Chapter 3

Public Debt and Optimal Taxes Without Commitment

3.1 Introduction

This paper studies the properties of the optimal time-consistent taxes for an economy with public debt. The previous literature on optimal taxation without commitment has focused only on economies without public debt, where the main result was that the optimal time-consistent capital taxes are different from zero at the steady state. In particular, Benhabib and Rustichini (1997) obtained that the optimal steady state policy is characterized by subsidies to capital. The intuition for this result is that capital subsidies encourage the accumulation of capital, which could become high enough to act as a commitment device against deviation from the announced policy. In this paper we explore whether the previous result holds when a market for public debt is present. To this end, we allow governments to issue debt. The implications of this new environment are twofold. On the one hand, the time-inconsistency problem could be worsened by the possibility of defaulting on debt payments. On the other hand, governments have a new instrument, debt, to affect the benefits and costs of deviating from the announced policy. Given that the government is benevolent and that the choice of not issuing bonds is still a sustainable outcome, the optimal management of public debt alleviates the consequences of the time-inconsistency problem. Our main result is that the optimal time-consistent capital tax turns out to be zero at the steady state. Thus, once governments have the possibility of issuing debt, public debt becomes the central commitment device against deviation.

One of the main concerns in the theory and practice of fiscal policy is how much capital income should be taxed. Currently, the capital income is taxed heavily in both the U.S. and the E.U. economies. However, the theory prescribes the opposite, that is, capital income tax rates should be set close to zero. This result was first shown by Chamley (1986) and Judd (1985), who proved the optimality of a zero capital income tax rate at the steady state for economies with identical and heterogeneous agents, respectively. Later on, Lucas (1990) and Chari, Christiano and Kehoe (1994) extended this result to economies with endogenous growth and to stochastic economies, respectively.[1] How to reconcile the observed taxes and the theory of capital taxation? An enlightening work by Aiyagari (1995) showed that for economies with incomplete insurance markets and borrowing constraints, the optimal tax rate on capital income is positive, even in the long-run. When capital investment yields pure profits, Jones, Manuelli and Rossi (1997) obtained that the asymptotic capital tax rate is not longer zero. For life-cycle economies, Garriga (2000) and Erosa

[1] Atkenson, Chari and Kehoe (1999) unified different extensions of the Chamley-Judd result.

and Gervais (2002) showed that, if the government has no access to age-dependent taxes, the optimal capital tax is different from zero both during the transition and at the steady state. Recently, Yakadina (2001) emphasized the role of public debt in the properties of optimal taxation for stochastic economies. She found that, when the amount of debt that a government can issue is restricted by lower and upper exogenous limits, the expected long-run capital taxes are different from zero. She argued that this result is due to the lack of complete insurance against future uncertainty.

A recent line of research has focused on the time-inconsistency problem of optimal taxation to account for actual capital tax rates. A limitation that previous studies share is the assumption that governments can commit to follow a prescribed policy plan. However, it is generally recognized that actual governments have no such a perfect commitment technology. Then, without commitment, and under a policy plan that sets capital taxes to zero and labor taxes to some positive distorsionary amount, the incentives to revise the policy plan and tax the pure rents generated by the capital input could be high.[2] In this sense, positive capital taxes could accrue from a zero capital tax announcement that is not credible. Thus, time-inconsistency problems could explain actual capital tax rates. Following this reasoning, we find the papers of Benhabib and Rustichini (1997) and Klein and Ríos-Rull (2000).

Benhabib and Rustichini (1997) characterized the optimal time-consistent capital taxes at the steady state. They first set the optimal taxation problem for an economy with debt. However, for simplicity, the analytical and the numerical characterization of capital

[2]In a finite-horizon economy, the incentives to revise the plan and tax heavily the long-run last period capital are very high (e.g., Fisher (1980)). However, this is not clear in an infinite-horizon setup, in which there is no last period and reputation forces come into play. There, the benefits from higher than announced capital tax rates could be outweighed by the loss of reputation.

taxes are carried out in a framework where governments cannot issue bonds.[3] To solve for the optimal time-consistent taxes, they introduced an incentive compatibility constraint for each period into the government optimization problem. This constraint says that the welfare value of continuing with the announced policy must be at least as large as the welfare value of deviating from it. The deviation value is endogenous and depends on the consequences of deviating, which are specified as follows: once the government deviates, individuals expect capital income to be taxed at a maximal rate (equal to one) at all future periods and, therefore, they decide not to save. The solution of this problem has a steady state. The steady state is called incentive-constrained, if the incentive compatibility constraints bind at the steady state. For utilities that are linear in consumption, Benhabib and Rustichini (1997) showed analytically that, when the steady state is incentive-constrained, the optimal time-consistent capital taxes are different from zero. In order to study concave utilities and determine the sign of the capital taxes, they calibrated the model and obtained that the optimal steady state policy is a subsidy to capital.

Following a different approach, Klein and Ríos-Rull (2000) studied the optimal policy for an economy without commitment. They focused on the properties of Markov perfect equilibria, which are time-consistent by construction. Throughout the paper, they imposed a balance budget so that governments cannot issue debt. This assumption is made to overcome theoretical and numerical problems associated with allowing issuing debt. They calibrated the model and their results can account for the magnitude of the empirical capital income tax rates.

[3] We show that the assumption on governments' inability to issue debt cannot be made for simplicity purposes since it turns out to be crucial for the results.

Summing up, from Benhabib and Rustichini (1997) and Klein and Ríos-Rull (2000), we know that time-inconsistency problems may explain the optimality of long-run capital taxes different from zero for economies without public debt. However, would this result hold in the presence of debt? In other words, is the time-inconsistency or the absence of a market for public debt the basis for this result?

The present paper explores the role of public debt and time-inconsistency problems in relation to the steady state optimal taxation. Our model follows the approach of Benhabib and Rustichini (1997). When choosing the optimal policy, governments face a constraint that embeds the trade-off between continuing with and deviating from the announced policy. We propose a deviation that implies reversion to a Markov equilibrium. This equilibrium specifies the consequences of deviating, which will be similar to those of Benhabib and Rustichini (1997). Our analysis is carried out both for utilities that are linear in consumption and for utilities that are concave in consumption. First, we consider an economy without a market for public debt. We obtain that, if the steady state is incentive-constrained, capital taxes are different from zero. The intuition for this result is that the incentives to deviate from the announced policy at the steady state depend only on the steady state level of capital. There is a steady state level of capital for which the optimal steady state capital tax is zero. If the incentive compatibility constraints are not satisfied for that level of capital, then the capital taxes can optimally adjust the steady state level of capital so as to satisfy the incentive compatibility constraints. Therefore, for economies without public debt, the optimal time-consistent capital tax is different from zero so as to have no incentives to deviate. Second, we allow governments to issue debt. We show that the optimal time-consistent capital tax rate is zero at the steady state. The explanation for this finding is as follows. For economies with public debt, the steady state depends on the steady state levels of both capital and debt. Moreover, the steady state capital and debt depend on the initial conditions and, hence, on the transition towards the steady state. Consequently, the incentives and costs of deviating from the announced policy at the steady state depend on both steady state levels of capital and debt, which in turn depend on the transition. We show that the optimal management of debt results in steady state levels of capital and debt, which are consistent with a zero capital tax rate. Thus, public debt can make the announcement of a zero capital tax at the steady state credible.

The plan of the paper is the following. Section 2 presents the model. Section 3 analyzes the steady state. Section 4 and 5 characterize the optimal time-consistent taxes for an economy without and with public debt, respectively. Section 6 concludes. The Appendices provide a general characterization of the time-consistent taxes and the policy after a deviation and contain the proofs of the results.

3.2 The Model

Our economy mimics that of Benhabib and Rustichini (1997). We consider an infinite-horizon economy with three agents: a representative consumer, a representative competitive firm and a benevolent government. The representative individual has an instantaneous utility function that depends on both consumption c_t and labor supply l_t so that his welfare takes the following form:

$$\sum_{t=0}^{\infty} \beta^t \left[u\left(c_t\right) - v\left(l_t\right) \right], \tag{3.1}$$

with $\beta \in (0,1)$. The utility functions $u(\cdot)$ and $v(\cdot)$ are

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}, \text{ and } v(l_t) = \frac{l_t^{1+\varepsilon}}{1+\varepsilon}, \tag{3.2}$$

with $\sigma \geq 0$ and $\varepsilon > 0$. Taking the government policy as given, the representative individual chooses consumption, labor, and assets a_{t+1} so as to maximize his welfare (3.1) subject to the budget constraint

$$r_t a_t + w_t l_t \geq a_{t+1} + c_t, \tag{3.3}$$

where r_t is the after-tax return on assets at date t and w_t is the wage rate net of labor taxes at date t. The first-order conditions for this optimization problem are

$$w_t u'\left(c_t\right) = v'\left(l_t\right), \tag{3.4}$$

$$u'\left(c_t\right) = \beta r_{t+1} u'\left(c_{t+1}\right), \tag{3.5}$$

where $u'\left(c_t\right)$ and $v'\left(l_t\right)$ denote the partial derivatives of the instantaneous utility functions (3.2) with respect to consumption and labor at date t, respectively. The individual's assets take the form of capital k_t and debt holdings b_t so that the following market equilibrium condition holds:

$$a_t = k_t + b_t. \tag{3.6}$$

The representative competitive firm produces output using capital and labor. The production function is $f\left(k_t, l_t\right)$, where f is increasing, concave, continuous and homogenous of degree one. The partial derivatives of $f(k_t, l_t)$ with respect to capital and labor are denoted $f_k(k_t, l_t)$ and $f_l(k_t, l_t)$, respectively. Taking factor prices as given, the firm chooses non-negative amounts of capital and labor so as to maximize profits. The resulting first-order conditions are

$$\left(1-\tau_t^k\right) f_k(k_t, l_t) = r_t, \tag{3.7}$$

$$\left(1-\tau_t^l\right) f_l(k_t, l_t) = w_t, \tag{3.8}$$

where τ_t^k and τ_t^l are the capital and labor income tax rates at date t, respectively.

The government finances debt payments and an exogenous public spending G per capita through taxes on labor and capital income and the issue of one-period bonds. Hence, the government budget constraint can be written as

$$b_{t+1} + (f_k(k_t, l_t) - r_t)(k_t + b_t) + (f_l(k_t, l_t) - w_t) l_t \geq G + r_t b_t. \tag{3.9}$$

The government can only sell bonds, that is, $b_{t+1} \geq 0$.[4] Those bonds are issued with the after-tax return rate r_t. In order to ensure distorsionary taxation, the first period after-tax return on assets is restricted by a lower bound. Note that this lower bound can be viewed as an upper bound on the initial capital tax rate. In the absence of that upper bound, the government could choose an initial capital tax rate so high that the resulting tax revenues could finance all future government expenditures. We consider an upper bound smaller or equal to one but sufficiently close to one. In particular, we assume that the initial after-tax return on assets satisfies

$$r_0 \geq r_{\min}, \tag{3.10}$$

[4]We assume non-negative government bonds in order to help identify a Markov equilibrium to revert to once a government deviates from the announced policy.

where $r_{\min} \geq 0$ is sufficiently close to zero.

The resource constraint is

$$k_{t+1} + c_t + G \leq f(k_t, l_t). \tag{3.11}$$

A competitive equilibrium for this economy is defined as follows:

Definition 5 *Given a sequence of tax rates* $\{\tau_t^k, \tau_t^l\}_{t=0}^{\infty}$, *the exogenous public spending* G *per period, and the initial assets* b_0 *and* k_0, *a competitive equilibrium is a sequence* $\{c_t, l_t, k_{t+1}, b_{t+1}, r_t, w_t\}_{t=0}^{\infty}$ *such that:* (i) *the representative individual maximizes his welfare* (3.1) *subject to the budget constraint* (3.3)*;* (ii) *factors are paid their marginal products according to equations* (3.7) *and* (3.8)*; and* (iii) *all markets clear (equations* (3.6) *and* (3.11) *hold, the latter with equality).*

3.2.1 The Government Optimization Problem

Without commitment, the government chooses among the time-consistent policies so as to maximize the welfare of the representative individual. In order to do so, the government takes explicitly into account an incentive compatibility constraint, which says that the welfare value of continuing with the announced policy must be higher than the welfare value of deviating from it. After a deviation, the economy reverts to some bad equilibrium, which provides a welfare called the deviation value. This deviation value depends on the consequences associated with the deviation from the announced policy. For the time being, we specify a general functional form for the deviation value that depends on the current levels of capital and bonds. Section 5.1 will provide a specific form, where after a deviation, individuals expect maximal capital taxes, which makes positive savings unattractive. The incentive compatibility constraints link current and future governments as follows:

$$\sum_{s=t}^{\infty} \beta^{s-t} \left(u\left(c_s\right) - v\left(l_s\right)\right) \geq V^D\left(k_t, b_t\right), \text{ for } t = 1, 2, ..., \tag{3.12}$$

where $V^D(k_t, b_t)$ is the value of welfare after a deviation at date t. We assume that $V^D(k_t, b_t)$ is differentiable in both variables.

The government chooses the sequences $\{c_t, l_t, k_{t+1}, b_{t+1}, w_t, r_t\}_{t=0}^{\infty}$ so as to maximize the welfare of the representative individual (3.1) subject to the first-order conditions (3.4) and (3.5), the resource constraint (3.11), the budget constraint (3.3), the lower bound on the initial after-tax return on assets (3.10), the incentive compatibility constraint (3.12) and the non-negativity constraint $b_{t+1} \geq 0$.[5] The Lagrangian for this optimization problem is

$$\begin{aligned}
L \equiv & \sum_{t=0}^{\infty} \beta^t \left[u\left(c_t\right) - v\left(l_t\right) + \lambda_t \left(\beta r_{t+1} u'\left(c_{t+1}\right) - u'\left(c_t\right)\right)\right. \\
& + \mu_t \left(w_t u'\left(c_t\right) - v'\left(l_t\right)\right) + \eta_t \left(f\left(k_t, l_t\right) - k_{t+1} - c_t - G\right) \\
& \left. + \xi_t \left(k_{t+1} + b_{t+1} + \left(f\left(k_t, l_t\right) - G - k_{t+1}\right) - r_t\left(k_t + b_t\right) - w_t l_t\right)\right] + \kappa_0 \left(r_0 - r_{\min}\right) \\
& + \sum_{t=1}^{\infty} \gamma_t \left(\sum_{s=t}^{\infty} \beta^{s-t} \left(u\left(c_s\right) - v\left(l_s\right)\right) - V^D\left(k_t, b_t\right)\right),
\end{aligned} \tag{3.13}$$

where $\beta^t \xi_t$, $\beta^t \mu_t$, $\beta^t \lambda_t$, $\beta^t \eta_t$, γ_t, and κ_0 are the Lagrange multipliers associated with constraints (3.3), (3.4), (3.5), (3.11), (3.12), and (3.10), respectively.

The first-order conditions for consumption, labor, capital, debt, return rate, and

[5] Given equations (3.3) and (3.11), this optimization problem also satisfies the government budget constraint (3.9).

wage rate are respectively as follows:[6]

$$\left[1+\beta^{-t}\left(\gamma*\beta\right)_t\right]u'\left(c_t\right)+u''\left(c_t\right)\left(\lambda_{t-1}r_t-\lambda_t+\mu_t w_t\right)=\eta_t, \tag{3.14}$$

$$\left[1+\beta^{-t}\left(\gamma*\beta\right)_t\right]v'\left(l_t\right)-\xi_t\left(f_l\left(k_t,l_t\right)-w_t\right)+\mu_t v''\left(l_t\right)=f_l\left(k_t,l_t\right)\eta_t, \tag{3.15}$$

$$\eta_t f_{k_t}-\beta^{-1}\eta_{t-1}+\xi_t\left(f_k\left(k_t,l_t\right)-r_t\right)=\gamma_t\beta^{-t}V_k^D\left(k_t,b_t\right), \tag{3.16}$$

$$\xi_t\beta r_t=\xi_{t-1}-\beta\gamma_t\beta^{-t}V_b^D\left(k_t,b_t\right), \tag{3.17}$$

$$\lambda_{t-1}u'\left(c_t\right)-\xi_t\left(k_t+b_t\right)=0, \tag{3.18}$$

$$\mu_t u'\left(c_t\right)-\xi_t l_t=0, \tag{3.19}$$

with

$$\left[1+\beta^{-t}\left(\gamma*\beta\right)_t\right]=1+\gamma_1\beta^{-1}+\gamma_2\beta^{-2}+...+\gamma_{t-1}\beta^{-(t-1)}+\beta^{-t}\gamma_t,$$

Notice that, since capital and debt are non-negative, the individual's total savings satisfy $a_{t+1}\geq 0$ in equilibrium. Moreover, taking into account the specific utility functions (3.2), the following transversality condition holds:

$$\lim_{t\to\infty}\beta^t u'\left(c_t\right)r_t\left(k_t+b_t\right)=0.$$

For an initial condition $a_0>0$, the first-order condition for the after-tax return implies $r_0=r_{\min}$. This means a very high capital tax at the initial date 0. The remaining tax rates

[6]These equations are the first-order conditions when the optimal debt is non-negative. Moreover, at the initial date 0, these conditions take a different form (see Appendix 3.7.1).

are obtained from the competitive equilibrium conditions (3.7) and (3.8). As usual in this literature, the second-order conditions are not clearly satisfied because they involve second and third derivatives of the utility function. Therefore, we assume that an optimal interior solution exists.

3.3 The Steady State

From now on, we will focus on the equilibrium at the steady state. First, a steady state is defined. Next, we distinguish between steady states where the incentive compatibility constraint (3.12) binds and those where this constraint does not bind. Finally, taking into account this distinction, we provide three lemmas that will help us to characterize the optimal taxes at the steady state. We define a steady state as follows:

Definition 6 *A steady state equilibrium is an optimal solution* $x = (c_t, l_t, k_t, b_t, w_t, r_t)$ *to the government optimization problem for some initial conditions* k_0 *and* b_0 *such that* c_t, l_t, k_t, b_t, w_t, *and* r_t *are constant.*

Taking into account conditions (3.7) and (3.8), the capital and labor tax rates τ_t^k and τ_t^l are also constant at a steady state. Consequently, at a steady state, constraints (3.4), (3.5) , (3.3), (3.11) and (3.12) can be written respectively as follows:

$$wu'(c) = v'(l), \tag{3.20}$$

$$\beta r = 1, \tag{3.21}$$

$$\left(\frac{1}{\beta} - 1\right)(k + b) + wl = c, \tag{3.22}$$

$$k + c + G = f\left(k, l\right), \tag{3.23}$$

$$\left(\frac{1}{1-\beta}\right)\left[\frac{c^{1-\sigma}}{1-\sigma} - \frac{l^{1+\varepsilon}}{1+\varepsilon}\right] \geq V^{D}\left(k, b\right), \tag{3.24}$$

Moreover, plugging equations (3.18) and (3.19) into $(3.14)-(3.17)$, the following optimality conditions must be satisfied at a steady state:

$$\left[1 + \beta^{-t}\left(\gamma * \beta\right)_{t}\right] u'\left(c\right) - \sigma\xi_{t} + \sigma\left[\xi_{t+1} - \xi_{t}\right]\left(\frac{k+b}{c}\right) = \eta_{t}, \tag{3.25}$$

$$\left[1 + \beta^{-t}\left(\gamma * \beta\right)_{t}\right] v'\left(l\right) - \xi_{t}\left(f_{l}\left(k, l\right) - w\right) + \xi_{t}\varepsilon w = f_{l}\left(k, l\right)\eta_{t}, \tag{3.26}$$

$$\left(\eta_{t} + \xi_{t}\right)\left(f_{k}\left(k, l\right) - \frac{1}{\beta}\right) = \gamma_{t}\beta^{-t}V_{k}^{D}\left(k, b\right) + \frac{1}{\beta}\left(-\eta_{t} + \eta_{t-1}\right), \tag{3.27}$$

$$\xi_{t} = \xi_{t-1} - \beta\gamma_{t}\beta^{-t}V_{b}^{D}\left(k, b\right). \tag{3.28}$$

Constraints $(3.20)-(3.24)$ and the first-order conditions $(3.25)-(3.28)$ yield a steady state for an economy with debt. For an economy without a market for public debt, a steady state $\tilde{x} = \left(\tilde{c}_{t}, \tilde{l}_{t}, \tilde{k}_{t}, \tilde{w}_{t}, \tilde{r}_{t}\right)$ is defined analogously and is determined by equations $(3.20)-(3.27)$ for $b = 0$.

The incentive compatibility constraint (3.24) could be binding or not at the steady state. The properties of the optimal taxes without commitment will depend crucially on this distinction. Consequently, we give the following definition:

Definition 7 *An incentive-constrained steady state is a steady state such that the incentive compatibility constraint* (3.24) *is binding. Otherwise, the steady state is incentive-unconstrained.*

This definition for incentive-constrained and unconstrained steady states and the analysis of the system (3.20)−(3.28) will allow us to present different lemmas. Those lemmas will become an important device in order to characterize the optimal time-consistent taxes for economies with public debt. The first result is the following:

Lemma 3 *If x is incentive-unconstrained, then* $\tau^k = 0$.

Proof. See Appendix 3.8. ■

As Lemma 3 states, the optimal capital tax rate is zero at incentive-unconstrained steady states. Therefore, we obtain the same long-run result as Chamley (1986) for his Ramsey problem. However, we must clarify that a solution to problem (3.13) that exhibits an incentive-unconstrained steady state is not equivalent to the solution of the Ramsey problem. A Lagrangian for the Ramsey problem would be that of (3.13) without the incentive compatibility constraints (3.12). On the one hand, if the incentive constraints (3.12) are binding during the transition, then, with a different history, the problem (3.13) and the Ramsey problem yield different transitions and different steady states.[7] On the other hand, if the incentive compatibility constraints (3.12) are not binding during the transition, then the solutions of the Ramsey problem and the problem (3.13) would coincide. However, problem (3.13) defines implicitly a reputation mechanism through equation (3.24), which makes the solution of problem (3.13) time-consistent, whereas the Ramsey solution is time-inconsistent.

