UNIVERSITAT DE BARCELONA

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RESUMEN. El teorema de equidistribución de Bilu establece que, dada una sucesión estricta de puntos en el toro algebraico N-dimensional cuya altura de Weil tiende a cero, las órbitas de Galois de los puntos se equidistribuyen con respecto a la medida de Haar de probabilidad del policírculo unidad. Para el caso unidimensional, versiones cuantitativas de este resultado fueron obtenidas independientemente por Petsche y por Favre y Rivera-Letelier.

Se presenta en esta tesis una versión cuantitativa del resultado de Bilu para el caso de dimensión cualquiera. Dado un punto en el toro algebraico de dimensión N de altura de Weil menor que 1, se proporciona una cota para la integral de una determinada función test en $\mathbb{P}^1(\mathbb{C})^N$ con respecto a la medida signada definida como la diferencia de la medida discreta de probabilidad asociada a la órbita de Galois del punto y la medida de probabilidad soportada en el policírculo unidad, donde coincide con la medida de Haar normalizada. Esta cota está dada en términos de una constante que depende únicamente de la función test, de la altura de Weil del punto, y de una noción que generaliza a dimensión superior el grado de un número algebraico.

Para la demostración de este resultado se utiliza el análisis de Fourier para la descomposición del problema y, a través de proyecciones, se reduce al caso unidimensional donde aplicamos la versión cuantitativa de Favre y Rivera-Letelier.

RESUM. El teorema d'equidistribució de Bilu estableix que, donat una successió de punts en el tor algebraic N-dimensional amb altura de Weil que tendeix cap a zero, les òrbites de Galois dels punts es equidistribueixen respecte de la mesura de Haar de probabilitat del policercle unitat. Per al cas unidimensional, versions quantitatives d'aquest resultat van ser obtingudes independentment per Petsche, i per Favre i Rivera-Letelier.

Es presenta en aquesta tesi una versió quantitativa del resultat de Bilu per al cas de dimensió qualsevol. Donat un punt en el tor algebraic de dimensió N d'altura de Weil més petita que 1, es proporciona una fita per a l'integral d'una determinada funció test en $\mathbb{P}^1(\mathbb{C})^N$ respecte de la mesura signada definida com la diferència de la mesura discreta de probabilitat associada a l'òrbita de Galois del punt i la mesura de probabilitat suportada en el policercle unitat, on coincideix amb la mesura de Haar normalitzada. Aquesta fita ve donada en termes d'una constant que depèn únicament de la funció test, de l'altura de Weil del punt, i d'una noció que generalitza a dimensió superior el grau d'un nombre algebraic.

Per a la demostració d'aquest resultat s'utilitza l'anàlisi de Fourier per la descomposició del problema i, mitjançant projeccions, es redueix al cas unidimensional on apliquem la versió quantitativa de Favre i Rivera-Letelier.

ABSTRACT. Bilu's equidistribution theorem establishes that, given a strict sequence of points on the N-dimensional algebraic torus whose Weil height tends to zero, the Galois orbits of the points are equidistributed with respect to the Haar probability measure of the unit polycircle. For the case of dimension one, quantitative versions of this result were independently obtained by Petsche and by Favre and Rivera-Letelier.

We present in this thesis a quantitative version of Bilu's result for the case of any dimension. Given a point on the algebraic torus of dimension N and Weil height less than 1, we give a bound for the integral of a suitable test function on $\mathbb{P}^1(\mathbb{C})^N$ with respect to the signed measure defined as the difference of the discrete probability measure associated to the Galois orbit of the point and the probability measure supported on the unit polycircle, where it coincides with the normalized Haar measure. This bound is given in terms of a constant depending only on the test function, the Weil height of the point, and a notion that generalizes to higher dimension the degree of an algebraic number.

For the proof of this result we use Fourier analysis techniques to decompose the problem and we reduce it, via projections, to the one-dimensional case where we apply the quantitative version by Favre and Rivera-Letelier.

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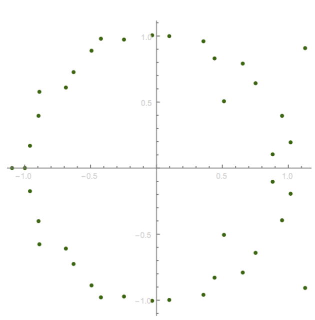
Introduction

1. Historical context of the equidistribution problem

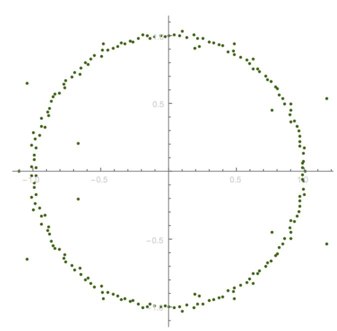
The distribution of the roots of a polynomial and, more generally, the distribution of the solutions of a system of N polynomial equations in N variables with rational coefficients have been studied by several authors through techniques that involve deep mathematical results.

Let us consider some examples in the one-dimensional case to get some intuition on the problem. For a given non-zero rational number a, consider the polynomials $x^k - a$, with $k \geq 1$. The roots of these polynomials are uniformly distributed in the circle of radius $a^{\frac{1}{k}}$ and, as k tends to infinity, they tend to the equidistribution on the unit circle. In contrast, if consider the family of polynomials $(x-1)^k$, with $k \geq 1$, we observe that they have a unique root at 1 of multiplicity k. The most obvious difference between these two families of polynomials is the growth of their coefficients with respect to their degree. Indeed, for the second family the coefficients grow exponentially with the degree, whereas for the first one they do not grow at all. We will soon see that the *size* of the coefficients of the polynomials in these families play a key role on the limit distribution of their roots.

Another interesting example where this asymptotic behavior can be observed is the following. For every $k \geq 1$, consider a polynomial f_k of degree k and coefficients in $\{-1, 0, 1\}$. By computing their roots and plotting them in the complex plane, one may verify experimentally that they tend to the equidistribution on the unit circle as k tends to infinity. The figure below shows a plot of the roots of two polynomials of respective degrees 40 and 200, for a random choice of coefficients in $\{-1, 0, 1\}$.



(A) $f_{40}(x) = 0$



(B) $f_{200}(x) = 0$

A significant result justifying this phenomenon and its analogue for the N-dimensional case is due to Bilu [**Bil97**]. It establishes the uniform distribution of Galois orbits of points of small *Weil height* in the algebraic torus towards the unit polycircle in terms of weak convergence of probability measures. In fact, we will see that its consequences go beyond the study of the distribution of the roots of polynomials. Before stating the theorem let us introduce some notation.

Fix an algebraic closure $\overline{\mathbb{Q}}$ of the field of rational numbers together with an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. By $\mathbb{C}^\times$ and $\overline{\mathbb{Q}}^\times$ we denote the multiplicative groups of $\mathbb{C}$ and $\overline{\mathbb{Q}}$, respectively. Let $T \subset (\mathbb{C}^\times)^N$ be a finite set, the discrete probability measure on $(\mathbb{C}^\times)^N$ associated to the set T is given by

$$\mu_T = \frac{1}{\#T} \sum_{\boldsymbol{\alpha} \in T} \delta_{\boldsymbol{\alpha}},$$

where $\#T$ denotes the cardinality of the set T and δ_α is the delta Dirac measure on $(\mathbb{C}^\times)^N$ supported on α. The *Galois orbit* of an element in $(\overline{\mathbb{Q}}^\times)^N$ is the orbit of the element under the action of the *absolute Galois group* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The unit polycircle $(S^1)^N$ is the set of points $(z_1, \ldots, z_N)$ in $\mathbb{C}^N$ such that $|z_1| = \ldots = |z_N| = 1$, it is a compact subgroup of $(\mathbb{C}^\times)^N$. We will denote by $\lambda_{(S^1)^N}$ the probability measure on $(\mathbb{C}^\times)^N$ supported on $(S^1)^N$, where it coincides with the normalized Haar measure. A sequence $\{\mu_k\}$ of probability measures on $(\mathbb{C}^\times)^N$ *converges weakly* to a probability measure μ on $(\mathbb{C}^\times)^N$ if, for every compactly supported continuous function $f : (\mathbb{C}^\times)^N \to \mathbb{R}$, we have

$$\lim_{k \to \infty} \int_{(\mathbb{C}^\times)^N} f d\mu_k = \int_{(\mathbb{C}^\times)^N} f d\mu.$$

A sequence of finite sets $\{T_k\}$ in $(\mathbb{C}^\times)^N$ is equidistributed with respect to a probability measure μ if the discrete probability measures associated to the sets T_k converge weakly to μ.

Let $\xi \in \overline{\mathbb{Q}}^\times$ and $f_\xi \in \mathbb{Z}[x]$ the minimal polynomial of ξ over $\mathbb{Z}$, i.e.: the polynomial with coprime integer coefficients vanishing at ξ of least degree. Then the degree of ξ over $\mathbb{Q}$, denoted by $\deg(\xi)$, is defined as the degree of its minimal polynomial and its *Weil height* as

$$\mathrm{h}(\xi) = \frac{m(f_\xi)}{\deg(\xi)},$$

where $m(f_\xi)$ is the *(logarithmic) Mahler measure* of the polynomial f_ξ,

$$m(f_\xi) = \frac{1}{2\pi} \int_0^{2\pi} \log |f_\xi(e^{i\theta})| d\theta.$$

The definition of Weil height is extended to $(\overline{\mathbb{Q}}^\times)^N$ as follows:

$$\mathrm{h}(\boldsymbol{\xi}) = \mathrm{h}(\xi_1) + \ldots + \mathrm{h}(\xi_N) \text{ for every } \boldsymbol{\xi} = (\xi_1, \ldots, \xi_N).$$

A sequence $\{\boldsymbol{\xi}_k\}$ in $(\overline{\mathbb{Q}}^\times)^N$ is *strict* if, for every proper algebraic subgroup $Y \subset (\overline{\mathbb{Q}}^\times)^N$, the cardinality of the set $\{k : \boldsymbol{\xi}_k \in Y\}$ is finite.

THEOREM ([**Bil97**], Theorem 1.1). *Let $\{\boldsymbol{\xi}_k\}$ be a strict sequence in $(\overline{\mathbb{Q}}^\times)^N$ such that $\lim_{k\to\infty} \mathrm{h}(\boldsymbol{\xi}_k) = 0$. Then the Galois orbits of $\boldsymbol{\xi}_k$ are equidistributed with respect to $\lambda_{(S^1)^N}$.*

As it was mentioned above, this result gives a satisfactory answer for the distribution of the roots of polynomials with rational coefficients. Indeed, for the one-dimensional case the theorem may be reformulated as follows. Let $\{f_k\}$ be a sequence of irreducible polynomials in $\mathbb{Q}[x]$. Assume that no cyclotomic polynomial is repeated an infinite number of times, and that $m(f_k) \in o(\deg(f_k))$. Then the roots of the polynomials f_k are equidistributed with respect to λ_{S^1}.

Observe that, for the first and last families of examples stated above the Mahler measures are $m(x^k - a) = \log^+ |a| \in o(k)$, where $\log^+ x = \max\{0, \log x\}$, and $m(f_k) \leq \log(1 + k) \in o(k)$. Hence, we can deduce the equidistribution of the roots towards the unit circle.

Other results can be found in the literature regarding the study of the distribution of roots of polynomials. Among them, there is a classical result due to Erdös and Turán [**ET50**] where the distribution of the arguments of the roots of polynomials with complex coefficients is proved under the assumption that the middle coefficients are not too big with respect to the extremal ones. This, together with the work of Hughes and Nikeghbali [**HN08**], provides a proof for the uniform distribution of the roots of these polynomials on the unit circle. The generalization to the multivariate case is given in [**DGS14**].

Bilu's equidistribution theorem belongs to a family of results concerning the distribution of Galois orbits of points of small height on algebraic varieties, a problem that has assumed a significant role over the last twenty years in Diophantine and Arithmetic Geometry. This result was inspired on a previous work of Szpiro, Ullmo and Zhang [**SUZ97**], where an equidistribution result for small points on *Abelian varieties* is stated. For every symmetric and ample line bundle L on an Abelian variety A we can define a height function on its algebraic points $\widehat{h}_L : A(\overline{\mathbb{Q}}) \to \mathbb{R}_+$, called the *Néron-Tate height* associated to L. A sequence in $A(\overline{\mathbb{Q}})$ is *generic* if every proper subvariety of A contains finitely many elements of the sequence. In [**SUZ97**], the authors prove that for every generic sequence in $A(\overline{\mathbb{Q}})$ whose Néron-Tate height tends to zero, the Galois orbits of the points of the sequence are equidistributed with respect to the Haar probability measure on $A(\mathbb{C})$.

This was the first step towards the proof of the *generalized Bogomolov conjecture* which was originally stated by Bogomolov [**Bog81**] for algebraic curves over $\mathbb{Q}$ of genus greater than one embedded in their associated Jacobian variety. The conjecture was later generalized for every non-torsion subvariety of an Abelian variety over $\mathbb{Q}$, where a torsion subvariety is the translation by a torsion point of an Abelian subvariety. It predicted that given an Abelian variety A over $\mathbb{Q}$ together with a symetric ample line bundle L, for every non-torsion subvariety $Y \subset A$ there is a constant $c_L(Y) > 0$ such that the set $\{P \in Y(\overline{\mathbb{Q}}) : \widehat{h}_L(P) < c_L(Y)\}$ is not Zariski-dense in Y. For the toric analogue, the result was proved in **[Zha95]**, later Bilu gave a simple proof based on his equidistribution theorem. The proof of the conjecture for the case of curves was given by Ullmo **[Ull98]** and, shortly after, Zhang demonstrated the general case in **[Zha98]**. In his paper, Zhang proved the equidistribution for strict sequences of algebraic points on Abelian varieties, rather than generic sequences, generalizing the results in **[SUZ97]**. In **[CL00]**, Chambert-Loir proved a generalization of Bogomolov's conjecture for *semiabelian varieties*.

Bilu's equidistribution theorem inspired several works on the subsequent years, specially for the one-dimensional case. Rumely **[Rum99]** translated Bilu's result to the language of complex potential theory and, in this setting, he gave a generalization for a class of heights on the complex projective line associated to compact sets of capacity one. Later, Baker and Hsia **[BH05]** proved, for any place of $\mathbb{Q}$, a general equidistribution property stated in the dynamical context for normalized canonical heights associated to polynomial maps with rational coefficients.

The strategy of the interpretation of heights in terms of the potential theory, which was developed by Rumely, Baker and Hsia, became a strong machinery for dealing with arithmetic equidistribution problems. This approach, and its generalization to the v-adic analyfication in the Berkovich sense for every finite place v, gave rise to the independent results of Favre and Rivera-Letelier **[FRL06]** and Baker and Rumely **[BR06]**. In parallel, following the so-called *variational principle* introduced in **[SUZ97]** and based on Arakelov Geometry, Chambert-Loir **[CL06]** proved a similar result. In these coetaneous but independent works, equidistribution results were given on the one dimensional case concerning normalized heights associated to dynamical systems for all places of $\mathbb{Q}$.

In **[CL06]**, Chambert-Loir studies the distribution at non-Archimedean places of orbits of points on varieties of any dimension considering ample line bundles with an associated algebraic metric. For the case of curves, he proves a stronger result considering heights associated to ample line bundles together with a more general class of adelic metric. Baker and Rumely in **[BR06]** associate to every rational function φ over $\mathbb{Q}$ a dynamical height h_φ on $\mathbb{P}^1(\overline{\mathbb{Q}})$. They prove that, for every place p of $\mathbb{Q}$, there is a probability measure $\mu_{\varphi,p}$ on the Berkovich projective line $\mathbb{P}^1_{Berk}(\mathbb{C}_p)$ such that, for every sequence of distinct points $\{\xi_k\}$ in $\mathbb{P}^1(\overline{\mathbb{Q}})$ with $\mathrm{h}_\varphi(\xi_k) \to 0$, the Galois orbits of ξ_k are equidistributed with respect to $\mu_{\varphi,p}$. Favre and Rivera-Letelier **[FRL06]** introduce the notion of *adelic measures* and associate to them an *adelic height*. They prove that these adelic heights are indeed Weil heights and such that they have non-negative essential minimum. They also give the following equidistribution result: given an adelic measure $\rho = \{\rho_p\}_{p\in M_{\mathbb{Q}}}$ and a sequence $\{\xi_k\}$ of distinct points in $\mathbb{P}^1(\overline{\mathbb{Q}})$ such that their adelic height h_ρ tends to zero, the Galois orbits of ξ_k are equidistributed with respect to ρ_p, for every place p of $\mathbb{Q}$. In addition, the authors give a quantitative version of the result.

The generalization, for every place, of the results on the distribution of the orbits of small points on varieties of any dimension was first given by [**Yua08**]. We will introduce some notions and notation before stating his result.

Let K be a number field and denote by M_K the set of places of K. For every $v \in M_K$, fix an embedding $\overline{K} \hookrightarrow \mathbb{C}_v$, where $\mathbb{C}_v$ is the completion of the algebraic closure of K_v.

Let X be a projective algebraic variety over K of dimension n. To every semipositive metrized line bundle $\overline{L}$ on X with L ample we can associate a height function $\mathrm{h}_{\overline{L}} : X(\overline{K}) \to \mathbb{R}$. Moreover, we can define $\mathrm{h}_{\overline{L}}(X)$, the height of X relative to $\overline{L}$, through the arithmetic intersection theory introduced in [**GS90**] and extended, by a limit process, to the general semipositive case in [**Zha95**]. We say that a sequence of points $\{\xi_k\}$ in $X(\overline{K})$ is $\overline{L}$-*small* if

$$\lim_{k\to\infty} \mathrm{h}_{\overline{L}}(\xi_k) = \frac{\mathrm{h}_{\overline{L}}(X)}{(n+1)\deg_L(X)}.$$

For every Archimedean place $v \in M_K$, the space X_v^{an} is the corresponding analytic space $X(\mathbb{C}_v)$. If $v \in M_K$ is non-Archimedean, X_v^{an} is the analyfication in the Berkovich sense of the projective scheme X over $\mathbb{C}_v$. Associated to the metrized line bundle $\overline{L}$, and for every place v, there is a v-adic canonical measure $c_1(\overline{L})_v^n$ of total mass $\deg_{\overline{L}}(X)$ called the v-adic Monge-Ampère measure.

Let $\xi \in X(\overline{K})$ and S its Galois orbit, i.e. the orbit of ξ under the action of $\mathrm{Gal}(\overline{K}/K)$. For every place $v \in M_K$, since S can be seen as a finite subset of X_v^{an}, we can define the discrete probability measure associated to the set S as

$$\mu_{S,v} = \frac{1}{\#S}\sum_{\alpha\in S}\delta_\alpha,$$

where δ_α is the delta Dirac measure on X_v^{an} supported on α. A sequence of probability measures $\{\mu_k\}$ in X_v^{an} converges weakly to a probability measure μ if, for every continuous function $f : X_v^{an} \to \mathbb{R}$, we have

$$\lim_{k\to\infty}\int_{X_v^{an}} f d\mu_k = \int_{X_v^{an}} f d\mu.$$

We can now state Yuan's result.

THEOREM ([**Yua08**], Theorem 3.1). *Let X be a projective variety of dimension n over a number field K, and $\overline{L}$ a semipositive metrized line bundle on X such that L is ample. Let $\{\xi_k\}$ be an infinite sequence in $X(\overline{K})$ such that it is generic and $\overline{L}$-small and let S_k be the Galois orbit of ξ_k. Then, for* *every place $v \in M_K$, the sequence of discrete probability measures $\{\mu_{S_k,v}\}$ converges weakly to* $\frac{1}{\deg_{\overline{L}}(X)} c_1(\overline{L})_v^n$.

The existence of sequences of points in $X(\overline{K})$ that are generic and $\overline{L}$-small is a very strong hypothesis. Nevertheless, it is satisfied for the Weil height in the algebraic torus, the Néron-Tate height in Abelian varieties and, more generally, for those heights coming from dynamical systems.

Gubler [**Gub08**] proved an analogous result when K is a function field. In the last years, several generalization of Yuan's result have been given, such as [**Che11**]. In [**BB10**], the equidistribution property is stated for big line bundles on Archimedean places. For the case of proper toric varieties, a stronger result is provided in [**BRLPS15**].

2. Statement of results and structure of the thesis

As a general fact, the results on the distribution of Galois orbits of points of small height are formulated in a qualitative way in the sense that no information is given about the rate of convergence of the weak limit of probability measures. An exception is given in [**FRL06**], where the authors provide, on every place of $\mathbb{Q}$ and for a fixed family of *test functions*, quantitative estimates for the rate of convergence in the one dimensional case. Independently, Petsche [**Pet05**] gives, for the particular setting of Bilu's result, a quantitative version for the one dimensional case using Erdös and Turán's result [**ET50**], and Fourier analysis techniques.

The aim of this thesis is to give a quantitative version of Bilu's equidistribution theorem for the case of dimension N. In particular, we provide a bound for the integral of a suitable test function with respect to the signed measure defined by the difference of the discrete probability measure associated to the Galois orbit of a point in $(\overline{\mathbb{Q}}^\times)^N$ and the measure $\lambda_{(S^1)^N}$. This estimate is given in terms of the height of the point, a generalization to higher dimension of the notion of the degree of an algebraic number and a constant depending linearly on the test function.

Let us introduce some notation.

Consider coordinates $((x_1 : y_1), \dots, (x_N : y_N))$ on $\mathbb{P}^1(\mathbb{C})^N$ and define the subvariety

$$\mathbb{H} = V\left(\prod_{j=1}^{N} x_j y_j\right).$$

The set of test functions $\mathcal{F}$ is defined as the set of all real-valued functions in $\mathscr{C}^{2N+1}(\mathbb{P}^1(\mathbb{C})^N)$ whose $2N$-jet vanishes at $\mathbb{H}$. This is, all functions whose partial derivatives up to order $2N$ vanish at $\mathbb{H}$, on every chart.

For every $\boldsymbol{n} = (n_1, \dots, n_N)$ in $\mathbb{Z}^N$, the monomial map $\chi^{\boldsymbol{n}} : (\overline{\mathbb{Q}}^\times)^N \to \overline{\mathbb{Q}}^\times$ is defined as

$$\chi^{\boldsymbol{n}}(\boldsymbol{\xi}) = \xi_1^{n_1} \cdots \xi_N^{n_N}, \text{ for any } \boldsymbol{\xi} = (\xi_1, \dots, \xi_N).$$

We define the *generalized degree* of $\boldsymbol{\xi} \in (\overline{\mathbb{Q}}^{\times})^N$ as

$$\mathscr{D}(\boldsymbol{\xi}) = \min_{\boldsymbol{n} \neq \boldsymbol{0}} \{ \|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) \},$$

where $\| \cdot \|_1$ stands for the 1-norm on $\mathbb{C}^N$.

We can now state the main theorem of this dissertation

THEOREM I. *There is a constant $C \approx 48.9897$ such that for every test function $f \in \mathcal{F}$ and every $\boldsymbol{\xi} \in (\overline{\mathbb{Q}}^{\times})^N$ with $\mathrm{h}(\boldsymbol{\xi}) \leq 1$, the following holds*

$$\left| \int_{\mathbb{P}^1(\mathbb{C})^N} f d\mu_S - \int_{\mathbb{P}^1(\mathbb{C})^N} f d\lambda_{(S^1)^N} \right| \leq c(f) \left(4\,\mathrm{h}(\boldsymbol{\xi}) + C \frac{\log(\mathscr{D}(\boldsymbol{\xi}) + 1)}{\mathscr{D}(\boldsymbol{\xi})} \right)^{\frac{1}{2}},$$

where S is the Galois orbit of $\boldsymbol{\xi}$, μ_S the discrete probability measure associated to it and $c(f)$ is a positive constant depending only on the function f.

The dependence of the constant $c(f)$ in terms of the function f is

$$c(f) \leq \sqrt{2}\pi \,\mathrm{Lip}(f) + 2 \sum_{l=1}^{N} \left\| \widehat{\frac{\partial (f \circ \phi)}{\partial u_l}} \right\|_{\mathrm{L}^1} + 16 \sum_{l=1}^{N} \left\| \widehat{\frac{\partial (f \circ \phi)}{\partial \theta_l}} \right\|_{\mathrm{L}^1},$$

where $\mathrm{Lip}(f)$ is the Lipschitz constant of the function f with respect to the spherical distance in $\mathbb{P}^1(\mathbb{C})^N$, $\| \cdot \|_{\mathrm{L}^1}$ stands for the L^1-norm on the locally compact Abelian group $\mathbb{Z}^N \times \mathbb{R}^N$ and ϕ is the map defined by

$$\begin{array}{rcl} \phi : (\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N & \longrightarrow & \mathbb{P}^1(\mathbb{C})^N \\ ((\theta_1, \dots, \theta_N), (u_1, \dots, u_N)) & \longmapsto & ((1 : e^{2\pi i \theta_1 + u_1}), \dots, (1 : e^{2\pi i \theta_N + u_N})). \end{array}$$

In order to prove this result, we use Fourier analysis techniques to discretize the problem. Once this is done, we are able to reduce the situation to the one dimensional case via monomial maps and, in this setting, we apply Favre and Rivera-Letelier's quantitative result.

We will now give an overview of the structure of this dissertation

The first chapter is devoted to the preliminaries needed in the remaining two chapters of the text. As the reader may soon appreciate, we will introduce very well-known notions. This allows us to fix notations and leads us to a more self-contained text. In Section 1.1 we give a summary of the classical measure theory, not only for positive measures but also for *complex-valued* ones and, in particular, for *signed measures*. Section 1.2 comprises an overview of the theory of Fourier analysis on locally compact Abelian groups. Later, in Chapter 3, we consider the particular case of $(\mathbb{C}^{\times})^N$. In Section 1.3 we deal with the problem of approximating locally integrable functions by smooth ones. We recall the notion of convolution and see that, given an integrable function we can build, by convolution with the so-called *mollifiers*, a sequence of smooth functions converging to the original one. In Section 1.4 we recall the theory of Riemannian manifolds in order to define the *Laplace operator* acting on the space of smooth functions. In the subsequent section, we give the definition of the *distributional Laplace operator* and, for this purpose, we summarize the theory of distributions on

Riemannian manifolds. Section 1.6 includes a short introduction to potential theory on the complex line, where we give the definitions of harmonic and subharmonic functions and of the potential of a compactly supported finite positive measure on $\mathbb{C}$. The last section of this preliminary chapter will deal with the notion and properties of the Weil height of points on $\mathbb{P}^n(\mathbb{C})$ and, in particular, of algebraic numbers.

In Chapter 2, we make an exhaustive study of Favre and Rivera-Letelier's quantitative equidistribution theorem for infinite places and the particular case of the Haar probability measure on the unit circle. In addition, we give an explicit computation of a constant appearing on their result. We follow essentially the same structure of the article; however, one may notice at first sight that, in contrast to the language of differential forms and currents that the authors use in the original paper, we rather work with the language of distributions.

The main theorem on Chapter 2, which corresponds to Corollary 1.4 in [**FRL06**], is the following:

THEOREM II. *There is a positive constant $C \approx 14.7628$ such that for every $\mathscr{C}^1$-function $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$ and every $\xi \in \overline{\mathbb{Q}}^\times$, the following holds*

$$\left| \int_{\mathbb{P}^1(\mathbb{C})} f d\mu_S - \int_{\mathbb{P}^1(\mathbb{C})} f d\lambda_{S^1} \right| \leq \mathrm{Lip}(f) \left(\frac{\pi}{d} + \left(4\,\mathrm{h}(\xi) + C \frac{\log(d+1)}{d} \right)^{\frac{1}{2}} \right),$$

where d is the degree of ξ over $\mathbb{Q}$ and S its Galois orbit.

In particular, if $\mathrm{h}(\xi) \leq 1$, we have

$$\left| \int_{\mathbb{P}^1(\mathbb{C})} f d\mu_S - \int_{\mathbb{P}^1(\mathbb{C})} f d\lambda_{S^1} \right| \leq \mathrm{Lip}(f) \left(4\,\mathrm{h}(\xi) + C' \frac{\log(d+1)}{d} \right)^{\frac{1}{2}},$$

with $C' \approx 48.9897$.

In Section 2. a generalization of the theory of potentials to the whole Riemann sphere is given. As an introduction to the section, we study the problem of determining under which conditions we can consider a *global* potential for a given signed measure on the complex projective line. The answer to this problem is provided by a well-known result in potential theory. In the first part of the section, following [**FRL06**], we give the definition of the mutual energy of signed measures and we study sufficient conditions for it to be well-defined. The main result of this section establishes hypotheses for a signed measure to have positive energy. In the second part of the section we give a way of regularizing compactly supported probability measures on $\mathbb{C}$ in such a way that they have smooth potentials. As a remark to this subsection, we should mention that in this text we choose a way of regularizing measures which is not exactly the one introduced by Favre and Rivera-Letelier. In our case we define the regularization by convolution with mollifiers on $\mathbb{C}$ and this allows us to slightly simplify some proofs.

In Section 2.2 we reproduce the proof of the quantitative equidistribution in [**FRL06**] for infinite places and the case of the probability Haar measure on the unit circle. In order to make explicit the constant appearing in their result, we make a specific choice of the mollifier used for regularizing the discrete probability measures associated to a finite Galois-invariant set.

The third and last Chapter of this thesis is dedicated to the proof of Theorem I. In Section 3.1 we prove some results on Fourier analysis for the particular case of the locally compact Abelian group $(\mathbb{C}^{\times})^N$. In Section 3.2 we study the set of test functions $\mathcal{F}$. In particular, we see that for every function $f \in \mathcal{F}$, the function $f \circ \phi$ and all its first order partial derivatives are Haar-integrable as well as all their Fourier transforms. In Section 3.3 we deal with Galois orbits of points in the N-dimensional algebraic torus, their cardinality and their height. Section 3.4 comprises a study of the generalized degree, were we see that it is bounded by the minimum of the degrees of each of the components of the point and that, in dimension one, it coincides with the algebraic degree of the element. In addition, we state sufficient conditions for a point to be such that its generalized degree is exactly the minimum of the degrees of the components. Finally, in the last section, we include a detailed proof of the main theorem based, as we previously mentioned, on the quantitative result of dimension one.

CHAPTER 1

Preliminaries

We will devote this first chapter to the introduction of the theory and techniques that will be used later in this dissertation. We prevent the reader that they may find, in some cases, definitions and notions that are generally well-known and assumed. This will give us the opportunity to introduce notation and it will lead to a more self-contained text.

As a general fact, most of the proofs of the results that will be stated on this preliminary chapter will not be included. In certain cases, we will give demonstrations that we either found quite representative or did not find a reference in the literature. Nevertheless, at the beginning of each section, we will give several references where detailed proofs can be found.

1. Measure theory

We will introduce here some of the basic concepts about measure theory and integration that will appear all along the text. There are many detailed references where all the results to be stated appear, among them [**Rud87**].

A *σ-algebra* Σ on a set X is a collection of subsets of X such that

(i) $X \in \Sigma$,
(ii) For every countable family $\{E_j\} \in \Sigma$, we have $\bigcup_j E_j \in \Sigma$,
(iii) For every $E \in \Sigma$, we have $X \setminus E \in \Sigma$.

The elements of Σ are called *measurable sets* and the pair (X, Σ) is called a *measurable space*. From the definition we deduce that the $\emptyset \in \Sigma$ and that any countable intersection of measurable sets is a measurable set. If X is a topological space, the *Borel σ-algebra* of X, that will be denoted by $\mathscr{B}(X)$, is the smallest σ-algebra containing all the open subsets of X. The elements in $\mathscr{B}(X)$ are called *Borel sets*.

A *positive measure* on a measurable space (X, Σ) is a set function $\mu : \Sigma \to [0, +\infty]$ which is not identically $+\infty$ and satisfies the σ-additivity condition:

$$\mu\left(\bigcup_{j\geq 0} E_j\right) = \sum_{j\geq 0} \mu(E_j), \ \forall \{E_j\} \subset \Sigma \text{ pairwise disjoint countable family.}$$

From the definition, we deduce that $\mu(\emptyset) = 0$ and that, given $E, F \in \Sigma$ such that $E \subset F$, then $\mu(E) \leq \mu(F)$.

We will often use the word measure when referring to positive measures. For a measure space (X, Σ) and a measure μ on it, the triplet (X, Σ, μ), or simply (X, μ), is called a *measure space*.

A positive measure μ on a measurable space (X, Σ) is *complete* if every subset F of a measurable set E with $\mu(E) = 0$ is measurable. Every measure μ can be extended to a complete measure as follows. Denote by $\overline{\Sigma}$ the family of elements of the form $E \cup N$, where $E \in \Sigma$ and N is a subset of a measurable set of measure 0. It is easy to see that $\overline{\Sigma}$ is a σ-algebra and that μ can be extended to $\overline{\Sigma}$ by setting $\mu(E \cup N) = \mu(E)$. Then the extension, that will also be denoted by μ, is a a complete measure on $(X, \overline{\Sigma})$. This fact will allow us, whenever it is convenient, to assume that any given measure is complete.

A *complex measure* on a measurable space (X, Σ) is a σ-additive set function $\mu : \Sigma \to \mathbb{C}$. As for positive measures, one can easily deduce that $\mu(\emptyset) = 0$. However, observe from the definition that complex measures only take finite values. A measure is said to be *real* or *signed* if it takes values on the real line $\mathbb{R}$.

Given a complex measure μ on a measurable space (X, Σ), we define its *total variation* $|\mu|$ by

$$|\mu|(E) = \sup \sum_i |\mu(E_i)| \text{ for every } E \in \Sigma,$$

where the supremum is taken over all finite collections $\{E_i\}$ of pairwise disjoint sets whose union is E. It can be proved that the total variation of a complex measure is a positive measure on X, which will also be referred to as the *trace measure* of μ. The total variation of a positive measure is the measure itself.

Let μ be a (positive, complex) measure on (X, Σ), its *total mass* is

$$\|\mu\| := |\mu|(X).$$

A positive measure μ is *finite* if its total mass is finite, i.e.:

$$\|\mu\| < \infty.$$

An important class of finite positive measures are *probability measures* which are those whose total mass is exactly 1. Another example of finite positive measures are total variations of complex measures.

From now on, we will assume that X is a topological space and we will consider the Borel σ-algebra $\mathscr{B}(X)$. A (positive, complex) measure μ in the measurable space $(X, \mathscr{B}(X))$ is *regular* if, for every $E \in \mathscr{B}(X)$, we have

$$|\mu|(E) = \sup_{\substack{K \subset E \\ \text{compact}}} |\mu|(K) = \inf_{\substack{E \subset V \\ \text{open}}} |\mu|(V).$$

We will denote by $M(X)$ the set of all complex-valued regular measures on X. In compact metric spaces every complex or finite positive measure is regular.

Let μ be a (complex, positive) measure on X, and let $Y \subset X$ be a Borel set. The restriction μ_Y of the measure μ to the the set Y is defined as

$$\mu_Y(E) = \mu(E \cap Y), \text{ for all } E \in \mathscr{B}(X).$$

Given a (complex, positive) measure μ on X, we say that it is supported on a Borel set $A \in \mathscr{B}(X)$ if $\mu(E) = 0$ for every $E \in \mathscr{B}(X)$ such that $E \cap A = \emptyset$. We define the *support* of the measure μ as the smallest closed subset F in X such that $\mu(X \setminus F) = 0$ and we will denote it by $\operatorname{supp}(\mu)$.

A measure μ on X is *discrete* if it is supported on a discrete set of X.

Given a measure $\mu \in M(X)$, there is a unique decomposition of the form

$$\mu = \mu_0^+ - \mu_0^- + i\mu_1^+ - i\mu_1^-,$$

with $\mu_0^+, \mu_0^-, \mu_1^+$ and μ_1^- positive measures in $M(X)$ such that

$$\operatorname{supp}(\mu_0^+) \cap \operatorname{supp}(\mu_0^-) = \emptyset = \operatorname{supp}(\mu_1^+) \cap \operatorname{supp}(\mu_1^-).$$

This decomposition is called the *Jordan decomposition* of μ.

Let μ and ν be two positive or complex regular measures on the topological spaces X and Y, the set function $\mu \otimes \nu$ on $\mathscr{B}(X) \times \mathscr{B}(Y)$ which is given by

$$(\mu \otimes \nu)(E \times F) = \mu(E)\nu(F), \text{ for every } E \in \mathscr{B}(X), F \in \mathscr{B}(Y)$$

can be naturally extended to a measure on the product measurable space $(X \times Y, \mathscr{B}(X \times Y))$ and it is called the *product measure*.

Let us consider the particular case of signed measures. For a signed measure $\mu \in M(X)$, its Jordan decomposition is given by

$$\mu = \mu^+ - \mu^-,$$

where μ^+ and μ^- are finite and regular positive measures on X with disjoint supports. In particular, we have that

$$\operatorname{supp}(\mu) = \operatorname{supp}(\mu^+) \cup \operatorname{supp}(\mu^-).$$

The trace measure of μ is

$$|\mu| = \mu^+ + \mu^-.$$

1.1. Measurable functions. Let X be a topological space. A complex-valued function f in the measurable space $(X, \mathscr{B}(X))$ is *measurable* if, for every $t \in \mathbb{R}$, the following sets are Borel sets

$$\{x \in X : \operatorname{Re}(f(x)) \leq t\} \text{ and } \{x \in X : \operatorname{Im}(f(x)) \leq t\}.$$

All algebraic operations with measurable functions are measurable functions provided they do not include indeterminacies of the form $\frac{0}{0}, \frac{\infty}{\infty}$ and $\infty - \infty$. Given a sequence of measurable functions $\{f_k\}$, we have that $\limsup f_k$ and $\liminf f_k$ are also measurable.

Let μ be a positive measure on $(X, \mathscr{B}(X))$, we say that two measurable complex-valued functions f and g on the measure space (X, μ) are equal *almost everywhere* if there is a Borel set $E \in \mathscr{B}(X)$, with $\mu(E) = 0$, such that $f(x) = g(x)$ for every $x \in X \setminus E$. It is easy to verify that this is an equivalence relation. Given a function f and a sequence of functions $\{f_k\}$, all of them being measurable on (X, μ), we say that $\{f_k\}$ converges almost everywhere to f if $\mu(\{x : \lim_k f_k(x) \neq f(x)\}) = 0$. We will write

$$f_k \xrightarrow[k\to\infty]{a.e} f.$$

1.2. Lebesgue integration. We will first recall the notion of Lebesgue integral of measurable functions with respect to positive measures on a measurable space. Afterwards, we will extend this definition to all complex-valued measures using the Jordan decomposition.

Let μ be a positive measure on a topological space X, and let $f : X \to [0, +\infty]$ be a measurable function. A partition of $[0, +\infty]$ is a finite sequence of increasing positive real numbers. Given a partition $\{t_1, \ldots, t_m\}$, we define the corresponding *Lebesgue integral sum* of the measurable functions f as

$$\sum_{k=1}^{m} t_k \mu(\{x : t_k \leq f(x) < f_{k+1}\}),$$

where $t_{m+1} = +\infty$. The *Lebesgue integral* of f with respect to the measure μ is defined as the supremum of all these Lebesgue integral sums over all partitions of $[0, +\infty]$ and it is denoted by

$$\int_X f d\mu.$$

Observe that the Lebesgue integral, or simply integral, of a measurable function with respect to a measure can be infinite.

An extended real-valued measurable function f on the measure space (X, μ) is *integrable* if

$$\int_X |f| d\mu < \infty.$$

Given a measurable function $f : X \to \mathbb{R} \cup \{\pm\infty\}$, its integral with respect to the measure μ is defined by

$$\int_X f d\mu = \int_X f_+ d\mu - \int_X f_- d\mu,$$

where $f_\pm = \frac{|f| \pm f}{2}$.

The set of all integrable functions on a measure space (X, μ) is a linear space and it is noted by $\mathrm{L}^1(X, \mu)$. It is easy to see that the integral defines a linear functional on this function space.

Given a Borel set $E \in \mathscr{B}(X)$ and an extended real-valued measurable function f on the measure space (X, μ), we have

$$\int_E f d\mu = \int_X \chi_E f d\mu,$$

where χ_E is the function that is equally 1 on E and vanishes otherwise.

We will now state a very useful result concerning measurable function on product measure spaces.

THEOREM 1.1 (Fubini's Theorem). *Let μ and ν be two regular positive measures on the topological spaces X and Y, respectively. If $f : X \times Y \to \mathbb{C}$ is a measurable function such that any of the following holds*

$$\int_Y \left(\int_X |f(x,y)| d\mu(x) \right) d\nu(y) < \infty,$$

$$\int_X \left(\int_Y |f(x,y)| d\nu(y) \right) d\mu(x) < \infty \text{ or}$$

$$\int_{X\times Y} |f(x,y)| d(\mu \otimes \nu)(x,y) < \infty.$$

Then we have

$$\int_{X\times Y} f(x,y) d(\mu \otimes \nu)(x,y) = \int_Y \left(\int_X f(x,y) d\mu(x) \right) d\nu(y) = \int_X \left(\int_Y f(x,y) d\nu(y) \right) d\mu(x).$$

PROOF. [**Rud87**], Theorem 8.8. □

Let us finally define the integral of an extended real-valued measurable function f on X with respect to a complex measure μ. Considering the Jordan decomposition of μ, we have

$$\int_X f d\mu = \int_X f d\mu_0^+ - \int_X f d\mu_0^- + i \int_X f d\mu_1^+ - i \int_X f d\mu_1^-.$$

As in the previous cases, this integral can be unbounded.

We may also extend the definition to measurable complex-valued functions on a measure space f by

$$\int_X f d\mu = \int_X \mathrm{Re}(f) d\mu + i \int_X \mathrm{Im}(f) d\mu,$$

where $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are the real and imaginary parts of the function f.

1.3. Convergence theorems. In this part of the section, we will state some important results that will allow us to consider and exchange limits under the integral sign. Let X be a topological space and assume that μ is a positive measure on $\mathscr{B}(X)$.

It can be shown that given a sequence $\{f_k\}$ and a function f, all of them measurable, if f_k converges uniformly on X to the function f, then we have

$$\lim_k \int_X f_k d\mu = \int_X \lim_k f_k d\mu = \int_X f d\mu. \tag{1.1}$$

However, uniform convergence is a very strong convergence condition and we will often have weaker ways of convergence, such as pointwise convergence. In general, pointwise convergence will not be enough to guarantee (1.1), but the following results will give us sufficient conditions under which it holds.

THEOREM 1.2 (Monotone convergence theorem). *Let* $\{f_k\}$ *be an increasing sequence of positive measurable functions on* X*, then*

$$\lim_k \int_X f_k d\mu = \int_X \lim_k f_k d\mu.$$

PROOF. [**Rud87**], Theorem 1.26. □

THEOREM 1.3 (Dominated convergence theorem). *Let* $\{f_k\}$ *be a sequence of measurable functions on* X *such that*

(1) *there is a positive integrable function* g *such that* $|f_k(x)| \leq g(x)$ *almost everywhere,*
(2) *the limit* $\lim_k f_k(x)$ *exists almost everywhere.*

Then we have

$$\lim_k \int_X f_k d\mu = \int_X \lim_k f_k d\mu.$$

PROOF. [**Rud87**], Theorem 1.34. □

1.4. Riesz representation theorem. Assume that X is a locally compact Hausdorff space and let $\mathscr{C}_0(X)$ be the set of continuous functions *vanishing at infinity*, this is, the set of all continuous complex-valued functions f on the topological space X such that, for every $\varepsilon > 0$, the set $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact. It can be proved that this space is contained in the space of integrable functions $\mathrm{L}^1(X, |\mu|)$, for every complex measure $\mu \in M(X)$. Hence, we deduce that the map that associates to each $f \in \mathscr{C}_0(X)$ the value $\int_X f d\mu$ is a bounded linear functional. The converse of this fact is given by the following classical result.

THEOREM 1.4 (Riesz representation theorem). *For every bounded linear functional* $T : \mathscr{C}_0(X) \to \mathbb{C}$ *there is a unique measure* $\mu \in M(X)$ *such that*

$$T(f) = \int_X f d\mu, \textit{ for all } f \in \mathscr{C}_0(X).$$

PROOF. [**Rud87**], Theorem 6.19. □

From this theorem, we deduce that the space $M(X)$ is a Banach space with the norm $\|\mu\| = |\mu|(X)$. There is another version of the Riesz representation theorem that will be of interest in the following sections. It is given for functionals on the set of compactly supported continuous functions on X, which is denoted by $\mathscr{C}_c^0(X)$. Before stating it, recall that a linear functional T is said to be positive if $T(f) \geq 0$ for every $f \geq 0$.

THEOREM 1.5. *For every positive linear functional* $T : \mathscr{C}_c^0(X) \to \mathbb{C}$ *there is a unique regular positive measure* μ *on* X *such that*

$$T(f) = \int_X f d\mu, \textit{ for all } f \in \mathscr{C}_c^0(X).$$

PROOF. [**Rud87**], Theorem 2.14. □

1.5. L^p **spaces.** In this part of the section we will define some function spaces that will be of great importance, the L^p-spaces. One of them, the space of integrable functions on a measure space, has already been introduced in the Lebesgue integration subsection.

For every $1 \le p < \infty$ and every measure space (X, μ), we define the space $\mathrm{L}^p(X, \mu)$ as the set of equivalence classes of measurable functions $f : X \to \mathbb{C}$ such that

$$\|f\|_{\mathrm{L}^p(X,\mu)} := \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}} < \infty.$$

The space $\mathrm{L}^\infty(X, \mu)$ consists on the equivalence classes of *essentially bounded* measurable functions, this is, measurable functions $f : X \to \mathbb{C}$ such that

$$\|f\|_{\mathrm{L}^\infty(X,\mu)} := \operatorname*{ess\,sup}_{x \in X} |f(x)| = \inf\{C \ge 0 : \mu(\{x : |f(x)| > C\}) = 0\}.$$

It can be shown that $\|\cdot\|_{\mathrm{L}^p(X,\mu)}$ and $\|\cdot\|_{\mathrm{L}^\infty(X,\mu)}$ are norms on $\mathrm{L}^p(X, \mu)$ and $\mathrm{L}^\infty(X, \mu)$, respectively, and that they are complete with respect to them. Hence, we have that $\mathrm{L}^p(X, \mu)$ is a Banach space for every $1 \le p \le \infty$. If there is no possible misunderstanding, we will write $\|\cdot\|_{\mathrm{L}^p}$.

For every $1 \le p, q \le \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and every pair of measurable functions f and g on a measure space (X, μ), *Hölder's inquality* establishes that

$$\|fg\|_{\mathrm{L}^1} = \int_X |fg| d\mu \le \|f\|_{\mathrm{L}^p} \|g\|_{\mathrm{L}^q}.$$

Given f and a sequence $\{f_k\}$ of measurable functions on X, we say that $\{f_k\}$ converges to f in L^p-norm, if $\lim_k \|f_k - f\|_{\mathrm{L}^p(X,\mu)} = 0$. We will write

$$f_k \xrightarrow[k\to\infty]{\mathrm{L}^p} f.$$

The space $\mathrm{L}^2(X, \mu)$ is particularly interesting since its norm is associated with the inner product

$$\langle f, g \rangle_{\mathrm{L}^2} := \int_X f \overline{g} d\mu, \text{ for every } f, g \in \mathrm{L}^2(X, \mu).$$

This fact makes $\mathrm{L}^2(X, \mu)$ into a Hilbert space. Recall that, in a Hilbert space there is another notion of convergence, the *weak convergence*. Let f and $\{f_k\}$ be functions in $\mathrm{L}^2(X, \mu)$, we say that the sequence $\{f_k\}$ converges weakly to f if

$$\lim_{k\to\infty} \langle f_k, g \rangle_{\mathrm{L}^2} = \langle f, g \rangle_{\mathrm{L}^2}, \text{ for every } g \in \mathrm{L}^2(X, \mu).$$

Observe that *strong convergence*, or convergence in L^2-norm, implies weak convergence.

1.6. Minkowski's integral inequality. In this last part of the section, we will give a proof of *Minkowski's integral inequality*, which will be used later in this chapter.

THEOREM 1.6. *Let* (X,μ) *and* (Y,ν) *be two measure spaces and let* $F : X \times Y \to \mathbb{R}$ *be a measurable function. Then, for every* $p > 1$, *the following holds*

$$\left[\int_Y \left|\int_X F(x,y)d\mu(x)\right|^p d\nu(y)\right]^{\frac{1}{p}} \leq \int_X \left[\int_Y |F(x,y)|^p d\nu(y)\right]^{\frac{1}{p}} d\mu(x).$$

PROOF. c.f. [**HLP52**], Theorem 202.

Let $p > 1$, we have

$$\begin{aligned}
&\int_Y \left|\int_X F(x,y)d\mu(x)\right|^p d\nu(y) \\
&\quad = \int_Y \left[\left|\int_X F(x,y)d\mu(x)\right|^{p-1} \left|\int_X F(x,y)d\mu(x)\right|\right] d\nu(y) \\
&\quad \leq \int_Y \left[\left|\int_X F(x,y)d\mu(x)\right|^{p-1} \int_X |F(x,y)|d\mu(x)\right] d\nu(y) \\
&\quad = \int_Y \left[\int_X \left|\int_X F(w,y)d\mu(w)\right|^{p-1} |F(x,y)|d\mu(x)\right] d\nu(y) \\
&\quad = \int_X \left[\int_Y \left|\int_X F(w,y)d\mu(w)\right|^{p-1} |F(x,y)|d\nu(y)\right] d\mu(x),
\end{aligned}$$

where the last equality is given by Fubini's theorem.

Let $q = \frac{p}{p-1}$, for every $x \in X$, by Hölder's inequality we obtain

$$\begin{aligned}
&\int_Y \left|\int_X F(w,y)d\mu(w)\right|^{p-1} |F(x,y)|d\nu(y) \\
&\quad \leq \left(\int_Y \left|\int_X F(w,y)d\mu(w)\right|^{q(p-1)} d\nu(y)\right)^{\frac{1}{q}} \left(\int_Y |F(x,y)|^p d\nu(y)\right)^{\frac{1}{p}} \\
&\quad = \left(\int_Y \left|\int_X F(w,y)d\mu(w)\right|^{p} d\nu(y)\right)^{\frac{1}{q}} \left(\int_Y |F(x,y)|^p d\nu(y)\right)^{\frac{1}{p}}.
\end{aligned}$$

Hence, putting both expressions together

$$\begin{aligned}
&\int_Y \left| \int_X F(x,y) d\mu(x) \right|^p d\nu(y) \\
&\le \int_X \left[\left(\int_Y \left| \int_X F(w,y) d\mu(w) \right|^p d\nu(y) \right)^{\frac{1}{q}} \left(\int_Y |F(x,y)|^p d\nu(y) \right)^{\frac{1}{p}} \right] d\mu(x) \\
&\quad = \left(\int_Y \left| \int_X F(w,y) d\mu(w) \right|^p d\nu(y) \right)^{\frac{1}{q}} \int_X \left[\int_Y |F(x,y)|^p d\nu(y) \right]^{\frac{1}{p}} d\mu(x).
\end{aligned}$$

Finally, if the first factor on the right-hand side of the last expression vanishes, the result follows trivially. Otherwise, we can divide both sides by it and obtain

$$\left[\int_Y \left| \int_X F(x,y) d\mu(x) \right|^p d\nu(y) \right]^{1-\frac{1}{q}} \le \int_X \left[\int_Y |F(x,y)|^p d\nu(y) \right]^{\frac{1}{p}} d\mu(x).$$

And the theorem follows since $\frac{1}{p} + \frac{1}{q} = 1$. □

2. Fourier analysis on locally compact Abelian groups

We will give in this section an introduction to the basic theorems of Fourier Analysis, following Rudin's classical reference *Fourier Analysis on Groups*, [**Rud62**]. As it is done there, we will introduce the theory for the general class of locally compact Abelian groups. Further on this text, in the last chapter, we will consider the particular cases of the unit circle, the integers and the real line.

