \to \boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times \mu) \\ \boldsymbol{\tau}_*(\alpha)&\colon \boldsymbol{Fm}_{\mathsf{Y}}(\kappa) \to \boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times \lambda) \\ x_{\alpha\approx\beta}&\colon \boldsymbol{Fm}_{\mathsf{X}}(1) \to \boldsymbol{Fm}_{\mathsf{X}}(\mu) \\ \alpha&\colon \boldsymbol{Fm}_{\mathsf{X}}(1) \to \boldsymbol{Fm}_{\mathsf{X}}(\lambda). \end{aligned}$$

Applying Claim 5.27.1 in the 2nd and 5th equalities below, we obtain that

$$\begin{aligned} \mathcal{F}(p_l) \circ \pi_\mu(x^i_{\alpha\approx\beta}) &= \mathcal{F}(p_l) \circ (\pi_\mu \circ \boldsymbol{\tau}_*(x_{\alpha\approx\beta}))(x^i) \\ &= \mathcal{F}(p_l) \circ (\mathcal{F}(x_{\alpha\approx\beta}) \circ \pi_1)(x^i) \\ &= \mathcal{F}(p_l \circ x_{\alpha\approx\beta}) \circ \pi_1(x^i) \\ &= \mathcal{F}(\alpha) \circ \pi_1(x^i) \\ &= \pi_\lambda \circ \boldsymbol{\tau}_*(\alpha)(x^i) \\ &= \pi_\lambda \circ q_l(x^i_{\alpha\approx\beta}). \end{aligned}$$

Thus we conclude that diagram (5.8) commutes.

Now, observe that we can define two maps p_r and q_r (dual to p_l and q_r) respectively by the rules:

$$x_{\alpha\approx\beta} \longmapsto \beta \text{ and } x^i_{\alpha\approx\beta} \longmapsto \boldsymbol{\tau}_*(\beta)(i).$$

An argument analogous to the one described above yields that $\pi_\lambda \circ q_r = \mathcal{F}(p_r) \circ \pi_\mu$. Hence we showed that

$$\pi_\lambda \circ q_l = \mathcal{F}(p_l) \circ \pi_\mu \quad \text{and} \quad \pi_\lambda \circ q_r = \mathcal{F}(p_r) \circ \pi_\mu. \tag{5.9}$$

Now, let ϕ be the X-congruence of $\boldsymbol{Fm}_{\mathsf{X}}(\lambda)$ generated by Φ. It is clear that π_ϕ is a coequalizer of p_l and p_r. Since $\mathcal{F}$ preserves colimits, this implies that $\mathcal{F}(\pi_\phi)$ is a coequalizer of $\mathcal{F}(p_l)$ and $\mathcal{F}(p_r)$. Keeping in mind that π_μ is surjective, this means that $\mathcal{F}(\pi_\phi)$ is also a coequalizer of $\mathcal{F}(p_l) \circ \pi_\mu$ and $\mathcal{F}(p_r) \circ \pi_\mu$. Finally, with an application of (5.9), we conclude that $\mathcal{F}(\pi_\phi)$ is a coequalizer of $\pi_\lambda \circ q_l$ and $\pi_\lambda \circ q_r$. In particular, this implies that the kernel of $\mathcal{F}(\pi_\phi) \circ \pi_\lambda$ is the Y-congruence of $\boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times \lambda)$ generated by

$$\boldsymbol{\tau}^*(\Phi) \cup \bigcup_{j<\lambda} \Theta(\vec{x}_j) \text{ where } \vec{x}_j = \langle x^i_j : i < \kappa \rangle. \tag{5.10}$$

Now observe that $\pi_\phi \circ \varepsilon = \pi_\phi \circ \delta$, where $\varepsilon, \delta \colon \boldsymbol{Fm}_\mathsf{X}(1) \rightrightarrows \boldsymbol{Fm}_\mathsf{X}(\lambda)$ since $\langle \varepsilon, \delta \rangle \in \phi$. By Claim 5.27.1 this implies that

$$\begin{aligned} \mathcal{F}(\pi_\phi) \circ \pi_\lambda \circ \boldsymbol{\tau}_*(\varepsilon) &= \mathcal{F}(\pi_\phi) \circ \mathcal{F}(\varepsilon) \circ \pi_1 \\ &= \mathcal{F}(\pi_\phi) \circ \mathcal{F}(\delta) \circ \pi_1 \\ &= \mathcal{F}(\pi_\phi) \circ \pi_\lambda \circ \boldsymbol{\tau}_*(\delta). \end{aligned}$$

Together with the description of the kernel of $\mathcal{F}(\pi_\phi) \circ \pi_\lambda$ given in (5.10), this implies that

$$\boldsymbol{\tau}^*(\Phi) \cup \bigcup_{j<\lambda} \Theta(\vec{x}_j) \vDash_\mathsf{Y} \boldsymbol{\tau}^*(\varepsilon \approx \delta).$$

This establishes Claim 5.27.2.

Claim 5.27.3. *$\langle \boldsymbol{\tau}, \Theta \rangle$ satisfies condition 2 of Definition 5.20.*

Consider an n-ary operation symbol $\psi \in \mathscr{L}_\mathsf{X}$ and $\varepsilon \approx \delta \in \Theta$. Claim 5.27.1 and the fact that the kernel of π_1 is the Y-congruence of $\boldsymbol{Fm}_\mathsf{Y}(\kappa)$ generated by Θ imply that

$$\begin{aligned} \pi_n(\varepsilon(\boldsymbol{\tau}_*(\psi)/\vec{x})) &= \pi_n \circ \boldsymbol{\tau}_*(\psi)(\varepsilon) = \mathcal{F}(\psi) \circ \pi_1(\varepsilon) = \mathcal{F}(\psi) \circ \pi_1(\delta) \\ &= \pi_n \circ \boldsymbol{\tau}_*(\psi)(\delta) = \pi_n(\delta(\boldsymbol{\tau}_*(\psi)/\vec{x})). \end{aligned}$$

Since π_n is the kernel of the Y-congruence of $\boldsymbol{Fm}_\mathsf{Y}(\kappa \times n)$ generated by $\Theta(\vec{x}_1) \cup \dots \cup \Theta(\vec{x}_n)$, we conclude that

$$\Theta(\vec{x}_1) \cup \dots \cup \Theta(\vec{x}_n) \vDash_\mathsf{Y} \varepsilon(\boldsymbol{\tau}_*(\psi)/\vec{x}) \approx \delta(\boldsymbol{\tau}_*(\psi)/\vec{x}).$$

This establishes Claim 5.27.3.

Claim 5.27.4. *$\langle \boldsymbol{\tau}, \Theta \rangle$ is a non-trivial translation.*

From Claims 5.27.2 and 5.27.3 it follows that $\langle \boldsymbol{\tau}, \Theta \rangle$ is a translation of $\vDash_\mathsf{X}$ into $\vDash_\mathsf{Y}$. It only remains to prove that $\langle \boldsymbol{\tau}, \Theta \rangle$ is non-trivial. Suppose that there is a tuple $\vec{\varphi} \in Fm(\mathscr{L}_\mathsf{Y}, 0)^\kappa$ such that $\mathsf{Y} \vDash \Theta(\vec{\varphi})$. Then let $\mathcal{G}$ be the functor right adjoint to $\mathcal{F}$. Since $\mathcal{F}$ is non-trivial, there is $\boldsymbol{A} \in \mathsf{Y}$ such that $\mathcal{G}(\boldsymbol{A})$ is non-trivial. Now observe that the solutions-set of Θ in $\boldsymbol{A}$ is in bijection with $\hom(\mathcal{F}(\boldsymbol{Fm}_\mathsf{X}(1)), \boldsymbol{A})$, since $\mathcal{F}(\boldsymbol{Fm}_\mathsf{X}(1))$ is the quotient of $\boldsymbol{Fm}_\mathsf{Y}(\kappa)$ under the Y-congruence generated by Θ. It is easy to see that $\vec{\varphi}^{\boldsymbol{A}}$ a solution of Θ in $\boldsymbol{A}$. Thus $\hom(\mathcal{F}(\boldsymbol{Fm}_\mathsf{X}(1)), \boldsymbol{A}) \neq \emptyset$. By the hom-set adjunction associated with $\mathcal{F} \dashv \mathcal{G}$ and the universal property of the free 1-generated algebra we have that

$$0 \neq |\hom(\mathcal{F}(\boldsymbol{Fm}_\mathsf{X}(1)), \boldsymbol{A})| = |\hom(\boldsymbol{Fm}_\mathsf{X}(1), \mathcal{G}(\boldsymbol{A}))| = |\mathcal{G}(\boldsymbol{A})|.$$

Since $\mathcal{G}(\boldsymbol{A})$ is non-trivial, we conclude that it has at least two elements. Again, this implies that there are two different solutions $\vec{a}, \vec{c} \in A^\kappa$ to the equations Θ. In particular, this shows that there is $i < \kappa$ such that

$$\Theta(\vec{x}) \cup \Theta(\vec{y}) \nvDash_\mathsf{Y} x^i \approx y^i.$$

This establishes Claim 5.27.4. ⊠

As an exemplification of the above construction, we will describe the translation associated with the adjunction between Kleene algebras and bounded distributive lattices.

Example 5.28 (Kleene Algebras)**.** Let $\mathcal{G}\colon \mathsf{DL}_{01} \to \mathsf{KA}$ be the functor described in Example 5.14. In [25] a functor $\mathcal{F}$ left adjoint to $\mathcal{G}$ is described. Let us recall briefly recall its behaviour. Given $\boldsymbol{A} \in \mathsf{KA}$, we let $\mathrm{Pr}(\boldsymbol{A})$ be the Priestley space dual to the bounded lattice reduct of $\boldsymbol{A}$ [33]. Moreover, we equip it with a map $g\colon \mathrm{Pr}(\boldsymbol{A}) \to \mathrm{Pr}(\boldsymbol{A})$ defined by the rule

$$g(F) \longmapsto A \smallsetminus \{\neg a : a \in F\}, \text{ with } F \in \mathrm{Pr}(\boldsymbol{A}).$$

Now observe that

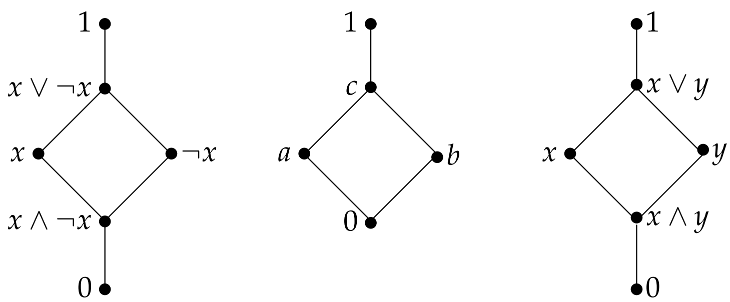
$$\mathrm{Pr}(\boldsymbol{A})^+ := \{F \in \mathrm{Pr}(\boldsymbol{A}) : F \subseteq g(F)\}$$

is the universe of a Priestley subspace of $\mathrm{Pr}(\boldsymbol{A})$. Keeping this in mind, we let $\mathcal{F}(\boldsymbol{A})$ be the bounded distributive lattice dual to $\mathrm{Pr}(\boldsymbol{A})^+$. Moreover, given a homomorphism $f\colon \boldsymbol{A} \to \boldsymbol{B}$ in KA, we let $\mathcal{F}(f)\colon \mathcal{F}(\boldsymbol{A}) \to \mathcal{F}(\boldsymbol{B})$ be the map defined by the rule

$$U \longmapsto \{F \in \mathrm{Pr}(\boldsymbol{B})^+ : f^{-1}(F) \in U\}, \text{ with } U \in \mathcal{F}(\boldsymbol{A}).$$

The map $\mathcal{F}\colon \mathsf{KA} \to \mathsf{DL}_{01}$ is the functor left adjoint to $\mathcal{G}$.

Now we turn to describe the translation associated with the adjunction $\mathcal{F} \dashv \mathcal{G}$. To this end, observe that the free Kleene algebra $\boldsymbol{Fm}_{\mathsf{KA}}(1)$, its image $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{KA}}(1))$ in DL_{01} and the free bounded distributive lattice $\boldsymbol{Fm}_{\mathsf{DL}_{01}}(2)$ are respectively the algebras depicted below.



Then let $\pi\colon \boldsymbol{Fm}_{\mathsf{DL}_{01}}(2) \to \mathcal{F}(\boldsymbol{Fm}_{\mathsf{KA}}(1))$ be the unique (surjective) homomorphism determined by the assignment $\pi(x) = a$ and $\pi(y) = b$. Following the general construction described above, we should identify Θ with the kernel of π viewed as a set of equations in 2 variables. But the only equation of this kind that is not vacuously satisfied is $x \wedge y \approx 0$. Hence we can set w.l.o.g. $\Theta := \{x \wedge y \approx 0\}$. The description of $\boldsymbol{\tau}$ is more complicated and we will detail it only for the case of negation. First observe that $\neg\colon \boldsymbol{Fm}_{\mathsf{KA}}(1) \to \boldsymbol{Fm}_{\mathsf{KA}}(1)$ is the unique endomorphism that sends x to $\neg x$. Then, applying the definition of $\mathcal{F}$, it is easy to see that $\mathcal{F}(\neg)$ is the endomorphism of $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{KA}}(1))$ that behaves as the identity except that it interchanges a and b. Now we have to choose an endomorphism $\boldsymbol{\tau}(\neg)$ of $\boldsymbol{Fm}_{\mathsf{DL}_{01}}(2)$ such that $\pi \circ \boldsymbol{\tau}(\neg) = \mathcal{F}(\neg) \circ \pi$. It is easy to see that the unique homomorphism $\boldsymbol{\tau}(\neg)$ determined by the assignment $\boldsymbol{\tau}(\neg)(x) = y$ and $\boldsymbol{\tau}(\neg)(y) = x$ fulfils this condition. Hence the translation of $\neg$ consists in the pair $\langle y, x \rangle$. The same idea allows to extend $\boldsymbol{\tau}$ to the other constant and binary basic symbols of KA as follows:¶

$$x \wedge y \longmapsto \langle x^1, x^2 \rangle \sqcap \langle y^1, y^2 \rangle := \langle x^1 \wedge y^1, x^2 \vee y^2 \rangle$$
$$x \vee y \longmapsto \langle x^1, x^2 \rangle \sqcup \langle y^1, y^2 \rangle := \langle x^1 \vee y^1, x^2 \wedge y^2 \rangle$$

and

$$\neg x \mapsto \neg \langle x^1, x^2 \rangle := \langle x^2, x^1 \rangle \qquad 1 \mapsto 1 := \langle 1, 0 \rangle \qquad 0 \mapsto 0 := \langle 0, 1 \rangle.$$

By Theorem 5.27 the pair $\langle \boldsymbol{\tau}, \Theta \rangle$ is a translation of $\vDash_{\mathsf{KA}}$ into $\vDash_{\mathsf{DL}_{01}}$.

For the reader familiar with the theory of algebraizable logics [19] it may be interesting to observe that this translation is not induced by a translation between two propositional logics (as was the case in Examples 5.21 and 5.22). This is due to the fact that DL_{01} and KA are not the equivalent algebraic semantics of any algebraizable logics. For what concerns (possibly unbounded) distributive lattices this was proved in [45, Theorem 2.1], however it is easy to adapt this result to the bounded case. For the sake of completeness we will sketch a proof of the fact that KA is not the equivalent algebraic semantics of any algebraizable logic, since we don't know any reference to this in the literature. Suppose towards a contradiction that there is an algebraizable logic $\mathcal{L}$ with equivalent algebraic semantics KA. Then let $\mathbf{2}$ be the two-element Boolean algebra. Since $\mathbf{2}$ is a Kleene algebra, either $\langle \mathbf{2}, \{1\} \rangle$ or $\langle \mathbf{2}, \{0\} \rangle$ is a reduced model of $\mathcal{L}$. Assume w.l.o.g. that this happens for $\langle \mathbf{2}, \{1\} \rangle$. Then consider the unique surjective homomorphism $f \colon \boldsymbol{A} \to \mathbf{2}$, where $\boldsymbol{A}$ is the four-element Kleene chain. The set $f^{-1}\{1\}$ is a deductive filter of $\mathcal{L}$. Clearly $\langle \boldsymbol{A}, f^{-1}\{1\} \rangle$ is not a reduced matrix. Then there must be a deductive filter $G \subsetneq f^{-1}\{1\}$ such that $\langle \boldsymbol{A}, G \rangle$ is a reduced matrix. This easily implies that $\mathcal{L}$ has no theorems. This contradicts the fact that algebraizable logics always have theorems. ⊠

5.5 Decomposition of right adjoints

In the preceding sections we drew a correspondence between adjunctions and translations, by showing how we can convert one into the other and vice-versa. Now we are ready to present the main outcome of this correspondence, namely the discovery that every every right adjoint functor between

¶At this stage the reader may find useful to compare the translation displayed here with the sublanguage $\mathscr{L}$ of the matrix power DL_{01} that we considered in Example 5.14.

generalized quasi-varieties can be decomposed into a combination of two canonical deformations, i.e., matrix powers with (possibly) infinite exponents and the $\theta_{\mathscr{L}}$ construction. More precisely, we have the following:

Theorem 5.29.

1. *Every non-trivial right adjoint between generalized quasi-varieties is naturally isomorphic to a functor of the form* $\theta_{\mathscr{L}} \circ [\kappa]$.
2. *Every functor of the form* $\theta_{\mathscr{L}} \circ [\kappa]$ *between generalized quasi-varieties is a right adjoint.*

Proof. 1. Consider a non-trivial right adjoint functor $\mathcal{G}\colon \mathsf{Y} \to \mathsf{X}$ between generalized quasi-varieties. Let $\mathcal{F}$ be the functor left adjoint to $\mathcal{G}$ and let η, ε be the unit and counit of the adjuntion respectively. In Theorem 5.27 we show that $\mathcal{F}$ gives rise to a translation $\langle \boldsymbol{\tau}, \Theta \rangle$ of $\vDash_{\mathsf{X}}$ into $\vDash_{\mathsf{Y}}$. Then consider the right adjoint functor $\theta_{\mathscr{L}} \circ [\kappa]\colon \mathsf{Y} \to \mathsf{X}$ associated with $\langle \boldsymbol{\tau}, \Theta \rangle$ as in Theorem 5.23. We will prove that $\mathcal{G}$ and $\theta_{\mathscr{L}} \circ [\kappa]$ are naturally isomorphic.