If the steady state is incentive-constrained, the properties of the steady state taxes could be quite different. In order to determine whether the steady state is incentive-

[7] Chamley (1986) discussed the dependence of steady state values on initial conditions when government bonds are present.

constrained, we analyze the system of constraints and first-order conditions (3.20) – (3.28). We find an optimality condition for being at incentive-constrained steady states. Under this condition, the Lagrange multipliers $\{\gamma\}_i^\infty$ associated with the incentive compatibility constraints (3.12) are positive, summable, and converging to zero, which are necessary conditions for optimality. Before stating that condition, we need to define $1+g$ and $1+j$ as follows:

$$1+g \equiv \left[\frac{V_k^D(k,b)-\frac{1}{\beta}}{V_k^D(k,b)-\frac{1}{\beta}-\left[(1+D)\left(f_k(k,l)-\frac{1}{\beta}\right)\right]}\right], \tag{3.29}$$

and

$$1+j \equiv \left[\frac{-\beta V_b^D(k,b)-D}{-\sigma V_b^D(k,b)\left(\frac{\left(\frac{k+b}{c}\right)f_l(k,l)}{u'(c)\left(f_l(k,l)-w\right)}\right)D}\right], \tag{3.30}$$

where

$$D \equiv \left[\frac{u'(c)\left(f_l(k,l)-w\right)}{(1+\varepsilon)w-(1-\sigma)f_l(k,l)+\frac{1}{\beta}\sigma\left(\frac{k+b}{c}\right)f_l(k,l)}\right]. \tag{3.31}$$

We can then present the next lemma:

Lemma 4 *An incentive-constrained steady state x satisfies the following properties:*

(*i*) *If* $\sigma = 0$, *then* $1+g \in \left(0, \frac{1}{\beta}\right)$. *Moreover,*

$$V_b^D(k,b) = -\frac{1}{\beta}D. \tag{3.32}$$

(*ii*) *If* $\sigma > 0$, *then* $1+j \in \left(0, \frac{1}{\beta}\right)$.

Proof. See Appendix 3.8. ■

Lemma 4 shows some optimality conditions for incentive-constrained steady states. These conditions help us to identify an important feature of the steady state, namely, whether the steady state is incentive-constrained or unconstrained. From Lemma 3, we know that capital taxes are optimally zero at incentive-unconstrained steady states. Are capital taxes different from zero at incentive-constrained steady states? As the next lemma shows, some incentive-constrained steady states will have the property of an optimal zero capital tax:

Lemma 5 *For* $\sigma > 0$*, if* x *is incentive-constrained and* $1 + j \in (0, 1)$*, then* $\tau^k = 0$.

Proof. See Appendix 3.8. ■

From Lemma 5, we can conclude that being at an incentive-constrained steady state is not a sufficient condition for optimal capital taxes different from zero. In fact, some incentive-constrained steady states will satisfy the Chamley-Judd result. For $\sigma = 0$, an incentive-constrained steady state always satisfies that $1+j \in \left[1, \frac{1}{\beta}\right)$. Therefore, we cannot apply Lemma 5 for utilities that are linear in consumption.

In this section we have developed a theoretical basis through Lemmas 3, 4, and 5, which will be crucial to characterize the steady state optimal taxation. From these lemmas, two features turn out to be important, namely, whether the utility is linear in consumption and whether governments can issue debt. First, the type of utility, either linear or strictly concave, is important for the properties of the optimal taxes both during the transition and at the steady state. When the utility is linear in consumption, individuals are indifferent about the timing of consumption. However, when the utility is strictly concave, individuals prefer to smooth consumption over time and, thus, they care about the deviations from a consumption pattern. This different behavior affects the properties of the optimal time-consistent taxes. Second, the possibility of issuing debt is also a major determinant of the taxation scheme. We have established Lemmas 3, 4, and 5 on the basis of the first-order condition for debt (3.28). Thus, those lemmas only hold for economies with public debt.[8] In addition, the presence of debt affects the intertemporal dynamics of the economy. First, through the issue of bonds governments seek to smooth consumption over time. Moreover, the steady state values depend on the initial conditions when government bonds are present. Given these intertemporal linkages, the current issues of debt will affect not only the present but also the future incentives to deviate. Hence, the existence of a market for public debt and the specific form of the utility do matter for the properties of optimal taxation. We will next proceed to characterize the optimal time-consistent taxes. First, we examine an economy without debt. Then, we consider an economy with public debt. The proofs of the properties of the optimal taxes are contained in the Appendices; Appendix 3.7.1 provides a general characterization regardless of the presence of debt and Appendix 3.8 includes the proofs of the specific taxation properties for economies without debt and for economies with debt.

3.4 An Economy without Public Debt

In an economy without bonds, governments run a balanced budget each date. The optimal allocation and policy are determined by equations (3.20)-(3.27) setting $b = 0$. For utilities that are linear in consumption, the steady state optimal taxation is characterized as follows:

[8]We cannot apply Lemmas 3, 4 and 5 for economies without a market for public debt. For economies without debt, Lemma 3 holds for utilities that are linear in consumption, but it does not need to hold for concave utilities. Regarding Lemma 4, statement (*ii*) and condition (3.32) of statement (*i*) hold, but the rest of Lemma 4 does not apply. Lemma 5 cannot be established.

Proposition 7 *For $\sigma = 0$, the optimal time-consistent taxes at a steady state $\tilde{x}$ satisfy the following properties:*

(i) $\tau^l \in \left(0, \frac{\varepsilon}{1+\varepsilon}\right)$.

(ii) If $\tilde{x}$ is incentive-constrained and $V_k^D(k,b) > \frac{1}{\beta}$, then $\tau^k > 0$. If $V_k^D(k,b) < \frac{1}{\beta}$, then $\tau^k < 0$.

Proof. See Appendix 3.8. ■

For strictly concave utilities, we state the following:

Proposition 8 *For $\sigma > 0$, the optimal time-consistent taxes at a steady state $\tilde{x}$ satisfy the following properties:*

(i) $\tau^l \in \left(0, \min\left[\frac{\sigma+\varepsilon}{1+\varepsilon}, 1\right]\right)$.

(ii) If $\tilde{x}$ is incentive-constrained and $V_k^D(k,b) > \frac{1}{\beta}u'(c_t)$, then $\tau^k > 0$.

Proof. See Appendix 3.8. ■

For an economy without bonds, the steady state is characterized by a positive and bounded tax rate on labor income. Moreover, if the incentive compatibility constraint (3.24) is binding, then the optimal capital tax rate is different from zero. Note that the benefits and costs of deviating at the steady state depend only on the steady level of capital. At incentive-constrained steady states, a capital tax different from zero provides the appropriate incentives to change the level of capital so as to satisfy the incentive compatibility constraint (3.24). Thus, the optimal time-consistent capital tax is different from zero for economies without public debt.

Propositions 7 and 8 capture the results on capital taxes of Benhabib and Rustichini (1997). However, we should reconsider the balanced budget constraint. Notice that public debt is an instrument that links past and future variables and makes the steady state dependent on the transition. Hence, the presence of debt could seriously affect the previous results.

3.5 An Economy with Public Debt

In this section we consider an economy with a market for government bonds. This means that, in order to finance the public spending, the government chooses tax rates on labor and capital income and the issue of debt. The optimal allocation and policy are now characterized by equations (3.20)-(3.28), that is, when governments can issue debt, the first-order condition for debt (3.28) must hold. This condition links past and future efforts to satisfy the budget constraint through the current incentives to deviate. Thus, the government will choose the issues of debt taking into account how the benefits of deviating are affected. As a result, the specific form of the deviation plays now a role that was absent without a market for debt. Before characterizing the optimal time-consistent taxes, we then need to specify a particular deviation form. To this end, we assume that the production function $f(k,l)$ satisfies an elasticity of substitution between capital and labor greater than one, which can be written as $\left[\left(f_l(k,l)\, f_k(k,l)\right)/\left(f(k,l)\, f_{kl}(k,l)\right)\right] > 1$ since $f(k,l)$ is homogenous of degree one. Notice that, under this assumption, capital is not essential for positive production. A specific example for this type of production functions is that of Benhabib and Rustichini (1997), which takes the following form:

$$f(k,l) = A(e)\,k + Bl + ek^{\alpha}l^{1-\alpha}, \tag{3.33}$$

where $\alpha \in (0,1)$, $B > 0$, and $A(e) = \frac{1}{\beta} - \varphi$ with φ continuous strictly increasing function, $e \geq 0$, and $\varphi(0) = 0$. This production function is a weighted average of a Cobb-Douglas and a linear function.[9]

3.5.1 A Deviation

So far we have specified a general functional form for the deviation value, for which we have only assumed dependence and differentiability with respect to the current levels of capital and public debt. In this section we will develop a specific deviation form. This deviation will be specified as the reversion to a Markov equilibrium. In order to do so, we first describe the economy without commitment. Then, we define both sustainable and Markov equilibria. Finally, following Chari and Kehoe (1993), we propose a Markov equilibrium.

For an economy without commitment, government and individuals take their corresponding decisions sequentially. At the beginning of date t, the government chooses a current policy as a function of history $h_{t-1} = (\pi_s |\, s = 0, ..., t-1)$, denoted $\Pi_t(h_{t-1})$, and a plan for future policies under all possible future histories, where $\pi_s = (r_s, w_s, b_{s+1})$. Given a history h_{t-1}, the policy plan Π induces future histories by $h_t = (h_{t-1}, \Pi_t(h_{t-1}))$ and so on. A continuation policy of Π is $(\Pi_t(h_{t-1}), \Pi_{t+1}(h_{t-1}, \Pi_t(h_{t-1})), ...)$. Once the government decides the current policy, the representative individual chooses consumption, labor and assets holdings at date t as a function of history h_t, denoted $F_t(h_t)$, and a plan for future allocations. An allocation at date s is denoted $d_s = (c_s, l_s, a_{s+1})$. Given a history h_t and a policy plan Π_t, a continuation allocation of F is $(F_t(h_t), F_{t+1}(h_t, \Pi_{t+1}(h_t)), ...)$. In order

[9]The linear component of the specific production function (3.33) can be viewed as home production.

to define a sustainable equilibrium, we will next frame both a continuation policy and a continuation allocation into an optimization problem.

Consider first the government at date t. Given some history h_{t-1} and given that future allocations evolve according to F, the government chooses a continuation policy that maximizes the welfare of the representative individual

$$\sum_{s=t}^{\infty} \beta^{s-t} \left[u\left(c_s\left(h_s\right)\right) - v\left(l_s\left(h_s\right)\right)\right], \tag{3.34}$$

subject to government budget constraint

$$\begin{aligned} & b_{s+1}\left(h_{s-1}\right) + \left(f_k\left(k_s\left(h_{s-1}\right), l_s\left(h_s\right)\right) - r_s\left(h_{s-1}\right)\right) a_s\left(h_{s-1}\right) + \\ & \left(f_l\left(k_s\left(h_{s-1}\right), l_s\left(h_s\right)\right) - w_s\left(h_{s-1}\right)\right) l_s\left(h_s\right) \geq G + r_s\left(h_{s-1}\right) b_s\left(h_{s-1}\right), \end{aligned} \tag{3.35}$$

for all dates $s \geq t$, the non-negativity constraint $b_{s+1} \geq 0$ for all dates $s \geq t$, and the lower bound on the initial after-tax return on assets $r_t \geq r_{\min}$, where future histories are induced by Π from h_{t-1}. The solution of this problem provides a value of welfare that is denoted $V\left(h_{t-1}; \Pi, F\right)$.

Consider the representative individual at date t. Given some history h_t and given that future policies evolve according to Π, the representative individual chooses a continuation allocation so as to maximize

$$\sum_{s=t}^{\infty} \beta^{s-t} \left[u\left(c_s\left(h_s\right)\right) - v\left(l_s\left(h_s\right)\right)\right], \tag{3.36}$$

subject to the individual budget constraint

$$r_t a_t\left(h_{t-1}\right) + w_t l_t\left(h_t\right) \geq a_{t+1}\left(h_t\right) + c_t\left(h_t\right), \tag{3.37}$$

for date t, and

$$r_s(h_{s-1})\, a_s(h_{s-1}) + w_s(h_{s-1})\, l_s(h_s) \geq a_{s+1}(h_s) + c_s(h_s), \tag{3.38}$$

for all dates $s > t$, where future histories are induced by Π from h_t. We denote the welfare resulting from this optimization problem by $W(h_t; \Pi, F)$. Using these programs, we define a sustainable equilibrium as follows:

Definition 8 *A sustainable equilibrium is a pair* (Π, F) *that satisfies the following conditions: (i) Given the allocation rule* F*, the continuation policy of* Π *solves the government's problem for every history* h_{t-1}*; (ii) given a policy plan* Π*, the continuation allocation of* F *solves the consumer's problem for every history* h_t*.*

A sustainable equilibrium is utility-Markov if the past history influences payoffs only to the extent that it changes the current state variables, namely, capital and public debt at date t. This definition can be formalized as

Definition 9 *A sustainable equilibrium is said to be utility-Markov if for any pair of histories* h_{t-1} *and* h'_{t-1} *such that* $(k_t(h_{t-1}), b_t(h_{t-1})) = (k_t(h'_{t-1}), b_t(h'_{t-1}))$*, then (i)* $V(h_{t-1}; \Pi, F) = V(h'_{t-1}; \Pi, F)$*, and (ii)* $W(h_{t-1}, \pi_t; \Pi, F) = W(h'_{t-1}, \pi_t; \Pi, F)$*, where* V *and* W *are defined in equations* (3.34) *and* (3.36)*, respectively.*

We will next follow Chari and Kehoe (1993) to propose a Markov equilibrium by solving two different programs. The first program defines the value function

$$V(k_t, b_t) = \max \sum_{s=t}^{\infty} \beta^{s-t} \left[u(c_s) - v(l_s)\right], \tag{3.39}$$

subject to the budget constraint (3.3) at date $s \geq t$, the government budget constraint (3.9) at date $s \geq t$, the first-order conditions (3.4) and (3.5) at date $s \geq t$, the market clearing condition (3.6) at date $s \geq t$, the incentive constraint

$$\sum_{s=r}^{\infty} \beta^{s-r} \left[u\left(c_s\right) - v\left(l_s\right) \right] \geq V\left(k_r, b_r\right), \tag{3.40}$$

at date $s \geq t$, the non-negativity constraint $b_{s+1} \geq 0$ at date $s \geq t$, and the lower bound on the initial after-tax return on assets $r_t \geq r_{\min}$, given the initial capital k_t and debt b_t. Here equation (3.40) ensures that the government will have no incentives to deviate from the announced policy. The sequence $\{d_s, \pi_s\}_{s=t}^{\infty}$ that solves problem (3.39) gives us a policy plan Π^m for the government.

The second program defines the value function

$$W\left(k_t, b_t; \pi_t\right) \equiv \max \sum_{s=t}^{\infty} \beta^{s-t} \left[u\left(c_s\right) - v\left(l_s\right) \right], \tag{3.41}$$

subject to the budget constraint (3.3) at date $s \geq t$, the government budget constraint (3.9) at date $s > t$, the first-order conditions (3.4) and (3.5) at date $s \geq t$, the market clearing condition (3.6) at date $s \geq t$, the incentive constraint (3.40) at date $s \geq t$, and the non-negativity constraint $b_{s+2} \geq 0$ at date $s \geq t$, given π_t and the initial capital k_t and debt b_t. We choose $\{d_s, \pi_{s+1}\}_{s=t}^{\infty}$ that solves the optimization problem (3.41), which in turn yields the consumer allocation rule F^m. This allocation rule is defined for all possible histories, including those in which the government deviates. To this end, the government budget constraint (3.9) is not required to hold at date t.

Given the way these two optimization problems (3.39) and (3.41) have been constructed, a solution for them will be sustainable, as the next lemma shows:

Lemma 6 *The pair* (Π^m, F^m) *is a sustainable equilibrium.*

Proof. See Appendix 3.8. ■

The programs (3.39) and (3.41) define a sustainable equilibrium thanks to equation (3.40). Under this incentive constraint, the government plans for any future date must provide a welfare from that date on that is at least as large as the welfare value of re-optimizing, that is, choosing the policy plan at that current date. Moreover, this sustainable equilibrium is utility-Markov by construction. On the one hand, if a unique solution exists, this is uniquely determined by the current capital and debt. On the other hand, if there is more than one solution, they should provide the same welfare. By analyzing problem (3.39), we can derive some properties of the Markov equilibrium (Π^m, F^m). It is obvious that, if the value of the total assets a_t is strictly positive, then it is optimal to set the initial after-tax return to its minimum value $r_{\min}$. This result holds for all dates $t+1$, $t+2$, ... Consequently, we state the following lemma:

Lemma 7 *The pair (Π^m, F^m) satisfies $a_s = 0$ and, thus, $k_s = b_s = 0$ for all dates $s \geq t+1$.*

Proof. See Appendix 3.8. ■

A deviation occurs at date t when a government chooses a policy plan from date t on different from the announced plan. In particular, when total assets are strictly positive at date t, a government will choose a lower than the announced initial after-tax return on assets, namely, $r_{\min}$. This minimum after-tax return can be viewed as a very high capital tax. Since individuals are aware of those incentives to tax heavily the assets income, they expect very high capital taxes for all future dates. Given the specific production function, if $r_{\min}$ is sufficiently close to zero, the first-order condition for assets holdings (3.5) dictates individuals not to save and, given the non-negativity constraints, both capital and debt become zero.

Summing up, when savings are strictly positive, governments have incentives to tax them as high as possible. Thus, if the after-tax return on savings is sufficiently low, individuals prefer not to save. Therefore, our Markov equilibrium is such that savings are never positive, that is, $a_s = 0$ for all dates $s \geq t+1$. In particular, our defection satisfies $k_s = b_s = 0$ for all dates $s \geq t+1$. Appendices 3.7.2 and 3.8 contain a general characterization of the policy after a deviation and the proofs of the lemmas, respectively.

In this section we have identified a Markov equilibrium to revert to after a deviation, which yields the value of welfare after deviating. This deviation value depends on the current stock of capital and debt and on the minimum after-tax return on assets $r_{\min}$. Two remarks are in order here. First, $\frac{1}{\beta} > r_{\min} \geq 0$ is satisfied because the announced after-tax return is such that $r = \frac{1}{\beta}$, thanks to equation (3.21), and the minimum after-tax return was assumed to satisfy $r_{\min} \geq 0$. Second, the numerical results of Benhabib and Rustichini (1997) are found for a deviation that specifies complete default and, thus, individuals decide not to save. This means $r_{\min} = 0$ at date t and $k_s = 0$ for all dates $s \geq t+1$. Therefore, our deviation clearly extends the environment of Benhabib and Rustichini (1997) to economies with public debt.

3.5.2 Optimal Time-Consistent Taxes

Once we have found a Markov equilibrium to revert to after a deviation, we turn to characterize the optimal taxes without commitment for an economy with government bonds. For utilities that are linear in consumption, we obtain the following result:

Proposition 9 *For $\sigma = 0$, the optimal time-consistent taxes at a steady state x satisfy the* *following properties:*

(*i*) $\tau^l \in \left(0, \frac{\varepsilon}{1+\varepsilon}\right)$.

(*ii*) *If* $r_{\min} \leq \frac{1}{\beta} - 1$, *then* x *is incentive-unconstrained and, thus,* $\tau^k = 0$.

Proof. See Appendix 3.8. ■

We have found in Proposition 7 that the long-run capital tax could be different from zero for an economy without bonds. In the presence of public debt, we can apply Lemmas 3, 4, and 5. We show that, if $r_{\min} \leq \frac{1}{\beta} - 1$, then condition (3.32) for incentive-constrained steady states of Lemma 4 does not hold and, therefore, the steady state must be incentive-unconstrained. Thus, using Lemma 3, the optimal capital tax rate is zero. Hence, the existence of a market for debt and its relation with the time-inconsistency problem affect clearly the properties of the optimal taxes. Moreover, we find that the smaller the steady state debt is, the wider the range of $r_{\min}$ that suffices to have an optimal zero capital tax rate. In particular, for zero steady state debt, we get the following result:

Corollary 2 *For* $\sigma = 0$, *if* $b = 0$, *then* x *is incentive-unconstrained and, thus,* $\tau^k = 0$.

Proof. See Appendix 3.8. ■

Suppose that the optimal steady state debt that solves the system (3.20) – (3.28) is zero. In that case, Corollary 2 states that the capital tax rate is zero. More precisely, if the optimal steady state debt is zero, then the incentive compatibility constraint (3.12) is not binding at the steady state. In turn, we can say that, if the incentive constraint (3.12) binds at the steady state, then the steady state debt must be strictly positive. This result can help us to understand that of Benhabib and Rustichini (1997). Without a market for debt, the steady state incentives to deviate depend on the steady state level of capital. There is a steady state level of capital consistent with an optimal capital tax rate equal to zero. When the incentive constraint (3.12) is not satisfied for that steady state capital, then Benhabib and Rustichini (1997) obtained that a subsidy to capital would promote a higher level of steady state capital in order to satisfy the incentive compatibility constraint (3.12). Let us now consider Corollary 2. When the incentive constraint (3.12) binds at the steady state, then we have a strictly positive level of debt, which makes the steady state level of total assets higher. Therefore, the optimal management of debt makes public debt become an endogenous commitment device.

For utilities that are strictly concave in consumption, we show the following:

Proposition 10 *For $\sigma > 0$, the optimal time-consistent taxes at a steady state x satisfy the following properties:*

(i) $\tau^l \in \left(0, \min\left[\frac{\sigma+\varepsilon}{1+\varepsilon}, 1\right]\right)$.