Unless something else is mentioned, we will consider additive locally compact Abelian groups.

On every locally compact Abelian group G there exists a positive regular measure μ_G, called the *Haar measure*, such that it is not identically 0 and it is translation-invariant. This is, for every Borel set $E \in \mathscr{B}(G)$ and every $x \in G$ we have

$$\mu_G(E) = \mu_G(x + E).$$

The existence of the Haar measure is proved in a constructive manner by building a translation-invariant linear functional over the set of compactly supported continuous functions on the group and then applying *Riesz representation theorem.*

An important fact about the Haar measure is that it is *unique* up to multiplication by a positive constant. In particular, when the group G is compact, we speak of the *probability or normalized Haar measure* which is such that $\mu_G(G) = 1$.

In the locally compact Abelian group $(\mathbb{R}, +)$ we consider a particular choice of Haar measure, called the *Lebesgue measure.* It is such that the measure of any interval $[a,b] \subset \mathbb{R}$ is $b-a$. For the n-dimensional case $(\mathbb{R}^n, +)$, we consider the product measure of n-th power of Lebesgue measure, which is also called the Lebesgue measure.

Let G be a locally compact Abelian group, a *character* on G is a group homomorphism $\gamma : G \to S^1$ such that $\gamma(x+y) = \gamma(x)\gamma(y)$, for every $x, y \in G$. The *dual group* of G in the sense of Pontryagin is the set of all continuous characters of G, and it is denoted by $\widehat{G}$. The additive structure of the group $\widehat{G}$ is given by

$$(\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x), \text{ for every } x \in G.$$

We can endow $\widehat{G}$ with a topology with respect to which it is itself a locally compact Abelian group. Indeed, for every compact $K \subset G$ and every $r > 0$, the open subsets

$$N(K, r) := \{\gamma \in \widehat{G} : \gamma(x) \in D(1, r) \text{ for all } x \in K\}$$

and their translates determine a basis for this topology of $\widehat{G}$.

We will mention some classical examples of locally compact Abelian groups and their duals that will be of interest in the future.

- Let $(\mathbb{R}, +)$ be the additive group of the real line with the natural topology. It can be proved that its dual group is isomorphic to $\mathbb{R}$.
- Let $(\mathbb{R}/\mathbb{Z}, +)$ be the additive group of the real numbers modulo the integers, which is homeomorphic to the multiplicative group $(S^1, \cdot)$. In this situation, the dual group of $\mathbb{R}/\mathbb{Z}$ can be identified with $\mathbb{Z}$.
- Let $(\mathbb{Z}, +)$ be the additive group of the integers, then its dual group is isomorphic to $\mathbb{R}/\mathbb{Z}$.

Let μ_G be a Haar measure on G, recall that for every $p \geq 1$ the space $\mathrm{L}^p(G, \mu_G)$ is defined as the space of functions $F : G \to \mathbb{C}$ such that

$$\int_G |F(x)|^p d\mu_G(x) < +\infty.$$

Since the Haar measure is unique up to multiplication by a positive constant, for any pair of Haar measures μ_G and μ'_G on G the spaces $\mathrm{L}^p(G, \mu_G)$ and $\mathrm{L}^p(G, \mu'_G)$ coincide and they will be denoted by $\mathrm{L}^p(G)$. In particular, we will say that the functions in $\mathrm{L}^1(G)$ are *Haar-integrable*.

Given a function $F \in \mathrm{L}^1(G)$, we define its *Fourier transform* relative to the Haar measure μ_G as the function $\widehat{F} : \widehat{G} \to \mathbb{C}$ given by

$$\widehat{F}(\gamma) = \int_G F(x)\gamma(-x) d\mu_G(x), \text{ for every } \gamma \in \widehat{G}.$$

It can be shown that the set of functions $\widehat{F}$ obtained this way is dense in $\mathscr{C}_0(\widehat{G})$, the set of continuous functions on $\widehat{G}$ vanishing at infinity.

Let $M(G)$ be the set of complex-valued regular measures on G. We define *Fourier-Stieltjes transform* of a measure $\mu \in M(G)$ as the function $\widehat{\mu} : \widehat{G} \to \mathbb{C}$ given by

$$\widehat{\mu}(\gamma) = \int_G \gamma(x) d\mu(x), \ \gamma \in \widehat{G}. \tag{1.2}$$

The set of all functions defined in this manner will be denoted by $B(\widehat{G})$.

Our goal now will be to state the so-called *Fourier inversion formula*. For this reason, we will introduce some previous concepts that will give us a hint on how this result is obtained.

Given a complex-valued function ϕ on G, we say that it is *positive definite* if it satisfies the inequality

$$\sum_{n,m=1}^{N} c_m \overline{c_m} \phi(x_n - x_m) \geq 0,$$

for every choice of elements $x_1, \ldots, x_N \in G$ and $c_1, \ldots, c_N \in \mathbb{C}$. There are many properties that can be deduced from this definition, among them we point out that ϕ is bounded and that it is uniformly continuous whenever ϕ is continuous at 0. Every character on G is a positive definite function. Moreover, every linear combination of characters with positive coefficients is positive definite. Another significant example of positive definite functions are given by

$$\phi(x) := \int_{\widehat{G}} \gamma(x) d\mu(\gamma), \text{ for every } x \in G,$$

where μ is a positive measure on $M(\widehat{G})$, the set of complex-valued regular measures on $\widehat{G}$.

Bochner's theorem gives an important characterization of positive definite functions. It establishes that a continuous function $\phi : G \to \mathbb{C}$ is positive definite if, and only if, there is a positive measure $\mu \in M(\widehat{G})$ such that

$$\phi(x) = \int_{\widehat{G}} \gamma(x) d\mu(\gamma), \text{ for every } x \in G.$$

Let us define the set $B(G)$ given by all functions $F : G \to \mathbb{C}$ such that there is some measure $\mu \in M(\widehat{G})$ with

$$F(x) = \int_{\widehat{G}} \gamma(x) d\mu(\gamma), \ x \in G.$$

Considering the Jordan decomposition of the complex-valued measure μ together with Bochner's theorem we are able to deduce that $B(G)$ coincides with the set of all finite linear combinations, with complex coefficients, of positive definite functions on G.

We are now able to state the Fourier inversion theorem.

THEOREM 1.7 (Fourier inversion theorem). *Let $F \in \mathrm{L}^1(G) \cap B(G)$, then the Fourier transform $\widehat{F}$, relative to a fixed Haar measure μ_G in G, is in $\mathrm{L}^1(\widehat{G})$. Moreover, there is a unique Haar measure on $\widehat{G}$, denoted by $\mu_{\widehat{G}}$, such that the following holds*

$$F(x) = \int_{\widehat{G}} \widehat{F}(\gamma)\gamma(x) d\mu_{\widehat{G}}(\gamma), \ \forall x \in G, \tag{1.3}$$

for all $F \in \mathrm{L}^1(G) \cap B(G)$.

The idea of the proof is to build, for a given function $F \in \mathrm{L}^1(G) \cap B(G)$, a positive linear translation-invariant functional T on the set of compactly supported continuous functions on $\widehat{G}$ given by

$$T(\psi \widehat{F}) = \int_{\widehat{G}} \psi(\gamma) d\mu_F(\gamma), \text{ for every } \psi \in \mathscr{C}^0_c(\widehat{G})$$

where μ_F is the measure on $M(\widehat{G})$ satisfying

$$F(x) = \int_{\widehat{G}} \gamma(x) d\mu_F(\gamma).$$

Thus, by a suitable version of the Riesz representation theorem, we deduce that there is a Haar measure on $\widehat{G}$, that we will denote by $\mu_{\widehat{G}}$, such that

$$T(\psi \widehat{F}) = \int_{\widehat{G}} \psi(\gamma) \widehat{F}(\gamma) d\mu_{\widehat{G}}(\gamma), \text{ for every } \psi \in \mathscr{C}^0_c(\widehat{G}).$$

From here we would obtain that $\mu_F = \widehat{F} \mu_{\widehat{G}}$ and deduce (1.3).

Using further techniques, one is able to prove the following. Fix a Haar measure μ_G on G, there is a unique Haar measure $\mu_{\widehat{G}}$ on $\widehat{G}$ such that, for every function $F \in \mathrm{L}^1(G)$ whose Fourier transform $\widehat{F}$ relative to μ_G is in $\mathrm{L}^1(\widehat{G})$, we have

$$F(x) = \int_{\widehat{G}} \widehat{F}(\gamma)\gamma(x) d\mu_{\widehat{G}}(\gamma) \text{ almost everywhere in } G.$$

To conclude this short introduction to the Fourier analysis on locally compact Abelian groups, we will state a result that will be useful afterwards.

THEOREM 1.8 (Plancherel's theorem). *The Fourier transform restricted to* $(\mathrm{L}^1 \cap \mathrm{L}^2)(G)$ *is an isometry with respect to the* L^2*-norms onto a dense linear subset space of* $\mathrm{L}^2(\widehat{G})$. *In fact, it can be uniquely extended to an isometry of* $\mathrm{L}^2(G)$ *onto* $\mathrm{L}^2(\widehat{G})$.

PROOF. **[Rud62]**, Theorem 1.6.1. □

As a corollary to this result, we have *Parseval's formula*

$$\int_G F_1(x)\overline{F_2(x)} d\mu_G(x) = \int_{\widehat{G}} \widehat{F}_1(\gamma)\overline{\widehat{F}_2(\gamma)} d\mu_{\widehat{G}}$$

for every $F_1, F_2 \in \mathrm{L}^2(G)$.

3. Smoothing of integrable functions

In this section we will introduce a way of approximating integrable functions on $\mathbb{R}^n$ by smooth ones. As it will soon be explained, we will do so by convolution with mollifiers. Before defining the convolution of functions, let us introduce some notation and recall the definition of certain function spaces.

Consider the Cartesian coordinates $x_1, \dots, x_n$ on $\mathbb{R}^n$. For every *multi-index* $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$ and every function f on $\mathbb{R}^n$, whenever it makes sense we will write

$$\frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}} = \frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

where $|\boldsymbol{\alpha}| = \alpha_1 + \ldots + \alpha_n$. If $\boldsymbol{\alpha} = \mathbf{0}$, we set $\frac{\partial^{|\mathbf{0}|} f}{\partial \boldsymbol{x}^{\mathbf{0}}} = f$.

For every open subset $U \subset \mathbb{R}^n$ and every integer $k \geq 0$, the space $\mathscr{C}^k(U)$ is the set of all functions $f : U \to \mathbb{C}$ such that $\frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}}$ is continuous for every $|\boldsymbol{\alpha}| \leq k$. The set of *smooth functions* on U is defined as

$$\mathscr{C}^\infty(U) = \bigcap_{k \geq 0} \mathscr{C}^k(U).$$

Let us, for a moment, denote the Lebesgue measure on $\mathbb{R}^n$ by λ. Given an open subset $U \subset \mathbb{R}^n$, the restriction of λ to U is defined as $\lambda_U(E) = \lambda(E \cap U)$ for every measurable set E on $\mathbb{R}^n$. In the first section, we gave the definition for the spaces $\mathrm{L}^p(U, \lambda_U)$, we will now define the *local* L^p-spaces. For every $1 \leq p \leq \infty$, the space $\mathrm{L}^p_{loc}(U, \lambda_U)$ is the set of all equivalence classes of measurable functions in U such that $f \in \mathrm{L}^p(V, \lambda_V)$ for some relatively compact subset $V \subset U$. Since we are considering $U \subset \mathbb{R}^n$, we are considering the corresponding restriction of the Lebesgue measure and it will induce no confusion if we write $\mathrm{L}^p_{loc}(U)$.

We have the following embeddings

$$\mathscr{C}^k(U) \hookrightarrow \mathscr{C}^0(U) \hookrightarrow \mathrm{L}^\infty_{loc}(U), \text{ for every } k \geq 0$$

and

$$\mathrm{L}^p_{loc}(U) \hookrightarrow \mathrm{L}^1_{loc}(U), \text{ for every } 1 \leq p \leq \infty.$$

The last one is given by Hölder's inequality.

As we said at the beginning of the section, we will approximate functions in $\mathrm{L}^1(U)$ and $\mathrm{L}^1_{loc}(U)$ by smooth ones. For this reason we will define the convolution product of measurable functions. From now on, we will denote by $d\boldsymbol{x} = dx_1 \ldots dx_n$ the Lebesgue measure on $\mathbb{R}^n$.

Let f and g be two measurable functions on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |f(\boldsymbol{y} - \boldsymbol{x}) g(\boldsymbol{x})| d\boldsymbol{x} < \infty \text{ for every } \boldsymbol{y} \in \mathbb{R}^n.$$

Then we define the *convolution* of f and g as the function on $\mathbb{R}^n$ given by

$$(f * g)(\boldsymbol{y}) = \int_{\mathbb{R}^n} |f(\boldsymbol{y} - \boldsymbol{x}) g(\boldsymbol{x})| d\boldsymbol{x}.$$

The convolution is a measurable function. It is easy to verify that it is commutative. This is done by making a suitable change of variables and recalling that the Lebesgue measure is a Haar measure, i.e. it is invariant under translations.

Let $\boldsymbol{x} \in \mathbb{R}^n$ and $r > 0$, we denote the open disc on $\mathbb{R}^n$ of centre $\boldsymbol{x}$ and radius r by

$$D(\boldsymbol{x}, r) = \{\boldsymbol{y} \in \mathbb{R}^n : \|\boldsymbol{y} - \boldsymbol{x}\| < r\},$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$.

For every open subset $U \subset \mathbb{R}^n$, we denote by $\mathscr{C}_c^\infty(U)$ the set of all compactly supported smooth functions on U. Recall that the support of a function $f : U \to \mathbb{C}$ is given by

$$\mathrm{supp}(f) = \overline{\{x \in U : f(x) \neq 0\}}.$$

The following lemma gives us enough conditions under which the convolution product is well defined. Moreover, these conditions will guarantee the smoothness of the convolution.

Lemma 1.9. *Let $f \in \mathrm{L}^1_{loc}(\mathbb{R}^n)$ and $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$. Then*

(1) $f * \varphi \in \mathscr{C}^\infty(\mathbb{R}^n)$,

(2) For every multi-index $\boldsymbol{\alpha}$ we have

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}}(f * \varphi) = f * \frac{\partial^{|\boldsymbol{\alpha}|}\varphi}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}}.$$

(3) If $\mathrm{supp}(\varphi) \subset D(0, r)$, then $\mathrm{supp}(f * \varphi)$ is contained in the r-neighborhood of $\mathrm{supp}(f)$.

Proof. **[Gri09]**, Lemma 2.1. □

A *mollifier* on $\mathbb{R}^n$ is a function $\varphi \in \mathscr{C}^\infty(\mathbb{R}^n)$ such that $\varphi \geq 0$ on $\mathbb{R}^n$, $\mathrm{supp}(\varphi) \subset D(\mathbf{0}, 1)$ and

$$\int_{\mathbb{R}^n} \varphi(\boldsymbol{x}) d\boldsymbol{x} = 1.$$

Given a mollifier φ and $\varepsilon > 0$, the function defined by

$$\varphi_\varepsilon(\boldsymbol{x}) = \frac{1}{\varepsilon^n}\varphi\left(\frac{\boldsymbol{x}}{\varepsilon}\right)$$

is also a mollifier and we have that $\mathrm{supp}(\varphi_\varepsilon) \subset D(0, \varepsilon)$.

Let us give a classical example of mollifiers, for any $a \in (0, 1]$

$$\varphi(\boldsymbol{x}) = \begin{cases} C \exp\left(-\frac{1}{(\|\boldsymbol{x}\|^2 - a^2)^2}\right), & \text{if } \|\boldsymbol{x}\| < a \\ 0, & \text{otherwise,} \end{cases} \tag{1.4}$$

where C is a positive constant such that $\int_{\mathbb{R}^n} \varphi(\boldsymbol{x}) d\boldsymbol{x} = 1$.

Theorem 1.10. *Let φ be a mollifier. Then the following holds*

(i) If f is uniformly continuous on $\mathbb{R}^n$, then

$$\lim_{\varepsilon \to 0} f * \varphi_\varepsilon = f \textit{ uniformly on } \mathbb{R}^n.$$

(ii) If $f \in \mathscr{C}^0(\mathbb{C})$, then

$$\lim_{\varepsilon \to 0} f * \varphi_\varepsilon = f \textit{ uniformly on compacts on } \mathbb{R}^n.$$

(iii) *Let $1 \leq p < \infty$, if $f \in \mathrm{L}^p(\mathbb{R}^n)$, then $f * \varphi \in \mathrm{L}^p(\mathbb{C})$ and*
$$\|f * \varphi\|_{\mathrm{L}^p} \leq \|f\|_{\mathrm{L}^p}.$$
Moreover,
$$f * \varphi_\varepsilon \xrightarrow[\varepsilon \to 0]{\mathrm{L}^p} f.$$
(iv) *Let $1 \leq p < \infty$, if $f \in \mathrm{L}^p_{loc}(\mathbb{C})$, then*
$$f * \varphi_\varepsilon \xrightarrow[\varepsilon \to 0]{\mathrm{L}^p_{loc}} f.$$

The first part of the theorem is a direct consequence of the uniform continuity of the function. The second part, after noticing that every continuous function on a compact subset of $\mathbb{R}^n$ is uniformly continuous, follows directly from the first one. The third part is deduced from Fubini's theorem and Hölder's inequality. The last part follows from the third one. The complete proof of this statement appears in [**Gri09**], Lemma 2.4, Theorem 2.11 and Exercise 2.18.

Observe that Theorem 1.10 gives us a method to build, from a given integrable function, a sequence of smooth functions whose (uniform, L^p) limit is the function itself. This way we will be able to use several tools and techniques for integrable functions that where designed for smooth functions.

4. Riemannian manifolds

We will give in this section a short introduction to the theory of Riemannian manifolds, recalling the basic notions and properties, in order to be able to give the definition of the *Laplace operator*. Afterwards, we will consider the particular case of the Riemann sphere: the projective complex line together with a certain metric, the so-called Fubini-Study metric.

As we have been doing in the previous parts of this preliminary chapter, in order to shorten this introduction we will omit all the proofs. However, we refer the reader to [**Gri09**], [**GHL04**], where all the details can be found.

A *smooth manifold* of dimension n is a connected Hausdorff second-countable topological space M together with a $\mathscr{C}^\infty$-atlas $\mathcal{A}$ of dimension n. Recall that an atlas of dimension n is a collection of charts $\{(U_i, \varphi_i)\}_{i \in I}$, where $U_i \subset M$ is open and φ_i is an homeomorphism between U_i and an open subset in $\mathbb{R}^n$ and they are such that $M = \bigcup_{i \in I} U_i$. An atlas $\mathcal{A}$ is $\mathscr{C}^\infty$ if for every pair of charts $(U_j, \varphi_j), (U_k, \varphi_k) \in \mathcal{A}$ with $U_j \cap U_k \neq \emptyset$, we have that $\varphi_j \circ \varphi_k^{-1}$ and $\varphi_k \circ \varphi_j^{-1}$ are smooth. It can be proved that smooth manifolds are locally compact. By abuse of notation, we will omit the atlas. Let M be a smooth manifold and k a positive integer, we denote by $\mathscr{C}^k(M)$ the set of functions $f : M \to \mathbb{R}$ such that, for every chart (U, φ) we have $f \circ \varphi^{-1} \in \mathscr{C}^k(\varphi(U))$. We will also denote $\mathscr{C}^\infty(M) = \bigcap_{k \geq 0} \mathscr{C}^k(M)$.

Let M be a smooth manifold of dimension n and $p \in M$ a point. The *tangent space* of M at p, denoted by T_pM, is the space of $\mathbb{R}$-differentiations at the point p. This is, the set of maps $\xi : \mathscr{C}^\infty(M) \to \mathbb{R}$ such that ξ is linear and $\xi(fg) = \xi(f)g(p) + \xi(g)f(p)$, for every $f, g \in \mathscr{C}^\infty(M)$. It is easy to see that T_pM is an $\mathbb{R}$-linear space of dimension n.

Consider a chart (U, φ) of M such that $p \in U$ and let $x_1, \dots, x_n$ its local coordinates. The partial derivative with respect to x_j evaluated at p is an element in T_pM defined by

$$\left.\frac{\partial}{\partial x_j}\right|_p (f) = \frac{\partial f}{\partial x_j}(p) := \frac{\partial (f \circ \varphi^{-1})}{\partial x_j}(\varphi(p)), \text{ for every } f \in \mathscr{C}^\infty(M).$$

It can be proved that $\left.\frac{\partial}{\partial x_1}\right|_p, \dots, \left.\frac{\partial}{\partial x_n}\right|_p$ are linearly independent and determine a basis for T_pM. For any $\xi \in T_pM$ there are $\xi_1, \dots, \xi_n \in \mathbb{R}$ with

$$\xi = \xi_1 \left.\frac{\partial}{\partial x_1}\right|_p + \dots + \xi_n \left.\frac{\partial}{\partial x_n}\right|_p$$

For every $f \in \mathscr{C}^\infty(M)$, we have

$$\xi(f) = \xi_1 \frac{\partial f}{\partial x_1}(p) + \dots + \xi_n \frac{\partial f}{\partial x_n}(p) =: \frac{\partial f}{\partial \xi}(p).$$

Let $f \in \mathscr{C}^\infty(M)$ and U be a chart of M containing the point p, we define the *differential* at the point p, $df(p)$, as the linear functional on T_pM given by

$$(df(p), \xi) := \xi(f), \text{ for any } \xi \in T_pM.$$

So $df(p)$ is an element on the dual space T_pM^*, called the *cotangent space* of M at p. It is easy to verify that T_pM^* is also an $\mathbb{R}$-linear space of dimension n and that the dual basis of $\left.\frac{\partial}{\partial x_1}\right|_p, \dots, \left.\frac{\partial}{\partial x_n}\right|_p$ is the basis $dx_1(p), \dots, dx_n(p)$. Using this notation, we can write locally for any $f \in \mathscr{C}^\infty(M)$

$$df(p) = \frac{\partial f}{\partial x_1}(p) dx_1(p) + \dots + \frac{\partial f}{\partial x_n}(p) dx_n(p).$$

A *Riemannian metric* (or metric tensor) g on a smooth n-dimensional manifold M is a familly $\{g(p)\}_{p \in M}$ such that $g(p)$ is a symmetric positive definite bilinear form on the tangent space T_pM that depends smoothly on $p \in M$. Observe that the bilinear form $g(p)$ defines an inner product $\langle \cdot, \cdot \rangle_g$ on T_pM given by

$$\langle \xi, \eta \rangle_g = g(p)(\xi, \eta), \text{ for every } \xi, \eta \in T_pM$$

which makes T_pM into an Euclidean space. Locally, on a chart U containing the point p, we have

$$\langle \xi, \eta \rangle_g = (\xi_1, \dots, \xi_n)(g_{i,j}(p))_{i,j}(\eta_1, \dots, \eta_n)^T,$$

where $g(p) = (g_{i,j}(p))_{i,j}$ is a $n \times n$ symmetric positive definite matrix and the components $g_{i,j}$ are smooth functions on the corresponding chart.

A *Riemannian manifold* is a pair (M, g) consisting of a smooth n-dimensional manifold M together with a Riemannian metric g.

An important fact about the metric tensor g is that it gives a canonical way of identifying the tangent and the cotangent spaces at a point p. Indeed, for any $\xi \in T_pM$, denote by $g(p)\xi$ the covector in T_pM^* such that, for every $\eta \in T_pM$

$$\langle g(p)\xi, \eta \rangle = g(p)(\xi, \cdot)(\eta) = \langle \xi, \eta \rangle_g.$$

Therefore, we can define the linear map $g(p) : T_pM \to T_pM^*$. It is easy to see that this map is injective and, since both the tangent and the cotangent space have the same dimension, we deduce that it is a bijection with inverse map $g^{-1}(p) : T_pM^* \to T_pM$.

The *gradient* of a function $f \in \mathscr{C}^\infty(M)$ at a point $p \in M$ is defined as

$$\nabla_g f(p) = g^{-1}(p) df(p).$$

Note that $\nabla_g f(p)$ is an element in T_pM and, for every $\eta \in T_pM$, satisfies

$$\langle \nabla_g f(p), \eta \rangle_g = \eta(f) = \frac{\partial f}{\partial \eta}(p).$$

Locally, on any chart U of M containing p, we can write

$$\nabla_g f(p) = \begin{pmatrix} g^{1,1}(p) & \dots & g^{1,n}(p) \\ \vdots & & \vdots \\ g^{n,1}(p) & \dots & g^{n,n}(p) \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x_1}(p) \\ \vdots \\ \frac{\partial f}{\partial x_n}(p) \end{pmatrix},$$

where $g^{-1}(p) = (g^{i,j}(p))_{i,j}$.

The following theorem establishes that any Riemannian manifold (M, g) has a canonical measure μ on $\mathscr{B}(M)$, the Borel σ-algebra of M. This measure is called the *Riemannian measure*.

THEOREM 1.11. *For Riemannian manifold (M, g) there is a measure μ on $\mathscr{B}(M)$ such that, in any chart U we have $d\mu = \sqrt{\det(g)}d\lambda$, where $\det(g)$ is the determinant of the metric matrix $g = (g_{i,j})_{i,j}$ and λ is the Lebesgue measure on $U \subset \mathbb{R}^n$. Moreover, μ is a complete and regular measure.*

Note that, since the Riemann measure is finite on compact sets, every compactly supported continuous function is integrable with respect to μ.

A very important fact about Riemannian manifolds is that they can be seen as metric spaces. For a Riemannian manifold, the *geodesic distance* between two points p_1 and p_2 is defined as the infimum of the lengths of all smooth paths in M connecting p_1 and p_2. It can be proved that it does define a distance and that its induced topology coincides with the original topology of the manifold.

A *vector field* in a Riemannian manifold (M, g) is a collection $\{\boldsymbol{v}(p)\}_{p\in M}$ such that $\boldsymbol{v}(p) \in T_pM$ for every $p \in M$. In local coordinates, we have

$$\boldsymbol{v}(p) = v_1(p) \left.\frac{\partial}{\partial x_1}\right|_p + \ldots + v_n(p) \left.\frac{\partial}{\partial x_n}\right|_p.$$

We will say that the vector field is smooth if, on every chart, the functions $v_j(p)$ are smooth.

Before giving the definition of the divergence of a smooth vector field on a Riemannian manifold let us recall its definition on the Euclidean space $\mathbb{R}^n$ and state the so-called *divergence theorem*. Choose coordinates $x_1, \ldots, x_n$ in $\mathbb{R}^n$, for a vector field $\boldsymbol{v} = (v_1, \ldots, v_n)$ with $v_j \in \mathscr{C}^1(\mathbb{R}^n)$, its divergence is given by

$$\operatorname{div}(\boldsymbol{v}) = \frac{\partial v_1}{\partial x_1} + \ldots + \frac{\partial v_n}{\partial x_n}.$$

Let $U \subset \mathbb{R}^n$ be a relatively compact open subset with smooth boundary ∂U, $V \subset \mathbb{R}^n$ such that $\overline{U} \subset V$ and let $\boldsymbol{v}$ a vector field on V of class $\mathscr{C}^1$. The divergence theorem states that

$$\int_U \operatorname{div}(\boldsymbol{v}) dx_1 \ldots dx_n = \int_{\partial U} \boldsymbol{v} \cdot \boldsymbol{n} d\sigma,$$

where $\boldsymbol{n}$ is the outward normal unit vector field on ∂U, σ the volume measure on ∂U and $\cdot$ indicates the Euclidean inner product in $\mathbb{R}^n$.

We give now the definition of the *divergence* of a smooth vector field on a Riemannian manifold in terms of the following result.

THEOREM 1.12. *Consider a smooth vector field $\boldsymbol{v} = \{\boldsymbol{v}(p)\}_{p \in M}$ on a Riemannian manifold (M, g). There is a unique smooth function on M, denoted by $\operatorname{div}_g \boldsymbol{v}$, such that the following holds*

$$\int_M (\operatorname{div}_g \boldsymbol{v}) u d\mu = -\int_M \langle \boldsymbol{v}, \nabla_g u \rangle_g d\mu, \text{ for any } u \in \mathscr{C}_c^\infty(M),$$

where $\mathscr{C}_c^\infty(M)$ denotes the set of smooth functions with compact support.

Alternatively, we can give a local definition for the divergence of a smooth vector field $\boldsymbol{v} = \{\boldsymbol{v}(p)\}_{p \in M}$ as follows. On every chart with local coordinates $x_1, \ldots, x_n$, we define

$$\operatorname{div}_g \boldsymbol{v} = \sum_{i=1}^n \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x_i} \left(\sqrt{\det(g)} v_i \right).$$

It can be proved that, in the intersection of any two charts, it defines the same function.

Observe that, for the particular case of $\mathbb{R}^n$ together with the Euclidean metric, Theorem 1.12 is a particular case of the divergence theorem. Indeed, let $U \subset \mathbb{R}^n$ be an open subset, consider a function $u \in \mathscr{C}_c^1(U)$ and a vector field $\boldsymbol{v}$ on U of class $\mathscr{C}^1$. By the divergence theorem applied to the vector field $u\boldsymbol{v}$, since u vanishes on ∂U and $\operatorname{div}(u\boldsymbol{v}) = \nabla u \cdot \boldsymbol{v} + u \operatorname{div} \boldsymbol{v}$, we obtain

$$\int_U \operatorname{div}(\boldsymbol{v}) u d\boldsymbol{x} = -\int_U \nabla u \cdot \boldsymbol{v} d\boldsymbol{x}.$$

Once the gradient and the divergence have been defined, we can proceed with the definition of the *(classical) Laplace operator* for smooth functions, also called the *Laplace-Beltrami operator*. Let us recall first the definition of the Laplacian on the Euclidean space $\mathbb{R}^n$ with coordinates $x_1, \ldots, x_n$ acting on the set of smooth functions $\mathscr{C}^\infty(\mathbb{R}^n)$, namely

$$\Delta = \frac{1}{2\pi}\left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}\right).$$

In this text, in order to normalize certain expressions, we will consider this particular multiple of the *usual* Laplace operator in $\mathbb{R}^n$. The reason of this normalization will be cleared out in section 6.

The Laplace operator on (M, g) is defined as $\Delta_g = \frac{1}{2\pi} \operatorname{div}_g \circ \nabla_g$. This is, for every smooth function f on M, its Laplacian is given by $\Delta_g f = \frac{1}{2\pi} \operatorname{div}_g(\nabla_g f)$. In local coordinates, we have

$$\Delta_g f(p) = \frac{1}{2\pi} \sum_{i=1}^{n} \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x_i} \left(\sqrt{\det(g)} \sum_{j=1}^{n} g^{i,j} \frac{\partial f}{\partial x_j} \right).$$

From the definition, since the divergence of a smooth vector field is smooth, we deduce that the Laplacian of a smooth function is necessarily smooth. Observe that the definition of the Laplacian in a Riemannian manifold (M, g) depends on the metric, hence we will write Δ_g. The notation Δ will always refer to the Laplace operator on the Euclidean space $\mathbb{R}^n$.

The following result, which will be of great interest in the future, is a consequence of Theorem 1.12.

THEOREM 1.13 (Green's formula). *Let $u, v \in \mathscr{C}^\infty(M)$ and such that at least one of them is compactly supported, then*

$$\int_M u \Delta_g v d\mu = -\frac{1}{2\pi} \int_M \langle \nabla_g u, \nabla_g v \rangle_g d\mu = \int_M v \Delta_g u d\mu.$$

PROOF. **[Gri09]**, Theorem 3.16. □

As we previously did, we will dedicate a couple of lines to this result for the particular case of $\mathbb{R}^n$ together with the Euclidean metric. Let $U \subset \mathbb{R}^n$ be an open subset and $u, v \in \mathscr{C}^2(U)$ be such that at least one of them has compact support. Applying the divergence theorem to the vector fields $u\nabla v$ and $v \nabla u$ yields to *Green's formula* 1.13:

$$\int_U v \Delta u d\boldsymbol{x} = -\frac{1}{2\pi} \int_U \nabla u \cdot \nabla v d\boldsymbol{x} = \int_U u \Delta v d\boldsymbol{x},$$

Let $U \subset \mathbb{R}^n$ relatively compact open subset and $V \subset \mathbb{R}^n$ an open subset such that $\overline{U} \subset V$. If φ and ψ are two functions in $\mathscr{C}^2(V)$, we have that $\operatorname{div}(\varphi \nabla \psi - \psi \nabla \varphi) = 2\pi(\varphi \Delta \psi - \psi \Delta \varphi)$ and, by the divergence theorem, we obtain the so-called *Green's identity*:

$$\int_U (\varphi \Delta \psi - \psi \Delta \varphi) d\boldsymbol{x} = \frac{1}{2\pi} \int_{\partial U} (\varphi \nabla \psi - \psi \nabla \varphi) \cdot \boldsymbol{n} d\sigma. \tag{1.5}$$

4.1. The Riemann sphere. Let us consider the particular case of $\mathbb{P}^1(\mathbb{C})$ together with the atlas given by the *usual charts* (U_0, α_0) and (U_1, α_1), where the open subsets are

$$U_0 = \{(z_0 : z_1) : z_0 \neq 0\} \text{ and } U_1 = \{(z_0 : z_1) : z_1 \neq 0\},$$

and the homeomorphisms are given by

$$\begin{array}{cccc} \alpha_0 : & U_0 & \longrightarrow & \mathbb{R}^2, \\ & (z_0 : z_1) & \mapsto & \left(\mathrm{Re}\frac{z_1}{z_0}, \mathrm{Im}\frac{z_1}{z_0}\right) \end{array}$$

$$\begin{array}{cccc} \alpha_1 : & U_1 & \longrightarrow & \mathbb{R}^2. \\ & (z_0 : z_1) & \mapsto & \left(\mathrm{Re}\frac{z_0}{z_1}, \mathrm{Im}\frac{z_0}{z_1}\right) \end{array}$$

The *Fubini-Study metric* g on $\mathbb{P}^1(\mathbb{C})$ is defined locally on any chart U_j with coordinates x, y by

$$g(x,y) = \begin{pmatrix} \frac{4}{(1+x^2+y^2)^2} & 0 \\ 0 & \frac{4}{(1+x^2+y^2)^2} \end{pmatrix}. \tag{1.6}$$

The Riemannian surface $\mathbb{P}^1(\mathbb{C})$ together with the Fubini-Study metric g is called the *Riemann sphere*. It is a well-known fact that the projective complex line can be identified with the unit sphere S^2 in $\mathbb{R}^3$. This is done via the *stereographic projection* $\rho : S^2 \setminus \{(0,0,1)\} \to \mathbb{C}$, that identifies the equator of S^2 with the unit circle $S^1 \subset \mathbb{C}$ and the point $(0,0,1)$ with the point at infinity $(0:1)$.

As it was mentioned above, there is a geodesic distance associated to every Riemannian manifold. For the particular case of the Riemann sphere, it is exactly the *spherical distance*, which is defined as

$$\mathrm{d}(p,p') := 2\arccos\left(\frac{|p_0\overline{p_0'} + p_1\overline{p_1'}|}{\sqrt{|p_0|^2+|p_1|^2}\sqrt{|p_0'|^2+|p_1'|^2}}\right),$$

for every $p = (p_0 : p_1)$ and $p' = (p_0' : p_1')$ in $\mathbb{P}^1(\mathbb{C})$. To simplify the computations, we will consider an equivalent distance, the *chordal distance*, defined as follows. For every $p = (p_0 : p_1)$ and $p' = (p_0' : p_1')$, we have

$$\mathrm{d_{ch}}(p,p') := \frac{2|p_0p_1' - p_1p_0'|}{\sqrt{|p_0|^2+|p_1|^2}\sqrt{|p_0'|^2+|p_1'|^2}}.$$

Indeed, we have the following result.

LEMMA 1.14. *For every $p, p' \in \mathbb{P}^1(\mathbb{C})$, we have*

$$\frac{2}{\pi}\,\mathrm{d}(p,p') \leq \mathrm{d_{ch}}(p,p') \leq \mathrm{d}(p,p').$$

PROOF. We will work on the sphere using the stereographic projection. Since the chordal distance $\mathrm{d_{ch}}$ between two points in the sphere is the length of the chord joining them and the spherical distance d is the angle between the vectors both points define, we have

$$\mathrm{d}_{\mathrm{ch}}(p,p') = 2\sin\left(\frac{\mathrm{d}(p,p')}{2}\right), \text{ for every } p,p' \in \mathbb{P}^1(\mathbb{C}).$$

For any choice of points we have that $\mathrm{d}(p,p') \leq \pi$, so we deduce

$$\mathrm{d}_{\mathrm{ch}}(p,p') \leq \mathrm{d}(p,p').$$

Now, let $\beta > 0$ be such that $\beta\,\mathrm{d}(p,p') \leq \mathrm{d}_{\mathrm{ch}}(p,p')$ for all $p,p' \in \mathbb{P}^1(\mathbb{C})$. This is equivalent to saying that $\beta x \leq 2\sin(\frac{x}{2})$ for every $0 \leq x \leq \pi$. By the convexity of the function $2\sin(\frac{x}{2})$, we deduce that the optimal value is $\beta = \frac{2}{\pi}$. □

Let us give explicitly the local expression of the Laplacian of a smooth function on the Riemann sphere. Let $f \in \mathscr{C}^\infty(\mathbb{P}^1(\mathbb{C}))$, on each chart U_j with coordinates x, y, the gradient of f is given by

$$\nabla_g f = \frac{1}{\sqrt{\det(g)}}\left(\frac{\partial f_i}{\partial x}, \frac{\partial f_i}{\partial y}\right),$$

where $f_i = f \circ \alpha_i^{-1}$. Therefore, the Laplacian of f is

$$\Delta_g f = \frac{1}{2\pi}\frac{1}{\sqrt{\det(g)}}\left(\frac{\partial^2 f_i}{\partial x^2} + \frac{\partial^2 f_i}{\partial y^2}\right) = \frac{1}{2\pi}\frac{(1+x^2+y^2)^2}{4}\left(\frac{\partial^2 f_i}{\partial x^2} + \frac{\partial^2 f_i}{\partial y^2}\right). \tag{1.7}$$

Finally, let us compute the volume of the Riemann sphere with respect to the Riemannian measure. Let A denote the Lebesgue measure on $\mathbb{C}$, since $\mu(\infty) = 0$, we have

$$\begin{aligned} \mu(\mathbb{P}^1(\mathbb{C})) &= \int_{\mathbb{P}^1(\mathbb{C})} d\mu = \int_{\mathbb{C}} \sqrt{\det(g(z))} dA(z) \\ &= \int_0^{2\pi}\int_0^{+\infty} \frac{4}{(1+r^2)^2} r dr d\theta = 2\pi \int_0^{+\infty} \frac{2ds}{(1+s)^2} = 4\pi. \end{aligned} \tag{1.8}$$

5. Distributions

The aim of this section is to give an introduction to the theory of distributions on a Riemannian manifold and, in particular, to give the definition of the *distributional Laplace operator*. This notion will be very relevant in the second chapter, where we will develop the theory of potentials in the Riemann sphere.

We will start first by defining the distributions on $\mathbb{R}^n$ and we will later consider the case of general Riemannian manifolds.

5.1. Distributions on $\mathbb{R}^n$. For any open $U \subset \mathbb{R}^n$, the set of *test functions* $\mathcal{D}(U)$ is given by the set $\mathscr{C}_c^\infty(U)$ endowed with the following convergence condition: a sequence $\{\phi_k\}$ converges to ϕ in $\mathcal{D}(U)$ if

(i) for any multi-index $\boldsymbol{\alpha} \in (\mathbb{Z}_{\geq 0})^n$, $\lim_{k\to\infty} \partial^{\boldsymbol{\alpha}}\phi_k = \partial^{\boldsymbol{\alpha}}\phi$ uniformly, and
(ii) there is a compact set $K \subset U$ such that $\operatorname{supp}(\phi_k) \subset K$ for every k.

We will write $\phi_k \xrightarrow{\mathcal{D}} \phi$. This notion of convergence defines a topology on $\mathcal{D}(U)$ and it is such that it makes it a linear topological space. Observe that if $\phi_k \xrightarrow{\mathcal{D}} \phi$, then $\partial^{\boldsymbol{\alpha}}\phi_k \xrightarrow{\mathcal{D}} \partial^{\boldsymbol{\alpha}}\phi$ for every multi-index $\boldsymbol{\alpha}$.

The *space of distributions* $\mathcal{D}'(U)$ on the open subset $U \subset \mathbb{R}^n$ is defined as the dual space of $\mathcal{D}(U)$, i.e. the space of all linear continuous functionals on $\mathcal{D}(U)$. Given a distribution $u \in \mathcal{D}'(U)$ and a test function $\phi \in \mathcal{D}(U)$, the action of u on ϕ is denoted by (u, ϕ). The continuity of u means that $\lim_k(u, \phi_k) = (u, \phi)$ whenever $\phi_k \xrightarrow{\mathcal{D}} \phi$. We deduce easily that $\mathcal{D}'(U)$ is a linear space. A sequence of distributions $\{u_k\}$ converges to u in $\mathcal{D}'(U)$ if

$$\lim_k (u_k, \phi) = (u, \phi), \text{ for every } \phi \in \mathcal{D}(U).$$

Let us give some classical examples of distributions.

- Locally integrable functions. A function $u \in \mathrm{L}^1_{loc}(U)$ may be identified as a distribution with the following rule
$$(u, \phi) = \int_U u\phi d\lambda, \text{ for any } \phi \in \mathcal{D}(U),$$
where λ is the Lebesgue measure on $\mathbb{R}^n$. This identification gives us the inclusion $\mathrm{L}^1_{loc}(U) \hookrightarrow \mathcal{D}'(U)$.
- Any regular complex measure ν on U, determines a distribution in $\mathcal{D}'(U)$ which is given by
$$(\nu, \phi) = \int_U \phi d\nu \text{ for any } \phi \in \mathcal{D}(U).$$
In particular, any finite linear combination of Dirac deltas: let $a_1, \dots, a_s \in U$ and $m_i \in \mathbb{R}$ for $i = 1, \dots, s$, then
$$\left(\sum_{i=1}^s m_i\delta_{a_i}, \phi\right) = \sum_{i=1}^s m_i\phi(a_i), \text{ for any } \phi \in \mathcal{D}(U).$$

For any $j = 1, \dots, n$, the *distributional partial derivative operator* ∂_j on $\mathcal{D}'(U)$ is defined as follows. Let $u \in \mathcal{D}'(U)$ be any distribution, then $\partial_j u$ is a distribution on U given by the identity

$$(\partial_j u, \phi) = -(u, \frac{\partial\phi}{\partial x_j}), \text{ for any } \phi \in \mathcal{D}(U),$$

which is well defined since $\frac{\partial\phi}{\partial x_j} \in \mathcal{D}(U)$.

Observe that if we consider $u \in \mathscr{C}^1(U)$, the integration by parts formula gives us

$$\int_U \frac{\partial u}{\partial x_j} \phi dA = -\int_U u \frac{\partial \phi}{\partial x_j} dA,$$

for every $\phi \in \mathscr{C}_c^\infty(U)$. Hence, the above definition is a generalization for distributions of the classical partial derivative.

More generally, for any multi-index $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, we define the operator $\partial^{\boldsymbol{\alpha}}$ on any $u \in \mathcal{D}'(U)$ as

$$(\partial^{\boldsymbol{\alpha}} u, \phi) = (-1)^{|\boldsymbol{\alpha}|} (u, \frac{\partial^{\boldsymbol{\alpha}} \phi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}), \text{ for any } \phi \in \mathcal{D}(U).$$

It is easy to see that a finite linear combination of operators of the form $\partial^{\boldsymbol{\alpha}_i}$ is also an operator on the space of distributions over U. In particular, we have the distributional Laplace operator $\Delta = \frac{1}{2\pi}\left(\partial_1^2 + \ldots + \partial_n^2\right)$ which is defined, for every $u \in \mathcal{D}'(U)$, as

$$(\Delta u, \phi) = (u, \Delta\phi), \text{ for any } \phi \in \mathcal{D}(U).$$

Recall that the Laplacian of a function $u \in \mathscr{C}^2(U)$ is defined as

$$\Delta u = \frac{1}{2\pi}\left(\frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2}\right).$$

And, in this situation, *Green's Formula* tells us that

$$\int_U \Delta u \phi dA = -\frac{1}{2\pi}\int_U \nabla u \cdot \nabla \phi dA = \int_U u \Delta \phi dA, \text{ for every } \phi \in \mathscr{C}_c^\infty(U)$$

where ∇u and $\nabla \phi$ are the gradient vectors of u and ϕ and $\cdot$ indicates the standard scalar product in $\mathbb{R}^n$. Hence, the definition of the distributional Laplacian as an operator on $\mathcal{D}'(U)$ is a generalization of the classical Laplacian.

5.2. Distributions on Riemannian manifolds. Consider a Riemannian manifold (M, g), the *space of test functions* on M, denoted by $\mathcal{D}(M)$, is defined as the set of all compactly supported smooth functions on M endowed with the following convergence. Let $\{\phi_k\}$ and ϕ in $\mathscr{C}_c^\infty(M)$, we say that ϕ_k converges to ϕ on $\mathcal{D}(M)$ if the following conditions are satisfied:

(i) On every chart $U \subset M$ and for every multi-index $\boldsymbol{\alpha}$, we have that

$$\lim_{k\to\infty} \partial^{\boldsymbol{\alpha}} \phi_k = \partial^{\boldsymbol{\alpha}} \phi \text{ uniformly on } U.$$

(ii) All supports $\operatorname{supp}(\phi_k)$ are contained on a compact subset of M.

Under these conditions, we will write $\phi_k \xrightarrow{\mathcal{D}} \phi$. As in the case of $\mathbb{R}^n$, this convergence induces a topology on the space $\mathcal{D}(M)$ with respect to which it is a linear topological space.

A *distribution* on (M, g) is a continuous linear functional on the space $\mathcal{D}(M)$. It is easy to verify that the set of distributions on M, denoted by $\mathcal{D}'(M)$, is a linear space. Given $u \in \mathcal{D}'(M)$ and $\phi \in \mathcal{D}(M)$, the action of u on ϕ is denoted by (u, ϕ). The continuity of a distribution $u \in \mathcal{D}'(M)$

is characterized by $\lim_k(u,\phi_k) = (u,\phi)$ for every $\phi_k \xrightarrow{\mathcal{D}} \phi$. We say that a sequence of distributions $\{u_k\}$ converges to u in $\mathcal{D}'(M)$, if $\lim_k(u_k,\phi) = (u,\phi)$ for every $\phi \in \mathcal{D}(M)$. We will write $u_k \xrightarrow{\mathcal{D}'} u$.

As it was established by Theorem 1.11, there is a complete and regular measure μ associated to the metric g, called the Riemannian measure. Thus, (M,μ) is a measure space and we can speak about measurable and integrable functions on M. For any $1 \leq p < \infty$, recall that $\mathrm{L}^p(M) = \mathrm{L}^p(M,\mu)$ is the spaces of equivalence classes of measurable functions $f : M \to \mathbb{R}$ such that

$$\int_M |f|^p d\mu < \infty.$$

And, for any $1 \leq p < \infty$, the spaces $\mathrm{L}^p_{loc}(M)$ are the set of equivalence classes of measurable functions f on M such that $f \in \mathrm{L}^p(K)$ for every compact subset $K \subset M$.

Given a function $u \in \mathrm{L}^1_{loc}(M)$, we can consider its associated distribution in $\mathcal{D}'(M)$ in the following way

$$(u,\phi) = \int_M u\phi d\mu, \text{ for every } \phi \in \mathcal{D}(M).$$

It can be proved that $u \in \mathrm{L}^1_{loc}(M)$ vanishes almost everywhere if, and only if, $u = 0$ in $\mathcal{D}'(M)$. Hence, we have that $\mathrm{L}^1_{loc}(M)$ is a subset of $\mathcal{D}'(M)$ and, since convergence in $\mathrm{L}^1_{loc}(M)$ implies convergence in $\mathcal{D}'(M)$, we have

$$\mathrm{L}^1_{loc}(M) \hookrightarrow \mathcal{D}'(M).$$

We will now define the *distributional Laplacian* Δ_g, an operator on the space of distributions of (M,g). Let $u \in \mathcal{D}'(M)$, its distributional Laplacian is given by the identity

$$(\Delta_g u,\phi) = (u,\Delta_g\phi), \text{ for every } \phi \in \mathcal{D}(M).$$

The Laplacian $\Delta_g u$ of a distribution $u \in \mathcal{D}'(M)$ is also a distribution. Indeed, it is a continuous linear functional on $\mathcal{D}(M)$ since, for every $a_1, a_2 \in \mathbb{R}$ and every $\phi_1,\phi_2 \in \mathcal{D}(M)$, we have

$$\begin{aligned}(\Delta_g u, a_1\phi_1 + a_2\phi_2) &= \int_M u\Delta_g(a_1\phi_1 + a_2\phi_2)d\mu \\ &= a_1\int_M u\Delta_g\phi_1 d\mu + a_2\int_M u\Delta_g\phi_2 d\mu = a_1(\Delta_g u,\phi_1) + a_2(\Delta_g u,\phi_2).\end{aligned}$$

And, given a sequence $\{\phi_k\}$ and a functions ϕ in $\mathcal{D}(M)$ such that $\phi_k \xrightarrow{\mathcal{D}} \phi$, we have that $\Delta_g\phi_k \xrightarrow{\mathcal{D}} \Delta_g\phi$ and

$$\begin{aligned}\lim_k(\Delta_g u,\phi_k) = \lim_k \int_M u\Delta_g\phi_k d\mu &= \int_M u \lim_k \Delta_g\phi_k d\mu \\ &= \int_M u\Delta_g\phi d\mu = (\Delta_g u,\phi).\end{aligned}$$

Let $u \in \mathscr{C}^\infty(M) \subset \mathrm{L}^1_{loc}(M)$, by Theorem 1.13, we have that

$$(\Delta_g u, \phi) = \int_M \Delta_g u \phi d\mu = \int_M u \Delta_g \phi d\mu = (u, \Delta_g \phi), \text{ for any } \phi \in \mathcal{D}(M).$$

Hence, for smooth functions on M the definitions of the distributional Laplacian and the classical Laplacian defined on the previous section agree.

