Coalitional Stability and Economic Efficiency

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A Elia, por todo.

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General Introduction

The maps showing countries[1] in the world have undergone dramatic changes during the last few decades. Since 1946 the number of independent countries has risen from 74 to 193. Several countries, like the former Soviet Union, Czechoslovakia or Yugoslavia have broken-up. Germany has re-unified. The European Union is moving towards more economic integration by introducing the Euro and it is not clear how far the political integration will go. In almost all continents of the world conflicts about regional power, autonomy or even independence still await a solution.

Inevitably such important changes and conflicts attract the attention of academics and among them economists who want to understand what drives these changes and attempt to predict the future trends they give rise to. The seminal paper by Alesina and Spolaore (*The Quart. J. of Econ.* **112**, 1997, 1027-1056) is a cornerstone of the recent political economy literature addressing these issues. (See the survey by Bolton, Roland and Spolaore in *Europ. Econ. Rev.* **40**, 1996, 697-705.) Alesina and Spolaore provide an analytical framework for the positive and normative analysis of country formation. They raise the fundamental question: What is the "optimal" number and size of countries and how does this structure compare to the

[1] Throughout this thesis we use the words "country", "jurisdiction" and "coalition" interchangeably.

one resulting from the democratic process? The basic model they consider has the following features:

They use a spatial model where the world's population is a continuum of agents distributed uniformly on the line segment $[0, 1]$. Each individual consumes a (local) public good and incurs a transportation cost proportional to the distance between her location and the location of the public good in her country. In each country the location of the public good is decided by majority rule and the costs to produce the public good are covered using a proportional tax scheme. Thus, driving force of the process of country formation is a trade-off between the benefits of large countries and the costs of heterogeneity in large populations. *The main implication of this model is that in equilibrium one generally observes an inefficiently large number of countries.*

The fact that individuals join each other in order to form a country can be seen as the formation of a "coalition" and the formation of countries can therefore be interpreted as an application of another economic literature - the one on coalition formation. (See the survey by Greenberg in *Handbook of Game Th. Vol. 2*, ed. by Aumann and Hart, 1994, 1305-1337.) However, the connections between the Alesina and Spolaore model and this literature are not immediately evident and its most important differences are the following two:

1. This literature traditionally assumes a finite number of consumers. The use of the logical construct of a continuum of agents is very convenient because it allows to apply very elegant and powerful mathematical tools. However, it is desirable that the conclusions derived in this way are confirmed for a large, but finite, number of agents.

2. The equilibrium concepts employed by both approaches are different. A concept frequently used in the literature on coalition formation has been

called by Greenberg and Weber (*J. of Econ. Th.* **38**, 1986, 101-117) strong Tiebout equilibrium. Alesina and Spolaore employ some ideas of this concept in an apparently weak form and complete it by some requirements special to the formation of countries which are usually not used in the literature on coalition formation.

The purpose of this thesis is to clarify these two differences and to carry out a test of robustness of the inefficiency result of Alesina and Spolaore to a *ceteris paribus* change to a finite number of consumers on one hand and to the concept of a strong Tiebout equilibrium on the other. The thesis is divided in three Chapters.

Chapter 1. We start by analyzing the second difference and apply the concept of a strong Tiebout equilibrium to the Alesina and Spolaore model. We find that the inefficiency result is robust to this change in the sense that one can understand their equilibrium as the unique outcome of a selection among all strong Tiebout equilibria where the selection is driven by specific rules of country formation. However, the concept of a strong Tiebout equilibrium in itself has no inherent forces that confirm the inefficiency result. We also modify the specific rules of country formation which Alesina and Spolaore employ in order to make them closer to the rules used in reality. This leads to the problem of nonexistence of an equilibrium.

Chapter 2. We proceed by analyzing the first difference and introduce a finite number of consumers in the model. We determine efficient coalition structures and employ a weak equilibrium concept that uses only ideas common to strong Tiebout equilibria and the Alesina and Spolaore concept. Since our notion is very weak, a multiplicity of equilibria arises for some of which the inefficiency result of Alesina and Spolaore is true while for others it is not.

Chapter 3. We extend the analysis of the previous Chapter. Since we show that it is not convenient to apply the equilibrium concept of Alesina and Spolaore in a model with a finite number of consumers, we are forced to employ another notion. We choose again the concept of a strong Tiebout equilibrium. While this helps to refine the multiplicity of equilibria found in the second Chapter, it does not lead to a unique equilibrium. Again the concept of strong Tiebout equilibria does not allow to confirm the inefficiency result of Alesina and Spolaore.

The general conclusion to draw from this thesis is rather ambiguous. On the one hand the formation of countries in itself is a very complicated field of study and the comparison of "optimal" configurations to the ones resulting from a democratic process is a very ambitious question. Alesina and Spolaore provide a very elegant analysis and their merits become even more visible in the light of this thesis. Their equilibrium has even more desirable properties. The simplifying assumptions in their analysis are carefully chosen and modifying them in order to make the model more realistic leads to problems of the existence of the equilibrium. But this shows on the other hand that there is much left for future research.

Chapter 1

Stability in the Alesina and Spolaore Country Formation Model

1.1 Introduction

The maps showing countries[1] in the world have undergone dramatic changes during the last few decades. Since 1946 the number of independent countries has risen from 74 to 193. Several countries, like the former Soviet Union, Czechoslovakia or Yugoslavia have broken-up. Germany has re-unified. The European Union is moving towards more economic integration by introducing the Euro and it is not clear how far the political integration will go. In almost all continents of the world conflicts about regional power, autonomy or even independence still await a solution.

[1] Throughout this paper we use the words "country", "jurisdiction" and "coalition" interchangeably.

In their seminal paper Alesina and Spolaore [1] - AS henceforth - address these issues.[2] They provide a simple positive and normative framework for the analysis of country formation. AS use a spatial model where the world's population is a continuum of agents distributed uniformly on the line segment $[0,1]$. Each individual consumes a (local) public good and incurs a transportation cost proportional to the distance between her location and the location of the public good in her country. In each country the location of the public good is decided by majority rule and the costs to produce the public good are covered using a proportional tax scheme. Thus, driving force of the process of country formation is a trade-off between the benefits of large countries and the costs of heterogeneity in large populations. *The main implication of this model is that in equilibrium one generally observes an inefficiently large number of countries.*

The purpose of this paper is to carry out a test of robustness of this result to *ceteris paribus* changes in the equilibrium concept. Our result is rather ambiguous. Some ingredients of the AS equilibrium concept can be strengthened while the modification of others leads either to a multiplicity or to the nonexistence of equilibria.[3]

The first ingredient of the AS-equilibrium concept is the idea that agents have, to some extend, the possibility to migrate between existing countries and to create new countries. These are frequent requirements for an equilibrium in the literature on coalition formation or on local public goods economies. A concept which strengthens considerably both requirements and assures therefore additional desirable properties has been called by

[2] For other papers offering an economic analysis of these issues see Le Breton and Weber [10] Haimanko, Le Breton and Weber [8], Bolton and Roland [2] or the survey of the literature in Bolton, Roland and Spolaore [3].

[3] In a multiplicity of equilibria there may be too many, too few or exactly the efficient number of countries.

Greenberg and Weber [5] strong Tiebout equilibrium.[4] We characterize strong Tiebout equilibria and show that there always exists an efficient one. However, the inefficiency result of AS is robust to the use of this concept in the sense that one can understand the unique inefficient equilibrium in AS as the outcome of a selection among strong Tiebout equilibria by means of specific rules of country formation. This is nice because it implies that the fact that all coalitions consist exactly of one interval on the line segment is an implication and not an assumption of the model.

These specific rules of country formation are the second ingredient of the AS-equilibrium concept. First, there is the requirement that after a perturbation at borders agents should move as to restore the initial situation. We show that this condition is very important because it rules out many reasonable strong Tiebout equilibria. Second, there is a rule that models international agreements over modifications of borders which must be ratified by simple majority rule. Because of the fact that in reality different countries specify very different rules that govern the ratification of these international agreements it is reasonable to ask for some robustness of the AS-inefficiency result to changes in the majority requirements. Unfortunately, under qualified majority rule an equilibrium does not exist.

The remainder of this paper is organized as follows. The next Section reviews the AS country formation model. Section 3 characterizes efficient coalition structures. The following Section deals with stable structures. It is divided in three parts which analyze the implications of free mobility, the consequences of the possibility to form new coalitions and the application of country formation. We conclude in the last Section. All proofs are relegated to the Appendix.

[4] For papers that use one or both requirements see also e.g. Greenberg and Weber [6], Demange [4], Jehiel and Scotchmer [9] or Haeringer [7] among others.

1.2 The Model

AS consider an economy with a population $\mathcal{P}$ consisting of a continuum of agents with mass one, uniformly distributed over the line segment $[0,1]$. We denote a generic member of $\mathcal{P}$ and its location by i.

A coalition S is any subset of $\mathcal{P}$ with $|S| > 0$, where $|S|$ denotes its size.[5] A coalition structure $\mathcal{W} = \{S_1, ..., S_N\}$ is any partition of $\mathcal{P}$ in coalitions, fulfilling that $|S_n \cap S_{n'}| = 0, \forall n \neq n'$ and $\cup_{n=1}^{N} S_n = \mathcal{P}$. This definition allows to think of coalitions as unions of closed intervals where boundary points of these intervals are intersections of coalitions.[6] We assign coalitions their subindex by the following procedure. Coalition S_1 has its left border at point 0. Starting at 0 and going to the right, the first agent belonging to another coalition than S_1 belongs to S_2. The next agent belonging to another coalition than S_2 forms part of either S_1 or S_3 and so on.

A ***connected coalition*** consists exactly of one interval and we say that a ***coalition structure is connected*** if all its coalitions are connected. If a coalition S_n consists of several closed intervals, we denote them by $S_{n,1}, ..., S_{n,K}$.[7]

We say that an agent i belongs to the border of two coalitions S_n and S_m if she belongs to both S_n and S_m. We denote these agents by $b(S_{n,k}, S_{m,l})$ (where $n < m$). Neighboring coalitions are those which share a border. The set $B(\mathcal{W})$ is the set of all border agents given $\mathcal{W}$.

For a given coalition structure $\mathcal{W}$, $S_i(\mathcal{W})$ is the coalition i belongs to. To save on notation, we often simply write S_i or even S for $S_i(\mathcal{W})$, b instead of $b(S_n, S_m)$ and B for $B(\mathcal{W})$. Also, given $\mathcal{W}$, we denote by $|S_{max}(\mathcal{W})|$ and

[5] The assumption that $|S| > 0$ assures a finite number of coalitions.

[6] This implies that every agent i belongs to at least one coalition but some agents (those located at boundary points) belong to more than one coalition. However, the measure of these agents is zero.

[7] Note that $S_{n,k}$ may be a singleton.

$|S_{min}(\mathcal{W})|$ the largest and smallest coalition size in $\mathcal{W}$. For simplicity we will write $|S_{max}|$ and $|S_{min}|$ whenever it is clear which coalition structure $\mathcal{W}$ is meant.

Coalitions have to provide a local public good bundle $l(S) \in [0,1]$. All agents have the same utility function, which is decreasing in the distance to $l(S)$ and increasing in $|S_i|$. This function can be represented by the individual cost function

$$c_i(S) = cd_i(S) + \frac{1}{|S_i|}, \tag{1.1}$$

where $d_i(S) = |i - l(S)|$ and the nonnegative parameter c measures the relative importance of the costs of being in a heterogeneous coalition with respect to the 'public good provision' costs.[8] For convenience we will also use the simplifying notation $c_i(\mathcal{W})$.

The decision over the location of the public good $l(S)$ is taken by majority voting. This implies, since individual utilities are single-peaked with respect to $l(S)$, that the median voter determines $l(S)$. In case of ties in unconnected coalitions we suppose that $l(S)$ coincides with the left median position.[9]

1.3 Efficient Coalition Structures

A coalition structure is efficient if it minimizes the sum of the individual cost functions subject to the constraint that the sum of individual contributions to public good provision must equal its total costs. This implies the following important result.

[8] AS postulate $U_i(S) = \alpha(1 - \beta d_i(S)) + y - \frac{\gamma}{|S_i|}$, where y represents income and α, β and γ are positive parameters. Hence $c = \frac{\alpha\beta}{\gamma}$.

[9] This assures that each location $l(S)$ of a local public good is an element of the union of intervals which constitute the coalition S.

PROPOSITION **1.1** *If $\mathcal{W}$ is an efficient coalition structure, then it is connected.*

The other properties of efficient coalition structures are characterized by proposition 1 in AS.

PROPOSITION **1.2** *[Alesina and Spolaore 1997] The social planner (i) locates the local public good in the middle of each coalition, and (ii) chooses N^* coalitions of equal size, such that*

$$N^* = \frac{\sqrt{c}}{2}, \tag{1.2}$$

provided that $\frac{\sqrt{c}}{2}$ is an integer. Otherwise the efficient number of coalitions N^* is given by either the largest integer smaller than $\frac{\sqrt{c}}{2}$, or the smallest integer larger than $\frac{\sqrt{c}}{2}$.

1.4 Stable Coalition Structures

1.4.1 Tiebout equilibria

The purpose of this Section is to analyze the consequences of free mobility of agents for coalition structures. A partition into coalitions is a Tiebout equilibrium if no individual wants to migrate to any other existing coalition.[10] More formally:

DEFINITION **1.1** *A coalition structure $\mathcal{W} = \{S_1, ..., S_N\}$ is a* ***Tiebout equilibrium [TE]*** *if for all $i \in [0, 1]$, we have,*

$$c_i(S_i) \leq c_i(S'), \forall S' \in \mathcal{W}. \tag{1.3}$$

[10] Different from AS or Jehiel and Scotchmer [9] we do not require a stability condition. But we will investigate the implications of such a requirement in a later Section.

Since we work with individual cost instead of utility, condition (1.3) states that each individual prefers his own coalition to any existing coalition. An important consequence of this requirement is the following result.

LEMMA **1.1** *If a coalition structure $\mathcal{W}$ is a Tiebout equilibrium, then $\mathcal{W}$ is connected.*

REMARK: An important consequence of this result and proposition 1.1 is that the implicit assumption of AS that coalition structures must be connected can in fact be derived as resulting from the requirements of efficiency and free mobility of agents.

The next proposition characterizes TE.

PROPOSITION **1.3** *A coalition structure $\mathcal{W} = \{S_1, ..., S_N\}$ is a Tiebout equilibrium if and only if $\mathcal{W}$ is connected and*

- *either all coalitions have the same size*
- *or there exist exactly two different sizes of coalitions with $|S_{min}||S_{max}| = \frac{2}{c}$.*

This result says that the requirement of free mobility of agents implies a very specific structure for $\mathcal{W}$. Coalitions are intervals and there can be at most two different coalition sizes.

REMARK: The definition of a Tiebout equilibrium is stronger than the definition of an A-equilibrium in AS because it allows any agent (and not only individuals located at borders) to move to any coalition (and not only to neighboring ones). However, the reader familiar with the work of AS will have noticed that the set of Tiebout equilibria coincides with the set of A-equilibria. Therefore, in the AS-model the fulfillment of a very weak free mobility condition (A-equilibrium) assures the desirable properties of a free mobility (Tiebout) equilibrium. This is true because in a given coalition S_n everyone is at least as well off as the agents at the border. Moreover, if an agent joins another coalition S_m she will be worse off than the border agents in S_m. Since in an A-equilibrium all border agents get the same utility, no agent has an incentive to move.

1.4.2 Strong Tiebout Equilibria

The purpose of this Section is to impose on Tiebout equilibria the additional requirement that there should be no group of agents that can all become better off by creating a new coalition.

One reason for the creation of a new coalition may be that, given c, coalitions in the initial structure may be so small that agents would like to form larger coalitions. Note that proposition 1.3 does not imply any minimum size for coalitions in a Tiebout equilibrium. A first step is to look if there exist two coalitions whose agents would unanimously agree to merge to one large coalition S^M. A coalition structure in which such a consent cannot be reached is pairwise-merger-proof.

DEFINITION **1.2** *A coalition structure $\mathcal{W} = \{S_1, ..., S_N\}$ is **pairwise-merger-proof [PMP]** if in any two coalitions S_n and S_m of $\mathcal{W}$ there is no unanimous consent to merge and form S^M, that is, there exists $i \in S_n \cup S_m$ with,*

$$c_i(S^M) \geq c_i(\mathcal{W}). \tag{1.4}$$

This requirement just says that in any merger of coalitions that can be proposed, there should be at least one agent who cannot (strictly) increase her utility. Note that in principle proposed mergers do not necessarily involve neighboring coalitions. But proving the following proposition, which characterize pairwise-merger-proof Tiebout equilibria, we show that if no neighbors merge non-neighbors do not merge either.

PROPOSITION **1.4** *A Tiebout equilibrium $\mathcal{W}$ is pairwise-merger-proof if and only if one of the following is true*
(i) $\mathcal{W}$ contains two coalitions of different sizes and there are no neighboring coalitions both of size $|S_{min}|$ or
(ii)

$$|S_{min}| \geq \sqrt{\frac{1}{c}}. \tag{1.5}$$

The requirement of pairwise-merger-proofness puts a lower bound on coalition sizes. There exist some Tiebout equilibria in which, given c, coalitions may be too small and it pays to merge. However, in some Tiebout equilibria the lower bound on coalition sizes is automatically fulfilled.[11]

A second reason for the creation of new coalitions may be that, given c, coalitions in the initial structure are too large and it is advantageous to create smaller ones. Again, proposition 1.3 puts no upper bound on the size of coalitions. We investigate now when agents from two neighboring coalitions do not want to secede and create a smaller (connected) coalition. This is the case which AS analyze. They call a coalition structure in which those secessions do not take place C'-stable.[12]

DEFINITION **1.3** *A coalition structure $\mathcal{W} = \{S_1, ..., S_N\}$ is **C'-stable** if in any two neighboring coalitions S_n and S_{n+1} of $\mathcal{W}$ there is no connected set*

[11] Note that the proof of proposition 1.4 focuses on the new border agents of the merged coalition S^M and that exactly one-half of the population in both coalitions experience the same change in utility as their border agent. This implies that the unanimous approval can be weakened to simple majority without affecting the result.

[12] AS also investigate the possibility of secessions involving only agents from one coalition, which they call C-stability. C-stability implies C'-stability.

of individuals z with $|z| \leq |S_{min}|$ that unanimously agrees to form z, that is, there exists $i \in z$ with,

$$c_i(z) \geq c_i(\mathcal{W}). \tag{1.6}$$

C'stability requires that in any proposed secession of a connected set of agents with cardinality smaller than $|S_{min}|$ there is at least one agent who puts her veto on the proposal because her utility will not (strictly) increase. We have the following result.[13]

PROPOSITION **1.5** *A Tiebout equilibrium $\mathcal{W}$ is C'-stable if and only if one of the following is true*
(i) $\mathcal{W}$ contains two coalitions of different sizes or
(ii)

$$|S| \leq \frac{\sqrt{2}+2}{\sqrt{c}}. \tag{1.7}$$

This result is very surprising. On one hand it says in part (ii), as we expected, that depending on c coalitions should not be too large. But on the other hand it says that Tiebout equilibria containing coalitions of different sizes do have a structure assuring that it is never beneficial to create smaller coalitions.[14]

We incorporate now the last two concepts into a stronger one. A coalition structure is a strong Tiebout equilibrium if it is a Tiebout equilibrium and no group of individuals can form a new coalition that makes all of them better off.

DEFINITION **1.4** *A coalition structure $\mathcal{W} = \{S_1, ..., S_N\}$ is a* ***strong Tiebout***

[13] If $\mathcal{W}$ is the grand coalition condition (1.7) becomes $|S| \leq \frac{\sqrt{6}+2}{\sqrt{c}}$.

[14] The crucial information in proposition 1.3 is that $|S_{min}||S_{max}| = \frac{2}{c}$. For Tiebout equilibria containing coalitions of the same size, $|S| \leq \sqrt{\frac{2}{c}}$ (which assures $|S|^2 \leq \frac{2}{c}$) is a sufficient condition for C'-stability.

equilibrium [STE] *if it is a Tiebout equilibrium and for any coalition* $z \notin \mathcal{W}$ *there exists* $i \in z$ *such that*

$$c_i(z) \geq c_i(\mathcal{W}). \tag{1.8}$$

REMARK: The definition of a strong Tiebout equilibrium is stronger than both pairwise-merger-proofness and C'-stability. On one hand it allows for secession proposals of any size and on the other these proposals may be unconnected.

