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Essays on assignment markets: Vickrey outcome and Walrasian Equilibrium

Francisco Javier Robles Jiménez

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Amparo y Francisco, gracias por el molino...
Karina y Edgar, gracias por el viento...
gracias totales.

“A bird flying about in a wood does not bump into the branches,
whereas in a room it will bump into the glass of the window.
This entitles us to say that the bird perceives the brunches but not the glass.
Do we ‘perceive’ the glass or do we merely know that it is there?”

Bertrand Russell
An outline of philosophy

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Contents

1 Introduction

This dissertation is framed in the study of the assignment problem involved in two-sided markets. A great variety of situations could be actually considered as an assignment problem. As a consequence, the term *assignment problem* seems to be vague. The initial attempt of this introduction is to illustrate what kind of situations we are going to consider, for our purposes, as an assignment problem. This will be followed by a review of the literature on assignment problems and an overview about the concerns of this Thesis.

As its name suggests, a two-sided market consists of a market with two sides or sectors. However, it is necessary to make clear that in these markets each agent belongs to one and only one side. Real life interactions provide valuable examples of two-sided markets: students must be placed in schools, doctors must be assigned to hospitals, buyers must find a house in the real estate market, agents must bid for some paintings in an auction. It can be hardly denied the importance of the study of these markets. On one hand, such a study gives insights for the understanding of more complex economic situations. On the other hand, many day transactions are, in fact, bilateral (Shapley & Shubik, 1972). Two essential features of the mentioned examples are the presence of indivisibilities and that money may or may not be involved in the possible allocations. The presence of indivisibilities shapes a central question, *what is the best possible way to make allocations in the market?* Clearly, the answer to this question is not unique since it depends on the properties we would like to be satisfied. Suppose that we can choose always an allocation such that there is no other alternative allocation available that makes some agent better off and no one worse off. This is precisely the notion of *(Pareto) efficiency*. As a starting point, efficiency seems to be an appealing property to choose an assignment. As Mas-Colell *et al.* (1995) points out, efficiency serves as an important minimal test for the desirability of an allocation. Hence we may consider here the assignment problem as the choice of an *efficient* allocation of resources to agents. Moreover, we will focus on those markets in which money, or a perfectly divisible good, is fully transferable.

The seminal paper Gale & Shapley (1962) presents the *marriage market* to study two-sided markets with indivisibilities and without money. In the marriage market, each agent is endowed only with a preference over agents on the other sector and each agent is interested in making at most one partnership. Gale and Shapley introduced *stability* as a solution concept. In a general way, we can describe the notion of stability as follows. A matching is stable if: (i) every agent will not be better off by leaving her/his current situation in order to become unmatched, and; (ii) there is no unmatched pair of agents who both would be better off by breaking their partnerships to be matched together. Gale & Shapley (1962) shows the existence of *stable matchings* with a simple and intuitive algorithm, the *deferred acceptance algorithm*.[1] This paper laid the foundations of what now is called in economics *matching theory*.[2]

Ten years later, Shapley & Shubik (1972) introduces a variant of the marriage market, the *assignment game*.[3] In this paper, the authors consider a buyer-seller market in which each agent desires to make at most one partnership. Unlike the marriage market, utility can be transferred by means of a perfectly divisible object, *money*.[4] It is assumed that agents' preferences are *quasi-linear with respect to* (w.r.t.) money. This assumption constitutes an advantageous starting point in the analysis of markets with transferable utility. One consequence of this assumption is that efficient assignments are found by solving the *linear program* that maximizes the value (in money) of the assignment. Stability is now a property of the payoff vector (in terms of money) that distributes the value attained by a matching. In an elegant way, existence of stable payoffs is proved by means of the *duality theory* of linear programming.[5] The marriage market and its counterpart with transferable utility, the assignment game, were both breaking points in the study of two-sided markets without and with money, respectively.

After the assignment game, some papers explored more general markets. Some of them kept bilateral exchanges but with more complex agents' preferences. The first generalization was made by Kaneko (1976). In this model, it is assumed that each seller could have more than one object on sale while each buyer is interested in at most one. In later works, mainly from the 1980s, it can be identified a research line focused on the following concerns. First, existence and properties of

[1] The deferred acceptance is considered as a successful story of the application of economic theory to real life problems (Roth, 2002, 2008). It has been applied to the student placement in New York and in Boston (Abdulkadiroğlu *et al.*, 2005b,a). Kojima & Manea (2010) offers two characterizations of the deferred acceptance algorithm.

[2] Roth & Sotomayor (1990) is definitely a reference for matching theory. Sönmez & Ünver (2011) is a detailed survey.

[3] Núñez & Rafels (2015) is a complete and recent survey on the assignment game and its extensions.

[4] For a discussion of the role played by money as a mean of payment, we refer to Shapley & Shubik (1977).

[5] Gale (1960) provides a wide treatment of linear programming and duality applied to economic systems.

several notions of stability are analysed. Once the existence of stable outcomes is guaranteed, many works tackle the following question, *how a stable outcome can be reached?* Assuming that we can apply a procedure to do so, it is natural to require agents' preferences as an input. In the case of quasi-linear preferences and unitary demands, it is sufficient *to know* the valuations of the agents over the objects. Suppose that we simply ask the agents to report their valuations to be able to propose a stable outcome. *Should we be concerned about misrepresentations of the true valuations?* It is a fact that the agents may attempt to manipulate the outcome by lying about their valuations. Therefore, in a decentralized information context, we must refine our previous question to *what game or procedure does produce a stable outcome and avoid incentives for agents to lie about their true valuations?* That is to say, we have to look for a game such that truth telling is an *equilibrium* of the game and every equilibrium leads to the same stable outcome. For the classical assignment game, these questions were addressed, for instance by Demange (1982) and Leonard (1983). A complementary approach to strengthen our understanding of the strategic behaviour of agents in the assignment game is given by Demange *et al.* (1986) and Pérez-Castrillo & Sotomayor (2002). Demange *et al.* (1986) introduces an ascending *multi-item auction* which leads to the best stable outcome for the buyers. Instead, Pérez-Castrillo & Sotomayor (2002) provides a *simple buying-selling procedure* which *implements in subgame perfect equilibrium* the best stable outcome for the sellers.

Some other works have relaxed some assumptions of the classical assignment model. Demange & Gale (1985) extends the assignment game by considering a domain of preferences that includes quasi-linear preferences. The authors show that some interesting properties of the classical model also hold in this more complex market. For a market in which agents on one side can make many partnerships while agents on the other sector can make at most one, we refer to Crawford & Knoer (1981) and Kelso & Crawford (1982).

Recently, in more general two-sided markets, the relationship of assignment problems, linear programming, auctions and substitutability has been investigated (Hatfield & Milgrom, 2005; Milgrom, 2009), see Milgrom (2007) for a discussion. All these works have strengthened the knowledge for a better market design.

The aim of this Thesis is to provide a contribution to the study, from a game theoretical approach, of the assignment problem involved in two-sided markets with money. This Thesis is divided in three chapters corresponding to three main parts. The purpose of this division is to consider different but complementary perspectives in the research agenda of assignment problems. In the following paragraphs, we are going to motivate and overview the three chapters.

The second chapter considers the following situation. There is only one seller in the market. This seller owns many indivisible objects on sale which are of the same type but heterogeneous, *e.g.* houses, cars, pieces of art. On the other sector of the market, many buyers meet and each one desires to acquire a certain number of objects. Preferences are quasi-linear w.r.t. money and each buyer values packages of objects additively up to a given capacity. Camiña (2006) considers a particular case in which each buyer can acquire at most one object. In our model, we explore the relationship between two main concepts used for the analysis of exchange economies: the *core* and the *Walrasian equilibrium*. The first concept, the core, captures a notion of stability in a context where agents may form coalitions and cooperate. The idea is that each agent is well informed about everyone's preferences. A payoff vector belongs to the core of the market if no coalition of agents can make a better agreement for them by trading only among themselves. On the other hand, the Walrasian equilibrium is confined to a more competitive view. It relays on the existence of a price system and an allocation of objects that guarantees that each agent obtains, according to her/his preferences, one of her/his most preferred bundle of objects given the prices. The relationship of these concepts has been widely addressed in different scenarios. In fact, Edgeworth[6] started this line of research as quoted by Mas-Colell *et al.* (1995). In the context of cooperative games, Mas-Colell (1988) provides an interpretation of the core and its relationship with the Walrasian equilibrium. For two-sided markets with indivisibilities, Shapley & Shubik (1972) shows that the *core equivalence theorem* holds, *i.e.* the core and the set of payoff vectors associated with Walrasian equilibria coincide. However, when assignment markets become more complex, even allowing only one agent to make more than one partnership, this coincidence is lost (Camiña, 2006; Sotomayor, 2007). Recently, Massó & Neme (2014) establishes that for two-sided markets where multiple partnership is allowed, the coincidence is obtained as a limit result for the replicated market.

The aim of Chapter 2 is to determine under which conditions some particular core elements are supported by Walrasian equilibria. That is to say, when *coalitionally rational agreements* can be coordinated by means of competitive prices. In fact, in this chapter we determine under which assumptions on buyers' valuations, the core equivalence theorem holds.

In Chapter 3, an implementation problem is considered. Similar to the previous part, in this chapter we consider a situation in which there is only one seller owning many heterogeneous objects on sale. Again, there are many buyers, each of them

[6]Edgeworth (1881).

desires to acquire many objects. Preferences are assumed to be quasi-linear w.r.t. money and buyers' valuations satisfy the gross-substitutes condition.[7] In an informal way, we say that objects satisfy the gross-substitutes condition if the following holds. Suppose that given a price vector p, an agent wants to acquire the package of objects R. Assume that some prices increase and now we have a new vector of prices $p' \geq p$. Then this agent wants to acquire at least a package R' at p' and R' contains each object belonging to R which price has not been increased in p'.

The aim of Chapter 3 is to provide a mechanism, that is a non-cooperative game, to assign indivisibilities when the gross-substitutes condition is satisfied. Without question, auctions are a powerful mechanism used in practice to allocate objects (Milgrom, 2004). In most common auctions, the auctioneer does not play any strategic role when the auction is carried out. Buyers (or bidders) are the only agents who play strategically. Nonetheless, in some instances, the seller could take an active role by bargaining with the buyers. Consider the next scenario inspired by the implementation of the core of convex cooperative games proposed by Serrano (1995). In a firm, one of the shareholders decides to sell its share of the assets. This agent decides to bargain the sale of his portfolio to the other shareholders by means of a sequence of offers and counteroffers. Another model which is closely related is the non-cooperative game used in Wilson (1978) for an exchange economy in which one agent centralizes the allocation of the goods.

We are interested in a particular outcome to be implemented in our setting, the so-called *Vickrey* outcome.[8] The Vickrey outcome consists of an efficient allocation of the objects and a payoff vector in which each buyer gets his marginal contribution to the whole market. In the context of gross-substitutes, the Vickrey outcome has appealing properties. It is an efficient outcome and it also belongs to the core of the cooperative game associated to the market (Ausubel & Milgrom, 2002). Even more, the Vickrey outcome gives to every buyer his maximum core payoff.[9] This chapter follows the same line of research of Pérez-Castrillo & Sotomayor (2002). We provide a two-stage mechanism which tries to mimic a simple bidding procedure with a seller's reply. This mechanism implements the Vickrey outcome in *subgame perfect equilibrium*.

The final part of this Thesis is devoted to an axiomatic approach. The assignment problem studied in the last section is the following. There are many objects to be al-

[7] See Kelso & Crawford (1982) and Gul & Stacchetti (1999) for a detailed discussion and applications.

[8] In fact, the mechanisms studied by Vickrey (1961), Clarke (1971) and Groves (1973). See Milgrom (2004) for details.

[9] When there are two or more sellers, in more general assignment games, this outcome may not be a core element (Sotomayor, 2002).

located. We assume that the owners of the objects centralize them in an institution. This institution is designated to distribute them according to some specified criterion. Each agent can receive at most one object and monetary transfers are allowed. Despite the fact that we are in a unitary demand case, we consider a quite general domain of preferences that includes quasi-linear preferences. We assume that each agent has a preference relation over bundles made of an object and some amount of money satisfying the following requirements: *money monotonicity*, agents always prefer to pay less money for each object; *finiteness*, the willingness to pay for each object is finite; *continuity*, for any bundle, its upper and the lower contour sets are both closed; and *weak preference for real objects*, it is weakly preferred to obtain any real object than nothing.

In previous works, it has been shown that the minimum Walrasian equilibrium has attractive features as a rule to distribute resources, see *e.g.* Demange & Gale (1985). It is a *fair*[10] rule and it also avoids incentives to be manipulated from the point of view of buyers. On the domain of quasi-linear preferences and unitary demands, the Vickrey allocation rule satisfies also these properties. However, on the domain we consider, this is not true, since the Vickrey rule can be manipulated (Morimoto & Serizawa, 2015). Hence, in our setting, a rule which always selects for each market the minimum Walrasian equilibrium price vector is a strong candidate to be an appealing allocation rule.

In Chapter 4, we analyse properties of the minimum Walrasian equilibrium rule. We mainly focus on monotonicity properties. The aim of this chapter is to study two-sided markets with money analogously to the study of Kojima & Manea (2010) for two-sided markets without money. Particularly, we investigate properties of the minimum Walrasian equilibrium rule in the next environments: general and quasi-linear preferences. For each of these environments, we provide a characterization of the minimum Walrasian equilibrium rule. It is important to remark that, unlike all previous characterizations of the Vickrey rule or of the minimum Walrasian equilibrium rule (Chew & Serizawa, 2007; Saitoh & Serizawa, 2008; Ashlagi & Serizawa, 2012; Sakai, 2012; Adachi, 2014; Morimoto & Serizawa, 2015), we do not impose any condition on the number of objects or on the number of agents. In particular, we allow for more objects than agents, which is not considered in the previous characterizations.

[10]Here, we say that a rule is fair in the sense of Varian (1974), *i.e.* the rule satisfies (Pareto) efficiency and *envy-freeness*. A rule is envy-free if every agent weakly prefers the bundle he is allocated rather than any other bundle allocated to some other agent.

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2 One-seller assignment markets with multi-unit demands: core and Walrasian equilibrium[§]

2.1 Introduction

In this model, we study markets with several buyers and only one seller. The seller owns many indivisible and potentially different objects on sale. Being heterogeneous, the objects are of the same type, for instance different houses or different tasks. On the other side of the market, each buyer has a non-negative valuation for each object and a desire to acquire a certain number of objects. This number is known as the capacity of the buyer. Since we are thinking of objects such as houses, cars or jobs, it makes sense to assume that a buyer, even if he values all of them positively, is not interested in acquiring more units than those that can be of use to him. We assume that buyers have quasi-linear preferences with respect to money and value packages of objects additively up to a given capacity. Side-payments are allowed. Our aim is to determine under which conditions all core allocations can be priced by means of Walrasian equilibrium prices.

This market is a particular case of the one considered in Jaume *et al.* (2012) and Massó & Neme (2014), where there are several sellers, each with a set of heterogeneous objects on sale. It is also a particular case of the package auction of Ausubel & Milgrom (2002), where there is only one seller, but buyers may not value packages additively. Two-sided markets with one seller have also been considered in Camiña (2006) and Stuart (2007). Two-sided markets with transferable utility are first considered from the viewpoint of coalitional games in the assignment game (Shapley & Shubik, 1972). In their market, buyers want to buy at most one unit and the objects on sale belong to different sellers. The core is non-empty and coincides with the set of Walrasian equilibrium payoff vectors (Gale, 1960). It has a lattice structure with two particular core elements, one of them optimal for all buyers and

[§]A paper based on this chapter has been accepted to be published at International Journal of Economic Theory.

the other one optimal for all sellers.

When the assumptions of the classical assignment model are relaxed, the lattice structure of the core and its coincidence with the set of Walrasian equilibrium payoff vectors do not hold in general. This is the case of many-to-many assignment markets where both buyers and sellers may be willing to trade more than one object and buyers value packages of objects additively up to their given capacity. The core of this game is always non-empty but has no lattice structure and it remains an open problem whether in this setting an optimal core element for each side of the market does exist. However, even under the assumptions that each seller has a set of homogeneous objects on sale, Sotomayor (2002) shows that a worst core element for each side of the market may not exist.

In the more encompassing many-to-many assignment model where each seller has several units of potentially different objects, the set of Walrasian equilibrium payoff vectors is non-empty and is strictly included in the core. However, let us point out that the definition of Walrasian equilibrium in Jaume *et al.* (2012) and Massó & Neme (2014) assumes that buyers demand as many copies of their preferred object as their capacities allow, being the prices given. Instead, we will follow the notion of Walrasian equilibrium used for more general markets in Gul & Stacchetti (1999), and also in Sotomayor (2007) and Arribillaga *et al.* (2014) for many-to-many assignment markets. There, Walrasian equilibrium is defined by means of a demand correspondence in which buyers maximize the utility of the packages they can buy given prices and their capacities. As a consequence, when buyers in a many-to-many assignment market value packages of objects additively, it may be the case that in a demanded package a buyer may obtain different utilities from the different objects that form the package.

In the present chapter, where we have only one seller with heterogeneous goods and multi-unit demands, we study the relationship between the core and the set of Walrasian equilibria. We say that a core allocation is supported by some Walrasian equilibrium if it is the payoff vector associated to some Walrasian equilibrium. Then, this core allocation is said to be supported by that vector of Walrasian equilibrium prices.

We first notice that the valuation functions of buyers in our model are monotone and satisfy the gross-substitutes property (Gul & Stacchetti, 1999). This implies that the characteristic function of the game is buyers-submodular and then, as an immediate consequence of Ausubel & Milgrom (2002), the core has a very simple structure: it is the non-empty set of efficient payoff vectors where each buyer gets a non-negative payoff bounded by his marginal contribution to the whole market. Hence, the core is endowed with a lattice structure by the partial order defined from the point of view of buyers and there exists one optimal core element for each side of the market. Moreover, as in the assignment game, in the buyers-optimal core allocation each buyer is paid his marginal contribution, that is, the Vickrey payoff (Vickrey, 1961). Our first aim is to analize under which conditions the two optimal core allocations are supported by Walrasian equilibria.

Also for valuations that are monotone and satisfy the gross-substitutes property, Gul & Stacchetti (1999) characterizes the maximum and minimum Walrasian equilibrium prices and show that even if the Vickrey outcome is not supported by the minimum Walrasian equilibrium price vector, it will become an allocation supported by a Walrasian equilibrium of the enlarged market obtained by a finite replication of the original market. Compared to that, we look for sufficient conditions on the market valuations that guarantee that the buyers-optimal core allocation (the Vickrey outcome) comes from a Walrasian equilibrium.

In the literature, the relationship between the whole core and the set of Walrasian equilibria has been addressed. In particular, Massó & Neme (2014) shows that for many-to-many assignment markets, the core converges to the set of Walrasian equilibrium payoff vectors in an infinite replication of the market. Although with a slightly different definition of Walrasian equilibrium, they also show that this coincidence may not be attained with a finite replication. Also in our setting, we show that the core of the one-seller assignment game may not coincide with the set of Walrasian equilibrium payoff vector when the original market is replicated finitely many times. In particular, different to Gul & Stacchetti (1999) results for the Vickrey outcome, we show that the seller-optimal core allocation may not come from a Walrasian equilibrium in any finite replication of the one-seller assignment market.

Further, we give necessary and sufficient conditions on the buyers' valuations so that the seller-optimal core allocation is supported by the maximum Walrasian equilibrium prices. Finally, we also characterize those buyers' valuations under which the set of Walrasian equilibrium payoff vectors coincides with the core.

The chapter is organized as follows. In the next section, the model is introduced and the necessary preliminaries are addressed. Section 2.3 is devoted to study under which conditions the buyers-optimal and the seller-optimal core allocations come from a Walrasian equilibrium. Finally, in Section 2.4 we characterize the coincidence between the set of Walrasian equilibrium payoff vectors and the core.

2.2 The model and some preliminaries

The *one-seller assignment market with multi-unit demands* is defined by a 5-tuple $(M, \{0\}, Q, A, r)$. The finite set of m buyers is denoted by M and the unique seller is denoted by 0. The seller owns a finite set Q of objects. These objects are indivisible and heterogeneous, but of a similar type, let us say different houses or different (maybe part-time) jobs.

Each buyer-object pair $(i,j) \in M \times Q$ has a potential gain $a_{ij} \in \mathbb{R}_+$, interpreted as the valuation of object j by buyer i, where $\mathbb{R}_+$ stands for the set of non-negative real numbers. Given a set S, we will denote by $|S|$ the cardinality of S and 2^S the set of all subsets of S. Without loss of generality, we normalize to zero the reservation value the seller has for each object. The valuation matrix, denoted by $A = (a_{ij})_{(i,j)\in M\times Q}$, captures each potential gain of all buyer-object pairs. Moreover, each buyer $i \in M$ can acquire $r_i \in \mathbb{N}$ objects and the vector $r = (r_i)_{i\in M} \in \mathbb{N}^M$ indicates the buyers' capacities. We assume that the seller owns some copies of a dummy object, as many as the sum of all buyers' capacities, $\sum_{i\in M} r_i$. With some abuse of notation, each copy of this dummy object is denoted by j_0 and each buyer values it at zero.

We assume that each buyer has a quasi-linear preference with respect to money and values packages of objects additively up to their given capacity. That is, buyer $i \in M$ values a package $R \subseteq Q$ by

$$\max_{\substack{R'\subseteq R\\ |R'|\leq r_i}} \left\{ \sum_{j\in R'} a_{ij} \right\}.$$

A *matching* μ between $S \subseteq M$ and Q in the market $(M,\{0\},Q,A,r)$, is a subset of $S \times Q$ such that each $j \in Q$ belongs to at most one pair and each $i \in S$ belongs to exactly r_i pairs. Notice that it is possible to match any buyer with dummy objects to complete his capacity. We denote by $\mathcal{M}(S,Q)$ the set of matchings between $S \subseteq M$ and Q, $\mu(S)$ is the set of objects matched by μ to some buyer in S, and when $S = \{i\}$ we simply write $\mu(i)$. We denote by $\mu^{-1}(j)$ the buyer matched to object $j \in Q$ by μ.

Let $(M,\{0\},Q,A,r)$ be a market. Given $S \subseteq M$, a matching $\mu \in \mathcal{M}(S,Q)$ is *optimal for* $S \cup \{0\}$ if

$$\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij} \quad \text{for all } \mu' \in \mathcal{M}(S,Q).$$

We denote by $\mathcal{M}_A(S,Q)$ the set of optimal matchings between S and Q in this market.

Let us introduce the definition of a coalitional game with transferable utility (a game). A game (N,v) is a pair formed by a finite set of players N and a characteristic function v that assigns a real number $v(S)$ to each coalition $S \subseteq N$, with

$v(\emptyset) = 0$. The core of a game (N, v) consists of

$$C(v) = \left\{ x \in \mathbb{R}^N \,\middle|\, \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq N \right\}.$$

Now, we consider a game associated to one-seller assignment markets. The *one-seller assignment game* related to a one-seller assignment market $(M, \{0\}, Q, A, r)$ is denoted by $(M \cup \{0\}, v_A)$. The worth of any coalition formed by only one type of agents is zero, because in these cases there is no trade. When a coalition is formed by a group of buyers $S \subseteq M$ and the seller, the worth is given by the following expression

$$v_A(S \cup \{0\}) = \max_{\mu \in \mathcal{M}(S,Q)} \left\{ \sum_{(i,j) \in \mu} a_{ij} \right\}.$$

Now, we define Walrasian equilibrium for a one-seller assignment market with multi-unit demands $(M, \{0\}, Q, A, r)$. We define by $2^Q_{r_i} = \{R \subseteq Q;\ |R| = r_i\}$ the set of allowable packages of objects for a buyer $i \in M$. A *price vector* $p = (p_j)_{j \in Q} \in \mathbb{R}^Q_+$ consists of one price for each object, with a price of zero for each dummy object. We denote by $\mathcal{P}$ the set of all price vectors. For each price vector $p \in \mathcal{P}$, we denote by $D_i(p) \subseteq 2^Q_{r_i}$ the *demand set of buyer i at level prices p*, that is

$$D_i(p) = \left\{ R \in 2^Q_{r_i} \,\middle|\, \sum_{j \in R} (a_{ij} - p_j) \geq \sum_{j \in R'} (a_{ij} - p_j) \text{ for all } R' \in 2^Q_{r_i} \right\}. \quad (2.1)$$

The demand set of any buyer is never empty, since at sufficiently high prices the demand set can be formed only by dummy objects.

Definition 2.1. A pair $(p, \mu) \in \mathcal{P} \times \mathcal{M}(M, Q)$ is a *Walrasian equilibrium* for a one-seller assignment market $(M, \{0\}, Q, A, r)$ if the following two conditions hold:
W.1 For all $i \in M$, $\mu(i) \in D_i(p)$,
W.2 For all $j \in Q \setminus \mu(M)$, $p_j = 0$.

If a pair (p, μ) is a Walrasian equilibrium, we say that p is a *Walrasian equilibrium price vector*. The *payoff vector associated to* (p, μ) is $(U(p, \mu), V(p, \mu)) \in \mathbb{R}^M \times \mathbb{R}$, where

$$\begin{array}{rcll} U_i(p, \mu) & = & \sum_{j \in \mu(i)} (a_{ij} - p_j) & \text{for each } i \in M, \text{ and} \\ V(p, \mu) & = & \sum_{j \in Q} p_j & \text{for the seller.} \end{array} \quad (2.2)$$

Gul & Stacchetti (1999) shows that when all buyers value packages additive up to a given capacity, these valuations satisfy the gross-substitutes condition[1] as well

[1]The gross-substitutes condition was introduced by Kelso & Crawford (1982) and it requires

as monotonicity.[2] Then, the following consequences regarding the set of Walrasian equilibria follow easily for one-seller assignment markets with multi-unit demands.

R.1 If (p, μ) is a Walrasian equilibrium, then μ is optimal and, for any optimal matching μ', (p, μ') is also a Walrasian equilibrium.

R.2 The set of Walrasian equilibrium price vectors of the market is non-empty and forms a complete lattice.

R.3 The maximum Walrasian equilibrium price for an object $k \in Q$ is

$$\overline{p}_k = \max_{\mu \in \mathcal{M}(M,Q)} \left\{ \sum_{(i,j)\in\mu} a_{ij} \right\} - \max_{\mu \in \mathcal{M}(M,Q\setminus\{k\})} \left\{ \sum_{(i,j)\in\mu} a_{ij} \right\}. \tag{2.3}$$

In order to express the minimum Walrasian equilibrium price of an object $k \in Q$ in a market $(M, \{0\}, Q, A, r)$, we need to consider a new type of matchings. We will allow only object k to be matched at most twice but not to the same buyer. This is equivalent to introducing an identical copy of object k and restrict to usual matchings that do not assign the two copies to the same buyer. With some abuse of notation, we will denote this set of matchings by $\mathcal{M}^k(M, Q)$. Now, we can give an expression for the minimum Walrasian equilibrium prices.

R.4 The minimum Walrasian equilibrium price for an object k is

$$\underline{p}_k = \max_{\mu \in \mathcal{M}^k(M,Q)} \left\{ \sum_{(i,j)\in\mu} a_{ij} \right\} - \max_{\mu \in \mathcal{M}(M,Q)} \left\{ \sum_{(i,j)\in\mu} a_{ij} \right\}. \tag{2.4}$$

Result **R.1** and the existence of Walrasian equilibria can also be found in Arribillaga *et al.* (2014) for a more general assignment model. With respect to Result **R.4**, which is proven by Gul & Stacchetti (1999) under the gross-substitutes condition, we provide an alternative proof in the Appendix of this chapter, for the case where valuations of packages are additive up to a capacity.

In the definition of Walrasian equilibrium, the owners of the objects do no play any role. Hence the set of Walrasian equilibria of the one-seller assignment market equals the set of Walrasian equilibria of the related many-to-one market where each object belongs to a different seller. Moreover, Walrasian equilibrium prices are

that for any two price vectors p and q such that $q \geqslant p$, and any $R \in D_i(p)$, there exists $R' \in D_i(q)$ such that $\{j \in R | p_j = q_j\} \subseteq R'$.

[2]Monotonicity simply says that if $R' \subseteq R$ then the valuation of package R is at least as high as the valuation of R'.

easily described by linear equalities and inequalities. A pair $(p,\mu) \in \mathcal{P} \times \mathcal{M}(M,Q)$ is a Walrasian equilibrium of $(M,\{0\},Q,A,r)$ if and only if

$$\begin{aligned}&0 \leq p_j \leq a_{ij} \text{ for all } (i,j) \in \mu, \ \ p_j = 0 \text{ for all } j \in Q \setminus \mu(M), \ \text{ and} \\ &a_{ij} - p_j \geq a_{ik} - p_k \ \text{ for each } \ i \in M, \ \text{ all } j \in \mu(i) \text{ and all } k \in Q \setminus \{\mu(i)\}. \quad (2.5)\end{aligned}$$

To better analyze the relationship between the core and the Walrasian equilibria for the one-seller assignment game, we first find a simple description of the core of this game. To this end, we use a result in Ausubel & Milgrom (2002). They introduce the notion of buyers-submodularity, which means that the marginal contribution of a buyer to a coalition containing the seller decreases as the coalition grows larger. A game $(M \cup \{0\}, v)$ is *buyers-submodular*[3] if for all $i \in M$,

$$v((T \cup \{i\}) \cup \{0\}) - v(T \cup \{0\}) \geq v((S \cup \{i\}) \cup \{0\}) - v(S \cup \{0\}), \qquad (2.7)$$

for all $T \subseteq S \subseteq M \setminus \{i\}$. The following result shows that the one-seller assignment game is buyers-submodular.[4]

Proposition 2.2. *Let* $(M,\{0\},Q,A,r)$ *be a one-seller assignment market and let* $(M \cup \{0\}, v_A)$ *be its related one-seller assignment game. Then* $(M \cup \{0\}, v_A)$ *is buyers-submodular.*

Proof. Recall that buyers' valuations satisfy the gross-substitutes condition and monotonicity (Gul & Stacchetti, 1999). Therefore, by Theorem 11 in Ausubel & Milgrom (2002), the game $(M \cup \{0\}, v_A)$ is buyers-submodular. □

Together with the fact that coalitions not containing the seller have null worth, buyers-submodularity implies that the one-seller assignment game $(M \cup \{0\}, v_A)$ is a big-boss game, as defined in Muto *et al.* (1988).[5] As a consequence of Theorem 7 in Ausubel & Milgrom (2002), or also Theorem 3.2 in Muto *et al.* (1988), the core of the one-seller assignment game is non-empty and can be described by

$$\left\{ (U,V) \in \mathbb{R}^M \times \mathbb{R} \;\middle|\; \sum_{i \in M} U_i + V = v_A(M \cup \{0\}),\ 0 \leq U_i \leq M_i^{v_A} \text{ for all } i \in M \right\}, \qquad (2.8)$$

[3]Let $(M \cup \{0\}, v)$ be a game. Expression (2.7) is equivalent to

$$v(S_1 \cup \{0\}) + v(S_2 \cup \{0\}) \geq v((S_1 \cup S_2) \cup \{0\}) + v((S_1 \cap S_2) \cup \{0\}), \qquad (2.6)$$

for all $S_1, S_2 \subseteq M$.