We will now introduce the vector field versions of the space of test functions and distributions on a Riemannian manifold. $\vec{\mathcal{D}}(M)$ will denote the space of smooth vector fields on a manifold M endowed with a convergence analogous to the convergence in $\mathcal{D}(M)$. It can be proved that $\vec{\mathcal{D}}(M)$ is a linear space and hence we can consider its dual space $\vec{\mathcal{D}}'(M)$, which is called the *space of distributional vector fields* on M.

We say that a vector field on (M, g) is measurable if, on every chart, all the components are measurable functions. We can then define, for every $1 \leq p < \infty$, the spaces

$$\vec{\mathrm{L}}^p(M) := \{v \text{ measurable vector field on } M : \|v\|_g \in \mathrm{L}^p(M)\}$$

and

$$\vec{\mathrm{L}}^p_{loc}(M) := \{v \text{ measurable vector field on } M : \|v\|_g \in \mathrm{L}^p_{loc}(M)\},$$

where

$$\|v\|_g = \langle v, v \rangle_g^{1/2}.$$

The space $\vec{\mathrm{L}}^2(M)$ is of particular interest since it is a Hilbert space with the inner product

$$(v, w)_{\vec{\mathrm{L}}^2} = \int_M \langle v, w \rangle_g d\mu, \text{ for any } v, w \in \vec{\mathrm{L}}^2(M).$$

Every vector field $v \in \vec{\mathrm{L}}^p_{loc}(M)$ defines a distributional vector field in $\vec{\mathcal{D}}'(M)$ which is given by

$$(v, \psi) = \int_M \langle v, \psi \rangle_g d\mu, \text{ for every } \psi \in \vec{\mathcal{D}}(M).$$

For any distribution $u \in \mathcal{D}'(M)$, the *distributional gradient* is defined as the distributional vector field $\nabla_g u$ given by the identity

$$(\nabla_g u, \psi) = -(u, \mathrm{div}_g(\psi)), \text{ for every } \psi \in \vec{\mathcal{D}}(M).$$

It is clear, by Theorem 1.12, that the distributional gradient of a vector field extends the definition of the classical gradient of a smooth vector field on a Riemannian manifold.

The following result will be useful in the future.

LEMMA 1.15. *Let $\{u_k\}$ and u in $\mathcal{D}'(M)$ such that $u_k \xrightarrow{\mathcal{D}'} u$. Then*

$$\nabla_g u_k \xrightarrow{\vec{\mathcal{D}}'} \nabla_g u.$$

Proof. For every $\phi \in \mathcal{D}(M)$, we have $\lim_k (u_k, \phi) = (u, \phi)$ and, by definition, for $k \geq 1$ we have $(\nabla_g u_k, \psi) = -(u_k, \operatorname{div}_g(\psi))$, for every $\psi \in \vec{\mathcal{D}}(M)$. Now, since the divergence of a smooth compactly supported vector field is in $\mathscr{C}_c^\infty(M)$, passing to the limit we obtain

$$\lim_k (\nabla_g u_k, \psi) = -\lim_k (u_k, \operatorname{div}_g(\psi)) = -(u, \operatorname{div}_g(\psi)) = (\nabla_g u, \psi),$$

for all $\psi \in \vec{\mathcal{D}}(M)$. □

6. Potential theory on the complex plane

We will now give a brief introduction to potential theory on the complex plane. We will start by giving the definitions of harmonic and subharmonic functions and some of the key results that will be needed later on this text. Afterwards, we will define the potential of a certain class of measures and state some important facts about it.

For a more detailed study of the potential theory, where the reader can find the proofs of the classical results appearing below, we refer to [**Ran95**], [**Tsu75**].

In the potential theory, the Laplace operator plays a significative role. As it has been mentioned previously, in order to obtain neat results, we normalize the Laplacian as follows

$$\Delta = \frac{1}{2\pi}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

6.1. Harmonic functions. Let $U \subset \mathbb{C}$ be an open subset, a function $h : U \to \mathbb{R}$ is *harmonic* if $h \in \mathscr{C}^2(U)$ and it is a solution of the *Laplace equation*, this is

$$\Delta h = 0 \text{ on } U.$$

There is a classical result relating harmonic functions and holomorphic ones. On a domain in the complex plane, the real part of any holomorphic function is harmonic. Conversely, every harmonic function on a simply connected domain is the real part of a holomorphic function and this function is unique up to addition of a constant.

In the converse, the assumption of a simply connected domain is required. However, we have that every harmonic function is locally the real part of a holomorphic function. Whence, every harmonic function on an open subset $U \subset \mathbb{C}$ is necessarily smooth.

Let $z \in \mathbb{C}$ and $r > 0$, we will denote

$$\overline{D}(z, r) = \{w \in \mathbb{C} : |z - w| \leq r\}.$$

Given a harmonic function h on an open neighborhood of the closed disc $\overline{D}(z, r)$, the *mean-value property* states that

$$h(z) = \frac{1}{2\pi}\int_0^{2\pi} h(z + re^{i\theta})d\theta.$$

This result is a consequence of *Cauchy's integral formula* and the fact that h is the real part of a holomorphic function on $D(z, r+\varepsilon)$ for some $\varepsilon > 0$.

Actually, the mean-value property characterizes harmonic functions in the following sense. Let U be an open subset of $\mathbb{C}$ and $h : U \to \mathbb{R}$ a continuous function such that, for every $z \in U$ there is some $r_z > 0$ satisfying

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} h(z + e^{i\theta}) d\theta, \text{ for every } 0 \leq r \leq r_z.$$

Then h is harmonic on U.

As a consequence of the mean-value property and its converse, we have the following result.

Corollary 1.16. *Let $\{h_n\}$ be a sequence of harmonic functions on a domain $D \subset \mathbb{C}$ converging locally uniformly to a function h. Then h is harmonic on D.*

Proof. [**Ran95**], Corollary 1.2.8. □

Let us consider now two principles that are a direct consequence of their corresponding holomorphic counterparts.

Let h_1 and h_2 be two harmonic functions on a domain $D \subset \mathbb{C}$. If $h_1 = h_2$ on a non-empty open subset of D, then $h_1 = h_2$ on D. This is the so-called *identity principle* of harmonic functions.

The *maximum principle* states that given a harmonic function h on a domain $D \subset \mathbb{C}$ the following holds.

(i) If h attains a local maximum on D, then h is constant.
(ii) If h can be continuously extended to the closure $\overline{D}$ of D on the Riemann sphere and $h \leq 0$ on the border ∂D, then $h \leq 0$ on D.

The following result will be of interest in the subsequent chapter. It is name *Liouville's theorem* for harmonic functions and it states that if a harmonic function on $\mathbb{C}$ is bounded either above or below, then it is constant. The result is a consequence of the maximum principle and *Harnack's inequality* which says that given a positive harmonic function on a disc $D(z, r)$, we have

$$\frac{r-s}{r+s} h(z) \leq h(z + se^{i\theta}) \leq \frac{r+s}{r-s} h(z).$$

for every $0 < s < r$ and every $0 \leq \theta < 2\pi$.

There is much more to be said about harmonic functions, for example the results about the *Dirichlet problem on the disc*. However, as we announced at the beginning of the section, this will be a short introduction to the theory of potentials.

6.2. Subharmonic functions. In this part of the section, we will give the definition of subharmonic functions on the complex plane. We will later see that these functions are of special interest in potential theory.

Let X be a topological space, a function $u : X \to [-\infty, +\infty)$ is *upper semicontinuous* if for every $\alpha \in \mathbb{R}$ the set $\{x \in X : u(x) < \alpha\}$ is open in X.

As a direct consequence of the definition, we have that every upper semicontinuous function u on a topological space X is bounded above on every compact $K \subset X$, and attains its bound.

Let $U \subset \mathbb{C}$ be an open, a function $u : U \to [-\infty, +\infty)$ is *subharmonic* if it is upper semicontinuous and it satisfies that for every $z \in U$ there is some $r_z > 0$ such that

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta, \text{ for every } 0 \leq r < r_z.$$

This is the so-called *local submean inequality.*

Observe that the integral on the right-hand side of the later expression is well defined. This follows from the fact that it is defined as the difference of two integrals corresponding to the positive and negative parts of the function u. Since the positive part is bounded, its integral is finite and, even if the integral of the negative part is infinite, the resulting difference is well-defined. Note that the function $u \equiv -\infty$ is subharmonic.

Given a holomorphic function f on an open $U \subset \mathbb{C}$, it can be easily proved that $\log |f|$ is subharmonic on U. Moreover, $\log |f|$ is harmonic on the open subset $\{x \in U : f(x) \neq 0\}$.

From the definition, we can deduce that any finite linear combination with positive coefficients of subharmonic functions is also subharmonic and that the maximum of two subharmonic functions is subharmonic.

The *maximum principle* for subharmonic functions states that a subharmonic function u on a domain $D \subset \mathbb{C}$ that attains a global maximum is necessarily constant. Moreover, if $\limsup_{z \to w} u(z) \leq 0$ for all $w \in \partial D$, then $u \leq 0$ on D. As for the case of harmonic functions, we are considering the closure of D on $\mathbb{P}^1(\mathbb{C})$. Hence, if D is not bounded, we have $\infty \in \partial D$.

There is a generalization of the last part of the maximum principle that holds for subharmonic function that do not grow too fast at infinity. This generalization would let us avoid considering the point at infinity as a point of the border of the domain and leads to the following version of *Liouville's theorem* for subharmonic functions: every subharmonic function on $\mathbb{C}$ that is bounded above is constant.

From the maximum principle and further results for harmonic functions, one can deduce that subharmonic functions satisfy the *global submean inequality*: let u be a subharmonic function on an open $U \subset \mathbb{C}$ and let $\overline{D}(z, r) \subset U$. Then

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta.$$

Given a decreasing sequence $\{u_n\}$ of subharmonic functions on an open $U \subset \mathbb{C}$, the pointwise limit $u(x) := \lim_{n \to \infty} u_n(x)$ is subharmonic on U. This is a consequence of the (global) submean inequality.

A significative fact about subharmonic functions is that they are locally integrable, as long as they are not identically $-\infty$. Recall that this means that they are integrable on every compact subset. As a consequence, we have that given a subharmonic function u on a domain $D \subset \mathbb{C}$, no identically $-\infty$, the set $\{z \in U : u(z) = -\infty\}$ has Lebesgue measure zero.

Let $U \subset \mathbb{C}$ be an open and $u \in \mathscr{C}^2(U)$. Then u is subharmonic if and only if $\Delta u \geq 0$ on U. We will see soon that this also holds for the distributional Laplacian of any subharmonic function, which is well defined since subharmonic functions are locally integrable.

The following result will show us how to build, from a given subharmonic function, a decreasing sequence of smooth subharmonic functions that converge pointwisely to the original one. As it was done in the section of smoothing of integrable functions, this is done by convolution with mollifiers.

THEOREM 1.17. *Let u be a subharmonic function on a domain $D \subset \mathbb{C}$, with $u \not\equiv -\infty$. Consider a mollifier φ such that $\varphi(z) = \varphi(|z|)$ and for every $\varepsilon > 0$ set $\varphi_\varepsilon(z) = \frac{1}{\varepsilon^2}\varphi\left(\frac{z}{\varepsilon}\right)$. Then $u_\varepsilon(z) := u * \varphi_\varepsilon(z)$ is smooth on $D_\varepsilon = \{z \in U : \operatorname{dist}(z, \partial D) > \varepsilon\}$ and for every $z \in D$ the following holds*

(i) $\lim_{\varepsilon \to 0} u_\varepsilon(z) = u(z)$,
(ii) $u_{\varepsilon_1}(z) \geq u_{\varepsilon_2}(z) \geq u(z)$ *for every* $0 < \varepsilon_1 \leq \varepsilon_2$.

PROOF. [**Ran95**], Theorem2.7.2. □

From this result we can deduce the *weak identity principle* for subharmonic functions: given subharmonic functions u_1 and u_2 on an open $U \subset \mathbb{C}$ such that $u_1 = u_2$ almost everywhere, we necessarily have that $u_1 = u_2$ on the whole U.

Let us see that the distributional Laplacian of a subharmonic function is indeed positive. Recall that the distributional Laplacian of a locally integrable function u is defined by the identity

$$\int_{\mathbb{C}} \Delta u \phi dA = \int_C u \Delta \phi dA, \text{ for every } \phi \in \mathscr{C}_c^\infty(\mathbb{C}),$$

where dA denotes the Lebesgue measure on $\mathbb{C}$. To prove that, given a subharmonic function u on $\mathbb{C}$, $\Delta u \geq 0$ we have to see that

$$\int_{\mathbb{C}} u \Delta \phi dA \geq 0, \text{ for every } \phi \in \mathscr{C}_c^\infty(\mathbb{C}) \text{ with } \phi \geq 0.$$

By the previous theorem, there is a decreasing sequence of smooth subharmonic functions $\{u_n\}$ such that $\lim_n u_n(z) = u(z)$ on $\mathbb{C}$ and, since $\Delta\phi$ is smooth and compactly supported, by the dominated convergence theorem we have

$$\int_{\mathbb{C}} u \Delta \phi dA = \int_{\mathbb{C}} \lim_n u_n \Delta \phi dA = \lim_n \int_{\mathbb{C}} u_n \Delta \phi dA.$$

Now, by Green's formula

$$\int_{\mathbb{C}} u_n \Delta \phi dA = \int_{\mathbb{C}} \Delta u_n \phi dA \geq 0.$$

Hence, we can conclude that

$$\int_{\mathbb{C}} u\Delta\phi dA \geq 0.$$

6.3. Potentials. At last, we will give the definition of the potential of finite compactly supported positive measures on $\mathbb{C}$. We will see that there is a close relation between potentials and subharmonic functions.

Let ρ be a finite measure on $\mathbb{C}$ with compact support. Its *potential* is the function $g_\rho : \mathbb{C} \to [-\infty, +\infty)$ defined as

$$g_\rho(z) = \int_{\mathbb{C}} \log|z-w| d\rho(w), \text{ for any } z \in \mathbb{C}.$$

The first remark about potentials is given by the following result.

THEOREM 1.18. *Given a finite and compactly supported measure ρ on $\mathbb{C}$, its potential g_ρ is subharmonic on $\mathbb{C}$ and harmonic on $\mathbb{C} \setminus \operatorname{supp}(\rho)$.*

PROOF. [**Ran95**], Theorem 3.1.2. □

The *minimum principle* for potentials establishes that given a finite measure ρ on $\mathbb{C}$ with compact support K, if $g_\rho \geq M$ on K, then $g_\rho \geq M$ on whole complex plane.

At the end of the previous section we saw that the distributional Laplacian of a subharmonic function is positive, whenever it is not identically $-\infty$. Actually, it can be proved that given a subharmonic function on a domain $D \subset \mathbb{C}$, the linear functional on $\mathscr{C}_c^\infty(D)$ defined by

$$\Lambda(\phi) := \int_D u\Delta\phi dA$$

is positive and bounded and it can be extended to a linear functional on the class of compactly supported continuous functions, $\mathscr{C}_c^0(D)$. Hence, by Riesz representation theorem, there is a unique positive finite regular measure $\Delta u dA$ on D such that

$$\Lambda(\phi) = \int_D \phi\Delta u dA \text{ for every } \phi \in \mathscr{C}_c^0(D).$$

If, in particular, we consider the potential of a finite compactly supported measure ρ on the complex plane we obtain the distributional equality[1] $\Delta g_\rho = \rho$. This is, for every $\phi \in \mathscr{C}_c^\infty(\mathbb{C})$ we have

$$\int_{\mathbb{C}} \phi\Delta g_\rho dA = \int_{\mathbb{C}} \phi d\rho. \tag{1.9}$$

The proof of this classical result is done by applying Green's identity (1.5), taking into account that the $\log|z-w|$ is harmonic on $|z-w| > \varepsilon$.

LEMMA 1.19 (Weyl's lemma). *Let u and v be two subharmonic functions on a domain $D \subset \mathbb{C}$, with $u, v \not\equiv -\infty$. If $\Delta u = \Delta v$, then there is a harmonic function h on D such that $u - v = h$.*

[1]It is in this context where the normalization of the Laplace operator makes sense.

PROOF. [**Ran95**], Theorem 3.7.10. □

This lemma together with the distributional identity that we gave above yields to the so-called *Riesz decomposition theorem*: for every subharmonic function u on a domain $D \subset \mathbb{C}$, with $u \not\equiv -\infty$ and every relatively compact open subset $U \subset D$ there is a harmonic function h on U such that

$$u(z) = -\int_U \log|z-w|\Delta u(w)dA(w) + h(z), \text{ for every } z \in U.$$

As an illustrative example let us write the potential of λ_{S^1}, the Lebesgue measure restricted to the unit circle, normalized in such a way that it is a probability measure. We have that

$$g_{\lambda_{S^1}}(z) = \log^+|z| := \max\{\log|z|, 0\}.$$

This is a consequence of the following lemma, which is a particular case of *Jensen's formula.*

LEMMA 1.20. *For every $z \in \mathbb{C}$ and every $r > 0$ the following holds*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|z - re^{i\theta}|d\theta = \max\{\log r, \log|z|\}.$$

Riesz decomposition theorem has very deep consequences for subharmonic functions, as the following lemma certifies. This result is mentioned in [**FRL06**] and its proof, which we will also include, is outlined in their paper.

LEMMA 1.21. *Let u be a subharmonic function on a domain $D \subset \mathbb{C}$ such that $u \not\equiv -\infty$. Then the following holds*

(i) $u \in \mathrm{L}^p_{loc}(D)$ *for every* $1 \leq p < \infty$,
(ii) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in \mathrm{L}^{2-\varepsilon}_{loc}(D)$ *for every* $\varepsilon > 0$.

Observe that, when we consider the partial derivatives of a subharmonic function, we are actually referring to the distributional partial derivatives. The fact that these partial derivatives belong to the space $\mathrm{L}^{2-\varepsilon}_{loc}(D)$ means that they are equal almost everywhere to a function on this space.

PROOF. By Riesz decomposition theorem, for every $z_0 \in D$ and every $R > 0$ with $D(z_0, R) \subset D$, there is a harmonic function h on $D(z_0, R)$ such that

$$u(z) = -\int_{D(z_0,R)} \log|z-w|\Delta u(w)dA(w) + h(z), \ \forall z \in D(z_0, R). \tag{1.10}$$

(i) Let us prove that $u \in \mathrm{L}^p(D(z_0, R))$ for every $1 \leq p < \infty$. Since, h is harmonic on $D(z_0, R)$ it is clear that $h \in \mathrm{L}^p(D(z_0, R))$ for every $1 \leq p \leq \infty$.

On the other hand, we have that $\Delta u dA$ is a positive finite regular measure and, by *Minkowski's Integral inequality* (c.f. Subsection 1.6), we obtain

$$\left(\int_{D(z_0,R)} \left|\int_{D(z_0,R)} \log|z-w|\Delta u(w)dA(w)\right|^p dA(z)\right)^{\frac{1}{p}}$$
$$\leq \int_{D(z_0,R)} \left(\int_{D(z_0,R)} |\log|z-w||^p dA(z)\right)^{\frac{1}{p}} \Delta u(w)dA(w)$$
$$= \int_{D(z_0,R)} \left(\int_{D(z_0-w,R)} |\log|z||^p dA(z)\right)^{\frac{1}{p}} \Delta u(w)dA(w)$$
$$\leq \int_{D(z_0,R)} \left(\int_{D(0,|z_0|+2R)} |\log|z||^p dA(z)\right)^{\frac{1}{p}} \Delta u(w)dA(w).$$

Hence, we only have to see that $\log|z|$ is in $\mathrm{L}^p(D(0,r))$ for every $r>0$ and every finite $p \geq 1$. We will then have proved that u is in $\mathrm{L}^p(D(z_0,R))$ for every disc $D(z_0,R) \subset D$.

We have

$$\int_{D(0,r)} |\log|z||^p dA(z)$$
$$= \int_{D(0,1)} (-\log|z|)^p dA(z) + \int_{1<|z|<r} (\log|z|)^p dA(z).$$

The second summand is clearly finite since $\log|z|$ is continuous on the bounded region $1<|z|<r$. For the first summand, recursively doing integration by parts we obtain

$$\int_{D(0,1)} (-\log|z|)^p dA(z) = (-1)^p 2\pi \int_0^2 (\log r)^p r dr$$
$$= (-1)^p 2\pi \left[\frac{r^2}{2}(\log r)^p\Big|_0^1 - \frac{p}{2}\int_0^1 (\log r)^{p-1} r dr\right]$$
$$= (-1)^{p+1}\pi p \int_0^1 (\log r)^{p-1} r dr$$
$$= (-1)^{p+1}\pi p \left[\frac{r^2}{2}(\log r)^{p-1}\Big|_0^1 - \frac{p-1}{2}\int_0^1 (\log r)^{p-2} r dr\right]$$
$$= (-1)^{p+2}\frac{\pi}{2}p(p-1)\int_0^1 (\log r)^{p-2} r dr$$
$$\cdots$$
$$= (-1)^{2p}\frac{\pi}{2^{p-1}}p(p-1)\cdots 2\int_0^1 r dr = \frac{\pi p!}{2^p}.$$

(ii) With the notation introduced in (1.10), we will see that the distributional partial derivative of u is equal almost everywhere to a function in $\mathrm{L}^{2-\varepsilon}(D(z_0,R))$, for every $\varepsilon>0$. Let us note, for $z\in D(z_0,R)$

$$g(z) := \int_{D(z_0,R)} \log|z-w|\Delta u(w)dA(w).$$

Since the distributional partial derivative operator is linear and it coincides with the classical partial derivative when applied to smooth functions, we only have to see that there is a function g_x in $\mathrm{L}^{2-\varepsilon}(D(z_0,R))$ such that $g_x=\frac{\partial g}{\partial x}$ on $\mathcal{D}'(D(z_0,R))$.

Consider the function

$$g_x(z) = \int_{D(z_0,R)} \frac{\operatorname{Re}(z-w)}{|z-w|}\Delta u(w)dA(w), \text{ with } z\in D(z_0,R).$$

By Minkowski's integral inequality, we have

$$\begin{aligned}\left(\int_{D(z_0,R)} |g_x(z)|^p\, dA(z)\right)^{\frac{1}{p}}&\\ &\leq \int_{D(z_0,R)}\left(\int_{D(z_0,R)} \frac{\operatorname{Re}(z-w)^p}{|z-w|^p}dA(z)\right)^{\frac{1}{p}}\Delta u(w)dA(w).\end{aligned}$$

For every $w\in D(z_0,R)$, we have

$$\begin{aligned}\int_{D(z_0,R)} \frac{\operatorname{Re}(z-w)^p}{|z-w|^p}dA(z) &\leq (2R)^p\int_{D(z_0,R)}\frac{1}{|z-w|^p}dA(z)\\ = (2R)^p\int_{D(z_0-w,R)}\frac{1}{|z|^p}dA(z) &\leq (2R)^p\int_{D(0,|z_0|+2R)}\frac{1}{|z|^p}dA(z)\\ &= 2\pi(2R)^p\int_0^{|z_0|+2R}\frac{dr}{r^{p-1}},\end{aligned}$$

which is finite if $p=2-\varepsilon$, for every $\varepsilon>0$. Hence, we deduce that the function g_x is in $\mathrm{L}^{2-\varepsilon}(D(z_0,R))$.

At last, we will verify that the distributional equality $g_x = \frac{\partial g}{\partial x}$ holds. Let ϕ be a smooth function with compact support on $D(z_0, R)$, we have

$$
\begin{aligned}
(\frac{\partial g}{\partial x}, \phi) = -(g, \frac{\partial \phi}{\partial x}) &= -\int_{D(z_0,R)} g(z) \frac{\partial \phi}{\partial x}(z) dA(z) \\
&= -\int_{D(z_0,R)} \left(\int_{D(z_0,R)} \log|z-w| \Delta u(w) dA(w) \right) \frac{\partial \phi}{\partial x}(z) dA(z) \\
&= -\int_{D(z_0,R)} \left(\int_{D(z_0,R)} \log|z-w| \frac{\partial \phi}{\partial x}(z) dA(z) \right) \Delta u(w) dA(w) \\
&= -\int_{D(z_0,R)} \left(-\int_{D(z_0,R)} \frac{\partial}{\partial x}(\log|z-w|) \phi(z) dA(z) \right) \Delta u(w) dA(w) \\
&= \int_{D(z_0,R)} \left(\int_{D(z_0,R)} \frac{\mathrm{Re}(z-w)}{|z-w|} \phi(z) dA(z) \right) \Delta u(w) dA(w) \\
&= \int_{D(z_0,R)} \left(\int_{D(z_0,R)} \frac{\mathrm{Re}(z-w)}{|z-w|} \Delta u(w) dA(w) \right) \phi(z) dA(z) = (g_x, \phi).
\end{aligned}
$$

$\square$

7. Height of algebraic numbers

In this last part of the chapter, we will introduce the notion of height of points in the n-th projective space $\mathbb{P}^n(\overline{\mathbb{Q}})$, where $\overline{\mathbb{Q}}$ is a fixed algebraic closure of $\mathbb{Q}$. As a particular case, we will give the definition of the height of algebraic numbers. After the definitions, we will present some important properties of the height, which will not be proved. For further detail on the topic, we refer the reader to [**BG07**].

An *absolute value* $|\cdot|$ on a field K is a real-valued function such that $|x| \geq 0$ for every $x \in K$, $|x| = 0$ if and only if $x = 0$, $|xy| = |x| \cdot |y|$ for every $x, y \in K$ and satisfies the *triangle inequality* $|x+y| \leq |x| + |y|$ for every pair of points x and y in K. If, in addition, it satisfies $|x+y| \leq \max\{|x|, |y|\}$ for every $x, y \in K$, the absolute value is called *non-Archimedean*. Otherwise, it is called *Archimedean*. An absolute value is *trivial* if it is identically 1 on $K^\times = K \setminus \{0\}$.

We can define a distance on K associated to a given absolute value $|\cdot|$ by $|x-y|$ for every $x, y \in K$ and this metric defines a topology on K. Two absolute values on K are *equivalent* if they define the same topology. It can be proved that this is an equivalence relation.

A *place* v of K is an equivalence class of non-trivial absolute values. Let M_K be the set of all places of K, for a given $v \in M_K$, $|\cdot|_v$ denotes an absolute value on the equivalence class of v. Given a field extension L/K and a place $v \in M_K$, we say that a place $w \in M_L$ *extends* v if any representative $|\cdot|_w$ restricted to K is a representative of v, we will write $w \mid v$.

We will denote by K_v the *completion* of the field K with respect to the place $v \in M_K$. It can be proved that there is a unique place of K_v extending v and such that it induces a topology with respect to which K_v is complete and K is dense in K_v. By abuse of notation we shall denote this place also by v.

Let K be a complete field with respect to an absolute value $|\cdot|_v$ and let L/K be a finite extension. Then there is a unique extension of $|\cdot|_v$ to an absolute value $|\cdot|_w$ on L, which is given by

$$|x|_w = |N_{L/K}(x)|_v^{\frac{1}{[L:K]}}, \text{ for every } x \in L,$$

where $[L:K]$ is the degree of the extension L/K and $N_{L/K}$ the norm. Moreover, L is complete with respect to $|\cdot|_w$. One can deduce from this fact that there is a unique extension to an absolute value on the algebraic closure $\overline{K}$ of a complete field K. However, since $\overline{K}/K$ is not finite in general, we cannot say that $\overline{K}$ is complete with respect to this absolute value.

In the field of the rational numbers $\mathbb{Q}$, there is only one Archimedean place ∞. A representative of ∞ is given by the ordinary absolute value $|\cdot|$ on $\mathbb{Q}$ which will also be denoted by $|\cdot|_\infty$. It can be shown that the remaining non-Archimedean places are in one-to-one correspondence with the prime numbers $p \in \mathbb{Q}$. Hence, we have

$$M_{\mathbb{Q}} = \{p : p \text{ prime or } p = \infty\}.$$

Given a prime number p, the *p-adic absolute value* $|\cdot|_p$ is defined for any prime $q \in \mathbb{Z}$ as

$$|q|_p = \begin{cases} 1 & \text{if } q \neq p, \\ \frac{1}{p} & \text{if } q = p. \end{cases}$$

Considering the factorization of any rational number into prime factors an the multiplicativity of absolute values, we can extend this definition to $\mathbb{Q}$.

From now on, for every non-Archimedean place $p \in M_{\mathbb{Q}}$, the representative $|\cdot|_p$ will correspond to the p-adic absolute value that we just defined. For the infinite place $p = \infty$, the representative $|\cdot|_\infty$ will always refer to the ordinary absolute value. As we introduced above, $\mathbb{Q}_p$ denotes the completion of the rational numbers with respect to the place p and, by abuse of notation we will also denote by $|\cdot|_p$ the extension of the p-adic absolute value to $\mathbb{Q}_p$. Moreover, since there is a unique extension of $|\cdot|_p$ to the algebraic closure $\overline{\mathbb{Q}_p}$, it will also be denoted by $|\cdot|_p$.

Let us consider the case of a number field K, i.e. a finite extension of the field of rational numbers $\mathbb{Q}$. Let $p \in M_{\mathbb{Q}}$ and $v \in M_K$ be such that $v \mid p$. The extension $K_v/\mathbb{Q}_p$ is finite and $\mathbb{Q}_p$ complete, hence there is a unique extension of $|\cdot|_p$ to K_v, which is given by

$$|x|_v = |N_{K_v/\mathbb{Q}_p}(x)|_p^{\frac{1}{[K_v:\mathbb{Q}_p]}}, \text{ for every } x \in K_v. \tag{1.11}$$

For a place v on a number field K, we will denote by $|\cdot|_v$ the representative of v corresponding to the restriction to K of (1.11).

We have chosen a certain normalization of the absolute values $|\cdot|_v$ representing a given place v on a number field K. We can now give the definition of the *Weil height* of a point $P = (x_0 : \ldots : x_n)$ in $\mathbb{P}^n(\overline{\mathbb{Q}})$,

$$\mathrm{h}(P) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} [K_v : \mathbb{Q}_v] \max\{\log|x_0|_v, \ldots, \log|x_n|_v\},$$

where K is a number field containing $x_0, \ldots, x_n$ and $\mathbb{Q}_v$ is the completion of $\mathbb{Q}$ with respect to the restriction of the absolute value $|\cdot|_v$. It can be proved that this definition does not depend on the choice of the field extension $K/\mathbb{Q}$.

The *product formula* states that, for every $x \in K^\times$

$$\prod_{v \in M_K} |x|_v^{[K_v:\mathbb{Q}_v]} = 1.$$

From it we can deduce that the definition of the height of a point in $\mathbb{P}^n(\overline{\mathbb{Q}})$ does not depend on the choice of the coordinates.

Roughly speaking, the height of a point in the projective space $\mathbb{P}^n(\overline{\mathbb{Q}})$ measures its algebraic complexity. In particular, if the coordinates of a given point $P \in \mathbb{P}^n(\overline{\mathbb{Q}})$ can be chosen in $\mathbb{Q}$, there are integers $x_0, \ldots, x_n$ with no common factor such that $P = (x_0 : \ldots : x_n)$ and

$$\mathrm{h}(P) = \max\{\log|x_0|, \ldots, \log|x_n|\}.$$

The definition of height can be extended to the affine space $\mathbb{A}^n(\overline{\mathbb{Q}})$. Given a point in $\mathbb{A}^n(\overline{\mathbb{Q}})$, its height is defined as the height of its image under the natural embedding of the affine space into $\mathbb{P}^n(\overline{\mathbb{Q}})$. This is, if $P = (x_1, \ldots, x_n)$, then $h(P) = h(1 : x_1 : \ldots : x_n)$. In particular, the Weil height of an algebraic number $\alpha \in \overline{\mathbb{Q}}$ is defined by

$$\mathrm{h}(\alpha) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} [K_v : \mathbb{Q}_v] \log^+ |\alpha|_v,$$

where recall $\log^+ |\alpha|_v = \max\{0, \log|\alpha|_v\}$.

It is clear from the definition that the height of a point $P \in \mathbb{P}^n(\overline{\mathbb{Q}})$ is always greater or equal than zero. For algebraic numbers, *Kronecker's theorem* gives a characterization of those elements whose height is exactly zero: let $\alpha \in \overline{\mathbb{Q}}^\times$, then $\mathrm{h}(\alpha) = 0$ if and only if α is a root of unity.

Given a collection of points $P_1, \ldots, P_r$ in $\mathbb{P}^n(\overline{\mathbb{Q}})$, we have that

$$\mathrm{h}(P_1 + \ldots + P_r) \leq \mathrm{h}(P_1) + \ldots + \mathrm{h}(P_r) + \log r.$$

Consider a point $P = (x_0 : \ldots : x_n)$ in $\mathbb{P}^n(\overline{\mathbb{Q}})$ and an element σ in the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then we have $\sigma P = (\sigma x_0 : \ldots : \sigma x_n)$ and $\mathrm{h}(P) = \mathrm{h}(\sigma P)$. In particular, if $\alpha, \beta \in \overline{\mathbb{Q}}$ are algebraic conjugates, their heights agree.

For two algebraic numbers α and β, we have that $\mathrm{h}(\alpha\beta) \leq \mathrm{h}(\alpha) + \mathrm{h}(\beta)$. Moreover, if β is a root of unity, then $\mathrm{h}(\alpha\beta) = \mathrm{h}(\alpha)$. For any integer n, we have $\mathrm{h}(\alpha^n) = |n|\,\mathrm{h}(\alpha)$ and, in particular, $\mathrm{h}(\alpha) = \mathrm{h}(\frac{1}{\alpha})$.

Let $S \subset \overline{\mathbb{Q}}$ be a finite set, its height is defined as $\mathrm{h}(S) = \sum_{\alpha \in S} \mathrm{h}(\alpha)$. If S is the Galois orbit of some element $\alpha \in \overline{\mathbb{Q}}$, i.e. the orbit of α under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, since the height of all algebraic conjugates coincide, we have $\mathrm{h}(S) = \#S\,\mathrm{h}(\alpha)$, where $\#S$ is the cardinality of the set S.

Finally, we will give an alternative definition for the Weil height of an algebraic number. In order to do so, let us introduce first several concepts. Let $f \in \overline{\mathbb{Q}}[x]$ be a non-zero polynomial, its *(logarithmic) Mahler measure* is defined as

$$m(f) := \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta.$$

Consider the factorization over $\overline{\mathbb{Q}}$ of the polynomial $f(x) = a_d x^d + \ldots + a_0$,

$$f(x) = a_d \prod_{j=1}^{d} (x - \alpha_j).$$

By *Jensen's formula* we obtain

$$m(f) = \log |a_d| + \sum_{j=1}^{d} \log^+ |\alpha_j|.$$

Given $\alpha \in \overline{\mathbb{Q}}$, its *minimal polynomial* over $\mathbb{Z}$ is defined as the polynomial $f \in \mathbb{Z}[x]$ of least degree such that $f(\alpha) = 0$. The *degree* of α (over $\mathbb{Q}$) is the degree of its minimal polynomial over $\mathbb{Z}$ and it will be denoted by $\deg(\alpha)$.

Let α be an algebraic number and f its minimal polynomial over $\mathbb{Z}$, it can be proved that

$$\mathrm{h}(\alpha) = \frac{m(f)}{\deg(\alpha)}.$$

An important result that can be derived from this definition is *Northcott's theorem.* It states that there are only finitely many algebraic numbers with bounded degree and bounded height.

CHAPTER 2

Quantitative equidistribution in the one-dimensional case

In this chapter we will study the quantitative result of the equidistribution of Galois orbits of points of small height on the projective complex line due to Charles Favre and Juan Rivera-Letelier [**FRL06**]. In this paper, they consider adelic measures and associate to them an *adelic height.* Then, for every place $v \in M_{\mathbb{Q}}$, they give an estimate for the rate of convergence of the discrete probability measure associated to a finite set towards the v-component of the adelic measure considered.

As we mentioned on the introduction, in this text we will only focus on the particular case of the classical Weil height and the Archimedean place. Before stating the result, which corresponds to Corollary 1.4 in [**FRL06**]r, we will recall some notation.

Let $S \subset \mathbb{C}$ be a finite set, the discrete probability measure associated to S is a measure on $\mathbb{C}$ which is given by

$$\mu_S = \frac{1}{\#S} \sum_{\alpha \in \mathbb{C}} \delta_\alpha,$$

where $\#S$ denotes the cardinality of the set S and δ_α is the delta Dirac measure supported on α.

We denote by λ_{S^1} the probability measure on $\mathbb{C}$ supported on the unit circle, where it coincides with the probability Haar measure.

When considering the Riemann sphere, we will be referring to the complex projective line $\mathbb{P}^1(\mathbb{C})$ together with the Fubini-Study metric, denoted by g, and whose local expression is given in (1.6). We will consider the natural embedding $\mathbb{C} \hookrightarrow \mathbb{P}^1(\mathbb{C})$, sending $z \mapsto (1 : z)$.

Given a real-valued function f on $\mathbb{P}^1(\mathbb{C})$, we say that it is a *Lipschitz function* if there is some $K > 0$ such that

$$|f(p) - f(p')| \leq K\, \mathrm{d}(p, p'), \text{ for every } p, p' \in \mathbb{P}^1(\mathbb{C}),$$

where $\mathrm{d}(p, p')$ is the spherical distance between p and p' in $\mathbb{P}^1(\mathbb{C})$. If the function $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$ is a Lipschitz function, its *Lipschitz constant* is defined as

$$\mathrm{Lip}(f) = \sup_{\substack{p, p' \in \mathbb{P}^1(\mathbb{C}) \\ p \neq p'}} \frac{|f(p) - f(p')|}{\mathrm{d}(p, p')}.$$

It is easy to see that every function f in $\mathscr{C}^1(\mathbb{P}^1(\mathbb{C}))$ is Lipschitz.

The main result of this chapter is the following.

THEOREM II. *There is a positive constant* $C \approx 14.7628$ *such that for every* $\mathscr{C}^1$*-function* $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$ *and every finite Galois-invariant set* $S \subset \overline{\mathbb{Q}}^\times$,

$$\left| \int_{\mathbb{P}^1(\mathbb{C})} f d\mu_S - \int_{\mathbb{P}^1(\mathbb{C})} f d\lambda_{S^1} \right| \\ \leq \mathrm{Lip}(f) \left(\frac{\pi}{\# S} + \left(4 \frac{h(S)}{\# S} + C \frac{\log(\# S + 1)}{\# S} \right)^{\frac{1}{2}} \right).$$

In particular, if $\frac{h(S)}{\# S} \leq 1$, *we have*

$$\left| \int_{\mathbb{P}^1(\mathbb{C})} f d\mu_S - \int_{\mathbb{P}^1(\mathbb{C})} f d\lambda_{S^1} \right| \leq \mathrm{Lip}(f) \left(4 \frac{h(S)}{\# S} + C' \frac{\log(\# S + 1)}{\# S} \right)^{\frac{1}{2}}, \tag{2.1}$$

with $C' \approx 48.9897$.

Let us fix some notation that will be used along the current chapter: dA will denote the Lebesgue measure on $\mathbb{C} \cong \mathbb{R}^2$. We will be considering the usual charts of the complex projective line (U_0, α_0) and (U_1, α_1), where the open subsets are

$$U_0 = \{(1 : z) \in \mathbb{P}^1(\mathbb{C}) : z \in \mathbb{C}\} \text{ and } U_1 := \{(z : 1) \in \mathbb{P}^1(\mathbb{C}) : z \in \mathbb{C}\},$$

and the homeomorphisms

$$\begin{array}{rccc} \alpha_0 : & U_0 & \longrightarrow & \mathbb{R}^2, \\ & (1:z) & \mapsto & (\mathrm{Re}(z), \mathrm{Im}(z)) \end{array} \qquad \begin{array}{rccc} \alpha_1 : & U_1 & \longrightarrow & \mathbb{R}^2 \\ & (z:1) & \mapsto & (\mathrm{Re}(z), \mathrm{Im}(z)) . \end{array}$$

On $\mathbb{P}^1(\mathbb{C})$ we consider the Riemannian measure μ associated to g.

As it was mentioned on the preliminaries, we will consider a suitable normalization of the Laplace operator on $\mathbb{R}^2$ with coordinates x, y, namely

$$\Delta = \frac{1}{2\pi} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

1. Potential theory on the Riemann sphere

The aim of this section is to extend the potential theory on the complex plane to the whole Riemann sphere. Given a signed measure ρ on $\mathbb{P}^1(\mathbb{C})$, the problem of finding an integrable function on an bounded proper open neighborhood of any point whose Laplacian is equal, as a distribution, to the measure restricted to the open set is essentially what was studied in Section 1.6. We will study under which conditions it is possible to consider a *global potential* for any given signed measure ρ. This is, when there is an integrable function $h : \mathbb{P}^1(\mathbb{C}) \to \overline{\mathbb{R}}$ such that $\rho = \Delta_g h$. The answer to this question is a well-know result on potential theory that will be stated and proved in this text (c.f. Theorem 2.3).

The first step will be to study the Laplacian of the extension of the *logarithmic kernel* to $\mathbb{P}^1(\mathbb{C})$. Let $\mathbf{\Delta} := \{(z,z) : z \in \mathbb{C}\}$, the logarithmic kernel is the function $K : \mathbb{C} \times \mathbb{C} \setminus \mathbf{\Delta} \to \mathbb{R}$ defined as

$$K(z,w) = \log|z-w|.$$

Observe that it can be naturally extended to a function

$$K : \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \setminus \mathbf{\Delta} \to \overline{\mathbb{R}},$$

by setting $K(\infty, w) = K(z, \infty) = \infty$. In order to study its Laplacian, let us see that, for any $z \in \mathbb{C}$, the function $K(z,\cdot)$ is integrable on the complex projective plane with respect to the Riemannian measure μ. This will allow us to think of $K(z,\cdot)$ as a distribution and we will then be able to consider its distributional Laplacian.

LEMMA 2.1. *For every $z \in \mathbb{C}$, we have that*

$$\int_{\mathbb{P}^1(\mathbb{C})} |K(z,w)| d\mu(w) < \infty.$$

PROOF. Recall that locally $\mu = \sqrt{\det(g)}dA$, and the local expression of g is given by (1.6). Hence, for every $z \in \mathbb{C}$ we have

$$\begin{aligned}
&\int_{\mathbb{P}^1(\mathbb{C})} |K(z,w)| d\mu(w) \\
&\qquad = \int_{\overline{D}(0,1)} \frac{4|\log|z-w||}{(1+|w|^2)^2} dA(w) + \int_{D(0,1)} \frac{4|\log|z-1/w||}{(1+|w|^2)^2} dA(w) \\
&\qquad\quad \leq 4\int_{\overline{D}(0,1)} |\log|z-w|| dA(w) + 4\int_{D(0,1)} |\log|z-1/w|| dA(w).
\end{aligned}$$

The first summand on the right-hand side of this last expression is clearly finite. Indeed, for every $z \in \mathbb{C}$, the function $\log|z-\cdot|$ is subharmonic on $\mathbb{C}$ and therefore, it is locally integrable. Let us see that the second summand is also finite. If $z = 0$ the finiteness follows directly using the previous argument. Suppose $z \neq 0$, then

$$\begin{aligned}
\int_{D(0,1)} |\log|z-1/w|| dA(w) &\leq \int_{D(0,1)} |\log|z|| dA(w) \\
&\quad + \int_{D(0,1)} |\log|w|| dA(w) + \int_{D(0,1)} |\log|w-1/z|| dA(w) < \infty.
\end{aligned}$$

□

LEMMA 2.2. *For any fixed $z \in \mathbb{C}$, we have the distributional equation*

$$\Delta_g K(z,\cdot) = \delta_z - \delta_\infty,$$

in the space of distributions of $\mathbb{P}^1(\mathbb{C})$.

PROOF. Let us prove first the result for $z = 0$. The function $k_0(w) := K(0,w)$ is integrable on $\mathbb{P}^1(\mathbb{C})$ and we can consider its distributional Laplacian. For every $\phi \in \mathcal{D}(\mathbb{P}^1(\mathbb{C}))$ we have

$$\begin{aligned}(\Delta_g k_0, \phi) &= \int_{\mathbb{P}^1(\mathbb{C})} k_0 \Delta_g \phi d\mu \\ &= \int_{\{(1:w):|w|\leq 1\}} k_0 \Delta_g \phi d\mu + \int_{\{(w:1):|w|<1\}} k_0 \Delta_g \phi d\mu \\ &= \int_{D(0,1)} \log|w| \Delta\phi_0(w) dA(w) - \int_{D(0,1)} \log|w| \Delta\phi_1(w) dA(w),\end{aligned}$$

where the third equality is given by the fact that $\Delta_g \phi d\mu$ restricted to the chart U_j coincides with $\Delta\phi_j dA$, setting $\phi_j(w) = \phi \circ \alpha_j^{-1}$.

We will study separately each summand on the previous expression. On one hand, by Green's identity (1.5) and the fact that $\log|w|$ is harmonic on $\mathbb{C} \setminus \{0\}$, we have

$$\begin{aligned}\int_{D(0,1)} \log|w| \Delta\phi_0(w) dA(w) &= \lim_{\varepsilon \to 0} \int_{\varepsilon < |w| \leq 1} \log|w| \Delta\phi_0(w) dA(w) \\ &= \lim_{\varepsilon \to 0} \int_{\varepsilon < |w| \leq 1} \left(\log|w| \Delta\phi_0(w) - \phi_0(w) \Delta \log|w|\right) dA(w) \\ = \lim_{\varepsilon \to 0} \int_{|w|=\varepsilon} \frac{1}{2\pi} &\left(\log|w| \nabla\phi_0(w) - \phi_0(w) \nabla \log|w|\right) \cdot (-\mathrm{Re}(w), -\mathrm{Im}(w)) d\sigma \\ + \int_{|w|=1} \frac{1}{2\pi} &\left(\log|w| \nabla\phi_0(w) - \phi_0(w) \nabla \log|w|\right) \cdot (\mathrm{Re}(w), \mathrm{Im}(w)) d\sigma \\ &= \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_0^{2\pi} \left(\phi_0(re^{i\theta}) - r \log r \frac{\partial \phi_0}{\partial r}(re^{i\theta}) \right) \Bigg|_{r=\varepsilon} d\theta \\ &\quad - \frac{1}{2\pi} \int_0^{2\pi} \left(\phi_0(re^{i\theta}) - r \log r \frac{\partial \phi_0}{\partial r}(re^{i\theta}) \right) \Bigg|_{r=1} d\theta \\ &= \phi_0(0) - \frac{1}{2\pi} \int_0^{2\pi} \phi_0(e^{i\theta}) d\theta,\end{aligned}$$

where σ denotes the volume measure on the border of $\{w \in \mathbb{C} : \varepsilon < |w| \leq 1\}$.

On the other hand, following an analogous argument, we obtain

$$\begin{aligned}\int_{D(0,1)} \log|w| \Delta\phi_1(w) dA(w) &= \lim_{\varepsilon \to 0} \int_{\varepsilon < |w| < 1} \log|w| \Delta\phi_1(w) dA(w) \\ &= \phi_1(0) - \frac{1}{2\pi} \int_0^{2\pi} \phi_1(e^{i\theta}) d\theta.\end{aligned}$$

Now, observe that

$$-\frac{1}{2\pi}\int_0^{2\pi}\phi_0(e^{i\theta})d\theta+\frac{1}{2\pi}\int_0^{2\pi}\phi_1(e^{i\theta})d\theta$$
$$=-\frac{1}{2\pi}\int_0^{2\pi}\phi_0(e^{i\theta})d\theta+\frac{1}{2\pi}\int_0^{2\pi}\phi_0(e^{-i\theta})d\theta=0.$$

So, putting everything together, we obtain

$$(\Delta_g k_0,\phi)=\int_{\mathbb{P}^1(\mathbb{C})}k_0\Delta_g\phi d\mu=\phi(0)-\phi(\infty)=(\delta_0-\delta_\infty,\phi).$$

Finally, let us prove the lemma for all $z\in\mathbb{C}$. Let $k_z(w)=K(z,w)$, for any $\phi\in\mathcal{D}(\mathbb{P}^1(\mathbb{C}))$, since $\mu(\infty)=0$ we have

$$(\Delta_g k_z,\phi)=\int_{\mathbb{P}^1(\mathbb{C})}k_z\Delta_g\phi d\mu=\int_{\mathbb{C}}\log|z-w|\Delta\phi_0(w)dA(w)$$
$$=\int_{\mathbb{C}}\log|v|(\Delta\phi_0)(v+z)dA(v)=\int_{\mathbb{C}}\log|v|\Delta\tilde{\phi}_0(v)dA(v)=(\Delta_g k_0,\tilde{\phi}),$$

where $\tilde{\phi}(w)=\phi(w+z)$. So, using the previous case, we deduce

$$(\Delta_g k_z,\phi)=(\Delta_g k_0,\tilde{\phi})=\tilde{\phi}(0)-\tilde{\phi}(\infty)=\phi(z)-\phi(\infty).$$

□

The next result gives a characterization of the signed measures on the projective complex plane for which we can consider a global potential.

THEOREM 2.3. *Let ρ be a signed measure on the Riemann sphere. Then $\|\rho\|=0$ if and only if there is an integrable function $h:\mathbb{P}^1(\mathbb{C})\to\overline{\mathbb{R}}$ such that $\Delta_g h=\rho$. Moreover, if this integrable function exists, it is unique up to addition of a constant.*

In order to prove the *unicity* of the solution, we need the following lemma.

LEMMA 2.4. *Consider two integrable functions u and v on $\mathbb{P}^1(\mathbb{C})$ such that $\Delta_g u=\Delta_g v$. Then $u-v$ is constant on $\mathbb{P}^1(\mathbb{C})$.*

The idea of the proof of this lemma is that there is a harmonic function $\tilde{h}$ on the Riemann sphere such that $u-v=\tilde{h}$ almost everywhere. By Liouville's Theorem, the only harmonic functions on $\mathbb{P}^1(\mathbb{C})$ are constants and the lemma follows.