To this end, it will be convenient to work with some substitutes of $\mathcal{G}$ and $\theta_{\mathscr{L}} \circ [\kappa]$. Let $\mathsf{ALG}_{\mathsf{X}}$ be the category of algebras of the type of X. Then let $\mathcal{G}^*\colon \mathsf{Y} \to \mathsf{ALG}_{\mathsf{X}}$ be the functor defined by the following rule:

$$\begin{aligned} \boldsymbol{A} &\longmapsto \mathrm{hom}(\boldsymbol{Fm}_{\mathsf{X}}(1), \mathcal{G}(\boldsymbol{A})) \\ f &\longmapsto \mathcal{G}(f) \circ (\cdot) \end{aligned}$$

for every algebra $\boldsymbol{A}$ and homomorphism f in Y. The operations of the algebra $\mathcal{G}^*(\boldsymbol{A})$ are defined as follows. Given an n-ary operation $\psi \in \mathscr{L}_{\mathsf{X}}$ with corresponding arrow $\psi\colon \boldsymbol{Fm}_{\mathsf{X}}(1) \to \boldsymbol{Fm}_{\mathsf{X}}(n)$, we set

$$\psi^{\mathcal{G}^*(\boldsymbol{A})}(f_1, \dots, f_n) \coloneqq \langle f_1, \dots, f_n \rangle \circ \psi$$

for every $f_1, \dots, f_n \in \mathcal{G}^*(\boldsymbol{A})$. Now observe that the map $\zeta_{\boldsymbol{A}}\colon \mathcal{G}(\boldsymbol{A}) \to \mathcal{G}^*(\boldsymbol{A})$ that takes an element $a \in \mathcal{G}(\boldsymbol{A})$ to the unique arrow $f \in \mathcal{G}^*(\boldsymbol{A})$ such that $f(x) = a$ is an isomorphism for every $\boldsymbol{A} \in \mathsf{Y}$. It is easy to see that the global application $\zeta\colon \mathcal{G} \to \mathcal{G}^*$ is a natural isomorphism between $\mathcal{G}, \mathcal{G}^*\colon \mathsf{Y} \to \mathsf{ALG}_{\mathsf{X}}$. As a consequence, we obtain the following:

Fact 5.29.1. *The map $\mathcal{G}^*$ can be viewed as a functor from Y to X naturally isomorphic to $\mathcal{G}$.*

Then we turn to construct our substitute for $\theta_{\mathscr{L}} \circ [\kappa]$. To do this, consider the functor

$$\mathrm{hom}(\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1)), \cdot)\colon \mathsf{Y} \to \mathsf{ALG}_{\mathsf{X}}.$$

In particular, given $\boldsymbol{A} \in \mathsf{Y}$, the operations on $\mathrm{hom}(\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1)), \boldsymbol{A})$, for short $\mathrm{hom}(\boldsymbol{A})$, are defined as follows:

$$\psi^{\mathrm{hom}(\boldsymbol{A})}(f_1, \dots, f_n) \coloneqq \langle f_1, \dots, f_n \rangle \circ \mathcal{F}(\psi)$$

for every $f_1, \dots, f_n \in \mathrm{hom}(\boldsymbol{A})$. Now, given $\boldsymbol{A} \in \mathsf{Y}$, we consider the map $\sigma_{\boldsymbol{A}} \colon \mathrm{hom}(\boldsymbol{A}) \to \boldsymbol{A}^{[\kappa]}(\theta, \mathscr{L})$ defined by the following rule:

$$f \longmapsto \langle f \circ \pi_1(x^i) : i < \kappa \rangle$$

where $\pi_1 \colon \boldsymbol{Fm}_{\mathsf{Y}}(\kappa) \to \mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1))$ is the map defined right before Definition 5.26. Keeping in mind that the kernel of π_1 is the Y-congruence of $\boldsymbol{Fm}_{\mathsf{Y}}(\kappa)$ generated by Θ, it is easy to see that $\sigma_{\boldsymbol{A}}$ is a well-defined bijection. It turns out that it is an isomorphism too: let $f_1, \dots, f_n \in \mathrm{hom}(\boldsymbol{A})$, we have that

$$\begin{aligned}
\sigma_{\boldsymbol{A}} \psi^{\mathrm{hom}(\boldsymbol{A})}(f_1, \dots, f_n) &= \sigma_{\boldsymbol{A}}(\langle f_1, \dots, f_n \rangle \circ \mathcal{F}(\psi)) \\
&= \langle \langle f_1, \dots, f_n \rangle \circ \mathcal{F}(\psi) \circ \pi_1(x^i) : i < \kappa \rangle \\
&= \langle \langle f_1, \dots, f_n \rangle \circ \pi_n \circ \boldsymbol{\tau}(\psi)(x^i) : i < \kappa \rangle \\
&= \psi^{\boldsymbol{A}^{[\kappa]}(\theta, \mathscr{L})}(\langle \langle \langle f_1, \dots, f_n \rangle \circ \pi_n(x^i_j) : i < \kappa \rangle : j < n) \\
&= \psi^{\boldsymbol{A}^{[\kappa]}(\theta, \mathscr{L})}(\langle \langle f_j \circ \pi_1(x^i) : i < \kappa \rangle : j < n) \\
&= \psi^{\boldsymbol{A}^{[\kappa]}(\theta, \mathscr{L})}(\sigma_{\boldsymbol{A}}(f_1), \dots, \sigma_{\boldsymbol{A}}(f_n)).
\end{aligned}$$

The third equality above follows from the commutation of diagram (5.6). This shows that the global map $\sigma \colon \mathrm{hom}(\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1)), \cdot) \to \theta_{\mathscr{L}} \circ [\kappa]$ is a natural isomorphism between $\mathrm{hom}(\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1)), \cdot), \theta_{\mathscr{L}} \circ [\kappa] \colon \mathsf{Y} \to \mathsf{ALG}_{\mathsf{X}}$. As a consequence we obtain the following:

Fact 5.29.2. *The map* $\mathrm{hom}(\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1)), \cdot)$ *can be viewed as a functor from* Y *to* X *naturally isomorphic to* $\theta_{\mathscr{L}} \circ [\kappa]$.

Thanks to Facts 5.29.1 and 5.29.2, to complete the proof it will be enough to construct a natural isomorphism

$$\mu \colon \mathcal{G}^* \to \mathrm{hom}(\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1)), \cdot).$$

This is what we do now. For every $\boldsymbol{A} \in \mathsf{Y}$, the component $\mu_{\boldsymbol{A}}$ of the natural transformation μ is the following map:

$$\varepsilon_{\boldsymbol{A}} \circ \mathcal{F}(\cdot) \colon \mathrm{hom}(\boldsymbol{Fm}_{\mathsf{X}}(1), \mathcal{G}(\boldsymbol{A})) \to \mathrm{hom}(\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1)), \boldsymbol{A}).$$

From the hom-set adjunction associated with $\langle \mathcal{F}, \mathcal{G}, \varepsilon, \eta \rangle$ it follows that $\mu_{\boldsymbol{A}}$ is a bijection. Then consider $f_1, \dots, f_n \in \mathrm{hom}(\boldsymbol{Fm}_{\mathsf{X}}(1), \mathcal{G}(\boldsymbol{A}))$. Applying the fact that $\mathcal{F}$ preserves coproducts, we have that:

$$\begin{aligned}
\mu_{\boldsymbol{A}} \psi^{\mathcal{G}^*(\boldsymbol{A})}(f_1, \dots, f_n) &= \mu_{\boldsymbol{A}}(\langle f_1, \dots, f_n \rangle \circ \psi) \\
&= \varepsilon_{\boldsymbol{A}} \circ \mathcal{F}(\langle f_1, \dots, f_n \rangle \circ \psi) \\
&= \varepsilon_{\boldsymbol{A}} \circ \langle \mathcal{F}(f_1), \dots, \mathcal{F}(f_n) \rangle \circ \mathcal{F}(\psi) \\
&= \langle \varepsilon_{\boldsymbol{A}} \circ \mathcal{F}(f_1), \dots, \varepsilon_{\boldsymbol{A}} \circ \mathcal{F}(f_n) \rangle \circ \mathcal{F}(\psi) \\
&= \langle \mu_{\boldsymbol{A}}(f_1), \dots, \mu_{\boldsymbol{A}}(f_n) \rangle \circ \mathcal{F}(\psi) \\
&= \psi^{\mathrm{hom}(\boldsymbol{A})}(\mu_{\boldsymbol{A}}(f_1), \dots, \mu_{\boldsymbol{A}}(f_n))
\end{aligned}$$

Therefore we conclude that $\mu_{\boldsymbol{A}}$ is an isomorphism.

It only remains to prove that the global map μ satisfies the commutative condition typical of natural transformations. In order to do this, consider any homomorphism $g \colon \boldsymbol{A} \to \boldsymbol{B}$ in Y and an element $f \in \mathcal{G}^*(\boldsymbol{A})$. From the hom-set adjuntion associated with $\langle \mathcal{F}, \mathcal{G}, \varepsilon, \eta \rangle$ it follows that

$$\begin{aligned} \hom(\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1)), g) \circ \mu_{\boldsymbol{A}}(f) &= g \circ \mu_{\boldsymbol{A}}(f) \\ &= \mu_{\boldsymbol{B}}(\mathcal{G}(g) \circ f) \\ &= (\mu_{\boldsymbol{B}} \circ \mathcal{G}^*(g))(f). \end{aligned}$$

Hence μ is a natural isomorphism as desired.

2. Suppose that $\theta_{\mathscr{L}} \circ [\kappa] \colon \mathsf{Y} \to \mathsf{X}$ is a functor between generalized quasi-varieties. Then consider an infinite cardinal $\lambda \geqslant \kappa$ such that $\mathbb{U}_{\lambda}(\mathsf{X}) = \mathsf{X}$ and define $\mathsf{K} := \mathbb{G}\mathbb{Q}_{\lambda}(\mathsf{Y}^{[\kappa]})$. It is not difficult to see that the application $\theta_{\mathscr{L}} \colon \mathsf{K} \to \mathsf{X}$ is well-defined. By Theorems 5.9 and 5.12 we know that the maps $[\kappa] \colon \mathsf{Y} \to \mathsf{K}$ and $\theta_{\mathscr{L}} \colon \mathsf{K} \to \mathsf{X}$ are right adjoint functors. As a consequence their composition $\theta_{\mathscr{L}} \circ [\kappa] \colon \mathsf{Y} \to \mathsf{X}$ is also a right adjoint. ⊠

Corollary 5.30. *Let $\mathcal{F} \colon \mathsf{X} \to \mathsf{Y}$ be a non-trivial left adjoint functor between generalized quasi-varieties and $\phi \in \mathrm{Con}_{\mathsf{X}} \boldsymbol{Fm}_{\mathsf{X}}(\lambda)$. Assume that the right adjoint of $\mathcal{F}$ decomposes as $\theta_{\mathscr{L}} \circ [\kappa]$. Then*

$$\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(\lambda)/\phi) \cong \boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times \lambda)/\mathrm{Cg}_{\mathsf{Y}}(\boldsymbol{\tau}^*(\phi) \cup \bigcup_{j<\lambda} \Theta(\vec{x}_j)).$$

The above theorem poses the following open question:

Problem 5. It would be nice to develop a kind of duality (see for example [3]) between right adjoints between generalized quasi-varieties and translations between relative equational consequences. The first step in this direction should be the following: define a notion of *equivalence* between translations and prove that the operation

$$\text{translation} > \text{right adjoint} > \text{translation}$$

respects this notion of equivalence. Observe that the same statement for the operation

$$\text{right adjoint} > \text{translation} > \text{right adjoint}$$

is exactly the contents of Theorem 5.29, where the notion of equivalence is that of *natural isomorphism*.

CHAPTER 6

Applications

At this stage of the work we know that every right adjoint functor between generalized quasi-varieties can be decomposed into a combination of two basic kinds of deformations, namely matrix powers with (possibly) infinite exponents and the $\theta_{\mathscr{L}}$ construction. With this machinery at hand, we will develop some applications related to the preservation of logico-algebraic properties in the presence of an adjunction. In particular, we find sufficient and necessary conditions under which a left adjoint functor $\mathcal{F}\colon \mathsf{X} \to \mathsf{Y}$ between generalized quasi-varieties induces a complete lattice embedding $\gamma_{\boldsymbol{A}}\colon \mathrm{Con}_{\mathsf{X}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Y}}\mathcal{F}(\boldsymbol{A})$ for every $\boldsymbol{A} \in \mathsf{X}$ (Theorem 6.5). Moreover, we study the preservation of EDPRC and its generalization known as ESPRC (resp. Theorems 6.11 and 6.17). In this context we also show that congruence regularity is not preserved by category equivalence in general (Lemma 6.19).

We provide a logical interpretation of these results within the framework of algebraizable logics. In particular, under the assumption that X and Y are respectively the equivalent algebraic semantics of two algebraizable logics $\mathcal{X}$ and $\mathcal{Y}$, we will find conditions under which the left adjoint $\mathcal{F}\colon \mathsf{X} \to \mathsf{Y}$ induces the preservation from $\mathcal{Y}$ to $\mathcal{X}$ of the (contextual) deduction theorem, of the existence of a generalized disjunctions and of the inconsistency lemma (Theorems 6.22, 6.21 and 6.23). Moreover, we will show how the matrix power construction can be applied to obtain an elementary proof of the fact that every prevariety of algebras is categorically equivalent to the equivalent algebraic semantics of an algebraizable logic expressed in enough variables (Theorem 6.26). The last section contains some remarks that relate the work done until now to the general theory of equivalence between structural closure operators [16].

6.1 Representation of congruence lattices

Consider a left adjoint functor $\mathcal{F}\colon \mathsf{X} \to \mathsf{Y}$ between generalized quasi-varieties. For every algebra $\boldsymbol{A} \in \mathsf{X}$ we define a map

$$\gamma_{\boldsymbol{A}}\colon \mathrm{Con}_{\mathsf{X}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Y}}\mathcal{F}(\boldsymbol{A})$$

by means of the following rule:

$$\theta \longmapsto \mathrm{Ker}\mathcal{F}(\pi_\theta), \text{ for } \theta \in \mathrm{Con}_{\mathsf{X}}\boldsymbol{A}.$$

Our goal is to find sufficient and necessary conditions under which the map $\gamma_{\boldsymbol{A}}$ becomes a complete lattice embedding (Theorem 6.5). The first fact that it is worth to remark about $\gamma_{\boldsymbol{A}}$ is the following:

Lemma 6.1. *Let $\mathcal{F}\colon \mathsf{X} \to \mathsf{Y}$ be a left adjoint between generalized quasi-varieties. The map $\gamma_{\boldsymbol{A}}$ is residuated (or, equivalently, a left adjoint between poset categories) for every $\boldsymbol{A} \in \mathsf{X}$.*

Proof. Since $\mathrm{Con}_{\mathsf{X}}\boldsymbol{A}$ is a complete lattice, to prove that $\gamma_{\boldsymbol{A}}$ is residuated it will be enough to show that it preserves arbitrary joins. First observe that $\pi_{0_{\boldsymbol{A}}}\colon \boldsymbol{A} \to \boldsymbol{A}/0_{\boldsymbol{A}}$ is an isomorphism. Since isomorphism are preserved by functors, we conclude that $\mathcal{F}(\pi_{0_{\boldsymbol{A}}})$ is an isomorphism too and, therefore, that $\gamma_{\boldsymbol{A}}(0_{\boldsymbol{A}}) = 0_{\mathcal{F}(\boldsymbol{A})}$. Then $\gamma_{\boldsymbol{A}}$ preserves joins of empty families. Consider a non-empty family $V \subseteq \mathrm{Con}_{\mathsf{X}}\boldsymbol{A}$ and denote by ϕ its join in $\mathrm{Con}_{\mathsf{X}}\boldsymbol{A}$. For every $\theta \in V$ we let $l_\theta, r_\theta\colon \boldsymbol{Fm}_{\mathsf{X}}(\theta) \to \boldsymbol{A}$ be the unique homomorphisms that send a pair $\langle a, b\rangle \in \theta$ respectively to its left and right components. It is clear that π_θ is the coequalizer of l_θ and r_θ. Now let $\biguplus_{\theta \in V}\theta$ be the disjoint union of the congruences in V. We consider the coproduct maps

$$\langle l_\theta : \theta \in V\rangle, \langle r_\theta : \theta \in V\rangle\colon \boldsymbol{Fm}_{\mathsf{X}}(\biguplus_{\theta \in V}\theta) \to \boldsymbol{A}.$$

It is easy to see that π_ϕ is the coequalizer of $\langle l_\theta : \theta \in V\rangle$ and $\langle r_\theta : \theta \in V\rangle$. This implies that $\mathcal{F}(\pi_\phi)$ is a coequalizer of $\mathcal{F}(\langle l_\theta : \theta \in V\rangle)$ and $\mathcal{F}(\langle r_\theta : \theta \in V\rangle)$. Keeping in mind that $\mathcal{F}$ preserves coproducts we obtain that:

$$\begin{aligned}
\gamma_{\boldsymbol{A}}(\phi) &= \mathrm{Cg}_{\mathsf{Y}}^{\mathcal{F}(\boldsymbol{A})} \bigcup_{\theta \in V} \{\langle \mathcal{F}(l_\theta)(a), \mathcal{F}(r_\theta)(a)\rangle : a \in \mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(\theta))\}\\
&= \mathrm{Cg}_{\mathsf{Y}}^{\mathcal{F}(\boldsymbol{A})} \bigcup_{\theta \in V} \gamma_{\boldsymbol{A}}(\theta)\\
&= \bigvee_{\theta \in V} \gamma_{\boldsymbol{A}}(\theta).
\end{aligned}$$

This concludes the proof that $\gamma_{\boldsymbol{A}}$ is residuated. Finally, observe that the complete lattices $\mathrm{Con}_{\mathsf{X}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Y}}\mathcal{F}(\boldsymbol{A})$ can be seen as categories whose objects are congruences and whose arrows mimic the inclusion relation. It is easy to see that the notion of a left adjoint functor from $\mathrm{Con}_{\mathsf{X}}\boldsymbol{A}$ to $\mathrm{Con}_{\mathsf{Y}}\mathcal{F}(\boldsymbol{A})$ coincides with that of a residuated map. ⊠

Until now we proved that γ_A preserves always joins. It is therefore natural to wonder under which conditions it preserves meets too. To this end, consider a subdirect representation

$$f\colon \boldsymbol{A} \to \prod_{i\in I} \boldsymbol{B}_i \text{ in } \mathsf{X}.$$

Then consider the homomorphism

$$\nu\colon \mathcal{F}(\prod_{i\in I} \boldsymbol{B}_i) \to \prod_{i\in I} \mathcal{F}(\boldsymbol{B}_i)$$

defined by the rule

$$\nu(b)(i) \longmapsto \mathcal{F}(\pi_i)(b), \text{ for every } b \in \mathcal{F}(\prod_{i\in I} \boldsymbol{B}_i).$$

It may happen that the composition $\nu \circ \mathcal{F}(f)$ is a subdirect representation. If this happens for every subdirect embedding f in X, then we say that $\mathcal{F}$ *preserves subdirect representations*. Therefore, considering subdirect products of *finite* families, we can speak of the preservation of *finite* subdirect representations.

