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Tesis doctoral

Control Predictivo de sistemas borrosos mediante teoría de conjuntos invariantes y programación copositiva

Model Predictive Control of fuzzy systems via invariant set theory and copositive programming

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Abstract

This thesis studies fuzzy model predictive control, looking for novel contributions in his field. The main idea of the thesis is to adapt the classic linear model predictive control to Takagi-Sugeno fuzzy systems. Although these systems can be dealt with nonlinear model predictive control, there are many advantages when their particular structure is taken into account.

The Takagi-Sugeno fuzzy systems are formed by a mixture of linear models that vary their importance weight (membership function) depending on the value of the system state. In the last 15 years, this structure has been used to develop a multitude of new applications and a whole control theory by adapting robust control through linear matrix inequalities (LMIs) to Takagi-Sugeno fuzzy systems. In these contributions the main idea is to design controllers valid for any known membership function. A priori, it seems very restrictive and too generic, but since controllers know the membership functions, the control action will depend on it, being a powerful design solution. This design philosophy has been called shape-independent design.

In pre-thesis developments, it was observed that this type of design philosophy for predictive control had not been studied in depth, so a first approach to the fuzzy predictive control was addressed. Unfortunately, the complexity of the problem increased exponentially with the control horizon and it was left for the present thesis to develop the methodologies, notation and theorems necessary to treat the problem in all its complexity, reaching solutions that increase its complexity through adjustable parameters. The obtained controller solves the shape independent fuzzy predictive control problem for a finite horizon, under some complexity assumptions.

Pursuing the above goal, interesting intermediate results have been

also developed: the design of a controller that (with adjustable degree of complexity) stabilizes the system in the largest possible region, and a direct adaptation of some iterative nonlinear predictive control techniques to the Takagi-Sugeno fuzzy systems. Considerations about invariant set theory for the Takagi-Sugeno problem have been developed to obtain the relevant feasible/terminal sets.

These developments are based, in addition on the linear model predictive control, on copositive programming. Understanding a copositivity problem, the study of whether a polynomial is positive for any positive value of its variables. In the case of fuzzy systems since the membership functions are always positive, they easily enter into this kind of problem with the proviso that there are other variables (the system state) that are not always positive. This problem has been addressed in the thesis by the application of Polya's theorem.

Resumen

En esta tesis se estudia el control predictivo para modelos borrosos Takagi-Sugeno, buscando aportaciones novedosas en este campo. La idea principal de la tesis es adaptar el control predictivo lineal clásico a los modelos borrosos Takagi-Sugeno. A pesar de que estos sistemas pueden tratarse mediante control predictivo no lineal, existen muchas ventajas si tenemos en cuenta su particular estructura.

Los sistemas borrosos Takagi Sugeno están formados por una ponderación de modelos lineales que van variando su peso (función de pertenencia) dependiendo del valor del estado del sistema. En los últimos quince años se ha aprovechado esta estructura para desarrollar multitud de nuevas aplicaciones y toda una teoría de control adaptando el control robusto mediante desigualdades matriciales lineales (LMIs) a sistemas borrosos Takagi-Sugeno. En estas aportaciones la idea principal es la de diseñar controladores que sean válidos para cualquier función de pertenencia conocida. Esto, a priori, parece muy restrictivo y demasiado genérico, pero dado que los controladores conocen la función de pertenencia, la acción de control dependerá de ella, siendo al final una solución de diseño muy potente. A esta filosofía de diseño se le ha llamado diseño independiente de la forma.

En desarrollos previos a la tesis, se observó que este tipo de filosofía de diseño para el control predictivo no se había estudiado en profundidad, entonces se abordó una primera aproximación al control predictivo borroso. Desgraciadamente, la complejidad del problema aumentaba de forma exponencial con el horizonte de control y se dejó para la presente tesis desarrollar las metodologías, la notación y los teoremas necesarios para tratar el problema en toda su complejidad, llegando a soluciones que aumentan su complejidad mediante parámetros ajustables. El controlador obtenido resuelve el problema de control predictivo borroso para

cualquier forma de las funciones de pertenencia para un horizonte finito, bajo algunas suposiciones de complejidad.

Para conseguir los anteriores objetivos, se han desarrollado resultados intermedios de interés: el diseño de un controlador que (con grado de complejidad ajustable) estabiliza al sistema en la región más grande posible, y la adaptación directa de algunas técnicas iterativas de control predictivo nolineal a los sistemas borrosos Takagi-Sugeno. Se han desarrollado consideraciones acerca de la teoría de conjuntos invariables para el problema Takagi-Sugeno con el fin de obtener los conjuntos factibles/terminales pertinentes.

Estos desarrollos están basados, además de en la teoría de control predictivo basada en modelos lineales, en la programación copositiva.Entendiendo como problema de copositividad, el estudio de si un polinomio es positivo para cualquier valor positivo de sus variables. En el caso de los sistemas borrosos al ser las funciones de pertenencia siempre positivas entran facilmente dentro de este tipo de problemas con la salvedad de que existen otras variables, (los estadosdel sistema) que no son siempre positivos. Este problema se ha abordado en la tesis mediante la aplicación del teorema de Polya.

Resum

En aquesta tesi s'estudia el control predictiu per models borrosos Takagi Sugeno, buscant aportacions noves en aquest camp. La idea principal de la tesi és adaptar el control predictiu lineal clàssic als models borrosos Takagi Sugeno. Tot i que aquests sistemes poden tractar-se mitjançant control predictiu no lineal, hi ha molts avantatges si tenim en compte la seua estructura particular.

Els sistemes borrosos Takagi-Sugeno estan formats per una ponderació de models lineals que van variant el seu pes (funció de pertinença) depenent del valor de l'estat del sistema. En l'última dècada s'ha aprofitat aquesta estructura per desenvolupar multitud de noves aplicacions i tota una teoria de control adaptant el control robust mitjançant desigualtats matricials lineals (LMIs) a sistemes borrosos Takagi-Sugeno. En aquestes aportacions la idea principal és la de dissenyar controladors que siguen vàlits per a qualsevol funció de pertinença coneguda. Això, a priori, sembla molt restrictiu i massa genèric, però atès que els controladors coneixen la funció de pertinença, l'acció de control dependrà d'ella, sent al final una solució de disseny molt potent. A aquesta filosofia de disseny se li ha anomenat disseny independent de la forma.

En desenvolupaments previs a la tesi, es va observar que aquest tipus de filosofia de disseny per al control predictiu no s'havia estudiat en profunditat, llavors es va abordar una primera aproximació al control predictiu borrós. Malauradament, la complexitat del problema augmentava de forma exponencial amb l'horitzó de control i es va deixar per a la present tesi desenvolupar les metodologies, la notació i els teoremes necessaris per tractar el problema en tota la seva complexitat, arribant a solucions que augmenten la seva complexitat mitjançant paràmetres ajustables. El controlador obtingut resol el problema de control predictiu borrós per a qualsevol forma de les funcions de pertinença per a un

horitzó finit, sota algunes suposicions de complexitat.

Per a assolir els objectius, s'han desenvolupat resultats intermedis interessants: el disseny d'un controlador que (amb grau de complexitat ajustable en el disseny) estabilitza al sistema en la regió més gran possible, i l'adaptació directa d'algunes tècniques iteratives de control predictiu nolineal als sistemes borrosos Takagi-Sugeno. S'han desenvolupat consideracions sobre la teoria de conjunts invariables per al problema Takagi-Sugeno per tal d'obtenir els conjunts factibles/terminals pertinents.

Aquests desenvolupaments estan basats, a més d'en la teoria de control predictiu basat en models lineals, en la programació copositiva. Entenent com problema de copositivitat, el estudi de si un polinomi és positiu per a qualsevol valor positiu de les seves variables. En el cas dels sistemes borrosos en ser les funcions de pertinença sempre positives entren fàcilment dins d'aquest tipus de problemes amb l'excepció que hi ha altres variables, (els estats del sistema) que no són sempre positius. Aquest problema s'ha abordat en la tesi mitjançant l'aplicació del teorema de Polya.

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Chapter 1

Introduction

Traditionally, both in academic and industry sectors, control of nonlinear systems has been dealt with linearization techniques, which allow us to work with classical control methods (Ogata, 1996), like root locus, frequency response and so on. Linearized systems can yield similar behaviour than the original one, but the basin of attraction and the performance of controller are limited by the employed model.

There exist several control methods to deal with nonlinear systems, such as backstepping, feedback linearization, passivity and so on (Slotine, Li, et al., 1991). But, these methods usually require an analytic reformulation of the system, hence, the complexity of the formulation increases significantly depending on the kind of control and system. So, if we only want to dealt with a system in specific region, Takagi-Sugeno fuzzy models allow modelling the system in a systematic way, and the developed control strategies can be, directly, applied for any TS fuzzy model. There are some sources of conservatism in the approach, discussed in (Sala, 2009).

The main idea of the thesis will be focused on obtaining a predictive control for TS systems, that were able to guarantee stability and minimize a cost index. Hence, it is interesting to develop a fuzzy predictive control based on the classical ones dating from the 1990s (Camacho & Bordons, 2013), which could be susceptible to be reformulated to the Takagi-Sugeno framework by means of techniques such as invariant set theory and copositive programming, since these tools have been applied for last years with successful results.

In the fuzzy model predictive control formulation, convex fuzzy summations of linear systems will appear, as this is the core of TS-fuzzy modelling. In order to deal with them, Polya theorem is employed. It allows approximating the nonlinear constraints to semidefinite ones (linear matrix inequalities, LMI) with a defined complexity degree. Moreover, for the shape independent problem, Polya theorem provides necessary and sufficient LMI constraints as the complexity degree increases, as discussed in (Sala & Ariño, 2007a).

A predictive control problem for TS systems must be solved by means of convex optimization algorithms, which are not able to dealt with these polynomial functions, thus the problem must be divided into easier to handle problems, which are not usually an exact formulation of the original problem, i.e. a set of feasible solutions can be ignored. This issue is the main workhorse of the present thesis, but Polya theorem can help us to reduce the gap between the original TS system and the posterior formulation of this one.

Takagi-Sugeno models are a particular case of multi-model systems in which they are interpolated via convex sums. On the other hand, another widely-used multi-model framework is the so-called Markov Jump Linear Systems (MJLS) (do Valle Costa, Fragoso, & Marques, 2006). They are a class of stochastic systems, where the system switches between a finite set of "modes" according to a specific probabilities, that are arranged in a matrix, which is called Markov transition probability matrix. We have considered that stochastic systems are susceptible to be formulated together with non linear fuzzy systems, combining concepts of both.

1.1 Objectives

The main goal of the current thesis is to achieve more relaxed stability conditions in fuzzy systems and less conservative (sub)-optimal results in predictive fuzzy control, by means of copositive programming.

The objectives of the current work are basically two:

The first one of these is enhancing conditions in order to guaranty the stability of the nonlinear systems, by means of new methodologies based on Polya Theorem as well as invariant set theory. So, nonlinear systems solutions are achieved by alternative methods, that can be more efficient.

The other goal is to obtain a methodology to carry out a nonlinear predictive control, understood in a minimax setting as future membership functions will be unknown, so that the outcomes may be more accurate than obtained ones by traditional methods in multi-model systems, either fuzzy or stochastic.

The presented contributions span several separated topics developed during the thesis work. The chapters are mostly based on the textual contents of some conference/journal papers so there are differences in notation and some repetition of preliminary material.

1.2 Thesis structure

Several chapters form the current thesis: the next one is the state of the art followed by two more chapters discussing some preliminary contributions; these three ones form **Part I**:

Chapter 2: State of the art. Presents the background material necessary to understand the thesis. Stability, non-linear fuzzy control, predictive control, LMIs, copositive programming and invariant set theory are presented, analysing their properties.