PROOF. Consider the function $h=u-v$, which is integrable on $\mathbb{P}^1(\mathbb{C})$. It is then easy to verify that $h_i=h\circ\alpha_i^{-1}$ is locally integrable on $\mathbb{C}$. Moreover, $\Delta h_i=0$ on $\mathcal{D}'(\mathbb{C})$. Indeed, every compactly supported smooth function ϕ on $\mathbb{C}$ can be naturally extended to a smooth function $\tilde{\phi}$ on the Riemann sphere and we obtain

$$(\Delta h_i,\phi)=\int_{\mathbb{C}}h_i\Delta\phi dA=\int_{\mathbb{P}^1(\mathbb{C})}h\Delta_g\tilde{\phi}d\mu=\int_{\mathbb{P}^1(\mathbb{C})}\Delta_g(u-v)\tilde{\phi}dA=0.$$

Let φ be a mollifier and set $\varphi_n(z) = n^2\varphi(nz)$. For every $n \geq 1$, we define the functions

$$h_{i,n}(z) = \int_{\mathbb{C}} h_i(z-w)\varphi_n(w)dA(w).$$

These functions are smooth on the complex plane and, for every $z \in \mathbb{C}$, they satisfy

$$\Delta h_{i,n}(z) = \int_{\mathbb{C}} h_i(w)\Delta\varphi_n(z-w)dA(w) = (\Delta h_i, \varphi_n(z-\cdot)) = 0.$$

Hence, they are harmonic on $\mathbb{C}$ and, by the mean-value property, for every $R > 0$ we obtain

$$h_{i,n}(z) = \frac{1}{\pi R^2}\int_{D(z,R)} h_{i,n}(w)dA(w).$$

On one hand, we know that h_i is locally integrable on $\mathbb{C}$ and, by Theorem 1.10 we have that $h_{i,n} \to h_i$ in $\mathrm{L}^1_{loc}(\mathbb{C})$. This is, for every compact $K \subset \mathbb{C}$, we have

$$\|h_{i,n} - h_i\|_{\mathrm{L}^1(K)} \to 0 \text{ as } n \to \infty.$$

On the other hand, we can define

$$\tilde{h}_i(z) = \frac{1}{\pi R^2}\int_{D(z,R)} h_i(w)dA(w).$$

Then we have

$$\begin{aligned}
|h_{i,n}(z) - \tilde{h}_i(z)| &= \left|\frac{1}{\pi R^2}\int_{D(z,R)} h_{i,n}(w)dA(w) - \frac{1}{\pi R^2}\int_{D(z,R)} h_i(w)dA(w)\right| \\
&\leq \frac{1}{\pi R^2}\int_{D(z,R)} |h_{i,n}(w) - h_i(w)|dA(w) \\
&= \frac{1}{\pi R^2}\|h_{i,n} - h_i\|_{\mathrm{L}^1(D(z,R))} \longrightarrow 0,
\end{aligned}$$

uniformly on compact sets as $n \to \infty$. By Corollary 1.16, this implies that $\tilde{h}_i$ is harmonic on $\mathbb{C}$.

Finally, since h_i is the L^1_{loc}-limit of $h_{i,n}$, we can conclude that $h_i = \tilde{h}_i$ almost everywhere. $\square$

We can now prove the theorem.

Proof of Theorem 2.3. Observe that the equality $\Delta_g h = \rho$ is actually referring to an equality of distributions in the sense that, for every smooth function ϕ on $\mathbb{P}^1(\mathbb{C})$, the following holds

$$\int_{\mathbb{P}^1(\mathbb{C})} h\Delta_g\phi d\mu = \int_{\mathbb{P}^1(\mathbb{C})} \phi d\rho. \tag{2.2}$$

As we mentioned above, the *unicity* of the solution follows directly from the previous lemma. Indeed, suppose that there are two different functions $h_1, h_2 : \mathbb{P}^1(\mathbb{C}) \to \overline{\mathbb{R}}$ such that $\Delta_g h_1 = \rho = \Delta_g h_2$. Then by Lemma 2.4, we obtain that $h_1 - h_2$ is constant on $\mathbb{P}^1(\mathbb{C})$.

We will assume now that there is an integrable function $h : \mathbb{P}^1(\mathbb{C}) \to \overline{\mathbb{R}}$ such that $\Delta_g h = \rho$. Taking in (2.2) the constant function $\phi \equiv 1$ on $\mathbb{P}^1(\mathbb{C})$, we obtain

$$\|\rho\| = \rho(\mathbb{P}^1(\mathbb{C})) = \int_{\mathbb{P}^1(\mathbb{C})} d\rho = \int_{\mathbb{P}^1(\mathbb{C})} h \Delta_g \phi d\mu = 0.$$

Finally, suppose that $\|\rho\| = 0$. For now, we assume that ρ has compact support contained on the chart U_0. If $\rho = \rho^+ - \rho^-$ is the Jordan decomposition of the signed measure ρ, we have that the supports of ρ^+ and ρ^- are compact in $U_0 \cong \mathbb{C}$. By Theorem 1.18 and (1.9), the functions

$$h_{\rho^+}(z) := \int_{\mathbb{C}} \log|z-w| d\rho^+(w)$$

and

$$h_{\rho^-}(z) := \int_{\mathbb{C}} \log|z-w| d\rho^-(w)$$

are subharmonic on $\mathbb{C}$ and are such that

$$\Delta h_{\rho^+} = \rho^+ \text{ and } \Delta h_{\rho^-} = \rho- .$$

Now, consider the function h_ρ given by

$$h_\rho(z) = \int_{\mathbb{P}^1(\mathbb{C})} \log|z-w| d\rho(w), \text{ for every } z \in \mathbb{C}.$$

Since the support of ρ is contained in $U_0 \cong \mathbb{C}$, we have $h_\rho(z) = h_{\rho^+}(z) - h_{\rho-}(z)$ for every $z \in \mathbb{C}$. We will now extend h_ρ to a function on $\mathbb{P}^1(\mathbb{C})$. Observe that, since $\rho(\mathbb{C}) = \rho(\mathbb{P}^1(\mathbb{C})) = 0$, if $z \neq 0$ we have

$$\begin{aligned} h_\rho(z) = \int_{\mathbb{C}} \log|z-w| d\rho(w) &= \int_{\mathbb{C}} \log|z| d\rho(w) + \int_{\mathbb{C}} \log\left|1 - \frac{w}{z}\right| d\rho(w) \\ &= \int_{\mathbb{C}} \log\left|1 - \frac{w}{z}\right| d\rho(w) \xrightarrow[z\to\infty]{} 0. \end{aligned}$$

By setting $h_\rho(\infty) = 0$, we claim that h_ρ is integrable on the Riemann sphere.

Assuming the claim is true, we can consider the distributional Laplacian of the function h_ρ. For every $\phi \in \mathcal{D}(\mathbb{P}^1(\mathbb{C}))$ we have

$$
\begin{aligned}
(\Delta_g h_\rho, \phi) &= \int_{\mathbb{P}^1(\mathbb{C})} h_\rho(z)\Delta_g\phi(z)d\mu(z) \\
&= \int_{\mathbb{P}^1(\mathbb{C})} \left(\int_{\mathbb{P}^1(\mathbb{C})} \log|z-w|d\rho(w)\right)\Delta_g\phi(z)d\mu(z) \\
&= \int_{\mathbb{P}^1(\mathbb{C})} \left(\int_{\mathbb{P}^1(\mathbb{C})} \log|z-w|\Delta_g\phi(z)d\mu(z)\right)d\rho(w) \\
&= \int_{\mathbb{P}^1(\mathbb{C})} (\phi(w)-\phi(\infty))d\rho(w) \\
&= \int_{\mathbb{P}^1(\mathbb{C})} \phi d\rho - \phi(\infty)\|\rho\| = (\rho, \phi),
\end{aligned}
$$

where the third equality is given by applying Fubini's theorem and the fourth equality is given by Lemma 2.2 and the fact that we were assuming ρ with vanishing total mass.

Let us prove the claim. Since $h_\rho(\infty) = 0$ and $\mu(\infty) = 0$, we have

$$
\int_{\mathbb{P}^1(\mathbb{C})} |h_\rho| d\mu = 4\int_{\mathbb{C}} \frac{|h_\rho(z)|}{(1+|z|^2)^2} dA(z) \le 4\int_{\mathbb{C}}\int_{\mathbb{C}} \frac{|\log|z-w||}{(1+|z|^2)^2} d\rho(w)dA(z).
$$

We have that ρ is compactly supported on $\mathbb{C}$, so there is some positive $R > 0$ such that $\operatorname{supp}(\rho) \subset D(0,R)$. Let $w \in D(0,R)$, then we have

$$
\begin{aligned}
\int_{\mathbb{C}} &\frac{|\log|z-w||}{(1+|z|^2)^2} dA(z) \\
&= \int_{|z|\le 1+R} \frac{|\log|z-w||}{(1+|z|^2)^2} dA(z) + \int_{|z|>1+R} \frac{|\log|z-w||}{(1+|z|^2)^2} dA(z)
\end{aligned}
\tag{2.3}
$$

On one hand, we have

$$
\begin{aligned}
\int_{|z|\le 1+R} \frac{|\log|z-w||}{(1+|z|^2)^2} dA(z) &\le \int_{|z|\le 1+R} |\log|z-w||dA(z) \\
&= \int_{D(w,1+R)} |\log|z||dA(z) \le \int_{D(0,2R+1)} |\log|z||dA(z),
\end{aligned}
$$

which is finite.

On the other hand, if $|z| > 1+R$, we have that $|z-w| > 1$ for every $w \in D(0,R)$ and

$$
\begin{aligned}
\int_{|z|>1+R} \frac{|\log|z-w||}{(1+|z|^2)^2} dA(z) &= \int_0^{2\pi}\int_{1+R}^{+\infty} \frac{\log|re^{i\theta}-w|}{(1+r^2)^2} r dr d\theta \\
&\le 2\pi\int_{1+R}^{+\infty} \frac{r\log(r+R)}{(1+r^2)^2} dr \le 2\pi\int_{1+R}^{+\infty} \frac{r\log(2r)}{(1+r^2)^2} dr < \infty.
\end{aligned}
$$

Hence, (2.3) is uniformly bounded for any $w \in D(0,R)$ and we deduce that h_ρ is integrable on $\mathbb{P}^1(\mathbb{C})$.

Now, we will consider the general situation where ρ is not necessarily compactly supported on one of the usual charts of the the Riemann sphere. We will see that, in this situation, we can decompose ρ as the sum of two finite signed measures ρ_0 and ρ_1 such that $\mathrm{supp}(\rho_i)$ is compact in U_i and $\rho_i(\mathbb{P}^1(\mathbb{C})) = 0$. Assuming this is true, we know that there are integrable functions $h_i : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$ with $\Delta_g h_i = \rho_i$ for $i = 0, 1$ and

$$\rho = \rho_0 + \rho_1 = \Delta_g h_0 + \Delta_g h_1 = \Delta_g(h_0 + h_1).$$

Consider the subsets

$$D_0 := \{(1 : z) : z \in D(0,1)\} \text{ and } \overline{D}_1 := \{(z : 1) : z \in \overline{D}(0,1)\}.$$

We define the signed measures $\tilde{\rho}_0$ and $\tilde{\rho}_1$ as the restrictions

$$\tilde{\rho}_0(A) := \rho(A \cap D_0) \text{ and } \tilde{\rho}_1(A) := \rho(A \cap \overline{D}_1),$$

for every A in the Borel σ-algebra $\mathscr{B}(\mathbb{P}^1(\mathbb{C}))$. Since $\mathbb{P}^1(\mathbb{C})$ equals the disjoint union of D_0 and $\overline{D}_1$, we have that $\rho = \tilde{\rho}_0 + \tilde{\rho}_1$ and, in particular, $\tilde{\rho}_0(\mathbb{P}^1(\mathbb{C})) + \tilde{\rho}_1(\mathbb{P}^1(\mathbb{C})) = 0$. Hence, we can write

$$\rho = \tilde{\rho}_0 + \tilde{\rho}_1(\mathbb{P}^1(\mathbb{C}))\delta_1 + \tilde{\rho}_0(\mathbb{P}^1(\mathbb{C}))\delta_1 + \tilde{\rho}_1.$$

It is enough to take $\rho_0 = \tilde{\rho}_0 + \tilde{\rho}_1(\mathbb{P}^1(\mathbb{C}))\delta_1$ and $\rho_1 = \tilde{\rho}_0(\mathbb{P}^1(\mathbb{C}))\delta_1 + \tilde{\rho}_1$. □

As we previously mentioned, Theorem 2.3 gives a characterization of those signed measures on $\mathbb{P}^1(\mathbb{C})$ for which we can consider the potential. We say that a signed measure ρ has *continuous potential* if there is a continuous function $h : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$ such that $\Delta_g h = \rho$. And, in particular, ρ has zero total mass. Observe that positive finite measures on the Riemann sphere or, more generally, signed finite measures with non-vanishing total mass do not have a global potential of any type. However, we can consider the potential of any finite measure locally. We will say that a measure has continuous potential if for every point, there is a neighborhood U containing it and there is a continuous function $h : U \to \mathbb{R}$ such that $\Delta_g h = \rho$ in $\mathcal{D}'(U)$.

Corollary 2.5. *If ρ is a signed measure on $\mathbb{P}^1(\mathbb{C})$ with continuous potential and there is a proper open set $U \subset \mathbb{P}^1(\mathbb{C})$ containing* $\mathrm{supp}(\rho)$*. Then there is a continuous function $h : U \to \mathbb{R}$ such that $\Delta_g h = \rho$ on $\mathcal{D}'(U)$.*

Proof. Let $p \in \mathbb{P}^1(\mathbb{C}) \setminus U$, the signed measure on $\mathbb{P}^1(\mathbb{C})$ defined by $\tilde{\rho} = \rho - \rho(U)\delta_p$ has vanishing total mass and, by Theorem 2.3, there is an integrable function $\tilde{h}$ on the Riemann sphere such that $\Delta_g \tilde{h} = \tilde{\rho}$. On the other hand, for every $q \in U$ there is an open neighborhood U_q and a continuous function $h_q : U_q \to \mathbb{R}$ satisfying $\Delta_g h_q = \rho$ on $\mathcal{D}'(U_q)$. Since the restriction of $\tilde{\rho}$ to U coincides with ρ, we have that $\Delta_g \tilde{h} = \Delta_g h_q$ on U_q. Hence $\tilde{h} - h_q$ is harmonic on U_q and, in particular, $\tilde{h}$ is continuous on U_q. Therefore, the function $h = \tilde{h}_{|U}$ is continuous and such that $\Delta_g h = \rho$ on $\mathcal{D}'(U)$. □

1.1. Energy. In this section we will define the mutual energy of signed measures. After introducing an examples that will later appear, we will state a result establishing sufficient conditions for a measure on the Riemann sphere to be such that its energy is positive.

Recall that, given a signed measure ρ, its trace measure is the finite positive measure given by

$$|\rho| = \rho^+ + \rho^-,$$

where $\rho = \rho^+ - \rho^-$ is the Jordan decomposition of ρ.

Definition 2.6. Consider two signed measures ρ and ρ' on $\mathbb{P}^1(\mathbb{C})$ such that $\log|z-w|$ is integrable on $\mathbb{C}\times\mathbb{C}\setminus\mathbf{\Delta}$ with respect to the product measure $|\rho|\otimes|\rho'|$. We define the mutual energy of ρ and ρ' by

$$(\rho, \rho') = -\int_{\mathbb{C}\times\mathbb{C}\setminus\mathbf{\Delta}} \log|z-w| d\rho(z) d\rho'(w). \tag{2.4}$$

Whenever it is well-defined, we will define the energy of a signed measure ρ by (ρ, ρ).

As a remark, observe that this definition of mutual energy does not coincide with the classical definition. Given two finite compactly supported measures on the complex plane, their energy is usually defined in a similar way but considering the whole $\mathbb{C}\times\mathbb{C}$ as the integration domain. Hence, it is possible to have infinite energy. As an example, every finite set on $\mathbb{C}$ is *polar* in the sense that the mutual energy of any non-zero measure supported on it has infinite energy. Therefore, Definition 2.6 will be much more suitable in our situation since, in particular, we will be considering discrete probability measures associated to finite sets on the complex plane.

Let $S \subset \mathbb{C}$ be a finite set and μ_S the discrete probability measure associated to it. It is clear that the energy of μ_S is well-defined and it is given by

$$(\mu_S, \mu_S) = \frac{1}{(\#S)^2} \sum_{\substack{\alpha,\beta\in S,\\ \alpha\neq\beta}} \log|\alpha-\beta|.$$

The next goal in this section will be to establish some conditions under which the mutual energy of two signed measures is well-defined. For this purpose, we will need the following technical result.

Lemma 2.7. *Let u be a subharmonic function on $\mathbb{C}$ and let ρ be a finite positive measure with continuous potential on the complex plane. Then u is locally integrable on $\mathbb{C}$ with respect to ρ.*

Proof. By Corollary 2.5, we know that there is a continuous subharmonic function $h : \mathbb{C} \to \mathbb{R}$ such that $\Delta h = \rho$ on $\mathcal{D}'(\mathbb{C})$.

Since the functions h and u are subharmonic, we can build sequences $\{h_m\}$ and $\{u_n\}$ of smooth subharmonic functions whose decreasing pointwise limits are h and u, respectively. Moreover, by the continuity of the function h, the sequence $\{h_m\}$ converges to h uniformly on compacts.

Let ϕ be a positive smooth function with compact support on $\mathbb{C}$ and denote $K_\phi = \operatorname{supp}(\phi)$. The monotone convergence theorem and the fact that $\phi \geq 0$ imply that

$$\int_{\mathbb{C}} u\phi d\rho = \int_{\mathbb{C}} \lim_n u_n \phi d\rho = \lim_n \int_{\mathbb{C}} u_n \phi d\rho.$$

Since $\Delta h = \rho$ on $\mathcal{D}'(\mathbb{C})$ and $u_n\phi$ is smooth and compactly supported on $\mathbb{C}$, for every $n \geq 1$ we have

$$\begin{aligned}\int_{\mathbb{C}} u_n\phi d\rho = \int_{\mathbb{C}} u_n \phi \Delta h dA = \int_{\mathbb{C}} \Delta(u_n\phi) h dA = \int_{\mathbb{C}} \Delta(u_n\phi) \lim_m h_m dA \\ = \lim_m \int_{\mathbb{C}} \Delta(u_n\phi) h_m dA = \lim_m \int_{\mathbb{C}} u_n \phi \Delta h_m dA,\end{aligned}$$

where the fourth equality is given by the fact that $\Delta(u_n\phi)h_m$ converges uniformly to $\Delta(u_n\phi)h$ on the support of ϕ.

Hence, we have

$$\int_{\mathbb{C}} u\phi d\rho = \lim_n \lim_m \int_{\mathbb{C}} u_n \phi \Delta h_m dA.$$

We claim that there is $c_\phi \geq 0$ such that, for every $m, n \geq 1$, we have

$$\left| \int_{\mathbb{C}} \phi u_n \Delta h_m dA \right| \leq c_\phi. \tag{2.5}$$

Therefore, for every positive smooth function ϕ with compact support we deduce that

$$\left| \int_{\mathbb{C}} u\phi d\rho \right| < \infty.$$

This is enough to obtain that u is locally integrable on $\mathbb{C}$ with respect to ρ. Indeed, For any compact K we can find a positive smooth function ϕ_K with compact support on $\mathbb{C}$ and such that $\phi_K \equiv 1$ on K. Since u is subharmonic and ϕ_K is a positive function with compact support, there is some real constant m such that $u\phi_K \leq m$ on $\mathbb{C}$ and

$$\begin{aligned}\int_K |u| d\rho \leq \int_{\mathbb{C}} |u\phi_K| d\rho \leq \int_{\mathbb{C}} |u\phi_K - m| d\rho + \int_{\mathbb{C}} |m| d\rho \\ = \int_{\mathbb{C}} (m - u\phi_K) d\rho + \int_{\mathbb{C}} |m| d\rho \leq -\int_{\mathbb{C}} u\phi_K d\rho + 2|m| \|\rho\| \\ \leq \left| \int_{\mathbb{C}} u\phi_K d\rho \right| + 2|m|\|\rho\| < \infty.\end{aligned}$$

We will finally prove the claim. Applying the divergence theorem to the smooth vector fields $\phi u_n \nabla h_m$ and $\phi h_m \nabla u_n$, for every $m, n \geq 1$ we obtain

$$\int_{\mathbb{C}} u_n \langle \nabla\phi, \nabla h_m \rangle dA + \int_{\mathbb{C}} \phi \langle \nabla h_m, \nabla u_n \rangle dA + 2\pi \int_{\mathbb{C}} \phi u_n \Delta h_m dA = 0,$$

$$\int_{\mathbb{C}} h_m \langle \nabla\phi, \nabla u_n \rangle dA + \int_{\mathbb{C}} \phi \langle \nabla h_m, \nabla u_n \rangle dA + 2\pi \int_{\mathbb{C}} \phi h_m \Delta u_n dA = 0.$$

This implies that

$$\begin{aligned}&\int_{\mathbb{C}} \phi u_n \Delta h_m dA\\ &\quad = \frac{1}{2\pi}\int_{\mathbb{C}} h_m \langle \nabla\phi, \nabla u_n\rangle dA + \int_{\mathbb{C}} \phi h_m \Delta u_n dA - \frac{1}{2\pi}\int_{\mathbb{C}} u_n \langle \nabla\phi, \nabla h_m\rangle dA.\end{aligned}$$

We will study the summands on the right-hand side of this last expression. Since the first and last of these summands are analogous, we will only study the boundedness of one of them. We have that ϕ is smooth and its support K_ϕ is compact on $\mathbb{C}$, hence its first order partial derivatives are bounded on $\mathbb{C}$, say by $c_1 \geq 0$, and we have

$$\begin{aligned}\left|\int_{\mathbb{C}} h_m \langle \nabla\phi, \nabla u_n\rangle dA\right| &\leq \int_{\mathbb{C}} \left| h_m \frac{\partial \phi}{\partial x}\frac{\partial u_n}{\partial x}\right| dA + \int_{\mathbb{C}} \left| h_m \frac{\partial \phi}{\partial y}\frac{\partial u_n}{\partial y}\right| dA\\ &\leq c_1 \int_{K_\phi} \left| h_m \frac{\partial u_n}{\partial x}\right| dA + c_1 \int_{K_\phi} \left| h_m \frac{\partial u_n}{\partial y}\right| dA\\ &= c_1 \left\| h_m \frac{\partial u_n}{\partial x}\right\|_{\mathrm{L}^1(K_\phi)} + c_1 \left\| h_m \frac{\partial u_n}{\partial y}\right\|_{\mathrm{L}^1(K_\phi)}.\end{aligned} \tag{2.6}$$

Now, by Lemma 1.21, since h_m and u_n are subharmonic functions, we have

$$h_m \in \mathrm{L}^p_{loc}(\mathbb{C}),\ \frac{\partial u_n}{\partial x}, \frac{\partial u_n}{\partial y} \in \mathrm{L}^{2-\varepsilon}_{loc}(\mathbb{C})$$

for every $1 \leq p < \infty$ and every $\varepsilon > 0$. This implies, by Theorem 1.10, that

$$h_m \xrightarrow[m\to\infty]{\mathrm{L}^p_{loc}(\mathbb{C})} h,\ \frac{\partial u_n}{\partial x} \xrightarrow[n\to\infty]{\mathrm{L}^{2-\varepsilon}_{loc}(\mathbb{C})} \frac{\partial u}{\partial x} \text{ and } \frac{\partial u_n}{\partial y} \xrightarrow[n\to\infty]{\mathrm{L}^{2-\varepsilon}_{loc}(\mathbb{C})} \frac{\partial u}{\partial y}.$$

Hence, by (2.6) and Hölder's inequality for some $1 \leq p < \infty$ and some $\varepsilon > 0$ such that $\frac{1}{p} + \frac{1}{2-\varepsilon} = 1$, we have

$$\begin{aligned}\left|\int_{\mathbb{C}} h_m \langle \nabla\phi, \nabla u_n\rangle dA\right| &\leq c_1 \left\| h_m \frac{\partial u_n}{\partial x}\right\|_{\mathrm{L}^1(K_\phi)} + c_1 \left\| h_m \frac{\partial u_n}{\partial y}\right\|_{\mathrm{L}^1(K_\phi)}\\ &\leq c_1 \|h_m\|_{\mathrm{L}^p(K_\phi)} \left\|\frac{\partial u_n}{\partial x}\right\|_{\mathrm{L}^{2-\varepsilon}(K_\phi)} + c_1 \|h_m\|_{\mathrm{L}^p(K_\phi)} \left\|\frac{\partial u_n}{\partial y}\right\|_{\mathrm{L}^{2-\varepsilon}(K_\phi)}\\ &\xrightarrow[m,n\to\infty]{} c_1 \|h\|_{\mathrm{L}^p(K_\phi)} \left\|\frac{\partial u}{\partial x}\right\|_{\mathrm{L}^{2-\varepsilon}(K_\phi)} + c_1 \|h\|_{\mathrm{L}^p(K_\phi)} \left\|\frac{\partial u}{\partial y}\right\|_{\mathrm{L}^{2-\varepsilon}(K_\phi)} < \infty.\end{aligned}$$

Finally, observe that Δu_n converges to Δu on $\mathcal{D}'(\mathbb{C})$ and we have that ϕh_m is smooth and compactly supported. Therefore,

$$\begin{aligned}\left|\int_{\mathbb{C}} \phi h_m \Delta u_n dA\right| \leq \int_{\mathbb{C}} \phi |h_m| \Delta u_n dA \xrightarrow[n\to\infty]{} \int_{\mathbb{C}} \phi |h_m| \Delta u dA\\ \leq c_0 \int_{\mathbb{C}} \phi \Delta u dA < \infty,\end{aligned}$$

where $c_0 \geq 0$ is such that $|h_m(z)| \leq c_0$ for every $z \in K_\phi$ and every $m \geq 1$. The existence of such c_0 is guarantied by the fact that h_m is a sequence of smooth functions converging uniformly on a compact set to a continuous function.

We can conclude that (2.5) holds for some $c_\phi \geq 0$ and the lemma follows. $\square$

Let us introduce a result providing sufficient conditions under which the energy of two signed measures is well defined.

PROPOSITION 2.8. *Let ρ and ρ' be two signed measures on $\mathbb{P}^1(\mathbb{C})$. Suppose that $|\rho|$ has continuous potential and ρ' is either finitely supported on $\mathbb{C}$ or such that $|\rho'|$ has continuous potential. Then*

$$\int_{\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})} |K(p,p')|d|\rho|(p)d|\rho'|(p') < \infty, \tag{2.7}$$

where $K : \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \to \overline{\mathbb{R}}$ is the logarithmic kernel.

Moreover, the mutual energy (ρ, ρ') is well-defined and we have

$$(\rho,\rho') = \int_{\mathbb{C}\times\mathbb{C}} \log|z-w|d\rho(z)d\rho'(w).$$

PROOF. Since every signed measure can be decomposed as the difference of two finite positive measures, it will be enough to prove the result for ρ and ρ' finite positive measures. The idea of the demonstration is to apply the previous lemma, where we saw that subharmonic functions are locally integrable with respect to positive measures with continuous potential. By the compacity of the Riemann sphere, we only need to study the integrability on a bounded neighborhood of every point.

Observe that once the first part of the result is proved, the rest follows directly since we necessarily have $(\rho \otimes \rho')(\mathbf{\Delta}) = 0$.

Let us denote by ρ_j and ρ'_j the restrictions of the measures ρ and ρ' to the charts U_j, for $j = 0, 1$. It is clear that ρ_j and ρ'_j are finite positive measures on $\mathbb{C}$ with continuous potential.

First suppose that ρ is such that its trace measure has continuous potential and ρ' has finite support $S \subset \mathbb{C}$, i.e.:

$$\rho' = \sum_{\alpha\in S} m_\alpha \delta_\alpha.$$

Then we have

$$\int_{\mathbb{P}^1(\mathbb{C})}\int_{\mathbb{P}^1(\mathbb{C})} |K(p,p')|d\rho'(p')d\rho(p) = \sum_{\alpha\in S} m_\alpha \int_{\mathbb{P}^1(\mathbb{C})} \left|\log\left|\frac{p_1}{p_0} - \alpha\right|\right| d\rho(p_0 : p_1)$$

In order to prove that this last integral is finite it will be enough to see that for every $q = (q_0 : q_1) \in \mathbb{P}^1(\mathbb{C})$ there is an open neighborhood V_q such that

$$\int_{V_p} \left|\log\left|\frac{p_1}{p_0} - \alpha\right|\right| d\rho(p_0 : p_1) < \infty.$$

Let $q = (1 : z_0)$ and $R > 0$. For every $\alpha \in S$, we have that $\log|z - \alpha|$ is a subharmonic function on $\mathbb{C}$ and, since ρ_0 has continuous potential, by Lemma 2.7 we obtain

$$\int_{D(z_0,R)} |\log|z-\alpha||d\rho_0(z) < \infty.$$

Suppose $q = (0 : 1)$ and consider its neighborhood $V_q = \{(z : 1) : |z| < 1\}$. From Lemma 2.7 we deduce

$$\begin{aligned}\int_{V_q} \left|\log\left|\frac{p_1}{p_0} - \alpha\right|\right| d\rho(p_0 : p_1) &= \int_{|z|<1} \left|\log\left|\frac{1}{z} - \alpha\right|\right| d\rho_1(z) \\ &\leq \int_{D(0,1)} |\log|z||d\rho_1(z) + \int_{D(0,1)} |\log|1 - z\alpha||d\rho_1(z) < \infty.\end{aligned}$$

Finally, suppose that both ρ and ρ' are finite positive measures with continuous potential. We will study the integrability of $\log|z - w|$ with respect to $\rho \otimes \rho'$ on neighborhoods of points in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. This is, given a point $(q, q') \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$, we will find open neighborhoods V_q and $V_{q'}$ of q and q', respectively, such that

$$\int_{V_q}\int_{V_{q'}} |K(p, p')|d\rho'(p')d\rho(p) < \infty.$$

Assume that $q, q' \in U_0$ and let $R, R' > 0$ be such that $q \in V_q = \{(1 : z) : |z| < R\}$ and $q' \in V_{q'} = \{(1 : w) : |w| < R'\}$. We have

$$\int_{V_q}\int_{V_{q'}} |K(p, p')|d\rho'(p')d\rho(p) = \int_{D(0,R)}\int_{D(0,R')} |\log|z-w||d\rho'_0(w)d\rho_0(z),$$

which is bounded since $\log|z - \cdot|$ is locally integrable with respect to ρ'_0 by Lemma 2.7, and ρ_0 is finite.

Suppose now that $q \in U_0$ and $q' = (0 : 1)$. Let $R > 0$ be such that $q \in V_q = \{(1 : z) : |z| < R\}$ and $V_{q'} = \{(w : 1) : |w| < 1\}$, we have

$$\begin{aligned}\int_{V_q}\int_{V_{q'}} |K(p, p')|d\rho'(p')d\rho(p) &= \int_{|z|<R}\int_{|w|<1} \left|\log\left|z - \frac{1}{w}\right|\right| d\rho'_1(w)d\rho_0(z) \\ &\leq \int_{|z|<R}\int_{|w|<1} |\log|wz - 1||d\rho'_1(w)d\rho(z) \\ &\quad + \int_{D(0,R)}\int_{D(0,1)} |\log|w||d\rho'(w)d\rho(z),\end{aligned}$$

which is also finite by an analogous argument as in the previous case.

At last, suppose $q = q' = (0:1)$ and let $V_q = V_{q'} = \{(z:1) : |z| < 1\}$, then

$$\begin{aligned}\int_{V_q}\int_{V_{q'}} |K(p,p')|d\rho'(p')d\rho(p) &= \int_{|z|<1}\int_{|w|<1}\left|\log\left|\frac{1}{z}-\frac{1}{w}\right|\right| d\rho_1'(w)d\rho_1(z)\\ &\leq \int_{|z|<1}\int_{|w|<1} |\log|z-w||d\rho_1'(w)d\rho_1(z)\\ &+\leq \int_{|z|<1}\int_{|w|<1} |\log|z||d\rho_1'(w)d\rho_1(z)\\ &+\int_{|z|<1}\int_{|w|<1} |\log|w||d\rho_1'(w)d\rho_1(z) < \infty.\end{aligned}$$

□

As an example, we will see that the energy of the measure λ_{S^1} vanishes. Recall that λ_{S^1} is the measure on $\mathbb{P}^1(\mathbb{C})$ supported on S^1 and such that its restriction to this compact subgroup coincides with the Haar probability measure. By Lemma 1.20, we have

$$\log^+|z| = \frac{1}{2\pi}\int_0^{2\pi} \log|z-e^{i\theta}|d\theta.$$

Hence, by Lemma 2.8 and Fubini's theorem, we have

$$(\lambda_{S^1},\lambda_{S^1}) = -\int_{\mathbb{C}\times\mathbb{C}} \log|z-w|d\lambda_{S^1}(w)d\lambda_{S^1}(z) = -\int_{\mathbb{C}} \log^+|z|d\lambda_{S^1}(z) = 0.$$

The following result will provide us sufficient conditions on the regularity of a signed measure on $\mathbb{P}^1(\mathbb{C})$ to be such that is has positive energy. It will be one of the key results for the proof of the main theorem of the chapter.

THEOREM 2.9. *Let ρ be a signed measure on $\mathbb{P}^1(\mathbb{C})$ with $\|\rho\| = 0$. Suppose that its trace measure $|\rho|$ has continuous potential. Then the following holds*

(1) There is a continuous function $h : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$ such that $\Delta_g h = \rho$.
(2) $\nabla_g h \in \vec{\mathrm{L}}^2(\mathbb{P}^1(\mathbb{C}))$.
(3) The energy of ρ is well-defined and we have

$$(\rho,\rho) = \frac{1}{2\pi}\int_{\mathbb{P}^1(\mathbb{C})} \langle\nabla_g h, \nabla_g h\rangle_g d\mu \geq 0, \tag{2.8}$$

where $\langle\cdot,\cdot\rangle_g$ is the inner product defined by the Fubini-Study metric g on the Riemann Sphere.

PROOF. (1) By Theorem 2.3, there is an integrable function h on $\mathbb{P}^1(\mathbb{C})$ satisfying the distributional equation $\Delta_g h = \rho$. And, without loss of generality, we can assume that $h(\infty) = 0$.

Consider the Jordan decomposition $\rho = \rho^+ - \rho^-$. We are assuming that the trace measure $|\rho|$ has continuous potential. This is, for every point

$p \in \mathbb{P}^1(\mathbb{C})$, there is a proper open neighborhood $V_p \subset \mathbb{P}^1(\mathbb{C})$ and a continuous subharmonic function $h_p : V_p \to \mathbb{R}$ such that

$$\Delta_g h_p = |\rho| \text{ in } \mathcal{D}'(V_p).$$

Without loss of generality we may assume that the neighborhood V_p is relatively compact in one of the open subsets U_0 or U_1.

On the other hand, we can consider the local potentials of the measures ρ^+ and ρ^- by restricting them to V_p. Hence, there are subharmonic functions u_p^+ and u_p^- on V_p such that

$$\Delta_g u_p^+ = \rho^+ \text{ and } \Delta_g u_p^- = \rho^- \text{ in } \mathcal{D}'(V_p).$$

Therefore, in $\mathcal{D}'(V_p)$ we have

$$\Delta_g h_p = |\rho| = \rho^+ + \rho^- = \Delta_g u_p^+ + \Delta_g u_p^- = \Delta_g (u_p^+ + u_p^-).$$

By Weyl's Lemma 1.19, we deduce that $(u_p^+ + u_p^-) - h_p$ is harmonic in V_p. From the continuity of h_p we deduce that, in particular, $u_p^+ + u_p^-$ is continuous on V_p. Now, if the sum of two upper semicontinuous functions is continuous, then both are necessarily continuous and we obtain that u_p^+ and u_p^- are continuous on V_p.

Finally, in $\mathcal{D}'(V_p)$ we have

$$\Delta_g h = \rho = \rho^+ - \rho^- = \Delta_g u_p^+ - \Delta_g u_p^- = \Delta_g (u_p^+ - u_p^-).$$

And, again by Weyl's Lemma, we deduce that h is continuous on V_p. Hence, we can conclude that h is continuous on $\mathbb{P}^1(\mathbb{C})$.

(2) Let us see now that $\nabla_g h \in \vec{\mathrm{L}}^2(\mathbb{P}^1(\mathbb{C}))$. We claim that, with the notation introduced above, for any $p \in \mathbb{P}^1(\mathbb{C})$

$$\nabla_g u_p^+, \nabla_g u_p^- \in \vec{\mathrm{L}}^2_{loc}(V_p).$$

Since $h - u_p^+ + u_p^-$ is harmonic on V_p, the claim implies that $\nabla_g h \in \vec{\mathrm{L}}^2_{loc}(V_p)$. Then we obtain that $\nabla_g h \in \vec{\mathrm{L}}^2(\mathbb{P}^1(\mathbb{C}))$ from the compactness of the Riemann sphere.

Let us prove the claim. Without loss of generality we may assume that V_p is a connected relatively compact subset in $\mathbb{R}^2$. Since u_p^+ is subharmonic on the domain V_p, Theorem 1.17 tells us that we can build by convolution a decreasing sequence of smooth functions $\{u_n\}$ with pointwise limit u_p^+. Moreover, since u_p^+ is continuous, by Theorem 1.10, we have that the convergence is uniform on compacts.

Let $\phi \in \mathcal{D}(V_p)$ be a positive function and set $K_\phi = \mathrm{supp}(\phi)$, by the Theorem 1.12 applied to the smooth vector field ∇u_n and the smooth compactly supported function ϕu_n, we obtain

$$\int_{V_p} \phi \langle \nabla u_n, \nabla u_n \rangle dA = -2\pi \int_{V_p} \phi u_n \Delta u_n dA - \int_{V_p} u_n \langle \nabla \phi, \nabla u_n \rangle dA. \tag{2.9}$$

Since u_p^+ is bounded on K_ϕ and the sequence $\{u_n\}$ converges uniformly to u_p^+ on compacts, we have that there is $c > 0$ such that $|u_n(z)| \leq c$ for every $z \in K_\phi$ and every $n \geq 1$. We obtain

$$\left| \int_{V_p} \phi u_n \Delta u_n dA \right| \leq c \int_{V_p} \phi \Delta u_n dA = c \int_{V_p} u_n \Delta \phi dA \xrightarrow[n\to\infty]{} c \int_{K_\phi} u_p^+ \Delta \phi dA < \infty \tag{2.10}$$

where the first inequality is given by the fact that ϕ is positive and u_n subharmonic. When considering the limit it is enough to see that $u_n \Delta \phi$ converges uniformly to $u_p^+ \Delta \phi$ on K_ϕ.

Now, we will study the second summand on the right-hand side of (2.9). Using a similar argument as in the proof of Lemma 2.7, we can prove that there is $\tilde{c}_\phi \geq 0$ such that, for every $n \geq 1$

$$\left| \int_{V_p} u_n \langle \nabla \phi, \nabla u_n \rangle dA \right| \leq \tilde{c}_\phi.$$

Hence, this together with (2.10) implies that, for every positive $\phi \in \mathcal{D}(V_p)$, there is a positive constant c_ϕ such that

$$0 \leq \int_{V_p} \phi \langle \nabla u_n, \nabla u_n \rangle dA \leq c_\phi, \text{ for every } n \geq 1.$$

Let $K \subset V_p$ compact. Then there is a compact $K' \subset V_p$ such that $K \subset K'$ and a positive smooth function ϕ on V_p such that $\phi \equiv 1$ on K and $\operatorname{supp}(\phi) \subset K'$. Hence, for every $n \geq 1$

$$\|\nabla u_n\|_{\vec{L}^2(K)} = \int_K \langle \nabla u_n, \nabla u_n \rangle dA \leq \int_{V_p} \phi \langle \nabla u_n, \nabla u_n \rangle dA \leq c_\phi.$$

So we have that the sequence $\{\nabla u_n\}$ is bounded in $\vec{\mathrm{L}}^2(K)$ and, since $\vec{\mathrm{L}}^2(K)$ is a Hilbert space, there is a subsequence $\{\nabla u_{n_k}\}$ converging weakly to some $v \in \vec{\mathrm{L}}^2(K)$. In particular, this implies that

$$\nabla u_{n_k} \xrightarrow{\mathcal{D}'(K)} v.$$

But, since u_n converges to u_p^+ in $\mathcal{D}'(K)$, by Lemma 1.15 we have

$$\nabla u_k \xrightarrow{\mathcal{D}'(K)} \nabla u_p^+.$$

Hence, we necessarily have $\nabla u_p^+ = v$ which is in $\vec{\mathrm{L}}^2(K)$ and the claim is proved.

(3) By Proposition 2.8, the energy of ρ is well-defined and we have

$$(\rho, \rho) = -\int_{\mathbb{C}\times\mathbb{C}} \log |z - w| d\rho(z) \otimes d\rho(w).$$

As we mentioned in the beginning of the proof, we may assume without loss of generality that $h(\infty) = 0$ and we have

$$h(z) = \int_{\mathbb{C}} \log|z-w| d\rho(w) \text{ for every } z \in \mathbb{C}.$$

Hence, by Fubini's theorem, we can write

$$(\rho,\rho) = -\int_{\mathbb{P}^1(\mathbb{C})} h(z) d\rho(z). \tag{2.11}$$

Let $h_0 = h \circ \alpha_0^{-1}$. We will see that, for every $R > 0$, the following holds (2.12)

$$\int_{|z|<R} h_0 d\rho = -\frac{1}{2\pi}\int_{|z|<R} \langle \nabla h_0, \nabla h_0\rangle dA + \frac{1}{2\pi}\int_{|z|=R} h_0 \langle \nabla h_0, n(z)\rangle d\sigma,$$

where $n(z)$ is the outward pointing unit normal vector to the curve $|z| = R$ and σ the corresponding volume measure.

We saw that h is continuous on the Riemann sphere and hence so is h_0 on $\mathbb{C}$. We can then build, by convolution, a sequence $\{h_n\}$ of smooth functions on $\mathbb{C}$ such that h_n converges locally uniformly to h_0. And we can write

$$\begin{aligned}\int_{|z|<R} h_0 d\rho &= \lim_n \int_{|z|<R} h_n d\rho \\ &= \lim_n \int_{|z|<R} h_n \Delta h_0 dA = \lim_n \int_{|z|<R} h_0 \Delta h_n dA \\ &= \lim_n \int_{|z|<R} \lim_m h_m \Delta h_n dA = \lim_{m,n} \int_{|z|<R} h_m \Delta h_n dA.\end{aligned}$$

For every $m, n \geq 1$, the divergence theorem applied to the smooth vector field $h_m \nabla h_n$ gives

$$\int_{|z|<R} h_m \Delta h_n dA = -\frac{1}{2\pi}\int_{|z|<R} \langle \nabla h_m, \nabla h_n\rangle dA + \frac{1}{2\pi}\int_{|z|=R} h_m \langle \nabla h_n, n(z)\rangle d\sigma.$$

Consider the limit as m goes to infinity. We have that $h_m \xrightarrow{\mathcal{D}'} h$ and, by Lemma 1.15, this implies $\nabla h_m \xrightarrow{\vec{\mathcal{D}}'} \nabla h$. Hence, since $\nabla h_n \in \vec{\mathcal{D}}$ and $\langle \nabla h_n, n(z)\rangle \in \mathcal{D}$, we obtain

$$\begin{aligned}&\lim_m \int_{|z|<R} h_m \Delta h_n dA \\ &\qquad = -\frac{1}{2\pi}\int_{|z|<R} \langle \nabla h_0, \nabla h_n\rangle dA + \frac{1}{2\pi}\int_{|z|=R} h_0 \langle \nabla h_n, n(z)\rangle d\sigma.\end{aligned}$$

Part (ii) of the theorem implies, in particular, that ∇h_0 is in $\vec{\mathrm{L}}^2_{loc}(\mathbb{C})$. This is equivalent to saying that $\frac{\partial h_0}{\partial x}$ and $\frac{\partial h_0}{\partial x}$ are in $\mathrm{L}^2_{loc}(\mathbb{C})$. Let φ be the mollifier such that $h_n = h_0 * \varphi_n$. We have

$$\frac{\partial h_n}{\partial x} = h_0 * \frac{\partial \varphi_n}{\partial x} = \frac{\partial h_0}{\partial x} * \varphi_n.$$

By Theorem 1.10, this implies that

$$\frac{\partial h_n}{\partial x} \xrightarrow{\mathrm{L}^2_{loc}(\mathbb{C})} \frac{\partial h_0}{\partial x} \text{ as } n \to \infty.$$

The same holds for the partial derivative with respect to y and thus we obtain

$$\nabla h_n \xrightarrow{\vec{\mathrm{L}}^2_{loc}(\mathbb{C})} \nabla h_0.$$

We know that $\vec{\mathrm{L}}^2(D(0,R))$ is a Hilbert space and therefore strong convergence implies weak convergence. Since ∇h_0 and $h \cdot n(z)$ are in $\vec{\mathrm{L}}^2(D(0,R))$, we have

$$\lim_{n,m} \int_{|z|<R} h_m \Delta h_n dA$$
$$= -\frac{1}{2\pi}\int_{|z|<R} \langle \nabla h_0, \nabla h_0 \rangle dA + \frac{1}{2\pi}\int_{|z|=R} h_0 \langle \nabla h_0, n(z)\rangle d\sigma.$$

Therefore (2.12) holds for every $R > 0$. Considering now the limit as R tends to infinity, since $h(\infty) = 0$, we obtain

$$\int_{\mathbb{P}^1(\mathbb{C})} h d\rho = -\frac{1}{2\pi}\int_{\mathbb{P}^1(\mathbb{C})} \langle \nabla_g h, \nabla_g h\rangle_g d\mu.$$

This last expression together with (2.11) concludes the proof of the theorem. □

1.2. Regularization of measures. In this section we present a method to *regularize* signed measures in such a way that they have smooth potential. This regularization is slightly different from the one appearing in [**FRL06**] and it will be done using convolutions with a mollifier on $\mathbb{C}$. For the record, a mollifier φ is a positive smooth function on $\mathbb{C}$ with support contained on the unit disc and such that

$$\int_{\mathbb{C}} \varphi dA = 1.$$

In addition, we will assume that $\varphi(z) = \varphi(|z|)$.

Let $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$ be a continuos function and $\varepsilon > 0$, we define

$$f_\varepsilon(z) = \begin{cases} f * \varphi_\varepsilon(z) & \text{if } z \in \mathbb{C}, \\ f(\infty) & \text{if } z = \infty. \end{cases}$$

where

$$\varphi_\varepsilon(z) = \frac{1}{\varepsilon^2}\varphi\left(\frac{z}{\varepsilon}\right).$$

We know, by Lemma 1.9 and Theorem 1.10, that f_ε is smooth on $\mathbb{C}$ and converges uniformly to f as $\varepsilon \to 0$. The continuity on the whole sphere follows directly from the definition of the function f_ε at the point at infinity.

Let us define now the convolution of finite measures on $\mathbb{P}^1(\mathbb{C})$.

DEFINITION 2.10. Let ρ be a signed measure on $\mathbb{P}^1(\mathbb{C})$ and $\varepsilon > 0$. We define its convolution $\rho_\varepsilon = \varphi_\varepsilon * \rho$ by

$$\int_{\mathbb{P}^1(\mathbb{C})} f d\rho_\varepsilon := \int_{\mathbb{P}^1(\mathbb{C})} (f * \varphi_\varepsilon) d\rho = \int_{\mathbb{P}^1(\mathbb{C})} f_\varepsilon d\rho,$$

for every real-valued continuous function f on $\mathbb{P}^1(\mathbb{C})$.

LEMMA 2.11. *Consider a regular signed measure ρ in $\mathbb{P}^1(\mathbb{C})$. Then for every $\varepsilon > 0$, the convolution ρ_ε is a regular signed measure. Moreover, if ρ is a probability measure, so is ρ_ε.*

PROOF. By linearity, we may assume ρ is a finite regular positive measure. For $\varepsilon > 0$, consider the functional

$$\begin{array}{cccc} \Lambda_\varepsilon : & \mathscr{C}^0(\mathbb{P}^1(C)) & \longrightarrow & \mathbb{R} \\ & f & \mapsto & \int_{\mathbb{P}^1(\mathbb{C})} f_\varepsilon d\rho. \end{array}$$

First of all, let us see that it is a linear functional. Let $f, f' \in \mathscr{C}^0(\mathbb{P}^1(\mathbb{C}))$ and $\lambda, \lambda' \in \mathbb{R}$, we can write

$$\begin{aligned} \Lambda_\varepsilon(\lambda f + \lambda' f') &= \int_{\mathbb{P}^1(\mathbb{C})} (\lambda f + \lambda' f')_\varepsilon d\rho \\ &= \int_{\mathbb{P}^1(\mathbb{C})} \left(\int_{\mathbb{C}} (\lambda f(z-w) + \lambda' f'(z-w)) \varphi_\varepsilon(w) dA(w) \right) d\rho(z) \\ &= \lambda \int_{\mathbb{P}^1(\mathbb{C})} \int_{\mathbb{C}} f(z-w)\varphi_\varepsilon(w) dA(w) d\rho(z) \\ &+ \lambda' \int_{\mathbb{P}^1(\mathbb{C})} \int_{\mathbb{C}} f'(z-w)\varphi_\varepsilon(w) dA(w) d\rho(z) \\ &= \lambda \int_{\mathbb{P}^1(\mathbb{C})} f_\varepsilon d\rho + \lambda' \int_{\mathbb{P}^1(\mathbb{C})} f'_\varepsilon d\rho = \lambda \Lambda_\varepsilon(f) + \lambda \Lambda_\varepsilon(f'). \end{aligned}$$

Now, we will prove that it is positive. Let $f \in \mathscr{C}^0(\mathbb{P}^1(\mathbb{C}))$ such that $f \geq 0$. Then since $\varphi_\varepsilon \geq 0$, we have

$$f_\varepsilon(z) = \int_{\mathbb{C}} f(z-w)\varphi_\varepsilon(w) dA(w) \geq 0$$

and thus, we deduce $\Lambda_\varepsilon(f) \geq 0$.

By *Riesz representation theorem*, there is a unique finite regular positive measure, that we will denote by ρ_ε, such that

$$\Lambda_\varepsilon(f) = \int_{\mathbb{P}^1(\mathbb{C})} f d\rho_\varepsilon.$$

Moreover, if ρ is a probability measure, then

$$\int_{\mathbb{P}^1(\mathbb{C})} d\rho_\varepsilon = \int_{\mathbb{P}^1(\mathbb{C})} d\rho = 1.$$

Hence, ρ_ε is a probability measure on $\mathbb{P}^1(\mathbb{C})$. $\square$

LEMMA 2.12. *Let ρ be a probability measure on $\mathbb{P}^1(\mathbb{C})$ with bounded support contained in $\mathbb{C}$ and $\varepsilon > 0$. Then*

(i) The convolution ρ_ε has bounded support contained in $\mathbb{C}$,
(ii) For every subharmonic function u on $\mathbb{P}^1(\mathbb{C})$ we can write

$$\int_{\mathbb{P}^1(\mathbb{C})} u d\rho_\varepsilon = \int_{\mathbb{P}^1(\mathbb{C})} u_\varepsilon d\rho.$$

PROOF. (i) Let $K = \operatorname{supp}(\rho) \subset \mathbb{C}$, which is compact. We define the subset

$$K_\varepsilon := \{z \in \mathbb{C} : \operatorname{dist}(z, K) \leq \varepsilon\},$$

where $\operatorname{dist}(z, K) = \min_{w \in K} |z - w|$.