Lemma 6.2. *Let $\mathcal{F}\colon \mathsf{X} \to \mathsf{Y}$ be a left adjoint between generalized quasi-varieties. The map γ_A preserves (finite) meets for every $\boldsymbol{A} \in \mathsf{X}$ if and only if $\mathcal{F}$ preserves (finite) subdirect representations.*

Proof. We begin by the "only if" part. Consider a subdirect representation $f\colon \boldsymbol{A} \to \prod_{i\in I} \boldsymbol{B}_i$ in X. We have that

$$0_{\mathcal{F}(\boldsymbol{A})} = \gamma_{\boldsymbol{A}}(0_{\boldsymbol{A}}) = \gamma_{\boldsymbol{A}} \bigcap_{i\in I} \mathrm{Ker}(\pi_i \circ f) = \bigcap_{i\in I} \gamma_{\boldsymbol{A}}(\mathrm{Ker}(\pi_i \circ f)).$$

Now consider $a, b \in \mathcal{F}(\boldsymbol{A})$. We have that

$$\begin{aligned} a = b &\iff \langle a, b\rangle \in \gamma_{\boldsymbol{A}}(\mathrm{Ker}(\pi_i \circ f)) \text{ for every } i \in I\\ &\iff \mathcal{F}(\pi_i \circ f)(a) = \mathcal{F}(\pi_i \circ f)(b) \text{ for every } i \in I\\ &\iff \pi_i \circ \nu \circ \mathcal{F}(f)(a) = \pi_i \circ \nu \circ \mathcal{F}(f)(b) \text{ for every } i \in I\\ &\iff \nu \circ \mathcal{F}(f)(a) = \nu \circ \mathcal{F}(f)(b). \end{aligned}$$

Hence we conclude that $\nu \circ \mathcal{F}(f)$ is injective. Consider $i \in I$ and observe that $\pi_i \circ f$ is surjective. Since $\mathcal{F}$ preserves surjectivity, we obtain that $\mathcal{F}(\pi_i \circ f)$ is surjective too. But clearly the last map is equal to $\pi_i \circ (\nu \circ \mathcal{F}(f))$. Therefore we conclude that $\nu \circ \mathcal{F}(f)$ is a subdirect embedding.

Now we turn to prove the "if" part. Consider an algebra $\boldsymbol{A} \in \mathsf{X}$. First observe that $\boldsymbol{A}/1_{\boldsymbol{A}}$ is a subdirect product of the empty family. From the assumption, we conclude that the trivial algebra $\mathcal{F}(\boldsymbol{A}/1_{\boldsymbol{A}})$ is a subdirect product of the empty family too and, therefore, that $\mathcal{F}(\boldsymbol{A}/1_{\boldsymbol{A}})$ is trivial. This implies that $\gamma_{\boldsymbol{A}}(1_{\boldsymbol{A}}) = 1_{\mathcal{F}(\boldsymbol{A})}$. In other words, $\gamma_{\boldsymbol{A}}$ preserves empty meets. Now consider a non-empty family $\{\theta_i : i \in I\} \subseteq \mathrm{Con}_{\mathsf{X}}\boldsymbol{A}$ and let ϕ be its meet. We have that the map $f \colon \boldsymbol{A}/\phi \to \prod_{i \in I} \boldsymbol{A}/\theta_i$ defined as

$$f(a)(i) := a/\theta_i, \text{ for } i \in I \text{ and } a \in A$$

is a subdirect embedding. Therefore we can apply the assumptions yielding that $\nu \circ \mathcal{F}(f)$ is a subdirect embedding too. Keeping this in mind, we obtain that for every $a, b \in \mathcal{F}(\boldsymbol{A})$,

$$\begin{aligned}
\langle a, b\rangle \in \gamma_{\boldsymbol{A}}(\phi) &\iff \mathcal{F}(\pi_\phi)(a) = \mathcal{F}(\pi_\phi)(b)\\
&\iff \pi_i \circ \nu \circ \mathcal{F}(f) \circ \mathcal{F}(\pi_\phi)(a) = \pi_i \circ \nu \circ \mathcal{F}(f) \circ \mathcal{F}(\pi_\phi)(b)\\
&\qquad \text{for every } i \in I\\
&\iff \mathcal{F}(\pi_i \circ f) \circ \mathcal{F}(\pi_\phi)(a) = \mathcal{F}(\pi_i \circ f) \circ \mathcal{F}(\pi_\phi)(b)\\
&\qquad \text{for every } i \in I\\
&\iff \mathcal{F}(\pi_{\theta_i})(a) = \mathcal{F}(\pi_{\theta_i})(b) \text{ for every } i \in I\\
&\iff \langle a, b\rangle \in \gamma_{\boldsymbol{A}}(\theta_i) \text{ for every } i \in I.
\end{aligned}$$

Hence we conclude that $\gamma_{\boldsymbol{A}}$ preserves meets. ⊠

Remarkably, the fact that $\gamma_{\boldsymbol{A}}$ preserves meets implies that it preserves also the generation of varieties:

Corollary 6.3. *Let $\mathcal{F} \colon \mathsf{X} \to \mathsf{Y}$ be a left adjoint between generalized quasi-varieties. Assume that $\gamma_{\boldsymbol{A}}$ preserves meets for every $\boldsymbol{A} \in \mathsf{X}$. If $\boldsymbol{B} \in \mathbb{V}(\mathsf{K})$, then $\mathcal{F}(\boldsymbol{B}) \in \mathbb{V}(\mathcal{F}(\mathsf{K}))$, for every $\mathsf{K} \cup \{\boldsymbol{B}\} \subseteq \mathsf{X}$.*

Proof. Suppose that $\boldsymbol{B} \in \mathbb{V}(\mathsf{K})$. Recall that $\mathbb{V}(\mathsf{K}) = \mathbb{H}\mathbb{P}_{\text{SD}}(\mathsf{K})$ and, therefore, that $\mathbb{V} = \mathbb{H}\mathbb{P}_{\text{SD}}$ (see for example [64]). This means that there is a subdirect embedding $f \colon \boldsymbol{D} \to \prod_{i \in I} \boldsymbol{C}_i$ with $\{\boldsymbol{C}_i : i \in I\} \subseteq \mathsf{K}$ and a surjective homomorphism $g \colon \boldsymbol{D} \to \boldsymbol{B}$. Now, observe that $\mathcal{F}(\boldsymbol{D}) \in \mathbb{P}_{\text{SD}}(\{\mathcal{F}(\boldsymbol{C}_i) : i \in I\})$ by Lemma 6.2. Finally $\mathcal{F}(\boldsymbol{B}) \in \mathbb{H}(\mathcal{F}(\boldsymbol{D}))$, since $\mathcal{F}$ preserves surjectivity. ⊠

Sometimes, in order to move properties of congruence lattices such as distributivity or modularity from Y to X, we need to assume that $\gamma_{\boldsymbol{A}}$ is an injective map. The next result provides a characterization of this condition, not only in terms of the behaviour of the left adjoint $\mathcal{F}$, but also in terms of the translation induced by $\mathcal{F}$ (compare with condition 1 of Definition 5.20).

Lemma 6.4. *Let $\mathcal{F} \colon \mathsf{X} \to \mathsf{Y}$ be a left adjoint between generalized quasi-varieties. The following conditions are equivalent:*

(i) *The map $\gamma_{\boldsymbol{A}}$ is injective for every $\boldsymbol{A} \in \mathsf{X}$.*

(ii) *The functor $\mathcal{F}$ is faithful.*

(iii) *For every cardinal* λ *and* $\Phi \cup \{\varepsilon \approx \delta\} \subseteq Eq(\mathscr{L}_{\mathsf{X}}, \lambda)$,

$$\Phi \vDash_{\mathsf{X}} \varepsilon \approx \delta \Longleftrightarrow \boldsymbol{\tau}^*(\Phi) \cup \bigcup_{j<\lambda} \Theta(\vec{x}_j) \vDash_{\mathsf{Y}} \boldsymbol{\tau}^*(\varepsilon \approx \delta),$$

where $\langle \boldsymbol{\tau}, \Theta \rangle$ *is the translation of* $\vDash_{\mathsf{X}}$ *into* $\vDash_{\mathsf{Y}}$ *induced by* $\mathcal{F}$.

Proof. (i)⇒(ii): We reason by contrapositive. Suppose that $\mathcal{F}$ is not faithful. Then there are two different arrows $f, g\colon \boldsymbol{A} \rightrightarrows \boldsymbol{B}$ in X such that $\mathcal{F}(f) = \mathcal{F}(g)$. Let $\pi_\theta\colon \boldsymbol{B} \to \boldsymbol{B}/\theta$ be their coequalizer and observe that $\theta \neq 0_{\boldsymbol{B}}$, since $f \neq g$. Now observe that $\mathcal{F}(\pi_\theta)$ is a coequalizer of $\mathcal{F}(f)$ and $\mathcal{F}(g)$ and, therefore, it is an isomorphism, since $\mathcal{F}(f) = \mathcal{F}(g)$. In particular, this means that $\gamma_{\boldsymbol{B}}(\theta) = \mathrm{Ker}(\mathcal{F}(\pi_\theta)) = 0_{\mathcal{F}(\boldsymbol{B})}$. From Lemma 6.1 it follows that also $\gamma_{\boldsymbol{B}}(0_{\boldsymbol{B}}) = 0_{\mathcal{F}(\boldsymbol{B})}$. Hence we conclude that $\gamma_{\boldsymbol{B}}\colon \mathrm{Con}_{\mathsf{X}}\boldsymbol{B} \to \mathrm{Con}_{\mathsf{Y}}\mathcal{F}(\boldsymbol{B})$ is not injective.

(ii)⇒(i): Consider $\boldsymbol{A} \in \mathsf{X}$ and two different congruences $\theta, \phi \in \mathrm{Con}_{\mathsf{X}}\boldsymbol{A}$. We can assume w.l.o.g. that there is a pair $\langle a, c\rangle \in \theta \smallsetminus \phi$. Then consider the arrows $\hat{a}, \hat{c}\colon \boldsymbol{Fm}_{\mathsf{X}}(1) \rightrightarrows \boldsymbol{A}$ that send x respectively to a and c. Clearly we have that

$$\pi_\theta \circ \hat{a} = \pi_\theta \circ \hat{c} \text{ and } \pi_\phi \circ \hat{a} \neq \pi_\phi \circ \hat{c}.$$

Since $\mathcal{F}$ is faithful, this means that $\mathcal{F}(\pi_\theta \circ \hat{a}) = \mathcal{F}(\pi_\theta \circ \hat{c})$ and $\mathcal{F}(\pi_\phi \circ \hat{a}) \neq \mathcal{F}(\pi_\phi \circ \hat{c})$. Then consider $b \in \mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1))$ such that $\mathcal{F}(\pi_\phi \circ \hat{a})(b) \neq \mathcal{F}(\pi_\phi \circ \hat{c})(b)$. We have that:

$$\langle \mathcal{F}(\hat{a})(b), \mathcal{F}(\hat{c})(b)\rangle \in \mathrm{Ker}(\mathcal{F}(\pi_\theta)) \smallsetminus \mathrm{Ker}(\mathcal{F}(\pi_\phi)) = \gamma_{\boldsymbol{A}}(\theta) \smallsetminus \gamma_{\boldsymbol{A}}(\phi).$$

Hence we conclude that $\gamma_{\boldsymbol{A}}$ is injective as desired.

(i)⇔(iii): Consider a cardinal λ. By Corollary 5.30 we can assume that the functor $\mathcal{F}$ sends $\boldsymbol{Fm}_{\mathsf{X}}(\lambda)$ to the quotient of $\boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times \lambda)$ under the Y-congruence generated by the pairs $\bigcup_{j<\lambda} \Theta(\vec{x}_j)$. By the Correspondence Theorem we can can lift the map $\gamma_{\boldsymbol{Fm}_{\mathsf{X}}(\lambda)}$ as follows:

$$\gamma_{\boldsymbol{Fm}_{\mathsf{X}}(\lambda)}\colon \mathrm{Con}_{\mathsf{X}}\boldsymbol{Fm}_{\mathsf{X}}(\lambda) \to \mathrm{Con}_{\mathsf{Y}}\boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times \lambda).$$

From Corollary 5.30 it follows that

$$\gamma_{\boldsymbol{Fm}_{\mathsf{X}}(\lambda)}\mathrm{Cg}_{\mathsf{X}}(\Phi) = \mathrm{Cg}_{\mathsf{Y}}(\boldsymbol{\tau}^*(\Phi) \cup \bigcup_{j<\lambda} \Theta(\vec{x}_j))$$

for every $\Phi \cup \{\varepsilon \approx \delta\} \subseteq Eq(\mathscr{L}_{\mathsf{X}}, \lambda)$. Now if (i) holds, then $\gamma_{\boldsymbol{Fm}_{\mathsf{X}}(\lambda)}$ is injective and, therefore, condition (iii) holds. Conversely, suppose that (i) fails. It is easy to show that there is a cardinal λ such that also $\gamma_{\boldsymbol{Fm}_{\mathsf{X}}(\lambda)}$ is not injective. Applying the above characterization of $\gamma_{\boldsymbol{Fm}_{\mathsf{X}}(\lambda)}$ we conclude that also condition (iii) fails. ⊠

Recall that a prevariety K is *relatively congruence distributive* (resp. *modular*) if $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ is a distributive (resp. modular) lattice for every $\boldsymbol{A} \in \mathsf{K}$. The following observation is a consequence of the work done until now:

Theorem 6.5. *Let $\mathcal{F}\colon \mathsf{X} \to \mathsf{Y}$ be a left adjoint between generalized quasi-varieties. The map $\gamma_{\boldsymbol{A}}$ is a complete lattice embedding for every $\boldsymbol{A} \in \mathsf{X}$ if and only if $\mathcal{F}$ is faithful and preserves subdirect representations. Assuming that these conditions hold, if Y is relatively congruence distributive (modular), then so is X.*

Example 6.6 (Kleene Algebras)**.** Let $\mathcal{F}\colon \mathsf{KA} \to \mathsf{DL}_{01}$ be the left adjoint described in Example 5.28. For every $\boldsymbol{A} \in \mathsf{KA}$ the lattices $\mathrm{Con}\boldsymbol{A}$ and $\mathrm{Con}\mathcal{F}(\boldsymbol{A})$ are isomorphic [25, Corollary 1.10]. In particular, we will show that $\gamma_{\boldsymbol{A}}\colon \mathrm{Con}\boldsymbol{A} \to \mathrm{Con}\mathcal{F}(\boldsymbol{A})$ is an isomorphism.

To prove this, recall from Example 5.28 that $\boldsymbol{A}$ can be associated with an expanded Priestley space $\langle \mathrm{Pr}(\boldsymbol{A}), g\rangle$. It turns out that $\mathrm{Con}\boldsymbol{A}$ is dually isomorphic to the lattice $\boldsymbol{P}$ of closed subsets $X \subseteq \mathrm{Pr}(\boldsymbol{A})$ such that $g(X) = X$ via the map defined by the rule

$$\theta \longmapsto \{F \in \mathrm{Pr}(\boldsymbol{A}) : F \text{ is a union of blocks of } \theta\}, \text{ for every } \theta \in \mathrm{Con}\boldsymbol{A}$$

see for example [26, pag. 215-216]. Let $\boldsymbol{V}$ be the lattice of closed subsets of $\mathrm{Pr}(\boldsymbol{A})^{+}$. The map $\mathrm{Pr}(\boldsymbol{A})^{+} \cap (\cdot)\colon \boldsymbol{P} \to \boldsymbol{V}$ is a lattice isomorphism. This is a consequence of the fact that $g\colon \mathrm{Pr}(\boldsymbol{A}) \to \mathrm{Pr}(\boldsymbol{A})$ is a homeomorphism that is also a dual order isomorphism such that $g \circ g$ is the identity. Finally it is well known that the lattices $\boldsymbol{V}$ and $\mathrm{Con}\mathcal{F}(\boldsymbol{A})$ are dually isomorphic under the map defined by the rule

$$X \longmapsto \{\langle U, W\rangle \in \mathcal{F}(\boldsymbol{A})^2 : U \cap X = W \cap X\}, \text{ for every } X \in V.$$

Now, let $\zeta\colon \mathrm{Con}\boldsymbol{A} \to \mathrm{Con}\mathcal{F}(\boldsymbol{A})$ be the composition of the three isomorphisms described above. For every $U, W \in \mathcal{F}(\boldsymbol{A})$ we have that:

$$\begin{aligned}
\langle U, W\rangle \in \gamma_{\boldsymbol{A}}(\theta) &\iff (\pi_\theta^{-1})^{-1}(U) = (\pi_\theta^{-1})^{-1}(W)\\
&\iff \{F \in \mathrm{Pr}(\boldsymbol{A}/\theta)^{+} : \pi_\theta^{-1}(F) \in U\} =\\
&\qquad \{F \in \mathrm{Pr}(\boldsymbol{A}/\theta)^{+} : \pi_\theta^{-1}(F) \in W\}\\
&\iff \{F \in \mathrm{Pr}(\boldsymbol{A})^{+} : F \text{ is a union of blocks of } \theta\} \cap U =\\
&\qquad \{F \in \mathrm{Pr}(\boldsymbol{A})^{+} : F \text{ is a union of blocks of } \theta\} \cap W\\
&\iff \langle U, W\rangle \in \zeta(\theta).
\end{aligned}$$

Hence we conclude that $\gamma_{\boldsymbol{A}} = \zeta$ and, therefore, that $\gamma_{\boldsymbol{A}}$ is a lattice isomorphism as desired. Moreover $\mathcal{F}$ is faithful by Lemma 6.4 and preserves subdirect representations by Lemma 6.2. ⊠

6.2 Equationally (semi)-definable principal congruences

A quasi-variety K has *equationally definable principal relative congruences* (EDPRC) when there is a finite set of equations $\Phi(x_1, x_2, y_1, y_2)$ in four variables such that for all $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c, d \in A$:

$$\langle a, b \rangle \in \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d) \iff \boldsymbol{A} \vDash \Phi(a, b, c, d).$$

In [84] this concept has been generalized as follows. A quasi-variety K has *equationally semi-definable principal relative congruences* (ESPRC) if, for each $n \in \omega$, there exists a finite set of equations $\Phi_n(x_1, x_2, y_1, y_2, \vec{z})$ in $4 + n$ variables such that, whenever $\vec{e} = e_1, \dots, e_n$ generates an algebra $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c, d \in A$, then

$$\langle a, b \rangle \in \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c, d) \iff \boldsymbol{A} \vDash \Phi_n(a, b, c, d, \vec{e}).$$

It is clear that EDPRC implies ESPRC, while the converse does not hold in general.