PROPOSITION **1.6** *A Tiebout equilibrium* $\mathcal{W}$ *is a strong Tiebout equilibrium if and only if it is pairwise-merger-proof and it is C'stable.*

This result establishes that saying $\mathcal{W}$ is a strong Tiebout equilibrium is equivalent to saying that its coalition sizes lie in an interval depending on c. The *lower bound* comes from the possibility to create larger coalitions. If coalitions are too small it pays to form larger ones. It turns out that pairwise-merger-proofness is a sufficient condition to prevent such a creation. The *upper bound* comes from the possibility to create smaller coalitions. If coalitions are too large they will break up in smaller ones. Here C'-stability is sufficient.

REMARK: Note that Tiebout equilibria containing coalitions of different sizes are almost always automatically strong Tiebout equilibria.

The last result of this Section concerns the existence of efficient strong Tiebout equilibria.

PROPOSITION **1.7** *An efficient strong Tiebout equilibrium always exists.*

1.4.3 Stable Countries

Strong Tiebout equilibria have many nice properties. However, given the specific application one has in mind, other additional properties may be desirable. One such application is the formation of countries. AS model specific rules for country formation and establish the existence of a unique stable equilibrium $\mathcal{W}(\widetilde{N})$ in which all coalitions are equally sized and the number of coalitions $\widetilde{N}$ is too large. Since this inefficiency result holds (strictly) if and only if $c > 50$, we assume in what follows that this holds for c.

There are three additional requirements AS impose. The first one is called "stability under rule A". For our purpose the following definition is sufficient.[15]

DEFINITION **1.5** *A coalition structure* $\mathcal{W} = \{S_1, ..., S_N\}$ *is* ***stable under rule A*** *if after any small perturbation that shifts any border between two neighboring coalitions and moves a small amount of agents from one coalition to the other one, these agents will move in such a way to restore the initial coalition structure.*

From proposition 2 in AS it follows directly the following corollary.

COROLLARY 1.1 *A coalition structure* $\mathcal{W} = \{S_1, ..., S_N\}$ *is a strong Tiebout equilibrium which is stable under rule A if and only if all coalitions are of the same size and*

$$\frac{\sqrt{c}}{\sqrt{2}+2} \leq N < \sqrt{\frac{c}{2}}.^{16} \tag{1.9}$$

[15] Jehiel and Scotchmer [9] also impose such a condition.

[16] Note that, since $c > 50$ implies $N \geq 2$, the lower bound is unambiguous.

REMARK: Note that stability under rule A has two important consequences. On one hand it strengthens pairwise-merger-proofness by establishing a lower upper bound for the number of coalitions. On the other hand and most importantly it excludes all coalition structures that contain coalitions of different sizes.

The second requirement that AS impose is called B-stability. It is intended to capture border rearrangements obtained by international agreements among existing countries, ratified by majority rule votes within each country. We test the robustness of the AS-inefficiency result to changes in the votes necessary to ratify an agreement.

DEFINITION **1.6** *Consider strong Tiebout equilibria which are stable under rule A. A coalition structure* $\mathcal{W} = \{S_1, ..., S_N\}$ *changes to another structure with* $N-1$ *or* $N+1$ *coalitions (of the same size) by applying* ***rule B(Q)*** *if the modification is approved by a population share of* Q *in each coalition of* $\mathcal{W}$.

DEFINITION **1.7** *A coalition structure* $\mathcal{W} = \{S_1, ..., S_N\}$ *is* ***B(Q)-equilibrium*** *if rule B(Q) is not applied.*

DEFINITION **1.8** *A B(Q)-equilibrium* $\mathcal{W} = \{S_1, ..., S_N\}$ *is* ***B(Q)-stable*** *if after any perturbation in the number of coalitions the system returns to* $\mathcal{W}$ *with repeated applications of rule B(Q).*

We analyze two different majority requirements.

Simple Majority Rule

AS analyze the case in which for ratification a simple majority is necessary. The following result follows directly from the two facts that AS apply B($\frac{1}{2}$)-stability to a larger set than we do and that their unique B($\frac{1}{2}$)-equilibrium is a strong Tiebout equilibrium.[17]

COROLLARY 1.2 *Let $\widetilde{N}$ be the largest integer strictly smaller than $\sqrt{\frac{c}{2}}$. The unique $B(\frac{1}{2})$-stable coalition structure has $\widetilde{N}$ coalitions of the same size. For $c > 50$, $\widetilde{N} > N^*$.*

REMARK: This shows that the inefficiency result of AS is robust to strengthening the ideas of giving agents the possibility to migrate freely among existing coalitions and to create new coalitions. The selection of $\mathcal{W}(\widetilde{N})$ can be understood as the selection of one specific strong Tiebout equilibrium motivated by the rules of country formation.

Qualified Majority Rule

In many countries changes of the territory, of the borders or the independence of regions require a modification of the constitution. For constitutional changes normally more than a simple majority is required. For simplicity in what follows we require $Q = \frac{2}{3}$.[18] We have the following result.

[17] The set of A-equilibria that are stable under rule A is strictly larger than the set of strong Tiebout equilibria that are stable under rule A, since the former does not include the requirement of C'-stability.

[18] In Spain the independence of any region needs a change of the constitution. The constitutional change requires *in this case* a majority of $\frac{2}{3}$ in both the Congress (Congreso de los diputados) and the Senate (Senado). Then there are general elections (to both

PROPOSITION **1.8** *All strong Tiebout equilibria which are stable under rule A, including the efficient coalition structure, are $B(\frac{2}{3})$-equilibria. There exists no $B(\frac{2}{3})$-stable coalition structure.*

This result has a simple intuition. When a higher quorum is needed fewer people can block a proposal. Thus, one should expect more B(Q)-equilibria. Once there are multiple B(Q)-equilibria any perturbation from one B(Q)-equilibrium to another persists *because* we reached a new B(Q)-equilibrium. Thus with multiple B(Q)-equilibria the requirement of $B(\frac{2}{3})$-stability is too strong.

REMARK: Note that proposition 1.8 is robust to the following two modifications of rule B(Q). First, enlargements of countries must be approved by simple majority while secessions need the approval of a qualified majority. Second, a rule B'(Q) in which the majority must be reached in each of the regions that would constitute new countries under the proposed modification.[19]

the congress and the senate), both newly elected organs have to ratify the decision and approve the new text of the constitution with a majority of $\frac{2}{3}$. Afterwards a referendum takes place. In Germany any change of the constitution must be approved by a majority of $\frac{2}{3}$ in both the Lower House of Parliament (Bundestag) and Upper House of Parliament (Bundesrat).

[19] This is true, since in the proof of proposition 1.8 we use the results of AS concerning the enlargements of countries. In secessions we show that some agents who in both coalition structures form part of the first country are enough to block the proposal. Since the new country is smaller than the old one, our result is robust. Moreover, since AS claim that their result holds through for rule B'(Q), ours does, too.

1.5 Conclusions

In this paper we have carried out a test of robustness to the equilibrium concept used in the Alesina and Spolaore [1] country formation model. This concept draws from two ideas. On the one hand agents have (to some extend) the possibility to migrate between existing coalitions or to create new coalitions. On the other hand it models specific features of the formation of countries.

(i) A large part of this paper dealt with strengthening the first ingredient. The concept of a Tiebout equilibrium in which no agent wants to move to any other existing coalition gives a rational for the implicit assumption in AS that coalitions consist of exactly one interval on the line segment. An interesting question for future research is to determine general conditions on the utility function and on the distribution of agents such that this nice result holds still through.

(ii) We have shown that the concepts of A-equilibria and C'-stability, that seem very weak on first sight, can be seen as a kind of "sufficient conditions" for more desirable properties. Together with another intuitive requirement that we called pairwise-merger-proofness they assure that an equilibrium has all the properties of a strong Tiebout equilibrium. Here also it is an interesting question (that we leave for future research) to ask under which more general conditions these three conditions are still sufficient for assuring the properties of strong Tiebout equilibria.

(iii) We have proved that an efficient strong Tiebout equilibrium always exists and have not identified a force inherent in this concept that goes in the direction of the AS inefficiency result. Therefore, in other applications of local public goods economies than the formation of countries there is no reason to expect the inefficiency result to hold true.

(iv) One drawback of the AS-model is that it predicts that in equilibrium all countries are of the same size. This comes from the requirement that after a perturbation of the border between any two countries the system should return to its initial position (stability under rule A). Our analysis has shown that this requirement rules out many equilibria with very nice properties in which countries are of different sizes.

(v) Our last result concerns international agreements which modify the partition of the world in countries. We have shown that the quorum necessary for such an agreement to become ratified in each country is crucial. Our change from simple to qualified majority is motivated by rules that exist in the real world for these questions and has two consequences. First, in order to assure the existence of an equilibrium one has to skip the part requiring immunity to perturbations. Second, one is left with a multiplicity of equilibria which moreover coincides with the set of strong Tiebout equilibria which are stable under rule A. This means that this concept provides no longer a refinement of the initial multiplicity of equilibria.

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Appendix: Proofs

Proof of Proposition **1.1**:

Let $\mathcal{W}$ be an unconnected coalition structure. We show that $\mathcal{W}$ cannot be efficient, since there exists another coalition structure $\mathcal{W}'$ with the same number of coalitions where the sum of distances between agents and its local public good is strictly lower. Let $S \in \mathcal{W}$ be the unconnected coalition with the lowest subindex in $\mathcal{W}$. Then there exists a line segment T such that for some agents $i \in T$ holds $i \notin S$ but their locations lie between the highest and the lowest agent in S. Note that T cannot be a singleton because then S were connected. Construct $\mathcal{W}'$ such that it is identical to $\mathcal{W}$ but all but the first part of S are moved to the left such that S is connected. Since $|T| > 0$, on one hand, the sum of distances between agents in coalition S and $l(S)$ must be strictly lower in $\mathcal{W}'$ than in $\mathcal{W}$. On the other, since all coalitions with a lower subindex than S are connected in $\mathcal{W}'$, for the remaining coalitions the sum of distances cannot have increased in comparison to $\mathcal{W}$. ■

Proof of Lemma **1.1**:

Let $\mathcal{W}$ be an unconnected Tiebout equilibrium and let $S \in \mathcal{W}$ be the unconnected coalition with the lowest subindex in $\mathcal{W}$. Denote the first part of S by S_L. Since S has at least two parts choose S_R in the following way. If $l(S) \in S_L$, then choose the second part. Otherwise choose S_R such that $l(S) \in S_R$. Call $b2$ and $b3$ the right border agent of S_L and the left border agent in S_R, respectively. Denote the coalition immediately on the right of S_L by S'. It is sufficient to distinguish four cases.

Case 1: Suppose $l(S) \leq b2 < l(S') \leq b3$. Denote $i = l(S')$. Observe that because $\mathcal{W}$ is a Tiebout equilibrium, $b2$ is indifferent between the two coalitions that are closest to her. This implies $c_i(S) > c_{b2}(S) = c_{b2}(S') > c_i(S')$. But $c_i(S) > c_i(S')$ if and only if $\frac{|S|-|S'|}{|S||S'|} < c(l(S') - l(S))$. Consider now

$j = b3$. Here $c_j(S) \leq c_j(S')$ if and only if $\frac{|S|-|S'|}{|S||S'|} \geq c(l(S') - l(S))$, which is a contradiction.

Case 2: Suppose $l(S) \leq b2 < b3 < l(S')$. Denote $i = b2$. We have that $c_i(S) = c_i(S')$ if and only if $\frac{|S'|-|S|}{|S||S'|} = c(l(S') + l(S) - 2b2)$. Consider now $j = b3$. Here $c_j(S) \leq c_j(S')$ if and only if $\frac{|S'|-|S|}{|S||S'|} \leq c(l(S') + l(S) - 2b3)$. Both conditions imply $b3 \leq b2$, which contradicts the definition of those agents.

Case 3: Suppose $l(S') \leq b3 < l(S)$. Denote $i = b2$. Now it holds that $c_i(S) = c_i(S')$ if and only if $\frac{|S|-|S'|}{|S||S'|} = c(l(S) - l(S'))$. Note that, since $l(S') < l(S)$, we have also that $|S| > |S'|$. Denote the agent with the highest position in S' whose position is on the left of $l(S)$ by j. Suppose $l(S') < j$. The fact that $c_j(S') \leq c_j(S)$ leads to $\frac{|S|-|S'|}{|S||S'|} \leq c(l(S) + l(S') - 2j)$. Both conditions imply $l(S') \geq j$, a contradiction. Suppose now $l(S') = j$. It follows that there must exist an agent $k \in S'$ with $k > l(S)$. We have that $c_k(S') \leq c_k(S)$ leads to $\frac{|S'|-|S|}{|S||S'|} \geq c(l(S) - l(S'))$. Here $l(S') < l(S)$, implies that $|S| < |S'|$, also a contradiction.

Case 4: The case $b3 < l(S) < l(S')$ is proven along the lines of the third case. One just has to interchange each S and S'. ■

PROOF OF PROPOSITION **1.3**:

Because of lemma 1.1 consider a TE $\mathcal{W}$ which is connected. Suppose there are at least two coalitions. Border agents are indifferent between the two coalitions they belong to if and only if [1.3.1] $c_b(S_n) = c_b(S_{n+1})$, $\forall b(S_n, S_{n+1}) \in B(\mathcal{W})$. This holds if either [1.3.2] $|S_n| = |S_{n+1}|$ or [1.3.3] $|S_n||S_{n+1}| = \frac{2}{c}$.

We show first that $\mathcal{W}$ contains at most two different coalition sizes. Suppose there are three or more different sizes $|S_l|$, $|S_m|$ and $|S_n|$. Then there exists at least two borders at which [1.3.3] must hold. That is, $|S_l||S_m| = \frac{2}{c}$ and $|S_m||S_n| = \frac{2}{c}$. This implies $|S_l| = |S_n|$, a contradiction.

We show now that if [1.3.1] $c_b(S_n) = c_b(S_{n+1})$, $\forall b \in B$, *then* $c_i(S_n) \leq c_i(S_m)$, $\forall i \in S_n$ *and* $\forall S_n, S_m \in \mathcal{W}$. Suppose [1.3.1], take (by symmetry) any $S_n \in \mathcal{W}$ with $n \leq N-1$, consider $b = b(S_n, S_{n+1})$ and choose S_m such that $m \geq n+1$. Observe that because of [1.3.1] and since all coalitions are symmetric, $c_b(\mathcal{W}) = \overline{c}$, $\forall b \in B(\mathcal{W})$. Define $b' = (S_{m-1}, S_m)$. Note that since $d_b(S_m) \geq d_{b'}(S_m)$, we have that $c_b(S_m) \leq \overline{c}$. hence $c_b(S_n) \leq c_b(S_m)$. On one hand, since $d_i(S_n) \leq d_b(S_n)$ for all $i \in S_n$, we have for all $i \in S_n$ that $c_i(S_n) \leq c_b(S_n)$. On the other hand it is true that $d_i(S_m) \geq d_b(S_m)$ for all $i \in S_n$. This implies $c_b(S_m) \leq c_i(S_m)$ for all $i \in S_n$. Thus the desired inequality $c_i(S_n) \leq c_i(S_m)$, for all $i \in S_n$ and for all $S_n, S_m \in \mathcal{W}$ follows. ■

PROOF OF PROPOSITION **1.4**:

Let $\mathcal{W}$ be a TE. *We show first that the proposition holds for neighboring pairs of coalitions.* Take any pair of neighboring coalitions S_n and S_{n+1} in $\mathcal{W}$ and focus first on S_n. If the merger occurs all agents in S_n have the same advantage of lower public good provision costs, but the left border agent $b = b(S_{n-1}, S_n)$ is among those agents who have to support the biggest increase of distance to the public good. Agent b refuses to form S^M if and only if $c_b(S_n) = \overline{c} = \frac{|S_n|}{2}c + \frac{1}{|S_n|} \leq \frac{|S_n|+|S_{n+1}|}{2}c + \frac{1}{|S_n|+|S_{n+1}|} = c_b(S^M)$. This gives [1.4.1] $|S_n|^2 + |S_n||S_{n+1}| \geq \frac{2}{c}$. Suppose $|S_n| \neq |S_{n+1}|$. By proposition 1.3 condition [1.4.1] is fulfilled. Suppose $|S_n| = |S_{n+1}| = |S|$. Condition [1.4.1] boils down to $|S| \geq \frac{1}{\sqrt{c}}$. Suppose $\mathcal{W}$ is a TE containing coalitions of different sizes and some coalitions of size $|S_{max}|$ are neighbors. Suppose $|S_{max}| < \frac{1}{\sqrt{c}}$. This contradicts $\sqrt{\frac{2}{c}} < |S_{max}|$, which must be true by proposition 1.3.

It remains to show that if neighbors do not merge, non-neighboring coalitions do not merge either. Take two non-neighboring coalitions S_n and S_m. Suppose $|S_n| \neq |S_m|$. Since now distances are higher than in the case of neighbors, we know that the merger does not take place. The same holds in the case that $|S_n| = |S_m| = |S_{max}|$. Suppose $|S_n| = |S_m| = |S_{min}|$. If there are only coalitions of size $|S_{min}|$ in between it is clear that the fact that neighboring coalitions do not merge implies that S_n and S_m do not merge either. Assume now that there is one coalition of size $|S_{max}|$ in between. Consider the agent with the highest position in the merger. Similar reasoning as above leads to $2|S_{max}||S_{min}| + |S_{min}|^2 \geq \frac{1}{c}$, which is fulfilled. ■

PROOF OF PROPOSITION **1.5**:

Since part (ii) follows from equation (11) on page 1036 in AS, we show (i) only. Suppose $\mathcal{W}$ is a TE containing two neighboring coalitions S_n and S_{n+1} of different sizes. Let $|S_n| > |S_{n+1}|$. We know from proposition 1.3 that [1.5.1] $|S_n||S_{n+1}| = \frac{2}{c}$ holds. We show that no $i \in \mathcal{W}$ wants to secede to any z with $|z| \leq |S_{n+1}|$ where i will be border agent. Consider $S \in \{S_n, S_{n+1}\}$. We have that $c_i(\mathcal{W}) \leq \frac{|S|}{2}c + \frac{1}{|S|} \leq \frac{|z|}{2}c + \frac{1}{|z|} \leq c_i(z)$ must hold (the last inequality is strict if z is unconnected). If $|S| = |z|$ this is trivially fulfilled, if not it implies [1.5.2] $|S||z| \leq \frac{2}{c}$. Since $|S| > |z|$ for all $S \in \{S_n, S_{n+1}\}$ and [1.5.1], [1.5.2] is true. For later reference note that the proof holds through for unconnected z. ■

PROOF OF PROPOSITION **1.6**:

Step 1: We show first that C'stability implies that there is no z with $|z| \leq |S_{min}|$ that blocks $\mathcal{W}$. If $\mathcal{W}$ contains two coalitions of different sizes this follows from the proof of proposition 1.5. Suppose $\mathcal{W}$ contains only coalitions of the same size. It remains to show that for any unconnected z, denoted by z_u with $|z_u| \leq |S|$, which blocks $\mathcal{W}$ there exists a connected, denoted by z_c, which also blocks $\mathcal{W}$. It is immediate that equation (1.8) is fulfilled for individuals with $d_i(S) = 0$. Denote the individuals with the lowest and highest position in a secession proposal by $b_1(z_u)$ and $b_2(z_u)$, respectively. If the interval between $b_1(z_u)$ and $b_2(z_u)$ contains at least one position of government, then there exists an individual in z_u with a higher or equal distance to government than in S. If the unconnected secession proposal

z_u does not contain any government location, then it is possible to construct a connected proposal z_c with the government at the same position and $|z_u| = |z_c|$. Consider the border agent who is closest to $l(S)$. The fact that $c_{b(z_c)}(z) > c_{b(z_c)}(S)$ implies $c_{b(z_u)}(z) > c_{b(z_u)}(S)$.

Step 2: For the reminder of this proof suppose $|z| > |S_{min}|$. Fix a secession proposal z and consider its border agents $b(z)$. Since individuals on the border of S have the minimum utility in $\mathcal{W}$, equation (1.8) is fulfilled for $b(z)$ if [1.6.1] $c_{b(z)}(z) \geq \frac{|z|}{2}c + \frac{1}{|z|} \geq \frac{|S|}{2}c + \frac{1}{|S|} \geq c_{b(z)}(S)$ holds (the first inequality is strict if and only if z is unconnected).

