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Contribution to the center and integrability problems in planar vector fields

Xavier Santallusia Esvert

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PhD Thesis

Contribution to the center and integrability problems in planar vector fields.

Xavier Santallusia Esvert

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Directors of the PhD Thesis:
Prof. Dr. Jaume Giné
Prof. Dr. Maite Grau

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Contribution to the center and integrability problems in planar vector fields.

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El Dr. Jaume Giné, catedràtic de l'Escola Politècnica Superior de la Universitat de Lleida i la Dra. Maite Grau, professora agregada de l'Escola Politècnica Superior de la Universitat de Lleida

CERTIFIQUEN:

Que la memòria "Contribution to the center and integrability problems in planar vector fields" presentada per Xavier Santallusia Esvert per optar al grau de Doctor s'ha realitzat sota la seva supervisió.

Lleida, 12 de desembre de 2016.

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Resum

Aquesta tesi consta d'un primer capítol introductori, set capítols amb diferents resultats i una bibliografia.

El primer capítol conté la definició i els resultats previs necessaris per abordar la resta de la memòria. El capítol 2 i 3 estan molt relacionats. En el primer es descriu un mètode alternatiu per al còmput de les constants de Poincaré–Liapunov. A diferència de mètodes anteriors, el mètode presentat no requereix el càlcul d'integrals i dóna de forma explícita les constants de Poincaré–Liapunov. En el tercer capítol es descriu com s'ha implementat aquest nou mètode i els resultats que dóna per a sistemes quadràtics i sistemes amb termes no lineals cúbics homogenis.

El quart capítol es centra en equacions d'Abel i la seva integrabilitat. Es descriu la forma d'una integral primera que sigui algebraica en funció de les variables dependents i es donen múltiples exemples d'equacions d'Abel integrables en aquest sentit. En el cinquè capítol també s'aborda el problema de la integrabilitat però per a equacions diferencials en el pla definides per funcions analítiques. Es fa un reescalat de les variables dependents i de la variable independent amb un paràmetre ε que està elevat a potències senceres (*blow-up* paramètric) de forma que el sistema resultant sigui analític en ε. Es dóna un mètode que aprofita que una integral primera, si existeix, ha de ser analítica en el paràmetre a fi de trobar condicions per a l'existència d'aquesta integral primera. D'aquesta manera es defineix el que s'anomenen variables essencials del sistema.

Els darrers tres capítols versen sobre les equacions d'Abel i el problema del centre. En general es consideren equacions d'Abel trigonomètriques. En el sisè capítol es donen algunes condicions necessàries i suficients per a que una equació d'Abel trigonomètrica definida per polinomis trigonomètrics de grau fins a 3 tingui un centre. Tots els exemples donats en aquest capítol tenen un centre universal. En el capítol setè es dóna un exemple d'una equació d'Abel trigonomètrica definida per polinomis trigonomètrics de grau 3 que té un centre que no és universal. D'aquesta manera es resol un problema obert: determinar el grau més petit pel qual un equació d'Abel trigonomètrica amb centre no és de composició. El darrer capítol tracta equacions d'Abel trigonomètriques i polinomials i dóna un compendi dels darrers resultats coneguts i conjectures sobre el problema del centre en aquestes equacions. També es donen exemples nous d'equacions d'Abel amb centre.

Resumen

Esta tesis consta de un primer capítulo introductorio, siete capítulos con diferentes resultados y una bibliografía.

El primer capítulo contiene la definición y los resultados previos necesarios para abordar el resto de la memoria. Los capítulos 2 y 3 están muy relacionados. En el primero se describe un método alternativo para el cómputo de las constantes de Poincaré–Liapunov. A diferencia de métodos anteriores, el método presentado no requiere el cálculo de integrales y da de forma explícita las constantes de Poincaré–Liapunov. En el tercer capítulo se describe cómo se ha implementado este nuevo método y los resultados que da para sistemas cuadráticos y sistemas con términos no lineales cúbicos homogéneos.

El cuarto capítulo se centra en ecuaciones de Abel y su integrabilidad. Se describe la forma de una integral primera que sea algebraica en función de las variables dependientes y se dan múltiples ejemplos de ecuaciones de Abel integrables en este sentido. En el quinto capítulo también se aborda el problema de la integrabilidad pero para ecuaciones diferenciales en el plano definidas por funciones analíticas. Se hace un reescalado de las variables dependientes y de la variable independiente con un parámetro ε que está elevado a poténcias enteras (*blow-up* paramétrico) de forma que el sistema resultante sea analítico en ε. Se da un método que aprovecha que una integral primera, si existe, debe ser analítica en el parámetro con el fin de encontrar condiciones para la existéncia de esta integral primera. De esta manera se define lo que se llaman variables esenciales del sistema.

Los últimos tres capítulos versan sobre las ecuaciones de Abel y el problema del centro. En general se consideran ecuaciones de Abel trigonométricas. En el sexto capítulo se dan algunas condiciones necesarias y suficientes para que una ecuación de Abel definida por polinomios trigonométricos de grado hasta 3 tenga un centro. Todos los ejemplos dados en este capítulo tienen un centro universal. En la capítulo séptimo se da un ejemplo de una ecuación de Abel definida por polinomios trigonométricos de grado 3 que tiene un centro que no es universal. De esta manera se resuelve un problema abierto: determinar el grado mas pequeño por el que un ecuación de Abel trigonométrica con centro no es de composición. El último capítulo trata ecuaciones de Abel trigonométricas y polinomiales y da un compendio de los últimos resultados conocidos y conjeturas sobre el problema del centro en estas ecuaciones. También se dan ejemplos nuevos de ecuaciones de Abel con centro.

Summary

This thesis consists of a first introductory chapter, seven chapters with different results and a bibliography.

The first chapter contains the definition and the previous results necessary to address the rest of the memory. Chapters 2 and 3 are closely related. In the first one, an alternative method is described for the computation of the Poincaré–Liapunov constants. Unlike previous methods, the presented method does not require the computation of primitives and gives an explicit expression of the Poincaré–Liapunov constants. The third chapter describes how this new method has been implemented and the results that it gives for quadratic systems and systems with homogeneous, cubic, non-linear terms.

The fourth chapter focuses on Abel equations and their integrability. We describe the form of a first integral that is algebraic in function of the dependent variables and give more examples of equations of Abel integrable from this point of view. The fifth chapter also discusses the integrability problem but for differential equations in the plane defined by analytical functions. A rescaling of the dependent and the independent variables with a parameter ε which is elevated to integer powers (parametrical blow up) so that the resulting system is analytical in ε. A method is given that takes advantage that a first integral, if it exists, it must be analytical in the parameter in order to find conditions for the existence of this first integral. In this way we define what are called essential variables of the system.

The last three chapters deal with Abel equations and the center problem. In general, we consider Abel trigonometric equations. In the sixth chapter some necessary and sufficient conditions for an Abel equation defined by trigonometric polynomials of degree up to 3 have a center are given. All the examples given in this chapter have a universal center. In the seventh chapter it is given an example of an Abel equation defined by trigonometric polynomials of degree 3 with a center which is not universal. In this way an open problem is solved: to determine the lowest degree such that a trigonometric Abel equation has a center which is not a composition center. The last chapter deals with trigonometric and polynomial Abel equations and gives a survey of the last known results and conjectures about the center problem for these equations. Besides some new examples of Abel differential equations with a center are given.

Contents

Chapter 1

Introduction

This thesis deals with planar differential systems of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1.1}$$

where $P(x, y)$ and $Q(x, y)$ are analytic functions defined in a neighborhood of the origin. Usually, we will consider that $P(x, y), Q(x, y) \in \mathbb{R}[x, y]$ are coprime polynomials, that is, there is no non-constant polynomial which divides both P and Q. The dot denotes derivation with respect to the independent variable t usually called *time*, that is $\dot{} = \frac{d}{dt}$.

When P and Q are polynomials, we call d the maximum degree of P and Q and we say that system (1.1) is of degree d. When d = 2, we say that (1.1) is a *quadratic system* . If p is a point such that $P(p) = Q(p) = 0$, then we say that p is a *singular point* of system (1.1).

A *periodic orbit* is an orbit $\Gamma = \{\phi(t) \in \mathbb{R}^2 : t \in \mathbb{R}\}$ different from a singular point such that $\phi(t)$ is a periodic solution of system (1.1), i.e., there exists a positive time $T \in \mathbb{R}^+$ called the period and satisfying $\phi(t+T) = \phi(t)$ for all t. Hence, a periodic orbit Γ is a closed invariant curve without singular points. An isolated periodic orbit Γ is called a *limit cycle* and always has a neighborhood free of other periodic orbits. Any non-isolated periodic orbit belongs always to a period annulus .

The *center problem* for a singular point p of an analytic differential system on the plane (1.1) consists on determining if p is a focus or if it is a center. We recall that p is a focus if there exists a neighborhood of it such that each trajectory spirals around p and we say that p is a center if there exists a neighborhood of it such that each trajectory is closed surrounding p.

We may consider $\mathcal{X}(x, y) = P(x, y)(\partial/\partial x)+Q(x, y)(\partial/\partial y)$ the vector field associated to system (1.1) and let us denote by $D\mathcal{X}(p)$ the differential matrix of $\mathcal{X}$ at p. It is well-known, see for instance [108], that if both eigenvalues of $D\mathcal{X}(p)$ are nonzero complex conjugates $\alpha \pm \beta i$ then the singular point p is a center or a focus. In case $\alpha \neq 0$, then p is a focus (stable if $\alpha < 0$ and unstable otherwise). If $\alpha = 0$, then p may be a center or a focus. In this case, we say that p is a *weak focus* for system (1.1). For example, it can be proved that system $\dot{x} = -y + xy + ay^2$, $\dot{y} = x$ has a focus at the origin if $a \neq 0$ (stable if $a < 0$ and unstable if $a > 0$). If $a = 0$, this system has a center at the origin.

The problem of distinguishing between a center and a focus in the case of purely imaginary eigenvalues is the classical *Poincaré–Liapunov center problem*.

1.1 Center problem and Poincaré's method

Let us consider the Poincaré–Liapunov center problem. We can always make an affine change of variables and a change of time to system (1.1) and assume, without loss of generality, that $p = (0, 0)$ and the system is of the form:

$$\dot{x} = -y + X(x, y), \quad \dot{y} = x + Y(x, y), \tag{1.2}$$

where X, Y are analytic functions in a neighborhood of the origin of order greater or equal than 2.

These systems have a linear center at the origin perturbed by analytic functions of order greater or equal than 2. In the local study of these systems we find three problems closely related to one another: the determination of the origin's stability, the existence and the number of local limit cycles around the origin and the determination of first integrals when they exist.

H. Poincaré [109] developed an important technique for the general solution of these problems: it consists on finding a formal power series of the form

$$H(x, y) = \sum_{n=2}^{\infty} H_n(x, y), \tag{1.3}$$

where $H_2(x,y) = (x^2+y^2)/2$ and $H_n(x,y)$ are homogeneous polynomials of degree n, so that

$$\dot{H} = \frac{\partial H}{\partial x}(-y+X) + \frac{\partial H}{\partial y}(x+Y) = \sum_{k=2}^{\infty} V_{2k}(x^2+y^2)^k,$$

where V_{2k}, $k \geq 1$, are real numbers called *Poincaré–Liapunov constants*. The determination of these constants allows the solution of the three mentioned problems. These constants are determined in a recursive way explained, for instance, in [120].

The vanishing of all Liapunov constants is a necessary and sufficient condition to have a center at the origin for system (1.2). If for some k we have $V_4 = V_6 = \ldots = V_{2k-2} = 0$ and $V_{2k} \neq 0$ then the origin of system (1.2) is a focus (stable if $V_{2k} < 0$ and unstable if $V_{2k} > 0$). We say that it is a focus of order k. In case all Liapunov constants are zero, the series $H(x,y)$ would be a first integral of the system if it was convergent. Poincaré proved that, if all Liapunov constants vanish, then it is always possible to find a power series of the form (1.3) convergent in some neighborhood of the origin. Then, this power series is an analytical first integral defined in some neighborhood of the origin. However, it is not always possible to express this first integral (convergent in some neighborhood of the origin) by means of elementary functions. This result is proved in [109].

Theorem 1.1 *System (1.2) has a center at the origin if, and only if, there exists a local analytical first integral of the form $H(x,y) = x^2+y^2+F(x,y)$ defined in a neighborhood of the origin, where $F(x,y)$ is an analytic function of order greater than* 2.

For a family of systems (1.2), Liapunov constants are polynomials whose variables are the coefficients of the terms of the functions $X(x,y)$ and $Y(x,y)$. Let us consider $\mathcal{I} =< V_{2k} >_{k\in\mathbb{N}}$ the ideal generated for these Liapunov constants. If $X(x,y)$ and $Y(x,y)$ are polynomials of degree d, then the ideal $\mathcal{I}$ has a finite number of generators by Hilbert's basis Theorem. Let $M(d)$ be the minimum number of generators. It was shown by Shi Songling [120] that, under certain hypothesis about Liapunov generator constants, the number of limit cycles around the origin is at least $M(d)$.

Poincaré's method solves the center problem only from a theoretical point of view. If $X(x,y)$ and $Y(x,y)$ are analytic functions, we have, in general, an infinite number of Liapunov constants to compute in order to deduce that the origin of system (1.2) is a center. If $X(x,y)$ and $Y(x,y)$ are polynomials, we have that the number of Liapunov constants that must be null to ensure that all of them are null is finite. Nevertheless, we do not know how many Liapunov constants must be computed to prove that the origin of system (1.3) is a center.

In Chapter 2, we consider an arbitrary analytic system which has a linear center at the origin of the form (1.2) and we compute recursively all its Poincaré–Liapunov constants in function of the coefficients of the system. In this way, we give an answer to the classical center problem. We also compute the coefficients of the Poincaré series in function of the same coefficients. The algorithm for these computations has an easy implementation. The method does not need the computation of any definite or indefinite integral. The algorithm is applied to some polynomial differential systems.

In the last years many papers have been published giving different methods to compute the Poincaré–Liapunov constants. In Chapter 2 it was given a new method to compute recursively all the Poincaré–Liapunov constants. In Chapter 3 we describe its implementation in two different ways, by means of a Computer Algebra System and making a specific program algorithm in any computer language. If this second alternative is used, later it is necessary to translate the results so that they can be manipulated with a Computer Algebra System. We describe also how the availability of symbolic manipulation procedures has recently led a significant progress in the resolution of the different problems related with the Poincaré–Liapunov constants as they are the center problem and the small-amplitude limit cycles.

1.2 Reversibility in the center problem

An analytic *involution* $R : U \to \mathbb{R}^2$ is an analytic diffeomorphism different from the identity such that $R \circ R = Id$, where Id is the identity map.

From the work of Montgomery and Zippin [101] we have that after a linear change of coordinates, any analytic involution different from the identity takes the form $R(x,y) = (R_1, R_2) = (x + r_1(x,y), -y + r_2(x,y))$ with analytic functions r_i without constant nor linear terms. Moreover, the analytic near–identity change of coordinates $\phi(x,y) = (u,v) = (x + \cdots, y + \cdots)$ with $u = (x + R_1)/2$, and $v = (y - R_2)/2$ linearizes the involution, that is, $R_0(u,v) =$

$\phi \circ R \circ \phi^{-1}(u,v) = (u,-v)$.

Definition 1.2 *Let R be a diffeomorphism $R: U \to U$ which is an involution such that $R_*\mathcal{X} = -\mathcal{X} \circ R$, where $\mathcal{X}$ is the vector field associated to system* (1.1) then the system is called reversible.

The first examples of reversible systems were given by Poincaré, see [109]. Systems which are reversible for the involution $R_0(x,y) = (x,-y)$ are called *time–reversible* systems. Notice that system (1.1) is time–reversible if and only if $P(x,-y) = -P(x,y)$ and $Q(x,-y) = Q(x,y)$.

We remark that the analytic near–identity change of coordinates $\phi(x,y) = (u,v)$ that linearizes the involution $R(x,y)$ transforms the reversible system (1.1) into a time–reversible system

$$\dot{u} = v\tilde{P}(u,v^2), \quad \dot{v} = \tilde{Q}(u,v^2). \tag{1.4}$$

According to Poincaré, system (1.2) has a center at the origin if, and only if, there exists a near–identity analytic change of coordinates

$$(u,v) = \phi(x,y) = (x + o(|(x,y)|), y + o(|(x,y)|)) \ ,$$

transforming system (1.2) into the normal form

$$\dot{u} = -v[1 + \psi(u^2+v^2)] \ , \ \dot{v} = u[1 + \psi(u^2+v^2)] \ , \tag{1.5}$$

with ψ an analytic function near the origin such that $\psi(0) = 0$. It is clear that the transformed system (1.5) is time–reversible. Then, the original system (1.2) is reversible by means of the involution $R = \phi^{-1} \circ R_0 \circ \phi$. Consequently, it follows that all systems (1.2) having a center are reversible.

1.3 Integrability problem

A $\mathcal{C}^j$ function $H : \mathcal{U} \to \mathbb{R}$ such that it is constant on each trajectory of (1.1) and it is not locally constant is called a *first integral* of system (1.1) of class j defined on $\mathcal{U} \subseteq \mathbb{R}^2$. The equation $H(x,y) = c$ for a fixed $c \in \mathbb{R}$ gives a set of trajectories of the system, but in an implicit way. When $j \geq 1$, these conditions are equivalent to

$$P(x,y)\frac{\partial H}{\partial x} + Q(x,y)\frac{\partial H}{\partial y} = 0,$$

and H not locally constant. The problem of finding such a first integral and the functional class it must belong to is what we call the *integrability problem* .

To find an integrating factor or an inverse integrating factor for system (1.1) is closely related to finding a first integral for it. When considering the integrability problem we are also addressed to study whether an (inverse) integrating factor belongs to a certain given class of functions.

Given $\mathcal{W}$ an open set of $\mathbb{R}^2$, the function $\mu : \mathcal{W} \to \mathbb{R}$ of class $\mathcal{C}^j(\mathcal{W})$, $j > 1$, that satisfies the linear partial differential equation

$$P(x,y)\frac{\partial \mu}{\partial x} + Q(x,y)\frac{\partial \mu}{\partial y} = -\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\mu(x,y) \tag{1.6}$$

is called an *integrating factor* of system (1.1) defined on $\mathcal{W}$. The expression $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ is called the *divergence* of system (1.1) and we denote it by $\text{div}(x,y)$.

An easier function to find which also gives additional properties for a differential system (1.1) is the inverse of an integrating factor, that is, $V = 1/\mu$, which is called *inverse integrating factor*.

We note that $\{V = 0\}$ is formed by orbits of system (1.1). If V is defined on $\mathcal{W}$, then the function $\mu = 1/V$ defines on $\mathcal{W} \setminus \{V = 0\}$ an integrating factor of system (1.1), which allows the computation of a first integral of the system on $\mathcal{W} \setminus \{V = 0\}$. The *first integral H associated to the inverse integrating factor V* can be computed through the integral $H(x,y) = \int (Q(x,y)dx - P(x,y)dy)/V(x,y)$, and the condition (1.6) for $\mu = 1/V$ ensures that this line integral is well defined. On the other hand, system (1.2) has a center at the origin if, and only if, there exists an analytic inverse integrating factor V well defined around the origin and with $V(0,0) \neq 0$, see [111].

One of the simplest ordinary differential equation is a linear one, which is of the form

$$\frac{dy}{dx} = A_0(x) + A_1(x)y. \tag{1.7}$$

where $A_1(x) \not\equiv 0$ and where A_i are meromorphic functions of x. The integrability of this kind of equations is straightforward. We integrate equation (1.7) and we have that any solution is given by

$$y(x) = e^{\int_0^x A_1(\sigma)d\sigma}\left(y_0 + \int_0^x e^{-\int_0^s A_1(\sigma)d\sigma} A_0(s)ds\right),$$

where y_0 is a real constant. From this expression we can deduce a first integral of (1.7) by isolating the constant y_0 so that $H(x,y) = y_0$.

The following order of difficulty ia a Riccati differential equation

$$\frac{dy}{dx} = A_0(x) + A_1(x)y + A_2(x)y^2,$$

where $A_2(x) \not\equiv 0$ and where A_i are meromorphic functions of x. For such equations it is not possible to find an explicit expression of a first integral. However, if we assume that $y = \varphi(x)$ is a solution of this differential equation, we are able to write a first integral. We define

$$\begin{aligned} \mathcal{D}_1(x) = & \int_0^x A_1(\sigma) + 2\varphi(\sigma)A_2(\sigma)d\sigma, \\ \mathcal{D}_2(x) = & \int_0^x e^{\mathcal{D}_1(s)}A_2(s)ds. \end{aligned}$$

Any other solution of the Riccati differential equation is of the form

$$y(x) = \varphi(x) + \frac{e^{\mathcal{D}_1(x)}}{z_0 - \int_0^x e^{\mathcal{D}_1(s)}A_2(s)ds}, \tag{1.8}$$

where z_0 is a real constant. We get a first integral by isolating the constant z_0 so that $H(x,y) = z_0$.

The following equation to be considered is the celebrated Abel differential equation , which is of the form

$$\frac{dy}{dx} = A_0(x) + A_1(x)y + A_2(x)y^2 + A_3(x)y^3,$$

where $A_3(x) \not\equiv 0$ and where A_i are meromorphic functions of x. There is no explicit expression of a first integral of this kind of equations. Indeed, in general, the knowledge of any finite number of solutions does not imply that this equation can be explicitly integrated.

In Chapter 4 algebro-geometric conditions to have a certain first integral for an Abel differential equation are given. These conditions establish a bridge with classical Galois theory because we transform the differential problem of finding a first integral for an Abel equation into an algebraic problem.

The existence of a first integral for a planar differential system of the form (1.1) in a neighborhood of a singular point gives much information about the behavior of the orbits in this neighborhood. Chapter 5 is devoted to the integrability problem of planar nonlinear differential equations. We introduce a new method to detect local analytic integrability or to construct a singular series expansion of the first integral around a singular point for planar vector fields. The method allows to find new variables (essential variables) where the integrability problem is more feasible. The new method can be used in different context and is an alternative to all the methods developed up to now for any particular case.

1.4 Darboux theory of integrability

An invariant algebraic curve of system (1.1) is an algebraic curve $f(x, y) = 0$ with $f \in \mathbb{C}[x, y]$, such that for some polynomial $M \in \mathbb{C}[x, y]$ we have $\mathcal{X}f = Mf$ and M is called the *cofactor* of the invariant algebraic curve $f = 0$. Here, $\mathbb{C}[x, y]$ is the ring of polynomials in the variables (x, y) with coefficients in $\mathbb{C}$. We remark that if the polynomial system (1.1) has degree m, then any cofactor has at most $m - 1$ as degree. We say that the curve $f = 0$ with $f \in \mathbb{C}[x, y]$ is an algebraic solution of system (1.1) if and only if it is an invariant algebraic curve and f is an irreducible polynomial over $\mathbb{C}[x, y]$.

An *exponential factor* of system (1.1) is a function of the form $f = e^{E/C}$ where E, $C \in \mathbb{C}[x, y]$ satisfying $\mathcal{X}f = Mf$ with M a polynomial of degree at most $m - 1$. We term M the cofactor of f as before.

A first integral of system (1.1) of the form $f_1^{\lambda_1}....f_q^{\lambda_q}$ with $\lambda_i \in \mathbb{C}$ and $f_i \in \mathbb{C}[x, y]$ or f_i an exponential factor is called a *Darboux first integral* .

An integrating factor of system (1.1) of the form $f_1^{\lambda_1}....f_q^{\lambda_q}$ with $\lambda_i \in \mathbb{C}$ and $f_i \in \mathbb{C}[x, y]$ or f_i an exponential factor is called a *Darboux integrating factor*.

We note that since polynomial system (1.1) is real, if $f = 0$ is a complex invariant curve, then its conjugate $\bar{f} = 0$ is also an invariant curve. Moreover we have the following result.

Proposition 1.3 *Suppose that $f \in \mathbb{C}[x,y]$ and let $f = f_1^{n_1}, \ldots, f_r^{n_r}$ be the factorization of f in irreducible factors over $\mathbb{C}[x,y]$. If $\mathcal{X}f = Mf$ then there exist M_i polynomials such that $\mathcal{X}f_i = M_i f_i$ for all $i = 1,...,r$ and $\sum_{i=1}^{r} n_i M_i = M$.*

We note that if (x_0, y_0) is a singular point of system (1.1), then either $M(x_0, y_0) = 0$ or $f(x_0, y_0) = 0$.

We present now the Darboux method of integrability [47] for system of the form (1.1).

Theorem 1.4 (Darboux) *Suppose that a polynomial system (1.1) of degree m admits q algebraic solutions $f_i = 0$ with cofactors M_i for $i = 1,...,q$.*

(a) If $q \geq m(m+1)/2 + 1$, then the function $f_1^{\lambda_1} .... f_q^{\lambda_q}$ for appropriate $\lambda_i \in \mathbb{C}$ not all zero is a first integral and $\sum_{i=1}^{q} \lambda_i M_i = 0$.

(b) If $q = m(m+1)/2$, then the function $f_1^{\lambda_1} .... f_q^{\lambda_q}$ for appropriate $\lambda_i \in \mathbb{C}$ not all zero is either a first integral and $\sum_{i=1}^{q} \lambda_i M_i = 0$, or $f_1^{\lambda_1} .... f_q^{\lambda_q}$ is an integrating factor and $\sum_{i=1}^{q} \lambda_i M_i = -\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)$.

(c) If $q < m(m+1)/2$, and there exist $\lambda_i \in \mathbb{C}$ not all zero such that $\sum_{i=1}^{q} \lambda_i M_i = 0$, then $f_1^{\lambda_1} .... f_q^{\lambda_q}$ is a first integral.

(d) If $q < m(m+1)/2$, and there exist $\lambda_i \in \mathbb{C}$ not all zero such that $\sum_{i=1}^{q} \lambda_i M_i = -\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)$, then $f_1^{\lambda_1} .... f_q^{\lambda_q}$ is an integrating factor.

Jouanolou proved in [85] that if the number of algebraic solutions is $q \geq m(m+1)/2+2$ then the exponents λ_i in the statement of Darboux Theorem may be chosen to be integers and hence we have a rational first integral in this case. From the Darboux Theorem and Jouanolou result it follows easily that for a polynomial system (1.1) of degree m one and only one of the following statements holds:

(a) System (1.1) has a finite number $q < m(m+1)/2 + 2$ of algebraic solutions.

(b) System (1.1) has an infinite number of algebraic solutions and admits a rational first integral of the form $f_1^{\lambda_1} ... f_q^{\lambda_q}$ with $\lambda_i \in \mathbb{Z}$.

Prelle and Singer proved in [110] that if system (1.1) has an elementary first integral, then it has an integrating factor of the form $f_1^{\lambda_1} ... f_q^{\lambda_q}$ with

$f_i \in \mathbb{C}[x, y]$ and $\lambda_i \in \mathbb{Z}$. Moreover each $f_i = 0$ is an algebraic solution. Roughly speaking an elementary first integral is a first integral which is composition of exponentials, logarithms and algebraic functions.

Singer [118] has extended the method to characterize the Liouville integrability. Recently several improvements of these theories of integrability have been obtained, see for instance [72, 94].

1.5 The center problem for Abel differential equations

Given a planar differential system of the form (1.2)

$$\dot{x} = -y + X(x, y), \quad \dot{y} = x + Y(x, y),$$

one can consider a change to polar coordinates, that is, $x = r\cos\theta$, $y = r\sin\theta$. In this way, one obtains an ordinary differential equation of the form

$$\frac{dr}{d\theta} = \mathcal{F}(r, \theta),$$

where $\mathcal{F}(r, \theta)$ is an analytic function in a neighborhood of $r = 0$, and where θ is a periodic variable of period 2π.

There are several equations of the form (1.2), for instance quadratic systems, which after another change of variables, lead to an ordinary differential equation of Abel type:

$$\frac{dr}{d\theta} = a_1(\theta)r^2 + a_2(\theta)r^3, \tag{1.9}$$

with $a_1(\theta)$ and $a_2(\theta)$ trigonometric polynomials. We will denote an equation of the former form *Abel trigonometric differential equation.*

The center problem for an Abel trigonometric differential equation asks whether all the solutions in a neighborhood of the constant solution $r = 0$ are periodic. That is, if $r(\theta; r_0)$ denotes the solution of the differential equation (1.9) with initial condition $r(0; r_0) = r_0$, we ask whether $r(2\pi; r_0) = r_0$ for all $|r_0|$ sufficiently small.

Chapter 6 deals with the center problem for Abel trigonometric differential equation (1.9). This problem is closely connected with the classical Poincaré center problem for planar polynomial vector fields.

In Chapter 7 we study the particular case in which $a_1(\theta)$ and $a_2(\theta)$ are cubic trigonometric polynomials in θ. A particular class of centers, the so-called universal centers or composition centers, is taken into account. An example of non-universal center and a characterization of all the universal centers for such equation are provided.

Polynomial Abel differential equations are also considered in the literature as a model problem. These equations are of the form

$$\frac{dy}{dx} = p(x)y^2 + q(x)y^3, \tag{1.10}$$

where y is real, x is a real independent variable considered in a real interval $[a, b]$ and $p(x)$ and $q(x)$ are real polynomials in $\mathbb{R}[x]$. The *center problem for a polynomial Abel equation* (1.10) is to characterize when all the solutions in a neighborhood of the solution $y = 0$ take the same value when $x = a$ and $x = b$, i.e. $y(a) = y(b)$. In this framework, given any real continuous function $c(x)$, we denote by $\tilde{c}(x) := \int_a^x c(\sigma)d\sigma$ and we will say that a real continuous function $w(x)$ is *periodic in* $[a, b]$ if $w(a) = w(b)$. In Chapter 8 a survey of the most important results in this context is made and the state of the art of several related conjectures is provided. Two new results on these conjectures are also given.

1.6 Publications

From the results obtained in this thesis have been derived the following publications:

The results obtained in chapter 2 and 3 gave rise to the publications:

J. Giné, X. Santallusia, *On the Poincaré–Liapunov constants and the Poincaré series*, Appl. Math. (Warsaw) **28** (2001), no. 1, 17–30. (38 cites in Google Scholar).

J. Giné, X. Santallusia, *Implementation of a new algorithm of computation of the Poincaré–Liapunov constants*, J. Comput. Appl. Math. **166** (2004), no. 2, 465–476. (20 cites in Google Scholar).

Some of the papers where the new algorithm is used are:

J. Chavarriga, I.A. García, J. Giné, *On integrability of differential equations defined by the sum of homogeneous vector fields with degenerate infinity*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 11 (2001), no. 3, 711–722. (30 cites in Google Scholar).

J. Giné, *Conditions for the existence of a center for the Kukles homogeneous systems*, Comput. Math. Appl. 43 (2002), 1261–1269. (28 cites in Google Scholar).

J. Giné, *Polynomial first integrals via the Poincaré series*, J. Comput. Appl. Math. 184 (2005), no. 2, 428–441. (7 cites in Google Scholar).

The results obtained in chapter 4 gave rise to the following publication:

J. Giné, X. Santallusia, *Abel differential equations admitting a certain first integral*, J. Math. Anal. Appl. **370** (2010), no. 1, 187–199. (4 cites in Google Scholar).

The most important paper citing our paper is:

Yufeng Xu, Zhimin He, *The short memory principle for solving Abel differential equation of fractional order*, Comput. Math. Appl. 62 (2011), no. 12, 4796–4805. (17 cites in Google Scholar).

The results obtained in chapter 5 gave rise to the following publication:

J. Giné, X. Santallusia, *Essential variables in the integrability problem of planar vector fields*, Phys. Lett. A **375** (2011), no. 3, 291–297. (7 cites in Google Scholar).

The most important papers citing our paper are:

J. Giné, On the degenerate center problem, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 21 (2011), no. 5, 1383–1392. (12 cites in Google Scholar).

Zhaoping Hu, M. Aldazharova, T.M. Aldibekov, V.G. Romanovski, *Integrability of 3-dim polynomial systems with three invariant planes* Nonlinear Dynam. 47 (2013), no. 4, 1077–1092. (4 cites in Google Scholar).

Zhaoping Hu, Maoan Han, V. G. Romanovski, *Local integrability of a family of three-dimensional quadratic systems* Physica D 265 (2013), 78–86. (4 cites in Google Scholar).

The results obtained in chapter 6 gave rise to the following publication:

J. Giné, M. Grau, X. Santallusia, *Composition conditions in the trigonometric Abel equation*, J. Appl. Anal. Comput. **3** (2013), no. 2, 133–144. (4 cites in Google Scholar).

The most important papers citing our paper are:

M Briskin, F Pakovich, Y Yomdin, *Algebraic geometry of the center-focus problem for Abel differential equations*, Ergodic Theory Dynam. Systems 36 (2016), no. 3, 714–744. (5 cites in Google Scholar).

J.F. Cariñena, J. de Lucas, *Quasi-Lie families, schemes, invariants and their applications to Abel equations*, J. Math. Anal. Appl. 430 (2015), no. 2, 648–671.

The results obtained in chapter 7 gave rise to the following publication:

J. Giné, M. Grau, X. Santallusia, *Universal centers in the cubic trigonometric Abel equation*, Electron. J. Qual. Theory Differ. Equ. **2014**, No. 1, 1–7. (5 cites in Google Scholar).

The most important papers citing our paper are:

A Brudnyi, *Universal curves in the center problem for Abel differential equations*, Ergodic Theory Dynam. Systems 36 (2016), no. 5, 1379–1395. (1 cites in Google Scholar).

A Brudnyi, *Shuffle and Faà di Bruno Hopf algebras in the center problem for ordinary differential equations* Bull. Sci. Math. 140 (2016), no. 7, 830–863.

The results obtained in chapter 8 gave rise to the following publication:

J. Giné, M. Grau, X. Santallusia, *The center problem and composition condition for Abel differential equations* Expo. Math. **34** (2016), no. 2, 210–222.

Chapter 2

On the Poincaré–Liapunov constants and the Poincaré series

2.1 Introduction

Many models of the nature use differential equation systems in the plane and using the qualitative theory of differential equations, introduced by Poincaré, it can be known the behavior of these systems in the majority of the cases. One of the problems that persists to control the behavior of this type of systems is to distinguish among a focus or a center (the center problem). The resolution of this problem goes through compute the so called Poincaré–Liapunov constants . Therefore to have a fast and easy method for the computation of such constants is of great usefulness to study this type of systems. Other very important problem is to determine systems that have centers at some singular points due to the fact that perturbations of these systems give rich bifurcations of limit cycles .

During the first half-part of this century the mathematicians were losing interest by these problems due to the computational problems, but after the facilities of using computers for the calculations these problems become of maximal interest. In the last years many papers have been published giving different methods to compute the Poincaré–Liapunov constant. In this work we compute recursively all the Poincaré–Liapunov constants in function of the coefficients of the system for an arbitrary analytic system which has a linear center at the origin, giving in this way an answer to the classical center problem. We also compute the coefficients of the Poincaré series in function of the same coefficients. Our method does not need the computation of any definite or indefinite integral as other methods require. The method we present is the simplest and easy, as far as we know, to implement on a computer.

Consider two-dimensional autonomous systems of differential equations of the form

$$\dot{x} = -y + X(x,y), \quad \dot{y} = x + Y(x,y), \tag{2.1}$$

where the nonlinearities are

$$X(x,y) = \sum_{s=2}^{\infty} X_s(x,y), \quad \text{and} \quad Y(x,y) = \sum_{s=2}^{\infty} Y_s(x,y),$$

with $X_s(x,y) = \sum_{k=0}^{s} a_k^s x^k y^{s-k}$ and $Y_s(x,y) = \sum_{k=0}^{s} b_k^s x^k y^{s-k}$ and a_k^s and b_k^s are arbitrary real coefficients.

For these systems Poincaré [109] developed an important technique that consists in finding a formal power series of the form

$$H(x,y) = \sum_{n=2}^{\infty} H_n(x,y), \tag{2.2}$$

where $H_2(x,y) = (x^2+y^2)/2$, and for each n, $H_n(x,y) = \sum_{k=0}^{n} C_k^n x^k y^{n-k}$ such that the derivative of H along the solutions of system (2.1) satisfies

$$\dot{H} = \sum_{k=2}^{\infty} V_{2k}(x^2+y^2)^k, \tag{2.3}$$

where V_{2k} are called the *Poincaré–Liapunov constants.*

In order to solve the problem of the stability at the origin of system (2.1), it is sufficient to consider the sign of the first Poincaré–Liapunov constant different from zero. If it is positive we have asymptotic stability for negative times, and if it is negative we have asymptotic stability for positive times. If all Poincaré–Liapunov constants are zero, then the origin is stable for all times, but there is no asymptotic stability for any time, see for instance [9]. In this last case, we have a center at the origin, i.e. there is an open neighborhood of the origin where all orbits are periodic, except of course the origin. The origin is said to be a fine focus of order k if V_{2k+2} is the first non-zero Poincaré–Liapunov constant. In this case at most k limit cycles can bifurcate from this fine focus [16]; these limit cycles are called *small-amplitude limit cycles*. Therefore to obtain the maximum number of limit cycles which can bifurcate from the origin for a given system, one has to find the maximum possible order of a fine focus. It is known that this maximum number is three for quadratic system [12] and it has been shown recently that it is greater or equal than eleven for cubic systems [126].