Example 6.7. The variety DL_{01} has EDPC witnessed by the equations

$$x_1 \wedge (y_1 \wedge y_2) \approx x_2 \wedge (y_1 \wedge y_2) \quad x_1 \vee (y_1 \vee y_2) \approx x_2 \vee (y_1 \vee y_2). \tag{6.1}$$

See for example [17, Example 7, pag. 201].

An example of a variety with ESPRC and without EDPRC comes from the study of relevance logic [7]. An algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \neg \rangle$ of type $\langle 2, 2, 2, 1 \rangle$ is a *relevant algebra* if $\langle A, \wedge, \vee \rangle$ is a distributive lattice, $\langle A, \cdot \rangle$ is a commutative semigroup and

$$\begin{gathered}
\neg\neg a = a \leqslant a \cdot a \\
a \leqslant b \iff \neg b \leqslant \neg a \\
a \cdot b \leqslant c \iff a \cdot \neg c \leqslant \neg b \\
a \leqslant a \cdot (\neg(b \cdot \neg b) \wedge \neg(c \cdot \neg c))
\end{gathered}$$

for every $a, b, c \in A$. Relevant algebras form a variety that we denote by RA. In relevant algebras one can define an implication setting $x \to y := \neg(x \cdot \neg y)$. The variety RA has ESPC witnessed by the following sets of equations [84, Example 9.5]:

$$\Phi_n := \{(z_1 \to z_1) \wedge \cdots \wedge (z_n \to z_n) \wedge (y_1 \leftrightarrow y_2) \leqslant (x_1 \leftrightarrow x_2)\}$$

for every $n \in \omega$. Moreover, RA lacks EDPC since it has not the congruence extension property [29, pag. 289]. ⊠

Remarkably EDPRC and ESPRC can be equivalently expressed as properties of congruence lattices. More precisely, we say that a join-semilattice $\langle A, \vee \rangle$ is *dually Brouwerian* when for every $a, b \in A$ there exists $\min\{c \in A : b \leqslant a \vee c\}$. The following result is a combination of [65, Theorems 5 and 8] and [84, Theorem 8.6].

Theorem 6.8 (Köhler, Pigozzi and Raftery). *A quasi-variety* K *has EDPRC (ESPRC) if and only if* $\langle \mathrm{Comp}_{\mathsf{K}} \boldsymbol{A}, \vee \rangle$ *is a dually Brouwerian join-semilattice for every (finitely generated)* $\boldsymbol{A} \in \mathsf{K}$.

The goal of this section is to identify some conditions under which EDPRC (and ESPRC) transfers in presence of an adjunction (Theorems 6.11 and 6.17). Given a left adjoint $\mathcal{F} \colon \mathsf{X} \to \mathsf{Y}$, we say that γ *preserves compact congruences* if the restriction

$$\gamma_{A} \colon \mathrm{Comp}_{\mathsf{X}} \boldsymbol{A} \to \mathrm{Comp}_{\mathsf{Y}} \mathcal{F}(\boldsymbol{A})$$

is well defined for every $\boldsymbol{A} \in \mathsf{X}$.

Lemma 6.9. *Let* $\mathcal{F} \colon \mathsf{X} \to \mathsf{Y}$ *be a left adjoint between quasi-varieties. The following conditions are equivalent:*

(i) *The map* γ *preserves compact congruences.*

(ii) *The map* $\gamma \colon \mathrm{Comp}_{\mathsf{X}} \boldsymbol{Fm}_{\mathsf{X}}(2) \to \mathrm{Comp}_{\mathsf{Y}} \mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(2))$ *is well defined.*

(iii) *There is a finite set* $J \subseteq \kappa$ *such that for every* $\boldsymbol{A} \in \mathsf{Y}$ *and* $\vec{a}, \vec{c} \in \boldsymbol{A}^{[\kappa]}(\theta, \mathscr{L})$

$$\vec{a} = \vec{c} \text{ if and only if } (\vec{a}(j) = \vec{c}(j) \text{ for every } j \in J)$$

where $\theta_{\mathscr{L}} \circ [\kappa]$ *is the functor that is right adjoint to* $\mathcal{F}$.

Proof. Part (i)⇒(ii) is straightforward. (ii)⇒(iii): Let $\theta_{\mathscr{L}} \circ [\kappa]$ be the decomposition of the right adjoint to $\mathcal{F}$. Moreover, let x and y be the free generators of $\boldsymbol{Fm}_{\mathsf{X}}(2)$. From Corollary 5.30 it follows that the algebra $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(2))$ can be identified with the quotient of $\boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times 2)$ under the Y-congruence generated by $\Theta(\vec{x}) \cup \Theta(\vec{y})$.

Clearly $\phi \coloneqq \mathrm{Cg}_{\mathsf{X}}(x, y)$ is compact. Thus by assumption also $\gamma(\phi)$ is compact. Now observe that the canonical projection $\pi_{\phi} \colon \boldsymbol{Fm}_{\mathsf{X}}(2) \to \boldsymbol{Fm}_{\mathsf{X}}(2)/\phi$ is the coequalizer of the arrows $x, y \colon \boldsymbol{Fm}_{\mathsf{X}}(1) \rightrightarrows \boldsymbol{Fm}_{\mathsf{X}}(2)$. In particular, this means that $\mathcal{F}(\pi_{\phi})$ is the coequalizer of $\mathcal{F}(x)$ and $\mathcal{F}(y)$. By Corollary 5.30 we can identify $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1))$ with the algebra $\boldsymbol{Fm}_{\mathsf{Y}}(\kappa)/\mathrm{Cg}_{\mathsf{Y}}(\Theta)$. Then we have

$$\mathcal{F}(x), \mathcal{F}(y) \colon \boldsymbol{Fm}_{\mathsf{Y}}(\kappa)/\mathrm{Cg}_{\mathsf{Y}}(\Theta) \to \boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times 2)/\mathrm{Cg}_{\mathsf{Y}}(\Theta(\vec{x}) \cup \Theta(\vec{y})).$$

Moreover, $\mathcal{F}(x)$ and $\mathcal{F}(y)$ are the unique maps that send the representatives of the κ variables $\vec{x}/\mathrm{Cg}_{\mathsf{Y}}(\Theta)$ of $\boldsymbol{Fm}_{\mathsf{Y}}(\kappa)/\mathrm{Cg}_{\mathsf{Y}}(\Theta)$ respectively to $\vec{x}/\mathrm{Cg}_{\mathsf{Y}}(\Theta(\vec{x}) \cup \Theta(\vec{y}))$ and to $\vec{y}/\mathrm{Cg}_{\mathsf{Y}}(\Theta(\vec{x}) \cup \Theta(\vec{y}))$. This last fact was shown in the Claim 5.27.1. Finally, since $\gamma(\phi)$ is compact, there is a finite set of equations $\Phi \subseteq Eq(\mathscr{L}_{\mathsf{Y}}, \kappa)$ such that

$$\Theta(\vec{x}) \cup \Theta(\vec{y}) \cup \Phi \vDash_{\mathsf{Y}} x^{i} \approx y^{i} \text{ for every } i < \kappa \tag{6.2}$$

$$\Theta(\vec{x}) \cup \Theta(\vec{y}) \cup \{x^{i} \approx y^{i} : i < \kappa\} \vDash_{\mathsf{Y}} \Phi. \tag{6.3}$$

Since Φ is finite, there is a finite set $J \subseteq \kappa$ such that the variables occurring in Φ are among $\{x^j : j \in J\} \cup \{y^j : j \in J\}$. For sake of simplicity, we will assume that $J = \{1, \ldots, n\}$. Accordingly we will write

$$\Phi(x^1, \ldots, x^n, y^1, \ldots, y^n)$$

to underline which are the variables occurring in Φ. Then consider $\boldsymbol{A} \in \mathsf{Y}$ and two elements $\vec{a}, \vec{c} \in \boldsymbol{A}^{[\kappa]}(\theta, \mathscr{L})$ such that $\vec{a}(j) = \vec{c}(j)$ for every $j \in J$. Observe that $\boldsymbol{A} \vDash \Theta(\vec{a})$, since $\vec{a} \in \boldsymbol{A}^{[\kappa]}(\theta, \mathscr{L})$. From (6.3) it follows that $\boldsymbol{A} \vDash \Phi(\vec{a}(1), \ldots, \vec{a}(n), \vec{a}(1), \ldots, \vec{a}(n))$. In particular, this implies that

$$\boldsymbol{A} \vDash \Phi(\vec{a}(1), \ldots, \vec{a}(n), \vec{c}(1), \vec{c}(n)) \tag{6.4}$$

since $\vec{a}(j) = \vec{c}(j)$. But observe that also $\boldsymbol{A} \vDash \Theta(\vec{c})$, since $\vec{c} \in \boldsymbol{A}^{[\kappa]}(\theta, \mathscr{L})$. Hence, with an application of (6.2) and (6.4), we conclude that $\vec{a} = \vec{c}$ as desired.

(iii)⇒(i): Consider $\boldsymbol{A} \in \mathsf{X}$ and $\phi \in \mathrm{Comp}_{\mathsf{X}}\boldsymbol{A}$. We know that ϕ is generated by a finite set $\{\langle a_i, b_i \rangle : i < n\} \subseteq A^2$. The canonical projection π_ϕ is the coequalizer of the homomorphisms $f, g \colon \boldsymbol{Fm}_{\mathsf{X}}(n) \rightrightarrows \boldsymbol{A}$ defined respectively as $f(x_i) = a_i$ and $g(x_i) = b_i$ for every $i < n$. Then $\mathcal{F}(\pi_\phi)$ is the coequalizer of $\mathcal{F}(f)$ and $\mathcal{F}(g)$. Now by Corollary 5.30 we can identify $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(n))$ with the quotient of $\boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times n)$ under

$$\eta := \mathrm{Cg}_{\mathsf{Y}}(\Theta(\vec{x}_1) \cup \cdots \cup \Theta(\vec{x}_n)).$$

Observe that $\mathcal{F}(\pi_\phi)$ is also a coequalizer for $\mathcal{F}(f) \circ \pi_\eta$ and $\mathcal{F}(g) \circ \pi_\eta$, since π_η is onto. In particular, this means that

$$\gamma(\phi) = \mathrm{Cg}_{\mathsf{Y}}^{\mathcal{F}(\boldsymbol{A})}\{\langle \mathcal{F}(f) \circ \pi_\eta(x^i_m), \mathcal{F}(g) \circ \pi_\eta(x^i_m) \rangle : i < \kappa, m < n\}. \tag{6.5}$$

Now define

$$\zeta := \mathrm{Cg}_{\mathsf{Y}}^{\mathcal{F}(\boldsymbol{A})}\{\langle \mathcal{F}(f) \circ \pi_\eta(x^i_m), \mathcal{F}(g) \circ \pi_\eta(x^i_m) \rangle : i \in J, m < n\}.$$

Observe that ζ is compact, since J is finite. Thus in order to prove that $\gamma(\phi)$ is compact too, it will be enough to show that $\gamma(\phi) = \zeta$. By (6.5) we know that $\zeta \subseteq \gamma(\phi)$. Then we move to the other inclusion. Let $\{x^i : i < \kappa\} \cup \{y^i : i < \kappa\}$ be the free generators of $\boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times 2)$. Consider $m < n$. We define a homomorphism $h \colon \boldsymbol{Fm}_{\mathsf{Y}}(\kappa \times 2) \to \mathcal{F}(\boldsymbol{A})$ by the following rule:

$$h(x^i) := \mathcal{F}(f) \circ \pi_\eta(x^i_m) \text{ and } h(y^i) := \mathcal{F}(g) \circ \pi_\eta(x^i_m)$$

for every $i < \kappa$. It is easy to see that $\mathcal{F}(\boldsymbol{A}) \vDash \Theta(h\vec{x}) \cup \Theta(h\vec{y})$. In particular, this implies that

$$\pi_\zeta \circ h(\vec{x}), \pi_\zeta \circ h(\vec{y}) \in (\mathcal{F}(\boldsymbol{A})/\zeta)^{[\kappa]}(\theta, \mathscr{L}).$$

From the definition of ζ it follows that the κ-sequences $\pi_\zeta \circ h(\vec{x})$ and $\pi_\zeta \circ h(\vec{y})$ agree on the indexes in J. Together with the assumptions, this implies that $\pi_\zeta \circ h(\vec{x}) = \pi_\zeta \circ h(\vec{y})$. Thus we obtain that:

$$\{\langle \mathcal{F}(f) \circ \pi_\eta(x^i_m), \mathcal{F}(g) \circ \pi_\eta(x^i_m)\rangle : i < \kappa\} \subseteq \zeta.$$

Applying (6.5), we conclude that $\gamma(\phi) \subseteq \zeta$. ⊠

The next example shows that in general the map γ need not to preserve compact congruences.

Example 6.10 (Modules)**.** Let $\boldsymbol{R}$ be a commutative ring with unit and $\boldsymbol{R}-\mathsf{Mod}$ be the class of $\boldsymbol{R}$-modules, seen as a variety of algebras. In other words, we assume that the language of $\boldsymbol{R}-\mathsf{Mod}$ includes a unary symbol $r(x)$ for every $r \in R$ in order to recover scalar multiplication as an internal operation. The forgetful functor $\mathcal{U}\colon \boldsymbol{R}-\mathsf{Mod} \to \mathsf{AG}$ into the variety of Abelian groups has a right adjoint $\mathcal{G}$ defined as follows. Recall that, given an Abelian group $\boldsymbol{A} \in \mathsf{AG}$, the set of homomorphisms $\hom(\boldsymbol{R}, \boldsymbol{A})$ equipped with addition and inverse defined component-wise is still an Abelian group. Then $\mathcal{G}(\boldsymbol{A})$ is the expansion of the Abelian group $\hom(\boldsymbol{R}, \boldsymbol{A})$ with an operation $r\colon \hom(\boldsymbol{R}, \boldsymbol{A}) \to \hom(\boldsymbol{R}, \boldsymbol{A})$ for every $r \in R$, defined for every $f \in \hom(\boldsymbol{R}, \boldsymbol{A})$ as:

$$r(f)(q) := f(r \cdot q), \text{ for every } q \in R.$$

We will briefly describe how to decompose the functor $\mathcal{G}$ into a combination of our two basic deformations. For the sake of simplicity, we will identify the universe of $\boldsymbol{R}$ with its cardinality. First consider the following sublanguage $\mathscr{L}$ of the matrix power $\mathsf{AG}^{[R]}$:

$$x + y := \langle x^i + y^i : i \in R\rangle \qquad -x := \langle -x^i : i \in R\rangle \qquad 0 := \langle 0^i : i \in R\rangle$$

$$r(x) := \langle x^{r \cdot i} : i \in R\rangle, \text{ for every } r \in R.$$

Then consider the following set of equations:

$$\theta := \{\overrightarrow{x^i + x^j} \approx \overrightarrow{x^{i+j}} : i, j \in R\}.$$

In the definition of θ we are assuming that $\vec{\varphi}$ denotes the R-sequence constant on φ. It is easy to see that $\mathcal{G}$ arises as the composition $\theta_{\mathscr{L}} \circ [R]$.