(ii) If $r_{\min} = 0$, *then x is incentive-unconstrained and, thus,* $\tau^k = 0$.

(iii) If x is incentive-constrained and $0 < r_{\min} \leq \left(\frac{1}{2-\beta}\right)\left(\frac{1}{\beta} - 1\right)$, *then* $\tau^k = 0$.

Proof. See Appendix 3.8. ■

Proposition 8 obtained that the optimal time-consistent capital tax at the steady state is different from zero for economies without debt. For economies with public debt, we first show that, if $r_{\min} = 0$, then condition $1 + j \in \left(0, \frac{1}{\beta}\right)$ for incentive-constrained steady states of Lemma 4 does not hold. Therefore, the steady state is incentive-unconstrained and, thus, $\tau^k = 0$. Moreover, for a range of $r_{\min}$, we obtain that incentive-constrained steady states satisfy $1+j \in (0, 1)$ and, thus, by Lemma 5, the optimal capital tax rate is zero. The intuition for these results is as follows. First, note that, since no issuing debt is still a sustainable outcome, then the introduction of a market for debt and the optimal management of that debt will be welfare improving. Second, in the presence of debt, the steady state values depend on the initial conditions and, hence, on the transition towards the steady state. Thus, the incentives and costs of deviating, which build the incentive compatibility constraint (3.12), depend on the steady state capital and debt, which in turn depend on the past decisions. Now, to minimize the costs of meeting the incentive constraints (3.12), the government can choose a debt sequence so that the transition leads to a steady state that is incentive-unconstrained or where the incentive compatibility constraints are alleviated. As a result, the optimal time-consistent capital tax rate is zero for an economy with debt.

All in all, the optimal time-consistent policy for economies with public debt is very different from that for economies without debt. Regarding labor taxation, we have found that the optimal long-run labor tax rate is positive and bounded both in economies without and with public debt. However, the labor taxes will take a different value in these economies. Moreover, we obtain that the optimal long-run capital tax rate is different from zero for economies without debt, but it is zero for economies with debt. That optimal zero capital tax rate is obtained under some conditions on the minimum after-tax return on assets $r_{\min}$. Note first that these conditions are sufficient but not necessary. Moreover, those conditions include $r_{\min} = 0$. Therefore, the steady states that exhibit a capital tax rate different from zero in Benhabib and Rustichini (1997) and in Propositions 7 and 8 have an optimal zero capital tax once a market for debt is introduced.

3.6 Conclusions

This paper has studied the optimal steady state taxes for an economy without commitment. We have found that the properties of the optimal time-consistent capital taxes hinge on the assumption of a balanced budget. For economies without government bonds, the optimal time-consistent capital tax rate could be different from zero at the steady state. In this framework the capital tax is the instrument to make the policy credible. However, for economies with public debt, we extend the Chamley-Judd result to economies without commitment showing that the optimal time-consistent tax rate on capital income is zero at the steady state. Therefore, once a market for debt is introduced, the main commitment device against deviation is public debt.

These results highlight the importance of debt in relation to time-inconsistency and optimal taxation. In a seminal paper Lucas and Stokey (1983) showed that a rich composition of debt can help overcome the time-inconsistency problem of optimal fiscal policy. In the present paper we have shown that the mere presence of a market for debt alleviates the long-run effects of the time-inconsistency problem. The optimal management of debt can make a long-run zero capital tax announcement credible. Moreover, while the previous literature on optimal taxes without commitment had focused on economies without debt, we have proved that the presence of a market for public debt is crucial for the properties of the optimal tax policy without commitment.

In this paper we have analyzed the best sustainable equilibrium. This equilibrium is non-Markov because governments take decisions based not only on the current state variables but also on the future reactions to its actions. Then, if a government deviates, the economy reverts to some specific bad equilibrium. In this paper we have identified a particular Markov equilibrium. However, under different assumptions, we would obtain a different equilibria to revert to after a deviation. For instance, if we had assumed a production function and a minimum after-tax return rate that allowed for positive savings, the Markov equilibrium would be different from the one in this paper. Nevertheless, we believe that our main result of an optimal zero capital tax at the steady state could still hold. The intuition for this conjecture is that we have established a theoretical basis for the optimality of zero capital taxes based on a general functional form for the deviation.

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3.7 Analytical Characterization

In this appendix we first characterize the optimal time-consistent taxes and, then, the properties of the economy after a deviation.

3.7.1 The Optimal Time-Consistent Taxes

In order to characterize the optimal time-consistent taxes at a steady state, we first study the transition of the economy towards the steady state so as to determine the sign of the Lagrange multipliers. To study the transition, we suppose that the economy is at the steady state and we do the following backward-looking exercise. The government chooses the next date initial debt such that the transition back to the steady state takes place in one date. Therefore, all the variables attain their respective steady state values from date 1 onwards. Next, we solve for the steady state values in conjunction with the conditions for the initial date.

The solution at date 0 of Lagrangian (3.13) is characterized by constraints (3.4), (3.5) , (3.3), (3.11), (3.12) , and (3.10) , which can be written, respectively, as follows:

$$w_0 u' (c_0) = v' (l_0) , \tag{3.42}$$

$$u' (c_0) = \beta r_1 u' (c_1) , \tag{3.43}$$

$$r_0 (k_0 + b_0) + w_0 l_0 = k_1 + b_1 + c_0, \tag{3.44}$$

$$k_1 + c_0 + G = f (k_0, l_0) , \tag{3.45}$$

$$\sum_{t=1}^{\infty} \beta^{t-1} \left[u\left(c_t\right) - v\left(l_t\right) \right] \geq V^D\left(k_1, b_1\right),$$

$$r_0 \geq r_{\min},$$

and by the following first-order conditions for consumption, labor, capital, bonds, return rate, and wage rate, respectively:

$$u'\left(c_0\right) + u''\left(c_0\right)\left(\mu_0 w_0 - \lambda_0\right) = \eta_0, \tag{3.46}$$

$$v'\left(l_0\right) - \xi_0\left(f_l\left(k_0, l_0\right) - w_0\right) + \mu_0 v''\left(l_0\right) = f_l\left(k_0, l_0\right)\eta_0, \tag{3.47}$$

$$\left(\eta_1 + \xi_1\right)\left(f_k\left(k_1, l_1\right) - r_1\right) = \gamma_1 \beta^{-1} V_k^D\left(k_1, b_1\right) + \frac{1}{\beta}\left(-\beta r_1 \eta_1 + \eta_0\right), \tag{3.48}$$

$$\xi_1 \beta r_1 = \xi_0 - \beta \gamma_1 \beta^{-1} V_b^D\left(k_1, b_1\right), \tag{3.49}$$

$$\kappa_0 - \xi_0\left(k_0 + b_0\right) = 0, \tag{3.50}$$

$$\mu_0 u'\left(c_0\right) - \xi_0 l_0 = 0. \tag{3.51}$$

Given an initial condition for total assets $a_0 > 0$, the initial after-tax return on assets is set optimally at its minimum value, that is, $r_0 = r_{\min}$. Moreover, the economy is already at the steady state at date 1, therefore $r_1 = r$. Using $\beta r = 1$, condition (3.43) implies $c_0 = c_1 = c$. Thus, equations $(3.20) - (3.24)$, (3.42), (3.44), and (3.45) yield $\{l_0, w_0, c, l, k, b, r, w\}$. We get expressions for τ^k and τ^l from the first-order conditions (3.7) and (3.8). We solve for the initial and the steady state Lagrange multipliers using equations $(3.25) - (3.28)$ and $(3.46) - (3.51)$.

By the Kuhn-Tucker theorem, the Lagrange multipliers for constraints (3.12) and (3.10) satisfy respectively $\gamma_t \geq 0$ and $\kappa_0 > 0$. Equation (3.50) implies $\xi_0 > 0$. Next, combining condition (3.28) and $V_b^D(k,b) \leq 0$ (as proved later), we obtain $\xi_t > 0$, which does not converge to zero.[10] Conditions (3.18) and (3.19) imply $\lambda_t > 0$ and $\mu_t > 0$. Moreover, combining conditions (3.25) and (3.26), it results that $(\eta_t + \xi_t) > 0$, which does not converge to zero.

We will next proceed to show that $\tau^l \in \left(0, \min\left[\frac{\sigma+\varepsilon}{1+\varepsilon}, 1\right]\right)$ and $\tau^k \in (-\infty, 1)$ for any σ regardless of whether debt is present or not. First, conditions (3.20) and (3.21) imply the unity upper bound on both tax rates. Next, using equations (3.18) and (3.19), we get

$$u''(c)\left[\lambda_t - \lambda_{t-1} r + \mu_t w\right] = \sigma\left[\xi_{t+1} - \xi_t\right]\left(\frac{k+b}{c}\right) - \sigma\xi_t \leq 0, \tag{3.52}$$

since $u''(c) \leq 0$, $\mu_t w_t > 0$, and $\lambda_t < \frac{1}{\beta}\lambda_{t-1}$ must hold in order to have a well-defined Lagrangian. Using conditions (3.20) and (3.26), equation (3.25) becomes

$$\begin{aligned} &\xi_t \varepsilon w - \left(\sigma\left[\xi_{t+1} - \xi_t\right]\left(\frac{k+b}{c}\right) - \sigma\xi_t\right) f_l(k,l) = \\ &\left[\left[1 + \beta^{-t}(\gamma * \beta)_t\right] u'(c) + \xi_t\right]\left(f_l(k,l) - w\right), \end{aligned} \tag{3.53}$$

whose LHS is positive thanks to inequality (3.52). Therefore, $f_l(k,l) - w > 0$ and, thus, $\tau^l > 0$. Moreover, we can write equation (3.53) as

$$\begin{aligned} &\xi_t\left[(1+\varepsilon) w - (1-\sigma) f_l(k,l)\right] - \sigma\left[\xi_{t+1} - \xi_t\right]\left(\frac{k+b}{c}\right) f_l(k,l) = \\ &\left[1 + \beta^{-t}(\gamma * \beta)_t\right] u'(c)\left(f_l(k,l) - w\right), \end{aligned}$$

whose RHS is positive. Hence, it follows that, if $\sigma\left[\xi_{t+1} - \xi_t\right] \geq 0$ holds, then we obtain $(1+\varepsilon) w - (1-\sigma) f_l(k,l) > 0$ and, in turn, $\frac{\sigma+\varepsilon}{1+\varepsilon} > \tau^l$.

[10] Note that $-\xi_t$, the Lagrange multiplier for the budget constraint (3.3), represents the marginal excess burden of taxation. Atkinson and Stern (1974) showed that $\xi_t > 0$ in a second-best framework.

We will next derive conditions to determine the sign of the optimal capital taxes. Combining equations (3.52) and (3.27), we get

$$\left(\eta_t+\xi_t\right)\left(f_k\left(k,l\right)-\frac{1}{\beta}\right)=\gamma_t\beta^{-t}\left[V_k^D\left(k,b\right)-\frac{1}{\beta}u'\left(c\right)\right]-\frac{1}{\beta}Z, \tag{3.54}$$

with

$$Z=\sigma\left[\xi_{t+1}-\xi_t\right]\left(\frac{k+b}{c}\right)-\sigma\left[\xi_t-\xi_{t-1}\right]\left(1+\left(\frac{k+b}{c}\right)\right).$$

From equation (3.54), it results that, if $Z\leq 0$, then $V_k^D\left(k,b\right)-\frac{1}{\beta}u'\left(c\right)\geq 0$ implies $\tau^k\geq 0$. Otherwise, if $V_k^D-\frac{1}{\beta}\left(u'\left(c\right)+\frac{1}{\gamma_t\beta^{-t}}Z\right)\leq 0$, then $\tau^k\leq 0$.

Notice that, for utilities that are linear in consumption, that is, $\sigma=0$, it results that $\sigma\left[\xi_{t+1}-\xi_t\right]\geq 0$ and $Z\leq 0$ always hold. Therefore, the optimal steady state taxes satisfy the following results. First, $\tau^l\in\left(0,\frac{\varepsilon}{1+\varepsilon}\right)$. Second, if $V_k^D\left(k,b\right)\geq\frac{1}{\beta}$, then $\tau^k\geq 0$. Otherwise, if $V_k^D\left(k,b\right)\leq\frac{1}{\beta}$, then $\tau^k\leq 0$.

3.7.2 The Economy After Deviation

We first compute $V_k^D\left(k_t,b_t\right)$ and $V_b^D\left(k_t,b_t\right)$ and, then, we study the optimality conditions after a deviation. The consequences of deviating have been specified in Lemma 7 as follows: once a government deviates, individuals expect a minimum return on assets after taxes at all future dates so that agents do not save. Hence, once the government deviates at date t, the economy is characterized by equations (3.3), (3.4) and (3.11), which take the following form:

$$r_{\min}\left(k_t+b_t\right)+w_t^d l_t^d=c_t^d, \tag{3.55}$$

$$w_t^d u'\left(c_t^d\right)=v'\left(l_t^d\right), \tag{3.56}$$

$$c_t^d + G = f\left(k_t, l_t^d\right), \tag{3.57}$$

at date t, and

$$w_{t+s}^d l_{t+s}^d = c_{t+s}^d,$$

$$w_{t+s}^d u'\left(c_{t+s}^d\right) = v'\left(l_{t+s}^d\right),$$

$$c_{t+s}^d + G = f\left(0, l_{t+s}^d\right),$$

at date $t+s$ for all $s > 0$.[11] Thus, the welfare after a deviation at date t is

$$V^D\left(k_t, b_t\right) = \left(\frac{\left(c_t^d\right)^{1-\sigma}}{1-\sigma}\right) - \left(\frac{\left(l_t^d\right)^{1+\varepsilon}}{1+\varepsilon}\right) + \left[\frac{\beta}{1-\beta}\right]\left(\left(\frac{\left(c_{t+s}^d\right)^{1-\sigma}}{1-\sigma}\right) - \left(\frac{\left(l_{t+s}^d\right)^{1+\varepsilon}}{1+\varepsilon}\right)\right), \tag{3.58}$$

which allows us to write

$$V_k^D\left(k_t, b_t\right) = \frac{\partial V^D\left(k_t, b_t\right)}{\partial k_t} = \left(c_t^d\right)^{-\sigma}\frac{\partial c_t^d}{\partial k_t} - \left(l_t^d\right)^{\varepsilon}\frac{\partial l_t^d}{\partial k_t},$$

$$V_b^D\left(k_t, b_t\right) = \frac{\partial V^D\left(k_t, b_t\right)}{\partial b_t} = \left(c_t^d\right)^{-\sigma}\frac{\partial c_t^d}{\partial b_t} - \left(l_t^d\right)^{\varepsilon}\frac{\partial l_t^d}{\partial b_t}. \tag{3.59}$$

To find these derivatives, we use the specific utility functions (3.2) and equations (3.55) – (3.57) to form the following set of equations:

$$r_{\min}\left(k_t + b_t\right) + \left(c_t^d\right)^{\sigma}\left(l_t^d\right)^{1+\varepsilon} - c_t^d = 0, \tag{3.60}$$

$$c_t^d + G - f\left(k_t, l_t^d\right) = 0. \tag{3.61}$$

[11]The variables with a superscript d are those after a deviation.

Let us first compute $V_b^D(k_t, b_t)$. If $r_{\min} = 0$, then $V_b^D(k, b) = 0$. For some $r_{\min} > 0$, using equations (3.60) – (3.61), we obtain the following system:

$$-\left[1 - \sigma\left(c_t^d\right)^{\sigma-1}\left(l_t^d\right)^{1+\varepsilon}\right]\left(\frac{\partial c_t^d}{\partial b_t}\right) + (1+\varepsilon)\left(c_t^d\right)^{\sigma}\left(l_t^d\right)^{\varepsilon}\left(\frac{\partial l_t^d}{\partial b_t}\right) = -r_{\min},$$

$$\left(\frac{\partial c_t^d}{\partial b_t}\right) - f_{l^d}\left(k_t, l_t^d\right)\left(\frac{\partial l_t^d}{\partial b_t}\right) = 0,$$

by implicit differentiation. Solving for $\dfrac{\partial l_t^d}{\partial b_t}$ and $\dfrac{\partial c_t^d}{\partial b_t}$, the derivative (3.59) becomes

$$V_b^D(k_t, b_t) = -r_{\min}\left[\frac{u'\left(c_t^d\right)\left(f_{l^d}\left(k_t, l_t^d\right) - w_t^d\right)}{(1+\varepsilon)\, w_t^d - (1-\sigma)\, f_{l^d}\left(k_t, l_t^d\right) - \sigma r_{\min}\left(\frac{k_t+b_t}{c_t^d}\right) f_{l^d}\left(k_t, l_t^d\right)}\right]. \quad (3.62)$$

Proceeding in the same way for $V_k^D(k_t, b_t)$, we obtain

$$V_k^D(k_t, b_t) = u'\left(c_t^d\right) f_k\left(k_t, l_t^d\right) + \left[\left[(1-\sigma) + \sigma r_{\min}\left(\frac{k_t + b_t}{c_t^d}\right)\right] f_k\left(k_t, l_t^d\right) - r_{\min}\right]$$
$$\left[\frac{u'\left(c_t^d\right)\left(f_{l^d}\left(k_t, l_t^d\right) - w_t^d\right)}{(1+\varepsilon)\, w_t^d - (1-\sigma)\, f_{l^d}\left(k_t, l_t^d\right) - \sigma r_{\min}\left(\frac{k_t+b_t}{c_t^d}\right) f_{l^d}\left(k_t, l_t^d\right)}\right].$$

We will next study the optimality conditions after a deviation at date t. Notice that Lemma 7 provides a necessary condition for the continuation allocations from any history of the Markov equilibrium (Π^m, F^m). Thus, it turns out that the allocations that solve problem (3.39) solve the same problem but replacing constraint (3.40) by the conditions provided in Lemma 7. From this problem, we obtain the following first-order conditions for consumption, labor, return rate, and wage rate at date t:

$$u'\left(c_t^d\right) + u''\left(c_t^d\right)\mu_t^d w_t^d = \eta_t^d, \quad (3.63)$$

$$v'\left(l_t^d\right) - \xi_t^d\left(f_{l^d}\left(k_t, l_t^d\right) - w_t^d\right) + \mu_t^d v''\left(l_t^d\right) = f_{l^d}\left(k_t, l_t^d\right)\eta_t^d, \quad (3.64)$$

$$\kappa_t^d - \xi_t^d \left(k_t + b_t\right) = 0, \tag{3.65}$$

$$\mu_t^d u'\left(c_t^d\right) - \xi_t^d l_t^d = 0. \tag{3.66}$$

Here μ_t^d, ξ_t^d, η_t^d, and κ_t^d are the Lagrange multipliers associated with constraints (3.4), (3.56), (3.57), and the lower bound on the initial after-tax return on assets, respectively.

The government sets $r_t = r_{\min}$, which implies $\kappa_t^d > 0$. From condition (3.65), we get $\xi_t^d > 0$. Using equation (3.66), condition (3.64) implies $\left(\eta_t^d + \xi_t^d\right) > 0$. Next, combining equations (3.55), (3.63) and (3.64), we obtain

$$\xi_t^d \varepsilon w_t^d + \xi_t^d \sigma \frac{w_t^d l_t^d}{c_t^d} w_t^d = \left(\eta_t^d + \xi_t^d\right)\left(f_{l^d}\left(k_t, l_t^d\right) - w_t^d\right), \tag{3.67}$$

whose LHS is positive. Therefore, $f_{l^d}\left(k_t, l_t^d\right) - w_t^d > 0$ and, thus, $\tau^{l^d} > 0$. Next, plugging equation (3.63) into (3.67), we get

$$\xi_t^d \left(\left(1+\varepsilon\right) w_t^d - \left(1-\sigma\right) f_{l^d}\left(k_t, l_t^d\right) - \sigma r_{\min}\left(\frac{k_t + b_t}{c_t^d}\right) f_{l^d}\left(k_t, l_t^d\right)\right) =$$
$$u'\left(c_t^d\right)\left(f_{l^d}\left(k_t, l_t^d\right) - w_t^d\right),$$

which implies

$$\left(1+\varepsilon\right) w_t^d - \left(1-\sigma\right) f_{l^d}\left(k_t, l_t^d\right) - \sigma r_{\min}\left(\frac{k_t + b_t}{c_t^d}\right) f_{l^d}\left(k_t, l_t^d\right) > 0.$$

Using the previous inequality, the optimal labor tax rate after a deviation satisfies that $\tau_t^{l^d} \in \left(0, \min\left[\frac{\sigma+\varepsilon}{1+\varepsilon}, 1\right]\right)$. From equation (3.62), it results that $V_b^D\left(k_t, b_t\right) \leq 0$.