In this work we are going to see that we can always determine C_k^n and V_{2k} from a_k^s and b_k^s, but C_k^n are not unique and consequence V_{2k} neither. Therefore, the Poincaré's formal series is not unique. Poincaré [109] proved, by boundedness, that there exists one which is convergent for polynomial systems, and Liapunov [91] generalized Poincaré's theorem to analytic systems. In [31] Chazy demonstrated using the theorem of analytical dependence respect to the initial parameters that there exists one which is convergent choosing adequately the arbitrary parameters that appear in the construction of Poincaré's series. For polynomial systems we have uniqueness for the V_{2k} in the sense of the following theorem due to Shi Songling [121].

Theorem 2.1 *Let* $\mathbf{A}$ *be the ring of real polynomials whose variables are the coefficients of the polynomial differential system. Given a set of Poincaré–Liapunov constants* $V_1, V_2, \ldots, V_i$*, let* $\mathbf{J}_{k-1}$ *be the ideal of* $\mathbf{A}$ *generated by* $V_1, V_2, \ldots, V_{k-1}$*. If* $V_1', V_2', \ldots, V_i'$ *is another set of Poincaré–Liapunov constants, then* $V_k \equiv V_k' \mod (\mathbf{J}_{k-1})$*.*

As it has said above the origin is a center if and only if all the V_i's are zero. Let $\mathbf{J} = (V_1, V_2, \ldots)$ be the ideal of $\mathbf{A}$ generated by all the V_i's. For polynomials systems, using the Hilbert's basis theorem, $\mathbf{J}$ is finitely generated; i.e. there exist $B_1, B_2, \ldots, B_q$ in $\mathbf{J}$ such that $\mathbf{J} = (B_1, B_2, \ldots, B_q)$. Such a set of generators is called a basis of $\mathbf{J}$.

Exist different algorithms to compute the Poincaré–Liapunov constants. The technique used by Bautin [12] is based on computing the derivatives of the return map from a nonlinear system of recursive differential equations.

There is another algorithm which involves the solution of a system of linear equations for the coefficients of H_n in terms of the coefficients of X_s, Y_s and H_k for $k = 2, \ldots, n-1$, see for instance [95] and [107]. Another method is to construct a Poincaré's formal power series in polar coordinates and the Poincaré–Liapunov constants can be computed from recursive linear formulas as definite integrals of trigonometric polynomials (see for example, [4] and [26, 29, 30]). In [54] the authors give a survey of different ways to compute the Poincaré–Liapunov constants.

Modifying the standard techniques explained in [9] for obtaining the Poincaré–Liapunov constants, it is given in [40] the first and the second Poincaré–Liapunov constants for an arbitrary analytic system using the return function and some algebraic properties of these constants. In [65] taking advantage of the complex structure that simplify their effective computation, have been found by hand V_3 and V_5. A development of the method presented in [65] is used in [64] to obtain V_7 for an arbitrary analytic system. Using a method based on the use of the Runge-Kutta-Fehlberg methods and the use of Richardson's extrapolation in [66] is given an analytic-numerical method for the computation of the Poincaré–Liapunov constants. Another algorithm to compute the Poincaré–Liapunov constants is developed in [56] and [55] where the method is based in the calculation of the successive derivatives of the first return map associated to the perturbations of some planar Hamiltonian systems. An important generalization of this last method is given in [123].

The chapter is organized as follows. In the next section we present a formula to compute the Poincaré–Liapunov constants (see Theorem 2.2) and we describe the algorithm that we have developed. As a particular case the formula is applied to quadratic systems. The last section is devoted to study the center problem for some particular systems as an application of the method.

2.2 The main result

We present a formula to compute the Poincaré–Liapunov constants and the Poincaré series for the general system (2.1) as a recurrence form, following the ideas of Shi Songling in [120] where he found the same expression for the

Poincaré–Liapunov constants, but he did not found the recursive relation with the Poincaré series to establish a method to compute at the same time the Poincaré–Liapunov constants and the Poincaré series.

The advantages of this method are:
(a) In all the process the unique calculations are products and sums without definite or indefinite integrals like the majority of the others methods.
(b) As a consequence of the first advantage (a) it is very easy and optimizable its implementation on a computer.
(c) The method gives, as we have said, at the same time the Poincaré–Liapunov constants and the Poincaré series. This fact allows us to find systems with a polynomial first integral imposing that the Poincaré series has a finite number of terms.

In the next chapter we planned to study if our method it is computationally more effective than the others methods and if it allows us to obtain new theoretical results.

Theorem 2.2 *The Poincaré-Liapunov constants of system (2.1) are*

$$V_n = \frac{\sum_{l=0}^{n/2} (n-(2l+1))!!\ (2l-1)!!\ d_{2l}^n}{\sum_{l=0}^{n/2} (n-(2l+1))!!\ (2l-1)!!\ \binom{n/2}{l}}, \qquad n = 4, 6, 8, \ldots$$

where $d_k^n = \sum_{m=1}^{n-2} \sum_{l=0}^{m+1} (l a_{k-l+1}^{n-m} + (m+1-l) b_{k-l}^{n-m})\, C_l^{m+1}$, $n \geq 3$, $k = 0, \ldots, n$, *with* $a_k^s = b_k^s = 0$ *for* $k < 0$ *or* $k > s$, $C_0^2 = C_2^2 = 1/2$ *and* $C_1^2 = 0$, *and*

$$C_k^n = \frac{\sum_{l=0}^{(k-1)/2} (n-(2l+1))!!\ (2l-1)!!\ \left(d_{2l}^n - \binom{n/2}{l} V_n\right)}{(n-k)!!\ k!!},$$

for $n \geq 3$, $k = 1, 3, 5, \ldots$ *and*

$$C_k^n = -\frac{\sum_{l=k/2}^{[(n-1)/2]} (n-(2l+2))!!\ (2l)!!\ d_{2l+1}^n + \lambda_n}{(n-k)!!\ k!!},$$

for $n \geq 3$, $k = 0, 2, 4, \ldots$ *where* λ_n *are arbitrary constants and* V_n *and* λ_n *are zero for* n *odd.*

Proof. From the evaluation of the derivate of $H(x, y)$ along the solutions of system (2.1) we have

$$\dot{H} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y}$$

$$= \left(\sum_{n=2}^{\infty} \frac{\partial H_n}{\partial x}\right)\left(-y+\sum_{s=2}^{\infty} X_s\right) + \left(\sum_{n=2}^{\infty} \frac{\partial H_n}{\partial y}\right)\left(x+\sum_{s=2}^{\infty} Y_s\right) =$$

$$\sum_{n=2}^{\infty}\left(-y\frac{\partial H_n}{\partial x}+x\frac{\partial H_n}{\partial y}\right)+\sum_{s=2}^{\infty} X_s \sum_{n=2}^{\infty}\frac{\partial H_n}{\partial x}+\sum_{s=2}^{\infty} Y_s \sum_{n=2}^{\infty}\frac{\partial H_n}{\partial y} =$$

$$\sum_{n=3}^{\infty}\left(-y\frac{\partial H_n}{\partial x}+x\frac{\partial H_n}{\partial y}+\sum_{m=1}^{n-2}\left(X_{n-m}\frac{\partial H_{m+1}}{\partial x}+Y_{n-m}\frac{\partial H_{m+1}}{\partial y}\right)\right) ,$$

then comparing with (2.3) we have

$$\begin{aligned} -y\frac{\partial H_n}{\partial x}+x\frac{\partial H_n}{\partial y} \quad &+\sum_{m=1}^{n-2}\left(X_{n-m}\frac{\partial H_{m+1}}{\partial x}+Y_{n-m}\frac{\partial H_{m+1}}{\partial y}\right) \\ &= \begin{cases} 0 & \text{if } n \text{ is odd,} \\ V_n(x^2+y^2)^{n/2} & \text{if } n \text{ is even,} \end{cases} \end{aligned} \tag{2.4}$$

For the second term of expression (2.4) we have

$$\sum_{m=1}^{n-2}\left(X_{n-m}\frac{\partial H_{m+1}}{\partial x}+Y_{n-m}\frac{\partial H_{m+1}}{\partial y}\right) =$$

$$\sum_{m=1}^{n-2}\left((\sum_{k=0}^{n-m} a_k^{n-m}x^k y^{n-m-k})(\sum_{l=0}^{m+1} lC_l^{m+1}x^{l-1}y^{m+1-l})\right)+$$

$$\sum_{m=1}^{n-2}\left((\sum_{k=0}^{n-m} b_k^{n-m}x^k y^{n-m-k})(\sum_{l=0}^{m+1}(m+1-l)C_l^{m+1}x^l y^{m-l})\right) =$$

$$= \sum_{m=1}^{n-2}\sum_{l=0}^{m+1}\sum_{k=0}^{n-m} la_k^{n-m}C_l^{m+1}x^{k+l-1}y^{n+1-l-k}$$

$$+\sum_{m=1}^{n-2}\sum_{l=0}^{m+1}\sum_{k=0}^{n-m}(m+1-l)b_k^{n-m}C_l^{m+1}x^{k+l}y^{n-l-k} =$$

$$= \sum_{m=1}^{n-2}\sum_{l=0}^{m+1}\sum_{k=l-1}^{n-m+l-1} la_{k-l+1}^{n-m}C_l^{m+1}x^k y^{n-k}$$

$$+\sum_{m=1}^{n-2}\sum_{l=0}^{m+1}\sum_{k=l}^{n-m+l}(m+1-l)b_{k-l}^{n-m}C_l^{m+1}x^k y^{n-k} =$$

Taking into account that $a_k^s = b_k^s = 0$ for $k < 0$ or $k > s$ this last expression takes the form

$$\sum_{m=1}^{n-2}\sum_{l=0}^{m+1}\sum_{k=0}^{n}(la_{k-l+1}^{n-m} + (m+1-l)b_{k-l}^{n-m})C_l^{m+1}x^k y^{n-k} =$$

$$\sum_{k=0}^{n}\sum_{m=1}^{n-2}\sum_{l=0}^{m+1}(la_{k-l+1}^{n-m} + (m+1-l)b_{k-l}^{n-m})C_l^{m+1}x^k y^{n-k}.$$

If we define $d_k^n = \sum_{m=1}^{n-2}\sum_{l=0}^{m+1}(la_{k-l+1}^{n-m} + (m+1-l)b_{k-l}^{n-m})C_l^{m+1}$, with $k = 0, \ldots, n$. We remark that in the computation of d_k^n contribute a_k^s, b_k^s and C_k^s for $s = 2, 3, \ldots, n-1$. Therefore we obtain

$$\sum_{m=1}^{n-2}\left(X_{n-m}\frac{\partial H_{m+1}}{\partial x} + Y_{n-m}\frac{\partial H_{m+1}}{\partial y}\right) = \sum_{k=0}^{n} d_k^n\ x^k y^{n-k}.$$

For the first term of expression (2.4) we have

$$y\frac{\partial H_n}{\partial x} - x\frac{\partial H_n}{\partial x} = y\frac{\partial}{\partial x}\sum_{k=0}^{n} C_k^n\ x^k y^{n-k} - x\frac{\partial}{\partial y}\sum_{k=0}^{n} C_k^n\ x^k y^{n-k} =$$

$$\sum_{k=0}^{n} kC_k^n x^{k-1}y^{n-k+1} - \sum_{k=0}^{n}(n-k)C_k^n x^{k+1}y^{n-k-1} =$$

$$\sum_{k=0}^{n-1}(k+1)C_{k+1}^n x^k y^{n-k} - \sum_{k=1}^{n}(n-k+1)C_{k-1}^n x^k y^{n-k} =$$

$$C_1^n y^n + \sum_{k=1}^{n-1}\left((k+1)C_{k+1}^n - (n-k+1)C_{k-1}^n\right)x^k y^{n-k} - C_{n-1}^n x^n.$$

On the other hand we have

$$V_n(x^2+y^2)^{n/2} = V_n\sum_{k=0}^{n/2}\binom{n/2}{k}x^{2k}y^{n-2k} = \sum_{\substack{k=0\\ 2\,\mid\,k}}\binom{n/2}{k/2}V_n x^k y^{n-k}.$$

Sustituting in expression (2.4) we obtain

$$\begin{aligned} C_1^n + \binom{n/2}{0} V_n &= d_0^n \,, \\ (k+1)C_{k+1}^n - (n-k+1)C_{k-1}^n + \binom{n/2}{k/2} V_n &= d_k^n \,, \quad k=1,\ldots,n-1, \\ -C_{n-1}^n + \binom{n/2}{n/2} V_n &= d_n^n \,, \end{aligned} \tag{2.5}$$

where the term $\binom{n/2}{k/2} V_n$ for $k = 0, \ldots, n$, are different from zero only for n and k even. We can rewrite expression (2.5) as follows

$$\begin{aligned} C_1^n &= d_0^n - \binom{n/2}{0} V_n, \qquad C_{n-1}^n = \binom{n/2}{n/2} V_n - d_n^n, \\ C_k^n &= \tfrac{1}{k}\left(d_{k-1}^n - \binom{n/2}{(k-1)/2} V_n + (n-k+2) C_{k-2}^n \right) \;, \quad k = 2, \ldots, n, \\ C_k^n &= -\tfrac{1}{n-k}\left(d_{k+1}^n - \binom{n/2}{(k+1)/2} V_n - (k+2) C_{k+2}^n \right) \;, \quad k = 0, \ldots, n-2, \end{aligned} \tag{2.6}$$

Expression (2.6) for n odd and k odd is

$$C_1^n = d_0^n, \qquad C_k^n = \frac{1}{k}\left(d_{k-1}^n + (n-k+2) C_{k-2}^n \right), \qquad k = 3, 5, \ldots, n.$$

In this case we claim that

$$C_k^n = \frac{\sum_{l=0}^{(k-1)/2} (n-(2l+1))!! \; (2l-1)!! \; d_{2l}^n}{(n-k)!! \; k!!}, \quad \text{for} \quad k = 1, 3, 5, \ldots, n.$$

We are going to prove the claim by induction. It is easy to see that is true for $k = 1$. Now, we suppose that it is true for $k - 2$, that is,

$$C_{k-2}^n = \frac{\sum_{l=0}^{(k-3)/2} (n-(2l+1))!! \; (2l-1)!! \; d_{2l}^n}{(n-k+2)!! \; (k-2)!!},$$

and then we have

$$C_k^n = \frac{1}{k}\left(d_{k-1}^n + (n-k+2) \frac{\sum_{l=0}^{(k-3)/2} (n-(2l+1))!! \; (2l-1)!! \; d_{2l}^n}{(n-k+2)!! \; (k-2)!!} \right) =$$

$$\frac{d^n_{k-1}}{k} + \frac{\sum_{l=0}^{(k-3)/2}(n-(2l+1))!!\ (2l-1)!!\ d^n_{2l}}{(n-k)!!\ k!!}$$
$$= \frac{\sum_{l=0}^{(k-1)/2}(n-(2l+1))!!\ (2l-1)!!\ d^n_{2l}}{(n-k)!!\ k!!}.$$

Expression (2.6) for n odd and k even is

$$C^n_{n-1} = -d^n_n, \qquad C^n_k = \frac{1}{n-k}\left(-d^n_{k+1} + (k+2)C^n_{k+2}\right), \qquad k = 0, 2, 4, \ldots, n-3.$$

In this case we claim that

$$C^n_k = -\frac{\sum_{l=k/2}^{(n-1)/2}(n-(2l+2))!!\ (2l)!!\ d^n_{2l+1}}{(n-k)!!\ k!!}, \quad \text{for} \quad k = 0, 2, 4, \ldots, n-1.$$

We are going to prove the claim by induction. It is easy to see that is true for $k = n-1$. Now, we suppose that it is true for $k+2$, that is,

$$C^n_{k+2} = -\frac{\sum_{l=(k+2)/2}^{(n-1)/2}(n-(2l+2))!!\ (2l)!!\ d^n_{2l+1}}{(n-k-2)!!\ (k+2)!!},$$

and then we have

$$C^n_k = \frac{1}{n-k}\left(-d^n_{k+1} + (k+2)\frac{-\sum_{l=(k+2)/2}^{(n-1)/2}(n-(2l+2))!!\ (2l)!!\ d^n_{2l+1}}{(n-k-2)!!\ (k+2)!!}\right) =$$
$$-\frac{d^n_{k+1}}{n-k} - \frac{\sum_{l=(k+2)/2}^{(n-1)/2}(n-(2l+2))!!\ (2l)!!\ d^n_{2l+1}}{(n-k)!!\ k!!}$$
$$= -\frac{\sum_{l=k/2}^{(n-1)/2}(n-(2l+2))!!\ (2l)!!\ d^n_{2l+1}}{(n-k)!!\ k!!}.$$

Expression (2.6) for n even and k odd is

$$C^n_1 = d^n_0 - \binom{n/2}{0}V_n, \qquad C^n_{n-1} = \binom{n/2}{n/2}V_n - d^n_n,$$

$$C^n_k = \frac{1}{k}\left(d^n_{k-1} - \binom{n/2}{(k-1)/2}V_n + (n-k+2)C^n_{k-2}\right), \qquad k = 3, 5, \ldots, n-1.$$

In this case we claim that

$$C_k^n = \frac{\sum_{l=0}^{(k-1)/2} (n-(2l+1))!!\ (2l-1)!!\ \left(d_{2l}^n - \binom{n/2}{l} V_n\right)}{(n-k)!!\ k!!},$$

for $k = 1, 3, 5, \ldots, n-1$. We are going to prove the claim by induction. It is easy to see that is true for $k = 1$. Now, we suppose that is true for $k-2$, that is,

$$C_{k-2}^n = \frac{\sum_{l=0}^{(k-3)/2} (n-(2l+1))!!\ (2l-1)!!\ \left(d_{2l}^n - \binom{n/2}{l} V_n\right)}{(n-k+2)!!\ (k-2)!!},$$

and then we have

$$C_k^n = \frac{1}{k}\left(d_{k-1}^n - \binom{n/2}{(k-1)/2} V_n + (n-k+2) C_{k-2}^n\right) =$$

$$= \frac{d_{k-1}^n - \binom{n/2}{(k-1)/2} V_n}{k}$$

$$+ \frac{\sum_{l=0}^{(k-3)/2} (n-(2l+1))!!\ (2l-1)!!\ \left(d_{2l}^n - \binom{n/2}{l} V_n\right)}{(n-k)!!\ k!!} =$$

$$\frac{\sum_{l=0}^{(k-1)/2} (n-(2l+1))!!\ (2l-1)!!\ \left(d_{2l}^n - \binom{n/2}{l} V_n\right)}{(n-k)!!\ k!!}.$$

From this last result we obtain that

$$C_{n-1}^n = \frac{\sum_{l=0}^{(n-2)/2} (n-(2l+1))!!\ (2l-1)!!\ \left(d_{2l}^n - \binom{n/2}{l} V_n\right)}{(n-1)!!},$$

but we know that $C_{n-1}^n = \binom{n/2}{n/2} V_n - d_n^n$, then we have

$$\sum_{l=0}^{(n-2)/2} (n-(2l+1))!!\ (2l-1)!!\ \left(d_{2l}^n - \binom{n/2}{l} V_n\right)$$

$$+(n-1)!!\left(d_n^n - \binom{n/2}{n/2} V_n\right) = 0,$$

which is equivalent to

$$\sum_{l=0}^{n/2}(n-(2l+1))!!\ (2l-1)!!\ \left(d_{2l}^n - \binom{n/2}{l} V_n\right) = 0, \tag{2.7}$$

From expression (2.7) we obtain

$$V_n = \frac{\sum_{l=0}^{n/2}(n-(2l+1))!!\ (2l-1)!!\ d_{2l}^n}{\sum_{l=0}^{n/2}(n-(2l+1))!!\ (2l-1)!!\ \binom{n/2}{l}}, \quad \text{for} \quad n = 4, 6, 8, \ldots$$

Finally, expression (2.6) for n even and k even is

$$C_k^n = \frac{1}{n-k}\left(-d_{k+1}^n + (k+2)C_{k+2}^n\right), \quad \text{for} \quad k = 0, 2, 4, \ldots, n-2.$$

Now we have only the recurrence between C_n^n, C_{n-2}^n, C_{n-4}^n, ..., C_4^n, C_2^n, C_0^n, and one of them is arbitrary. If we chose $C_n^n = -\frac{\lambda_n}{n!!}$, with λ_n arbitrary, then we claim that

$$C_k^n = -\frac{\sum_{l=k/2}^{n/2-1}(n-(2l+2))!!\ (2l)!!d_{2l+1}^n + \lambda_n}{(n-k)!!\ k!!}, \quad \text{for} \quad k = 0, 2, 4, \ldots, n.$$

We are going to prove the claim by induction. It is easy to see that is true for $k = n$. Now, we suppose that is true for $k+2$, that is,

$$C_{k+2}^n = -\frac{\sum_{l=(k+2)/2}^{n/2-1}(n-(2l+2))!!\ (2l)!!d_{2l+1}^n + \lambda_n}{(n-k-2)!!\ (k+2)!!},$$

and then we have

$$C_k^n = \frac{1}{n-k}\left(-d_{k+1}^n + (k+2)\frac{-\sum_{l=(k+2)/2}^{n/2-1}(n-(2l+2))!!\ (2l)!!\ d_{2l+1}^n + \lambda_n}{(n-k-2)!!\ (k+2)!!}\right) =$$

$$-\frac{d_{k+1}^n}{n-k} - \frac{\sum_{l=(k+2)/2}^{n/2-1}(n-(2l+2))!!\ (2l)!!\ d_{2l+1}^n + \lambda_n}{(n-k)!!\ k!!} =$$

$$-\frac{\sum_{l=k/2}^{n/2-1}(n-(2l+2))!!\ (2l)!!\ d_{2l+1}^n+\lambda_n}{(n-k)!!\ k!!},$$

which completes the proof of the theorem. ■

The method works as follows. From the first terms of the Poincaré series (2.2), i.e. $C_0^2 = C_2^2 = 1/2$ and $C_1^2 = 0$ it is possible calculated d_k^3 for $k = 0, 1, 2, 3$ and from here C_k^3 for $k = 0, 1, 2, 3$. Therefore the next step is calculated d_k^4 for $k = 0, 1, 2, 3, 4$ and finally we obtain V_4 and C_k^4 for $k = 0, 1, 2, 3, 4$. The process continues in an analogous way. The method has been implemented using the computer algebra system `Mathematica`.

A particular case: The quadratic systems

We are going to apply the above expressions for quadratic systems . In this case all a_k^s and b_k^s are zero except a_0^2, a_1^2, a_2^2 and b_0^2, b_1^2, b_2^2. Therefore in the expression

$$d_k^n = \sum_{m=1}^{n-2}\sum_{l=0}^{m+1}(la_{k-l+1}^{n-m}+(m+1-l)b_{k-l}^{n-m})\ C_l^{m+1},$$

we have $n-m=2$; i.e. $m=n-2$, and the previous expression takes the form

$$d_k^n = \sum_{l=0}^{n-1}(la_{k-l+1}^2+(n-1-l)b_{k-l}^2)\ C_l^{n-1},$$

where we can omit the upper index of a_0^2, a_1^2, a_2^2 and b_0^2, b_1^2, b_2^2 because always is 2. Taking into account that the subindex of a_{k-l+1} must be $k-l+1 = 0, 1, 2$ and the subindex of b_{k-l} must be $k-l = 0, 1, 2$, we have that $l = k+1$, $l = k$, $l = k-1$ and $l = k$, $l = k-1$, $l = k-2$ respectively with $0 \le l \le n-1$. Then d_k^n is

$$d_k^n = (k+1)a_0C_{k+1}^{n-1} + (ka_1+(n-1-k)b_0)C_k^{n-1}+$$
$$((k-1)a_2+(n-k)b_1)C_{k-1}^{n-1}+(n+1-k)b_2C_{k-2}^{n-1},$$

and the restriction $0 \le l \le n-1$ implies that $C_l^{n-1} = 0$ if it is not satisfied. Then the Poincaré–Liapunov constant for quadratic systems is

$$V_n = \frac{\sum_{l=0}^{n/2}\ (n-(2l+1))!!\ (2l-1)!!\ d_{2l}^n}{\sum_{l=0}^{n/2}\ (n-(2l+1))!!\ (2l-1)!!\ \binom{n/2}{l}},\qquad n = 4, 6, 8, \ldots$$

where

$$d_{2l}^n = (2l+1)a_0 C_{2l+1}^{n-1} + (2la_1 + (n-1-2l)b_0) C_{2l}^{n-1} + ((2l-1)a_2 + (n-2l)b_1) C_{2l-1}^{n-1} + (n+1-2l) b_2 C_{2l-2}^{n-1},$$

The application to more general systems is based in to find the expression d_k^n and it is easy to see that the contributions to d_k^n of each homogeneous terms of the system are independent.

2.3 Applications

When we apply our method to particular cases of system (2.1) we can arrive further in the determination of the Poincaré–Liapunov constants. The system of Proposition 2.3 it was studied in [40] and [64] with $a_4 = b_4 = 0$ and in [123] with $b_2 = a_2$. Here we present the following result.

Proposition 2.3 *Consider the system*

$$\left\{ \begin{array}{rcl} \dot{x} & = & -y + a_2 x^2 + a_3 x^3 + a_4 x^4, \\ \dot{y} & = & x + b_2 y^2 + b_3 y^3 + b_4 y^4, \end{array} \right. \tag{2.8}$$

where a_i and b_i are real numbers. Then, the origin is a center if and only if one of the following conditions holds $a_2 - b_2 = a_3 + b_3 = a_4 - b_4 = 0$, $a_2 + b_2 = a_3 + b_3 = a_4 + b_4 = 0$, $a_2 = a_3 = a_4 = b_3 = 0$ and $b_2 = b_3 = b_4 = a_3 = 0$.

Proof. (a) **Sufficiency**. Every group of conditions give the necessary symmetries to show that system (2.8) is time-reversible and then the origin is a center (the symmetry principle, see [102], page 135).

(b) **Necessity**. The first Poincaré–Liapunov constant is $V_4 = a_3 + b_3$ so taking $b_3 = -a_3$ we obtain that the second Poincaré–Liapunov constant takes the form

$$V_6 = 5a_2^2 a_3 - 6a_2^3 b_2 - 22a_4 b_2 - 5a_3 b_2^2 + 6a_2 b_2^3 + 22a_2 b_4,$$

If a_2 is different from zero we can express b_4 in function of the rest of parameters. In this case the third Poincaré–Liapunov constant is

$$V_8 = \frac{1}{a_2}(b_2 + a_2)(b_2 - a_2)(-235a_2^3 a_3 - 1254a_3 a_4 + 84a_2^4 b_2 -$$

$$285a_3^2b_2 + 1100a_2a_4b_2 + 357a_2a_3b_2^2 - 216a_2^2b_2^3),$$

the vanish of the first factor, that is, $b_2 = a_2$ gives the first condition of Proposition 2.3. If $b_2 = -a_2$ corresponds to the second condition. From the last factor of V_8 we can isolate a_4, if $57a_3 - 50a_2b_2$ is different from zero, and the vanishing of the following Poincaré–Liapunov constants imply $a_2 = b_2 = 0$. In case that $57a_3 - 50a_2b_2$ is zero, that is, $a_3 = 50a_2b_2/57$ the last factor of V_8 takes the form $a_2b_2(a_2^2 + b_2^2)$ which implies $b_2 = 0$, and we obtain the fourth condition of Proposition 2.3.

If a_2 is zero then the second Poincaré–Liapunov constant is $V_6 = b_2(22a_4 + 5a_3b_2)$. Let $22a_4 + 5a_3b_2$ be zero with $b_2 \neq 0$, that is, $a_4 = -5a_3b_2/22$ in this case $V_8 = a_3b_2(235b_2^2 + 1254b_4)$. The case $a_3 = 0$ corresponds to the third condition of Proposition 2.3. In the case $b_4 = -235b_2^3/1254$ the following Poincaré–Liapunov constants imply $a_3 = b_2 = 0$. Finally, if $b_2 = 0$ the next Poincaré–Liapunov constant V_8 is zero and $V_{10} = a_3(a_4-b_4)(a_4+b_4)$. The vanishing of the factors $a_4 - b_4$ and $a_4 + b_4$ correspond to particular cases of first and second conditions respectively. When $a_3 = 0$ with $(a_4 - b_4)(a_4 + b_4) \neq 0$ we have $V_{12} = 0$ and $V_{14} = a_4b_4(a_4-b_4)(a_4+b_4)$. The cases $a_4 = 0$ and $b_4 = 0$ correspond to particular cases of third and fourth conditions respectively. ■

Now we consider the following system

$$\dot{x} = -y + xf(x,y), \quad \dot{y} = x + yf(x,y),$$

with $f(x,y) = \sum_{i=1}^{3} f_i(x,y)$ where $f_i(x,y)$ are homogeneous polynomials of degree i. Any center at the origin of this type of systems is necessarily isochronous (the closed orbits of the center have the same period), since, in polar coordinates (r,φ), the angle φ satisfies the equation $\dot{\varphi} = 1$. This type of isochronous centers are called *uniformly isochronous centers*, see [45]. If $f_2 = f_3 = 0$ the origin is automatically a center because the system has $R(x,y) = (1 - a_2x + a_1y)^{-3}$ as integrating factor. These class of systems have been studied in [44] with $f_3 = 0$. Here we present the center conditions for $f_2 = 0$ and $f_3 \neq 0$.

Proposition 2.4 *Consider the system*

$$\left\{ \begin{array}{rcl} \dot{x} &=& -y + x\left(a_1x + a_2y + a_6x^3 + a_7x^2y + a_8xy^2 + a_9y^3\right), \\ \dot{y} &=& x + y\left(a_1x + a_2y + a_6x^3 + a_7x^2y + a_8xy^2 + a_9y^3\right), \end{array} \right. \tag{2.9}$$

where a_i are real numbers. Then, the origin is a center if and only if

$$a_1(a_7+3a_9)-a_2(a_8+3a_6)=0 \textit{ and } (3a_1a_2^2-3a_2^3)a_6+(3a_1a_2^2-a_1^3)a_7-2a_1^2a_2a_8=0.$$

Proof. (a) **Sufficiency**. Suppose that the two conditions of Proposition 2.4 hold. If $a_1 = a_2 = 0$ then the system has an integrating factor of the form

$$R(x,y) = (1-(a_7+2a_9)x^3+3a_6x^2y-3a_9xy^2+(2a_6+a_8)y^3)^{-5/3},$$

which is defined at the origin and therefore the origin is a center. If $a_1 = 0$ with $a_2 \neq 0$ the first condition of Proposition 2.4 results $a_2(3a_6+a_8) = 0$, which implies $a_8 = -3a_6$. In this case the second condition is $a_2^3a_6 = 0$ and therefore $a_6 = 0$. System (2.9) with $a_1 = a_6 = a_8 = 0$ is invariant by the change of variables $(x,y,t) \to (x,-y,-t)$ and this symmetry ensures that the origin is a center. If $a_2 = 0$ with $a_1 \neq 0$ the first condition of Proposition 2.4 results $a_1(3a_9+a_7) = 0$, which implies $a_7 = -3a_9$. In this case the second condition is $a_1^3a_9 = 0$ and therefore $a_9 = 0$. System (2.9) with $a_2 = a_7 = a_9 = 0$ is invariant by the change of variables $(x,y,t) \to (-x,y,-t)$ and this symmetry ensures that the origin is a center also. Finally if $a_1a_2 \neq 0$ from the first condition of Proposition 2.4 we can isolate a_9, and from the second condition of Proposition 2.4 we obtain a_8 in function of the rest of parameters. In this case making a rotation with $\tan\alpha = -a_2/a_1$, system (2.9), in the new variables (X,Y), is invariant by the change of variables $(X,Y,t) \to (-X,Y,-t)$ and therefore has a center at the origin.

(b) **Necessity**. The first Poincaré–Liapunov constant V_4 is zero. The second and the third Poincaré–Liapunov constants are the two conditions of Proposition 2.4. ■

To prove which is the maximum number of small-amplitude limit cycles which can bifurcate from the origin, the method used is, like it is habitual, the one of finding a fine focus of maximum order. From our calculations it is easy to see that if $a_2 = b_2 = a_3 = b_3 = 0$ we obtain $V_4 = V_6 = V_8 = V_{10} = V_{12} = 0$ and $V_{14} = a_4b_4(a_4-b_4)(a_4+b_4)$ which is different from zero if $a_4b_4 \neq 0$ and $a_4 \neq b_4$ and $a_4 \neq -b_4$ and therefore we obtain a fine focus of order six for system (2.8). In the same way if $a_1 = 0$ and $a_8 = -3a_6$ we have $V_4 = V_6 = 0$ and $V_8 = a_2^3a_6$ which is different from zero if a_2 and a_6 are different from zero, and therefore we obtain a fine focus of order three for system (2.9). Therefore we can conclude the following result

Proposition 2.5 *The maximum number of small-amplitude limit cycles which can bifurcate from the origin is at least six for system (2.8) and three for system (2.9).*

The Poincaré–Liapunov constants of systems (2.8) and (2.9) can be easily computed from the algorithm given in the previous section.

Chapter 3

Implementation of a new algorithm for the computation of the Poincaré–Liapunov constants

3.1 Introduction

In chapter 2 have given an easy algorithm for the computation of the Poincaré–Liapunov constants and at the same time the Poincaré series in order to solve theoretically the center problem. The knowledge of systems with a center is very important because perturbations of these systems give rich bifurcations of limit cycles. The knowledge of systems with limit cycles it is a part of the second part of the 16th Hilbert problem.

The second part of the 16th Hilbert problem concerns on the qualitative theory of differential systems of equations and it is the following. Consider systems of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{3.1}$$

where P and Q are polynomials and x and y are real unknown functions. Systems of the form (3.1) are called *polynomial systems*. Among trajectories of a polynomial system one can single out some which correspond to isolated periodic solutions. These trajectories, as we have said in the introduction, are called *limit cycles* . Let $\pi(P, Q)$ be the number of limit cycles of (3.1) and define

$$H_n = \sup\{\pi(P, Q);\ \partial P, \partial Q \leq n\}.$$

The question of the second part of the 16th Hilbert problem is the maximal possible number of limit cycles, estimates the value of H_n in terms of n, and their location. The first part of the 16 Hilbert problem deals with an estimation of number of ovals of an algebraic curve. Very important connections exist among both parts as the limit cycles of a system with a polynomial inverse integrating factor $V(x, y)$ correspond to ovals of the curve $V(x, y) = 0$, see [68] . Therefore if we know an estimation for the number of limit cycles of system (3.1) we can know an estimation for the number of ovals of the algebraic curve $V(x, y) = 0$, if this algebraic curve exist, if we control the degree of the polynomial inverse integrating factor $V(x, y)$ in function of n.

In the present chapter we consider some questions, those that are related with the Poincaré–Liapunov constants and with the second part of the 16th Hilbert problem. In fact, there exists a whole area of the subject, it is rather misleading to think of it as a single problem; its history and present status are described in detail in [116]. Much of the recent progress has been achieved by consideration of various kinds of bifurcation. One of them in which the Poincaré–Liapunov constants intervene is the limit cycles which bifurcate out of a critical point, as we said, the so-called *small amplitude limit cycles.*

Very briefly, the position is that remarkably little is known about the value of H_n in terms of n. It has not even been established that there exist and upper bound for such value. However it has been proved that a given polynomial system cannot have infinitely many limit cycles by Ecalle [52] and Ilyashenko [84]. The first major contribution was that of Bautin [12], who proved that $H_2 \geq 3$ and this work is classical in the theory of limit cycles bifurcations and his ideas have been very influential in the development of the subject. Afterwards, Landis and Petrovskii published two papers, in one of which it was suggested that $H_2 = 3$ and in the other precise bounds were given for H_n with $n \geq 3$. However, the proofs of these results were soon withdrawn, but nevertheless it appears to have been widely believed for several years that $H_2 = 3$. It was until 1979 that the first examples of quadratic systems with at least four limit cycles appeared given by Shi Songling [119] and Cheng and Wang [33]. These developments stimulated renewed interest in 16th Hilbert's problem. Only very recently some lower bounds were also obtained in the case where P and Q are polynomials of degree three, in what follows *cubic systems*. It was showed by Żołądek that $H_3 \geq 11$. Considerable results in the direction to prove that H_2 is finite were obtained by Dumortier, Roussarie and Rousseau trying to investigate limit cycles which appear from singular trajectories, mainly, from a center or focus type equilibrium point or from a separatrix cycle.

In Section 3.2 we explain the computational problems that appear once we know the Poincaré–Liapunov constants using the algorithmic procedure given in Chapter 1. The implementation of this algorithm is described in Section 3.3. In Section 3.4 we explain the basic idea of the bifurcation of limit cycles out of critical point and see on that consists the center problem. Finally, in Section 3.5 we have concentrate on the computational problems that arise using a Computer Algebra System to solve the above problems and in Section 3.6 some applications are commented.