[4]In order to make this chapter more selfcontained, an alternative proof of buyers-submodularity of the one-seller assignment game is provided in the Appendix.

[5]Buyers submodularity clearly implies condition B2** in page 312 of Muto *et al.* (1988), which implies B2 in the definition of big-boss game.

where $M_i^{v_A} = v_A(M \cup \{0\}) - v_A((M \setminus \{i\}) \cup \{0\})$ denotes the marginal contribution of buyer $i \in M$ to the grand coalition, which is also known as the Vickrey payoff for this agent. Furthermore, the core has a lattice structure with respect to the usual order defined on buyers' payoffs. Therefore, we can guarantee the existence of one optimal core allocation for each side of the market. In the buyers-optimal core allocation $(\overline{U}, \underline{V}) \in \mathbb{R}^M \times \mathbb{R}$, each buyer gets his marginal contribution, *i.e.* $\overline{U}_i = M_i^{v_A}$ for all $i \in M$ and $\underline{V} = v_A(M \cup \{0\}) - \sum_{i \in M} M_i^{v_A}$. On the other hand, in the seller-optimal core allocation $(\underline{U}, \overline{V}) \in \mathbb{R}^M \times \mathbb{R}$, each buyer $i \in M$ gets $\underline{U}_i = 0$ and $\overline{V} = v_A(M \cup \{0\})$. Thus, the core of the one-seller assignment game has an optimal core allocation for each market sector as it happens in the classical Shapley & Shubik (1972) assignment game. This is not known to be true for other many-to-many assignment models (see *e.g.* Sotomayor, 2002).

A first relationship between core and Walrasian equilibria for many-to-many assignment markets is well known (see *e.g.* Theorem 36 in Arribillaga *et al.*, 2014): the payoff vector $(U(p,\mu), V(p,\mu))$ associated to any Walrasian equilibrium (p,μ) of the one-seller market $(M, \{0\}, Q, A, r)$ (or of any many-to-many assignment market) belongs to the core of the associated game. However, there may be core allocations not supported by Walrasian equilibrium prices. In particular, we ask when the two optimal core allocations are Walrasian equilibrium payoff vectors.

2.3 When optimal core allocations are supported by Walrasian equilibria?

To begin the study of the relationship between optimal core allocations and the Walrasian equilibria, we focus on conditions on the buyers' valuations so that optimal core allocations are supported by Walrasian equilibrium prices.

As it was remarked in the previous section, the set of Walrasian equilibrium payoff vectors is a subset of the core. Notwithstanding, the one-seller assignment market has an interesting property which does not hold in more general many-to-many assignment markets: the Vickrey outcome coincides with the buyers-optimal core allocation. The relationship between the Vickrey outcome and the set of Walrasian equilibria was addressed in Gul & Stacchetti (1999). They show that when the original market is replicated finitely many times, the Vickrey outcome is supported by a Walrasian equilibrium in the enlarged market. In fact, they show that it is sufficient to replicate the market at least as many times as the number of objects. In spite of that, we are interested in conditions to guarantee that optimal core allocations are supported by a Walrasian equilibrium in the original market with no need of replication.

The next proposition concerns the buyers-optimal core allocation and characterizes those markets in which it is possible to allocate to each buyer (one of) his most preferred package of objects. The feasibility of such an allocation characterizes when the game is convex[6] and implies the existence of a Walrasian equilibrium supporting the buyers-optimal core allocation.

Proposition 2.3. *Let* $(M,\{0\},Q,A,r)$ *be a one-seller assignment market and let* $(M\cup\{0\},v_A)$ *be its related one-seller assignment game. The following assertions are equivalent:*

i. $(M\cup\{0\},v_A)$ *is convex,*

ii. There is a matching $\mu\in\mathcal{M}(M,Q)$ *such that for each* $i\in M$*,*

$$\sum_{j\in\mu(i)} a_{ij} \geq \sum_{j\in R} a_{ij} \text{ for all } R\in 2^{Q}_{r_i}. \tag{2.10}$$

iii. The minimum Walrasian equilibrium price vector is $\underline{p}=(0,...,0)\in\mathcal{P}$*.*

Proof. $i.\Rightarrow ii.$ First, it can be deduced from Proposition 3.4 in Muto *et al.* (1988) that $(M\cup\{0\},v_A)$ is convex if and only if for any $S\subseteq M$,

$$v_A(S\cup\{0\}) = v_A(\{0\}) + \sum_{i\in S} M_i^{v_A}.$$

Now, assume that $(M\cup\{0\},v_A)$ is convex. Take any $\mu\in\mathcal{M}_A(M,Q)$, we have that

$$\sum_{(i,j)\in\mu} a_{ij} = v_A(M\cup\{0\}) = v_A(\{0\}) + \sum_{i\in M} M_i^{v_A} = \sum_{i\in M} M_i^{v_A}. \tag{2.11}$$

Since $\sum_{j\in\mu(i)} a_{ij} \geq M_i^{v_A}$ for all $i\in M$, expression (2.11) implies that $\sum_{j\in\mu(i)} a_{ij} = M_i^{v_A}$ for all $i\in M$. Moreover, for all $i\in M$, let $R_i\in 2^{Q}_{r_i}$ be such that

$$\sum_{j\in R_i} a_{ij} \geq \sum_{j\in R} a_{ij} \text{ for all } R\in 2^{Q}_{r_i}.$$

Since $(M\cup\{0\},v_A)$ is convex, then

$$\sum_{j\in R_i} a_{ij} = v_A(\{i,0\}) = M_i^{v_A} \text{ for all } i\in M,$$

[6] A game (N,v) is convex if for all $S,T\subseteq N$

$$v(S)+v(T)\leq v(S\cup T)+v(S\cap T). \tag{2.9}$$

and as a consequence, for each $i \in M$, we have $\sum_{j\in\mu(i)} a_{ij} = \sum_{j\in R_i} a_{ij} \geq \sum_{j\in R} a_{ij}$ for all $R \in 2^Q_{r_i}$.

$ii. \Rightarrow iii.$ Take matching μ of statement $ii.$ and the price vector $p = (0,...,0) \in \mathcal{P}$. It is immediate that by (2.10), $\mu(i) \in D_i(p)$ for each $i \in N$. Indeed, (p,μ) is a Walrasian equilibrium. The minimality of the price vector p is also immediate.

$iii. \Rightarrow i.$ Since $\underline{p} = (0,...,0) \in \mathcal{P}$ is a Walrasian equilibrium price vector, then there is a matching $\mu \in \mathcal{M}(M,Q)$ such that $\mu(i) \in D_i(\underline{p})$ for all $i \in N$. In fact, because of null prices, notice that such $\mu \in \mathcal{M}(M,Q)$ satisfies

$$\sum_{i\in S}\sum_{j\in\mu(i)} a_{ij} = v_A(S\cup\{0\}) \text{ for all } S \subseteq M.$$

As a consequence, it is easy to see that $M_i^{v_A} = \sum_{j\in\mu(i)} a_{ij}$ for every $i \in M$. Then, we have that for any $S \subseteq M$,

$$\sum_{i\in S} M_i^{v_A} = \sum_{i\in S}\sum_{j\in\mu(i)} a_{ij} = v_A(S\cup\{0\}),$$

which shows that the game $(M\cup\{0\}, v_A)$ is convex. □

Notice that if the above equivalent conditions hold, and μ is a matching that allocates to each buyer one of his preferred packages, then the Vickrey payoff to each $i \in M$ is $\sum_{j\in\mu(i)} a_{ij}$ and hence it is attained at the minimum Walrasian equilibrium. However, there are instances (see Example 2.4) in which the game is not convex but the buyers-optimal core allocation still comes from a Walrasian equilibrium. To see that, we only need to check whether for all $i \in M$ it holds $M_i^{v_A} = \sum_{j\in\mu(i)}(a_{ij} - \underline{p}_j)$, where μ is an optimal matching and $\underline{p} \in \mathcal{P}$ can be computed following expression (2.3).

The next example also shows that the fact the buyers-optimal core allocation is supported by Walrasian equilibirum prices does not guarantee that all other core allocations come from Walrasian equilibria.

Example 2.4. Consider a market with unitary capacities $(M,\{0\},Q,A,r)$ given by $M = \{1,2\}$, $Q = \{1',2'\}$ and $r = (1,1)$. For the purposes of this example, we show no dummy objects. The valuation matrix A is the following

$$\begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} 5 & \textcircled{4} \\ \textcircled{4} & 2 \end{pmatrix} \end{array}.$$

Consider the one-seller assignment game $(M\cup\{0\}, v_A)$. By (2.8), the core can be described by the set of payoff vectors $(U,V) \in \mathbb{R}^2_+ \times \mathbb{R}_+$ such that $U_1 + U_2 + V = 8$, $U_1 \leq 4$ and $U_2 \leq 3$.

Take the unique optimal matching $\mu = \{(1,2'),(2,1')\}$. By (2.5), a price vector $p \in \mathbb{R}^2_+$ is a Walrasian equilibrium price vector if and only if $0 \le p_{1'} \le 4$, $0 \le p_{2'} \le 4$ and $1 \le p_{1'} - p_{2'} \le 2$. Since the corresponding payoff vector $(U_1,U_2;V)$ satisfies $U_1 = 4 - p_{2'}$, $U_2 = 4 - p_{1'}$ and $V = p_{1'} + p_{2'}$, we have $p_{1'} - p_{2'} = (4-U_2)-(4-U_1) = U_1 - U_2$. Hence, the Walrasian equilibrium payoff vectors are described by $U_1 + U_2 + V = 8$, $1 \le U_1 - U_2 \le 2$, $0 \le U_1 \le 4$ and $0 \le U_2 \le 4$. Figure 2.1 depicts the core and the set of Walrasian equilibrium payoff vectors.

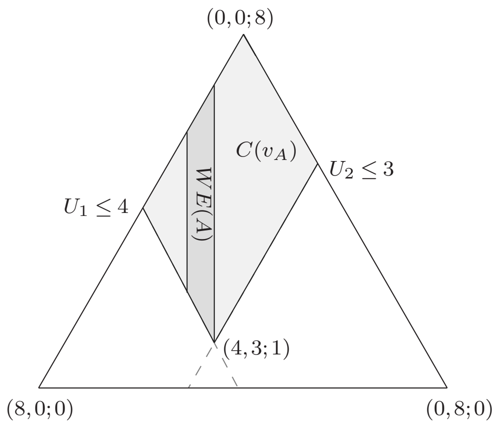

Figure 2.1: The set of Walrasian equilibrium payoff vectors $WE(A)$ is strictly included in the core $C(v_A)$

In the above example the game is not convex and the buyers-optimal core allocation is supported by a Walrasian equilibrium while the seller-optimal core allocation is not. Moreover, although convexity is a strong condition, it is not sufficient to guarantee that the seller-optimal core allocation is supported by a Walrasian equilibrium, as the following example illustrates.

Example 2.5. Consider $(M,\{0\},Q,A,r)$ given by $M = \{1,2\}$, $Q = \{1',2',3'\}$ and $r = (1,1)$. The valuation matrix A is the following

$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \begin{pmatrix} 2 & \textcircled{4} & 1 \\ \textcircled{2} & 1 & 1 \end{pmatrix} \end{array}.$$

Notice that the game is convex and the seller-optimal core allocation is $(\underline{U},\overline{V}) = (0,0;6)$. We can obtain the maximum Walrasian equilibrium prices $\overline{p} = (1,3,0)$ by means of formula (2.4) and we see that the corresponding payoff vector is $(U,V) = (1,1;4)$ which is not the seller-optimal core allocation.

An interesting fact about the seller-optimal core allocation is the following one. When the seller-optimal core allocation is not a Walrasian equilibrium payoff vector, even if the economy is replicated finitely many times *à la* Gul & Stacchetti (1999), it may not be supported by any Walrasian equilibrium in the enlarged market. To see this, consider the previous example. If we replicate the market finitely many times, the replicas of object $3'$ will be unmatched in any optimal matching. As a consequence, the price of these replicas in any Walrasian equilibrium will be zero. Assume there is a Walrasian equilibrium that supports the seller-optimal core allocation in the enlarged market. In this equilibrium each buyer will pay for each of his matched objects his own valuation of the object. But then, any buyer strictly prefers a replica of object $3'$ for free to his matched objects at the described prices, and this contradicts these prices are Walrasian equilibrium prices.

In order to characterize when the seller-optimal core allocation is a Walrasian equilibrium payoff vector, let us first define the set of desirable objects, Q^*_A. We say that an object is desirable if at least one buyer valuates it positively

$$Q^*_A = \{j \in Q \mid \text{there is some } i \in M \text{ such that } a_{ij} > 0\}.$$

The following result gives a characterization of markets so that the seller-optimal core allocation is a Walrasian equilibrium payoff vector.

Proposition 2.6. *Let $(M,\{0\},Q,A,r)$ be a one-seller assignment market. The seller-optimal core allocation is a Walrasian equilibrium payoff vector if and only if there is an optimal matching $\mu \in \mathcal{M}_A(M,Q)$ and the following two conditions are satisfied:*

(a) For each $j \in \mu(M)$, $a_{\mu^{-1}(j)j} \geq a_{ij}$ for all $i \in M \setminus \{\mu^{-1}(j)\}$,

(b) $Q^*_A \subseteq \mu(M)$.

Proof. We first prove the 'if' part. Assume that $\mu \in \mathcal{M}_A(M,Q)$ satisfies conditions (a) and (b). Define $p_j = a_{\mu^{-1}(j)j}$ for all $j \in \mu(M)$ and $p_j = 0$ for all $j \in Q \setminus \mu(M)$. We show that $\mu(i) \in D_i(p)$ for all $i \in M$. Take any $i \in M$ and consider any $R \in 2^Q_{r_i}$. Since μ satisfies (a) and (b), and by definition of p, we have

$$\sum_{j\in R}(a_{ij}-p_j) = \sum_{j\in R\cap\mu(M)}(a_{ij}-a_{\mu^{-1}(j)j}) + \sum_{j\in R\setminus\mu(M)}(a_{ij}-0) \leq 0 = \sum_{j\in\mu(i)}(a_{ij}-p_j),$$

and thus $\mu(i) \in D_i(p)$ for all $i \in M$. Besides, by definition of p, we get $p_j = 0$ for each $j \in Q \setminus \mu(M)$. Notice that $(U(p,\mu),V(p,\mu))$ is the seller-optimal core allocation.

Now, we prove the 'only if' part. Assume that (p,μ) is a Walrasian equilibrium and $(U(p,\mu),V(p,\mu))$ is the seller-optimal core allocation. By property **R.1** in page

16, we have that $\mu \in \mathcal{M}_A(M,Q)$. Moreover, in the seller-optimal core allocation the seller's payoff is equal to $v_A(M \cup \{0\})$.

We claim that

$$p_j = a_{\mu^{-1}(j)j} \text{ for all } j \in \mu(M). \tag{2.12}$$

If $p_j > a_{\mu^{-1}(j)j}$ for some $j \in \mu(M)$, then for all $R \in D_{\mu^{-1}(j)}(p)$ we have $j \notin R$, and as a consequence (p,μ) is not a Walrasian equilibrium. On the other hand, if $p_j < a_{\mu^{-1}(j)j}$ for some $j \in \mu(M)$ then $\sum_{j\in Q} p_j < v_A(M \cup \{0\})$ and the seller-optimal core allocation is not the payoff vector of (p,μ).

Now, taking (2.12) into account, we shall prove that μ satisfies condition (a) of the statement. Assume on the contrary that there is some $i \in M$ such that $a_{ij_1} > a_{\mu^{-1}(j_1)j_1}$ for some $j_1 \in Q$ with $i \in M \setminus \{\mu^{-1}(j_1)\}$. Let $R \in 2^Q_{r_i}$ be the package formed by object j_1 and copies of the dummy object, *i.e.* $R = \{j_1, j_0^1, j_0^2, ..., j_0^{r_i-1}\}$. Since $\sum_{j\in R}(a_{ij} - p_j) > 0 = \sum_{j\in\mu(i)}(a_{ij} - p_j)$, we obtain that $\mu(i) \notin D_i(p)$ in contradiction with (p,μ) being a Walrasian equilibrium. Then μ satisfies (a). In order to show (b), assume on the contrary that, there is some $j_2 \in Q_A^* \setminus \mu(M)$. By definition of Walrasian equilibrium, the price of this object is $p_{j_2} = 0$. Since $j_2 \in Q_A^*$, there is some $i \in M$ such that $a_{ij_2} > 0$. This implies that $\mu(i) \notin D_i(p)$ because $\sum_{j\in R}(a_{ij} - p_j) > \sum_{j\in\mu(i)}(a_{ij} - p_j)$ where $R = \{j_2, j_0^1, j_0^2, ..., j_0^{r_i-1}\}$. This contradicts (p,μ) being a Walrasian equilibrium. Hence, μ satisfies (b). □

Condition (a) above requires that each object must be allocated to the buyer who values it the most, while condition (b) simply says that each desirable object must be allocated. Notice that condition (a) is not satisfied in Example 2.4, and condition (b) is not satisfied in Example 2.5.

2.4 The coincidence of the core and the set Walrasian equilibrium payoff vectors

In this section, we address the relationship between the whole core and the set of Walrasian equilibria. The aim is to obtain conditions so that the core coincides with the set of Walrasian equilibrium payoff vectors. For more general many-to-many assignment markets but with a different definition of the demand set, Massó & Neme (2014) shows that the sequence of cores of replicated markets converges to the set of Walrasian equilibrium payoffs when the number of replicas tends to infinity. Moreover, for any number of replicas there is a market with a core payoff that is not a Walrasian equilibrium payoff. Also, for our definition of Walrasian equilibrium that follows Gul & Stacchetti (1999), Example 2.5 shows that the process of finite replication does not guarantee that the seller-optimal core allocation is a Walrasian equilibrium payoff vector.

Since no core coincidence result can be achieved for arbitrary one-seller assignment markets after a finite replication of the economy, we focus on the search of conditions on buyers' valuations that guarantee that all core allocations are supported by Walrasian equilibria. Taking into account Propositions 2.3 and 2.6, one might wonder if convexity together with conditions (a) and (b) of Proposition 2.6 are sufficient to obtain the coincidence between the core and the set of Walrasian equilibrium payoff vectors. The answer is negative as the next example shows.

Example 2.7. Consider a market $(M,\{0\},Q,A,r)$ given by $M=\{1,2\}$, $Q=\{1',2'\}$ and $r=(1,1)$. The valuation matrix A is the following

$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} 2 & \textcircled{4} \\ \textcircled{2} & 1 \end{pmatrix} \end{array}.$$

Notice that the game is convex and conditions (a) and (b) of Proposition 2.6 hold. Consider the core allocation $(4,0;2)$. Notice that the unique optimal matching in this market assigns object $1'$ to buyer 2 and object $2'$ to buyer 1. Because of the unitary demands, the unique price vector that supports the core allocation $(4,0;2)$ is $p=(2,0)$. However, $p=(2,0)$ is not a Walrasian equilibrium price vector since $\{1'\}\notin D_2(p)$.

The following theorem is the main result of this chapter and characterizes the coincidence between the set of Walrasian equilibrium payoff vectors and the core. This theorem provides three conditions that together characterize when each core element is supported by Walrasian equilibrium prices. First of all, it requires the existence of an optimal matching satisfying those two conditions stated in Proposition 2.6, *i.e.* each real object is assigned to a buyer who values it the most and each desirable object must be assigned. Secondly, a new condition is required and it is related to the idea of second best valuation. This condition in the next theorem establishes that for each buyer, his marginal contribution to the market is not higher than the difference between how much he values his assigned package of objects and the sum of the highest valuation of all objects in that package when he is not in the market. Hence the third condition also resembles the idea of the social opportunity cost of allocating efficiently a package to a buyer.

Theorem 2.8. *Let $(M,\{0\},Q,A,r)$ be a one-seller assignment market and let $(M\cup\{0\},v_A)$ be its associated one-seller assignment game. The core of $(M\cup\{0\},v_A)$ coincides with the set of Walrasian equilibrium payoff vectors if and only if there is an optimal matching $\mu\in\mathcal{M}_A(M,Q)$ that satisfies the following three conditions:*

(a) For each $j \in \mu(M)$, $a_{\mu^{-1}(j)j} \geq a_{ij}$ *for all* $i \in M \setminus \{\mu^{-1}(j)\}$,

(b) $Q_A^* \subseteq \mu(M)$,

(c) $M_i^{v_A} \leq \sum_{j\in\mu(i)} \left(a_{ij} - \max_{t\in M\setminus\{i\}}\{a_{tj}\}\right)$ *for all* $i \in M$.

Proof. We first prove the 'if' part. Assume that some $\mu \in \mathcal{M}_A(M,Q)$ satisfies (a), (b) and (c). We show that any $(U,V) \in C(v_A)$ is the payoff vector of some Walrasian equilibrium. By conditions (a) and (c), for each $i \in M$, we can find some $(\alpha_{ij})_{j\in\mu(i)} \in \mathbb{R}^{r_i}$ such that $a_{ij} \geq \alpha_{ij} \geq \max_{t\in M\setminus\{i\}}\{a_{tj}\}$ for all $j \in \mu(i)$ and

$$M_i^{v_A} = \sum_{j\in\mu(i)} (a_{ij} - \alpha_{ij}). \tag{2.13}$$

Take any $(U,V) \in C(v_A)$ and define $b_i = M_i^{v_A} - U_i$ for all $i \in M$. Since for all $i \in M$ we have $M_i^{v_A} \geq U_i \geq 0$, then $M_i^{v_A} \geq b_i \geq 0$.

Let us define $p \in \mathbb{R}^Q$ by

$$p_j = \begin{cases} \alpha_{\mu^{-1}(j)j} + \dfrac{a_{\mu^{-1}(j)j} - \alpha_{\mu^{-1}(j)j}}{M^{v_A}_{\mu^{-1}(j)}}\, b_{\mu^{-1}(j)} & \text{if } j \in \mu(M) \text{ and } M^{v_A}_{\mu^{-1}(j)} > 0, \\ a_{\mu^{-1}(j)j} & \text{if } j \in \mu(M) \text{ and } M^{v_A}_{\mu^{-1}(j)} = 0, \\ 0 & \text{if } j \in Q \setminus \mu(M) \text{ or } j = j_0. \end{cases} \tag{2.14}$$

Notice that $p \in \mathcal{P}$ (recall the definition of $\mathcal{P}$ in page 15). We show that $\mu(i) \in D_i(p)$ for all $i \in M$. It is sufficient to see that $a_{ij} - p_j \geq a_{ik} - p_k$ for all $j \in \mu(i)$ and all $k \in Q \setminus \mu(i)$. To this end, let us see that for all $i \in M$ and all $j \in \mu(i)$ it holds $a_{ij} - p_j \geq 0$ while $a_{ik} - p_k \leq 0$ for all $k \in Q \setminus \mu(i)$. On one hand, take $i \in M$ such that $M_i^{v_A} > 0$. Then $a_{ij} - p_j = a_{ij} - \alpha_{ij} - \frac{a_{ij}-\alpha_{ij}}{M_i^{v_A}} b_i = (a_{ij} - \alpha_{ij})(1 - \frac{b_i}{M_i^{v_A}}) \geq 0$ for all $j \in \mu(i)$. Take $i \in M$ such that $M_i^{v_A} = 0$. Then $a_{ij} - p_j = a_{ij} - a_{ij} = 0$ for all $j \in \mu(i)$. On the other hand, take $k \in \mu(M)$ such that $M^{v_A}_{\mu^{-1}(k)} > 0$. Then, for any $i \in M \setminus \{\mu^{-1}(k)\}$, we have $a_{ik} - p_k = a_{ik} - \alpha_{\mu^{-1}(k)k} - \frac{a_{\mu^{-1}(k)k} - \alpha_{\mu^{-1}(k)k}}{M^{v_A}_{\mu^{-1}(k)}} b_{\mu^{-1}(k)} \leq 0$ because $a_{\mu^{-1}(k)k} \geq \alpha_{\mu^{-1}(k)k} \geq a_{ik}$. Take $k \in \mu(M)$ such that $M^{v_A}_{\mu^{-1}(k)} = 0$. Then for any $i \in M \setminus \{\mu^{-1}(k)\}$, we have $a_{ik} - p_k = a_{ik} - a_{\mu^{-1}(k)k} \leq 0$ because of assumption (a). Finally, consider $k \in Q \setminus \mu(M)$. Then for any $i \in M$, $a_{ik} - p_k = 0$ because of (b). Thus $\mu(i) \in D_i(p)$ for all $i \in M$ and $p \in \mathcal{P}$ defined in (2.14). Hence, (p,μ) is a

Walrasian equilibrium. Then, the payoffs in this Walrasian equilibrium are

$$U_i(p,\mu) = \sum_{j\in\mu(i)} (a_{ij} - p_j) = \sum_{j\in\mu(i)} \left(a_{ij} - \alpha_{ij} - \frac{a_{ij}-\alpha_{ij}}{M_i^{v_A}} b_i \right)$$
$$= \sum_{j\in\mu(i)} (a_{ij} - \alpha_{ij}) \left(1 - \frac{b_i}{M_i^{v_A}} \right) = M_i^{v_A} - b_i = U_i,$$

for all $i \in M$ such that $M_i^{v_A} > 0$, where the last equality comes from expression (2.13). Take now any $i \in M$ such that $M_i^{v_A} = 0$. From the definition of p_j in (2.14), we have $U_i(p,\mu) = \sum_{j\in\mu(i)}(a_{ij} - p_j) = \sum_{j\in\mu(i)}(a_{ij} - a_{ij}) = 0 = U_i$. Since $(U(p,\mu), V(p,\mu)) \in C(v_A)$ for any Walrasian equilibrium (p,μ), by efficiency, the seller's payoff is $V(p,\mu) = v_A(M \cup \{0\}) - \sum_{i\in M} U_i(p,\mu) = v_A(M \cup \{0\}) - \sum_{i\in M} U_i = V$. This completes the proof of the 'if' part.

Now, we prove the 'only if' part. Assume that the core and the set of payoff vectors associated with the Walrasian equilibria coincide. By Proposition 2.6, conditions (a) and (b) hold for some optimal matching $\mu \in \mathcal{M}_A(M,Q)$. Then, we only have to prove (c). Assume on the contrary that for this μ, there is some buyer $i' \in M$ such that $M_{i'}^{v_A} > \sum_{j\in\mu(i')}(a_{i'j} - \max_{t\in M\setminus\{i'\}}\{a_{tj}\})$. Recall the description of the core in (2.8) and consider $(U,V) \in C(v_A)$ with $U_{i'} = M_{i'}^{v_A}$ for buyer i' and $U_i = 0$ for all $i \in M \setminus \{i'\}$. By assumption, there is a Walrasian equilibrium (p,μ') such that (U,V) is its payoff vector. Take this Walrasian equilibrium price vector p and matching $\mu \in \mathcal{M}_A(M,Q)$ such that $M_{i'}^{v_A} > \sum_{j\in\mu(i')}(a_{i'j} - \max_{t\in M\setminus\{i'\}}\{a_{tj}\})$. Then (p,μ) is a Walrasian equilibrium (recall R.1 in page 16). Therefore $p_j = a_{\mu^{-1}(j)j}$ for all $j \in \mu(M \setminus \{i'\})$ and $M_{i'}^{v_A} = \sum_{j\in\mu(i')}(a_{i'j} - p_j)$. We obtain $\sum_{j\in\mu(i')}(a_{i'j} - p_j) = M_{i'}^{v_A} > \sum_{j\in\mu(i')}(a_{i'j} - \max_{i\in M\setminus\{i'\}}\{a_{ij}\})$. As a consequence, $\sum_{j\in\mu(i')} \max_{i\in M\setminus\{i'\}}\{a_{ij}\} > \sum_{j\in\mu(i')} p_j$ which implies that there is some $i \in M \setminus \{i'\}$ such that $a_{ij} > p_j$ for some $j \in \mu(i')$. We then have that $\mu(i) \notin D_i(p)$ because $a_{ik} - p_k = 0 < a_{ij} - p_j$ for all $k \in \mu(i)$ and the above $j \notin \mu(i)$. This contradicts that (p,μ) is a Walrasian equilibrium. Hence, condition (c) holds. □

The above theorem gives a characterization of the coincidence of the core and the Walrasian equilibria in one-seller assignment markets. Notice that the core and Walrasian equilibrium payoff vectors do not coincide in Example 2.7 because condition (c) is not satisfied.