3.8 Proofs of the Propositions and Lemmas

Proof of Lemma 3

If the steady state is incentive-unconstrained, then the first-order condition for debt (3.28) implies $\xi_t = \xi_{t-1}$. Given equation (3.25), η_t is also constant, that is, $\eta_t = \eta_{t-1}$. Thus, condition (3.27) implies $f_k(k,l) = \frac{1}{\beta}$ and, thus, $\tau^k = 0$. ■

Proof of Lemma 4

We will next show that, for utilities that are linear in consumption, that is, $\sigma = 0$, there exists a real number g such that $\gamma_t \beta^{-t} = (1+g)\gamma_{t-1}\beta^{-(t-1)}$, or equivalently, $\gamma_t \beta^{-t} = (1+g)^t \gamma_0$. Solving for ξ_t in equation (3.53), we obtain

$$\xi_t = \left[1 + \beta^{-t}(\gamma * \beta)_t\right]\left[\frac{f_l(k,l) - w}{(1+\varepsilon)w - f_l(k,l)}\right], \tag{3.68}$$

which can be written as

$$\xi_t = \xi_{t-1} + \gamma_t \beta^{-t}\left[\frac{f_l(k,l) - w}{(1+\varepsilon)w - f_l(k,l)}\right]. \tag{3.69}$$

Plugging equations (3.25), (3.31) and (3.68) into (3.27), we get

$$\left[1 + \beta^{-t}(\gamma * \beta)_t\right][1+D]\left(f_k(k,l) - \frac{1}{\beta}\right) = \gamma_t \beta^{-t}\left(V_k^D(k,b) - \frac{1}{\beta}\right). \tag{3.70}$$

If $V_k^D = \frac{1}{\beta}$, then $f_k = \frac{1}{\beta}$. Thus, equations (3.20) – (3.24) yield an incentive-unconstrained steady state. Since we try to identify incentive-constrained steady states, we consider $V_k^D \neq \frac{1}{\beta}$. Solving for $\gamma_t \beta^{-t}$ in equation (3.70) and rearranging terms, we get

$$\gamma_t \beta^{-t} = (1+g)\gamma_{t-1}\beta^{-(t-1)},$$

where $1+g$ takes the value defined in equation (3.29). Moreover, if the steady state is incentive-constrained, then $1+g$ belongs to $\left(0, \frac{1}{\beta}\right)$. First, $1+g$ must be greater than 0 so that γ_t is positive. Second, if $1+g$ were greater or equal to $\frac{1}{\beta}$, then Lagrangian (3.13) would not be well-defined. In addition, when public debt is present, equations (3.28) and (3.69) imply that condition (3.32) must hold at incentive-constrained steady states. Therefore, for $\sigma = 0$, an incentive-constrained steady state of an economy with public debt must satisfy conditions (3.29) and (3.32).

We will next show that, for utilities that are concave in consumption, that is, $\sigma > 0$, there exits a real number j such that $\gamma_t \beta^{-t} = (1+j)\,\gamma_{t-1}\beta^{-(t-1)}$. Using the necessary conditions (3.25), (3.26) and (3.28), ξ_t can be written as follows:

$$\xi_t = (1 + \beta^{-t}\,(\gamma * \beta)_t)E + \gamma_{t+1}\beta^{-(t+1)}F. \tag{3.71}$$

Here $E > 0$ and $F \geq 0$ take the following forms:

$$\begin{aligned} E &= \frac{u'(c)\,(f_l(k,l) - w)}{(1+\varepsilon)\,w - (1-\sigma)\,f_l(k,l)}, \\ F &= \frac{-\sigma V_b^D(k,b)\left(\frac{k+b}{c}\right) f_l(k,l)}{(1+\varepsilon)\,w - (1-\sigma)\,f_l}. \end{aligned}$$

Considering equation (3.71) at date $t-1$ and rearranging terms, we obtain

$$\gamma_{t+1}\beta^{-(t+1)} = (1+j)\,\gamma_t\beta^{-t},$$

where

$$1 + j \equiv \left[\frac{F - E - \beta V_b^D(k,b)}{F}\right],$$

which becomes equation (3.30). By arguments analogous to those for $\sigma = 0$, if the steady state is incentive-constrained, then $1 + j \in \left(0, \frac{1}{\beta}\right)$. Notice next that, if $V_b^D(k,b) = 0$, then condition (3.71) can be written as

$$\xi_t = \xi_{t-1} + \gamma_t \beta^{-t} E,$$

which implies $\gamma_t = 0$ through equation (3.28). Therefore, $V_b^D(k,b)$ must be strictly negative at incentive-constrained steady states. In particular, for $1+j \in \left(0, \frac{1}{\beta}\right)$, $V_b^D(k,b)$ must satisfy

$$-\frac{1}{\beta}(E-F) > V_b^D(k,b) > -\frac{1}{\beta}\left[E + \left(\frac{1}{\beta}-1\right)F\right]. \tag{3.72}$$

More precisely, in order to have $1+j>0$ and $1+j<\frac{1}{\beta}$, we require that

$$-\frac{1}{\beta}\left[\frac{u'(c)(f_l(k,l)-w)}{(1+\varepsilon)w-(1-\sigma)f_l(k,l)+\frac{1}{\beta}\sigma\left(\frac{k+b}{c}\right)f_l(k,l)}\right] > V_b^D(k,b), \tag{3.73}$$

and

$$V_b^D(k,b) > -\frac{1}{\beta}\left[\frac{u'(c)(f_l(k,l)-w)}{(1+\varepsilon)w-(1-\sigma)f_l(k,l)-\left(\frac{1}{\beta}-1\right)\sigma\left(\frac{k+b}{c}\right)f_l(k,l)}\right], \tag{3.74}$$

respectively. ■

Proof of Lemma 5

The proof of Lemma 4 shows that a real number j exists such that $\gamma_t\beta^{-t} = (1+j)^t\gamma_0$. If $1+j$ belongs to $(0,1)$, then $\gamma_t\beta^{-t}$ approaches zero and, in turn, $\left[1+\beta^{-t}(\gamma*\beta)_t\right]$ converges to a positive constant. Using conditions (3.25) and (3.28), we obtain that both ξ_t and η_t become constant. Moreover, thanks to condition (3.27), $(\eta_t+\xi_t)\left(\frac{1}{\beta}-f_k(k,l)\right)$ approaches zero and, thus, $\tau^k = 0$ at the steady state. The condition $1+j\in(0,1)$ for zero capital taxes at incentive-constrained steady states can be written as

$$-\frac{1}{\beta}(E-F) > V_b^D(k,b) > -\frac{1}{\beta}E,$$

where E and F are defined in equation (3.71). More precisely, in order to have $1+j>0$ and $1+j<1$, we require condition (3.73) and

$$V_b^D(k,b) > -\frac{1}{\beta}\left[\frac{u'(c)(f_l(k,l)-w)}{(1+\varepsilon)w-(1-\sigma)f_l(k,l)}\right], \tag{3.75}$$

respectively. ■

Proof of Lemma 6

First, we must show that given a policy plan Π^m, the continuation allocation of F^m solves the individual's problem (3.36) for every history h_t. Note that the solution of problem (3.41) for $(k_t, b_t; \pi_t)$ at date $t+j$ for $j \geq 1$ coincides with the solution of (3.39) for (k_{t+1}, b_{t+1}) from date $t+1$ on. Then, the policies that solve problem (3.41) are π_t and, by the recursivity of (3.39), those generated by Π^m for date $s>t$. As a result, the policies that solve problem (3.41) are exactly the policies that the consumer faces when solving (3.36). Moreover, given that constraint (3.3) holds for all dates $s \geq t$ for problem (3.41), then equations (3.37) and (3.38) are both satisfied. Hence, the allocations generated from F^m are the optimal response to that policy. Thus, they solve problem (3.36).

Second, we must prove that, given the allocation rule F^m, the continuation policy of Π^m solves the government's problem (3.34) for every history h_{t-1}. It suffices to show that no deviation improves welfare. This means that, if individuals follow the allocation rule F^m and the government policies from date $t+1$ onwards are generated from Π^m, then there is no policy π_t at date t which satisfies the budget constraint (3.35) and improves welfare. By construction of Π^m and F^m, this is the case since they satisfy constraint (3.40). ■

Proof of Lemma 7

In order to characterize the Markov equilibrium (Π^m, F^m), we will focus on the government problem (3.39) yielding Π^m. Since the pair (Π^m, F^m) is a sustainable equilibrium, the sequence $\{d_s, \pi_s\}_{s=t+1}^{\infty}$ that solves the government problem at date t must solve the government problem at date $t+1$, and at any arbitrary date $s \geq t$. We will show that the equilibrium (Π^m, F^m) satisfies $a_s = 0$ for all dates $s \geq t+1$. If the initial savings are strictly positive, that is, $a_t > 0$, the optimality conditions imply that $r_t = r_{\min}$. First, suppose that the government at date t chooses $r_{\min}$ at all future dates $s \geq t+1$. If $r_{\min}$ is sufficiently close to zero, the first-order condition (3.5) of the individual implies $a_s = 0$ for all dates $s \geq t+1$. This result holds because the production function satisfies an elasticity of substitution between capital and labor greater than one. Second, suppose that the government problem at date s chooses some $r_s > r_{\min}$, which gives rise to an optimal $a_s > 0$. When the economy is at date s, the government problem (3.39) yields a first-order condition for the initial after-tax return implying that $r_s = r_{\min}$, which violates the incentive constraint (3.40) for the government problem (3.39) at date t. Hence, for an arbitrary date t, the pair (Π^m, F^m) satisfies $a_s = 0$ for all dates $s \geq t+1$. Moreover, given the non-negativity constraints for capital and debt, this implies $k_s = b_s = 0$ for all dates $s \geq t+1$. ■

Proof of Proposition 7

For utilities that are linear in consumption, that is, $\sigma = 0$, Appendix 3.7.1 showed the following results. First, $\tau^l \in \left(0, \frac{\varepsilon}{1+\varepsilon}\right)$ holds. Second, if $V_k^D(k,b) > \frac{1}{\beta}$, then $\tau^k > 0$ at incentive-constrained steady states. If $V_k^D(k,b) \leq \frac{1}{\beta}$, the inequality is reversed. ■

Proof of Proposition 8

We show first that $\sigma\left[\xi_{t+1}-\xi_{t}\right]\geq 0$. Solving for ξ_{t+1} in equation (3.53), we obtain

$$\xi_{t+1}=P^{t}\left[\xi_{0}-R_{t}Q\right]. \tag{3.76}$$

Here we have

$$P=\left[\frac{(1+\varepsilon)\,w-(1-\sigma)\,f_{l}\,(k,l)+\sigma\left(\frac{k}{c}\right)f_{l}\,(k,l)}{\sigma\left(\frac{k}{c}\right)f_{l}\,(k,l)}\right],$$

$$Q=\left[\frac{u'\,(c)\,(f_{l}\,(k,l)-w)}{\sigma\left(\frac{k}{c}\right)f_{l}\,(k,l)}\right],$$

$$R_{t}=P^{-1}\left[1+\beta^{-t}\,(\gamma*\beta)_{0}\right]+P^{1}\left[1+\beta^{-t}\,(\gamma*\beta)_{t}\right]+...+P^{-t}\left[1+\beta^{-t}\,(\gamma*\beta)_{t}\right],$$

with $P>0$, $Q>0$, and $R_{t}>0$, which is increasing in time. If $P\leq 1$, then ξ_{t+1} converges to a negative number. Therefore, $P>1$ must be satisfied. Next, equation (3.76) implies $(1+\varepsilon)\,w-(1-\sigma)\,f_{l}\,(k,l)>0$ and, thus, $\tau^{l}\in\left(0,\min\left[\frac{\sigma+\varepsilon}{1+\varepsilon},1\right]\right)$.

We next characterize the capital tax. First, given that $\left[\xi_{t+1}-\xi_{t}\right]\geq 0$, a solution for Lagrangian (3.13) requires $\left[\xi_{t+1}-\xi_{t}\right]\leq\frac{1}{\beta}\left[\xi_{t}-\xi_{t-1}\right]$. Second, from equation (3.22), we get that $\left(1+\frac{1}{\left(\frac{k}{c}\right)}\right)>\frac{1}{\beta}$. Thus, $Z\leq 0$ holds, where Z is defined in equation (3.54). From Appendix 3.7.1, it results that, if $V_{k}^{D}\,(k,b)-\frac{1}{\beta}u'\,(c)\geq 0$, then $\tau^{k}\geq 0$. ■

Proof of Proposition 9

For $\sigma=0$, Appendix 3.7.2 showed that $\tau^{l}\in\left(0,\frac{\varepsilon}{1+\varepsilon}\right)$.

We will next show that, if $r_{\min}\leq\frac{1}{\beta}-1$, then the necessary condition (3.32) for incentive-constrained steady states does not hold. As a result, Lemma 3 implies $\tau^{k}=0$.

Using equations (3.8), (3.31) and (3.62) , we can write condition (3.32) as

$$r_{\min}\left(\frac{\tau^{l^d}}{\varepsilon-(1+\varepsilon)\,\tau^{l^d}}\right)=\frac{1}{\beta}\left(\frac{\tau^{l}}{\varepsilon-(1+\varepsilon)\,\tau^{l}}\right). \tag{3.77}$$

First, we know that $r_{\min}<\frac{1}{\beta}$. Second, we will show that, if $r_{\min}\leq\frac{1}{\beta}-1$, then $l^d\geq l$, which implies $\tau^{l^d}<\tau^l$. Combining equations (3.22) and (3.23), we obtain

$$\frac{1}{\beta}k+\left(\frac{1}{\beta}-1\right)b+l^{\varepsilon+1}+G=f\left(k,l\right). \tag{3.78}$$

Using constraints (3.60) and (3.61), we find

$$r_{\min}\left(k+b\right)+\left(l^{d}\right)^{\varepsilon+1}+G=f\left(k,l^{d}\right). \tag{3.79}$$

From inspection of equations (3.78) and (3.79) , it results that, if

$$r_{\min}\left(k+b\right)\leq\frac{1}{\beta}k+\left(\frac{1}{\beta}-1\right)b, \tag{3.80}$$

then $l^d\geq l$, which implies $\tau^{l^d}\leq\tau^l$ through equation (3.8). Hence, if $r_{\min}\leq\frac{1}{\beta}-1$, then $\tau^{l^d}\leq\tau^l$ and, therefore, condition (3.77) does not hold. As a result, it follows that the steady state must be incentive-unconstrained and, thus, $\tau^k=0$. ■

Proof of Corollary 2

Since $r_{\min}<\frac{1}{\beta}$, if $b=0$, then equation (3.80) is satisfied for all $r_{\min}$. ■

Proof of Proposition 10

Equation (3.28) implies that $\sigma\left[\xi_{t+1}-\xi_t\right]\geq 0$ holds and, thus, $\tau^l\in(0,\min\left[\frac{\sigma+\varepsilon}{1+\varepsilon},1\right])$.

The proof of Lemma 4 showed that incentive-constrained steady states require $V_b^D\left(k,b\right)$ be strictly negative and belong to the range specified by (3.72). If $r_{\min}=0$, the system $(3.60)-(3.61)$ implies that $V_b^D(k,b)=0$. As a result, the steady state is incentive-unconstrained and, by Lemma 3, $\tau^k=0$.

We proceed next to show that, if $\left(\frac{1}{2-\beta}\right)\left(\frac{1-\beta}{\beta}\right)\geq r_{\min}$, then incentive-constrained steady states satisfy $1+j\in(0,1)$. Using equations (3.62) and (3.8), condition (3.75) for $1+j<1$ at incentive-constrained steady states becomes

$$\frac{1}{\beta}\left(\frac{u'(c)\,\tau^l}{(\varepsilon+\sigma)-(1+\varepsilon)\,\tau^l}\right)>r_{\min}\left(\frac{u'\left(c_t^d\right)\tau^{l^d}}{(\varepsilon+\sigma)-(1+\varepsilon)\,\tau^{l^d}-\sigma r_{\min}\left(\frac{k_t+b_t}{c_t^d}\right)}\right). \tag{3.81}$$

Inequality (3.81) can be written as

$$\frac{1}{\beta}\left(\frac{u'(c)\,\tau^l}{(1+\varepsilon)\left(1-\tau^l\right)-(1-\sigma)}\right)>\frac{1}{\beta}\left(\frac{u'(c)\,\tau^{l^d}}{(1+\varepsilon)\left(1-\tau^{l^d}\right)-(1-\sigma)}\right)*Y,$$

where

$$Y=\left[\frac{r_{\min}u'\left(c_t^d\right)}{\frac{1}{\beta}u'(c)}\right]\left[\frac{(1+\varepsilon)\left(1-\tau^{l^d}\right)-(1-\sigma)}{(1+\varepsilon)\left(1-\tau^{l^d}\right)-(1-\sigma)-\sigma r_{\min}\left(\frac{k_t+b_t}{c_t^d}\right)}\right].$$

In order to show that inequality (3.81) holds at incentive-constrained steady states, that is, $1+j\in(0,1)$, we proceed in two steps. First, we prove by total differentiation that, if $\frac{1}{\beta}-1\geq r_{\min}$, then $c<c^d$ and $\tau^l>\tau^{l^d}$. Second, we show that $Y<1$ holds whenever $\left(\frac{1}{2-\beta}\right)\left(\frac{1-\beta}{\beta}\right)\geq r_{\min}$. Let us denote by $\triangle_c^*$ the change in c from the best sustainable consumption to the consumption after a deviation at date t. Thus, $\triangle_c^*>0$ means $c<c_t^d$. Combining equations (3.23) and (3.61), we get

$$c_t^d-f\left(k,l_t^d\right)>c-f(k,l). \tag{3.82}$$

Since $r_{\min}\leq\frac{1}{\beta}-1$, equations (3.22) and (3.60) allow us to write

$$c_t^d-\left(c_t^d\right)^{\sigma}\left(l_t^d\right)^{1+\varepsilon}\leq c_t-c^{\sigma}l^{1+\varepsilon}. \tag{3.83}$$

Equations (3.82) and (3.83) imply the following two inequalities:

$$\triangle_c^* - f_l(k,l)\,\triangle_l^* > 0, \tag{3.84}$$

$$\left(1-\sigma\frac{wl}{c}\right)\triangle_c^* - (1+\varepsilon)\,w\triangle_l^* \leq 0. \tag{3.85}$$

If $1-\sigma\dfrac{wl}{c} \geq 0$, then inequalities (3.84) and (3.85) imply $l \leq l^d$ and $c < c^d$ and, in turn, $\tau^l > \tau^{l^d}$. Consider next $1-\sigma\dfrac{wl}{c} < 0$. Since $\tau^l = 1-\left(c^{\sigma}l^{\varepsilon}/f_l(k,l)\right)$, we can write

$$\triangle_{\tau^l}^* = -\left(\frac{1}{f_l(k,l)\,l}\right)\left[\sigma\frac{wl}{c}\triangle_c^* + \left(\varepsilon - \left(\frac{f_{ll}(k,l)\,l}{f_l(k,l)}\right)\right)w\triangle_l^*\right],$$

with the following sign:

$$sign\left(d\tau^l\right) = -sign\left(\sigma\frac{wl}{c}dc + \left(\varepsilon - \left(\frac{f_{ll}(k,l)\,l}{f_l(k,l)}\right)\right)wdl\right). \tag{3.86}$$

Here $-\left(\left(f_{ll}(k,l)\,l\right)/f_l(k,l)\right) < 1$ because the elasticity of substitution between capital and labor is greater than one. From $1-\sigma\dfrac{wl}{c} < 0$, it follows that

$$\triangle_l^* > -\left[\frac{\sigma\frac{wl}{c}-1}{(1+\varepsilon)\,w}\right]\triangle_c^*. \tag{3.87}$$

Combining equations (3.84) and (3.87), we can write

$$f_l(k,l)\left[\frac{\sigma\frac{wl}{c}-1}{(1+\varepsilon)\,w}\right]\triangle_c^* > -f_l(k,l)\,\triangle_l^* > -\triangle_c^*,$$

which implies

$$\left(\frac{1}{(1+\varepsilon)\,w}\right)\left[f_l(k,l)\left(\sigma\frac{wl}{c}-1\right)+(1+\varepsilon)\,w\right]\triangle_c^* > 0.$$

The previous inequality means that $\triangle_c^* > 0$, that is, $c < c^d$. Concerning labor, there are two possibilities, namely, $l \leq l^d$ or $l > l^d$. If $l \leq l^d$, then it is clear that $\tau^l > \tau^{l^d}$. Consider

$l > l^d$, that is, $\triangle_l^* < 0$. Using equations (3.86) and (3.87), we can write

$$sign\left(-\triangle_{\tau^l}^*\right) = sign\left(\sigma\frac{wl}{c}\triangle_c^* + \left(\varepsilon - \left(\frac{f_{ll}(k,l)\,l}{f_l(k,l)}\right)\right)w\triangle_l^*\right) \geq$$
$$sign\left(\left[-(1+\varepsilon)\left(\frac{\sigma\frac{wl}{c}}{\sigma\frac{wl}{c}-1}\right) + \left(\varepsilon - \left(\frac{f_{ll}(k,l)\,l}{f_l(k,l)}\right)\right)\right]w\triangle_l^*\right).$$

If $\triangle_l^* < 0$, then $sign\left(-\triangle_{\tau^l}^*\right) > 0$ and, thus, again $\tau^l > \tau^{l^d}$. Hence, if $\frac{1}{\beta} - 1 \geq r_{\min}$, then $c < c^d$ and $\tau^l > \tau^{l^d}$.

We will next show that, given previous results, $\left[\frac{1}{2-\beta}\right]\left[\frac{1-\beta}{\beta}\right] \geq r_{\min}$ implies that $Y < 1$. We can write condition $Y < 1$ as

$$\sigma r_{\min}\left(\frac{k_t + b_t}{c_t^d}\right) < \left[\frac{r_{\min}u'\left(c_t^d\right) - \frac{1}{\beta}u'(c)}{\frac{1}{\beta}u'(c)}\right]\left[(1+\varepsilon)\left(1-\tau^{l^d}\right) - (1-\sigma)\right]. \tag{3.88}$$

Using equation (3.74), we get

$$\left[(1+\varepsilon)\left(1-\tau^{l^d}\right) - (1-\sigma)\right] > \sigma\left[\frac{1}{\beta} - 1\right]\left(\frac{k_t + b_t}{c}\right), \tag{3.89}$$

as $\tau^l > \tau^{l^d}$. Combining equations (3.88) and (3.89), we obtain

$$r_{\min} < \left(\frac{c^d}{c}\right)[1-\beta]\left[\frac{1}{\beta} - r_{\min}\left(\frac{u'\left(c^d\right)}{u'(c)}\right)\right],$$

which is satisfied whenever

$$r_{\min} < \left[\frac{1-\beta}{\beta}\right]\left[\left(\frac{c}{c^d}\right) + (1-\beta)\left(\frac{u'\left(c^d\right)}{u'(c)}\right)\right]^{-1}.$$

Since $c < c^d$, it results that $\left(\frac{1}{2-\beta}\right)\left(\frac{1-\beta}{\beta}\right) \geq r_{\min}$ is a sufficient condition for $Y < 1$ and, thus, for zero capital taxes at incentive-constrained steady states. ■

Chapter 4

Time-Consistency of Optimal Fiscal Policy in an Endogenous Growth Model

4.1 Introduction

This paper studies the time-inconsistency problem of optimal fiscal policy and the role of debt restructuring for an economy with private capital and endogenous growth achieved via public capital. This is an interesting framework for addressing time-consistency issues. In a growth model future allocations are, first, mostly driven by the future possibilities of growth and, second, they are more important for welfare. Given the relevant role that the government plays in the growth process, it is crucial to make the announced policy time-consistent. Since the capital levy problem makes debt restructuring unable to solve the time-inconsistency problem, we consider a zero tax rule on capital income. More precisely, the current and future governments choose the optimal policy subject to a zero capital tax constraint at all dates. In this case, the time-inconsistency problem of capital taxation prevails since the efforts of satisfying the zero capital tax rule are very high in the short-run but low in the long-run. We show that the careful management of public debt can make the policy subject to this restriction on capital taxes time-consistent. We characterize the distinctive properties of debt restructuring for economies with private capital and endogenous growth.