3.2 Computation of the Poincaré–Liapunov constants

In the previous chapter we have given a formula to compute the Poincaré–Liapunov constants and the Poincaré series for the general systems (2.1) as a recurrence form following the ideas of Shi Songling in [120] where he found the same expression for the Poincaré–Liapunov constants, but he did not found the recursive relation with the Poincaré series to establish a method to compute them. The advantages of this method are that in all the process the unique calculations are products and sums without indefinite/definite integrals as most of the others methods. For instance the original Bautin's method is effectively costly in computer time because it involves computations of indefinite/definite integrals. Consequently this new algorithm it is very easy and optimizable its implementation on a computer. Others methods display also this advantage, this is certainly the case of the successive derivatives approach, see for instance [109] and [56]. But our method gives simultaneously the Poincaré–Liapunov constants and the Poincaré series. The knowledge of the Poncaré series is very useful for applications, for example the study of systems that have a polynomial first integral, which have a finite

Poincaré series.

Although there exist different recursive methods for the determination of the Poincaré–Liapunov constants, as we have see in the previous chapter, and the development of the algebraic manipulators has allowed to approach the computation of the first Poincaré–Liapunov constants, other important computational problems appear as we will see in this section.

Consider system (2.1) and assume that is polynomial. Let $\mathbf{A}$ be the ring of real polynomials whose variables are the coefficients of the polynomial differential system (2.1). Let $\mathbf{J} = (V_1, V_2, \ldots)$ be the ideal of $\mathbf{A}$ generated by all the Poincaré–Liapunov contants V_i's. For such polynomial systems, using the Hilbert's basis theorem, $\mathbf{J}$ is finitely generated; i.e. there exist $B_1, B_2, \ldots, B_q$ in $\mathbf{J}$ such that $\mathbf{J} = (B_1, B_2, \ldots, B_q)$ because $\mathbf{A}$ is Noetherian. Such a set of generators is called a basis of $\mathbf{J}$.

Notice that Hilbert's basis theorem assures us the existence of a generators basis, but it does not provide us a constructive method to find it. The existent methods to solve this problem are based in the Buchberger's algorithm to find a Gröebner basis , but it is only applicable for very simple cases. Therefore it is a computational problem of algebraic nature due to the appearance, already for simple systems, of massive Poincaré–Liapunov constants that are polynomials with rational coefficients and efficient algorithms do not exist that allow to determine simple groups of generators. One of the main difficulties comes ultimately on the decomposition in prime numbers of a big integer number. Therefore the resolution of the computational problem goes to have efficient algorithms that work with big integers and in decomposition in primes numbers of big numbers, a classical problem in computational mathematics. The procedure is know as decomposition of the ideal $\mathbf{J}$ in primary ideals and sometimes this is possible using modular arithmetic as we will see along the next chapters.

Now we apply Theorem 2.2 to the most simple polynomial systems.

2.1 The quadratic and cubic homogeneous perturbations

We are going to apply the expressions of Theorem 2.2 to quadratic and cubic homogeneous perturbations, i.e. systems with a linear center perturbed by quadratic polynomials , in what follows *quadratic systems* , and cubic homogeneous polynomials respectively. For quadratic systems all a_k^s and b_k^s are zero except a_0^2, a_1^2, a_2^2 and b_0^2, b_1^2, b_2^2. Therefore in the expression

$$d_k^n = \sum_{m=1}^{n-2} \sum_{l=0}^{m+1} (l a_{k-l+1}^{n-m} + (m+1-l) b_{k-l}^{n-m}) \; C_l^{m+1}, \tag{3.2}$$

we have $n - m = 2$; i.e. $m = n - 2$, and the previous expression takes the form

$$d_k^n = \sum_{l=0}^{n-1} (l a_{k-l+1}^2 + (n-1-l) b_{k-l}^2) \; C_l^{n-1}.$$

Taking into account that the subindex of a_{k-l+1} must be $k - l + 1 = 0, 1, 2$ and the subindex of b_{k-l} must be $k - l = 0, 1, 2$, we have that $l = k+1$, $l = k$, $l = k - 1$ and $l = k$, $l = k - 1$, $l = k - 2$ respectively with $0 \le l \le n - 1$. Then d_k^n is

$$\begin{aligned} d_k^n &= (k+1)a_0^2 C_{k+1}^{n-1} + (k a_1^2 + (n-1-k) b_0^2) C_k^{n-1} \\ &+ ((k-1)a_2^2 + (n-k) b_1^2) C_{k-1}^{n-1} + (n+1-k) b_2^2 C_{k-2}^{n-1}, \end{aligned}$$

and the restriction $0 \le l \le n - 1$ implies that $C_l^{n-1} = 0$ if it is not satisfied.

For cubic homogeneous perturbations all a_k^s and b_k^s are zero except a_0^3, a_1^3, a_2^3, a_3^3 and b_0^2, b_1^2, b_2^2, b_3^3. Since $n - m = 3$, i.e. $m = n - 3$ therefore expression (3.2) takes the form

$$d_k^n = \sum_{l=0}^{n-2} (l a_{k-l+1}^3 + (n-2-l) b_{k-l}^3) \; C_l^{n-2},$$

Taking into account that the subindex of a_{k-l+1} must be $k - l + 1 = 0, 1, 2, 3$ and the subindex of b_{k-l} must be $k - l = 0, 1, 2, 3$, we have that $l = k + 1$, $l = k$, $l = k - 1$, $l = k - 2$ and $l = k$, $l = k - 1$, $l = k - 2$, $l = k - 3$ respectively with $0 \le l \le n - 2$. Then d_k^n is

$$\begin{aligned} d_k^n &= (k+1)a_0^3 C_{k+1}^{n-2} + (k a_1^3 + (n-2-k) b_0^3) C_k^{n-2} \\ &+ ((k-1)a_2^3 + (n-k-1) b_1^3) C_{k-1}^{n-2} \\ &+ ((k-2)a_3^3 + (n-k) b_2^3) C_{k-2}^{n-2} + (n+1-k) b_3^3 C_{k-3}^{n-2}, \end{aligned}$$

and the restriction $0 \le l \le n - 2$ implies that $C_l^{n-2} = 0$ if it is not satisfied.

The application to more general systems is based in to find the expression d_k^n and it is easy to see that the contributions to d_k^n of each homogeneous terms of system (2.1) are independent.

3.3 Implementation of the algorithm

The implementation of an algorithm can be approached in two different ways. On one hand they can be used the commercial versions of the algebraic manipulators as Axiom (the comercial version of Scratchpad), Maple, Mathematica, Reduce, Macsyma and specialized programs as Macaulay, Cocoa, Mas, Magma, Posso and Singular. However, these manipulators are not even enough powerful for very extensive calculations or the programming of the algorithms is not foreseen. On the other hand, these programs are of very general character and written in language LISP generally. They require the use of big computers or computers specially designed for their use that consume a great quantity of memory and a lot of time of CPU, what hinders their use considerably for certain problems. Another form of approaching these problems is by means of the use of algebraic manipulators specially designed for the resolution of concrete problems and not with a general purpose. Implementing the algorithm using a programming language and building a specific program for the resolution of the concrete problem.

When we first became involved in computations relating to the center problem and small-amplitude limit cycles, we used the method that it consists in to construct a Poincaré's formal power series in polar coordinates and the Poincaré–Liapunov constants that can be computed from recursive linear formulas as definite integrals of trigonometric polynomials, see [29] and [30]. The algorithm was written in C++ and was used to obtain results for linear centers perturbed by quartic and quintic homogeneous polynomials. The program's operation is controlled by a PSP (Poincaré Series Processor) command file running under the LINUX operating system, and different files identifier of LINUX are exploited to give the user a simple method of distinguishing between the various files relating to a particular system of differential equations. Initially, the user is required to provide information in one file. In <filename> LYCONFIG the user enters the degree of the polynomial system, the type of system, i.e. homogeneous or complete, the range of k for which V_{2k} is computed together with the optional relations between the coefficients of the polynomial system that we want to introduce.

The program is organized so that in the kth "round", the polynomial V_{2k} is computed. When the nominated terminal value of k is reached, different files are produced and their contents we are going to describe. On the first, the initial value of k is 2. The program runs as far as round the terminal value of k and the Poincaré–Liapunov constants $V_4, V_6, \ldots, V_{2k}$ are stored in different files <filemanes> PLCVk (Poincaré–Liapunov constant k) and if there are restrictions introduced in the file LYCONFIG, the constants are stored in <filenames> SPLCVk (Simplified Poincaré–Liapunov constant k). The Poincaré series are stored in different files <filenames> TMPnCk (kth homogeneous part of the Poincaré series for a polynomial system of degree n). In this way, the calculations can be restarted at $k = k + 1$. In practice, the program is first run from $k = 2$ to $k = 2r - 1$, then substitutions from $V_4, V_6, \ldots, V_{2r}$ are decided and the program is called again, but now the initial value of k is $2r$ with these relations between coefficients in the LYCONFIG. This is a valuable facility, for the appropriate substitutions cannot usually be seen in advance of knowing the first Poincaré–Liapunov constants. It is a matter of judgement how many Poincaré–Liapunov constants should be calculated before entering further substitutions. As a rough guide one would not normally compute more than two or three, and often only one. After the Poincaré–Liapunov constants are computed, it is possible to do their translation to Mathematica format by the program translation CONVERT, which give the Poincaré–Liapunov constants in the general Mathematica format. The reduction procedure is heavily interactive. We have not sought to automate it, for experience suggests that some of the information which we require later would be lost if we did. Like many other computer implementations, it has evolved with changes being made in response to user requirements as well as the continuing efforts to improve its efficiency.

As it has been seen in the previous section our investigations developed a more sophisticated algorithm (Theorem 2.2) which we have initially implemented in `Mathematica 3.0` on a Pentium III with 450 Mhz and 64 Mb. RAM. Our current implementation of the algorithm, and that which we describe here, uses C++ on the same computer. The program's operation is also controlled by a PSP (Poincaré Series Processor) command file running under the LINUX operating system. Initially, the user is required to provide information. The program ask for if the user wants to stored the d_k^n and C_k^n

in different files <filenames> Dk and Ck, respectively. After the user enters the range of k for which V_{2k} is computed and the degree of the polynomial system. As the coefficients of the Poincaré–Liapunov constants are rational numbers, the implementation uses a library for doing number theory. The NTL library v. 3.6b, freely available for research and educational purposes. The latest version of NTL is available at `www.shoup.net`. The output of the algorithm, the V_{2k}, is directly in Mathematica format. The obtained timings have been controlled by the function *Timing* of mathematica and by function *gettimeofday* of C++. The implementations versions are available to anyone who is interested.

We planned to study if our method it is computationally more effective than others methods. The comparisons among these methods are very difficult because each method uses different coordinates system and therefore the number of terms of the coefficients of the Poincaré series and the Poincaré–Liapunov constants varies according to the used coordinates. Therefore we present the following tables for quadratic and cubic homogeneous perturbations giving the times of calculation, the width in bytes and the number of terms for the methods developed in [123], [29], [30] and our algorithm. The computations of the method developed in [123] have been implemented with Maple VR4 on a Workstation (SUN Ultra E-450) with three processor Pentium II with 250 Mhz and 256 Mb. RAM, which makes not possible a direct comparative.

Algorithm	constants	time	width in bytes	number of terms
Method [123]	k=2 to k=4	1.17	57; 543; 2104	2; 14; 44
Maple V	k=5	4.34	6075	110
Method [29]	k=2 to k=4	1.20	24; 204; 772	1; 7; 24
C++	k=5	14.45	2240	58
Our Method	k=2 to k=4	51.25	87; 1329; 7092	6; 56; 220
mathematica	k=5	711.67	25413	628
Our Method	k=2 to k=4	2.24	87; 1329; 7092	6; 56; 220
C++	k=5	34.47	25413	628

Table 3.1: Quadratic perturbations

Algorithm	const.	time	width in bytes	number of terms
Method [123]	k=2..6	1.71	21; 60; 281; 1214; 2895	2; 2; 14; 30; 82
complex c.	k=7	3.92	7540	150
Maple V	k=8	15.15	13555	302
Method [29]	k=2..6	1.09	10; 24; 163; 382; 1181	1; 1; 7; 14; 41
polar c.	k=7	1.81	2427	74
C++	k=8	13.23	5306	151
Our Method	k=2..6	197.62	39; 285; 1456; 4650; 13880	4; 16; 60; 160; 396
cartesian c.	k=7	1516.15	36321	848
Mathematica	k=8	10697.4	85432	1716
Our Method	k=2..6	2.64	39; 285; 1456; 4650; 13880	4; 16; 60; 160; 396
cartesian c.	k=7	5.53	36321	848
C++	k=8	36.47	85432	1716

Table 3.2: Cubic homogeneous perturbations

3.4 Small-amplitude limit cycles and the center problem

In this case we consider systems in which the origin is a critical point of focus type, and show how to bifurcate limit cycles out of it. Thus we investigate systems of the form

$$\dot{x} = \lambda x - y + X(x,y) \ , \quad \dot{y} = x + \lambda y + Y(x,y) \ , \tag{3.3}$$

where the nonlinearities are

$$X(x,y) = \sum_{s=2}^{n} X_s(x,y), \quad \text{and} \quad Y(x,y) = \sum_{s=2}^{n} Y_s(x,y),$$

with X_s and Y_s are homogeneous polynomials of degree s. The linear part is in canonical form, and the stability of the origin is determined by the sign of λ. If $\lambda = 0$, the origin is a center for the linearized system, and is said to be a *fine focus* of the nonlinear system. In order to solve the problem of the stability at the origin of system (3.3), it is sufficient to consider the sign of the first Poincaré–Liapunov constant different from zero. The origin is a nonlinear center, i.e. there is an open neighborhood of the origin where all orbits are periodic except of course the origin, if and only if all Poincaré–Liapunov constants are zero. The idea is to perturb the coefficients arising in the X_s and Y_s so that limit cycles bifurcate out of the origin. Such limit cycles are said to be of *small amplitude*. The origin is said to be a fine focus of order k if V_{2k+2} is the first non-zero Poincaré–Liapunov constant. In this case at most k limit cycles can bifurcate from this fine focus, see for instance [16]. To maximize the number of limit cycles which can bifurcate, we start with a fine focus which is as close to being a center for the nonlinear system as possible. Therefore to obtain the maximum number of limit cycles which can bifurcate from the origin for a given system, one has to find the maximum possible order of a fine focus.

Suppose that the origin is a fine focus of order k. The first step is to perturb the coefficients in X and Y so that $V_{2k} \neq 0$ with $V_{2l} = 0$ for $l < 2k$ and $V_{2k}V_{2k+2} < 0$; if this can be achieved, the stability of the origin is reversed, and a limit cycle Γ_1 bifurcates. Next, further perturbations are introduced so that $V_{2k-2}V_{2k} < 0$ with $V_{2l} = 0$ for $l < 2k-2$. The stability of the origin is again reversed, and another limit cycle Γ_2 appears. Provided that V_{2k-2} is small enough, Γ_1 persists, and there are therefore two limit cycles. Proceeding in this way, k limit cycles bifurcate provided perturbations can be so arranged that $V_{2k}V_{2k+2} < 0$ for $1 \leq l \leq k$, see [120].

Since it is the first non-zero Poincaré–Liapunov constant that is of significance, what we really need are the non-zero expressions obtained by calculating each V_{2k} under the conditions $V_2 = \cdots = V_{2k-2} = 0$. It can happens that a reduced Poincaré–Liapunov constant was zero, in which case it does not contribute in the process of bifurcation of limit cycles. For a given class of systems, the aim is to maximize the number of limit cycles which can bifurcate from the origin. Thus, it is necessary to find k_1, the maximum possible order of a fine focus. This k_1 is characterized by the fact that the origin is a center if $V_{2k} = 0$ for $k \leq 1 + k_1$, but not if any of these constants is non-zero. In practice, one proceeds with the computation of the Poincaré–Liapunov constants until appears that k_1 has been reached. Then it is necessary to prove independently that the origin is a center. This is often difficult, and developing criteria for the existence of a center is a significant and substantive problem. The different techniques as reversibility of the system, existence of a first integral or an integrating factors defined in a neighborhood of the critical point and existence of analytical changes to simplified systems are those most used ones.

3.5 Summary of the computational problems

Assisting to the previous section there are four steps to the procedure each one of them with concrete computational problems.

1. Calculation of the Poincaré–Liapunov constants. In the calculation of this constants, very large expressions arise. It is here that Computer Algebra System have proved so valuable. We have a recurrent formula to compute the polynomials and the limitations that it imposes us the RAM of the computer.

2. Reduction of the Poincaré–Liapunov constants. In the reduction of the V_{2k} it can be used the direct substitutions from the relations $V_2 = \cdots = V_{2k-2} = 0$ which involve rational functions of the coefficients arising in X and Y. This contrasts with the formal calculation of a basis for the ideal generated by the Poincaré–Liapunov constants applying the Buchberger's Gröbner basis method or variations of this one, where in that case all substitutions are polynomial. These methods are based on defining a division algorithm using some monomial ordering, see [46]. The division algorithm it is used also to make the reduction of the Poincaré–Liapunov constants. Calculations based on Buchberger's algorithm can be done only for sufficient simple polynomials with the program packages of a Computer Algebra System. Unfortunately, we deal with very massive polynomials that even if very powerful computers are used it is not possible to solve the ideal membership problem. There are variations of the Buchberger's algorithm taking into account some special properties of the Poincaré–Liapunov constants, see [113].

3. Establishing the value of k_1 by proving that the origin is center if $V_{2k} = 0$ for $k \leq 1 + k_1$. In this case different paths have to be analyzed and also there is a computational cost.

4. Beginning with a fine focus of maximal order, finding a sequence of perturbations each of which reverses the stability of the origin. Numerical values of the variables of the system must be fixed to get the result.

3.6 Some applications

We give a summary of some of the results which have been obtained using the techniques described in these two first chapters. Let $\tilde{H}$ denote the maximum number of limit cycles which can bifurcate out of a fine focus. It has has long been known that $\tilde{H} = 3$ for quadratic systems; this was shown by Bautin [12]. In [16], Blows and Lloyd proved that $\tilde{H} = 5$ for cubic systems in which the quadratic terms are absent. For general cubic systems the last result due to Żołądek [126] is $\tilde{H} \geq 11$.

Certain systems with quartic and quintic homogeneous nonlinearities have been recently investigated using the described method, see [69]. For systems of the form

$$\dot{x} = -y, \quad \dot{y} = x + Q_n(x, y), \tag{3.4}$$

where $Q_n(x, y)$ is homogeneous polynomial of degree n, for $n = 4$ and $n = 5$, it is proved that the maximum number of small-amplitude limit cycles which can bifurcate from the origin is at least four for system (3.4) with $n = 4$ and five for system (3.4) with $n = 5$.

Other type of systems which we have studied in detail are the so-called "homogeneous systems"; these systems are of the form

$$\dot{x} = -y + P_n(x, y), \quad \dot{y} = x + Q_n(x, y), \tag{3.5}$$

where $P_n(x, y)$ and $Q_n(x, y)$ are homogeneous polynomial of degree n. These type of systems have been studied using the method with polar coordinates for $n = 4$ and $n = 5$. The discussion about the number of small-amplitude limit cycles which can bifurcate from the origin for system (3.5) have given which is greater or equal seven for $n = 4$ and greater or equal nine for $n = 5$, see [29] and [30].

Much of activity has been concerned with systems of Liénard type. i.e., differential systems of the form

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x), \tag{3.6}$$

where $g(0) = 0$ and $g'(0) < 0$. The value of $\tilde{H}$ is obtained in a large number of cases for such systems, see [16]. The BiLiénard systems are systems of the form

$$\dot{x} = y - F(x), \quad \dot{y} = -x - G(y). \tag{3.7}$$

The case with $F(x) = a_2x^2 + a_3x^3 + a_4x^4$ and $G(y) = b_2y^2 + b_3y^3 + b_4y^4$ have been studied in the previous chapter and the maximum number of small-amplitude limit cycles which can bifurcate from the origin is at least six.

Other systems recently investigated using the techniques described in this chapter are the systems of the form

$$\dot{x} = y + xf(x,y)\ , \quad \dot{y} = -x + yf(x,y)\ . \tag{3.8}$$

This type of systems are called *uniformly isochronous centers* because they have an isochronous center at the origin and in polar coordinates (r, φ) the angle φ satisfies the equation $\dot{\varphi} = 1$. As we have seen in Chapter 1, for system (3.8) the maximum number of small-amplitude limit cycles which can bifurcate from the origin is at least three when $f(x,y) = f_1(x,y) + f_3(x,y)$ where $f_i(x,y)$ are homogeneous polynomials of degree i.

Chapter 4

Abel differential equations admitting a certain first integral

4.1 Introduction

In this chapter we study the integrability of the *Abel differential equations* i.e., differential equations of the form

$$\frac{dy}{dx} = a(x)y^3 + b(x)y^2 + c(x)y + d(x)\,, \tag{4.1}$$

with $a(x) \not\equiv 0$ and where a, b, c and d are meromorphic functions of x. Abel equations appear in the reduction of order of many second and higher order differential equations, and hence are frequently found in the modeling of real problems in varied areas. There are only a few families of Abel equations in which a complete classification of their solutions is known. For instance, in [32] a classification of the integrable rational Abel differential equations according to invariant theory, i.e., the integrable Abel equations where a, b, c and d are rational functions was presented. As far as we know, this is the most general method available at the moment to solve Abel equations, already described by E. Kamke [86], page 26, as sub-method (g) due to M. Chini [36]. The importance of the Abel equations, as we have said, lies in that appear in the reduction of order of other several differential equations, see [32, 86, 112]. For instance, in [112], Abel equations are analyzed from the point of view of the study of the superposition rules. These nonlinear superposition rules are not derived from the classical Lie's theorem, see also [60].

4.2 Abel differential equations of second kind

The Abel differential equations of second kind are differential equations of the form

$$\frac{d\tilde{y}}{dx} = \frac{\tilde{a}(x)\tilde{y}^3 + \tilde{b}(x)\tilde{y}^2 + \tilde{c}(x)\tilde{y} + \tilde{d}(x)}{\tilde{p}(x)\tilde{y} + \tilde{q}(x)}, \tag{4.2}$$

where $\tilde{q}(x) \not\equiv 0$. Obviously, the case $\tilde{p}(x) \equiv 0$ includes the Abel equations (4.1) of first kind. Moreover, if $\tilde{p}(x) \not\equiv 0$ we can transform any Abel equation of second kind to the first kind through the change $\tilde{y} = 1/y - \tilde{q}(x)/\tilde{p}(x)$, where

$$a = \frac{\tilde{a}\tilde{q}^3}{\tilde{p}^4} - \frac{\tilde{b}\tilde{q}^2}{\tilde{p}^3} + \frac{\tilde{a}\tilde{q}}{\tilde{p}^2} - \frac{\tilde{d}}{\tilde{p}},\ b = -\frac{3\tilde{a}\tilde{q}^2}{\tilde{p}^3} + \frac{2\tilde{b}\tilde{q}}{\tilde{p}^2} - \frac{\tilde{c}}{\tilde{p}} - \left(\frac{\tilde{q}}{\tilde{p}}\right)',\ c = \frac{3\tilde{a}\tilde{q}}{\tilde{p}^2} - \frac{\tilde{b}}{\tilde{p}},\ d = -\frac{\tilde{a}}{\tilde{p}}.$$

4.3 Algebraic first integral in the dependent variable

A first integral for a differential equation

$$\frac{dy}{dx} = f(x, y), \tag{4.3}$$

is a non-constant scalar–valued function $h = h(x, y)$ such that $\frac{d}{dx}h(x, y(x)) \equiv 0$ whenever $y = y(x)$ is a solution of (4.3). We say that such a first integral h is *algebraic first integral in the dependent variable* y if it can be expressed into the form

$$h(x, y) = \left(\prod_{i=1}^{n}(y - g_i(x))^{\alpha_i}\right)h_0(x) \tag{4.4}$$

where $\alpha_i \in \mathbb{C} - \{0\}$ for $i = 1, \ldots, n$, the functions $x \mapsto g_i(x)$ are particular solutions of (4.3) for $i = 1, \ldots, n$ and the function $h_0(x) \neq 0$. This type of first integral was proposed by Painlevé in [103], see also [67]. and references therein. Notice that, if the differential equation is real then there is a local first integral which is also real. We can first work for a first integral in $\mathbb{C}$ and find α_i, $g_i(x)$ and $h_0(x)$ complex, but at the end we can look for a real first integral from the complex one.

4.4 Algebraic first integral in the dependent variable for an Abel equation

The following proposition gives the algebraic conditions that must be satisfied by the particular solutions of an Abel equation (4.1) in order to have a first integral of the form (4.4).

Proposition 4.1 *The Abel equation* (4.1) *admits a first integral of the form* (4.4) *if and only if there are* n *particular solutions* $\{g_i\}_{i=1,\dots,n}$ *of* (4.1), *and* n *non-zero complex numbers* $\{\alpha_i\}_{i=1,\dots,n}$ *such that*

$$\sum_{i=1}^{n} \alpha_i = 0, \qquad \textit{and} \qquad \sum_{i=1}^{n} \alpha_i g_i = 0. \tag{4.5}$$

Moreover, when this is the case one has $h_0(x) = \alpha_0 e^{-\int (a \sum_{i=1}^{n} \alpha_i g_i^2) dx}$, *where* $\alpha_0 \in \mathbb{C} - \{0\}$ *is an arbitrary constant.*

Proof. For f as in (4.3) we introduce the differential operator

$$D := \frac{\partial}{\partial x} + f(x,y)\frac{\partial}{\partial y}.$$

For any function $h = h(x,y)$ and any solution $y = y(x)$ of (4.3) the chain-rule gives

$$\frac{d}{dx} h(x, y(x)) = Dh(x,y)|_{y=y(x)}.$$

From the definition of first integral $h(x,y)$ is a first integral if h is constant on each solution $y = y(x)$ of (4.3), i.e. $h(x, y(x)) = cte$. Therefore $Dh(x, y(x)) \equiv 0$ for all x. Consequently, we see that h is a first integral of (4.3) if and only if $Dh \equiv 0$. If $g(x)$ is a particular solution of the Abel equation (4.1) then

$$D(y-g) = (ay^2 + by + c + (ay+b)g + ag^2)(y-g),$$

where $D := \partial/\partial x + (a(x)y^3 + b(x)y^2 + c(x)y + d(x))\partial/\partial y$, see [60, 61].

If (4.4) is a first integral of (4.1) then we have

$$\begin{aligned} Dh &= h\left[\sum_{i=1}^{n} \alpha_i (ay^2 + by + c + (ay+b)g_i + ag_i^2) + \frac{h_0'}{h_0}\right] \\ &= h\left[(ay^2 + by + c)\sum_{i=1}^{n} \alpha_i + (ay+b)\sum_{i=1}^{n} \alpha_i g_i + a\sum_{i=1}^{n} \alpha_i g_i^2 + \frac{h_0'}{h_0}\right] \equiv 0. \end{aligned}$$

If we view the bracketed expression as a polynomial in y with coefficients as a functions of x which must vanish, and this easily leads to conditions (4.5) and the expression for $h_0(x)$. The converse is obvious. ■

Consider the vector space of functions and the associated affine space having the zero function as a distinguished point, i.e., we think any function as a point in this affine space. The geometric interpretation of condition (4.5) is that we must have n particular solutions $g_i(x)$ inside an hyperplane of dimension $n-2$. If we choose n arbitrary points in an affine space of sufficiently high dimension they define a hyperplane of dimension $n-1$. Only when these points verify condition (4.5) they define a subspace of lower dimension. In this case we will say that these points are *coplanar* for dimension $n \geq 4$ and are *collinear* for $n = 3$.

Obviously, we cannot talk about coplanarity for $n = 1$ or $n = 2$. Moreover, the case $n = 1$ reduce to $\alpha_1 = 0$ and the case $n = 2$ implies $\alpha_1 = \alpha_2 = 0$. This fact does not mean it would be impossible to construct a first integral of an Abel equation using one or two particular solution; only that such a first integral cannot take the form (4.4). To achieve that form we need $n \geq 3$.

4.4.1 Computation of $h_0(x)$

In this subsection we are going to see how to compute the factor $h_0(x)$ of the first integral (4.4) for an Abel equation (4.1) when the independent term $d(x)$ is null.

Proposition 4.2 *For an Abel equation with $d(x) = 0$ and with n not null coplanar particular solutions $y = g_i(x)$ for $i = 1, \ldots, n$ we have $h_0(x) = 1/\prod_{i=1}^n g_i^{\alpha_i}$.*

Proof. From ${g_i}' = ag_i^3 + bg_i^2 + cg_i$, we get $(\ln g_i)' = ag_i^2 + bg_i + c$, an taking into account (4.5), we obtain that $\sum_{i=1}^n \alpha_i (\ln g_i)' = (\sum_{i=1}^n \alpha_i \ln g_i)' = (\ln \prod_{i=1}^n g_i^{\alpha_i})' = a \sum_{i=1}^n \alpha_i g_i^2$, and replacing in the original expression of h_0 the claim follows. ■

Proposition 4.3 *For an Abel equation with $d(x) = 0$ and with $n-1$ not null coplanar particular solutions $y = g_i(x)$ for $i = 2, \ldots, n$ and $y = g_1(x) = 0$, we have $h_0(x) = e^{-\alpha_1 \int c(x)dx} / \prod_{i=2}^n g_i^{\alpha_i}$.*

Proof. In the same way as before we have $(\ln g_i)' = ag_i^2 + bg_i + c$, for $i = 2, \ldots, n$, and therefore $a\sum_{i=2}^n \alpha_i g_i^2 + b\sum_{i=2}^n \alpha_i g_i + c\sum_{i=2}^n \alpha_i = (\ln \prod_{i=2}^n g_i^{\alpha_i})'$, but for $g_1 = 0$, $\sum_{i=2}^n \alpha_i g_i = -\alpha_1 g_1 = 0$, and $\sum_{i=2}^n \alpha_i = -\alpha_1$, from where $a\sum_{i=1}^n \alpha_i g_i^2 = a\sum_{i=2}^n \alpha_i g_i^2 = (\ln \prod_{i=2}^n g_i^{\alpha_i})' + \alpha_1 c$, and replacing in the original expression of h_0 the proposition is proved. ∎

Some seminal works had already been studied Abel integral equations with a first integral of the form (4.4), without reaching the geometric interpretation given in this chapter, see [35, 53]. In particular, the case $d(x) = 0$ studied in Proposition 4.2 and 4.3, which is equivalent the Abel equation of second kind (4.2) with $\tilde{a}(x) = 0$, is studied in [53].

4.5 Admissible invariant change of variables for Abel equations

Two Abel equations are defined to be equivalent if one can be obtained from the other through the transformation

$$\{x = t(X),\, y(x) = R(X)Y + S(X)\}, \tag{4.6}$$

where X and Y are respectively the new independent and dependent variables and t, R and S are arbitrary meromorphic functions of X satisfying $t' \neq 0$ and $R \neq 0$. These transformations form a non abelian group. We can break down these transformation into two consecutive changes. The first one $\{x = x,\, y(x) = r(x)Y + s(x)\}$ which is a dilatation plus a translation and the second one a scaling of the independent variable $\{x = t(X), Y = Y\}$. Integration strategies were discussed in [93, 10, 32], around objects called *invariant* under the transformation (4.6) which can be built with the coefficients $\{a, b, c, d\}$ of the Abel equation (4.1) and their derivatives. The transformation (4.6) preserves the structure of the Abel equation (4.1) and also the coplanarity condition (4.5) of any set of functions. Moreover, in the new variables the coplanarity condition admits the same coefficients $\alpha_i \in \mathbb{C} - \{0\}$. These coefficients are unique except a not null multiplicative constant factor as the following proposition shows.

Proposition 4.4 *If n is the minimum number of coplanar particular solutions of an Abel equation then $\alpha_i \in \mathbb{C} - \{0\}$ are unique except for a non–zero constant multiplicative factor.*

Proof. Suppose that n is the minimum number of coplanar particular solutions $g_i(x)$ and there exist $\alpha_i \in \mathbb{C} - \{0\}$ and $\beta_i \in \mathbb{C} - \{0\}$, such that $\sum_{i=1}^n \alpha_i = 0$, $\sum_{i=1}^n \alpha_i g_i = 0$, $\sum_{i=1}^n \beta_i = 0$, and $\sum_{i=1}^n \beta_i g_i = 0$. From these relations we have $\sum_{i=2}^n (\alpha_i/\alpha_1 - \beta_i/\beta_1) = 0$ and $\sum_{i=2}^n (\alpha_i/\alpha_1 - \beta_i/\beta_1) g_i = 0$. If there are no null coefficients inside $\{\alpha_i/\alpha_1 - \beta_i/\beta_1\}$, we will have a smaller subset of coplanar particular solutions in contradiction with the hypothesis. Hence, we have that all the coefficients $\alpha_i/\alpha_1 - \beta_i/\beta_1$ must be null, i.e., $\beta_i = \beta_1 \alpha_i/\alpha_1$ and the claim follows. ∎

4.6 An illustrative example

Consider the Abel equation of second kind

$$\frac{d\tilde{y}}{dx} = \frac{3\tilde{y}^2 - 3\tilde{y} - x}{8x\tilde{y} - 9x},$$

which possesses the first integral

$$\tilde{h}(x, \tilde{y}) = \frac{x^3(4x^2 + (8\tilde{y}^2 - 36\tilde{y} + 27)x + 4\tilde{y}^4 - 4\tilde{y}^3)}{(x^2 + 2x(\tilde{y}^2 - 3\tilde{y}) + \tilde{y}^4)^3}.$$

We can transform this Abel equation of second kind to the first kind through the change $y = 1/(8\tilde{y} - 9)$ and we get

$$\frac{dy}{dx} = \left(8 - \frac{27}{8x}\right) y^3 - \frac{15}{4x} y^2 - \frac{3}{8x} y, \tag{4.7}$$

which correspond to a case with $d(x) = 0$ and the transformed first integral is

$$h(x, y) = \frac{x^3 y^8 ((4096x^2 - 3456x + 729)y^4 + (972 - 2304x)y^3 + (128x + 270)y^2 + 28y + 1)}{((4096x^2 - 17280x + 6561)y^4 + (2916 - 768x)y^3 + (128x + 486)y^2 + 36y + 1)^3}.$$

This first integral is composed by 9 particular solutions, one solution is $y = g_1(x) = 0$, four particular solutions $g_2(x)$, $g_3(x)$, $g_4(x)$, $g_5(x)$ coming from the algebraic curve in the numerator and four particular solutions $g_6(x)$, $g_7(x)$, $g_8(x)$, $g_9(x)$ coming from the algebraic curve in the denominator. Moreover, we have $\alpha_1 = 8$, $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 1$, $\alpha_6 = \alpha_7 = \alpha_8 = \alpha_9 = -3$. Obviously, the condition $\sum_{i=1}^n \alpha_i = 0$ is satisfied. Hence, the Abel equation

(8.1) is a case with $n = 9$. It is straightforward to see that $\sum_{i=1}^{n} \alpha_i g_i = 0$, because using the Cardano–Vieta formulae we obtain that

$$g_2 + g_3 + g_4 + g_5 = \frac{2304x - 972}{4096x^2 - 3456x + 729},$$

$$g_6 + g_7 + g_8 + g_9 = \frac{768x - 2916}{4096x^2 - 17280x + 6561},$$

and therefore we have that $8g_1 + g_2 + g_3 + g_4 + g_5 - 3(g_6 + g_7 + g_8 + g_9) = 0$. The computation of $h_0(x)$, is through the expression $h_0(x) = e^{-\alpha_1 \int c(x)dx} / \prod_{i=2}^{n} g_i^{\alpha_i}(x)$ because $d(x) = 0$ and $g_1(x) = 0$ and we must apply Proposition 4.3. Taking into account the Cardano–Vieta formulae, we know that

$$g_2\, g_3\, g_4\, g_5 = \frac{1}{4096x^2 - 3456x + 729}, \quad g_6\, g_7\, g_8\, g_9 = \frac{1}{4096x^2 - 17280x + 6561}.$$

Hence, the computations give

$$h_0(x) = \frac{e^{-\alpha_1 \int c(x)dx}}{\prod_{i=2}^{n} g_i^{\alpha_i}(x)} = \frac{x^3(4096x^2 - 3456x + 729)}{(4096x^2 - 17280x + 6561)^3},$$

which is the correct $h_0(x)$ because the polynomials of fourth degree in the first integral are not monic. Although the Abel equation (4.7) is an example of $n = 9$, it can serve as an example for $n < 9$ if such a set of particular solutions exist. From this example it is natural to consider whether there exist examples of other values of n and what is more important, how to find necessary and sufficient conditions to establish that an Abel equation admits a first integral of the form (4.4) for a given n.