Step 2.1: Suppose $\mathcal{W}$ contains two coalitions of different sizes and $|z| \geq |S_{max}|$. Note that [1.6.1] is trivially fulfilled if $|S| = |S_{max}| = |z|$. Let $|S| = |S_{max}| < |z|$. Equation [1.6.1] becomes [1.6.2] $|S||z| \geq \frac{2}{c}$ which is fulfilled since $|z| > |S_{min}|$ and $|S_{min}||S_{max}| = \frac{2}{c}$. If $|S| = |S_{min}|$ we reach again [1.6.2] which is true again, since $|z| \geq |S_{max}|$.

Step 2.2: Suppose $\mathcal{W}$ contains two coalitions of different sizes and $|S_{max}| > |z| > |S_{min}|$. Assume furthermore that z is such that $l(z) = l(S)$ for some coalition S.

Step 2.2.1: Suppose there exists $b \in b(z)$ with $|S_b| = |S_{max}|$. Note that, since if $|S| = |S_{max}|$ then b can not be closer to $l(z)$ than she is to $l(S)$, we focus on $|S| = |S_{min}|$. For a secession proposal z of size $|z|$, denote by $|z| + u$ the line segment between its lowest and highest agent. Note that if z is connected, then $u = 0$. Suppose the position of b lies between $l(z)$ and $l(S)$ (otherwise his distance increases in z). Agent b rejects z if and only if $c_b(z) = \frac{|z|+u}{2}c + \frac{1}{|z|} \geq \frac{|S_{max}|-|z|-u-|S_{min}|}{2}c + \frac{1}{|S_{max}|} = c_{b(z)}(S)$ which implies that $|z|c + \frac{1}{|z|} \geq \frac{|S_{max}|-2u-|S_{min}|}{2}c + \frac{1}{|S_{max}|}$. Hence it is sufficient that $min_{|z|}\{|z|c + \frac{1}{|z|}\} \geq \frac{|S_{max}|-|S_{min}|}{2}c + \frac{1}{|S_{max}|}$. Since the solution to the minimization problem is $\frac{1}{\sqrt{c}}$, it is enough to show that $2\sqrt{c} \geq \frac{|S_{max}|}{2}c + \frac{1}{|S_{min}|}$. By PMP $2\sqrt{c} \geq \frac{2}{|S_{min}|} \geq \frac{|S_{max}|}{2}c + \frac{1}{|S_{min}|}$, where the last inequality is holds by $|S_{min}||S_{max}| = \frac{2}{c}$.

Step 2.2.2: Suppose that for at least one $b \in b(z)$ holds $|S_b| = |S_{min}|$ or that $\mathcal{W}$ contains only coalitions with size $|S_{min}|$. Denote $|z| = |S_{min}| + m$ with $0 < m < 1$. Condition [1.6.2] $|S||z| \geq \frac{2}{c}$ can be written as $|S_{min}|^2 + |S_{min}|m \geq \frac{2}{c}$. If $m \geq |S_{min}|$ this is true by PMP. Suppose now $0 < m < |S_{min}|$.

Note that if $|S| = |S_{max}|$, then there exists an agent who rejects z if $\frac{|S_{min}|+|S_{max}|}{2}c + \frac{1}{|z|} \geq \frac{1}{|S_{min}|}$. This gives $|z||S_{min}| \geq \frac{1}{c}\frac{2(|z|-|S_{min}|)}{|S_{min}|+|S_{max}|}$ and is fulfilled by PMP since $4|S_{min}| > 2|z|$.

Define the interval between both border agents in z as $n + |S_{min}|$. We have that z is connected if and only if $n = m$. Let $n < |S_{min}|$. Equation [1.6.1] can be written as [1.6.3] $c_{b(z)}(z) = \frac{|S_{min}|+n}{2}c + \frac{1}{|S_{min}|+m} \geq \frac{|S_{min}|-n}{2}c + \frac{1}{|S_{min}|} = c_{b(z)}(S)$. This gives $\frac{n}{m}|S_{min}|(|S_{min}| + m) \geq \frac{1}{c}$, which is true because of PMP, $\frac{n}{m} \geq 1$ and $|S_{min}|m \geq 0$. Let $2|S_{min}| > n \geq |S_{min}|$. Equation [1.6.3] becomes [1.6.4] $c_{b(z)}(z) = \frac{|S_{min}|+n}{2}c + \frac{1}{|S_{min}|+m} \geq (|S_{min}| - \frac{n}{2})c + \frac{1}{|S_{min}|} = c_{b(z)}(S)$. From this we get $(n - \frac{|S_{min}|}{2})c \geq \frac{|S_{min}|}{2}c \geq \frac{m}{(|S_{min}|+n)|S_{min}|}$ or $|S_{min}|^2\frac{|S_{min}|+m}{2m} \geq \frac{1}{c}$. This is true since PMP and $|S_{min}| + m > 2m$. It is clear that z with $n \geq 2|S_{min}|$ are also rejected.

Step 2.2.3: Note that by construction both border agents in z reject z. Therefore, for other locations $l(z) \neq l(S)$ there is always one border agent closer to $l(S)$ than in the numerical expressions above. This implies that such a z does not block $\mathcal{W}$.

Step 2.2.4: It remains to show that if $\mathcal{W}$ contains two coalitions of different sizes and $|S_{max}| > |z| > |S_{min}|$, all z that lie entirely in a coalition of size $|S_{max}|$ or between two of the same size are not successful. We use the results of AS which say that it suffices that $|S_{max}| \leq \frac{\sqrt{2}+2}{\sqrt{c}}$. We show that $\frac{2}{c} \leq \frac{\sqrt{2}+2}{\sqrt{c}}|S_{min}|$. Since $|S_{min}| \geq \frac{1}{\sqrt{c}}$, we need that $\frac{2}{c} \leq \frac{\sqrt{2}+2}{c}$, which is true. For unconnected z one can argue as in step 1. ■

PROOF OF PROPOSITION **1.7**:

We show that the efficient coalition structure determined in AS (which we review in proposition 1.2) is always a STE. From the proof of proposition 1 in AS (page 1047) we know that for $c < 8$ the grand coalition is efficient. Note that in this case it is not possible to form a larger coalition. Hence the lower bound for STE does not apply. For this range of values of c the condition $1 \leq \frac{\sqrt{6}+2}{\sqrt{c}}$ is always fulfilled. For $c \geq 8$ take the efficient coalition structure consisting of N^* coalitions of the same size. We express the conditions for N^* to be a STE also in the number of coalitions and not in its size. We know that $N^* \in [max\{2, \frac{\sqrt{c}}{2} - 1\}, \frac{\sqrt{c}}{2} + 1)$. Note that $2 \geq \frac{\sqrt{c}}{2} - 1$ if and only if $c \leq 36$. For this values of c it is true that $\frac{\sqrt{c}}{\sqrt{2}+2} < 2$. On one hand we have $\frac{\sqrt{c}}{\sqrt{2}+2} < \frac{\sqrt{c}}{2} - 1$, which is also fulfilled for $c > 36$. Consider now the upper bound for N^*. We have that $\frac{\sqrt{c}}{2} + 1 < \sqrt{c}$ if and only if $c > 4$, which is also fulfilled. ■

PROOF OF PROPOSITION **1.8**:

Consider strong Tiebout equilibria which are stable under rule A.

Step 1: Application of rule $B(\frac{2}{3})$ in order to increase the number of coalitions. A change from a coalition structure $\mathcal{W}$ with N coalitions of size $|S| = \frac{1}{N}$ to $\mathcal{W}'$ with $N+1$ coalitions of size $|S'| = \frac{1}{N+1}$ is carried out if there is a majority of $\frac{2}{3}$ in every coalition of $\mathcal{W}$. Hence this change is *not* carried out if there is a coalition with $\frac{1}{3}$ of its population against it.[20] Consider the first coalition in $\mathcal{W}$. If the change is carried out, coalitions get smaller, which implies that all agents have to pay more for public good provision. Denote the population share between the old local public good $l(S_1)$ and the new border $b(S_1', S_2')$ as P_1. Note that all agents in $P_1 = |S'| - \frac{|S|}{2}$ get

[20] We assume here that exactly $\frac{1}{3}$ is enough. This is motivated by the fact that in most countries $\frac{2}{3}$ of the parliament is not an integer and therefore strictly more than $\frac{2}{3}$ is needed. This assumption is not crucial for our result because the only change in which we have ties is the one from $N = 2$ to $N' = 3$.

unambiguously worse off by the change. We have that $P_1 \geq \frac{1}{3}|S|$ if and only if $N \geq 5$. Note that for $c > 50$, the grand coalition is not a STE. We are left with $N \in \{2,3,4\}$. Consider the change from $N = 2$ to $N' = 3$ and the first country. The agents who get unambiguously worse off are those located in the interval $[\frac{5}{24}, \frac{3}{8}]$. This is a total of $\frac{1}{6}$ or $\frac{1}{3}$ of the population of the first country. Consider the change from $N = 3$ to $N' = 4$ and the second country. The agents in $[\frac{7}{16}, \frac{9}{16}]$ get unambiguously worse off. Their cardinality is $\frac{1}{8}$ which is more than the necessary $\frac{1}{9}$. For the change from $N = 4$ to $N' = 5$ consider again the first country. The agents in $[\frac{1}{8}, \frac{17}{80}]$ get unambiguously worse off. Their cardinality is $\frac{7}{80}$ which is again more than the necessary $\frac{1}{12}$. Thus, there are always enough agents against the change.

Step 2: Application of rule $B(\frac{2}{3})$ in order to decrease the number of coalitions. From the analysis of AS (lemma 4, page 1053) we know that stability under rule A implies that there is always a majority (in at least one coalition) against the change. A majority is more than enough under rule B($\frac{2}{3}$).

Step 3: $B(\frac{2}{3})$-equilibria. Because of step 1 and 2 all STE which are stable under rule A are B($\frac{2}{3}$)-equilibria.

Step 4: $B(\frac{2}{3})$-stability. Note that for $c > 50$ there are multiple B($\frac{2}{3}$)-equilibria. Suppose a perturbation to any STE which is stable under rule A. Since the latter is a B($\frac{2}{3}$)-equilibrium (step 3), rule B($\frac{2}{3}$) is not applied and the system does not return to its initial position. Thus, there exists no B($\frac{2}{3}$)-stable coalition structure. ∎

Chapter 2

Efficiency and Stability in a Discrete Model of Country Formation

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2.1 Introduction

This paper studies efficient and stable country configurations in a simple model of country formation. Driving force of the model is a trade-off between the benefits of large countries and the costs of heterogeneity of large and diverse populations. Large jurisdictions bring several benefits with them. For

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example there may be economies of scale in public good provision or simply the fact that the (fixed) costs of government provision are shared among many citizens. On the other hand there is an opposite force. Since larger populations are less homogeneous, the choice of the type of government may become less similar to the individuals' most preferred type when the size of the country increases.

The focus of our model on the trade-off between benefits and costs of large groups is a frequent feature of the literature seeking to explain the formation of groups in societies see e.g. Demange (1994). For a detailed exposition of the benefits and costs concerning the organization of the world in countries see Alesina and Spolaore (1997) (henceforth AS) or Le Breton and Weber (2000).

Both our model and the questions we want to answer are inspired by AS. Alesina and Spolaore assume that the world population is one-dimensional and uniformly distributed on the unit interval. Individuals join together to form 'countries', that are intervals with the 'government' located in its middle. Each individual has the utility function of the same form, piecewise linear, directly proportional to the negative of its distance from the government and indirectly proportional to the negative of the size of its country. AS then consider two criteria for country formation: efficiency (the average utility of a citizen should be maximized) and stability (for precise definitions see the paper in question). They show that under efficiency as well as under stability criterion all countries are intervals of the same size. One of their main conclusions is that the efficient number of countries is always smaller than the stable one, from which they conclude that

> '... The democratic process leads to an inefficiently large number of countries. Namely, when countries are formed through

> a democratic process, more countries are created than with a social planner who maximizes world average utility. ...'

Our aim is to test how reliable the conclusions of AS are. The main difference of our model and that of AS is that they work in a continuous setting and our model is discrete. Further, we slightly modified the stability notion to make a better sense for the discrete model. We show that for each value of the parameter determining the exact form of the utility function (more precisely, expressing the relative importance of its 'distance dependent' part to its 'country-size dependent' part) efficient configurations as well as stable configurations exist, however, there is no unambiguous relation between them. Moreover, although in an efficient configuration the country sizes are 'almost equal', in a stable one they may assume any integer value in a certain (not very small) interval.

One explanation of the differences obtained may be that in a discrete model the forces at play are changed considerably, since adding a player to a coalition alters substantially the coalition, whereas in the continuous model the measure (and the power) of a single player is zero. Is some situations however, a discrete model may be more appropriate. One reason may be that the real world is discrete and world population is large but not infinite. Another consideration may be that the formation of countries is driven by regions and a region may be a set of individuals with similar geografic position and utility (consider the disintegration of the Former Soviet Union, Yogoslavia, Czechoslovakia, as well as the movements for regional authonomy or even independence in e.g. Canada, Spain, France or Italy).

The first discretizations of AS have been attempted by Dahm (1999) and (2000). Dahm uses the same stability notions as AS and shows that in the discrete case a stable configuration may fail to exist. Moreover, in situations where a stable configuration exists, the relation to efficient configurations is not so clear as in AS.

Other papers dealing with relations between efficiency and stability of coalitional structures are Drèze and Greenberg (1980) and LeBreton and Weber (2000). In the first one the authors consider a model of an economy as a cooperative game where the utility function of each individual has two arguments: his consumption bundle and the coalition to which he belongs. An example is presented with three agents and two commodities where efficient allocations are not stable. The second paper focuses on one country and analyses when it is efficient to maintain a unified country as well as when this country is stable, that is, no region wants to secede. The paper shows that it is possible to reconcile both requirements.

The AS paper belongs to political economy literature on country formation which is reviewed in Bolton, Roland and Spolaore (1996). The AS model has recently been extended by Le Breton and Weber (2000). Unlike our paper, the latter focuses on group deviations. Another related work is Haeringer (2000). He also uses a discrete model, but deals only with stable structures. On the other hand, his model is more general than ours. However this generality makes it impossible to describe stable structures in more detail.

Our model can also be understood as a hedonic coalition formation game. This notion was introduced by Drèze and Greenberg (1980), who defined a hedonic game as such in which the utility of a member of a given coalition is not affected by the way outsiders organize themselves but depends only on the identity of the other members of her coalition. Recent contributions include Banerjee, Konishi and Sönmez (1998) and Bogomolnaia and Jackson (1998). These papers deal solely with various definitions of stable coalition structures and restrictions of the admissible preferences.

Inspired by the seminal work of Tiebout (1956) are the papers on local public good economies like Guesnerie and Oddou (1981), Greenberg and Weber (1986), (1993), Weber and Zamir (1986) and Jehiel and Scotchmer (1997). They mainly investigate the existence and characterization of stable partitions of individuals into jurisdictions rather than providing a detailed comparison of stable and efficient structures.

This paper is organized as follows. In Section 2 we present the basic model. Section 3 characterizes efficient configurations and Section 4 is devoted to our main stability notion which is slightly extended in Section 5. Section 6 concludes.

2.2 The Model

We suppose that the world population is finite, its cardinality will be denoted by W. Individuals are located in discrete points on the line, the distance between any two neighbouring locations is 1. Individuals join together in order to provide a (local) public good which is nonrival and excludable. We interpret the public good as 'government'. It may represent a bundle of administrative, judicial, economic services and public policies. Each government identifies a country. (We will use from now on the terminology *country* for coalitions that form and *government* for the public good that must be provided.)

The cost of the public good provision is constant and is independent from the size of the country. Each individual has to belong exactly to one country. Because of excludability, the benefits arise only to citizens of a given country, who in turn have to finance their government.

The policy space we analyze is unidimensional. Each point in this interval represents a type of government and for each individual there exists a unique ideal policy. This allows us to identify each individual unambiguously with a point in the policy space. Hence we consider the individual's ideal point to be its 'location'.

Each individual has the same utility function, separable in the public good and money which takes the piecewise linear form similar to the one in AS

$$u(i) = u(1 - a\ell_i) + y - \frac{k}{|\mathcal{P}_i|}, \tag{2.1}$$

where u, a, k are nonnegative parameters, y is an exogeneous income of the individual (the same for all individuals), u measures the maximum utility derived when the location of individual coincides with the location of the government, ℓ_i is the distance of individual i from his government and $|\mathcal{P}_i|$ the number of citizens of the country of individual i.

Since the constant terms in the utility function are irrelevant, its maximization is equivalent to minimization of the individuals' cost function

$$w(i) = ua\ell_i + \frac{k}{|\mathcal{P}_i|}. \tag{2.2}$$

In order to make our treatment more concise, we shall consider the normalized cost

$$w_n(i) = c\ell_i + \frac{1}{|\mathcal{P}_i|}, \tag{2.3}$$

where the nonnegative parameter $c = \frac{ua}{k}$ measures the relative importance of the 'heterogeneity' costs with respect to the 'government provision' costs.

It will turn out that it is sufficient to consider 'connected countries'. We say that a country $\mathcal{P}$ is connected if $\mathcal{P}$ contains with any two citizens i, j all the intermediate citizens. Hence, a country $\mathcal{P}$ is connected if and only if the distance between any two neighbouring citizens of $\mathcal{P}$ is equal 1. Therefore we shall denote a configuration of countries as a vector $(p_1, p_2, \ldots, p_N)$ of country sizes, assuming that the first country from the left has p_1 citizens, the second one p_2, etc.

We shall look at two different criteria for the formation of countries:

- *Efficiency.* How many countries should be created and of what size if the sum of costs of all the world inhabitants is to be minimized?
- *Stability.* We consider the following three possibilities for deviations:
  - A citizen leaves his original country and forms a new country of his own.
  - A citizen leaves one country to join a neighbouring country.
  - A citizen of one country and a citizen of a neighbouring country leave their countries to form a new country.

2.3 Efficiency

Let us call the sum of distances of all the citizens of a country $\mathcal{P}$ from its government the heterogeneity $H(\mathcal{P})$ of $\mathcal{P}$. The heterogeneity $H(\mathcal{E})$ of a configuration $\mathcal{E}$ is the sum of heterogeneities of all countries in $\mathcal{E}$. The minimum possible heterogeneity of a country with p inhabitants will be denoted by $h(p)$.

In the proof of the following assertions we shall call each citizen located to the left of the government of his country a *left citizen* and the one located to the right a *right citizen*. For a real number a, $\lfloor a \rfloor$ and $\lceil a \rceil$ denote the greatest integer not greater than a and the smallest integer not smaller than a, respectively.

Lemma 1 *Let $\mathcal{P}$ be any country. Then the heterogeneity of $\mathcal{P}$ is minimal if its government is located at a median position, i.e. when the number of the left citizens of $\mathcal{P}$ is equal to the number of the right citizens of $\mathcal{P}$.*

Proof. Let $|\mathcal{P}| = p$ and let the government be located at a median position. We shall denote by $H^*(\mathcal{P})$ the heterogeneity of $\mathcal{P}$ and by p_L^* and p_R^* the number of the left and right citizens of $\mathcal{P}$. Clearly, $p_L^* = p_R^* = \lfloor \frac{p}{2} \rfloor$. Now, let the position of the government be w.l.o.g. in the distance x to the left of the original position. Let the heterogeneity of $\mathcal{P}$ be now $H'(\mathcal{P})$. Now, all the right citizens remain right, but each one of them has now his distance from the government increased by x. Of the left citizens some remain left (and the distance from the government of each one of them will be decreased by x) and some may be 'jumped' over. Let us denote the number of jumped-over citizens by j. If their distances from the government were originally $y, d_1 + y, d_2 + y, \ldots, d_{j-1} + y$ (where $y \in (0, d_1)$) from the rightmost one to the leftmost one, now they are $x - y, x - y - d_1, \ldots, x - y - d_{j-1}$ in the same order. So

$$H^*(\mathcal{P}) - H'(\mathcal{P}) = -p_R^* x + (p_L^* - j)x + j(2y - x) = 2j(y - x),$$

if the government was originally not located in any citizen and

$$H^*(\mathcal{P}) - H'(\mathcal{P}) = -p_R^* x + (p_L^* - j)x + j(2y - x) - x = 2j(y - x) - x$$

otherwise. Since $y - x < 0$ if $j \geq 1$ and $x > 0$, we have $H^*(\mathcal{P}) - H'(\mathcal{P}) \leq 0$, which implies the claim. ■

Let us notice here that in case $|\mathcal{P}|$ is even, the minimum heterogeneity is achieved also if the government is located in anyone of middle citizens, which are not median positions according to the definition.