Now, consider the free commutative ring with unit $\mathbb{Z}[x_1, x_2, \dots]$ with countably many free generators $\{x_n : n \in \omega\}$. Let γ be the map between congruence lattices induced by the forgetful functor $\mathcal{U}\colon \mathbb{Z}[x_1, x_2, \dots]-\mathsf{Mod} \to \mathsf{AG}$. We want to prove that γ does not preserve compact congruences. Suppose the contrary towards a contradiction. Let $\theta_{\mathscr{L}} \circ [\mathbb{Z}[x_1, x_2, \dots]]$ be the decomposition of the right adjoint to $\mathcal{U}$ described above. By Lemma 6.9 there is a finite set $J \subseteq \{x_n : n \in \omega\} \subseteq [\mathbb{Z}[x_1, x_2, \dots]]$ such that for every $\boldsymbol{A} \in \mathsf{AG}$ and $\vec{a}, \vec{c} \in \boldsymbol{A}^{[\mathbb{Z}[x_1,x_2,\dots]]}(\theta, \mathscr{L})$:

$$\vec{a} = \vec{c} \text{ if and only if } (\vec{a}(n) = \vec{c}(n) \text{ for every } x_n \in J). \tag{6.6}$$

Then consider any non-trivial commutative ring with unit $\boldsymbol{A}$. There are two different element $a, b \in A$. We consider the group homomorphisms $f, g \colon \mathbb{Z}[x_1, x_2, \dots] \rightrightarrows \boldsymbol{A}$ defined by the following rule:

$$f(x_n) := a \quad \text{and} \quad g(x_n) := \begin{cases} a & \text{if } x_n \in J \\ b & \text{otherwise} \end{cases}$$

for every $n \in \omega$. The sequences $\langle f(c) : c \in \mathbb{Z}[x_1, x_2, \dots]\rangle$ and $\langle g(c) : c \in \mathbb{Z}[x_1, x_2, \dots]\rangle$ belong to $\boldsymbol{A}^{[\mathbb{Z}[x_1,x_2,\dots]]}(\theta, \mathscr{L})$. Moreover, they are different but agree on the components in J. But this contradicts condition (6.6). Hence we conclude that γ does not preserve compact congruences. ⊠

As expected, the preservation of compact congruences plays a fundamental role in the preservation of EDPRC:

Theorem 6.11. *Let $\mathcal{F} \colon \mathsf{X} \to \mathsf{Y}$ be a faithful left adjoint between quasi-varieties that preserves subdirect representations and let γ preserve compact congruences. If* Y *has EDPRC, then also* X *has it.*

Proof. By Theorem 6.8 it will be enough to show that $\mathrm{Comp}_{\mathsf{X}}\boldsymbol{A}$ forms a dually Brouwerian join-semilattice for every $\boldsymbol{A} \in \mathsf{K}$. Consider an algebra $\boldsymbol{A} \in \mathsf{X}$. It is clear that $\mathrm{Comp}_{\mathsf{X}}\boldsymbol{A}$ forms a join-semilattice. Pick $\theta, \phi \in \mathrm{Comp}_{\mathsf{X}}\boldsymbol{A}$. Our goal will be to prove that the set

$$W := \{\eta \in \mathrm{Comp}_{\mathsf{X}}\boldsymbol{A} : \theta \leqslant \eta \vee \phi\}$$

has a minimum in $\mathrm{Comp}_{\mathsf{X}}\boldsymbol{A}$. By assumption we have $\gamma_{\boldsymbol{A}}(\theta), \gamma_{\boldsymbol{A}}(\phi) \in \mathrm{Comp}_{\mathsf{Y}}\mathcal{F}(\boldsymbol{A})$. Then consider the following set

$$Z := \{\eta \in \mathrm{Comp}_{\mathsf{Y}}\mathcal{F}(\boldsymbol{A}) : \gamma_{\boldsymbol{A}}(\theta) \leqslant \eta \vee \gamma_{\boldsymbol{A}}(\phi)\}$$

and observe that by Theorem 6.5

$$W = \{\eta \in \mathrm{Comp}_{\mathsf{X}}\boldsymbol{A} : \gamma_{\boldsymbol{A}}(\eta) \in Z\}.$$

By assumption (and Theorem 6.8) Z has a minimum $\alpha \in \mathrm{Comp}_{\mathsf{Y}}\mathcal{F}(\boldsymbol{A})$. Together with Theorem 6.5, this implies that

$$\gamma_{\boldsymbol{A}}(\theta) \leqslant \gamma_{\boldsymbol{A}}(\phi) \vee \alpha \leqslant \gamma_{\boldsymbol{A}}(\phi) \vee \bigcap_{\eta \in W} \gamma_{\boldsymbol{A}}(\eta) = \gamma_{\boldsymbol{A}}(\phi) \vee \gamma_{\boldsymbol{A}}(\bigcap W)$$

and, therefore, that

$$\theta \leqslant \phi \vee \bigcap W.$$

To complete the proof it will be enough to show that $\bigcap W \in \mathrm{Comp}_{\mathsf{X}}\boldsymbol{A}$. Recall that θ is finitely generated and that $\theta \leqslant \phi \vee \bigcap W$. Together with the fact that $\mathrm{Cg}_{\mathsf{X}}^{\boldsymbol{A}}$ is a finitary closure operator, this implies that there is a congruence $\eta \in \mathrm{Comp}_{\mathsf{X}}\boldsymbol{A}$ such that $\eta \leqslant \bigcap W$ and $\theta \leqslant \phi \vee \eta$. But this implies that $\eta \in W$ and, therefore, that $\bigcap W = \eta \in \mathrm{Comp}_{\mathsf{X}}\boldsymbol{A}$ as desired. ⊠

Example 6.12 (Kleene Algebras)**.** Let $\mathcal{F} \colon \mathsf{KA} \to \mathsf{DL}_{01}$ the left adjoint functor described in Example 5.28. Recall that DL_{01} has EDPC (Example 6.7). This means that $\mathrm{Comp}\mathcal{F}(\boldsymbol{A})$ is a dually Brouwerian join-semilattice for every $\boldsymbol{A} \in \mathsf{KA}$ by Theorem 6.8. Moreover, recall from Example 6.6 that $\gamma_{\boldsymbol{A}}$ is a lattice isomorphism for every $\boldsymbol{A} \in \mathsf{KA}$ (and, therefore, preserves compcat congruences). Thus also $\mathrm{Con}\mathcal{F}(\boldsymbol{A})$ is a dually Brouwerian join-semilattice for every $\boldsymbol{A} \in \mathsf{KA}$. This is to say that also KA has EDPC [93, Theorem 2.2]. ⊠

Now we move our attention to the study of the preservation of ESPRC. Since this notion can be characterized in terms of the behaviour of compact congruences of finitely generated algebras (Theorem 6.8), it will be useful to spend a few words on these concepts. We begin by observing that they are related as follows:

Lemma 6.13. *Let $\mathcal{F} \colon \mathsf{X} \to \mathsf{Y}$ be a left adjoint between quasi-varieties. If $\mathcal{F}$ preserves finitely generated algebras, then γ preserves compact congruences. When epimorphims in Y are surjective, the converse holds too.*

Proof. Suppose that $\mathcal{F}$ preserves finitely generated algebras. In particular, this means that $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1))$ is finitely generated. This implies that the exponent κ of the matrix power in the decomposition of the right adjoint to $\mathcal{F}$ can be chosen finite. Thus, with an application of part (iii)⇒(i) of Lemma 6.9, we conclude that γ preserves compact congruences.

Now suppose that epimorphisms in Y are surjective and that γ preserves compact congruences. Let also $\theta_{\mathscr{L}} \circ [\kappa]$ be the decomposition of the right adjoint to $\mathcal{F}$. By the assumptions we can find a finite $J \subseteq \kappa$ that fulfils condition (iii) of Lemma 6.9. Now by Corollary 5.30 we can identify $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1))$ with $\boldsymbol{Fm}_{\mathsf{Y}}(\kappa)/\mathrm{Cg}_{\mathsf{Y}}(\Theta)$. Then let $\boldsymbol{A}$ be the subalgebra of $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1))$ generated by $\{x^i/\mathrm{Cg}_{\mathsf{Y}}(\Theta) : i \in J\}$. Clearly $\boldsymbol{A}$ is finitely generated.

We claim that the inclusion map of $\boldsymbol{A}$ into $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1))$ is an epimorphism. To this end, consider two homomorphisms $f, g \colon \mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1)) \to \boldsymbol{B}$ that agree on A. First observe that

$$\boldsymbol{B} \vDash \Theta\langle f(x^i/\mathrm{Cg}_{\mathsf{Y}}(\Theta)) : i < \kappa\rangle \cup \Theta\langle g(x^i/\mathrm{Cg}_{\mathsf{Y}}(\Theta)) : i < \kappa\rangle$$

and, therefore, that the sequences

$$\langle f(x^i/\mathrm{Cg}_{\mathsf{Y}}(\Theta)) : i < \kappa\rangle \text{ and } \langle g(x^i/\mathrm{Cg}_{\mathsf{Y}}(\Theta)) : i < \kappa\rangle$$

belong to $\boldsymbol{B}^{[\kappa]}(\theta, \mathscr{L})$. Since f and g agree on A, we can apply condition (iii) of Lemma 6.9 obtaining $f(x^i/\mathrm{Cg}_{\mathsf{Y}}(\Theta)) = g(x^i/\mathrm{Cg}_{\mathsf{Y}}(\Theta))$ for every $i < \kappa$. Since the set $\{x^i/\mathrm{Cg}_{\mathsf{Y}}(\Theta) : i < \kappa\}$ generates $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1))$, we conclude that $f = g$. This establishes our claim.

Together with the fact that epimorphisms in Y are surjective, our claim implies that $\boldsymbol{Fm}_{\mathsf{Y}}(\kappa)/\Theta$ is finitely generated. This means that $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1))$ is finitely generated and, therefore, that the exponent κ of the matrix power in the decomposition of the right adjoint to $\mathcal{F}$ can be chosen finite. Then consider a new decomposition $\theta'_{\mathscr{L}'} \circ [n]$ with $n \in \omega$ of the right adjoint to $\mathcal{F}$. By Corollary 5.30 $\mathcal{F}$ preserves finitely generated algebras. ⊠

From our point of view, the reference to epimorphism surjectivity in Lemma 6.13 is very interesting since this condition is the algebraic equivalent of the Beth definability property in propositional logics [14, 15, 59, 60]. The next example shows that, without the assumption of epimorphism surjectivity, the preservation of compact congruences does not imply the preservation of finitely generated algebras.

Example 6.14 (Ring Hom-Functor)**.** Consider a generalized quasi-variety X and an algebra $\boldsymbol{A} \in \mathsf{X}$. Then let $\hom(\boldsymbol{A}, \cdot) \colon \mathsf{X} \to \mathsf{Set}$ be the functor defined by the following rule:

$$\begin{aligned} \boldsymbol{B} &\longmapsto \hom(\boldsymbol{A}, \boldsymbol{B}) \\ f \colon \boldsymbol{B} \to \boldsymbol{C} &\longmapsto f \circ (\cdot) \colon \hom(\boldsymbol{A}, \boldsymbol{B}) \to \hom(\boldsymbol{A}, \boldsymbol{C}). \end{aligned}$$

The functor $\hom(\boldsymbol{A}, \cdot)$ has a left adjoint $\mathcal{F} \colon \mathsf{Set} \to \mathsf{X}$ defined as follows. Given a set I, the algebra $\mathcal{F}(I)$ is the copower of $\boldsymbol{A}$ indexed by I. Moreover, given a function $f \colon I \to J$ between sets, we let $\mathcal{F}(f) \colon \mathcal{F}(I) \to \mathcal{F}(J)$ be the map $\langle p_{f(i)} : i \in I \rangle$ induced by the universal propery of the coproduct $\mathcal{F}(I)$, where $\{p_j \colon \boldsymbol{A} \to \mathcal{F}(J) : j \in J\}$ are the maps associated with the copower $\mathcal{F}(J)$.

Now consider the special case where X is the variety R of commutative rings with unit. Then consider the functor $\mathcal{F}$ that is left adjoint to $\hom(\boldsymbol{Q}, \cdot) \colon \mathsf{R} \to \mathsf{Set}$, where $\boldsymbol{Q}$ is the ring of rational numbers. First observe that $\mathcal{F}$ does not preserve finitely generated algebras. This is a consequence of the fact that the singletons are finitely generated in Set, while their image $\boldsymbol{Q}$ is not. It only remains to prove that γ preserves compact congruences. In order to do this, it will be convenient to decompose canonically the right adjoint $\hom(\boldsymbol{Q}, \cdot)$. For the sake of simplicity, let us identify the universe of $\boldsymbol{Q}$ with its cardinality. Then define a set of equations in Q variables as follows:

$$\theta := \{\overrightarrow{x^i + x^j} \approx \overrightarrow{x^{i+j}} : i, j \in Q\} \cup \{\overrightarrow{x^i \cdot x^j} \approx \overrightarrow{x^{i \cdot j}} : i, j \in Q\} \cup \{\overrightarrow{x^1} \approx \overrightarrow{1}\}.$$

In the definition of θ we are assuming that $\vec{\varphi}$ denotes the Q-sequence constant on φ. Finally, let $\mathscr{L} := \emptyset$. It is easy to see that $\boldsymbol{A}^{[Q]}(\theta, \mathscr{L}) = \hom(\boldsymbol{Q}, \boldsymbol{A})$ for every $\boldsymbol{A} \in \mathsf{R}$. Thus $\theta_{\mathscr{L}} \circ [Q] = \hom(\boldsymbol{Q}, \cdot)$. Now consider a pair $\vec{a}, \vec{c} \in$

$\boldsymbol{A}^{[\mathbb{Q}]}(\theta, \mathscr{L})$. Since the inclusion of the integers $\boldsymbol{Z}$ into the rationals $\boldsymbol{Q}$ is an epimorphism in R, we have that

$$\begin{aligned} \vec{a} = \vec{c} & \text{ if and only if } \vec{a} \text{ and } \vec{c} \text{ agree on } Z \\ & \text{ if and only if } \vec{a}(1) = \vec{c}(1). \end{aligned}$$

Therefore we can apply part (iii)⇒(i) of Lemma 6.9, obtaining that γ preserves compact congruences. ⊠

It is therefore natural to ask under which condition a left adjoint preserves finitely generated algebras in general. We have the following:

Lemma 6.15. *Let $\mathcal{F}\colon \mathsf{X} \longleftrightarrow \mathsf{Y}\colon \mathcal{G}$ be an adjunction $\mathcal{F} \dashv \mathcal{G}$ between quasi-varieties. The following conditions are equivalent:*

(i) *$\mathcal{F}$ preserves finitely generated algebras.*

(ii) *$\mathcal{F}$ preserves finitely presentable algebras.*

(iii) *$\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1))$ is finitely generated.*

(iv) *$\mathcal{G}$ preserves directed colimits.*

(v) *$\mathcal{G}$ can be decomposed as $\theta_{\mathscr{L}} \circ [\kappa]$ with κ finite.*

(vi) *$\mathcal{G}$ can be decomposed as $\theta_{\mathscr{L}} \circ [\kappa]$ with κ and θ finite.*

Proof. Parts (i)⇒(iii), (iii)⇒(v), (vi)⇒(v) are straightforward. Part (v)⇒(i) has been already proved in the last paragraph of the proof of Lemma 6.13. (i)⇒(ii): Recall that finitely presentable algebras are exactly the quotients of finitely generated algebras under compact congruences. From the assumption and Lemma 6.13 it follows that $\mathcal{F}$ preserves finitely generated algebras and that γ preserves compact congruences. It is easy to infer that $\mathcal{F}$ preserves finitely presentable algebras too.

(ii)⇒(vi): From the assumption it follows that $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1))$ is finitely presentable. Then there is $n \in \omega$ and a compact Y-congruence Θ such that $\mathcal{F}(\boldsymbol{Fm}_{\mathsf{X}}(1)) = \boldsymbol{Fm}_{\mathsf{Y}}(n)/\Theta$. Now, Θ is generated by a finite set $\Phi \subseteq \Theta$. This means that the right adjoint to $\mathcal{F}$ can be decomposed as $\theta_{\mathscr{L}} \circ [n]$, where $\theta := \{\vec{\varepsilon} \approx \vec{\delta} : \langle \varepsilon, \delta \rangle \in \Phi\}$ and $\vec{\varepsilon}, \vec{\delta}$ are sequences of length n.

(vi)⇒(iv): According to Theorem 5.29, the functor $\mathcal{G}$ can be written as the composition of two functors $[\kappa]\colon \mathsf{Y} \to \mathsf{K}$ and $\theta_{\mathscr{L}}\colon \mathsf{K} \to \mathsf{X}$, where K is a generalized quasi-variety. Moreover, by assumption θ and κ are finite. The fact that θ is finite, together with the fact that X and Y are quasi-varieties, implies that the map $\theta_{\mathscr{L}}\colon \mathbb{Q}(\mathsf{Y}^{[\kappa]}) \to \mathsf{X}$ is a well-defined right adjoint functor (this was explained in the paragraph right before Theorem 5.23). Now, recall from Example 5.10 that $\mathbb{Q}(\mathsf{Y}^{[\kappa]}) = \mathsf{Y}^{[\kappa]}$, since Y is a quasi-variety and κ is finite. In particular, we have that $[\kappa]\colon \mathsf{Y} \to \mathsf{Y}^{[\kappa]}$ is a category equivalence and, therefore, preserves directed colimits. Therefore it only remains to show that $\theta_{\mathscr{L}}\colon \mathsf{Y}^{[\kappa]} \to \mathsf{X}$ preserves directed colimits too. Taking a look at the proof of

Theorem 5.12, we see that $\theta_{\mathscr{L}}$ preserves λ-directed colimits for every regular cardinal larger than the number of variables that occur in the generalized quasi-equations axiomatizing $\mathsf{Y}^{[\kappa]}$. Since $\mathsf{Y}^{[\kappa]}$ is a quasi-variety, we can take $\lambda = \aleph_0$, yielding that $\theta_{\mathscr{L}}$ preserves directed colimits.

(iv)⇒(ii): In Lemma 5.2 we showed that finitely X-presentable algebras admit a categorical description, i.e., they are the algebras $\boldsymbol{A} \in \mathsf{X}$ for which the functor $\mathrm{hom}(\boldsymbol{A}, \cdot)\colon \boldsymbol{A} \to \mathsf{Set}$ preserves directed colimits. Then let $\boldsymbol{A} \in \mathsf{X}$ be finitely presentable. From the fact that $\mathcal{F} \dashv \mathcal{G}$ it follows that the functors $\mathrm{hom}(\mathcal{F}(\boldsymbol{A}), \cdot)$ and $\mathrm{hom}(\boldsymbol{A}, \mathcal{G}(\cdot))$ are naturally isomorphic. Observe that $\mathrm{hom}(\boldsymbol{A}, \mathcal{G}(\cdot)) = \mathrm{hom}(\boldsymbol{A}, \cdot) \circ \mathcal{G}$. From the assumption we know that both $\mathrm{hom}(\boldsymbol{A}, \cdot)$ and $\mathcal{G}$ preserve directed colimits. Thus we conclude that $\mathrm{hom}(\mathcal{F}(\boldsymbol{A}), \cdot)$ preserves colimits too, i.e., that $\mathcal{F}(\boldsymbol{A})$ is finitely presentable.*

⊠

The next result can be compared with Corollary 6.3.