As a consequence of Theorem 2.8, when the buyers have a sufficiently large capacity, the core coincides with the set of Walrasian equilibrium payoff vectors. Indeed, when there are no capacity constraints (or each buyer has a capacity greater than the number of non-dummy objects), an optimal matching assigns each object to one of the buyers who values it the most. Hence, conditions (a) and (b) are satisfied. Moreover, when a buyer leaves the market, his objects are assigned to the buyers with the second highest valuation, and this implies that for all $i \in M$, condition (c) holds with an equality.

It is also quite straightforward to check that if for some capacities $r \in \mathbb{N}^M$ the core of the market $(M, \{0\}, Q, A, r)$ coincides with the set of Walrasian equilibrium payoff vectors, then they also coincide if capacities are increased to r' where $r'_i \geq r_i$ for all $i \in M$.

Finally, the following corollary allows us to obtain a stronger condition for the coincidence of the core and the set of Walrasian equilibrium payoff vectors.

Corollary 2.9. Let $(M, \{0\}, Q, A, r)$ be a one-seller assignment market and $\mu \in \mathcal{M}_A(M, Q)$ be such that: (d) if $(i, j) \notin \mu$, then $a_{ij} = 0$. The core coincides with the set of Walrasian equilibrium payoff vectors.

Proof. It is immediate to see that (d) implies conditions (a) and (b) of Theorem 2.8. Now, notice that for any $i \in M$,

$$\sum_{j \in \mu(i)} a_{ij} = \sum_{(t,j) \in \mu} a_{tj} - \sum_{t \in M \setminus \{i\}} \sum_{j \in \mu(t)} a_{tj}$$
$$\geq v_A(M \cup \{0\}) - v_A((M \setminus \{i\}) \cup \{0\}) = M_i^{v_A}. \tag{2.15}$$

Since (d) holds, expression (2.15) implies condition (c) of Theorem 2.8. Therefore, (d) implies the coincidence between the core and the set of Walrasian equilibria. □

Note that property (d) is stronger than convexity of the game and it represents those markets where agents only value positively those objects optimally assigned to them. The next example shows that if a buyer places a small positive valuation on some object not assigned to him, the coincidence between core and Walrasian equilibrium payoff vectors may be lost.

Example 2.10. Consider a market $(M, \{0\}, Q, A, r)$ given by $M = \{1, 2\}$, $Q = \{1', 2'\}$, $r = (1, 1)$ and the following valuation matrix A, where $0 < \varepsilon < 4$:

$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} \varepsilon & \textcircled{4} \\ \textcircled{2} & 0 \end{pmatrix} \end{array}.$$

Since the marginal contributions of the two buyers are 4 and 2 respectively, $(0, 2; 4)$ belongs to the core. This core element can only be supported by prices $p = (0, 4)$ but at the price of 0, buyer 1 would prefer object $1'$ at price 0 rather than object $2'$ at price 4. Hence, for any $0 < \varepsilon < 4$, the core of this market does not coincide with the set of Walrasian equilibrium payoff vectors.

2.5 Appendix

Now, we provide an alternative proof of Proposition 2.2 for the one-seller assignment game.

Proof. First, consider $r_i = 1$ for all $i \in M$. Let $(M \cup \{0\}, v_A)$ be the one-seller assignment game. We deduce from Theorem 1 in Shapley (1962), that for all $i, i' \in M$ and all $S \subseteq M \setminus \{i, i'\}$

$$v_A((S \cup \{0\}) \cup \{i\}) - v_A(S \cup \{0\}) \geq v_A((S \cup \{0\}) \cup \{i, i'\}) - v_A((S \cup \{0\}) \cup \{i'\})$$

and, by repeatedly applying this, we obtain that $(M \cup \{0\}, v_A)$ satisfies (2.7).

Now, consider $r_i \geq 1$ for all $i \in M$. We prove that $(M \cup \{0\}, v_A)$ satisfies (2.7). Define a related market in which each buyer $i \in M$ is *replicated* r_i times. Denote by $i(s)$ the s-th copy of i and by $\widetilde{M}$ the new set of buyers formed by replication of all buyers in M. Notice that now each buyer has capacity one. Define the valuation matrix $\widetilde{A} = (a_{i(s)j})_{(i(s),j) \in \widetilde{M} \times Q}$ by $a_{i(s)j} = a_{ij}$ for all $(i, j) \in M \times Q$ and all $s \in \{1, ..., r_i\}$. In this way, we obtain $(\widetilde{M}, \{0\}, Q, \widetilde{A}, \widetilde{r})$ with $\widetilde{r}_{i(s)} = 1$ for all $i(s) \in \widetilde{M}$. Notice that $(M \cup \{0\}, v_A)$ and $(\widetilde{M} \cup \{0\}, v_{\widetilde{A}})$ are related:

$$v_A(S \cup \{0\}) = v_{\widetilde{A}}(\widetilde{S} \cup \{0\}) \text{ for all } S \subseteq M,$$

where $\widetilde{S}$ is formed by the replica of all buyers in S. Then, inequality (2.7) for the game $(M \cup \{0\}, v_A)$ is equivalent to

$$\begin{aligned} v_{\widetilde{A}}((\widetilde{T} \cup \{i(1), ..., i(r_i)\}) \cup \{0\}) - v_{\widetilde{A}}(\widetilde{T} \cup \{0\}) \\ \geq v_{\widetilde{A}}((\widetilde{S} \cup \{i(1), ..., i(r_i)\}) \cup \{0\}) - v_{\widetilde{A}}(\widetilde{S} \cup \{0\}), \end{aligned} \tag{2.16}$$

where $\widetilde{T}$, $\widetilde{S}$ and $\{i(1), ..., i(r_i)\}$ are obtained by replicating all buyers in T, S and $\{i\}$, respectively. Define $S_1 = \widetilde{T} \cup \{i(1), ..., i(r_i)\}$, $S_2 = \widetilde{S}$ and notice that $S_1 \cup S_2 = \widetilde{S} \cup \{i(1), ..., i(r_i)\}$ and $S_1 \cap S_2 = \widetilde{T}$. Since demands of agents in $\widetilde{M}$ are unitary, the game $(\widetilde{M} \cup \{0\}, v_{\widetilde{A}})$ satisfies (2.7), which is equivalent to (2.6). Then,

$$v_{\widetilde{A}}(S_1 \cup \{0\}) + v_{\widetilde{A}}(S_2 \cup \{0\}) \geq v_{\widetilde{A}}((S_1 \cup S_2) \cup \{0\}) + v_{\widetilde{A}}((S_1 \cap S_2) \cup \{0\}),$$

and rearranging terms leads to (2.16). Therefore (2.7) holds for the game $(M \cup \{0\}, v_A)$. □

In this Appendix, we provide an alternative proof for expression (2.4) of the minimum Walrasian equilibrium price. Here, we are going to denote objects by

Greek letters. Given a market $(M,\{0\},Q,A,r)$ and any object $\beta \in Q$, denote by $\tilde{\beta}$ the replica of β, that is, a copy of β such that each buyer values β and $\tilde{\beta}$ the same. Let us consider the market $(M,\{0\},Q\cup\{\tilde{\beta}\},A^{\beta},r)$ with this replicated object: the set of objects is $Q\cup\{\tilde{\beta}\}$ and the valuation matrix is A^{β} with $a^{\beta}_{ij} = a_{ij}$ for all $(i,j) \in M \times Q$ and $a^{\beta}_{i\tilde{\beta}} = a_{i\beta}$ for all $i \in M$. Now, from the set M to the set of objects $Q\cup\{\tilde{\beta}\}$, we define a restricted set of matchings: those matchings that do not assign both objects β and $\tilde{\beta}$ to the same buyer. This set of matchings have been denoted by $\mathcal{M}^{\beta}(M,Q)$ in page 16. Now for the sake of clarity of the rest of the proof we need to make more evident the copy of β and hence, we write

$$\widetilde{\mathcal{M}}(M,Q\cup\{\tilde{\beta}\}) = \{\mu \in \mathcal{M}(M,Q\cup\{\tilde{\beta}\}) \mid \text{if } \beta \in \mu(i) \text{ and } \tilde{\beta} \in \mu(i'), \text{ then } i \neq i'\}. \tag{2.17}$$

We denote by $\widetilde{\mathcal{M}}_{A^{\beta}}(M,Q\cup\{\tilde{\beta}\})$ the set of optimal matchings in $\widetilde{\mathcal{M}}(M,Q\cup\{\tilde{\beta}\})$. That is, a matching $\mu' \in \widetilde{\mathcal{M}}(M,Q\cup\{\tilde{\beta}\})$ belongs to $\widetilde{\mathcal{M}}_{A^{\beta}}(M,Q\cup\{\tilde{\beta}\})$ if $\sum_{(i,j)\in\mu'} a^{\beta}_{ij} \geq \sum_{(i,j)\in\mu} a^{\beta}_{ij}$ for all $\mu \in \widetilde{\mathcal{M}}(M,Q\cup\{\tilde{\beta}\})$. Since $a^{\beta}_{ij} = a_{ij}$ for all $(i,j) \in M\times Q$ and $a^{\beta}_{i\tilde{\beta}} = a_{i\beta}$ for all $i \in M$, with some abuse of notation, we write a_{ij} for all $(i,j) \in M\times Q$ and $a_{i\beta}$ for all $i \in M$, instead of a^{β}_{ij} and $a^{\beta}_{i\tilde{\beta}}$, respectively.

The following remark will be useful in the sequel.

Remark 2.11. For all $\mu \in \widetilde{\mathcal{M}}_{A^{\beta}}(M,Q\cup\{\tilde{\beta}\})$ such that $\tilde{\beta} \in \mu(i_1)$ and $\beta \in \mu(i_2)$, matching $\overline{\mu} = (\mu\setminus\{(i_1,\tilde{\beta}),(i_2,\beta)\})\cup\{(i_1,\beta),(i_2,\tilde{\beta})\}$ also belongs to $\widetilde{\mathcal{M}}_{A^{\beta}}(M,Q\cup\{\tilde{\beta}\})$. That is, we can interchange β with $\tilde{\beta}$.

The next lemma, that is needed in the proof of Theorem 2.13, relates the original market $(M,\{0\},Q,A,r)$ and the market with a replica $(M,\{0\},Q\cup\{\tilde{\beta}\},A^{\beta},r)$ whenever the object $\beta \in Q$ is optimally assigned in the original market. We see that in the market $(M,Q\cup\{\tilde{\beta}\},A^{\beta},r)$, there exists a restricted optimal matching such that those objects that were unassigned remain unassigned and the object β remains assigned to the same buyer.

Lemma 2.12. *Let (M,Q,A,r) be a many-to-one assignment market, $\mu \in \mathcal{M}_A(M,Q)$ and $\beta \in \mu(M)$. Then there exists $\mu' \in \widetilde{\mathcal{M}}_{A^{\beta}}(M,Q\cup\{\tilde{\beta}\})$ such that*

(a) if $\gamma \in Q\setminus\mu(M)$, then $\gamma \in Q\setminus\mu'(M)$, and

(b) if $\beta \in \mu(i)$, then $\beta \in \mu'(i)$.

Proof. First, we prove condition (a) of the statement. Consider any matching $\mu_1 \in \widetilde{\mathcal{M}}_{A^{\beta}}(M,Q\cup\{\tilde{\beta}\})$ and if $\beta \notin \mu_1(M)$ and $\tilde{\beta} \in \mu_1(M)$ just recall Remark 2.11 and interchange β and $\tilde{\beta}$ in μ_1. So, we can assume without loss of generality that $\beta \in \mu_1(M)$.

If $Q \setminus \mu(M) = Q \setminus \mu_1(M)$ then trivially $\mu' = \mu_1$ satisfies condition (a).

Otherwise, if there is some $\gamma \in \mu_1(M) \setminus \mu(M)$, then there is a buyer i_1 such that $\gamma \in \mu_1(i_1) \setminus \mu(i_1)$. Because of the capacity constraint of i_1, there is some object $\alpha_1 \in \mu(i_1) \setminus \mu_1(i_1)$. Notice that $\alpha_1 \neq \tilde{\beta}$ because $\tilde{\beta} \notin \mu(M)$. If $\alpha_1 \notin \mu_1(M)$, define $\mu' = (\mu_1 \setminus \{(i_1,\gamma)\}) \cup \{(i_1,\alpha_1)\}$. Notice that $\mu' \in \widetilde{\mathcal{M}}(M, Q \cup \{\tilde{\beta}\})$. Then

$$0 \leq a_{i_1\alpha_1} - a_{i_1\gamma} = \sum_{j \in \mu'(i_1)} a_{i_1 j} - \sum_{j \in \mu_1(i_1)} a_{i_1 j} \Rightarrow \sum_{j \in \mu'(i_1)} a_{i_1 j} \geq \sum_{j \in \mu_1(i_1)} a_{i_1 j},$$

where the first inequality holds by the optimality of μ. Then $\mu' \in \widetilde{\mathcal{M}}_{A^\beta}(M, Q \cup \{\tilde{\beta}\})$ and $\gamma \notin \mu'(M)$.

If otherwise $\alpha_1 \in \mu_1(M)$, then there is a buyer $i_2 \neq i_1$ such that $\alpha_1 \in \mu_1(i_2)$. Since $\alpha_1 \in \mu_1(i_2) \setminus \mu(i_2)$ there is some $\alpha_2 \in \mu(i_2) \setminus \mu_1(i_2)$ and $\alpha_1 \neq \alpha_2$. If $\alpha_2 \notin \mu_1(M)$, we finish as above by taking $\mu' = (\mu_1 \setminus \{(i_1,\gamma),(i_2,\alpha_1)\}) \cup \{(i_1,\alpha_1),(i_2,\alpha_2)\}$. Otherwise, if $\alpha_2 \in \mu_1(M)$, there is some i_3 such that $\alpha_2 \in \mu_1(i_3) \setminus \mu(i_3)$ and some $\alpha_3 \in \mu(i_3) \setminus \mu_1(i_3)$. We continue and obtain a sequence $\gamma, \alpha_1, ..., \alpha_l$ such that $\gamma \in \mu_1(i_1) \setminus \mu(i_1)$, $\alpha_t \in \mu(i_t) \setminus \mu_1(i_t)$, $\alpha_t \in \mu_1(i_{t+1}) \setminus \mu(i_{t+1})$ for $t \in \{1, ..., l-1\}$ and $\alpha_l \in \mu(i_l) \setminus \mu_1(i_l)$.

Although buyers in the above sequence can be repeated, the objects $\alpha_1, ..., \alpha_l$ can be taken to be all different. Indeed, assume as induction hypothesis that for some $2 \leq s < l$, $\alpha_1, ..., \alpha_s$ are all different (we already know that $\alpha_1 \neq \alpha_2$). Assume that $i_s = i_k$ for some $k \in \{0, ..., s-1\}$, that is both $\alpha_k, \alpha_s \in \mu_1(i_k) \setminus \mu(i_k)$. By assumption on $\alpha_1, ..., \alpha_s$, we have $\alpha_k \neq \alpha_s$. Hence there exists $\alpha_{s+1} \in \mu(i_k) \setminus \mu_1(i_k)$ different from α_{k+1}. The fact that $\alpha_1, ..., \alpha_l$ can be taken to be all different guarantees that the sequence finishes with some $l \geq 1$ such that $\alpha_l \notin \mu_1(M)$.

Take then $\mu' = (\mu_1 \setminus \{(i_1,k),(i_2,\alpha_1),...,(i_l,\alpha_{l-1})\}) \cup \{(i_1,\alpha_1),(i_2,\alpha_2),...,(i_l,\alpha_l)\}$. Moreover, we can assume without loss of generality that μ' satisfies that if $\beta \in \mu'(i)$ and $\tilde{\beta} \in \mu'(i')$ then $i \neq i'$. Indeed, since $\tilde{\beta} \neq \alpha_t$ for all $t \in \{1, ..., l\}$, we have that $\beta, \tilde{\beta} \in \mu'(i)$ can only happen if for some $t \in \{1, ..., l\}$, $\beta = \alpha_t$ and $\tilde{\beta} \in \mu_1(i_t)$. But by definition of α_t, this means that $\alpha_t \in \mu_1(i_{t+1})$ and then by Remark 2.11, we can interchange β and $\tilde{\beta}$ in μ_1 in such a way that $\beta \in \mu_1(i_t)$. This means, because of $\beta \in \mu(i_t)$, that β will be different from any α_t for all $t \in \{1, ..., l\}$.

It only remains to prove that $\mu' \in \widetilde{\mathcal{M}}_{A^\beta}(M, Q \cup \{\tilde{\beta}\})$. To see this, denote $\alpha_0 = \gamma$. Then, by optimality of μ, we have

$$\sum_{t=1}^{l} \sum_{j \in \mu(i_t)} a_{i_t j} \geq \sum_{t=1}^{l} \sum_{j \in (\mu(i_t) \setminus \{\alpha_t\}) \cup \{\alpha_{t-1}\}} a_{i_t j},$$

which leads to

$$0 \le \sum_{t=1}^{l}(a_{i_t\alpha_t} - a_{i_t\alpha_{t-1}}) = \sum_{t=1}^{l}\left(\sum_{j\in\mu'(i_t)} a_{i_tj} - \sum_{j\in\mu_1(i_t)} a_{i_tj}\right).$$

As a consequence $\mu' \in \widetilde{\mathcal{M}}_{A^\beta}(M, Q\cup\{\tilde{\beta}\})$ and $\gamma \notin \mu'(M)$.

If there is more than one object γ such that $\gamma \in \mu_1(M)\setminus\mu(M)$, repeat the above procedure starting now from μ' to construct μ'' and so on, in order to get a matching under the desired requirements.

Now, we prove condition (b) of the statement. Take $\mu \in \mathcal{M}_A(M,Q)$ and let $i_1 \in M$ be such that $\beta \in \mu(i_1)$. Let $\mu_1 \in \widetilde{\mathcal{M}}_{A^\beta}(M, Q\cup\{\tilde{\beta}\})$ be a matching that satisfies the requirements of condition (a). Trivially, if $\beta \in \mu_1(i_1)$ we are done. Otherwise, we have $\beta \notin \mu_1(i_1)$. If $\tilde{\beta} \notin \mu_1(M)$ notice that $\mu' = \{(i,j) \in M\times(Q\cup\{\tilde{\beta}\}) \mid (i,j) \in \mu\}$ belongs to $\widetilde{\mathcal{M}}_{A^\beta}(M, Q\cup\{\tilde{\beta}\})$, $\beta \in \mu'(i_1)$ and condition (a) is also satisfied. If $\tilde{\beta} \in \mu_1(i_1)$ interchange β and $\tilde{\beta}$ in μ_1 and we are done. Finally, consider that $\tilde{\beta} \in \mu_1(M)\setminus\{\mu_1(i_1)\}$. Since $\beta \in \mu(i_1)\setminus\mu_1(i_1)$ there is some object $\alpha_1 \in \mu_1(i_1)\setminus\mu(i_1)$. Notice that $\alpha_1 \neq \beta$ and $\alpha_1 \neq \tilde{\beta}$. Since μ_1 satisfies the requirements of condition (a) and $\alpha_1 \in \mu_1(M)$, we have that $\alpha_1 \in \mu(M)$. Then there is some buyer $i_2 \neq i_1$ such that $\alpha_1 \in \mu(i_2)$.

If $\beta \in \mu_1(i_2)$, define $\mu' = (\mu_1\setminus\{(i_1,\alpha_1),(i_2,\beta)\})\cup\{(i_1,\beta),(i_2,\alpha_1)\}$. Then

$$\begin{aligned}\sum_{k=1}^{2}\left(\sum_{j\in\mu'(i_k)} a_{i_kj} - \sum_{j\in\mu_1(i_k)} a_{i_kj}\right) &= a_{i_1\beta} + a_{i_2\alpha_1} - a_{i_1\alpha_1} - a_{i_2\beta} \\ &= \sum_{k=1}^{2}\sum_{j\in\mu(i_k)} a_{i_kj} - \sum_{j\in(\mu(i_1)\setminus\{\beta\})\cup\{\alpha_1\}} a_{i_1j} - \sum_{j\in(\mu(i_2)\setminus\{\alpha_1\})\cup\{\beta\}} a_{i_2j} \\ &\ge 0, \end{aligned} \tag{2.18}$$

where the inequality comes from the optimality of μ. Therefore,

$$\sum_{k=1}^{2}\sum_{j\in\mu'(i_k)} a_{i_kj} \ge \sum_{k=1}^{2}\sum_{j\in\mu_1(i_k)} a_{i_kj} \tag{2.19}$$

implies that $\mu' \in \widetilde{\mathcal{M}}_{A^\beta}(M, Q\cup\{\tilde{\beta}\})$ and it satisfies all the requirements. The case where $\tilde{\beta} \in \mu_1(i_2)$ is analogous.

If $\beta \notin \mu_1(i_2)$ and $\tilde{\beta} \notin \mu_1(i_2)$, since $\alpha_1 \in \mu(i_2)\setminus\mu_1(i_2)$, there is $\alpha_2 \in \mu_1(i_2)\setminus\mu(i_2)$. Because μ_1 satisfies the requirements of condition (a), there is some $i_3 \in M$ such that $\alpha_2 \in \mu(i_3)$. We continue with this procedure and we obtain a sequence of objects $\beta, \alpha_1, ..., \alpha_{l-1}$ with $l > 1$, each one different from $\tilde{\beta}$, such that $\beta \in \mu(i_1)\setminus\mu_1(i_1)$, $\alpha_t \in \mu_1(i_t)\setminus\mu(i_t)$, $\alpha_t \in \mu(i_{t+1})\setminus\mu_1(i_{t+1})$ with $t \in \{1,...,l-1\}$ and $\beta \in$

$\mu_1(i_l)$ or $\tilde{\beta} \in \mu_1(i_l)$. This end can be guaranteed since, by an argument similar to the one used in the proof of part (a), the elements $\alpha_1, ..., \alpha_{l-1}$ can be chosen to be all different. Assume $\beta \in \mu_1(i_l)$ (similarly for $\tilde{\beta} \in \mu_1(i_l)$), then define $\mu' = (\mu_1 \setminus \{(i_1,\alpha_1),(i_2,\alpha_2),...,(i_l,\beta)\}) \cup \{(i_1,\beta),(i_2,\alpha_1),...,(i_l,\alpha_{1-1})\}$ and by applying the same argument of (2.18) and (2.19) we obtain that $\mu' \in \widetilde{\mathcal{M}}_{A^\beta}(M, Q \cup \{\tilde{\beta}\})$ and agents unassigned by μ remain unassigned by μ' and moreover $\beta \in \mu'(i_1)$. □

The next characterization of the minimum Walrasian equilibrium price vector is inspired by a similar result of Beviá *et al.* (1999) in a model without capacity limitations. However, notice that when there are no capacity limitations an optimal matching consists of assigning each object to the buyer who values it the most. Compared to that, when buyers are limited by capacity constraints, optimal matchings need to be carefully computed and the elimination of an object may cause a reshuffling of all other objects.

Theorem 2.13. *Let* $(M,\{0\},Q,A,r)$ *be a one-seller assignment market, the minimum Walrasian equilibrium price vector of* $(M,\{0\},Q,A,r)$ *is* $\underline{p} = (\underline{p}_\beta)_{\beta \in Q}$ *where*

$$\underline{p}_\beta = \max_{\mu \in \widetilde{\mathcal{M}}(M,Q\cup\{\tilde{\beta}\})} \left\{ \sum_{(i,j)\in\mu} a_{ij} \right\} - \max_{\mu \in \mathcal{M}(M,Q)} \left\{ \sum_{(i,j)\in\mu} a_{ij} \right\} \quad \textit{for each } \beta \in Q. \tag{2.20}$$

Proof. Given $(M,\{0\},Q,A,r)$, let us prove that $(\underline{p},\mu)$ is a Walrasian equilibrium for $(M,\{0\},Q,A,r)$ where $\mu \in \mathcal{M}_A(M,Q)$ is any optimal matching and $\underline{p}$ is defined by expression (2.20).

Assume that there is some buyer i^* such that $\mu(i^*) \notin D_{i^*}(\underline{p})$. Then, $R \in D_{i^*}(\underline{p})$ with $R \in 2^Q_{r_{i^*}} \setminus \{\mu(i^*)\}$. We will prove that any object $\alpha \in R \setminus \mu(i^*)$ can be replaced with any $\beta \in \mu(i^*) \setminus R$ to obtain $(R \setminus \{\alpha\}) \cup \{\beta\} \in D_{i^*}(\underline{p})$. By way of contradiction, assume that there exists $\alpha \in R \setminus \mu(i^*)$ and $\beta \in \mu(i^*) \setminus R$ such that $(R \setminus \{\alpha\}) \cup \{\beta\} \notin D_{i^*}(\underline{p})$. Then

$$\sum_{j\in R} (a_{i^*j} - \underline{p}_j) > \sum_{j \in (R\setminus\{\alpha\})\cup\{\beta\}} (a_{i^*j} - \underline{p}_j) \text{ which implies } \underline{p}_\beta - \underline{p}_\alpha > a_{i^*\beta} - a_{i^*\alpha},$$

and from (2.20) we obtain

$$\max_{\mu \in \widetilde{\mathcal{M}}(M,Q\cup\{\tilde{\beta}\})} \left\{ \sum_{(i,j)\in\mu} a_{ij} \right\} - \max_{\mu \in \widetilde{\mathcal{M}}(M,Q\cup\{\tilde{\alpha}\})} \left\{ \sum_{(i,j)\in\mu'} a_{ij} \right\} > a_{i^*\beta} - a_{i^*\alpha}. \tag{2.21}$$

Take any $\mu_1 \in \widetilde{\mathcal{M}}_{A^\beta}(M, Q \cup \{\tilde{\beta}\})$ such that $\beta \in \mu_1(i^*)$ and if $k \in Q \setminus \mu(M)$ then

$k \notin \mu_1(M)$. Notice that such a matching does exist because of Lemma 2.12 in this Appendix. We consider the following cases.

Case 1. $\alpha \notin \mu_1(i^*)$. Define $\mu' = (\mu_1 \setminus \{(i^*, \beta)\}) \cup \{(i^*, \tilde{\alpha})\}$ where $\tilde{\alpha}$ is the replica of object $\alpha \in Q$. Then $\mu' \in \widetilde{\mathcal{M}}(M, Q \cup \{\tilde{\alpha}\})$ and

$$\max_{\mu \in \widetilde{\mathcal{M}}(M, Q \cup \{\tilde{\alpha}\})} \left\{ \sum_{(i,j) \in \mu} a_{ij} \right\} \geq \sum_{(i,j) \in \mu_1} a_{ij} - \sum_{j \in \mu_1(i^*)} a_{i^*j} + \sum_{j \in \mu'(i^*)} a_{i^*j}$$
$$= \sum_{(i,j) \in \mu_1} a_{ij} - a_{i^*\beta} + a_{i^*\tilde{\alpha}},$$

and since $a_{i^*\tilde{\alpha}} = a_{i^*\alpha}$, this contradicts (2.21).

Case 2. $\alpha \in \mu_1(i^*)$. Since $\alpha \in \mu_1(M)$, because of the properties of μ_1, then $\alpha \in \mu(M)$. Let $i' \in M$ be such that $\alpha \in \mu(i')$. Since $\alpha \in R \setminus \mu(i^*)$, we deduce $i' \neq i^*$.
Case 2.1. $\tilde{\beta} \notin \mu_1(M)$. Define $\overline{\mu} = \{(i,j) \in M \times (Q \cup \{\tilde{\beta}\}) | (i,j) \in \mu\}$ and notice that $\overline{\mu} \in \widetilde{\mathcal{M}}_{A^\beta}(M, Q \cup \{\tilde{\beta}\})$. Define $\mu' = (\overline{\mu} \setminus \{(i^*, \beta)\}) \cup \{(i^*, \tilde{\alpha})\}$. Notice that $\mu' \in \widetilde{\mathcal{M}}(M, Q \cup \{\tilde{\alpha}\})$. Then we have

$$\sum_{(i,j) \in \mu_1} a_{ij} - \max_{\mu \in \widetilde{\mathcal{M}}(M, Q \cup \{\tilde{\alpha}\})} \left\{ \sum_{(i,j) \in \mu} a_{ij} \right\}$$
$$\leq \sum_{i \in M} \left(\sum_{j \in \mu(i)} a_{ij} - \sum_{j \in \mu'(i)} a_{ij} \right) = a_{i^*\beta} - a_{i^*\tilde{\alpha}},$$

which contradicts (2.21).

Case 2.2. $\tilde{\beta} \in \mu_1(M)$. Let i'' be such that $\tilde{\beta} \in \mu_1(i'')$.
Case 2.2.1. $i' = i''$. Define $\mu' = (\mu_1 \setminus \{(i', \tilde{\beta})\}) \cup \{(i', \tilde{\alpha})\}$. We then have that, $\mu' \in \widetilde{\mathcal{M}}(M, Q \cup \{\tilde{\alpha}\})$,

$$\max_{\mu \in \widetilde{\mathcal{M}}(M, Q \cup \{\tilde{\alpha}\})} \left\{ \sum_{(i,j) \in \mu} a_{ij} \right\} \geq \sum_{(i,j) \in \mu_1} a_{ij} - \sum_{j \in \mu_1(i')} a_{i'j} + \sum_{j \in \mu'(i')} a_{i'j}$$
$$= \sum_{(i,j) \in \mu_1} a_{ij} - a_{i'\tilde{\beta}} + a_{i'\tilde{\alpha}}.$$

By the optimality of μ, we have $a_{i^*\beta} + a_{i'\alpha} \geq a_{i^*\alpha} + a_{i'\beta}$. As a consequence,

$$a_{i^*\beta} - a_{i^*\alpha} \geq a_{i'\beta} - a_{i'\alpha} = a_{i'\tilde{\beta}} - a_{i'\tilde{\alpha}} \geq \sum_{(i,j) \in \mu_1} a_{ij} - \max_{\mu \in \widetilde{\mathcal{M}}(M, Q \cup \{\tilde{\alpha}\})} \left\{ \sum_{(i,j) \in \mu} a_{ij} \right\},$$

which contradicts (2.21).