Let $\tilde{K}_\varepsilon \subset \mathbb{C}$ compact and such that $K_\varepsilon \subsetneq \tilde{K}_\varepsilon$. Consider a non-zero continuous function $f : \mathbb{P}^1(\mathbb{C}) \to [0, 1]$ such that $\operatorname{supp}(f) \subset \tilde{K}_\varepsilon$ and $f \equiv 1$ on K_ε. Since, for every $z \in K$ we have $D(z, \varepsilon) \subset K_\varepsilon$, we obtain

$$f_\varepsilon(z) = \int_{D(z,\varepsilon)} f(w)\varphi_\varepsilon(z - w) dA(w) = \int_{D(z,\varepsilon)} \varphi_\varepsilon(z - w) dA(w) = 1.$$

Hence,

$$\begin{aligned} 1 = \int_K f_\varepsilon d\rho = \int_{\mathbb{P}^1(\mathbb{C})} f_\varepsilon d\rho = \int_{\mathbb{P}^1(\mathbb{C})} f d\rho_\varepsilon \\ = \int_{\tilde{K}_\varepsilon} f d\rho_\varepsilon \leq \int_{\tilde{K}_\varepsilon} d\rho_\varepsilon \leq \int_{\mathbb{P}^1(\mathbb{C})} d\rho_\varepsilon = 1. \end{aligned}$$

Therefore, we necessarily have that $\operatorname{supp}(\rho_\varepsilon) \subset \tilde{K}_\varepsilon$ which is bounded in $\mathbb{C}$.

(ii) We just saw that, if ρ is compactly supported on the complex plane then so is ρ_ε. Therefore, we can restrict the integration domain to $\mathbb{C}$.

For any subharmonic function u on $\mathbb{C}$ there is a sequence of smooth subharmonic functions $\{u_n\}$ whose pointwise limit is u by decreasing. By the monotone convergence theorem, we can write

$$\begin{aligned} \int_{\mathbb{C}} u(z) d\rho_\varepsilon(z) = \int_{\mathbb{C}} \lim_{n\to\infty} u_n(z) d\rho_\varepsilon(z) \\ = \lim_{n\to\infty} \int_{\mathbb{C}} u_n(z) d\rho_\varepsilon(z) = \lim_{n\to\infty} \int_{\mathbb{C}} u_{n,\varepsilon}(z) d\rho(z). \end{aligned}$$

Now observe that, since for every $\varepsilon > 0$ we have $\varphi_\varepsilon \geq 0$, the sequence $\{\varphi_\varepsilon u_n\}$ converges pointwisely to $\varphi_\varepsilon u$ by decreasing. Thus, we are still under the hypothesis of the monotone convergence theorem and we obtain

$$\begin{aligned} \lim_{n\to\infty} u_{n,\varepsilon}(z) = \lim_{n\to\infty} \int_{\mathbb{C}} u_n(z - w)\varphi_\varepsilon(w) dA(w) \\ = \int_{\mathbb{C}} \lim_{n\to\infty} u_n(z - w)\varphi_\varepsilon(w) dA(w) = \int_{\mathbb{C}} u(z - w)\varphi_\varepsilon(w) dA(w) = u_\varepsilon(z). \end{aligned}$$

Finally, since $u_{1,\varepsilon}(z) \geq u_{2,\varepsilon}(z) \geq \ldots \geq u_\varepsilon(z)$, we have

$$\int_{\mathbb{C}} u(z)d\rho_\varepsilon(z) = \lim_{n\to\infty}\int_{\mathbb{C}} u_{n,\varepsilon}(z)d\rho(z)$$
$$= \int_{\mathbb{C}} \lim_{n\to\infty} u_{n,\varepsilon}(z)d\rho(z) = \int_{\mathbb{C}} u_\varepsilon(z)d\rho(z).$$

□

PROPOSITION 2.13. *Let ρ be a probability measure on $\mathbb{P}^1(\mathbb{C})$ with compact support contained in $\mathbb{C}$ and $\varepsilon > 0$. Then ρ_ε has smooth potential.*

Before proving this result, let us make some remarks. As for the case of signed measures with continuous potential, those with smooth potential satisfy a local condition. We say that a signed measure ρ on $\mathbb{P}^1(\mathbb{C})$ has smooth potential if for every point $p \in \mathbb{P}^1(\mathbb{C})$ there is a neighborhood U_p and a smooth function $h_p : U_p \to \mathbb{R}$ such that $\Delta_g h_p = \rho$. An analogous result to Corollary 2.5 also holds for signed measures with smooth potential.

As a second comment to this proposition, we mention that in order to prove the main theorem of this chapter, the authors in **[FRL06]** only ask for a regularization with continuous potential. Hence, it would be enough to regularize by convolution with a continuous function.

PROOF. By the previous lemma, the probability measure ρ_ε is compactly supported on $\mathbb{C}$ and therefore we can consider its potential, which is given by

$$u_{\rho_\varepsilon}(z) = \int_{\mathbb{C}} \log|z-w|d\rho_\varepsilon(w).$$

For any fixed $z \in \mathbb{C}$, the logarithmic kernel $K(z,\cdot) = \log|z-\cdot|$ is subharmonic on $\mathbb{C}$ and, by Fubini's theorem, we can write

$$\begin{aligned} u_{\rho_\varepsilon}(z) = \int_{\mathbb{C}} K(z,w)d\rho_\varepsilon(w) &= \int_{\mathbb{C}} K_\varepsilon(z,w)d\rho(w) \\ &= \int_{\mathbb{C}}\left(\int_{\mathbb{C}} K(z,w-v)\varphi_\varepsilon(v)dA(v)\right)d\rho(w) \\ &= \int_{\mathbb{C}}\left(\int_{\mathbb{C}} \log|z-w+v|d\rho(w)\right)\varphi_\varepsilon(v)dA(v) \\ &= \int_{\mathbb{C}} u_\rho(z+v)\varphi_\varepsilon(v)dA(v) = u_\rho * \varphi_\varepsilon(z), \end{aligned}$$

where u_ρ is the potential associated to ρ. This implies that u_{ρ_ε} is smooth. □

Consider a finite set $S \subset \mathbb{C}$ and the discrete probability measure associated to it, μ_S. Since it is a finitely supported measure, we can consider its potential, which is given by

$$u_S(z) = \frac{1}{\#S}\sum_{\alpha\in S}\log|z-\alpha|.$$

Observe that this is a non-continuous subharmonic function on $\mathbb{C}$. The previous proposition establishes that, for any $\varepsilon > 0$, the probability measure $\mu_{S,\varepsilon} = \mu_S * \varphi_\varepsilon$ has smooth potential.

2. Quantitative equidistribution

As we just saw, for any finite set $S \subset \mathbb{C}$, the potential of the measure μ_S is not even locally bounded and thus little can be said about the energy of the measure $\mu_S - \lambda_{S^1}$. However, once a mollifier has been fixed, for every $\varepsilon > 0$ the regularization $\mu_{S,\varepsilon} - \lambda_{S^1}$ is such that its trace measure has continuous potential and, by Theorem 2.9, we have that

$$(\mu_{S,\varepsilon} - \lambda_{S^1}, \mu_{S,\varepsilon} - \lambda_{S^1}) \geq 0.$$

The first step towards the proof of the main theorem of this chapter will be to give an estimate of the difference of the energies of $\mu_S - \lambda_{S^1}$ and its regularization. In order to make the estimations explicit, we will consider a specific mollifier φ given by

$$\varphi(z) = \begin{cases} \mathfrak{c} \exp\left(\frac{-1}{(|z|^2-1)^2}\right) & \text{if } |z| < 1, \\ 0 & \text{if } |z| \geq 1, \end{cases}$$

where $\mathfrak{c}$ is such that $\int_{\mathbb{C}} \varphi dA = 1$. This is,

$$\mathfrak{c} = \left(2\pi \int_0^1 \exp\left(\frac{-1}{(r^2-1)^2}\right) rdr\right)^{-1} \approx 3.57355.$$

We would like to point out that all the explicit constants appearing from now on will depend on the particular choice we made for φ.

Let us consider two technical lemmas.

LEMMA 2.14. *For every finite set $S \subset \mathbb{C}$ and every $\varepsilon > 0$ we have*

$$|(\mu_S, \lambda_{S^1}) - (\mu_{S,\varepsilon}, \lambda_{S^1})| \leq \varepsilon. \tag{2.13}$$

PROOF. For any $\alpha \in \mathbb{C}$, the measure $\delta_{\alpha,\varepsilon}$ is compactly supported on $\mathbb{C}$ and has smooth potential. Since λ_{S^1} is also a compactly supported signed measure on $\mathbb{C}$ and it has continuous potential, from Proposition 2.8 we deduce that $(\delta_\alpha, \lambda_{S^1})$ and $(\delta_{\alpha,\varepsilon}, \lambda_{S^1})$ are well-defined. This, together with Fubini's theorem leads to

$$(\delta_\alpha, \lambda_{S^1}) = -\int_{\mathbb{C}} \left(\int_{\mathbb{C}} \log|z-w| d\lambda_{S^1}(w)\right) d\delta_\alpha(z) = -\log^+|\alpha|$$

and

$$\begin{aligned}(\delta_{\alpha,\varepsilon}, \lambda_{S^1}) &= -\int_{\mathbb{C}} \left(\int_{\mathbb{C}} \log|z-w| d\lambda_{S^1}(w)\right) d\delta_{\alpha,\varepsilon}(z) \\ &= -\int_{\mathbb{C}} \log^+|z| d\delta_{\alpha,\varepsilon}(z) = -\int_{\mathbb{C}} \left(\int_{\mathbb{C}} \log^+|z-w|\varphi_\varepsilon(w) dA(w)\right) d\delta_\alpha(z) \\ &= -\int_{\mathbb{C}} \log^+|\alpha - w|\varphi_\varepsilon(w) dA(w).\end{aligned}$$

We will see that for any $\varepsilon > 0$ and any $w \in D(0,\varepsilon)$, we have

$$|\log^+|\alpha - w| - \log^+|\alpha|| < \varepsilon.$$

Suppose there is $w \in D(0,\varepsilon)$ such that $|\alpha - w| < 1$. Then we have

$$|\log^+|\alpha - w| - \log^+|\alpha|| = \log^+|\alpha|.$$

If $|\alpha| \leq 1$ it follows trivially. If $|\alpha| > 1$, since we are assuming there is $w \in D(0,\varepsilon)$ such that $|\alpha - w| < 1$, we have $|\alpha| < 1 + \varepsilon$ and therefore $\log^+|\alpha| = \log|\alpha| < \log(1+\varepsilon) < \varepsilon$.

Assume now that there is $w \in D(0,\varepsilon)$ such that $|\alpha - w| > 1$. If α is in the unit disc, we have $|\log^+|\alpha - w| - \log^+|\alpha|| = \log|\alpha - w| < \log(1+\varepsilon) \leq \varepsilon$ and we are done. Finally, suppose α is not in the unit disc so we have

$$|\log^+|\alpha - w| - \log^+|\alpha|| = |\log|\alpha - w| - \log|\alpha|| \leq |w| < \varepsilon.$$

Putting everything together, we obtain

$$\begin{aligned}
|(\delta_{\alpha,\varepsilon}, \lambda_{S^1}) - (\delta_\alpha, \lambda_{S^1})| &= \left| \int_{\mathbb{C}} \log^+|\alpha - w| \varphi_\varepsilon(w) dA(w) - \log^+|\alpha| \right| \\
&\leq \int_{D(0,\varepsilon)} \left| \log^+|\alpha - w| - \log^+|\alpha| \right| \varphi_\varepsilon(w) dA(w) \\
&< \int_{D(0,\varepsilon)} \varepsilon \varphi_\varepsilon(w) dA(w) = \varepsilon.
\end{aligned}$$

At last, for any finite set $S \subset \mathbb{C}$ we can conclude

$$|(\mu_{S,\varepsilon}, \lambda_{S^1}) - (\mu_S, \lambda_{S^1})| = \left| \frac{1}{\#S} \sum_{\alpha \in S} (\delta_{\alpha,\varepsilon}, \lambda_{S^1}) - \frac{1}{\#S} \sum_{\alpha \in S} (\delta_\alpha, \lambda_{S^1}) \right| < \varepsilon.$$

□

LEMMA 2.15. *For every $\varepsilon > 0$ and every finite set $S \subset \mathbb{C}$, we have*

$$(\mu_{S,\varepsilon}, \mu_{S,\varepsilon}) \leq (\mu_S, \mu_S) + \frac{1}{\#S} \left(2\pi \log \frac{1}{\varepsilon} + C \right), \tag{2.14}$$

where $C \approx 1.10559$.

PROOF. Let $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$. Since, for every $\varepsilon > 0$, $\delta_{\alpha,\varepsilon}$ and $\delta_{\beta,\varepsilon}$ are compactly supported on the complex plane and have smooth potentials, by Proposition 2.8, their mutual energy is well-defined and we have

$$(\delta_{\alpha,\varepsilon}, \delta_{\beta,\varepsilon}) = -\int_{\mathbb{C} \times \mathbb{C}} K(w,z) d\delta_{\alpha,\varepsilon}(z) d\delta_{\beta,\varepsilon}(w),$$

where recall $K(z,w) = \log|z - w|$.

By Theorem 1.17, we have

$$\begin{aligned}\int_{\mathbb{C}\times\mathbb{C}} K(z,w)d\delta_{\alpha,\varepsilon}(z)d\delta_{\beta,\varepsilon}(w) &= \int_{\mathbb{C}\times\mathbb{C}} (K(\cdot,w)*\varphi_\varepsilon)(z)d\delta_\alpha(z)d\delta_{\beta,\varepsilon}(w)\\ &= \int_{\mathbb{C}} (K(\cdot,w)*\varphi_\varepsilon)(\alpha)d\delta_{\beta,\varepsilon}(w) \geq \int_{\mathbb{C}} K(\alpha,w)d\delta_{\beta,\varepsilon}(w)\\ &= \int_{\mathbb{C}} (K(\alpha,\cdot)*\varphi_\varepsilon)(w)d\delta_\beta(w) = (K(\alpha,\cdot)*\varphi_\varepsilon)(\beta) \geq K(\alpha,\beta).\end{aligned}$$

So, we can write

$$(\delta_{\alpha,\varepsilon},\delta_{\beta,\varepsilon}) \leq -\log|\alpha-\beta| = (\delta_\alpha,\delta_\beta).$$

On the other hand, by Proposition 2.8, for every $\alpha \in \mathbb{C}$ we have that $K(z,w)$ in integrable with respect to $\delta_{\alpha,\varepsilon}\otimes\delta_{\alpha,\varepsilon}$ and by Fubini's theorem we have

$$\begin{aligned}-(\delta_{\alpha,\varepsilon},\delta_{\alpha,\varepsilon}) &= \int_{\mathbb{C}\times\mathbb{C}} K(z,w)d\delta_{\alpha,\varepsilon}(z)d\delta_{\alpha,\varepsilon}(w)\\ &= \int_{\mathbb{C}\times\mathbb{C}} (K(\cdot,w)*\varphi_\varepsilon)(z)d\delta_\alpha(z)d\delta_{\alpha,\varepsilon}(w) = \int_{\mathbb{C}} (K(\cdot,w)*\varphi_\varepsilon)(\alpha)d\delta_{\alpha,\varepsilon}(w)\\ &\geq \int_{\mathbb{C}} K(\alpha,w)d\delta_{\alpha,\varepsilon}(w) = \int_{\mathbb{C}} (K(\alpha,\cdot)*\varphi_\varepsilon)(w)d\delta_\alpha(w) = (K(\alpha,\cdot)*\varphi_\varepsilon)(\alpha)\\ &= \int_{\mathbb{C}} \log|z|\varphi_\varepsilon(z)dA(z).\end{aligned}$$

Since $\operatorname{supp}(\varphi_\varepsilon) \subset D(0,\varepsilon)$ and $\varphi_\varepsilon(z) = \varphi_\varepsilon(|z|)$, we obtain

$$\begin{aligned}\int_{\mathbb{C}} \log|z|\varphi_\varepsilon(z)dA(z) &= \int_0^{2\pi}\int_0^{\varepsilon} \log r\varphi_\varepsilon(r)rdrd\theta\\ &= 2\pi\int_0^{\varepsilon} \frac{1}{\varepsilon^2}\log r\varphi\left(\frac{r}{\varepsilon}\right)rdr = 2\pi\int_0^1 (\log\varepsilon+\log s)\varphi(s)sds\\ &= 2\pi\log\varepsilon - C,\end{aligned}$$

where

$$C = -\int_{\mathbb{C}} \log|z|\varphi(z)dA(z) \approx 1.10559.$$

So we can write

$$(\delta_{\alpha,\varepsilon},\delta_{\alpha,\varepsilon}) \leq 2\pi\log\frac{1}{\varepsilon} + C$$

Finally, for every finite set $S \subset \mathbb{C}$ we have

$$\begin{aligned}(\mu_{S,\varepsilon}, \mu_{S,\varepsilon}) &= \frac{1}{(\#S)^2} \sum_{\substack{\alpha,\beta\in S\\ \alpha\neq\beta}} (\delta_{\alpha,\varepsilon}, \delta_{\beta,\varepsilon}) + \frac{1}{(\#S)^2} \sum_{\alpha\in S} (\delta_{\alpha,\varepsilon}, \delta_{\alpha,\varepsilon}) \\ &\leq \frac{1}{(\#S)^2} \sum_{\substack{\alpha,\beta\in S\\ \alpha\neq\beta}} (\delta_{\alpha}, \delta_{\beta}) + \frac{1}{(\#S)^2} \sum_{\alpha\in S} \left(2\pi \log \frac{1}{\varepsilon} + C\right) \\ &= (\mu_S, \mu_S) + \frac{1}{\#S} \left(2\pi \log \frac{1}{\varepsilon} + C\right).\end{aligned}$$

□

We can now give an estimate for the difference of the energy $\mu_S - \lambda_{S^1}$ and its regularization.

Proposition 2.16. *For every $\varepsilon > 0$ and every finite set $S \subset \mathbb{C}$, we have*

$$(\mu_{S,\varepsilon} - \lambda_{S^1}, \mu_{S,\varepsilon} - \lambda_{S^1}) - (\mu_S - \lambda_{S^1}, \mu_S - \lambda_{S^1}) \leq \frac{1}{\#S} \left(2\pi \log \frac{1}{\varepsilon} + C\right) + 2\varepsilon,$$

where $C \approx 1.10559$.

Proof. We know that both λ_{S^1} and $\mu_{S,\varepsilon}$ are probability measures with compact support on $\mathbb{C}$ and continuous potential. Hence, by Theorem 2.9, the measure $\mu_{S,\varepsilon} - \lambda_{S^1}$ is such that its energy is well-defined and positive.

Since $(\lambda_{S^1}, \lambda_{S^1}) = 0$, we have

$$(\mu_S - \lambda_{S^1}, \mu_S - \lambda_{S^1}) = (\mu_S, \mu_S) - 2(\mu_S, \lambda_{S^1})$$

and, by the previous lemmas, we obtain

$$\begin{aligned}(\mu_S - \lambda_{S^1}, \mu_S - \lambda_{S^1}) &\geq (\mu_{S,\varepsilon}, \mu_{S,\varepsilon}) - \frac{1}{\#S} \left(2\pi \log \frac{1}{\varepsilon} + C\right) - 2(\mu_{S,\varepsilon}, \lambda_{S^1}) + 2\varepsilon \\ &= (\mu_{S,\varepsilon} - \lambda_{S^1}, \mu_{S,\varepsilon} - \lambda_{S^1}) - \frac{1}{\#S} \left(2\pi \log \frac{1}{\varepsilon} + C\right) - 2\varepsilon.\end{aligned}$$

□

We will now define a pairing for $\mathscr{C}^1$-functions on the Riemann Sphere. It is a generalization to $\mathbb{P}^1(\mathbb{C})$ of the *Dirichlet form* of functions of Class $\mathscr{C}^1$ in an open bounded domain in the complex plane.

Definition 2.17. For any two real-valued functions $f, h \in \mathscr{C}^1(\mathbb{P}^1(\mathbb{C}))$, we define their *Dirichlet form* as

$$\langle f, h \rangle = \frac{1}{2\pi} \int_{\mathbb{P}^1(\mathbb{C})} \langle \nabla_g f, \nabla_g h \rangle_g d\mu.$$

Let us see that the Dirichlet form is well-defined. Recall that locally on any chart U_j with coordinates x, y, the Fubini-Study metric is given by

$$g(x,y) = \begin{pmatrix} \frac{4}{(1+x^2+y^2)^2} & 0 \\ 0 & \frac{4}{(1+x^2+y^2)^2} \end{pmatrix}.$$

When restricted to U_j, we have that $\mu = \sqrt{\det(g(x,y))}dxdy$ and

$$\langle \nabla f, \nabla h \rangle_g = \frac{(1+x^2+y^2)^2}{4} \left(\frac{\partial f_i}{\partial x}\frac{\partial h_i}{\partial x} + \frac{\partial f_i}{\partial y}\frac{\partial h_i}{\partial y} \right),$$

where $f_i = f \circ \alpha_i^{-1}$ and $h_i = h \circ \alpha_i^{-1}$.

Hence, we obtain

$$\begin{aligned} \int_{\mathbb{P}^1(\mathbb{C})} \langle \nabla f, \nabla h \rangle_g d\mu = \int_{\overline{D}(0,1)} & \left(\frac{\partial f_0}{\partial x}\frac{\partial h_0}{\partial x} + \frac{\partial f_0}{\partial y}\frac{\partial h_0}{\partial y} \right) dxdy \\ & + \int_{D(0,1)} \left(\frac{\partial f_1}{\partial x}\frac{\partial h_1}{\partial x} + \frac{\partial f_1}{\partial y}\frac{\partial h_1}{\partial y} \right) dxdy < \infty. \end{aligned}$$

As it was mentioned on the beginning of the chapter, it is easy to see that $\mathscr{C}^1$-functions are Lipschitz. The following proposition will provide us with a relation between the Dirichlet form of a given function and its Lipschitz constant.

Proposition 2.18. *For every real-valued $f \in \mathscr{C}^1(\mathbb{P}^1(\mathbb{C}))$ we have*

$$\langle f, f \rangle \leq 2\,\mathrm{Lip}(f)^2.$$

Proof. Let us assume the following claim:

$$\langle \nabla_g f(p), \nabla_g f(p) \rangle_g^{\frac{1}{2}} \leq \mathrm{Lip}(f), \text{ for every } p \in \mathbb{P}^1(\mathbb{C}). \tag{2.15}$$

Hence, by (1.8), we deduce

$$\begin{aligned} \langle f, f \rangle = \frac{1}{2\pi} \int_{\mathbb{P}^1(\mathbb{C})} \langle \nabla_g f(p), \nabla_g f(p) \rangle_g d\mu(p) & \leq \frac{1}{2\pi} \int_{\mathbb{P}^1(\mathbb{C})} \mathrm{Lip}(f)^2 d\mu \\ & = \frac{\mathrm{Lip}(f)^2}{2\pi} \mu(\mathbb{P}^1(\mathbb{C})) = 2\,\mathrm{Lip}(f)^2. \end{aligned}$$

So we are only left with the proof of (2.15).

Let $p \in \mathbb{P}^1(\mathbb{C})$, we will see that

$$\langle \nabla_g f(p), \xi \rangle_g \leq \mathrm{Lip}(f) \|\xi\|_g \text{ for every } \xi \in T_p\mathbb{P}^1(\mathbb{C}),$$

where $\|\xi\|_g = \langle \xi, \xi \rangle_g^{\frac{1}{2}}$. Hence, taking $\xi = \nabla_g f(p)$ would prove the claim.

Without loss of generality, we may assume $p = (1 : z_0) \in U_0$. Let $\xi \in T_p\mathbb{P}^1(\mathbb{C})$, with $\xi = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y}$ and consider the path $\gamma_0 : [0,1] \to \mathbb{R}^2$ given by

$$\gamma_0(s) = (x_0 + s\xi_1, y_0 + s\xi_2),$$

where $z_0 = x_0 + iy_0$. It is then obvious that $\gamma = \alpha_0^{-1} \circ \gamma_0$ is a smooth path in U_0 such that $\gamma(0) = p$ and $\dot{\gamma}(0) = \xi$.

For every $t \in [0,1]$, by the *mean value theorem*, there is some $c \in (0,t)$ such that

$$f_0(\gamma_0(t)) - f_0(\gamma_0(0)) = (f_0 \circ \gamma_0)'(c) = \langle \nabla f_0(\gamma_0(c)), (\xi_1, \xi_2)\rangle,$$

where $f_0 = f \circ \alpha_0^{-1}$ and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^2$. Note that $f_0 \circ \gamma_0 = f \circ \gamma$ and

$$\langle \nabla f_0(\gamma_0(c)), (\xi_1, \xi_2)\rangle = \langle \nabla_g f(\gamma(c)), \xi\rangle_g.$$

Therefore, we can rewrite

$$f(\gamma(t)) - f(\gamma(0)) = \langle \nabla_g f(\gamma(c)), \xi\rangle_g.$$

On the other hand, since f is a Lipschitz function of Lipschitz constant $\mathrm{Lip}(f)$ with respect to the spherical distance, and the spherical distance is the infimum of the lengths of all smooth paths joining two points in the sphere, we obtain

$$\begin{aligned} |f(\gamma(t)) - f(\gamma(0))| \leq \mathrm{Lip}(f)\, \mathrm{d}(\gamma(t), \gamma(0)) &\leq \mathrm{Lip}(f)\, \mathrm{length}(\gamma_{|_{[0,t]}}) \\ &= \mathrm{Lip}(f) \int_0^t \|\dot\gamma(s)\|_g ds. \end{aligned}$$

Putting everything together and letting $t \to 0$, we can conclude

$$\langle \nabla_g f(p), \xi\rangle_g = \lim_{t\to 0} \langle \nabla_g f(\gamma(c)), \xi\rangle_g \leq \lim_{t\to 0} \mathrm{Lip}(f) \int_0^t \|\dot\gamma(s)\|_g ds = \mathrm{Lip}(f)\|\xi\|_g.$$

□

We will now give a bound for the integral of a function in $\mathscr{C}^1(\mathbb{P}^1(\mathbb{C}))$ with respect to a signed measure satisfying the hypothesis of Theorem 2.9. In fact, the bound is given in terms of the Dirichlet form of the function and the energy of the measure.

PROPOSITION 2.19. *For every function $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$ of class $\mathscr{C}^1$ and every signed measure ρ on the Riemann sphere with vanishing total mass and such that its trace measure has continuous potential, we have*

$$\left| \int_{\mathbb{P}^1(\mathbb{C})} f d\rho \right|^2 \leq \langle f, f\rangle (\rho, \rho).$$

PROOF. By Theorem 2.9, there is a continuous real-valued function h on the Riemann sphere such that $\Delta_g h = \rho$, $\nabla_g h$ is in $\vec{\mathrm{L}}^2(\mathbb{P}^1(\mathbb{C}))$ and

$$(\rho, \rho) = \frac{1}{2\pi} \int_{\mathbb{P}^1(\mathbb{C})} \langle \nabla_g h, \nabla_g h\rangle_g d\mu.$$

Since f is in $\mathscr{C}^1(\mathbb{P}^1(\mathbb{C}))$, we have that $\nabla_g f$ is in $\vec{\mathrm{L}}^2(\mathbb{P}^1(\mathbb{C}))$. Therefore, if we proceed as in the proof of the last part of Theorem 2.9, we obtain

$$\int_{\mathbb{P}^1(\mathbb{C})} f d\rho = -\frac{1}{2\pi} \int_{\mathbb{P}^1(\mathbb{C})} \langle \nabla_g f, \nabla_g h\rangle_g d\mu.$$

Finally, by Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&\left|\int_{\mathbb{P}^1(\mathbb{C})} f d\rho\right| \\
&\quad \leq \left(\frac{1}{2\pi}\int_{\mathbb{P}^1(\mathbb{C})}\langle\nabla_g f,\nabla_g f\rangle_g d\mu\right)^{\frac{1}{2}}\left(\frac{1}{2\pi}\int_{\mathbb{P}^1(\mathbb{C})}\langle\nabla_g h,\nabla_g h\rangle_g d\mu\right)^{\frac{1}{2}} \\
&\qquad\qquad = \langle f,f\rangle^{\frac{1}{2}}(\rho,\rho)^{\frac{1}{2}}.
\end{aligned}$$

□

Let us, before giving the proof of Theorem II, state a very nice result that relates the energy of the signed measure $\mu_S - \lambda_{S^1}$ and the height of the elements in the finite set S.

Recall that, given a finite set $S \subset \overline{\mathbb{Q}}$, its height is given by

$$\mathrm{h}(S) = \sum_{\alpha\in S}\mathrm{h}(\alpha).$$

LEMMA 2.20. *Let $S \subset \overline{\mathbb{Q}}^{\times}$ be a finite Galois-invariant set. Then we have*

$$(\mu_S - \lambda_{S^1}, \mu_S - \lambda_{S^1}) \leq 2\frac{\mathrm{h}(S)}{\#S}.$$

PROOF. Since S is a finite Galois-invariant set in $\overline{\mathbb{Q}}^{\times}$, it is a finite union of different Galois orbits. This is, $S = S_1 \cup \ldots \cup S_r$ where S_i is the orbit of an algebraic number under the action of the absolute Galois group.

For every $i = 1, \ldots, r$, we will denote by $P_i(x) \in \mathbb{Z}[x]$ the minimal polynomial over $\mathbb{Z}$ of the orbit S_i and a_i its leading coefficient. We can then write

$$P_i(x) = a_i \prod_{\alpha\in S_i}(x-\alpha),$$

and we have $\deg(P_i) = \#S_i$.

Now, consider the polynomial $P(x) = P_1(x)\cdots P_r(x) = A\prod_{\alpha\in S}(x-\alpha)$, where $A = \prod_{i=1}^r a_i$, and observe that it has degree $d := \#S$. Recall the definition of the discriminant of the polynomial $P(x)$,

$$\Delta_P = (-1)^{d(d-1)/2}A^{2d-2}\prod_{\alpha,\beta\in S,\alpha\neq\beta}(\alpha-\beta)$$

Since it is a symmetric function in the roots of $P(x)$, which are all different, it can be expressed in terms of its coefficients and hence, we deduce that Δ_P is a non-zero integer.

Let us study the mutual energy of $\mu_S - \lambda_{S^1}$. By Proposition 2.8, we know that (μ_S, μ_S) and (μ_S, λ_{S^1}) are well-defined and, since the mutual energy of λ_{S^1} vanishes, we can write

$$(\mu_S - \lambda_{S^1}, \mu_S - \lambda_{S^1}) = (\mu_S, \mu_S) - 2(\mu_S, \lambda_{S^1}).$$

On one hand, we have that

$$(\mu_S, \mu_S) = -\int_{\mathbb{C}\times\mathbb{C}\setminus\boldsymbol{\Delta}} \log|z-w| d\mu_S(w) \otimes d\mu_S(z)$$
$$= -\frac{1}{(\#S)^2} \sum_{\substack{\alpha,\beta\in S\\ \alpha\neq\beta}} \log|\alpha-\beta| = -\frac{1}{(\#S)^2} \log\left(\prod_{\substack{\alpha,\beta\in S\\ \alpha\neq\beta}} |\alpha-\beta|\right)$$
$$= -\frac{1}{d^2}\log\left|\frac{\Delta_P}{A^{2d-2}}\right| = -\frac{1}{d^2}\log|\Delta_P| + \frac{1}{d^2}(2d-2)\log|A| \leq \frac{2}{d}\log|A|.$$

On the other hand,

$$(\mu_S, \lambda_{S^1}) = -\int_{\mathbb{C}\times\mathbb{C}\setminus\boldsymbol{\Delta}} \log|z-w| d\mu_S(z) \otimes d\lambda_{S^1}(w)$$
$$= -\frac{1}{\#S}\int_{\mathbb{C}} \sum_{\alpha\in S} \log|z-\alpha| d\lambda_{S^1}(z)$$
$$= -\frac{1}{\#S}\int_{\mathbb{C}} \log\left|\prod_{\alpha\in S}(z-\alpha)\right| d\lambda_{S^1}(z)$$
$$= -\frac{1}{\#S}\int_{\mathbb{C}} \log\left|\frac{P(z)}{A}\right| d\lambda_{S^1}(z)$$
$$= -\frac{1}{d}\int_{\mathbb{C}} \log|P_1(z)\cdots P_r(z)| d\lambda_{S^1}(z) + \frac{1}{d}\log|A|$$
$$= -\frac{1}{d}\left(\int_{\mathbb{C}} \log|P_1(z)| d\lambda_{S^1}(z) + \ldots + \int_{\mathbb{C}} \log|P_r(z)| d\lambda_{S^1}(z)\right) + \frac{1}{d}\log|A|$$
$$= -\frac{1}{d}(m(P_1) + \ldots + m(P_r)) + \frac{1}{d}\log|A|,$$

where $m(P_i)$ is the Mahler measure of the polynomial P_i.

Putting everything together, we obtain

$$(\mu_S - \lambda_{S^1}, \mu_S - \lambda_{S^1}) \leq \frac{2}{d}(m(P_1) + \ldots + m(P_r))$$
$$= \frac{2}{d}(\mathrm{h}(S_1) + \ldots + \mathrm{h}(S_r)) = 2\frac{\mathrm{h}(S)}{\#S}.$$

$\square$

We will now give the proof of the main theorem of the chapter.

Proof of Theorem II. Let $\varepsilon > 0$, then we have

$$
\begin{aligned}
(2.16) \qquad \left|\int_{\mathbb{P}^1(\mathbb{C})} f d\mu_S - \int_{\mathbb{P}^1(\mathbb{C})} f d\lambda_{S^1}\right| &= \left|\int_{\mathbb{P}^1(\mathbb{C})} f d(\mu_S - \lambda_{S^1})\right| \\
&= \left|\int_{\mathbb{P}^1(\mathbb{C})} f d(\mu_S - \mu_{S,\varepsilon}) + \int_{\mathbb{P}^1(\mathbb{C})} f d(\mu_{S,\varepsilon} - \lambda_{S^1})\right| \\
&\leq \left|\int_{\mathbb{P}^1(\mathbb{C})} f d(\mu_S - \mu_{S,\varepsilon})\right| + \left|\int_{\mathbb{P}^1(\mathbb{C})} f d(\mu_{S,\varepsilon} - \lambda_{S^1})\right|.
\end{aligned}
$$

The proof of the result will be divided into two parts, corresponding to each one of the summands on the right-hand side of (2.16).

We will begin with the second summand. For every $\varepsilon > 0$, the measure $\mu_{S,\varepsilon}$ is a probability measure with compact support on $\mathbb{C}$ and smooth potential. Hence, the signed measure $\mu_{S,\varepsilon} - \lambda_{S^1}$ on $\mathbb{P}^1(\mathbb{C})$ has vanishing total mass and its trace measure has continuous potential. By Proposition 2.19, we have

$$
\left|\int_{\mathbb{P}^1(\mathbb{C})} f d(\mu_{S,\varepsilon} - \lambda_{S^1})\right| \leq \langle f, f\rangle^{\frac{1}{2}} (\mu_{S,\varepsilon} - \lambda_{S^1}, \mu_{S,\varepsilon} - \lambda_{S^1})^{\frac{1}{2}}.
$$

Let us study the energy of $\mu_{S,\varepsilon} - \lambda_{S^1}$. By Proposition 2.16 and Lemma 2.20, there is a positive constant $C_0 \approx 1.10559$ such that

$$
\begin{aligned}
(\mu_{S,\varepsilon} - \lambda_{S^1}, \mu_{S,\varepsilon} - \lambda_{S^1}) &\leq (\mu_S - \lambda_{S^1}, \mu_S - \lambda_{S^1}) + 2\varepsilon + \frac{1}{\#S}\left(2\pi \log\frac{1}{\varepsilon} + C_0\right) \\
&\leq 2\frac{\mathrm{h}(S)}{\#S} + 2\varepsilon + \frac{1}{\#S}\left(2\pi \log\frac{1}{\varepsilon} + C_0\right).
\end{aligned}
$$

Letting $\varepsilon = \frac{1}{\#S}$, we obtain

$$
\begin{aligned}
(\mu_{S,\varepsilon} - \lambda_{S^1}, \mu_{S,\varepsilon} - \lambda_{S^1}) &\leq 2\frac{\mathrm{h}(S)}{\#S} + \frac{2}{\#S} + \frac{1}{\#S}\left(2\pi \log \#S + C_0\right) \\
&\leq 2\frac{\mathrm{h}(S)}{\#S} + C_1 \frac{\log(\#S + 1)}{\#S},
\end{aligned}
$$

where $C_1 = \frac{2 + 2\pi\log 6 + C_0}{\log 7}$.

Hence, we obtain

$$
(2.17) \qquad \left|\int_{\mathbb{P}^1(\mathbb{C})} f d(\mu_{S,\varepsilon} - \lambda_{S^1})\right| \leq \langle f, f\rangle^{\frac{1}{2}} \left(2\frac{\mathrm{h}(S)}{\#S} + C_1 \frac{\log(\#S + 1)}{\#S}\right)^{\frac{1}{2}}.
$$

We will study now the first summand in (2.16). Let $f_0 = f \circ \alpha_0^{-1}$, since $S \subset \mathbb{C}$, for every $\varepsilon > 0$ we can write

$$
\begin{aligned}
(2.18) \quad \left| \int_{\mathbb{P}^1(\mathbb{C})} f d(\mu_S - \mu_{S,\varepsilon}) \right| &= \left| \int_{\mathbb{C}} f_0 d\mu_S - \int_{\mathbb{C}} f_0 d\mu_{S,\varepsilon} \right| \\
&= \left| \int_{\mathbb{C}} f_0 d\mu_S - \int_{\mathbb{C}} f_{0,\varepsilon} d\mu_S \right| = \left| \int_{\mathbb{C}} (f_0 - f_{0,\varepsilon}) d\mu_S \right| \\
&\leq \frac{1}{\#S} \sum_{\alpha \in S} |f_0(\alpha) - f_{0,\varepsilon}(\alpha)| \\
&= \frac{1}{\#S} \sum_{\alpha \in S} \left| \int_{\mathbb{C}} (f_0(\alpha) - f_0(\alpha - w)) \varphi_\varepsilon(w) dA(w) \right| \\
&\leq \frac{1}{\#S} \sum_{\alpha \in S} \int_{\mathbb{C}} |f_0(\alpha) - f_0(\alpha - w)| \, \varphi_\varepsilon(w) dA(w)
\end{aligned}
$$

Let $z, z' \in \mathbb{C}$, by Lemma 1.14 we have

$$
\begin{aligned}
\mathrm{d}((1:z),(1:z')) &\leq \frac{\pi}{2} \mathrm{d}_{\mathrm{ch}}((1:z),(1:z')) \\
&= \pi \frac{|z - z'|}{\sqrt{1+|z|^2}\sqrt{1+|z'|^2}} \leq \pi |z - z'|.
\end{aligned}
$$

Hence, for every $\alpha \in S$ and every $w \in \mathbb{C}$ we have

$$
\frac{|f_0(\alpha) - f_0(\alpha - w)|}{|w|} \leq \pi \frac{|f(1:\alpha) - f(1:\alpha - w)|}{\mathrm{d}((1:\alpha),(1:\alpha - w))} \leq \pi \operatorname{Lip}(f).
$$

So, from (2.18) and the fact that we are taking $\varepsilon = \frac{1}{\#S}$, we deduce

$$
\begin{aligned}
(2.19) \quad & \left| \int_{\mathbb{P}^1(\mathbb{C})} f d(\mu_S - \mu_{S,\varepsilon}) \right| \\
&\leq \frac{1}{\#S} \sum_{\alpha \in S} \int_{\mathbb{C}} |f_0(\alpha) - f_0(\alpha - w)| \, \varphi_\varepsilon(w) dA(w) \\
&\leq \frac{1}{\#S} \sum_{\alpha \in S} \pi \operatorname{Lip}(f) \int_{D(0,\varepsilon)} |w| \varphi_\varepsilon(w) dA(w) \\
&\leq \frac{1}{\#S} \sum_{\alpha \in S} \pi \operatorname{Lip}(f) \varepsilon = \pi \operatorname{Lip}(f) \varepsilon = \pi \frac{\operatorname{Lip}(f)}{\#S}.
\end{aligned}
$$

Therefore by (2.16), (2.17), (2.19) and Proposition 2.18 we obtain

$$\begin{aligned}&\left|\int_{\mathbb{P}^1(\mathbb{C})} f d\mu_S - \int_{\mathbb{P}^1(\mathbb{C})} f d\lambda_{S^1}\right| \\ &\qquad \le \pi \frac{\mathrm{Lip}(f)}{\#S} + \langle f, f\rangle^{\frac{1}{2}} \left(2\frac{\mathrm{h}(S)}{\#S} + C_1 \frac{\log(\#S+1)}{\#S}\right)^{\frac{1}{2}} \\ &\qquad\qquad \le \mathrm{Lip}(f)\left(\frac{\pi}{\#S} + \left(4\frac{\mathrm{h}(S)}{\#S} + C\frac{\log(\#S+1)}{\#S}\right)^{\frac{1}{2}}\right),\end{aligned}$$

where $C = 2C_1 \approx 14.7628$.

Finally, if we assume that $\frac{\mathrm{h}(S)}{\#S}$ is bounded by 1, taking

$$C' = \frac{\pi^2 + C_1 \log 2 + 2\pi\sqrt{4 + C_1 \log 2}}{\log 2} \approx 48.9897$$

we obtain (2.1). □

CHAPTER 3

Quantitative equidistribution in the N-dimensional case

In this final chapter we will give a generalization to the N-dimensional case of the quantitative equidistribution of Galois orbits of small height. As it was done in the previous chapter, for a certain set of test functions, we will give a bound for the rate of convergence in terms of a constant depending on the function, the height of the Galois orbit, and a generalization to higher dimension of the degree of an algebraic number.

Before stating the main result, we will introduce some notations. Consider the subvariety

$$\mathbb{H} := \{(p_1, \dots, p_N) \in \mathbb{P}^1(\mathbb{C})^N : p_k = (0:1) \text{ or } p_k = (1:0) \text{ for some } k\}.$$

The set of test functions will be denoted by $\mathcal{F}$. We will say that a function $f : \mathbb{P}^1(\mathbb{C})^N \to \mathbb{R}$ is in $\mathcal{F}$ if it satisfies the following

(i) f is of class $\mathscr{C}^{2N+1}$,
(ii) The $2N$-jet of f vanishes on $\mathbb{H}$. This is, on every chart of $\mathbb{P}^1(\mathbb{C})^N$, the partial derivatives of f up to order $2N$ vanish on $\mathbb{H}$. In particular, f vanishes on $\mathbb{H}$.

For any $\boldsymbol{n} = (n_1, \dots, n_N) \in \mathbb{Z}^N$ we will consider the *monomial map* given by

$$\begin{array}{rcl} \chi^{\boldsymbol{n}} : (\overline{\mathbb{Q}}^{\times})^N & \longrightarrow & \overline{\mathbb{Q}}^{\times} \\ \boldsymbol{z} = (z_1, \dots, z_N) & \longmapsto & \chi^{\boldsymbol{n}}(\boldsymbol{z}) = z_1^{n_1} \dots z_N^{n_N}. \end{array}$$

Given an element $\boldsymbol{\xi} \in (\overline{\mathbb{Q}}^{\times})^N$, we define its *generalized degree* by

$$\mathscr{D}(\boldsymbol{\xi}) = \min_{\boldsymbol{n} \neq \boldsymbol{0}} \{\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))\}, \tag{3.1}$$

where $\|\cdot\|_1$ is the 1-norm in $\mathbb{C}^N$ and $\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))$ stands for the degree of the algebraic number $\chi^{\boldsymbol{n}}(\boldsymbol{\xi})$ over $\mathbb{Q}$.

These notions that we just introduced will be studied in detail in the following sections.

Consider a finite set $S \subset \mathbb{C}^N$, we recall that the discrete probability measure associated to S is given by $\mu_S = \frac{1}{\# S} \sum_{\boldsymbol{\alpha} \in S} \delta_{\boldsymbol{\alpha}}$. We will also consider the measure $\lambda_{(S^1)^N}$ in $\mathbb{C}^N$ supported on the unit polycircle $(S^1)^N$, where it coincides with the normalized Haar measure.

We can now state the main theorem of the chapter.

THEOREM I. *There is a constant $C \approx 48.9897$ such that for every test function $f \in \mathcal{F}$ and every $\boldsymbol{\xi} \in (\overline{\mathbb{Q}}^{\times})^N$ with $\mathrm{h}(\boldsymbol{\xi}) \leq 1$, the following holds*

$$\left| \int_{\mathbb{P}^1(\mathbb{C})^N} f d\mu_S - \int_{\mathbb{P}^1(\mathbb{C})^N} f d\lambda_{(S^1)^N} \right| \leq c(f) \left(4\,\mathrm{h}(\boldsymbol{\xi}) + C \frac{\log(\mathscr{D}(\boldsymbol{\xi}) + 1)}{\mathscr{D}(\boldsymbol{\xi})} \right)^{\frac{1}{2}},$$

where S is the Galois orbit of $\boldsymbol{\xi}$ as a subset of $(\mathbb{C}^{\times})^N$, μ_S the discrete probability measure associated to it, and $c(f)$ is a positive constant depending on the function f, to be specified in (3.3).

To show the dependence of $c(f)$ with respect to the function f, we identify $(\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N$ with $(\mathbb{C}^{\times})^N$ via the isomorphism

$$(\boldsymbol{\theta}, \boldsymbol{u}) = ((\theta_1, \ldots, \theta_N), (u_1, \ldots, u_N)) \mapsto (e^{2\pi i\theta_1 + u_1}, \ldots, e^{2\pi i \theta_N + u_N}),$$

and set

$$\begin{array}{rccc} (3.2) & \phi : (\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N & \longrightarrow & \mathbb{P}^1(\mathbb{C})^N \\ & (\boldsymbol{\theta}, \boldsymbol{u}) & \longmapsto & ((1 : e^{2\pi i\theta_1 + u_1}), \ldots, (1 : e^{2\pi i\theta_N + u_N})). \end{array}$$

Let $F := f \circ \phi$, this is, F is the function

$$\begin{array}{rcl} F : (\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N & \longrightarrow & \mathbb{R} \\ (\boldsymbol{\theta}, \boldsymbol{u}) & \longmapsto & f((1 : e^{2\pi i\theta_1 + u_1}), \ldots, (1 : e^{2\pi i\theta_N + u_N})). \end{array}$$

We will see in Section 2.2 that if f is a test function in $\mathcal{F}$, then both F and its Fourier transform $\widehat{F}$ are Haar-integrable as well as all the first order partial derivatives of F and also their Fourier transforms.

In $\mathbb{P}^1(\mathbb{C})^N$ we consider the spherical distance d^N, given by

$$\mathrm{d}^N(P, P') = \sqrt{\sum_{j=1}^{N} \mathrm{d}(p_j, p'_j)},$$

for every $P = (p_1, \ldots, p_N)$ and $P' = (p'_1, \ldots, p'_N)$.

With the notation as above, the constant in Theorem I can be bounded by

$$(3.3) \qquad c(f) \leq \sqrt{2}\pi\, \mathrm{Lip}(f) + 2 \sum_{l=1}^{N} \left\| \widehat{\frac{\partial F}{\partial u_l}} \right\|_{\mathrm{L}^1} + 16 \sum_{l=1}^{N} \left\| \widehat{\frac{\partial F}{\partial \theta_l}} \right\|_{\mathrm{L}^1},$$

where $\mathrm{Lip}(f)$ is the Lipschitz constant of f with respect to the spherical distance d^N on $\mathbb{P}^1(\mathbb{C})^N$ and $\|\cdot\|_{\mathrm{L}^1}$ stands for the L^1-norm on the locally compact Abelian group $\mathbb{Z}^N \times \mathbb{R}^N$.

As a corollary to Theorem I we obtain Bilu's equidistribution theorem:

COROLLARY 3.1. *Let $\{\boldsymbol{\xi}_k\} \subset (\overline{\mathbb{Q}}^{\times})^N$ be a strict sequence such that $\lim_k \mathrm{h}(\boldsymbol{\xi}_k) = 0$. Then the Galois orbits of $\boldsymbol{\xi}_k$ are equidistributed with respect to $\lambda_{(S^1)^N}$.*

1. Fourier analysis on $(\mathbb{C}^\times)^N$

In the preliminaries of this dissertation, we dedicated a section to the theory of Fourier Analysis on locally compact Abelian groups. Now, we will consider the particular case of the group $(\mathbb{C}^\times)^N$. As we described above, it will be identified with $(\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N$ via the isomorphism mapping $\boldsymbol{z} \in (\mathbb{C}^\times)^N$ to its logarithmic-polar coordinates $(\boldsymbol{\theta}, \boldsymbol{u}) \in (\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N$. In $(\mathbb{C}^\times)^N$, we will consider the Haar measure induced by the product of the probability Haar measure on $(\mathbb{R}/\mathbb{Z})^N$ and the Lebesgue measure on $\mathbb{R}^N$.

The dual group of $(\mathbb{C}^\times)^N \cong (\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N$ in the sense of Pontryagin is

$$\widehat{(\mathbb{C}^\times)^N} \cong \mathbb{Z}^N \times \mathbb{R}^N,$$

which implies that for any $\gamma \in \widehat{(\mathbb{C}^\times)^N}$, there is a unique $(\boldsymbol{n}, \boldsymbol{t}) \in \mathbb{Z}^N \times \mathbb{R}^N$ such that

$$\gamma(\boldsymbol{\theta}, \boldsymbol{u}) = e^{2\pi i \boldsymbol{n}\cdot\boldsymbol{\theta}} e^{2\pi i \boldsymbol{t}\cdot\boldsymbol{u}},$$

with the notation

$$\boldsymbol{n} \cdot \boldsymbol{\theta} = (n_1, \ldots, n_N) \cdot (\theta_1, \ldots, \theta_N) = n_1\theta_1 + \ldots + n_N\theta_N$$

and similarly for $\boldsymbol{t} \cdot \boldsymbol{u}$.

The measure on $(\mathbb{C}^\times)^N$ induces a unique Haar measure on its dual group $\mathbb{Z}^N \times \mathbb{R}^N$ which is given by the product of the discrete measure on $\mathbb{Z}^N$ and the Lebesgue measure on $\mathbb{R}^N$.