Moreover, the above theorem is true for any, not only uniform distances between citizens, but the next one already uses this uniformity.

Lemma 2 *Let a country $\mathcal{P}$ have p citizens. Then $H(\mathcal{P}) = h(p)$ if and only if $\mathcal{P}$ is connected and in this case*

$$h(p) = \begin{cases} \frac{p^2}{4}, & \textit{if } p \textit{ is even;} \\ \frac{p^2-1}{4}, & \textit{if } p \textit{ is odd.} \end{cases}$$

Proof. The necessity of connectedness of a country is implied by the fact that the distances of citizens from the government used in the following expressions for the heterogeneity of a country are obtained when $\mathcal{P}$ is connected and they are lower bounds for the distances in a country that is not connected.

Case $p = 2k$. Let the government be located in the segment between the two middle citizens, at distance $x \in \langle 0, 1 \rangle$ from the left one of them. Then

$$h(2k) = \sum_{m=1}^{k} (m - 1 + x) + \sum_{m=1}^{k} (m - x) = k(k+1) - k = k^2 = \frac{p^2}{4}.$$

For $p = 2k + 1$ we have

$$h(2k+1) = 2 \sum_{m=1}^{k} m = k(k+1) = \frac{p^2 - 1}{4}.$$

■

In an efficient configuration the heterogeneity of each country $\mathcal{P}$ with p citizens must be equal to $h(p)$. Now we shall prove that the sequence

$h(p), p = 1, 2, \ldots$ is convex, which will ensure that in an efficient configuration the sizes of countries are approximately equal.

Lemma 3 *For each $p \geq 2$ and $q, 0 < q < p$ we have*

$$h(p+q) + h(p-q) \geq 2h(p),$$

while equality occurs only if p is even and $q = 1$.

Proof. From Theorem 2 we have

$$h(p+q)+h(p-q) \geq \frac{(p+q)^2-1}{4} + \frac{(p-q)^2-1}{4} = \frac{p^2+q^2-1}{2} \geq \frac{p^2}{2} \geq 2h(p).$$

The first inequality is fulfilled as equality if and only if both $p+q$ and $p-q$ are odd; the second one is equality if and only if $q = 1$ and the third one if and only if p is even. Hence the desired result follows. ■

Theorem 1 *In an efficient configuration $\mathcal{E}$ the sizes of two countries may differ by at most 2, and if this occurs, the maximum and the minimum sizes of a country in $\mathcal{E}$ are odd.*

In general, it is now easy to see how to generate an efficient configuration for the world population of W citizens and a fixed number $N \leq W$ of countries. First of all, we divide W by N. If W *mod* $N = M$ is zero, then all the countries will have the same size $\frac{W}{N}$. Otherwise M is nonzero but smaller than N, hence we will have M countries of size $\lceil \frac{W}{N} \rceil$ and $N - M$ countries of size $\lfloor \frac{W}{N} \rfloor$. Configurations obtained in this way will be called *standard configurations* and for given W and N they will be denoted by $\mathcal{E}(W, N)$.

Moreover, if M is at least 2 and $\lceil \frac{W}{N} \rceil$ is even or if $N - M$ is at least 2 and $\lfloor \frac{W}{N} \rfloor$ is even, we can choose several pairs of countries with equal even size $q = \lceil \frac{W}{N} \rceil$ or $q = \lfloor \frac{W}{N} \rfloor$, replace each such pair by two countries with sizes $q-1$ and $q+1$ and obtain another efficient configuration.

Example. If $W = 6$ and $N = 3$, the sizes of countries in an efficient configuration are (2,2,2) or (3,2,1); for $W = 8$ and $N = 3$ the only possible efficient configurations are (3,3,2) and its permutations.

Now consider $W = 100, N = 8$. Here $\lfloor \frac{100}{8} \rfloor = 12$ and $100 \ mod \ 8 = 4$, hence

$$\mathcal{E}(100, 8) = (13, 13, 13, 13, 12, 12, 12, 12), \tag{2.4}$$

other possibilities are

$$(13, 13, 13, 13, 11, 13, 12, 12) \text{ and } (13, 13, 13, 13, 11, 13, 11, 13), \tag{2.5}$$

so all the efficient configurations for $W = 100$ and $N = 8$ ca be expressed as permutations of (2.4) and (2.5).

It is clear that for a given W and each value of parameter c there exists at least one efficient configuration. In what follows we shall describe an approach for finding the efficient configurations for given W and c with the number of countries not prescribed in advance .

The total cost of the standard configuration $\mathcal{E}(W, N)$ is a linear function of c

$$C_{W,N}(c) = N + H(\mathcal{E}(W, N))c. \tag{2.6}$$

To find the efficient number of countries N^* for a given c, we have to find the global minimum of functions of the form (2.6) for $N = 1, 2, \dots, W$. Fortunately, it is not necessary to compare $C_{W,N^*}(c)$ with $C_{W,N}(c)$ for all values of N, as we will just prove. We shall denote by $c_N^M(W)$ for $M < N$ the point where the functions $C_{W,M}(c)$ and $C_{W,N}(c)$ intersect. To begin with, we first compute the intersection $c^{N}_{N+1}(W)$ of two 'adjacent' costs, which has to fulfill

$$N + H(\mathcal{E}(W,N)).c^{N}_{N+1}(W) = N + 1 + H(\mathcal{E}(W,N+1)).c^{N}_{N+1}(W). \quad (2.7)$$

From equation 2.7 we get

$$c^{N}_{N+1}(W) = \frac{1}{H(\mathcal{E}(W,N)) - H(\mathcal{E}(W,N+1))} \quad (2.8)$$

Lemma 4 *The sequence* $c^{0}_{1}(W) = 0, c^{1}_{2}(W), c^{2}_{3}(W) \ldots, c^{W-1}_{W}(W), c^{W}_{W+1}(W) = 1$ *defined by (2.8) is nondecreasing for each* W.

Proof. We need to show that $c^{N-1}_{N}(W) \leq c^{N}_{N+1}(W)$ for all $N = 1, 2, \ldots, W-1$, which is equivalent, since $H(\mathcal{E}(W,N)) - H(\mathcal{E}(W,N+1)) > 0$ for all N, to

$$H(\mathcal{E}(W,N)) + H(\mathcal{E}(W,N)) \leq H(\mathcal{E}(W,N-1)) + H(\mathcal{E}(W,N+1)). \quad (2.9)$$

Now both sides of (2.9) correspond to the heterogeneity of some configuration with $2N$ countries of the world population $2W$; but in the left hand side we have in fact the heterogeneity of an efficient configuration $\mathcal{E}(2W, 2N)$ with $2N$ countries and in the right hand side we have heterogeneity of some other configuration for $2W$ citizens. Therefore the desired inequality follows. ■

Theorem 2 *For given* W *and* $c \in \langle 0, 1\rangle$, *an efficient configuration exists and it has* $N^* \leq W$ *countries if and only if* $c \in \langle c^{N^*-1}_{N^*}(W), c^{N^*}_{N^*+1}(W)\rangle$. *If* $c > 1$ *then there is a unique efficient configuration having* W *one-citizen countries.*

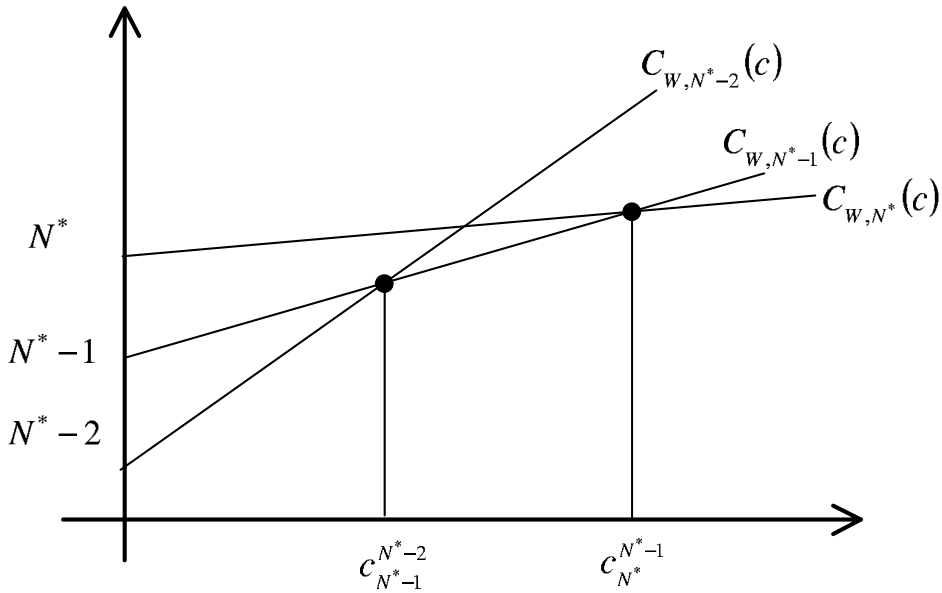

Figure 2.1:

Proof. Since $C_{W,1}(0) = 1 < C_{W,2}(0) = 2 < ... < C_{W,W}(0) = W$ and $C_{W,1}(1) \geq C_{W,2}(1) \geq \cdots \geq C_{W,W}(1)$, we immediately have that $0 < c_{N+1}^{N}(W) \leq 1$ and it remains to show that

$$(\forall N < N^*) c_{N^*}^{N}(W) \leq c_{N^*}^{N^*-1}(W)$$

and

$$(\forall N > N^*) c_{N}^{N^*}(W) \geq c_{N^*+1}^{N^*}(W).$$

We shall show the first inequality, the proof of the second one is similar.

Due to Lemma 4 we have $c_{N^*-1}^{N^*-2}(W) \leq c_{N^*}^{N^*-1}(W)$. If this inequality is fulfilled strictly, we have for $c = c_{N^*-1}^{N^*-2}(W)$ that $C_{W,N^*}(c) > C_{W,N^*-1}(c) = C_{W,N^*-2}(c)$ and since the rate of growth of $C_{W,N^*-2}(c)$ is greater than that of $C_{W,N^*-1}(c)$, the former function must intersect $C_{W,N^*}(c)$ earlier than the latter. (See Figure 2.1 for illustration.) Hence we get $c_{N^*}^{N^*-2}(W) \leq c_{N^*}^{N^*-1}(W)$ and the desired inequality follows by induction. ■

To simplify the search for efficient configuration even further, we have the following assertions.

Lemma 5 *Let W and N be such that $\lceil \frac{W}{N} \rceil \leq 3$. Then $N + H(\mathcal{E}(W,N)) = W + H(\mathcal{E}(W,W))$.*

Proof. If $\lceil \frac{W}{N} \rceil \leq 3$ then the only possible sizes of countries in $\mathcal{E}(W,N)$ are 1,2 and 3. Let us denote their numbers n_1, n_2 and n_3 respectively. Then $C_{W,N}(c) = n_1 + n_2 + n_3 + (n_1.0 + n_2.1 + n_3.2)c$, hence $N + H(\mathcal{E}(W,N)) = n_1 + 2n_2 + 3n_3 = W = W + H(\mathcal{E}(W,W))$. ■

Corollary 1 *For a given W and N such that $N < W$ and $\lceil \frac{W}{N} \rceil \leq 3$, if there exists $N' < N$ such that $\lceil \frac{W}{N'} \rceil \leq 3$, then $\mathcal{E}(W,N)$ is efficient if and only if $c = 1$.*

For illustration, we consider $W = 24$. Table 1 gives the efficient structures and Figure 2.2 depicts the cost functions $C_{24,N}(c)$ for various N. Corollary 1 ensures that it is not necessary to consider $N > 8$, since configurations $\mathcal{E}(24,N)$ for $N = 9, 10, \ldots, 23$ are efficient if and only if $c = 1$ and $\mathcal{E}(24,24)$ is efficient for all $c \geq 1$.

2.4 Stable configurations

The notion of stability is based on an assumption that an individual can leave his country and/or join another country without obtaining the consent of any of the countries affected (see Tiebout (1956), Westhof (1977),

N	Efficient configurations	$C_{W,N}(c)$	Global optimum for
1	(24)	$1 + 144c$	$c \in \langle 0, \frac{1}{72} \rangle$
2	(12,12), (11,13)	$2 + 72c$	$c \in \langle \frac{1}{72}, \frac{1}{24} \rangle$
3	(8,8,8), (7,9,8)	$3 + 48c$	$c \in \langle \frac{1}{24}, \frac{1}{12} \rangle$
4	(6,6,6,6), (5,7,6,6), (5,7,5,7)	$4 + 36c$	$c \in \langle \frac{1}{12}, \frac{1}{8} \rangle$
5	(5,5,5,5,4)	$5 + 28c$	$c \in \langle \frac{1}{8}, \frac{1}{4} \rangle$
6	(4,4,4,4,4,4), (3,5,4,4,4,4) (3,5,3,5,4,4), (3,5,3,5,3,5)	$6 + 24c$	$c = \frac{1}{4}$
7	(4,4,4,3,3,3,3), (4,5,3,3,3,3,3)	$7 + 20c$	$c = \frac{1}{4}$
8	(3,3,3,3,3,3,3,3)	$8 + 16c$	$c \in \langle \frac{1}{4}, 1 \rangle$

Table 2.1: Efficient configurations for $W = 24$.

Haeringer (2000)). We suppose that the position of the government in a country is always in the middle of the country (which is one of the efficient locations of the government for that country, as proved in Lemma 1). In case a change in the configuration occurs, the location of the government in each of the affected countries will move to recover the minimal possible heterogeneity (this may be justified by voting of the citizens over the location of the government, see Alesina and Spolaore (1997) for a justification of this assumption). Formally we shall define a stable configuration as follows:

Definition 1 *A configuration $\mathcal{E}$ of countries is said to be stable if none of the players can obtain a higher utility when leaving his country in $\mathcal{E}$ and either*

1. *creating a country of his own or*

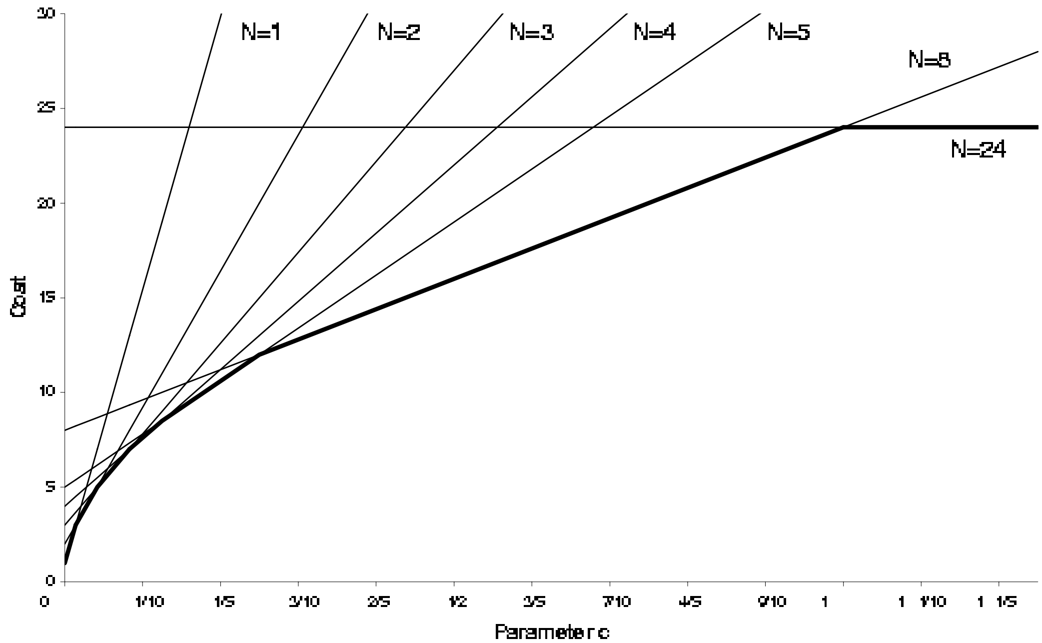

Figure 2.2:

2. joining a neighbouring country.

Haeringer (2000) proved in a more general model (with a monotone distribution of individuals on the line, identical concave utility function for each citizen and under the rule that an individual can join any country he wishes) the following assertion:

Lemma 6 *If a configuration $\mathcal{E}$ is stable then all the countries in $\mathcal{E}$ are connected.*

In our model, increasing utility is equivalent to decreasing cost. Moreover, it is easy to see that the citizens with the highest cost in a country are its border citizens. So the necessary conditions for a stable configuration can be summarized in the following two assertions.

Lemma 7 *Let $\mathcal{P}$ be a connected country with $p > 1$ citizens. Then no citizen of $\mathcal{P}$ wants to separate and create a country of his own if and only if*

$$c \leq \frac{2}{p}. \tag{2.10}$$

Proof. We have to compare the cost of a border citizen of the original country with the cost of a citizen in a one-citizen country, i.e. $\frac{1}{p} + \frac{p-1}{2}c$ should not be higher than 1. Suitable algebraic rearrangements yield the desired result. ■

Lemma 8 *Let two neighbouring countries, $\mathcal{P}$ and $\mathcal{Q}$, have sizes p and $q, p \leq q$ respectively. Then*

1. *the border citizen of $\mathcal{Q}$, who is closest to $\mathcal{P}$, does not want to jump to $\mathcal{P}$ if and only if $q = p + 1$; in case $q > p + 1$ it must hold*

$$c \leq \frac{2}{(p+1)q}; \tag{2.11}$$

2. *the border citizen of $\mathcal{P}$, who is closest to $\mathcal{Q}$ does not want to jump to $\mathcal{Q}$ if and only if*

$$c \geq \frac{2}{p(q+1)}. \tag{2.12}$$

Proof.

1. Consider the border citizen from the bigger country, $\mathcal{Q}$. If he prefers to stay in his original country than to join the smaller country, then his old cost is lower than his cost in the smaller country which will have $p+1$ citizens and its government located in its middle:

$$\frac{1}{q} + \frac{q-1}{2}c \leq \frac{1}{p+1} + \frac{p}{2}c,$$

The above inequality is equivalent to

$$\frac{q-p-1}{2}c \leq \frac{q-p-1}{(p+1)q}.$$

Clearly, if $q = p+1$, this inequality is trivially fulfilled and further manipulations give the desired inequality.

2. If the border citizen from the smaller country $\mathcal{P}$ with $p > 1$ citizens prefers to stay in $\mathcal{P}$ rather than to join $\mathcal{Q}$, then the following inequality must be fulfilled:

$$\frac{1}{p} + \frac{p-1}{2}c \leq \frac{1}{q+1} + \frac{q}{2}c,$$

which is equivalent to

$$\frac{p-q-1}{2}c \leq \frac{p-q-1}{p(q+1)}.$$

Since $p < q+1$, after dividing by $p-q-1$ we get the desired inequality. (If $p = 1$, the starting inequality $1 \leq \frac{1}{q+1} + \frac{q}{2}c$ also leads to the desired claim.)