Case 2.2.2. $i' \neq i''$. Since $\alpha \in \mu(i') \setminus \mu_1(i')$, there is an object $\beta_0 \in \mu_1(i') \setminus \mu(i')$.

Case 2.2.2.a. $\beta_0 \in \mu(i^*)$. Define $\mu' = (\mu_1 \setminus \{(i',\beta_0),(i^*,\beta)\}) \cup \{(i',\tilde{\alpha}),(i^*,\beta_0)\}$. Let us continue denoting by μ' the matching that results by interchanging the roles of β and $\tilde{\beta}$ above. Then trivially $\mu' \in \widetilde{\mathcal{M}}(M,Q\cup\{\tilde{\alpha}\})$ and we get

$$\sum_{(i,j)\in\mu_1} a_{ij} - \max_{\mu\in\widetilde{\mathcal{M}}(M,Q\cup\{\tilde{\alpha}\})} \left\{\sum_{(i,j)\in\mu} a_{ij}\right\} \le \sum_{i\in M}\left(\sum_{j\in\mu_1(i)} a_{ij} - \sum_{j\in\mu'(i)} a_{ij}\right)$$
$$= (a_{i'\beta_0} - a_{i'\tilde{\alpha}}) + (a_{i^*\beta} - a_{i^*\beta_0}). \quad (2.22)$$

By optimality of μ, we have $a_{i'\beta_0} + a_{i^*\alpha} \le a_{i^*\beta_0} + a_{i'\alpha}$. Then, the following inequality $(a_{i'\beta_0} - a_{i'\alpha}) + (a_{i^*\beta} - a_{i^*\beta_0}) \le a_{i^*\beta} - a_{i^*\alpha}$ together with (2.22) contradicts (2.21).

Case 2.2.2.b. $\beta_0 \in \mu(i'')$. Define $\mu' = (\mu_1 \setminus \{(i',\beta_0),(i'',\tilde{\beta})\}) \cup \{(i',\tilde{\alpha}),(i'',\beta_0)\}$. Then

$$\sum_{(i,j)\in\mu_1} a_{ij} - \max_{\mu\in\widetilde{\mathcal{M}}(M,Q\cup\{\tilde{\alpha}\})} \left\{\sum_{(i,j)\in\mu} a_{ij}\right\} \le \sum_{i\in M}\left(\sum_{j\in\mu_1(i)} a_{ij} - \sum_{j\in\mu'(i)} a_{ij}\right)$$
$$= (a_{i'\beta_0} - a_{i'\tilde{\alpha}}) + (a_{i''\tilde{\beta}} - a_{i''\beta_0})$$
$$= (a_{i'\beta_0} - a_{i'\tilde{\alpha}}) + (a_{i''\tilde{\beta}} - a_{i''\beta_0}) + (a_{i^*\alpha} - a_{i^*\beta}) + (a_{i^*\beta} - a_{i^*\alpha}). \quad (2.23)$$

By optimality of matching μ, we have $a_{i'\beta_0} + a_{i''\beta} + a_{i^*\alpha} \le a_{i'\alpha} + a_{i''\beta_0} + a_{i^*\beta}$. Then $(a_{i'\beta_0} - a_{i'\alpha}) + (a_{i''\beta} - a_{i''\beta_0}) + (a_{i^*\alpha} - a_{i^*\beta}) + (a_{i^*\beta} - a_{i^*\alpha}) \le (a_{i^*\beta} - a_{i^*\alpha})$ together with (2.23) contradicts (2.21).

Case 2.2.2.c. $\beta_0 \notin \mu(i^*)$ and $\beta_0 \notin \mu(i'')$. Recall that $\beta_0 \in \mu_1(i')$ and by the assumptions on μ and the properties of μ_1, there is some i_1 such that $\beta_0 \in \mu(i_1)$. Since $\beta_0 \notin \mu_1(i_1)$ there is some $\beta_1 \in \mu_1(i_1) \setminus \mu(i_1)$. If $\beta_1 \in \mu(i^*)$ or $\beta_1 \in \mu(i'')$ we finish by an argument similar to cases 2.2.2.a and 2.2.2.b, respectively. Otherwise, we continue. Denote $i_0 = i'$ and assume we reach a sequence $\beta_0, \beta_1, ..., \beta_l$ such that $\beta_t \in \mu_1(i_t) \setminus \mu(i_t)$, $\beta_t \in \mu(i_{t+1}) \setminus \mu_1(i_{t+1})$ for $t \in \{0,1,...,l-1\}$. The same argument as in the proof of Lemma 2.12 allows us to choose all $\beta_0, \beta_1, ..., \beta_l$ to be different.

Hence, this procedure stops at some step $l \ge 1$ such that either object $\beta_l \in \mu_1(i^*)$ or $\beta_l \in \mu_1(i'')$. Assume that the object $\beta_l \in \mu(i^*)$ and define the following matching $\mu' = (\mu_1 \setminus \{(i',\beta_0),(i^*,\beta),(i_1,\beta_1),...,(i_l,\beta_l)\}) \cup \{(i',\tilde{\alpha}),(i_1,\beta_0),...,(i_l,\beta_{l-1}),(i^*,\beta_l)\}$.

Then

$$\sum_{(i,j)\in\mu_1} a_{ij} - \max_{\mu\in\widetilde{\mathcal{M}}(M,Q\cup\{\tilde{\alpha}\})}\left\{\sum_{(i,j)\in\mu} a_{ij}\right\} \leq \sum_{i\in M}\left(\sum_{j\in\mu_1(i)} a_{ij} - \sum_{j\in\mu'(i)} a_{ij}\right)$$

$$= (a_{i'\beta_0} - a_{i'\tilde{\alpha}}) + (a_{i^*\beta} - a_{i^*\beta_l}) + \sum_{k=1}^{l}(a_{i_k\beta_k} - a_{i_k\beta_{k-1}}). \tag{2.24}$$

By optimality of μ, we have $a_{i'\beta_0} + a_{i^*\alpha} + \sum_{k=1}^{l} a_{i_k\beta_k} \leq a_{i'\alpha} + a_{i^*\beta_l} + \sum_{k=1}^{l} a_{i_k\beta_{k-1}}$. Then,

$$(a_{i'\beta_0} - a_{i'\alpha}) + (a_{i^*\beta} - a_{i^*\beta_l}) + \sum_{k=1}^{l}(a_{i_k\beta_k} - a_{i_k\beta_{k-1}}) \leq a_{i^*\beta} - a_{i^*\alpha},$$

together with (2.24) contradicts (2.21).

For the object $\beta_l \in \mu(i'')$, we are going to take the following matching $\mu' = (\mu_1 \setminus \{(i',\beta_0),(i'',\tilde{\beta}),(i_i,\beta_1),...,(i_l,\beta_l)\}) \cup \{(i',\tilde{\alpha}),(i_1,\beta_0),...,(i_l,\beta_{l-1}),(i'',\beta_l)\}$ for $M \cup Q \cup \{\tilde{\alpha}\}$ and proceed analogously.

Therefore, we have proved that $(R\setminus\{\alpha\})\cup\{\beta\} \in D_{i^*}(\underline{p})$. By repeatedly applying this procedure that replaces an element of $R\setminus\mu(i^*)$ with another of $\mu(i^*)\setminus R$, we obtain a sequence of sets of cardinality r_{i^*} such that $R_t \in D_{i^*}(\underline{p})$ for all $t \in \{1,...,l\}$ and $R_l = \mu(i^*)$, which proves that $\mu(i^*) \in D_{i^*}(\underline{p})$. Notice finally that, from (2.20), if $\beta \in Q\setminus\mu(M)$ then $\underline{p}_\beta = 0$. This concludes that $(\underline{p},\mu)$ is a Walrasian equilibrium for the market $(M,\{0\},Q,A,r)$.

Now, we see that indeed $\underline{p} = (\underline{p}_\beta)_{\beta\in Q}$ defined in (2.20) is the minimum Walrasian equilibrium price vector. Let p be any Walrasian equilibrium price vector. Consider any $\beta \in Q$ and its replica $\tilde{\beta}$. Take an optimal matching $\mu \in \mathcal{M}_A(M,Q)$ and let $\mu' \in \widetilde{\mathcal{M}}_{A^\beta}(M,Q\cup\{\tilde{\beta}\})$ be such that if $\beta \in \mu(i)$ for some $i \in M$ and $k \in Q\setminus\mu(M)$, then $\beta \in \mu'(i)$ and $k \notin \mu'(M)$. The existence of such $\mu' \in \widetilde{\mathcal{M}}_{A^\beta}(M,Q\cup\{\tilde{\beta}\})$ is guaranteed by Lemma 2.12. Notice that if $\beta,\tilde{\beta} \notin \mu'(M)$; or $\beta \in \mu'(M)$ and $\tilde{\beta} \notin \mu'(M)$; or $\beta \notin \mu'(M)$ and $\tilde{\beta} \in \mu'(M)$, then

$$\max_{\mu\in\widetilde{\mathcal{M}}(M,Q\cup\{\tilde{\beta}\})}\left\{\sum_{(i,j)\in\mu} a_{ij}\right\} = \max_{\mu\in\mathcal{M}(M,Q)}\left\{\sum_{(i,j)\in\mu} a_{ij}\right\},$$

which implies that $\underline{p}_\beta = 0$ and, hence $\underline{p}_\beta \leq p_\beta$.

Otherwise, if $\beta,\tilde{\beta} \in \mu'(M)$, then there exists some $\alpha \in \mu(M)\setminus\mu'(M)$. Let $i' \in M$ be such that $\tilde{\beta} \in \mu'(i')$. Define $R_i = \mu'(i)$ for all $i \in M\setminus\{i'\}$ and $R_{i'} = (\mu'(i')\setminus\{\tilde{\beta}\})\cup\{\beta\}$. Since (p,μ) is a Walrasian equilibrium and because of the properties of μ', we obtain

$$\sum_{i\in M}\left(\sum_{j\in\mu(i)} a_{ij}-p_j\right)\geq\sum_{i\in M}\left(\sum_{j\in R_i} a_{ij}-p_j\right)$$
$$=\sum_{(i,j)\in\mu'} a_{ij}-\left(p_\beta+\sum_{j\in\mu'(M)\setminus\{\tilde{\beta}\}} p_j\right).$$

Then

$$\max_{\mu\in\mathcal{M}_A(M,Q)}\left\{\sum_{(i,j)\in\mu} a_{ij}\right\}-p_\alpha\geq\max_{\mu\in\widetilde{\mathcal{M}}_{A^\beta}(M,Q\cup\{\tilde{\beta}\})}\left\{\sum_{(i,j)\in\mu} a_{ij}\right\}-p_\beta,$$

which implies

$$p_\beta\geq\max_{\mu\in\widetilde{\mathcal{M}}_{A^\beta}(M,Q\cup\{\tilde{\beta}\})}\left\{\sum_{(i,j)\in\mu} a_{ij}\right\}-\max_{\mu\in\mathcal{M}_A(M,Q)}\left\{\sum_{(i,j)\in\mu} a_{ij}\right\}+p_\alpha$$
$$\geq\max_{\mu\in\widetilde{\mathcal{M}}_{A^\beta}(M,Q\cup\{\tilde{\beta}\})}\left\{\sum_{(i,j)\in\mu} a_{ij}\right\}-\max_{\mu\in\mathcal{M}_A(M,Q)}\left\{\sum_{(i,j)\in\mu} a_{ij}\right\}=\underline{p}_\beta,$$

where the last inequality holds because any Walrasian equilibrium price is non-negative, *i.e.* $p_\alpha\geq 0$ for all $\alpha\in Q$. □

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3 An implementation of the Vickrey outcome with gross-substitutes

3.1 Introduction

In this chapter, we consider a market in which many buyers and only one seller meet. The seller owns many indivisible and heterogeneous objects on sale. On the other side of the market, each buyer is interested in packages of objects and has a non-negative valuation for each of them. Preferences are assumed to be quasi-linear with respect to money and buyers' valuations satisfy monotonicity and the gross-substitutes condition.[1] An outcome for this market specifies an allocation of the objects to some buyers and the payment each of these buyers makes for his assigned package of objects.

An outstanding outcome for this market is the *Vickrey outcome*[2]which has the following interesting properties, the allocation of the objects is efficient and if a buyer gets a package, he pays the social opportunity cost of being allocating to him that package. In spite of its properties, the Vickrey outcome may generate a low revenue for the seller. To deal with this fact, it has been considered in the literature,[3] as a competitive standard, the belonging of the (Vickrey) payoff vector associated with the Vickrey outcome to the core of a related coalitional game. Ausubel & Milgrom (2002) shows that if monotonicity and the gross-substitutes condition holds, then the Vickrey payoff vector belongs to the core. Even more, it is the best core allocation for the buyers. In a recent paper, Goeree & Lien (2016) shows an impossibility result for core-selecting auctions: if the Vickrey payoff vector does not belong to the core, then no core-selecting auction exists.

In this chapter, we study whether the strategic interaction of all agents leads to the buyers-optimal core allocation. In particular, we introduce a simple mechanism which resembles a bidding procedure. While in standard auctions only buyers play, a key feature of our mechanism is that all buyers and the seller interact in a com-

[1]Condition introduced by Kelso & Crawford (1982).

[2]In fact, VCG mechanisms (Vickrey, 1961; Clarke, 1971; Groves, 1973). See *e.g.* Milgrom (2004) for details.

[3]See *e.g.* Day & Raghavan (2007) and Day & Milgrom (2008).

plete information environment. The mechanism works as follows. First, each buyer requests (for instance, bidding in a sealed envelope) a package he would like to buy and how much he would pay for it. Then, the seller decides the final allocation and the prices. In more detail, she chooses a group of buyers, and she sells a package at a price to each of these buyers in such a way that no buyer is worse off than with his initial request.

A usual requirement for allocating objects is efficiency. When buyers request packages of objects simultaneously, an overlapping problem may arise. Then, this overlapping problem may produce a loss of efficiency in the allocation. In particular, if the seller is restricted to choose only among requested packages, the outcome of a subgame perfect equilibrium (SPE) in pure strategies may not be efficient due to a coordination problem among buyers' requests. As a consequence, the outcome of this equilibrium does not belong to the core. In order to avoid this problem, in our mechanism, the seller is allowed to allocate non-requested packages as long as this does not make any buyer worse off. We prove then that in any SPE, the final allocation of the objects is efficient for the whole market. In a second result, we prove that every SPE outcome of the game coincides with the Vickrey outcome of the market.

If each buyer can acquire at most one object, Demange *et al.* (1986) proposes an allocation mechanism described as follows. Selling prices start at reservation prices; then every buyer requests the objects he would like to buy at the announced prices; if it is possible to allocate each object to a buyer who requests it, the procedure is done; otherwise, the price of the overdemanded objects is increased and the procedure is iterated with new prices. The mechanism leads to the Vickrey outcome. When each object belongs to a different seller and each agent can make at most one partnership, we are in the setting of the *assignment game* (Shapley & Shubik, 1972). For this market, Pérez-Castrillo & Sotomayor (2002) considers a buying and selling procedure to implement in SPE the best core element for the sellers (which is supported by the maximum Walrasian equilibrium price vector). The mechanism works as follows. Simultaneously, each seller announces the price of her object. Then, each buyer sequentially reports his preferred matchings, taking into account what the previous buyer has reported. If buyers play a dominant strategy consisting of truly reporting their indifferences, then the SPE outcomes correspond to the best core element for the sellers.

Our work is also related to the pay-as-bid auction of Bernheim & Whinston (1986). These authors consider a setting in which buyers want to buy packages of heterogeneous objects. In the mechanism they propose, each buyer reports how much he would pay for each package and the seller chooses an allocation of the packages. If a buyer receives a package, then he pays his bid. This game has multiple equilibria, some of them non-efficient. To overcome that, the authors restrict the strategies of the buyers, the so-called truthful strategies to obtain SPE with good properties. Notwithstanding, reporting a price for each package seems too complex for real-world markets. This is why, by assuming that not only the buyers (as in Bernheim & Whinston, 1986) but also the seller has complete information, our mechanism only requires that each buyer reports one package he would like to buy and how much he would pay for it. By allowing the seller to allocate non-requested packages, as long as no buyer is made worse off, we show that all SPE of our game is not only efficient but also leads to the Vickrey outcome of the market.

For an exchange economy, Wilson (1978) considers a mechanism in which all but one agent play as buyers, the remaining agent plays as an auctioneer. First, all buyers play simultaneously by requesting a set of feasible trades to the auctioneer. In the second stage, the auctioneer chooses for each buyer at most one trade. If a trade is chosen from a buyer, then he participates in the exchange. Otherwise, he stays with his initial resources. The author shows that there exists a (principal) Nash equilibrium which leads to a core allocation. It is shown that if the market is replicated, then the outcome given by a (principal) Nash equilibrium is a Walrasian equilibrium outcome. Though our mechanism has some resemblance with the game in Wilson (1978), nevertheless, we are in a different setting.

Our approach to the implementation problem is more similar to that in Pérez-Castrillo & Sotomayor (2002). The seller is the owner of all objects and they are all indivisible. When buyers request, they only choose one package to buy. Moreover, the seller can choose among non-requested packages. The mechanism provided tries to capture a natural bidding procedure in which all agents play in complete information. It produces efficient allocations in SPE. Moreover, it implements in SPE, the Vickrey outcome. Since the gross-substitutes condition is satisfied, this outcome is in the core, *i.e.* no coalition of players can improve its payoff by trading only among themselves.

The chapter is divided as follows. Next section is devoted to an introduction of the market and the cooperative game associated with it. In section 3, the mechanism is presented and we characterize its set of SPE outcomes. Finally, the Appendix contains some technical lemmas needed to establish the implementation result.

3.2 The market and some preliminaries

Consider a market with m buyers and only one seller. The finite set of buyers is denoted by $M = \{1, 2, ..., m\}$ and the seller is denoted by 0. She owns a finite set of indivisible objects on sale, denoted by Q. The set of objects Q includes copies of a dummy object j_0, as many as the number of buyers. Each buyer $i \in M$ has a *valuation* for each package of objects,[4] $w_i : 2^Q \to \mathbb{R}$ such that $w_i(\emptyset) = 0$ and we assume that for each buyer i and for each dummy object j_0, $w_i(R \cup \{j_0\}) = w_i(R)$ for all $R \subseteq Q \setminus \{j_0\}$. Moreover, each agent has a quasi-linear preference with respect to money on $2^Q \times \mathbb{R}$. A price vector $\rho \in \mathbb{R}^Q_+$ consists of a non-negative price $\rho_j \in \mathbb{R}_+$ for each object $j \in Q$. We interpret this price as the quantity of money to be paid by any buyer $i \in M$ if he acquires object $j \in Q$.[5] Therefore, the utility of a buyer $i \in M$ when he acquires a package $R \subseteq Q$ given a price vector $\rho \in \mathbb{R}^Q_+$ is[6] $w_i(R) - \sum_{j \in R} \rho_j$. Given a price vector $\rho \in \mathbb{R}^Q_+$, the demand set of buyer $i \in M$ consists of

$$D_i(\rho) = \left\{ R \subseteq Q \,\middle|\, w_i(R) - \sum_{j \in R} \rho_j \geq w_i(R') - \sum_{j \in R'} \rho_j \text{ for all } R' \subseteq Q \right\}.$$

Notice that the demand set of any buyer $i \in M$ is never empty. Even at sufficiently high prices, the demand of the buyer will include a package of null objects.

In all this chapter, we will assume some properties on the buyers' valuations.

Assumption 3.1. For each buyer $i \in M$, his valuation w_i satisfies

i. *Monotonicity*: $w_i(S) \geq w_i(T)$ for all $T \subseteq S \subseteq Q$.

ii. *Gross-substitutes* condition: for any two price vectors $\rho, \rho' \in \mathbb{R}^Q_+$ such that $\rho' \geq \rho$, and any $R \in D_i(\rho)$, there exists $R' \in D_i(\rho')$ such that $\{j \in R | \rho_j = \rho'_j\} \subseteq R'$.

Monotonicity says that for any buyer, the more objects in a package, the better. In particular, we have that for each $i \in M$, $w_i(S) \geq 0$ for all $S \subseteq Q$. The gross-substitutes condition was introduced by Kelso & Crawford (1982). This property has been also widely used in Gul & Stacchetti (1999). When buyers' valuations do not satisfy it, market clearing prices may not exist. In an informal way, we say that a buyer i's valuation satisfies the gross-substitutes condition if the following holds. Suppose that given a price vector ρ, buyer i wants to acquire the package of objects R. Assume that some prices increase and now we have a new vector of prices $\rho' \geq \rho$. Then the buyer i wants to acquire at least a package R' at ρ' and R' contains each object belonging to R which price has not been increased in ρ'. Different valuation functions that satisfy the two above properties can be found in Gul & Stacchetti (1999). Take for instance, a buyer i with a k_i-satiation valuation,

[4] For each set S, we will denote by $|S|$ the cardinality of S and by 2^S the power set of S.
[5] We assume that the price of each null object is always zero.
[6] We will assume that any sum over the empty set is equal to zero.

that is, i values packages up to a given capacity $k_i \in \mathbb{N}$, *i.e.* the valuation of buyer $i \in M$ on any package $Q' \subseteq Q$ satisfies

$$w_i(Q') = \max_{\substack{Q'' \subseteq Q': \\ |Q''| \leq k_i}} \left\{ \sum_{j \in Q''} w_i(\{j\}) \right\}.$$

In this case, the valuation of a buyer over a package of objects is given by the addition of values the buyer gives to objects up to capacity k_i. Notice that when $k_i = 1$ for every $i \in M$ we are in the setting of Demange *et al.* (1986).

To sum up, our market is described by $(M, \{0\}, Q, w)$ where w stands for buyers' valuations $w = (w_i)_{i \in M}$ and they satisfy Assumption 3.1.

Given a subset of buyers $S \subseteq M$, an allocation of Q to S consists of a partition of the set of all objects among agents in S, that is, $(A_i)_{i \in S}$ such that $\emptyset \neq A_i \subseteq Q$ is the set of objects allocated to $i \in S$, $\bigcup_{i \in S} A_i = Q$ and $A_i \cap A_{i'} = \emptyset$ if $i \neq i'$. We denote by $\mathcal{A}(S)$ the set of all allocations of Q to S. We say that an allocation $A \in \mathcal{A}(S)$ is efficient for S if

$$\sum_{i \in S} w_i(A_i) \geq \sum_{i \in S} w_i(A'_i) \text{ for all } A' \in \mathcal{A}(S).$$

We denote by $\mathcal{A}^*(S)$ the set of efficient allocations for S. Notice that $\mathcal{A}^*(S)$ is never empty for any non-empty coalition of buyers $S \subseteq M$.

Given a market $(M, \{0\}, Q, w)$, let us consider the coalitional game[7] associated with $(M, \{0\}, Q, w)$ as in Ausubel & Milgrom (2002). This game is denoted by $(M \cup \{0\}, v)$ where the set of players is the set of agents of the market $M \cup \{0\}$ and the worth of each coalition is given as follows. The worth of the empty coalition and the worth of any coalition formed by only one type of agents is zero because in these cases there is no trade. When a coalition is formed by a group of buyers $\emptyset \neq S \subseteq M$ and the seller, the worth is given by

$$v(S \cup \{0\}) = \max_{A \in \mathcal{A}(S)} \left\{ \sum_{i \in S} w_i(A_i) \right\}. \quad (3.1)$$

[7] A *game in coalitional form with transferable utility* is a pair (N, v) formed by a finite set of players N and a characteristic function v that assigns a real number $v(S)$ to each coalition $S \subseteq N$, with $v(\emptyset) = 0$. The core of a game (N, v) is $C(v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq N\}$. We say that a game (N, v) satisfies monotonicity if $v(T) \leq v(S)$ for all $T \subseteq S \subseteq N$.

A payoff vector $u \in \mathbb{R}^{M \cup \{0\}}$ consists of a payoff for each agent of the market. That is, u_i is the payoff associated to buyer $i \in M$ and u_0 is the seller's payoff. Following Ausubel & Milgrom (2002), a payoff vector $u^* \in \mathbb{R}^{M \cup \{0\}}$ is the *Vickrey payoff vector*[8] of the market $(M, \{0\}, Q, w)$ if for each buyer $i \in M$, we have that

$$u_i^* = v(M \cup \{0\}) - v((M \setminus \{i\}) \cup \{0\}) \tag{3.2}$$

and for the seller,

$$u_0^* = v(M \cup \{0\}) - \sum_{i \in M} u_i^*.$$

A drawback of the Vickrey payoff vector is that it may generate a low payoff for the seller (Milgrom, 2004). In order to determine when the seller's payoff is too low, we will consider the criterion used in Ausubel & Milgrom (2002), Day & Raghavan (2007) and Day & Milgrom (2008). We say that seller's payoff is unacceptably low if the Vickrey payoff vector does not belong to the core of the associated coalitional game (see expression (3.1). Ausubel & Milgrom (2002) shows that the Vickrey payoff vector belongs to the core of the game $(M \cup \{0\}, v)$ if monotonicity and the gross-substitutes condition are satisfied by each buyer valuation function. In that case, the coalitional game is *buyers-submodular*.[9] This means that the marginal contribution of any buyer to any coalition containing the seller decreases as the coalition grows larger. More precisely, the game $(M \cup \{0\}, v)$ is buyers-submodular if for all $i \in M$ and all $T \subseteq S \subseteq M \setminus \{i\}$, it holds that

$$v((T \cup \{0\}) \cup \{i\}) - v(T \cup \{0\}) \geq v((S \cup \{0\}) \cup \{i\}) - v(S \cup \{0\}). \tag{3.3}$$

An equivalent expression to (3.3) is the following one:

$$v(S \cup \{0\}) - v(T \cup \{0\}) \geq \sum_{i \in S \setminus T} \Big(v(S \cup \{0\}) - v((S \setminus \{i\}) \cup \{0\}) \Big), \tag{3.4}$$

for all $T \subseteq S \subseteq M$.

The aim of the next section is to provide a mechanism for our market such that the payoff vector in any *Subgame Perfect Equilibrium* is the Vickrey payoff vector of the market.

[8] Notice that the Vickrey payoff vector is unique. The Vickrey payoff vector is the payoff vector associated to the *Vickrey auction* or VCG mechanisms (Vickrey, 1961; Clarke, 1971; Groves, 1973). See *e.g.* Ausubel & Milgrom (2002) and Milgrom (2004) for details.

[9] In the literature, this condition is sometimes called *bidders-submodularity*.

3.3 A mechanism to implement the Vickrey outcome

In this section, we introduce a two-phase mechanism Γ in a complete information setting to implement the Vickrey payoff vector of our market with m buyers and only one seller. Let us first describe the mechanism in an informal way. First, each buyer announces a package of objects he wants to acquire and the price he would pay for it. All these requests are made simultaneously. In the second phase, the final allocation and the prices are determined: with the information of buyers' requests, the seller chooses a coalition of buyers and assigns to each of these buyers a package at a price. The seller is allowed to allocate the requested package to a buyer at his proposed price or a different package at a price that makes this buyer not worse off than with his initial request.

In more detail, let $(M,\{0\},Q,w)$ be a market that satisfies Assumption 3.1 and assume all agents have complete information. The two phases of the mechanism Γ are:

1. Buyers play simultaneously. Each buyer $i \in M$ announces a tentative package $\emptyset \neq B_i \subseteq Q$ and how much he would pay for it, $(B_i,x_i) \in 2^Q \times \mathbb{R}_+$.

   We denote by (B,x) the requests of all buyers to the seller, where $B = (B_i)_{i\in M}$ and $x = (x_i)_{i\in M}$.

2. Once the seller receives the requests of all buyers (B,x), the seller chooses a triple (S,A,p) where: a) $S \subseteq M$ is a non-empty coalition of buyers; b) $A \in \mathcal{A}(S)$ is an allocation of Q to S; and c) $p = (p_i)_{i\in S} \in \mathbb{R}^S_+$ under the constraint[10]

$$w_i(A_i) - p_i \geq w_i(B_i) - x_i \quad \text{for each } \; i \in S. \tag{3.5}$$

Once the seller has played, the mechanism Γ ends. The payoff of each agent is the following. If a buyer $i \in M$ belongs to S, he receives the package A_i, he pays p_i and his payoff is $w_i(A_i) - p_i$. If a buyer $i \in M$ does not receive a package, that is $i \in M \setminus S$, he pays nothing and his payoff is zero. The seller's payoff is $\sum_{i\in S} p_i$.

Once the mechanism Γ ends, its outcome is $(A,p) \in \mathcal{A}(S) \times \mathbb{R}^S_+$, that is, the coalition $S \subseteq M$ of buyers, the allocation chosen by the seller and the payment p_i each buyer $i \in S$ has to make for the package allocated to him. We say that an outcome $(A,p) \in \mathcal{A}(S) \times \mathbb{R}^S_+$ of the mechanism Γ is a Vickrey outcome[11] if the payoff vector

[10] Notice that the seller can at least choose (S,A,p) where $S = \{i\}$ for some $i \in M$, the allocation is $A = (A_i)$ with $A_i = Q$ and $p_i = x_i$.

[11] It is known that different allocations may produce the Vickrey payoff vector, see *e.g.* Gul & Stacchetti (1999).

associated to (A, p) is the Vickrey payoff vector of the market.

Notice that in the mechanism Γ, once buyers have made requests, the seller is able to allocate non-requested packages to buyers as long as these buyers are not made worse off than with their initial request. The following example shows that if the seller is restricted to choose which packages to allocate only among those requested that do not overlap, then in a SPE the allocation of the objects may not be efficient for the entire set of buyers.