Corollary 6.16. *Let $\mathcal{G}\colon \mathsf{Y} \to \mathsf{X}$ be a right adjoint functor between quasi-varieties that preserves directed colimits. If $\boldsymbol{A} \in \mathbb{Q}(\mathsf{K})$, then $\mathcal{G}(\boldsymbol{A}) \in \mathbb{Q}(\mathcal{G}(\mathsf{K}))$, for every $\mathsf{K} \cup \{\boldsymbol{A}\} \subseteq \mathsf{Y}$.*

Proof. Suppose that $\boldsymbol{A} \in \mathbb{Q}(\mathsf{K})$. A characterization of the operator $\mathbb{Q}(\cdot)$ says that $\boldsymbol{A}$ is a directed union of subalgebras of direct products of K. Since direct products are limits, they are preserved by $\mathcal{G}$. Moreover, right adjoints preserve embeddings. This can be proven in different ways: one of them is to think that $\mathcal{G}$ can be decomposed as $\theta_{\mathscr{L}} \circ [\kappa]$ and that the deformations $\theta_{\mathscr{L}}$ and $[\kappa]$ preserve embeddings. Thus it only remains to prove that $\mathcal{G}$ preserves directed unions.

To this end, consider the directed union $\boldsymbol{A}$ of a directed diagram $\{\boldsymbol{B}_j : j \in J\}$ in Y. Since Y is a quasi-variety, we know that $\boldsymbol{A} \in \mathsf{Y}$. Moreover $\boldsymbol{A}$ is the colimit of the diagram. By the assumption we know that $\mathcal{G}(\boldsymbol{A})$ is the colimit of the diagram $\{\mathcal{G}(\boldsymbol{B}_j) : j \in J\}$. Since $\mathcal{G}$ preserves embeddings, we conclude that $\mathcal{G}(\boldsymbol{A})$ is also the directed union of this diagram. ⊠

The preservation of finitely generated algebras is related to the preservation of ESPRC as follows:

Theorem 6.17. *Let $\mathcal{F}\colon \mathsf{X} \to \mathsf{Y}$ be a faithful left adjoint between quasi-varieties that preserves subdirect representations and finitely generated algebras. If Y has ESPRC, then also X has it.*

Proof. From Lemma 6.13 we know that γ preserves compact congruences. Then we can carry on a proof analogous to the one of Theorem 6.11, obtaining that if the compact congruences of finitely generated algebras of Y form a

*The proof of part (iv)⇒(ii) given here was suggested to me by Matěj Dostál in a private communication.

dually Brouwerian join-semilattices, then the same holds in X. Thus with an application of Theorem 6.8 we are done. ⊠

6.3 The case of category equivalence

The properties considered in the two previous sections are preserved by category equivalence. In particular, if $\mathcal{F}\colon \mathsf{X} \to \mathsf{Y}$ is a category equivalence between generalized quasi-varieties, then $\gamma_{\boldsymbol{A}}\colon \mathrm{Con}_{\mathsf{X}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Y}}\mathcal{F}(\boldsymbol{A})$ is an isomorphism for every $\boldsymbol{A} \in \mathsf{X}$. Thus all lattice theoretic properties of congruence lattices transfer from Y to X (and vice-versa), e.g., this happens for congruence distributivity (modularity) and EDPRC. The same holds for ESPRC, since category equivalences preserve finitely generated algebras [73, Theorem 3.1.(5)]. For general information about categorical properties in prevarieties we refer the reader to the survey [63].

In the rest of this section we will assume that empty algebras do not exist. Given two varieties X and Y, we write $\mathsf{X} \leqslant \mathsf{Y}$ to denote the fact that X can be interpreted in Y in the sense of Definition 5.16. If $\langle \mathsf{X}_n : n \in \omega \rangle$ is a decreasing sequence w.r.t. $\leqslant$ of finitely axiomatized varieties of finite type, then the class of varieties

$$\{\mathsf{Y} : \mathsf{X}_n \leqslant \mathsf{Y} \text{ for some } n \in \omega\}$$

is called the *Maltsev class* defined by this sequence, and the associated condition on varieties Y is called a *Maltsev condition*. A Maltsev condition is *linear* if the axioms $t_1 \approx t_2$ of each X_n can be chosen in a way that each t_i contains at most one occurrence of a basic operation symbol. Examples of properties characterized by linear Maltsev conditions in varieties are congruence distributivity, modularity, permutability and near unanimity. McKenzie applied his combinatorial decomposition of category equivalences to obtain a new and direct proof of the following result [73, Theorem 6.10], that first appeared in [34].

Theorem 6.18 (Davey and Werner)**.** *Categorically equivalent varieties satisfy the same linear Maltsev conditions.*

Remarkably, the combinatorial description of category equivalence can be applied also to disprove that some property is a linear Maltsev condition. We will provide an example of this application. Recall that a variety K is *congruence regular* if for every $\boldsymbol{A} \in \mathsf{K}$, $a \in A$ and $\theta, \phi \in \mathrm{Con}\boldsymbol{A}$, if $a/\theta = a/\phi$, then $\theta = \phi$.

Lemma 6.19. *Congruence regularity is not categorical in varieties.*

Proof. Let $\mathbb{Z}_4$ be the additive group of integers modulo 4 (with universe $\{0, 1, 2, 3\}$). Then let σ, π and μ be the operations defined by the following tables:

	σ	π
0	0	0
1	1	0
2	2	2
3	1	2

μ	0	1	2	3
0	0	1	0	1
1	0	0	0	0
2	0	3	2	1
3	0	0	0	0

Let $\boldsymbol{A}$ be the expansion of $\mathbb{Z}_4$ with σ, π and μ. The following equations hold in $\boldsymbol{A}$:

$$\sigma\sigma(x) \approx \sigma x \text{ and } \mu(\sigma\pi(x), \sigma(x)) \approx x.$$

Hence σ is idempotent and invertible in $\mathbb{V}(\boldsymbol{A})$, and its invertibility is witnessed by the unary terms $\pi(x)$ and x and the binary term $\mu(x, y)$. As a consequence $\mathbb{V}(\boldsymbol{A})$ is categorically equivalent to $\mathbb{V}(\boldsymbol{A})(\sigma)$ (see Example 5.15).

We know that $\mathbb{V}(\boldsymbol{A})$ is congruence regular, since $\boldsymbol{A}$ has a group reduct and congruence regularity is a Maltsev condition that holds in groups. Therefore it will be enough to show that $\mathbb{V}(\boldsymbol{A})(\sigma)$ is not congruence regular. To this end observe that the partition $\theta = \{\{0, 2\}, \{1, 3\}\}$ is a congruence of $\boldsymbol{A}$. To see this, observe that clearly θ is a congruence of $\mathbb{Z}_4$. Moreover, θ clearly preserves σ. Since θ identifies elements according to whether they are odd or even, it is compatible with π. Therefore it only remains to prove that θ is compatible with μ. Consider the following sets:

$$\begin{aligned}
X_1 &:= \{\langle 0, 0\rangle, \langle 0, 2\rangle, \langle 2, 0\rangle, \langle 2, 2\rangle\}\\
X_2 &:= \{\langle 1, 1\rangle, \langle 1, 3\rangle, \langle 3, 1\rangle, \langle 3, 3\rangle\}\\
X_3 &:= \{\langle 0, 1\rangle, \langle 0, 3\rangle, \langle 2, 1\rangle, \langle 2, 3\rangle\}\\
X_4 &:= \{\langle 1, 0\rangle, \langle 1, 2\rangle, \langle 3, 0\rangle, \langle 3, 2\rangle\}.
\end{aligned}$$

It will be enough to check that if two pairs belong to the same X_i, then their values under μ are identified by θ. But this can be easily checked taking a look at the table defining μ. Hence $\theta \in \mathrm{Con}\boldsymbol{A}$.

Then we consider the canonical projection $p_\theta \colon \boldsymbol{A} \to \boldsymbol{A}/\theta$. Applying our category equivalence, we obtain a surjective homomorphism $p_\theta \colon \boldsymbol{A}(\sigma) \to (\boldsymbol{A}/\theta)(\sigma)$. Now, its kernel $\theta(\sigma)$ can be identified with the partition $\{\{0, 2\}, \{1\}\}$ (observe that the universe of $\boldsymbol{A}(\sigma)$ is $\{0, 1, 2\}$). Therefore the congruence classes of 1 relative to $\theta(\mu)$ and to the identity relation of $A(\sigma)$ coincide. In particular, this shows that $\boldsymbol{A}(\sigma)$ is not congruence regular. As a consequence $\mathbb{V}(\boldsymbol{A})(\sigma)$ is not congruence regular too. ⊠

Corollary 6.20. *Congruence regularity is not a linear Maltsev condition.*

Proof. This is a direct consequence of Theorem 6.18 and Lemma 6.19. ⊠

6.4 Algebraizable logics

Until now we focused on the preservation of properties (especially of congruences lattices) in the presence of an adjunction between two generalized quasi-varieties X and Y. It may happen that X and Y are the equivalent algebraic semantics of two algebraizable logics. In this case it is possible to give a logical interpretation to these preservation results, by means of some transfer theorems that typically relate a metalogical property with an algebraic one. In order to explain how, let us recall some definitions. A logic $\mathcal{L}$ has a *generalized disjunction* if there is a set of formulas $\Delta(x, y, \vec{z})$ such that for every $\Gamma \cup \{\varphi, \psi, \gamma\} \subseteq Fm$:

$$(\Gamma, \varphi \vdash_{\mathcal{L}} \gamma \text{ and } \Gamma, \psi \vdash_{\mathcal{L}} \gamma) \iff \Gamma, \Delta(\varphi, \psi, \vec{\alpha}) \vdash_{\mathcal{L}} \gamma \text{ for every } \vec{\alpha} \in Fm.$$

We have the following [27, Theorem 2.5.17]:

Theorem 6.21 (Czelakowski)**.** *A finitary algebraizable logic has a generalized disjunction if and only if its equivalent algebraic semantics is relatively congruence distributive.*[†]

An algebraizable logic is *elementary algebraizable* if its equivalent algebraic semantics is a quasi-variety. A logic $\mathcal{L}$ has a *deduction-detachment theorem* (DDT) if there is a finite set of formulas $\Phi(x, y)$ in two variables such that for every $\Gamma \cup \{\psi, \varphi\} \subseteq Fm$:

$$\Gamma, \psi \vdash_{\mathcal{L}} \varphi \iff \Gamma \vdash_{\mathcal{L}} \Phi(\psi, \varphi).$$

A logic $\mathcal{L}$ has a *contextual deduction-detachment theorem* (CDDT) if for each $n \in \omega$ there is a finite set of formulas $\Phi(x, y, \vec{z})$ in $n + 2$ variables such that for every $\Gamma \cup \{\varphi, \psi\} \subseteq Fm(\vec{z})$:

$$\Gamma, \psi \vdash_{\mathcal{L}} \varphi \iff \Gamma \vdash \Phi_n(\psi, \varphi, \vec{z}).$$

The following transfer result is a combination of [20, Theorem 5.5] and [84, Theorem 9.2].

Theorem 6.22 (Blok, Pigozzi and Raftery)**.** *A finitary elementary algebraizable logic has the DDT (CDDT) if and only if its equivalent algebraic semantics has EDPRC (ESPRC).*

A logic $\mathcal{L}$ has an *inconsistency lemma* if for each $n \in \omega$ there is a set of formulas $\Phi(x_1, \dots, x_n)$ in n variables such that for every $\Gamma \cup \{\varphi_1, \dots, \varphi_n\} \subseteq Fm$:

$$\Gamma \cup \{\varphi_1, \dots, \varphi_n\} \text{ is inconsistent in } \mathcal{L} \iff \Gamma \vdash_{\mathcal{L}} \Phi_n(\varphi_1, \dots, \varphi_n).$$

[†]Metalogical properties that correspond to relative congruence modularity have been obtained in [27, 84]. Even if for sake of simplicity we chose not to discuss them here, they can be easily integrated in the following discussion.

The inconsistency lemma can be characterized by means of the behaviour of compact congruences. To explain how, recall that a join-semilattice $\langle A, \vee \rangle$ is *dually pseudo-complemented* if it has a greatest element 1 and for every $a \in A$ there exists $\min\{c \in A : b \vee a = 1\}$. We have the following:

Theorem 6.23 (Raftery). *A finitary logic algebraized by a quasi-variety* K *has the inconsistency lemma if and only if the compact* K*-congruences of* $\boldsymbol{A}$ *form a dually pseudo-complemented join-semilattice for every* $\boldsymbol{A} \in \mathsf{K}$.

The above result was proved in [86, Theorem 3.10], but cfr. also [87].

Drawing consequences from the work done until now, we can obtain the following informations on preservation of metalogical properties:

Theorem 6.24. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be two finitary logics algebraized respectively by the generalized quasi-varieties* X *and* Y*, which moreover are related by a faithful left adjoint* $\mathcal{F} \colon \mathsf{X} \to \mathsf{Y}$ *that preserves subdirect representations.*

1. *If* $\mathcal{Y}$ *has a generalized disjunction, then the same holds for* $\mathcal{X}$*.*

Moreover, if X *and* Y *are quasi-varieties:*

2. *If* γ *preserves compact congruences and* $\mathcal{Y}$ *has the inconsistency lemma, then the same holds for* $\mathcal{X}$*.*
3. *If* γ *preserves compact congruences and* $\mathcal{Y}$ *has the DDT, then the same holds for* $\mathcal{X}$*.*
4. *If* $\mathcal{F}$ *preserves finitely generated algebras and* $\mathcal{Y}$ *has the CDDT, then the same holds for* $\mathcal{X}$*.*

Proof. 1 follows from Theorems 6.5 and 6.21. 3 follows from Theorems 6.11 and 6.22. 4 follows from Theorems 6.17 and 6.22. It only remains to prove 2. Suppose that γ preserves compact congruences and that $\mathcal{Y}$ has the inconsistency lemma. From Theorem 6.23 it follows that $\mathrm{Comp}_{\mathsf{Y}}\mathcal{F}(\boldsymbol{A})$ is a dually pseudo-complemented join-semilattice for every $\boldsymbol{A} \in \mathsf{X}$. It will be enough to prove that the same holds for $\mathrm{Comp}_{\mathsf{X}}\boldsymbol{A}$.

We claim that the total congruence $1_{\boldsymbol{A}}$ is compact. To see this, recall that the total congruence $1_{\mathcal{F}(\boldsymbol{A})}$ is a join of compact congruences, since $\mathrm{Con}_{\mathsf{Y}}\mathcal{F}(\boldsymbol{A})$ is an algebraic lattice. Together with the fact that there is a greatest Y-compact congruence of $\mathcal{F}(\boldsymbol{A})$, this implies that $1_{\mathcal{F}(\boldsymbol{A})}$ is compact. Applying the fact that $\gamma_{\boldsymbol{A}}$ preserves joins, we obtain that:

$$\bigvee\{\gamma_{\boldsymbol{A}}(\theta) : \theta \in \mathrm{Comp}_{\mathsf{X}}\boldsymbol{A}\} = \gamma_{\boldsymbol{A}} \bigvee \mathrm{Comp}_{\mathsf{X}}\boldsymbol{A} = \gamma_{\boldsymbol{A}}(1_{\boldsymbol{A}}) = 1_{\mathcal{F}(\boldsymbol{A})}.$$

Since $1_{\mathcal{F}(\boldsymbol{A})}$ is compact there are $\theta_1, \dots, \theta_n \in V$ such that

$$\gamma_{\boldsymbol{A}}(\theta_1) \vee \dots \vee \gamma_{\boldsymbol{A}}(\theta_n) = 1_{\mathcal{F}(\boldsymbol{A})} = \gamma(1_{\boldsymbol{A}}).$$

Since $\gamma_{\boldsymbol{A}}$ is a lattice embedding, this means that $\theta_1 \vee \dots \vee \theta_n = 1_{\boldsymbol{A}}$. Hence $1_{\boldsymbol{A}}$ is a finite join of compact elements and, therefore, it is compact too. This establishes our claim.

As a consequence $\boldsymbol{A}$ has a greatest compact X-congruence, i.e., the total congruence $1_{\boldsymbol{A}}$. Keeping this in mind, it is almost straightforward to adapt the argument used in the proof of Theorem 6.11 and conclude that $\mathrm{Comp}_{\mathsf{X}}\boldsymbol{A}$ is dually pseudo-complemented. ⊠

An algebraizable logic $\mathcal{L}$ has the *strong finite model property* (SFMP) when for every finite set of formulas $\Gamma \cup \{\varphi\}$:

$$\Gamma \nvdash_{\mathcal{L}} \varphi \iff \begin{array}{l} \text{there is a finite model } \langle \boldsymbol{A}, F\rangle \text{ of } \mathcal{L} \\ \text{and a homomorphism } f\colon \boldsymbol{Fm} \to \boldsymbol{A} \text{ s.t.} \\ f[\Gamma] \subseteq F \text{ and } f(\varphi) \notin F. \end{array}$$

Theorem 6.25. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two finitary logics algebraized respectively by the quasi-varieties X and Y, which moreover are related by a faithful left adjoint $\mathcal{F}\colon \mathsf{X} \to \mathsf{Y}$ that preserves finitely generated algebras. If $\mathcal{Y}$ has the SFMP, then the same holds for $\mathcal{X}$.*

Proof. From the general theory of algebraizable logics it follows that the structural transformers that witness the algebraizability of $\mathcal{X}$ and $\mathcal{Y}$ can be chosen finite. Under this condition the fact that $\mathcal{Y}$ (resp. $\mathcal{X}$) has the SFMP is equivalent to the fact that Y (resp. X) is generated as a quasi-variety by its finite members. By the assumption we know that Y is generated as a quasi-variety by its finite members. It will be enough to prove that the same holds for X.