■

Corollary 2 *A configuration containing two neighbouring countries of sizes p and $q \geq p+2$ can never be stable.*

Proof. According to Lemma 8, instability on the border of two neighbouring countries of sizes p and $q \geq p+2$ will not occur if and only if

$$\frac{2}{p(q+1)} \leq c \leq \frac{2}{(p+1)q},$$

but since $(p+1)q > p(q+1)$ holds for each pair of integers p, q with $q > p$, such c can never exist. ■

Since in a configuration with just one country Lemma 8 is irrelevant, Lemma 7 implies

Corollary 3 *For a given world population W and a value of parameter c, the configuration $\mathcal{E} = (W)$ is stable if and only if $W.c \leq 2$.*

Corollary 4 *The configuration in which all countries have size 1 is the only stable configuration for $c > 1$. Conversely, this configuration cannot be stable if $c < 1$.*

Proof. If $c > 1$ and there exists a country with size $p \geq 2$ in a configuration, then the inequality $c \leq \frac{2}{p}$ cannot be fulfilled. Conversely, if in a configuration $\mathcal{E}$ all countries have size 1, then the only constraint that applies is the converse of inequality (2.12), which in this case implies $c \geq 1$. ■

Corollary 5 *A configuration $\mathcal{E}$ containing at least two countries, but all with the same size $p > 1$ is stable if and only if*

$$\frac{2}{p(p+1)} \leq c \leq \frac{2}{p}.$$

Corollary 6 *A configuration $\mathcal{E}$ containing countries with sizes $p_1 < p_2 < \cdots < p_k$ for $k \geq 2$, but such that the size difference of two neighbouring countries is never more than 1 and there are no neighbouring countries with size p_1 is stable if and only if*

$$\frac{2}{p_1(p_1+2)} \leq c \leq \frac{2}{p_k}.$$

If there exist neighbouring countries with size p_1 then p_1 has to fulfill the stronger condition

$$\frac{2}{p_1(p_1+1)} \leq c.$$

We shall interpret the statements of Corollaries 5 and 6 from a different angle, namely enabling us to compute the possible country sizes for a given

c. Let us therefore denote

$$t_1^L = \sqrt{\frac{2}{c} + \frac{1}{4}} - \frac{1}{2} \tag{2.13}$$

$$t_2^L = \sqrt{\frac{2}{c} + 1} - 1 \tag{2.14}$$

$$t^U = \frac{2}{c} \tag{2.15}$$

and Corollary 6 can now be reformulated as:

Theorem 3 *For a given value of parameter $c \leq 1$, the size p of any country in a stable configuration $\mathcal{E}$ with at least two countries must fulfill $p \in \langle t_1^L, t^U \rangle$; if there are no neighbouring countries both with their sizes equal to the minimum size of a country in $\mathcal{E}$, then the lower bound has to be weakened to t_2^L.*

A simple algebraic procedure will show that $t_1^L \leq t^U - 1$ for each $c \leq 1$, which means that there is always an integer p in interval $\langle t_1^L, t^U \rangle$; however this does not automatically imply that for each c and each W there will always exist a stable configuration.

Theorem 4 *For each W and $c \leq 1$ a stable configuration exists, namely it is the configuration $\mathcal{E}(W, N)$ for suitable N.*

Proof. If $W \leq t^U = \frac{2}{c}$ then configuration (W) is stable. So we have to deal with the case $W > \frac{2}{c}$ and for each value of such W and $c < 1$ we construct a stable configuration.

For $c \in \langle \frac{2}{3}, 1)$ we have $t_2^L \leq 1$ and country sizes 1 and 2 are possible. So for W even we set $\mathcal{E} = (2, 2, \ldots, 2)$ and for W odd we shall have $\mathcal{E} = (1, 2, \ldots, 2)$. For $c \in \langle \frac{1}{3}, \frac{2}{3} \rangle$ we have $t_1^L \leq 2$ and $t^U \geq 3$, so for W even we have again a configuration consisting of solely two-citizen countries and for W odd we have $\mathcal{E} = (3, 2, \ldots, 2)$ (remember, that now $W > 3$). If $c \leq \frac{1}{3}$, then there are always at least two feasible country sizes if and only if $t_1^L \leq t^U - 2$, which is equivalent to the quadratic inequality $c^2 - 4c + 2 \geq 0$ with the solution set equal to $(-\infty, 2 - \sqrt{2}\rangle \cup \langle 2 + \sqrt{2}, \infty)$. This solution set contains the interval $(0, \frac{1}{3}\rangle$. So suppose that we choose for the size of a country $k = \lceil t_1^L \rceil$, so $k+1$ is also feasible. Now consider the divisibility of W by k. If $W \ mod \ k = 0$, we set $\mathcal{E} = (k, k, \ldots, k)$; if $W \ mod \ k = 1$, there will be just one country of size $k+1$ in $\mathcal{E}$, the rest will have size k. In general, if $W \ mod \ k = r$ for some $r < k$, $\mathcal{E}$ will contain r countries of size $k+1$ and the rest will have size k.

This construction will be possible for all values of k and r if W is sufficiently large, i.e. at least equal to $(k-1)(k+1) = k^2 - 1$. Due to our choice of k we know that $k \leq t_1^L + 1$ and since $W \geq \frac{2}{c}$, we must have $(t_1^L + 1)^2 - 1 \leq \frac{2}{c}$, which trivially holds. ■

The following example shows that the set of stable configurations may be quite diverse for some values of parameters.

Example. If $c = \frac{1}{4}$ then $p \in \{3, 4, 5, 6, 7, 8\}$, or $p \in \{2, 3, 4, 5, 6, 7, 8\}$ if there are no neighbouring countries with size 2.

If $W = 5$ then the sizes $6, 7, 8$ are irrelevant, so we get that the only stable configurations are (5) and $(2, 3)$. However, if $W = 24$, the number of stable configurations is already very high. All of them, apart from suitable permutations of countries, are listed in Table 2, sorted according to the number of countries N. Those, that are also efficient for $c = \frac{1}{4}$, are marked with a star. On the other hand, there exist configurations efficient for $c = \frac{1}{4}$, which cannot be stable, even with a suitable permutation of countries, for example $(3,3,3,5,5,5)$.

N	Stable configurations
3	$(8,8,8)$
4	$(6,6,6,6),(5,6,6,7)$
5	$(4,5,5,5,5)^*,(4,4,5,5,6),(3,4,5,6,6)$
6	$(4,4,4,4,4,4)^*,(3,4,4,4,4,5)^*,(3,3,4,4,5,5)^*,(2,3,4,5,5,5),(2,3,4,4,5,6)$
7	$(3,3,3,3,4,4,4)^*,(2,3,3,3,4,4,5),(2,3,2,3,4,5,5),(3,3,3,3,3,4,5)^*$
8	$(3,3,3,3,3,3,3,3)^*,(2,3,3,3,3,3,3,4),(2,3,2,3,3,3,4,4)$
9	$(2,3,2,3,2,3,2,3,4)$

Table 2.2: Stable configurations for $c = \frac{1}{4}$ and $W = 24$.

2.5 Group deviations

Since the set of stable configurations may be so large, we try to see what happens, if we allow for group deviations. The simplest possibility is that two people, who originally were not in the same country, leave their countries and form a new country. Now again we suppose that the original countries cannot prevent the secession, but both deviating citizens must strictly increase their utilities (or, equivalently, strictly decrease their cost).

Definition 2 *A configuration is called strongly stable if it is stable and no two citizens from two different neighbouring countries want to leave their original countries and form a country of their own.*

If a configuration contains just one country, then only Lemma 7 applies and we have similarly as in Section 4:

Corollary 7 *For a given world population* W *and a value of parameter* c*, the configuration* $\mathcal{E} = (W)$ *is strongly stable if and only if* $W.c \leq 2$*.*

Clearly, no citizen of a two-citizen country will be willing to participate in a deviation according to Definition 2. On the other hand, a deviation of this sort brings the greatest improvement compared to their original situation to a pair of neighbouring citizens. Therefore it is sufficient to consider only the following case:

Lemma 9 *Let* $\mathcal{E}$ *be a stable configuration and* $\mathcal{P}$ *and* $\mathcal{Q}$ *two neigbouring countries in* $\mathcal{E}$ *with sizes* p *and* $q, p \leq q$ *respectively. Then the two neighbours, one from* $\mathcal{P}$ *and the other from* $\mathcal{Q}$ *will not unanimously want to leave* $\mathcal{P}$ *and* $\mathcal{Q}$ *respectively and form a country of their own if and only if one of the following cases occurs:*

1. $p = q = 1$ *and* $c \geq 1$*,*
2. $p = 2$ *or* $q = 2$*;*
3. $2 < p$ *and* $c \leq \frac{1}{p}$*.*

Proof. A border citizen from a country with $p > 1$ citizens prefers staying in his original country before creating a two-citizens country if and only if

$$\frac{1}{p} + \frac{p-1}{2}c \leq \frac{1}{2} + \frac{1}{2}c,$$

which is equivalent to

$$\frac{p-2}{2}c \leq \frac{p-2}{2p}.$$

This inequality is trivialy fulfilled if $p = 2$; if $p > 2$ we get $c \leq \frac{1}{p}$. On the other hand, the 'nondeviation' inequality for a citizen from a one-citizen country is $1 \leq \frac{1}{2} + \frac{1}{2}c$, equivalent to $c \geq 1$. ■

Corollary 8 *For a given c, the size $p > 2$ of a country in a strongly stable configuration which has a neighbouring country with size $q \geq p$ cannot exceed $\frac{1}{c}$.*

Now we can summarize the conditions for the country sizes in a strongly stable configuration in the following assertion:

Lemma 10 *For a given value of parameter c, the configuration $(1, 1, \ldots, 1)$ is strongly stable if and only if $c \geq 1$. For $c < 1$ in a strongly stable configuration $\mathcal{E}$ with at least two countries each country $\mathcal{P}$ either must have a neighbour of size 2 or $p \geq t_1^L$ if $\mathcal{P}$ has no neighbour of size p, otherwise it is sufficient to have $p \geq t_2^L$; and $p \leq \frac{1}{c} + 1$ if $\mathcal{P}$ has no neighbour with size exceeding $\frac{1}{c}$, otherwise $p \leq \frac{1}{c}$.*

Theorem 5 *For each value of W and c there exists a strongly stable configuration.*

Proof. If $c \geq 1$ then the configuration containing only countries of size 1 is strongly stable. For $c < 1$ we can repeat the construction in the proof of Theorem 4. For $c \geq \frac{1}{3}$ in the constructed configuration each country has a neighbouring country of size 2, so the construction gives a strongly stable configuration. For $c < \frac{1}{3}$ we need to ensure that $k + 1 \leq \frac{1}{c}$, which means that we need $t_1^L + 2 \leq \frac{1}{c}$, which is equivalent to the quadratic inequality $2c^2 - 5c + 1 \geq 0$. Its solution set $(-\infty, \frac{5-\sqrt{17}}{4}\rangle \cup \langle\frac{5+\sqrt{17}}{4}, \infty)$ however does not include the whole interval $(0, \frac{1}{3}\rangle$, but at least $(0, \frac{1}{5}\rangle$ is covered by this case.

For $c \in \langle\frac{1}{4}, \frac{1}{3}\rangle$ we have $2 \leq t_1^L \leq 3 \leq \frac{1}{c}$ and $t_2^L \leq 2$, so 2 is also an admissible country size, if no neighbours of size 2 exist. So we can take configuration $(3, 3, \ldots, 3)$ if $W \equiv 0(mod\ 3)$ and for cases $W \equiv 1(mod\ 3)$ and $W \equiv 2(mod\ 3)$ configurations $(2, 3, \ldots, 3, 2)$ and $(2, 3, \ldots, 3)$ respectively. This will be possible for all W that fulfill $W \geq \frac{2}{c} \geq 6$.

For $c \in \langle\frac{1}{5}, \frac{1}{4})$ the smallest possible country size is 3, but now $\frac{1}{c} \geq 4$, so 4 as the size of a country is possible. So we can take configuration $(3, 3, \ldots, 3)$, $(3, 3, \ldots, 3, 4)$ and $(4, 3, \ldots, 3, 4)$ for if $W \equiv 0, 1, 2(mod\ 3)$ respectively. Here again W is sufficiently large, since $W \geq \frac{2}{c} > 8$. ■

However, the set of strongly stable configurations is smaller than that of stable configurations. Again for $c = \frac{1}{4}$, the feasible intervals of country sizes are $\{3, 4\}$ and $\{2, 3, 4\}$ if there are no neighbouring countries with sizes 2. However, countries of size 5 are also admissible, as long as all their neighbours have size at most 4. All the possible configurations for $W = 24$ are summarized in Table 3.

N	Strongly stable configurations
6	$(4,4,4,4,4,4)^*, (3,4,4,4,4,5)^*, (3,3,4,5,4,5)^*$
7	$(3,3,3,3,4,4,4)^*, (2,3,3,3,4,4,5), (3,3,3,3,3,4,5)^*$
8	$(3,3,3,3,3,3,3,3)^*, (2,3,3,3,3,3,3,4), (2,3,2,3,3,3,4,4)$
9	$(2,3,2,3,2,3,2,3,4)$

Table 2.3: Strongly stable configurations for $c = \frac{1}{4}$ and $W = 24$.

2.6 Conclusion and directions for further research

In this paper we studied a simple one-dimensional model of country formation. We considered optimality as well as stability critera and have shown that configurations that arise in the two cases may be quite different.

For each parameter value of a particular instance of our problem, the set of efficient configurations is always nonempty and its structure is quite simple. On the other hand, although stable configurations always exists, they are very diverse if we only allow individual moves, and are still quite variable if the simplest possibility of group deviations is allowed.

It would be interesting to see how these results change if some of the characteristics of the model are modified, for example, we may have:

- the distances between two neigbours arbitrary, and/or
- in each discrete point an arbitrary finite population size, and/or
- secessions of larger groups of citizens allowed, and/or
- possible only if the majority of citizens in affected countries approve them,
- different forms of utility functions.

We expect that some algorithms for obtaining efficient and stable configurations in each case could be derived, possibly using methods of discrete location theory, described in [6] and [17].

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Chapter 3

Optimality of Strong Tiebout Equilibria in a Finite Local Public Goods Economy

3.1 Introduction

The Tiebout [18] hypothesis asserts that an equilibrium in local public good economies exists, and moreover that it is Pareto optimal. The mechanism for attaining this equilibrium is that of agents who reveal their true preferences for the public goods by "voting with the feet" among a *given* (large) number of jurisdictions which solves the "free rider problem".[1]

The purpose of this paper is to study this assertion in a location model

[1] We use the words "jurisdiction" and "coalition" interchangeably. Also, we apologize that although Chapter 2 and 3 analyze the same model, they use different notation. The reason for this is that, when Chapter 2 was already published, it turned out that its notation was not convenient for the analysis carried out in Chapter 3.

of local public good economies in which the number of jurisdictions is not exogenously fixed but endogenously determined, the number of consumers is finite and the decision over the location of the public goods is taken by majority voting. In particular we want to answer the question whether or not there are forces that always lead to the creation of too many jurisdictions, as shown in the seminal work by Alesina and Spolaore [1] (AS henceforth) who used a model very similar to ours but with a *continuum* of agents.

In economic modeling a continuum of agents is used to approximate large numbers of agents. Its main advantage is that it allows the use of nice mathematical tools. When the number of agents is finite each agent has a non-negligible impact on the economy which depending on the type of application one has in mind may be more realistic.[2] Moreover, it is desirable that the conclusions derived using the logical construct of a continuum of agents are confirmed for a large, but finite, number of agents. In this sense our model provides a test of robustness of the AS inefficiency result at the costs of technical complications.

The main result of this paper is that with a finite number of agents, even if this number is very large, efficiency and stability may be reconciled. In other words, our results do not "converge" to the ones of AS when the number of agents becomes large.[3] However, inefficiencies may follow much

[2] A finite number of agents allows another interpretation concerning the application of country formation, analyzed by AS. In our model a single agent may be understood as an indivisible region or ethnic group. In this interpretation a region represents types of individuals whose distinguishing feature may be a common history, language, religion or any other group specific characteristic that implies similar preferences over possible country membership. This captures the ethnic division of many countries (e.g. Afghanistan is formed by 55 ethnic groups) and that in reality those groups (want to) separate from existing countries (e.g. Basque region, Catalunya, north and south Italy, Québec, former Yugoslavia, former Tchecoslovakia) or join each other in order to create a new one (e.g. reunified Germany, the European Union).

[3] As we will explain below our change from a continuum of agents to a finite number of agents requires also modifications in the equilibrium concept. Otherwise an equilibrium

more irregular pattern as in AS. On one hand in equilibrium there may be less coalitions than efficiency requires. On the other hand, while in the AS model in equilibrium all coalitions have the same size, this is no longer true in our model. We find that the size of coalitions may differ too much.

We carry out our analysis in the finite version of the AS-model developed in Cechlárová et al. [3]. This means that our model follows Demange [5], Le Breton and Weber [17], Haeringer [12] or Haimanko et al. [13] and models coalition formation as the result of a specific trade-off between the benefits of large jurisdictions and the costs of heterogeneity in large populations. Like Greenberg and Weber [9] or Jehiel and Scotchmer [14] jurisdictions consist of consumers who choose the same public project and share the costs equally. Individuals have identical quasi-linear utility functions but differ in their most preferred location of the public good. We assume that there is a finite set of equidistant points on a line at each of which exactly one agent is located.

The combination of a finite model with the presence of majority voting in order to determine the location of the local public good is an important element of our model. It implies that adding a single individual to an existing coalition creates two types of externalities. A positive externality arises since public good provision costs decrease for all (initial) members of the jurisdiction, while a negative externality stems from the fact that the decision over the location of the local public good is taken by majority voting. The latter causes (under our tie-breaking rule) the location of the public good to shift necessarily, implying that for a part of the (initial) members of the jurisdiction the public good becomes farther away. In a continuous model with decision scheme, like AS or Westhoff [19], both externalities are absent.[4] In finite models without decision scheme like Greenberg and We-

may not exist.

[4] An exception is the paper by Jehiel and Scotchmer [15] who also use a continuum of agents but only allow a positive measure of agents to move.

ber [9] the negative externality is absent.[5]

Our stability analysis builds on the notion of a stable configuration investigated in Cechlárová et al. [3] and uses their results concerning efficient coalition structures. The concept of stable configurations generates multiple equilibria for some of which the AS-inefficiency result is true while for others it is not. The purpose of this paper is to refine this concept, to sharpen the prediction concerning equilibrium structures and compare them again to efficient structures.

We adapt first the AS-stability concept to our model. This approach is not fruitful, since in many situations an equilibrium does not exist.[6]

Instead, we follow the literature on coalition formation and local public good economies and consider the concept that Greenberg and Weber [9] have called strong Tiebout equilibrium.[7] This means on one hand that there is no set of agents who unanimously decide to leave their coalition and join each other in a new coalition because all become better-off. And on the other hand, there is no agent who prefers to leave her coalition and to join another existing one. At least one of these ideas underlies the work in Westhoff [19], Greenberg [7] and [8], Guesnerie and Oddou [11], Drèze and Greenberg [6], Greenberg and Weber [9] and [10], Demange [5], Konishi et al. [16], Haeringer [12], or recently Bogomolnaia and Jackson [2], among others.

We prove that a strong Tiebout equilibrium exists, provided either the

[5] Haeringer [12] shows the importance of majority voting (or more in general a decision scheme) in a model with a finite number of agents by pointing out that an important result of Greenberg and Weber [9] is no longer true.

[6] Interestingly, we also show that in situations in which an equilibrium exists, too few jurisdictions may be created.

[7] See the discussion of these authors on the formalization of Tiebout's idea by this concept.

number of agents or heterogeneity costs are high enough.[8] Moreover, we show that these conditions are not necessary. However, a slight weakening of our concept allows to prove general existence.

Strong Tiebout equilibria may contain coalitions of different sizes. Although our model is very symmetric, the size differences between coalitions in this coalition structures may be too large and generate an inefficiency. Independently of this type of inefficiency the number of coalitions in strong Tiebout equilibria may be too large or too small. Both types of inefficiencies contrast with the results in AS.

Our result that a strong Tiebout equilibrium may be efficient confirms the result in Dahm [4], where this equilibrium concept is analyzed in the original AS-model with a continuum of agents. Both paper together suggest that in order to establish unambiguously the inefficiency result of AS one needs to combine both the continuous model *and* the AS-stability concept.

To our best knowledge the only other paper combining an endogenously determined number of jurisdictions with a finite number of agents and majority voting over the locations of the public goods is Haeringer [12]. Our model is a special case of his' but his focus of analysis is very different. Rather than analyzing the efficiency of stable equilibria, his main concern is to find general conditions for the existence of stable equilibria.[9]

The remainder of this paper is organized as follows. In Section 2, we describe the basic model and definitions. We review what is known from Cechlárová et al. [3] about stable configurations in Section 3. Then we discuss the AS-stability concepts. Section 5 deals with the generalization of stable configurations to free mobility equilibria, called Tiebout equilibria. The last Section characterizes and discusses existence and optimality

[8] This type of conditions are not uncommon in local public good economies with a finite number of agents. Compare e.g. to Wooders [20], where an equilibrium is shown to exist provided the number of consumers of each type is sufficiently large.

[9] Note that in the general model a strong Tiebout equilibrium does not always exist.

of strong Tiebout equilibria. In Appendix A we show how our model can be adapted e.g. in order to have a continuum of consumers distributed uniformly on a finite set of locations. All proofs are relegated to Appendix B.