Chapter 3: Improved stability for Takagi-Sugeno systems by applying Polya's Theorem with multi-indices. In this chapter, a copositive methodology is developed to carry out a Polya relaxation according to the value of antecedents, which are available in the TS modeling stage. Similar methodology is proposed in the paper (Ariño & Sala, 2007) but it just employs two antecedents and two rules and it does not use Polya expansion. So, at this chapter the methodology is extended to any case. The contribution of this chapter is presented in the paper (Querol, Ariño, Hernández-Mejías, & Sala, 2014).

Chapter 4: Guaranteed Cost Control for Discrete Stochastic Fuzzy Systems via LMIs. This chapter formulates a guaranteed quadratic cost control method for a stochastic fuzzy system via LMIs. The problem is carried out with restrictions on the states and inputs.

Then, **Part II** contains three chapters, where, the main contributions of this work are discussed.

Chapter 5: Asymptotically Exact Stabilization for Constrained Discrete Takagi-Sugeno Systems via Set-Invariance. With the purpose of calculating the maximal polytopic set where a fuzzy system can be stabilizable, a methodology is developed at this chapter. The basic idea lies in developing a polytopic approximation to the maximal control invariant set, which induces, too, a piecewise linear Lyapunov function (polyhedral level sets). In addiction, copositive programming is employed to maximize the size of the referred polytopic set. The contributions of this chapter were submitted in paper (Ariño, Sala, Pérez, Bedate, & Querol, 2017).

Chapter 6: Model Predictive Control for Discrete Fuzzy Systems via Iterative Quadratic Programming. A heuristic methodology is discussed in order to undertake a fuzzy predictive control problem by means of an iterative algorithm, in the framework of well-known sequential quadratic programming (SQP) ideas. Thus, we minimize a cost index and achieve more accurate results than shape-independent methods based on LMIs, because the final result is shape-dependent. Moreover this technique employees a terminal cost and a convex terminal set. If we enforce some additional Lyapunov decrease constraints, this predictive control method can obtain suboptimal control inputs, that are able to stabilize the system, although algorithm convergence has not been reached yet. The main contributions of this chapter are presented in the paper (Ariño, Pérez, Querol, & Sala, 2014).

Chapter 7: Shape-Independent Model Predictive Control for Takagi-Sugeno Fuzzy Systems. This chapter presents a minimax predictive control method, to obtain a guaranteed cost in fuzzy systems, based on generic copositive programming. A formulation of the model predictive control of TS systems is discussed, so that results are valid for any membership value (shape-independent) with a suitable account of causality (control can depend on current and past memberships and state), and a family of progressively better controllers can be obtained by increasing Polya-related complexity parameters. So, optimal control actions are a parametric fuzzy polynomial equations, which are formulated specifically for each model and time instant. The contributions of this chapter are presented in the work (Ariño, Querol, & Sala, 2017, submitted).

To close the thesis, **Part III** is presented where the conclusions and future research (chapter 8), and the bibliography are included.

Part I

Preliminaries

Chapter 2
State of the art

2.1 Introduction

In this chapter, the background to understand later developments in the current thesis is going to be presented. Thus, first of all, the Lyapunov stability concept will be discussed.

Linear Matrix Inequalities (LMIs) concept, as well as their properties, will be presented in the next section. LMIs are employed to deal with nonlinear fuzzy systems and associated conditions, where the copositive programming will be applied.

After, as expected, the concepts of fuzzy and stochastic systems will be explained, besides the main features of these, like copositive conditions and Polya theorem.

The predictive control is another matter discussed in the following section, because it is one of the main topics of the current thesis, terms like cost index, model predictive control (MPC), terminal set and cost, receding horizon and optimizer are explained.

And finally, the set theory is going to be presented in the last section, where concepts like invariant set and one-step set, employed in the current thesis, will be presented.

2.2 Lyapunov Stability

Stability is one of the most important property employed in systems analysis within the control engineering field. Nonlinear systems may have more complex and exotic behavior than linear systems, the mere notion of stability is not enough to describe the essential features of their motion. A number of more refined stability concepts, such as asymptotic stability, exponential stability and global asymptotic stability, are needed. In this section, we define these stability concepts formally, for autonomous systems, and explain their practical meanings.

Prior to formulate the Lyapunov equation features, it is necessary to define a equilibrium point and what properties it has.

Considering an autonomous system $\dot{x} = f(x, t)$, state x' is an equilibrium state (or equilibrium point) of the system, if once $x(t)$ is equal to x' it remains equal to x' over time. Mathematically, this means that the constant vector x' satisfies:

$$0 = f(x') \tag{2.1}$$

So, for a linear time-invariant system $\dot{x} = A(t)x$, provided that A is not singular; if $0 = Ax'$, then $x' = 0$ is the single equilibrium point. On the other hand, nonlinear systems can have multiple isolated equilibrium points.

Essentially, stability in the sense of Lyapunov means that the system trajectory can be kept close to the equilibrium point, starting sufficiently near to it. So, let us define local stability as:

Definition 2.1 *(Slotine et al., 1991) The equilibrium state $x = 0$ is said to be stable if, for any $e > 0$, there exists $h > 0$, such that if $||x_0|| < h$ then trajectories $||x(t, x_0)|| < e$ for all $t > 0$. Otherwise, the equilibrium point is unstable.*

Also, asymptotic stability in the sense of Lyapunov (Slotine et al., 1991) means that the equilibrium is stable, and besides, if the states started around the equilibrium point, these ones are going to converge to the equilibrium point when $t \to \infty$.

Definition 2.2 *(Slotine et al., 1991) An equilibrium point $x = 0$ is asymptotically stable if it is stable, and if in addition there exists some $e > 0$ such that $||x(0)|| < e$ implies that $x(t) \to 0$ as $t \to \infty$.*

In any system, an equilibrium point $x = 0$ is locally asymptotically stable, if there exists an open region $\mathbb{X}$ containing the origin and a scalar function $V(x)$ continuous and differentiable so that:

$$\begin{aligned} V(x) > 0 \quad & \forall x \in \mathbb{X},\ x \neq 0 \\ \frac{dV}{dt} < 0 \quad & t \geq 0 \quad \forall x \in \mathbb{X} \\ & V(0) = 0 \end{aligned} \tag{2.2}$$

The function $V(x)$ is called Lyapunov function (Khalil, 1996).

For instance, if a quadratic function $V(x) = x^T P x$ is proposed as Lyapunov function where P is a definite positive matrix, i.e. $P > 0$, condition $\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} < 0$ ensures stability for a continuous-time system (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994; Khalil, 1996).

In the case of discrete-time systems $x_{t+1} = f(x_t)$ dealing with sequences of signals x_t, $t = 0, \ldots, \infty$, equilibrium points fulfill $x' = f(x')$, but stability definition is identical (considering t to be an integer). The Lyapunov stability theorem now is stated as:

Theorem 2.1 *(Kalman & Bertram, 1960) A discrete-time system* $x_{t+1} = f(x_t)$ *so that* $0 = f(0)$ *is locally asymptotically stable if there exists an open region* $\mathbb{X}$ *containing the origin and a continuous scalar function* $V(x)$ *such that:*

$$\begin{aligned} V(x) > 0 \quad & \forall x \in \mathbb{X},\ x \neq 0 \\ V(f(x)) - V(x) < 0 \quad & \forall x \in \mathbb{X} \\ & V(0) = 0 \end{aligned} \tag{2.3}$$

For instance, for a discrete-time system with a candidate Lyapunov function $V(x_k) = x_k^T P x_k$, the stability condition becomes $V(x_{k+1}) - V(x_k) = x_{k+1}^T P x_{k+1} - x_k^T P x_k < 0$.

Apart from quadratic Lyapunov functions, other more generic ones have been proposed in literature, such as polynomial ones of degree greater than 2 (Prajna, Papachristodoulou, & Parrilo, 2004), parameter-dependent ones (Ding, 2010), etc. Some of them will be reviewed in later sections.

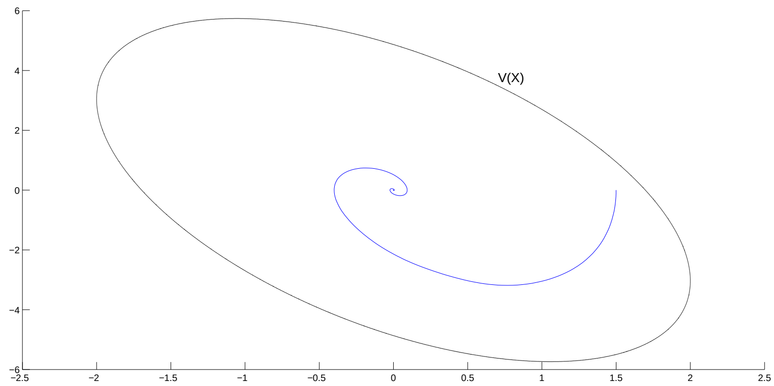

Figure 2.1: Lyapunov function example

2.3 Convex Optimization

The so-called convex optimization algorithms are a class of computationally efficient (polynomial time) optimization techniques that apply to a class of problems to be discussed in this section.

A convex optimization problem is one in the form (Boyd & Vandenberghe, 2004):

$$\begin{aligned} &\min f_0(x) \quad s.t. \\ &\qquad\qquad f_i(x) \leq b_i; \quad i = 1, \ldots, m \end{aligned} \tag{2.4}$$

where the functions $f_0, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ are convex, i.e., satisfy

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y) \tag{2.5}$$

for all $x, y \in \mathbb{R}^n$ and all $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta = 1$, $\alpha \geq 0$, $\beta \geq 0$.

Although powerful generic results can be estabilished from the above setup, basically, are three particular classes of convex optimization problems which are widely used in the control engineering field (Boyd & Vandenberghe, 2004):

- **Linear Programming (LP)**: A general linear program has the form:

$$\begin{aligned} \min c^T x + d \\ s.t. \\ Gx \leq h \\ Ax = b \end{aligned} \tag{2.6}$$

where $G \in \mathbb{R}^{m \times n}$ and $A \in \mathbb{R}^{p \times n}$. Linear programs are, of course, convex optimization problems. It is common to omit the constant d in the objective function, since it does not affect the optimal (or feasible) set.

- **Quadratic Programming (QP)**: This convex optimization problem can be expressed in the form:

$$\begin{aligned} \min \frac{1}{2} x^T P x + q^T x + r \\ s.t. \\ Gx \leq h \\ Ax = b \end{aligned} \tag{2.7}$$

where symmetric positive semi-definite matrix $P = P^T \in \mathbb{R}^n$, $G \in \mathbb{R}^{m \times n}$, and $A \in \mathbb{R}^{p \times n}$. In a quadratic program, we minimize a convex quadratic function over a polyhedron. Evidently, if $P = 0$, then QP reduces to LP. Note that, if P is not positive semi-definite, the above problem might be non convex, posing additional computational difficulties (Audet, Hansen, Jaumard, & Savard, 2000; Pardalos & Schnitger, 1988). A particular case (copositive programming), relevant in the control problems objective of this thesis, will be discussed later on in Section 2.4.

- **Semidefinite programming (SDP)**: Following the analogy to LP, a standard form SDP has linear equality constraints, and a (matrix) nonnegativity constraint on the symmetric matrix variable $X \in \mathbb{R}^n$:

$$\begin{aligned} \min tr(CX) \\ s.t. \\ tr(A_i X) = b_i \quad i = 1, \ldots, p \\ X \geq 0 \end{aligned} \tag{2.8}$$

where $C, A_1, \ldots, A_p \in \mathbb{R}^n$ are a symmetric matrices. Recall that $tr(CX) = \sum_{i,j=1}^{n} C_{ij} X_{ij}$ is the form of a general real-valued linear function on $\mathbb{R}^n$. Actually, LP and QP can be conceived as a particular case of the SDP setup (Boyd et al., 1994). In fact, in control engineering an equivalent formulation of the above SDP problem is usually cast as optimisation over Linear Matrix Inequalites, which will be briefly outlined in Section 2.3.1.