Example 3.2. Consider a market $(M, \{0\}, Q, w)$ where $M = \{1, 2\}$, $Q = \{q_1, q_2\}$ and the valuations are $w_1(\{q_1\}) = 3$, $w_1(\{q_2\}) = 4$, $w_1(\{q_1, q_2\}) = 6$, $w_2(\{q_1\}) = 5$, $w_2(\{q_2\}) = 4$ and $w_2(\{q_1, q_2\}) = 5$.[12] Suppose that both buyers request package $\{q_1, q_2\}$ at price 5. Assume that the seller can make the final allocation only among requested packages with the proposed prices by the buyers. There is a SPE where the seller allocates the requested package to buyer 1 at the proposed price and nothing to buyer 2. The payoff vector is then $(1, 0, 5) \in \mathbb{R}^{M \cup \{0\}}$. Consider the allocation $A \in \mathcal{A}(M)$ where $A_1 = \{q_2\}$ and $A_2 = \{q_1\}$, buyer 1 pays 2.5 for A_1 and buyer 2 pays 3.5 for A_2. The payoff vector is then $(1.5, 1.5, 6)$. Each agent is strictly better off and hence $(1, 0, 5)$ does not belong to the core of the game $(M \cup \{0\}, v)$.

The previous example shows that simultaneous requests, in general, could generate a coordination problem which may damage the efficiency of the final allocation. As a consequence of that situation, the payoff vector does not belong to the core of the associated coalitional game. Also in the pay-as-bid auction of Bernheim & Whinston (1986), although agents place bids on every possible package of objects, non-efficient equilibria appear. Nevertheless, in the mechanism Γ efficiency may be improved because the seller can allocate packages that have been not requested at a price that makes the buyers who receive them no worse off than with their initial request.

The following lemma remarks that when the seller chooses the outcome that maximizes her payoff given any buyers' strategy profile, she will price packages as high as possible given constraint (3.5). As a consequence, in any SPE, inequality in (3.5) is satisfied as an equality.

Lemma 3.3. *Consider any market $(M, \{0\}, Q, w)$ and let (B, x) be an arbitrary buyers' strategy profile in the mechanism Γ. Then, in any best reply to (B, x), the*

[12] For the purposes of this example, we include no dummy objects.

seller chooses (S,A,p) *such that* $\emptyset \neq S \subseteq M$, $A \in \mathcal{A}(S)$ *and* $p \in \mathbb{R}^S_+$ *satisfies*

$$w_i(A_i) - p_i = w_i(B_i) - x_i \text{ for all } i \in S. \tag{3.6}$$

Proof. Given (B,x), let (S,A,p) be a best reply of the seller *i.e.* $\emptyset \neq S \subseteq M$, $A \in \mathcal{A}(S)$ and $w_i(A_i) - p_i \geq w_i(B_i) - x_i$ for each $i \in S$. By way of contradiction, suppose that $w_{i^*}(A_{i^*}) - p_{i^*} > w_{i^*}(B_{i^*}) - x_{i^*}$ for some $i^* \in S$. Consider the triple (S,A,p') where $\emptyset \neq S \subseteq M$, $A \in \mathcal{A}(S)$ and $p' \in \mathbb{R}^S_+$, that satisfies $p'_i = p_i$ for all $i \in S \setminus \{i^*\}$ and $p'_{i^*} = w_{i^*}(A_{i^*}) - (w_{i^*}(B_{i^*}) - x_{i^*})$. Notice that p'_{i^*} satisfies constraint (3.5), $p'_{i^*} > p_{i^*}$ and $\sum_{i \in S} p'_i > \sum_{i \in S} p_i$ which contradicts that the seller was maximizing her payoff at (S,A,p). □

Another lemma is needed before being able to present a SPE of our mechanism. This lemma says that, given any market, if the objects are efficiently allocated to a coalition S of buyers, then each buyer $i \in S$ values the package he receives above his marginal contribution to $S \cup \{0\}$ in the game $(M \cup \{0\}, v)$, see expression (3.1).

Lemma 3.4. *Consider any market* $(M,\{0\},Q,w)$ *and the related game* $(M \cup \{0\}, v)$*, see expression (3.1). For any set of buyers* $\emptyset \neq S \subseteq M$ *and any* $A = (A_i)_{i \in S} \in \mathcal{A}^*(S)$*, we have that*

$$w_i(A_i) \geq v(S \cup \{0\}) - v((S \setminus \{i\}) \cup \{0\}) \text{ for all } i \in S. \tag{3.7}$$

Proof. Take any set of buyers $\emptyset \neq S \subseteq M$, any $A = (A_i)_{i \in S} \in \mathcal{A}^*(S)$ and any $i_1 \in S$. If $S = \{i_1\}$, then $A_{i_1} = Q$ and the result follows immediately. Otherwise, if $|S| > 1$, choose $i_2 \in S \setminus \{i_1\}$ and define the following allocation $A' \in \mathcal{A}(S \setminus \{i_1\})$ where $A'_{i_2} = A_{i_2} \cup A_{i_1}$ and $A'_i = A_i$ for each $i \in S \setminus \{i_1, i_2\}$. Notice that, $w_{i_2}(A_{i_2} \cup A_{i_1}) \geq w_{i_2}(A_{i_2})$ because of the monotonicity assumption on buyers' valuations. Then, we have

$$\begin{aligned} w_{i_1}(A_{i_1}) = \sum_{i \in S} w_i(A_i) - \sum_{i \in S \setminus \{i_1\}} w_i(A_i) &\geq \sum_{i \in S} w_i(A_i) - \sum_{i \in S \setminus \{i_1\}} w_i(A'_i) \\ &\geq v(S \cup \{0\}) - v((S \setminus \{i\}) \cup \{0\}). \end{aligned}$$

□

Now, we start the analysis of the mechanism Γ. We are interested in the SPE of this mechanism in pure strategies. The following result contains the description of a SPE in which the payoff vector is the Vickrey payoff vector of the market. The following result guarantees the non-emptiness of the set of SPE of Γ.

Theorem 3.5. *For any market* $(M,\{0\},Q,w)$*, a Vickrey payoff vector is attained in a Subgame Perfect Equilibrium of the mechanism* Γ.

Proof. We show there exists a SPE of the mechanism Γ in which its payoff vector is the Vickrey payoff vector of the market $(M,\{0\},Q,w)$.

Firstly, we describe the strategy of each buyer. We denote by $M_i^v = v(M\cup\{0\}) - v((M\setminus\{i\})\cup\{0\})$ the marginal contribution of buyer $i\in M$ to the grand coalition in the game $(M\cup\{0\},v)$. Take any $A=(A_i)_{i\in M}\in\mathcal{A}^*(M)$ an efficient allocation for the coalition M of buyers. By expression (3.7) in Lemma 3.4, we have that

$$w_i(A_i)\geq M_i^v \;\text{ for all }\; i\in M.$$

As a consequence, fix an efficient allocation $A=(A_i)_{i\in M}\in\mathcal{A}^*(M)$ and, for each buyer $i\in M$, define his strategy as (B_i,x_i) where $B_i=A_i$ and $x_i\in\mathbb{R}_+$ satisfies

$$w_i(A_i)-x_i = M_i^v \;\text{ for all }\; i\in M. \tag{3.8}$$

We denote this buyers' strategy profile by (A,x).

Now, we describe the seller's strategy. On one hand, if the buyers' strategy profile is (A,x), then the seller replies with (M,A,p) where $p=(p_i)_{i\in M}\in\mathbb{R}_+^M$ satisfies $p_i=x_i$ for each $i\in M$. Otherwise, in any other strategy profile of the buyers, we assume the seller replies with an action that maximizes her payoff.

Let us argue that the above strategies form a SPE of Γ. Firstly, we have to prove that the action (M,A,p) maximizes the seller's payoff given the buyers' strategy profile (A,x). Consider any other seller's action (S',A',p') where $\emptyset\neq S'\subseteq M$, $A'\in\mathcal{A}(S')$ and $p'\in\mathbb{R}_+^{S'}$ satisfies

$$w_i(A'_i)-p'_i\geq w_i(A_i)-x_i \text{ for all } i\in S'. \tag{3.9}$$

We obtain

$$\begin{aligned}\sum_{i\in M}p_i=\sum_{i\in M}x_i=\sum_{i\in M}\Big(w_i(A_i)-M_i^v\Big)&=v(M\cup\{0\})-\sum_{i\in M}M_i^v\\ \geq v(S'\cup\{0\})-\sum_{i\in S'}M_i^v\geq\sum_{i\in S'}\Big(w_i(A'_i)-M_i^v\Big)&\geq\sum_{i\in S'}p'_i,\end{aligned}\tag{3.10}$$

where the first inequality comes from buyers-submodularity (3.4) with $S=M$ and $T=S'$, the second one by expression (3.1) and the third one by expression (3.9). This concludes that the seller's action (M,A,p) is a best reply for the seller to the buyers' strategy profile (A,x). Since by definition of the seller's strategy, she is maximizing in any other different strategy profile of the buyers, the first part of the backward induction is done.

Now, we see that any buyer $i \in M$, by requesting (A_i, x_i) where $w_i(A_i) - x_i = M_i^v$, is playing a best reply to the other agents' strategies in the simultaneous phase of the mechanism Γ. Assume buyer $i \in M$ unilaterally modifies his request to (B_i, x_i'). Recall that the seller's strategy is to maximize her payoff. Let (S', A', p') be her reply to the buyers' strategy profile in which buyer $i \in M$ requests (B_i, x_i') and the remaining buyers $k \in M \setminus \{i\}$ play (A_k, x_k) where $w_k(A_k) - x_k = M_k^v$. Two cases must be analyzed.

Case 1: $w_i(B_i) - x_i' \leq w_i(A_i) - x_i$.

If $i \notin S'$, then the payoff for buyer i is 0 which does not exceed his previous payoff $w_i(A_i) - p_i = w_i(A_i) - x_i = M_i^v \geq 0$. In case $i \in S'$, by Lemma 3.3 we know that the payoff for buyer $i \in M$ under this deviation (B_i, x_i') is $w_i(A_i') - p_i'$. Notice that $w_i(A_i') - p_i' = w_i(B_i) - x_i' \leq w_i(A_i) - x_i$, where the last inequality comes from the assumption of the Case 1. As a consequence, buyer i is not better off and we are done.

Case 2: $w_i(B_i) - x_i' > w_i(A_i) - x_i$.

Let $\varepsilon > 0$ be such that $w_i(B_i) - x_i' = (w_i(A_i) - x_i) + \varepsilon$. We first point out that for all $\overline{A} = (\overline{A}_k)_{k \in M \setminus \{i\}} \in \mathcal{A}^*(M \setminus \{i\})$ and all $k \in M \setminus \{i\}$, it holds

$$M_k^v \leq v((M \setminus \{i\}) \cup \{0\}) - v((M \setminus \{i,k\}) \cup \{0\}) \leq w_k(\overline{A}_k),$$

where the first inequality is due to buyers-submodularity (3.3) and the second one because of expression (3.7) in Lemma 3.4. As a consequence, we define $\overline{p} = (\overline{p}_k)_{k \in M \setminus \{i\}}$ such that $\overline{p}_k = w_k(\overline{A}_k) - M_k^v \geq 0$ for all $k \in M \setminus \{i\}$. By using these prices, the seller obtains

$$\begin{aligned} \sum_{k \in M \setminus \{i\}} \overline{p}_k &= \sum_{k \in M \setminus \{i\}} \left(w_k(\overline{A}_k) - M_k^v \right) = v((M \setminus \{i\}) \cup \{0\}) - \sum_{k \in M \setminus \{i\}} M_k^v \\ &= v(M \cup \{0\}) - \sum_{k \in M} M_k^v = \sum_{k \in M} p_k. \end{aligned} \tag{3.11}$$

In fact, this guarantees that by excluding the deviating buyer i and choosing $(M \setminus \{i\}, \overline{A}, \overline{p})$ the seller can achieve the same payoff as previous to the deviation.

We see now that given any best reply (S', A', p') of the seller to the buyers' strategy profile in which buyer i requests (B_i, x_i') and the remaining buyers $k \in M \setminus \{i\}$ play (A_k, x_k) where $w_k(A_k) - x_k = M_k^v$, the deviating buyer i does not belong to S'. Assume on the contrary that the seller's best reply is (S', A', p') with $i \in S'$. By Lemma 3.3 we know that $w_k(A_k') - p_k' = w_k(A_k) - x_k$ for all $k \in S' \setminus \{i\}$ and $w_i(A_i') - p_i' = w_i(B_i) - x_i'$. Define the following $p'' \in \mathbb{R}_+^{S'}$ by $p_k'' = p_k'$ for all $k \in S' \setminus \{i\}$ and $p_i'' = p_i' + \varepsilon$ where ε is defined at the beginning of Case 2. Notice that

$$\begin{aligned}p_i'' &= p_i' + \varepsilon = p_i' + ((w_i(B_i) - x_i') - (w_i(A_i) - x_i)) \\ &= (w_i(A_i') - (w_i(B_i) - x_i')) + (w_i(B_i) - x_i') - (w_i(A_i) - x_i)) \\ &= w_i(A_i') - (w_i(A_i) - x_i).\end{aligned}$$

Hence, $w_i(A_i') - p_i'' = w_i(A_i) - x_i$ and the triple (S', A', p'') satisfies expression (3.5) and hence it was also available for the seller as a reply to the buyers initial strategy (A, x). Since we already know that (M, A, p) was a best reply of the seller to the buyers' strategy profile (A, x), see expression (3.10), we have $\sum_{k \in S'} p_k'' \leq \sum_{k \in M} p_k$ and we obtain

$$\sum_{k \in S'} p_k' < \left(\sum_{k \in S'} p_k' \right) + \varepsilon = \sum_{k \in S'} p_k'' \leq \sum_{k \in M} p_k = \sum_{k \in M \setminus \{i\}} \overline{p}_k,$$

where the last equality follows from expression (3.11). Therefore, we see that the seller gets a higher payoff by choosing $(M \setminus \{i\}, \overline{A}, \overline{p})$, that excludes buyer i, instead of choosing any (S', A', p'). Hence, the unilateral deviation of buyer i in Case 2 does not make him better off.

By expression (3.8) and the fact that $p_i = x_i$ for all $i \in M$, this shows that the payoff vector in the described SPE is the Vickrey payoff vector. This concludes the proof that there is a SPE that yields a Vickrey outcome. □

Our aim now is to prove that in fact, in any SPE, each buyer gets his marginal contribution. The next proposition proves that in any SPE, the final allocation of the goods is efficient for the whole market. In fact, we prove that in any SPE, the best reply of the seller (S, A, P) to the buyers' requests is efficient for the coalition of agents, that is $A \in \mathcal{A}^*(S)$.

Proposition 3.6. *Let (B, x) be the buyers' strategy profile in an arbitrary Subgame Perfect Equilibrium of Γ and let (S, A, p) be the reply of the seller to (B, x) in this SPE. Then*

$$\sum_{i \in S} w_i(A_i) = v(S \cup \{0\}) = v(M \cup \{0\}).$$

Proof. Consider any SPE of Γ. Let (S, A, p), where $\emptyset \neq S \subseteq M$, $A = (A_i)_{i \in S} \in \mathcal{A}(S)$ and $p \in \mathbb{R}_+^S$ satisfies (3.5), be the reply of the seller to the buyers' strategy profile (B, x) in this SPE. First, we prove $\sum_{i \in S} w_i(A_i) = v(S \cup \{0\})$. Notice that by the definition of the game $(M \cup \{0\}, v)$, see (3.1), we have $\sum_{i \in S} w_i(A_i) \leq v(S \cup$

$\{0\})$. Assume on the contrary that

$$\sum_{i\in S} w_i(A_i) < v(S\cup\{0\}). \tag{3.12}$$

Case 1. There is an allocation $A' = (A'_i)_{i\in S} \in \mathcal{A}^*(S)$ such that $w_i(A'_i) \geq w_i(B_i) - x_i$ for all $i \in S$. We then define $p'_i = w_i(A'_i) - (w_i(B_i) - x_i)$ for each $i \in S$. We have

$$\begin{aligned}\sum_{i\in S} p'_i = \sum_{i\in S}\Big(w_i(A'_i) - (w_i(B_i) - x_i)\Big) &= v(S\cup\{0\}) - \sum_{i\in S}\Big(w_i(B_i) - x_i\Big)\\ > \sum_{i\in S}\Big(w_i(A_i) - (w_i(B_i) - x_i)\Big) &= \sum_{i\in S} p_i,\end{aligned}$$

where the last equality follows from Lemma 3.3. This contradicts the fact that (S,A,p) maximizes the seller's payoff given (B,x).

Case 2. For every allocation $A' = (A'_i)_{i\in S} \in \mathcal{A}^*(S)$, there is some buyer $i \in S$ such that $w_i(A'_i) < w_i(B_i) - x_i$.

By applying Lemma 3.8, in the Appendix, there exist $\emptyset \neq T \subseteq S$ and an allocation $\bar{A} = (\bar{A}_i)_{i\in T} \in \mathcal{A}^*(T)$ such that $w_i(\bar{A}_i) \geq w_i(B_i) - x_i$ for all $i \in T$. Moreover, by the assumption of Case 2 $T \neq S$, and then Lemma 3.8 guarantees

$$\sum_{i\in S\setminus T}\Big(w_i(B_i) - x_i\Big) > v(S\cup\{0\}) - v(T\cup\{0\}). \tag{3.13}$$

Define $\bar{p}_i = w_i(\bar{A}_i) - (w_i(B_i) - x_i)$ for all $i \in T$. Notice that $(T,\overline{A},\overline{p})$ satisfies the requirement in expression (3.5) given (B,x). Thus, since (S,A,p) maximizes the seller's payoff, given (B,x), we obtain

$$\begin{aligned}\sum_{i\in S}\Big(w_i(A_i) - (w_i(B_i) - x_i)\Big) = \sum_{i\in S} p_i \geq \sum_{i\in T}\bar{p}_i &= \sum_{i\in T}\Big(w_i(\bar{A}_i) - (w_i(B_i) - x_i)\Big)\\ &= v(T\cup\{0\}) - \sum_{i\in T}\Big(w_i(B_i) - x_i\Big),\end{aligned}$$

where the first equality follows from Lemma 3.3. Since $T \subsetneq S$, then we have

$$v(S\cup\{0\}) - v(T\cup\{0\}) > \sum_{i\in S} w_i(A_i) - v(T\cup\{0\}) \geq \sum_{i\in S\setminus T}\Big(w_i(B_i) - x_i\Big),$$

which contradicts (3.13). Hence $\sum_{i\in S} w_i(A_i) = v(S\cup\{0\})$.

Now, we prove $v(S\cup\{0\}) = v(M\cup\{0\})$. If $S = M$, we are done. Otherwise, by monotonicity of the game v, we have that $v(S\cup\{0\}) \leq v(M\cup\{0\})$. Assume on the contrary that $v(S\cup\{0\}) < v(M\cup\{0\})$. Let $\mathcal{I} = \{I \subseteq M\setminus S \mid v(S\cup\{0\}) < v((S\cup I)\cup\{0\})\}$. Notice that $\mathcal{I}$ is non-empty since $M\setminus S \in \mathcal{I}$. Let I be a minimal coalition in $\mathcal{I}$ with respect to the inclusion relation $\subseteq$, notice that $\emptyset \neq I$. Take any $i_1 \in I$ and all $R \subseteq S$

$$\begin{aligned} v((R\cup\{i_1\})\cup\{0\}) - v(R\cup\{0\}) &\geq v((S\cup\{i_1\})\cup\{0\}) - v(S\cup\{0\}) \\ &\geq v((S\cup I)\cup\{0\}) - v((S\cup I\setminus\{i_1\})\cup\{0\}) > 0, \end{aligned} \tag{3.14}$$

where the two first inequalities come from buyers-submodularity (3.3) and the strcit inequality from the minimality of I.

Case 1: There exists $A' \in \mathcal{A}^*(S\cup\{i_1\})$ such that $w_i(A'_i) \geq w_i(B_i) - x_i$ for all $i \in S$.

Define $p'_i = w_i(A'_i) - (w_i(B_i) - x_i)$ for each $i \in S$. Since by assumption, (S,A,p) is a best reply to (B,x) in the selected SPE, by Lemma 3.3 we have that

$$\sum_{i\in S} p_i = \sum_{i\in S} w_i(A_i) - \sum_{i\in S}\Big(w_i(B_i) - x_i\Big).$$

Now, since we have already proved $\sum_{i\in S} w_i(A_i) = v(S\cup\{0\})$, we have

$$\begin{aligned} \sum_{i\in S} p_i &= v(S\cup\{0\}) - \sum_{i\in S}\Big(w_i(B_i) - x_i\Big) < v((S\cup\{i_1\})\cup\{0\}) - \sum_{i\in S}\Big(w_i(B_i) - x_i\Big) \\ &= \sum_{i\in S\cup\{i_1\}} w_i(A'_i) - \sum_{i\in S}\Big(w_i(B_i) - x_i\Big) = \sum_{i\in S} p'_i + w_{i_1}(A'_{i_1}), \end{aligned} \tag{3.15}$$

where the inequality follows from expression (3.14). Moreover, notice that $w_{i_1}(A'_{i_1}) \geq v((S\cup\{i_1\})\cup\{0\}) - v(S\cup\{0\}) > 0$ where the first inequality follows from Lemma 3.4 and the second one from (3.14). Let $\varepsilon > 0$ be such that

$$w_{i_1}(A'_{i_1}) > \varepsilon > 0 \text{ and } \sum_{i\in S} p_i < \sum_{i\in S} p'_i + w_{i_1}(A'_{i_1}) - \varepsilon. \tag{3.16}$$

Claim: Buyer i_1 has incentives to unilaterally deviate from (B_{i_1}, x_{i_1}) to (A'_{i_1}, x'_{i_1}), where $x'_{i_1} = w_{i_1}(A'_{i_1}) - \varepsilon$.

To prove the claim, let $(\tilde{S},\tilde{A},\tilde{p})$ be the reply of the seller when only buyer i_1 deviates. Notice that if $i_1 \notin \tilde{S}$, since also $i_1 \in I \subseteq M\setminus S$, then he has no incentives to deviate because both payoffs are zero. We show that $i_1 \in \tilde{S}$. By way of contradiction, assume that $i_1 \notin \tilde{S}$. By Lemma 3.3, we know that $\tilde{p}_i = w_i(\tilde{A}_i) - (w_i(B_i) - x_i)$ for all $i \in \tilde{S}$. Recall that (S,A,p) is a best reply of the seller to the original buyers' strategies (B,x). Then, since $i_1 \notin \tilde{S}$, notice that

$$\sum_{i \in S} p_i \geq \sum_{i \in \tilde{S}} \tilde{p}_i = \sum_{i \in \tilde{S}} w_i(\tilde{A}_i) - \sum_{i \in \tilde{S}} \Big(w_i(B_i) - x_i \Big). \tag{3.17}$$

Nevertheless, consider $(S \cup \{i_1\}, A', p')$ where $A' = (A'_i)_{i \in S \cup \{i_1\}} \in \mathcal{A}^*(S \cup \{i_1\})$ as defined previously and $p'_i = w_i(A'_i) - (w_i(B_i) - x_i)$ for each $i \in S$ and $p'_{i_1} = x'_{i_1}$. Making use of expressions (3.17) and (3.16), we have

$$\sum_{i \in S \cup \{i_1\}} p'_i = \sum_{i \in S} p'_i + x'_{i_1} = \sum_{i \in S} p'_i + \Big(w_{i_1}(A'_{i_1}) - \varepsilon \Big) > \sum_{i \in S} p_i \geq \sum_{i \in \tilde{S}} \tilde{p}_i.$$

This shows that the triple $(S \cup \{i_1\}, A', p')$ is a better reply when only buyer i_1 deviates and contradicts that $(\tilde{S}, \tilde{A}, \tilde{p})$ is a best reply with $i_1 \notin \tilde{S}$. Hence, in a best reply of the seller $(\tilde{S}, \tilde{A}, \tilde{p})$ when only buyer i_1 deviates, we have that $i_1 \in \tilde{S}$. By Lemma 3.3 his payoff is $w_{i_1}(\tilde{A}_{i_1}) - \tilde{p}_{i_1} = w_{i_1}(A'_{i_1}) - x'_{i_1} = \varepsilon > 0$. This shows then that buyer i_1 has incentives to unilaterally deviate as it was claimed.

Since buyer i_1 has incentives to deviate, this contradicts that the agents follow a SPE.

Case 2: For all $A' \in \mathcal{A}^*(S \cup \{i_1\})$, there is some $i \in S$ such that $w_i(A'_i) < w_i(B_i) - x_i$.

By applying Lemma 3.9, in the Appendix, there exist $T \subsetneq S$ and $\bar{A} \in \mathcal{A}^*(T \cup \{i_1\})$ such that $w_i(\bar{A}_i) \geq w_i(B_i) - x_i$ for all $i \in T$ and

$$\begin{aligned} \sum_{i \in S \setminus T} \Big(w_i(B_i) - x_i \Big) &> v((S \cup \{i_1\}) \cup \{0\}) - v((T \cup \{i_1\}) \cup \{0\}) \\ &\geq v(S \cup \{0\}) - v((T \cup \{i_1\}) \cup \{0\}), \end{aligned} \tag{3.18}$$

where the last inequality comes from the monotonicity of v. Define $\bar{p}_i = w_i(\bar{A}_i) - (w_i(B_i) - x_i)$ for each $i \in T$. Taking (3.18) into account, we get

$$\begin{aligned} \sum_{i \in S} p_i = \sum_{i \in S} w_i(A_i) - \sum_{i \in S} \Big(w_i(B_i) - x_i \Big) &= v(S \cup \{0\}) - \sum_{i \in S} \Big(w_i(B_i) - x_i \Big) \\ < v((T \cup \{i_1\}) \cup \{0\}) - \sum_{i \in T} \Big(w_i(B_i) - x_i \Big) &= \sum_{i \in T} \bar{p}_i + w_{i_1}(A'_{i_1}), \end{aligned}$$

where the first equality follows from Lemma 3.3. Therefore, buyer i_1 has incentives to deviate (the argument follows similarly as in the previous Claim in this proof). This completes the proof and hence $v(S \cup \{0\}) = v(M \cup \{0\})$. □

The following theorem is the main result of this chapter. It shows that in any Subgame Perfect Equilibrium, the game Γ always leads to the Vickrey payoff vector of the market. This mechanism involves the strategic behavior of all agents in the market. First, we saw in the previous result that in any SPE, all objects are efficiently allocated among all buyers. Now, Theorem 3.7 shows that in any SPE each buyer will get a payoff equal to his marginal contribution to the whole market. As a consequence of the assumption on the buyers' valuations, it turns out that the Vickrey payoff vector belongs to the core of the associated coalitional game. Hence, once the agents have played any SPE of the game Γ, no coalition of agents can improve their current payoff by trading only among themselves.

Theorem 3.7. *The outcome of any SPE of* Γ *is a Vickrey outcome of the market* $(M,\{0\},Q,w)$.

Proof. Fix any SPE of Γ. Let (B,x) be the buyers' strategy profile in this SPE and denote by (S,A,p) the seller's reply to (B,x). First, take any $i_1 \in S$ and let $(S^{i_1},A^{i_1},p^{i_1})$ be as stated in Lemma 3.10 in Appendix taking $t = i_1$. Define $D \subseteq M$ by $D = S \cup S^{i_1}$.

Firstly, we show that for any $\tilde{A} \in \mathcal{A}^*(D\setminus\{i_1\})$, we have $w_i(\tilde{A}_i) \geq w_i(B_i) - x_i$ for all $i \in D\setminus\{i_1\}$. To this end, assume on the contrary there exists $\overline{A} \in \mathcal{A}^*(D\setminus\{i_1\})$ and some $i_2 \in D\setminus\{i_1\}$ such that $w_{i_2}(B_{i_2}) - x_{i_2} > w_{i_2}(\overline{A}_{i_2})$. Notice that

$$\begin{aligned} w_{i_2}(B_{i_2}) - x_{i_2} > w_{i_2}(\overline{A}_{i_2}) &\geq v((D\setminus\{i_1\})\cup\{0\}) - v((D\setminus\{i_1,i_2\})\cup\{0\}) \\ &\geq v(D\cup\{0\}) - v((D\setminus\{i_2\})\cup\{0\}) \geq 0, \end{aligned} \tag{3.19}$$

where the second inequality comes from Lemma 3.4 and the third one follows from buyers-submodularity of v.

Take an arbitrary $A' \in \mathcal{A}^*(D\setminus\{i_2\})$. If $w_i(A'_i) \geq w_i(B_i) - x_i$ for all $i \in D\setminus\{i_2\}$, define $p'_i = w_i(A'_i) - (w_i(B_i) - x_i)$ for all $i \in D\setminus\{i_2\}$. Therefore

$$\begin{aligned} \sum_{i\in S} p_i = \sum_{i\in S} w_i(A_i) - \sum_{i\in S}\Big(w_i(B_i) - x_i\Big) &= v(S\cup\{0\}) - \sum_{i\in S}\Big(w_i(B_i) - x_i\Big) \\ &= v(D\cup\{0\}) - \sum_{i\in D}\Big(w_i(B_i) - x_i\Big) \\ &< v((D\setminus\{i_2\})\cup\{0\}) - \sum_{i\in D\setminus\{i_2\}}\Big(w_i(B_i) - x_i\Big) = \sum_{i\in D\setminus\{i_2\}} p'_i, \end{aligned}$$

where the first equality comes from Lemma 3.3, the second equality from Proposition 3.6, the third equality follows from Proposition 3.6, monotonicity of v and $(w_i(B_i) - x_i) = 0$ for all $i \in S^{i_1}\setminus S$ (see Remark 3.11 in Appendix) and the inequality from (3.19). This contradicts the fact that (S,A,p) maximizes the seller's payoff. Then for all $A' \in \mathcal{A}^*(D \setminus \{i_2\})$, there is a buyer $i \in D \setminus \{i_2\}$ such that $w_i(A'_i) < w_i(B_i) - x_i$.