To this end, consider the translation $\langle \boldsymbol{\tau}, \Theta\rangle$ associated with $\mathcal{F}$. By Lemma 6.15 we can assume that $\boldsymbol{\tau}$ is an n-translation for some $n \in \omega$ and that Θ is finite. Then suppose that

$$\varphi_1 \approx \psi_1, \ldots, \varphi_n \approx \psi_n \nvDash_{\mathsf{X}} \varepsilon \approx \delta. \tag{6.7}$$

Let $x_1, \ldots, x_m$ be the variables occurring in the deduction above. By Lemma 6.4 we know that

$$\bigcup_{1 \leqslant i \leqslant n} \boldsymbol{\tau}^*(\varphi_i \approx \psi_i) \cup \bigcup_{1 \leqslant j \leqslant m} \Theta(\vec{x}_j) \nvDash_{\mathsf{Y}} \boldsymbol{\tau}^*(\varepsilon \approx \delta).$$

Since this is a quasi-equation, it fails in a finite algebra $\boldsymbol{A} \in \mathsf{Y}$. Now, the right adjoint to $\mathcal{F}$ has the form $\theta_{\mathscr{L}} \circ [n]$. Since $\boldsymbol{A}$ and the exponent of the matrix power are finite, also $\boldsymbol{A}^{[n]}(\theta, \mathscr{L}) \in \mathsf{X}$ is finite. It is easy to see that the deduction (6.7) fails in $\boldsymbol{A}$. We conclude that X is generated as a quasi-variety by its finite members. ⊠

To describe another kind of application to the theory of algebraizable logics, let us recall that there are varieties of algebras that are not the equivalent algebraic semantics of any algebraizable logic, e.g., the variety of Kleene algebras (see Example 5.28). This problem is easily overcome if we consider varieties up to category equivalence.

Theorem 6.26. *Every prevariety is categorically equivalent to the equivalent algebraic semantics of a finitely algebraizable logic formulated in enough variables. In particular, every quasi-variety is categorically equivalent to the equivalent algebraic semantics of a finitary and finitely algebraizable logic.*

Proof. Let K be a prevariety and consider the matrix power $\mathsf{K}^{[2]}$. We consider the following basic operations of the matrix power $\mathsf{K}^{[2]}$:

$$\begin{aligned} \langle x^1, x^2 \rangle \to \langle y^1, y^2 \rangle &:= \langle x^1, y^1 \rangle \\ \langle x^1, x^2 \rangle \leftarrow \langle y^1, y^2 \rangle &:= \langle x^2, y^2 \rangle \\ \Box \langle x^1, x^2 \rangle &:= \langle x^2, x^1 \rangle. \end{aligned}$$

Let $\mathcal{L}$ be the logic expressed with a proper class of variables[‡] determined by the class of matrices

$$\{ \langle \boldsymbol{A}^{[2]}, \{ \langle a, a \rangle : a \in A \} \rangle : \boldsymbol{A} \in \mathsf{K} \}.$$

We will show that $\mathcal{L}$ is algebraizable with equivalent algebraic semantics the prevariety $\mathsf{K}^{[2]}$. First observe that for every $\boldsymbol{A}^{[2]} \in \mathsf{K}^{[2]}$ and $\langle b, c \rangle \in A^2$ we have that:

$$\begin{aligned} b = c \quad &\Longleftrightarrow \quad \langle b, c \rangle = \langle c, b \rangle \\ &\Longleftrightarrow \quad \langle b, c \rangle = \Box \langle b, c \rangle. \end{aligned}$$

This implies that for every set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash_{\mathcal{L}} \varphi \Longleftrightarrow \{\Box \gamma \approx \gamma : \gamma \in \Gamma\} \vDash_{\mathsf{K}^{[2]}} \Box \varphi \approx \varphi.$$

Moreover for every $\boldsymbol{A}^{[2]} \in \mathsf{K}^{[2]}$ and $\langle a, b \rangle, \langle c, d \rangle \in A^2$ we have that:

$$\begin{aligned} \langle a, b \rangle = \langle c, d \rangle \quad &\Longleftrightarrow \quad a = c \text{ and } b = d \\ &\Longleftrightarrow \quad \langle a, c \rangle = \langle c, a \rangle \text{ and } \langle b, d \rangle = \langle d, b \rangle \\ &\Longleftrightarrow \quad \langle a, b \rangle \to \langle c, d \rangle = \Box(\langle a, b \rangle \to \langle c, d \rangle) \text{ and} \\ &\qquad\quad \langle a, b \rangle \leftarrow \langle c, d \rangle = \Box(\langle a, b \rangle \leftarrow \langle c, d \rangle). \end{aligned}$$

This means that the following condition holds:

$$x \approx y \dashv\vDash_{\mathsf{K}^{[2]}} \{x \to y \approx \Box(x \to y), x \leftarrow y \approx \Box(x \leftarrow y)\}.$$

We conclude that $\mathcal{L}$ is algebraizable through the following structural transformers

$$\boldsymbol{\tau}(x) := \{\Box x \approx x\} \text{ and } \boldsymbol{\rho}(x, y) := \{x \to y, x \leftarrow y\}.$$

Moreover, the equivalent algebraic semantics of $\mathcal{L}$ is the prevariety generated by $\mathsf{K}^{[2]}$. By Example 5.10 we know that this prevariety is $\mathsf{K}^{[2]}$ itself and that

[‡]For the precise definition of a logic expressed in a proper class of variables see [15].

$\mathsf{K}^{[2]}$ is categorically equivalent to K. Thus we conclude that K is categorically equivalent to the equivalent algebraic semantics of a finitely algebraizable algebraizable logic.

It only remains to prove that every quasi-variety K is categorically equivalent to the equivalent algebraic semantics of a *finitary* and finitely algebraizable logic. To this end consider a quasi-variety K. Recall from Example 5.10 that the matrix power $\mathsf{K}^{[2]}$ is a quasi-variety categorically equivalent to K. The above construction shows that $\mathsf{K}^{[2]}$ is the equivalent algebraic semantics of a finitely algebraizable logic $\mathcal{L}$ that is algebraizable through two finite structural transformers $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$. The fact that $\boldsymbol{\tau}$ is finite, together with the fact that $\mathsf{K}^{[2]}$ is a quasi-variety, implies that $\mathcal{L}$ is finitary.[§] ⊠

Corollary 6.27. *The property of being the equivalent algebraic semantics of an algebraizable logic is not categorical in prevarieties.*

Proof. This is a consequence of Theorem 6.26, together with the fact that there are a lot of prevarieties that are not the equivalent algebraic semantics of any algebraizable logic. ⊠

Drawing consequences from Theorem 6.26 one can obtain an elementary proof of the following fact, which was discovered in [21].

Corollary 6.28 (Blok-Raftery)**.** *There is a variety* K *such that:*

1. K *is the equivalent algebraic semantics of a finitary finitely algebraizable logic.*
2. *If a lattice equation fails in the variety of lattices, then it fails in the congruence lattice of some algebra in* K.

Proof. Observe that sets form a variety X of algebras (with empty language). In the proof of Theorem 6.26 we showed that $\mathsf{X}^{[2]}$ is the equivalent algebraic semantics of a finitary finitely algebraizable logic. Moreover, $\mathsf{X}^{[2]}$ is a variety categorically equivalent to X (see Example 5.10). Thus $\mathsf{X}^{[2]}$ satisfies condition 1. To prove condition 2 we reason as follows. Whitman proved in [95] that every lattice can be embedded into the lattice of equivalence relations of some set. Thus Whitman's theorem implies that if a lattice equation fails in the variety of lattices, then it fails in the congruence lattice of some algebra in X. Since X and $\mathsf{X}^{[2]}$ are categorically equivalent, the congruence lattices of algebras in $\mathsf{X}^{[2]}$ are isomorphic to the congruence lattices of algebras in X. Therefore we conclude that if a lattice equation fails in the variety of lattices, then it fails in the congruence lattice of some algebra in the variety $\mathsf{X}^{[2]}$. ⊠

We conclude this section remarking that the property of being axiomatized by generalized quasi-equations of a certain kind is preserved by category equivalence between prevarieties. More precisely, we have the following:

[§] I want to thank James G. Raftery for attracting my attention to the fact that the logic $\mathcal{L}$ arising from this construction is finitary.

Theorem 6.29. *In prevarieties the fact of being a generalized quasi-variety, a quasi-variety or a variety is categorical.*

Proof. For varieties and quasi-varieties this was proved respectively in [10] and [11]. Then it will be enough to prove that the property of being a generalized quasi-variety is categorical. We only provide a sketch of the proof. Let X and Y be two categorically equivalent prevarieties and suppose that X is closed under $\mathbb{U}_\kappa$ for some regular cardinal κ. It is easy to show that X is closed under the formation of κ-directed unions. This property is preserved by category equivalence, thus we conclude that the same holds for Y. But this means that Y is closed under the operator $\mathbb{U}_\kappa$, showing that Y is a generalized quasi-variety too. ⊠

6.5 A digression on structurality

In the previous sections we saw that every right adjoint $\mathcal{G} \colon \mathsf{Y} \to \mathsf{X}$ between generalized quasi-varieties arises from a translation $\langle \boldsymbol{\tau}, \Theta \rangle$ between relative equational consequences. In particular it arises from a map

$$\boldsymbol{\tau}^* \colon \mathcal{P}(Eq(\mathscr{L}_\mathsf{X}, \lambda)) \to \mathcal{P}(Eq(\mathscr{L}_\mathsf{Y}, \kappa \times \lambda)) \text{ for every } \lambda$$

that commutes with joins and, therefore, is residuated. Moreover $\boldsymbol{\tau}^*$ commutes with substitutions in a sense that we will make clear later on (Theorem 6.32). This observation sets translations such as $\boldsymbol{\tau}^*$ inside the general theory of equivalence between structural closure operators on $\mathcal{M}$-sets, as developed in [16], [44] and [52], where residuated maps that commute with substitutions are called *structural transformers*.¶ Accordingly, the goal of this section will be to review (for the expert reader) the work done until now from the point of view of this general theory. All the relevant definitions can be found in the references given above.

Consider the map

$$\zeta \colon End(\mathcal{L}_\mathsf{X}, \lambda) \to End(\mathcal{L}_\mathsf{Y}, \kappa \times \lambda)$$

defined as follows: given a substitution $\sigma \in End(\mathcal{L}_\mathsf{X}, \lambda)$, we let $\zeta(\sigma) \in End(\mathcal{L}_\mathsf{Y}, \kappa \times \lambda)$ be the substitution defined by

$$\zeta(\sigma)(x_j^i) \coloneqq \boldsymbol{\tau}_*(\sigma x_j)(i)$$

for every $j < \lambda$ and $i < \kappa$.

Lemma 6.30. *The map ζ is a monoid homomorphism.*

¶The theory of equivalence between structural closure operators has been generalized beyond the framework of $\mathcal{M}$-sets in different ways in [47, 49, 50, 51, 77, 91, 92].

Proof. It is clear that ζ preserves the neutral element. Therefore we turn to prove the same for composition. First observe that for every $\sigma \in End(\mathscr{L}_X, \lambda)$ and $\varphi \in Fm(\mathscr{L}_X, \lambda)$ we have that

$$\zeta(\sigma)(\boldsymbol{\tau}_*(\varphi)(i)) = \boldsymbol{\tau}_*(\sigma\varphi)(i) \text{ for } i < \kappa.$$

This can be proved easily by induction on the formula φ. Then consider $\sigma, \sigma' \in End(\mathcal{L}_X, \lambda)$. Let $\varphi := \sigma'(x_j)$. We have that:

$$\begin{aligned}\zeta(\sigma \circ \sigma')(x_j^i) &= \boldsymbol{\tau}_*(\sigma(\sigma'(x_j)))(i) = \boldsymbol{\tau}_*(\sigma\varphi)(i) = \zeta(\sigma)(\boldsymbol{\tau}_*(\varphi)(i)) \\ &= \zeta(\sigma)(\boldsymbol{\tau}_*(\sigma' x_j)(i)) = \zeta(\sigma)(\zeta(\sigma')(x_j^i)).\end{aligned}$$

Since $\zeta(\sigma \circ \sigma')$ and $\zeta(\sigma) \circ \zeta(\sigma')$ are substitutions, we conclude that they coincide. Hence σ is a monoid homomorphism. ⊠

Now observe that the monoid $End(\mathcal{L}_X, \lambda)$ induces two maps

$$\begin{aligned}&*_X \colon End(\mathcal{L}_X, \lambda) \times Eq(\mathscr{L}_X, \lambda) \to Eq(\mathscr{L}_X, \lambda) \\ &*_Y \colon End(\mathcal{L}_X, \lambda) \times Eq(\mathscr{L}_Y, \kappa \times \lambda) \to Eq(\mathscr{L}_Y, \kappa \times \lambda)\end{aligned}$$

defined respectively as:

$$\sigma *_X (\varepsilon \approx \delta) := \sigma(\varepsilon) \approx \sigma(\delta) \text{ and } \sigma *_Y (\varphi \approx \psi) := \zeta(\sigma)(\varphi) \approx \zeta(\sigma)(\psi)$$

for every $\sigma \in End(\mathcal{L}_X, \lambda)$, $\varepsilon \approx \delta \in Eq(\mathscr{L}_X, \lambda)$ and $Fm(\mathscr{L}_Y, \kappa \times \lambda)$. The first fact it is worth to remark is that the maps $*_X$ and $*_Y$ define actions of the same monoid on both sets of equations:

Lemma 6.31. *$\langle Eq(\mathscr{L}_X, \lambda), *_X\rangle$ and $\langle Eq(\mathscr{L}_Y, \kappa \times \lambda), *_Y\rangle$ are $End(\mathcal{L}_X, \lambda)$-sets.*

Proof. It is clear that $*_X$ is an action. The same fact for $*_Y$ follows from Lemma 6.30. ⊠

Keeping this in mind, we are now ready to prove that $\boldsymbol{\tau}^*$ commutes with substitutions.

Theorem 6.32. *The map $\boldsymbol{\tau}^* \colon \mathcal{P}(Eq(\mathscr{L}_X, \lambda)) \to \mathcal{P}(Eq(\mathscr{L}_Y, \kappa \times \lambda))$ is a structural transformer.*

Proof. First observe that $\boldsymbol{\tau}^*$ is a residuated map by definition. Then consider a substitution $\sigma \in End(\mathcal{L}_X, \lambda)$ and an equation $\varepsilon \approx \delta \in Eq(\mathscr{L}_X, \lambda)$. Recall from the proof of Lemma 6.30 that

$$\zeta(\sigma)(\boldsymbol{\tau}_*(\varepsilon)(i)) = \boldsymbol{\tau}_*(\sigma\varepsilon)(i) \text{ and } \zeta(\sigma)(\boldsymbol{\tau}_*(\delta)(i)) = \boldsymbol{\tau}_*(\sigma\delta)(i)$$

for every $i < \kappa$. Keeping this in mind, we have that:

$$\begin{aligned}\boldsymbol{\tau}^*(\sigma *_{\mathsf{X}} (\varepsilon \approx \delta)) &= \boldsymbol{\tau}^*(\sigma\varepsilon \approx \nu\delta)\\ &= \{\boldsymbol{\tau}_*(\sigma\varepsilon)(i) \approx \boldsymbol{\tau}_*(\sigma\delta)(i) : i < \kappa\}\\ &= \{\zeta(\sigma)(\boldsymbol{\tau}_*(\varepsilon)(i)) \approx \zeta(\sigma)(\boldsymbol{\tau}_*(\delta)(i)) : i < \kappa\}\\ &= \{\sigma *_{\mathsf{Y}} \boldsymbol{\tau}_*(\varepsilon)(i) \approx \sigma *_{\mathsf{Y}} \boldsymbol{\tau}_*(\delta)(i) : i < \kappa\}\\ &= \sigma *_{\mathsf{Y}} \boldsymbol{\tau}^*(\varepsilon \approx \delta).\end{aligned}$$

This concludes the proof. ⊠

In the general theory of equivalence between closure operators special attention has been devoted to *structural representation*, i.e., residuated injections between the closed sets of two structural closure operators that, moreover, commute with substitutions. Our goal will be to prove that, when the left adjoint $\mathcal{F}$ is faithful, the map γ induces a special structural representation (Theorem 6.34). In order to do this, we will need to go through some more observations. First consider the consequence relation

$$\vDash^{\Theta}_{\mathsf{Y}} \text{ on } Eq(\mathscr{L}_{\mathsf{Y}}, \kappa \times \lambda)$$

determined by the closure system of all theories of $\vDash_{\mathsf{Y}}$ that contain

$$\bigcup_{j<\lambda} \Theta\{x^i_j : i < \kappa\}.$$

Let also $\mathrm{Cg}^{\Theta}_{\mathsf{Y}}$ be the associated closure operator. In general $\vDash^{\Theta}_{\mathsf{Y}}$ need not to be structural with respect to the original substitution monoid $\mathrm{end}(\mathcal{L}_{\mathsf{Y}}, \kappa \times \lambda)$. Nevertheless we have the following:

Lemma 6.33. *The consequence* $\vDash^{\Theta}_{\mathsf{Y}}$ *is structural on* $\langle Eq(\mathscr{L}_{\mathsf{Y}}, \kappa \times \lambda), *_{\mathsf{Y}}\rangle$.