There are several techniques to solve the above problems (simplex, interior point, duality, etc.) which are out of the objectives of this thesis. The reader is referred to, for instance, (Boyd & Vandenberghe, 2004) for further detail.

Software implementations of solutions for the above optimization problems are available in most scientific computation packages (Matlab, Mathematica, ...) in the case of linear and quadratic programming. Code for optimising SDPs is less widespread in a general case, but there are some tools which are widely used in a control engineering enviromnent, some of them commercial (Gahinet, Nemirovskii, Laub, & Chilali, 1994), and some of them freely available, such as (Löfberg, 2004; Sturm, 1999). Other tools/variations have appeared later on, but the enumeration and comparison between them is not in the objectives of this thesis.

Sometimes, optimizing a particular objective funcion is not necessary, i.e. just a feasible solution is required. Thus, obviously, if one of the previous methods can achieve the optimal result, it is also able to obtain a feasible one,actually in a faster way.

2.3.1 Linear Matrix Inequalities (LMIs)

Linear matrix inequalities (LMIs) is the name that SDP is often given in control theory contexts. Convex optimization is used to numerically solve LMIs. In this section, the LMI concept is going to be presented together with its properties, which will be used in the present thesis to solve optimal control problems further on.

A Linear Matrix Inequality (LMI) is an expression with the following form:

$$A(x) := A_0 + x_1 A_1 + x_2 A_2 + \cdots + x_m A_m > 0 \tag{2.9}$$

where:

- $x = \{x_i\ ,\ i{=}1,\ldots,m\}$ is a real vector,
- $A_0, A_1, A_2, \ldots, A_m \in \mathbb{R}^{n\times n}$ are symmetric matrices, and
- notation $A(x) > 0$ denotes that $A(x)$ is a positive definite matrix, i.e. where all its eigenvalues $\lambda(A(x))$ are positive.

An inequality form SDP, has no equality constraints, and one LMI[1]:

$$\begin{aligned} \min c^T x & \\ s.t. & \\ A_0 + x_1A_1 + x_2A_2 + \cdots + x_mA_m > 0 \end{aligned} \tag{2.10}$$

with decision variable $x \in \mathbb{R}^n$, and symmetric matrices $A_0, A_1, \ldots, A_m \in \mathbb{R}^{n\times n}$, $c \in \mathbb{R}^n$.

Let us discuss some relevant properties of LMIs.

Schur Complement

From (Cottle, 1974), let $A(x)$ be a matrix partitioned as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{2.11}$$

then, $A(x) < 0$ is equivalent to:

$$A_{22} < 0 \quad A_{11} - A_{12}^T A_{22}^{-1} A_{12} < 0 \tag{2.12}$$

and

$$A_{11} < 0 \quad A_{22} - A_{12} A_{11}^{-1} A_{12}^T < 0 \tag{2.13}$$

Congruence transformation

Taking a square matrix $F \in \mathbb{R}^{n\times n}$. Let U be a nonsingular matrix with appropriated size, then the following two statements are equivalent:

$$F > 0 \tag{2.14}$$

$$U^T F U > 0 \tag{2.15}$$

[1]Indeed, several matrix inequalities can be trivially rewritten as a single block-diagonal one, and equalities can be supressed via variable ellimination.

Cross Product Lemma

Following (Guerra, Kruszewski, Vermeiren, & Tirmant, 2006), let X, Y and $F = F^T$ be definite positive matrices of appropriate dimensions. Then:

$$X^TY + Y^TX \leq X^TFX + Y^TF^{-1}Y \tag{2.16}$$

Xie and de Souza property

Another property is going to be presented in the current section, which lies in the fact that; considering matrices $\Pi < 0$, X and a scalar α, the following statement holds (L. Xie & de Souza, 1992):

$$(X + \alpha\Pi^{-1})^T\Pi(X + \alpha\Pi^{-1}) \leq 0 \Leftrightarrow X^T\Pi X \leq -\alpha(X^T + X) - \alpha^2\Pi^{-1} \tag{2.17}$$

Finsler lemma

Taking $x \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times n}$. The next four statements are equivalent (Boyd et al., 1994; Skelton, Iwasaki, & Grigoriadis, 1997):

1. $x^TQx < 0 \quad \forall x \neq 0$ such that $Rx = 0$
2. $R_\perp^TQR_\perp$ being $RR_\perp = 0$
3. $Q - \sigma R^TR < 0$ for a scalar $\sigma \in \mathbb{R}$
4. $Q + XR + R^TX^T < 0$ for some matrix $X \in \mathbb{R}^{n\times m}$

S-procedure

Let $F_i(u)$, $i = 1, \dots p$ be quadratic functions in the variable $u \in \mathbb{R}^n$ defined as:

$$F_i(u) = u^TT_iu + 2b_i^Tu + v_i \quad i = 0, \dots, p \tag{2.18}$$

The S-procedure states that a sufficient condition for $F_0(u) > 0\ \forall u$ under constraints $F_i(u) \geq 0$, $i = 1, \dots, p$ is the existence of scalars

$\{\alpha_1 \geq 0, \ldots, \alpha_p \geq 0\}$, such that the next expression is satisfied for all u:

$$F_0(u) - \sum_{i=1}^{p} \alpha_i F_i(u) \geq 0 \qquad (2.19)$$

Considering the vector $\xi := (u^T 1)^T$, we can express the above in matrix form:

$$\begin{pmatrix} T_0 & b_0 \\ b_0^T & v_0 \end{pmatrix} - \sum_{i=1}^{p} \alpha_i \begin{pmatrix} T_i & b_i \\ b_i^T & v_i \end{pmatrix} \geq 0 \qquad (2.20)$$

2.3.2 Sum of Squares (SOS)

The so-called sum-of-squares technique was introduced in the control systems community by (Prajna, Papachristodoulou, & Parrilo, 2004). Let us summarise a handful of basic ideas on it.

Given a polynomial $p(x)$, it may be expressed as SOS if it can be formulated in the following way:

$$p(x) = \sum_{i=1}^{m} f_i^2(x) \qquad (2.21)$$

Definition 2.3 *Given a set of multivariable polynomials $f_1, \ldots, f_m$ the ideal associated with this set can be defined as:*

$$\mathbf{ideal}(f_1, \ldots, f_m) = \{f | f = \sum_{i=1}^{m} t_i f_i \quad t_i \in \mathbb{R}[x]\} \qquad (2.22)$$

Definition 2.4 *Given a set of multivariable polynomials $g_1, \ldots, g_p$, the cone associated with such set can be defined as:*

$$\mathbf{cone}(g_1, \ldots, g_p) = \{g | g = s_0 + \sum_{i} s_i g_i + \sum_{i,j} s_{ij} g_i g_j + \sum_{i,j,k} s_{ijk} g_i g_j g_k + \ldots\} \qquad (2.23)$$

where, $s_\alpha \in \mathbb{R}[x]$ are SOS polynomials.

Other properties:

- Let a x point be, for which $(f_1(x) = 0, \dots, f_m(x) = 0)$. Then for any polynomial t_i, $t_i(x)f_i(x) = 0$ holds and thus, if $f \in$ **ideal**$(f_1, \dots, f_m)$ then $f(x) = 0$
- Let a x point be, for which $(g_1(x) \geq 0, \dots, g_p(x) \geq 0)$ for any SOS polynomial s_i, $s_i(x)g_i(x) \geq 0$ holds and thus, if $g \in$ **cone**$(g_1, \dots, g_p)$ then $g(x) \geq 0$

SOS Optimization:

SOS optimization program is a problem with a linear cost function and a special type of restrictions on the decision variables (Prajna, Papachristodoulou, & Wu, 2004)(Jarvis-Wloszek, 2003): the decision variables appear linearly as coefficients of some polynomials which are constrained to be SOS.

This problem may be formulated as:

$$\min_{u_k \in \mathbb{R}^n} c^T u \tag{2.24}$$

subject to:

$$a_{k,0}(x) + a_{k,1}(x)u_1 + \cdots + a_{k,n}(x)u_n \in SOS(k = 1, \dots, N_s) \tag{2.25}$$

The vector $c \in \mathbb{R}^n$ and monomials $\{a_{k,j}\}$ are given as part of the data for the optimization problem, and the values of $u \in \mathbb{R}^n$ are the decision variables.

Quadratic forms can be expressed as $p(x) = x^T Q x$ where Q is a symmetric matrix. Similarly, polynomials of degree $\leq 2d$ can be formulated as:

$$p(x) = z(x)^T Q z(x) \tag{2.26}$$

In the vector z, all monomials degree are $\leq d$. This is known as the Gram matrix form. An important fact is that p is SOS, if and only if, there exists a symmetric and positive semidefinite matrix Q such that $p(x) = z(x)^T Q z(x)$ (Chesi, 2010). This provides a interesting connection between SOS polynomials and positive semidefinite matrices, which allows to state that SOS optimisation problems are a particular case of the SDP ones, so they can be solved with the earlier presented SDP/LMI software after a pre-processing layer forming such Gram matrices. Packages such as SOSTOOLS (Prajna, Papachristodoulou, & Parrilo, 2002) or the Yalmip SOS module (Löfberg, 2004) can be used to take care of such steps.

Example 2.1 *Taking a polynomial* $F(x) = 2x^4 + x^3 + 2x^2 - 3x$, *in order to apply a optimization it can be formulated as LMI/SDP, such as it has been explained above, so that,* $F(x)$ *will be become* $z^T(x)Qz(x)$*:*

$$F(x) = \begin{pmatrix} 1 & x & x^2 \end{pmatrix} Q \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} \quad (2.27)$$

Now, it is possible to apply a SDP optimization with the following coefficients and restrictions.

$$q_{11} = 0$$

$$q_{12} + q_{21} = -3$$

$$q_{13} + q_{22} + q_{31} = 2$$

$$q_{23} + q_{32} = 1$$

$$q_{33} = 2$$

The ideal and cone concepts are used for local SOS optimization via the so-called Positivstellensatz argumentations. For brevity, the reader is referred to the above-cited works or the thesis (Jarvis-Wloszek, 2003).

The advent of sum-of-squares allowed to express nonlinear systems as a convex combination of polynomial vertex models, based on a Taylor-series expansion (Sala & Ariño, 2009). Fuzzy polynomial systems appear in, for instance (Tanaka, Yoshida, Ohtake, & Wang, 2009; Sala, 2009; Bernal, Sala, Jaadari, & Guerra, 2011) generalising earlier Takagi-Sugeno approaches. Nevertheless, as these issues will not be used in this thesis, detail of them is omitted for brevity.

2.4 Copositive programming

A function $f(\lambda) : \mathbb{R}^r \mapsto \mathbb{R}$, mapping $\lambda \in \mathbb{R}^r$ to a real number, will be denoted as copositive, if the next condition holds.

$$f(\lambda) \geq 0 \quad \forall\ \lambda \geq 0 \quad (2.28)$$

where $\lambda \geq 0$ must be understood as element-wise nonnegativity.

Proving copositiveness for a generic function might be difficult. However, some results exist for homogeneous polynomials $p(\lambda)$ of, say, degree d, i.e., such that $p(b\lambda) = b^d p(\lambda)$ for any $b \in \mathbb{R}$.