3.2 The Model

We consider an economy consisting of a finite number T of agents, the set of which is denoted by $\mathcal{T} = \{1, ..., T\}$. Typical members of $\mathcal{T}$ are denoted by i. A coalition S is any subset of $\mathcal{T}$, and $|S|$ denotes its cardinality or "size". A coalition structure $\mathcal{W} = \{S_1, ..., S_N\}$ is any partition of $\mathcal{T}$ in coalitions,

$$S_n \cap S_{n'} = \emptyset, \forall n \neq n', \text{and} \cup_{n=1}^{N} S_n = \mathcal{T}.$$

We denote by Π the set of all coalition structures and by $\mathcal{P}$ the set of all coalitions that can form. For each coalition structure $\mathcal{W}$ we denote by $S^i(\mathcal{W})$ the coalition i belongs to. When it is clear which coalition structure is meant, we write S^i and sometimes S instead of $S^i(\mathcal{W})$.

We consider a spatial model in which at each integer on the line segment $[1, T]$ one agent is located.[10] To simplify notation we denote by i the location of agent i. We assign coalitions their subindex by the following procedure. Agent 1 belongs to coalition S_1. Agent 2 belongs either to coalition S_1 or S_2. The next agent who does not belong to neither S_1 nor S_2 forms part of S_3 and so on.

We say that a coalition is connected if, when considering two agents in it, all agents with intermediate positions between those agents belong to the

[10] For simplicity we choose $[1, T]$ and the finite number T of agents. As shown in Appendix A, the model can be easily adjusted to accommodate for a continuum of agents partitioned equally over the finite number of locations T as in Greenberg [8] and/or for the line interval $[0, 1]$ as in AS.

same coalition. Moreover, a coalition structure $\mathcal{W}$ is connected, provided the previous property holds across all coalitions. More formally:

DEFINITION **3.1** ***Connectedness.***
A coalition S is connected if $i, i'' \in S$ implies that $i' \in S$ for all i' such that $i \leq i' \leq i''$. A coalition structure $\mathcal{W}$ is connected if all coalitions $S \in \mathcal{W}$ are connected.

For any connected coalition structure $\mathcal{W}$, the vector $w(\mathcal{W}) = (|S_1|, ..., |S_N|)$ indicates coalition sizes. For coalition structures with this property we will say that coalitions with subsequent subindexes, like S_n and S_{n+1}, are "neighboring coalitions". Also, we will say that an agent i is a "border agent" in a coalition S if she is either the agent with the highest or the lowest position in S.

Coalitions have to provide a local public good bundle $l(S) \in [1, T]$. Each agent $i \in \mathcal{T}$ is endowed with the same positive amount y of a private good and has preferences which are represented by the quasi-linear utility function

$$U_i(S) = \alpha(1 - \beta d_i(S)) + y - \frac{\gamma}{|S^i|}, \tag{3.1}$$

where S is the coalition to which i belongs, α, β and γ are positive parameters and $d_i(S) = |i - l(S)|$ is the distance from agent i to its local public good. The parameter α measures the maximum utility of the public good, when $i = l(S)$. The parameter β measures the loss in utility that an agent suffers when the location of government is far from her's. Finally the parameter γ represents the provision costs for the local public good. These costs are divided by the cardinality of the coalition S^i. This represents an equal cost sharing rule which, since income is the same for all agents, may be interpreted as proportional taxation.[11]

[11] The same functional form of the utility function is used in AS. As in their model our results generalize (without qualitative changes) to the case where government costs are

Since it is completely tantamount but more convenient to use instead of (3.1) the normalized individual cost function

$$c_i(S) = cd_i(S) + \frac{1}{|S^i|}, \tag{3.2}$$

where $c = \frac{\alpha\beta}{\gamma}$, we shall do so. The nonnegative parameter c measures the relative importance of "heterogeneity" costs with respect to the "public good provision" costs. For simplicity, we will also use the notation $c_i(\mathcal{W})$, instead of $c_i(S)$.

The decision over the location of the public good $l(S)$ is taken by majority voting. This implies, since individual utilities are single-peaked with respect to $l(S)$, that the median voter determines $l(S)$. In case of ties we suppose that $l(S)$ is located exactly in the middle of the two median voters.[12]

3.3 Stable Configurations

We start by reviewing what is already known from previous work about stable coalition structures in this model. Cechlárová et al. [3] use the concept of a stable configuration.[13]

Definition **3.2** *A coalition structure $\mathcal{W} = \{S_1, ..., S_N\}$ is a **stable configuration** if for all $i \in T$, we have, denoting S^i by S_n, that $c_i(S_n) \leq c_i(S' \cup i), \forall S' \in \{\emptyset, S_{n-1}, S_{n+1}\}$.*

$\gamma = \delta + \xi|S|$, with $\delta > 0$, $\xi > 0$.

[12] Formally the median voter in S is the agent i_m such that $|\{i \in S : i \leq i_m\}| = |\{i \in S : i \geq i_m\}|$. The so defined decision scheme coincides in this model with what Haeringer [12] calls the mean of the extremes and is a straightforward extension of the scheme used in AS for a continuum of agents. We choose this tie breaking rule in order to insure the existence of stable equilibria, see Haeringer [12].

[13] These authors consider also strongly stable configurations, which are immune to deviations of two neighboring agents of two different coalitions. This is as a special case included in the concept of $\mathcal{C}$-stability which we consider below.

Since we work with individual costs instead of utilities, this definition just says that any agent should have at least the utility she could get by joining a neighboring coalition or standing on her own.

For the corresponding result it is necessary to introduce some notations. We denote by $|S_{max}(\mathcal{W})|$ and by $|S_{min}(\mathcal{W})|$ the largest and the smallest coalition size in $\mathcal{W}$, respectively. Again, when it is clear which coalition structure $\mathcal{W}$ is meant, we simply write $|S_{max}|$ and $|S_{min}|$. With this conventions we define $\overline{c(\mathcal{W})}^{SC} = \frac{2}{|S_{max}|}$. We define $\underline{c(\mathcal{W})}^{SC}$ to be equal to $\frac{2}{|S_{min}|(|S_{min}|+1)}$ if there exist two neighboring coalitions with size $|S_{min}|$ in $\mathcal{W}$ and to $\frac{2}{|S_{min}|(|S_{min}|+2)}$ otherwise.[14] Cechlárová et al. [3] show that the following is true.

PROPOSITION **3.1** *(Cechlárová et al. [3]) A connected coalition structure* $\mathcal{W} = \{S_1, ..., S_N\}$ *is a stable configuration if and only if*
(1) neighboring coalitions have a size difference of at most one and
(2) $\underline{c(\mathcal{W})}^{SC} \leq c \leq \overline{c(\mathcal{W})}^{SC}$.

The intuition behind this proposition is that neighboring coalitions can not be too different in size and c must lie in an interval depending on $\mathcal{W}$. If the size difference between two neighboring coalitions is larger than one, then it pays for one of the two border agents to join the other coalition. If the size difference is at most one but we have that $c < \underline{c(\mathcal{W})}^{SC}$, then it pays for the border agent in the smallest coalition to join the neighboring coalition. If $\overline{c(\mathcal{W})}^{SC} < c$, then the worst off in the largest coalition can become better off by creating a singleton coalition.

Since the notion of a stable configuration is a weak requirement, in general there exists a multiplicity of stable configurations. As a consequence

[14] If $\mathcal{W}$ is the grand coalition, then, since there is no neighboring coalition to which agents can deviate, we define $\underline{c(\mathcal{W})}^{SC}$ to be zero. Similarly, if $\mathcal{W}$ contains only singleton coalitions, we define $\overline{c(\mathcal{W})}^{SC}$ to be infinity.

of this for some configurations the inefficiency result of AS is true while for others it is not. In order to have a sharper prediction we refine stable configurations by other concepts in the following Sections.

3.4 The Alesina and Spolaore Stability Concepts

Since the purpose of this work is to analyze the AS-inefficiency result in a model with a finite number of consumers, it is straightforward to employ the AS stability concepts to our model. We will show that these concepts are not appropriate when the number of agents is finite.

AS refine the multiplicity of equilibria derived so far by means of a notion which is called A-stability.[15] Denote by $B(\mathcal{W})$ all border agents in a coalition structure $\mathcal{W}$, that is, the agents with the highest and lowest position in each coalition. AS require that after a 'small' perturbation that moves a 'small' set of individuals to a neighboring coalition, the system returns to its initial position. Since in our model at each location we have exactly one individual, this individual is the smallest set we can think of. With this convention the requirement of A-stability can be formalized as follows.[16]

[15] To be fully precise, the first stability concept AS employ is not the notion of a stable configuration but that of A-equilibria. The former differs from the latter in two aspects. First, it allows all agents (and not only agents at borders) to move to neighboring coalitions. However, it turns out that it is sufficient to check that no agent at a border wants to deviate. Second, it includes, since we work with finite model, the requirement that no agents prefers to stand on its own. Skipping this requirement does not imply a qualitative change in what follows. Because of both differences, the concept of a stable configuration is slightly stronger than the notion of an A-equilibrium.

[16] In the case of a continuum of agents, which is uniformly distributed over the T locations, one can define smaller sets (than all agents located at a given point). See Appendix A for the necessary modifications of the model. Still, under other definitions no richer structures than A-equilibria can arise and what we will say about the refinement of A-equilibria will be true for these equilibria, too.

DEFINITION **3.3** *A coalition structure $\mathcal{W} = \{S_1, ..., S_N\}$ is **A-stable** if for all $i \in B(\mathcal{W})$, we have, denoting S^i by S_n, that $c_i(S_n) < c_i(S' \cup i), \forall S' \in \{S_{n-1}, S_{n+1}\}$ if $|S_n| = 1$ and $\forall S' \in \{\emptyset, S_{n-1}, S_{n+1}\}$ otherwise.*

That is to say that all agents should get a strictly higher utility (lower costs) in $\mathcal{W}$ than in neighboring coalitions or standing on their own.

REMARK: Note that this requirement is very strong with a finite number of consumers, since it implies - as in Westhoff [19] - that an individual switches from one coalition to another even when she is indifferent.

The corresponding result is very similar to proposition 2 in AS.[17]

PROPOSITION **3.2** *A connected coalition structure $\mathcal{W} = \{S_1, ..., S_N\}$ is A-stable if and only if all coalitions are of the same size and*

$$\frac{cT}{2} < N < \frac{T}{\sqrt{1+\frac{2}{c}}-1}.^{18} \tag{3.3}$$

REMARK: Note that as in the model with a continuum of agents coalitions must be of the same size. This is already a strong result in the model with a continuum of agents but there it is still feasible. In the AS-model there exists always an infinity of possibilities to partition equally. This is no longer true if we change to a model with a finite number of consumers. The possibilities to partition equally depend on the integer T – but in general there are only a few. The limitations arising from this are clearest if T is a prime number. In this case there are only two coalition structures possible: the

[17] The left part of the numerical condition comes from the requirement of individual rationality, which is absent in AS. If we modify the model as explained in Appendix A and let $T \to \infty$, then the right part of the condition becomes the one in AS.

[18] Again, if $\mathcal{W}$ is the grand coalition the right inequality does not apply. If $\mathcal{W}$ consists only of singleton coalitions, then the left inequality must be skipped.

grand coalition or singletons. While this gives a sharp prediction concerning equilibrium coalition structures, we believe that it is too sharp. Moreover, this implies that the inefficiency result of AS is no longer true. By choosing c appropriately one can induce that only the grand coalition is A-stable, while efficiency requires a strictly larger number of coalitions.

Since in general A-stable coalition structures are not unique, AS refine this set of equilibria via the notion of B-stability. An informal discussion of this concept will suffice for our purposes. The underlying idea is that **exactly one coalition** can be created or eliminated by means of an international agreement which must be ratified by simple majority rule within each coalition affected. A B-equilibrium is then an A-stable structure $\mathcal{W}$ which confronted with another A-stable coalition structure $\mathcal{W}'$, containing exactly one coalition more or less than $\mathcal{W}$, is preferred by a majority in at least one coalition of $\mathcal{W}$. Such a B-equilibrium $\mathcal{W}$ is B-stable if after a perturbation in the number of coalitions to any other A-stable structure the system returns to $\mathcal{W}$ by repeatedly creating or eliminating exactly one coalition.

In the model with a finite number of consumers it is very often not possible to shift to another A-stable structure with exactly one coalition more or less. If larger shifts are considered, problems of existence of a B-stable coalition structure arise. Consider the following simple example.

Example Let $T = 15$ and $c = \frac{1}{4}$. Proposition 3.2 implies that coalitions are of equal size and that $1,8 < N < 7,5$. Hence we have that $N \in \{3, 5\}$. Denote the corresponding coalition structures by $\mathcal{W}(3)$ and $\mathcal{W}(5)$, respectively. The configuration $\mathcal{W}(5)$ is not B-stable since agents 7, 8 and 9 prefer $\mathcal{W}(3)$ and these agents are a majority in the second coalition of $\mathcal{W}(3)$. The second structure $\mathcal{W}(3)$ is not B-stable either. Consider the first coalition in $\mathcal{W}(5)$, where agent 1 and 2 form a majority. We have that

$c_1(\mathcal{W}(5)) \simeq \frac{7}{12} < \frac{7}{10} \simeq c_1(\mathcal{W}(3))$ and $c_2(\mathcal{W}(5)) \simeq \frac{1}{3} < \frac{9}{20} \simeq c_2(\mathcal{W}(3))$.[19] ■

All this suggests that maybe the notion of A-stability is too strong and that one may apply B-stability directly to stable configurations. But then coalition structures can no longer be unambiguously identified with coalition numbers and one is forced to modify the AS-stability concepts so much that they become very different from the concepts AS use in their model.

Since this approach is not straightforward, we prefer to go another way and refine stable configurations by the concept of a strong Tiebout equilibrium - a concept merging ideas frequently used in the literature on coalition formation. We develop this notion step by step and consider in the next Section Tiebout equilibria.

3.5 Tiebout Equilibria

The idea of this Section is to strengthen the concept of a stable configuration to a free mobility equilibrium in which agents can deviate to all existing coalitions. We follow Haeringer [12] and call a free mobility equilibrium Tiebout equilibrium. His definition applied to our setting reads as follows:

Definition 3.4 *A coalition structure* $\mathcal{W} = \{S_1, ..., S_N\}$ *is a* ***Tiebout equi-***

[19] Note that this existence problem comes from the fact that we compare two coalition structures with an odd number of coalitions and not directly from having a finite number of agents. It would also arise if the number of agents were a continuum and two odd numbers are compared. In this example the grand coalition is excluded by individual rationality, which is absent in AS. However, it can be shown that in general no other coalition structure than the one with the largest coalition number, in the example $\mathcal{W}(5)$, may be B-stable. Note that this is also true in AS, where this coalition structure turns out to be the unique stable one.

librium [TE] *if for all* $i \in \mathcal{T}$, *we have,* $c_i(S^i) \leq c_i(S' \cup i), \forall S' \in \mathcal{W} \cup \emptyset, S' \neq S^i$.

Intuitively, a coalition structure $\mathcal{W}$ is a Tiebout equilibrium if no agent wants to move to another existing coalition, because she can not increase her utility (or in our formulation decrease her costs). From Theorem 1 in Haeringer [12] we know that a Tiebout equilibrium always exists. The next proposition characterizes these equilibria in our model.

PROPOSITION **3.3** *A coalition structure* $\mathcal{W} = \{S_1, ..., S_N\}$ *is a Tiebout equilibrium if and only if*
(1) it is connected,
(2) neighboring coalitions have a size difference of at most one and
(3) $\underline{c(\mathcal{W})}^{SC} \leq c \leq \overline{c(\mathcal{W})}^{SC}$.

Proposition 3.3 differs from proposition 3.1 only by the fact that connectedness is no longer an assumption. Because "heterogeneity costs" are an important element in our model, the requirement of free mobility becomes a 'local' stability condition. Stability depends on pairs of neighboring coalitions. The fact that a coalition structure $\mathcal{W}$ is a stable configuration implies a minimum size for neighboring coalitions, which in turn implies that deviations to coalitions further away are not profitable.

3.6 Strong Tiebout Equilibria

3.6.1 Definition of Strong Tiebout Equilibria

Strong Tiebout equilibria add to Tiebout equilibria the requirement that there should not exist a group of agents who by creating a new coalition could all become strictly better off.

DEFINITION **3.5** *A coalition structure $\mathcal{W} = \{S_1, ..., S_N\}$ is a* ***strong Tiebout equilibrium*** *if it is a Tiebout equilibrium and for any coalition $z \in \mathcal{P} \backslash \mathcal{W}$ there exists $i \in z$ such that $c_i(z) \geq c_i(\mathcal{W})$.*[20]

3.6.2 Characterization of Strong Tiebout Equilibria

Again, for our next result we need to introduce some more notation. Define $|S'_{max}(\mathcal{W})|$ to be the size of the largest coalition which is a neighbor of a coalition with size $|S_{max}(\mathcal{W})|$. Note that this is not automatically the second largest coalition size. We may have that $|S_{max}(\mathcal{W})| = |S'_{max}(\mathcal{W})|$. Now we will define $\overline{c(\mathcal{W})}^{STE}$. If $\mathcal{W}$ is the grand coalition, $\overline{c(\mathcal{W})}^{STE}$ is equal to $min_{|z| \in [1, \frac{T+2}{3})}[\frac{2}{|z|T} \frac{T-|z|}{T-3|z|+2}]$. Again, if $\mathcal{W}$ consists only of singletons, we set $\overline{c(\mathcal{W})}^{STE}$ equal to infinity. Otherwise it is equal to the minimum of, on one hand $\frac{2}{|z||S_{max}|} \frac{|S_{max}|-|z|}{|S_{max}|-2|z|+1}$ over all $|z| \in [1, \frac{|S_{max}|+1}{2})$ which are odd, and on the other $\frac{2}{|z||S'_{max}|} \frac{|S'_{max}|-|z|}{|S'_{max}|-2|z|+2}$ over all $|z| \in [2, \frac{|S'_{max}|+2}{2})$ which are even. Note that $\overline{c(\mathcal{W})}^{STE} \leq \overline{c(\mathcal{W})}^{SC}$. With this conventions we can state the corresponding result.

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[20] We require all blocking agents to become strictly better off in order to prevent problems of existence, see Haeringer [12].

PROPOSITION **3.4** *A coalition structure* $\mathcal{W} = \{S_1, ..., S_N\}$ *is a strong Tiebout equilibrium if and only if*

(1) it is connected,

(2) neighboring coalitions have a size difference of at most one and

(3) $\underline{c(\mathcal{W})}^{SC} \leq c \leq \overline{c(\mathcal{W})}^{STE}$.

We see from the comparison of proposition 3.3 and proposition 3.4 that adding to Tiebout equilibria the requirement that there should not be a group of agents who could all become better off by creating a new coalition has only the effect to lower the upper bound for c to $\overline{c(\mathcal{W})}^{STE}$. Saying a coalition structure $\mathcal{W}$ is a strong Tiebout equilibrium is equivalent to saying that $\mathcal{W}$ is connected, neighboring coalitions are similar in size and that the parameter c, which shows the relative importance of "heterogeneity costs" to public good provision costs, lies in an interval depending on $\mathcal{W}$.

The possibility of free mobility of agents generates the first two conditions and the lower bound for c. In any pair of neighboring coalitions there exists an agent i who can induce living in a larger coalition. Since agent i will be the worst-off in this new coalition, such a move is not advantageous if the "heterogeneity costs" are high enough. The upper bound for c comes from the possibility that agents may create new coalitions. If c is too high, then it pays for the individuals who form part of the largest coalitions (and suffer the highest "heterogeneity costs"), to form smaller coalitions. Surprisingly, it never pays to create larger coalitions. The fact that $\mathcal{W}$ is a Tiebout equilibrium already implies a structure on $\mathcal{W}$ that prevents such a creation to be beneficial.