For any complex-valued Haar-integrable function F on $(\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N$, its Fourier transform $\widehat{F} : \mathbb{Z}^N \times \mathbb{R}^N \to \mathbb{C}$ is given by

$$\widehat{F}(\boldsymbol{n}, \boldsymbol{t}) = \int_{(\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N} F(\boldsymbol{\theta}, \boldsymbol{u}) e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{\theta}} e^{-2\pi i \boldsymbol{t}\cdot\boldsymbol{u}} d\boldsymbol{\theta} d\boldsymbol{u}.$$

If, in addition, we assume that $\widehat{F}$ is Haar-integrable, then for every $(\boldsymbol{\theta}, \boldsymbol{u}) \in (\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N$, the Fourier inversion formula (1.3) gives

$$F(\boldsymbol{\theta}, \boldsymbol{u}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^N} \int_{\mathbb{R}^N} \widehat{F}(\boldsymbol{n}, \boldsymbol{t}) e^{2\pi i \boldsymbol{\theta}\cdot\boldsymbol{n}} e^{2\pi i \boldsymbol{u}\cdot\boldsymbol{t}} d\boldsymbol{t}.$$

By abuse of notation, we will write

$$\|F\|_{\mathrm{L}^p}^p = \int_{(\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N} |F(\boldsymbol{\theta}, \boldsymbol{u})|^p d\boldsymbol{\theta} d\boldsymbol{u},$$

for any $F \in \mathrm{L}^p((\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N)$ and

$$\|G\|_{\mathrm{L}^p}^p = \sum_{\boldsymbol{n} \in \mathbb{Z}^N} \int_{\mathbb{R}^N} |G(\boldsymbol{n}, \boldsymbol{t})|^p d\boldsymbol{t},$$

for any $G \in \mathrm{L}^p(\mathbb{Z}^N \times \mathbb{R}^N)$.

We will now study some general results that will be useful for the proof of the main result of this chapter.

LEMMA 3.2. *Let $F : (\mathbb{C}^\times)^N \longrightarrow \mathbb{C}$ be a Haar-integrable function such that its Fourier-transform $\widehat{F}$ is also Haar-integrable. For any finite regular measure λ on $(\mathbb{C}^\times)^N$ such that F is integrable with respect to λ, we have that $\widehat{F}\overline{\widehat{\lambda}}$ is Haar-integrable. Moreover, the following holds*

$$\int_{(\mathbb{C}^\times)^N} F d\lambda = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \int_{\mathbb{R}^N} \widehat{F}(\boldsymbol{n},\boldsymbol{t})\overline{\widehat{\lambda}(\boldsymbol{n},\boldsymbol{t})} d\boldsymbol{t}.$$

PROOF. Let λ be a finite regular measure on $(\mathbb{C}^\times)^N$. Recall that the Fourier-Stieltjes transform of the measure λ is defined by

$$\widehat{\lambda}(\boldsymbol{n},\boldsymbol{t}) = \int_{(\mathbb{R}/\mathbb{Z})^N\times\mathbb{R}^N} e^{-2\pi i \boldsymbol{t}\cdot\boldsymbol{u}} e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{\theta}} d\lambda(\boldsymbol{\theta},\boldsymbol{u}).$$

Since both F and $\widehat{F}$ are Haar-integrable, we can apply the Fourier inversion formula that, together with Fubini's theorem, leads to

$$\begin{aligned}
\int_{(\mathbb{C}^\times)^N} F d\lambda &= \int_{(\mathbb{R}/\mathbb{Z})^N\times\mathbb{R}^N} F(\boldsymbol{\theta},\boldsymbol{u}) d\lambda(\boldsymbol{\theta},\boldsymbol{u}) \\
&= \int_{(\mathbb{R}/\mathbb{Z})^N\times\mathbb{R}^N} \left(\sum_{\boldsymbol{n}\in\mathbb{Z}^N} \int_{\mathbb{R}^N} \widehat{F}(\boldsymbol{n},\boldsymbol{t}) e^{2\pi i \boldsymbol{u}\cdot\boldsymbol{t}} e^{2\pi i \boldsymbol{\theta}\cdot\boldsymbol{n}} d\boldsymbol{t} \right) d\lambda(\boldsymbol{\theta},\boldsymbol{u}) \\
&= \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \int_{\mathbb{R}^N} \widehat{F}(\boldsymbol{n},\boldsymbol{t}) \left(\int_{(\mathbb{R}/\mathbb{Z})^N\times\mathbb{R}^N} e^{2\pi i \boldsymbol{u}\cdot\boldsymbol{t}} e^{2\pi i \boldsymbol{\theta}\cdot\boldsymbol{n}} d\lambda(\boldsymbol{\theta},\boldsymbol{u}) \right) d\boldsymbol{t} \\
&= \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \int_{\mathbb{R}^N} \widehat{F}(\boldsymbol{n},\boldsymbol{t})\overline{\widehat{\lambda}(\boldsymbol{n},\boldsymbol{t})} d\boldsymbol{t}.
\end{aligned}$$

□

LEMMA 3.3. *Let $F : (\mathbb{C}^\times)^N \longrightarrow \mathbb{C}$ be a Haar-integrable function such that its Fourier transform $\widehat{F}$ is also Haar-integrable and let λ be a finite regular measure on $(\mathbb{C}^\times)^N$. Then*

$$\begin{aligned}
&\left| \int_{(\mathbb{C}^\times)^N} F d\lambda - \int_{(\mathbb{C}^\times)^N} F d\lambda_{(S^1)^N} \right| \\
&\qquad \leq \left| \int_{\mathbb{R}^N} \widehat{F}(\boldsymbol{0},\boldsymbol{t}) \left(\overline{\widehat{\lambda}(\boldsymbol{0},\boldsymbol{t})} - 1 \right) d\boldsymbol{t} \right| + \left| \sum_{\boldsymbol{n}\neq 0} \int_{\mathbb{R}^N} \widehat{F}(\boldsymbol{n},\boldsymbol{t})\overline{\widehat{\lambda}(\boldsymbol{n},\boldsymbol{t})} d\boldsymbol{t} \right|.
\end{aligned}$$

PROOF. First of all note that, since $\lambda_{(S^1)^N}$ is the measure on $(\mathbb{C}^\times)^N$ supported on $(S^1)^N$ where it coincides with the normalized Haar measure, for any $(\boldsymbol{n},\boldsymbol{t}) \in \mathbb{Z}^N \times \mathbb{R}^N$ we have

$$\widehat{\lambda_{(S^1)^N}}(\boldsymbol{n},\boldsymbol{t}) = \int_{(\mathbb{R}/\mathbb{Z})^N} e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{\theta}} d\boldsymbol{\theta} = \begin{cases} 1 & \text{if } \boldsymbol{n} = \boldsymbol{0}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, by Lemma 3.2 we obtain

$$\int_{(\mathbb{C}^\times)^N} F d\lambda_{(S^1)^N} = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \int_{\mathbb{R}^N} \widehat{F}(\boldsymbol{n},\boldsymbol{t})\overline{\widehat{\lambda_{(S^1)^N}}(\boldsymbol{n},\boldsymbol{t})}d\boldsymbol{t} = \int_{\mathbb{R}^N} \widehat{F}(\boldsymbol{0},\boldsymbol{t})d\boldsymbol{t}. \tag{3.4}$$

Then we have

$$\begin{aligned}
&\left|\int_{(\mathbb{C}^\times)^N} F d\lambda - \int_{(\mathbb{C}^\times)^N} F d\lambda_{(S^1)^N}\right| \\
&\qquad\qquad = \left|\sum_{\boldsymbol{n}\in\mathbb{Z}^N} \int_{\mathbb{R}^N} \widehat{F}(\boldsymbol{n},\boldsymbol{t})\overline{\widehat{\lambda}(\boldsymbol{n},\boldsymbol{t})}d\boldsymbol{t} - \int_{\mathbb{R}^N} \widehat{F}(\boldsymbol{0},\boldsymbol{t})d\boldsymbol{t}\right| \\
&\qquad\qquad \le \left|\int_{\mathbb{R}^N} \widehat{F}(\boldsymbol{0},\boldsymbol{t})\left(\overline{\widehat{\lambda}(\boldsymbol{0},\boldsymbol{t})} - 1\right)d\boldsymbol{t}\right| + \left|\sum_{\boldsymbol{n}\neq\boldsymbol{0}} \int_{\mathbb{R}^N} \widehat{F}(\boldsymbol{n},\boldsymbol{t})\overline{\widehat{\lambda}(\boldsymbol{n},\boldsymbol{t})}d\boldsymbol{t}\right|.
\end{aligned}$$

□

For every function $F : (\mathbb{C}^\times)^N \longrightarrow \mathbb{C}$ and every $\boldsymbol{w} \in (\mathbb{C}^\times)^N$, the translation of F by ω is the function $\tau_{\boldsymbol{w}}F : (\mathbb{C}^\times)^N \longrightarrow \mathbb{C}$ given by

$$\tau_{\boldsymbol{w}}F(\boldsymbol{z}) := F(\boldsymbol{w}\cdot\boldsymbol{z}), \text{ for any } \boldsymbol{z} \in (\mathbb{C}^\times)^N. \tag{3.5}$$

Let us fix some notation, for $\boldsymbol{w} = (w_1, \dots, w_N)$ in $(\mathbb{C}^\times)^N$, we will write

$$|\boldsymbol{w}| = (|w_1|, \dots, |w_N|), \ \arg(\boldsymbol{w}) = (\arg(w_1), \dots, \arg(w_N))$$

and, for every $\boldsymbol{t} = (t_1, \dots, t_N)$ in $\mathbb{R}^N$, we will note

$$|\boldsymbol{w}|^{\boldsymbol{t}} = \prod_{j=1}^{N} |w_j|^{t_j}.$$

Lemma 3.4. *Let $\boldsymbol{w} \in (\mathbb{C}^\times)^N$ and let $F : (\mathbb{C}^\times)^N \longrightarrow \mathbb{C}$ be a Haar-integrable function. Then for any $(\boldsymbol{n},\boldsymbol{t}) \in \mathbb{Z}^N \times \mathbb{R}^N$, the following holds*

$$\widehat{\tau_{\boldsymbol{w}}F}(\boldsymbol{n},\boldsymbol{t}) = |\boldsymbol{w}|^{2\pi i\boldsymbol{t}} e^{i\boldsymbol{n}\cdot\arg(\boldsymbol{w})}\widehat{F}(\boldsymbol{n},\boldsymbol{t}).$$

Proof. By abuse of notation, which is justified by the identification $(\mathbb{C}^\times)^N \cong (\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N$, we can write

$$\tau_{\boldsymbol{w}}F(\boldsymbol{\theta},\boldsymbol{u}) = F\left(\boldsymbol{\theta} + \frac{\arg(\boldsymbol{w})}{2\pi}, \boldsymbol{u} + \log|\boldsymbol{w}|\right).$$

Hence, after some suitable change of variables, we obtain

$$\begin{aligned}\widehat{\tau_{\boldsymbol{w}}F}(\boldsymbol{n},\boldsymbol{t}) &= \int_{(\mathbb{R}/\mathbb{Z})^N\times\mathbb{R}^N} \tau_{\boldsymbol{w}}F(\boldsymbol{\theta},\boldsymbol{u})e^{-2\pi i\boldsymbol{t}\cdot\boldsymbol{u}}e^{-2\pi i\boldsymbol{n}\cdot\boldsymbol{\theta}}d\boldsymbol{\theta}d\boldsymbol{u}\\ &= \int_{(\mathbb{R}/\mathbb{Z})^N\times\mathbb{R}^N} F\left(\boldsymbol{\theta}+\frac{\arg(\boldsymbol{w})}{2\pi},\boldsymbol{u}+\log|\boldsymbol{w}|\right)e^{-2\pi i\boldsymbol{t}\cdot\boldsymbol{u}}e^{-2\pi i\boldsymbol{n}\cdot\boldsymbol{\theta}}d\boldsymbol{\theta}d\boldsymbol{u}\\ &= \int_{(\mathbb{R}/\mathbb{Z})^N\times\mathbb{R}^N} F(\boldsymbol{\theta},\boldsymbol{u})e^{-2\pi i\boldsymbol{t}\cdot\boldsymbol{u}}e^{-2\pi i\boldsymbol{n}\cdot\boldsymbol{\theta}}e^{2\pi i\boldsymbol{t}\cdot(\log|\boldsymbol{w}|)}e^{2\pi i\boldsymbol{n}\cdot\frac{\arg(\boldsymbol{w})}{2\pi}}d\boldsymbol{\theta}d\boldsymbol{u}\\ &= e^{2\pi i\boldsymbol{t}\cdot(\log|\boldsymbol{w}|)}e^{2\pi i\boldsymbol{n}\cdot\frac{\arg(\boldsymbol{w})}{2\pi}}\widehat{F}(\boldsymbol{n},\boldsymbol{t}).\end{aligned}$$

□

2. The set of test functions

On the beginning of the chapter, we defined the set of test functions $\mathcal{F}$ that was later used for the statement of the main result. The aim of this section will be to give a justification for this definition which, at first sight, might seem rather unnatural.

As it has been already mentioned, Fourier Analysis is among the techniques used on the proof of Theorem I. For this reason, we will need to make some assumptions on the Haar-integrability of the function and its Fourier transform, as well as for the first order partial derivatives. Let us first recall the definition of the set of test functions and make some significant remarks about them. Afterwards, we will be able to state a result for these test functions establishing the desired properties.

Let us recall the definition of the set of test functions. Every function $f\in\mathcal{F}$ is such that

(i) $f:\mathbb{P}^1(\mathbb{C})^N\to\mathbb{R}$ is $\mathscr{C}^{2N+1}$,

(ii) The $2N$-jet of f vanishes on $\mathbb{H}$, where the subvariety $\mathbb{H}$ is given by

$$\mathbb{H}:=\{(p_1,\dots,p_N)\in\mathbb{P}^1(\mathbb{C})^N : p_k=(0:1)\text{ or }p_k=(1:0)\text{ for some }k\}.$$

Consider a point $P\in\mathbb{P}^1(\mathbb{C})^N$ and a system of local coordinates of $\mathbb{P}^1(\mathbb{C})^N$ around it. We say that the $2N$-jet of f vanishes on P if all the partial derivatives up to order $2N$ of the coordinate expression of f vanish at P.

Under the natural inclusion

$$\begin{array}{rcl}\iota:(\mathbb{C}^\times)^N & \hookrightarrow & \mathbb{P}^1(\mathbb{C})^N\\ (z_1,\dots,z_N) & \mapsto & ((1:z_1),\dots,(1:z_N)),\end{array}$$

we can think of the functions in $\mathcal{F}$ as functions supported on $(\mathbb{C}^\times)^N$. Identifying $\mathbb{R}/\mathbb{Z}\times\mathbb{R}$ and $\mathbb{C}^\times$ via the logarithmic-polar coordinate map and composing with the previous inclusion, we can define the map ϕ given by (3.2).

With this notation, we can now state the main theorem of this section.

THEOREM 3.5. *For any $f\in\mathcal{F}$, the function $F:(\mathbb{R}/\mathbb{Z})^N\times\mathbb{R}^N\longrightarrow\mathbb{R}$, given by $F=f\circ\phi$, satisfies the following properties*

(i) F is Haar-integrable,

(ii) $\widehat{F}$ *is Haar-integrable,*
(iii) For every $l = 1, \dots, N$, $\frac{\partial F}{\partial u_l}$ *and* $\frac{\partial F}{\partial \theta_l}$ *are Haar-integrable,*
(iv) For every $l = 1, \dots, N$, $\widehat{\frac{\partial F}{\partial u_l}}$ *and* $\widehat{\frac{\partial F}{\partial \theta_l}}$ *are Haar-integrable,*

The proof of this result will be divided in several propositions and lemmas. But, before doing so, we will state some important facts of $F = f \circ \phi$, with $f \in \mathcal{F}$.

Remark 3.6. The differentiability of the natural inclusion ι together with the differentiability of the logarithmic-polar coordinate change-of-variables implies that the map ϕ is differenciable

$$\begin{tikzcd}
(\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N \arrow[r, "\phi"] \arrow[d, "\cong"] & \mathbb{P}^1(\mathbb{C})^N \\
(\mathbb{C}^\times)^N. \arrow[ur, "\iota"'] &
\end{tikzcd}$$

Hence, the function $F = f \circ \phi$ is in $\mathscr{C}^{2N+1}\big((\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N\big)$.

We will recall the notation that has been previously given for the usual charts and their homeomorphisms in the projective complex line. Let

$$U_0 := \{(1 : z) : z \in \mathbb{C}\} \text{ and } U_1 := \{(z : 1) : z \in \mathbb{C}\},$$

be the usual open subsets on $\mathbb{P}^1(\mathbb{C})$ and consider the homeomorphisms,

$$\begin{array}{rccc} \alpha_0 : & U_0 & \longrightarrow & \mathbb{R}^2, \\ & (1:z) & \mapsto & (\mathrm{Re}(z), \mathrm{Im}(z)) \end{array} \qquad \begin{array}{rccc} \alpha_1 : & U_1 & \longrightarrow & \mathbb{R}^2 \\ & (z:1) & \mapsto & (\mathrm{Re}(z), \mathrm{Im}(z))\,. \end{array}$$

For any choice of indexes $j_1, \dots, j_N \in \{0, 1\}$, we will consider the open subset $U_{j_1} \times \ldots \times U_{j_N}$ in $\mathbb{P}^1(\mathbb{C})^N$ and the homeomorphism

$$\alpha_{j_1,\dots,j_N} = (\alpha_{j_1}, \dots, \alpha_{j_N}) : U_{j_1} \times \ldots \times U_{j_N} \to \mathbb{R}^{2N}.$$

Observe that, for any $(\theta, u) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$, the point $(1 : e^{2\pi i\theta + u}) \in U_0 \cap U_1$. Hence, for every $(j_1, \dots, j_N) \in \{0, 1\}^N$, the image of ϕ is contained in the open subset $U_{j_1} \times \ldots \times U_{j_N}$. As a consequence, we can define

$$\phi_{j_1,\dots,j_N} = \alpha_{j_1,\dots,j_N} \circ \phi,$$

and we have the following diagram

$$\begin{tikzcd}
 & \mathbb{P}^1(\mathbb{C})^N \arrow[dr, "f"] & \\
(\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N \arrow[ur, "\phi"] \arrow[r, "\phi"] \arrow[dr, "\phi_{j_1,\dots,j_N}"'] & U_{j_1} \times \ldots \times U_{j_N} \arrow[u, hook] \arrow[r, "f"] \arrow[d, "\alpha_{j_1,\dots,j_N}"] & \mathbb{R} \\
 & \mathbb{R}^2 \times \ldots \times \mathbb{R}^2 \arrow[ur, "f_{j_1,\dots,j_N}"'] &
\end{tikzcd}$$

where

$$f_{j_1,\dots,j_N} = f \circ \alpha^{-1}_{j_1,\dots,j_N}.$$

We set

$$\phi_{j_1,\dots,j_N}(\boldsymbol{\theta},\boldsymbol{u}) = (\phi_{j_1}(\theta_1,u_1),\dots,\phi_{j_k}(\theta_k,u_k)),$$

where, for any $j \in \{0,1\}$ and any $(\theta,u) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$, we have

$$\phi_j(\theta,u) = (\phi_j^1(\theta,u),\phi_j^2(\theta,u)) = (e^{(-1)^j u}\cos(2\pi\theta),(-1)^j e^{(-1)^j u}\sin(2\pi\theta)).$$

Remark 3.7. From the diagram we deduce

$$F = f_{j_1,\dots,j_N} \circ \phi_{j_1,\dots,j_N},$$

for any choice of indexes $j_1,\dots,j_N \in \{0,1\}$.

We will now write explicitly the second property on the definition of the set $\mathcal{F}$ using the coordinate expression $f_{j_1,\dots,j_N}$ of the function f on every chart of $\mathbb{P}^1(\mathbb{C})^N$. On $\mathbb{R}^2 \times \dots \times \mathbb{R}^2$, choose a coordinate system $((x_1,y_1),\dots,(x_N,y_N))$. Let $\boldsymbol{a} = (a_1,\dots,a_N)$ and $\boldsymbol{b} = (b_1,\dots,b_N)$ be two multi-indexes, this is two N-tuples with positive integer entries. Let $|\boldsymbol{a}| = a_1 + \dots + a_N$, $|\boldsymbol{b}| = b_1 + \dots + b_N$. We will denote

$$\frac{\partial^{|\boldsymbol{a}|+|\boldsymbol{b}|} f_{j_1,\dots,j_N}}{\partial \boldsymbol{x}^{\boldsymbol{a}} \partial \boldsymbol{y}^{\boldsymbol{b}}} = \frac{\partial^{|\boldsymbol{a}|+|\boldsymbol{b}|} f_{j_1,\dots,j_N}}{\partial x_1^{a_1} \dots \partial x_N^{a_N} \partial y_1^{b_1} \dots \partial y_N^{b_N}},$$

for any $j_1,\dots,j_N \in \{0,1\}$, whenever it makes sense.

Hence, condition (ii) on the definition of $\mathcal{F}$ can be written as follows. For every $\boldsymbol{a},\boldsymbol{b} \in (\mathbb{Z}_{\geq 0})^N$ such that $|\boldsymbol{a}| + |\boldsymbol{b}| \leq 2N$, every $P \in \mathbb{H}$ and every $j_1,\dots,j_N \in \{0,1\}$ such that $P \in U_{j_1} \times \dots \times U_{j_N}$

$$\frac{\partial^{|\boldsymbol{a}|+|\boldsymbol{b}|} f_{j_1,\dots,j_N}}{\partial \boldsymbol{x}^{\boldsymbol{a}} \partial \boldsymbol{y}^{\boldsymbol{b}}}\big(\alpha_{j_1,\dots,j_N}(P)\big) = 0.$$

We will prove in the following lemmas that, for every $f \in \mathcal{F}$, the functions $F = f \circ \phi$ satisfy very strong properties. Among them, we will see that all the partial derivatives of F up to a certain order tend to zero when any of the coordinates approach either $-\infty$ or $+\infty$. We will also prove that these partial derivatives are bounded. These two important facts will allow us to prove Theorem 3.5.

Lemma 3.8. *Let $F = f \circ \phi$, for some $f \in \mathcal{F}$. Then, for every $\boldsymbol{\alpha},\boldsymbol{\beta} \in (\mathbb{Z}_{\geq 0})^N$ such that $|\boldsymbol{\alpha}| + |\boldsymbol{\beta}| \leq 2N$ and every $l = 1,\dots,N$*

$$\lim_{u_l \to \pm\infty} \frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} F}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}}}(\boldsymbol{\theta},\boldsymbol{u}) = 0.$$

Proof. We will study the limit

$$\lim_{u_l \to +\infty} \frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} F}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}}}(\boldsymbol{\theta},\boldsymbol{u}) = 0.$$

The limit when $u_l \to -\infty$ can be done with a similar argument.

When considering $u_l \to +\infty$, we are approaching a point in $\mathbb{H}$ whose l-th coordinate is $(0:1)$. Hence, we will work on some chart $U_{j_1} \times \dots \times U_{j_N}$ with

$\boldsymbol{J} = (j_1, \ldots, j_N) \in \{0,1\}^N$ such that $j_l = 1$. To simplify the notations, we will write

$$f_{\boldsymbol{J}} = f_{j_1,\ldots,j_N} \text{ and } \phi_{\boldsymbol{J}} = \phi_{j_1,\ldots,j_N}.$$

The partial derivatives up to a certain order $|\boldsymbol{\alpha}| + |\boldsymbol{\beta}| \leq 2N$ of the function $F = f_{\boldsymbol{J}} \circ \phi_{\boldsymbol{J}}$ are computed by recursively applying the chain rule. It is easy to see that when we do so, we obtain an expression on the partial derivatives up to order $|\boldsymbol{\alpha}| + |\boldsymbol{\beta}|$ of the function $f_{\boldsymbol{J}}$ evaluated on $\phi_{\boldsymbol{J}}(\boldsymbol{\theta}, \boldsymbol{u})$ and coefficients concerning partial derivatives up to the same order of the coordinate functions of $\phi_{\boldsymbol{J}}$.

On one hand, the limit of the partial derivatives of order less than or equal to $2N$ of the function $f_{\boldsymbol{J}}$ evaluated on $\phi_{\boldsymbol{J}}(\boldsymbol{\theta}, \boldsymbol{u})$ vanishes due to the fact that the $2N$-jet of f vanishes on $\mathbb{H}$. Thus, if we verify that the limit of the partial derivatives of the coordinate functions of $\phi_{\boldsymbol{J}}$ are finite, we would have proved the lemma.

The only coordinate functions of $\phi_{\boldsymbol{J}}$ depending on the variable u_l are

$$\phi^1_{j_l}(\theta_l, u_l) = e^{(-1)^{j_l} u_l} \cos(2\pi\theta_l),$$

$$\phi^2_{j_l}(\theta_l, u_l) = (-1)^{j_l} e^{(-1)^{j_l} u_l} \sin(2\pi\theta_l).$$

Hence, we only have to study the behavior when $u_l \to +\infty$ of the partial derivatives of $\phi^1_{j_l}$ and $\phi^2_{j_l}$ up to order $2N$. Recall that $j_l = 1$, so all these partial derivatives are of the form

$$\pm(2\pi)^k e^{-u_l} \cos(2\pi\theta_l) \text{ or } \pm (2\pi)^k e^{-u_l} \sin(2\pi\theta_l),$$

for some positive integer k. Therefore, their limit when $u_l \to +\infty$ is zero. $\square$

LEMMA 3.9. *Let $F = f \circ \phi$, for some $f \in \mathcal{F}$, and let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \{0,1\}^N$. Then there is a positive constant $K(\boldsymbol{\alpha}, \boldsymbol{\beta}, f)$, depending on the multindexes $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and the function f, such that*

$$\left| \frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} F}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}}}(\boldsymbol{\theta}, \boldsymbol{u}) \right| \leq K(\boldsymbol{\alpha}, \boldsymbol{\beta}, f) \prod_{l=1}^{N} \left(\frac{e^{u_l}}{1 + e^{2u_l}} \right)^{\max\{\alpha_l, \beta_l\}},$$

for any $(\boldsymbol{\theta}, \boldsymbol{u}) \in (\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N$.

Moreover, there are positive constants $K_1(k, \boldsymbol{\alpha}, \boldsymbol{\beta}, f)$ and $K_2(k, \boldsymbol{\alpha}, \boldsymbol{\beta}, f)$ such that

$$\left| \frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+1} F}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}} \partial \theta_k}(\boldsymbol{\theta}, \boldsymbol{u}) \right| \leq K_1(k, \boldsymbol{\alpha}, \boldsymbol{\beta}, f) \left(\frac{e^{u_k}}{1 + e^{2u_k}} \right) \prod_{l \neq k} \left(\frac{e^{u_l}}{1 + e^{2u_l}} \right)^{\max\{\alpha_l, \beta_l\}}$$

and

$$\left| \frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+1} F}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}} \partial u_k}(\boldsymbol{\theta}, \boldsymbol{u}) \right| \leq K_2(k, \boldsymbol{\alpha}, \boldsymbol{\beta}, f) \left(\frac{e^{u_k}}{1 + e^{2u_k}} \right) \prod_{l \neq k} \left(\frac{e^{u_l}}{1 + e^{2u_l}} \right)^{\max\{\alpha_l, \beta_l\}},$$

for any $(\boldsymbol{\theta}, \boldsymbol{u}) \in (\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N$ and any $k = 1, \ldots, N$.

PROOF. As it was done in the previous proof, by recursively applying the chain rule, we can obtain an expression for the partial derivative $\frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}F}{\partial\boldsymbol{\theta}^{\boldsymbol{\alpha}}\partial\boldsymbol{u}^{\boldsymbol{\beta}}}$, whenever $|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|\leq 2N+1$. For any choice of $\boldsymbol{J}\in\{0,1\}^N$, this expression is a sum of partial derivatives of the function $f_{\boldsymbol{J}}$ up to order $|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|$ evaluated on $\phi_{\boldsymbol{J}}(\boldsymbol{\theta},\boldsymbol{u})$ times a certain suitable product of partial derivatives of the coordinate functions of $\phi_{\boldsymbol{J}}$ of order 1 or 2.

Let us consider the first case, with $\boldsymbol{\alpha},\boldsymbol{\beta}\in\{0,1\}^N$ and $\boldsymbol{\alpha},\boldsymbol{\beta}\neq\mathbf{0}$. In this situation, it is easy to see that

$$
\begin{aligned}
\frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}F}{\partial\boldsymbol{\theta}^{\boldsymbol{\alpha}}\partial\boldsymbol{u}^{\boldsymbol{\beta}}}(\boldsymbol{\theta},\boldsymbol{u})
&= \sum_{\substack{\boldsymbol{a},\boldsymbol{b}\in\{0,1\}^N\\ a_l+b_l=\beta_l}} \frac{\partial^{|\boldsymbol{a}|+|\boldsymbol{b}|}f_{\boldsymbol{J}}}{\partial\boldsymbol{x}^{\boldsymbol{a}}\partial\boldsymbol{y}^{\boldsymbol{b}}}(\phi_{\boldsymbol{J}}(\boldsymbol{\theta},\boldsymbol{u}))\prod_{l=1}^{N}\psi_0(l)\\
&\quad+\sum_{\substack{\boldsymbol{a}',\boldsymbol{b}'\in\{0,1\}^N\\ a'_l+b'_l=\alpha_l}} \frac{\partial^{|\boldsymbol{a}|+|\boldsymbol{b}|+|\boldsymbol{a}'|+|\boldsymbol{b}'|}f_{\boldsymbol{J}}}{\partial\boldsymbol{x}^{\boldsymbol{a}+\boldsymbol{a}'}\partial\boldsymbol{y}^{\boldsymbol{b}+\boldsymbol{b}'}}(\phi_{\boldsymbol{J}}(\boldsymbol{\theta},\boldsymbol{u}))\prod_{l=1}^{N}\psi_1(l),
\end{aligned}
\tag{3.6}
$$

where

$$
\psi_0(l)=\begin{cases}
\left(\frac{\partial^2\phi^1_{j_l}}{\partial u_l\partial\theta_l}\right)^{a_l}\left(\frac{\partial^2\phi^1_{j_l}}{\partial u_l\partial\theta_l}\right)^{b_l} & \text{if } \alpha_l=\beta_l=1,\\
\left(\frac{\partial\phi^1_{j_l}}{\partial u_l}\right)^{a_l}\left(\frac{\partial\phi^2_{j_l}}{\partial u_l}\right)^{b_l} & \text{if } \alpha_l=0,\beta_l=1,\\
1 & \text{if } \alpha_l=\beta_l=0,\\
0 & \text{if } \alpha_l=1,\beta_l=0.
\end{cases}
$$

And

$$
\psi_1(l)=\left(\frac{\partial\phi^1_{j_l}}{\partial u_l}\right)^{a_l}\left(\frac{\partial\phi^2_{j_l}}{\partial u_l}\right)^{b_l}\left(\frac{\partial\phi^1_{j_l}}{\partial\theta_l}\right)^{a'_l}\left(\frac{\partial\phi^2_{j_l}}{\partial\theta_l}\right)^{b'_l}.
$$

For a given $(\boldsymbol{\theta},\boldsymbol{u})\in(\mathbb{R}/\mathbb{Z})^N\times\mathbb{R}^N$, the idea is to chose the right multi-index $\boldsymbol{J}\in\{0,1\}^N$ in such a way that $\phi_{\boldsymbol{J}}(\boldsymbol{\theta},\boldsymbol{u})\in[-1,1]^{2N}$. Since the partial derivatives of the function $f_{\boldsymbol{J}}$ up to order $2N$ are continuous, there are positive constants $\tilde{K}(\boldsymbol{a},\boldsymbol{b},f)$ such that

$$
\left|\frac{\partial^{|\boldsymbol{a}|+|\boldsymbol{b}|}f_{\boldsymbol{J}}}{\partial\boldsymbol{x}^{\boldsymbol{a}}\partial\boldsymbol{y}^{\boldsymbol{b}}}(\phi_{\boldsymbol{J}}(\boldsymbol{\theta},\boldsymbol{u}))\right|\leq\tilde{K}(\boldsymbol{a},\boldsymbol{b},f).
$$

Let us define the map $J:\mathbb{R}^N\to\{0,1\}^N$ given by

$$
J(\boldsymbol{u})=(j(u_1),\dots,j(u_N)),
$$

where

$$
j(u)=\begin{cases}0 & \text{if } u\leq 0,\\ 1 & \text{if } u>0.\end{cases}\tag{3.7}
$$

We will choose $\boldsymbol{J}$ depending on the point $(\boldsymbol{\theta}, \boldsymbol{u})$ by taking $\boldsymbol{J} = J(\boldsymbol{u})$. Observe that, once this choice has been made, we obtain

$$0 < e^{(-1)^{j_l} u_l} = e^{(-1)^{j(u_l)} u_l} = e^{-|u_l|} \leq 1,$$

for every $l = 1, \ldots, N$. And, consequently, we can deduce

$$\phi_{j_l}(\theta_l, u_l) = (e^{(-1)^{j_l} u_l} \cos(2\pi\theta_l), (-1)^{j_l} e^{(-1)^{j_l} u_l} \sin(2\pi\theta_l)) \in [-1, 1]^2.$$

Then we would only be left to study the functions ψ_0 and ψ_1, for every $l = 1, \ldots, N$. For any $(\theta, u) \in (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$, taking $j = j(u)$, the partial derivatives of first and second order of the functions ϕ_j^1 and ϕ_j^2 are of the form

$$\pm(2\pi)^k e^{-|u|} \cos(2\pi\theta) \text{ or } \pm (2\pi)^k e^{-|u|} \sin(2\pi\theta),$$

for some $k \in \{0, 1\}$, and they can all be bounded by $2\pi e^{-|u|}$. Note that

$$\frac{e^{-|u|}}{2} \leq \frac{e^{-|u|}}{1 + e^{-2|u|}} = \frac{e^u}{1 + e^{2u}}.$$

Hence, we can write

$$\begin{aligned} &\left| \frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} F}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}}} (\boldsymbol{\theta}, \boldsymbol{u}) \right| \\ &\quad \leq \sum_{\substack{\boldsymbol{a}, \boldsymbol{b} \in \{0,1\}^N \\ a_l + b_l = \beta_l}} \left| \frac{\partial^{|\boldsymbol{a}|+|\boldsymbol{b}|} f_{\boldsymbol{J}}}{\partial \boldsymbol{x}^{\boldsymbol{a}} \partial \boldsymbol{y}^{\boldsymbol{b}}} (\phi_{\boldsymbol{J}}(\boldsymbol{\theta}, \boldsymbol{u})) \right| (4\pi)^{|\boldsymbol{\beta}|} \prod_{l=1}^N \left(\frac{e^{u_l}}{1 + e^{2u_l}} \right)^{\max\{\alpha_l, \beta_l\}} \\ &+ \sum_{\substack{\boldsymbol{a}', \boldsymbol{b}' \in \{0,1\}^N \\ a'_l + b'_l = \alpha_l}} \left| \frac{\partial^{|\boldsymbol{a}+\boldsymbol{a}'|+|\boldsymbol{b}+\boldsymbol{b}'|} f_{\boldsymbol{J}}}{\partial \boldsymbol{x}^{\boldsymbol{a}+\boldsymbol{a}'} \partial \boldsymbol{y}^{\boldsymbol{b}+\boldsymbol{b}'}} (\phi_{\boldsymbol{J}}(\boldsymbol{\theta}, \boldsymbol{u})) \right| (4\pi)^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} \prod_{l=1}^N \left(\frac{e^{u_l}}{1 + e^{2u_l}} \right)^{\max\{\alpha_l, \beta_l\}} \\ &\qquad \leq K(\boldsymbol{\alpha}, \boldsymbol{\beta}, f) \prod_{l=1}^N \left(\frac{e^{u_l}}{1 + e^{2u_l}} \right)^{\max\{\alpha_l, \beta_l\}}, \end{aligned}$$

taking

$$K(\boldsymbol{\alpha}, \boldsymbol{\beta}, f) = (4\pi)^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} \sum_{\substack{\boldsymbol{a}, \boldsymbol{b} \in \{0,1\}^N \\ a_l + b_l = \beta_l}} \tilde{K}(\boldsymbol{a}, \boldsymbol{b}, f) + \sum_{\substack{\boldsymbol{a}', \boldsymbol{b}' \in \{0,1\}^N \\ a'_l + b'_l = \alpha_l}} \tilde{K}(\boldsymbol{a}+\boldsymbol{a}', \boldsymbol{b}+\boldsymbol{b}', f).$$

Finally, we will prove the last part of the lemma. Observe that now we are allowed to differentiate twice with respect to the same variable. Consider some $k = 1, \ldots, N$, we will study

$$\frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+1} F}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}} \partial \theta_k} (\boldsymbol{\theta}, \boldsymbol{u}).$$

Without loss of generality, we may assume that $\alpha_k \neq 0$. Otherwise we would be on the previous situation taking $\boldsymbol{\alpha}'$ instead of $\boldsymbol{\alpha}$, with $\alpha'_l = \alpha_l$ for every

$l \neq k$ and $\alpha'_k = 1$. Deriving the expression in (3.6), we obtain

$$\begin{aligned}
&\frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+1} F}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}} \partial \theta_k}(\boldsymbol{\theta}, \boldsymbol{u}) \\
&\qquad = \sum_{\substack{\boldsymbol{a},\boldsymbol{b} \in \{0,1\}^N \\ a_j+b_j=\beta_j}} \frac{\partial^{|\boldsymbol{a}|+|\boldsymbol{b}|+1} f_{\boldsymbol{J}}}{\partial \boldsymbol{x}^{\boldsymbol{a}} \partial \boldsymbol{y}^{\boldsymbol{b}} \partial x_k}(\phi_{\boldsymbol{J}}(\boldsymbol{\theta}, \boldsymbol{u})) \frac{\partial \phi^1_{j_k}}{\partial \theta_k}(\theta_k, u_k) \prod_{l=1}^N \psi_0(l) \\
&\qquad\qquad + \frac{\partial^{|\boldsymbol{a}|+|\boldsymbol{b}|+1} f_{\boldsymbol{J}}}{\partial \boldsymbol{x}^{\boldsymbol{a}} \partial \boldsymbol{y}^{\boldsymbol{b}} \partial y_k}(\phi_{\boldsymbol{J}}(\boldsymbol{\theta}, \boldsymbol{u})) \frac{\partial \phi^2_{j_k}}{\partial \theta_k}(\theta_k, u_k) \prod_{l=1}^N \psi_0(l) \\
&\qquad + \beta_l \frac{\partial^{|\boldsymbol{a}|+|\boldsymbol{b}|} f_{\boldsymbol{J}}}{\partial \boldsymbol{x}^{\boldsymbol{a}} \partial \boldsymbol{y}^{\boldsymbol{b}}}(\phi_{\boldsymbol{J}}(\boldsymbol{\theta}, \boldsymbol{u})) \left(\frac{\partial^2 \phi^1_{j_k}}{\partial u_k \partial \theta_k} \right)^{a_k} \left(\frac{\partial^2 \phi^1_{j_k}}{\partial u_k \partial \theta_k} \right)^{b_k} \prod_{l \neq k} \psi_0(l) \\
&\quad + \sum_{\substack{\boldsymbol{a},\boldsymbol{b} \in \{0,1\}^N \\ a_j+b_j=\alpha_j}} \frac{\partial^{|\boldsymbol{a}|+|\boldsymbol{b}|+|\boldsymbol{a}'|+|\boldsymbol{b}'|+1} f_{\boldsymbol{J}}}{\partial \boldsymbol{x}^{\boldsymbol{a}+\boldsymbol{a}'} \partial \boldsymbol{y}^{\boldsymbol{b}+\boldsymbol{b}'} \partial x_k}(\phi_{\boldsymbol{J}}(\boldsymbol{\theta}, \boldsymbol{u})) \frac{\partial \phi^1_{j_k}}{\partial \theta_k}(\theta_k, u_k) \prod_{l=1}^N \psi_1(l) \\
&\qquad + \frac{\partial^{|\boldsymbol{a}|+|\boldsymbol{b}|+|\boldsymbol{a}'|+|\boldsymbol{b}'|+1} f_{\boldsymbol{J}}}{\partial \boldsymbol{x}^{\boldsymbol{a}+\boldsymbol{a}'} \partial \boldsymbol{y}^{\boldsymbol{b}+\boldsymbol{b}'} \partial y_k}(\phi_{\boldsymbol{J}}(\boldsymbol{\theta}, \boldsymbol{u})) \frac{\partial \phi^2_{j_k}}{\partial \theta_k}(\theta_k, u_k) \prod_{l=1}^N \psi_1(l) \\
&\qquad\qquad + \frac{\partial^{|\boldsymbol{a}|+|\boldsymbol{b}|+|\boldsymbol{a}'|+|\boldsymbol{b}'|} f_{\boldsymbol{J}}}{\partial \boldsymbol{x}^{\boldsymbol{a}+\boldsymbol{a}'} \partial \boldsymbol{y}^{\boldsymbol{b}+\boldsymbol{b}'}}(\phi_{\boldsymbol{J}}(\boldsymbol{\theta}, \boldsymbol{u})) \frac{\partial \psi_1(k)}{\partial \theta_k} \prod_{l \neq k} \psi_1(l).
\end{aligned}$$

Following an analogous argument as before, the result can be easily obtained. Observe that in this situation, since we are deriving twice with respect to θ_k, we should consider a bigger bound for the partial derivatives of the functions ϕ^1_j and ϕ^2_j of orders 1 and 2. Namely, they can all be bounded by $4\pi^2 e^{-|u|}$. $\square$

The following proposition implies part (i) and (iii) of Theorem 3.5.

PROPOSITION 3.10. *Let $F = f \circ \phi$, for some $f \in \mathcal{F}$. Then for any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \{0,1\}^N$ we have*

$$\frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} F}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}}} \in (\mathrm{L}^1 \cap \mathrm{L}^2)((\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N).$$

Moreover, for any $k = 1, \ldots, N$, we have

$$\frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+1} F}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}} \partial \theta_k}, \frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+1} F}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}} \partial u_k} \in (\mathrm{L}^1 \cap \mathrm{L}^2)((\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N).$$

PROOF. First of all, observe that if we are considering any partial derivative of the function F such that we are, at least, deriving once with respect to either θ_j or u_j for every j, then the integrability of the absolute value or the square of the function is a direct consequence of Lemma 3.9. Indeed, let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_N)$ in $\{0,1\}^M$ and suppose that $\alpha_j + \beta_j \neq 0$ for every $j = 1, \ldots, N$. Then for some positive constant $K(\boldsymbol{\alpha}, \boldsymbol{\beta}, f)$ we have

$$\left| \frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} F}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}}}(\boldsymbol{\theta}, \boldsymbol{u}) \right| \leq K(\boldsymbol{\alpha}, \boldsymbol{\beta}, f) \prod_{l=1}^{N} \left(\frac{e^{u_l}}{1 + e^{2u_l}} \right).$$

In this cases, the proposition follows from Fubini's theorem and the fact that

$$\int_{\mathbb{R}} \frac{e^u}{1 + e^{2u}} du = \frac{\pi}{2} \text{ and } \int_{\mathbb{R}} \left(\frac{e^u}{1 + e^{2u}} \right)^2 du = \frac{1}{2}.$$

Let us consider now the remaining cases where the partial derivatives of F are such that there is at least some $k \in \{1, \ldots, N\}$ for which we are not deriving with respect to neither θ_k nor u_k. It is easy to see that in these situations, Lemma 3.9 is not enough to guarantee the integrability.

We will prove the result for the partial derivative

$$\frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} F}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}}},$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \{0,1\}^N$ are such that $\alpha_l + \beta_l = 0$ for some l. The same argument can be used for proving the integrability of

$$\frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+1} F}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}} \partial \theta_k}, \frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+1} F}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}} \partial u_k},$$

with $\alpha_l + \beta_l = 0$ for some $l \neq k$.

Fix some $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_N)$ in $\{0,1\}^N$ such that $\alpha_l + \beta_l = 0$ for some l. Define the set $\mathcal{I} = \{l : \alpha_l + \beta_l = 0\}$, which is non-empty, and take $\boldsymbol{\beta}' = (\beta_1', \ldots, \beta_N')$, where $\beta_l' = 1$ for every $l \in \mathcal{I}$ and $\beta_l' = \beta_l$ otherwise. We claim that, for every $(\boldsymbol{\theta}, \boldsymbol{v}) \in (\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N$

$$\left| \frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} F}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}}}(\boldsymbol{\theta}, \boldsymbol{v}) \right| \leq K(\boldsymbol{\alpha}, \boldsymbol{\beta}', f) \prod_{l \notin \mathcal{I}} \left(\frac{e^{v_l}}{1 + e^{2v_l}} \right) \prod_{l \in \mathcal{I}} \arctan e^{-|v_l|}.$$

Assuming this is true, since both $e^v/(1+e^{2v})$ and $\arctan e^{-|v|}$ are in $(\mathrm{L}^1 \cap \mathrm{L}^2)(\mathbb{R})$, we have

$$\frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} F}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}}} \in (\mathrm{L}^1 \cap \mathrm{L}^2)((\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N).$$

Hence, we only have to prove the claim. Suppose $\mathcal{I} = \{l_1, \ldots, l_s\}$ and consider the sets

$$V_l = \begin{cases} (-\infty, v_l] & \text{if } j(v_l) = 0, \\ [v_l, +\infty) & \text{if } j(v_l) = 1. \end{cases}$$

We can write

$$\int_{V_{l_1} \times \ldots \times V_{l_s}} \frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}'|} F}{\partial \theta^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}'}} du_{l_1} \ldots du_{l_s}$$
$$= \int_{V_{l_1} \times \ldots \times V_{l_s}} \frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+s} F}{\partial \theta^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}} \partial u_{l_1} \ldots \partial u_{l_s}} du_{l_1} \ldots du_{l_s}.$$

Since $|\boldsymbol{\alpha}| + |\boldsymbol{\beta}'| \leq 2N-1$, by Lemma 3.8 we have

$$\int_{V_{l_s}} \frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+s} F}{\partial \theta^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}} \partial u_{l_1} \ldots \partial u_{l_s}}(\boldsymbol{\theta}, (\ldots, u_{l_s}, \ldots)) du_{l_s}$$
$$= \frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+s-1} F}{\partial \theta^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}} \partial u_{l_1} \ldots \partial u_{l_{s-1}}}(\boldsymbol{\theta}, (\ldots, u_{l_s}, \ldots))\Big|_{V_{l_s}}$$
$$= (-1)^{j(v_{l_s})} \frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+s-1} F}{\partial \theta^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}} \partial u_{l_1} \ldots \partial u_{l_{s-1}}}(\boldsymbol{\theta}, (\ldots, v_{l_s}, \ldots)).$$

Hence, applying recursively what we just obtained, we deduce

$$\left|\frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} F}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}}}(\boldsymbol{\theta}, \boldsymbol{v})\right| = \left|\int_{V_{l_1} \times \ldots \times V_{l_s}} \frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}'|} F}{\partial \theta^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}'}} du_{l_1} \ldots du_{l_s}\right|$$
$$\leq K(\boldsymbol{\alpha}, \boldsymbol{\beta}', f) \prod_{l \notin \mathcal{I}} \left(\frac{e^{v_l}}{1+e^{2v_l}}\right) \int_{V_{l_1} \times \ldots \times V_{l_s}} \prod_{j=1}^{s} \left(\frac{e^{u_{l_j}}}{1+e^{2u_{l_j}}}\right) du_{l_1} \ldots du_{l_s}$$
$$= K(\boldsymbol{\alpha}, \boldsymbol{\beta}', f) \prod_{l \notin \mathcal{I}} \left(\frac{e^{v_l}}{1+e^{2v_l}}\right) \prod_{l \in \mathcal{I}} \int_{V_l} \left(\frac{e^{u_l}}{1+e^{2u_l}}\right) du_l.$$

Finally, observe that

$$\int_{V_l} \left(\frac{e^{u_l}}{1+e^{2u_l}}\right) du_l = \arctan e^{-|v_l|}.$$

□

In order to prove parts (ii) and (iv) of Theorem 3.5, we will need the following lemma.

Lemma 3.11. *Let $F = f \circ \phi$, for some $f \in \mathcal{F}$. Then for every $(\boldsymbol{n}, \boldsymbol{t}) \in \mathbb{Z}^N \times \mathbb{R}^N$ and every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \{0,1\}^N$, the following holds*

$$\frac{\widehat{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} F}}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}}}(\boldsymbol{n}, \boldsymbol{t}) = \left[\prod_{l=1}^{N} (2\pi i n_l)^{\alpha_l} (2\pi i t_l)^{\beta_l}\right] \widehat{F}(\boldsymbol{n}, \boldsymbol{t}),$$

$$\frac{\widehat{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+1} F}}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}} \partial \theta_k}(\boldsymbol{n}, \boldsymbol{t}) = \left[\prod_{l=1}^{N} (2\pi i n_l)^{\alpha_l} (2\pi i t_l)^{\beta_l}\right] \widehat{\frac{\partial F}{\partial \theta_k}}(\boldsymbol{n}, \boldsymbol{t})$$

and

$$\frac{\widehat{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+1} F}}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}} \partial u_k}(\boldsymbol{n}, \boldsymbol{t}) = \left[\prod_{l=1}^{N} (2\pi i n_l)^{\alpha_l} (2\pi i t_l)^{\beta_l}\right] \widehat{\frac{\partial F}{\partial u_k}}(\boldsymbol{n}, \boldsymbol{t}).$$

Proof. The idea of the proof is to do integration by parts. This and the fact that the function F and all its partial derivatives up to order $2N$ tend to zero as any of the u_l goes to either $-\infty$ or $+\infty$ will be enough to prove the result.

We will give a detailed proof for partial derivatives of first order, the remaining cases are obtained by applying the same argument recursively.

Let us start first with a partial derivative of the function F with respect to the variable θ_l. In Proposition 3.10, we saw that any of these partial derivatives are indeed Haar-integrable and, consequently, we can consider their Fourier transform. For any $(\boldsymbol{n},\boldsymbol{t}) \in \mathbb{Z}^N \times \mathbb{R}^N$, we have

$$\widehat{\frac{\partial F}{\partial \theta_l}}(\boldsymbol{n},\boldsymbol{t}) = \int_{(\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N} \frac{\partial F}{\partial \theta_l}(\boldsymbol{\theta},\boldsymbol{u}) e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{\theta}} e^{-2\pi i \boldsymbol{t}\cdot\boldsymbol{u}} d\boldsymbol{\theta} d\boldsymbol{u}.$$

Integrating by parts, we obtain

$$\int_{\mathbb{R}/\mathbb{Z}} \frac{\partial F}{\partial \theta_l}(\boldsymbol{\theta},\boldsymbol{u}) e^{-2\pi i n_l \theta_l} d\theta_l = (2\pi i n_l) \int_{\mathbb{R}/\mathbb{Z}} F(\boldsymbol{\theta},\boldsymbol{u}) e^{-2\pi i n_l \theta_l} d\theta_l.$$

Hence, from the definition of the Fourier transform of the partial derivative $\frac{\partial F}{\partial \theta_l}$ together with Fubini's theorem, we deduce

$$\begin{aligned}\widehat{\frac{\partial F}{\partial \theta_l}}(\boldsymbol{n},\boldsymbol{t}) = \int_{(\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N} (2\pi i n_l) F(\boldsymbol{\theta},\boldsymbol{u}) e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{\theta}} e^{-2\pi i \boldsymbol{t}\cdot\boldsymbol{u}} d\boldsymbol{\theta} d\boldsymbol{u} \\ = (2\pi i n_l) \widehat{F}(\boldsymbol{n},\boldsymbol{t}).\end{aligned}$$

We will now consider the partial derivative of the function F with respect to u_l. Since it is Haar-integrable, we can consider its Fourier transform, which is given by

$$\widehat{\frac{\partial F}{\partial u_l}}(\boldsymbol{n},\boldsymbol{t}) = \int_{(\mathbb{R}/\mathbb{Z})^N \times \mathbb{R}^N} \frac{\partial F}{\partial u_l}(\boldsymbol{\theta},\boldsymbol{u}) e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{\theta}} e^{-2\pi i \boldsymbol{t}\cdot\boldsymbol{u}} d\boldsymbol{\theta} d\boldsymbol{u},$$

for any $(\boldsymbol{n},\boldsymbol{t}) \in \mathbb{Z}^N \times \mathbb{R}^N$.