4.7 Reduced, Bernoulli and Resolvent Abel equations

4.7.1 Reduced Abel equation

The reduced Abel equation is obtained through the change of dependent variable $y = Y - b(x)/(3a(x))$ which is an admissible transformation according to Section 4.5 and preserves the coplanarity condition (4.5) of any set of particular solutions. The Abel equation (4.1) becomes

$$\frac{dY}{dx} = a(x)(Y^3 + p(x)Y + q(x)), \tag{4.8}$$

where $p(x) = c(x)/a(x) - b^2(x)/(3a^2(x))$, and

$$\begin{aligned} q(x) = \; & \tfrac{1}{a^3(x)}\Big[a^2(x)d(x) + \tfrac{1}{3}(a(x)b'(x) \\ & -a'(x)b(x) - a(x)b(x)c(x)) + \tfrac{2}{27}b^3(x)\Big]. \end{aligned}$$

4.7.2 Bernoulli Abel equation

The reduced Abel equation with $q(x) = 0$ is a Bernoulli equation, and therefore it is integrable. We call this type of equations as *Bernoulli Abel equations*. This case has an algebraic first integral and consequently has coplanar particular solutions. We are going to see that the value of n is always 3. Notice that Bernoulli Abel equations always have the particular solution $Y = 0$, and if $Y = G(x)$ is a particular solution then $Y = -G(x)$ is also a particular solution. Hence, we have a set of three coplanar particular solution choosing $G_1 = 0$, $G_2 = G(x)$, an arbitrary solution of the considered Bernoulli Abel equation, and $G_3 = -G_2$, with $\alpha_1 = -2$ and $\alpha_2 = \alpha_3 = 1$, and from here we can build the algebraic first integral (4.4). In the following we consider the case $q(x) \not\equiv 0$. We also remark that Bernoulli Abel equation is a particular example of Lie system (see e.g. [25]) with associated Lie group isomorphic to the affine group: the vector fields $\mathcal{X}_1 = y\partial/\partial y$, $\mathcal{X}_2 = y^3\partial/\partial y$ are such that $[\mathcal{X}_1, \mathcal{X}_2] = 2\mathcal{X}_2$. Therefore solvable by quadratures.

4.7.3 Resolvent Abel equation

In the case $q(x) \not\equiv 0$, we can construct the *resolvent Abel equation*. It is obtained through the change of dependent variable $Y = q^{\frac{1}{3}}(x)\mathcal{Y}$ which is an admissible transformation according to Section 4.5 and also preserves the coplanarity condition (4.5) of any set of particular solutions. The reduced Abel equation (4.8) with $q(x) \not\equiv 0$ becomes

$$\frac{d\mathcal{Y}}{dx} = J(x)(\mathcal{Y}^3 - K(x)\mathcal{Y} + 1), \tag{4.9}$$

with $J(x) = a(x)q^{\frac{2}{3}}(x)$, and $K(x) = q^{-\frac{5}{3}}(x)\,[q'(x)/(3a(x)) - p(x)q(x)]$. Notice, that in the resolvent Abel equation we only have two functions of x, but they are, in general, irrational functions. In the following, we will work when convenient with the reduced Abel equation or with the resolvent Abel equation. For the reduced Abel equation we will use the following notation $p_i(x) := {p_{i-1}}'(x)/a(x)$, where $p_0(x) := p(x)$ and $q_i(x) := {q_{i-1}}'(x)/a(x)$, where $q_0(x) := q(x)$. For the resolvent Abel equation we also define $L(x) = K'(x)/J(x)$, $M(x) = L'(x)/K'(x) = L'(x)/(J(x)L(x))$ and $N(x) = M'(x)/J(x)$ when $K'(x) \not\equiv 0$. Moreover, if $y = g(x)$ is a particular solution in the original Abel equation (4.1), we define by $Y = G(x)$ the corresponding particular solution in the reduced Abel equation (4.8) and by $\mathcal{Y} = \mathcal{G}(x)$ the particular solution in the resolvent Abel equation (4.9) and similarly for other built elements from the mentioned particular solutions. Finally, depending on the context, if we are working with the original Abel equation (4.1) we will understand $D := \partial/\partial x + (ay^3 + by^2 + cy + d)\partial/\partial y$, if we are working with the reduced Abel equation (4.8) we will understand $\hat{D} := \partial/\partial x + a(Y^3 + pY + q)\partial/\partial Y$ and if we are working with the resolvent Abel equation (4.9) we will understand $\mathcal{D} := \partial/\partial x + J(\mathcal{Y}^3 - K\mathcal{Y} + 1)\partial/\partial\mathcal{Y}$.

4.8 Abel equations with three collinear particular solutions

The next theorem characterizes when an Abel equation with $q(x) \neq 0$ (i.e., which is not a Bernoulli Abel equation) has three collinear particular solutions and consequently has an algebraic first integral (4.4) with $n = 3$.

Theorem 4.5 *An Abel equation with $q(x) \neq 0$ has three distinct collinear particular solutions if and only if $K(x)$ is a constant ($K' = 0$) and $K^3 \neq 27/4$. Moreover, on the context of the resolvent Abel equation, these three solutions are constants $\mathcal{Y} = \mathcal{G}_1(x) = k_1$, $\mathcal{Y} = \mathcal{G}_2(x) = k_2$ and $\mathcal{Y} = \mathcal{G}_3(x) = k_3$, where k_1, k_2 and k_3, are the simple roots of the cubic algebraic equation $k^3 - Kk + 1 = 0$.*

Proof. Sufficiency: If $K' = 0$ and $K^3 \neq 27/4$, the three roots k_1, k_2 and k_3 of the equation $k^3 - Kk + 1 = 0$ are constants and simple. Hence, are also particular solutions of the resolvent Abel equation. They are also collinear, because taking $\alpha_1 = k_2 - k_3$, $\alpha_2 = k_3 - k_1$ and $\alpha_3 = k_1 - k_2$, then we have $\alpha_1 + \alpha_2 + \alpha_3 = 0$ and $\alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3 = 0$.

Necessity: We consider that we have three particular solutions $\mathcal{Y} = \mathcal{G}_1(x)$, $\mathcal{Y} = \mathcal{G}_2(x)$ and $\mathcal{Y} = \mathcal{G}_3(x)$ of the resolvent Abel equation (4.9) and there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C} - \{0\}$, such that

$$\alpha_1 + \alpha_2 + \alpha_3 = 0 \quad \text{and} \quad \alpha_1\mathcal{G}_1 + \alpha_2\mathcal{G}_2 + \alpha_3\mathcal{G}_3 = 0. \tag{4.10}$$

The derivative of the last equality is $\alpha_1\mathcal{G}_1' + \alpha_2\mathcal{G}_2' + \alpha_3\mathcal{G}_3' = 0$. Taking into account that $\mathcal{G}_i' = J(\mathcal{G}_i^3 - K\mathcal{G}_i + 1)$ and the conditions (4.10) we get $\alpha_1\mathcal{G}_1^3 + \alpha_2\mathcal{G}_2^3 + \alpha_3\mathcal{G}_3^3 = 0$. Hence, we obtain the homogeneous system

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= 0, \\ \alpha_1\mathcal{G}_1 + \alpha_2\mathcal{G}_2 + \alpha_3\mathcal{G}_3 &= 0, \\ \alpha_1\mathcal{G}_1^3 + \alpha_2\mathcal{G}_2^3 + \alpha_3\mathcal{G}_3^3 &= 0. \end{aligned}$$

To have a non trivial solution it must happen that

$$\begin{vmatrix} 1 & 1 & 1 \\ \mathcal{G}_1 & \mathcal{G}_2 & \mathcal{G}_3 \\ \mathcal{G}_1^3 & \mathcal{G}_2^3 & \mathcal{G}_3^3 \end{vmatrix} = (\mathcal{G}_1 - \mathcal{G}_2)(\mathcal{G}_3 - \mathcal{G}_1)(\mathcal{G}_2 - \mathcal{G}_3)(\mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3) = 0,$$

which implies $\mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 = 0$. The derivative of this last equality is $\mathcal{G}_1' + \mathcal{G}_2' + \mathcal{G}_3' = 0$, and substituting each derivative $\mathcal{G}_i' = J(\mathcal{G}_i^3 - K\mathcal{G}_i + 1)$, we get $\mathcal{G}_1^3 + \mathcal{G}_2^3 + \mathcal{G}_3^3 + 3 = 0$. Taking into account that $\mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 = 0$ we have $\mathcal{G}_1^3 + \mathcal{G}_2^3 + \mathcal{G}_3^3 = 3\mathcal{G}_1\mathcal{G}_2\mathcal{G}_3$, which implies $\mathcal{G}_1\mathcal{G}_2\mathcal{G}_3 = -1$. The derivative of this last equality is $\mathcal{G}_1'\mathcal{G}_2\mathcal{G}_3 + \mathcal{G}_1\mathcal{G}_2'\mathcal{G}_3 + \mathcal{G}_1\mathcal{G}_2\mathcal{G}_3' = 0$, and substituting each derivative $\mathcal{G}_i'$, we obtain $\mathcal{G}_1^3\mathcal{G}_2\mathcal{G}_3 + \mathcal{G}_2^3\mathcal{G}_1\mathcal{G}_3 + \mathcal{G}_3^3\mathcal{G}_1\mathcal{G}_2 - 3K\mathcal{G}_1\mathcal{G}_2\mathcal{G}_3 + \mathcal{G}_1\mathcal{G}_2 + \mathcal{G}_1\mathcal{G}_3 + \mathcal{G}_2\mathcal{G}_3 = 0$. On the other hand $\mathcal{G}_1^3\mathcal{G}_2\mathcal{G}_3 + \mathcal{G}_2^3\mathcal{G}_1\mathcal{G}_3 + \mathcal{G}_3^3\mathcal{G}_1\mathcal{G}_2 = -2\mathcal{G}_1\mathcal{G}_2\mathcal{G}_3(\mathcal{G}_1\mathcal{G}_2 + \mathcal{G}_1\mathcal{G}_3 + \mathcal{G}_2\mathcal{G}_3)$, due to $\mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 = 0$. Hence, we have $-2\mathcal{G}_1\mathcal{G}_2\mathcal{G}_3(\mathcal{G}_1\mathcal{G}_2 + \mathcal{G}_1\mathcal{G}_3 + \mathcal{G}_2\mathcal{G}_3) - 3K\mathcal{G}_1\mathcal{G}_2\mathcal{G}_3 + \mathcal{G}_1\mathcal{G}_2 + \mathcal{G}_2\mathcal{G}_3 + \mathcal{G}_2\mathcal{G}_3 = 0$, and from $\mathcal{G}_1\mathcal{G}_2\mathcal{G}_3 = -1$, we arrive to $\mathcal{G}_1\mathcal{G}_2 + \mathcal{G}_1\mathcal{G}_3 + \mathcal{G}_2\mathcal{G}_3 = -K$.

Hence, the three solutions $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are simple roots (because are different by hypothesis) of the cubic algebraic equation $\mathcal{Y}^3 - K\mathcal{Y} + 1 = 0$, but if we recall that $\mathcal{G}_i' = J(\mathcal{G}_i^3 - K\mathcal{G}_i + 1)$, we get that $\mathcal{G}_i' = 0$. Therefore, the three $\mathcal{G}_i$ are not null constants due to $\mathcal{G}_1\mathcal{G}_2\mathcal{G}_3 = -1$. Moreover, since $K = (\mathcal{G}_1^3 + 1)/\mathcal{G}_1$, then K is also a constant. Moreover, the discriminant of the cubic algebraic equation $K^3 - 27/4 \neq 0$, because $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are the simple roots of it, which completes the proof. ∎

The condition $K'(x) = 0$ expressed in terms of the coefficients of the reduced Abel equation is

$$9p_1q^2 - 6pqq_1 + 5q_1^2 - 3qq_2 = 0 \tag{4.11}$$

This condition, if $q \not\equiv 0$ allows us to determine if we have or not a case with $n = 3$. Obviously, we must also compute K to verify if $K^3 \neq 27/4$. For $n = 3$ the collinear solutions of the reduced Abel equation are $G_i(x) = k_i\, q^{\frac{1}{3}}(x)$ and for the original Abel equation are given by $g_i(x) = k_i\, q^{\frac{1}{3}}(x) - b(x)/(3a(x))$. While $\mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 = k_1 + k_2 + k_3 = 0$ and $G_1 + G_2 + G_3 = 0$, we will have that $g_1 + g_2 + g_3 = -b(x)/a(x)$.

Notice, that the conditions $q(x) \equiv 0$ and $K'(x) = 0$ already appear in the classical book of Kamke [86] page 26 as integrability conditions for an Abel equation. The first case, $q(x) \equiv 0$ corresponds to $\Phi(x) = q(x)a^3(x) \equiv 0$ and there exist a transformation into a Bernoulli equation. The second case $K'(x) = 0$ is $K(x) = \alpha$, where α is an arbitrary constant, and in this case there exist a transformation into a separated variables equation. Hence, we have reobtained these cases, but in a new general framework where we have reinterpreted these conditions in an algebro–geometric way which allow us to generalize these conditions to obtain new cases of integrable Abel equations.

4.8.1 Algebraic first integral for the case $n = 3$

From what we have seen in Section 4.4 and in Section 4.8, for $K' = 0$ and $K \neq 27/4$, the first integral of the resolvent Abel will be

$$\mathcal{H}(x, \mathcal{Y}) = (\mathcal{Y} - k_1)^{(k_2-k_3)}(\mathcal{Y} - k_2)^{(k_3-k_1)}(\mathcal{Y} - k_3)^{(k_1-k_2)} e^{(k_2-k_3)(k_3-k_1)(k_1-k_2)\int J(x)dx},$$

where we have used that $k_1^2(k_2 - k_3) + k_2^2(k_3 - k_1) + k_3^2(k_1 - k_2) = -(k_2 - k_3)(k_3 - k_1)(k_1 - k_2)$. Recall that k_1, k_2 and k_3 are roots of the cubic algebraic equation $k^3 - Kk + 1 = 0$. The first integral for the reduced Abel equation is given by

$$\begin{aligned} H(x, Y) &= (Y - k_1q^{\frac{1}{3}}(x))^{(k_2-k_3)}(Y - k_2q^{\frac{1}{3}}(x))^{(k_3-k_1)}(Y - k_3q^{\frac{1}{3}}(x))^{(k_1-k_2)} \\ &\quad \times\, e^{(k_2-k_3)(k_3-k_1)(k_1-k_2)\int a(x)q^{\frac{2}{3}}(x)dx}. \end{aligned}$$

4.8.2 First integral for the case $K^3 = 27/4$

For $K'(x) = 0$, the resolvent Abel equation is of separable variables, and therefore integrable. But, if $K^3 = 27/4$, there are not three collinear particular solutions. Let $\omega_0 = 1$, $\omega_1 = -\frac{1}{2} + \frac{i}{2}\sqrt{3}$, and $\omega_{-1} = -\frac{1}{2} - \frac{i}{2}\sqrt{3}$ be, the three cubic roots of unity. The possible cubic algebraic equations in k are $k^3 - \frac{3}{2}\sqrt[3]{2}\omega_j k + 1 = 0$, for $j = -1, 0, 1$. For each one, the simple root is $k_s = -\sqrt[3]{4}\omega_{-j}$ and the double root is $k_d = \frac{\sqrt[3]{4}}{2}\omega_{-j}$, being satisfied that $k_s = -2k_d$. The first integral in this case is

$$\mathcal{H}(x, \mathcal{Y}) = \frac{(\mathcal{Y} - k_d)e^{\frac{3k_d}{\mathcal{Y}-k_d}} e^{9k_d^2 \int J(x)dx}}{\mathcal{Y} + 2k_d}. \tag{4.12}$$

It is important to note that, in this degenerate case a Darboux exponential factor appears due to the appearance of a multiple algebraic curve, see definitions in [27, 28]. For $K^3 = 27/4$, the resolvent Abel equation is $d\mathcal{Y}/dx = J(x)(\mathcal{Y} - k_s)(\mathcal{Y} - k_d)^2$. We have three Darboux factors $\mathcal{F}_1 = \mathcal{Y} - k_s$, $\mathcal{F}_2 = \mathcal{Y} - k_d$ and $\mathcal{F}_3 = e^{-\frac{1}{\mathcal{Y}-k_d}}$ with cofactors $\mathcal{K}_1 = J(x)(\mathcal{Y} - k_d)^2$, $\mathcal{K}_2 = J(x)(\mathcal{Y} - k_s)(\mathcal{Y} - k_d)$ and $\mathcal{K}_3 = J(x)(\mathcal{Y} - k_s)$. Keeping in mind that $-\mathcal{K}_1 + \mathcal{K}_2 - 3k_d\mathcal{K}_3 + 9k_d^2 J(x) = 0$ this prove that (4.12) is a first integral in this case. Consequently, the first integral for the reduced Abel equation is

$$H(x, Y) = \frac{(Y - k_d q^{\frac{1}{3}}(x))e^{\frac{3k_d q^{\frac{1}{3}}(x)}{Y - k_d q^{\frac{1}{3}}(x)}} e^{9k_d^2 \int a(x) q^{\frac{2}{3}}(x)dx}}{Y + 2k_d q^{\frac{1}{3}}(x)}.$$

4.9 Abel equations with four coplanar particular solutions

4.9.1 Characterization of the Abel equations with four coplanar particular solutions

Let us assume that $q(x) \not\equiv 0$. Given four particular solutions not necessary different between them, $\mathcal{Y} = \mathcal{G}_1(x)$, $\mathcal{Y} = \mathcal{G}_2(x)$, $\mathcal{Y} = \mathcal{G}_3(x)$ and $\mathcal{Y} = \mathcal{G}_4(x)$, we can consider the associated elementary symmetric polynomials given by $\mathcal{S}_1 = \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 + \mathcal{G}_4$, $\mathcal{S}_2 = \mathcal{G}_1\mathcal{G}_2 + \mathcal{G}_1\mathcal{G}_3 + \mathcal{G}_1\mathcal{G}_4 + \mathcal{G}_2\mathcal{G}_3 + \mathcal{G}_2\mathcal{G}_4 + \mathcal{G}_3\mathcal{G}_4$, $\mathcal{S}_3 = \mathcal{G}_1\mathcal{G}_2\mathcal{G}_3 + \mathcal{G}_1\mathcal{G}_2\mathcal{G}_4 + \mathcal{G}_1\mathcal{G}_3\mathcal{G}_4 + \mathcal{G}_2\mathcal{G}_3\mathcal{G}_4$ and $\mathcal{S}_4 = \mathcal{G}_1\mathcal{G}_2\mathcal{G}_3\mathcal{G}_4$. Next theorem, the most important result of this chapter, gives the characterization of the Abel equations with four different coplanar particular solutions.

Theorem 4.6 *An Abel equation with $q(x) \not\equiv 0$ and $K'(x) \not\equiv 0$, has four different coplanar particular solutions $\mathcal{Y} = \mathcal{G}_1(x)$, $\mathcal{Y} = \mathcal{G}_2(x)$, $\mathcal{Y} = \mathcal{G}_3(x)$ and $\mathcal{Y} = \mathcal{G}_4(x)$, if and only if, there exist one root of the quintic algebraic equation*

$$315\mathcal{Y}^5 - 35(10K + M)\mathcal{Y}^3 + 63\mathcal{Y}^2 + 5(15K^2 - 7L + 5KM)\mathcal{Y} + 7M - 11K = 0, \tag{4.13}$$

which is a particular solution of the resolvent Abel equation, and all the roots of the quartic algebraic equation $\mathcal{Y}^4 - \mathcal{S}_1\mathcal{Y}^3 + \mathcal{S}_2\mathcal{Y}^2 - \mathcal{S}_3\mathcal{Y} + \mathcal{S}_4 = 0$ are simple, where $\mathcal{S}_1 = \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 + \mathcal{G}_4$ is the particular solution of (4.13) *and $\mathcal{S}_4 = 3L/(35\mathcal{S}_1^3 - 25K\mathcal{S}_1 - 7) - \mathcal{S}_1$, $\mathcal{S}_2 = (5\mathcal{S}_1(\mathcal{S}_1 + \mathcal{S}_4))/3 - K$ and $\mathcal{S}_3 = \mathcal{S}_1\mathcal{S}_2 - 1$. The four different roots of the quartic algebraic equation are the coplanar particular solutions, and therefore, the quartic algebraic equation is an algebraic invariant curve of the resolvent Abel equation even if the four roots are not different.*

The condition that all the roots of the quartic algebraic equation are simple is given by the discriminant $\Delta = \mathcal{S}_1^2\mathcal{S}_2^2\mathcal{S}_3^2 - 4\mathcal{S}_2^3\mathcal{S}_3^2 - 4\mathcal{S}_1^3\mathcal{S}_3^3 + 18\mathcal{S}_1\mathcal{S}_2\mathcal{S}_3^3 - 27\mathcal{S}_3^4 - 4\mathcal{S}_1^2\mathcal{S}_2^3\mathcal{S}_4 + 6\mathcal{S}_2^4\mathcal{S}_4 + 18\mathcal{S}_1^3\mathcal{S}_2\mathcal{S}_3\mathcal{S}_4 - 80\mathcal{S}_1\mathcal{S}_2^2\mathcal{S}_3\mathcal{S}_4 - 6\mathcal{S}_1^2\mathcal{S}_3^2\mathcal{S}_4 + 144\mathcal{S}_2\mathcal{S}_3^2\mathcal{S}_4 - 27\mathcal{S}_1^4\mathcal{S}_4^2 + 144\mathcal{S}_1^2\mathcal{S}_2\mathcal{S}_4^2 - 128\mathcal{S}_2^2\mathcal{S}_4^2 - 192\mathcal{S}_1\mathcal{S}_3\mathcal{S}_4^2 + 256\mathcal{S}_4^3 \neq 0$. To verify if one root of (4.13) is a particular solution of the resolvent Abel equation (4.9) can be done by computing the algebraic resultant

$$\operatorname{res}(\mathcal{V}, \mathcal{D}\mathcal{V}, \mathcal{Y}), \tag{4.14}$$

where $\mathcal{V} = 315\mathcal{Y}^5 - 35(10K + M)\mathcal{Y}^3 + 63\mathcal{Y}^2 + 5(15K^2 - 7L + 5KM)\mathcal{Y} + 7M - 11K$. We recall that the resultant of two polynomials P and Q over a field k is defined as the product

$$\operatorname{res}(P, Q, y) = c_1^{\deg Q} c_2^{\deg P} \prod_{(y_1,y_2):\, P(y_1)=0,\, Q(y_2)=0} (y_1 - y_2),$$

of the differences of their y-roots, where y_1 and y_2 take on values in the algebraic closure of k and c_1 and c_2 are the leading coefficients of P and Q, respectively. In fact the resultant is the determinant of the Sylvester matrix, see for instance [125]. If resultant (4.14) is null we have found four distinct coplanar solutions. Subsequently, we must determine what root of (4.13) is also solution of (4.9) and then to construct the quartic algebraic equation to see if all its roots are simple. If all its roots are not simple, i.e., multiple roots appear, then at most three roots will be solutions of (4.9). In this degenerate case, we do not have a similar situation to the one described in Section 4.8.2. A multiple root does not imply, in general, the existence of a multiple Darboux factor of the form $\mathcal{F} = \mathcal{Y} - \mathcal{G}(x)$, when $\mathcal{Y} = \mathcal{G}(x)$ is a solution of (4.9). If there is only one root of (4.13) which is solution of (4.9) then the four simple roots of the quartic algebraic equation are the unique coplanar particular solutions.

Proof. Necessity: Consider that the resolvent Abel equation (4.9) has four different particular solutions $\mathcal{Y} = \mathcal{G}_1(x)$, $\mathcal{Y} = \mathcal{G}_2(x)$, $\mathcal{Y} = \mathcal{G}_3(x)$ and $\mathcal{Y} = \mathcal{G}_4(x)$, and there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{C} - \{0\}$, such that

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0, \quad \text{and} \quad \alpha_1\mathcal{G}_1 + \alpha_2\mathcal{G}_2 + \alpha_3\mathcal{G}_3 + \alpha_4\mathcal{G}_4 = 0.$$

Following the same reasonings as in the case $n = 3$ we have

$$\begin{array}{rcl} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &=& 0, \\ \alpha_1\mathcal{G}_1 + \alpha_2\mathcal{G}_2 + \alpha_3\mathcal{G}_3 + \alpha_4\mathcal{G}_4 &=& 0, \\ \alpha_1\mathcal{G}_1^3 + \alpha_2\mathcal{G}_2^3 + \alpha_3\mathcal{G}_3^3 + \alpha_4\mathcal{G}_4^3 &=& 0, \\ \alpha_1(\mathcal{G}_1^5 + \mathcal{G}_1^2) + \alpha_2(\mathcal{G}_2^5 + \mathcal{G}_2^2) + \alpha_3(\mathcal{G}_3^5 + \mathcal{G}_3^2) + \alpha_4(\mathcal{G}_4^5 + \mathcal{G}_4^2) &=& 0. \end{array}$$

To have a non trivial solution it must happen that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \mathcal{G}_1 & \mathcal{G}_2 & \mathcal{G}_3 & \mathcal{G}_4 \\ \mathcal{G}_1^3 & \mathcal{G}_2^3 & \mathcal{G}_3^3 & \mathcal{G}_4^3 \\ \mathcal{G}_1^5 + \mathcal{G}_1^2 & \mathcal{G}_2^5 + \mathcal{G}_2^2 & \mathcal{G}_3^5 + \mathcal{G}_3^2 & \mathcal{G}_4^5 + \mathcal{G}_4^2 \end{vmatrix} = 0.$$

which implies, eliminating the 6 factors of the form $\mathcal{G}_i - \mathcal{G}_j$, that $-1 + \mathcal{G}_1^2\mathcal{G}_2 + \mathcal{G}_1\mathcal{G}_2^2 + \mathcal{G}_1^2\mathcal{G}_3 + 2\mathcal{G}_1\mathcal{G}_2\mathcal{G}_3 + \mathcal{G}_2^2\mathcal{G}_3 + \mathcal{G}_1\mathcal{G}_3^2 + \mathcal{G}_2\mathcal{G}_3^2 + \mathcal{G}_1^2\mathcal{G}_4 + 2\mathcal{G}_1\mathcal{G}_2\mathcal{G}_4 + \mathcal{G}_2^2\mathcal{G}_4 + 2\mathcal{G}_1\mathcal{G}_3\mathcal{G}_4 + 2\mathcal{G}_2\mathcal{G}_3\mathcal{G}_4 + \mathcal{G}_3^2\mathcal{G}_4 + \mathcal{G}_1\mathcal{G}_4^2 + \mathcal{G}_2\mathcal{G}_4^2 + \mathcal{G}_3\mathcal{G}_4^2 = 0$. In terms of the elementary symmetric polynomials defined previously, we get $\mathcal{S}_1\mathcal{S}_2 - \mathcal{S}_3 - 1 = 0$. The derivative of this expression is $\mathcal{S}_1'\mathcal{S}_2 + \mathcal{S}_1\mathcal{S}_2' - \mathcal{S}_3' = 0$. If we compute $\mathcal{S}_1'$, $\mathcal{S}_2'$, $\mathcal{S}_3'$ i $\mathcal{S}_4'$ we obtain

$$\begin{array}{l} \mathcal{S}_1' = J(\mathcal{S}_1^3 - 3\mathcal{S}_1\mathcal{S}_2 + 3\mathcal{S}_3 - K\mathcal{S}_1 + 4), \\ \mathcal{S}_2' = J(\mathcal{S}_1^2\mathcal{S}_2 - 2\mathcal{S}_2^2 - \mathcal{S}_1\mathcal{S}_3 + 4\mathcal{S}_4 - 2K\mathcal{S}_2 + 3\mathcal{S}_1), \\ \mathcal{S}_3' = J(\mathcal{S}_1^2\mathcal{S}_3 - 2\mathcal{S}_2\mathcal{S}_3 - \mathcal{S}_1\mathcal{S}_4 - 3K\mathcal{S}_3 + 2\mathcal{S}_2), \\ \mathcal{S}_4' = J(\mathcal{S}_1^2\mathcal{S}_4 - 2\mathcal{S}_2\mathcal{S}_4 - 4K\mathcal{S}_4 + \mathcal{S}_3). \end{array} \tag{4.15}$$

Using that $\mathcal{S}_3 = \mathcal{S}_1\mathcal{S}_2 - 1$ the first two equations of (4.15) are

$$\begin{aligned} \mathcal{S}_1' &= J(\mathcal{S}_1^3 - K\mathcal{S}_1 + 1), \\ \mathcal{S}_2' &= J(4\mathcal{S}_4 - 2\mathcal{S}_2^2 - 2K\mathcal{S}_2 + 4\mathcal{S}_1). \end{aligned} \tag{4.16}$$

Notice that the sum of the four coplanar solutions $\mathcal{S}_1 = \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 + \mathcal{G}_4$ of the resolvent Abel equation is also a particular solution. This condition is also true in the context of the reduced Abel equation, i.e., $Y = S_1 = G_1 + G_2 + G_3 + G_4$ is also a particular solution of (4.8), and in the original Abel equation the condition has the form $y = g_1 + g_2 + g_3 + g_4 + b/a = s_1 + b/a$, which is also particular solution of (4.1). Another way to obtain (4.15) is to think that if $\mathcal{Y} = \mathcal{G}_i(x)$, are particular solutions of (4.9) then the quartic algebraic equation $\mathcal{U} = \mathcal{Y}^4 - \mathcal{S}_1\mathcal{Y}^3 + \mathcal{S}_2\mathcal{Y}^2 - \mathcal{S}_3\mathcal{Y} + \mathcal{S}_4 = 0$ is an invariant algebraic curve of (4.9). Consequently, the Euclidean division of $\mathcal{DU}$ and $\mathcal{U}$ must have a null polynomial remainder. This polynomial remainder is a polynomial whose coefficients are exactly the equations described in (4.15).

Substituting the derivatives $\mathcal{S}_1'$, $\mathcal{S}_2'$, $\mathcal{S}_3'$ and $\mathcal{S}_4'$ in $\mathcal{S}_1'\mathcal{S}_2 + \mathcal{S}_1\mathcal{S}_2' - \mathcal{S}_3' = 0$, eliminating the factor J which is not null because $q(x) \not\equiv 0$, and using $\mathcal{S}_3 = \mathcal{S}_1\mathcal{S}_2 - 1$, we get $5\mathcal{S}_1^2 - 3\mathcal{S}_2 + 5\mathcal{S}_1\mathcal{S}_4 - 3K = 0$. The derivative of this last expression is $10\mathcal{S}_1\mathcal{S}_1' - 3\mathcal{S}_2' + 5\mathcal{S}_1'\mathcal{S}_4 + 5\mathcal{S}_1\mathcal{S}_4' - 3K' = 0$, dividing by J, using the computed derivatives, the conditions $\mathcal{S}_3 = \mathcal{S}_1\mathcal{S}_2 - 1$ and $\mathcal{S}_2 = (5\mathcal{S}_1(\mathcal{S}_1 + \mathcal{S}_4))/3 - K$, we get $(-7 - 25K\mathcal{S}_1 + 35\mathcal{S}_1^3)(\mathcal{S}_1 + \mathcal{S}_4) - 3L = 0$. Taking into account that $K' \not\equiv 0$, which implies $L \not\equiv 0$, we conclude that the factor $-7 - 25K\mathcal{S}_1 + 35\mathcal{S}_1^3 \not\equiv 0$ and we can isolate $\mathcal{S}_4$, which takes the form $\mathcal{S}_4 = 3L/(-7 - 25K\mathcal{S}_1 + 35\mathcal{S}_1^3) - \mathcal{S}_1$.

Derivating $(-7-25K\mathcal{S}_1+35\mathcal{S}_1^3)(\mathcal{S}_1+\mathcal{S}_4)-3L = 0$, we obtain $(-25K'\mathcal{S}_1 - 25K\mathcal{S}_1' + 105\mathcal{S}_1^2\mathcal{S}_1')(\mathcal{S}_1+\mathcal{S}_4) + (-7-25K\mathcal{S}_1+35\mathcal{S}_1^3)(\mathcal{S}_1'+\mathcal{S}_4') - 3L' = 0$, and doing the opportune substitutions we arrive to $315JL\mathcal{S}_1^5 - 35(10JKL + L')\mathcal{S}_1^3 + 63JL\mathcal{S}_1^2 + 5(15JK^2L - 5K'L - 2JL^2 + 5KL')\mathcal{S}_1 + 7L' - 11JKL = 0$ which dividing by $J\,L$ is $\mathcal{V} = 0$ or condition (4.13). This proves that $\mathcal{S}_1$ is a root of this quintic algebraic equation (4.13). Moreover, due to the construction of the quartic algebraic equation with four different solutions, obviously it cannot have multiple roots. This condition (4.13), in general cannot be solvable by radicals, unless the Galois group is solvable. Hence, we have transformed the differential problem of finding solutions for an Abel differential equation into an algebraic problem.

Sufficiency: Consider now that when we solve equation (4.13), we obtain a root which is also a particular solution of (4.9), we construct the quartic algebraic equation $\mathcal{U} = \mathcal{Y}^4 - \mathcal{S}_1\mathcal{Y}^3 + \mathcal{S}_2\mathcal{Y}^2 - \mathcal{S}_3\mathcal{Y} + \mathcal{S}_4 = 0$, where $\mathcal{S}_1$ is the root of (4.13), and where $\mathcal{S}_i$ are the ones enunciated in the theorem. First, we want to prove that this quartic algebraic equation is an invariant algebraic curve. Hence, we compute $\mathcal{DU}$, using that $K' = JL$ and $L' = JLM$, and also that $\mathcal{S}_1' = J(\mathcal{S}_1^3 - K\mathcal{S}_1 + 1)$. If we compute the polynomial remainder of the Euclidean division between $\mathcal{DU}$ and $\mathcal{U}$ with respect to $\mathcal{Y}$ we will obtain

$$\frac{JL(-5\mathcal{S}_1\mathcal{Y}^2 + 5\mathcal{S}_1^2\mathcal{Y} - 3)(315\mathcal{S}_1^5 - 35(10K + M)\mathcal{S}_1^3 + 63\mathcal{S}_1^2 + 5(15K^2 - 7L + 5KM)\mathcal{S}_1 + 7M - 11K)}{(-7 - 25K\mathcal{S}_1 + 35\mathcal{S}_1^3)^2},$$

which is obviously null because $\mathcal{S}_1$ is a root of (4.13). Therefore the quartic algebraic equation has four roots $\mathcal{Y} = \mathcal{G}_1(x)$, $\mathcal{Y} = \mathcal{G}_2(x)$, $\mathcal{Y} = \mathcal{G}_3(x)$ and $\mathcal{Y} = \mathcal{G}_4(x)$ which are also solutions of the resolvent Abel equation (4.9). By construction the elementary symmetric polynomials associated to the four roots are $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ and $\mathcal{S}_4$ the coefficients of the quartic equation. If these four roots are coplanar, there should exist $\alpha_i \in \mathbb{C} - \{0\}$ such that

$$\begin{array}{rcl} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 & = & 0, \\ \alpha_1\mathcal{G}_1 + \alpha_2\mathcal{G}_2 + \alpha_3\mathcal{G}_3 + \alpha_4\mathcal{G}_4 & = & 0, \\ \alpha_1\mathcal{G}_1^3 + \alpha_2\mathcal{G}_2^3 + \alpha_3\mathcal{G}_3^3 + \alpha_4\mathcal{G}_4^3 & = & 0, \\ \alpha_1(\mathcal{G}_1^5 + \mathcal{G}_1^2) + \alpha_2(\mathcal{G}_2^5 + \mathcal{G}_2^2) + \alpha_3(\mathcal{G}_3^5 + \mathcal{G}_3^2) + \alpha_4(\mathcal{G}_4^5 + \mathcal{G}_4^2) & = & 0. \end{array} \quad (4.17)$$

Moreover, if this system has a non trivial solution then $\mathcal{S}_1\mathcal{S}_2 - \mathcal{S}_3 - 1 = 0$ which is verified by hypothesis. Notice also that the minor of the form

$$\left| \begin{array}{ccc} 1 & 1 & 1 \\ \mathcal{G}_1 & \mathcal{G}_2 & \mathcal{G}_3 \\ \mathcal{G}_1^3 & \mathcal{G}_2^3 & \mathcal{G}_3^3 \end{array} \right|,$$

is not null, because in the opposite case we will have $\mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 = 0$ and we will remain into a case with $n = 3$ which implies $K'(x) \equiv 0$ in contradiction with the hypothesis. Taking into account that the determinant of the homogeneous system (4.17) is null, we can consider the last equation is a linear combination of the rest. Hence, we must find a vector $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ orthogonal to the vectors $(1, 1, 1, 1)$, $(\mathcal{G}_1, \mathcal{G}_1, \mathcal{G}_3, \mathcal{G}_4)$, $(\mathcal{G}_1^3, \mathcal{G}_1^3, \mathcal{G}_3^3, \mathcal{G}_4^3)$. This vector must be of the form $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \lambda(x)(A_1(x), A_2(x), A_3(x), A_4(x))$, where $A_1(x), A_2(x), A_3(x), A_4(x)$ are these not null determinants

$$A_1 = \left| \begin{array}{ccc} 1 & 1 & 1 \\ \mathcal{G}_2 & \mathcal{G}_3 & \mathcal{G}_4 \\ \mathcal{G}_2^3 & \mathcal{G}_3^3 & \mathcal{G}_4^3 \end{array} \right|, \quad A_2 = - \left| \begin{array}{ccc} 1 & 1 & 1 \\ \mathcal{G}_1 & \mathcal{G}_3 & \mathcal{G}_4 \\ \mathcal{G}_1^3 & \mathcal{G}_3^3 & \mathcal{G}_4^3 \end{array} \right|,$$

$$A_3 = \begin{vmatrix} 1 & 1 & 1 \\ \mathcal{G}_1 & \mathcal{G}_2 & \mathcal{G}_4 \\ \mathcal{G}_1^3 & \mathcal{G}_2^3 & \mathcal{G}_4^3 \end{vmatrix}, \quad A_4 = - \begin{vmatrix} 1 & 1 & 1 \\ \mathcal{G}_2 & \mathcal{G}_2 & \mathcal{G}_3 \\ \mathcal{G}_1^3 & \mathcal{G}_2^3 & \mathcal{G}_3^3 \end{vmatrix}.$$

Since $A_1+A_2+A_3+A_4=0$ and $A_1\mathcal{G}_1+A_2\mathcal{G}_2+A_3\mathcal{G}_3+A_4\mathcal{G}_4=0$, to guarantee the coplanarity of the roots we only need to prove that λ is not null and that α_i are constants. Taking into account that

$$\frac{d}{dx}(\mathcal{G}_i - \mathcal{G}_j) = J(\mathcal{G}_i^2 + \mathcal{G}_i\mathcal{G}_j + \mathcal{G}_j^2 - K)(\mathcal{G}_i - \mathcal{G}_j),$$

we have that

$$\frac{dA_i}{dx} = J(3(\mathcal{S}_1^2 - \mathcal{S}_2) - 4K)A_i, \quad \text{for} \quad i = 1,2,3,4.$$

Since $\alpha_i = \lambda A_i$, if we want to prove that $d\alpha_i/dx = 0$, it should happen that $\lambda' A_i + \lambda A_i' = 0$, but this fact is guaranteed taking

$$\lambda = \lambda_0 e^{-\int J(3(\mathcal{S}_1^2 - \mathcal{S}_2) - 4K)dx},$$

where $\lambda_0 \in \mathbb{C} - \{0\}$ is an arbitrary constant, which completes the proof of the sufficiency of the theorem. ∎

In this case it is possible that more than one root of (4.13) is a particular solution of (4.9), and also to have more than one quartic algebraic equation which is an invariant algebraic curve of (4.9). We recall that to see if a root of (4.13) is a particular solution of (4.9) it is necessary and sufficient that the algebraic resultant (4.14) to be null. If we define $\mathcal{W} = J^{-1}\mathcal{D}\mathcal{V} = 1575\mathcal{Y}^7 - 105(25K+M)\mathcal{Y}^5 + 1701\mathcal{Y}^4 + 5(225K^2 - 77L + 26KM - 7N)\mathcal{Y}^3 - 21(56K+5M)\mathcal{Y}^2 + (126 - 75K^3 + 185KL - 25K^2M - 10LM + 25KN)\mathcal{Y} + 75K^2 - 46L + 25KM + 7N$. If the algebraic resultant (4.14) is null this implies that the polynomials $\mathcal{V}$ and $\mathcal{W}$ with respect to $\mathcal{Y}$ have common factors. The first one is of fifth degree and the second of seventh degree. If we apply the Euclidean division algorithm we obtain a remainder of degree at most 4. In particular, the remainder is of degree 4 and the coefficient of $\mathcal{Y}^4$ is 12474. Hence, this remainder never vanishes and consequently the common divisor is never (4.13). Therefore, (4.13) is never an invariant algebraic curve of (4.9), which implies that 5 roots cannot be particular solutions of the resolvent Abel equation (4.9).