By applying Lemma 3.8, in the Appendix, taking $S = D \setminus \{i_2\}$, there exist $\emptyset \neq T \subsetneq D \setminus \{i_2\}$ and $\bar{A} \in \mathcal{A}^*(T)$ such that $w_i(\bar{A}_i) \geq w_i(B_i) - x_i$ for all $i \in T$ and

$$\sum_{i \in (D \setminus \{i_2\}) \setminus T} \Big(w_i(B_i) - x_i \Big) > v((D \setminus \{i_2\}) \cup \{0\}) - v(T \cup \{0\}).$$

Making use of (3.19) notice that

$$\begin{aligned} \sum_{i \in D \setminus T} \Big(w_i(B_i) - x_i \Big) &> v((D \setminus \{i_2\}) \cup \{0\}) - v(T \cup \{0\}) \\ &\quad + v(D \cup \{0\}) - v((D \setminus \{i_2\}) \cup \{0\}) = v(D \cup \{0\}) - v(T \cup \{0\}). \end{aligned} \tag{3.20}$$

Define $\bar{p}_i = w_i(\bar{A}_i) - (w_i(B_i) - x_i)$ for each $i \in T$. We have

$$\begin{aligned} v(D \cup \{0\}) - \sum_{i \in D} \Big(w_i(B_i) - x_i \Big) &= v(S \cup \{0\}) - \sum_{i \in S} \Big(w_i(B_i) - x_i \Big) \\ &= \sum_{i \in S} w_i(A_i) - \sum_{i \in S} \Big(w_i(B_i) - x_i \Big) = \sum_{i \in S} p_i \\ &\geq \sum_{i \in T} \bar{p}_i = v(T \cup \{0\}) - \sum_{i \in T} \Big(w_i(B_i) - x_i \Big), \end{aligned}$$

where the first equality follows from Proposition 3.6, monotonicity of v and $(w_i(B_i) - x_i) = 0$ for all $i \in S^{i_1} \setminus S$ (see Remark 3.11 in Appendix), the second equality comes from Proposition 3.6, the third equality comes from Lemma 3.3 and the first inequality follows from the fact that (S, A, p) maximizes the seller's payoff. Then,

$$v(D \cup \{0\}) - v(T \cup \{0\}) \geq \sum_{i \in D \setminus T} \Big(w_i(B_i) - x_i \Big).$$

This contradicts (3.20). Hence for every $i_1 \in S$ and any allocation $\tilde{A} \in \mathcal{A}^*(D \setminus \{i_1\})$, we have $w_i(\tilde{A}_i) \geq w_i(B_i) - x_i$ for all $i \in D \setminus \{i_1\}$.

Now, we prove that the outcome of any SPE is a Vickrey outcome. For any $i_1 \in S$, fix $\tilde{A} \in \mathcal{A}^*(D \setminus \{i_1\})$. Now, define a price vector $\tilde{p} = (\tilde{p}_i)_{i \in D \setminus \{i_1\}} \in \mathbb{R}_+^{D \setminus \{i_1\}}$ such that $\tilde{p}_i = w_i(\tilde{A}_i) - (w_i(B_i) - x_i)$ for all $i \in D \setminus \{i_1\}$. We have

$$\begin{aligned} v(M \cup \{0\}) - \sum_{i \in S} \Big(w_i(B_i) - x_i \Big) &= \sum_{i \in S} w_i(A_i) - \sum_{i \in S} \Big(w_i(B_i) - x_i \Big) = \sum_{i \in S} p_i \\ &\geq \sum_{i \in D \setminus \{i_1\}} \tilde{p}_i = v((D \setminus \{i_1\}) \cup \{0\}) - \sum_{i \in D \setminus \{i_1\}} \Big(w_i(B_i) - x_i \Big) \\ &= v((M \setminus \{i_1\}) \cup \{0\}) - \sum_{i \in D \setminus \{i_1\}} \Big(w_i(B_i) - x_i \Big), \end{aligned}$$

where the first equality follows from Proposition 3.6, the second equality from Lemma 3.3, the inequality since (S, A, p) maximizes the seller's payoff and the last equality from Lemma 3.13 (in Appendix). Then,

$$v(M \cup \{0\}) - v((M \setminus \{i_1\}) \cup \{0\}) \geq \sum_{i \in S} \Big(w_i(B_i) - x_i \Big) - \sum_{i \in D \setminus \{i_1\}} \Big(w_i(B_i) - x_i \Big).$$

Since $D = S \cup S^{i_1}$ and $w_i(B_i) - x_i = 0$ for all $i \in S^{i_1} \setminus S$ (see Remark 3.11 in Appendix), we obtain

$$v(M \cup \{0\}) - v((M \setminus \{i_1\}) \cup \{0\}) \geq w_{i_1}(B_{i_1}) - x_{i_1}. \tag{3.21}$$

We have proved then that $M_i^v \geq w_i(B_i) - x_i$ for all $i \in S$. We see now that $w_i(B_i) - x_i \geq M_i^v$ for all $i \in S$.

Take any buyer $i_1 \in S$, let $(S^{i_1}, A^{i_1}, p^{i_1})$ be as in the statement of Lemma 3.10 taking $t = i_1$. Then

$$\begin{aligned} v(M \cup \{0\}) - \sum_{i \in S} \Big(w_i(B_i) - x_i \Big) &= \sum_{i \in S} w_i(A_i) - \sum_{i \in S} \Big(w_i(B_i) - x_i \Big) = \sum_{i \in S} p_i \\ &= \sum_{i \in S^{i_1}} p_i^{i_1} = v(S^{i_1} \cup \{0\}) - \sum_{i \in S^{i_1}} \Big(w_i(B_i) - x_i \Big), \end{aligned}$$

where the first equality follows from Proposition 3.6, the second equality follows from Lemma 3.3, the third equality follows from Lemma 3.10 and the last equality follows from Lemma 3.10 and Lemma 3.12 in Appendix. Then,

$$v(M \cup \{0\}) - v(S^{i_1} \cup \{0\}) = \sum_{i \in S \setminus S^{i_1}} \Big(w_i(B_i) - x_i \Big) - \sum_{i \in S^{i_1} \setminus S} \Big(w_i(B_i) - x_i \Big).$$

By expression (3.33), we know that $w_i(B_i) - x_i = 0$ for all $i \in S^{i_1} \setminus S$ (see Remark

3.11 in Appendix). On one hand, we obtain

$$v(M \cup \{0\}) - v(S^{i_1} \cup \{0\}) = \sum_{i \in S \setminus S^{i_1}} \Big(w_i(B_i) - x_i \Big). \tag{3.22}$$

On the other hand, by buyers-submodularity (3.4), we have

$$v(M \cup \{0\}) - v(S^{i_1} \cup \{0\}) \geq \sum_{i \in M \setminus S^{i_1}} M_i^v \geq \sum_{i \in S \setminus S^{i_1}} M_i^v, \tag{3.23}$$

where the last inequality follows since $M_i^v \geq 0$ for each $i \in M$. Making use of expressions (3.22) and (3.23), we obtain

$$\sum_{i \in S \setminus S^{i_1}} \Big(w_i(B_i) - x_i \Big) \geq \sum_{i \in S \setminus S^{i_1}} M_i^v. \tag{3.24}$$

Making use of expressions (3.21) and (3.24), we conclude that $w_i(B_i) - x_i = M_i^v$ for all $i \in S$. This shows that in any SPE of the mechanism Γ, if a buyer i obtains a package of objects, *i.e.* $i \in S$, he requests (B_i, x_i) such that $w_i(B_i) - x_i = M_i^v$. By Lemma 3.3, we obtain that the payoff for each buyer $i \in S$ under any SPE is his marginal contribution M_i^v. Moreover, the payoff for each buyer $i \in M \setminus S$ is zero which is exactly his marginal contribution M_i^v (see Proposition 3.6). Since the reply of the seller in any SPE includes an efficient allocation for the market, we conclude that the payoff vector given in any SPE is the Vickrey payoff vector of the market. This completes the proof. □

3.4 Appendix

The following lemmas are used in the main results of this chapter.

Lemma 3.8. *Let (B, x) be the buyers' strategy profile in any SPE of the mechanism Γ, defined in page 45. For any non-empty coalition of buyers $S \subseteq M$, we have either:*

1. *there exists an efficient allocation $A = (A_i)_{i \in S} \in \mathcal{A}^*(S)$ such that $w_i(A_i) \geq w_i(B_i) - x_i$ for all $i \in S$, or*

2. *there exist a non-empty subcoalition of buyers $T \subsetneq S$ and an efficient allocation $A = (A_i)_{i \in T} \in \mathcal{A}^*(T)$ such that $w_i(A_i) \geq w_i(B_i) - x_i$ for all $i \in T$*

and

$$\sum_{i\in S\setminus T}\left(w_i(B_i)-x_i\right) > v(S\cup\{0\})-v(T\cup\{0\}).$$

Proof. First, notice that whenever $S=\{i\}$ for some $i\in M$, then for the allocation $A=(A_i)\in\mathcal{A}^*(S)$ where $A_i=Q$, it holds that

$$w_i(A_i)\geq w_i(B_i)\geq w_i(B_i)-x_i. \tag{3.25}$$

Now, we proceed to prove this lemma. Take any non-empty coalition of buyers $S\subseteq M$. If there exists an efficient allocation $A^1=(A_i^1)_{i\in S}\in\mathcal{A}^*(S)$ such that $w_i(A_i^1)\geq w_i(B_i)-x_i$ for all $i\in S$, we are done. Otherwise if for some $i\in S$ it holds that $w_i(A_i^1)<w_i(B_i)-x_i$, let $T_1=S$. Fix one $A^1=(A_i^1)_{i\in T_1}\in\mathcal{A}^*(T_1)$. We know that there is some $i_1\in T_1$ such that

$$w_{i_1}(A_{i_1}^1)<w_{i_1}(B_{i_1})-x_{i_1}. \tag{3.26}$$

Denote now $T_2=T_1\setminus\{i_1\}$ and notice that $T_2\neq\emptyset$ since otherwise $T_1=\{i_1\}$ and expression (3.26) contradicts (3.25). By inequality (3.26) and by Lemma 3.4, we have

$$\begin{aligned} w_{i_1}(B_{i_1})-x_{i_1}>w_{i_1}(A_{i_1}^1)&\geq v(T_1\cup\{0\})-v((T_1\setminus\{i_1\})\cup\{0\})\\ &=v(S\cup\{0\})-v(T_2\cup\{0\}). \end{aligned} \tag{3.27}$$

If there is an allocation $A^2=(A_i^2)_{i\in T_2}\in\mathcal{A}^*(T_2)$ such that $w_i(A_i^2)\geq w_i(B_i)-x_i$ for all $i\in T\setminus\{i_1\}$, we are done taking $T=T_2$. Otherwise, fix one $A^2=(A_i^2)_{i\in T_2}\in\mathcal{A}^*(T_2)$, we know that there is some $i_2\in T_2$ such that

$$w_{i_2}(A_{i_2}^2)<w_{i_2}(B_{i_2})-x_{i_2}. \tag{3.28}$$

Denote now $T_3=T_2\setminus\{i_2\}$ and notice that $T_3\neq\emptyset$ since otherwise $T_2=\{i_2\}$ and expression (3.28) contradicts (3.25). By inequality (3.28) and by Lemma 3.4, we have

$$w_{i_2}(B_{i_2})-x_{i_2}>w_{i_2}(A_{i_2}^2)\geq v(T_2\cup\{0\})-v((T_2\setminus\{i_2\})\cup\{0\}). \tag{3.29}$$

By adding (3.27) and (3.29), we get

$$\sum_{i\in S\setminus T_3}\left(w_i(B_i)-x_i\right) > v(S\cup\{0\})-v(T_3\cup\{0\}).$$

By proceeding recursively, we construct a sequence of sets $\{T_1, ..., T_{k+1}\}$ such that $T_1 = S$, $T_l \setminus T_{l+1} = \{i_l\}$ for $l \in \{1, ..., k\}$, $A^l \in \mathcal{A}^*(T_l)$ for $l \in \{1, ..., k+1\}$, $w_{i_l}(A^l_{i_l}) < w_{i_l}(B_{i_l}) - x_{i_l}$ for $l \in \{1, ..., k\}$ and

$$\sum_{i \in S \setminus T_{l+1}} \Big(w_i(B_i) - x_i \Big) > v(S \cup \{0\}) - v(T_{l+1} \cup \{0\}) \text{ for } l \in \{1, ..., k\}.$$

Now if there is an efficient allocation $A^{k+1} \in \mathcal{A}^*(T_{k+1})$ such that $w_i(A^{k+1}_i) \geq w_i(B_i) - x_i$ for all $i \in T_{k+1}$, we are done taking $T = T_{k+1}$. Otherwise, we continue the procedure one more step. Notice that, since S is finite, we will eventually reach T_r with $|T_r| = 1$. In that case, this procedure ends because of expression (3.25). $\square$

The following lemma proceeds similarly to the previous one. Then, we state it without a proof.

Lemma 3.9. *Let (B, x) be the buyers' strategy profile in any SPE of the mechanism Γ defined in page 45. For any non-empty coalition of buyers $S \subsetneq M$ and any $i' \in M \setminus S$, we have either:*

1. *there exists an efficient allocation $A = (A_i)_{i \in S \cup \{i'\}} \in \mathcal{A}^*(S \cup \{i'\})$ such that $w_i(A_i) \geq w_i(B_i) - x_i$ for all $i \in S$, or*
2. *there exist a subcoalition of buyers $T \subsetneq S$ and an efficient allocation $A = (A_i)_{i \in T \cup \{i'\}} \in \mathcal{A}^*(T \cup \{i'\})$ such that $w_i(A_i) \geq w_i(B_i) - x_i$ for all $i \in T$ and*

$$\sum_{i \in S \setminus T} \Big(w_i(B_i) - x_i \Big) > v((S \cup \{i'\}) \cup \{0\}) - v((T \cup \{i'\}) \cup \{0\}).$$

The next lemma shows that in any SPE, and for any buyer who receives a package, there is an alternative action that makes the seller indifferent.

Lemma 3.10. *Consider any SPE of Γ. Let (B, x) be the buyers' strategy profile in this SPE and let (S, A, p) be the reply of the seller to (B, x) in this SPE. For each buyer $t \in S$, there is a triple (S^t, A^t, p^t) such that $S^t \subseteq M \setminus \{t\}$, $A^t \in \mathcal{A}(S^t)$, $p^t = (p^t_i)_{i \in S^t} \in \mathbb{R}^{S^t}_+$ satisfies $w_i(A^t_i) - p^t_i \geq w_i(B_i) - x_i$ for all $i \in S^t$ and*

$$\sum_{i \in S^t} p^t_i = \sum_{i \in S} p_i. \tag{3.30}$$

Proof. Notice that if $p_t = 0$, it is straightforward to find a triple $(S \setminus \{t\}, A', p')$ that satisfies equality (3.30). Assume now that there is a buyer $t \in S$ such that

$p_t > 0$ and that for all $S^t \subseteq M \setminus \{t\}$, all $A^t \in \mathcal{A}(S^t)$ and all $(p_i^t)_{i \in S^t} \in \mathbb{R}_+^{S^t}$ such that $w_i(A_i^t) - p_i^t \geq w_i(B_i) - x_i$ for all $i \in S^t$, it holds

$$\sum_{i \in S^t} p_i^t < \sum_{i \in S} p_i. \tag{3.31}$$

Since (3.31) holds for all $S^t \subseteq M \setminus \{t\}$, all $A^t \in \mathcal{A}(S^t)$ and all $(p_i^t)_{i \in S^t} \in \mathbb{R}_+^{S^t}$ such that $w_i(A_i^t) - p_i^t \geq w_i(B_i) - x_i$ for all $i \in S^t$, buyer t has incentives to deviate by slightly decreasing the price he proposed to pay for the package B_t in such a way that the inequality (3.31) is still maintained. This contradicts that the agents follow a SPE. □

An immediate consequence of the previous lemma is the following. Since (S,A,p) is a best reply of the seller to (B,x), we have that (S^t,A^t,p^t) is also a best reply of the seller to (B,x) and a direct consequence of Lemma 3.3 is that

$$w_i(A_i^t) - p_i^t = w_i(B_i) - x_i \text{ for all } i \in S^t. \tag{3.32}$$

The next remark easily follows from Lemma 3.10.

Remark 3.11. Consider any SPE of Γ. Let (B,x) be the buyers' strategy profile in this SPE and let (S,A,p) be the reply of the seller to (B,x) in this SPE. Consider any $t \in S$ and let (S^t,A^t,p^t) be as in the statement of Lemma 3.10. Then

$$w_i(B_i) - x_i = 0 \text{ for all } i \in S^t \setminus S. \tag{3.33}$$

Otherwise, if for some $i \in S^t \setminus S$, $w_i(B_i) - x_i > 0$, buyer i has incentives to increase a bit x_i, to make the seller choose (S^t,A^t,p^t) instead of (S,A,p), so that the buyer i gets a positive payoff.

Lemma 3.12 is related with the previous lemma. It says that for each buyer t who gets a package in equilibrium, if we consider (S^t,A^t,p^t) as stated in Lemma 3.10, then A^t is efficient for S^t.

Lemma 3.12. *Consider any SPE of* Γ. *Let* (B,x) *be the buyers' strategy profile in this SPE and let* (S,A,p) *be the reply of the seller to* (B,x) *in this SPE. For each buyer* $t \in S$, *let* (S^t,A^t,p^t) *be as in the statement of Lemma 3.10. Then*

$$\sum_{i \in S^t} w_i(A_i^t) = v(S^t \cup \{0\}).$$

Proof. Let (S,A,p) be the reply of the seller to (B,x) under a SPE. Take any buyer $t \in S$ and let (S^t,A^t,p^t) be as in the statement of Lemma 3.10, *i.e.* $S^t \subseteq M \setminus \{t\}$,

$A^t = (A_i^t)_{i\in S^t} \in \mathcal{A}^*(S^t)$ and $p^t \in \mathbb{R}_+^{S^t}$ satisfies (3.5). Assume on the contrary that $\sum_{i\in S^t} w_i(A_i^t) < v(S^t \cup \{0\})$, see expression (3.1) for the definition of $(M \cup \{0\}, v)$.

If there is an allocation $A' \in \mathcal{A}^*(S^t)$ such that $w_i(A'_i) \geq w_i(B_i) - x_i$ for all $i \in S^t$, then define $p'_i = w_i(A'_i) - (w_i(B_i) - x_i)$ for each $i \in S^t$. We have

$$\begin{aligned}\sum_{i\in S} p_i = \sum_{i\in S^t} p_i^t = \sum_{i\in S^t} w_i(A_i^t) - \sum_{i\in S^t}\Big(w_i(B_i) - x_i\Big)\\ < v(S^t \cup \{0\}) - \sum_{i\in S^t}\Big(w_i(B_i) - x_i\Big) = \sum_{i\in S^t} p'_i,\end{aligned}$$

where the first equality comes from Lemma 3.10 and the second equality from expression (3.32). This contradicts the fact that (S, A, p) maximizes the seller's payoff. Therefore, for every $A' \in \mathcal{A}^*(S^t)$ there is some $i \in S^t$ such that $w_i(A'_i) < w_i(B_i) - x_i$.

By applying Lemma 3.8 to $S = S^t$, there exist $\emptyset \neq T \subsetneq S^t$ and an efficient allocation $\bar{A} \in \mathcal{A}^*(T)$ such that $w_i(\bar{A}_i) \geq w_i(B_i) - x_i$ for all $i \in T$ and

$$\sum_{i\in S^t\setminus T}\Big(w_i(B_i) - x_i\Big) > v(S^t \cup \{0\}) - v(T \cup \{0\}). \tag{3.34}$$

Define $\bar{p}_i = w_i(\bar{A}_i) - (w_i(B_i) - x_i)$ for all $i \in T$. Since (S, A, p) maximizes the seller's payoff, we obtain

$$\begin{aligned}\sum_{i\in S^t}\Big(w_i(A_i^t) - (w_i(B_i) - x_i)\Big) &= \sum_{i\in S^t} p_i^t = \sum_{i\in S} p_i\\ &\geq \sum_{i\in T} \bar{p}_i = \sum_{i\in T} w_i(\overline{A}_i) - \sum_{i\in T}\Big(w_i(B_i) - x_i\Big)\\ &= v(T \cup \{0\}) - \sum_{i\in T}\Big(w_i(B_i) - x_i\Big),\end{aligned}$$

where the first equality comes from expression (3.32) in Appendix and the second equality from Lemma 3.10 in Appendix. Since $T \subseteq S^t$, we have

$$\sum_{i\in S^t} w_i(A_i^t) - v(T \cup \{0\}) \geq \sum_{i\in S^t\setminus T}\Big(w_i(B_i) - x_i\Big),$$

This contradicts (3.34). Hence $\sum_{i\in S^t} w_i(A_i^t) = v(S^t \cup \{0\})$. □

The next lemma relates Lemma 3.10 and 3.12. For any equilibrium of Γ, let S be the set of buyers who get a package, $t \in S$ and S^t be as stated in Lemma 3.10. Then the worth attained by the coalitions $(S \setminus \{t\}) \cup S^t$ and $M \setminus \{t\}$ is the same.

Lemma 3.13. *Consider any SPE of* Γ. *Let* (B,x) *be the buyers' strategy profile in this SPE and let* (S,A,p) *be the reply of the seller to* (B,x) *in this SPE. For each buyer* $t \in S$, *let* (S^t,A^t,p^t) *be as in the statement of Lemma 3.10 and let* $D = S \cup S^t$. *Then*

$$v((D \setminus \{t\}) \cup \{0\}) = v((M \setminus \{t\}) \cup \{0\}).$$

Proof. Fix a SPE. Let (S,A,p) be the reply of the seller to (B,x) in this SPE. Take any buyer $t \in S$. Let (S^t,A^t,p^t) be as in the statement of Lemma 3.10 and let $D = S \cup S^t$. First, we show that $v((S^t \cup \{i_1\}) \cup \{0\}) = v(S^t \cup \{0\})$ for any $i_1 \in M \setminus D$. Assume on the contrary that there is some $i_1 \in M \setminus D$ such that

$$v((S^t \cup \{i_1\}) \cup \{0\}) > v(S^t \cup \{0\}). \tag{3.35}$$

If there is an allocation $A' \in \mathcal{A}^*(S^t \cup \{i_1\})$ such that $w_i(A'_i) \geq w_i(B_i) - x_i$ for all $i \in S^t$, then define $p'_i = w_i(A'_i) - (w_i(B_i) - x_i)$ for each $i \in S^t$. Since (S,A,p) maximizes the seller's payoff and because of Lemma 3.10, we have

$$\begin{aligned}\sum_{i\in S} p_i = \sum_{i\in S^t} p^t_i &= v(S^t \cup \{0\}) - \sum_{i\in S^t}\Big(w_i(B_i) - x_i\Big)\\ &< v((S^t \cup \{i_1\}) \cup \{0\}) - \sum_{i\in S^t}\Big(w_i(B_i) - x_i\Big)\\ &= \sum_{i\in S^t\cup\{i_1\}} w_i(A'_i) - \sum_{i\in S^t}\Big(w_i(B_i) - x_i\Big) = \sum_{i\in S^t} p'_i + w_{i_1}(A'_{i_1}),\end{aligned}$$

where the first equality comes from Lemma 3.10 in Appendix, the second equality comes from expression (3.32) and Lemma 3.12 both in Appendix. Therefore, buyer i_1 has incentives to deviate following a similar argument as in expression (3.15). This contradicts that the agents play a SPE. Therefore, for any $A' \in \mathcal{A}^*(S^t \cup \{i_1\})$ there is some $i \in S^t$ such that $w_i(A'_i) < w_i(B_i) - x_i$.

By applying Lemma 3.9 to $S = S^t$ and $i' = i_1$, there exist $T \subsetneq S$ and $\bar{A} \in \mathcal{A}^*(T \cup \{i_1\})$ such that $w_i(\bar{A}_i) \geq w_i(B_i) - x_i$ for all $i \in T$ and

$$\begin{aligned}\sum_{i\in S^t\setminus T}\Big(w_i(B_i) - x_i\Big) &> v((S^t \cup \{i_1\}) \cup \{0\}) - v((T \cup \{i_1\}) \cup \{0\})\\ &\geq v(S^t \cup \{0\}) - v((T \cup \{i_1\}) \cup \{0\}),\end{aligned} \tag{3.36}$$

where the second inequality comes from monotonicity of v. Define $\bar{p}_i = w_i(\bar{A}_i) -$

$(w_i(B_i)-x_i)$ for each $i \in T$. Taking (3.36) into account, we obtain

$$\begin{aligned}\sum_{i\in S} p_i &= \sum_{i\in S^t} p_i^t = v(S^t \cup \{0\}) - \sum_{i\in S^t}\Big(w_i(B_i)-x_i\Big)\\ &< v((T\cup\{i_1\})\cup\{0\}) - \sum_{i\in T}\Big(w_i(B_i)-x_i\Big)\\ &= \sum_{i\in T\cup\{i_1\}} w_i(\bar{A}_i) - \sum_{i\in T}\Big(w_i(B_i)-x_i\Big) = \sum_{i\in T}\bar{p}_i + w_{i_1}(\bar{A}_{i_1}),\end{aligned}$$

where the first equality comes from Lemma 3.10 in Appendix, the second equality comes from expression (3.32) and Lemma 3.12 both in Appendix. Therefore, buyer i_1 has incentives to deviate following a similar argument as in expression (3.15). This contradicts that the agents play a SPE. Hence, we have shown that for any $i_1 \in M\setminus D$

$$v((S^t\cup\{i_1\})\cup\{0\}) = v(S^t\cup\{0\}) \tag{3.37}$$

Now, we prove $v((D\setminus\{t\})\cup\{0\}) = v((M\setminus\{t\})\cup\{0\})$. Assume on the contrary that $v((D\setminus\{t\})\cup\{0\}) < v((M\setminus\{t\})\cup\{0\})$. Define the following set of buyers $\mathcal{I} = \{I \subseteq M\setminus D \mid v((D\setminus\{t\})\cup\{0\}) < v(((D\setminus\{t\})\cup I)\cup\{0\})\}$. Notice that $\mathcal{I}$ is non-empty and $I \neq \emptyset$ for all $I \in \mathcal{I}$. Take any $I \in \mathcal{I}$ and any $i_1 \in I$, we have that

$$\begin{aligned}& v((S^t\cup\{i_1\})\cup\{0\}) - v(S^t\cup\{0\})\\ &\geq v(((D\setminus\{t\})\cup\{i_1\})\cup\{0\}) - v((D\setminus\{t\})\cup\{0\})\\ &\geq v(((D\setminus\{t\})\cup I)\cup\{0\}) - v(((D\setminus\{t\})\cup I\setminus\{i_1\})\cup\{0\}) > 0,\end{aligned} \tag{3.38}$$

where the first two inequalities come from buyers-submodularity (3.3) and the last inequality since $i_1 \in I \in \mathcal{I}$. However, expression (3.38) contradicts (3.37). This completes the proof. Hence, $v((D\setminus\{t\})\cup\{0\}) = v((M\setminus\{t\})\cup\{0\})$. □

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4 Axioms for the minimum Walrasian equilibrium in assignment problems with unitary demands

4.1 Introduction

We consider assignment problems in which a set of indivisible objects needs to be allocated to a group of agents. Each object can be allocated to at most one agent and monetary transfers are allowed. Every agent has a preference over bundles made of one object and money to be paid. The objects may have one or several owners, but the owners do not play any strategic role. It may be assumed that they handle their objects to a centralized institution. This institution will choose an allocation rule to distribute the objects among the agents according to some specified criterion.

An allocation rule specifies, for each preference profile of the agents, an assignment of objects and the corresponding price to be paid for each object. *Efficiency* and *fairness* can be identified as usual requirements to distribute resources.[1] In this chapter, we study these and other properties for such rules.

In assignment problems where agents have (strict) preferences over the objects, the objects have priorities over the agents and monetary transfers are not allowed, an allocation rule specifies only an assignment of the objects to the agents for each preference profile. In this setting, an outstanding rule is the one that selects the allocation determined by Gale & Shapley (1962) deferred acceptance algorithm. In Kojima & Manea (2010) a first axiomatization of the deferred acceptance is obtained with unspecified priorities. The aim of this chapter is to perform a similar analysis for markets with indivisibilities in which money is allowed.

The purpose of this chapter is to study the minimum Walrasian equilibrium rule in two settings. The first one considers general preferences and in the second setting agents have quasi-linear preferences.

[1] See for instance Maskin (1987) and Alkan *et al.* (1991).

Each agent has a preference relation over bundles made of an object and some amount of money, satisfying the following requirements: *money monotonicity*, agents always prefer to pay less money for each object; *finiteness*, the willingness to pay for each object is finite; *continuity*, for any bundle, the upper and the lower contour sets are both closed; and *weak preference for real objects*, it is always weakly preferred to obtain any real object than nothing. The first three requirements have been considered in the literature, see for instance Alkan *et al.* (1991) or Morimoto & Serizawa (2015).[2]

The related two-sided assignment market has been widely studied in Demange & Gale (1985). It is known that the set of Walrasian equilibria is non-empty. Even more, the set of Walrasian equilibrium price vectors has a complete lattice structure which guarantees the existence of the minimum Walrasian equilibrium price vector. The authors prove also that each buyer (agent) has no incentive to manipulate the minimum Walrasian equilibrium rule. In Miyake (1998), the strategic behaviour of the agents is analysed. It is shown that among all Walrasian equilibrium selecting rules, the unique that is not manipulable by any buyer (agent) is the one that produces a minimum Walrasian equilibrium.