Unfortunately, the definition of $\overline{c(\mathcal{W})}^{STE}$ is tedious. This is because it is not possible to determine a global minimum $|z^*|$ independent of c. In general $|z^*|$ depends on c. For example if $c \geq \frac{1}{3}$, then one can show that the grand coalition is immune to secessions if and only if $c \leq \bar{c}$, where $\bar{c}$

$(\lvert S_{max}\rvert, \lvert S'_{max}\rvert)$	(2,1)	(2,2)	(3,2)	(3,3)	(4,3)	(4,4)	(5,4)
$\overline{c(\mathcal{W})}^{STE}$	1	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$
$(\lvert S_{max}\rvert, \lvert S'_{max}\rvert)$	(5,5)	(6,5)	(6,6)	(7,6)	(7,7)	(8,7)	(8,8)
$\overline{c(\mathcal{W})}^{STE}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{5}{36}$	$\frac{1}{8}$
$(\lvert S_{max}\rvert, \lvert S'_{max}\rvert)$	(9,8)	(9,9)	(10,9)	(10,10)	(11,10)	(11,11)	(12,11)
$\overline{c(\mathcal{W})}^{STE}$	$\frac{1}{9}$	$\frac{5}{54}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{3}{40}$	$\frac{7}{110}$	$\frac{7}{110}$

Table 3.1: Upper bounds for c depending on $\mathcal{W}$

is determined using the highest $|z|$ for which $T - 3|z| + 2 > 0$. Since for other values of c this is not true, one has to compute all upper bounds for c corresponding to each value of $|z|$ for which $T - 3|z_{max}| + 2 \geq 0$ and take the lowest bound. Table 3.1 displays some of those bounds for configurations other than the grand coalition.[21]

Remark: Note that although the characterizations of strong Tiebout equilibria seem to be similar on first sight in both the model with a finite number of agents and the one with a continuum, there are substantial differences. While with a continuum of consumers there are at most two different coalition sizes which can be quite different, with a finite number of consumers more coalition sizes are allowed as long as neighboring coalitions do not differ by more than one agent. Since in principle in the model with a finite number of agents the bounds for c could be solved for $|S|$, both models establish bounds for $|S|$ depending on c. It is interesting to note that with a continuum of agents these bounds are very often automatically fulfilled if there are two different coalition sizes. In the finite model such a tendency does not exist. Furthermore, with a finite number of agents the lower bound comes from the possibility of free mobility, while with a continuum this

[21] Note that these bounds are independent of T.

bound comes from the possibility to create larger coalitions.

3.6.3 Existence of Strong Tiebout Equilibria

In this Section, we give sufficient conditions for the existence of a strong Tiebout equilibrium and show that these conditions are not necessary.

We define now a lower bound $\underline{T(c)}$ for the number of agents T. Note that the lower bound for c can be reformulated to give a minimum size for coalitions. Call this size $|\underline{S}|$. Since $|\underline{S}|$ may not be an integer denote by $\lceil|\underline{S}|\rceil$ the smallest integer not smaller than $|\underline{S}|$. We have that $\underline{T(c)}$ is equal to the number of positions on the line segment contained in $\lceil|\underline{S}|\rceil - 1$ coalitions of size $\lceil|\underline{S}|\rceil$.[22] With this convention we have the following proposition.

PROPOSITION **3.5** *If $T \geq \underline{T(c)}$ or $c \geq \frac{1}{3}$, then there exists a strong Tiebout equilibrium.*

Some comments are in order. Firstly, for all but a relatively small range of values for c, proposition 3.5 assures the existence of a strong Tiebout equilibrium. Secondly, provided T is high enough, for the remaining parameter values a strong Tiebout equilibrium also exists. Only if both c and T are very small, our proof technics do not allow to proof a general existence result. The proof of proposition 3.5 relies on a construction using the two smallest coalition sizes of a Tiebout equilibrium bigger than $|\underline{S}|$. Combinatorial problems arise when those values cannot be combined such that the sum of coalition sizes gives exactly T.

Note that these combinatorial problems do not imply that for $c < \frac{1}{3}$ and $T < \underline{T(c)}$ a strong Tiebout equilibrium does not exist. The opposite may

[22] More formally, we have that the most restrictive bound for c implied by stable configurations can be written as $|\underline{S}| = \sqrt{\frac{2}{c} + \frac{1}{4}} - \frac{1}{2}$. Hence $\underline{T(c)} = \lceil|\underline{S}|\rceil(\lceil|\underline{S}|\rceil - 1)$.

be true for three reasons. Firstly, although T is small, the sum of coalition sizes may give exactly T. Secondly, the grand coalition may be stable and thirdly, other coalition sizes may be feasible. For example one can show that for $8 < \lceil |\underline{S}| \rceil$ the size $\lceil |\underline{S}| \rceil + 2$ is also possible. Moreover, depending on $|\underline{S}|$ the size $\lceil |\underline{S}| \rceil - 1$ may be feasible. The following example illustrates these points.

Example Let $c \in [\frac{1}{21}, \frac{1}{15})$. The proof of proposition 3 uses coalition sizes 6 and 7 and shows that a strong Tiebout equilibrium exists for all $T \geq 30$. This example shows that for all other T there exists also a strong Tiebout equilibrium.
The combinatorial problems do not arise for all $T < 30$ which can be obtained as the sum of coalition sizes 6 and 7. Hence we are left with $T \leq 11$ and $T \in \{15, 16, 17, 22, 23, 29\}$. From table 3.1 we see that for these values of c a Tiebout equilibrium is "strong" if and only if $|S'_{max}| \leq 10$. Therefore we can use the coalition sizes $6, 7, 8, 9$ and 10 freely as well as the pair $(11, 10)$. Furthermore, it is immediate that this implies that the grand coalition consisting of at most 11 agents is also a strong Tiebout equilibrium. Thus, for $T \in \{1, 2, ..., 11\}$ choose the grand coalition, for $T = 15$ (7,8), for $T = 16$ (8,8), for $T = 17$ (8,9), for $T = 22$ (7,7,8), for $T = 23$ (7,8,8) and for $T = 29$ (7,7,7,8). ■

However, although this example indicates a way to construct other strong Tiebout equilibria than the ones used in the proof and makes the existence very likely, this may not always be possible. Similar reasoning as in the example guarantees existence at least for all $c \geq \frac{1}{21}$. We introduce now a slightly weaker concept which guarantees existence. The idea is to give still all agents the possibility to create new coalitions but now the agents in the last two coalitions cannot join an already existing coalition.[23] More formally:

Definition **3.6** *A coalition structure $\mathcal{W} = \{S_1, ..., S_N\}$ is a* ***strong Tiebout equilibrium with rest [STEWR]*** *if*
(i) for all $i \in \mathcal{T} \backslash S_N \cup S_{N-1}$, we have, $c_i(S^i) \leq c_i(S' \cup i), \forall S' \in \mathcal{W}, S' \neq S^i$
(ii) and for any coalition $z \in \mathcal{P} \backslash \mathcal{W}$ there exists $i \in z$ such that $c_i(z) \geq c_i(\mathcal{W})$.

Given c, we can calculate the minimum size of a coalition implied by free mobility and divide T by this size. If T divides this size only with non-zero rest, we put these agents in the last coalition. The resulting coalition structure is not a strong Tiebout equilibrium, because the agents in the last coalition would prefer to join the second from last coalition. Since this is not possible in a strong Tiebout equilibrium with rest, existence is given. Note that the agent in the "rest" do not want to create a new coalition.

Proposition **3.6** *For all values of c and T, there exists a strong Tiebout equilibrium with rest.*

3.6.4 Optimality of Strong Tiebout Equilibria

As in AS, we consider a coalition structure $\mathcal{W}$ to be efficient if it maximizes the sum of individual utilities subject to the constraint that the sum of

[23] Introducing the restriction that an agent can only deviate if the coalition she joins is willing to absorb her (individual stability in the terminology of Bogomolnaia and Jackson [2]), can also help to overcome the computational problems. This is because it relaxes both the restriction on the minimum size of a coalition and on the size difference of neighboring coalitions implied by proposition 3.1. One can show for instance that for $T = 100$ and $c = \frac{3}{10000}$ the configuration $(99, 1)$ is individual stable.

N	(strong) Tiebout equilibria and efficiency
3	$(8,8,8)_{TE}$
4	$(6,6,6,6)_{TE}$, $(5,6,6,7)_{TE}$
5	$(4,5,5,5,5)^*_{TE}$, $(4,4,5,5,6)_{TE}$, $(3,4,5,6,6)_{TE}$
6	$(4,4,4,4,4,4)^*_{STE}$, $(3,4,4,4,4,5)^*_{STE}$, $(3,3,4,5,4,5)^*_{STE}$, $(2,3,4,5,5,5)_{TE}$, $(2,3,4,4,5,6)_{TE}$
7	$(3,3,3,3,4,4,4)^*_{STE}$, $(2,3,3,3,4,4,5)_{STE}$, $(2,3,2,3,4,5,5)_{TE}$, $(3,3,3,3,3,4,5)^*_{STE}$
8	$(3,3,3,3,3,3,3,3)^*_{STE}$, $(2,3,3,3,3,3,3,4)_{STE}$, $(2,3,2,3,3,3,4,4)_{STE}$
9	$(2,3,2,3,2,3,2,3,4)_{STE}$, $(2,3,2,3,3,3,3,3,2)_{STE}$

Table 3.2: (strong) Tiebout equilibria and efficiency for $c = \frac{1}{4}$ and $T = 24$.

individual contributions to public good provision must equal its total costs.

Informally speaking, a coalition structure $\mathcal{W}$ is efficient if

(1) the local public good is located in the middle of the coalition,

(2) $|S_{min}| \geq |S_{max}| - 2$, with equality if $|S_{max}|$ is odd, and

(3) N lies within an interval that depends on c.[24]

In our model condition (1) is always fulfilled. We illustrate the second and the third condition with an example for $c = \frac{1}{4}$ and $T = 24$ in table 3.2.[25] Strong Tiebout equilibria and Tiebout equilibria have the subindexes STE and TE, respectively. Efficient configurations are marked with a star and permutations of the coalition structures contained are left out.

The first thing we learn from this table is that there exist strong Tiebout equilibria that are efficient. Since this is important we show now that there exist high values of T (as high as desired) for which one can find values of c

[24] For more detailed and precise discussion see the analysis in Cechlárová et al. [3].

[25] In order to facilitate the comparison with the work of Cechlárová et al. [3], we continue their example. Note that, since in this example coalition sizes are relatively small, the size of z which is most likely to block $\mathcal{W}$ is one or two. This is why the strongly stable configurations in Cechlárová et al. [3] are strong Tiebout equilibria.

such that an efficient strong Tiebout equilibrium exists. This is true because we can take the structure $(4, 4, 4, 4, 4, 4)$ with $N = 6$ coalitions for $T = 24$ and replicate it r times. The coalition structure consisting of rN coalitions of size 4 is an efficient strong Tiebout equilibrium for an economy with rT agents. In the following proposition we use the notation $\mathcal{W} = (T, N, |S|)$ for a coalition structure containing N coalitions of equal size $|S|$ in an economy with T individuals.

PROPOSITION **3.7** *Given c let $\mathcal{W} = (T, N, |S|)$ be an efficient and strongly Tiebout stable coalition structure. Then for all $r \in \mathbb{N}$ the coalition structure $\mathcal{W}_r = (rT, rN, |S|)$ is also an efficient and strongly Tiebout stable coalition structure for an economy with rT individuals and the same c.*

This result is important, since our model can be adjusted, without inducing any qualitative change in the results, for having a continuum of agents distributed uniformly on an equally distant set of points in the line segment $[0, 1]$. Letting $T \to \infty$ is like letting our model approach the AS-model. Note that the inefficiency result of AS does not mean that there is a discontinuity in the limit, because both papers use different equilibrium concepts. But it suggests that the unique efficient configuration in the continuous model of AS may be a strong Tiebout equilibrium.[26]

We come back now to table 3.2. It contains strong Tiebout equilibria that are not efficient because the size difference between coalitions is too large.[27] Note that this source of inefficiencies is much less sever in strong Tiebout equilibria than in Tiebout equilibria. This is true because if some

[26] Dahm [4] shows that the last conjecture is correct. Details about a continuum of agents located on a finite set of locations can be found in Appendix A.

[27] Although our model is very homogeneous, the notion of a strong Tiebout equilibrium allows to explain considerable differences in coalition sizes, like in $(2, 3, 3, 3, 4, 4, 5)$. The AS-stability concept (A-stability) implies equally sized coalitions. Note that locating agents on a circle helps to reduce the number of equilibria, since the maximal size difference

coalitions are very large it pays to create smaller ones and such a coalition structure is not a strong Tiebout equilibrium.

From table 3.2 we also see that there may be strong Tiebout equilibria that are not efficient because they involve too many coalitions. This is a general tendency of strong Tiebout equilibria, because requiring that it should not be beneficial to create new coalitions puts an upper bound on coalition sizes and implies therefore a relatively large number of coalitions. This upper bound on coalition sizes is in general too high to conclude unambiguously that the stable number of coalitions is always strictly larger than the efficient one. Moreover, there may be too few coalitions as the following example shows.

Example Let $T = 6$ and $c = \frac{1}{3}$. By proposition 3.4 the grand coalition is stable (since $c \leq \frac{1}{3}$). The grand coalition implies a total cost of $C(N = 1) = 4$, while the structure $(3, 3)$ inherits a cost of only $C(N = 2) = \frac{10}{3}$. ■

Our last example shows that there does not always exists an efficient (strong) Tiebout equilibrium, as the preceding analysis may suggest.[28]

Example Let $T = 3$ and $c = 0,8$. There are only four configurations possible $(1, 1, 1)$, $(1, 2)$, $(2, 1)$ and (3). From proposition 3.3 we see that the only Tiebout equilibria are $(1, 2)$ and $(2, 1)$. From table 3.1, we see that these Tiebout equilibria are also "strong". However, the grand coalition is uniquely efficient, since the overall costs implied by these configurations are $C(N = 3) = 3$, $C(N = 2) = 2 + c \simeq 2.8$ and $C(N = 1) = 1 + 2c \simeq 2.6$. ■

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between the first and the last coalition becomes then one. Still, both types of inefficiencies may arise.

[28] This is an important difference to the continuous model, see Dahm [4].

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Appendix A: A Continuum of Agents located on a finite set of points

In this Section we modify the model in such a way that world population is a continuum of agents with mass one and agents are uniformly distributed over the set of points $\{0, \frac{1}{T-1}, \frac{2}{T-1}, ..., \frac{T-2}{T-1}, 1\}$. The utility function (3.1) can then be written as

$$U_i(S) = \alpha(1 - \beta p_i(S)) + y - \frac{\gamma}{s_i}, \tag{3.4}$$

where α, β, γ and y are defined as before. The distance from agent i to her local public good is denoted by $p_i(S) = |i - l(S)|$. We define $p_i(S) = \frac{d_i(S)}{T-1}$. This implies that $d_i(S)$ represents the number of locations between i and $l(S)$ plus one. The cardinality of a coalition S^i is interpreted as the integer $|S^i|$ which indicates the number of locations contained in S^i and $s_i = \frac{|S^i|}{T}$ is the population share contained in coalition S^i. This implies that s_i is a fraction of one and not an integer. Hence the notion of the size of a coalition s_i coincides with AS. Equation 3.4 is equivalent to the individual cost function (3.2) if and only if $c = \frac{\alpha\beta}{\gamma T(T-1)}$. With c defined in this way all qualitative results in Cechlárová et al. [3] and in this paper hold through.[29]

Appendix B: Proofs

The implications of the different stability notions are driven by comparisons of coalition membership. We define $\triangle c_i(S, S') = c_i(S) - c_i(S')$. Hence $\triangle c_i(S, S') > 0$ means that membership in coalition S' implies a strictly lower

[29] Note e.g. that in proposition 3.7 the structure in the replicate economy is efficient for other values of c than the one in the initial structure.

cost or higher utility than in S. Usually S indicates the status quo. The gains or losses from the change in tax payments $\triangle t_i(S,S') = \frac{|S'|-|S|}{|S||S'|}$ and distance $\triangle d_i(S,S')$ are defined analogously. We denote the border agents of a given coalition S, that is, the agents with the lowest and the highest position in S by $\underline{b}(S)$ and $\bar{b}(S)$, respectively. When it is clear to which coalition we refer we write $b = \{\underline{b}, \bar{b}\}$.

Proof of proposition **3.2:**

Suppose $|S_n| = |S_{n+1}| + 1$. Denote the agent in S_n which is closest to S_{n+1} by j. We have that $c_j(S_n) = c_j(S_{n+1} \cup j)$, violating the condition for A-stability. Assume now that $|S_n| = |S|, \forall S_n \in \mathcal{W}$. It is straightforward that A-stability requires the inequalities in proposition 3.1 to become strict. Taking into account that $N = \frac{T}{|S|}$ gives the condition. ■

Proof of Proposition **3.3**: Note that, since our model is a special case of the model in Haeringer [12], we use his Lemma 1 which says that if $\mathcal{W}$ is a Tiebout equilibrium, then it is connected.

It remains to show that if $\mathcal{W}$ is a stable configuration, then it is a Tiebout equilibrium. Let $\mathcal{W}$ be a stable configuration and $S_k, S_{k+m} \in \mathcal{W}, m > 1$. W.l.o.g suppose $|S_k| \geq |S_{k+m}|$. It suffices to consider $b1 = \bar{b}(S_k)$ and $b2 = \underline{b}(S_{k+m})$, since those agents have the lowest utilities in S_k and S_{k+m} and are closest to the other coalition. Suppose $|S_k| = |S_{k+m}| = |S|$. By symmetry focus on $b1$ who deviates if and only if $c_{b1}(S_k) = \frac{|S|-1}{2}c + \frac{1}{|S|} > c_{b1}(S_{k+m} \cup b1) \geq \frac{2+|S|}{2}c + \frac{1}{|S|+1}$. This implies $c < \frac{2}{3|S|(|S|+1)}$ contradicting $c \geq \frac{2}{|S|(|S|+2)}$. Suppose $|S_k| > |S_{k+m}|$. Note that we can restrict to the case in which $|S_{k+m-1}| > |S_{k+m}|$. Because if $|S_{k+m-1}| \leq |S_{k+m}|$, then $\exists S_n \in \mathcal{W}$ such that $k < n \leq k+m-1$ with $|S_n| = |S_{k+m}|$ and if no one wants to deviate from S_k to S_n or the other way around, then, since distances are higher, the same holds for the pair S_k, S_{k+m}. Consider b2.

Since $\mathcal{W}$ is a stable configuration we have that $\triangle c_{b2}(S_{k+m}, S_{k+m-1} \cup b2) = c_{b2}(S_{k+m}) - c_{b2}(S_{k+m-1} \cup b2) \leq 0$. Suppose by way of contradiction that $\triangle c_{b2}(S_{k+m}, S_k \cup b2) = c_{b2}(S_{k+m}) - c_{b2}(S_k \cup b2) > 0$. This implies $\frac{|S_k|}{2}c + \frac{1}{|S_k|+1} < c_{b2}(S_k \cup b2) < c_{b2}(S_{k+m}) \leq c_{b2}(S_{k+m-1} \cup b2) = \frac{|S_{k+m-1}|}{2}c + \frac{1}{|S_{k+m-1}|+1}$ or $c < \frac{2}{(|S_k|+1)(|S_{k+m-1}|+1)}$. We know, since $\mathcal{W}$ is a stable configuration, that $c \geq \frac{2}{(|S_{k+m}|)(S_{|k+m-1|}+1)}$. Both conditions together imply that $|S_k| < |S_{k+m}|$ which is a contradiction. Consider b1. It is true that $\triangle t_{b1}(S_k, S_{k+m} \cup b1) \leq 0$ and $\triangle d_{b1}(S_k, S_{k+m} \cup b1) < 0$. Therefore we have that $\triangle c_{b1}(S_k, S_{k+m} \cup b1) < 0$. ■

Proof of Proposition **3.4**:

We begin by introducing some notation and definitions.

Notation and definitions: Define $Z(S_n, S_{n+1}) = \{z \in \mathcal{P} \mid l(S_n) < \underline{b}(z) \leq \bar{b}(z) < l(S_{n+1})\}$ and denote by $Z_{|z|}(S_n, S_{n+1}) \subset Z(S_n, S_{n+1})$ all $z \in Z(S_n, S_{n+1})$ with cardinality $|z|$. When it is clear which pair of coalitions in $\mathcal{W}$ is meant we write simply Z and $Z_{|z|}$. Define $\tilde{i}(z) \in z$ such that $\triangle c_{\tilde{i}(z)}(\mathcal{W}, z) \leq \triangle c_i(\mathcal{W}, z), \forall i \in z$ and $\tilde{z}_{|z|} \subset Z_{|z|}$ such that $\triangle c_{\tilde{i}(\tilde{z})}(\mathcal{W}, \tilde{z}) \geq \triangle c_{\tilde{i}(z)}(\mathcal{W}, z), \forall z \in Z_{|z|}$. When the context is clear we will write $\tilde{i}$ and $\tilde{z}$.