If we continue applying the Euclidean division the next remainder is of third degree. It is straightforward to see, using algebraic resultants between the coefficients, that the vanishing of this remainder implies $K' = 0$ in contradiction with the hypothesis of Theorem 4.6 (in fact, we would be in a case with $n = 3$). Hence, the previous remainder of fourth degree cannot be the common divisor and the common divisor will be this last remainder of degree 3 or the following remainders of lower degree. The conclusion is that condition (4.13) can have at most three roots which are also particular solutions of (4.9). It is necessary to point out that $\mathcal{S}_1$ will be also a root of the mentioned remainder of degree at most 3 whose coefficients are algebraic expressions in K, L, M and N. Hence, $\mathcal{S}_1$ will be, in general, also algebraic in K, L, M and N. Following the classical Galois theory, we can think with the extension $\mathbb{Q}[K, L, M]$, which is the field of the coefficients of (4.13), where $\mathbb{Q}$ is the field of rational numbers, and $\mathbb{Q}[K, L, M]$ is the field obtained from $\mathbb{Q}$ by adjoining K, L and M. Moreover, the extension of the previous field $\mathbb{Q}[K, L, M, N]$ contains the roots of (4.13) which are also solutions of (4.9) if the common divisor is of first degree. If the common divisor is of degree 2 or 3 we should use an algebraic extension of $\mathbb{Q}[K, L, M, N]$.

Condition (4.13) expressed in terms of the coefficients of the reduced Abel equation (4.8), doing the change $\mathcal{Y} = Yq^{-\frac{1}{3}}(x)$, is

$$\begin{aligned} V = {} & (2835p_1q^2 - 1890pqq_1 + 1575q_1^2 - 945qq_2)Y^5 + (3150pp_1q^2 - 315p_2q^2 \\ & -2100p^2qq_1 - 420p_1qq_1 + 1960pq_1^2 - 840pqq_2 - 245q_1q_2 + 105qq_3)Y^3 \\ & +(567p_1q^3 - 378pq^2q_1 + 315qq_1^2 - 189q^2q_2)Y^2 + (675p^2p_1q^2 + 315p_1^2q^2 \\ & -225pp_2q^2 - 450p^3qq_1 - 420pp_1qq_1 + 75p_2qq_1 + 465p^2q_1^2 + 275p_1q_1^2 \\ & -75p^2qq_2 - 210p_1qq_2 - 185pq_1q_2 + 35q_2^2 + 75pqq_3 - 25q_1q_3)Y + 99pp_1q^3 \\ & +63p_2q^3 - 66p^2q^2q_1 - 159p_1q^2q_1 + 175pqq_1^2 - 135q_1^3 - 75pq^2q_2 \\ & +130qq_1q_2 - 21q^2q_3 = 0 \end{aligned} \tag{4.18}$$

In the same way that for the resolvent Abel equation (4.9), computing the resultant with respect to Y between this quintic polynomial V and its derivative $\hat{D}V$, we can check if there is a root of (4.18) which is also a particular solution of (4.8). If this root r_0 exists we can construct the quartic equation $Y^4 - S_1Y^3 + S_2Y^2 - S_3Y + S_4 = 0$, where $S_1 = r_0$ and

$$\begin{aligned} S_4 &= \frac{105q^2S_1^4 + 25(3pq^2 - qq_1)S_1^2 - 21q^3S_1 + 9p_1q^2 - 6pqq_1 + 5q_1^2 - 3qq_2}{-105qS_1^3 + 25(q_1 - 3pq)S_1 + 21q^2}, \\ S_2 &= \frac{5S_1S_4 + 5qS_1^2 + 3pq - q_1}{3q}, \\ S_3 &= S_1S_2 - q, \end{aligned} \tag{4.19}$$

where the denominators are different of zero because we are in the case $q \neq 0$ and $K' \neq 0$. We recall that $S_i = q^{\frac{i}{3}}(x)\mathcal{S}_i$. Once the quartic algebraic equation is built, if it has not multiple roots, we know that its simple roots are coplanar. Let G_1, G_2, G_3 and G_4 be these roots, the coplanarity coefficients are

$$\alpha_i = (-1)^i q^{-\frac{4}{3}} \left(\prod_{1\leq j<k\leq 4,\ j,k\neq i} (G_j - G_k) \right) (S_1 - G_i) e^{-\int a(3(S_1^2 - S_2)+4p-\frac{4q'}{3aq})dx}, \tag{4.20}$$

for $i = 1, 2, 3, 4$. These 4 coefficients α_i and the 4 particular solutions $G_i(x)$ allow us to construct the first integral (4.4) taking into account what we have seen in Subsection 4.4. Another way to compute the α_i from the knowledge of the $G_i(x)$ is by using the homogeneous linear system with respect to the α_i

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= 0, \\ G_1\alpha_1 + G_2\alpha_2 + G_3\alpha_3 + G_4\alpha_4 &= 0, \\ G_1^3\alpha_1 + G_2^3\alpha_2 + G_3^3\alpha_3 + G_4^3\alpha_4 &= 0, \end{aligned} \tag{4.21}$$

where the $G_i(x)$ can be evaluated in an arbitrary value of x for which they are simultaneously defined. When particularizing the value of x we must look for a value such that the rank of the homogeneous linear system (4.21) is 3, because in an opposite case we would be in the case $K' = 0$.

Finally, if we know a quartic invariant algebraic curve of a reduced Abel equation, we can consider the coefficient of the cubic term divided by the coefficient of the quartic term and changed of sign. If this function (located, in principle, in the resolvent or in the reduced Abel equation or adapted as it has already been seen if we are in the original Abel equation) is a particular solution of the Abel equation and is a root of the quintic algebraic condition ((4.13) if we are working in the resolvent Abel equation or (4.18) if we are working in the reduced Abel equation) and the quartic algebraic curve has not multiple roots, then we are in the case described by Theorem 4.6. In this case we can construct the coplanar coefficients and the first integral.

For the illustrative example given the Subsection 4.6, none of the two algebraic invariant curves satisfies these conditions and consequently this example does not fall in this case.

4.10 Abel equations with more than four coplanar particular solutions

It seems natural the extension of the method of the previous section to Abel equations with more than four coplanar particular solutions, i.e. for $n > 4$. First, we must construct the system linear in the coplanarity coefficients (like system (4.17) for $n = 4$) that allows us to find the first integral by successive derivation. Later, we must demand that this system has null determinant. This condition is symmetric with respect to the candidates coplanar particular solutions and therefore expressible in terms of elementary symmetric polynomials of these coplanar particular solutions.

Derivating this condition, taking into account the derivatives of the elementary symmetric polynomials, we arrive to n equations in function of the elementary symmetric polynomials. Eliminating each symmetric polynomial until arriving to a unique condition (as condition (4.13) for $n = 4$) with a supplementary condition (in the case $n = 3$ that the first elementary symmetric polynomial ($\mathcal{S}_1$) is null or in the case $n = 4$ that the first elementary symmetric polynomial ($\mathcal{S}_1$) is a particular solution of the resolvent Abel equation). These two conditions allows us to establish when we have n different coplanar particular solutions. We recall here the illustrative example given in Subsection 4.6 which is a case with $n = 9$. They will also appear degenerate cases that correspond to the case that the equation of degree n

$$\mathcal{Y}^n + \sum_{i=1}^{n} (-1)^i \ \mathcal{S}_i \ \mathcal{Y}^{n-i} = 0, \tag{4.22}$$

constructed from the elementary symmetric polynomials, which is an invariant algebraic curve of the resolvent Abel equation, has multiple roots (as the cubic equation for $n = 3$ and the quartic equation for $n = 4$). Obviously, this technique to approach this differential problem has a great analogy with the algebraic problems solved with the classical Galois theory. Hence, the method developed in this work constitute a new approach different from the non linear Galois differential theory recently developed by Malgrange, see [96, 97].

4.11 The Chini Equation

In [36], Chini studied the equations of the form

$$\frac{dY}{dx} = a(x)(Y^m + p(x)Y + q(x)). \tag{4.23}$$

For $m = 3$ we have the reduced Abel equation. Doing the change $Y = q^{1/m}\mathcal{Y}$, we have $d\mathcal{Y}/dx = J(\mathcal{Y}^m - K\mathcal{Y} + 1)$. If $Y = G(x)$ is a particular solution of equation (4.23) then $D_m(Y - G) = (a(x)(\sum_{i=0}^{m-1} Y^{m-1-i}G^i + p(x))(Y - G)$, where $D_m := \partial/\partial x + a(x)(Y^m + p(x)Y + q(x))$. Imposing to have a first integral of the form (4.4), we obtain $m - 1$ conditions

$$\sum_{i=1}^{n} \alpha_i = 0, \quad \sum_{i=1}^{n} \alpha_i g_i = 0, \quad \sum_{i=1}^{n} \alpha_i g_i^2 = 0, \quad \ldots, \quad \sum_{i=1}^{n} \alpha_i g_i^{m-2} = 0. \tag{4.24}$$

For $K' = 0$ (except for some values of K which correspond to $\operatorname{res}(\mathcal{Y}^m - K\mathcal{Y} + 1, m\mathcal{Y}^{m-1} - K, \mathcal{Y}) = 0$), we have that the equation $\mathcal{Y}^m - K\mathcal{Y} + 1 = 0$ has m constant roots, and therefore we will have m different particular solutions which satisfy (4.24). This implies that the minimum number of coplanar solutions must be m and perhaps it is also the maximum. For instance, for $m = 4$, it is easy to see that it is not possible to have $n = 5$ solutions satisfying (4.24) without implying that they contain a subset of four solutions $n = 4$ satisfying (4.24). Therefore, a straightforward generalization of our method for Abel equations can be done for the Chini equations.

4.12 Concluding remarks

We have reinterpreted the known cases of solvable Abel equations which appeared in Kamke's book and other works (see [32, 75, 86]) reducing all the cases to a unique case from a geometric point of view. Using this new approach we obtain new cases of solvable Abel equations.

This unique case correspond for $n = 3$ to have three *collinear* solutions in the affine space of solutions of the Abel equation which is characterized in Theorem 4.5. We have also solved the degenerate case when two of these three solutions coincide, see Subsection 4.8.2. We have characterize the case $n = 4$ i.e., the case when we have four *coplanar* solutions in the affine space of solutions of the Abel equation, see Theorem 4.6. The degenerate case for $n = 4$, which is the case when the quartic algebraic equation (4.22) has not all its roots simple, is still open and may be an achievable objective for a future investigation. Obviously, we can also consider the case when we have more than four *coplanar* solutions, i.e. the case $n > 4$, which remains open its full characterization. In characterizing the cases with $n > 4$ can receive the same limitations as in the classical Galois theory. A possible continuation of this line of research is to find similar characterizations for higher-degree equations such as the Chini's equations.

4.13 Examples

It is straightforward to construct examples in the context of Theorem 4.6 because given four different arbitrary functions $y = g_1(x)$, $y = g_2(x)$, $y = g_3(x)$ and $y = g_4(x)$, with the unique condition of being coplanar, we can solve the following linear system with respect to functions $a(x)$, $b(x)$, $c(x)$ and $d(x)$

$$\begin{aligned} g_1^3 a + g_1^2 b + g_1 c + d &= g_1', \\ g_2^3 a + g_2^2 b + g_2 c + d &= g_2', \\ g_3^3 a + g_3^2 b + g_3 c + d &= g_3', \\ g_4^3 a + g_4^2 b + g_4 c + d &= g_4'. \end{aligned}$$

which has a unique solution. The associated Abel equation will be an example of Theorem 4.6 except if $q = 0$ or $K' = 0$ and it can also happen that the associated quartic has not simple roots. The examples are described working with the reduced Abel equation to avoid fractional exponents.

Example 1. Consider the Abel equation

$$\frac{dy}{dx} = a(x)y^3 + b(x)y^2, \tag{4.25}$$

with $(a/b)' = \gamma b$, and $\gamma \in \mathbb{C} - \{0\}$, equivalent to have $b(x) = B'(x)$ and $a(x) = \gamma B(x)B'(x)$ with $B(x)$ an non constant arbitrary function. This Abel equation is studied in [86], page 26, as sub-method (f). Through the change of dependent variable $y = Y - b(x)/(3a(x)) = Y - 1/(3\gamma B(x))$, we obtain the reduced Abel equation (4.8) with $p(x) = -1/(3\gamma^2 B(x)^2)$ and $q(x) =$

$(2-9\gamma)/(27\gamma^3 B(x)^3)$. For $\gamma = 2/9$ is a Bernoulli equation, hence from now on we consider $\gamma \neq 2/9$. If we compute K we get $K = (3(1-3\gamma))/(9\gamma-2)^{\frac{2}{3}}$ which is a constant ($K' = 0$) and we are in the conditions of Theorem 4.5 if $K^3 \neq 27/4$. For simplicity we consider the case $\gamma = 1/3$ and $B(x) = x$ which has $K = 0$ and $q = -1/x^3$. The resolution of the cubic algebraic equation gives $k_1 = -\omega_0 = -1$, $k_2 = -\omega_1$ and $k_3 = -\omega_{-1}$, where ω_j are the cubic roots the unity, see Section 4.8.2. From Section 4.8.1, a first integral for the reduced Abel equation is

$$\begin{aligned} H(x,Y) &= \frac{x^2(Y-\frac{\omega_1}{x})(Y-\frac{\omega_{-1}}{x})}{(Y-\frac{1}{x})^2}\left(\frac{Y-\frac{\omega_{-1}}{x}}{Y-\frac{\omega_1}{x}}\right)^{\sqrt{3}i} \\ &= \frac{x^2(Y^2+\frac{1}{x}Y+\frac{1}{x^2})}{(Y-\frac{1}{x})^2}\left(\frac{Y+\frac{1}{2x}+\frac{\sqrt{3}i}{2x}}{Y+\frac{1}{2x}-\frac{\sqrt{3}i}{2x}}\right)^{\sqrt{3}i}. \end{aligned}$$

Going back to the original Abel equation through the inverse change $Y = y + 1/x$, the first integral of the Abel equation (4.25) with $\gamma = 1/3$ and $B(x) = x$ is

$$\begin{aligned} h(x,y) &= \tfrac{x^2y^2+3xy+3}{y^2}\left(\tfrac{2xy+3+\sqrt{3}i}{2xy+3-\sqrt{3}i}\right)^{\sqrt{3}i} \\ &= \tfrac{x^2y^2+3xy+3}{y^2}\, e^{-2\sqrt{3}\arctan\left(\frac{\sqrt{3}}{2xy+3}\right)}. \end{aligned}$$

The case $K^3 = 27/4$, which implies in this case $\gamma = 1/4$, is a degenerate case studied in Section 4.8.2.

In [57], the Abel equation (4.25) where $a(x)$ and $b(x)$ are polynomials in x is considered. For two given points a and b in $\mathbb{C}$, the "Poincaré mapping" of the above equation transforms the values of its solutions at a into their values at b. In [57], the global analytic properties of the Poincaré mapping are studied, in particular, its analytic continuation, its singularities and its fixed points (which correspond to the periodic solutions such that $y(a) = y(b)$). In Section 5 and 6 of [57] the equation $dy/dx = \gamma xy^3 + y^2$ is studied as a local model of the previous Abel equation near a simple fixed singularity, and for the discriminant value $\gamma = 1/4$ it is proved the existence of an infinite number of periodic solutions. This particular case is easier investigate than that of generic γ.

Example 2. We construct an example using the introduction of this section. We can take $g_1(x) = x-2$, $g_2(x) = x$, $g_3(x) = 1$ and $g_4(x) = 0$, which are coplanar particular solutions taking $\alpha_1 = 1$, $\alpha_2 = -1$ $\alpha_3 = 2$ and $\alpha_4 = -2$. The Abel equation turns out to be

$$\frac{dy}{dx} = \frac{(-2x+3)y^3 + (3x^2 - 6x + 3)y^2 + (-3x^2 + 8x - 6)y}{x(x-1)(x-2)(x-3)}, \qquad (4.26)$$

and taking into account what we have seen in Subsection 4.4 and 4.4.1, a first integral for the Abel equation (4.26) is

$$h(x, y) = \frac{x^3(x-2)(y-1)^2(y-x+2)}{(x-3)^3(x-1)y^2(y-x)}.$$

Now we are going to check how our method works for the Abel equation (4.26). Computing $p(x)$, $q(x)$ following Subsection 4.7.1 and also $p_i(x)$ for $i = 1, 2$ and $q_i(x)$ for $i = 1, 2, 3$, and substituting in (4.18), eliminating a common factor in x, we get the quintic equation $V = 0$ with $V = ((2x - 3)Y + 1)((2x-3)Y + x^2 - 4x + 4)((2x-3)Y + x^2 - 2x + 1)((4x^4 - 24x^3 + 57x^2 - 63x + 27)Y^2 + (-4x^5 + 30x^4 - 96x^3 + 162x^2 - 144x + 54)Y + x^6 - 9x^5 + 35x^4 - 75x^3 + 95x^2 - 69x + 23)$. The resultant with respect to Y between this quintic polynomial V and its derivative $\hat{D}V$ is null which implies that there is a root of (4.18) which is also a particular solution of (4.26). The polynomial remainder of the Euclidean division between $\hat{D}V$ and V with respect to Y is the product of the first linear factors of V. In this case we have that the maximum common divisor is the highest possible. Hence, we have 3 roots of the quintic polynomial which are also particular solution of (4.26). These roots are $r_0 = Y = -1/(2x-3)$, $r_1 = Y = -(x^2 - 4x + 4)/(2x - 3)$ and $r_2 = Y = -(x^2 - 2x + 1)/(2x - 3)$. For each particular solution we can construct the quartic invariant algebraic $Y^4 - S_1Y^3 + S_2Y^2 - S_3Y + S_4 = 0$, where S_1 is the root and S_2, S_3 and S_4 are given in (4.19). Taking the first root $S_1 = r_0$, the roots of this quartic polynomial are $G_1(x) = (x^2 - 5x + 5)/(2x-3)$, $G_2(x) = (x^2 - x - 1)/(2x - 3)$, $G_3(x) = -(x^2 - 4x + 4)/(2x - 3)$ and $G_4(x) = -(x^2 - 2x + 1)/(2x - 3)$. Through the inverse change $Y = y - (x^2 - 2x + 1)/(2x + 3)$ (to pass from the reduced Abel equation to the original Abel equation (4.26)) we obtain the four coplanar particular solutions $g_1(x) = x - 2$, $g_2(x) = x$, $g_3(x) = 1$ and $g_4(x) = 0$. Moreover, $r_1 = G_3$ and $r_2 = G_4$. Using r_1 or r_2 to construct the quartic invariant algebraic curve we arrive to similar results. To determine the coplanar coefficients we can use (4.20) or the homogeneous linear system (4.21) evaluating the $G_i(x)$ in an arbitrary value of x for which they are defined simultaneously. In this case we get $G_1 - G_2 + 2G_3 - 2G_4 = 0$ and using the Subsection 4.4 and 4.4.1, we obtain the first integral of the Abel equation (4.26).

Chapter 5

Essential variables in the integrability problem of planar vector fields

5.1 Introduction

Differential equations appear naturally in the description of many phenomena of nature. Once the local equations are formulated in a particular context, the next problem is to solve these equations. The first attempt to solve differential equations either explicitly or by series expansions goes back to Euler, Newton and Leibniz and the subsequent works of Lagrange, Poisson, Hamilton and Liouville. The notion of *integrability* was introduced to describe the property of equations for which all local and global information can be obtained either from the solutions or implicitly from the constants of motion. The local analysis of the differential equations, close to its complex time singularities by Kovlevskaya and Painlevé and its phase space singularities by Poincaré allows us to find global properties of the differential systems and the research has shifted its interest away from the theory of integrability. In fact, the innovative idea of Painlevé was to generalize the notion of small parameters in perturbation theory by introducing an artificial parameter α in the equation in such away that, if the equation is single-valued for all $\alpha \neq 0$, then it is also single-valued for $\alpha = 0$ (the Painlevé α-method). The main difficulty is to choose a good parameter α such that the equation for $\alpha = 0$ can be immediately integrated (find its solution). This can be achieved, for instance, by using a scaling symmetry. The exact solution of the equation for $\alpha = 0$ provides a starting point in a perturbative expansion of the solution in powers of α. This idea is applied in this chapter but in a different context to find the perturbative expansion or the singular series expansion of the first integral (not necessarily perturbative) of a planar differential system. In [83] it was studied the connection between the existence of first integrals (invariants) and the Painlevé property in two-dimensional Lotka-Volterra and quadratic systems. The conclusion was that the Painlevé property is a too strong condition for the existence of the first integrals (invariants).

The success of dynamical systems theory was so overwhelming that exacts methods of integration were considered for years useless and non-generic. However last years, solitons, pattern formation and ordered structures are the key features of systems with infinite degrees of freedom and have show how crucial is the understanding of the phenomena of integrability and non-integrability in dynamical systems. For dynamical systems a universal definition of integrability seems elusive. Integrability in dynamical systems has different meanings for different contexts, see [79]. In this chapter we focus our attention in the the integrability problem of planar nonlinear differential equations and in this context the notion of integrability is based on knowledge of a first integral which can be represented by the combination of known functions or by its series expansion.

In [27] it was proved that every local flow on a two-dimensional manifold M has continuous first integral on every canonical region. Moreover in [90] it was improved this result establishing that every C^r local flow on a two-dimensional manifold M has a C^r (respectively C^∞, C^w) first integral for $r \in \mathbb{N}$ (respectively $r = \infty$, ω) on every canonical region. Hence, around every regular point of a planar analytic differential system there exists an analytic first integral. This is not in general true for a singular point and the objective of present chapter is to investigate when this happens or to obtain information about the singular series expansion of the first integral.

5.2 The method

We consider two–dimensional autonomous systems of real differential equations of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{5.1}$$

where $P(x, y)$ and $Q(x, y)$ are analytic functions defined in a neighborhood of the origin such that $P(0, 0) = Q(0, 0) = 0$ and there is no $d(x, y)$, non–unit element of the ring of analytic functions defined in a neighborhood of the origin, which divides both $P(x, y)$ and $Q(x, y)$.

Definition 5.1 *A point* $(x_0, y_0) \in \mathbb{R}^2$ *is a* singular *point for system (5.1) if both* $P(x_0, y_0) = 0$ *and* $Q(x_0, y_0) = 0$.

Without loss of generality, by translating to the origin any singular point, we may assume $(x_0, y_0) = (0, 0)$. It is clear that the origin is a singular point for (5.1) and since $P(x, y)$ and $Q(x, y)$ are coprime elements of the ring of analytic functions defined on a neighborhood of the origin, the set of singular points for (5.1) is a discrete set in the domain of (5.1). Thus, we may always assume that the neighborhood of the origin considered contains no singular points except for the origin.

To implement the method we introduce a scaling of the variables and the time given by $(x, y, t) \to (\varepsilon^p x, \varepsilon^q y, \varepsilon^r t)$ where $\varepsilon > 0$ and p, q and $r \in \mathbb{Z}$ and system (5.1) takes the form

$$\dot{x} = \varepsilon^{r-p} P(\varepsilon^p x, \varepsilon^q y), \quad \dot{y} = \varepsilon^{r-q} Q(\varepsilon^p x, \varepsilon^q y), \tag{5.2}$$

We choose p, q, r in such away that system (5.2) will be analytic in ε. Hence, by the classical theorem of the analytic dependence with respect to the parameters we have that system (5.2) admits a first integral which can be developed in power series of ε because it is analytic with respect to this parameter. Hence we can propose the following development for the first integral

$$H(x, y) = \sum_{k=0}^{\infty} \varepsilon^k h_k(x, y), \tag{5.3}$$

where $h_k(x, y)$ are arbitrary functions. We notice that $P(x, y)$ and $Q(x, y)$ are analytic functions, none of them can be null and $P(0, 0) = Q(0, 0) = 0$, so we can develop them, in a neighborhood of the origin, as convergent series of x and y of the form

$$\begin{aligned} P(x, y) &= p_n(x, y) + p_{n+1}(x, y) + \ldots + p_j(x, y) + \ldots, \\ Q(x, y) &= q_n(x, y) + q_{n+1}(x, y) + \ldots + q_j(x, y) + \ldots, \end{aligned} \tag{5.4}$$

with $n = \min\{\text{subdeg}_{|(0,0)} P(x, y), \text{subdeg}_{|(0,0)} Q(x, y)\} \geq 1$. We recall that given an analytic function $f(x, y)$ defined in a neighborhood of a point

(x_0, y_0), we define $\mathrm{subdeg}_{|(x_0,y_0)} f(x,y)$ as the least positive integer j such that some derivative $(\partial^j f/\partial x^i \partial y^{j-i})(x_0, y_0)$ is not zero. We notice that this computation depends on the variables (x, y) which the function $f(x, y)$ depends on, so we will explicit the variables used in each computation of subdeg. For instance, $\mathrm{subdeg}_{|(x_0,y_0)} f(x, y) = 0$, if and only if, $f(x_0, y_0) \neq 0$. In (5.4), $p_j(x, y)$ and $q_j(x, y)$ denote homogeneous polynomials of x and y of degree $j \geq n$. It is possible that $p_n(x, y)$ or $q_n(x, y)$ is null but, by definition, not both of them can be null.

As the simplest case we consider the case with $p = q = 1$. To develop the method we introduce the rescaling of the variables $(x, y, t) \to (\varepsilon x, \varepsilon y, \varepsilon^{1-n} t)$ and system (5.4) takes the form

$$\begin{array}{rcl} \dot{x} = P(x,y) & = & p_n(x,y) + \varepsilon p_{n+1}(x,y) + \cdots + \varepsilon^{j-n} p_j(x,y) + \cdots, \\ \dot{y} = Q(x,y) & = & q_n(x,y) + \varepsilon q_{n+1}(x,y) + \cdots + \varepsilon^{j-n} q_j(x,y) + \cdots. \end{array} \tag{5.5}$$

Now we propose a formal series in ε as a first integral of the form (5.3) and we obtain the following straightforward result:

Proposition 5.2 *If system (5.5) has the formal series (5.3) as a formal first integral in ε, i.e., imposing the condition $\dot{H} = PH_x + QH_y \equiv 0$ and developing in power series expansion of ε, we obtain that h_0 is the first integral of the homogeneous system*

$$\dot{x} = p_n(x, y), \quad \dot{y} = q_n(x, y). \tag{5.6}$$

and the functions $h_k(x, y)$ satisfy the following recursive differential equations

$$p_n \frac{\partial h_k}{\partial x} + q_n \frac{\partial h_k}{\partial y} + \sum_{m=1}^{k} \left(p_{n+m} \frac{\partial h_{k-m}}{\partial x} + q_{n+m} \frac{\partial h_{k-m}}{\partial y} \right) = 0. \tag{5.7}$$

In fact, the method works at the same form that if we impose that system (5.4) has a first integral which can be expanded as a formal series of homogeneous parts, imposing $\dot{H} \equiv 0$ we arrive also to condition (5.7). The difference is that now, using the parameter ε, the functions need not be homogeneous parts and we can construct also a singular series expansion in the variables x and y. What it is also important is that we can use different scalings of variables $(x, y, t) \to (\varepsilon^p x, \varepsilon^q y, \varepsilon^r t)$ where $\varepsilon > 0$ and p, q and $r \in \mathbb{Z}$. On the other hand, the method depends strongly on h_0 the first integral of the homogeneous system (5.6) and as more simpler is h_0 more we can go beyond with the method. However, h_0 can be chosen using different scalings of variables $(x, y, t) \to (\varepsilon^p x, \varepsilon^q y, \varepsilon^r t)$ where $\varepsilon > 0$ and p, q and $r \in \mathbb{Z}$, in such a way that h_0 will be as simple as possible. The following straightforward result gives the form of this h_0.

Proposition 5.3 *The first integral h_0 of the homogeneous system (5.6) is of the form*

$$h_0 = x \exp\left(-\int \frac{du}{\phi(u) - u}\right), \tag{5.8}$$

where $\phi(u) = q_n(1, u)/p_n(1, u)$ and $u = y/x$.

Moreover, we will see through the examples that the form of h_0 suggests the natural changes where the initial system (5.4) takes the simplest form. We remark that not always we obtain, after the scaling of variables and time, a homogeneous system for $\varepsilon = 0$. The method gives necessary conditions to have analytic integrability or a singular series expansion around a singular point and information about what we call the *essential variables* of a system.

Definition 5.4 *The essential variables of system (5.4) at a singular point translated to the origin, are the variables where h_0 takes its simplest form.*

The more easy situation is that once the first term h_0 is computed, the change of variables $(x, z) = (x, h_0(x, y))$ reduces the system into a new one for which the scaling method presented in this chapter is more effective. In these essential variables, the system also takes its simplest form. In some cases, when we write system (5.4) in its essential variables, the system reduces to a linear equation or to a Riccati equation . Therefore, the method is also an alternative method to detect if a system reduces to some integrable equation. The essential variables of a given system are not unique because the choice of the values p, q and r is neither unique nor automatic. Therefore, when we apply the scaling method presented and we transform the system into its essential variables, or the system reduces to an integrable equation or we have recursive differential equations for the functions h_k that give us information about the series expansion that admits system (5.6) as a first integral. See the examples presented in the following sections.

In a similar way that in the analysis of the Painlevé α-method, see for instance [89], we have the following result respect to our method for the case when ε is a small parameter and the series (5.3) is a perturbative expansion. We omit the proof which is formally the same that in the Painlevé case.

Theorem 5.5 *Let* $\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X};\varepsilon)$ *the transformed system of (5.1) via the scaling of variables and time, which is analytic in a path connected domain* $(\mathbf{X},\varepsilon) \in (D_{\mathbf{X}} \subset \mathbb{R}^2, D_\varepsilon \subset \mathbb{R})$. *For a given singular point and* $\delta \in \mathbb{R}^+$ *sufficiently small, the first integral (5.3) is single-valued for all* ε *such that* $|\varepsilon| < \delta$ *if and only if each* $h_k(x,y)$ *of (5.3) is single-valued for all* k.

However, for the method developed in this chapter the parameter ε needs not be small. The parameter ε may be relatively large (for instance $\varepsilon \to 1$). The convergence of series (5.3) must be analyzed in each particular case, and the convergent rate depends upon the nonlinear terms of the system (5.4).

In the following sections we present how works the method to obtain the essential variables where the original equation takes the most simplified form and to obtain the necessary conditions of analytic integrability or how to obtain a singular series expansion of the first integral.

In our analysis when we consider the above expansions from a formal perspective, we will ignore questions of convergence. As system (5.6) is analytic, according with the result of Mattei and Moussu [98], once the existence of a formal first integral has been established, we can ensure that there is also an analytical first integral around the isolated singular point.

When the authors had a first version of the present work they received a version of the paper [1]. In [1], it is considered the perturbations of quasi-homogeneous planar Hamiltonian systems, where the Hamiltonian function does not contain multiple factors. For such a kind of systems it is characterized the integrability problem, by connecting it with the normal form theory. The authors of [1] remark that the most interesting cases (linear saddle, linear center, nilpotent case, etc) fall into the category studied in their work. In [1] quasi-homogeneous systems are considered, but as they remark in their work any vector field $\mathbf{F}$ can be expanded in quasi-homogeneous terms of type $\mathbf{t}$ of successive degrees. The key idea is to introduce a parameter ε by means of the the scaling $\mathbf{x} = E\,\tilde{\mathbf{x}}$, with $\tilde{\mathbf{x}} \in \mathbb{R}^2$ and where E is the diagonal matrix $E = \mathrm{diag}\{\varepsilon^{t_1}, \varepsilon^{t_2}\}$. Then, they get the system $\dot{\tilde{\mathbf{x}}} = E^{-1}\mathbf{F}(E\,\tilde{\mathbf{x}})$ and expanding in ε, it is possible to write the system into the form

$$\dot{\tilde{\mathbf{x}}} = \mathbf{F}_r(\tilde{\mathbf{x}})\varepsilon^r + \mathbf{F}_{r+1}(\tilde{\mathbf{x}})\varepsilon^{r+1} + \cdots \tag{5.9}$$

where each term $\mathbf{F}_k \in \mathcal{P}_k^t$ is quasi-homogeneous of type $\mathbf{t}$ and degree k. Finally, it is enough to put $\varepsilon = 1$ to recover the original vector field $\mathbf{F}$. Hence, in some sense, the key idea is the same but here using directly the scaling of variables and also a time rescaling and using the method not only to detect formal integrability but also to construct singular series expansion of the first integral and the inverse integrating factor. The main result in [1] is stated as follows:

Theorem 5.6 *Assume that system (5.9) satisfies that the lowest degree term* $\mathbf{F}_r$ *in (5.9) is Hamiltonian, that is,* $\mathbf{F}_r = \mathbf{X}_h$ *for certain* $h \in \mathcal{P}^t_{r+|t|}$ *and the decomposition of* h *has only simple factors (they are* x*,* y *and* $y^{t_1} - \alpha x^{t_2}$ *with* $\alpha \in \mathbb{C}$*). Then (5.9) is formally integrable if and only if it is formally conjugated to a Hamiltonian system.*

This theorem shows the important results that can be found using the scaling method presented in this chapter.