Proof. From condition 2 of Definition 5.20 it follows that

$$\bigcup_{j<\lambda} \Theta(\vec{x}_j) \vDash_{\mathsf{Y}} \zeta(\sigma)(\Theta)$$

for every $\sigma \in \mathrm{end}(\mathcal{L}_{\mathsf{X}}, \lambda)$. This means that the theory generated by $\bigcup_{j<\lambda} \Theta\{x^i_j : i < \kappa\}$ is invariant under the substitutions of the form $\zeta(\sigma)$. It is well known that this implies that the structurality condition holds for $\vDash^{\Theta}_{\mathsf{Y}}$, when restricted to the substitutions $\zeta(\sigma)$. ⊠

Consider a cardinal λ. As in the proof of Lemma 6.4, we can consider the lifting

$$\gamma\colon \mathcal{Th}(\vDash_{\mathsf{X}}) \to \mathcal{Th}(\vDash^{\Theta}_{\mathsf{Y}})$$

defined for every $\Phi \subseteq Eq(\mathscr{L}_{\mathsf{Y}}, \kappa \times \lambda)$ as $\gamma\mathrm{Cg}_{\mathsf{X}}(\Phi) := \mathrm{Cg}^{\Theta}_{\mathsf{Y}}\boldsymbol{\tau}^*(\Phi)$. In particular, we have the following:

Theorem 6.34. *If $\mathcal{F}$ is faithful, then the map $\gamma\colon \mathcal{T}h(\vDash_{\mathsf{X}}) \to \mathcal{T}h(\vDash^{\Theta}_{\mathsf{Y}})$ is a structural representation which, moreover, is induced by the structural transformer $\boldsymbol{\tau}^*$.*

Proof. From condition 1 of Definition 5.20 it follows that γ is induced by $\boldsymbol{\tau}^*$. Then we turn to prove that γ is a structural representation. From Lemma 6.1 and 6.4 it follows that γ is residuated and injective. Therefore it only remains to check that γ commutes with substitutions. To this end, consider $\sigma \in \mathrm{end}(\mathcal{L}_{\mathsf{X}}, \lambda)$ and $\Phi \in \mathcal{T}h(\vDash_{\mathsf{X}})$. We have that:

$$\begin{aligned}\gamma \mathrm{Cg}_{\mathsf{X}}\sigma(\Phi) &= \mathrm{Cg}^{\Theta}_{\mathsf{Y}}\boldsymbol{\tau}^*\mathrm{Cg}_{\mathsf{X}}\sigma(\Phi) = \mathrm{Cg}^{\Theta}_{\mathsf{Y}}\boldsymbol{\tau}^*\sigma(\Phi) = \mathrm{Cg}^{\Theta}_{\mathsf{Y}}\zeta(\sigma)\boldsymbol{\tau}^*(\Phi)\\ &= \mathrm{Cg}^{\Theta}_{\mathsf{Y}}\zeta(\sigma)\mathrm{Cg}^{\Theta}_{\mathsf{Y}}\boldsymbol{\tau}^*(\Phi) = \gamma\zeta(\sigma)\mathrm{Cg}^{\Theta}_{\mathsf{Y}}\boldsymbol{\tau}^*(\Phi).\end{aligned}$$

In the above series of equalities the second one follows from the fact that γ is induced by $\boldsymbol{\tau}^*$, the third one from the fact that $\boldsymbol{\tau}^*$ is structural and the fourth one from the fact that $\mathrm{Cg}^{\Theta}_{\mathsf{Y}}$ is structural with respect to $\zeta(\sigma)$. ⊠

Bibliography

[1] J. Adámek. How many variables does a quasivariety need? *Algebra Universalis*, (27):44–48, 1990.

[2] J. Adámek, H. Herrlich, and G. E. Strecker. Abstract and concrete categories: the joy of cats. *Reprints in Theory and Applications of Categories*, (17):1–507 (electronic), 2006. Reprint of the 1990 original [Wiley, New York; MR1051419].

[3] J. Adámek and H.-E. Porst. Algebraic theories of quasivarieties. *Journal of Algebra*, 208:379–398, 1998.

[4] J. Adámek and J. Rosický. *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994.

[5] J. Adámek, J. Rosický, and E. M. Vitale. *Algebraic Theories: A Categorical Introduction to General Algebra*. Cambridge Tracts in Mathematics. Cambridge University Press, 2011.

[6] H. Albuquerque, J. M. Font, R. Jansana, and T. Moraschini. Assertional logics, truth-equational logics, and the hierarchies of abstract algebraic logic. In J. Czelakowski, editor, *Don Pigozzi on Abstract Algebraic Logic and Universal Algebra*, Outstanding Contributions. Springer-Verlag., To appear.

[7] A. Ross Anderson and N. D. Belnap, Jr. *Entailment*. Princeton University Press, Princeton, N. J., 1975. Volume I: The logic of relevance and necessity, With contributions by J. Michael Dunn and Robert K. Meyer, and further contributions by John R. Chidgey, J. Alberto Coffa, Dorothy L. Grover, Bas van Fraassen, Hugues LeBlanc, Storrs McCall, Zane Parks, Garrel Pottinger, Richard Routley, Alasdair Urquhart and Robert G. Wolf.

[8] S. Awodey. *Category theory*, volume 49 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York, 2006.

[9] S. V. Babyonyshev. Fully Fregean logics. *Reports on Mathematical Logic*, 37:59–77, 2003.

[10] B. Banaschewski. On categories of algebras equivalent to a variety. *Algebra Universalis*, (16):264–267, 1983.

[11] P. Bankston and R. Fox. On categories of algebras equivalent to a quasivariety. *Algebra Universalis*, (16):153–158, 1983.

[12] J. L. Bell and M. Machover. *A Course in Mathematical Logic*. North-Holland, Amsterdam, 1977.

[13] C. Bergman. *Universal Algebra: Fundamentals and Selected Topics*. Chapman &; Hall Pure and Applied Mathematics. Chapman and Hall/CRC, 2011.

[14] G. Bezhanishvili, T. Moraschini, and J. G. Raftery. Epimorphism surjectivity in varieties of residuated structures. *Submitted*, 2016.

[15] W. J. Blok and E. Hoogland. The Beth property in Algebraic Logic. *Studia Logica*, 83(1–3):49–90, 2006.

[16] W. J. Blok and B. Jónsson. Equivalence of consequence operations. *Studia Logica*, 83(1–3):91–110, 2006.

[17] W. J. Blok and D. Pigozzi. On the structure of varieties with equationally definable principal congruences I. *Algebra Universalis*, 15:195–227, 1982.

[18] W. J. Blok and D. Pigozzi. Protoalgebraic logics. *Studia Logica*, 45:337–369, 1986.

[19] W. J. Blok and D. Pigozzi. *Algebraizable logics*, volume 396 of *Mem. Amer. Math. Soc.* A.M.S., Providence, January 1989.

[20] W. J. Blok and D. Pigozzi. Abstract algebraic logic and the deduction theorem. *Available in internet http://orion.math.iastate.edu/dpigozzi/*, 1997. Manuscript.

[21] W. J. Blok and J. G. Raftery. On congruence modularity in varieties of logic. *Algebra Universalis*, 45(1):15–21, 2001.

[22] W. J. Blok and J. Rebagliato. Algebraic semantics for deductive systems. *Studia Logica, Special Issue on Abstract Algebraic Logic, Part II*, 74(5):153–180, 2003.

[23] F. Bou and U. Rivieccio. The logic of distributive bilattices. *Logic Journal of the I.G.P.L.*, 19(1):183–216, 2011.

[24] S. Burris and H. P. Sankappanavar. *A course in Universal Algebra*. Available in internet https://www.math.uwaterloo.ca/~snburris/htdocs/ualg.html, the millennium edition, 2012.

[25] R. Cignoli. The class of Kleene algebras satisfying an interpolation property and Nelson algebras. *Algebra Universalis*, 23(3):262–292, 1986.

[26] W. H. Cornish and P. R. Fowler. Coproducts of Kleene algebras. *J. Austral. Math. Soc. Ser. A*, 27(2):209–220, 1979.

[27] J. Czelakowski. *Protoalgebraic logics*, volume 10 of *Trends in Logic—Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 2001.

[28] J. Czelakowski. The Suszko operator. Part I. *Studia Logica, Special Issue on Abstract Algebraic Logic, Part II*, 74(5):181–231, 2003.

[29] J. Czelakowski and W. Dziobiak. Deduction theorems within RM and its extensions. *The Journal of Symbolic Logic*, 64(1):279–290, 1999.

[30] J. Czelakowski and R. Jansana. Weakly algebraizable logics. *The Journal of Symbolic Logic*, 65(2):641–668, 2000.

[31] J. Czelakowski and D. Pigozzi. Fregean logics. *Annals of Pure and Applied Logic*, 127(1-3):17–76, 2004.

[32] J. Czelakowski and D. Pigozzi. Fregean logics with the multiterm deduction theorem and their algebraization. *Studia Logica*, 78(1-2):171–212, 2004.

[33] B. A. Davey and H. A. Priestley. *Introduction to lattices and order*. Cambridge University Press, New York, second edition, 2002.

[34] B. A. Davey and H. Werner. Dualities and equivalences for varieties of algebras. In *Contributions to lattice theory (Szeged, 1980)*, volume 33 of *Colloq. Math. Soc. János Bolyai*, pages 101–275. North-Holland, Amsterdam, 1983.

[35] L. Descalço and M. A. Martins. On the injectivity of the Leibniz operator. *Bulletin of the Section of Logic*, 34(4):203–211, 2005.

[36] J. J. Dukarm. Morita equivalence of algebraic theories. *Colloquium Mathematicum*, 55:11–17, 1988.

[37] M. Dummett and E. J. Lemmon. Modal logics between S4 and S5. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 5:250–264, 1959.

[38] J. M. Font. *Abstract Algebraic Logic - An Introductory Textbook*. College Publications, 2016. Forthcoming.

[39] J. M. Font, F. Guzmán, and V. Verdú. Characterization of the reduced matrices for the $\{\wedge, \vee\}$-fragment of classical logic. *Bulletin of the Section of Logic*, 20:124–128, 1991.

[40] J. M. Font and R. Jansana. *A general algebraic semantics for sentential logics*, volume 7 of *Lecture Notes in Logic*. A.S.L., 2009. First edition 1996. Electronic version freely available through Project Euclid at `projecteuclid.org/euclid.lnl/1235416965`.

[41] J. M. Font, R. Jansana, and D. Pigozzi. A survey on abstract algebraic logic. *Studia Logica, Special Issue on Abstract Algebraic Logic, Part II*, 74(1–2):13–97, 2003. With an "Update" in 91 (2009), 125–130.

[42] J. M. Font and T. Moraschini. Logics of varieties, logics of semilattices, and conjunction. *Logic Journal of the IGPL*, 22:818–843, 2014.

[43] J. M. Font and T. Moraschini. On the logics associated with a given variety of algebras. In A. Indrzejczak, J. Kaczmarek, and M. Zawidzki, editors, *Trends in Logic XIII: Gentzen's and Jaśkowski's heritage. 80 years of natural deduction and sequent calculi*, pages 67–80, Łódź, 2014. Wydawnictwo Uniwersytetu Łódzkiego.

[44] J. M. Font and T. Moraschini. M-sets and the representation problem. *Studia Logica*, 103(1):21–51, 2015.

[45] J. M. Font and V. Verdú. Algebraic logic for classical conjunction and disjunction. *Studia Logica, Special Issue on Algebraic Logic*, 50:391–419, 1991.

[46] R. Freese, E. Kiss, and M. Valeriote. Universal Algebra Calculator. Available at: www.uacalc.org, 2011.

[47] N. Galatos and J. Gil-Férez. Modules over quantaloids: Applications to the isomorphism problem in algebraic logic and π-institutions. *Journal of Pure and Applied Algebra*, To appear.

[48] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated Lattices: an algebraic glimpse at substructural logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, Amsterdam, 2007.

[49] N. Galatos and C. Tsinakis. Equivalence of consequence relations: an order-theoretic and categorical perspective. *The Journal of Symbolic Logic*, 74(3):780–810, 2009.

[50] J. Gil-Férez. Multi-term π-institutions and their equivalence. *Mathematical Logic Quarterly*, 52(5):505–526, 2006.

[51] J. Gil-Férez. *Categorical Applications to Abstract Algebraic Logic*. Ph. D. Dissertation, University of Barcelona, 2009.

[52] J. Gil-Férez. Representations of structural closure operators. *Archive for Mathematical Logic*, 50(1):45–73, 2011.

[53] M. L. Ginsberg. Multivalued logics: A uniform approach to inference in artificial intelligence. *Computational Intelligence*, 4:265–316, 1988.

[54] K. Gödel. Eine Interpretation des intuitionistischen Aussagenkalküls. *Ergebnisse eines mathematisches Kolloquiums*, 4:39–40, 1933.

[55] V. A. Gorbunov. *Algebraic theory of quasivarieties*. Siberian School of Algebra and Logic. Consultants Bureau, New York, 1998. Translated from the Russian.

[56] B. Herrmann. Algebraizability and Beth's theorem for equivalential logics. *Bulletin of the Section of Logic*, 22(2):85–88, 1993.

[57] B. Herrmann. Equivalential and algebraizable logics. *Studia Logica*, 57:419–436, 1996.

[58] B. Herrmann. Characterizing equivalential and algebraizable logics by the Leibniz operator. *Studia Logica*, 58:305–323, 1997.

[59] E. Hoogland. Algebraic characterizations of various Beth definability properties. *Studia Logica, Special Issue on Abstract Algebraic Logic*, 65:91–112, 2000.

[60] E. Hoogland. *Definability and interpolation: model-theoretic investigations*. Ph. D. Dissertation DS-2001-05, Institute for Logic, Language and Computation, University of Amsterdam, 2001.

[61] R. Jansana. Selfextensional logics with implication. In J.-Y. Beziau, editor, *Logica universalis*, pages 65–88. Birkhäuser, Basel, 2005.

[62] J. A. Kalman. Lattices with involution. *Transactions of the Americal Mathematical Society*, 87:485–491, 1958.

[63] E. W. Kiss, L. Márki, P. Pröhle, and W. Tholen. Categorical algebraic properties. a compendium on amalgamation, congruence extension, epimorphisms, residual smallness, and injectivity. *Studia Sci. Math. Hungar.*, 18:79–141, 1983.

[64] S. R. Kogalovski. On Birkhoff's theorem. *Uspekhi Mat. Nauk*, 20:206–207, 1965.

[65] P. Köhler and D. Pigozzi. Varieties with equationally definable principal congruences. *Algebra Universalis*, 11:213–219, 1980.

[66] A. N. Kolmogorov. *From Frege to Gödel. A source book in mathematical logic*, chapter On the principle of the excluded middle, 1925, pages 414–437. Cambridge: Harvard University Press, 1977.

[67] M. Kracht. *Tools and techniques in modal logic*, volume 142 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1999.

[68] M. Kracht. *Modal consequence relations*, chapter Part 2: Advanced Theory. Elsevier Science Inc., New York, NY, USA, 2006.

[69] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[70] R. D. Maddux. Undecidable semiassociative relation algebras. *The Journal of Symbolic Logic*, 59(2):398–418, 1994.

[71] L. L. Maksimova and V. V. Rybakov. A lattice of normal modal logics. *Algebra and Logic*, 13:105–122, 1974.

[72] A. I. Mal'cev. *The metamathematics of algebraic systems, collected papers: 1936-1967*. Amsterdam, North-Holland Pub. Co., 1971.

[73] R. McKenzie. An algebraic version of categorical equivalence for varieties and more general algebraic categories. In *Logic and algebra (Pontignano, 1994)*, volume 180 of *Lecture Notes in Pure and Appl. Math.*, pages 211–243. Dekker, New York, 1996.

[74] R. N. McKenzie, G. F. McNulty, and W. F. Taylor. *Algebras, lattices, varieties. Vol. I*. The Wadsworth &; Brooks/Cole Mathematics Series. Wadsworth &; Brooks/Cole Advanced Books &; Software, Monterey, CA, 1987.

[75] J. C. C. McKinsey and A. Tarski. Some theorems about the sentential calculi of Lewis and Heyting. *The Journal of Symbolic Logic*, 13:1–15, 1948.

[76] T. Moraschini. An algebraic study of exactness in partial contexts. *International Journal of Approximate Reasoning*, 55:457–468, 2014.

[77] T. Moraschini. The semantic isomorphism theorem in abstract algebraic logic. Submitted, 2015.

[78] T. Moraschini. A computational glimpse at the Leibniz and Frege hierarchies. Manuscript, January 2016.

[79] T. Moraschini and A. Vidal. Leibniz classifier: a software for abstract algebraic logic. `iiia.csic.es/~amanda/files/LClassifier.zip`, February 2015.

[80] C. Noguera and P. Cintula. The proof by cases property and its variants in structural consequence relations. *Studia Logica*, 101:713–747, 2013.

[81] H.-E. Porst. Equivalence for varieties in general and for $\mathcal{BOOL}$ in particular. *Algebra Universalis*, 43:157–186, 2000.

[82] H.-E. Porst. Generalized Morita Theories. *Notices of the South African Mathematical Society*, 32:4–16, 2001.

[83] J. G. Raftery. The equational definability of truth predicates. *Reports on Mathematical Logic*, (41):95–149, 2006.

[84] J. G. Raftery. Contextual deduction theorems. *Studia Logica*, 99(1):279–319, 2011.

[85] J. G. Raftery. A perspective on the algebra of logic. *Quaestiones Mathematicae*, 34:275–325, 2011.

[86] J. G. Raftery. Inconsistency lemmas in algebraic logic. *Mathematical Logic Quaterly*, 59(6):393–406, 2013.

[87] J. G. Raftery. Relative congruence schemes and decompositions in quasivarieties. *Manuscript*, 2016.

[88] H. Rasiowa. *An algebraic approach to non-classical logics*, volume 78 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, Amsterdam, 1974.

[89] W. Rautenberg. On reduced matrices. *Studia Logica*, 52:63–72, 1993.

[90] U. Rivieccio. *An Algebraic Study of Bilattice-based Logics*. Ph. D. Dissertation, University of Barcelona, 2010.

[91] C. Russo. *Quantale Modules, with Applications to Logic and Image Processing*. Ph. D. Dissertation, University of Salerno, 2007.

[92] C. Russo. An order-theoretic analysis of interpretations among propositional deductive systems. *Annals of Pure and Applied Logic*, 164(2):112–130, 2013.

[93] H. P. Sankappanavar. A characterization of principal congruences of De Morgan algebras and its applications. In A. I. Arruda, R. Chuaqui, and N. C. A. da Costa, editors, *Mathematical logic in Latin America (Proc. IV Latin Amer. Sympos. Math. Logic, Santiago, 1978)*, volume 99 of *Stud. Logic Foundations Math.*, pages 341–349. North-Holland, Amsterdam, 1980.

[94] A. Tarski and S. Givant. *A formalization of set theory without variables*, volume 41 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1987.

[95] P. M. Whitman. Lattices, equivalence relations, and subgroups. *Bulletin of the American Mathematical Society*, 52:507–522, 1946.

[96] R. Wójcicki. *Theory of logical calculi. Basic theory of consequence operations*, volume 199 of *Synthese Library*. Reidel, Dordrecht, 1988.

Index