For instance (Bundfuss & Dür, 2008), consider $f(\lambda)$ to be an homogeneous polynomial function of degree 2,

$$p(\lambda) = \sum_{i,j=1}^{r} \lambda_i \lambda_j p_{ij} \tag{2.29}$$

it is clear that, if $p_{ij} \geq 0$ and $\lambda_i \geq 0$ for all $i, j \in \{1, \ldots, r\}$, then $p(\lambda) \geq 0$. An analogous statement can be asserted for a parametric polynomial copositive function $p(\lambda, u) \geq 0,\ u \in \mathbb{U}$. Indeed, consider $p(\lambda, u)$ as:

$$p(\lambda, u) = \sum_{i,j=1}^{r} \lambda_i \lambda_j p_{ij}(u) \tag{2.30}$$

if $p_{ij}(u) \geq 0$ and $\lambda_i \geq 0$ for all $i, j \in \{1, \ldots, r\}$; then $p(\lambda, u) \geq 0$. Note that, these previous copositive conditions are sufficient but not necessary, i.e. although some $p_{ij}(u)$ or p_{ij} were less than zero, functions $p(\lambda)$ and $p(\lambda, u)$ could be still copositive[2].

Note that equations (2.29) and (2.30) can be formulated in the following way:

$$p(\lambda) = \lambda^T P \lambda \tag{2.31}$$

$$p(\lambda, u) = \lambda^T P(u) \lambda \tag{2.32}$$

where polynomial coefficients are arranged in matrices P and $P(u)$, so that $p_{ij} = \{P\}_{ij}$ and $p_{ij}(u) = \{P(u)\}_{ij}$.

Theorem 2.2 (Polya) *Let $p(\lambda) \geq \epsilon > 0$ be a homogeneous polynomial which is copositive when vector $\lambda \geq 0$. Then, there exists a finite $d \in \mathbb{N}$, where all the coefficients of the homogeneous polynomial below are positive (Pólya, 1928):*

$$p(\lambda) \left(\sum_{i=1}^{r} \lambda_i \right)^d \tag{2.33}$$

[2]For instance, in degree 2 polynomials expressed as (2.31), there exist positive-definite matrices P which have negative entries; see also Example 2.2.

Theorem 2.3 *An homogeneous polynomial $p(\lambda)$ is copositive if and only if there exists $d \geq 1$ such that $\left(\sum_{i=1}^{r} \xi_i^2\right)^d p(\xi_i^2) \in SOS$ (Reznick, 1995).*

Example 2.2 *Taking the polynomial $p(\lambda) = \sum_{i=1}^{2} \sum_{j=2}^{2} \lambda_i \lambda_j \Xi_{ij}$, with the next coefficients $\Xi_{11} = 3, \Xi_{12} = -1.1, \Xi_{21} = -1.6, \Xi_{22} = 2.5$, being $\lambda_i \geq 0$ and $\sum_{i=1}^{r} \lambda_i = 1$, this one can be formulated as:*
$\sum_{i=1}^{2} \sum_{j=2}^{2} \lambda_i \lambda_j \Xi_{ij} = \lambda_1 \lambda_1 \Xi_{11} + \lambda_1 \lambda_2 (\Xi_{12} + \Xi_{21}) + \lambda_2 \lambda_2 \Xi_{22} = \lambda_1 \lambda_1 3$
$+ \lambda_1 \lambda_2 (-1.1 - 1.6) + \lambda_2 \lambda_2 2.5$
note that, not all the coefficients are positive, nevertheless:
$\left(\sum_{i=1}^{2} \lambda_i\right)^2 \sum_{i=1}^{2} \sum_{j=2}^{2} \lambda_i \lambda_j \Xi_{ij} = \lambda_1 \lambda_1 \lambda_1 \lambda_1 \Xi_{11} + \lambda_1 \lambda_1 \lambda_1 \lambda_2 (2\Xi_{11} + \Xi_{12}$
$+ \Xi_{21}) + \lambda_1 \lambda_1 \lambda_2 \lambda_2 (\Xi_{11} + 2\Xi_{12} + 2\Xi_{21} + \Xi_{22}) + \lambda_1 \lambda_2 \lambda_2 \lambda_2 (\Xi_{12} + \Xi_{21}$
$+ 2\Xi_{22}) + \lambda_2 \lambda_2 \lambda_2 \lambda_2 \Xi_{22} = \lambda_1 \lambda_1 \lambda_1 \lambda_1 3 + \lambda_1 \lambda_1 \lambda_1 \lambda_2 (2 * 3 - 1.1 - 1.6)$
$+ \lambda_1 \lambda_1 \lambda_2 \lambda_2 (3 + 2 * (-1.1) + 2 * (-1.6) + 2.5) + \lambda_1 \lambda_2 \lambda_2 \lambda_2 (-1.1 - 1.6$
$+ 2 * 2.5) + \lambda_2 \lambda_2 \lambda_2 \lambda_2 2.5$
all the coefficients of the above polynomial are positive.

In section 2.5.7, these methodologies will be applied on TS systems.

From the above considerations, a symmetric matrix $P \in \mathbb{R}^n$ is called copositive if $\lambda^T P \lambda \geq 0$, for all $\lambda \geq 0$ (Dür, 2010). Obviously, all positive-semidefinite matrices are copositive.

As described in (Dür, 2010), copositive programming is closely related to the general (possibly non-convex) quadratic programming problem, as one can be converted to the other one via some manipulations.

2.5 Takagi-Sugeno Models and Fuzzy Control

This section discuses the concept of Takagi and Sugeno (TS) fuzzy systems (Takagi & Sugeno, 1985) (Tanaka & Wang, 2004). Followed by the **IF-THEN** procedures in order to construct such models. Then a model-based fuzzy controller design using the concept of "parallel distributed compensation" (PDC) is described. Moreover, in this chapter, the stability analysis and control design issues are shown, which can be reduced to linear matrix inequalities (LMIs) problems. And finally, the relaxing methodologies are presented in order to achieve a less conservative solution for LMI conditions.

The fuzzy model proposed by Takagi and Sugeno is described by fuzzy **IF-THEN** rules which represent local linear input-output relations of a nonlinear system. The main feature of a TS fuzzy model is to express the local dynamics of each fuzzy implication (rule) by a linear system model.

The *ith* rules of the TS fuzzy models have the following form, where the acronyms: CFS (Continuous Fuzzy System) and DFS (Discrete Fuzzy System) denote the continuous and the discrete fuzzy systems, respectively:

CFS:

$$\textbf{IF}\ z_1(t)\ is\ M_{i1}\quad and\quad \dots\quad z_p(t)\ is\ M_{ip}$$

$$\textbf{THEN} \begin{cases} \dot{x}(t) = A_i x(t) + B_i u(t) \\ y(t) = C_i x(t) \end{cases} \tag{2.34}$$

DFS:

$$\textbf{IF}\ z_1(k)\ is\ M_{i1}\quad and\quad \dots\quad z_p(k)\ is\ M_{ip}$$

$$\textbf{THEN} \begin{cases} x_{k+1} = A_i x_k + B_i u_k \\ y_k = C_i x_k \end{cases} \tag{2.35}$$

Here, M_{ij} is the fuzzy set and r is the number of model rules; $x(t) \in \mathbb{R}^n$ and $x_k \in \mathbb{R}^n$ are the state vectors, $u(t) \in \mathbb{R}^m$ and $u_k \in \mathbb{R}^m$ are the input vectors, $y(t) \in \mathbb{R}^q$ and $y_k \in \mathbb{R}^q$ are the output vectors, $A_i \in \mathbb{R}^{nn}$, $B_i \in \mathbb{R}^{nm}$, and $C_i \in \mathbb{R}^{qn}$; $z_1(t), \dots, z_p(t)$ are known as premise variables that may be functions of the state variables, external disturbances, and/or time. We will use $z(t)$ to denote the vector containing all the individual elements $z_1(t), \dots, z_p(t)$.

The TS systems can be modeled by means of sector nonlinearity methodology, which ensures that, the interpolated models are an exact formulation of the nonlinear system, in a limited local region on the state space. Evidently, the models are interpolated by membership functions.

In the sector nonlinearity, there exists an area where, such as it is depicted in the figure 2.2, for the system $\dot{x}(t) = f(x(t)) \in [a_1, a_2]x(t)$ with $x(0) = 0$. It is possible to achieve TS models with their corresponding membership functions in a bounded sector defined by the lines $-d_1 < x(t) < d_2$. The methodology to obtain all the elements for these fuzzy models is going to be explained in this section.

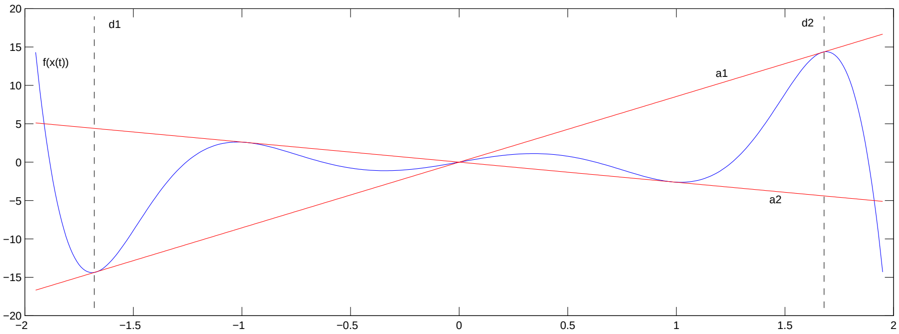

Figure 2.2: sector nonlinearity

Given a pair of $x(t), u(t)$, the final outputs of the fuzzy systems can be obtained as follows: **CFS**:

$$\dot{x}(t) = \frac{\sum_{i=1}^{r} w_i(z_t)(A_i x(t) + B_i u(t))}{\sum_{i=1}^{r} w_i(z_t)} = \sum_{i=1}^{r} \mu_i(z_t)(A_i x(t) + B_i u(t)) \tag{2.36}$$

$$y(t) = \frac{\sum_{i=1}^{r} w_i(z_t) C_i x(t)}{\sum_{i=1}^{r} w_i(z_t)} = \sum_{i=1}^{r} \mu_i(z_t) C_i x(t) \tag{2.37}$$

The same for a discrete fuzzy system: **DFS**:

$$x_{t+1} = \frac{\sum_{i=1}^{r} w_i(z_t)(A_i x_t + B_i u_t)}{\sum_{i=1}^{r} w_i(z_t)} = \sum_{i=1}^{r} \mu_i(z_t)(A_i x_t + B_i u_t) \tag{2.38}$$

$$y_t = \frac{\sum_{i=1}^{r} w_i(z_t) C_i x_t}{\sum_{i=1}^{r} w_i(z_t)} = \sum_{i=1}^{r} \mu_i(z_t) C_i x_t \tag{2.39}$$

where:

$$z_t = [z_1(t), z_2(t), \ldots, z_p(t)] \tag{2.40}$$

$$w_i(z_t) = \prod_{j=1}^{p} M_{i,j}(z_j(t)) \tag{2.41}$$

$$\mu_i(z_t) = \frac{w_i(z_t)}{\sum_{i=1}^{r} w_i(z_t)} \tag{2.42}$$

for all t. The term $M_{ij}(z_j(t))$ is the grade of membership of $z_j(t)$ in M_{ij}. Since:

$$\begin{cases} \sum_{i=1}^{r} w_i(z_t) > 0 \\ w_i(z_t) \geq 0 \quad i = 1, 2, \ldots, r \end{cases} \tag{2.43}$$

we have:

$$\begin{cases} \sum_{i=1}^{r} \mu_i(z_t) = 1 \\ \mu_i(z_t) \geq 0 \quad i = 1, 2, \ldots, r \end{cases} \tag{2.44}$$

for all t.

The sector-nonlinearity technique can be conservative in some multivariable cases (Sala, 2009). Also, models and controller analysis can be extended to piecewise setups (González, Sala, Bernal, & Robles, 2015; Gonzalez, Sala, Bernal, & Robles, 2017). Some proposals to reduce conservatism in the modelling step appear in (Robles, Sala, Bernal, & González, 2016; Robles, Sala, Bernal, & Gonzalez, 2016).