In Morimoto & Serizawa (2015) it is shown that, under the assumption that there are strictly more agents than objects, the minimum Walrasian equilibrium rule is characterized by: *individual rationality*, *efficiency*, *strategy-proofness* and *no subsidy for losers*. An allocation rule is *individually rational* if each agent would be no worse off if he had received no object and paid nothing. An assignment of objects is *efficient* if there is no other assignment that makes some agent strictly better off and each remaining agent no worse off. A rule is *efficient* if it always selects efficient assignments. *Strategy-proofness* states that no agent has incentives to unilaterally misrepresent his true preference. *No subsidy for losers* requires that if an agent gets a null object, then he will pay a non-negative price for it. Notwithstanding, as they point out, this characterization does not hold when the number of objects exceeds the number of agents. This is also true with quasi-linear preferences. Consider the case with only one agent with quasi-linear preferences and two objects on sale. Take into account the following rule: assign to the agent his most valued object at a price of zero and let the other object with a strictly positive price. It is easy to see that the four axioms are satisfied but the price vector given by the rule is not a Walrasian equilibrium price vector. Hence, the minimum Walrasian equilibrium cannot be characterized with those four axioms in a general setting that does not impose any condition on the number of objects with respect to the number of agents.

The first result of the present chapter is a characterization of the minimum Wal-

[2] Morimoto & Serizawa (2015) imposes also a strict preference for real objects.

rasian equilibrium rule. We allow for any number of objects and any number of agents. We prove that the minimum Walrasian equilibrium rule is characterized by *desirability of positively priced objects, non-wastefulness, envy-freeness* and *monotonicity with respect to willingness to pay*. *Desirability of positively priced objects* requires that if an object j has a strictly positive price, it is because there is an agent who strictly prefers to acquire j for free to his assigned bundle. *Non-wastefulness* is a weak condition of efficiency. It is inspired by *non-wastefulness* used in Kojima & Manea (2010). This axiom says that if an object is not assigned, it is because every agent weakly prefers the object he gets at its given price to this non-assigned object for free. *Envy-freeness*[3] reflects a notion of fairness and means that every agent weakly prefers his allocation to any other object at its given price. It is known that when the number of agents exceeds the number of objects, *envy-freeness* implies *efficiency* (Svensson, 1983). However, this is not true when there are more objects than agents. We prove that *efficiency* is implied by *envy-freeness*, *non-wastefulness* and *desirability of positively priced objects*. The last axiom of our characterization is *monotonicity with respect to willingness to pay*: if an agent misrepresents his true preference by weakly decreasing his willingness to pay for all objects, he will not be better off. That is, *monotonicity with respect to willingness to pay* avoids incentives to declare a lower willingness to pay for the objects.

An additional axiom is introduced with the name of *antimonotonicity*. This property avoids incentives to lie in the following way. If an agent misrepresents his true preference by weakly decreasing or increasing his willingness to pay for all objects, he will not be better off. It is clear that *strategy-proofness* implies *antimonotonicity* and *antimonotonicity* implies *monotonicity with respect to willingness to pay*. Similar to the result shown in Miyake (1998), we prove that among Walrasian equilibrium selecting rules, the unique that satisfies *monotonicity with respect to willingness to pay* is the one that produces the minimum one.

The second part of this chapter assumes that each agent has a preference that is quasi-linear with respect to money. That is to say, each agent has a non-negative valuation for each object in terms of money. These valuation functions are a particular case of the model with gross-substitutes[4] considered in Gul & Stacchetti (1999), where it is proved the non-emptiness of the set of Walrasian equilibria and the existence of the minimum Walrasian equilibrium prices.

In this setting, a well-known mechanism that produces the minimum Walrasian equilibrium is an ascending multi-item auction introduced by Demange *et al.* (1986). Even more, the minimum Walrasian equilibrium rule is characterized by *strategy-proofness, individual rationality* and *efficiency*, see for instance Holmström (1979)

[3]As considered in Alkan *et al.* (1991).

[4]Condition introduced by Kelso & Crawford (1982).

and Chew & Serizawa (2007). However, the assumption that the number of objects is lower than the number of agents is crucial for this characterization.

The characterization given in the first part of the present chapter, holds for the domain of quasi-linear preferences. However, in order to obtain a better understanding of the minimum Walrasian equilibrium rule, we provide an alternative characterization in the setting of quasi-linear preferences without *strategy-proofness* and without making use of *envy-freeness*. We prove that an allocation rule is the minimum Walrasian equilibrium rule if and only if it satisfies *desirability of positively priced objects*, *efficiency*, *antimonotonicity* and *non-wastefulness*. This characterization holds for any number of objects and any number of agents.

The chapter is organized as follows. In the next section, the model with general preferences is introduced and a characterization of the minimum Walrasian equilibrium on this domain is provided. Section 3 is devoted to study the case with quasi-linear preferences.

4.2 General preferences

The problem concerns the allocation of a finite set O of indivisible objects (or indivisibilities) to a finite set N of agents. We will denote by n the number of agents and by o the number of objects. The set O of objects includes at least one real object and a null object q_0 with as many copies of it as the number of agents. We denote by o^* the number of real objects. Each indivisible object can be assigned to at most one agent. There is also a perfectly divisible object, called money. The endowment of each agent $i \in N$ consists of enough money to buy any object.

Each agent i has a complete and transitive binary relation over bundles made of an object $j \in O$ and money $m \in \mathbb{R}$. That is to say, a preference (relation) R_i on $O \times \mathbb{R}$. Let P_i and I_i be the strict preference and the indifference relations associated with R_i, respectively. Given a preference relation R_i and a bundle $(j,m) \in O \times \mathbb{R}$, the upper contour set of (j,m) at R_i consists of $\overline{C}(R_i,(j,m)) = \{(j',m') \in O \times \mathbb{R} \mid (j',m')\ R_i\ (j,m)\}$. Similarly for the lower contour set of (j,m) at R_i, $\underline{C}(R_i,(j,m)) = \{(j',m') \in O \times \mathbb{R} \mid (j,m)\ R_i\ (j',m')\}$. We assume that for each agent $i \in N$, each preference R_i satisfies the following properties.

(A.1) Money monotonicity: For each $j \in O$ and each $m, m' \in \mathbb{R}$, if $m > m'$, then $(j,m')\ P_i\ (j,m)$.

(A.2) Finiteness: For each $j, j' \in O$ and each $m \in \mathbb{R}$, there exist $m', m'' \in \mathbb{R}$ such that $(j',m')\ R_i\ (j,m)$ and $(j,m)\ R_i\ (j',m'')$.

(A.3) Continuity: For each bundle $(j,m) \in O \times \mathbb{R}$, $\overline{C}(R_i,(j,m))$ and $\underline{C}(R_i,(j,m))$ are both closed sets.

(A.4) Weak preference for real objects: For each $j \in O$, $(j,0)\ R_i\ (q_0,0)$.

Conditions (A.1), (A.2) and (A.3) have been considered in the literature, see for instance Alkan *et al.* (1991) and Morimoto & Serizawa (2015). In Morimoto & Serizawa (2015), a strong version of (A.4) is used. If a preference R_i satisfies properties (A.1), (A,2), (A.3) and (A.4), then R_i will be called a classical preference. We denote by $\mathcal{R}_\mathcal{C}$ the set of classical preferences. Given a preference $R_i \in \mathcal{R}_\mathcal{C}$, similar to Alkan *et al.* (1991), we say that m represents the *willingness to pay* of i for object j at R_i when $(j,m)\ I_i\ (q_0,0)$ and it will be denoted by $WP(R_i,j)$. Because of (A.2) and (A.3), such an amount m does exist, and, by (A.1), it is unique.

A profile of classical preferences consists of an $n-$tuple of classical preferences, one for each agent $i \in N$ and it will be denoted by $R=(R_i)_{i\in N} \in \mathcal{R}_\mathcal{C}^n$, where $\mathcal{R}_\mathcal{C}^n$ stands for the set of all $n-$tuples of classical preferences. For any $S \subseteq N$, R_{-S} stands for $(R_i)_{i\in N\setminus S}$, if $S=\{i'\}$, we write $R_{-i'}$.

A preference R_i is a *quasi-linear preference* if for each object $j \in O$ there is a "valuation"[5] a_{ij} such that: (i) $a_{ij} \geq 0$, (ii) $a_{iq_0}=0$ and for each $(j,m) \in O \times \mathbb{R}$ and each $(j',m') \in O \times \mathbb{R}$, $(j,m)\ R_i\ (j',m')$ if and only if $a_{ij}-m \geq a_{ij'}-m'$. We will denote by $\mathcal{R}_\mathcal{Q}$ the set of quasi-linear preferences. Notice that $\mathcal{R}_\mathcal{Q} \subsetneq \mathcal{R}_\mathcal{C}$.

An *assignment* of the objects is an $n-$tuple $z=(z_i)_{i\in N}=(z_1,...,z_n) \in O^n$ such that if $i \neq i'$, then $z_i \neq z_{i'}$. We denote by $O_z \subseteq O$ the set of objects assigned under z and let $\mathcal{Z}$ be the set of all assignments. A *price vector*, $p=(p_j)_{j\in O} \in \mathbb{R}_+^o$, consists of a price $p_j \geq 0$ for each object $j \in O$ and $p_{q_0}=0$ for each copy of the null object q_0. We denote by $\mathcal{P}$ the set of all price vectors. Given an assignment $z \in \mathcal{Z}$ and a price vector $p \in \mathcal{P}$, an *allocation* consists of a bundle (z_i,p_{z_i}) for each agent i, and it will be denoted by $((z_i,p_{z_i}))_{i\in N}$.

Now, we are going to define the Walrasian equilibrium. Let us first introduce the notion of demand set. Given $R_i \in \mathcal{R}_\mathcal{C}$ and $p \in \mathcal{P}$, we denote by $D(R_i,p) \subseteq O$ the *demand set of* i *with* R_i *at* p, that is

$$D(R_i,p)=\left\{j \in O \;\middle|\; (j,p_j)\ R_i\ (j',p_{j'}) \text{ for all } j' \in O\right\}.$$

[5] The valuation of agent i for object j is given by the willingness to pay.

The demand set of any agent is never empty. At sufficiently high prices, the demand set can be formed only by null objects.

A *Walrasian equilibrium* is a price vector together with an assignment of objects such that each agent obtains an object belonging to his demand set given the prices, and the price of each non–assigned object is zero.

Definition 4.1. Given a preference profile $R \in \mathcal{R}^n_{\mathcal{C}}$, a pair $(p,z) \in \mathcal{P} \times \mathcal{Z}$ is a *Walrasian equilibrium* at R, if the following two conditions hold:

- $z_i \in D(R_i, p)$ for all $i \in N$,
- $p_j = 0$ for all $j \in O \setminus O_z$.

If (p,z) is a Walrasian equilibrium at $R \in \mathcal{R}^n_{\mathcal{C}}$, we say that p is a Walrasian equilibrium (WE) price vector. We denote by $\mathcal{P}^{W(R)} \subseteq \mathcal{P}$ the set of WE price vectors at R.

It is known by Demange & Gale (1985), that for any $R \in \mathcal{R}^n_{\mathcal{C}}$:

- The set of Walrasian equilibria is not empty.
- $\mathcal{P}^{W(R)}$ has a complete lattice structure.[6]
- There exists a unique $\underline{p} \in \mathcal{P}^{W(R)}$ such that $\underline{p} \leq p$ for all $p \in \mathcal{P}^{W(R)}$, named the minimum WE (MWE) price vector. Then, a Walrasian equilibrium $(\underline{p}, z)$ will be called a minimum Walrasian equilibrium.

It is known by Morimoto & Serizawa (2015) that any minimum Walrasian equilibrium satisfies the *demand connectedness* property for any $R \in \mathcal{R}^n_{\mathcal{C}}$:

- *Demand connectedness*. Let $(\underline{p}, z)$ be a minimum Walrasian equilibrium at R. For each $j \in O$ with $\underline{p}_j > 0$, there is a sequence of different agents $\{1, 2, ..., i\}$ such that: $z_1 = q_0$ or $\underline{p}_{z_1} = 0$; $z_l \neq q_0$ and $\underline{p}_{z_l} > 0$ for each $l \in \{2, ..., i-1\}$; $z_i = j$ and $\{z_l, z_{l+1}\} \subseteq D(R_l, \underline{p})$ for each $l \in \{1, ..., i-1\}$.[7]

Mechanisms used to allocate objects which produce Walrasian equilibria have been widely considered in the literature, see for instance Demange *et al.* (1986) and Pérez-Castrillo & Sotomayor (2002). These types of mechanisms, which satisfy

[6] Let $p, p' \in \mathcal{P}$ be two price vectors. Their *join* $q = p \vee p'$ and *meet* $s = p \wedge p'$ are the price vectors defined by $q_j = \max\{p_j, p'_j\}$ and $s_j = \min\{p_j, p'_j\}$, respectively. A set of price vectors $\mathcal{P}' \subseteq \mathcal{P}$ is a lattice if for any two $p, p' \in \mathcal{P}'$, both $p \vee p' \in \mathcal{P}'$ and $p \wedge p' \in \mathcal{P}'$. The lattice is complete if for any $\mathcal{P}'' \subseteq \mathcal{P}'$, $inf \mathcal{P}'' \in \mathcal{P}'$ and $sup \mathcal{P}'' \in \mathcal{P}'$.

[7] See Miyake (1998) and Roth & Sotomayor (1990) for a detailed discussion of this property.

interesting properties related to fairness and efficiency, are examples of (allocation) rules that produce assignments and prices according to the preferences of the agents.

An *(allocation) rule* f consists of a pair of maps (f^o, f^m) from preference profiles to assignments and price vectors. That is, for any $R \in \mathcal{R}^n_{\mathcal{C}}$, the rule f produces an assignment $f^o(R) \in \mathcal{Z}$ and a price vector $f^m(R) \in \mathcal{P}$. In other words, given $R \in \mathcal{R}^n_{\mathcal{C}}$, $j = f^o_i(R)$ is agent i's assignment and $f^m_j(R)$ the price agent i has to pay for his assigned object j at R. We will use the following notation to denote the bundle given by f to agent i at R, $f_i(R) = (f^o_i(R), f^m_{f^o_i(R)}(R))$. With some abuse of notation and when no confusion arises, we will denote the price of object $f^o_i(R)$ assigned to i by $f^m_i(R)$. Hence the allocation will be denoted by $(f_i(R))_{i\in N} = ((f^o_i(R), f^m_i(R)))_{i\in N}$. Notice that when there are more objects than agents some objects are not assigned by the rule, but the rule specifies a price for each object. Therefore, any agent i is able to compare his bundle $(f^o_i(R), f^m_i(R))$ with any other bundle, including those made of a non-assigned object j at its given price, that is, $(j, f^m_j(R))$.

A rule f is a *Walrasian equilibrium (WE) rule* if for each preference profile $R \in \mathcal{R}^n_{\mathcal{C}}$, $(f^m(R), f^o(R)) \in \mathcal{P} \times \mathcal{Z}$ is a Walrasian equilibrium at R. A rule f is the *minimum Walrasian equilibrium (MWE) rule* if for each $R \in \mathcal{R}^n_{\mathcal{C}}$, $(f^m(R), f^o(R)) \in \mathcal{P} \times \mathcal{Z}$ is a minimum Walrasian equilibrium at R.[8]

In the following, we introduce some classical and new properties (axioms) for allocation rules. A natural requirement in allocation problems is efficiency. It entails the maximization of the welfare of all agents as a whole. Given $R \in \mathcal{R}^n_{\mathcal{C}}$, $z, z' \in \mathcal{Z}$ and $p, p' \in \mathcal{P}$, the allocation $((z_i, p_{z_i}))_{i\in N}$ Pareto-dominates $((z'_i, p'_{z'_i}))_{i\in N}$ at R if

- $(z_i, p_{z_i})\ R_i\ (z'_i, p'_{z'_i})$ for all $i \in N$,
- $(z_i, p_{z_i})\ P_i\ (z'_i, p'_{z'_i})$ for some $i \in N$,
- $\sum_{i\in N} p_{z_i} \geq \sum_{i\in N} p'_{z'_i}$.

We say that an allocation $((z_i, p_{z_i}))_{i\in N}$ satisfies *efficiency* at $R \in \mathcal{R}^n_{\mathcal{C}}$ if no allocation Pareto-dominates it at R. Now, we introduce efficiency as an axiom.

[8] The MWE rule is not unique because there might be different assignments of objects compatible with the MWE price vector. Notwithstanding, for each pair of Walrasian equilibria $(z, \underline{p})$ and $(z', \underline{p})$ where $\underline{p}$ is the MWE price vector, we have that $(z_i, \underline{p}_{z_i})\ I_i\ (z'_i, \underline{p}_{z'_i})$ for all $i \in N$. Therefore, more precisely, we characterize the class of MWE rules. For a similar treatment of rules that are uniquely defined up to some indifference, see for instance Adachi (2014) or Morimoto & Serizawa (2015).

Definition 4.2. *Efficiency*: A rule f satisfies efficiency (EFF) if for each $R \in \mathcal{R}_{\mathcal{C}}^n$, $(f_i(R))_{i \in N}$ satisfies efficiency at R.

Morimoto & Serizawa (2015) shows that the MWE rule is efficient. The following axiom, individual rationality, implies that every agent will pay for his assignment, at most, his willingness to pay for it.

Definition 4.3. *Individual rationality*: A rule f satisfies individual rationality (IR) if for all $R \in \mathcal{R}_{\mathcal{C}}^n$ and all $i \in N$,

$$f_i(R)\ R_i\ (q_0, 0).$$

Definition 4.4. *Desirability of positively priced objects*: A rule f satisfies desirability of positively priced objects (DO) if for all $R \in \mathcal{R}_{\mathcal{C}}^n$ and all $j \in O$,

$$f_j^m(R) > 0 \Rightarrow (j, 0)\ P_i\ f_i(R) \text{ for some } i \in N.$$

Desirability of positively priced objects says that whenever an object j has a positive price, there is an agent i who is getting a bundle (k, p_k) but strictly prefers j for free.

Definition 4.5. *Non–wastefulness*: A rule f satisfies non–wastefulness (NW) if for all $R \in \mathcal{R}_{\mathcal{C}}^n$, all $i \in N$ and all $j \in O \setminus O_{f^o(R)}$,

$$f_i(R)\ R_i\ (j, 0).$$

The notion of non-wastefulness is used in Kojima & Manea (2010) for an allocation problem in which money is not allowed. It is a weak condition of efficiency. Our axiom says that if an object j is not assigned, it is because every agent weakly prefers his allocation rather than acquiring j for free.

The following property, envy-freeness, has been widely studied in Svensson (1983) and in Alkan *et al.* (1991). We consider the definition of envy-freeness used in Alkan *et al.* (1991). It requires that every agent weakly prefers his allocation to any other object at its given price.

Definition 4.6. *Envy–freeness*: A rule f satisfies envy–freeness (EF) if for all $R \in \mathcal{R}_{\mathcal{C}}^n$ and all $i \in N$,

$$f_i(R)\ R_i\ (j, f_j^m(R)) \text{ for all } j \in O.$$

Envy-freeness reflects an idea of fairness. When there are as many agents as objects, it is known that envy-freeness implies efficiency (Svensson, 1983). Nonetheless, that implication does not hold when there are strictly more real objects than agents. A natural question is then under which conditions envy-freeness implies efficiency. The following result shows that if an allocation rule satisfies EF, NW and DO, then it satisfies also EFF and IR.

Proposition 4.7. *On the domain $\mathcal{R}_{\mathcal{C}}^n$, if a rule f is EF, NW and DO then f is EFF and IR.*

Proof. It is straightforward to see that EF implies IR because we have as many null objects as the number of agents. Suppose that f satisfies EF, NW and DO but not EFF. This implies that there is a preference profile $R \in \mathcal{R}_{\mathcal{C}}^n$ such that $(f(R)_i)_{i\in N}$ is Pareto-dominated by some allocation $((z_i, p_i))_{i\in N}$ at R. We are going to consider two cases.

Case 1: $O_{f^o(R)} = O_z$. By EF, we have that for every $i \in N$

$$f_i(R)\ R_i\ (j, f_j^m(R)) \text{ for all } j \in O.$$

Now, since $((z_i, p_i))_{i\in N}$ Pareto-dominates $(f_i(R))_{i\in N}$, we have that $(z_i, p_{z_i})\ R_i\ f_i(R)$ for all $i \in N$ and $(z_{i'}, p_{z_{i'}})\ P_{i'}\ f_{i'}(R)$ for some agent i'. Therefore, for each agent i, we have

$$(z_i, p_{z_i})\ R_i\ f_i(R)\ R_i\ (j, f_j^m(R)) \text{ for all } j \in O,$$

in particular $(z_i, p_{z_i})\ R_i\ (z_i, f_{z_i}^m(R))$ with a strict preference for some agent i'. This implies that

$$\sum_{i\in N} p_{z_i} < \sum_{i\in N} f_{f_i^o(R)}^m(R),$$

which contradicts that $((z_i, p_i))_{i\in N}$ Pareto-dominates $(f_i(R))_{i\in N}$ at R.

Case 2: $O_{f^o(R)} \neq O_z$. By NW of f, for each $j \in O \setminus O_{f^o(R)}$, $f_i(R)\ R_i\ (j, 0)$ for every $i \in N$. Now, DO of f implies $f_j^m(R) = 0$ for each $j \in O \setminus O_{f^o(R)}$. Define $\overline{N} = \{i \in N | f_i^o(R) \in O_{f^o(R)} \cap O_z\}$. Take any $j \in O_z \setminus O_{f^o(R)}$ which means that $j = z_{i_1}$ for some $i_1 \in N$. Since $((z_i, p_i))_{i\in N}$ Pareto-dominates $(f_i(R))_{i\in N}$ at R, we have that

$$(j, p_j)\ R_{i_1}\ f_{i_1}(R)\ R_{i_1}\ (j, 0),$$

and from money monotonicity, $p_j = 0$. That is, $p_j = 0$ for each $j \in O_z \setminus O_{f^o(R)}$.

For any $i \in \overline{N}$, making use of EF, we have

$$(z_i, p_{z_i})\ R_i\ f_i(R)\ R_i\ (j, f_j^m(R)) \text{ for all } j \in O,$$

with a strict preference for some agent $i' \in \overline{N}$. This implies that

$$\sum_{i\in N} p_{z_i} = \sum_{i\in \overline{N}} p_{z_i} < \sum_{i\in \overline{N}} f^m_{f^o_i(R)}(R), \tag{4.1}$$

where the first equality comes from the fact that $p_j = 0$ for each $j \in O_z \setminus O_{f^o(R)}$. Therefore, (4.1) contradicts that $((z_i, p_{z_i}))_{i\in N}$ Pareto-dominates $(f_i(R))_{i\in N}$ at R. $\square$

In order to introduce the following axiom, we say that $R_i \in \mathcal{R_C}$ is a monotonic transformation of $R'_i \in \mathcal{R_C}$ (R_i m.t. R'_i) if the willingness to pay for any real object j increases in some amount $\varepsilon_j \geq 0$ at R_i with respect to his willingness to pay for it at R'_i. That is, for all $j \in Q$ with $j \neq q_0$ there is $\varepsilon_j \geq 0$ such that

$$(j,m)\ I_i\ (q_0,0) \Rightarrow (j,m-\varepsilon_j)\ I'_i\ (q_0,0).$$

The next property, monotonicity with respect to willingness to pay, avoids incentives to misrepresents the willingness to pay by decreasing it.

Definition 4.8. *Monotonicity with respect to willingness to pay*: A rule f satisfies monotonicity with respect to willingness to pay (MWP) if for all $R \in \mathcal{R}^n_{\mathcal{C}}$, all $i \in N$ and all $R'_i \in \mathcal{R_C}$ such that R_i m.t. R'_i, then

$$f_i(R)\ R_i\ f_i(R_{-i}, R'_i).$$

An immediate consequence of monotonicity with respect to willingness to pay is the following: the price of any object j assigned to some agent i does not decrease if object j is again assigned to i when he weakly decreases his willingness to pay. A strong version of the monotonicity with respect to willingness to pay is strategy-proofness.

Definition 4.9. *Strategy-proofness*: A rule f satisfies strategy-proofness (SP) if for all $R \in \mathcal{R}^n_{\mathcal{C}}$, all $i \in N$ and all $R'_i \in \mathcal{R_C}$,

$$f_i(R)\ R_i\ f_i(R_{-i}, R'_i).$$

It is shown in Demange & Gale (1985) that the MWE rule is not manipulable, that is to say, it satisfies strategy-proofness. For the domain of quasi-linear preferences, see Leonard (1983) for a similar result.

Under the assumption that there are strictly more agents than objects to be allocated, Morimoto & Serizawa (2015) characterizes the rule that produces the minimum Walrasian equilibrium with strategy-proofness, individual rationality, efficiency and no subsidy for losers.[9] Nevertheless, the assumption that there are more agents than objects is crucial in their characterization. As the authors point out, when the number of objects exceeds the number of agents, the characterization does not hold. Without restriction on the number of objects in relationship with the number of agents, the following result gives a characterization of the minimum Walrasian equilibrium in which envy-freeness and monotonicity with respect to willingness to pay play a fundamental role.

Theorem 4.10. *On the domain $\mathcal{R}^n_{\mathcal{C}}$, a rule f is the MWE rule if and only if f satisfies MWP, EF, NW and DO.*

Proof. The "if" part. First, we prove that if f satisfies EF, DO and NW, then for any $R \in \mathcal{R}^n_{\mathcal{C}}$, $(f^m(R), f^o(R))$ is a Walrasian equilibrium at R.

Take any $R \in \mathcal{R}^n_{\mathcal{C}}$. Notice that EF of f implies that $f^o_i(R) \in D(R_i, f^m(R))$ for each $i \in N$ and NW of f implies that if $j \in O \setminus O_{f^o(R)}$, then $f_i(R)\ R_i\ (j,0)$ for all $i \in N$. Therefore, DO of f implies that, if $j \in O \setminus O_{f^o(R)}$, then $f^m_j(R) = 0$. Hence $(f^m(R), f^o(R))$ is a Walrasian equilibrium at R. It remains to see that it is a minimum Walrasian equilibrium at R.

Let f satisfy the four axioms and $\underline{p} \in \mathcal{P}^{W(R)}$ be the MWE price vector at R. Assume by contradiction that the price vector $f^m(R)$ is not the MWE price vector at R. Let $(\underline{p}, \underline{z})$ be a minimum Walrasian equilibrium at R. Therefore, $\underline{p}_j \leq f^m_j(R)$ for all $j \in O$ and for some object k, $\underline{p}_k < f^m_k(R)$. Since $f^m(R) \in \mathcal{P}^{W(R)}$, we have that $f^m_k(R) > 0$.

Since $f^m(R) \in \mathcal{P}^{W(R)}$ and $f^m_k(R) > 0$, by definition of Walrasian equilibrium, k is assigned to some agent. Let i be the agent such that $f^o_i(R) = k$. On one hand, notice that if $\underline{z}_i \neq k$, then

$$(\underline{z}_i, \underline{p}_{\underline{z}_i})\ R_i\ (k, \underline{p}_k)\ P_i\ (k, f^m_k(R)) = f_i(R).$$

On the other hand, if $\underline{z}_i = k$, then

$$(\underline{z}_i, \underline{p}_{\underline{z}_i})\ P_i\ (k, f^m_k(R)) = f_i(R).$$

By continuity of R_i, there exists a small $\varepsilon > 0$ such that

$$(\underline{z}_i, \underline{p}_{\underline{z}_i} + \varepsilon)\ P_i\ (k, f^m_k(R)).$$

[9] No subsidy for losers means that if an agent gets a null object, she/he will pay a non-negative price for it.

Consider the preference $R'_i \in \mathcal{R}_Q$ for agent i such that $(j,0)\ I'_i\ (q_0,0)$ for every $j \in O \setminus \{\underline{z}_i\}$ and $(\underline{z}_i, \underline{p}_{\underline{z}_i} + \varepsilon)\ I'_i\ (q_0,0)$. Since $(\underline{z}_i, \underline{p}_{\underline{z}_i} + \varepsilon)\ P_i\ (k, f^m_k(R))\ R_i\ (q_0,0)$, there is an $\alpha > 0$ such that $(\underline{z}_i, \underline{p}_{\underline{z}_i} + \varepsilon + \alpha)\ I_i\ (q_0,0)$. Hence the willingness to pay for object $\underline{z}_i$ at R_i is greater than the willingness to pay for $\underline{z}_i$ at R'_i. Moreover, the willingness to pay for any other object at R'_i is 0, hence R_i m.t. R'_i. Denote by $R' = (R_{-i}, R'_i)$ the new preference profile. Notice that $(\underline{z}, \underline{p})$ is still a Walrasian equilibrium at R'. Even more, we see below that in every Walrasian equilibrium at R', agent i gets object $\underline{z}_i$, which follows from the Decomposition Lemma in Demange & Gale (1985).

Suppose by way of contradiction that there exists a Walrasian equilibrium (p^*, z^*) at R' such that $z^*_i \neq \underline{z}_i$. By definition of R'_i we know that $WP(R'_i, j) = 0$ for all $j \in O \setminus \{\underline{z}_i\}$. Notice that if $z^*_i \neq q_0$ there is another Walrasian equilibrium (p', z') in which $z'_i = q_0$ and for all remaining agents $t \in N \setminus \{i\}$, $z'_t = z^*_t$ and the price vector is the same $p' = p^*$. Hence, w.l.o.g. assume that in (p^*, z^*), $z^*_i = q_0$. Making use of the Decomposition Lemma (see Appendix for a proof applied to our setting), Corollary 4.26 (in Appendix) establishes that if for some Walrasian equilibrium $(\underline{p}, \underline{z})$ there is an agent $i \in N$ such that $(\underline{z}_i, \underline{p}_{\underline{z}_i})\ P'_i\ (q_0,0)$, then in each Walrasian equilibrium he will obtain a real object j, which is a contradiction with $z^*_i = q_0$.