An optimal policy selected by a government at a given date is time-inconsistent when it is not longer optimal when reconsidered at some later date, although no relevant information has been revealed. The time-inconsistency problem of optimal fiscal policy arises in very general frameworks. In particular, in a representative agent model with a benevolent government, the optimal policy is time-inconsistent when the government has no lump-sum taxes at its disposal. As Faig (1994) made clear, different endowments call for different policy plans and, once these endowments have changed, the government has incentives to change the policy plan in order to set a less distorsionary taxation. Capital taxes illustrate very well this problem. As Chamley (1986) showed, a government should promise low future capital taxes in order to encourage investment. However, once this investment has taken place, the current capital income is a pure rent and should be taxed heavily. Therefore, taxing the future capital income is distorsionary, whereas taxing the current capital income is not distorsionary. This is known as the capital levy problem. Moreover, labor income taxes have a different degree of distortion depending on the planning date. The optimal labor tax rate for future dates must take into account how this policy affects the capital accumulation in the time interval. However, once the capital investment is bygone, these effects are not taken into account. Therefore, the optimal labor tax rate is different from the announced policy. In view of these incentives to deviate from the previously selected policy, governments face a credibility problem. In the absence of full-commitment, the optimal policy cannot be implemented and this time-inconsistency leads to a welfare loss.

In a seminal paper Lucas and Stokey (1983) showed that the optimal policy could be made time-consistent if governments commit to honoring debt and this debt is issued with a sufficiently rich maturity structure. For a barter economy with exogenous public spending and no capital, they showed how the careful selection of the maturity of debt indexed to consumption could provide the right incentives to future governments so as to continue with the announced policy. This method has been called debt restructuring. Persson and Svensson (1986) and Faig (1991) extended this method to open economies. Alvarez, Kehoe and Neumeyer (2002) solved the time-inconsistency problem of labor taxes and monetary policy through debt restructuring. Faig (1994) made the optimal fiscal policy time-consistent through restructuring debt indexed to consumption and indexed to leisure for an economy with endogenous government consumption and public capital. Zhu (1995) showed that the careful management of the maturity of debt indexed to consumption, to the after-tax wage, and to the after-tax return on capital could solve the time-inconsistency problem for an economy with private capital. However, this private capital was not the standard stock of private capital, but it was assumed to have an endogenous rate of utilization so that it was never in inelastic supply. For an economy with a standard stock of private capital, can debt restructuring solve the time-inconsistency problem? The issues of debt with different maturities can outweigh the different degree of distortion that the labor tax has depending on the planning date. However, the management of debt cannot change the non-distorsionary nature of the capital tax at the initial date. Therefore, debt restructuring cannot solve the time-inconsistency of optimal policy for economies with a stock of private capital. This problem has been widely recognized by the literature, and it has led to limit the debt restructuring method to quite simple models. Among them, we find models that have no private capital and display no growth. However, no solution and no appropriate measure of how important is this problem have been provided yet.

This paper investigates the time-inconsistency problem of optimal fiscal policy, abstracting from reputational issues, for an economy with private and public capital. The economy is modeled in an endogenous growth framework where public capital is not only essential for production, but it is also the engine of growth.[1] We assume that governments commit to honoring debt. We first study the policy under full-commitment. In the absence of full-commitment, this policy cannot be made time-consistent through debt restructuring because of the capital levy problem. Following Kydland and Prescott (1977), a natural solution could be to rely on a constant tax rate at all dates. In particular, we consider a zero capital tax rule. We find the full-commitment policy subject to a zero tax rate on capital income at all dates and show that this policy can be made time-consistent through debt restructuring. Therefore, we solve the time-inconsistency problem of fiscal policy through debt restructuring for an economy with private capital and endogenous growth. In

[1] Aschauer (1989) showed the empirical importance of public capital in private production.

this framework we characterize how debt restructuring should be conducted. We find three distinctive properties. First, the presence of the initial capital income makes debt with one-period maturity have a different behavior with respect to debt with longer maturities. Second, given that public capital is endogenous, the issues of debt are not contingent on the stream of government spending. Finally, we find that the amounts of debt can grow over the issuing date but not along the maturing date.

We find that debt restructuring cannot make the optimal capital tax policy time-consistent, but it can make a zero capital tax rule credible. The implications of the zero capital tax constraint are twofold. First, the initial zero capital tax enters into the government problem by shifting the amount of resources that must be raised through distorsionary taxation all over the life-period. Second, the future zero capital taxes translate into an additional constraint fixing a relationship between current and future consumption. This constraint is binding in the short but not in the long-run. These implications affect differently the current and the next government. The next zero capital tax has a first effect on the next government, but a second one on the current government. Moreover, the future zero capital taxes distort more the next than the current government. As this paper shows, the current government can outweigh the incentives to deviate that come from the next zero capital tax by issuing negative debt indexed to consumption with one-period maturity, which enters directly into the present value budget constraint of the next government. In addition, we show that the incentives to deviate from the announced policy for the future dates can be balanced through the issues of debt indexed to consumption and to after-tax wage for all future maturities.

The restriction of zero capital taxes at all dates allows us to find a policy plan that is time-consistent. However, this policy implies a welfare loss by comparison with the full-commitment policy without the zero capital tax restriction. The results of Chari, Christiano and Kehoe (1994) suggest that this welfare loss could be large. In a model with exogenous government spending and no growth, they showed that about 80% of the welfare gains in a Ramsey system comes from the high taxes on the initial capital income. To compute the welfare differential in our framework, we use numerical solution methods. We find that the time-consistent policy is quite close to the full-commitment policy without the restriction of zero capital taxes in terms of both growth and welfare.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 solves the policy plans under full-commitment and under debt restructuring. Section 4 concludes with a summary of the main findings. Finally, the Appendices include proofs and explain the numerical solution method.

4.2 The Model

Our economy is a version of the endogenous growth model with public spending of Barro (1990). This version departs from the original model in modelling the government policy. We consider that government spending takes the form of public investment, which can be financed through time-variant tax rates on labor and capital income and through debt.[2] We assume that public debt can be issued with a sufficiently rich structure in terms of maturity calendar and debt-type variety. More precisely, the government at date t can

[2] The first-best allocation would be attainable if the government could levy taxes on consumption, capital income, and labor income. In that case, the time-inconsistency problem would obviously disappear. As usual in this literature, consumption taxes are excluded so as to ensure distorsionary taxation (see Zhu (1995)).

issue sequences $\{ {}_{t+1}b^c_s,\ {}_{t+1}b^w_s\}_{s=t+1}^{\infty}$, which enter in the economy at the end of date t, of claims on debt indexed to consumption and to after-tax wage at date $s \geq t+1$, respectively.[3] Through the issue of these types of bonds, the government promises debt payments, interest and principal, which can be respectively viewed as additional units of consumption and net labor income that the individual receives at some future date.[4]

We consider an infinite-horizon economy populated by identical individuals. Each individual is endowed with a given initial capital k_0, initial debt claims maturing at date $t \geq 0$, and one unit of time per period that can be either devoted to leisure $1 - l_t$ or to output production l_t. The representative individual derives utility from consumption c_t and leisure so that his objective is to maximize the sum of discounted utilities

$$\sum_{t=0}^{\infty} \beta^t U\left(c_t, 1 - l_t\right), \tag{4.1}$$

with $\beta \in (0, 1)$. The utility function $U(\cdot, \cdot)$ takes the following form:

$$U(c_t, 1 - l_t) = \theta \ln c_t + (1 - \theta) \ln\left(1 - l_t\right), \tag{4.2}$$

where $\theta \in (0, 1)$ measures the importance of consumption relative to leisure. Taking prices and the government policy as given, the consumer maximizes his welfare (4.1) subject to the budget constraint

$$\begin{aligned} & p_t \left[c_t + k_{t+1} + \sum_{s=t+1}^{\infty} \frac{p_s}{p_t} \left({}_{t+1}b^c_s - {}_t b^c_s \right) + \sum_{s=t+1}^{\infty} \frac{p_s}{p_t} q_s \left({}_{t+1}b^w_s - {}_t b^w_s \right) \right] \leq \\ & p_t \left[{}_t b^c_t + (1 - \tau^l_t) w_t \left[l_t + {}_t b^w_t \right] + R_t k_t \right], \end{aligned} \tag{4.3}$$

[3] If debt were indexed to before-tax wage, a government could default on debt payments through changing the labor income tax rate. Hence, another source of time-inconsistency would appear. Therefore, we consider debt indexed to after-tax wage. This variety of debt can be also found in Faig (1994) and Zhu (1995).

[4] Debt indexed to consumption can be identified with Treasury Inflation-Protected Securities that are issued with a 5-, 10-, and 30-year maturity by the U.S. Treasury since 1997. These securities vary with the consumer price index. We may identify debt indexed to after-tax wage with the promise of future social security pensions, which are closely linked to the wage rate.

and the no-Ponzi-game conditions

$$\lim_{t\to\infty}\sum_{s=t}^{\infty} p_s\ {}_tb_s^c = 0,\ \lim_{t\to\infty}\sum_{s=t}^{\infty} p_s q_s\ {}_tb_s^w = 0,\ \lim_{t\to\infty} p_t k_{t+1} = 0. \tag{4.4}$$

Here p_t is the price of a final good at date t, q_t is the price of a bond indexed to after-tax wage in terms of final goods at date t, w_t is the real wage received for the fraction of time that the individual devotes to work at date t, τ_t^l is the labor income tax rate at date t, R_t is the gross return on capital, after tax τ_t^k and depreciation δ_k rates, and r_t is the net return on capital at date t, that is, $R_t = \left(1 + \left(1 - \tau_t^k\right) r_t - \delta_k\right)$. The first-order conditions for this optimization problem are the following:

$$\frac{U_x\left(c_t, 1 - l_t\right)}{U_c\left(c_t, 1 - l_t\right)} = \left(1 - \tau_t^l\right) w_t, \tag{4.5}$$

$$\frac{U_c\left(c_t, 1 - l_t\right)}{U_c\left(c_{t+1}, 1 - l_{t+1}\right)} = \beta\left(1 + \left(1 - \tau_{t+1}^k\right) r_{t+1} - \delta_k\right), \tag{4.6}$$

$$\beta^t \frac{U_c\left(c_t, 1 - l_t\right)}{U_c\left(c_0, 1 - l_0\right)} = \frac{p_t}{p_0},\ \text{and } q_t = \left(1 - \tau_t^l\right) w_t, \tag{4.7}$$

where $U_c\left(c_t, 1 - l_t\right)$ and $U_x\left(c_t, 1 - l_t\right)$ denote the marginal utility with respect to consumption and leisure, respectively. Following this notation, second-order derivatives of the utility function will be denoted by $U_{c_tc_t}$, $U_{c_tx_t}$ and $U_{x_tx_t}$.

The public investment accumulates over time and amounts to the stock of public capital g_t that depreciates at a rate δ_g. This public capital satisfies the next assumptions. First, public capital is publicly-provided. Second, for the purpose of ongoing growth, public investment is tied to production in the following way:

$$g_{t+1} - \left(1 - \delta_g\right) g_t = \varphi_t y_t, \tag{4.8}$$

with $0 \leq \varphi_t \leq 1$ set optimally by the government. Condition (4.8) provides an accumulation rule for public capital that is independent of the accumulation of private capital, allowing thus for transitional dynamics. In addition, public capital must satisfy the government intertemporal budget constraint

$$\sum_{t=0}^{\infty} p_t z_{0t} \geq 0, \tag{4.9}$$

where

$$z_{0t} \equiv \left[\tau_t^l w_t l_t + \tau_t^k r_t k_t - (g_{t+1} - (1-\delta_g)\, g_t) - {}_0b_t^c - q_t\, {}_0b_t^w\right]. \tag{4.10}$$

which is the government "cash-flow". This cash-flow (4.10) at date 0 equals

$$\sum_{s=1}^{\infty} \frac{p_s}{p_0} \left({}_0b_s^c - {}_1b_s^c\right) + \sum_{s=1}^{\infty} \frac{p_s}{p_0} q_s \left({}_0b_s^w - {}_1b_s^w\right), \tag{4.11}$$

which can be viewed either as the excess of tax revenues over public spending and debt payments or as the real value of the net issue of new debt at date 0. In order to allow for sustained constant growth in the long-run, the initial inherited debt $\{ {}_0b_t^c, {}_0b_t^w\}_{t=0}^{\infty}$ must satisfy that both $\frac{{}_0b_t^c}{c_t}$ and ${}_0b_t^w$ become constant in the long-run. Given the initial exogenous conditions ${}_0b_t^c$ and ${}_0b_t^w$ and the endogenous variable $\frac{{}_0b_t^c}{c_t}$, we assume that $\lim_{t\to\infty} {}_0b_t^w = \kappa_w$ and $\lim_{t\to\infty} {}_0b_t^c = \kappa_c$, where κ_c and κ_w are arbitrary finite numbers. The latter is assumed in order to assure that $\frac{{}_0b_t^c}{c_t}$ becomes constant under any possible growth process.[5]

In this economy there is a final good that is produced through the following technology:

$$y_t = f(k_t, l_t, g_t) = A k_t^{\alpha} \left(l_t g_t\right)^{1-\alpha}, \tag{4.12}$$

where $A > 0$ and $\alpha \in (0,1)$. This production function exhibits diminishing returns with respect to each factor but constant returns with respect to k_t and g_t together. Hence, public

[5]Notice that, if we had assumed that the initial inherited debt had a maximum finite maturity, ${}_0b_t^c$ would have become zero in the long-run, and so would have done $\frac{{}_0b_t^c}{c_t}$. Under our assumption of $\lim_{t\to\infty} {}_0b_t^w = \kappa_w$ we allow for asymptotic ${}_0b_t^c$ different from zero that are consistent with sustained constant growth.

capital is an essential input that enhances both private capital and labor marginal products and allows for endogenous growth.

A representative firm produces the final good and maximizes profits given factor prices. The necessary conditions for this optimization program are

$$r_t = f_{k_t} \text{ and } w_t = f_{l_t}, \tag{4.13}$$

where f_{k_t} and f_{l_t} denote the marginal products of capital and labor at date t, respectively. A similar notation will be used for other derivatives of the production function.

Given that the final good can be either consumed or invested, the resource constraint can be written as

$$c_t + k_{t+1} + g_{t+1} \leq Ak_t^{\alpha} \left(l_t g_t\right)^{1-\alpha} + \left(1-\delta_k\right) k_t + \left(1-\delta_g\right) g_t. \tag{4.14}$$

In what follows we define a competitive equilibrium for this economy:

Definition 10 *Given the policy* $\left\{g_{t+1}, \tau_t^k, \tau_t^l\right\}_{t=0}^{\infty}$, *the initial debt* $\{\,{}_0b_t^c,\ {}_0b_t^w\}_{t=0}^{\infty}$ *and the initial public capital* g_0 *and private capital* k_0, *an allocation* $\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}$ *is a competitive equilibrium allocation if and only if there exists a price sequence* $\{p_t, q_t, r_t, w_t\}_{t=0}^{\infty}$ *such that: (i) the representative individual maximizes his welfare* (4.1) *subject to the budget constraint* (4.3) *and the no-Ponzi game conditions* (4.4)*; (ii) factors are paid their marginal products according to equation* (4.13)*; and (iii) all markets clear (equation* (4.14) *holds with equality).*

4.3 The Policy Selection

Once the behavior of private agents has been described, we turn to the policy selection. First, we analyze the policy under full-commitment. Next, we compute the optimal policy under some restrictions and show that debt restructuring can make this policy time-consistent. Both policies are characterized analytically in the short and long-run. Finally, in order to make growth and welfare comparisons, we use numerical solution methods.

4.3.1 The Full-Commitment Policy

For the time being, we assume that future governments commit to the policy chosen by the initial government. Once the government at date 0 selects a plan for all dates $t \geq 0$, future governments will be bound to set the policy that is the continuation of the original plan chosen at date 0. This assumption can be viewed as a full-commitment among the successive governments that makes the optimal policy planned at date 0 sustainable.

Before we setup the government optimization problem, we describe the policy and the competitive equilibrium. When the government chooses a policy, capital and labor tax rates can be positive or negative and can vary over time. However, we assume that capital tax rates are bounded upward by unity.[6] This restriction translates into $\tau_0^k \leq 1$ at date 0 and the following equation:[7]

$$U_{c_t} \geq \beta U_{c_{t+1}} \left(1 - \delta_k\right), \tag{4.15}$$

at all dates $t \geq 1$. Among the conditions for a competitive equilibrium, the following

[6] This upper bound can be justified by means of limited-liability, that is, there is a limit to the capital income that can be taxed.

[7] This equation results from combining $\tau_t^k \leq 1$ and the first-order condition for capital (4.6).

transversality conditions must be satisfied:

$$\lim_{t\to\infty}\sum_{s=t}^{\infty}\beta^s U_{c_s}\ {}_t b_s^c = 0,\ \lim_{t\to\infty}\sum_{s=t}^{\infty}\beta^s U_{c_s}\left(1-\tau_s^l\right) f_{l_s}\ {}_t b_s^w = 0,\ \lim_{t\to\infty}\beta^t U_{ct} k_{t+1} = 0, \tag{4.16}$$

$$\lim_{t\to\infty}\beta^t U_{ct} g_{t+1} = 0. \tag{4.17}$$

Adding the budget constraint (4.3) over time and plugging the transversality conditions (4.16) and the first-order conditions $(4.5)-(4.7)$ and (4.13), we obtain the implementability condition

$$\sum_{t=0}^{\infty}\beta^t\left[\left(c_t - {}_0b_t^c\right)U_c\left(c_t, 1-l_t\right) - \left(l_t + {}_0b_t^w\right)U_x\left(c_t, 1-l_t\right)\right] \leq W_0 U_c\left(c_0, 1-l_0\right), \tag{4.18}$$

where W_0 is the individual's initial capital income, that is $W_0 = R_0 k_0$.

The government at date 0 chooses the initial tax rate on capital income τ_0^k and the sequences $\{c_t, l_t, k_{t+1}, g_{t+1}\}_{t=0}^{\infty}$ so as to maximize the welfare of the representative individual (4.1) subject to the resource constraint (4.14), the implementability condition (4.18), the upper bound on capital tax rates (4.15) and $\tau_0^k \leq 1$, the accumulation rule for public capital (4.8), and the transversality condition (4.17), given initial values for debt $\{\ {}_0b_s^c,\ {}_0b_s^w\}_{s=0}^{\infty}$ and for private k_0 and public capital g_0.[8]

The solution of this problem satisfies constraints (4.14), (4.15), (4.17) and (4.18), and the next first-order conditions for consumption, labor, private and public capital:

$$\begin{aligned}
\mu_{0t} &= W_c\left(c_t, l_t,\ {}_0b_t^c,\ {}_0b_t^w, \Theta_t, \lambda_0\right), &(4.19)\\
f_{l_t}\mu_{0t} &= W_x\left(c_t, l_t,\ {}_0b_t^c,\ {}_0b_t^w, \Theta_t, \lambda_0\right), &(4.20)\\
\mu_{0t} &= \beta\mu_{0t+1}\left(1 + f_{k_{t+1}} - \delta_k\right), &(4.21)\\
\mu_{0t} &= \beta\mu_{0t+1}\left(1 + f_{g_{t+1}} - \delta_g\right). &(4.22)
\end{aligned}$$

[8]When equations (4.14), (4.17) and (4.18) hold, the government budget constraint (4.9) is satisfied.

In these first-order conditions we have

$$W_{c_t} = (1+\lambda_0)\,U_{c_t} + \lambda_0\left[U_{c_tc_t}\left(c_t - {}_0b_t^c + \Theta_t\right) - U_{c_tx_t}\left(l_t + {}_0b_t^w\right)\right],$$

$$W_{x_t} = (1+\lambda_0)\,U_{x_t} + \lambda_0\left[U_{x_tc_t}\left(c_t - {}_0b_t^c + \Theta_t\right) - U_{x_tx_t}\left(l_t + {}_0b_t^w\right)\right],$$

and

$$\Theta_t = \begin{cases} -W_0 + \frac{\phi_{00}}{\lambda_0}, \text{ for } t = 0, \\ \frac{1}{\lambda_0}\left(\phi_{0t} - (1-\delta_k)\,\phi_{0t-1}\right), \text{ for } t > 0, \end{cases}$$

where μ_{0t}, ϕ_{0t} and λ_0 are the Lagrange multipliers associated with constraints (4.14), (4.15) and (4.18) at date t, respectively.[9] Since initial capital revenues are pure rents, the initial capital tax rate is obviously one. The remaining optimal tax rates are obtained from the first-order conditions (4.5) and (4.6).