2.5.1 PDC Compensator

The history of the so-called *parallel distributed compensation* (PDC) began with a model-based design procedure proposed by Kang and Sugeno (Sugeno & Kang, 1986), however, the stability of the control systems was not approached in the design procedure. This controller was called *parallel distributed compensation* (Wang, Tanaka, & Griffin, 1995) (PDC).

A PDC fuzzy controller is formulated as follows:

$$u_k = \sum_{i=1}^{r} \mu_i(z_t) F_i x_k \tag{2.45}$$

for brevity:

$$u_k = \sum_{i=1}^{r} \mu_i F_i x_k \tag{2.46}$$

The equation (2.46) is substituted in (2.38), and for a DFS we obtain:

$$x_{t+1} = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j (A_i + B_i F_j) x_t \tag{2.47}$$

The same for a CFS:

$$\dot{x}_t = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j (A_i + B_i F_j) x_t \tag{2.48}$$

2.5.2 TS Stability

The TS systems stability was developed by (Tanaka & Sugeno, 1990)(Tanaka & Sugeno, 1992), such as it is explained in the next theorems, which are based on the Lyapunov stability definition, i.e. $V(x) > 0$, $\dot{V}(x) < 0$.

There are the following sufficient conditions for stability on DFS and CFS (Tanaka & Wang, 2004), with a closed loop controller:

Theorem 2.4 *A DFS as* (2.47) *is globally asymptotically stable if there exists a common positive definite matrix P so that:*

$$Q_{ij} = (A_i + B_i F_j)^T P (A_i + B_i F_j) - P \tag{2.49}$$

for $\sum_i^r \sum_j^r \mu_i \mu_j Q_{ij} < 0$ and $i, j = 1, \ldots, r$

Theorem 2.5 *A CFS as* (2.48) *is globally asymptotically stable if there exists a common positive definite matrix P so that:*

$$Q_{ij} = (A_i + B_i F_j)^T P - P(A_i + B_i F_j) \tag{2.50}$$

for $\sum_i^r \sum_j^r \mu_i \mu_j Q_{ij} < 0$ and $i, j = 1, \ldots, r$

Techniques to relax the inequality $\sum_i^r \sum_j^r \mu_i \mu_j Q_{ij} < 0$ will be presented in the following section 2.5.7.

2.5.3 Decay rate

Such as is indicated in (Tanaka, Ikeda, & Wang, 1998)(Tanaka, Taniguchi, & Wang, 1998b), the speed of response is related to decay rate, which consists in the largest Lyapunov exponent. Moreover, the decay rate is a parameter employed usually in the controller design.

In a DFS the condition $\Delta V(x(t)) \leq (\alpha^2 - 1)V(x(t))$ is the same as:

being $\beta = \alpha^2$

$$\min \beta \tag{2.51}$$

$$s.t.$$

$$X > 0 \tag{2.52}$$

$$\sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j Q_{ij} \geq 0 \tag{2.53}$$

being:

$$X = P^{-1} \quad M_i = F_i X \quad 0 < \beta \leq 1 \tag{2.54}$$

$$Q_{ij} = \begin{pmatrix} \beta X & (XA_i + B_i M_j)^T \\ XA_i + B_i M_j & X \end{pmatrix} \tag{2.55}$$

For a CFS, such as it is indicated in (Tanaka & Wang, 2004), the decay rate condition is $\dot{V}(x(t)) = -2\alpha V(x(t))$ for all trajectories, what is the same:

$$Q_{ij} = G_{ij}^T P + P G_{ij} + 2\alpha P \tag{2.56}$$

and to achieve a maximal decay rate with $\alpha > 0$:

$$\max \alpha \tag{2.57}$$

$$s.t.$$

$$X > 0 \tag{2.58}$$

$$\textstyle\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j Q_{ij} \leq 0 \tag{2.59}$$

where:

$$X = P^{-1} \quad M_i = F_i X \tag{2.60}$$

$$Q_{ij} = A_i X + B_i M_j + X A_j^T + M_i^T B_j^T + 2\alpha X \tag{2.61}$$

The above decay-rate optimizations are in the class of quasi-convex problems known as Generalized eigenvalue problems (GEVP) (Boyd & El Ghaoui, 1993).

2.5.4 Guaranteed Cost

Sometimes, in control engineering, it is interesting to ensure an upper bound for a cost index, perhaps dependent on the initial conditions.Usually, the cost index is a quadratic performance function, such as is written below.

Such as is written in (Tanaka, Nishimura, & Wang, 1998) (Tanaka, Taniguchi, & Wang, 1998a) (Tanaka, Taniguchi, & Wang, 1999), the cost index for a discrete fuzzy system can be defined in the following way:

$$J_{N\to\infty} = \sum_{k=N}^{\infty} x_k^T H x_k + u_k^T F u_k \tag{2.62}$$

Taking into account the Bellman theorem, which will be explained in the section 2.7.2. The cost index can be limited by applying the following condition in one step, i.e.

$$x_{N+1}^T P x_{N+1} - x_N^T P x_N < -(x_N^T H x_N + u_N^T F u_N) \tag{2.63}$$

With the initial state x_0, the cost index can be bounded by $\lambda > x_0^T P x_0$. This latter condition holds if there exist the matrices M_i and $X > 0$ such that:

$$\min \lambda \tag{2.64}$$

$$s.t.$$

$$\begin{pmatrix} \lambda & x_0^T \\ x_0 & X \end{pmatrix} > 0 \tag{2.65}$$

$$\textstyle\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j Q_{ij} > 0 \tag{2.66}$$

being

$$Q_{ij} = \begin{pmatrix} X & XA_i^T + M_j^T B_i^T & X & M_j^T \\ A_i X + B_i M_j & X & 0 & 0 \\ X & 0 & H & 0 \\ M_j & 0 & 0 & F \end{pmatrix} \tag{2.67}$$

where $P = X^{-1}$ and the PDC controller is formulated as $F_i = M_i X^{-1}$

2.5.5 Constraint on the Control Input

Such as, it is indicated in (Tanaka & Wang, 2004), the next theorem explains the conditions to bound the fuzzy system input:

Theorem 2.6 *Assume that the initial condition $x(0)$ is known. The constraint $||u(t)||_2 \leq \mu$ is enforced at all times $t \geq 0$, if the next LMI conditions hold:*

$$\begin{pmatrix} 1 & x(0)^T \\ x(0) & X \end{pmatrix} \geq 0 \tag{2.68}$$

$$\begin{pmatrix} X & M_i^T \\ M_i & \mu^2 I \end{pmatrix} \geq 0 \tag{2.69}$$

where $X = P^{-1}$ and $M_i = F_i X$

2.5.6 Constraint on the Output

Such as is discussed in (Tanaka & Wang, 2004), the next theorem explains the conditions in order to bound the output of a fuzzy system:

Theorem 2.7 *Assume that the initial condition $x(0)$ is known. The constraint $||y(t)||_2 \leq \lambda$ is enforced at all times $t \geq 0$, if the following LMI conditions hold:*

$$\begin{pmatrix} 1 & x(0)^T \\ x(0) & X \end{pmatrix} \geq 0 \tag{2.70}$$

$$\begin{pmatrix} X & XC_i^T \\ C_iX & \lambda^2 I \end{pmatrix} \geq 0 \tag{2.71}$$

hold, where $X = P^{-1}$

2.5.7 Fuzzy summations and Polya Expansion

In previous sections, conditions in the form:

$$\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j x^T Q_{ij} x \geq 0 \tag{2.72}$$

have appeared in control design setups. These conditions are nonlinear expressions, as μ_i are nonlinear function of the states. However, as $\mu_i \geq 0$, $\sum_{i=1}^{r} \mu_i = 1$ some sufficient LMI conditions have been proposed in literature.

The simplest ones (Tanaka & Wang, 2004) are:

$$Q_{ij} + Q_{ji} \geq 0, \quad Q_{ii} \geq 0$$

In (Tuan, Apkarian, Narikiyo, & Yamamoto, 2001), another sufficient conditions for (2.72) are formulated:

$$Q_{ii} > 0 \tag{2.73}$$

$$\frac{2}{r-1} Q_{ii} + Q_{ij} + Q_{ji} \geq 0 \quad i \neq j \tag{2.74}$$

In (Xiaodong & Qingling, 2003), a fuzzy continuous system is stable if satisfies the next theorem:

Theorem 2.8 *The fuzzy continuous system* (2.48) *is stable, if there exist matrices* M_i, P, Y_{ij}*; where* P *is a symmetric positive definite matrix, and* Y_{ii} *is symmetric, for* $i \neq j, Y_{ji} = Y_{ij}^T$*,* $i, j = 1, 2 \ldots, r$*. And the following LMIs are satisfied:*

$$PA_i^T + M_i^T B_i^T + A_i P + B_i M_i < Y_{ii} \tag{2.75}$$

$$\begin{aligned} PA_i^T + M_j^T B_i^T + A_i P + B_i M_j + PA_j^T + M_i^T B_j^T \\ + A_j P + B_j M_i < Y_{ij} + Y_{ij}^T \end{aligned} \tag{2.76}$$

Where $[Y_{ij}] < 0$ *and* $F_i = M_i P^{-1}, i = 1, \ldots, r$

Theorem 2.9 *The equilibrium of the DFS* (2.48) *is quadratically stabilizable via PDC controller* (2.46) *if there exist matrices* $P > 0$*,* M_i*,* $i = 1, 2, \ldots, r$*;* Y_{iii}*,* $i = 1, 2, \ldots, r$*;* $Y_{jii} = Y_{iij}^T$*, and* Y_{iji}*,* $i = 1, 2, \ldots, r$*,* $j \neq i$*,* $j = 1, 2, \ldots, r$ *and* $Y_{ijl} = Y_{lji}^T$*,* $Y_{ilj} = Y_{jli}^T$*,* $Y_{jil} = Y_{lij}^T$*,* $i = 1, 2, \ldots, r-2$*,* $j = i+1, \ldots, r-1$*,* $l = j+1, \ldots, r$*; satisfying the next LMIs:*

$$\begin{pmatrix} -P & * \\ A_i P + B_i M_i & -P \end{pmatrix} < Y_{iii} \quad i = 1, 2, \ldots, r \tag{2.77}$$

$$\begin{aligned} &\begin{pmatrix} -3P & * \\ 2A_i P + A_j P + B_i(M_i + M_j) + B_j M_i & -3P \end{pmatrix} \\ &\qquad \leq Y_{iij} + Y_{iji} + Y_{iij}^T \\ &\qquad i = 1, 2, \ldots, r-2, \ i \neq j, \ j = 1, 2, \ldots, r \end{aligned} \tag{2.78}$$

$$\begin{aligned} &\begin{pmatrix} -6P & * \\ 2(A_i + A_j + A_l)P + B_i(M_j + M_l) + B_j(M_i + M_l) + B_l(M_i + M_j) & -6P \end{pmatrix} \\ &\qquad \leq Y_{ijl} + Y_{ilj} + Y_{jil} + Y_{ijl}^T + Y_{ilj}^T + Y_{jil}^T \\ &\qquad i = 1, 2, \ldots, r-2, j = i+1, \ldots, r-1, l = j+1, \ldots, r \end{aligned} \tag{2.79}$$

$$\begin{pmatrix} Y_{1,i,1} & Y_{1,i,2} & \cdots & Y_{1,i,r} \\ Y_{2,i,1} & Y_{2,i,2} & \cdots & Y_{2,i,r} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{r,i,1} & Y_{r,i,2} & \cdots & Y_{r,i,r} \end{pmatrix} \leq 0 \quad i = 1, 2, \ldots, r \tag{2.80}$$

being $F_j = M_i P^{-1}, j = 1, 2, \ldots, r$

However, the problem was finally closed with an asymptotically exact (necessary and sufficient) condition as some complexity parmameter d grows, called Polya relaxation in (Sala & Ariño, 2007a).