Since in every Walrasian equilibrium at R' agent i will obtain $\underline{z}_i$, then the maximum-WE price of object $\underline{z}_i$ at R' is at most $\underline{p}_{\underline{z}_i} + \varepsilon$. Hence

$$f_i(R')\ R_i\ (\underline{z}_i, \underline{p}_{\underline{z}_i} + \varepsilon)\ P_i\ f_i(R).$$

Therefore, agent i has incentives to misrepresent his preference, by decreasing, his willingness to pay. This completes the proof of the "if" part.

The "only if" part. Let f be the MWE rule. It easily follows that f satisfies EF. Since the price of any non-assigned object is zero, envy-freeness implies NW. As it was remarked, f satisfies SP, then f satisfies also MWP. Finally, we will see that DO is satisfied. Assume by way of contradiction that there is an object $j \in O$ such that $f^m_j(R) > 0$ and $f_i(R)\ R_i\ (j,0)$ for all $i \in N$. Consider $(p,z) \in \mathcal{P} \times \mathcal{Z}$ such that the price for each object $k \in O \setminus \{j\}$ is $p_k = f^m_k(R)$ and $p_j = 0$. The assignment of the objects is given by $z_i = f^o_i(R)$ for each $i \in N$. Notice that (p,z) is a Walrasian equilibrium and contradicts the minimality of $f^m(R)$. Therefore, there is an agent i such that $(j,0)\ P_i\ f_i(R)$ and DO holds. □

The previous result gives a new characterization for an allocation rule which always selects, for any preference profile, a minimum Walrasian equilibrium. This characterization relies on the axiom MWP. Intuitively, the result of Theorem 4.10 is divided in two parts. First, we show that if a rule satisfies EF, NW and DO, then it always selects a Walrasian equilibrium for any preference profile. Taking this into account, we show that uniqueness is provided by MWP. This axiom captures the attempt of the agents to misrepresent the maximum amount of money they do want to pay for each object. That is to say, MWP requires that no agent will be better off if he decreases the willingness to pay for some or all objects.

Since SP implies MWP, an immediate consequence of the previous theorem is the following corollary.

Corollary 4.11. For any $R \in \mathcal{R}_{\mathcal{C}}^n$, a rule f is the MWE rule if and only if f satisfies SP, EF, NW and DO.

The next property takes into account monotonic transformation of the willingness to pay. Suppose that an agent tries to manipulate an allocation rule by weakly increasing or decreasing the willingness to pay for all objects, the next axiom *antimonotonicity* avoids this behavior. It is clear that monotonicity with respect to willingness to pay is a necessary condition for antimonotonicity.

Definition 4.12. *Antimonotonicity*: A rule f satisfies antimonotonicity (AM) if for all $R \in \mathcal{R}_{\mathcal{C}}^n$, all $i \in N$ and all $R_i' \in \mathcal{R}_{\mathcal{C}}$, such that R_i m.t. R_i' then

$$f_i(R)\ R_i\ f_i(R_{-i}, R_i') \text{ and } f_i(R_{-i}, R_i')\ R_i'\ f_i(R).$$

If we focus only on WE rules, the following corollary of the previous theorem gives a characterization of the MWE rule.

Corollary 4.13. On the domain $\mathcal{R}_{\mathcal{C}}^n$, a WE rule f is the MWE rule if and only if f satisfies Antimonotonicity.

The next examples show the independence of the axioms used in Theorem 4.10.

Example 4.14. Given any $R \in \mathcal{R}_{\mathcal{C}}^n$, define $\lambda = \max_{i \in N, k \in O}\{WP(R_i, k)\}$. Consider a rule f such that for each $R \in \mathcal{R}_{\mathcal{C}}^n$, each agent gets a null object and

$$f_j^m(R) = \begin{cases} \lambda & \text{if } j \in \{k \in O | \text{ there is some } i \in N \text{ such that } WP(R_i, k) > 0\}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Notice that the rule satisfies DO, MWP and EF, but NW is violated.

Example 4.15. Consider a rule f that when there are more than two agents and only one object coincides with the maximum Walrasian equilibrium rule and otherwise with the minimum Walrasian equilibrium rule. Then this rule f satisfies EF, NW and DO, but MWP is violated.

Notice that in the previous example, the maximum Walrasian equilibrium price of the object is given by the highest willingness to pay of the object. This agent has incentives to announce a lower willingness to pay.

Example 4.16. Consider a rule f such that for each $R \in \mathcal{R}_{\mathcal{C}}^n$, $(\underline{p}, f^o(R)) \in \mathcal{P} \times \mathcal{Z}$ is a minimum Walrasian equilibrium. Now we are going to define $f^m(R)$. For each assigned object $j \in O_{f^o(R)}$, $f_j^m(R) = \underline{p}_j$ and $f_j^m(R) = 1$ for each $j \in O \setminus O_{f^o(R)}$. Notice that the rule satisfies EF, NW and MWP, but DO is violated.

Example 4.17. Given any $R \in \mathcal{R}_{\mathcal{C}}^n$, label every agent and every object. When there are more real objects than agents, consider the MWE rule. When there are more agents than real objects, let f be the rule that assigns object 1 to agent 1, object 2 to agent 2, and so on. Each real object at a price of zero. This rule satisfies MWP, NW, DO but EF is not satisfied.

4.3 Quasi-linear preferences

In this section, it is assumed that agents have quasi-linear preferences in money. It is known that when there are strictly more agents than objects, the MWE rule is characterized by strategy-proofness, efficiency and individual rationality, see Holmström (1979) and Chew & Serizawa (2007). Notwithstanding, this characterization does not hold when the number of objects exceeds the number of agents. Our characterization for the MWE rule in Section 2 also holds for the domain of quasi-linear preferences. However, we now provide a new characterization for the quasi-linear domain without making use of envy-freeness or strategy-proofness. Therefore, we perform a broader analysis for the MWE rule which does not depend on the number of agents and the number of objects.

A preference R_i of an agent $i \in N$ is a *quasi-linear preference* if for each object $j \in O$ there is a “valuation" a_{ij} such that: (i) $a_{ij} \geq 0$, (ii) $a_{iq_0} = 0$ and for each $(j,m) \in O \times \mathbb{R}$ and each $(j',m') \in O \times \mathbb{R}$, $(j,m)\ R_i\ (j',m')$ if and only if $a_{ij} - m \geq a_{ij'} - m'$. In this section, we write $a_i = (a_{ij})_{j \in O}$ to represent the preference of agent i and denote by $\mathcal{A}$ the set of all (vector of) valuations. Then, $a = (a_i)_{i \in N} \in \mathcal{A}^n$ denotes a profile of valuations and $\mathcal{A}^n$, the set of all possible profiles. Moreover, for each $t \in N$, (a_{-t}) stands for $(a_i)_{i \in N \setminus \{t\}}$. In this section, we write $f(a)$ instead of $f(R)$. That is to say, $f^o(a) \in \mathcal{Z}$ and $f^m(a) \in \mathcal{P}$.

In Gul & Stacchetti (1999), the Walrasian equilibrium has been widely studied. In particular, the non-emptiness of the set of Walrasian equilibria and the complete lattice structure of the set of WE price vectors are proved under the presence of gross-substitutes.[10] In fact, the authors establish an expression to obtain the MWE prices in terms of agents' valuations. Since the quasi-linear preferences considered in this section satisfy the gross-substitutes condition and are monotonic, the results in Gul & Stacchetti (1999) apply in our setting. Therefore, in order to be able to express and make use of the expression for the MWE price given in Gul & Stacchetti (1999), for an arbitrary object k in this section, we will introduce an identical copy of object k, denoted by $\tilde{k}$.[11]

With some abuse of notation, we will denote the set of assignments of $O \cup \{\tilde{k}\}$ by $\widetilde{\mathcal{Z}}^k$. Now, we can give an expression for the MWE prices. The MWE price of object k is given by

$$\underline{p}_k = \max_{z \in \widetilde{\mathcal{Z}}^k} \left\{ \sum_{i \in N} a_{iz_i} \right\} - \max_{z \in \mathcal{Z}} \left\{ \sum_{i \in N} a_{iz_i} \right\}. \tag{4.3}$$

Since preferences are quasi-linear, an assignment z of the objects is efficient for $a \in \mathcal{A}^n$ if

$$z \in argmax_{z' \in \mathcal{Z}} \left\{ \sum_{i \in N} a_{iz'_i} \right\}. \tag{4.4}$$

Following Chew & Serizawa (2007), a rule on the domain of quasi-linear preferences $\mathcal{A}$ satisfies efficiency if for every $a \in \mathcal{A}^n$, it selects an efficient assignment at a.

Remark 4.18. In the Appendix of the Chapter 2 of this Thesis it is shown that if for some efficient assignment $z \in \mathcal{Z}$, $k = z_t$ for some agent $t \in N$, then there exists an assignment $z' \in \widetilde{\mathcal{Z}}^k$ such that $k = z'_t$ and

$$\sum_{i \in N} a_{iz'_i} = \max_{z \in \widetilde{\mathcal{Z}}^k} \left\{ \sum_{i \in N} a_{iz_i} \right\}. \tag{4.5}$$

When the number of agents exceeds the number of objects, it is well known that on the domain of quasi-linear preferences with unitary demands, the MWE prices coincide with the prices given by the Vickrey rule.[12] Even more, the MWE rule is characterized by strategy-proofness, individual rationality and efficiency, see for instance, Holmström (1979) and Chew & Serizawa (2007). However, the assump-

[10] Introduced by Kelso & Crawford (1982).

[11] We assume that this copy $\tilde{k}$ of object $k \in O$ satisfies that each agent is always indifferent to acquire, at any quantity of money, the original object k or its copy $\tilde{k}$.

[12] See Gul & Stacchetti (1999), Demange *et al.* (1986) and Leonard (1983).

tion that the number of objects is lower than the number of agents is crucial for that characterization.

The following result is a characterization of the minimum Walrasian equilibrium for quasi-linear preferences with no restriction on the number of objects. In this new characterization, antimonotonicity is used instead of strategy-proofness.

Theorem 4.19. *On the domain of quasi-linear preferences, a rule f is the MWE rule if and only if f satisfies AM, EFF, NW and DO.*

Proof. The "if" part. Suppose that f satisfies AM, EFF, NW and DO, then we show that f is a MWE rule. Take any $a \in \mathcal{A}^n$ and let $\underline{p}$ be its MWE price vector. Assume by way of contradiction that $f_k^m(a) \neq \underline{p}_k$ for some $k \in O$. Consider two cases.

Case 1: $f_k^m(a) > \underline{p}_k$. Consider that object k is not allocated. NW and DO of f imply that $f_k^m(a) = 0$. Hence $f_k^m(a) = 0 = \underline{p}_k$. Assume now that $k = f_t^o(a)$ for some $t \in N$. Take $a' = (a_{-t}, a'_t)$ where $a'_{tj} = 0$ for all $j \in O \setminus \{k\}$ and $\underline{p}_k < a'_{tk} < f_k^m(a)$. Then

$$\begin{aligned}\max_{z \in \mathcal{Z}: z_t = k} \left\{ \sum_{i \in N} a'_{iz_i} \right\} &= \sum_{i \in N} a'_{if_i^o(a)} = \sum_{i \in N \setminus \{t\}} a_{if_i^o(a)} + a'_{tk} > \sum_{i \in N \setminus \{t\}} a_{if_i^o(a)} + \underline{p}_k \\ &= \sum_{i \in N \setminus \{t\}} a_{if_i^o(a)} + \max_{z \in \widetilde{\mathcal{Z}}^k} \left\{ \sum_{i \in N} a_{iz_i} \right\} - \sum_{i \in N} a_{if_i^o(a)} \\ &= \max_{z \in \widetilde{\mathcal{Z}}^k} \left\{ \sum_{i \in N} a_{iz_i} \right\} - a_{tk} = \max_{z \in \mathcal{Z}: z_t \neq k} \left\{ \sum_{i \in N} a'_{iz_i} \right\},\end{aligned}$$

where the third equality comes from (4.3) and the last one comes from Remark 4.18. Therefore, by EFF of f, $f_t^o(a') = f_t^o(a) = k$. Because of NW, f is IR, which implies $f_k^m(a') \leq a'_{tk}$ because $a'_{tk} - f_k^m(a') \geq 0$. Hence $f_k^m(a') < f_k^m(a)$ which contradicts AM of f.

Case 2: $f_k^m(a) < \underline{p}_k$. If k is not allocated, then $\underline{p}_k = 0$. Since the prices are non–negative, $f_k^m(a) < 0$ is a contradiction, which implies that k must be allocated.

Assume now that $k = f_t^o(a)$ for some $t \in N$. Take $a' = (a_{-t}, a'_t)$ where $a'_{tj} = 0$

for all $j \in O \setminus \{k\}$ and $f_k^m(a) < a'_{tk} < \underline{p}_k$. Notice that

$$\max_{z \in \mathcal{Z}: z_t = k} \left\{ \sum_{i \in N} a'_{iz_i} \right\} = \sum_{i \in N} a'_{if_i^o(a)} = \sum_{i \in N \setminus \{t\}} a_{if_i^o(a)} + a'_{tk} < \sum_{i \in N \setminus \{t\}} a_{if_i^o(a)} + \underline{p}_k$$
$$= \sum_{i \in N \setminus \{t\}} a_{if_i^o(a)} + \max_{z \in \widetilde{\mathcal{Z}}^k} \left\{ \sum_{i \in N} a_{iz_i} \right\} - \sum_{i \in N} a_{if_i^o(a)}$$
$$= \max_{z \in \widetilde{\mathcal{Z}}^k} \left\{ \sum_{i \in N} a_{iz_i} \right\} - a_{tk} = \max_{z \in \mathcal{Z}: z_t \neq k} \left\{ \sum_{i \in N} a'_{iz_i} \right\}.$$

Therefore by EFF of f, it holds that $f_t^o(a') \neq k$. Let $k' = f_t^o(a')$, then $a'_{tk'} = 0$. By IR of f, because of NW, $f_{k'}^m(a') = 0$. Then

$$0 = a'_{tk'} - f_{k'}^m(a') < a'_{tk} - f_k^m(a),$$

which contradicts AM of f. This completes the proof of the "if" part.

The "only if" part. Since it is known that the MWE rule f satisfies SP, it satisfies also AM. It is clear that f satisfies EFF, NW and DO. □

The following examples show that each of the axioms used in Theorem 4.19 is independent from the other axioms.

Example 4.20. Consider an efficient rule such that the price of each object is zero. This rule trivially satisfies EFF, NW and DO but AM is violated.

Example 4.21. Consider an efficient rule such that the price of each assigned object coincides with its MWE price and set a positive price for each non-assigned real object. Notice that the rule satisfies SP, NW and EFF. Since SP implies AM, then it also satisfies AM, but DO is not satisfied.

Example 4.22. If $o^* \neq n$, consider the minimum Walrasian equilibrium rule. If $o^* = n$, consider the following rule. All objects are assigned efficiently and every agent gets a real object. The price of each object is its MWE price plus 1. This rule satisfies AM, DO and EFF but NW is not satisfied.

Example 4.23. First, label every agent and every object. When there are more objects than agents, consider the MWE rule. When there are more agents than real objects, let f be the rule such that assigns object 1 to agent 1, object 2 to agent 2, and so on each real object at a price of zero. This rule satisfies AM, NW, DO but EFF is not satisfied.

4.4 Appendix

In this Appendix, we provide a proof of the Decomposition Lemma, Corollary 1 and Property 1: (A) introduced in Demange & Gale (1985) applied to our setting. For the sake of comprehensiveness, see also Roth & Sotomayor (1990).

Lemma 4.24. *(Decomposition Lemma, Demange & Gale, 1985) Let $R \in \mathcal{R}_C^n$ be a preference profile and let (z^*, p^*) and (z, p) be two Walrasian equilibria at R. Let $N^* = \{t \in N | (z_t^*, p^*_{z_t^*}) \ P_t \ (z_t, p_{z_t})\}$, $\underline{N} = \{t \in N | (z_t, p_{z_t}) \ P_t \ (z_t^*, p^*_{z_t^*})\}$ and $N^0 = \{t \in N | (z_t, p_{z_t}) \ I_t \ (z^*, p^*_{z_t^*})\}$. Let $O^* = \{j \in O | p_j^* > p_j\}$, $\underline{O} = \{j \in O | p_j > p_j^*\}$ and $O^0 = \{j \in O | p_j^* = p_j\}$. If $i \in N^*$, then $z_i^* \in \underline{O}$. Similarly, if $i \in \underline{N}$, then $z_i \in O^*$. Moreover, if $j \in O^*$, then j is assigned to some $i \in N$ under z^* and $i \in \underline{N}$. Similarly, if $j \in \underline{O}$, then j is assigned to some $i \in N$ under $\underline{z}$ and $i \in N^*$.*

Proof. Let (z^*, p^*) and (z, p) be two Walrasian equilibria at R. By Walrasian equilibrium of (z, p) we have that for each $i \in N$, $(z_i, p_{z_i}) \ R_i \ (j, p_j)$ for any $j \in O$. If $i \in N^*$, then $(z_i^*, p^*_{z_i^*}) \ P_i \ (z_i, p_{z_i})$. By transitivity, $(z_i^*, p^*_{z_i^*}) \ P_i \ (z_i, p_{z_i}) \ R_i \ (z_i^*, p_{z_i^*})$. By money monotonicity, we have that $p^*_{z_i^*} < p_{z_i^*}$ which implies that $z_i^* \in \underline{O}$. A symmetrical argument choosing $i \in \underline{N}$ shows that $z_i \in O^*$.

Assume now there is some $j \in O^*$. This means that $p_j^* > p_j$ and then $p_j^* > 0$ which implies that j is assigned to some $i \in N$ under z^*, *i.e.* $j = z_i^*$. Hence we have that $(j, p_j) \ P_i \ (j, p_j^*)$. By Walrasian equilibrium of (z, p), we have that $(z_i, p_{z_i}) \ R_i \ (j, p_j)$. By transitivity, $(z_i, p_{z_i}) \ R_i \ (j, p_j) \ P_i \ (j, p_j^*)$. Which shows that $i \in \underline{N}$. A symmetrical argument choosing $j \in \underline{O}$, shows that j is assigned to some $i \in N$ under $\underline{z}$ and $i \in N^*$. □

The next corollary is a consequence of the previous lemma. In Demange & Gale (1985), the corollary is stated as Corollary 1, we provide a proof for our particular setting.

Corollary 4.25. (Corollary 1, Demange & Gale, 1985) Let $R \in \mathcal{R}_C^n$ be a preference profile and let (z^*, p^*) and (z, p) be two Walrasian equilibria at R. Let N^*, $\underline{N}$, N^0, O^*, $\underline{O}$ and O^0 be as in the statement of Lemma 4.24. Then $\bigcup_{i \in \underline{N}} z_i = O^* = \bigcup_{i \in \underline{N}} z_i^*$. Similarly, $\bigcup_{i \in N^*} z_i^* = \underline{O} = \bigcup_{i \in N^*} z_i$.

Proof. Because of the previous lemma, for every $i \in \underline{N}$, we know that $z_i \in O^*$. Then

$$\bigcup_{i \in \underline{N}} z_i \subseteq O^*. \tag{4.6}$$

Moreover, we know that for every $j \in O^*$, there is an agent $i \in N$ such that $j = z_i^*$ and $i \in \underline{N}$, then

$$O^* \subseteq \bigcup_{i \in \underline{N}} z_i^*. \tag{4.7}$$

Making use of expressions (4.6) and (4.7), we have that

$$\left| \bigcup_{i \in \underline{N}} z_i \right| \leq |O^*| \leq \left| \bigcup_{i \in \underline{N}} z_i^* \right| = \left| \bigcup_{i \in \underline{N}} z_i \right|. \tag{4.8}$$

Therefore, expression (4.8) implies that $\bigcup_{i \in \underline{N}} z_i = O^* = \bigcup_{i \in \underline{N}} z_i^*$. By a similar argument, we can prove that $\bigcup_{i \in N^*} z_i^* = \underline{O} = \bigcup_{i \in N^*} z_i$. □

The next property is stated in Demange & Gale (1985) as Property 1: (A). This property states that if in a Walrasian equilibrium there is an agent who strictly prefers his allocation in this equilibrium to a null object at zero price, then in every Walrasian equilibrium he will be assigned to a real object.

Corollary 4.26. (Property 1: (A), Demange & Gale, 1985) Let $R \in \mathcal{R}_C^n$ be a preference profile and let (z, p) be a Walrasian equilibrium at R. If $(z_i, p_{z_i})\ P_i\ (q_0, 0)$ for some $i \in N$, then in every Walrasian equilibrium (p^*, z^*) at R, $z_i^* \neq q_0$.

Proof. Let (z, p) be a Walrasian equilibrium at R. Assume that for some $i \in N$, $(z_i, p_{z_i})\ P_i\ (q_0, 0)$ and let (p^*, z^*) be another Walrasian equilibrium at R with $z_i^* = q_0$. Define $\underline{N}$ as in the Decomposition Lemma, then $i \in \underline{N}$. But z_i^* is a null object and it cannot belong to O^* because its price in every Walrasian equilibrium is zero, and this contradicts Corollary 4.25. □

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5 Concluding remarks

This dissertation is devoted to the study of the assignment problem framed in two-sided markets with indivisibilities where money is allowed to be transferable. The Thesis is divided in three parts, the underlying aim of this division is to provide distinct perspectives of the assignment problem. We explore three main processes to determine allocations: the Walrasian equilibrium, the cooperative exchange among coalitions of agents, and a non-cooperative approach.[1] The second chapter of this dissertation analyses the relationship of the core and the Walrasian equilibrium. The third chapter is confined to the implementation problem: we provide a non-cooperative game to implement the best core allocation for the agents belonging to one sector of the market. Finally, the last chapter considers an axiomatic approach of the minimum Walrasian equilibrium. In the following paragraphs, we will present and overview the main results of each chapter. Moreover, we will conclude with some implications and possible further research.

The second chapter is devoted to the study of markets with the presence of a single seller and many buyers. Each buyer wants to acquire a fix quantity of objects. The first result shows that the cooperative game associated to the market is *buyers-submodular*. As a consequence, the core of this game can be easily described: a payoff vector belongs to the core if each buyer gets a non-negative payoff not exceeding his marginal contribution to the whole market and the worth of the market is distributed among all agents. We study then under which valuations the game is convex. We have an interesting result: we show that if the game is convex, then the marginal contribution of any buyer to any pair of coalitions formed by any group of buyers and the seller is the same. Moreover, when this happens, the buyers-optimal core allocation is always supported by a minimum Walrasian equilibrium of the market and it turns out that the price of each object is zero under this situation. Convexity of the game implies the existence of a matching in which each buyer obtains one of his most preferred packages of objects. Although this interesting property is meaningful for the relationship between the buyers-optimal core allocation and Walrasian equilibria, it is not sufficient to guarantee that the seller-optimal

[1] Wilson (1978) points out these three processes to allocate resources in exchange economies.

core allocation will be supported by the maximum Walrasian equilibrium prices. We provide a characterization of those markets for which the seller-optimal core allocation is the maximum Walrasian equilibrium payoff vector. This characterization says that the seller will obtain the worth of the entire market under Walrasian equilibrium prices if and only if all desired objects are allocated and each object is, in fact, allocated to one of the buyers who value it the most. Finally, we undertake the main question of this model: under which conditions all coalitionally rational agreements can be coordinated by means of Walrasian equilibrium prices? We provide a characterization of such situation. The core coincides with the set of Walrasian equilibrium payoff vectors if and only if there is a matching such that all desired objects are allocated, each object is allocated to one of the buyers who values it the most and the marginal contribution of each buyer to the entire market is bounded by the difference between how much he values the package he obtained and the second best valuation for each object in that package. Moreover, we provide an alternative proof of the known formula to compute the minimum Walrasian equilibrium price of each object.

In the third chapter, we analyze exchanges in markets with a single seller and many buyers as in the previous setting. However, this model considers more general preferences for the buyers: since we only require that the gross-substitutes condition is satisfied. This exchanges are carried out in a non-cooperative framework. The aim of this chapter is to study the strategic behavior of the agents, when all of them trade in a competitive environment.

The chapter provides a simple mechanism in which all buyers and the seller play. Initially, buyers submit requests or bids. Then the seller decides the allocation of the objects and the final prices. This mechanism relates the SPE outcomes with the core. A key point of this mechanism is the role played by the seller, which may improve the efficiency of the final allocation. There are two remarks to note from this mechanism. First, although the seller has the final decision, in any outcome of this mechanism, if a buyer gets a package he has not requested, he will get at least the same utility provided by his request. In particular, in any SPE outcome, in spite of the market power of the seller, every buyer gets his maximum core payoff. This means that even in the case in which only some buyers get a package, there is no coalition of agents that can improve the outcome trading only by themselves.

We have shown that all SPE yield the best core element for buyers. A natural question related to this result is whether Walrasian equilibrium prices may support this core outcome. In general, the answer is in the negative. Notwithstanding, when the gross-substitutes condition is satisfied, the existence of Walrasian equilibria is guaranteed.[2] Even more, if we replicate the market *à la* Gul & Stacchetti (1999),[3] similar to the result in Wilson (1978), the best core element for buyers becomes competitive and hence, the outcome of any SPE of our mechanism is supported by some Walrasian equilibrium. That is, in the replicated market, the selling prices of packages given by the mechanism are supported by the minimum Walrasian equilibrium price vector of the market.

In the last chapter, we consider the problem arising from the following situation. In a two-sided market, a group of buyers desire to acquire objects owned only by agents on the other side of the market. It is assumed that each buyer can acquire at most one object and has a preference in a domain that includes quasi-linear preferences. The owners of the objects will centralize all of them in an institution or in a clearing house for their allocation. The institution will then determine a procedure to allocate the indivisibilities. A natural question is then *how to allocate the objects?* Some criterion could be required to be satisfied by the allocation rule, *e.g.* Pareto-efficiency, fairness, egalitarian or Rawl's criteria.[4] In the economic problem just described, if Pareto-efficiency is required to the allocation rule that distributes resources, then, because of the First Theorem of the Welfare Economics, we know that every Walrasian equilibrium leads to a Pareto-efficient allocation. We can then add some other properties that lead to some particular Walrasian equilibrium, or otherwise we could restrict our problem to choose only among the set of Walrasian equilibria, hence *which Walrasian equilibrium rule should be chosen?* In previous works Demange & Gale (1985) and Morimoto & Serizawa (2015) show that the minimum Walrasian equilibrium satisfies desirable properties as an outstanding rule to allocate indivisibilities, *e.g. group strategy-proofness* and *fairness*. The Chapter 4 of this Thesis can be understood as an extension of the previous works in which we study properties of the minimum Walrasian equilibrium under different environments. We provide the following characterizations. When the objects are heterogeneous, the minimum Walrasian equilibrium is the unique rule satisfying *non-wastefulness, envy-freeness, desirability of positively priced objects* and *monotonicity with respect to willingness to pay*. For the quasi-linear domain, the minimum Walrasian equilibrium rule is characterized by *antimonotonicity, non-wastefulness, efficiency* and *desirability of positively priced objects*.

Properties of the minimum Walrasian equilibrium are approached in the last chapter of this Thesis, in the context of general agents' preferences. Due to the features

[2] In fact, the set of Walrasian equilibrium price vectors has a complete lattice structure (Gul & Stacchetti, 1999)

[3] See Section 5 in Gul & Stacchetti (1999).

[4] See Feldman & Serrano (2006) for a discussion.

of these preferences, it is not possible to report all indifference sets for each agent. This leads to the following situation. Suppose that a government or a company is, in fact, interested in applying the minimum Walrasian equilibrium to allocate indivisibilities. The first question is then, *can this task be carried out?* In the case of quasi-linear preferences, the answer is in the positive: the agents need to report the willingness to pay for each object. In the domain of preferences considered in Chapter 4, reporting the entire preference is impossible. Hence, *what information is needed to compute the minimum Walrasian equilibrium of the market?* Let me provide some ideas for simple cases. In the case of a single object, it is sufficient to report the willingness to pay. However, suppose that there are two objects to be allocated, *a* and *b* and two agents. In this case, reporting the willingness to pay may not be sufficient to compute the minimum Walrasian equilibrium. In this particular case, each agent has to report how much she/he would pay for object *a* in order to be indifferent with respect to object *b* for free and vice versa. In situations with more objects and agents, providing an answer becomes more complex. In order to offer an answer, in future research we would investigate a procedure to guarantee the computation of the minimum Walrasian equilibrium with general preferences.

Following the same line of research of Chapter 4, we propose as future research to study the following situation. Consider a market in which buyers and sellers, or firms and workers, meet. A feature of the market is that each agent is interested in at most one partnership with agents on the other market sector. For each partnership there is a non-negative gain that can be split between the partners. The natural solution concept introduced to analyse these market situations is *stability*. An outcome of the market specifies a matching and a price vector. It is known that an outcome is stable if it is individually rational and there is no buyer-seller partnership and a price for the object owned by this seller so that, at this price, both buyer and seller are better off making a new partnership than under the previous outcome. It is known that stable outcomes do exist and there is one for each market sector in which all its members get their best stable payoff. We address the following point, among stable rules, which are the properties that characterize the best stable outcome for each sector of the market?

A possible extension of the study of two sided markets under an axiomatic approach is the following. In some market situations, agents on the first sector may be interested in making more than one partnership with the agents on the second sector (or in acquiring a package of objects in the case of buyers) while each agent on the second sector may be interested in only one partnership (or owning only one object on sale in the case of sellers). This markets are called *many-to-one* markets (see *e.g.*

Sotomayor, 2002) and it is known that under quasi-linear preferences, monotonicity and gross-substitutes condition, the minimum Walrasian equilibrium does exist (Gul & Stacchetti, 1999). As future research, we also focus on axioms in order to characterize the minimum Walrasian equilibrium in this setting.

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