5.3 The abel equation

In [32] a classification of the Abel equations known as solvable in the literature was presented. In [75] it is showed that all the integrable rational Abel differential equations that appear in [32] can be reduced to a Riccati differential equation or to a first–order linear differential equation through a change defined by a rational map. The change is given explicitly for each class and it is found in a unified way from the knowledge of the explicitly first integral. We will see in this section through an example how the method presented in this chapter gives the rational map which reduces to a Riccati differential equation or to a first–order linear differential equation. In [32], it appears for instance the Class 2, given first by Liouville [92], which is the Abel equation

$$\frac{dy}{dx} = y^3 - 2xy^2. \tag{5.10}$$

Equation (5.10) has the associated differential system

$$\dot{y} = y^3 - 2xy^2, \qquad \dot{x} = 1.$$

Now, doing the rescaling of variables $(x, y) \to (\varepsilon^p x, \varepsilon^q y)$ with $p, q \in \mathbb{Z}$ we have

$$\dot{y} = \varepsilon^{2q} y^3 - 2\varepsilon^{p+q} xy^2, \qquad \dot{x} = \varepsilon^{-p},$$

which has the associated Abel equation

$$\frac{dy}{dx} = \varepsilon^{p+2q} y^3 - 2\varepsilon^{2p+q} xy^2.$$

Taking $q = -2p$ and $p = -1$ we have

$$\frac{dy}{dx} = \varepsilon^3 y^3 - 2xy^2. \tag{5.11}$$

Now we propose a formal series in ε as a first integral of the form (5.3) where h_0 is the first integral of the system $\dot{y} = -2xy^2$, $\dot{x} = 1$. Imposing the condition $\dot{H} \equiv 0$ and developing in power series expansion of ε, the first equation for ε^0 is a partial differential equation for h_0 given by

$$\frac{\partial h_0}{\partial x} - 2xy^2 \frac{\partial h_0}{\partial y} = 0,$$

whose solution is $h_0(x, y)$ an arbitrary function of $(x^2y - 1)/y$. Just taking this value as a new variable $z = (x^2y - 1)/y$ we obtain that the original equation (6.14) is transformed to the Riccati equation $dx/dz = -x^2 - z$. Therefore the essential variables in this case are $(x, z) = (x, (x^2y - 1)/y)$.

Sometimes the process must be repeated because the essential variables are not any of the original ones. Hence, the change is given by a rational map in both variables.

5.4 Some examples of nonlinear differential systems

Example 1. Consider the following system

$$\dot{x} = -y, \quad \dot{y} = ax + by + y^2. \tag{5.12}$$

First we are going to see what happens if we do the rescaling of variables $(x, y) \to (\varepsilon x, \varepsilon y)$ which is equivalent to impose that system (5.12) has a first integral which can be expanded as a formal series of homogeneous parts. The system after the rescaling $(x, y) \to (\varepsilon x, \varepsilon y)$ takes the form

$$\dot{x} = -y, \quad \dot{y} = ax + by + \varepsilon y^2.$$

We now propose a formal series in ε as a first integral of the form (5.3) where h_0 is the first integral of the homogeneous system $\dot{y} = -y$, $\dot{x} = ax + by$. Imposing the condition $\dot{H} \equiv 0$ and developing in power series expansion of ε, the first equation for ε^0 is a partial differential equation for h_0 given by

$$-y\frac{\partial h_0}{\partial x} + (ax + by)\frac{\partial h_0}{\partial y} = 0,$$

whose solution is $h_0(x,y)$ an arbitrary function of

$$-\frac{b\ \arctan\left[\frac{b+2y/x}{\sqrt{4a-b^2}}\right]}{\sqrt{4a-b^2}}+\frac{1}{2}\log\left[a+b\,y/x+y^2/x^2\right]+\log x,$$

and it is no possible go further. Therefore the homogeneous rescaling does not give information about the integrability problem of system (5.12).

Now we apply the following rescaling of variables $(x,y)\to(\varepsilon^p x,\varepsilon^q y)$ with $p,q\in\mathbb{Z}$ we have

$$\dot{y}=a\,\varepsilon^{p-q}x+b\,y+\varepsilon^q\,y^2,\qquad \dot{x}=-\varepsilon^{-p+q}y,$$

which has the associated orbital equation

$$\frac{dy}{dx}=-b\,\varepsilon^{p-q}-\frac{a\,\varepsilon^{2p-2q}x}{y}-\varepsilon^p\,y.$$

Taking $p=0$ and $q=-1$ we have

$$\frac{dy}{dx}=-b\,\varepsilon-\frac{a\,\varepsilon^2 x}{y}-y. \tag{5.13}$$

We propose a formal series in ε as a first integral of the form (5.3) where h_0 is the first integral of the system $\dot{y}=-y$, $\dot{x}=1$. Imposing the condition $\dot{H}\equiv 0$ and developing in power series expansion of ε, the first equation for ε^0 is a partial differential equation for h_0 given by

$$-\frac{\partial h_0}{\partial x}+y\frac{\partial h_0}{\partial y}=0,$$

whose solution is $h_0(x,y)$ an arbitrary function of $e^x y$. Just taking this value as a new variable $z=e^x y$ we obtain that the original system (5.12) is transformed to

$$\dot{z}=-e^x(a\,e^x x+b\,z),\qquad \dot{x}=z. \tag{5.14}$$

In these new essential variables (x,z), the system becomes simpler. It is straightforward to see that if $a=0$ the orbital equation is of separate variables and a first integral is $H(x,z)=z+b\,e^x$, and if $b=0$ the orbital equation is also of separate variables and a first integral is given $H(x,z)=z^2/2+a\,e^{2x}(2x-1)/4$. Hence in the following we consider the case $ab\neq 0$.

Now, we apply the another rescaling of variables $(x,z) \to (\varepsilon^{k_1}x, \varepsilon^{k_2}z)$ with $k_1, k_2 \in \mathbb{Z}$ and we obtain

$$\dot{z} = -a\, e^{2\varepsilon^{k_1}x}\varepsilon^{k_1-k_2}x - b\, e^{\varepsilon^{k_1}x}z, \qquad \dot{x} = \varepsilon^{k_2-k_1}z,$$

which has the associated orbital equation

$$\frac{dz}{dx} = -b\, e^{\varepsilon^{k_1}x}\varepsilon^{k_1-k_2} - \frac{a\, e^{2\varepsilon^{k_1}x}\varepsilon^{2k_1-2k_2}x}{z}.$$

Taking $k_1 = 0$ and $k_2 = -1$ we have

$$\frac{dz}{dx} = -\frac{\varepsilon\, e^x(a\,\varepsilon\, e^x x + bz)}{z}. \tag{5.15}$$

Now we propose a formal series in ε as a first integral of the form

$$H(x,z) = h_0(x,z) + \varepsilon h_1(x,z) + \cdots + \varepsilon^j h_j(x,z) + \cdots, \tag{5.16}$$

where h_0 is the first integral of the homogeneous system $\dot{z} = 0$, $\dot{x} = 1$, i.e., $h_0(x,y) = z$. Imposing the condition $\dot{H} \equiv 0$ and developing in power series expansion of ε, we obtain the recursive partial differential equation given by

$$z\frac{\partial h_k}{\partial x} = a\, e^{2x}\frac{\partial h_{k-2}}{\partial z} + b\, e^x z\frac{\partial h_{k-1}}{\partial z}.$$

Hence, we have

$$h_k = \frac{1}{z}\int \left(a\, e^{2x}\frac{\partial h_{k-2}}{\partial z} + b\, e^x z\frac{\partial h_{k-1}}{\partial z}\right) dx.$$

In fact, it is easy to see that h_k has the form $h_k = ae^{kx}P_k(x)/z^{k-1}$ where P_k is a polynomial of degree $\le k$. Therefore the first integral for system (5.14) is $H(x,z) = \sum_{k=0}^{\infty} ae^{kx}P_k(x)/z^{k-1}$ and taking into account that $z = e^x y$ system (5.12) has a singular first integral of the form

$$H(x,y) = \sum_{k=0}^{\infty} \frac{a\, e^x}{y^{k-1}}\, P_k(x).$$

Example 2. Consider the following differential system

$$\begin{aligned} \dot{x} &= -y\left[2x^2 + y^2 + (x^2+y^2)^2\right], \\ \dot{y} &= x\left[2x^2 + y^2 + 2(x^2+y^2)^2\right]. \end{aligned} \tag{5.17}$$

System (5.17) appears in the book of Nemytskii and Stepanov in [102, p.122] and it has a degenerate center at the origin because for this system we know the C^∞ first integral

$$H(x,y) = (2x^2 + y^2)\exp(-1/(x^2 + y^2)), \tag{5.18}$$

and this first integral determines a family of closed curves each of which surrounds the origin. Notice that system (5.17) has the symmetry $(x, y, t) \to (x, -y, -t)$. But this system is not analytically (nor formally) integrable in a neighborhood of the origin because the equation $\dot{H} \equiv 0$ implies that $H = H_{2m} + H_{2m+2} + \cdots$, where $H_{2m} = (x^2 + y^2)^m$ with $m \geq 1$, and this is in contradiction with the terms of degree $2m + 4$ of $\dot{H} \equiv 0$. On the other hand system (5.17) has the simple inverse integrating factor $V = (x^2+y^2)^2(2x^2+y^2)$ defined in the whole plane. We are going to see how the method developed in this work gives the above first integral and the above inverse integrating factor. System (5.17) after the rescaling $(x, y) \to (\varepsilon x, \varepsilon y)$ and the adequate time rescaling takes the form

$$\dot{x} = -y\left[2x^2 + y^2 + \varepsilon^2(x^2 + y^2)^2\right], \qquad \dot{y} = x\left[2x^2 + y^2 + 2\varepsilon^2(x^2 + y^2)^2\right].$$

We now propose a formal series in ε as a first integral of the form (5.3) where h_0 is the first integral of the homogeneous system $\dot{y} = -y(2x^2 + y^2)$, $\dot{x} = x(2x^2 + y^2)$, i.e. $h_0 = x^2 + y^2$. Imposing the condition $\dot{H} \equiv 0$ and developing in power series expansion of ε, the next equation for ε^1 is a partial differential equation for h_1 given by

$$-y(2x^2 + y^2)\frac{\partial h_1}{\partial x} + x(2x^2 + y^2)\frac{\partial h_1}{\partial y} = 0,$$

whose solution is $h_1(x, y)$ an arbitrary function of $x^2 + y^2$. Taking $h_1(x, y) \equiv 0$, the next equation for ε^2 is a partial differential equation for h_2 given by

$$-y(2x^2 + y^2)\frac{\partial h_2}{\partial x} + x(2x^2 + y^2)\frac{\partial h_2}{\partial y} + 2xy(x^2 + y^2)^2 = 0,$$

whose solution is given by $h_2(x, y) = (x^2 + y^2)\log(2x^2 + y^2) + f_2(x^2 + y^2)$ where f_2 is an arbitrary function. Taking all the arbitrary functions equal zero, we obtain the following results for the recursive differential system

$$h_{2s-1} \equiv 0, \qquad h_{2s} = (x^2 + y^2)^{s+1}(\log(2x^2 + y^2))^s.$$

Hence we obtain the formal series

$$H(x,y) = (x^2+y^2)\sum_{s=0}^{\infty}(x^2+y^2)^s(\log(2x^2+y^2))^s,$$

which for (x, y) near the origin gives

$$H(x,y) = \frac{x^2+y^2}{1-(x^2+y^2)\log(2x^2+y^2)} = \frac{1}{\frac{1}{x^2+y^2}-\log(2x^2+y^2)}. \tag{5.19}$$

From here we have that

$$\tilde{H}(x,y) = \frac{1}{x^2+y^2} - \log(2x^2+y^2),$$

is also a first integral and taking exponentiation we obtain the first integral (5.18) given also in the book of Nemytskii and Stepanov for this case.

It is also possible, instead of use a development of the first integral (5.3), a development of the inverse integrating factor of the form

$$V(x,y) = V_0(x,y) + \varepsilon V_1(x,y) + \cdots + \varepsilon^j V_j(x,y) + \cdots, \tag{5.20}$$

where V_0 is the inverse integrating factor of the homogeneous system (5.6). Imposing the condition $PV_x + QV_y - (P_x + Q_y)V \equiv 0$, and developing in power series expansion of ε, we obtain a recursive differential equations for the $V_k(x, y)$ in the same way that for a first integral.

Hence, system (5.17) after the rescaling $(x, y) \to (\varepsilon x, \varepsilon y)$ and the adequate time rescaling takes the form given above. If we now propose a formal series in ε as an inverse integrating factor of the form (5.20) where V_0 is the inverse integrating factor of the homogeneous system $\dot{y} = -y(2x^2 + y^2)$, $\dot{x} = x(2x^2 + y^2)$, i.e., $V_0 = 2x^2 + y^2$. Now, imposing the condition $PV_x + QV_y - (P_x + Q_y)V \equiv 0$ and developing in power series expansion of ε, the next equation for ε^1 is a partial differential equation for V_1 given by

$$-y(2x^2+y^2)\frac{\partial V_1}{\partial x} + x(2x^2+y^2)\frac{\partial V_1}{\partial y} = -2xyV_1,$$

whose solution is given by $V_1(x, y) = (2x^2 + y^2)g_1(x^2 + y^2)$ where g_1 is an arbitrary function of $x^2 + y^2$. Taking $V_1(x, y) \equiv 0$, the next equation for ε^2 is a partial differential equation for V_2 given by

$$-y(2x^2+y^2)\frac{\partial V_2}{\partial x} + x(2x^2+y^2)\frac{\partial V_2}{\partial y} = -2xyV_2 + 4xy(x^2+y^2)(2x^2+y^2),$$

whose solution is given by $V_2(x,y) = -2(x^2+y^2)(2x^2+y^2)\log(2x^2+y^2) + (2x^2+y^2)g_2(x^2+y^2)$ where g_2 is an arbitrary function. Taking $g_2(x,y) \equiv 0$, the next equation for ε^3 is a partial differential equation for V_3 whose solution is given by $V_3(x,y) = (2x^2+y^2)g_3(x^2+y^2)$ where g_3 is an arbitrary function. We take $g_3(x,y) \equiv 0$ and the next equation for ε^4 is a partial differential equation for V_4 whose solution is given by $V_4(x,y) = (x^2+y^2)^2(2x^2+y^2)(\log(2x^2+y^2))^2 + (2x^2+y^2)g_4(x^2+y^2)$ where g_4 is an arbitrary function. We also take $g_4(x,y) \equiv 0$. Taking all the arbitrary functions equal zero, we obtain the following results for the following recursive differential system $V_i(x,y) \equiv 0$ for $i \geq 5$. Hence we obtain the formal series

$$V(x,y) = (2x^2+y^2)(-1+(x^2+y^2)\log(2x^2+y^2))^2,$$

which is indeed and inverse integrating factor of system (5.17). Finally, we obtain a polynomial inverse integrating factor doing the computation

$$\tilde{V} = V\,H^2 = (x^2+y^2)^2(2x^2+y^2)$$

where H is given by (5.19).

Example 3. In [99] Moussu gives another example of a real polynomial differential system having a degenerate center for which a local analytic first integral does not exists. The example of Moussu is

$$\dot{x} = y^3, \qquad \dot{y} = -x^3 + \frac{x^2y^2}{2}\ . \tag{5.21}$$

The above system has a center at the origin because the origin is a monodromy singular point and the system is invariant under the symmetry: $(x,y,t) \to (x,-y,-t)$. However, system (5.21) is not analytically (nor formally) integrable in a neighborhood of the origin. This conclusion is obtained quickly if one tries to impose that system (5.21) has a first integral that will be developed in power series. First, it is important to remark that the map $(x,y) \to (x,z^2)$ transforms system (5.21) into the system

$$\dot{x} = -z, \qquad \dot{z} = 2x^3 + x^2z\ ,$$

which has a nilpotent singular point at the origin of node type.

We now apply the method to system (5.21). Hence, doing the rescaling of variables $(x,y) \to (\varepsilon^p x, \varepsilon^q y)$ with $p,q \in \mathbb{Z}$ we have

$$\dot{x} = \varepsilon^{3q-p}y^3, \qquad \dot{y} = -\varepsilon^{3p-q}x^3 + \varepsilon^{2p+q}\frac{x^2y^2}{2},$$

which has the associated orbital equation

$$\frac{dy}{dx} = -\,\varepsilon^{4p-4q}\frac{x^3}{y^3} + \varepsilon^{3p-2q}\frac{x^2}{2y}.$$

Taking $p = 2q/3$ and $q = -3/4$ we have

$$\frac{dy}{dx} = -\,\varepsilon\frac{x^3}{y^3} + \frac{x^2}{2y}. \tag{5.22}$$

We propose a formal series in ε as a first integral of the form (5.3) where h_0 is the first integral of the system $\dot{y} = x^2$, $\dot{x} = 2y$. Imposing the condition $\dot{H} \equiv 0$ and developing in power series expansion of ε, the first equation for ε^0 is a partial differential equation for h_0 given by

$$2y^3\frac{\partial h_0}{\partial x} + x^2y^2\frac{\partial h_0}{\partial y} = 0,$$

whose solution is $h_0(x, y)$ an arbitrary function of $3y^2 - x^3$. Just taking this value as a new variable $z = 3y^2 - x^3$ we obtain that the original system (5.21) is transformed to

$$\dot{z} = -18x^3, \qquad \dot{x} = x^3 + z. \tag{5.23}$$

In these new essential variables (x, z), the system becomes simpler. Now, we apply the another rescaling of variables $(x, z) \to (\varepsilon^{k_1}x, \varepsilon^{k_2}z)$ with $k_1, k_2 \in \mathbb{Z}$ and we obtain

$$\dot{z} = -18\varepsilon^{3k_1-k_2}x^3, \qquad \dot{x} = \varepsilon^{2k_1}x^3 + \varepsilon^{k_2-k_1}z,$$

which has the associated orbital equation

$$\frac{dz}{dx} = \frac{-18\varepsilon^{3k_1-k_2}x^3}{\varepsilon^{2k_1}x^3 + \varepsilon^{k_2-k_1}z}.$$

Taking $k_1 = k_2 = 1$ we have

$$\frac{dz}{dx} = \frac{-18\varepsilon^2x^3}{\varepsilon^2x^3 + z}. \tag{5.24}$$

Now we propose a formal series in ε as a first integral of the form

$$H(x, z) = h_0(x, z) + \varepsilon h_1(x, z) + \cdots + \varepsilon^j h_j(x, z) + \cdots, \tag{5.25}$$

where h_0 is the first integral of the system $\dot{z} = 0$, $\dot{x} = z$, i.e. $h_0(x, y) = z$. Imposing the condition $\dot{H} \equiv 0$ and developing in power series expansion of ε, we obtain the recursive partial differential equation given by

$$6z\frac{\partial h_k}{\partial x} = 18x^3\frac{\partial h_{k-1}}{\partial z} - x^3\frac{\partial h_{k-1}}{\partial x},$$

for k even because it is possible to choose $h_k \equiv 0$ for k odd. Hence, we have

$$h_k = \frac{1}{6z}\int \left(18x^3\frac{\partial h_{k-1}}{\partial z} - x^3\frac{\partial h_{k-1}}{\partial x}\right) dx,$$

for k even. In fact, it is easy to see that h_k has the form

$$h_k = \frac{C_{k-1}\, x^{2k}}{z^{k-1}} + \frac{C_{k-2}\, x^{2k-1}}{z^{k-2}} + \cdots + \frac{C_{k/2}\, x^{3k/2+1}}{z^{k/2}},$$

for k even and where C_k are fixed constants. Therefore the first integral for system (5.23) is

$$H(x,z) = \sum_{k=0}^{\infty}\left(\frac{C_{k-1}\, x^{2k}}{z^{k-1}} + \frac{C_{k-2}\, x^{2k-1}}{z^{k-2}} + \cdots + \frac{C_{k/2}\, x^{3k/2+1}}{z^{k/2}}\right).$$

for k even. Taking into account that $z = 3y^2 - x^3$ system (5.21) has the singular first integral

$$H(x,y) = \sum_{k=0}^{\infty}\left(\frac{C_{k-1}\, x^{2k}}{(3y^2 - x^3)^{k-1}} + \frac{C_{k-2}\, x^{2k-1}}{(3y^2 - x^3)^{k-2}} + \cdots + \frac{C_{k/2}\, x^{3k/2+1}}{(3y^2 - x^3)^{k/2}}\right).$$

5.5 On the center problem

Consider a real quadratic system of ordinary differential equations on $\mathbb{R}^2$ with an isolated singular point at the origin, at which the linear part are non-zero pure imaginary numbers. By analytic change of coordinates and a constant time rescaling the system takes the form

$$\dot{u} = -v + \cdots, \qquad \dot{v} = u + \cdots. \tag{5.26}$$

The classical Poincaré–Liapunov center theorem states that the origin is a center if and only if the system admits an analytic first integral of the form

$\phi(u,v) = u^2 + v^2 + \cdots$, see for instance [70, 76, 77, 78, 122] and references therein. The method developed in this work can also be applied in this case, where $h_0 = u^2 + v^2$ and we obtain a recursive differential equation for the h_k of the form (5.7). Clearly the method described in this chapter can also be used to detect nilpotent centers and also degenerate centers. In this way, in [1], it is used to rediscover the Poincaré theorem for nondegenerate centers and the Strozyna and Zoladek theorem for nilpotent centers with analytic first integral.

5.6 On the resonant center problem

If system (5.26) is complexified in a natural way by setting $z = u + iv$ then we have a differential equation of the form $\dot{z} = iz + \cdots$. In this case one constructs, step-by-step, the formal first integral $\Phi = z\bar{z} + \cdots$ satisfying the equation $\dot{\Phi} = v_3|z|^4 + v_5|z|^6 + \cdots$, where the coefficients v_i called the focus quantities are polynomials of the coefficients of the original system. The theorem of Poincaré–Liapunov [91, 109] says that when all the $v_i = 0$ then the point $z = 0$ is a center. Existence of a first integral ϕ is equivalent to existence of an analytic first integral for the complexified equation of the form $\Phi = z\bar{z} + \cdots$. Taking the complex conjugated equation there arises an analytic system of ordinary differential equations on $\mathbb{C}^2$ of the form $\dot{z} = iz + \cdots$, $\dot{w} = -iw + \cdots$. Hence, after the complexification the system is transformed into an analytic system with eigenvalues $+i$ and $-i$. This is the $[1 : -1]$ resonant singular point and the numbers v_i become the coefficients before the resonant terms in its orbital normal form. This was the way chosen by Dulac [51] to approach the center problem for quadratic systems, see also [38, 117].

The next natural generalization of the above theory is to consider the case of a polynomial vector field in $\mathbb{C}^2$ with $[p : -q]$ resonant elementary singular point

$$\dot{x} = p\,x + \cdots, \quad \dot{y} = -q\,y + \cdots, \tag{5.27}$$

where $p, q \in \mathbb{Z}$. These facts motivate the generalization of the concept of real center to certain classes of systems of ordinary differential equations on $\mathbb{C}^2$. In this case we have the following definition of a resonant center or focus, coming from Dulac [51] see also [127].

Definition 5.7 *A $[p : -q]$ resonant elementary singular point of an analytic* *system is a center if, an only if, there exists a local meromorphic first integral* $\Phi = h_0 + \cdots$, *with* $h_0 = x^p y^q$. *This singular point is a resonant focus of order* k *if, an only if, there is a formal power series* $\Phi = x^p y^q + \cdots$ *with the property* $\dot{\Phi} = g_k h_0^{k+1} + \cdots$.

The saddle–node case, i.e., the $[1:0]$ resonance, and the node case, i.e., the case with $q < 0 < p$, were studied in [127]. We focus our attention in the resonant saddle case, i.e., $1 \leq p < q$ assuming $(p, q) = 1$. In this case we have that $p > 0$, $q > 0$. Hence, the linear part has analytic first integral $h_0 = x^q y^p$ and we can seek the conditions for the existence of an analytic first integral of the form $\Phi = h_0 + \cdots$ such that $\dot{\Phi} = g_1 h_0^2 + g_2 h_0^3 + \cdots$ and the $[p : -q]$ resonant focus numbers g_i are also polynomials of the coefficients of the system. The resonant saddle case is also studied in [39, 59, 82, 115, 127]. The method developed in this work can also applied in this case, where $h_0 = x^q y^p$ and we also obtain a recursive differential equation for the next h_k. The obstructions to the analyticity of each h_k are the necessary conditions to have an analytic first integral.

Chapter 6

Composition conditions in the trigonometric Abel equation

6.1 Introduction and statement of the main results

We consider the ordinary differential equation

$$\frac{d\rho}{d\theta} = a_1(\theta)\rho^2 + a_2(\theta)\rho^3, \tag{6.1}$$

where ρ is a real variable and $a_i(\theta)$ are trigonometric polynomials in θ for $i = 1, 2$. When $a_1(\theta)$ and $a_2(\theta)$ are identically zero, we say that (6.1) is a *trivial* center. We shall denote the derivative of ρ with respect θ by $d\rho/d\theta$ or ρ'. We can solve equation (6.1) by the Picard iteration and find a solution which is unique with the prescribed initial value $\rho(0) = \rho_0$. We say that equation (6.1) determines a *center* if for any sufficiently small initial values $\rho(0)$ the solution of (6.1) satisfies $\rho(0) = \rho(2\pi)$. The center problem for equation (6.1) is to find conditions on the coefficients a_i under which this equation determines a center.

The original center problem arises from the study of the planar analytic differential systems first studied by Poincaré [109] and later by Liapunov [91] and other authors, see [13, 51, 58, 87, 88]. In the case of a non-degenerate singular point the system can be written into the form

$$\dot{x} = -y + P(x, y), \quad \dot{y} = x + Q(x, y), \tag{6.2}$$

where P and Q are analytic functions without constant and linear terms, i.e., $P(x,y) = \sum_{i=2}^{\infty} P_i(x,y)$ and $Q(x,y) = \sum_{i=2}^{\infty} Q_i(x,y)$, where P_i and Q_i are homogeneous polynomials of degree i. Poincaré proved that the origin of system (6.2) is a center if and only if the coefficients of P and Q satisfy a certain infinite system of algebraic equations called the Poincaré–Liapunov constants . We note that taking polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ system (6.2) takes the form

$$\dot{r} = \sum_{s=2}^{\infty} f_s(\theta) r^s, \quad \dot{\theta} = 1 + \sum_{s=2}^{\infty} g_s(\theta) r^{s-1}, \tag{6.3}$$

where

$$f_i(\theta) = \cos\theta P_i(\cos\theta, \sin\theta) + \sin\theta Q_i(\cos\theta, \sin\theta),$$
$$g_i(\theta) = \cos\theta Q_i(\cos\theta, \sin\theta) - \sin\theta P_i(\cos\theta, \sin\theta).$$

We remark that f_i and g_i are homogeneous polynomials of degree $i+1$ in the variables $\cos\theta$ and $\sin\theta$. In the region $\mathcal{R} = \{(r,\theta) : \dot{\theta} > 0\}$ the differential system (6.3) is equivalent to the differential equation

$$\frac{dr}{d\theta} = \frac{\sum_{s=2}^{\infty} f_s(\theta)\, r^s}{1 + \sum_{s=2}^{\infty} g_s(\theta)\, r^{s-1}} = \sum_{i=1}^{\infty} a_i(\theta)\, r^{i+1}, \tag{6.4}$$

where, since P and Q are analytic functions, we have expanded as an analytic series in r to obtain equation (6.4) whose coefficients $a_i(\theta)$ are trigonometric polynomials. This reduces the center problem for the planar differential system (6.2) to the center problem for the class of equations (6.4).

In the particular case that P and Q are homogeneous polynomials of degree n then equation (6.4) takes the form

$$\frac{dr}{d\theta} = \frac{f(\theta) r^n}{1 + g(\theta) r^{n-1}}, \tag{6.5}$$

using the Cherkas transformation (see [34])

$$\rho = \frac{r^{n-1}}{1 + r^{n-1} g(\theta)}, \quad \text{whose inverse is} \quad r = \frac{\rho^{1/(n-1)}}{(1 - \rho g(\theta))^{1/(n-1)}}, \tag{6.6}$$

the differential equation (6.5) becomes the Abel differential equation

$$\frac{d\rho}{d\theta} = ((n-1)f(\theta) - g'(\theta))\,\rho^2 - (n-1)f(\theta)g(\theta)\rho^3. \tag{6.7}$$

which corresponds to equation (6.1) with $a_1(\theta) = ((n-1)f(\theta) - g'(\theta))$ and $a_2(\theta) = -(n-1)f(\theta)g(\theta)$. Notice that in this case $a_1(\theta)$ and $a_2(\theta)$ are trigonometric polynomials of degree $n+1$ and $2(n+1)$ respectively. By the regularity of the Cherkas transformation and its inverse at $r = \rho = 0$, equation (6.5) has a center if and only if equation (6.7) has a center.

In [21, 23, 71] it is studied the center problem for the analytic ordinary differential equation

$$\frac{d\rho}{d\theta} = \sum_{i=1}^{\infty} a_i(\theta)\rho^{i+1}, \tag{6.8}$$

on the cylinder $(\rho, \theta) \in \mathbb{R} \times \mathbb{S}^1$ in a neighborhood of $\rho = 0$ and where $a_i(\theta)$ are trigonometric polynomials in θ. An explicit expression for the first return map of equation (6.8) is given in [21], see also [23]. The expression of the first return map is given in terms of the following iterated integrals of order k,

$$I_{i_1,\ldots,i_k}(a) := \int \cdots \int_{0 \le s_1 \le \cdots \le s_k \le 2\pi} a_{i_k}(s_k) \cdots a_{i_1}(s_1)\, ds_k \cdots ds_1,$$

where, by convention, for $k = 0$ we assume that this equals 1. Actually, iterated integrals appear historically in the study of Abel equations, see for instance [3, 49, 48]. Let $\rho(\theta; \rho_0; a)$, $\theta \in [0, 2\pi]$, be the solution of equation (6.8) corresponding to a with initial value $\rho(0; \rho_0; a) = \rho_0$. Then $P(a)(\rho_0) := \rho(2\pi; \rho_0; a)$ is the first return map of this equation and in [21, 23] it is proved the following result.

Theorem 6.1 *For sufficiently small initial values ρ_0 the first return map $P(a)$ is an absolute convergent power series $P(a)(\rho_0) = \rho_0 + \sum_{n=1}^{\infty} c_n(a)\rho_0^{n+1}$, where*

$$c_n(a) = \sum_{i_1+\cdots+i_k=n} c_{i_1,\ldots,i_k} I_{i_1,\ldots,i_k}(a), \quad \textit{and}$$

$$c_{i_1,\ldots,i_k} = (n - i_1 + 1) \cdot (n - i_1 - i_2 + 1) \cdot (n - i_1 - i_2 - i_3 + 1) \cdots 1.$$

By Theorem 6.1 the center set $\mathcal{C}$ of equation (6.8) is determined by the system of polynomial equations $c_n(a) = 0$, for $n = 1, 2, \ldots$.

In [23] it is given the definition of *universal center* in terms of the monodromy group associated to equation (6.8). In fact we have a *universal center* when the monodromy group is trivial. Hence, the set $\mathcal{U}$ of universal centers is, in a sense, a stable part of the center set $\mathcal{C}$. It is also well-known that, in general, $\mathcal{U} \neq \mathcal{C}$, see for instance [71]. The following proposition establishes the characterization of the universal centers in terms of iterated integrals and it is also given in [23].

Proposition 6.2 *Equation* (6.8) *determines a universal center if and only if for all positive integers* $i_1, \ldots, i_k$ *with* $k \geq 1$ *the iterated integral* $I_{i_1,\ldots,i_k}(a) = 0$.

In [23] it is also considered the case when equation (6.8) has a finite number of terms, i.e.

$$\frac{dv}{d\theta} = \sum_{i=1}^{n} a_i(\theta) v^{i+1}. \tag{6.9}$$

It is proved that equation (6.9) with all a_i trigonometric polynomials has a universal center if and only if there are trigonometric polynomials q and polynomials $p_1, \ldots, p_n \in \mathbb{C}[z]$ such that

$$\tilde{a}_i = p_i \circ q, \qquad 1 \leq i \leq n, \qquad \tilde{a}_i(x) = \int_0^x a_i(s) ds. \tag{6.10}$$

Conditions (6.10) are called *composition conditions*. The vanishing of all iterated integrals $I_{i_1,\ldots,i_k}(a) = 0$ for all positive integers $i_1, \ldots, i_k$ with $k \geq 1$ is equivalent to composition conditions for equation (6.9), as it is proved in [23]. This result is generalized to equation (6.8) in [71] where the following theorem is established.

Theorem 6.3 *Any center of the differential equation* (6.8) *is universal if and only if equation* (6.8) *satisfies the composition condition.*

The composition conditions have been studied in several papers in the last years in different contexts, see for instance [8, 2, 5, 6, 7, 20, 37, 42] and references therein.

Given an angle $\alpha \in [0, \pi)$, we say that the differential equation (6.8) is *α-symmetric* if its flow is symmetric with respect to the straight line $\theta = \alpha$. Obviously, this is equivalent to that equation (6.8) is invariant under the change of variables $\theta \mapsto 2\alpha - \theta$. Any differential equation (6.8) which is α-symmetric has a center, due to the symmetry.

We say that the differential equation (6.8) is of *separable variables* if the function on the right-hand side of equation (6.8) splits as product of two functions of one variable, one depending on ρ and the other on θ, that is,

$$\frac{d\rho}{d\theta} = a(\theta)\, b(\rho).$$

In such a case there is only one center condition which is $\int_0^{2\pi} a(\theta)\, d\theta = 0$.

In [71] it is also proved the following result for equation (6.8).

Theorem 6.4 *If the differential equation* (6.8) *has a center which is either α-symmetric, or of separable variables, then it is universal.*

This last result gives two big families of universal centers also for the Abel equation (6.1).

6.2 Universal centers of the Abel equation (6.1)

In this section we study the universal centers of equation (6.1). It is well-know that not all the centers of equation (6.1) are universal due to the following fact. Any quadratic system in the plane, i.e., system (6.2) with homogeneous P and Q of degree at most 2, can be transformed to an Abel equation of the form (6.7) where $a_1(\theta)$ and $a_2(\theta)$ are trigonometric polynomials of degree 3 and 6 respectively. Moreover in [71] it is proved that there are centers of the quadratic system (6.2) which are not universal (for instance the Darboux component except its intersection with the symmetric one). In [71] it is proved that these non-universal centers of the quadratic system (6.2) give non-universal centers of the associated Abel equation (6.7). In [41] there is another example of a center of an Abel equation which is not universal and where $a_1(\theta)$ and $a_2(\theta)$ are trigonometric polynomials of degree 3 and 6 respectively. Hence, the following open problem can be established:

Open problem: To determine the lowest degree of the trigonometric polynomials $a_1(\theta)$ and $a_2(\theta)$ such that the Abel equation (6.1) has a center which is not universal .

Blinov in [14] proved the following result which shows that the lowest possible degree such that an Abel equation can have a non-universal center is at least 3.

Proposition 6.5 *All the centers of equation* (6.1) *when* $a_1(\theta)$ *and* $a_2(\theta)$ *are trigonometric polynomials of degree* 1 *and* 2 *are universal centers and, in consequence, verify the composition condition.*

For sake of completeness we give a short proof of Proposition 6.5 in the appendix. The proof given in [14] and ours consist in solving the center problem for equation (6.1) with $a_1(\theta)$ and $a_2(\theta)$ of degree at most 2 and to check that all the center cases are universal. However, this procedure is unapproachable for higher degrees due to the cumbersome computations needed to solve the center problem.

In this paper we study the centers of equation (6.1) when $a_1(\theta)$ and $a_2(\theta)$ are trigonometric polynomials of degree 3, i.e.,

$$\begin{aligned} a_1(\theta) &= b_{00} + b_{10}\cos\theta + b_{01}\sin\theta + b_{20}\cos(2\theta) + b_{02}\sin(2\theta) \\ &\quad + b_{30}\cos(3\theta) + b_{03}\sin(3\theta), \\ a_2(\theta) &= c_{00} + c_{10}\cos\theta + c_{01}\sin\theta + c_{20}\cos(2\theta) + c_{02}\sin(2\theta) \\ &\quad + c_{30}\cos(3\theta) + c_{03}\sin(3\theta), \end{aligned} \tag{6.11}$$

where b_{ij} and c_{ij} are real constants. We remark that if $a_1(\theta)$ and $a_2(\theta)$ are both identically null, then we have a trivial center. If $a_1(\theta)$ or $a_2(\theta)$ is identically null, then all the centers are of separable variables and, consequently, all the centers are universal. Thus, we can assume that none $a_1(\theta)$ or $a_2(\theta)$ is identically null. Indeed, the first two center conditions $c_1(a) = 0$ and $c_2(a) = 0$ imply that $b_{00} = c_{00} = 0$, see Theorem 6.1. In order to make a systematic study of the problem for the Abel equation (6.1) with $a_1(\theta)$ and $a_2(\theta)$ of the form (6.11), we assume that the subdegree of $a_1(\theta)$ is either 1, 2 or 3. In each case, we can make an affine change of the variable θ and a rescaling of ρ such that $a_1(\theta)$ takes one of the following forms:

Case I. $a_1(\theta) = \sin\theta + h.o.t.$,
Case II. $a_1(\theta) = \sin(2\theta) + h.o.t.$,
Case III. $a_1(\theta) = \sin(3\theta) + h.o.t.$,

where *h.o.t.* means higher order terms. We have not been able to completely study Case I. Theorem 6.6 deals with Cases II and III.