As it was done in the previous case, using integration by parts, we have

$$\begin{aligned}&\int_{\mathbb{R}} \frac{\partial F}{\partial u_l}(\boldsymbol{\theta},\boldsymbol{u}) e^{-2\pi i t_l u_l} du_l \\ &\qquad = F(\boldsymbol{\theta},\boldsymbol{u}) e^{-2\pi i t_l u_l}\Big|_{-\infty}^{+\infty} + (2\pi i t_l) \int_{\mathbb{R}} F(\boldsymbol{\theta},\boldsymbol{u}) e^{-2\pi i t_l u_l} du_l \\ &\qquad = (2\pi i t_l) \int_{\mathbb{R}} F(\boldsymbol{\theta},\boldsymbol{u}) e^{-2\pi i t_l u_l} du_l.\end{aligned}$$

The last equality is given by the fact that, by Lemma 3.8, for every $t_l \in \mathbb{R}$ we have

$$\lim_{u_l \to -\infty} F(\boldsymbol{\theta},\boldsymbol{u}) e^{-2\pi i t_l n_l} = 0 = \lim_{u_l \to +\infty} F(\boldsymbol{\theta},\boldsymbol{u}) e^{-2\pi i t_l n_l}.$$

Therefore, by Fubini's theorem we obtain

$$\widehat{\frac{\partial F}{\partial u_l}}(\boldsymbol{n},\boldsymbol{t}) = (2\pi i t_l) \widehat{F}(\boldsymbol{n},\boldsymbol{t}).$$

Before, we mentioned that the remaining cases would be done by recursively applying this method. Note that it is possible since we would always be under the hypothesis of Lemma 3.8 and Proposition 3.10. $\square$

The following result completes the proof of the main theorem of this section.

PROPOSITION 3.12. *Let $F = f \circ \phi$, with $f \in \mathcal{F}$. Then we have*

$$\widehat{F} \in \mathrm{L}^1(\mathbb{Z}^N \times \mathbb{R}^N)$$

and

$$\widehat{\frac{\partial F}{\partial u_l}}, \widehat{\frac{\partial F}{\partial \theta_l}} \in \mathrm{L}^1(\mathbb{Z}^N \times \mathbb{R}^N),$$

for any $l = 1, \ldots, N$.

PROOF. Let us prove the Haar-integrability of the Fourier Transform $\widehat{F}$:

$$\sum_{\boldsymbol{n} \in \mathbb{Z}^N} \int_{\mathbb{R}^N} \left| \widehat{F}(\boldsymbol{n}, \boldsymbol{t}) \right| d\boldsymbol{t} < +\infty.$$

In order to do so, we will divide the integration domain in 2^{2N} pairwise disjoint subspaces. Let $\alpha, \beta \in \{0, 1\}$ and consider the sets

$$W(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0, \\ \mathbb{Z} \setminus \{0\} & \text{if } \alpha = 1. \end{cases} \quad \text{and } V(\beta) = \begin{cases} (-1, 1) & \text{if } \beta = 0, \\ \mathbb{R} \setminus (-1, 1) & \text{if } \beta = 1. \end{cases}$$

For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_N)$ in $\{0, 1\}^N$, let us define

$$\boldsymbol{W}(\boldsymbol{\alpha}) = W(\alpha_1) \times \ldots \times W(\alpha_N) \text{ and } \boldsymbol{V}(\boldsymbol{\beta}) = V(\beta_1) \times \ldots \times V(\beta_N).$$

With this notation, it is easy to verify that

$$\mathbb{Z}^N \times \mathbb{R}^N = \bigsqcup_{\boldsymbol{\alpha}, \boldsymbol{\beta} \in \{0,1\}^N} \boldsymbol{W}(\boldsymbol{\alpha}) \times \boldsymbol{V}(\boldsymbol{\beta}).$$

Therefore, we can write

$$\sum_{\boldsymbol{n} \in \mathbb{Z}^N} \int_{\mathbb{R}^N} \left| \widehat{F}(\boldsymbol{n}, \boldsymbol{t}) \right| d\boldsymbol{t} = \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta} \in \{0,1\}^N} \sum_{\boldsymbol{n} \in \boldsymbol{W}(\boldsymbol{\alpha})} \int_{\boldsymbol{V}(\boldsymbol{\beta})} \left| \widehat{F}(\boldsymbol{n}, \boldsymbol{t}) \right| d\boldsymbol{t}. \tag{3.8}$$

Fix $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in $\{0, 1\}^N$. By Lemma 3.11, for every $(\boldsymbol{n}, \boldsymbol{t}) \in \boldsymbol{W}(\boldsymbol{\alpha}) \times \boldsymbol{V}(\boldsymbol{\beta})$ we have

$$\left| \widehat{F}(\boldsymbol{n}, \boldsymbol{t}) \right| = \prod_{l : \alpha_l \neq 0} (2\pi n_l)^{-1} \prod_{l : \beta_l \neq 0} (2\pi t_l)^{-1} \left| \widehat{\frac{\partial^{|\boldsymbol{\alpha}| + |\boldsymbol{\beta}|} F}{\partial \boldsymbol{\theta}^{\boldsymbol{\alpha}} \partial \boldsymbol{u}^{\boldsymbol{\beta}}}}(\boldsymbol{n}, \boldsymbol{t}) \right|.$$

If $\alpha_l \neq 0$ and $\beta_l \neq 0$, we have $0 \notin W(\alpha_l)$ and $0 \notin V(\beta_l)$. Thus, this last expression is well-defined.

From Lemma 3.10, we know that $\frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}F}{\partial\boldsymbol{\theta}^{\boldsymbol{\alpha}}\partial\boldsymbol{u}^{\boldsymbol{\beta}}}$ is in $(\mathrm{L}^1\cap\mathrm{L}^2)((\mathbb{R}/\mathbb{Z})^N\times\mathbb{R}^N)$ and *Plancherel Theorem* 1.8 implies that

$$\left\|\widehat{\frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}F}{\partial\boldsymbol{\theta}^{\boldsymbol{\alpha}}\partial\boldsymbol{u}^{\boldsymbol{\beta}}}}\right\|_{\mathrm{L}^2(\mathbb{Z}^N\times\mathbb{R}^N)} = \left\|\frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}F}{\partial\boldsymbol{\theta}^{\boldsymbol{\alpha}}\partial\boldsymbol{u}^{\boldsymbol{\beta}}}\right\|_{\mathrm{L}^2((\mathbb{R}/\mathbb{Z})^N\times\mathbb{R}^N)}.$$

In particular, $\widehat{\frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}F}{\partial\boldsymbol{\theta}^{\boldsymbol{\alpha}}\partial\boldsymbol{u}^{\boldsymbol{\beta}}}}$ is in $\mathrm{L}^2(\mathbb{Z}^N\times\mathbb{R}^N)$.

By Cauchy-Schwartz's inequality, we have

$$\begin{aligned}
&\left(\sum_{\boldsymbol{n}\in\boldsymbol{W}(\boldsymbol{\alpha})}\int_{\boldsymbol{V}(\boldsymbol{\beta})}\left|\widehat{F}(\boldsymbol{n},\boldsymbol{t})\right|d\boldsymbol{t}\right)^2\\
&=\left(\sum_{\boldsymbol{n}\in\boldsymbol{W}(\boldsymbol{\alpha})}\int_{\boldsymbol{V}(\boldsymbol{\beta})}\prod_{l:\alpha_l\neq0}(2\pi n_l)^{-1}\prod_{l:\beta_l\neq0}(2\pi t_l)^{-1}\left|\widehat{\frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}F}{\partial\boldsymbol{\theta}^{\boldsymbol{\alpha}}\partial\boldsymbol{u}^{\boldsymbol{\beta}}}}(\boldsymbol{n},\boldsymbol{t})\right|d\boldsymbol{t}\right)^2\\
&\qquad\leq\left(\sum_{\boldsymbol{n}\in\boldsymbol{W}(\boldsymbol{\alpha})}\int_{\boldsymbol{V}(\boldsymbol{\beta})}\prod_{l:\alpha_l\neq0}\frac{1}{4\pi^2n_l^2}\prod_{l:\beta_l\neq0}\frac{1}{4\pi^2t_l^2}d\boldsymbol{t}\right)\cdot\\
&\qquad\qquad\cdot\left(\sum_{\boldsymbol{n}\in\boldsymbol{W}(\boldsymbol{\alpha})}\int_{\boldsymbol{V}(\boldsymbol{\beta})}\left|\widehat{\frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}F}{\partial\boldsymbol{\theta}^{\boldsymbol{\alpha}}\partial\boldsymbol{u}^{\boldsymbol{\beta}}}}(\boldsymbol{n},\boldsymbol{t})\right|^2d\boldsymbol{t}\right).
\end{aligned}$$

On the one hand,

$$\begin{aligned}
&\sum_{\boldsymbol{n}\in\boldsymbol{W}(\boldsymbol{\alpha})}\int_{\boldsymbol{V}(\boldsymbol{\beta})}\prod_{l:\alpha_l\neq0}\frac{1}{4\pi^2n_l^2}\prod_{l:\beta_l\neq0}\frac{1}{4\pi^2t_l^2}d\boldsymbol{t}\\
&\qquad=\left(\sum_{\boldsymbol{n}\in\boldsymbol{W}(\boldsymbol{\alpha})}\prod_{l:\alpha_l\neq0}\frac{1}{4\pi^2n_l^2}\right)\left(\int_{\boldsymbol{V}(\boldsymbol{\beta})}\prod_{l:\beta_l\neq0}\frac{1}{4\pi^2t_l^2}d\boldsymbol{t}\right)\\
&\quad=\left(\prod_{l:\alpha_l\neq0}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{4\pi^2n^2}\right)\left(\prod_{l:\beta_l\neq0}\int_{\mathbb{R}\setminus(-1,1)}\frac{1}{4\pi^2t}dt\prod_{l:\beta_l=0}\int_{-1}^{1}dt\right)\\
&\qquad\qquad=\left(\frac{1}{12}\right)^{|\boldsymbol{\alpha}|}\left(\frac{1}{4\pi^2}\right)^{|\boldsymbol{\beta}|}2^N.
\end{aligned}$$

And, on the other hand

$$\sum_{\boldsymbol{n}\in\boldsymbol{W}(\boldsymbol{\alpha})}\int_{\boldsymbol{V}(\boldsymbol{\beta})}\left|\widehat{\frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}F}{\partial\boldsymbol{\theta}^{\boldsymbol{\alpha}}\partial\boldsymbol{u}^{\boldsymbol{\beta}}}}(\boldsymbol{n},\boldsymbol{t})\right|^2d\boldsymbol{t}\leq\left\|\widehat{\frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}F}{\partial\boldsymbol{\theta}^{\boldsymbol{\alpha}}\partial\boldsymbol{u}^{\boldsymbol{\beta}}}}\right\|^2_{\mathrm{L}^2(\mathbb{Z}^N\times\mathbb{R}^N)}.$$

Putting everything together we can conclude

$$\sum_{\boldsymbol{n}\in\boldsymbol{W}(\boldsymbol{\alpha})}\int_{\boldsymbol{V}(\boldsymbol{\beta})}\left|\widehat{F}(\boldsymbol{n},\boldsymbol{t})\right|d\boldsymbol{t}<\infty$$

for every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \{0,1\}^N$ and therefore, by (3.8), we deduce that $\widehat{F}$ is Haar-integrable.

For proving the Haar-integrability of the transform of the partial derivatives of F of first order, we follow the same technique. Observe that this is possible since, on Lemma 3.8 and Proposition 3.10, we obtained the desired properties. $\square$

As a final observation about the set of test functions $\mathcal{F}$, we will see that every compactly supported continuous function on $(\mathbb{C}^\times)^N$ is the limit of a sequence of functions $\{f_k \circ \phi\}$ with $f_k \in \mathcal{F}$. Thanks to this result, we are able to deduce from the main theorem in the N-dimensional case Bilu's classical equidistribution theorem.

LEMMA 3.13. *Let $F : (\mathbb{C}^\times)^N \to \mathbb{R}$ be a compactly supported continuous function. Then there is a sequence $\{f_m\} \subset \mathcal{F}$ such that*

$$\lim_{m\to\infty} f_m \circ \phi = F \text{ uniformly.}$$

PROOF. Every compactly supported function $F : (\mathbb{C}^\times)^N \to \mathbb{R}$ can be naturally extended to a continuous function $F : \mathbb{C}^N \to \mathbb{R}$ with compact support on $(\mathbb{C}^\times)^N$ by setting $F(z_1, \ldots, z_N) = 0$ whenever $z_j = 0$ for some j. Let φ be a mollifier on $\mathbb{C}^N$ and, for every $m \geq 1$, set

$$\varphi_m(\boldsymbol{z}) = m^{2N}\varphi(m\boldsymbol{z}) \text{ with } \boldsymbol{z} \in \mathbb{C}^N.$$

The mollifier φ_m is supported on the disc $D(\mathbf{0}, \frac{1}{m})$ so, by Lemma 1.9, the function $F * \varphi_m$ is smooth and its support is contained in a $\frac{1}{m}$-neighborhood of the support of F, which is compact and contained in $(\mathbb{C}^\times)^N$. Therefore, there is $M > 0$ such that for all $m \geq M$, the function $F * \varphi_m$ is compactly supported on $(\mathbb{C}^\times)^N$.

For every $m \geq M$, let $f_m : \mathbb{P}^1(\mathbb{C})^N \to \mathbb{R}$ such that $f_m(z_1, \ldots, z_N) = 0$ whenever $z_j = 0$ or $z_j = \infty$ for some j and $f_m = F_m$ otherwise. Hence, f_m is a smooth function on $\mathbb{P}^1(\mathbb{C})^N$ with compact support on $(\mathbb{C}^\times)^N$. In particular, $\{f_m\}_{m\geq M} \subset \mathcal{F}$.

The function F is uniformly continuous on $\mathbb{C}^N$ hence, by Theorem 1.10, we deduce that $F * \varphi_m$ converges uniformly to F. $\square$

3. Galois orbits and heights of elements in $(\overline{\mathbb{Q}}^\times)^N$

Consider an element $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N)$ in $(\overline{\mathbb{Q}}^\times)^N$. Recall that the Galois orbit of $\boldsymbol{\xi}$ is the orbit under the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We say that a finite set in $(\overline{\mathbb{Q}}^\times)^N$ is Galois-invariant if it is invariant under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In particular, every finite Galois-invariant set is a finite union of Galois orbits.

Given a finite set $S \subset (\overline{\mathbb{Q}}^\times)^N$, its height is given by the sum of the heights of all its elements, $\mathrm{h}(S) = \sum_{\boldsymbol{\xi}\in S} \mathrm{h}(\boldsymbol{\xi})$. In particular, given a Galois orbit $S \subset (\overline{\mathbb{Q}}^\times)^N$ of cardinality D, we have $\mathrm{h}(S) = D\,\mathrm{h}(\boldsymbol{\xi})$, for any $\boldsymbol{\xi} \in S$.

LEMMA 3.14. *Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N)$ in $(\overline{\mathbb{Q}}^{\times})^N$, S its Galois orbit and set $D := \#S$. Then*

(1) $D = [\mathbb{Q}(\xi_1, \dots, \xi_N) : \mathbb{Q}]$,

(2) For every $\boldsymbol{n} = (n_1, \dots, n_N)$ in $\mathbb{Z}^N$, consider the monomial map

$$\begin{array}{rcl} \chi^{\boldsymbol{n}} : (\overline{\mathbb{Q}}^{\times})^N & \longrightarrow & \overline{\mathbb{Q}}^{\times} \\ \boldsymbol{z} = (z_1, \dots, z_N) & \longmapsto & \chi^{\boldsymbol{n}}(\boldsymbol{z}) = z_1^{n_1} \dots z_N^{n_N}. \end{array}$$

Then $\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))$ divides D.

PROOF. Let M be the normal closure of the extension $\mathbb{Q}(\xi_1, \dots, \xi_N)$ of $\mathbb{Q}$, i.e. the smallest normal extension of $\mathbb{Q}$ containing $\mathbb{Q}(\xi_1, \dots, \xi_N)$. Since the extension M over $\mathbb{Q}$ is Galois, we have that its Galois group $G = \mathrm{Gal}(M/\mathbb{Q})$ has cardinality $[M : \mathbb{Q}]$.

The orbit S of $\boldsymbol{\xi}$ under the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ coincides with the orbit of $\boldsymbol{\xi}$ under the action of G,

$$S = \{\sigma\boldsymbol{\xi} = (\sigma\xi_1, \dots, \sigma\xi_N) : \sigma \in G\}.$$

For any element $\boldsymbol{\alpha} \in S$, its *isotropy group* is defined as

$$G_{\boldsymbol{\alpha}} := \{\sigma \in G : \sigma\boldsymbol{\alpha} = \boldsymbol{\alpha}\}.$$

Since the set S is an orbit, the isotropy subgroups of its elements are conjugate and whence, they have the same cardinality. From this fact, we can easily deduce the classical *orbit-stabilizer theorem* that states

$$\#S = \#G/\#G_{\boldsymbol{\alpha}}, \text{ for any } \boldsymbol{\alpha} \in S.$$

Since $M \hookleftarrow \mathbb{Q}$ is a normal extension, for any intermediate extension $M \hookleftarrow L \hookleftarrow \mathbb{Q}$ we have that $M \hookleftarrow L$ is normal. Thus, $M \hookleftarrow \mathbb{Q}(\xi_1, \dots, \xi_N)$ is a Galois extension whose Galois group has cardinality

$$\#\,\mathrm{Gal}(M/\mathbb{Q}(\xi_1, \dots, \xi_N)) = [M : \mathbb{Q}(\xi_1, \dots, \xi_N)].$$

We claim that $G_{\boldsymbol{\xi}} = \mathrm{Gal}(M/\mathbb{Q}(\xi_1, \dots, \xi_N))$ and, assuming this claim, we have

$$\begin{aligned} \#G = [M : \mathbb{Q}] &= [M : \mathbb{Q}(\xi_1, \dots, \xi_N)] \cdot [\mathbb{Q}(\xi_1, \dots, \xi_N) : \mathbb{Q}] \\ &= \#\,\mathrm{Gal}(M/\mathbb{Q}(\xi_1, \dots, \xi_N)) \cdot [\mathbb{Q}(\xi_1, \dots, \xi_N) : \mathbb{Q}] \\ &= \#G_{\boldsymbol{\xi}} \cdot [\mathbb{Q}(\xi_1, \dots, \xi_N) : \mathbb{Q}]. \end{aligned}$$

We can then deduce

$$D = \#S = \frac{\#G}{\#G_{\boldsymbol{\xi}}} = [\mathbb{Q}(\xi_1, \dots, \xi_N) : \mathbb{Q}].$$

Let us prove the claim.

$$\begin{aligned} \sigma \in G_{\boldsymbol{\xi}} &\iff \sigma \in G \text{ is such that } \sigma\boldsymbol{\xi} = \boldsymbol{\xi} \\ &\iff \sigma \in G \text{ and } \sigma\xi_j = \xi_j,\ \forall j = 1, \dots, N \\ &\iff \sigma \in G \text{ and } \sigma(\alpha) = \alpha\ \forall \alpha \in \mathbb{Q}(\xi_1, \dots, \xi_N) \\ &\iff \sigma \in \mathrm{Gal}(M/\mathbb{Q}(\xi_1, \dots, \xi_N)). \end{aligned}$$

Finally, we will see that the second part of the lemma is a direct consequence of what we just proved. For any $\boldsymbol{n} \in \mathbb{Z}^N$, the element $\chi^{\boldsymbol{n}}(\boldsymbol{\xi}) = \xi_1^{n_1} \cdots \xi_N^{n_N}$ is an element in the field extension $\mathbb{Q}(\xi_1, \ldots, \xi_N)$. Hence, we have an intermediate extension $\mathbb{Q}(\xi_1, \ldots, \xi_N) \hookleftarrow \mathbb{Q}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) \hookleftarrow \mathbb{Q}$ and, by the multiplicative formula for the degree of field extensions, we have

$$\begin{aligned} D = [\mathbb{Q}(\xi_1, \ldots, \xi_N) : \mathbb{Q}] &= [\mathbb{Q}(\xi_1, \ldots, \xi_N) : \mathbb{Q}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))] \cdot [\mathbb{Q}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) : \mathbb{Q}] \\ &= [\mathbb{Q}(\xi_1, \ldots, \xi_N) : \mathbb{Q}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))] \cdot \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})). \end{aligned}$$

□

LEMMA 3.15. *Let $\xi \in \overline{\mathbb{Q}}^\times$, $d = \deg(\xi)$ and S its Galois orbit. Then*

$$\frac{1}{d} \sum_{\alpha \in S} |\log |\alpha|| \leq 2\,\mathrm{h}(\xi).$$

PROOF. We have

$$\begin{aligned} \frac{1}{d}\sum_{\alpha \in S} |\log|\alpha|| &= \frac{1}{d}\sum_{\alpha \in S} \max\{-\log|\alpha|, \log|\alpha|\} \\ &= \frac{1}{d}\sum_{\alpha \in S} \log\max\left\{\frac{1}{|\alpha|}, |\alpha|\right\} = \frac{1}{d}\sum_{\alpha \in S} \log\max\{1, |\alpha|^2\} - \log|\alpha|. \end{aligned}$$

Let $P_\xi(x) = a_d x^d + \ldots + a_0 \in \mathbb{Z}[x]$ be the minimal polynomial of ξ over $\mathbb{Z}$. Since S is the orbit of ξ, we have

$$P_\xi(x) = a_d \prod_{\alpha \in S} (x - \alpha)$$

and hence

$$(-1)^d a_d \prod_{\alpha \in S} \alpha = a_0.$$

Since $|a_0|$ is a non-zero positive integer, we can write

$$\begin{aligned} \frac{1}{d}\sum_{\alpha \in S} \log\max\{1, |\alpha|^2\} - \log|\alpha| &= \frac{1}{d}\sum_{\alpha \in S} \log\max\{1, |\alpha|^2\} + \log\frac{|a_d|}{|a_0|} \\ \leq \frac{1}{d}\sum_{\alpha \in S} \log\max\{1, |\alpha|^2\} + \log|a_d| &\leq 2\left(\frac{1}{d}\sum_{\alpha \in S} \log\max\{1, |\alpha|\} + \log|a_d|\right) \\ &= 2\frac{m(P_\xi)}{\deg(\xi)} = 2\,\mathrm{h}(\xi). \end{aligned}$$

□

LEMMA 3.16. *Let $\boldsymbol{\xi}_1 \in (\overline{\mathbb{Q}}^\times)^N$ and consider its Galois orbit $\{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_D\}$, where $\boldsymbol{\xi}_j = (\xi_{j,1}, \ldots, \xi_{j,N})$, for every $j = 1, \ldots, D$. Then*

$$\frac{1}{D} \sum_{l=1}^{N} \sum_{j=1}^{D} |\log|\xi_{j,l}|| \leq 2\,\mathrm{h}(\boldsymbol{\xi}_1).$$

PROOF. Observe that, for every $l = 1, \cdots, N$, the elements $\xi_{j,l}$ and $\xi_{k,l}$ are algebraic conjugates. Let us denote by S_l the Galois orbit of $\xi_{1,l}$. By Lemma 3.14 we have that $\#S_l = \deg(\xi_{1,l})$ divides D. This is, there is a positive integer k_l such that $D = \deg(\xi_{1,l})k_l$, were k_l is exactly the number of times each element of the orbit is repeated in $\{\xi_{1,l}, \ldots, \xi_{D,l}\}$. We obtain,

$$\begin{aligned}\frac{1}{D}\sum_{l=1}^{N}\sum_{j=1}^{D}|\log|\xi_{j,l}|| &= \sum_{l=1}^{N}\frac{1}{k_l\deg(\xi_{1,l})}\sum_{j=1}^{D}|\log|\xi_{j,l}|| \\ &= \sum_{l=1}^{N}\frac{1}{\deg(\xi_{1,l})}\sum_{\alpha\in S_l}|\log|\alpha|| \leq \sum_{l=1}^{N}2\,\mathrm{h}(\xi_{1,l}) = 2\,\mathrm{h}(\boldsymbol{\xi}_1),\end{aligned}$$

where the inequality follows from Lemma 3.15 □

LEMMA 3.17. *Let $S \subset \overline{\mathbb{Q}}^{\times}$ be a Galois-invariant set of cardinality D. For every $0 < \delta < 1$, we have*

$$\#S_\delta < 2\left(\log\frac{1}{\delta}\right)^{-1}\mathrm{h}(S),$$

where $S_\delta = \{\alpha \in S : |\log|\alpha|| > \log\frac{1}{\delta}\}$.

PROOF. We know that S is a finite disjoint union of Galois orbits, say

$$S = S_1 \cup \ldots \cup S_m.$$

And observe that, by definition, for any $\alpha \in S_\delta$ we have that

$$1 < \left(\log\frac{1}{\delta}\right)^{-1}|\log|\alpha||.$$

Hence, we obtain

$$\begin{aligned}\#S_\delta &< \sum_{\alpha\in S_\delta}\left(\log\frac{1}{\delta}\right)^{-1}|\log|\alpha|| \leq \left(\log\frac{1}{\delta}\right)^{-1}\sum_{\alpha\in S}|\log|\alpha|| \\ &= \left(\log\frac{1}{\delta}\right)^{-1}\sum_{l=1}^{m}\sum_{\alpha\in S_l}|\log|\alpha|| \leq \left(\log\frac{1}{\delta}\right)^{-1}\sum_{l=1}^{m}2\,\mathrm{h}(S_l) \\ &= 2\left(\log\frac{1}{\delta}\right)^{-1}\mathrm{h}(S),\end{aligned}$$

where the last inequality holds by Lemma 3.15. □

4. The generalized degree

In this section, we will study what we previously defined as the *generalized degree*. Recall that, given a non-zero algebraic N-tuple $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N)$, its generalized degree is given by

$$\mathscr{D}(\boldsymbol{\xi}) = \min_{\boldsymbol{n}\neq\boldsymbol{0}}\{\|\boldsymbol{n}\|_1\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))\}.$$

As a first remark, we note that the set $\{\deg(\xi_1),\dots,\deg(\xi_N)\}$ is contained in the set $\{\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) : \boldsymbol{n} \neq \boldsymbol{0}\}$. Hence, we deduce that

$$\mathscr{D}(\boldsymbol{\xi}) \leq \min\{\deg(\xi_1),\dots,\deg(\xi_N)\}. \tag{3.9}$$

This upper bound for the generalized degree implies that it can be computed after a finite number of operations, considering all $\boldsymbol{n} \neq \boldsymbol{0}$ such that $\|\boldsymbol{n}\|_1 \leq \min\{\deg(\xi_1),\dots,\deg(\xi_N)\}$.

The following example shows that we can have the strict inequality in (3.9). In dimension 2, let $\boldsymbol{\xi} = (\alpha, \alpha^{-1})$ for some $\alpha \in \mathbb{Q}^\times$ with $\deg(\alpha) > 2$. Then taking $\boldsymbol{n} = (1,1)$, we obtain $\mathscr{D}(\boldsymbol{\xi}) \leq \|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) = 2 < \deg(\alpha)$.

Let us see now that it is indeed a generalization to higher dimension of the notion of the degree of an algebraic number over $\mathbb{Q}$. Consider $\xi \in \overline{\mathbb{Q}}^\times$, in the one-dimensional case we have

$$\mathscr{D}(\xi) = \min_{n\neq 0}\{|n| \deg(\xi^n)\}.$$

For every non-zero integer n, denote by $Q_n(x)$ the minimal polynomial of $\xi^{|n|}$ over $\mathbb{Q}$, of degree $\deg(\xi^{|n|}) = \deg(\xi^n)$. By setting $R_n(x) = Q_n(x^{|n|}) \in \mathbb{Q}[x]$ we obtain that $R_n(\xi) = 0$ and this implies that

$$\deg(\xi) \leq \deg(R_n(x)) = |n| \deg(\xi^n).$$

Then, we can conclude that $\mathscr{D}(\xi) = \deg(\xi)$.

On the following, we will study some properties about the generalized degree. Recall that a strict sequence in the algebraic torus $(\overline{\mathbb{Q}}^\times)^N$ is a sequence such that every proper algebraic subgroup on $(\overline{\mathbb{Q}}^\times)^N$ contains finitely many elements of the sequence.

In the one dimensional situation, a strict sequence $\{\xi_k\}$ in $\overline{\mathbb{Q}}^\times$ such that $\lim_{k\to\infty} \mathrm{h}(\xi_k) = 0$ satisfies that $\lim_{k\to\infty} \deg(\xi_k) = \infty$. Indeed, suppose there is $c > 0$ such that $\deg \xi_k \leq c$, for every $k \geq 0$. By *Northcott's theorem*, there are only finitely many algebraic numbers with bounded degree and bounded height. Whence, there is some $\alpha \in \overline{\mathbb{Q}}^\times$ such that $\xi_k = \alpha$ for infinitely many k's. Since $\lim_k \mathrm{h}(\xi_k) = 0$, we necessarily have $\mathrm{h}(\alpha) = 0$ which implies that α is a root of unity. In particular, there is a proper algebraic subgroup in $\overline{\mathbb{Q}}^\times$ containing an infinite subsequence of $\{\xi_k\}$.

The following lemma is a generalization to higher dimension of this fact.

LEMMA 3.18. *Let $\{\boldsymbol{\xi}_k\}$ be a strict sequence in $(\overline{\mathbb{Q}}^\times)^N$ such that $\mathrm{h}(\boldsymbol{\xi}_k) \to 0$ as $k \to \infty$. Then*

$$\lim_{k\to\infty} \mathscr{D}(\boldsymbol{\xi}_k) = \infty.$$

PROOF. First of all, observe that for every $\boldsymbol{n} \neq 0$ the sequence $\{\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_k)\}$ is a strict sequence in $\overline{\mathbb{Q}}^\times$.

Now, set $\boldsymbol{\xi}_k = (\xi_{k,1}, \ldots, \xi_{k,N})$ and let $\boldsymbol{n} \neq \boldsymbol{0}$, then we have

$$\begin{aligned}\mathrm{h}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_k)) &= \mathrm{h}(\xi_{k,1}^{n_1} \cdots \xi_{k,N}^{n_N}) \leq \mathrm{h}(\xi_{k,1}^{n_1}) + \ldots + \mathrm{h}(\xi_{k,N}^{n_N}) \\ &= |n_1| \,\mathrm{h}(\xi_{k,1}) + \ldots |n_N| \,\mathrm{h}(\xi_{k,N}) \leq \|\boldsymbol{n}\|_1 \,\mathrm{h}(\boldsymbol{\xi}_k) \xrightarrow[k\to\infty]{} 0.\end{aligned}$$

Thus, as we saw before, we have that for every $\boldsymbol{n} \neq \boldsymbol{0}$

$$\lim_{k\to\infty} \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_k)) = \infty.$$

Finally, we know that for every $k \geq 0$ there is some $\boldsymbol{n}_k \neq \boldsymbol{0}$ such that $\mathscr{D}(\boldsymbol{\xi}_k) = \|\boldsymbol{n}_k\| \deg(\chi^{\boldsymbol{n}_k}(\boldsymbol{\xi}_k))$ and hence

$$\lim_{k\to\infty} \mathscr{D}(\boldsymbol{\xi}_k) = +\infty.$$

$\square$

We will now state some partial results that give sufficient conditions for the generalized degree to be maximal.

LEMMA 3.19. *Let* $\xi_1, \ldots, \xi_N \in \overline{\mathbb{Q}}^\times$ *such that* $(\deg(\xi_j), \deg(\xi_k)) = 1$ *for every* $j \neq k$. *Then*

$$[\mathbb{Q}(\xi_1, \ldots, \xi_N) : \mathbb{Q}] = \deg(\xi_1) \cdots \deg(\xi_N).$$

PROOF. It is easy to see that, for any algebraic numbers $\xi_1, \ldots, \xi_N$, not necessarily of pairwise coprime degree, we have

$$[\mathbb{Q}(\xi_1, \ldots, \xi_N) : \mathbb{Q}] \leq \deg(\xi_1) \cdots \deg(\xi_N). \tag{3.10}$$

For every $j = 1, \ldots, N$, we have

$$[\mathbb{Q}(\xi_1, \ldots, \xi_N) : \mathbb{Q}] = [\mathbb{Q}(\xi_1, \ldots, \xi_N) : \mathbb{Q}(\xi_j)][\mathbb{Q}(\xi_j) : \mathbb{Q}],$$

from where we deduce that $\deg(\xi_j)$ divides $[\mathbb{Q}(\xi_1, \ldots, \xi_N) : \mathbb{Q}]$. Since, for all $j \neq i$, we have $(\deg(\xi_i), \deg(\xi_j)) = 1$, the degree of the extension $\mathbb{Q}(\xi_1, \ldots, \xi_N)$ is a multiple of $\deg(\xi_1,) \ldots \deg(\xi_N)$ and, by (3.10), the lemma follows. $\square$

LEMMA 3.20. *Let* $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N)$ *in* $(\overline{\mathbb{Q}}^\times)^N$ *and set* $p_j = \deg(\xi_j)$. *If the* p_j*'s are all pairwise different primes, we have*

$$\mathscr{D}(\boldsymbol{\xi}) = \min\{p_1, \ldots, p_N\}.$$

PROOF. Without loss of generality, suppose $p_1 = \min\{p_1, \ldots, p_N\}$. We will see that, for every $\boldsymbol{n} \neq \boldsymbol{0}$ such that $\|\boldsymbol{n}\|_1 < p_1$, the degree of $\chi^{\boldsymbol{n}}(\boldsymbol{\xi})$ over $\mathbb{Q}$ is at least p_1. This is enough to prove the result.

Suppose there is some $0 < \|\boldsymbol{n}\|_1 < p_1$ such that $\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) < p_1$. Since $\chi^{\boldsymbol{n}}(\boldsymbol{\xi})$ is in the field extension $\mathbb{Q}(\xi_1, \ldots, \xi_N)$, which has degree $p_1 \cdots p_N$ over $\mathbb{Q}$, we deduce that

$$\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) \mid p_1 \cdots p_N.$$

Thus, we necessarily have $\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) = 1$. We will see that this is not possible and we will be done. For every $n \in \mathbb{Z}$ satisfying $|n| < p_1$, we have

$$\deg(\xi_j^n) = \deg(\xi_j^{|n|}) = p_j, \text{ with } j = 1, \ldots, N.$$

If $\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) = 1$ for some $\boldsymbol{n} = (n_1, \dots, n_N) \neq \boldsymbol{0}$ with $\|\boldsymbol{n}\|_1 < p_1$. Then there is some $a \in \mathbb{Q}$ satisfying $\xi_1^{n_1} \cdots \xi_N^{n_N} = a$. Since $\boldsymbol{n} \neq \boldsymbol{0}$, there is at least some k such that $n_k \neq 0$ and we have

$$p_k = \deg(\xi_k^{n_k}) = \deg\left(\frac{a}{\prod_{j \neq k} \xi_j^{n_j}}\right) = \deg\left(\prod_{j \neq k} \xi_j^{n_j}\right).$$

Observe that $\prod_{j \neq k} \xi_j^{n_j} \in \mathbb{Q}(\xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_N)$, which is a field extension of degree $p_1 \cdots p_{k-1} p_{k+1} \cdots p_N$ over $\mathbb{Q}$. Hence, $\deg\left(\prod_{j \neq k} \xi_j^{n_j}\right) = p_k$ divides $\prod_{j \neq k} p_j$, which is not possible. $\square$

LEMMA 3.21. *Let $\alpha \in \overline{\mathbb{Q}}^\times$, if there is some $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\sigma(\alpha) = -\alpha$, then $\deg(\alpha)$ is even.*

PROOF. Let $\alpha \in \overline{\mathbb{Q}}^\times$, we have the tower of field extensions

$$\mathbb{Q} \hookrightarrow \mathbb{Q}(\alpha^2) \hookrightarrow \mathbb{Q}(\alpha)$$

and hence

$$[\mathbb{Q}(\alpha) : \mathbb{Q}] = [\mathbb{Q}(\alpha) : \mathbb{Q}(\alpha^2)][\mathbb{Q}(\alpha^2) : \mathbb{Q}].$$

The degree of the field extension $\mathbb{Q}(\alpha^2) \hookrightarrow \mathbb{Q}(\alpha)$ is at most 2 since the polynomial $x^2 - \alpha^2$ is in $\mathbb{Q}(\alpha^2)[x]$ and vanishes at α. Suppose there is $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\sigma(\alpha) = -\alpha$, then $\sigma(\alpha^2) = \sigma(\alpha)^2 = \alpha^2$ and $\sigma_{|_{\mathbb{Q}(\alpha^2)}} = id$. If there was $\beta \in \mathbb{Q}(\alpha^2)$ such that $\alpha + \beta = 0$, applying σ we would obtain $-\alpha + \beta = 0$ and necessarily $\alpha = 0$, which is not possible. Hence, $x^2 - \alpha^2$ is the minimal polynomial of α over $\mathbb{Q}(\alpha^2)$, $[\mathbb{Q}(\alpha) : \mathbb{Q}(\alpha^2)] = 2$ and the result follows. $\square$

LEMMA 3.22. *Let $\alpha, \beta \in \overline{\mathbb{Q}}^\times$ be such that $(\deg(\alpha), \deg(\beta)) = 1$ and $F_\alpha \cap F_\beta = \mathbb{Q}$, where F_α and F_β are the splitting fields of the minimal polynomials of α and β over $\mathbb{Q}$. Then*

$$\deg(\alpha\beta) = \deg(\alpha)\deg(\beta).$$

PROOF. Denote by F the splitting field of the product of the minimal polynomials of α and β over $\mathbb{Q}$. Then F is a finite Galois extension over $\mathbb{Q}$ of Galois group $G = \mathrm{Gal}(F/\mathbb{Q})$. We have the tower of field extensions

$$\mathbb{Q} \hookrightarrow \mathbb{Q}(\alpha\beta) \hookrightarrow \mathbb{Q}(\alpha, \beta) \hookrightarrow F,$$

from where we deduce that $F/\mathbb{Q}(\alpha, \beta)$ and $F/\mathbb{Q}(\alpha\beta)$ are Galois with respective Galois groups

$$G_{\alpha,\beta} := \mathrm{Gal}(F/\mathbb{Q}(\alpha, \beta)) = \{\sigma \in G : \sigma(\alpha) = \alpha, \sigma(\beta) = \beta\},$$

$$G_{\alpha\beta} := \mathrm{Gal}(F/\mathbb{Q}(\alpha\beta)) = \{\sigma \in G : \sigma(\alpha\beta) = \alpha\beta\}.$$

And we have

$$\#G_{\alpha,\beta} = [F : \mathbb{Q}(\alpha, \beta)], \ \#G_{\alpha\beta} = [F : \mathbb{Q}(\alpha\beta)].$$

We claim that $G_{\alpha,\beta} = G_{\alpha\beta}$, hence by Lemma 3.19 we obtain

$$[\mathbb{Q}(\alpha\beta):\mathbb{Q}] = \frac{[F:\mathbb{Q}]}{[F:\mathbb{Q}(\alpha\beta)]} = \frac{[F:\mathbb{Q}]}{[F:\mathbb{Q}(\alpha,\beta)]}$$
$$= [\mathbb{Q}(\alpha,\beta):\mathbb{Q}] = \deg(\alpha)\deg(\beta).$$

Let us prove the claim. The inclusion $G_{\alpha,\beta} \subset G_{\alpha\beta}$ is trivial so we are left with the reciprocal. Consider an element $\sigma \in G_{\alpha\beta}$ of order $m > 1$, then we have $\alpha\beta = \sigma(\alpha\beta) = \sigma(\alpha)\sigma(\beta)$. Define $k := \frac{\sigma(\alpha)}{\alpha} = \frac{\beta}{\sigma(\beta)}$ and observe that $\frac{\sigma(\alpha)}{\alpha} \in F_\alpha$ and $\frac{\beta}{\sigma(\beta)} \in F_\beta$, thus $k \in F_\alpha \cap F_\beta = \mathbb{Q}$.

On the other hand, since σ acts trivially on $\mathbb{Q}$ and $\sigma^m = id$, we obtain

$$k^m = k \cdot \sigma(k) \cdots \sigma^{m-1}(k) = \frac{\sigma(\alpha)}{\alpha}\frac{\sigma^2(\alpha)}{\sigma(\alpha)} \cdots \frac{\sigma^{m-1}(\alpha)}{\sigma^{m-2}(\alpha)}\frac{\alpha}{\sigma^{m-1}(\alpha)} = 1.$$

So necessarily we have $k = \pm 1$. If $k = -1$, we have $\sigma(\alpha) = -\alpha$ and $\sigma(\beta) = -\beta$ which, by Lemma 3.21, implies that both $\deg(\alpha)$ and $\deg(\beta)$ are even. This is not possible since the degrees are coprime. Hence we have $k = 1$ and $\sigma(\alpha) = \alpha$, $\sigma(\beta) = \beta$. So we can conclude that $\sigma \in G_{\alpha,\beta}$. □

Observe that this result is no longer true if we drop the hypothesis of the linearly disjoint splitting fields. Indeed, if $\alpha = \sqrt[3]{2}$ and β is a primitive root of unity of order 3. Then we have that $\deg(\alpha) = 3$ and $\deg(\beta) = 2$ are coprime, $F_\alpha = \mathbb{Q}(\alpha,\beta)$ and $F_\beta = \mathbb{Q}(\beta)$ do not intersect trivially and $\deg(\alpha\beta) = 3$.

LEMMA 3.23. *Let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N) \in (\overline{\mathbb{Q}}^\times)^N$ be such that*

(i) $(\deg(\xi_j), \deg(\xi_k)) = 1$ *for all $j \neq k$,*
(ii) There is a permutation $\tau \in S_N$, with

$$F_{\xi_{\tau(j)}} \cap F_{\prod_{k>j} \xi_{\tau(k)}} = \mathbb{Q} \text{ for all } j < N,$$

where the field $F_{\prod_{k>j} \xi_{\tau(k)}}$ is the splitting field of the minimal polynomial of $\prod_{k>j} \xi_{\tau(k)}$ over $\mathbb{Q}$.

Then

$$\mathscr{D}(\boldsymbol{\xi}) = \min\{\deg(\xi_1), \ldots, \deg(\xi_N)\}.$$

PROOF. We may assume without loss of generality that (ii) holds for $\tau = id$, otherwise we can re-order the components of $\boldsymbol{\xi}$.

Let $k \in \{1, \ldots, N\}$ be such that $\deg(\xi_k) < \deg(\xi_j)$ for all $j \neq k$ and suppose $\mathscr{D}(\boldsymbol{\xi}) < \deg(\xi_k)$.

Then, there is at least some non-zero $\boldsymbol{n} = (n_1, \ldots, n_N) \in \mathbb{Z}^N$ such that $\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) < \deg(\xi_k)$. In particular $\|\boldsymbol{n}\|_1 = |n_1| + \ldots + |n_N| < \deg(\xi_k)$. Let us study the degree of $\chi^{\boldsymbol{n}}(\boldsymbol{\xi}) = \xi_1^{n_1} \cdots \xi_N^{n_N}$ over $\mathbb{Q}$.

For all $j < N$ we have $\xi_j^{n_j} \in \mathbb{Q}(\xi_j)$ and $\xi_{j+1}^{n_{j+1}} \cdots \xi_N^{n_N} \in \mathbb{Q}(\xi_{j+1}, \ldots, \xi_N)$, which is an extension over $\mathbb{Q}$ of degree $\deg(\xi_{j+1}) \cdots \deg(\xi_N)$. Then we deduce that $\deg(\xi_j^{n_j}) \mid \deg(\xi_j)$ and $\deg(\xi_{j+1}^{n_{j+1}} \cdots \xi_N^{n_N}) \mid \deg(\xi_{j+1}) \cdots \deg(\xi_N)$ respectively. By condition (i), we obtain

$$(\deg(\xi_j^{n_j}), \deg(\xi_{j+1}^{n_{j+1}} \cdots \xi_N^{n_N})) = 1, \text{ for all } j < N.$$

Hence, by (ii) and Lemma 3.22

$$\deg(\xi_j^{n_j} \cdots \xi_N^{n_N}) = \deg(\xi_j^{n_j}) \deg(\xi_{j+1}^{n_{j+1}} \cdots \xi_N^{n_N}), \text{ for all } j < N.$$

Therefore, applying this method recursively, we can conclude that

$$\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) = \deg(\xi_1^{n_1}) \cdots \deg(\xi_N^{n_N}).$$

Observe that for every $j = 1, \ldots, N$, the polynomial $x^{|n_j|} - \xi_j^{|n_j|} \in \mathbb{Q}(\xi_j^{n_j})[x]$ vanishes at ξ_j and thus

$$\deg(\xi_j^{n_j}) = [\mathbb{Q}(\xi_j^{n_j}) : \mathbb{Q}] = \frac{[\mathbb{Q}(\xi_j) : \mathbb{Q}]}{[\mathbb{Q}(\xi_j) : \mathbb{Q}(\xi_j^{n_j})]} \geq \frac{\deg(\xi_j)}{|n_j|}.$$

Putting everything together, we obtain

$$\begin{aligned} \|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) &= (|n_1| + \ldots + |n_N|) \deg(\xi_1^{n_1}) \cdots \deg(\xi_N^{n_N}) \\ &\geq (|n_1| + \ldots + |n_N|) \frac{\deg(\xi_1)}{|n_1|} \cdots \frac{\deg(\xi_N)}{|n_N|} > \deg(\xi_k), \end{aligned}$$

which cannot hold. Therefore, we necessarily have $\mathscr{D}(\boldsymbol{\xi}) = \deg(\xi_k)$. $\square$

5. Bounds for the Lipschitz constant of the function f_δ

In this section, we will give a bound for the Lipschitz constant of the function $f_\delta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{C}$ defined by

$$f_\delta(0:1) = 0, \; f_\delta(1:z) = \rho_\delta(|z|) \frac{z}{|z|} \text{ for any } z \in \mathbb{C},$$

where $\rho_\delta : \mathbb{R} \to [0,1]$, with $0 < \delta < 1$, is given by

$$\rho_\delta(r) = \begin{cases} 0 & \text{if } r < \frac{\delta}{2} \\ \frac{(5\delta - 4r)(\delta - 2r)^2}{\delta^3} & \text{if } \frac{\delta}{2} \leq r \leq \delta \\ 1 & \text{if } \delta < r < \frac{1}{\delta} \\ (-2 + \delta r)^2(-1 + 2\delta r) & \text{if } \frac{1}{\delta} \leq r \leq \frac{2}{\delta} \\ 0 & \text{if } r > \frac{2}{\delta}. \end{cases}$$

The first thing we will do is to prove that f_δ is in $\mathscr{C}^1(\mathbb{P}^1(\mathbb{C}))$. Afterwards, we will study the Lipschitz constant of its real and imaginary parts.

It is easy to see that the function f_δ is compactly supported on the intersection of the usual charts, $U_0 \cap U_1$. In fact, we have that

$$\operatorname{supp}(f_\delta) = \left\{ (1:z) : \frac{\delta}{2} \leq |z| \leq \frac{2}{\delta} \right\}.$$

For this reason, in order to prove that f_δ is in $\mathscr{C}^1(\mathbb{P}^1(\mathbb{C}))$, it will be enough to prove that the function $\rho_\delta(|z|)\frac{z}{|z|}$ is of class $\mathscr{C}^1$ in a neighborhood of the set $\left\{z : \frac{\delta}{2} \leq |z| \leq \frac{2}{\delta}\right\}$.

The piecewise-defined function ρ_δ is continuous, as well as its derivative, which is given by

$$\rho'_\delta(r) = \begin{cases} -\frac{24}{\delta^3}(\delta - 2r)(\delta - r) & \text{if } \frac{\delta}{2} \leq r \leq \delta \\ 6\delta(-2 + \delta r)(-1 + \delta r) & \text{if } \frac{1}{\delta} \leq r \leq \frac{2}{\delta} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, since $|z|$ and $z/|z|$ are smooth at $\mathbb{C}^\times$, we can conclude that $\rho_\delta(|z|)\frac{z}{|z|}$ is indeed of class $\mathscr{C}^1$.

Let us compute now a bound for the Lipschitz constants, with respect to the spherical distance, of the real and imaginary parts of the function f_δ, that will be denoted by u_δ and v_δ respectively. In order to do so, we will choose coordinates (x, y) in $\mathbb{R}^2 \cong \mathbb{C}$. Let

$$\tilde{u}_\delta(x, y) := u_\delta(1 : x + iy) = \frac{\rho_\delta(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}x,$$

$$\tilde{v}_\delta(x, y) := v_\delta(1 : x + iy) = \frac{\rho_\delta(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}y.$$

Since the computations are symmetric for both the real and imaginary parts of f_δ, it will be enough to study the Lipschitz constant of one of them.

To simplify these computations, we will study the Lipschitz constant with respect to the chordal distance in the Riemann Sphere. Once this is done, and since both the spherical and the chordal distance are equivalent, we will be done.

First of all, recall that the chordal distance restricted to the open subset $U_0 \subset \mathbb{P}^1(\mathbb{C})$ is given by

$$\mathrm{d_{ch}}((1 : x_0 + iy_0), (1 : x_1 + iy_1)) = \frac{2\|(x_0, y_0) - (x_1, y_1)\|}{\sqrt{1 + m(x_0, y_0)^2}\sqrt{1 + m(x_1, y_1)^2}},$$

where $\|\cdot\|$ denotes the Euclidean metric on $\mathbb{R}^2$ and $m(x, y) = \sqrt{x^2 + y^2}$. Now, since the function u_δ is supported on U_0, we have

$$\begin{aligned} &\sup_{z_0, z_1 \in \mathbb{C}} \frac{|u_\delta(1 : z_0) - u_\delta(1 : z_1)|}{\mathrm{d_{ch}}((1 : z_0), (1 : z_1))} \\ &= \sup_{(x_0, y_0), (x_1, y_1) \in \mathbb{R}^2} \frac{|\tilde{u}_\delta(x_0, y_0) - \tilde{u}_\delta(x_1, y_1)|}{\|(x_0, y_0) - (x_1, y_1)\|} \frac{\sqrt{1 + m(x_0, y_0)^2}\sqrt{1 + m(x_1, y_1)^2}}{2}. \end{aligned}$$

We will consider different cases.