The relaxation consist in proving positiveness of the coefficients of the expanded polynomial below, equal to that in (2.72) as $\sum_{i=1}^{r} \mu_i = 1$:

$$\left(\sum_{l=1}^{r} \mu_l\right)^d \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j x^T Q_{ij} x > 0 \quad \forall x \neq 0 \tag{2.81}$$

In order to compute the Polya relaxation, a multi-indices set $\mathbb{I}_d$, with d-dimension, will be defined for an easy adjustment of polynomial coefficients. So, this one is expressed as:

$$\mathbb{I}_d = \left\{\mathbf{i} = (i_1, i_2, \ldots, i_d) \in \mathbb{N}^d | 1 \leq i_j \leq r \quad j = 1, \ldots, d\right\} \tag{2.82}$$

with the above index, it is possible to write the following equation as:

$$\sum_{\mathbf{i}\in\mathbb{I}_d} Q_{\mathbf{i}} = \sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\cdots\sum_{i_d=1}^{r} Q_{i_1,i_2,\ldots,i_d} \tag{2.83}$$

and, each $\mu_{\mathbf{i}}$ will have the next formulation:

$$\mu_{\mathbf{i}} = \prod_{l=1}^{d} \mu_{i_l} = \mu_{i_l}\mu_{i_2}\ldots\mu_{i_d} \quad i \in \mathbb{I}_d \tag{2.84}$$

With the goal of simplifying the index $\mathbb{I}_d$, we use the index $\mathbb{I}_d^+$, being $\mathbb{I}_d = \cup\mathcal{P}(\mathbb{I}_d^+)$, which will ease computation of Polya's relaxation.

$$\mathbb{I}_d^+ = \{\mathbf{i} \in \mathbb{I}_d | i_p \leq i_{p+1} \quad p = 1, \ldots, d-1\} \tag{2.85}$$

Evidently:

$$\sum_{\mathbf{i}\in\mathbb{I}_d} \mu_{\mathbf{i}} Q_{\mathbf{i}} = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j Q_{ij} \tag{2.86}$$

And moreover:

$$\sum_{\mathbf{i}\in\mathbb{I}_d} \mu_{\mathbf{i}} Q_{i_1 i_2} = \sum_{\mathbf{i}\in\mathbb{I}_d^+} \mu_{\mathbf{i}} \sum_{\mathbf{j}\in\mathcal{P}(\mathbf{i})} Q_{j_1 j_2} = \sum_{\mathbf{i}\in\mathbb{I}_d^+} \mu_{\mathbf{i}} \widetilde{Q}_{\mathbf{i}} \tag{2.87}$$

being $\widetilde{Q}_{\mathbf{i}} = \sum_{\mathbf{j}\in\mathcal{P}(\mathbf{i})} Q_{j_1 j_2}$, and $\mathcal{P}(\mathbf{i})$ denotes the permutations of the vector $\mathbf{i}$.

For instance, the $\widetilde{Q}_{\mathbf{i}}$ with $\mathbf{i} = 1122$ will be:

$$\widetilde{Q}_{\mathbf{i}} = Q_{11} + 2Q_{12} + 2Q_{21} + Q_{22} \tag{2.88}$$

2.5.8 Non-quadratic stabilization

From (de Oliveira, Bernussou, & Geromel, 1999) and (Daafouz & Bernussou, 2001), a new stability condition for a fuzzy systems was developed, with the following Lyapunov function:

$$V(x_k) = x_k^T \mathrm{P}(x_k) x_k \tag{2.89}$$

with $\mathrm{P}(x_k) = \sum_{i=1}^{r} \mu_i(x_k) P_i$, a TS system is poly-quadratically stable if satisfies the theorem 2.10:

Theorem 2.10 *System as* $\sum_{i=1}^{r} \mu_i(x_k)(A_i+BF_i)x_k$ *is poly-quadratically stable, if and only if, there exist symmetric positive definite matrices* S_i*,* S_j*,* G_i *and* F_i *of appropriate dimensions such that:*

$$\begin{pmatrix} G_i + G_i^T - S_i & G_i^T Q_i^T \\ Q_i G_i & S_k \end{pmatrix} > 0 \tag{2.90}$$

being $Q_i = (A_i + BF_i)$ *with* $S_i = P_i^{-1}$

Proof: Such as is discussed in (de Oliveira et al., 1999) and (Daafouz & Bernussou, 2001), for a TS system formulated as:

$$\sum_{i=1}^{r} \mu_i(x_k)(A_i + BF_i)x_k \tag{2.91}$$

and with the following Lyapunov equation $V(x_k) = x_k^T \sum_{i=1}^{r} \mu_i(x_k) P_i x_k$ with $P_i > 0 \quad \forall i$, the system is poly-quadratically stable if it satisfies:

$$V(x_k) - V(x_{k+1}) > 0 \Leftrightarrow P_i - Q_i^T \left(P_k\right) Q_i > 0 \tag{2.92}$$

with $Q_i = A_j + BF_i$, for $i = 1, \ldots, r$.

Thus, by applying Schur complement twice:

$$S_k - Q_i^T \left(S_i\right) Q_i > 0 \tag{2.93}$$

where $S_k = P_k^{-1}$, taking $G_i = S_i + g_i I$ with g_i a positive scalar, we can deduce:

$$g_i^{-2}(S_i + 2g_i I) > Q_i^T T_{ij}^{-1} Q_i \quad T_{ij} = S_j - Q_i S_i Q_i^T > 0 \tag{2.94}$$

Such as it is done in (de Oliveira et al., 1999) and (Daafouz & Bernussou, 2001), by employing the Schur complement in the previous equation, it is possible replace g_i, so that:

$$\begin{pmatrix} G_i + G_i^T - S_i & (S_i - G_i)Q_i^T \\ Q_i(S_i - G_i) & S_j - Q_i S_i Q_i^T \end{pmatrix} > 0 \tag{2.95}$$

and finally, the equation (2.95) can be expressed as:

$$\begin{pmatrix} I & 0 \\ -Q_i & I \end{pmatrix} \begin{pmatrix} G_i + G_i^T - S_i & G_i^T Q_i^T \\ Q_i G_i & S_k \end{pmatrix} \begin{pmatrix} I & -Q_i^T \\ 0 & I \end{pmatrix} > 0 \tag{2.96}$$

being $Q_i G_i = A_i G_i + B M_i$ with $M_i = F_i G_i$. ■

Such as is presented in (Guerra & Vermeiren, 2004), a non-quadratic Lyapunov function is formulated as:

$$V(x) = x_k^T \left(P_z(x_k)\right)^{-1} x_k \tag{2.97}$$

being $P_z(x_k) = \sum_{i=1}^r \mu_i(x_k) P_i$, with the control action $u_k = K_z(x_k) P_z(x_k)^{-1}$, where $K_z(x_k) = \sum_{i=1}^r \mu_i(x_k) K_i$

With the previous definitions for the Lyapunov function and the control action, a TS system is stabilizable if it satisfies the theorem 2.11

Theorem 2.11 *A system* $\sum_{i=1}^r \mu_i(x_k)(A_i x_k + B_i u_k)$*, with the input* $u_k = K_z(x_k) P_z(x_k)^{-1}$*, is non-quadratically stabilizable, if there exist symmetric positive definite matrices* P_i *of appropriate dimensions, such that:*

$$\sum_{i=1}^r \sum_{j=1}^r \sum_{l=1}^r \mu_i(x_k)\mu_j(x_k)\mu_l(x_{k+1}) \begin{pmatrix} P_i & (A_i P_j + B_i K_j)^T \\ A_i P_j + B_i K_j & P_l \end{pmatrix} > 0 \tag{2.98}$$

Proof: From the Lyapunov equation (2.97), the condition $V(x_k) - V(x_{k+1}) > 0$ must be satisfied to guaranty the stability, so:

$$x_k^T \left(P_z(x_k)\right)^{-1} x_k - x_k^T \left(A_z(x_k) + B_z(x_k) K_z(x_k) \left(P_z(x_k)\right)^{-1}\right)^T \left(\left(P_z(x_{k+1})\right)^{-1}\right) \left(A_z(x_k) + B_z(x_k) K_z(x_k) \left(P_z(x_k)\right)^{-1}\right) x_k > 0 \tag{2.99}$$

by taking out common factor, now, it is possible to apply the congruence matrix $P_z(x_k)$, thus:

$$P_z(x_k) - (A_z(x_k)P_z(x_k) + B_z(x_k)K_z(x_k))^T (P_z(x_{k+1}))^{-1} (A_z(x_k)P_z(x_k) + B_z(x_k)K_z(x_k)) > 0 \quad (2.100)$$

Finally, the theorem 2.11 can be obtained by means of the Schur complement application. ■

Another extended polynomial conditions for TS systems stability are written in (Lendek, Guerra, & Lauber, 2012). Where $z \in \mathbb{R}^{n_z}$ is the scheduling vector, and it is assumed that the scheduling variables $z(k)$ are available at the time instant k. The subscript $z-$ stands for the sum being evaluated at time $k-1$, e.g., $A_{z-} = \sum_{i=1}^{r} \mu_i(z(k-1))A_i$, $z+$ means evaluation at time $k+1$, e.g., $A_{z+} = \sum_{i=1}^{r} \mu_i(z(k+1))A_i$; and multiple subscripts imply multiple sums, e.g., $A_{zz+} = \sum_{i=1}^{r} \mu_i(z(k)) \sum_{j=1}^{r} \mu_j(z(k+1))A_{ij}$.

Theorem 2.12 *A system $x_{k+1} = (A_z x_k + B_z u_k)$, with the input $u_k = K_f {G_h}^{-1}$, is asymptotically stable, if there exist symmetric positive definite matrix P_p; and G_h, K_f matrices of appropriate dimensions, such that:*

$$\begin{pmatrix} G_h + G_h^T - P_p & (*) \\ A_z G_z + B_z K_f & P_{p+}(x_{k+1}) \end{pmatrix} > 0 \quad (2.101)$$

Theorem 2.13 *A system $x_{k+1} = (A_z x_k + B_z u_k)$, with the input $u_k = K_f {G_h}^{-1}$, is asymptotically stable, if there exist symmetric positive definite matrix P_p; and G_h, K_f ones of appropriate dimensions, such that:*

$$\begin{pmatrix} \Omega_{1,1} + P_p & (*) & \dots & 0 & 0 \\ \Omega_{2,1} & \Omega_{2,2} & \dots & 0 & 0 \\ 0 & \Omega_{3,2} & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & \Omega_{\alpha-1,\alpha-1} & 0 \\ 0 & 0 & \dots & \Omega_{\alpha,\alpha-1} & -P_{p-\alpha} \end{pmatrix} < 0 \quad (2.102)$$

where $\Omega_{i,i-1} = A_{z+i-2}G_{h+i-2} - B_{z+i-2}K_{f+i-2}$; $\Omega_{i,i-1} = -G_{h+i-1} - G_{h+i-1}^T$, $i = 1, 2, \dots, \alpha$

The next p-degree Homogeneous Polynomially Non-Quadratic Lyapunov (HPNQL) function, (Ding, 2010), is expressed as follows:

$$V(x_k) = x_k^T S_{p,z}^{-1} x_k \tag{2.103}$$

being $S_{p,z} = \sum_{\mathbf{p}\in\mathcal{K}(p)} \mu^{\mathbf{p}} S_{\mathbf{p}} > 0$, where $\mu^{\mathbf{p}}$ denotes the vector-product of membership functions $\mu^{\mathbf{p}} = \mu_1^{p_1}\mu_1^{p_1}\mu_2^{p_2}\dots\mu_r^{p_r}$, and $\mathbf{p} = p_1 p_2 \dots p_r$; by definition, $\mathcal{K}(p)$ is the set of r-tuples, which are obtained as all possible combinations of nonnegative integers $p_i, i \in \{1,\dots,r\}$, such that $p_1 + p_2 + \dots + p_r = p$.