We shall prove five lemmatas first.

Lemma 3.4.1: *Let $\mathcal{W} \in \Pi$ be a TE with $N > 1$ and $z \in \mathcal{P} \backslash \mathcal{W}$. The coalition z does not block $\mathcal{W}$ if*

(a) there exists $b \in b(z)$ with $|z| \geq |S^b|$;

(b) z is connected and there exists $i \in z$ with $|z| \geq |S^i|$;

(c) $\frac{2}{|S_{min}|(|S_{min}|+1)} > c \geq \frac{2}{|S_{min}|(|S_{min}|+2)}$ and there exists $b \in b(z)$ with $|z| < |S^b| \leq |S_{min}| + 1$;

(d) $\frac{2}{|S_{min}|(|S_{min}|+1)} > c \geq \frac{2}{|S_{min}|(|S_{min}|+2)}$, $|z| \leq |S_{min}| + 1$, z is connected and there exists $i \in z$ with $|S^i| \leq |S_{min}| + 1$.

Proof of Lemma 3.4.1: Let $\mathcal{W} \in \Pi$ be a TE with $N > 1$ and $z \in \mathcal{P} \backslash \mathcal{W}$.

(a) We have that $\triangle c_{b(z)}(\mathcal{W},z) = d_{b(z)}(\mathcal{W})c + t_{b(z)}(\mathcal{W}) - d_{b(z)}(z)c - t_{b(z)}(z) \leq \frac{|S^{b(z)}|-1}{2}c + \frac{1}{|S^{b(z)}|} - \frac{|z|-1}{2}c - \frac{1}{|z|} \leq 0$. The last inequality is equivalent to [3.4.1] $\frac{|S^{b(z)}|-|z|}{2}c + \frac{|z|-|S^{b(z)}|}{|S^{b(z)}||z|} \leq 0$. If $\exists b \in b(z)$ with $|z| = |S^b|$ [3.4.1] is trivially fulfilled. For $|z| > |S^b|$ we obtain from [3.4.1] that $c \geq \frac{2}{|S^b||z|}$. The only case in which this is not immediately fulfilled is if $|z| = |S_{min}| + 1$, for all $b \in b(z)$ holds that $|S^b| = |S_{min}|$ and each S with $|S| = |S_{min}|$ has only neighboring coalitions of size $|S_{min}| + 1$. Since this implies that z is unconnected, we have that for all $b \in b(z)$ holds that $d_b \geq \frac{|S_{min}|+2}{2}$. It follows that $\triangle c_b(\mathcal{W},z) \leq 0 \Leftrightarrow c \geq \frac{2}{3|S_{min}|(|S_{min}|+1)}$. This is true since $\frac{2}{|S_{min}|(|S_{min}|+2)} > \frac{2}{3|S_{min}|(|S_{min}|+1)}$.

(b) Suppose z is connected and contains i with $|z| \geq |S^i|$. Note that it cannot be true that $\forall b \in b(z)$ holds $|z| < |S^b|$, since $|z| \geq |S^i| + 2 + 2|k|$ and $|S^b| \leq |S^i| + |k|$, where k is the number is defined by $S^i = S_n$ and $S^b = S_{n+k}$. Hence $\exists b \in b(z)$ such that $|z| \geq |S^{b(z)}|$ and, by part (a), z cannot block $\mathcal{W}$.

(c) Suppose $\frac{2}{|S_{min}|(|S_{min}|+1)} > c \geq \frac{2}{|S_{min}|(|S_{min}|+2)}$. If $\exists b \in b(z)$ with $|z| < |S^b| \leq |S_{min}| + 1$, then [3.4.1] becomes $\frac{2}{|S^b||z|} \geq c$ which is fulfilled.

(d) Let $\frac{2}{|S_{min}|(|S_{min}|+1)} > c \geq \frac{2}{|S_{min}|(|S_{min}|+2)}$, z be connected, $|z| \leq |S_{min}|+1$ and $i \in z$ with $|S^i| \leq |S_{min}| + 1$. If $|S^i| \leq |z|$, the conclusion follows from part (b). If $|S^i| > |z|$, then $\exists b \in b(z) \cap S^i$ and we conclude by part (c).

LEMMA 3.4.2: *Let $\mathcal{W} \in \Pi$ be a TE. For any unconnected $z \in \mathcal{P} \backslash \mathcal{W}$ it is true that either z does not block $\mathcal{W}$ or there exists $z' \in \mathcal{P} \backslash \mathcal{W}$ which is connected and also blocks $\mathcal{W}$.*

PROOF OF LEMMA 3.4.2: Let $\mathcal{W}$ be a TE and $z \in \mathcal{P} \backslash \mathcal{W}$ be unconnected. It follows that $|z| < |S^b|, \forall b \in b(z)$ either from lemma 3.4.1 (a) or from $N = 1$. Assume $\exists S_n, S_{n+1} \in \mathcal{W}$ such that $z \subset S_n \cup S_{n+1}$. If $l(S_n)$ or $l(S_{n+1})$ lie in the interval $[\underline{b}(z), \bar{b}(z)]$, then $\exists i \in z$ with $\triangle t_i(S,z) < 0$ and $\triangle d_i(S,z) \leq 0$ which implies $\triangle c_i(S,z) < 0$. If not, then $l(S_n) < \underline{b}(z) < \bar{b}(z) < l(S_{n+1})$. Construct a connected coalition z' such that $|z| = |z'|$ and $l(z') \in [l(z), l(z) + \frac{1}{2}]$. It follows that $\triangle c_{\tilde{i}}(\mathcal{W}, z') \geq \triangle c_{\tilde{i}}(\mathcal{W}, z)$. Suppose now that z contains agents from two not neighboring coalitions or from more than three coalitions. Since z is unconnected $\exists b \in b(z)$ such that $d_b(z) \geq \frac{|S^b|}{2} > \frac{|S^b|-1}{2} \geq d_b(S^b)$. Hence $\triangle c_b(S^b, z) < 0$.

LEMMA 3.4.3: *Let $\mathcal{W} \in \Pi$ be a TE containing S_n and S_{n+1}. For any connected $z \subseteq S_n \cup S_{n+1}$, with $|S_n| \geq |S_{n+1}| > |z|$ and $z \notin Z(S_n, S_{n+1})$ it is true that either z does not block $\mathcal{W}$ or there exists $z' \in Z(S_n, S_{n+1})$ which also blocks $\mathcal{W}$.*

PROOF OF LEMMA 3.4.3: Let $\mathcal{W} \in \Pi$ be a TE and $z \subseteq S_n \cup S_{n+1}$, with $|S_n| \geq |S_{n+1}| > |z|$ and $z \notin Z(S_n, S_{n+1})$ be connected. Note that it is impossible that both $l(S_n)$ and $l(S_{n+1})$ lie in z. Suppose $l(S_n)$ or $l(S_{n+1})$ lie in z. W.l.o.g. assume $l(S_{n+1}) \in z$ and $l(z) \leq l(S_{n+1})$. Then there exists $i \in [l(S_{n+1}), l(S_{n+1}) + \frac{1}{2}]$ with $\triangle t_i(\mathcal{W}, z) < 0$ and $\triangle d_i(\mathcal{W}, z) \leq 0$ which implies $\triangle c_i(\mathcal{W}, z) < 0$. If neither $l(S_n)$ nor $l(S_{n+1})$ lie in z, we have that either $\bar{b}(z) < l(S_n)$ or $l(S_{n+1}) < \underline{b}(z)$. W.l.o.g. assume the latter. By symmetry consider $z' \in Z(S_n, S_{n+1})$ with $|z'| = |z|$ and $l(z) - l(S_{n+1}) = l(S_{n+1}) - l(z')$.

LEMMA 3.4.4: *Let $\mathcal{W} \in \Pi$ be a TE containing S_n and S_{n+1} with $|S_n| \geq |S_{n+1}|$ and $c \geq \frac{2}{|S_{n+1}|(|S_{n+1}|+1)}$. If $z \in Z(S_n, S_{n+1})$ with $|S_{n+1}| > |z|$ blocks $\mathcal{W}$, then there exists $z' \in Z(S_{max}, S'_{max})$ that also blocks $\mathcal{W}$.*

PROOF OF LEMMA 3.4.4: Let $\mathcal{W} \in \Pi$ be a TE containing S_n and S_{n+1} with $|S_n| \geq |S_{n+1}|$, $c \geq \frac{2}{|S_{n+1}|(|S_{n+1}|+1)}$ and let $z \in Z(S_n, S_{n+1})$ with $|S_{n+1}| > |z|$. Suppose $|S_{max}| + |S'_{max}| = |S_{n+1}| + |S_n| + k$, $k > 0$. Assume w.l.o.g. that $|S_{max}| = |S_n| + k$. Construct $z' \in Z(S_{max}, S'_{max})$ such that $|z| = |z'|$ and $l(z) - \bar{b}(S_n) = l(z') - \bar{b}(S_{max})$. Order the agents for which $S_n \cap z \neq \emptyset$ and $S_{max} \cap z' \neq \emptyset$ from the right to the left. For the agents with the same number q in z and z' holds that $\triangle c_q(S_n, z) \leq \triangle c_q(S_{max}, z')$, since

$\triangle c_q(S_n, z) - \triangle c_q(S_{max}, z') = \frac{-k}{2}c + \frac{k}{|S_n|(|S_n|+k)} \leq 0 \Leftrightarrow c \geq \frac{2}{|S_n|(|S_n|+k)}$ which is true by assumption.

LEMMA 3.4.5: *Let $\mathcal{W} \in \Pi$ be a TE containing S_n and S_{n+1} with $|S_n| \geq |S_{n+1}| + k, k \geq 0$ and $c \geq \frac{2}{|S_{n+1}|(|S_{n+1}|+1)}$. If $|z|$ is odd, then $\tilde{z}$ is such that $l(\tilde{z}) = \bar{b}(S_n)$ and $\tilde{i}(\tilde{z}) = \underline{b}(\tilde{z})$. If $|z|$ is even, then $\tilde{z}$ is such that $l(\tilde{z}) = \bar{b}(S_n) + \frac{1}{2}$ and $\tilde{i}(\tilde{z}) = \bar{b}(\tilde{z})$.*

PROOF OF LEMMA 3.4.5: Let $\mathcal{W} \in \Pi$ be a TE containing S_n and S_{n+1} with $|S_n| \geq |S_{n+1}| + k, k \geq 0$. From the definition of $\triangle c_i(\mathcal{W}, z)$ it is immediate that $\tilde{i}(z) \in b(z)$. Suppose $|z|$ is odd and $|z| > 1$. Consider $\hat{z} \in Z_{|z|}$ such that $l(\hat{z}) = \bar{b}(S_n)$. Since $\triangle c_{\bar{b}(\hat{z})}(\mathcal{W}, \hat{z}) - \triangle c_{\underline{b}(\hat{z})}(\mathcal{W}, \hat{z}) = \frac{2-k}{2}c + \frac{k}{|S_n|(|S_n|+k)} \geq 0$, we have that $\tilde{i}(\hat{z}) = \underline{b}(\hat{z})$. Note that for $|z| = 1$ this is trivially true. In order to prove that $\tilde{z} = \hat{z}$, we have to show that $\forall z \in Z_{|\hat{z}|}, z \neq \hat{z}, \exists b \in b(z)$ such that [3.4.5] $\triangle c_b(\mathcal{W}, z) \leq \triangle c_{\tilde{i}(\hat{z})}(\mathcal{W}, \hat{z})$. Choose $b = \underline{b}(z)$ if $l(z) < l(\hat{z})$ and $b = \bar{b}(z)$ otherwise. Define $m = |l(z) - l(\hat{z})|$. Equation [3.4.5] becomes [3.4.5a] $\frac{k+2(m-1)}{2}c + \frac{k}{|S_{n+1}|(|S_{n+1}|+k)} \geq 0$, if $l(z) > l(\hat{z})$ and [3.4.5b] $mc \geq 0$ otherwise. Both expressions are fulfilled, since $m > 0$. Suppose $|z|$ is even. Consider $\hat{z} \in Z_{|z|}$ such that $l(\hat{z}) = \bar{b}(S_n) + \frac{1}{2}$. Here it holds that $\triangle c_{\bar{b}(\hat{z})}(\mathcal{W}, \hat{z}) - \triangle c_{\underline{b}(\hat{z})}(\mathcal{W}, \hat{z}) \leq 0$, since the latter is either fulfilled with equality or equivalent to $c \geq \frac{2}{|S_{n+1}|(|S_{n+1}|+1)}$, which is true. Hence $\tilde{i}(\hat{z}) = \bar{b}(\hat{z})$. In order to prove that $\tilde{z} = \hat{z}$ choose b and define m as above. Here equation [3.4.5] becomes [3.4.5b], if $l(z) > l(\hat{z})$ and [3.4.5c] $\frac{2m-k}{2}c + \frac{k}{|S_{n+1}|(|S_{n+1}|+k)} \geq 0$ otherwise. Again both expressions are fulfilled, since $m > 0$.

We are now in a position to prove proposition 3.4.

Let $\mathcal{W} \in \Pi$ be a TE. Because of lemma 3.4.2 consider connected $z \in \mathcal{P} \backslash \mathcal{W}$. Suppose $N > 1$. Lemma 3.4.1 (b) implies that $z \subseteq S_n \cup S_{n+1}$ and that $|z| < min\{|S_n|, |S_{n+1}|\}$. Assume w.l.o.g. $|S_n| \geq |S_{n+1}| > |z|$. Lemma 3.4.3 implies that it suffices to consider $z \in Z$. Note that we can suppose that

$c \geq \frac{2}{|S_{n+1}|(|S_{n+1}|+1)}$. This is because if not lemma 3.4.1 (d) applies. Hence from lemma 3.4.4 we know that we can restrict to $z \in Z(S_{max}, S'_{max})$. With lemma 3.4.5, we identify the "less secession prone" agent in the "most secession prone" coalition for every $Z_{|z|}$ in Z. The coalition structure $\mathcal{W}$ is not blocked by any $z \in Z_{|z|}$ if and only if $\triangle c_{\tilde{i}(\tilde{z})}(\mathcal{W}, \tilde{z}) \leq 0$, with $\tilde{z} \in Z_{|z|}$. This is equivalent to the functional form in proposition 3.4, provided $|\tilde{z}| < \frac{|S|+h}{2}$ with $S \in \{S_{max}, S'_{max}\}$ and $h \in \{1, 2\}$ as in the proposition. If $|\tilde{z}| \geq \frac{|S|+h}{2}$, then $d_{\tilde{i}(\tilde{z})}(\mathcal{W}, \tilde{z}) \leq 0$. Since $\triangle t_{\tilde{i}(\tilde{z})}(\mathcal{W}, \tilde{z}) < 0$ this implies $\triangle c_{\tilde{i}(\tilde{z})}(\mathcal{W}, \tilde{z}) < 0$. Suppose $N = 1$. It follows necessarily that $|z| < |S|$. It is immediate that no connected z containing $l(\mathcal{W})$ can block $\mathcal{W}$. By symmetry focus on $z \subseteq [1, \frac{T}{2}]$. We have that for each $|z|$, $\tilde{z} = \{1, 2, ..., |z|\}$ and $\tilde{i} = |z|$. Therefore $\triangle c_{\tilde{i}(\tilde{z})}(\mathcal{W}, \tilde{z}) \leq 0$ implies the functional form in proposition 3.4, provided $|z| < \frac{T+2}{3}$. By a similar reasoning as above $|\tilde{z}| \geq \frac{T+2}{3}$ leads to $\triangle c_{\tilde{i}(\tilde{z})}(\mathcal{W}, \tilde{z}) < 0$. ■

Proof of Proposition **3.5**:

We introduce first some notation. For a real number a, denote by $\lceil a \rceil$ the smallest integer not smaller than a. Similarly, denote by $\lfloor a \rfloor$ the highest integer not higher than a. It is useful to recall the following bounds for coalition sizes from Cechlárová et al. [3]: $|\underline{S}| = \sqrt{\frac{2}{c} + \frac{1}{4}} - \frac{1}{2}$, $|\underline{\underline{S}}| = \sqrt{\frac{2}{c} + 1} - 1$ and $|\bar{S}| = \frac{2}{c}$.[30] Using these bounds, condition (2) in proposition 3.1 reads as

$$(2') \ \forall S \in \mathcal{W} \text{ holds } |S| \in \begin{cases} [|\underline{S}|, |\bar{S}|] \text{ if } \exists S_n, S_{n+1} \in \mathcal{W} \text{ s. th. } |S_n| = |S_{n+1}| = |S_{min}| \\ [|\underline{\underline{S}}|, |\bar{S}|] \text{ otherwise.} \end{cases}$$

If $c \geq 1$ then the configuration $(1, 1, ..., 1)$ is a STE. For $[\frac{2}{3}, 1)$, we have $\{1, 2\} \subset [|\underline{S}|, |\bar{S}|]$. Hence when T is even choose $(2, 2, ..., 2)$ and when T is odd choose $(1, 2, ..., 2)$. This works for all T and the resulting structure is a STE. For $[\frac{1}{3}, \frac{2}{3})$, we have $\{2, 3\} \subset [|\underline{S}|, |\bar{S}|]$. Hence when T is even choose $(2, 2, ..., 2)$ and when T is odd choose $(3, 2, ..., 2)$. Again, this works for all T

[30] Note that $|\underline{\underline{S}}| < |\underline{S}|$ and that all bounds are independent of T.

and the resulting structure is a STE. For the remaining values of c proceed as follows. For $c \in [\frac{2}{(|S|+1)|S|}, \frac{2}{(|S|-1)|S|})$ take $|S|$ (starting with $|S| = 3$) and $|S|+1$. Suppose $T \geq \lceil|\underline{S}|\rceil(\lceil|\underline{S}|\rceil - 1)$. Then coalition sizes can be combined such that $\sum_n |S_n| = T$ for all T. The lower bound on coalition sizes coming from a TE is fulfilled by construction. Since $\frac{2}{|z||S|} \leq \frac{2}{|z||S|}\frac{|S|-|z|}{|S|-2|z|+h}, \forall |z|$ and $h \in \{1, 2\}$ such that $1 < |z| < \frac{|S|+2}{2}$, no new coalition is created if $c \leq \frac{4}{|S|(|S|+2)}$. For the bigger size $|S|+1$ the latter becomes $c \leq \frac{4}{(|S|+1)(|S|+3)}$ and we have that $\frac{2}{(|S|-1)|S|)} \leq \frac{4}{(|S|+1)(|S|+3)}$ holds if $|S| \geq 7$. For the remaining sizes $3 \leq |S| \leq 6$ we see from table 3.1 that these configurations are STE if and only if $c \leq \frac{1}{|S|}$ which is implied by $c < \frac{2}{(|S|-1)|S|)}$. ■

Proof of Proposition **3.6**:

Suppose proposition 3.5 does not apply. Consider the coalition structure $\mathcal{W} = (\lceil|\underline{S}|\rceil, \lceil|\underline{S}|\rceil, ..., S_{rest})$, where S_{rest} with $|S_{rest}| < \lceil|\underline{S}|\rceil$ contains all remaining agents. Suppose $\exists z \in \mathcal{P}\backslash\mathcal{W}$, containing $i \in S_{rest}$, that blocks $\mathcal{W}$. Note that for any such z and any $i \in S_{rest}$, there exists a symmetric z' and i' in the first coalition. This agent i' has bigger incentives to block than i. Thus, z can not block $\mathcal{W}$. ■

Proof of Proposition **3.7**:

It suffices to prove efficiency. Let $\mathcal{W} = (T, N, |S|)$ be efficient. Note that one can think of any $\mathcal{W}_r$ as $(rT, (r-1)N + N, |S|)$. Compare $\mathcal{W}_r$ to any $\mathcal{W}'_r$ containing $(r-1)N$ coalitions of size $|S|$ and $k \neq N$ coalitions of other sizes. Since $\mathcal{W}$ is efficient, $\mathcal{W}_r$ is at least as efficient as $\mathcal{W}'_r$. Overall efficiency follows then from theorem 2 in Cechlárová et al. [3] (we are just choosing different units in figure 2.1, p. 43). ■