The procedure is to compute a set of necessary conditions $c_n(a) = 0$ for $n = 3, \ldots, M$, with M large, which are the coefficients of the first return map, see Theorem 6.1. In general, these necessary conditions are very long. Therefore, it is computationally very difficult to determine the irreducible components of the variety $V := V(\langle c_3, c_4, \ldots, c_M \rangle)$. We are using the classical notation of computational algebra given for instance in the textbook [46]. If the center conditions are smaller, as for instance in the proof of Proposition 6.5 given in the appendix, one can use resultants between polynomials of several variables to find the points of this variety. When this computations cannot be overcome, we look for the irreducible decomposition of the variety V. This is an extremely difficult computational problem. We have followed the algorithm described in [114] which makes use of modular arithmetics. The last step of this algorithm has not been verified. This step ensures that all the points of the variety V have been found. That is, we know that all the encountered points belong to the decomposition of V but we do not know whether the given decomposition is complete. We remark that, nevertheless, it is practically sure that the given list is complete, see for instance [11, 114]. Therefore, in the following we provide sufficient conditions to have a center, which are practically necessary. We denote this situation by the expression *with probability close to* 1.

Theorem 6.6 *All the centers of equation* (6.1) *when* $a_1(\theta)$ *and* $a_2(\theta)$ *are trigonometric polynomials of degree* 3 *of the form* (6.11) *with either*
- $b_{10} = b_{01} = b_{20} = 0$ *and* $b_{02} = 1$ (Case II)*, with probability close to* 1*, or;*
- $b_{10} = b_{01} = b_{20} = b_{02} = b_{30} = 0$ *and* $b_{03} = 1$ (Case III)

are universal centers and, consequently, verify the composition condition.

Proof of Theorem 6.6. To proof this result we have computed eleven necessary conditions $c_n(a) = 0$ for $n = 3, \ldots, 13$. These necessary conditions are very long, so we do not present them here. However, one can check our computations with the help of any available computer algebra system. In this case, in order to obtain the families of centers we look for the irreducible decomposition of the variety $V(I)$ of the ideal $I = \langle c_3, c_4, \ldots, c_{13} \rangle$. We have used the routine *minAssGTZ* of the computer algebra system Singular [81] and we have found the irreducible decomposition of the variety of the ideal

I over the field of rational numbers for (Case III) and over the finite field $\mathbb{Z}/(p)$, with $p = 32003$, for (Case II).

The obtained decomposition for the case $b_{10} = b_{01} = b_{20} = 0$ and $b_{02} = 1$ (Case II) consists of 3 components defined by the following ideals

1) $\langle b_{03}, c_{03}, c_{20}, c_{01}\rangle$;

2) $\langle b_{30}, c_{30}, c_{20}, c_{10}\rangle$;

3) $\langle c_{20}, c_{01}, c_{10}, b_{03}c_{02} - c_{03}, c_{02}b_{30} - c_{30}\rangle$;

In the first case 1) we have $a_1(\theta) = \sin(2\theta) + b_{30}\cos(3\theta)$ and $a_2(\theta) = c_{10}\cos\theta + c_{02}\sin(2\theta) + c_{30}\cos(3\theta)$. Therefore equation (6.1) is invariant under the change of variables $\theta \mapsto \pi - \theta$ and the differential equation (6.1) is α-symmetric with $\alpha = \pi/2$ and, thus, it is universal by Theorem 6.4.
In the second case 2) we have $a_1(\theta) = \sin(2\theta) + b_{03}\sin(3\theta)$ and $a_2(\theta) = c_{01}\sin\theta + c_{02}\sin(2\theta) + c_{03}\sin(3\theta)$. Therefore equation (6.1) is invariant under the change of variables $\theta \mapsto -\theta$ and the differential equation (6.1) is also α-symmetric with $\alpha = 0$.
The third case 3) corresponds to a particular case studied in Theorem 6.7 given by $b_{10} = b_{01} = c_{10} = c_{01} = 0$.

Finally, we take the eleven necessary conditions $c_n(a) = 0$ for $n = 3, \ldots, 13$ and we impose the case $b_{10} = b_{01} = b_{20} = b_{02} = b_{30} = 0$ and $b_{03} = 1$ (Case III). Here we can obtain the irreducible decomposition of the variety $V(I)$ over the field $\mathbb{Q}$. To show that all the obtained families are universal centers for equation (6.1) we refer to the case studied in Theorem 6.7 given by $b_{10} = b_{01} = b_{30} = 0$. $\square$

Moreover, although we cannot completely solve Case I, we present the following result.

Theorem 6.7 *All the centers (with probability close to 1) of equation (6.1) when $a_1(\theta)$ and $a_2(\theta)$ are trigonometric polynomials of degree 3 of the form (6.11) with either*

- $b_{10} = b_{01} = c_{10} = c_{01} = 0$ *or;*
- $b_{20} = b_{02} = c_{20} = c_{02} = 0$ *or;*
- $b_{10} = b_{01} = b_{30} = 0$ *or;*

- $b_{10} = b_{01} = b_{03} = 0$

are universal centers and, consequently, verify the composition condition.

Proof of Theorem 6.7. To proof this result we have followed the same computations than in the previous theorem to obtain eleven center conditions $c_n(a)$ for $n = 3, \ldots, 13$, and we have proceeded analogously.

The obtained decomposition for the case $b_{10} = b_{01} = c_{10} = c_{01} = 0$ consists of 3 components defined by the following ideals

1) $\langle c_{30}, c_{03}, b_{20}, b_{02}, b_{30}, b_{03}\rangle$;

2) $\langle b_{03}c_{30} - b_{30}c_{03}, b_{02}c_{20} - b_{20}c_{02}, b_{02}c_{30} - b_{30}c_{02}\rangle$;

3) $\langle b_{03}c_{30} - b_{30}c_{03}, b_{02}c_{20} - b_{20}c_{02}, -3b_{02}^2b_{03}^2b_{20} + b_{03}^2b_{20}^3 + 2b_{02}^3b_{03}b_{30}$
$-6b_{02}b_{03}b_{20}^2b_{30} + 3b_{02}^2b_{20}b_{30}^2 - b_{20}^3b_{30}^2\rangle$.

We now show that equation (6.1) has a universal center under these conditions. In the first case 1) we have that $a_1(\theta) = 0$ and $a_2(\theta) = c_{20}\sin(2\theta) + c_{02}\cos(2\theta)$. Therefore equation (6.1) is in this case of separable variables and by Theorem 6.4 it has a universal center. In the second case we have $b_{20}a_2(\theta) = c_{20}a_1(\theta)$. Hence we have composition condition and equation (6.1) has a universal center. In the third case 3) we take $b_{20} = r_0\sin\beta$ and $b_{02} = r_0\cos\beta$ and it is easy to see that equation (6.1) is invariant under the change of variables $\theta \mapsto \pi - \beta - \theta$. Hence the differential equation (6.1) is α-symmetric with $\alpha = (\pi - \beta)/2$ and by Theorem 6.4 it has a universal center.

The obtained decomposition for the case $b_{20} = b_{02} = c_{20} = c_{02} = 0$ consists of 4 components defined by the following ideals

1) $\langle b_{03}c_{30} - b_{30}c_{03}, b_{01}c_{10} - b_{10}c_{01}, b_{01}c_{03} - b_{03}c_{01}\rangle$;

2) $\langle b_{03}c_{30} - b_{30}c_{03}, b_{01}c_{10} - b_{10}c_{01}, -3b_{01}^2b_{03}b_{10} + b_{03}b_{10}^3 + b_{01}^3b_{30} - 3b_{01}b_{10}^2b_{30}\rangle$;

3) $\langle b_{03}c_{30} - b_{30}c_{03}, b_{10}, b_{01}, -b_{30}c_{01}^3 + 3b_{03}c_{01}^2c_{10} + 3b_{30}c_{01}c_{10}^2 - b_{03}c_{10}^3\rangle$;

4) $\langle b_{01}c_{10} - b_{10}c_{01}, b_{30}, b_{03}, -3b_{01}^2b_{10}c_{03} + b_{10}^3c_{03} + b_{01}^3c_{30} - 3b_{01}b_{10}^2c_{30}\rangle$;

We now show that equation (6.1) has a universal center under these conditions. In the first case 1) we have that $b_{01}a_2(\theta) = c_{01}a_1(\theta)$. Hence we have composition condition and equation (6.1) has a universal center. In the second case 2) and fourth case 4) we take $b_{10} = r_1 \sin\beta_1$ and $b_{01} = r_1 \cos\beta_1$ and it is easy to see that equation (6.1) is invariant under the change of variables $\theta \mapsto -2\beta_1 - \theta$. Hence the differential equation (6.1) is α-symmetric with $\alpha = -\beta_1$ and by Theorem 6.4 it has a universal center. In the third case 3), if we take $c_{10} = r_2 \sin\beta_2$ and $c_{01} = r_2 \cos\beta_2$, it is easy to see that equation (6.1) is invariant under the change of variables $\theta \mapsto -2\beta_2 - \theta$. Therefore the differential equation (6.1) is also α-symmetric with $\alpha = -\beta_2$ and by Theorem 6.4 it has a universal center.

The decomposition for the case $b_{10} = b_{01} = b_{30} = 0$ consists of 4 components defined by the following ideals

1) $\langle b_{03}, b_{02}, b_{20}\rangle$;

2) $\langle b_{03}, b_{02}c_{20} - b_{20}c_{02}, -3c_{01}^2c_{03}c_{10} + c_{03}c_{10}^3 + c_{01}^3c_{30} - 3c_{01}c_{10}^2c_{30}\rangle$;

3) $\langle c_{30}, c_{01}, c_{10}, b_{02}c_{20} - b_{20}c_{02}, c_{20}b_{03} - b_{20}c_{03}\rangle$;

4) $\langle c_{30}, c_{20}, c_{10}, b_{20}\rangle$.

In the first case 1) we have that $a_1(\theta) = 0$. Therefore equation (6.1) is of separable variables and by Theorem 6.4 it has a universal center. In the second case 2) we take $c_{10} = r_4 \sin\beta_4$ and $c_{01} = r_4 \cos\beta_4$ and it is easy to see that equation (6.1) is invariant under the change of variables $\theta \mapsto -2\beta_4 - \theta$. Hence the differential equation (6.1) is α-symmetric with $\alpha = -\beta_4$ and by Theorem 6.4 it has a universal center. The third case 3) correspond to case 2) of the decomposition studied in the case $b_{10} = b_{01} = c_{10} = c_{01} = 0$ (first paragraph of this proof). In the last case 4) we have $a_1(\theta) = b_{02}\sin(2\theta) + b_{03}\sin(3\theta)$ and $a_2(\theta) = c_{01}\sin\theta + c_{02}\sin(2\theta) + c_{03}\sin(3\theta)$. Therefore equation (6.1) is invariant under the change of variables $\theta \mapsto -\theta$ and the differential equation (6.1) is also α-symmetric with $\alpha = 0$.

The decomposition for the case $b_{10} = b_{01} = b_{03} = 0$ also consists of 4 components defined by the following ideals

1) $\langle b_{30}, b_{02}, b_{20}\rangle$;

2) $\langle b_{30}, b_{02}c_{20} - b_{20}c_{02}, -3c_{01}^2c_{03}c_{10} + c_{03}c_{10}^3 + c_{01}^3c_{30} - 3c_{01}c_{10}^2c_{30}\rangle$;

3) $\langle c_{03}, c_{01}, c_{10}, b_{02}c_{20} - b_{20}c_{02}, c_{20}b_{03} - b_{20}c_{03}\rangle$;

4) $\langle c_{03}, c_{20}, c_{01}, b_{20}\rangle$.

The first case 1) and the second case 2) are studied in the decomposition of the case $b_{10} = b_{01} = b_{30} = 0$ (previous paragraph of this proof). The third case 3) corresponds to case 2) of the decomposition studied in the case $b_{10} = b_{01} = c_{10} = c_{01} = 0$ (first part of this proof). In the last case 4) we have $a_1(\theta) = b_{02}\sin(2\theta)+b_{30}\cos(3\theta)$ and $a_2(\theta) = c_{10}\cos\theta+c_{02}\sin(2\theta)+c_{30}\cos(3\theta)$. Therefore equation (6.1) is invariant under the change of variables $\theta \mapsto -\theta-\pi$ and the differential equation (6.1) is again α-symmetric with $\alpha = -\pi/2$. □

Appendix

Proof of Proposition 6.5. First we study the case when $a_1(\theta)$ and $a_2(\theta)$ are trigonometric polynomials of degree 1, therefore we have

$$\begin{aligned} a_1(\theta) &= b_{00} + b_{10}\cos\theta + b_{01}\sin\theta, \\ a_2(\theta) &= c_{00} + c_{10}\cos\theta + c_{01}\sin\theta, \end{aligned}$$

where b_{ij} and c_{ij} are real constants. We recall that the first two center conditions imply that $b_{00} = 0$ and $c_{00} = 0$. The next center condition is $c_3(a) = 0$ with $c_3(a) = b_{01}c_{10} - b_{10}c_{01}$. We take $b_{10} = kc_{10}$ and $c_{01} = kc_{01}$, with $k \in \mathbb{R}$, and some of the next center conditions are zero. In this case equation (6.1) takes the form

$$\dot{r} = r^2(k + r)(c_{10}\cos\theta + c_{01}\sin\theta). \tag{6.12}$$

Equation (6.12) is of separable variables and by Theorem 6.4 has a universal center.

Second, in the case where $a_1(\theta)$ and $a_2(\theta)$ are trigonometric polynomials of degree 2 we have

$$\begin{aligned} a_1(\theta) &= b_{00} + b_{10}\cos\theta + b_{01}\sin\theta + b_{20}\cos(2\theta) + b_{02}\sin(2\theta), \\ a_2(\theta) &= c_{00} + c_{10}\cos\theta + c_{01}\sin\theta + c_{20}\cos(2\theta) + c_{02}\sin(2\theta), \end{aligned}$$

where b_{ij} and c_{ij} are real constants. The first conditions to have a center are, as before, that $b_{00} = 0$ and $c_{00} = 0$. Applying a rotation and a rescaling we can divide the study in two separate cases:

$$\text{(i) } b_{10} = 1 \text{ and } b_{01} = 0 \quad \text{and} \quad \text{(ii) } b_{10} = b_{01} = 0.$$

We begin to study case (i). In this case the next center condition is $c_3(a) = 0$ with $c_3(a) = -2c_{01} - b_{20}c_{02} + b_{02}c_{20} = 0$. From this condition we isolate $c_{01} = (b_{02}c_{20} - b_{20}c_{02})/2$. The next center conditions take the form

$$\begin{aligned} c_4(a) &= b_{02}b_{20}c_{02} - 2b_{20}c_{10} + 2c_{20} - b_{02}^2c_{20}, \\ c_5(a) &= -12b_{20}c_{02} - 5b_{02}^2b_{20}c_{02} + 3b_{20}^3c_{02} - 4b_{20}c_{02}^2 + 2b_{20}^3c_{02}^2 + 16b_{02}b_{20}c_{10} \\ &\quad +8b_{02}b_{20}c_{02}c_{10} - 8b_{20}c_{10}^2 - 4b_{02}c_{20} + 5b_{02}^3c_{20} - 3b_{02}b_{20}^2c_{20} \\ &\quad +4b_{02}c_{02}c_{20} - 4b_{02}b_{20}^2c_{02}c_{20} + 8c_{10}c_{20} - 8b_{02}^2c_{10}c_{20} + 2b_{02}^2b_{20}c_{20}^2. \end{aligned}$$

The resultant between these two polynomials with respect to c_{10} gives the following result

$$\mathrm{res}(c_4(a), c_5(a), c_{10}) = b_{20}(b_{20}c_{02} - b_{02}c_{20})C_{56},$$

where $C_{56} = -12b_{20} + 3b_{02}^2b_{20} + 3b_{20}^3 - 4b_{20}c_{02} + 2b_{02}^2b_{20}c_{02} + 2b_{20}^3c_{02} + 4b_{02}c_{20} - 2b_{02}^3c_{20} - 2b_{02}b_{20}^2c_{20}$.

a) Case $b_{20} = 0$. In this case the condition $c_4(a) = 0$ with $c_4(a) = (b_{02}^2 - 2)c_{20}$. The cases $b_{02} = \pm\sqrt{2}$ do not satisfy the next center conditions, so they do not give rise to centers. In the case $c_{20} = 0$ equation (6.1) takes the form

$$\dot{r} = r^2 \cos\theta(1 + c_{10}r + 2(b_{02} + c_{02}r)\sin\theta). \tag{6.13}$$

System (6.13) has an α-symmetric center, with $\alpha = \pi/2$, because it has the symmetry $\theta \to \pi - \theta$ and in virtue of Theorem 6.4 it is a universal center.

b) Case $b_{20}c_{02} - b_{02}c_{20} = 0$ and $b_{20} \neq 0$. In this case we take $b_{20} = c_{20}k$ and $b_{02} = c_{02}k$ and the next center condition is $c_4(a) = 0$ with $c_4(a) = c_{20}(c_{10}k - 1)$. The case $c_{02} = 0$ implies $b_{20} = 0$ which is out of our assumptions in this case. Hence we must take $c_{10} = 1/k$. In this case equation (6.1) has the form

$$\dot{r} = r^2(k + r)(\cos\theta + c_{20}k\cos 2\theta + c_{02}k\sin 2\theta)$$

which is of separable variables and by Theorem 6.4 it has a universal center.

c) Case $C_{56} = 0$ with $b_{20}c_{02} - b_{02}c_{20} \neq 0$ and $b_{20} \neq 0$. In this case we compute the following resultants:

$$\begin{aligned} \text{res}(c_4(a), c_6(a), c_{10}) &= b_{20}(-b_{20}c_{02} + b_{02}c_{20})C_{57}, \\ \text{res}(c_4(a), c_7(a), c_{10}) &= b_{20}(-b_{20}c_{02} + b_{02}c_{20})C_{58}, \\ \text{res}(c_4(a), c_8(a), c_{10}) &= b_{20}(-b_{20}c_{02} + b_{02}c_{20})C_{59}, \\ \text{res}(c_4(a), c_9(a), c_{10}) &= b_{20}(-b_{20}c_{02} + b_{02}c_{20})C_{510}, \end{aligned}$$

where C_{57}, C_{58}, C_{59} and C_{510} are polynomials in the variables b_{20}, b_{02}, c_{20} and c_{02}. The next step is to make the following resultants with respect to c_{02}.

$$\begin{aligned} \text{res}(C_{56}, C_{57}, c_{02}) &= b_{20}C_{67}, \quad \text{res}(C_{56}, C_{58}, c_{02}) = b_{20}^2 C_{68}, \\ \text{res}(C_{56}, C_{59}, c_{02}) &= b_{20}^2 C_{69}, \quad \text{res}(C_{56}, C_{510}, c_{02}) = b_{20}^3 C_{610}, \end{aligned}$$

where C_{67}, C_{68}, C_{69} and C_{610} are polynomials in the variables b_{20}, b_{02} and c_{20}. Now we perform the following resultants with respect to b_{02}.

$$\begin{aligned} \text{res}(C_{67}, C_{68}, b_{02}) &= b_{20}^{10} C_{78}, \\ \text{res}(C_{67}, C_{69}, b_{02}) &= b_{20}^8 (b_{20}^2 - 2) c_{20} C_{79}, \\ \text{res}(C_{67}, C_{610}, b_{02}) &= b_{20}^{12} C_{710}, \end{aligned}$$

where C_{78}, C_{79} and C_{710} are polynomials in the variables b_{20} and c_{20}. The cases $b_{20}^2 - 2 = 0$ and $c_{20} = 0$ give no common root. Hence, we make the following resultants with respect to c_{20}.

$$\begin{aligned} \text{res}(C_{78}, C_{79}, c_{02}) &= b_{20}^{48}(b_{20} + 2)^2(b_{20} - 2)^2 C_{89}, \\ \text{res}(C_{78}, C_{710}, c_{02}) &= b_{20}^{72}(b_{20} + 2)^2(b_{20} - 2)^2 C_{810}, \end{aligned}$$

where C_{89}, and C_{810} are polynomials uniquely in the variable b_{20}. The cases $b_{20}^2 - 4 = 0$ give no common root. Therefore we make the last resultant with respect to b_{20} which gives the result

$$\text{res}(C_{89}, C_{810}, b_{20}) \neq 0.$$

Therefore, there is no common root and consequently there are no more cases.

Now we study the case (ii) $b_{10} = b_{01} = 0$. In this case the first center condition has the form $c_3(a) = 0$ with $c_3(a) = b_{02}c_{20} - b_{20}c_{02}$. We take $b_{20} = c_{20}k$ and $b_{02} = c_{02}k$ and the next center conditions are $c_4(a) = 0$ and $c_5(a) = k(-2c_{01}c_{02}c_{10} + c_{01}^2c_{20} - c_{10}^2c_{20})$.

a) Case $k = 0$. In this case $a_2(\theta) = 0$ and (6.1) is of separable variables and by Theorem 6.4 it has a universal center.

b) Case $-2c_{01}c_{02}c_{10} + c_{01}^2c_{20} - c_{10}^2c_{20} = 0$. We take $c_{20} = 2c_{01}c_{10}m$ and $c_{02} = (c_{01}^2 - c_{10}^2)m$, with $m \in \mathbb{R}$, and equation (6.1) takes the form

$$\dot{r} = r^2\psi(\theta)(r + 2\psi'(\theta)m(k + r)), \tag{6.14}$$

where $\psi(\theta) = c_{10}\sin\theta + c_{01}\cos\theta$. In this case equation (6.14) has an α-symmetric center, with $\alpha = -\tau$, because it has the symmetry $\theta \to -2\tau - \theta$ where $\tau = \arctan(c_{01}/c_{10})$. Hence, by Theorem 6.4 it is also a universal center.

Chapter 7

Universal centers in the cubic trigonometric Abel equation

7.1 Introduction and statement of the main results

In this chapter we also consider the Abel trigonometric differential equation

$$\frac{d\rho}{d\theta} = a_1(\theta)\,\rho^2 + a_2(\theta)\,\rho^3, \tag{7.1}$$

defined on the cylinder $(\rho,\theta) \in \mathbb{R} \times \mathcal{S}^1$ and where $a_1(\theta)$ and $a_2(\theta)$ are real trigonometric polynomials in θ of degree $\max\{\deg a_1, \deg a_2\} = d$.

Equation (7.1) is a particular case of the analytic ordinary differential equation

$$\frac{d\rho}{d\theta} = \mathcal{F}(\rho,\theta) = \sum_{i\geq 1} a_i(\theta)\,\rho^{i+1}, \tag{7.2}$$

defined on the cylinder $(\rho,\theta) \in \mathbb{R}\times\mathcal{S}^1$ and where $a_i(\theta)$ are real trigonometric polynomials in θ. We denote by $\rho = \rho(\theta;\rho_0)$ the general solution of (7.2) with initial condition $\rho(0;\rho_0) = \rho_0$. We remark that $\rho = 0$ is a particular solution and that, as a consequence, we have that $\rho(\theta;\rho_0)$ is defined for all $\theta \in \mathcal{S}^1$ for $|\rho_0|$ small enough.

We recall that equation (7.2) has a *center* when $\rho(2\pi;\rho_0) = \rho_0$ for $|\rho_0|$ small enough, that is, when all the orbits in a neighborhood of the particular solution $\rho = 0$ are 2π-periodic. The center problem for equation (7.2) is to find conditions on the coefficients $a_i(\theta)$ under which this equation determines a center. The original center problem arises from the study of the planar analytic differential systems, see for instance [73] and references therein.

Classically, there exist two ways to characterize centers in equation (7.2). The first one is to prove the existence of a first integral $H(\rho, \theta)$ which is 2π-periodic in θ. A function $H(\rho, \theta)$ defined in a neighborhood of $\rho = 0$, of class $\mathcal{C}^1$ and non locally constant, is a *first integral* of equation (7.2) if $H(\rho(\theta; \rho_0), \theta)$ does not depend on θ. Equivalently, $(\partial H/\partial \rho)\, \mathcal{F}(\rho, \theta) + \partial H/\partial \theta \equiv 0$.

The second way is to consider the first return map $P(a)$ associated to equation (7.2) $P(a)(\rho_0) := \rho(2\pi; \rho_0)$ and to verify that it is the identity map for $|\rho_0|$ small enough. In [21] (see also [23]), an explicit expression for the first return map $P(a)(\rho_0)$ was given. We remark that $P(a)(\rho_0)$ is an absolute convergent power series for sufficiently small initial values $|\rho_0|$ whose development takes the form

$$P(a)(\rho_0) \;=\; \rho_0 + \sum_{n\geq 1} c_n(a)\rho_0^{n+1}. \tag{7.3}$$

We recall here the expression of this first return map and the definition of universal center.

Theorem 7.1 [23] *For sufficiently small initial values $|\rho_0|$ the first return map $P(a)$ is an absolute convergent power series* (7.3)*, where*

$$c_n(a) = \sum_{i_1+\cdots+i_k=n} c_{i_1,\ldots,i_k} I_{i_1,\ldots,i_k}(a), \quad \textit{and}$$

$$c_{i_1,\ldots,i_k} = (n - i_1 + 1)\cdot(n - i_1 - i_2 + 1)\cdot(n - i_1 - i_2 - i_3 + 1)\cdots 1,$$

and where $I_{i_1\ldots i_k}(a)$ is the following iterated integral of order k

$$I_{i_1\ldots i_k}(a) \;:=\; \int\cdots\int_{0\leq s_1\leq\cdots\leq s_k\leq 2\pi} a_{i_k}(s_k)\cdots a_{i_1}(s_1)\, ds_k\cdots ds_1.$$

Of course, equation (7.2) has a center if and only if $c_n(a) = 0$, for all $n \geq 1$. From the form of the first return map $P(a)$, the following definition, given in [23], follows in a natural way.

Definition 7.2 [23] *The differential equation* (7.2) *has a universal center if for all positive integers $i_1, \ldots, i_k$ with $k \geq 1$ the iterated integral $I_{i_1\ldots i_k}(a) = 0$.*

The expression of the coefficients of the first return map $P(a)(\rho_0) := \rho(2\pi;\rho_0)$ for the Abel differential equation $d\rho/d\theta = a_0(\theta)\rho + a_1(\theta)\rho^2 + a_2(\theta)\rho^3$, and thus for equation (7.1), was given by [48, 49, 3].

We also recall that differential equation (7.2) satisfies the *composition conditions* if there is a nonconstant trigonometric polynomial q and there are polynomials $p_i \in \mathbb{R}[z]$, for $i \geq 1$ such that

$$\tilde{a}_i = p_i \circ q, \quad i \geq 1, \quad \text{where} \quad \tilde{a}_i(\theta) = \int_0^\theta a_i(s)ds.$$

The first time that this definition appears was in the work Alwash and Lloyd [7]. The composition conditions have been studied by several authors in different contexts, see for instance [6, 7, 73] and references therein.

Universal centers of equation (7.2) were characterized in [71] through the following result.

Theorem 7.3 [71] *Any center of the differential equation* (7.2) *is universal if and only if equation* (7.2) *satisfies the composition conditions.*

In [23] the same result was proved when equation (7.2) has a finite number of terms.

The aim of this chapter is to study universal and non-universal centers of an Abel differential equation (7.1) in relation with the degree of the trigonometric polynomials $a_1(\theta)$ and $a_2(\theta)$. From the results of the previous chapter it appears to infer that if the degree is less or equal three all the centers are universal centers. But we are going to see that this is not true.

Recall that equation (7.1) has a universal center when all the iterated integrals $I_{i_1\ldots i_k}(a) = 0$, for all $i_1,\ldots,i_k$. Now, each of the indexes $i_1,\ldots,i_k$ can only take the values 1 or 2. Besides the characterization of universal centers as composition centers for the Abel trigonometric equation (7.1) proved in [23, 71], in [43] another characterization is provided in terms of the vanishing of a finite set of *double moments*. We assume that the minimal common period of a_1 and a_2 is $2\pi/k$, with $k \in \mathbb{N}^+$.

Theorem 7.4 [43] *Equation* (7.1) *has a universal center if and only if for all* $i, j \in \mathbb{N}$ *satisfying* $i + j \leq 4d/k - 3$*,*

$$\int_0^{2\pi} \tilde{a}_1^i(s)\tilde{a}_2^j(s)a_2(s)\,ds = \int_0^{2\pi} a_1(s)\,ds = 0.$$

These type of integrals are known as the double moments.

It is well-known that not all the centers of equation (7.1), and thus of equation (7.2), are universal, see [2]. Any quadratic system in the plane can be transformed to an Abel equation of the form (7.1) where $a_1(\theta)$ and $a_2(\theta)$ are trigonometric polynomials of degree 3 and 6 respectively. Moreover in [71] it is proved that there are centers of a quadratic system which are not universal (for instance the Darboux component except its intersection with the symmetric one). Indeed, in [71] it is proved that these non-universal centers of some quadratic systems give non-universal centers of their associated Abel equation. A previous and different example of a center of an Abel equation which is not universal and where $a_1(\theta)$ and $a_2(\theta)$ are also trigonometric polynomials of degree 3 and 6 respectively, is provided in [41]. Hence, the following open problem is established in the previous chapter.

Open problem: To determine the lowest degree of the trigonometric polynomials $a_1(\theta)$ and $a_2(\theta)$ such that the Abel equation (7.1) has a center which is not universal.

In this paper we solve this open problem, see Theorem 7.6. We recall that Blinov in [14] proved that the lowest possible degree such that an Abel equation can have a non-universal center is at least 3.

Proposition 7.5 [14] *All the centers of equation* (7.1) *when* $a_1(\theta)$ *and* $a_2(\theta)$ *are trigonometric polynomials of degree* 1 *and* 2 *are universal centers and, in consequence, verify the composition condition.*

The proof given in [14] (see also the Appendix Chapter 6) consists in solving the center problem for equation (7.2) with $a_1(\theta)$ and $a_2(\theta)$ of degree at most 2 and to check that all the center cases are universal. However, this procedure is unapproachable for higher degrees due to the cumbersome computations needed to solve the center problem. Indeed, Blinov's result solves the center and the universal center problem for Abel differential equations (7.1) up to degree 2. The next equations to be studied are the cubic ones, i.e., $d = 3$. This was made partially in the previous chapter.

The following result concludes that the lowest degree of a trigonometric Abel equation (7.1) with a non-universal center is 3.

Theorem 7.6 *The cubic* ($d = 3$) *trigonometric Abel differential equation*

$$\frac{d\rho}{d\theta} = (\cos\theta + 2\cos 2\theta)\,\rho^2 + (\sin\theta - \sin 2\theta + \sin 3\theta)\,\rho^3, \qquad (7.4)$$

has a center which is not universal.

The proof of this result is given in Section 8.2.

We recall that there are two big families of universal centers of equation (7.2) as the following result establishes.

Theorem 7.7 [71] *If the differential equation* (7.2) *has a center which is either α-symmetric , or of separable variables, then it is universal.*

For the case of the Abel trigonometric equation (7.1), we give the following result about the universal centers which belong to the classes of α-symmetric or of separable variables differential equations. To simplify notation, we consider 1 as a prime number.

Proposition 7.8 *If the degrees of $a_1(\theta)$ and $a_2(\theta)$ are both prime numbers or they are coprime and the Abel differential equation* (7.1) *has a universal center then the differential equation is either α-symmetric or of separable variables.*

As a direct consequence of this result, we have that any universal center of equation (7.1) with $d = 3$ is either α-symmetric or of separable variables.

The chapter is organized as follows. Section 7.2 contains the proofs of the two main results, namely Theorem 7.6 and Proposition 7.8, together with some preliminary results.

7.2 Preliminary results and proofs of the main results

As we have stated in the previous section, a way to characterize that equation (7.2) has a center is to prove the existence of a first integral $H(\rho, \theta)$ which is defined in a neighborhood of $\rho = 0$ and it is 2π-periodic in θ. A function which is closely related to a first integrals is the inverse integrating factor. A function $V(\rho, \theta)$ defined in a neighborhood of $\rho = 0$, of class $\mathcal{C}^1$ and non locally null, is an *inverse integrating factor* of equation (7.2) if

$$\frac{\partial V}{\partial \rho}\, \mathcal{F}(\rho, \theta) \,+\, \frac{\partial V}{\partial \theta} \,=\, \frac{\partial \mathcal{F}}{\partial \rho}\, V(\rho, \theta)$$

and $V(\rho, \theta)$ is 2π-periodic in θ. Given an inverse integrating factor $V(\rho, \theta)$ of (7.2), one can construct a first integral $H(\rho, \theta)$ of (7.2) through the following line integral:

$$H(\rho,\theta) \;=\; \int_{(\rho_0,\theta_0)}^{(\rho,\theta)} \frac{d\rho - \mathcal{F}(\rho,\theta)d\theta}{V(\rho,\theta)}$$

along any curve connecting an arbitrarily chosen point (ρ_0,θ_0) (such that $V(\rho_0,\theta_0) \neq 0$) and the point (ρ,θ). The following result reads for Corollary 5 in [62] written with our notation and our assumptions, see also [63].

Lemma 7.9 [62] *Let $V(\rho,\theta)$ be an inverse integrating factor of equation* (7.2) *whose leading term in the development around $\rho = 0$:*

$$V(\rho,\theta) \;=\; \rho^{\mu}\, v(\theta) \;+\; o(\rho^{\mu}),$$

where $v(\theta) \not\equiv 0$, is such that either $\mu = 0$ or $\mu > 1$ and μ is not a natural number, then equation (7.2) *has a center, that is $\rho = 0$ belongs to a continuum of periodic orbits.*

Now we are in conditions to prove our first result.

7.2.1 Proof of Theorem 7.6

For the particular Abel differential equation (7.4), we denote by $a_1(\theta) := \cos\theta + 2\cos 2\theta$, $a_2(\theta) := \sin\theta - \sin 2\theta + \sin 3\theta$ and

$$\tilde{a}_1(\theta) := \int_0^{\theta} a_1(s)ds, \quad \tilde{a}_2(\theta) := \int_0^{\theta} a_2(s)ds.$$

We have that the iterated integral

$$\begin{aligned} I_{221}(a) &= \int_{0\le s_1 \le s_2 \le s_3 \le 2\pi} a_2(s_3)\, a_2(s_2)\, a_1(s_1)\; ds_3\, ds_2\, ds_1 \\ &= -\int_0^{2\pi} \tilde{a}_1(s)\, \tilde{a}_2(s)\, a_2(s)\; ds \;=\; \frac{\pi}{2}. \end{aligned}$$

Therefore and on account of Theorem 7.3, if equation (7.4) has a center, it cannot be universal. Moreover, the function

$$H(\rho,\theta) := \frac{g^2 - (\cos\theta + \sin\theta - 1)g + 1 - \cos\theta}{g^2 + (\cos\theta + \sin\theta - 1)g + 1 - \cos\theta} \cdot e^{-4g + 2\arctan\left(\frac{(\cos\theta - \sin\theta - 1)g}{g^2 + \cos\theta - 1}\right)},$$

with

$$g(\rho,\theta)=\sqrt{\frac{1}{\rho}-\sin\theta+\sin 2\theta}$$

is a first integral of equation (7.4). This is because the function $H(\rho,\theta)$, for $\rho>0$ small enough, is of class $\mathcal{C}^1$; is not constant; is periodic in θ of period 2π; and satisfies $(\partial H/\partial\rho)\,\mathcal{F}(\rho,\theta)\,+\,\partial H/\partial\theta\,\equiv\,0$. Therefore, equation (7.4) has a center.

Another way to prove this statement is to note that the algebraic function

$$V(\rho,\theta)\;=\;\frac{\rho^{3/2}\left[2+2\sin(2\theta)\rho+\left(2-3\cos\theta+2\cos(2\theta)-\cos(3\theta)\right)\rho^2\right]}{2\sqrt{1-\left(\sin\theta-\sin(2\theta)\right)\rho}},$$

is an inverse integrating factor of equation (7.4). On account of Lemma 7.9 and since the leading term of the development of $V(\rho,\theta)$ around $\rho=0$ is $V(\rho,\theta)\;=\;\rho^{3/2}\;+\;o(\rho^{3/2})$ (that is $\mu=3/2$) we have that equation (7.4) has a center. ■

Our second result, Proposition 7.8, relies on the degrees of trigonometric polynomials. The following result is Lemma 16 of [71] deals with the relation between degrees of trigonometric polynomials.