1. If $(x_0, y_0), (x_1, y_1) \notin D(0, \frac{2}{\delta})$, we trivially obtain

$$\frac{|\tilde{u}_\delta(x_0, y_0) - \tilde{u}_\delta(x_1, y_1)|}{\mathrm{d_{ch}}((1 : x_0 + iy_0), (1 : x_1 + iy_1))} = 0.$$

2. Suppose $(x_0, y_0), (x_1, y_1) \in D(0, \frac{2}{\delta})$. For $t \in [0,1]$, consider the function $g(t) = \tilde{u}_\delta((1-t)(x_0, y_0) + t(x_1, y_1))$. By the *mean value theorem*, we know that there is some $c \in (0,1)$ such that $g(1) - g(0) = g'(c)$. Applying the chain rule, we obtain

$$\tilde{u}_\delta(x_1, y_1) - \tilde{u}_\delta(x_0, y_0) = \nabla \tilde{u}_\delta((1-c)(x_0, y_0) + c(x_1, y_1)) \cdot (x_1 - x_0, y_1 - y_0).$$

Hence, we deduce

$$\frac{|\tilde{u}_\delta(x_0, y_0) - \tilde{u}_\delta(x_1, y_1)|}{\|(x_0, y_0) - (x_1, y_1)\|} \leq \sup_{(x,y) \in D(0, \frac{2}{\delta})} \|\nabla \tilde{u}_\delta(x, y)\|. \tag{3.11}$$

Let us study the gradient of $\tilde{u}_\delta$. For every $(x, y) \in \mathbb{R}^2$ we have

$$\frac{\partial \tilde{u}_\delta}{\partial x}(x, y) = \left(\frac{x}{m(x,y)}\right)^2 \rho_\delta'(m(x,y)) + \left(\frac{y}{m(x,y)}\right)^2 \frac{\rho_\delta(m(x,y))}{m(x,y)},$$

$$\frac{\partial \tilde{u}_\delta}{\partial y}(x, y) = \frac{xy}{m(x,y)^2} \left(\rho_\delta'(m(x,y)) - \frac{\rho_\delta(m(x,y))}{m(x,y)}\right).$$

Without loss of generality we may restrict ourselves to the situation were (x, y) is such that $\frac{\delta}{2} \leq m(x, y) \leq \frac{2}{\delta}$, otherwise both partial derivatives would vanish. It can be easily shown that $|\rho_\delta'(r)| \leq \frac{3}{\delta}$ for every $r \geq 0$. This, together with the fact that $0 \leq \rho_\delta \leq 1$, $x \leq m(x, y)$, $y \leq m(x, y)$ and $m(x, y) \geq \frac{\delta}{2}$, leads to

$$\left|\frac{\partial \tilde{u}_\delta}{\partial x}(x, y)\right|, \left|\frac{\partial \tilde{u}_\delta}{\partial y}(x, y)\right| \leq \frac{4}{\delta}.$$

We can then conclude that, for any $(x, y) \in \mathbb{R}^2$

$$\|\nabla \tilde{u}_\delta(x, y)\| \leq \frac{4\sqrt{2}}{\delta}.$$

On the other hand, given $(x_0, y_0), (x_1, y_1) \in D(0, \frac{2}{\delta})$ we have that

$$\frac{\sqrt{1 + m(x_0, y_0)^2}\sqrt{1 + m(x_1, y_1)^2}}{2} \leq \frac{\delta^2 + 4}{2\delta^2}.$$

Therefore, we obtain

$$\frac{|\tilde{u}_\delta(x_0, y_0) - \tilde{u}_\delta(x_1, y_1)|}{\mathrm{d_{ch}}((1 : x_0 + iy_0), (1 : x_1 + iy_1))} \leq 2\sqrt{2}\frac{\delta^2 + 4}{\delta^3}.$$

3. Suppose now that $(x_0, y_0) \in D(0, \frac{2}{\delta})$ and $(x_1, y_1) \in D(0, \frac{3}{\delta}) \setminus D(0, \frac{2}{\delta})$. As we did in the previous case, we can deduce that

$$\frac{|\tilde{u}_\delta(x_0, y_0) - \tilde{u}_\delta(x_1, y_1)|}{\|(x_0, y_0) - (x_1, y_1)\|} \leq \frac{4\sqrt{2}}{\delta}$$

and

$$\frac{\sqrt{1 + m(x_0, y_0)^2}\sqrt{1 + m(x_1, y_1)^2}}{2} \leq \frac{\delta^2 + 9}{2\delta^2}.$$

Hence, we obtain

$$\frac{|\tilde{u}_\delta(x_0,y_0)-\tilde{u}_\delta(x_1,y_1)|}{\mathrm{d_{ch}}((1:x_0+iy_0),(1:x_1+iy_1))}\leq 2\sqrt{2}\frac{\delta^2+9}{\delta^3}.$$

4. Finally suppose that $(x_0,y_0)\in D(0,\frac{2}{\delta})$ and $(x_1,y_1)\notin D(0,\frac{3}{\delta})$. In this situation, we have $\tilde{u}_\delta(x_1,y_1)=0$ and

$$|\tilde{u}_\delta(x_1,y_1)|=|\rho_\delta(m(x_0,y_0))|\frac{|x_0|}{m(x_0,y_0)}\leq 1.$$

Since

$$\mathrm{d_{ch}}((1:x_0+iy_0),(1:x_1+iy_1))\geq \mathrm{d_{ch}}((1:\frac{2}{\delta}),(1:\frac{3}{\delta}))=\frac{2\delta}{\sqrt{(\delta^2+9)(\delta^2+4)}},$$

we can conclude

$$\frac{|\tilde{u}_\delta(x_0,y_0)-\tilde{u}_\delta(x_1,y_1)|}{\mathrm{d_{ch}}((1:x_0+iy_0),(1:x_1+iy_1))}\leq \frac{2\delta}{\sqrt{(\delta^2+9)(\delta^2+4)}}2\delta.$$

After all this cases have been studied, we can deduce that

$$\sup_{(x_0,y_0),(x_1,y_1)\in\mathbb{R}^2}\frac{|\tilde{u}_\delta(x_0,y_0)-\tilde{u}_\delta(x_1,y_1)|}{\mathrm{d_{ch}}((1:x_0+iy_0),(1:x_1+iy_1))}\leq 2\sqrt{2}\frac{\delta^2+9}{\delta^3}.$$

However, as we mentioned above, we were looking for a bound of the Lipschitz constant of u_δ with respect to the spherical distance. By Lemma 1.14, we know that $\mathrm{d}(P,P')\geq \mathrm{d_{ch}}(P,P')$ for any pair of points $P,P'\in\mathbb{P}^1(\mathbb{C})$ and we obtain

$$\begin{aligned}\mathrm{Lip}(u_\delta)&=\sup_{P,P'\in\mathbb{P}^1(\mathbb{C})}\frac{|u_\delta(P)-u_\delta(P')|}{\mathrm{d}(P,P')}\\&\leq\sup_{(x_0,y_0),(x_1,y_1)\in\mathbb{R}^2}\frac{|\tilde{u}_\delta(x_0,y_0)-\tilde{u}_\delta(x_1,y_1)|}{\mathrm{d_{ch}}((1:x_0+iy_0),(1:x_1+iy_1))}\leq 2\sqrt{2}\frac{\delta^2+9}{\delta^3}.\end{aligned}$$

Analogously, we deduce that

$$\mathrm{Lip}(v_\delta)\leq 2\sqrt{2}\frac{\delta^2+9}{\delta^3}.$$

6. Quantitative equidistribution in dimension N

In this last section we will give the proof for Theorem I. It will be done by applying Fourier analysis techniques that allow us to discretize the problem. Then, we reduce the problem, via projections, to the one-dimensional case where the result follows from Favre and Rivera-Letelier's Theorem II.

Let $\boldsymbol{\xi}$ be an element in $(\overline{\mathbb{Q}}^\times)^N$, S be its Galois orbit and μ_S the discrete probability measure on $(\mathbb{C}^\times)^N$ associated to S. By Lemma 3.3, for every

Haar-integrable function $F : (\mathbb{C}^\times)^N \to \mathbb{R}$ whose Fourier transform $\widehat{F}$ is also Haar-integrable, we have

$$\left| \int_{(\mathbb{C}^\times)^N} F d\mu_S - \int_{(\mathbb{C}^\times)^N} F d\lambda_{(S^1)^N} \right| \leq \left| \int_{\mathbb{R}^N} \widehat{F}(\mathbf{0}, \boldsymbol{t}) \left(\overline{\widehat{\mu}_S(\mathbf{0}, \boldsymbol{t})} - 1 \right) d\boldsymbol{t} \right| + \left| \sum_{\boldsymbol{n} \neq 0} \int_{\mathbb{R}^N} \widehat{F}(\boldsymbol{n}, \boldsymbol{t}) \overline{\widehat{\mu}_S(\boldsymbol{n}, \boldsymbol{t})} d\boldsymbol{t} \right|. \tag{3.12}$$

We will divide the proof of the main result into two parts corresponding to the estimates of the two terms on the right-hand side of this inequality.

First, we will consider the integral not depending on the angle. Before studying it, we will introduce a technical result.

Lemma 3.24. *For every $x > 0$, $|\log x| \geq \frac{|x-1|}{\sqrt{1+x^2}}$.*

Proof. Set

$$g(x) = \begin{cases} \log x - \frac{x-1}{\sqrt{1+x^2}} & \text{if } x \geq 1, \\ -\log x - \frac{1-x}{\sqrt{1+x^2}} & \text{if } 0 < x < 1. \end{cases}$$

Then

$$g'(x) = \begin{cases} \frac{-x-x^2+(1+x^2)^{\frac{3}{2}}}{x(1+x^2)^{\frac{3}{2}}} & \text{if } x \geq 1, \\ -\frac{-x-x^2+(1+x^2)^{\frac{3}{2}}}{x(1+x^2)^{\frac{3}{2}}} & \text{if } 0 < x < 1. \end{cases}$$

It is easy to see that $(1+x^2)^{\frac{3}{2}} - x - x^2$ has no real roots. Hence, the function g is monotonic on each interval $(0,1)$ and $(1, +\infty)$. Since we have $g(1) = 0$, $\lim_{x \to 0^+} g(x) = +\infty$ and $\lim_{x \to +\infty} g(x) = +\infty$, we know that g is monotonically decreasing on $(0,1)$ and monotonically increasing on $(1, +\infty)$. Thus, we deduce that $g(x) \geq 0$ for every $x \in (0, +\infty)$. $\square$

Now, we can proceed with the study of the first term on the right-hand side of (3.12).

Proposition 3.25. *Let $\boldsymbol{\xi} \in (\overline{\mathbb{Q}}^\times)^N$ and S its Galois orbit. For every Lipschitz function $f : \mathbb{P}^1(\mathbb{C})^N \longrightarrow \mathbb{R}$ such that $F = f \circ \phi$, and $\widehat{F}$ are Haar-integrable we have*

$$\left| \int_{\mathbb{R}^N} \widehat{F}(\mathbf{0}, \boldsymbol{t}) \left(\overline{\widehat{\mu}_S(\mathbf{0}, \boldsymbol{t})} - 1 \right) d\boldsymbol{t} \right| \leq \sqrt{2}\pi \operatorname{Lip}(f) \operatorname{h}(\boldsymbol{\xi}), \tag{3.13}$$

where $\operatorname{Lip}(f)$ *is the Lipschitz constant of f with respect to the spherical distance in $\mathbb{P}^1(\mathbb{C})^N$.*

Proof. For any $(\boldsymbol{n}, \boldsymbol{t}) \in \mathbb{Z}^N \times \mathbb{R}^N$, the Fourier-Stieltjes transform (1.2) of μ_S is given by

$$\widehat{\mu}_S(\boldsymbol{n}, \boldsymbol{t}) = \frac{1}{D} \sum_{j=1}^{D} e^{-2\pi i \boldsymbol{t} \cdot \boldsymbol{u}_j} e^{-2\pi i \boldsymbol{n} \cdot \boldsymbol{\theta}_j}, \tag{3.14}$$

where $S=\{\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_D\}$ and

$$\boldsymbol{\xi}_j=e^{2\pi i\boldsymbol{\theta}_j+\boldsymbol{u}_j}=(e^{2\pi i\theta_{j,1}+u_{j,1}},\ldots,e^{2\pi i\theta_{j,N}+u_{j,N}})$$

for every $j=1,\ldots,D$.

In this situation, by Lemma 3.4 and (3.4), we obtain

$$\begin{aligned}(3.15)\quad &\left|\int_{\mathbb{R}^N}\widehat{F}(\mathbf{0},\boldsymbol{t})\left(\overline{\widehat{\mu}_S(\mathbf{0},\boldsymbol{t})}-1\right)d\boldsymbol{t}\right|\\
&=\left|\frac{1}{D}\sum_{j=1}^{D}\int_{\mathbb{R}^N}\widehat{F}(\mathbf{0},\boldsymbol{t})(e^{2\pi i\boldsymbol{t}\cdot\boldsymbol{u}_j}-1)d\boldsymbol{t}\right|\\
&=\left|\frac{1}{D}\sum_{j=1}^{D}\int_{\mathbb{R}^N}\left(\widehat{\tau_{|\boldsymbol{\xi}_j|}F}(\mathbf{0},\boldsymbol{t})-\widehat{F}(\mathbf{0},\boldsymbol{t})\right)d\boldsymbol{t}\right|\\
&=\left|\frac{1}{D}\sum_{j=1}^{D}\int_{(\mathbb{C}^\times)^N}\left(\tau_{|\boldsymbol{\xi}_j|}F(\boldsymbol{z})-F(\boldsymbol{z})\right)d\lambda_{(S^1)^N}(\boldsymbol{z})\right|\\
&\leq\frac{1}{D}\sum_{j=1}^{D}\int_{(\mathbb{C}^\times)^N}\left|\tau_{|\boldsymbol{\xi}_j|}F(\boldsymbol{z})-F(\boldsymbol{z})\right|d\lambda_{(S^1)^N}(\boldsymbol{z})\\
&=\frac{1}{D}\sum_{j=1}^{D}\int_{(\mathbb{C}^\times)^N}\left|F(|\boldsymbol{\xi}_j|\boldsymbol{z})-F(\boldsymbol{z})\right|d\lambda_{(S^1)^N}(\boldsymbol{z}),\end{aligned}$$

where $\tau_{|\boldsymbol{\xi}_j|}F$ is the translation of F by the element $|\boldsymbol{\xi}_j|=(|\xi_{j,1}|,\ldots,|\xi_{j,N}|)$ defined by (3.5).

Since f is a Lipschitz function on $\mathbb{P}^1(\mathbb{C})^N$ of constant $\mathrm{Lip}(f)$ with respect to the spherical distance d^N, for any $\boldsymbol{z},\boldsymbol{z}'\in(\mathbb{C}^\times)^N$ we have

$$(3.16)\quad \left|F(\boldsymbol{z})-F(\boldsymbol{z}')\right|=|f(\phi(\boldsymbol{z}))-f(\phi(\boldsymbol{z}'))|\leq\mathrm{Lip}(f)\,\mathrm{d}^N(\phi(\boldsymbol{z}),\phi(\boldsymbol{z}')),$$

where recall $\phi(\boldsymbol{z})=((1:z_1),\ldots,(1:z_N))$.

For any $\boldsymbol{z}=(z_1,\ldots,z_N)$ and $\boldsymbol{z}'=(z_1',\ldots,z_N')$ in $(\mathbb{C}^\times)^N$, by Lemma 1.14 we have

$$\mathrm{d}^N(\phi(\boldsymbol{z}),\phi(\boldsymbol{z}'))=\sqrt{\sum_{l=1}^{N}\mathrm{d}((1:z_l),(1:z_l'))^2}\leq\frac{\pi}{2}\sqrt{\sum_{l=1}^{N}\mathrm{d}_{\mathrm{ch}}((1:z_l),(1:z_l'))^2}.$$

So, for $\boldsymbol{z}\in(\mathbb{C}^\times)^N$ and every $j=1,\ldots,D$

$$\mathrm{d}^N(\phi(|\boldsymbol{\xi}_j|\boldsymbol{z}),\phi(\boldsymbol{z}))\leq\frac{\pi}{2}\sqrt{\sum_{l=1}^{N}\mathrm{d}_{\mathrm{ch}}((1:|\xi_{j,l}|z_l),(1:z_l))^2}.$$

Since $\lambda_{(S^1)^N}$ is supported on $(S^1)^N$, it will be enough to consider $\boldsymbol{z}=(z_1,\ldots,z_N)$ such that $|z_l|=1$ for every $l=1,\ldots,N$.

By Lemma 3.24, if $z_l \in S^1$ we obtain

$$\mathrm{d}_{\mathrm{ch}}((1:|\xi_{j,l}|z_l),(1:z_l)) = \frac{2|\,|\xi_{j,l}|z_l - z_l|}{\sqrt{1+|\xi_{j,l}|^2|z_l|^2}\sqrt{1+|z_l|^2}}$$
$$= \frac{\sqrt{2}\,|\,|\xi_{j,l}|-1|}{\sqrt{1+|\xi_{j,l}|^2}} \leq \sqrt{2}\,|\log|\xi_{j,l}||,$$

for $j=1,\dots,D$.

Hence, for $\boldsymbol{z} \in (S^1)^N$ and $j=1,\dots,D$, we have

$$\mathrm{d}^N(\phi(|\boldsymbol{\xi}_j|\boldsymbol{z}),\phi(\boldsymbol{z})) \leq \frac{\pi}{2}\sqrt{\sum_{l=1}^{N}\mathrm{d}_{\mathrm{ch}}((1:|\xi_{j,l}|z_l),(1:z_l))^2} \tag{3.17}$$
$$\leq \frac{\pi}{2}\sum_{l=1}^{N}\mathrm{d}_{\mathrm{ch}}((1:|\xi_{j,l}|z_l),(1:z_l)) \leq \frac{\pi}{\sqrt{2}}\sum_{l=1}^{N}|\log|\xi_{j,l}||.$$

So we can conclude from (3.15), (3.16) and (3.17) that

$$\left|\int_{\mathbb{R}^N}\widehat{F}(\boldsymbol{0},\boldsymbol{t})(\overline{\widehat{\mu}_S(\boldsymbol{0},\boldsymbol{t})}-1)d\boldsymbol{t}\right|$$
$$\leq \mathrm{Lip}(f)\frac{1}{D}\sum_{j=1}^{D}\int_{(\mathbb{C}^\times)^N}\mathrm{d}^N(\phi(|\boldsymbol{\xi}_j|\boldsymbol{z}),\phi(\boldsymbol{z}))d\lambda_{(S^1)^N}(\boldsymbol{z})$$
$$\leq \frac{\pi}{\sqrt{2}}\mathrm{Lip}(f)\frac{1}{D}\sum_{j=1}^{D}\sum_{l=1}^{N}|\log|\xi_{j,l}|| \leq \sqrt{2}\pi\,\mathrm{Lip}(f)\,\mathrm{h}(\boldsymbol{\xi}),$$

where the last inequality is given by Lemma 3.16. □

Let us study now the second term in (3.12).

PROPOSITION 3.26. *There is a constant $C \approx 48.9897$ such that for every $\boldsymbol{\xi} \in (\overline{\mathbb{Q}}^\times)^N$ with $\mathrm{h}(\boldsymbol{\xi}) < 1$, every $0<\delta<1$ and $F = f\circ\phi$, with $f \in \mathcal{F}$, the following holds*

$$\left|\sum_{\boldsymbol{n}\neq\boldsymbol{0}}\int_{\mathbb{R}^N}\widehat{F}(\boldsymbol{n},\boldsymbol{t})\overline{\widehat{\mu}_S(\boldsymbol{n},\boldsymbol{t})}d\boldsymbol{t}\right| \leq 2\,\mathrm{h}(\boldsymbol{\xi})\sum_{l=1}^{N}\left\|\widehat{\frac{\partial F}{\partial u_l}}\right\|_{\mathrm{L}^1}$$
$$+\frac{1}{2\pi}\left(\frac{-2}{\log\delta}+\frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\right)\left(4\,\mathrm{h}(\boldsymbol{\xi})+C\frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}}\sum_{l=1}^{N}\left\|\widehat{\frac{\partial F}{\partial \theta_l}}\right\|_{\mathrm{L}^1},$$

where S is the Galois orbit of $\boldsymbol{\xi}$, μ_S the discrete probability measure associated to it and $\mathscr{D}(\boldsymbol{\xi})$ the generalized degree of $\boldsymbol{\xi}$.

To prove Proposition 3.26, we will first state and prove some previous results.

LEMMA 3.27. *Let $\boldsymbol{\xi} \in (\overline{\mathbb{Q}}^{\times})^N$ and S its Galois orbit. Denote by μ_S the discrete probability measure associated to S. Then, for every $\boldsymbol{t}, \boldsymbol{t}' \in \mathbb{R}^N$ and for every $\boldsymbol{n} \in \mathbb{Z}^N \setminus \{\boldsymbol{0}\}$, we have*

$$|\widehat{\mu}_S(\boldsymbol{n}, \boldsymbol{t}) - \widehat{\mu}_S(\boldsymbol{n}, \boldsymbol{t}')| \leq 4\pi \|\boldsymbol{t} - \boldsymbol{t}'\|_1 \operatorname{h}(\boldsymbol{\xi}). \tag{3.18}$$

PROOF. Let us write $S = \{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_D\}$, with $D = \#S$.

As we saw in (3.14), if we write $\boldsymbol{\xi}_j = e^{\boldsymbol{u}_j} e^{2\pi i \boldsymbol{\theta}_j}$ for some $\boldsymbol{u}_j \in \mathbb{R}^N$ and some $\boldsymbol{\theta}_j \in (\mathbb{R}/\mathbb{Z})^N$, then

$$\widehat{\mu}_S(\boldsymbol{n}, \boldsymbol{t}) = \frac{1}{D} \sum_{j=1}^{D} e^{-2\pi i \boldsymbol{t} \cdot \boldsymbol{u}_j} e^{-2\pi i \boldsymbol{n} \cdot \boldsymbol{\theta}_j}.$$

Therefore, we can write

$$\begin{aligned}
|\widehat{\mu}_S(\boldsymbol{n}, \boldsymbol{t}) - \widehat{\mu}_S(\boldsymbol{n}, \boldsymbol{t}')| &= \left| \frac{1}{D} \sum_{j=1}^{D} \left(e^{-2\pi i \boldsymbol{u}_j \cdot \boldsymbol{t}} e^{-2\pi i \boldsymbol{\theta}_j \cdot \boldsymbol{n}} - e^{-2\pi i \boldsymbol{u}_j \cdot \boldsymbol{t}'} e^{-2\pi i \boldsymbol{\theta}_j \cdot \boldsymbol{n}} \right) \right| \\
\leq \frac{1}{D} \sum_{j=1}^{D} \left| e^{-2\pi i \boldsymbol{u}_j \cdot \boldsymbol{t}} - e^{-2\pi i \boldsymbol{u}_j \cdot \boldsymbol{t}'} \right| &= \frac{1}{D} \sum_{j=1}^{D} \left| -2\pi i \int_{\boldsymbol{u}_j \cdot \boldsymbol{t}'}^{\boldsymbol{u}_j \cdot \boldsymbol{t}} e^{-2\pi i x} dx \right| \\
\leq \frac{2\pi}{D} \sum_{j=1}^{D} \left| \boldsymbol{u}_j \cdot \boldsymbol{t} - \boldsymbol{u}_j \cdot \boldsymbol{t}' \right| &\leq \frac{2\pi}{D} \sum_{j=1}^{D} \sum_{l=1}^{N} |u_{j,l}| |t_l - t_l'| \\
\leq 2\pi \|\boldsymbol{t} - \boldsymbol{t}'\|_1 \frac{1}{D} \sum_{j=1}^{D} \sum_{l=1}^{N} |\log|\boldsymbol{\xi}_{j,l}|| &\leq 4\pi \|\boldsymbol{t} - \boldsymbol{t}'\|_1 \operatorname{h}(\boldsymbol{\xi}),
\end{aligned}$$

where the last inequality is given by Lemma 3.16. □

PROPOSITION 3.28. *There is a constant $C \approx 48.9897$ such that for every $\boldsymbol{\xi} \in (\overline{\mathbb{Q}}^{\times})^N$, every $0 < \delta < 1$ and every $\boldsymbol{n} \neq \boldsymbol{0}$ the following holds*

$$\begin{aligned}
&|\widehat{\mu}_S(\boldsymbol{n}, \boldsymbol{0})| \\
&\quad \leq \|\boldsymbol{n}\|_1 \left(\frac{-2}{\log \delta} \operatorname{h}(\boldsymbol{\xi}) + \frac{4\sqrt{2}(\delta^2 + 9)}{\delta^3} \left(4 \operatorname{h}(\boldsymbol{\xi}) + C \frac{\log(\mathscr{D}(\boldsymbol{\xi}) + 1)}{\mathscr{D}(\boldsymbol{\xi})} \right)^{\frac{1}{2}} \right),
\end{aligned}$$

where S is the Galois orbit of $\boldsymbol{\xi}$ and μ_S the discrete probability measure associated to it.

PROOF. Let D be the cardinality of S and set $S = \{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_D\}$, where

$$\boldsymbol{\xi}_j = e^{\boldsymbol{u}_j} e^{2\pi i \boldsymbol{\theta}_j},$$

for some $\boldsymbol{u}_j \in \mathbb{R}^N$ and $\boldsymbol{\theta}_j \in (\mathbb{R}/\mathbb{Z})^N$.

Observe that, for any $\boldsymbol{n} \in \mathbb{Z}^N$ such that $\boldsymbol{n} \neq \boldsymbol{0}$, we have

$$\widehat{\mu}_S(\boldsymbol{n}, \boldsymbol{0}) = \frac{1}{D} \sum_{j=1}^{D} e^{-2\pi i \boldsymbol{n} \cdot \boldsymbol{\theta}_j} = \frac{1}{D} \sum_{j=1}^{D} \frac{\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)}{|\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)|}.$$

For $0 < \delta < 1$, let $f_\delta : \mathbb{P}^1(\mathbb{C}) \longrightarrow \mathbb{C}$ be the $\mathscr{C}^1$-function defined in Section 5 and write $f_\delta = u_\delta + i v_\delta$. It was there proved that

$$\mathrm{Lip}(u_\delta), \mathrm{Lip}(v_\delta) \leq \frac{2\sqrt{2}(\delta^2 + 9)}{\delta^3}.$$

Then, for $\boldsymbol{n} \neq \boldsymbol{0}$, we have that

$$\begin{aligned} &\left| \widehat{\mu}_S(\boldsymbol{n}, \boldsymbol{0}) - \frac{1}{D} \sum_{j=1}^{D} f_\delta(1 : \chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)) \right| \\ &\quad = \left| \frac{1}{D} \sum_{j=1}^{D} \frac{\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)}{|\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)|} - \frac{1}{D} \sum_{j=1}^{D} \rho_\delta(|\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)|) \frac{\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)}{|\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)|} \right| \\ &\quad = \left| \frac{1}{D} \sum_{j=1}^{N} \frac{\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)}{|\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)|} \left(1 - \rho_\delta(|\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)|)\right) \right| \leq \frac{1}{D} \sum_{j=1}^{D} \left|1 - \rho_\delta(|\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)|)\right| \end{aligned} \tag{3.19}$$

For $\boldsymbol{n} \in \mathbb{Z}^N \setminus \{\boldsymbol{0}\}$ and $0 < \delta < 1$, define the set

$$J_{\boldsymbol{n},\delta} = \left\{ j : \delta \leq |\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)| \leq \frac{1}{\delta} \right\}.$$

If $j \in J_{\boldsymbol{n},\delta}$, then by definition of the function ρ_δ we have $\rho_\delta(|\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)|) = 1$. If $j \notin J_{\boldsymbol{n},\delta}$, then we have $0 \leq \rho_\delta(|\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)|) < 1$. Hence,

$$\frac{1}{D} \sum_{j=1}^{D} \left|1 - \rho_\delta(|\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)|)\right| = \frac{1}{D} \sum_{j \notin J_{\boldsymbol{n},\delta}} 1 - \rho_\delta(|\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)|) \leq \frac{1}{D} \sum_{j \notin J_{\boldsymbol{n},\delta}} 1. \tag{3.20}$$

Let $S_{\boldsymbol{n}}$ be the Galois orbit in $\overline{\mathbb{Q}}^\times$ of $\chi^{\boldsymbol{n}}(\boldsymbol{\xi})$, of cardinality $\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))$. By Lemma 3.14, we know that there is an integer $l_{\boldsymbol{n}}$ such that $\# S_{\boldsymbol{n}} l_{\boldsymbol{n}} = D$. If we set $S_{\boldsymbol{n},\delta} := \left\{ \alpha \in S_{\boldsymbol{n}} : |\log|\alpha|| > \log \frac{1}{\delta} \right\}$, we have

$$\frac{1}{D} \sum_{j \notin J_{\boldsymbol{n},\delta}} 1 = \frac{1}{\# S_{\boldsymbol{n}}} \sum_{\alpha \in S_{\boldsymbol{n},\delta}} 1 \leq 2 \left(\log \frac{1}{\delta} \right)^{-1} \mathrm{h}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})), \tag{3.21}$$

where the last inequality is given by Lemma 3.17.

As we saw in the proof of Lemma 3.18, for $\boldsymbol{n} \neq \boldsymbol{0}$ we have

$$\mathrm{h}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) \leq \|\boldsymbol{n}\|_1 \, \mathrm{h}(\boldsymbol{\xi}).$$

Hence, putting this together with (3.19), (3.20) and (3.21), we obtain

$$\left| \widehat{\mu}_S(\boldsymbol{n}, \boldsymbol{0}) - \frac{1}{D} \sum_{j=1}^{D} f_\delta(1 : \chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)) \right| \leq 2 \left(\log \frac{1}{\delta} \right)^{-1} \|\boldsymbol{n}\|_1 \, \mathrm{h}(\boldsymbol{\xi}). \tag{3.22}$$

On the other hand, we have

$$\frac{1}{D}\sum_{j=1}^{D} f_\delta(1:\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)) = \frac{1}{l_{\boldsymbol{n}}\#S_{\boldsymbol{n}}}\sum_{\alpha\in S_{\boldsymbol{n}}} l_{\boldsymbol{n}} f_\delta(1:\alpha) = \int_{\mathbb{P}^1(\mathbb{C})} f_\delta d\mu_{S_{\boldsymbol{n}}}, \tag{3.23}$$

where $\mu_{S_{\boldsymbol{n}}}$ is the discrete probability measure on $\mathbb{P}^1(\mathbb{C})$ associated to the Galois orbit $S_{\boldsymbol{n}}$.

Let λ_{S^1} be the measure on $\mathbb{C}^\times$ supported on the unit circle, where it coincides with the Haar probability measure. By the definition of the function f_δ, we have that $f_\delta(1:z)=z$ if $|z|=1$ and thus, we have

$$\int_{\mathbb{P}^1(\mathbb{C})} f_\delta d\lambda_{S^1} = \int_{\mathbb{C}^\times} z d\lambda_{S^1}(z) = 0.$$

By Theorem II, there is a constant $C_0 \approx 14.7628$ such that

$$\begin{aligned}
&\left|\int_{\mathbb{P}^1(\mathbb{C})} f_\delta d\mu_{S_{\boldsymbol{n}}}\right| = \left|\int_{\mathbb{P}^1(\mathbb{C})} f_\delta d\mu_{S_{\boldsymbol{n}}} - \int_{\mathbb{P}^1(\mathbb{C})} f_\delta d\lambda_{S^1}\right| \\
&\quad\le \left|\int_{\mathbb{P}^1(\mathbb{C})} u_\delta d\mu_{T_{\boldsymbol{n}}} - \int_{\mathbb{P}^1(\mathbb{C})} u_\delta d\lambda_{S^1}\right| + \left|\int_{\mathbb{P}^1(\mathbb{C})} v_\delta d\mu_{T_{\boldsymbol{n}}} - \int_{\mathbb{P}^1(\mathbb{C})} v_\delta d\lambda_{S^1}\right| \\
&\le (\mathrm{Lip}(u_\delta)+\mathrm{Lip}(v_\delta))\left(\frac{\pi}{\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))} + \left(4\,\mathrm{h}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) + C_0\frac{\log(\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))+1)}{\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))}\right)^{\frac{1}{2}}\right) \\
&\le \frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\left(\frac{\pi}{\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))} + \left(4\,\mathrm{h}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) + C_0\frac{\log(\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))+1)}{\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))}\right)^{\frac{1}{2}}\right).
\end{aligned} \tag{3.24}$$

Since $\mathrm{h}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) \le \|\boldsymbol{n}\|_1\,\mathrm{h}(\boldsymbol{\xi})$ for every $\boldsymbol{n}\in\mathbb{Z}^n\setminus\{\boldsymbol{0}\}$, we can write

$$\begin{aligned}
&\left(4\,\mathrm{h}(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) + C_0\frac{\log(\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))+1)}{\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))}\right)^{\frac{1}{2}} \\
&\qquad\le \left(4\|\boldsymbol{n}\|_1\,\mathrm{h}(\boldsymbol{\xi}) + C_0\frac{\|\boldsymbol{n}\|_1\log(\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))+1)}{\|\boldsymbol{n}\|_1\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))}\right)^{\frac{1}{2}} \\
&\qquad\qquad\le \|\boldsymbol{n}\|_1\left(4\,\mathrm{h}(\boldsymbol{\xi}) + C_0\frac{\log(\|\boldsymbol{n}\|_1\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))+1)}{\|\boldsymbol{n}\|_1\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))}\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence, this fact together with (3.23) and (3.24)

$$\begin{aligned}
&\left|\frac{1}{D}\sum_{j=1}^{D} f_\delta(1:\chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j))\right| \\
&\le \frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\|\boldsymbol{n}\|_1\left(\frac{\pi}{\|\boldsymbol{n}\|_1\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))} + \left(4\,\mathrm{h}(\boldsymbol{\xi}) + C_0\frac{\log\left(\|\boldsymbol{n}\|_1\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))+1\right)}{\|\boldsymbol{n}\|_1\deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))}\right)^{\frac{1}{2}}\right).
\end{aligned} \tag{3.25}$$

We have $\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) \geq 1$ for every $\boldsymbol{n} \neq \boldsymbol{0}$ and, since we are assuming that $\mathrm{h}(\boldsymbol{\xi}) < 1$, there is a constant $C > 0$ such that

$$\left| \frac{1}{D} \sum_{j=1}^{D} f_\delta(1 : \chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)) \right| \leq \frac{4\sqrt{2}(\delta^2+9)}{\delta^3} \|\boldsymbol{n}\|_1 \left(4\,\mathrm{h}(\boldsymbol{\xi}) + C \frac{\log(\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) + 1)}{\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))} \right)^{\frac{1}{2}}. \tag{3.26}$$

As it was done in the proof of Theorem II, we can take

$$C = \frac{\pi^2 + C_0 \log 2 + 2\pi\sqrt{4 + C_0 \log 2}}{\log 2} \approx 48.9897.$$

Finally, note that the function $\frac{\log(x+1)}{x}$ defined for $x \geq 1$ is monotonically decreasing on its domain and hence, we deduce that for every $\boldsymbol{n} \neq \boldsymbol{0}$

$$\frac{\log(\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi})) + 1)}{\|\boldsymbol{n}\|_1 \deg(\chi^{\boldsymbol{n}}(\boldsymbol{\xi}))} \leq \frac{\log(\mathscr{D}(\boldsymbol{\xi}) + 1)}{\mathscr{D}(\boldsymbol{\xi})}.$$

This together with (3.26) implies that

$$\left| \frac{1}{D} \sum_{j=1}^{D} f_\delta(1 : \chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)) \right| \leq \frac{4\sqrt{2}(\delta^2+9)}{\delta^3} \|\boldsymbol{n}\|_1 \left(4\,\mathrm{h}(\boldsymbol{\xi}) + C \frac{\log(\mathscr{D}(\boldsymbol{\xi}) + 1)}{\mathscr{D}(\boldsymbol{\xi})} \right)^{\frac{1}{2}},$$

for every $\boldsymbol{n} \neq \boldsymbol{0}$.

We can then finish the proof of the proposition:

$$\begin{aligned} |\widehat{\mu}_S(\boldsymbol{n}, \boldsymbol{0})| &\leq \left| \widehat{\mu}_S(\boldsymbol{n}, \boldsymbol{0}) - \frac{1}{D} \sum_{j=1}^{D} f_\delta(1 : \chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)) \right| + \left| \frac{1}{D} \sum_{j=1}^{D} f_\delta(1 : \chi^{\boldsymbol{n}}(\boldsymbol{\xi}_j)) \right| \\ &\leq \frac{-2}{\log \delta} \|\boldsymbol{n}\|_1 \mathrm{h}(\boldsymbol{\xi}) + \frac{4\sqrt{2}(\delta^2+9)}{\delta^3} \|\boldsymbol{n}\|_1 \left(4\,\mathrm{h}(\boldsymbol{\xi}) + C \frac{\log(\mathscr{D}(\boldsymbol{\xi}) + 1)}{\mathscr{D}(\boldsymbol{\xi})} \right)^{\frac{1}{2}}. \end{aligned}$$

□

Corollary 3.29. *There is a constant $C \approx 48.9897$ such that for every $\boldsymbol{\xi} \in (\overline{\mathbb{Q}}^\times)^N$ with $\mathrm{h}(\boldsymbol{\xi}) < 1$, every $0 < \delta < 1$ and every $(\boldsymbol{n}, \boldsymbol{t}) \in \mathbb{Z}^N \times \mathbb{R}^N$ with $\boldsymbol{n} \neq \boldsymbol{0}$, the following holds*

$$\begin{aligned} |\widehat{\mu}_S(\boldsymbol{n}, \boldsymbol{t})| \leq 4\pi \|\boldsymbol{t}\|_1 \mathrm{h}(\boldsymbol{\xi}) &+ \frac{-2}{\log \delta} \|\boldsymbol{n}\|_1 \mathrm{h}(\boldsymbol{\xi}) \\ &+ \frac{4\sqrt{2}(\delta^2+9)}{\delta^3} \|\boldsymbol{n}\|_1 \left(4\,\mathrm{h}(\boldsymbol{\xi}) + C \frac{\log(\mathscr{D}(\boldsymbol{\xi}) + 1)}{\mathscr{D}(\boldsymbol{\xi})} \right)^{\frac{1}{2}}, \end{aligned}$$

where S is the Galois orbit of $\boldsymbol{\xi}$ and μ_S the discrete probability measure associated to it.

PROOF. Note that, for any $(\boldsymbol{n},\boldsymbol{t})\in\mathbb{Z}^N\times\mathbb{R}^N$

$$|\widehat{\mu}_S(\boldsymbol{n},\boldsymbol{t})|\le|\widehat{\mu}_S(\boldsymbol{n},\boldsymbol{t})-\widehat{\mu}_S(\boldsymbol{n},\boldsymbol{0})|+|\widehat{\mu}_S(\boldsymbol{n},\boldsymbol{0})|.$$

So the result follows directly from Lemma 3.27 and Proposition 3.28. □

PROOF OF PROPOSITION 3.26. Let $(\boldsymbol{n},\boldsymbol{t})\in\mathbb{Z}^N\times\mathbb{R}^N$ be such that $\boldsymbol{n}\neq\boldsymbol{0}$. Corollary 3.29, together with the fact that we are assuming $\mathrm{h}(\boldsymbol{\xi})<1$, implies that there is a constant $C\approx 48.9897$ with

$$\begin{aligned}|\widehat{\mu}_S(\boldsymbol{n},\boldsymbol{t})|&\le 4\pi\|\boldsymbol{t}\|_1\,\mathrm{h}(\boldsymbol{\xi})\\&\quad+\left(\frac{-2}{\log\delta}+\frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\right)\|\boldsymbol{n}\|_1\left(4\,\mathrm{h}(\boldsymbol{\xi})+C\frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}}.\end{aligned}$$

By Theorem 3.5, given $f\in\mathcal{F}$, we know that the function $F=f\circ\phi$ is Haar-integrable as well as its Fourier transform $\widehat{F}$. Hence, we can write

$$\begin{aligned}&\left|\sum_{\boldsymbol{n}\neq\boldsymbol{0}}\int_{\mathbb{R}^N}\widehat{F}(\boldsymbol{n},\boldsymbol{t})\overline{\widehat{\mu}_S(\boldsymbol{n},\boldsymbol{t})}d\boldsymbol{t}\right|\\&\quad\le\sum_{\boldsymbol{n}\neq\boldsymbol{0}}\int_{\mathbb{R}^N}|\widehat{F}(\boldsymbol{n},\boldsymbol{t})||\widehat{\mu}_S(\boldsymbol{n},\boldsymbol{t})|d\boldsymbol{t}\le 4\pi\,\mathrm{h}(\boldsymbol{\xi})\sum_{\boldsymbol{n}\neq\boldsymbol{0}}\int_{\mathbb{R}^N}|\widehat{F}(\boldsymbol{n},\boldsymbol{t})|\|\boldsymbol{t}\|_1d\boldsymbol{t}\\&+\left(\frac{-2}{\log\delta}+\frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\right)\left(4\,\mathrm{h}(\boldsymbol{\xi})+C\frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}}\sum_{\boldsymbol{n}\neq\boldsymbol{0}}\int_{\mathbb{R}^N}|\widehat{F}(\boldsymbol{n},\boldsymbol{t})|\|\boldsymbol{n}\|_1d\boldsymbol{t}.\end{aligned}\tag{3.27}$$

By Lemma 3.11, for every $l=1,\dots,N$ we have that

$$\widehat{\frac{\partial F}{\partial u_l}}(\boldsymbol{n},\boldsymbol{t})=2\pi it_l\widehat{F}(\boldsymbol{n},\boldsymbol{t})\text{ and }\widehat{\frac{\partial F}{\partial\theta_l}}(\boldsymbol{n},\boldsymbol{t})=2\pi in_l\widehat{F}(\boldsymbol{n},\boldsymbol{t}).$$

Using this, we obtain

$$\begin{aligned}\sum_{\boldsymbol{n}\neq\boldsymbol{0}}\int_{\mathbb{R}^N}|\widehat{F}(\boldsymbol{n},\boldsymbol{t})|\cdot\|\boldsymbol{t}\|_1d\boldsymbol{t}&=\frac{1}{2\pi}\sum_{l=1}^N\sum_{\boldsymbol{n}\neq\boldsymbol{0}}\int_{\mathbb{R}^N}|\widehat{F}(\boldsymbol{n},\boldsymbol{t})|\cdot|2\pi t_l|d\boldsymbol{t}\\&\le\frac{1}{2\pi}\sum_{l=1}^N\sum_{\boldsymbol{n}\in\mathbb{Z}^N}\int_{\mathbb{R}^N}\left|\widehat{\frac{\partial F}{\partial u_l}}(\boldsymbol{n},\boldsymbol{t})\right|d\boldsymbol{t}=\frac{1}{2\pi}\sum_{l=1}^N\left\|\widehat{\frac{\partial F}{\partial u_l}}\right\|_{\mathrm{L}^1}\end{aligned}$$

and

$$\begin{aligned}\sum_{\boldsymbol{n}\neq\boldsymbol{0}}\int_{\mathbb{R}^N}|\widehat{F}(\boldsymbol{n},\boldsymbol{t})|\cdot\|\boldsymbol{n}\|_1d\boldsymbol{t}&=\frac{1}{2\pi}\sum_{l=1}^N\sum_{\boldsymbol{n}\neq\boldsymbol{0}}\int_{\mathbb{R}^N}|\widehat{F}(\boldsymbol{n},\boldsymbol{t})|\cdot|2\pi n_l|d\boldsymbol{t}\\&\le\frac{1}{2\pi}\sum_{l=1}^N\sum_{\boldsymbol{n}\in\mathbb{Z}^N}\int_{\mathbb{R}^N}\left|\widehat{\frac{\partial F}{\partial\theta_l}}(\boldsymbol{n},\boldsymbol{t})\right|d\boldsymbol{t}=\frac{1}{2\pi}\sum_{l=1}^N\left\|\widehat{\frac{\partial F}{\partial\theta_l}}\right\|_{\mathrm{L}^1}.\end{aligned}$$

Finally, by (3.27), we can conclude

$$\left|\sum_{\boldsymbol{n}\neq\boldsymbol{0}}\int_{\mathbb{R}^N}\widehat{F}(\boldsymbol{n},\boldsymbol{t})\overline{\widehat{\mu}_S(\boldsymbol{n},\boldsymbol{t})}d\boldsymbol{t}\right| \leq 2\,\mathrm{h}(\boldsymbol{\xi})\sum_{l=1}^{N}\left\|\widehat{\frac{\partial F}{\partial u_l}}\right\|_{\mathrm{L}^1}$$
$$+\frac{1}{2\pi}\left(\frac{-2}{\log\delta}+\frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\right)\left(4\,\mathrm{h}(\boldsymbol{\xi})+C\frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}}\sum_{l=1}^{N}\left\|\widehat{\frac{\partial F}{\partial \theta_l}}\right\|_{\mathrm{L}^1}.$$

□

Proof of Theorem I. Let $f\in\mathcal{F}$ and set $F=f\circ\phi$. The measures μ_S and $\lambda_{(S^1)^N}$ are measures in $\mathbb{P}^1(\mathbb{C})^N$ and they are compactly supported on $(\mathbb{C}^\times)^N\hookrightarrow\mathbb{P}^1(\mathbb{C})^N$. Therefore, we can write

$$\left|\int_{\mathbb{P}^1(\mathbb{C})^N} f d\mu_S-\int_{\mathbb{P}^1(\mathbb{C})^N} f d\lambda_{(S^1)^N}\right|=\left|\int_{(\mathbb{C}^\times)^N} F d\mu_S-\int_{(\mathbb{C}^\times)^N} F d\lambda_{(S^1)^N}\right|.$$

By Theorem 3.5, the function F and its Fourier transform $\widehat{F}$ are Haar-integrable and thus, as we already saw in equation (3.12), we have

$$\left|\int_{(\mathbb{C}^\times)^N} F d\mu_S-\int_{(\mathbb{C}^\times)^N} F d\lambda_{(S^1)^N}\right|$$
$$\leq\left|\int_{\mathbb{R}^N}\widehat{F}(\boldsymbol{0},\boldsymbol{t})\left(\overline{\widehat{\mu}_S(\boldsymbol{0},\boldsymbol{t})}-1\right)d\boldsymbol{t}\right|+\left|\sum_{\boldsymbol{n}\neq\boldsymbol{0}}\int_{\mathbb{R}^N}\widehat{F}(\boldsymbol{n},\boldsymbol{t})\overline{\widehat{\mu}_S(\boldsymbol{n},\boldsymbol{t})}d\boldsymbol{t}\right|$$

Since any test function $f\in\mathcal{F}$ is Lispchitz, by Propostions 3.25 and 3.26, there is a constant $C\approx 48.9897$ such that

$$\left|\int_{\mathbb{P}^1(\mathbb{C})^N} f d\mu_S-\int_{\mathbb{P}^1(\mathbb{C})^N} f d\lambda_{(S^1)^N}\right|$$
$$\leq\sqrt{2}\pi\,\mathrm{Lip}(f)\,\mathrm{h}(\boldsymbol{\xi})+2\,\mathrm{h}(\boldsymbol{\xi})\sum_{l=1}^{N}\left\|\widehat{\frac{\partial F}{\partial u_l}}\right\|_{\mathrm{L}^1}$$
$$+\frac{1}{2\pi}\left(\frac{-2}{\log\delta}+\frac{4\sqrt{2}(\delta^2+9)}{\delta^3}\right)\left(4\,\mathrm{h}(\boldsymbol{\xi})+C\frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}}\sum_{l=1}^{N}\left\|\widehat{\frac{\partial F}{\partial \theta_l}}\right\|_{\mathrm{L}^1}.$$

We search numerically for the minimum of the function $\frac{-2}{\log\delta}+\frac{4\sqrt{2}(\delta^2+9)}{\delta^3}$, for $0<\delta<1$, and we obtain 94.9591, attained at $\delta\approx 0.9071$. Hence, we have

$$\left|\int_{\mathbb{P}^1(\mathbb{C})^N} f d\mu_S - \int_{\mathbb{P}^1(\mathbb{C})^N} f d\lambda_{(S^1)^N}\right| \leq \sqrt{2}\pi \operatorname{Lip}(f)\operatorname{h}(\boldsymbol{\xi})$$
$$+ 2\operatorname{h}(\boldsymbol{\xi}) \sum_{l=1}^{N} \left\|\widehat{\frac{\partial F}{\partial u_l}}\right\|_{\mathrm{L}^1} + 16 \left(4\operatorname{h}(\boldsymbol{\xi}) + C\frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}} \sum_{l=1}^{N} \left\|\widehat{\frac{\partial F}{\partial \theta_l}}\right\|_{\mathrm{L}^1}$$
$$\leq \left(\sqrt{2}\pi \operatorname{Lip}(f) + 2\sum_{l=1}^{N} \left\|\widehat{\frac{\partial F}{\partial u_l}}\right\|_{\mathrm{L}^1} + 16\sum_{l=1}^{N} \left\|\widehat{\frac{\partial F}{\partial \theta_l}}\right\|_{\mathrm{L}^1}\right) \left(4\operatorname{h}(\boldsymbol{\xi}) + C\frac{\log(\mathscr{D}(\boldsymbol{\xi})+1)}{\mathscr{D}(\boldsymbol{\xi})}\right)^{\frac{1}{2}}.$$

□

Finally, let us deduce from this result Bilu's equidistribution Theorem.

Proof of Corollary 3.1. Let S_k be the Galois orbit of $\boldsymbol{\xi}_k$ for every $k \geq 0$. By Theorem I and Lemma 3.18, for every $f \in \mathcal{F}$ we have

$$\left|\int_{\mathbb{P}^1(\mathbb{C})^N} f d\mu_{S_k} - \int_{\mathbb{P}^1(\mathbb{C})^N} f d\lambda_{(S^1)^N}\right|$$
$$\leq c(f)\left(4\operatorname{h}(\boldsymbol{\xi}_k) + C\frac{\log(\mathscr{D}(\boldsymbol{\xi}_k)+1)}{\mathscr{D}(\boldsymbol{\xi}_k)}\right)^{\frac{1}{2}} \xrightarrow[k\to\infty]{} 0.$$

Since, by Lemma 3.13, for every continuous compactly supported function $F : (\mathbb{C}^\times)^N \to \mathbb{R}$ there is a sequence in $\mathcal{F}$ whose uniform limit is F, the Corollary follows. □

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