Given that we will next study the transition and the properties of the balanced growth path (BGP), we first define the following:

Definition 11 *A balanced growth path is an optimal solution* $\{c_t, l_t, k_{t+1}, g_{t+1}\}$ *of the government optimization problem for some initial conditions of debt* $\{ {}_0b_t^c,\ {}_0b_t^w\}_{t=0}^{\infty}$ *and public* g_0 *and private capital* k_0 *such that* l_t *is constant and* c_t*,* k_{t+1}*, and* g_{t+1} *grow all at the same constant rate.*

The transitional dynamics of this model can be attributed to a number of factors, namely, the individual's initial wealth, the upper bound on capital tax rates, and the initial private to public capital ratio. The first two factors are clearly embodied in the first-order conditions $(4.19)-(4.22)$. First, the initial capital income W_0 affects directly decisions that involve variables at date 0 through conditions (4.19) and (4.20) and indirectly decisions at

[9] As pointed out by Lucas and Stokey (1983), second-order conditions are not clearly satisfied because they involve third and second-derivatives of the utility function. Therefore, we assume that an optimal solution interior exists.

all dates through equation (4.18). The initial debt structure $\{\, {}_0b_t^c,\ {}_0b_t^w\}_{t=0}^{\infty}$ affects directly decisions at all dates. Second, the decisions are taken differently depending on whether the restriction on capital taxation (4.15) is binding or not. Finally, the ratio of private to public capital generates transitional dynamics because this ratio cannot adjust instantaneously to its steady state value. This property comes from the specific production function (4.12) and from the fact that private and public capital follow independent accumulation rules.

In the following proposition we describe the dynamics of the labor tax rate, the consumption growth rate γ_{c_t} and labor to their corresponding steady state values (marked with an ss subscript):

Proposition 11 *If* $\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}$ *and* $\left\{g_{t+1}, \tau_t^l, \tau_t^k\right\}_{t=0}^{\infty}$ *are respectively the optimal allocation and the optimal policy under full-commitment, then* $\gamma_{c_t} \leq \gamma_{c_{ss}}$ *for all dates* $t > 1$ *where* $\tau_t^k = 1$.

Moreover, let $(k_0\,/g_0)$ *be close enough to its steady state value and* ${}_0b_t^w = \kappa_w$ *and* ${}_0b_t^c = \kappa_c = 0$ *for all dates* $t > 1$, *then*

(i) $l_t \leq l_{ss}$ *and* $\tau_t^l \leq \tau_{ss}^l$ *when* $(k_t\,/g_t) \geq (k_{ss}\,/g_{ss})$; $l_t \geq l_{ss}$ *when* $(k_t\,/g_t) \leq (k_{ss}\,/g_{ss})$, *provided* $1 \geq \delta_g > \delta_k \geq 0$.

(ii) $\tau_t^l \leq \tau_{ss}^l$ *when* $l_t \leq l_{ss}$, *and* $(k_t\,/g_t) = \frac{\alpha}{1-\alpha}$ *for any* l_t, *provided* $1 \geq \delta_g = \delta_k \geq 0$.

(iii) $l_t \leq l_{ss}$ *and* $\tau_t^l \leq \tau_{ss}^l$ *when* $(k_t\,/g_t) \leq (k_{ss}\,/g_{ss})$; $l_t \geq l_{ss}$ *when* $(k_t\,/g_t) \geq (k_{ss}\,/g_{ss})$, *provided* $1 \geq \delta_k > \delta_g \geq 0$.

Proof. See Appendix 4.5. ■

Proposition 11 characterizes the dynamics of the full-commitment policy and allocation under different assumptions. Those requirements reduce the sources of transition to the upper bound on capital tax rates (4.15). During the first periods of transition, this restriction is binding and the capital tax rate is equal to one. In that time interval the economy exhibits a lower growth rate. The transition also includes some periods in which this upper bound is not binding because lag values of ϕ_{0t} enter into the necessary conditions $(4.19)-(4.22)$. In this case we can also describe the dynamics of the labor income tax rate and labor. However, neither the sign of the capital tax nor the dynamics of the consumption growth rate can be determined for those periods. These transitional dynamics lead to a BGP, which is characterized as follows:

Proposition 12 *If* $\left\{\tau_t^k, \tau_t^l\right\}_{t=0}^{\infty}$ *is the optimal tax policy under full-commitment, then* $\tau_{ss}^k = 0$ *and*

$$\tau_{ss}^l = \frac{\lambda_0 \left[\frac{1+\ {}_0b_{ss}^w}{1-l_{ss}} - \left(\frac{{}_0b^c}{c}\right)_{ss}\right]}{1+\lambda_0\left[\frac{1+\ {}_0b_{ss}^w}{1-l_{ss}}\right]},$$

at the BGP.

Proof. See Appendix 4.5. ■

We obtain the Chamley's (1986) result through Propositions 11 and 12. In the short-run the capital income should be taxed heavily. However, the optimal capital tax rate is zero at the BGP. Propositions 11 and 12 describe the optimal policy planned by the government at date 0. Is this policy sustainable? Under full-commitment, the optimal policy is sustainable independently of the debt structure. The government decides the amount of debt to issue such that the budget constraint (4.9) is satisfied, but it is indifferent about how to allocate this amount of debt among the different maturities or types. Indeed, the government could conduct the optimal policy by issuing only one-period debt indexed to consumption. However, in the absence of full-commitment, future governments will have incentives to select a continuation policy different from the announced plan. Thus, the policy plan chosen at date 0 becomes time-inconsistent.

The incentives to deviate from the announced policy come from the possibility of re-optimizing taking into account the new endowments. These incentives take two forms. First, once savings decisions have been taken, the government would find optimal to default on the current debt payments and to tax the initial capital income as much as possible. Second, given that taxes have a different degree of distortion and that the debt obligations are different, the optimal policy does not longer coincide with the announced plan. We assume that governments commit to honoring debt, but the other remaining forces are still active. Given these incentives to deviate, the optimal policy planned at date 0 is time-inconsistent. Moreover, as we have already argued, debt restructuring cannot make the optimal capital taxation credible because of the capital levy problem.

4.3.2 The Policy under Debt-Commitment

From this section on, we assume that future governments can reconsider both taxation and spending plans, but they commit to honoring debt and are free to redesign the public debt that will be inherited by the next period government. In this framework we investigate under which conditions an optimal policy can be made time-consistent through debt restructuring. The resulting policy will be named the policy under debt-commitment.

A possible and natural solution is to restrict the policy to a capital tax rule. Then, we find the full-commitment policy under this rule and study whether this policy can be made time-consistent or not. In principle, the capital tax rule could be set at any arbitrary value. Following Lucas (1990) and the efficiency of zero capital taxes in the long-run, we propose a zero tax rule on capital income. Therefore, the current and future governments choose their policy subject to a constant capital tax rate equal to zero for all dates. This restriction on capital taxation can be written as $\tau_0^k = 0$ at date 0 and as follows:[10]

$$U_{c_t} = \beta U_{c_{t+1}} \left(1 + f_{k_{t+1}} - \delta_k\right), \tag{4.23}$$

at all dates $t \geq 1$. The initial zero capital tax affects the implementability condition (4.18) by increasing the revenues through distorsionary taxation that the government must raise over its life-period. Moreover, given the different incentives to tax capital in the short and in the long-run, the costs of satisfying the capital tax restriction (4.23) are very high in the short but low in the long-run. Therefore, under a capital tax rule, the time-inconsistency problem of capital taxation does not disappear. We know from the previous section that debt restructuring cannot make the optimal capital taxes credible. The question now is whether debt restructuring can make a zero tax rule on capital credible or not.

We next define the government optimization problem. The government at date 0 chooses the sequences $\{c_t, l_t, k_{t+1}, g_{t+1}\}_{t=0}^{\infty}$ so as to maximize the welfare of the representative individual (4.1) subject to the resource constraint (4.14), the implementability condition (4.18), the zero tax rate constraint on capital income (4.23) and $\tau_0^k = 0$, the accumulation rule for public capital (4.8), and the transversality condition (4.17), given initial debt $\{\,{}_0b_s^c,\ {}_0b_s^w\}_{s=0}^{\infty}$ and initial private k_0 and public capital g_0.

The solution of this problem is determined by constraints (4.14), (4.17), (4.18) and (4.23), and the following necessary conditions for consumption, labor, private capital,

[10]Introduce $\tau_t^k = 0$ and equation (4.13) into the first-order condition for capital (4.6).

and public capital, respectively:

$$\mu_{0t} = W_c\left(c_t, l_t, {}_0b_t^c, {}_0b_t^w, \Theta_{c_t}, \lambda_0\right), \tag{4.24}$$

$$f_{l_t}\mu_{0t} = W_x\left(c_t, l_t, {}_0b_t^c, {}_0b_t^w, \Theta_{x_t}, \lambda_0\right), \tag{4.25}$$

$$\mu_{0t} = \beta\mu_{0t+1}\left(1 + f_{k_{t+1}} - \delta_k\right) - \beta\xi_{0t} f_{k_{t+1}k_{t+1}} U_{c_{t+1}}, \tag{4.26}$$

$$\mu_{0t} = \beta\mu_{0t+1}\left(1 + f_{g_{t+1}} - \delta_g\right) - \beta\xi_{0t} f_{k_{t+1}g_{t+1}} U_{c_{t+1}}, \tag{4.27}$$

where

$$W_{c_t} = \left(1+\lambda_0\right) U_{c_t} + \lambda_0\left[U_{c_tc_t}\left(c_t - {}_0b_t^c + \Theta_{c_t}\right) - U_{c_tx_t}\left(l_t + {}_0b_t^w\right)\right],$$

$$W_{x_t} = \left(1+\lambda_0\right) U_{x_t} + \lambda_0\left[U_{x_tc_t}\left(c_t - {}_0b_t^c + \Theta_{x_t}\right) - U_{x_tx_t}\left(l_t + {}_0b_t^w\right)\right],$$

with

$$\Theta_{c_t} = \begin{cases} -W_0 + \frac{\xi_{00}}{\lambda_0}, \text{ for } t = 0, \\ \frac{1}{\lambda_0}\left(\xi_{0t} - \left(1 + f_{k_t} - \delta_k\right)\xi_{0t-1}\right), \text{ for } t > 0, \end{cases}$$

and

$$\Theta_{x_t} = \begin{cases} -W_0 + \frac{\xi_{00}}{\lambda_0} - f_{kl_0}\frac{U_{c_t}}{U_{c_tx_t}}k_0, \text{ for } t = 0, \\ \frac{1}{\lambda_0}\left(\xi_{0t} - \left(1 + f_{k_t} - \delta_k\right)\xi_{0t-1} - f_{k_tl_t}\frac{U_{c_t}}{U_{c_tx_t}}\xi_{0t-1}\right), \text{ for } t > 0. \end{cases}$$

Here μ_{0t}, λ_0 and ξ_{0t} are the Lagrange multipliers associated with constraints (4.14), (4.18) and (4.23), respectively.[11] We obtain the optimal labor tax rates from the first-order condition (4.5).

We describe next the transition towards the steady state. This transition is driven by the effect of the initial wealth, the zero tax rate constraint on capital income, and the private to public capital ratio. We characterize the transition as follows:

[11] As in the previous section, we assume that an optimal interior solution exists.

Proposition 13 *If* $\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}$ *and* $\left\{g_{t+1}, \tau_t^l\right\}_{t=0}^{\infty}$ *are respectively the optimal allocation and the optimal policy under the zero tax rate constraint on capital income* (4.23)*, then close enough to a BGP it holds that*

(i) $\tau_t^l \leq \tau_{ss}^l$*,* $\gamma_{c_t} \leq \gamma_{c_{ss}}$ *when* $(k_t/g_t) \geq (k_{ss}/g_{ss})$ *and* $l_t \leq l_{ss}$*; and* $l_t \geq l_{ss}$*,* $\gamma_{c_t} \geq \gamma_{c_{ss}}$ *when* $(k_t/g_t) \leq (k_{ss}/g_{ss})$ *provided* $1 \geq \delta_g > \delta_k \geq 0$*.*

(ii) $\tau_t^l \leq \tau_{ss}^l$ *when* $l_t \leq l_{ss}$ *provided* $1 \geq \delta_g = \delta_k \geq 0$*.*

(iii) $\tau_t^l \leq \tau_{ss}^l$ *and* $l_t \leq l_{ss}$ *when* $(k_t/g_t) \leq (k_{ss}/g_{ss})$ *provided* $1 \geq \delta_k > \delta_g \geq 0$*.*

Proof. See Appendix 4.5. ∎

Proposition 13 describes the dynamics of the allocation and policy around a BGP. Here the sources of transitional dynamics amount to the zero tax rate restriction on capital income (4.23). The dynamics of labor and the labor tax rate for this policy are similar to the dynamics of these variables for the policy of the previous section, described respectively in Propositions 13 and 11. By contrast, in this case we cannot conclude whether the economy will exhibit a lower growth rate during the transition or not. This difference clearly hinges on that now capital income cannot be taxed. At the BGP we obtain the following result:

Proposition 14 *If* $\left\{\tau_t^l\right\}_{t=0}^{\infty}$ *are the optimal labor tax rates under a zero tax rate on capital income, then*

$$\tau_{ss}^l = \frac{\lambda_0 \left[\frac{1+ \ {}_0b_{ss}^w}{1-l_{ss}} - \left(\frac{{}_0b^c}{c}\right)_{ss}\right]}{1 + \lambda_0 \left[\frac{1+ \ {}_0b_{ss}^w}{1-l_{ss}}\right]},$$

at the BGP. Moreover, the zero tax rate restriction on capital income (4.23) *is not binding at the BGP, that is,* $\xi_{ss} = 0$*.*

Proof. See Appendix 4.5. ■

We characterize the transition and steady state for an economy under a zero tax rate restriction on capital income in Propositions 13 and 14. An interesting result is that the restriction on capital taxes (4.23) is not binding at the BGP. This result clearly comes from the optimality of a zero capital tax in the long-run.

Let us turn now to the time-inconsistency problem. In this section we have obtained the full-commitment policy under a capital tax rule. We wonder if, in the absence of full-commitment, this policy can be made time-consistent through debt restructuring. We have assumed that governments commit to honoring debt. However, the changes in the debt obligations and in the elasticity of the labor supply over time generate incentives to deviate from the announced policy. Moreover, by restricting our analysis to a zero tax rate on capital income, the capital taxation problem does not vanish. First, a current capital tax is non-distorsionary. Therefore, a zero current capital tax increases the amount of revenues that the government needs to raise by distorsionary taxation over its life-period. This effect is captured by the increase in the value of the Lagrange multiplier λ_0 for the implementability constraint (4.18), which can be viewed as the present value of the government intertemporal budget constraint. Second, future capital taxes are distorsionary, but they are less distorsionary in the short than in the long-run. As a result, the Lagrange multiplier ξ_{0t} for the zero capital tax rate constraint (4.23) is positive in the short-run but zero at the BGP, as Proposition 14 shows. The different efforts to satisfy the capital tax rule enlarge the asymmetry between short and long-run decisions. In the absence of full-commitment, the next period government would reconsider the spending and taxation plans, and the policy plan at date 0 will be time-inconsistent. This time-inconsistency implies that the allocation and the policy described in this section cannot be implemented and the final result would involve a welfare reduction. In order to prevent this welfare loss, we should study whether debt restructuring can make this policy time-consistent or not. We can then state the following:

Proposition 15 *If the sequences $\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}$ and $\left\{g_{t+1}, \tau_t^l\right\}_{t=0}^{\infty}$ are respectively the optimal allocation and the optimal policy under the zero tax rate constraint on capital income* (4.23)*, then it is always possible to choose a debt structure $\{\ {}_1b_t^c,\ {}_1b_t^w\}_{t=1}^{\infty}$ at market prices* (4.7) *such that the continuation sequences $\{c_t, l_t, k_{t+1}\}_{t=1}^{\infty}$ and $\left\{g_{t+1}, \tau_t^l\right\}_{t=1}^{\infty}$ of the same allocation and policy are a solution for the government problem when it is reconsidered at date* 1*. This could be done through the following debt structure:*

$$ {}_1b_t^c - {}_0b_t^c = \left[\frac{\lambda_0}{\lambda_1} - 1\right] {}_0b_t^c + \Gamma_t^c, \tag{4.28}$$

$$ {}_1b_t^w - {}_0b_t^w = \left[\frac{\lambda_0}{\lambda_1} - 1\right] \left({}_0b_t^w + 1\right) + \Gamma_t^w, \tag{4.29}$$

where

$$\Gamma_t^c = \begin{cases} -\left[\frac{\lambda_1 k_1 - \xi_{00}}{\lambda_1}\right]\left(1 + f_{k_t} - \delta_k\right),\ \textit{for date } t = 1, \\ 0,\ \textit{for all dates } t > 1, \end{cases}$$

and

$$\Gamma_t^w = \begin{cases} \left[\frac{\lambda_1 k_1 - \xi_{00}}{\lambda_1}\right]\left(\frac{\theta}{1-\theta}\right)\frac{(1-l_t)^2}{c_t} f_{k_t l_t},\ \textit{for date } t = 1. \\ 0,\ \textit{for all dates } t > 1. \end{cases}$$

By induction, the same is true for all later periods.

Proof. See Appendix 4.5. ■

The present value of the new issues of debt at date 0 is obtained from the government budget constraint (4.11). There are infinite pairs of sequences $\{ {}_1b_t^c, {}_1b_t^w\}_{t=1}^{\infty}$ that satisfy this constraint, however just one enables the government to make the optimal policy time-consistent. Proposition 15 guarantees that under that debt structure, the policy plan chosen by the government at date 0 will be sustainable. Hence, for economies with private capital and endogenous growth, the full-commitment policy under a zero tax rate constraint on capital income can be made time-consistent through debt restructuring.

How is the debt structure that makes the optimal policy time-consistent? For an economy without capital, Lucas and Stokey (1983) presented some examples to characterize the optimal issues of debt. They obtained that the debt structure ensuring time-consistency was contingent on the exogenous government spending and had a constant pattern with respect to the inherited debt independently of the maturity. In our model this debt structure cannot be completely characterized because the sign of $(\lambda_1 k_1 - \xi_{00})$ is unknown. However, we can infer three distinctive properties. First, given that our public spending is endogenous, the debt structure is not contingent on the stream of government spending. Second, from inspection of conditions (4.24) and (4.25), it results that the issues of debt indexed to consumption ${}_rb_t^c$ at date r with maturity $t \geq r$ can grow over time r. In principle, this debt could also grow along its maturity t. However, from condition (4.28), we know that ${}_rb_t^c$ becomes constant for long maturities t given that we assumed $\lim_{t\to\infty} {}_0b_t^w = \kappa_w$ to allow for constant sustained growth. Third, equations (4.28) and (4.29) show that debt indexed to consumption and to after-tax-wage maturing at all dates $t > 1$ should follow a constant pattern with respect to the inherited debt. However, debt maturing at date 1 should follow a different pattern. These asymmetric issues of debt reflect the existing asymmetry in the set of first-order conditions (4.24) − (4.27). This different behavior in the debt maturing at date 1 shows that, for economies with private capital, the debt structure that ensures time-consistency must take into account the effect of the initial capital income.

In this section the optimal plan subject to a capital tax rule has been made time-consistent. However, it is not known yet how desirable is this plan. A related paper to ours is that by Chari, Christiano and Kehoe (1994), who compared an economy with and without a zero tax rate restriction on capital income. The economy without this constraint exhibits high initial capital taxes followed by a zero tax from then on. They found that most of the welfare gains come from the initial capital taxes. Consequently, they argued that, since the temptation to deviate from the announced future zero capital taxes is so large, the time-inconsistency problem of capital taxation must be quantitatively very severe. In our paper the policy plan cannot make use of capital taxation. This may suggest that our restricted tax policy, though time-consistent, could imply high welfare losses. In the next section we use numerical solution methods to answer this question.

4.3.3 Numerical Solution

In order to compare the debt-commitment policy with the full-commitment policy, we solve the two policies numerically.[12] We use the eigenvalue-eigenvector decomposition method, suggested by Novales et al. (1999) and based on Sims (1998). This method consists of the following steps. First, we choose some parameter and initial values. Second,

[12] All simulations are carried out with the program GAUSS-386.

the conditions and constraints of the economy are transformed so that they are functions of either variables or ratios that are constant at the BGP. We find the steady state values, as Table 4 shows. We linearize all equations around the steady state and find the unstable eigenvalues. Next, by imposing orthogonality between each eigenvector associated with an unstable eigenvalue of the linear system and the variables of the system, we obtain some equations, known as stability conditions. Finally, we impose these conditions into the original non-linear model and compute a numerical solution.

[Insert Table 4 about here.]

From the parameter values used in the literature, we choose the following. The discount rate β is 0.99. The coefficient A in the production function (4.12) equals 0.48. The parameter α is 0.25 as in Barro (1990). Depreciation rates for private δ_k and public capital δ_g are 0.025 and 0.03, respectively.[13] The preference parameter θ is 0.3 so as to have reasonable values for leisure. Initial values for private and public capital are respectively 15 and 45. The initial debt takes the value zero for all maturities, that is, ${}_0b_s^c = 0$ and ${}_0b_s^w = 0$ for all maturities $s \geq 0$.

For the same parameters and initial conditions, we obtain series for the full-commitment and for the debt-commitment policy.[14] Here we report the main results. As Figure 5 shows, the policy under debt-commitment yields a higher growth rate of consumption in the short-run. In the long-run, the growth rate under debt restructuring approaches from below the rate attained under full-commitment. The growth rates at the BGP are 3.83 and 3.84 per cent for the full-commitment and for the debt-commitment policy, respectively.

[13]Since public capital is provided without charge, it is expected to suffer a faster depreciation.