At time $k+1$, it is appropriate to express $V(x_{k+1})$ as:

$$V(x_k) = x_k^T S_{p,z_+}^{-1} x_k \tag{2.104}$$

where $S_{p,z_+} = \sum_{\mathbf{p}\in\mathcal{K}(p)} \mu_+^{\mathbf{p}} S_{\mathbf{p}} > 0$

Theorem 2.14 *(Ding, 2010) System $\sum_{i=1}^r \mu_i(x_k)(A_i x_k + B_i u_k)$ subject to $u_k = -Y_{p,z}S_{p,z}^{-1}$ is HPNQL stable if and only if there exist symmetric matrices S_p and any matrices Y_p, $\mathbf{p} \in \mathcal{K}(p)$, a degree $p \geq 1$, and sufficiently large $d \geq 0$, $d_+ \geq 0$ such that*

$$\sum_{\mathbf{k}\in\mathcal{K}(p+d_+)} \mathbf{h}_+^{\mathbf{k}} \frac{(p+d_+)!}{\pi(\mathbf{k})} = 1$$

$$\begin{aligned} L_{\mathbf{i},\mathbf{k}}^p := \frac{(p+d_+)!}{\pi(\mathbf{k})} \sum_{\mathbf{j}\in\mathcal{K}(d),\mathbf{i}\succeq\mathbf{j})} \left(\frac{d!}{\pi(\mathbf{j})} \sum_{s\in\{1,\dots,r\},i_s>j_s} \right. \\ \left. \times \begin{pmatrix} S_{\mathbf{i}-\mathbf{i}-\varepsilon_s} & (*) \\ A_s S_{\mathbf{i}-\mathbf{i}-\varepsilon_s} - B_s Y_{\mathbf{i}-\mathbf{i}-\varepsilon_s} & 0 \end{pmatrix} \right) \\ + \frac{(p+d+1)!}{\pi(\mathbf{i})} \sum_{\mathbf{l}\in\mathcal{K}(d_+),\mathbf{k}\succeq\mathbf{l}} \frac{(d_+)!}{\pi(\mathbf{l})} \begin{pmatrix} 0 & 0 \\ 0 & S_{\mathbf{k}-\mathbf{l}} \end{pmatrix} > 0 \end{aligned} \tag{2.105}$$

being $\pi(p) = (p_1!)(p_2!)\dots(p_r!)$ and ε_s as the r-tuple with its s-th element being 1 and all other elements being 0.

For simplicity, we use the following shortenings in the sequel:

$$\begin{cases} \mu_i = \mu_i(x_t),\ \mu = [\mu_1,\dots,\mu_r]^T, \mu_i^+ = \mu_i(x_{t+1}) \\ \mu^+ = [\mu_1^+,\dots,\mu_r^+]^T,\ \mu^k = \mu_1^{k_1}, \mu_2^{k_2},\dots,\mu_r^{k_r} \\ \mu^{+^k} = \mu_1^{+^{k_1}}\mu_2^{+^{k_2}}\dots\mu_r^{+^{k_r}},\ P(\mu) = \sum_{i=1}^r \mu_i P_i \\ P_g(\mu) = \sum_{k\in\mathcal{K}(g)} \mu^k P_k,\ P_g^+(\mu) = \sum_{k\in\mathcal{K}(g)} \mu^{+^k} P_k \end{cases} \tag{2.106}$$

The fuzzy Lyapunov function, named homogeneous polynomially parameter-dependent Lyapunov function (HPPD-LF) and associated control input (X. Xie, Ma, Zhao, Ding, & Wang, 2013), are designed as follows:

$$u_t = F_{g_1}(\mu)G_{g_2}(\mu)^{-1}x_t \tag{2.107}$$
$$V(x_t) = x^T(\mu)G_{g_2}(\mu)^{-T}P_{g_2}(\mu)G_{g_2}(\mu)^{-1}x_t \tag{2.108}$$

Theorem 2.15 *(X. Xie et al., 2013) Consider the discrete-time T–S fuzzy system $x_{t+1} = \sum_{i=1}^{r}\mu(x_t)(A_i x_t + B_i u_t)$ and the input $u_k = F_{g_1}(\mu)$ $G_{g_2}(\mu)^{-1}x_t$. For given $g \in \mathbb{Z}_+, d_1 \in \mathbb{Z}_+$, and $d_2 \in \mathbb{Z}_+$, the closed-loop fuzzy system is globally asymptotically stable, if there exist symmetric matrices $P_k \in \mathbb{R}^{n\times n}(k \in \mathcal{K}(g))$, matrices $F_k \in \mathbb{R}^{q\times n}, G_k \in \mathbb{R}^{n\times n}(k \in \mathcal{K}(g))$, symmetric matrices R^{ii}_{pq} with $q \in \mathcal{K}(g+d_1), p \in \mathcal{K}(g+d_2-1)$ and $i = 1,\ldots,r$, matrices $R^{ij}_{pq} = (R^{ji}_{pq})^T$ with $q \in \mathcal{K}(g+d_1), p \in \mathcal{K}(g+d_2-1)$, and $1 \le i < j \le r$, such as the following LMIs hold.*

$$\begin{aligned}&\left(\sum_{k'\in\mathcal{K}(d_2),k\ge k'}\ \sum_{i\in\{1,2,\ldots,r\};k_i>k'_i}\frac{(g+d_1)!}{\pi(k''')}\frac{d_2!}{\pi(k')}\begin{pmatrix}P_{k-k'-e_i} & *\\ A_iG_{k-k'-e_i}+B_iF_{k-k'-e_i} & 0\end{pmatrix}\right.\\
&\left.+\sum_{k''\in\mathcal{K}(g),k'''\ge k''}\frac{(d_1)!}{\pi(k'''-k'')}\frac{(g+d_2+1)!}{\pi(k)}\begin{pmatrix}0 & *\\ 0 & G_{k''}+G^T_{k''}-P_{k''}\end{pmatrix}\right)\\
&+\sum_{1\le i\le r}R^{ii}_{(k-2e_i)k'''}+He\left(\sum_{1\le i\le j\le r}R^{ij}_{(k-e_i-e_j)k'''}\right)>0\\
&\forall k'''\in\mathcal{K}(g+d_1), k\in\mathcal{K}(g+d_2+1)\end{aligned} \tag{2.109}$$

$$\begin{aligned}&\begin{cases}R^{ii}_{(k-2e_i)k'''}=0, & for\ k_i-2<0\\ R^{ij}_{(k-e_i-e_j)k'''}=0, & for\ k_i-1<0\ or\ k_j-1<0\end{cases}\\
&\begin{pmatrix}R^{11}_{pq} & \ldots & R^{1r}_{pq}\\ \vdots & \ddots & \vdots\\ R^{r1}_{pq} & \ldots & R^{rr}_{pq}\end{pmatrix}<0,\ \forall q\in\mathcal{K}(g+d_1), p\in\mathcal{K}(g+d_2-1)\end{aligned} \tag{2.110}$$

The control action and Lyapunov function (multiinstant HPPD-LF)] can be designed with the following forms (X. Xie, Yue, Zhang, & Xue, 2016):

$$u_t = F_g(t-m+1,\ldots,t)G_g^{-1}(t-m+1,\ldots,t)x_t \tag{2.111}$$

$$V(x_t) = x_t^T \left(G_g^{-T}(t-m+1,\ldots,t) P_s(t-m+1,\ldots,t) G_g^{-1}(t-m+1,\ldots,t) \right) x_t \quad (2.112)$$

where $F_g(t-m+1,\ldots,t) = \sum_{\left(\substack{k^i \in \mathcal{K}(g_i), \\ i \in \{1,2,\ldots,m\}}\right)} \prod_{i=1}^m \mu^{k^i}(t-m+i) F_{k^1\ldots k^m}$ and $G_g(t-m+1,\ldots,t) = \sum_{\left(\substack{k^i \in \mathcal{K}(g_i), \\ i \in \{1,2,\ldots,m\}}\right)} \prod_{i=1}^m \mu^{k^i}(t-m+i) G_{k^1\ldots k^m}$, $F_{k^1\ldots k^m} \in \mathbb{R}^{n_2 \times n_1}$, and $G_{k^1\ldots k^m} \in \mathbb{R}^{n_1 \times n_1}$ are control gain matrices to be determined, $g_1,\ldots,g_m \in \mathbb{Z}_+$. $P_s(t-m+1,\ldots,t) = \sum_{\left(\substack{k^i \in \mathcal{K}(g_i), \\ i \in \{1,2,\ldots,m\}}\right)} \prod_{i=1}^m \mu^{k^i}(t-m+i) P_{s^1 s^m}$, and $P_{s^1\ldots s^m} \in \mathbb{R}^{n_1 \times n_1}$, $s_1,\ldots,s_m \in \mathbb{Z}_+$. the closed-loop fuzzy system can be represented as follows:

$$x_{t+1} = \left(A(\mu) + B(\mu) F_g(t-m+1,\ldots,t) G^{-1} g(t-m+1,\ldots,t) \right) x_t. \quad (2.113)$$

Theorem 2.16 *(X. Xie et al., 2016) Consider the discrete-time T–S fuzzy system* $x_{t+1} = \sum_{i=1}^r \mu(x_t)(A_i x_t + B_i u_t)$ *and the input* $u_k = F_{g_1}(\mu) G_{g_2}(\mu)^{-1} x_t$*. For given* $g_1,\ldots,g_m, s_1,\ldots,s_m \in \mathbb{Z}_+$ *and* $d_1,\ldots,d_{m+1} \in \mathbb{Z}_+$*, the closed-loop fuzzy system is globally asymptotically stable, if there exist symmetric matrices* $P_{s^1 s^2\ldots s^m} \in \mathbb{R}^{n_1 \times n_1} (s^i \in \mathcal{K}(s_i), i \in 1,2,\ldots,m)$*, matrices* $F_{k^1 k^2\ldots k^m} \in \mathbb{R}^{n_2 \times n_1}, G_{k^1 k^2\ldots k^m} \in \mathbb{R}^{n_1 \times n_1} (k^i \in \mathcal{K}(gi), i \in \{1,2,\ldots \ldots,m\})$*, symmetric matrices* $R^{ii}_{f^1 f^2\ldots f_{m+1}} \in \mathbb{R}^{2n_1\ldots 2n_1}$ *with* $f^l \in \mathcal{K}(f_l + d_{m+2-l}), l \in \{1,2,\ldots,m-1\}, f^m \in \mathcal{K}(f_m + d_2 - 2), f^{m+1} \in \mathcal{K}(f_{m+1} + d_1)$*, and* $i = 1,\ldots,r$*, matrices* $R^{ij}_{f^1 f^2\ldots f^{m+1}} = (R^{ji}_{f^1 f^2\ldots f^{m+1}})^T \in \mathbb{R}^{2n_1 \times 2n_1}$ *with* $f^l \in \mathcal{K}(f_l + d_{m+2-l}), l \in \{1,2,\ldots,m-1\}, f^m \in \mathcal{K}(f_m + d_2 - 2), f^{m+1} \in \mathcal{K}(f_{m+1} + d_1)$*, and* $1 \le i < j \le r$*, such as the linear matrix inequalities (LMIs) in terms of following equations hold.*

$$\Upsilon_{f^1 f^2\ldots f^{m+1}} + \sum_{1 \le i \le r} R^{ii}_{f^1 f^2\ldots f^{m-1}(f^m - 2e_i) f^{m+1}} + He\left(\sum_{1 \le i \le j \le r} R^{ij}_{f^1 f^2\ldots f^{m-1}(f^m - e_i - e_j) f^{m+1}} \right) > 0 \quad (2.114)$$