Lemma 7.10 [71] *Let $A(\theta)$ and $B(\theta)$ be two trigonometric polynomials of degrees d and $\bar{d}$, respectively. The following statements hold.*

(a) *The trigonometric polynomial $A'(\theta)$ is of degree d.*

(b) *The trigonometric polynomial $A(\theta)B(\theta)$ is of degree $d+\bar{d}$.*

(c) *Let $N(z)$ be a polynomial in $\mathbb{R}[z]$ of degree k, then $N(A(\theta))$ is a trigonometric polynomial of degree $k\,d$.*

7.2.2 Proof of Proposition 7.8

If the Abel differential equation (7.1) has a universal center then we have that $\tilde{a}_1(\theta)$ and $\tilde{a_2}(\theta)$ satisfy the composition conditions, i.e., there exist a nonconstant trigonometric polynomial $q(\theta)$ and two real polynomials $p_1,p_2\in\mathbb{R}[z]$ such that $\tilde{a}_1(\theta)\;=\;p_1\left(q(\theta)\right)$ and $\tilde{a}_2(\theta)\;=\;p_2\left(q(\theta)\right)$. Let $d_i=\deg a_i$ for $i=1,2$. By Lemma 7.10(a), we have that $d_i\;=\;\deg\tilde{a}_i$ for $i=1,2$.

Assume first that d_1 and d_2 are both prime numbers. Then, by Lemma 7.10(c) we have that either $\deg q\;=\;1$ or $\deg p_1\;=\;\deg p_2\;=\;1$. In the case that $\deg q = 1$ the differential equation (7.1) has a center which is α-symmetric, see [71]. In the case that $\deg p_1 = \deg p_2 = 1$, we have $\tilde{a}_1(\theta) = \alpha_1 q(\theta) + \beta_1$ and $\tilde{a}_2(\theta) = \alpha_2 q(\theta) + \beta_2$ with α_i and β_i real numbers, $i = 1, 2$. As we can take without loss of generality that $q(0) = 0$, and since $\tilde{a}_1(0) = \tilde{a}_2(0) = 0$, we get that $\beta_1 = \beta_2 = 0$. Hence in this case equation (7.1) takes the form

$$\frac{d\rho}{d\theta} = q'(\theta)(\alpha_1 \rho^2 + \alpha_2 \rho^3),$$

which is an equation of separable variables.

Assume now that d_1 and d_2 are coprime. Again by Lemma 7.10(c), we have that $\deg q = 1$ (or it would be a common divisor of d_1 and d_2). Thus, the differential equation (7.1) has a center which is α-symmetric, see [71]. ■

Chapter 8

The composition condition for Abel differential equations

8.1 Introduction

Consider a planar differential system

$$\dot{x} = -y + P(x, y), \qquad \dot{y} = x + Q(x, y), \tag{8.1}$$

where the dot denotes derivation with respect to an independent real variable t, x and y are real and where P and Q are real analytic functions without constant nor linear terms. We recall that a singular point is a center if in a neighborhood of the singular point all the solutions are periodic. In this chapter we only consider the singular point at the origin of coordinates in system (8.1). The *center problem* consists in determining necessary and sufficient conditions on P and Q such that system (8.1) has a center at the origin.

In the particular case that P and Q are homogeneous polynomials system (8.1) can be transformed into an Abel trigonometric differential equation. More specifically if P and Q are homogeneous polynomials of degree n, with $n \geq 2$, the process is to take polar coordinates (r, θ) and system (8.1) becomes

$$\dot{r} = f(\theta) r^n, \qquad \dot{\theta} = 1 + g(\theta) r^{n-1},$$

where

$$\begin{aligned} f(\theta) &= P(\cos\theta, \sin\theta)\cos\theta + Q(\cos\theta, \sin\theta)\sin\theta, \\ g(\theta) &= Q(\cos\theta, \sin\theta)\cos\theta - P(\cos\theta, \sin\theta)\sin\theta. \end{aligned}$$

Now, as we have made in previous chapters, applying the Cherkas transformation [34] given by

$$\rho = \frac{r^{n-1}}{1+g(\theta)r^{n-1}} \quad \text{whose inverse is} \quad r = \frac{\rho^{1/(n-1)}}{(1-\rho g(\theta))^{1/(n-1)}},$$

system (8.1) becomes the Abel trigonometric differential equation

$$\frac{d\rho}{d\theta} = ((1-n)f(\theta) + g'(\theta))\,\rho^2 + ((1-n)f(\theta)g(\theta))\,\rho^3.$$

By the regularity of the Cherkas transformation and its inverse at $r = \rho = 0$, system (8.1) has a center at the origin if and only if the former ordinary differential equation has a center. Hence we have transformed the center-focus problem of system (8.1) into a center problem for an Abel differential equation. Other examples of systems of the form (8.1) which can be transformed into an Abel differential equation can be found in [50].

In this context a *trigonometric Abel differential equation* is an ordinary differential equation of the form

$$\frac{d\rho}{d\theta} = a_1(\theta)\rho^2 + a_2(\theta)\rho^3, \tag{8.2}$$

where ρ is real, θ is a real and periodic independent variable with $\theta \in [0, 2\pi]$, and $a_1(\theta)$ and $a_2(\theta)$ are real trigonometric polynomials. We recall that the *center problem for a trigonometric Abel differential equation* (8.2) is to characterize when all the solutions in a neighborhood of the solution $\rho = 0$ are periodic of period 2π.

Some authors also consider polynomial Abel differential equations as a model to tackle the center problem for a trigonometric Abel differential equation, see [17, 18, 19]. We denote as a *polynomial Abel differential equation* an ordinary differential equation of the form

$$\frac{dy}{dx} = p(x)y^2 + q(x)y^3, \tag{8.3}$$

where y is real, x is a real independent variable considered in a real interval $[a, b]$ and $p(x)$ and $q(x)$ are real polynomials in $\mathbb{R}[x]$. The *center problem for a polynomial Abel equation* (8.3) is to characterize when all the solutions in a neighborhood of the solution $y = 0$ take the same value when $x = a$ and $x = b$, i.e., $y(a) = y(b)$. In this framework, given any real continuous function $c(x)$, we denote by $\tilde{c}(x) := \int_a^x c(\sigma)d\sigma$ and we will say that a real continuous function $w(x)$ is *periodic in* $[a,b]$ if $w(a) = w(b)$.

Alwash and Lloyd in [7] provided a sufficient condition for an equation (8.2) to have a center in $[0, 2\pi]$. Inspired by this work, Briskin, Françoise and Yomdin in [17] provided the following sufficient condition for the polynomial Abel equation (8.3).

Theorem 8.1 *[17] If there exists a real differentiable function w periodic in $[a,b]$ and such that*

$$\tilde{p}(x) = p_1(w(x)) \quad \textit{and} \quad \tilde{q}(x) = q_1(w(x))$$

for some real differentiable functions p_1 and q_1, then the polynomial Abel equation (8.3) *has a center in $[a,b]$.*

In [43] it is shown that if the sufficient condition stated in Theorem 8.1 is satisfied then there is a countable set of definite integrals which need to vanish. In [43] it is also shown that this is equivalent to the existence of a real polynomial $w(x)$ with $w(a) = w(b)$ and two real polynomials $p_1(x)$ and $q_1(x)$ such that $\tilde{p}(x) = p_1(w(x))$ and $\tilde{q}(x) = q_1(w(x))$. This sufficient condition is known as the *composition condition*.

To see that the composition condition implies that equation (8.3) has a center in $[a,b]$ one can consider the transformation $y(x) = Y(w(x))$ in equation (8.3) in order to obtain the following Abel differential equation

$$\frac{dY}{dw} = p_1'(w)Y^2 + q_1'(w)Y^3. \tag{8.4}$$

Hence, there is a bijection between the solutions $Y = Y(w)$ of equation (8.4) and the solutions $y = Y(w(x))$ of equation (8.3). Since w is periodic in $[a,b]$, we get that equation (8.3) has a center in $[a,b]$ because $y(a) = Y(w(a)) = Y(w(b)) = y(b)$.

It turns out that all the known polynomial Abel differential equations which have a center in $[a,b]$ satisfy the composition condition. The *composition conjecture*, see Conjecture 8.3, is that the sufficient condition given in Theorem 8.1 is also necessary. That is, if a polynomial Abel equation (8.3) has a center in $[a,b]$, the conjecture states that the composition condition is satisfied.

For a trigonometric Abel differential equation (8.2), Alwash in [2] showed that this conjecture is not true, see also [6, 41]. The *composition condition* for a trigonometric Abel differential equation (8.2) is that there exist real polynomials $p_1(x), p_2(x) \in \mathbb{R}[x]$ and a trigonometric polynomial $\omega(\theta)$ such that $\tilde{a}_i(\theta) = p_i(\omega(\theta))$, for $i = 1, 2$. Recall that $\tilde{a}_i(\theta) := \int_0^\theta a_i(s)ds$. The fact that $\omega(\theta)$ and p_1, p_2 can be taken to be polynomials is proved in [43, 71]. There exist several counterexamples of the fact that the composition conjecture is not satisfied in the trigonometric case. The authors of [2, 6, 41] provide examples of trigonometric polynomials $a_1(\theta)$ and $a_2(\theta)$ for which the corresponding trigonometric Abel differential equation (8.2) has a center and does not verify the composition condition.

The chapter is organized as follows. The following section contains a summary of some conjectures related to the composition conjecture and the corresponding results. Section 8.3 is devoted to the known results about the composition conjecture together with two new statements, cf. Theorems 8.5 and 8.7. These statements are proved in sections 8.4 and 8.6, respectively. The last section 8.7 contains an appendix with the code of two programs, written in the language of Mathematica and used in these proofs.

8.2 Some other composition conjectures

In this section we consider the polynomial Abel differential equation

$$\frac{dy}{dx} = p(x)y^2 + \varepsilon q(x)y^3, \tag{8.5}$$

where y is real, x is a real variable considered on the real interval $[a, b]$, $\varepsilon \in \mathbb{R}$ and $p(x)$ and $q(x)$ are real polynomials. We also assume that $\int_a^b p(s)ds = 0$.

One of the problems that can be tackled is to characterize when equation (8.5) has a center in $[a, b]$ for all ε with $|\varepsilon|$ small enough. This type of centers are called *infinitesimal centers* or *persistent centers* , see [6, 41].

The following computations were first performed in [17]. We include them for the sake of completeness. Given real values ε and y_0, we denote by $Y_\varepsilon(x; y_0)$ the solution of equation (8.5) for the value of the parameter ε and with initial condition y_0, that is, the real function $Y_\varepsilon(x; y_0)$ satisfies

$$\frac{\partial}{\partial x} Y_\varepsilon(x; y_0) = p(x)Y_\varepsilon(x; y_0)^2 + \varepsilon q(x)Y_\varepsilon(x; y_0)^3, \quad Y_\varepsilon(a; y_0) = y_0. \tag{8.6}$$

We remark that, with this notation, a persistent center is when $Y_\varepsilon(b; y_0) = y_0$ for all ε with $|\varepsilon|$ small enough and for all y_0 with $|y_0|$ small enough.

Recall that we denote $\tilde{p}(x) = \int_a^x p(s)ds$. We note that when $\varepsilon = 0$, equation (8.5) has a center in $[a,b]$ because

$$Y_0(x;y_0) = \frac{y_0}{1 - y_0\tilde{p}(x)},$$

and clearly for all $|y_0| < \mu_p$, where

$$\mu_p := \min_{x\in[a,b]} \frac{1}{|\tilde{p}(x)|},$$

we have that $Y_0(x;y_0)$ is continuous in $[a,b]$ and $Y_0(a;y_0) = Y_0(b;y_0) = y_0$ due to the assumption $\tilde{p}(b) = \int_a^b p(s)ds = 0$. Note that $\mu_p > 0$ and therefore all the solutions in a neighborhood of $y = 0$ are defined for all $x \in [a,b]$. Equation (8.5) with $\varepsilon = 0$ has a center in $[a,b]$. As we have said, this means that there exists a family of periodic orbits in $[a,b]$ for equation (8.5) with $\varepsilon = 0$ in a neighborhood of the solution $y = 0$. The underlying idea when considering equation (8.5) is to determine which orbits in this family persist for values of ε with $|\varepsilon|$ small enough. Since the dependence of equation (8.5) in ε is analytic (indeed linear), we have that the dependence of $Y_\varepsilon(x;y_0)$ in ε is analytic. Thus, we can develop this function in ε in a neighborhood of $\varepsilon = 0$ as

$$Y_\varepsilon(x;y_0) = Y_0(x;y_0) + \pi_1(x;y_0)\varepsilon + o(\varepsilon).$$

Since, from (8.6), $Y_\varepsilon(a;y_0) = y_0$ and $Y_0(a;y_0) = y_0$, we deduce that $\pi_1(a;y_0) = 0$. Indeed, we can develop the first equation of (8.6) in powers of ε and equating the coefficients of ε^1 we deduce that

$$\frac{\partial}{\partial x}\pi_1(x;y_0) = 2p(x)Y_0(x;y_0)\pi_1(x;y_0) + q(x)Y_0(x;y_0)^3.$$

Integrating this linear ordinary differential equation for $\pi_1(x;y_0)$ we get that

$$\pi_1(x;y_0) = \frac{y_0^3}{(1 - y_0\tilde{p}(x))^2} \int_a^x \frac{q(\sigma)}{1 - y_0\tilde{p}(\sigma)}\, d\sigma. \tag{8.7}$$

Therefore, the necessary and sufficient condition for equation (8.5) to have a center in $[a,b]$ *at first order in* ε is that $\pi_1(b;y_0) = 0$. From (8.7), we deduce that this is to say that

$$\int_a^b \frac{q(\sigma)}{1 - y_0\tilde{p}(\sigma)}\, d\sigma \equiv 0,$$

for all y_0 with $|y_0|$ close enough to 0. We can develop this integral in powers of y_0 in a neighborhood of $y_0 = 0$ and we get that this condition is equivalent to

$$\int_a^b q(\sigma)\, \tilde{p}^{\,n}(\sigma)\, d\sigma \;=\; 0, \tag{8.8}$$

for all natural numbers $n \in \mathbb{N} \cup \{0\}$, see [6]. Conditions (8.8) are called the *moment conditions*. The *composition conjecture for moments* is that the moments conditions imply the composition condition. Moreover, in [20] it is proved that "at infinity" the center conditions are reduced to the moment conditions.

A counterexample to the composition conjecture for moments in the polynomial case was given in [104]. We reproduce here this example, see also [5, 6, 41].

In equation (8.5) we take $p(x) = T_6'(x)$ and $q(x) = T_2'(x) + T_3'(x)$ where $T_i(x)$ denotes the i-th Chebyshev polynomial and $T_i'(x)$ its derivative. We have that $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$ and $T_6(x) = (T_3 \circ T_2)(x) = (T_2 \circ T_3)(x) = 32x^6 - 48x^4 + 18x^2 - 1$. We take also $a = -\sqrt{3}/2$ and $b = \sqrt{3}/2$. Under these conditions the moment conditions (8.8) are zero taking into account that $T_2(\sqrt{3}/2) - T_2(-\sqrt{3}/2) = T_3(\sqrt{3}/2) - T_3(-\sqrt{3}/2) = 0$. We note that if an equation (8.5) satisfies the composition condition then the moment conditions are satisfied. Indeed, if an equation (8.5) satisfies the composition condition then the following conditions

$$\int_a^b p(\sigma) \tilde{q}^{\,n}(\sigma) d\sigma \;=\; 0, \tag{8.9}$$

are satisfied for all natural numbers $n \in \mathbb{N} \cup \{0\}$. This is due to the fact that if $p(x)$ and $q(x)$ satisfy the composition condition then the integrands of the integrals (8.8) and (8.9) are functions of $w(\sigma)$ multiplied by $w'(\sigma)$ and since $w(\sigma)$ is periodic in $[a, b]$, we deduce that they all need to be zero. Now we see that there are integrals in (8.9) for this example that are not zero. For instance,

$$\int_a^b p(\sigma) \tilde{q}^{\,2}(\sigma) d\sigma \;=\; \int_{-\sqrt{3}/2}^{\sqrt{3}/2} T_6'(\sigma)(T_2(\sigma) + T_3(\sigma))^2 d\sigma \;\neq\; 0.$$

Hence, equation (8.5) with $p(x) = T_6'(x)$ and $q(x) = T_2'(x) + T_3'(x)$ does not satisfy the composition condition.

We even have a stronger result. If one considers the differential equation of this example with $\varepsilon = 1$, this equation does not have a center in $[-\sqrt{3}/2, \sqrt{3}/2]$. Easy computations show that the sixth Poincaré–Liapunov constant is $v_6 = 432\sqrt{3}/385$. We have used the method explained in section 8.4 to compute the Poincaré–Liapunov constants v_2, v_3, v_4, v_5 and v_6. Therefore, this example shows that if the equation has a center at first order of ε, then it is not necessary that the equation has a center when $\varepsilon = 1$.

In the trigonometric case, that is, if one considers a trigonometric Abel differential equation of the form

$$\frac{d\rho}{d\theta} = a_1(\theta)\rho^2 + \varepsilon a_2(\theta)\rho^3, \tag{8.10}$$

where ρ is real, θ is a real and periodic independent variable with $\theta \in [0, 2\pi]$ and ε is a real value close to 0, one can define the composition conjecture for moments analogously to the polynomial case. The moment conditions in this case write as

$$\int_0^{2\pi} \tilde{a}_1^n(\theta)a_2(\theta)d\theta = 0, \tag{8.11}$$

with $n \in \mathbb{N} \cup \{0\}$. It is also possible to construct a counterexample of the composition conjecture for moments as the following example shows. We take in equation (8.10) $a_1(\theta) = \sin 3\theta$ and $a_2(\theta) = \cos\theta$. In this case $\tilde{a}_1(\theta) = (1-\cos 3\theta)/3$. It is easy to see that $\tilde{a}_1^n(\theta)$ will be a linear combination of the trigonometric functions $1, \cos 3\theta, \cos 6\theta, \ldots$ all of them orthogonal to $a_2(\theta) = \cos\theta$. Hence all the moment conditions (8.11) are satisfied. However the integrals

$$\int_0^{2\pi} a_1(\theta)\tilde{a}_2^n(\theta)d\theta, \tag{8.12}$$

with $n \in \mathbb{N} \cup \{0\}$, are in general not zero.

In [100], the characterization of all the pairs of real polynomials $p(x)$ and $q(x)$ for which the moment conditions (8.8) are satisfied is given. We note that this result characterizes all the Abel differential equations (8.5) with a center at first order of ε.

Theorem 8.2 *[100] Given $p(x)$ and $q(x) \in \mathbb{R}[x]$ and $a < b \in \mathbb{R}$. The moment conditions*

$$\int_a^b \tilde{q}(\sigma)\, p^{\,n}(\sigma)\, d\sigma = 0,$$

for all $n \in \mathbb{N} \cup \{0\}$ *are satisfied if and only if there exist* $w_1(x)$, $w_2(x)$, $\ldots$, $w_m(x) \in \mathbb{R}[x]$ *with* $m \geq 1$ *and* $w_i(a) = w_i(b)$ *for* $i = 1, 2, \ldots, m$ *such that*

$$\tilde{p}(x) = p_1(w_1(x)) = \cdots = p_m(w_m(x)) \text{ and } \tilde{q}(x) = \sum_{i=1}^{m} q_i(w_i(x)),$$

where $p_i(x)$ *and* $q_i(x) \in \mathbb{R}[x]$ *for* $i = 1, 2, \ldots, m$.

In [100] several examples are given for which the conditions stated in Theorem 8.2 are given and the composition condition is not satisfied.

However, in [41] it is shown that the natural translation of Theorem 8.2 to the trigonometric case does not hold. That is, it can be shown that there are differential equations of the form (8.10) with a center at first order of ε which do not satisfy the thesis of Theorem 8.2. The characterization of the trigonometric Abel differential equations (8.10) with a center at first order of ε is an open problem.

In [41] it is proved that the existence of a center in $[a, b]$ for all ε small enough of equation (8.5) (that is, a persistent center) implies the conditions (8.8) and (8.9). In the trigonometric case, it is also shown that if equation (8.10) has a persistent center then the conditions (8.11) and (8.12) need to be verified.

Recently in [106] it is proved that if conditions (8.8) and (8.9) for a polynomial Abel differential equation (8.5) are verified, then the composition condition is satisfied.

In the trigonometric case, that is for equation (8.10), if all moments conditions (8.11) and (8.12) are satisfied then equation (8.2) does not necessarily satisfy the composition condition, see [41]. However, under these hypothesis, it may happen that the equation (8.10) with $\varepsilon = 1$ has a center as the example of section 3 in [41] shows, see also [42].

The *generalized moment conditions* are

$$\int_a^b \tilde{p}^{\,n}(\sigma)\tilde{q}^{\,m}(\sigma)q(\sigma)d\sigma = 0 \quad \text{and} \quad \int_a^b \tilde{p}^{\,n}(\sigma)\tilde{q}^{\,m}(\sigma)p(\sigma)d\sigma = 0$$

for all $n, m \in \mathbb{N} \cup \{0\}$. A proof that the generalized moment conditions imply that the polynomial Abel equation satisfy the composition condition is given in [24, 105, 41]. A proof of the translation of this fact for the trigonometric Abel equation is given in [42].

In [43] the authors provide an explicit bound of the number of generalized moments (also called double moments) that have to vanish to ensure that an Abel differential equation, either in the trigonometric form (8.2) or in the polynomial form (8.3), satisfies the composition condition. This result allows to recognize the centers which satisfy the composition condition or simply *composition centers* for polynomial and trigonometric Abel differential equation. This last result is used in the next section to computationally approach the composition conjecture. In [22] the definition of *universal center* was introduced, which coincides with the definition of composition center, see [23, 71].

8.3 Composition conjecture

Given a polynomial Abel differential equation (8.3), the *center variety* is the set of polynomials $p(x)$ and $q(x)$ for which the equation has a center in $[a, b]$ and the *composition center variety* is the set of polynomials $p(x)$ and $q(x)$ for which the equation has a universal or composition center (that is the composition condition is verified) in $[a, b]$. After all that we have said in the previous sections the statement of the composition conjecture is the following.

Conjecture 8.3 *For any polynomial Abel differential equation* (8.3) *the center variety and the composition center variety coincide.*

We recall that this conjecture is not true for trigonometric Abel differential equations, see [2, 6, 41, 71, 73, 74]. Moreover in [74] was proved that the lowest degree of a trigonometric Abel differential equation (8.2) with a non-composition center is 3. Conjecture 8.3 is satisfied under certain restrictions of the coefficients of the polynomial Abel differential equation, see for instance Theorem 2 in [6] and Theorem 2 in [15].

However a systematic verification of Conjecture 8.3 has not been done. The aim of this section is to verify if all the centers of the polynomial Abel differential equation (8.3) for lower degrees of p and q are composition centers. We also analyze the case in which the number of monomials in $p(x)$ and $q(x)$ is up to 2.

As we have said, in [42] another characterization of the composition centers is provided in terms of the vanishing of a finite set of generalized moments or double moments. As usual for a polynomial $p(x) \in \mathbb{R}[x]$, δp denotes the degree of p.

Theorem 8.4 *[43] Given p, $q \in \mathbb{R}[x]$ with $\max(\delta p, \delta q) = n$, equation (8.3) has a composition center if and only if for all $i, j \in \mathbb{N} \cup \{0\}$ satisfying $i + j \leq 2n - 3$,*

$$\int_a^b \tilde{p}^i(x)\tilde{q}^j(x)q(x)\,dx \;=\; \int_a^b p(x)\,dx \;=\; 0. \tag{8.13}$$

This characterization of the composition centers allows to discriminate the composition centers from other centers and approaches the conjecture from a computational point of view. The main results of the paper are the following.

Theorem 8.5 *For any polynomial Abel differential equation with degree given by $\max(\delta p, \delta q) \leq 3$ the center variety and the composition center variety coincide.*

The proof of Theorem 8.5 is given in section 8.4.

We have also dealt with the case in which $\max(\delta p, \delta q) = 4$. In this case we cannot end up with all the computations to ensure that the center variety and the composition center variety coincide. In section 8.5, we will make use of modular arithmetics and the algorithm described in [114] which provide the center variety with a probability close to 1. All the pairs $p(x)$ and $q(x)$ that we find using this algorithm give rise to composition centers. Therefore, we can state the following conjecture.

Conjecture 8.6 *For any polynomial Abel differential equation with degree $\max(\delta p, \delta q) = 4$ the center variety and the composition center variety coincide.*

The computations motivating this conjecture are given in section 8.5.

Given a polynomial Abel differential equation (8.3) defined in the real interval $[a, b]$, with $a < b$, we can make the following change of the independent variable $x \to (x - a)/(b - a)$. This leads to an Abel differential equation defined on the real interval $[0, 1]$.

Theorem 8.7 *Consider a polynomial Abel differential equation* (8.3) *defined on the real interval* $[0, 1]$. *Assume that* p *and* q *only have two monomials, that is,*

$$p(x) = a_i x^i + a_j x^j \quad \text{and} \quad q(x) = a_m x^m + a_n x^n,$$

with $a_i, a_j, a_m, a_n \in \mathbb{R}$ *and* $i, j, m, n \in \mathbb{N} \cup \{0\}$. *Then the center variety and the composition center variety coincide.*

The proof of Theorem 8.7 is given in section 8.6.

8.4 Proof of Theorem 8.5

Given an ordinary differential equation of the form (8.3), there is a well known general method to compute center conditions which was proved by Poincaré. We will denote the center conditions as the *Poincaré–Liapunov constants* for equation (8.3). In order to compute them we propose a formal first integral of the form $H(y, x) = y + \sum_{k=2}^{\infty} h_k(x) y^k$, where $h_k(x)$ are polynomials. We recall that a first integral for an equation (8.3) satisfies that $\dot{H} = \dot{y}\, \partial H/\partial y + \dot{x}\, \partial H/\partial x \equiv 0$, where $\dot{y} = p(x)y^2 + q(x)y^3$, $\dot{x} = 1$. By imposing that $\dot{H} = 0$, we obtain the following recursive system of linear differential equations

$$h_k'(x) + (k-1)p(x)h_{k-1}(x) + (k-2)q(x)h_{k-2}(x) = 0, \tag{8.14}$$

for $k \geq 2$ and with $h_0(x) \equiv 0$ and $h_1(x) \equiv 1$. From the recursive system (8.14) we compute the polynomials $h_k(x)$ and we obtain the Poincaré–Liapunov constant $v_k := h_k(b) - h_k(a)$. The equation has a center in $[a, b]$ if $v_k = 0$ for all $k \geq 2$. We note that v_k is a polynomial in the coefficients of $p(x)$ and $q(x)$.

We denote the coefficients of $p(x)$ and $q(x)$ in the following way $p(x) = \sum_{i=0}^{3} b_i x^i$, $q(x) = \sum_{i=0}^{3} c_i x^i$.

In order to proof the result we have computed fifteen necessary conditions $v_k = 0$ for $k = 2, \ldots, 16$. These necessary conditions are very long, so we do not present them here. However, one can check our computations with the help of any available computer algebra system. In this case, in order to obtain the families of centers we look for the irreducible decomposition of the variety $V(I)$ of the ideal $I = \langle v_2, v_3, \ldots, v_{16} \rangle$. This is an extremely difficult computational problem. We have used the routine *minAssGTZ* of the computer algebra system Singular [81] and we have found the irreducible decomposition of the variety of the ideal I over the field of rational numbers when $\max(\delta p, \delta q) \leq 3$.

The obtained decomposition consists of 2 components defined by the following ideals

1) $\langle 2c_2 + 3c_3, 4c_0 + 2c_1 - c_3, 2b_2 + 3b_3, 4b_0 + 2b_1 - b_3 \rangle$;

2) $\langle b_3c_2 - b_2c_3, b_3c_1 - b_1c_3, b_2c_1 - b_1c_2, 12c_0 + 6c_1 + 4c_2 + 3c_3, 12b_0 + 6b_1 + 4b_2 + 3b_3 \rangle$;

The generalized moment conditions u_i are obtained computing the integrals (8.13) and in this case we have found the irreducible decomposition of the variety of the ideal $J = \langle u_1, u_2, \ldots, u_{17} \rangle$ over the field of rational numbers. To deduce if all the centers are composition centers we must only compare both decompositions and in both cases they are the same.

8.5 On the Conjecture 8.6

As before, we denote the coefficients of $p(x)$ and $q(x)$ in the following way $p(x) = \sum_{i=0}^{4} b_i x^i$, $q(x) = \sum_{i=0}^{4} c_i x^i$. We have computed fifteen necessary conditions $v_k = 0$ for $k = 2, \ldots, 16$, that we do not present here. In order to obtain the families of centers we look for the irreducible decomposition of the variety $V(I)$ as in the previous section. In this case, however, we cannot find the irreducible decomposition of the variety of the ideal I over the field of rational numbers due to the computational difficulty. We try to find this irreducible decomposition over a finite field. We take the prime $p = 32003$ and we have found this decomposition over the finite field $\mathbb{Z}/(p)$. We have chosen this prime because the algorithm turned out to be very efficient and goes to a reasonable speed when using it.

We have followed the algorithm described in [114] which makes use of modular arithmetics. The modular approach used to obtain center conditions consists on the following five steps.

Step 1. Choose a prime number p and from the ideal I compute the minimal associated primes $\tilde{I}_1, \ldots, \tilde{I}_s$ with coefficients in $\mathbb{Z}_p$,

Step 2. Using the rational reconstruction algorithm of Wang et al. [124], we obtain the ideals I_i, $i = 1, \ldots, s$, with coefficients in $\mathbb{Q}$, Step 3. For each i, using the radical membership test, check whether the polynomials v_k for $k = 2, \dots, 16$ are in the radicals of the ideals I_i, that is, whether the reduced Gröbner basis of the ideal $< 1 - wv_j, I_i >$ is equal to $\{1\}$, where w is a mute variable. If yes, then go to *Step 4*, otherwise take another prime p and go to *Step 1*.

Step 4. Compute the intersection over the rational numbers $Q = \cap_{i=1}^{s} I_i$,

Step 5. Check that $\sqrt{Q} = \sqrt{I}$, that is, that for any $q_i \in Q$, the reduced Gröbner basis of the ideal $< 1 - wq_i, I >$ is equal to $\{1\}$ and for any $v_j \in I$, the reduced Gröbner basis of the ideal $< 1 - wv_j, Q >$ is equal to $\{1\}$. Recall that $I = \langle v_2, v_3, \dots, v_{16} \rangle$. If this is the case, then $V(I) = \cup_{i=1}^{s} V(I_i)$. If not, then go to *Step 1* and choose another prime p.

We note that whenever we compute the Gröbner basis of an ideal, we must to do it over the field of rational numbers.

The last step of this algorithm has not been verified into the field of rational numbers. However, we have checked it over finite fields $\mathbb{Z}/(p)$, with different prime numbers p. This last step ensures that all the points of the variety $V(I)$ have been found. That is, we know that all the encountered points belong to the decomposition of $V(I)$ but we do not know whether the given decomposition is complete. We remark that, nevertheless, it is practically sure that the given list is complete, see for instance [11, 80, 114]. Therefore, in the following we provide sufficient conditions to have a center, which are practically necessary. We denote this situation by the expression *with probability close to* 1.

The obtained decomposition for the case $\max(\delta p, \delta q) = 4$ consists of 2 components defined by the following ideals

1) $\langle c_4, 2c_2 + 3c_3, 4c_0 + 2c_1 - c_3, b_4, 2b_2 + 3b_3, 4b_0 + 2b_1 - b_3 \rangle$;

2) $\langle b_4c_3 - b_3c_4, b_4c_2 - b_2c_4, b_3c_2 - b_2c_3, b_4c_1 - b_1c_4, b_3c_1 - b_1c_3, b_2c_1 - b_1c_2, 60c_0 + 30c_1 + 20c_2 + 15c_3 + 12c_4, 60b_0 + 30b_1 + 20b_2 + 15b_3 + 12b_4 \rangle$;

The generalized moment conditions u_i are obtained computing the integrals (8.13) and in this case we have found the irreducible decomposition of the variety of the ideal $J = \langle u_1, u_2, \dots, u_{17} \rangle$ over the field of rational numbers. To deduce if all the centers are composition centers we must only compare both decompositions and in both cases they are the same.

For the case $\max(\delta p, \delta q) = 5$ without loss of generality we can divide the study in two cases: either $b_5 = 1$ or $b_5 = 0$ and $c_5 = \pm 1$. We get these cases by a rescaling of the form $y = kY$ with $k \neq 0$. Even in the simple case that $b_5 = 0$ and $c_5 = \pm 1$, and using modular arithmetics, we have not been able to find the irreducible decomposition of the variety $V(\langle v_2, v_3, \ldots, v_{16}\rangle)$.

8.6 Proof of Theorem 8.7

In equation (8.3) we write $p(x) = a_i x^i + a_j x^j$ and $q(x) = a_m x^m + a_n x^n$. Using the method of construction of a formal first integral described at the beginning of section 8.4, we obtain that the first Poincaré–Liapunov constant is $v_2 = -(1+j)a_i - (1+i)a_j$. All the Poincaré–Liapunov constants computed in this section have been obtained by using the algorithm described in the appendix. The vanish of v_2 gives us $a_i = (1+i)a_j/(1+j)$. The second Poincaré–Liapunov constant is $v_3 = -(1+n)a_m - (1+m)a_n$. Vanishing this constant we obtain $a_n = (1+n)a_m/(1+m)$. We note that at this moment we have that $\tilde{p}(x) = \int_0^x p(\sigma)d\sigma$ and $\tilde{q}(x) = \int_0^x q(\sigma)d\sigma$ satisfy that $\tilde{p}(0) = \tilde{p}(1) = 0$ and $\tilde{q}(0) = \tilde{q}(1) = 0$.

The third Poincaré–Liapunov constant is given by $v_4 = -a_j a_m (i-j)(m-n)(i+j+ij-m-n-mn)$. We divide the study of the vanishing of v_4 in three cases.

First case: $a_j a_m = 0$. When $a_j = 0$ we have that $a_i = 0$ and then $p(x) \equiv 0$. This case gives a differential equation with separated variables which forms a composition center (recall that $\tilde{q}(0) = \tilde{q}(1) = 0$). In the case that $a_m = 0$ we get an analogous result.

Second case: $(i-j)(m-n) = 0$. If $i = j$ then $p(x)$ has a single monomial $p(x) = a_j x^j$ and then $\tilde{p}(x) = a_j x^{j+1}/(j+1)$. The condition $\tilde{p}(1) = 0$ implies that $a_j = 0$ and, hence, $p(x) \equiv 0$. We get again a differential equation with separated variables which forms a composition center (recall that $\tilde{q}(0) = \tilde{q}(1) = 0$). In the case that $m = n$ we get an analogous result.

Third case: $i + j + ij - m - n - mn = 0$. We take $i = (-j + m + n + mn)/(1+j)$. The next Poincaré–Liapunov constant is

$$v_5 = -2a_j^2 a_m (j-m)(j-n)(m-n)(2j + j^2 - m - n - mn)^2.$$

Excluding the previous cases, we get that either $j = n$ or $(2j + j^2 - m - n - mn) = 0$. In the case that $j = n$ we get that there exists a constant

C such that $p(x) = Cq(x)$ which forms a composition center. In the latter case $(2j + j^2 - m - n - mn) = 0$, we obtain that $i = j$ and, thus, $p(x) \equiv 0$.

8.7 Appendix

Program to compute the Poincaré–Liapunov constants for an equation (8.3) defined in the interval $[0, 1]$ and with p and q polynomials up to degree 5.

```
p = b0 + b1x + b2x^2 + b3x^3 + b4x^4 + b5x^5;
q = c0 + c1x + c2x^2 + c3x^3 + c4x^4 + c5x^5;
h = -Apply[Plus, Integrate[Cases[Expand[p],_], x]];
Numerator[Factor[(h/.x-> 1) - (h/.x-> 0)]] >> v2.txt
hh = h; h = -Apply[Plus, Integrate[Cases[Expand[2 * p * hh + q],_], x]];
Numerator[Factor[(h/.x-> 1) - (h/.x-> 0)]] >> v3.txt
For[k = 4, k < 16, k++, hhh = hh; hh = h;
h = -Apply[Plus, Integrate[Cases[Expand[(k - 1) * p * hh
+(k - 2) * q * hhh],_], x]];
Put[Numerator[Factor[(h/.x-> 1) - (h/.x-> 0)]],
StringJoin["v", ToString[k], ".txt"]]]
```

Program to compute the moment conditions for an equation (8.3) defined on $[0, 1]$.

```
p = b0 + b1x + b2x^2 + b3x^3 + b4x^4 + b5x^5;
q = c0 + c1x + c2x^2 + c3x^3 + c4x^4 + c5x^5;
P = Integrate[p/.x-> z, z, 0, x];
Q = Integrate[q/.x-> z, z, 0, x];
Numerator[Factor[Integrate[p, x, 0, 1]]] >> u1.txt;
Numerator[Factor[Integrate[q, x, 0, 1]]] >> u2.txt;
For[i = 1, i < 8, i++, For[j = 1, j < i + 1, j++,
Put[Numerator[Factor[Apply[Plus, Integrate[Cases[
Expand[P^(i - j) * Q^j * p],_], x, 0, 1]]]],
StringJoin["u", ToString[2 + (i - 1)i/2 + j], ".txt"]]]]
```

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