[14]We check the second-order conditions of the corresponding optimization problems in this numerical exercise.

[Insert Figure 5 about here.]

This similarity between long-run growth rates suggests that the differences in welfare may not be so dramatic. Numerically, the welfare (4.1) takes values of 76.32 and 73.43 for the full-commitment and for the debt-commitment policy, respectively. By comparison with full-commitment, the debt-commitment policy involves a welfare reduction of only 3.79%. Therefore, the debt-commitment policy seems quite close to the full-commitment in terms of both growth and welfare.[15] We could question if this result rests on the existence of the upper bound on capital tax rates (4.15) of the full-commitment policy. Taking the same parameter values and initial conditions, the full-commitment policy without that upper bound yields a 4.27% growth rate and welfare sized by 84.37. In addition, the first-best policy could allow us to test whether this numerical closeness is significant or not. For the same set of parameter and initial values, the first-best policy yields a 13.87% growth rate and a welfare value of 249.21. These results confirm that the debt-commitment policy and the full-commitment policy are very close both in growth and in welfare terms.

Our results contrast with Chari, Christiano and Kehoe's (1994) findings. For an economy with exogenous government spending and no growth, they found that about 80% of the welfare gains are due to the high initial capital taxes. However, we obtain a welfare gain of only 3.79%. This difference may come from two facts. First, our model allows for endogenous growth, so future allocations play a more important role for welfare. Second, our taxes finance a productive public investment rather than an exogenous stream of government spending. By comparison with lump-sum taxation, our tax structure distorts the individual decisions and reduces welfare. The zero tax rate restriction on capital income

[15]This result holds for different changes in parameter and initial values.

makes the existing tax structure more distorsionary. However, if the government spending financed through these taxes is endogenous rather than exogenous, the final tax structure is, intuitively, less distorsionary. Moreover, in the present model public spending may be more important for welfare than the way of financing it. In fact, the stream of public investment behaves similarly for the full-commitment and the debt-commitment policy; in the short-run, the public capital rate of growth is higher under full-commitment, this inequality reverses in the medium term and, as Figure 6 shows, the two become quite similar in the long-run.

[Insert Figure 6 about here.]

As Figures $7-9$ show, the way of financing this spending differs under the two policies. The tax rates on capital income under full-commitment are very high in the first periods and zero from then on. Labor taxes are negative in the initial period and positive afterwards. For the debt-commitment policy, capital taxes are zero at all dates. Labor tax rates are higher in the short-run and approach the labor tax rates under full-commitment in the long-run. In order to spread the excess of the burden, governments issue debt in the following way. In the first period, the policy under full-commitment involves a cash flow deficit that becomes surplus after few periods. For the debt-commitment policy, the government has a cash flow surplus, which vanishes in the medium and the long-run.

[Insert Figures 7, 8 and 9 about here.]

Finally, it would be interesting to understand the properties of the debt structure that ensures time-consistency. As we have just mentioned, the government under debt-commitment runs a surplus at date 0. Since we depart from zero initial debt holdings, a surplus implies that the government issues negative claims. Figure 10 and Table 5 show how these claims are allocated among the different types of debt and maturities. First, the government should issue positive debt indexed to after-tax-wage maturing at all dates $s \geq 1$. These issues are higher for debt with one-period maturity and take the same value for debt maturing at date 2 and later on. Therefore, the surplus is carried out through issues of negative bonds indexed to consumption. These issues consist of only one-period debt. Therefore, the government at date 0 issues a negative amount of debt indexed to consumption that matures at date 1. Thus, the incentives to deviate from the policy announced for the current date 1 are balanced by the issues of one-period bonds indexed to consumption and indexed to after-tax wage. Given the way the initial capital income and the debt indexed to consumption enter into the implementability condition of the government at date 1, equation (4.39), these issues of negative debt indexed to consumption are intended to compensate the initial wealth effect. Finally, the issues of positive debt indexed to after-tax wage maturing at all dates $s \geq 2$ outweigh the incentives to deviate from the announced policy for those future dates.

[Insert Figure 10 and Table 5 about here.]

4.4 Conclusions

This paper has investigated the role of debt restructuring in the time-inconsistency problem of optimal fiscal policy for an economy with private capital and endogenous growth achieved via public capital. Given the important role that the fiscal policy plays in the growth process, providing a solution to the time-inconsistency problem is crucial. In order to overcome the capital levy problem, governments choose the optimal policy subject to a zero capital tax at all dates. Since the costs of meeting this restriction are much higher in the short than in the long-run, the capital levy problem does not disappear. We have shown that debt restructuring can make the optimal policy subject to this constraint on capital taxes time-consistent. Moreover, we have characterized how debt restructuring should be conducted for economies with private capital and endogenous growth.

Given that the debt-commitment policy has been found subject to a restriction on capital taxes, an important question is how far this policy is with respect to the full-commitment policy. We have solved both policies numerically and we have found that the debt-commitment policy is quite close both in growth and in welfare terms to the full-commitment policy without the restriction of zero capital taxes. In this sense, we can also argue that the time-inconsistency of capital taxation is not be quantitatively so severe.

We have shown the importance of both rules and debt in relation to the time-inconsistency problem. The previous literature showed that debt restructuring could not provide time-consistency for economies with a stock of private capital. We have reconsidered this issue and we have established that, for economies with private capital, a time-consistent policy plan requires both a debt-commitment and a rule on capital taxation. Thus, an implication of this analysis is that of Kydland and Prescott (1977). In relation to the debt restructuring method, we can state as they do that "reliance on policies such as ... constant tax rates constitute a safer course of action." Regarding the importance of debt, we have shown that the optimal management of debt constitutes an incentive device to implement the announced policy.

We will next discuss two possible extensions. First, the main source of time-inconsistency for our economy is capital taxation. For the purposes of gaining time-consistency, governments have been constrained to a constant zero capital tax at all dates. Another approach could be to impose a one-period commitment to capital income tax rates, that is, that the initial capital tax is inherited from the previous government. Then, we could study whether debt restructuring could make the resulting policy time-consistent or not. From our results and those of Zhu (1995), it could be the case that debt restructuring can solve the time-inconsistency problem, but another debt instrument may be necessary, namely, debt indexed to the after-tax return to capital.

Second, we have assumed that governments can issue debt with a sufficiently rich debt structure. However, given that the government's assets menu could be limited, it would be interesting to analyze what the government could do better without access to a rich composition of debt. For a finite-horizon economy, Rogers (1989) studied the time-inconsistency problem when the structure of debt was not sufficiently rich. In this context, she found that the enrichment of the debt structure could limit the costs of time-inconsistency. However, an infinite-horizon economy is very different from a finite-horizon one in relation to time-inconsistency issues. For an infinite-period economy, reputational forces come into play. There it would be very interesting to see how reputation and debt restructuring interrelate. The properties of the best sustainable policy would illustrate whether debt restructuring matters or not and how much.

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4.5 Proofs of the Propositions

Proof of Proposition 11

Equate the RHS of equations (4.21) and (4.22) and solve for labor

$$l_t = \left[\frac{\delta_g - \delta_k}{A}\right]^{\frac{1}{1-\alpha}} * X_t^{-\frac{1}{1-\alpha}},$$

with

$$X_t = \left[(1-\alpha)\left(\frac{k_t}{g_t}\right)^{\alpha} - \alpha\left(\frac{k_t}{g_t}\right)^{-(1-\alpha)}\right]. \tag{4.30}$$

If $\delta_g > \delta_k$, it results that $\frac{\partial X_t}{\partial(k_t/g_t)} > 0$ and $\frac{\partial l_t}{\partial X_t} < 0$. Hence, $\frac{\partial l_t}{\partial(k_t/g_t)} < 0$ by the chain rule. Otherwise, if $\delta_g < \delta_k$, we obtain $\frac{\partial l_t}{\partial(k_t/g_t)} < 0$. If $\delta_g = \delta_k$, $(k_t/g_t) = \frac{\alpha}{1-\alpha}$. Combining equations (4.19) and (4.20) with (4.5), the labor income tax rate is

$$\tau_t^l = \frac{\lambda_0\left[\frac{1}{1-l_t} + \frac{{}_0b_t^w}{1-l_t} - \frac{{}_0b_t^c}{c_t}\right] - \left[\frac{1}{\beta}\frac{\phi_{ot-1}}{c_{t-1}} - \frac{\phi_{ot}}{c_t}\right]}{1+\lambda_0\left[\frac{1}{1-l_t} + \frac{{}_0b_t^w}{1-l_t}\right]}, \tag{4.31}$$

which becomes

$$\tau_t^l = \frac{\lambda_0\left[\frac{1}{1-l_t} + \frac{\kappa_w}{1-l_t}\right] - \left[\frac{1}{\beta}\frac{\phi_{ot-1}}{c_{t-1}} - \frac{\phi_{ot}}{c_t}\right]}{1+\lambda_0\left[\frac{1}{1-l_t} + \frac{\kappa_w}{1-l_t}\right]},$$

where $\left[\frac{1}{\beta}\left(\phi_{ot-1}/c_{t-1}\right) - \left(\phi_{ot}/c_t\right)\right] \geq 0$ given the initial debt and that (ϕ_{ot}/c_t) approaches zero from above. Now, it is clear that when l_t approaches its steady state value from below so does τ_t^l. Finally, by simple inspection of equation (4.15), it results that, if $\tau_t^k = 1$, we get $\gamma_{c_t} < \gamma_{c_{ss}}$. ■

Proof of Proposition 12

First, we show that the capital tax rate is zero at the BGP. Considering equations (4.2) and (4.19), we can write the first-order condition for capital (4.21) as

$$W_{c_t} = \beta W_{c_{t+1}} \left(1 + f_{kt+1} - \delta_k\right), \tag{4.32}$$

where $W_{c_t} = U_{c_t} H_{c_t}$, with

$$H_{c_t} = 1 - \lambda_0 \left[\frac{{}_0b_t^c}{c_t} - \frac{1}{\lambda_0} \left(\frac{\phi_{0t}}{c_t} - (1-\delta_k) \frac{\phi_{0t-1}}{c_{t-1}} \frac{c_{t-1}}{c_t} \right) \right]. \tag{4.33}$$

All terms in equation (4.33) are either variables or ratios that are constant at the BGP. Therefore, $H_{c_t} = H_c$ and, thus, $\left(W_{c_t}/W_{c_{t+1}}\right) = \left(U_{c_t} H_c / U_{c_{t+1}} H_c\right) = \left(U_{c_t}/U_{c_{t+1}}\right)$ in the long-run. Comparing conditions (4.6) and (4.32), it is obvious that the capital tax rate is zero at the BGP. Note also that, as the capital tax rate becomes zero, the upper bound on capital tax rates will not be binding and the multiplier ϕ_{0t} will be zero. Finally, given that (ϕ_{ot}/c_t) is zero in the long-run, the labor income tax rate (4.31) becomes

$$\tau_{ss}^l = \frac{\lambda_0 \left[\frac{1+ {}_0b_{ss}^w}{1-l_{ss}} - \left(\frac{{}_0b^c}{c} \right)_{ss} \right]}{1 + \lambda_0 \left[\frac{1+ {}_0b_{ss}^w}{1-l_{ss}} \right]}. \ \blacksquare \tag{4.34}$$

Proof of Proposition 13

We equate conditions (4.26) and (4.27) and solve for labor

$$l_{t+1} = \left[\frac{\delta_g - \delta_k}{A} \right]^{\frac{1}{1-\alpha}} * Q_t^{-\frac{1}{1-\alpha}},$$

where

$$Q_t = X_t - \frac{\frac{\xi_{0t-1}}{c_{t-1}}}{\left(1 + \frac{1}{\beta}\frac{\xi_{0t-1}}{c_{t-1}} - \frac{\xi_{0t}}{c_t}\right)} \left[\alpha(1-\alpha) \frac{c_{t-1}}{c_t} \frac{c_t}{k_t} \left(\left(\frac{k_t}{g_t}\right)^{\alpha} + \left(\frac{k_t}{g_t}\right)^{-(1-\alpha)} \right) \right].$$

We know from equation (4.30) that $\frac{\partial X_t}{\partial(k_t/g_t)} > 0$. Since (ξ_{0t}/c_t) approaches zero from above, we can guarantee that around a BGP, if $\delta_g > (<)\ \delta_k$ and $(k_t/g_t) \leq (k_{ss}/g_{ss})$, then $Q_t \leq Q_{ss}$ and $l_t \geq (\leq)\ l_{ss}$. From equation (4.23), if $\delta_g > \delta_k$, we get $\frac{\partial \gamma_{c_t}}{\partial(k_t/g_t)} < 0$. When $\delta_g = \delta_k$, the dynamics of (k_t/g_t) cannot be determined. Combining equations (4.24) and (4.25) with (4.5), the labor income tax rate is

$$\tau_t^l = \frac{\lambda_0 \left[\frac{1+\ {}_0b_t^w}{1-l_t} - \frac{{}_0b_t^c}{c_t}\right] - \left[\frac{1}{\beta}\frac{\xi_{ot-1}}{c_{t-1}} - \frac{\xi_{ot}}{c_t}\right]}{1+\lambda_0 \left[\frac{1+\ {}_0b_t^w}{1-l_t}\right] - \frac{\xi_{ot-1}}{c_{t-1}}\frac{c_{t-1}}{c_t}\left(\frac{\theta}{1-\theta}\right)(1-l_t)\,\alpha\,(1-\alpha)\,A\left(\frac{k_t}{g_t}\right)^{\alpha-1} l_t^{-\alpha}}, \tag{4.35}$$

which, under the assumptions on the initial debt, becomes

$$\tau_t^l = \frac{\lambda_0 \left[\frac{1+\kappa_w}{1-l_t}\right] - \left[\frac{1}{\beta}\frac{\xi_{ot-1}}{c_{t-1}} - \frac{\xi_{ot}}{c_t}\right]}{1+\lambda_0 \left[\frac{1+\ \kappa_w}{1-l_t}\right] - \frac{\xi_{ot-1}}{c_{t-1}}\frac{c_{t-1}}{c_t}\left(\frac{\theta}{1-\theta}\right)(1-l_t)\,\alpha\,(1-\alpha)\,A\left(\frac{k_t}{g_t}\right)^{\alpha-1} l_t^{-\alpha}},$$

where $\left[\frac{1}{\beta}\left(\xi_{0t-1}/c_{t-1}\right) - \left(\xi_{0t}/c_t\right)\right] \geq 0$ given that (ξ_{0t}/c_t) approaches zero from above. Obviously, when l_t approaches its steady state value from below so does τ_t^l. ■

Proof of Proposition 14

First, we prove that ξ_{0t} is zero at the BGP. Given the utility function (4.2) and condition (4.19), the first-order condition for capital (4.26) can be written as

$$W_{c_t} = \beta W_{c_{t+1}}\left(1 + f_{kt+1} - \delta_k\right) - \beta \xi_{0t} f_{k_{t+1}k_{t+1}} U_{c_{t+1}}, \tag{4.36}$$

where $W_{c_t} = U_{c_t} H_{c_t}$, with

$$H_{c_t} = 1 + \lambda_0 \left[\frac{{}_0b_t^c}{c_t} - \frac{1}{\lambda_0}\left(\frac{\xi_{0t}}{c_t} - \left(1 + f_{k_t} - \delta_k\right)\frac{\xi_{0t-1}}{c_{t-1}}\frac{c_{t-1}}{c_t}\right)\right].$$

It can be checked that $H_{c_t} = H_c$ at the BGP. Thus, equation (4.36) becomes

$$U_{c_t} = \beta U_{c_{t+1}}\left[R_{t+1} + \frac{\xi_{0t}}{c_t}\frac{c_t}{c_{t+1}}\frac{c_{t+1}}{k_{t+1}}\alpha\,(1-\alpha)\,\frac{A}{H_c}\left(\frac{k_{t+1}}{g_{t+1}}\right)^{\alpha-1} l_{t+1}^{1-\alpha}\right]. \tag{4.37}$$

Note that (c_{t+1}/k_{t+1}),(c_{t+1}/c_t), (k_{t+1}/g_{t+1}), and l_{t+1} are different from zero. Combining equations (4.23) and (4.37), the ratio (ξ_{0t}/c_t) must be zero at the BGP. Using $(\xi_{0t}/c_t) = 0$, the zero tax rate constraint (4.23) is already satisfied through equation (4.37). Hence, ξ_{0t} is zero at the BGP.

When (ξ_{0t}/c_t) approaches zero, the labor income tax rate (4.35) becomes

$$\tau^l_{ss} = \frac{\lambda_0 \left[\frac{1+\ {}_0b^w_{ss}}{1-l_{ss}} - \left(\frac{{}_0b^c}{c}\right)_{ss}\right]}{1 + \lambda_0 \left[\frac{1+\ {}_0b^w_{ss}}{1-l_{ss}}\right]}, \tag{4.38}$$

at the BGP. Note that, since the steady state labor and the Lagrange multiplier λ_0 for the implementability constraint (4.18) for the debt-commitment policy are different from those for the full-commitment policy, then their corresponding steady state labor tax rates (4.38) and (4.34) are also different. ■

Proof of Proposition 15

We consider the policy plans for the governments at date 0 and at date 1. If both plans can be solved for the same allocation, the solution can be generalized for all later dates and, hence, the policy plan at date 0 is made time-consistent. The government at date 0 can make its policy plan time-consistent by selecting a debt structure $\{\ {}_1b^c_t,\ {}_1b^w_t\}_{t=1}^{\infty}$ such that the same allocation and policy $\left\{c_t, l_t, k_{t+1}, g_{t+1}, \tau^l_t\right\}_{t=1}^{\infty}$ solve both optimization problems at date 0 and at date 1. Therefore, the issues of debt at date 0 can be chosen in such a way that the sequence $\left\{c_t, l_t, k_{t+1}, g_{t+1}, \tau^l_t\right\}_{t=1}^{\infty}$ that solves the constraints and the first-order conditions of the plan at date 0 solves also the corresponding constraints and the first-order conditions of the plan at date 1. Let us now present the two policy plans. When the government at date 0 plans a policy for all dates $t \geq 0$, the policy and allocation solve the set of constraints

$$\sum_{t=0}^{\infty} \beta^t \left[U_{c_t} \left(c_t - {}_0b_t^c \right) - U_{x_t} \left[l_t + {}_0b_t^w \right] \right] \leq U_{c_0} W_0, \tag{4.39}$$

$$c_t + k_{t+1} + g_{t+1} \leq A k_t^{\alpha} \left(l_t g_t \right)^{1-\alpha} + \left(1 - \delta_k \right) k_t + \left(1 - \delta_g \right) g_t, \text{ for all } t \geq 0, \tag{4.40}$$

$$U_{c_t} = \beta U_{c_{t+1}} \left(1 + f_{k_{t+1}} - \delta_k \right), \text{ for all } t \geq 0, \tag{4.41}$$

the first-order conditions for the individual

$$U_{x_t} = \left(1 - \tau_t^l \right) U_{c_t} f_{l_t}, \text{ for all } t \geq 0, \tag{4.42}$$

and for the government

$$W_x \left(c_t, l_t, {}_0b_t^c, {}_0b_t^w, \Theta_{x_t}, \lambda_0 \right) = f_{l_t} W_c \left(c_t, l_t, {}_0b_t^c, {}_0b_t^w, \Theta_{c_t}, \lambda_0 \right), \text{ for all } t \geq 0, \tag{4.43}$$

$$\mu_{0t} = \beta \mu_{0t+1} \left(1 + f_{k_{t+1}} - \delta_k \right) - \beta \xi_{0t} f_{k_{t+1} k_{t+1}} U_{c_{t+1}}, \text{ for all } t \geq 0, \tag{4.44}$$

$$\mu_{0t} = \beta \mu_{0t+1} \left(1 + f_{g_{t+1}} - \delta_g \right) - \beta \xi_{0t} f_{k_{t+1} g_{t+1}} U_{c_{t+1}}, \text{ for all } t \geq 0. \tag{4.45}$$

Equations (4.39)-(4.45) form the system that the government at date 0 solves when announcing its policy plan.

When the government at date 1 selects a policy for all dates $t \geq 1$, this policy and the corresponding allocation satisfy constraints

$$\sum_{t=1}^{\infty} \beta^t \left[U_{c_t} \left(c_t - {}_1b_t^c \right) - U_{x_t} \left[l_t + {}_1b_t^w \right] \right] \leq U_{c_1} W_1, \tag{4.46}$$

$$c_t + k_{t+1} + g_{t+1} \leq A k_t^{\alpha} \left(l_t g_t \right)^{1-\alpha} + \left(1 - \delta_k \right) k_t + \left(1 - \delta_g \right) g_t, \text{ for all } t \geq 1, \tag{4.47}$$

$$U_{c_t} = \beta U_{c_{t+1}} \left(1 + f_{k_{t+1}} - \delta_k \right), \text{ for all } t \geq 1, \tag{4.48}$$

the first-order conditions for the individual

$$U_{x_t} = \left(1 - \tau_t^l \right) U_{c_t} f_{l_t}, \text{ for all } t \geq 1, \tag{4.49}$$

and the first-order conditions for the government

$$W_x\left(c_t, l_t,\ {}_1b_t^c,\ {}_1b_t^w, \Theta_{x_t}, \lambda_1\right) = f_{l_t} W_c\left(c_t, l_t,\ {}_1b_t^c,\ {}_1b_t^w, \Theta_{c_t}, \lambda_1\right), \text{ for all } t \geq 1, \tag{4.50}$$

$$\mu_{1t} = \beta\mu_{1t+1}\left(1 + f_{k_{t+1}} - \delta_k\right) - \beta\xi_{1t} f_{k_{t+1}k_{t+1}} U_{c_{t+1}}, \text{ for all } t \geq 1, \tag{4.51}$$

$$\mu_{1t} = \beta\mu_{1t+1}\left(1 + f_{g_{t+1}} - \delta_g\right) - \beta\xi_{1t} f_{k_{t+1}g_{t+1}} U_{c_{t+1}}, \text{ for all } t \geq 1. \tag{4.52}$$

The government at date 1 solves equations (4.46)-(4.52) when choosing its policy plan.

We show next that the sequence $\left\{c_t, l_t, k_{t+1}, g_{t+1}, \tau_t^l\right\}_{t=1}^{\infty}$ that solves the policy plan at date 0 for all dates $t \geq 1$ can solve the policy plan at date 1. First, since the sequence $\left\{c_t, l_t, k_{t+1}, g_{t+1}, \tau_t^l\right\}_{t=1}^{\infty}$ solves constraints $(4.40) - (4.42)$ for all dates $t \geq 0$, it is also a solution for equations $(4.47) - (4.49)$. Second, to solve equation (4.50) for the same allocation, we need one debt instrument at each period. Let us make equations (4.51) and (4.52) time-consistent. Equating conditions (4.51) and (4.52), ξ_t can be expressed as $\mu_{t+1}\left[\frac{R_{t+1} - \left(1 + f_{g_{t+1}} - \delta_g\right)}{U_{c_{t+1}}\left(f_{k_{t+1}k_{t+1}} - f_{k_{t+1}g_{t+1}}\right)}\right]$. Then, equations (4.51) and (4.52) become

$$\mu_t = \beta\mu_{t+1}\left[R_{t+1} - f_{k_{t+1}k_{t+1}}\left[\frac{R_{t+1} - \left(1 + f_{g_{t+1}} - \delta_g\right)}{f_{k_{t+1}k_{t+1}} - f_{k_{t+1}g_{t+1}}}\right]\right],$$

and

$$\mu_t = \beta\mu_{t+1}\left[R_{t+1} - f_{k_{t+1}g_{t+1}}\left[\frac{R_{t+1} - \left(1 + f_{g_{t+1}} - \delta_g\right)}{f_{k_{t+1}k_{t+1}} - f_{k_{t+1}g_{t+1}}}\right]\right],$$

respectively. Therefore, once μ_{1t} and μ_{0t} take the same value, the same allocation solves the two equations. In order to make μ_{1t} take that value, an extra debt instrument for each date is needed. Note that, when $\mu_{1t} = \mu_{0t}$, we get $\xi_{1t} = \xi_{0t}$ for the same allocation. So far, two debt instruments are needed in order to solve all equations but constraint (4.46). The path for this debt is a function of λ_1. Once the government at date 0 finds this function, it is imposed into the budget constraint (4.3), which leads to a specific debt structure. Hence, by Walras' law, the implementability condition (4.46) also holds. Thus, under that debt structure, the continuing allocation and policy planned at date 0 solves the policy plan at date 1.

Let us now find the debt structure $\{\ {}_1b_t^c,\ {}_1b_t^w\}_{t=1}^{\infty}$ that provides time-consistency. The issues of four types of debt must be found: (i) debt indexed to consumption maturing at the first date ${}_1b_1^c$; (ii) debt indexed to consumption maturing at the second date and later $\{\ {}_1b_t^c\}_{t=2}^{\infty}$; (iii) debt indexed to after-tax-wage with one-period maturity ${}_1b_1^w$; and (iv) the issues of debt indexed to after-tax-wage maturing at the second and later dates $\{\ {}_1b_t^w\}_{t=2}^{\infty}$.

We will next find the issues of debt indexed to after-tax-wage with two-period and higher maturities, that is, $\{\ {}_1b_t^w\}_{t=2}^{\infty}$. The first-order conditions for consumption and leisure under plans (4.43) and (4.50) can be written respectively as follows:

$$W_x\left(c_t, l_t,\ {}_0b_t^c,\ {}_0b_t^w, \Theta_{x_t}, \lambda_0\right) - f_{l_t} W_c\left(c_t, l_t,\ {}_0b_t^c,\ {}_0b_t^w, \Theta_{c_t}, \lambda_0\right) \;=\; 0. \tag{4.53}$$

$$W_x\left(c_t, l_t,\ {}_1b_t^c,\ {}_1b_t^w, \Theta_{x_t}, \lambda_1\right) - f_{l_t} W_c\left(c_t, l_t,\ {}_1b_t^c,\ {}_1b_t^w, \Theta_{c_t}, \lambda_1\right) \;=\; 0. \tag{4.54}$$

Equating the LHS of equations (4.53) and (4.54), we find

$$\begin{aligned}
&[\lambda_0 - \lambda_1]\left(\left(U_{x_t} - f_{l_t}U_{c_t}\right) + \left(U_{c_t x_t} - f_{l_t}U_{c_t c_t}\right)c_t - \left(U_{x_t x_t} - f_{l_t}U_{c_t x_t}\right)l_t\right. \\
&-\left[\tfrac{\xi_{0t-1}-\xi_{1t-1}}{\lambda_0-\lambda_1}\right] U_{c_t} f_{k_t l_t}\Big) = -\left(U_{c_t x_t} - f_l U_{c_t c_t}\right)\left(\lambda_1\ {}_1b_t^c - \lambda_0\ {}_0b_t^c + \left(\left(\xi_{0t} - \xi_{1t}\right)\right.\right. \\
&-\left(1 + f_{k_{t-1}} - \delta_k\right)\left(\xi_{0t-1} - \xi_{1t-1}\right)\big)\big) - \left(U_{x_t x_t} - f_{l_t}U_{c_t x_t}\right)\left[\lambda_1\ {}_1b_t^w - \lambda_0\ {}_0b_t^w\right].
\end{aligned} \tag{4.55}$$

We divide equation (4.55) by $-\left(U_{x_t x_t} - f_{l_t}U_{c_t x_t}\right)\lambda_1$. Taking into account that $\xi_{0t} = \xi_{1t}$ and, then, adding ${}_0b_t^w\left[\frac{\lambda_0}{\lambda_1} - 1\right]$, we obtain

$$\begin{aligned}
&\left[\tfrac{U_{x_t x_t} - f_{l_t}U_{c_t x_t}}{U_{c_t x_t} - f_{l_t}U_{c_t c_t}}\right]\left[\left[\tfrac{\lambda_0}{\lambda_1} - 1\right]\left[l_t +\ {}_0b_t^w - \tfrac{U_{x_t} - f_{l_t}U_{c_t}}{U_{x_t x_t} - f_{l_t}U_{c_t x_t}}\right] - \left[{}_1b_t^w -\ {}_0b_t^w\right]\right] = \\
&\qquad\left[{}_1b_t^c - \tfrac{\lambda_0}{\lambda_1}\ {}_0b_t^c + \left[\tfrac{\lambda_0}{\lambda_1} - 1\right]c_t\right].
\end{aligned} \tag{4.56}$$

We find now the equation for $\mu_{0t} = \mu_{1t}$. Substituting μ_t by its value, equation (4.24), dividing by $-U_{x_t x_t}\lambda_1$ and adding ${}_0b_t^w \left[\frac{\lambda_0}{\lambda_1} - 1\right]$, we get

$$\begin{aligned} &\left[\frac{U_{x_t x_t}}{U_{c_t x_t}}\right] \left[\left[\frac{\lambda_0}{\lambda_1} - 1\right] \left[l_t + {}_0b_t^w - \left[\frac{U_{x_t}}{U_{x_t x_t}}\right]\right] - \left[{}_1b_t^w - {}_0b_t^w\right]\right] = \\ &\left[{}_1b_t^c - \frac{\lambda_0}{\lambda_1}\,{}_0b_t^c + \left[\frac{\lambda_0}{\lambda_1} - 1\right] c_t\right]. \end{aligned} \tag{4.57}$$

Equating the LHS of equations (4.56) and (4.57), it results that

$${}_1b_t^w - {}_0b_t^w = \left[\frac{\lambda_0}{\lambda_1} - 1\right] \left[{}_0b_t^w + l_t - \left[\frac{U_{c_t x_t} U_{c_t} - U_{c_t c_t} U_{x_t}}{U_{c_t c_t} U_{x_t x_t} - (U_{c_t x_t})^2}\right]\right].$$

Taking into account the specific instantaneous utility function (4.2), we have

$${}_1b_t^w - {}_0b_t^w = \left[\frac{\lambda_0}{\lambda_1} - 1\right] \left[{}_0b_t^w + 1\right],$$

which are the issues of debt indexed to after-tax-wage maturing at date $t \geq 2$.

This procedure needs to be replicated for the three remaining debt types. ■

4.6 Numerical Solution Method

In this appendix we describe the eigenvalue-eigenvector decomposition method in more detail. Through this appendix, as an example, we apply this procedure to solve for the debt-commitment policy. The eigenvalue-eigenvector decomposition method consists of the following steps:

i) We choose some reasonable values for the parameters and the initial conditions.

ii) The model is solved for a steady state. The solution of the debt-commitment model $\left\{c_t, l_t, g_{t+1}, k_{t+1}, \tau_t^l, \mu_{0t}, \xi_{0t}\right\}_{t\geq 0}$ and λ_0 is characterized by constraints (4.14), (4.18) and (4.23) and the necessary conditions (4.5) and $(4.24) - (4.27)$. If the number of total periods were T, the system would have $4T + 3(T-1) + 1$ equations and the same number of unknown variables. However, our economy extends over an infinite-horizon. For each period, there are four equations, (4.5), (4.14), (4.24), and (4.25), involving variables at that date; three equations, (4.23), (4.26), and (4.27), linking current to future variables; and one, the implementability condition (4.18), that is a function of variables at all dates. For the purpose of solving the system, the equations that link current to future variables need to be replaced by the stability conditions that depend on variables at the current date. In an endogenous growth model the steady state levels of the variables change over time. These variables are transformed so as to take constant values at the steady state, that is, $w_t^{kg} = \frac{k_t}{g_t}$, $w_t^{cc} = \frac{c_t}{c_{t-1}}$, $w_t^{ck} = \frac{c_t}{k_t}$, and $w_t^{\xi c} = \frac{\xi_{0t}}{c_t}$. However, notice that to find the value of λ_0, that is independent of time, we need to know the whole series of variables and plug them into the implementability condition (4.18). To solve this, a steady state is computed for a given value of λ_0 and, afterwards, we search for the value of λ_0 that solves condition (4.18).

Next, equations (4.5), (4.14) , (4.23), and (4.24) – (4.27) are written as functions of the new set of variables. Finally, since all new variables are constant at the BGP, we can take away the t index and find a steady state.

iii) The constraints and conditions of the economy are linearized around the steady state. These equations can be viewed as a function $f\left(w_t^{ck}, l_t, w_t^{kg}, w_t^{cc}, w_t^{\phi c}\right)$. We define $y_t = \left(w_t^{ck} - w_{ss}^{ck}, l_t - l_{ss}, w_t^{kg} - w_{ss}^{kg}, w_t^{cc} - w_{ss}^{cc}, w_t^{\phi c} - w_{ss}^{\phi c}\right)$ and do a first-order Taylor approximation around the steady state

$$\left.\frac{\partial f}{\partial y_t}\right|_{ss} y_t + \left.\frac{\partial f}{\partial y_{t-1}}\right|_{ss} y_{t-1} = Ay_t + By_{t-1} = 0.$$

iv) We compute the unstable eigenvalues of the linear system. An unstable eigenvector is defined as one that takes an absolute value greater than $\beta^{-\frac{1}{2}}$. This number is chosen so that the objective function is bounded above. We find the set of eigenvalues and eigenvectors of the matrix $-(A^{-1})B$.

v) The stability conditions are obtained. We find these conditions by imposing orthogonality between each eigenvector associated with an unstable eigenvalue and the variables of the system, that is,

$$Cy_t = 0,$$

where C is the matrix of eigenvectors associated with unstable eigenvalues. These stability conditions guarantee that the transversality conditions hold.

vi) The stability conditions are imposed into the original non-linear model. We replace the equations that linked current to future variables by the stability conditions and, now, we can compute a solution. Notice that, since the stability conditions are computed for the linearized system, the solution involves some numerical error.

For a more complete review, see Novales et al. (1999) and Sims (1998).

4.7 Figures and Tables

TABLE 4. Summary of Results from the Numerical Solution Method

Variables	Policy 0	Policy 1	Policy 2	Policy 3
$\left(\frac{c}{k}\right)_{ss}$	0.03956	0.03858	0.03838	0.03849
$\mathbf{l}_{ss}$	0.83333	0.29043	0.27352	0.27523
$\left(\frac{k}{g}\right)_{ss}$	0.34284	0.35462	0.35589	0.35578
$\boldsymbol{\tau}_{ss}^{l}$	–	0.83320	0.84134	0.84092
$\boldsymbol{\tau}_{ss}^{k}$	–	0	0	0
M. E. K.	2.49e-7	1.95e-9	4.94e-5	0.71765
M. E. G.	2.49e-7	3.45e-10	4.94e-5	0.70100
M. E. T.	–	–	0	0.03586
E. I.	–	-5.2e-10	-1.83e-7	-2.17e-6
Growth rate	13.87%	4.274%	3.838%	3.849%
Welfare	249.207	84.3657	76.3204	73.4250

Policy 0 is the first-best policy.

Policy 1 is the policy under full-commitment.

Policy 2 is the policy under full-commitment with $\tau_t^k \leq 1$.

Policy 3 is the policy under debt-commitment (with $\tau_t^k = 0$).

M.E.K. and M.E.G. stand for the maximum error at satisfying the first-order condition for private and public capital, respectively.

M.E.T. stands for the maximum error at satisfying the restriction on capital tax rates.

E.I. stands for the error that is made at satisfying the implementability condition.

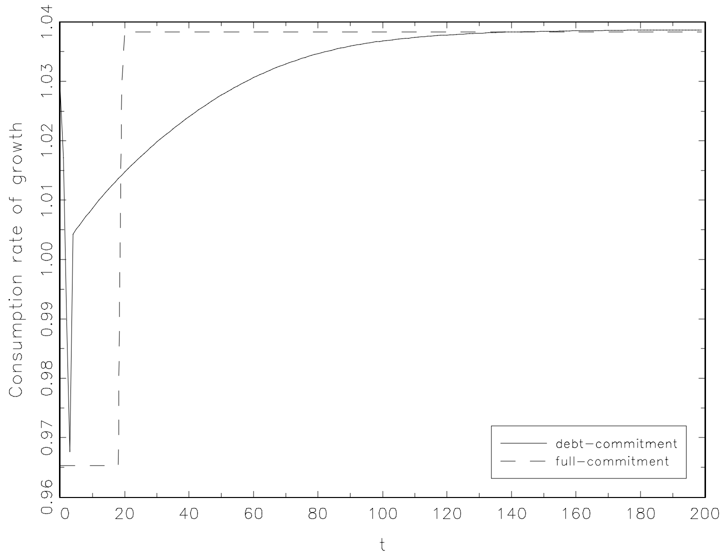

FIGURE 5. The Growth Rate of Consumption

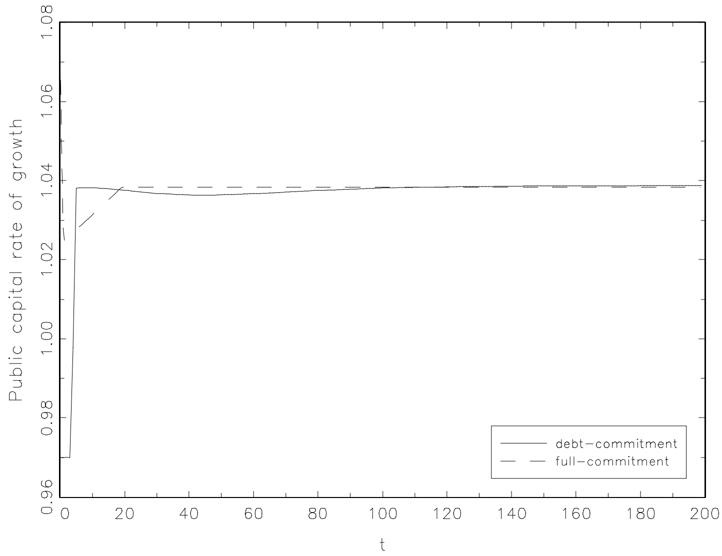

FIGURE 6. The Growth Rate of Public Capital

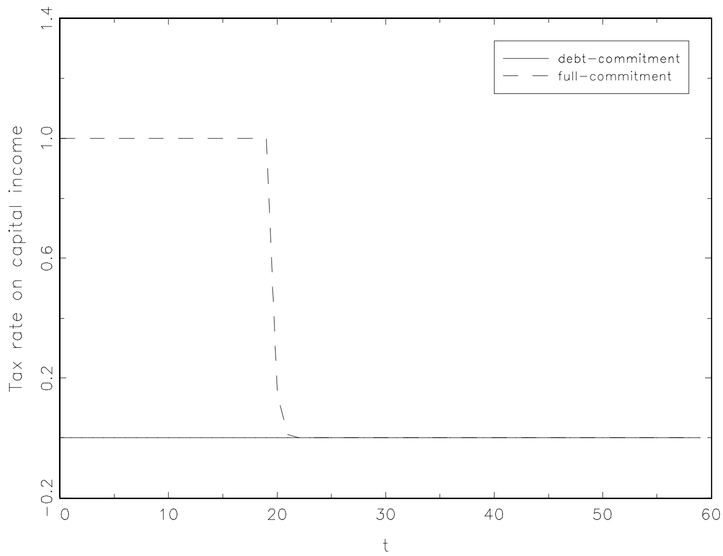

FIGURE 7. The Tax Rate on Capital Income

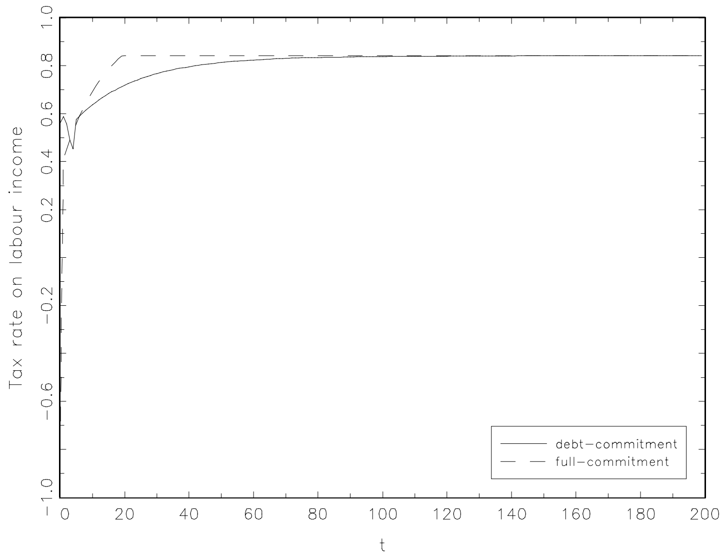

FIGURE 8. The Tax Rate on Labor Income

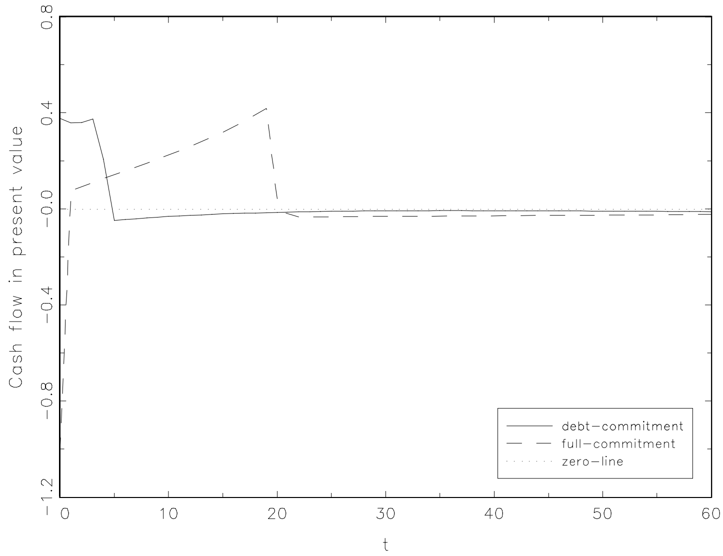

FIGURE 9. Cash Flow in Present Value

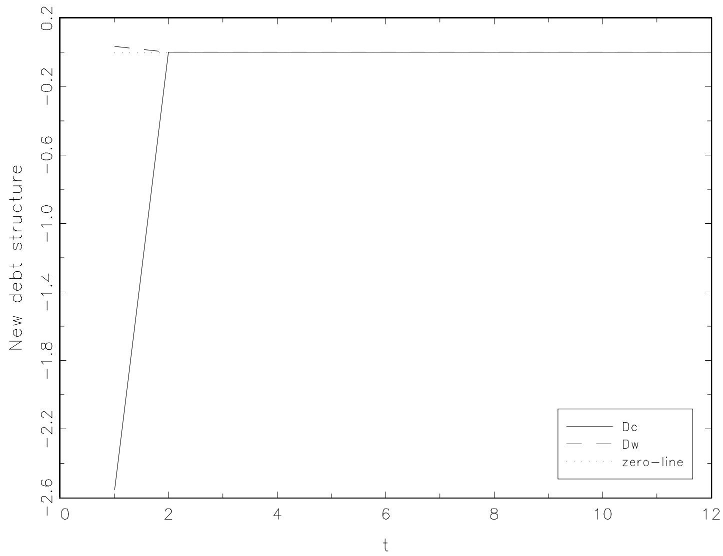

FIGURE 10. The Optimal Debt Structure

DC and DW stand for debt indexed to consumption and to after-tax-wage, respectively.

TABLE 5. Debt Restructuring: The Optimal Issues of Debt at Date 0

s	**1**	**2**	**3**	**4**	...	∞
${}_1\mathbf{b}_s^c$	-2.55260	0	0	0	...	0
${}_1\mathbf{b}_s^w$	0.03177	0.00039	0.00039	0.00039	...	0.00039