$$\forall f^l \in \mathcal{K}(f_l + d_{m+2-l}), l \in \{1,2,\ldots,m-1\}, \quad f^m \in \mathcal{K}(f_m + d_2), f^{m+1} \in \mathcal{K}(f_{m+1} + d_1) \quad (2.115)$$

$$f_1 = \max\{g_1, s_1\}, f_i = \max\{g_i, s_i, g_{i-1}, s_{i-1}\}, (i = 2,\ldots,m-1), \quad f_m = \max\{g_m + 1, s_m, g_{m-1}, s_{m-1}\}, f_{m+1} = \max g_m, s_m \quad (2.116)$$

$$\Upsilon_{f^1 f^1 \ldots f^{m+1}} = \left(\sum_{\substack{s^j \in \mathcal{K}(s_j),\, f^j \geq s^j \\ j \in \{1,2,\ldots,m\}}} \phi^{s^1 s^2 \ldots s^m}_{f^1 f^2 \ldots f^{m+1}} + \sum_{\substack{k^j \in \mathcal{K}(k_j),\, f^j \geq k^j,\, j \in \{1,2,\ldots,m-1\} \\ f^m - k^m - e_l \geq 0,\, l \in \{1,2,\ldots,m\}}} \varphi^{k^1 k^2 \ldots k^{m_l}}_{f^1 f^2 \ldots f^{m+1}} - \sum_{\substack{s^j \in \mathcal{K}(s_j),\, f^{j+1} \geq s^j \\ j \in \{1,2,\ldots,m\}}} \psi^{s^1 s^2 \ldots s^m}_{f^1 f^2 \ldots f^{m+1}} + \sum_{\substack{k^j \in \mathcal{K}(k_j),\, f^{j+1} \geq k^j, \\ j \in \{1,2,\ldots,m\}}} \omega^{k^1 k^2 \ldots k^m}_{f^1 f^2 \ldots f^{m+1}} \right) \tag{2.117}$$

$$\phi^{s^1 s^2 \ldots s^m}_{f^1 f^2 \ldots f^{m+1}} = \left(\prod_{i=1}^{m-1} \frac{(f_i + d_{m-i+2} - s_i)!}{\pi(f^i - s^i)} \right) \frac{(f_m + d_2 - s_m)!}{\pi(f^m - s^m)} \frac{(f_{m+1} + d_1)!}{\pi(f^{m+1})} \begin{pmatrix} P_{s^1 \ldots s_m} & * \\ 0 & 0 \end{pmatrix} \tag{2.118}$$

$$\varphi^{k^1 k^2 \ldots k^{m_l}}_{f^1 f^2 \ldots f^{m+1}} = \left(\prod_{i=1}^{m-1} \frac{(f_i + d_{m-i+2} - g_i)!}{\pi(f^i - k^i)} \right) \frac{(f_m + d_2 - 1 - g_m)!}{\pi(f^m - k^m - e_l)} \frac{(f_{m+1} + d_1)!}{\pi(f^{m+1})} \begin{pmatrix} 0 & * \\ A_l G_{k^1 \ldots k^m} + B_l F_{k^1 \ldots k^m} & 0 \end{pmatrix} \tag{2.119}$$

$$\psi^{s^1 s^2 \ldots s^m}_{f^1 f^2 \ldots f^{m+1}} = \frac{(f_1 + d_{m+1})!}{\pi(f^1)} \left(\prod_{i=1}^{m-2} \frac{(f_{i+1} + d_{m-1-i} - s_i)!}{\pi(f^{i+1} - s^i)} \right) \frac{(f_m + d_2 - s_{m-1})!}{\pi(f^m - s^{m-1})} \frac{(f_{m+1} + d_1 - s_m)!}{\pi(f^{m+1} - s^m)} \begin{pmatrix} 0 & * \\ 0 & P_{s^1 \ldots, s^m} \end{pmatrix} \tag{2.120}$$

$$\omega^{k^1 k^2 \ldots k^m}_{f^1 f^2 \ldots f^{m+1}} = \frac{(f_1 + d_{m+1})!}{\pi(f^1)} \left(\prod_{i=1}^{m-2} \frac{(f_{i+1} + d_{m-1-i} - g_i)!}{\pi(f^{i+1} - k^i)} \right) \frac{(f_m + d_2 - g_{m-1})!}{\pi(f^m - k^{m-1})} \frac{(f_{m+1} + d_1 - g_m)!}{\pi(f^{m+1} - k^m)} \begin{pmatrix} 0 & * \\ 0 & He(G_{k^1 \ldots k^m}) \end{pmatrix} \tag{2.121}$$

$$
\begin{cases}
R^{ii}_{f^1 f^2 \dots f^{m-1}(f^m - 2e_i) f^{m+1}} = 0, \; for \; f^m_i - 2 < 0 \\
R^{ij}_{f^1 f^2 \dots f^{m-1}(f^m - e_i - e_j) f^{m+1}} = 0, \; for \; f^m_i - 1 < 0 \; or \; f^m_j - 1 < 0
\end{cases}
$$
$$
\begin{pmatrix}
R^{11}_{f^1 f^2 \dots f^{m+1}} & \dots & R^{1r}_{f^1 f^2 \dots f^{m+1}} \\
\vdots & \ddots & \vdots \\
R^{r1}_{f^1 f^2 \dots f^{m+1}} & \dots & R^{rr}_{f^1 f^2 \dots f^{m+1}}
\end{pmatrix} < 0
$$
$$
\forall f^l \in \mathcal{K}(f_l + d_{m+2-l}), l \in \{1, 2, \dots, m-1\},
$$
$$
f^m \in \mathcal{K}(g + d_2 - 1), f^{m+1} \in \mathcal{K}(f_{m+1} + d_1) \tag{2.122}
$$

2.5.9 Other approaches

The previous subsections have reviewed the main issues which will be of use in later developments. Nevertheless, fuzzy control is a mature field nowadays and significant contributions in various settings have been made. The review articles (Sala, Guerra, & Babuska, 2005) and (Feng, 2006) and references therein focus the main aspects prior to 2006, and the recent review paper (Guerra, Sala, & Tanaka, 2015) discusses the main novelties and perspectives of the field in the last decade.

Some issues worth pinpointing are, for instance:

The linear-fractional approach to fuzzy systems modelling (Tuan, Apkarian, Narikiyo, & Kanota, 2004), which can in some cases provide models with a lower number of rules, rational in the membership functions.

The work (Lendek, Guerra, Babuska, & De Schutter, 2011) discusses other non-quadratic control tools and the application to observers (which have not been considered here as the results of the thesis focus on state feedback).

Another issue is the fact that there might exist different Takagi-Sugeno models for the same nonlinear system (say $f(x) = x_1 x_2$ can be either modelled as $\psi(x_1)x_2$, bounding $\psi(x_1)$ with consequents and membership functions depending on x_1, or as $\psi(x_2)x_1$ with bounds and memberships depending on x_2. Some guidelines to choose the "best" model appear in (Robles, Sala, Bernal, & Gonzalez, 2016).

In fact, there is also the option of handling piecewise TS fuzzy models by independently applying sector-nonlinearity techniques in several regions. These ideas, introduced in (Johansson, Rantzer, & Arzen, 1999), have been developed in, for instance (Qiu, Feng, & Gao, 2012; Campos, Souza, Tôrres, & Palhares, 2013); recently, the work (Gonzalez et al., 2017) provides asymptotically exact stability conditions for such setting. Nevertheless, stabilization conditions for piecewise models are non-convex (bilinear matrix inequalities) so their application is limited due to such fact.

Also, there has been heavy work on the so-called adaptive fuzzy control in which a controller is tuned based on data, gradients of Lyapunov functions, etc. instead of LMIs. The interested reader can consult, for instance (Labiod, Boucherit, & Guerra, 2005; Boulkroune, Tadjine, M'Saad, & Farza, 2008; Tong, Huo, & Li, 2014) and references therein.

Nevertheless, further details on the above ideas is omitted as it will not be used in the contributions of this thesis.

2.6 Discrete-time Stochastic Markov-Jump Linear Systems

As is discussed in (do Valle Costa et al., 2006), a dynamical system is, in a certain moment, described by a model G_1. But it supposes that this system is subject to abrupt changes that cause it to be described, after a certain amount of time, by a different model, say G_2. More generally we can imagine that the system is subject to a series of possible changes that make it switch, over time, among a countable set $\mathcal{I} = 1, \ldots, m$ of models, for example, $\{G_1, G_2, \ldots, G_m\}$.

We will assume that the jumps evolve stochastically according to a Markov chain, that is, given that at a certain instant k the system lies in mode i, and we can know the jump probability for each of the other modes, and also the probability of remaining in mode i. Generally these systems are called Markov Jump Linear System (MJLS). We know the jump probability for each of the other modes, and also the probability of remaining in mode i (these probabilities depend only on the current operation mode). Notice that we assume only that the jump probability is known: in general, we do not know a priori when, if ever, jumps will occur. Summarizing the Markov chain transition probability matrix will can be builded as (L. Zhang & Boukas, 2009) (Boukas, 2007):

For a discrete system, the probability of being at time $k+1$ in mode j, conditioned to being at time k in mode i will be denoted by π_{ji}, forming a matrix:

$$\Pi = [\pi_{ji}] \quad where \quad \theta(k) = i \quad and \quad \theta(k+1) = j \tag{2.123}$$

being $\pi_{ji} \geq 0$ and $\sum_{j=1,i}^{m} \pi_{ji} = 1$

This class of systems are known in the international literature as discrete-time Markov jump linear systems (MJLS). The Markov state (or operation mode) will be denoted by $\theta(k)$.

A Stochastic discrete system MJLS is formulated as follows:

$$x_{k+1} = A_{\theta(k)} x_k + B_{\theta(k)} u_k \tag{2.124}$$

$$\theta(k) \in [1, \dots, m] \tag{2.125}$$

defining $\mathcal{I} = [1, \dots, m]$

2.6.1 Stochastic Stability

Concept of stochastic stability will be introduced at the present section, this one consists in a specific Lyapunov function for a stochastic systems, that was developed in (Daafouz, Riedinger, & Iung, 2002) (do Valle Costa et al., 2006) (Xiong, Lam, Gao, & Ho, 2005) among others.

The stability definition of a stochastic discrete system was presented by (do Valle Costa et al., 2006) as follows:

Definition 2.5 *System* (2.124) *is said to be stochastically stable if, for* $u_k \equiv 0$ *and every initial condition* $x_0 \in \mathbb{R}^n$ *and* $\theta_0 \in \mathcal{I}$*, the following holds:*

$$E\left\{\sum_{k=0}^{\infty} ||x_k||^2 \quad |x_0, \theta_0\right\} < \infty$$

And subject to the following theorem:

Theorem 2.17 *(do Valle Costa et al., 2006) The unforced system* (2.124) *is stochastically stable if, and only if, there exists a set of symmetric and positive-definite matrices P_i; $i \in \mathcal{I}$ satisfying*

$$A_i^T \mathbb{P}^i A_i - P_i < 0 \tag{2.126}$$

where $\mathbb{P}^i = \sum_{j \in \mathcal{I}} \pi_{ji} P_j$

A discrete MJLS is stabilizable if satisfies the next theorem:

Theorem 2.18 *(L. Zhang & Boukas, 2009) Consider a closed-loop discrete system* (2.124)*, the corresponding system is stochastically stable, if there exist matrices $X_i > 0$, $M_i \quad \forall i \in \mathcal{I}$, such that:*

$$\begin{pmatrix} -\mathcal{X} & \mathcal{L}^i Q_i \\ * & -X_i \end{pmatrix} < 0 \tag{2.127}$$

being:

$$\begin{aligned} Q_i &= A_i X_i + B_i M_i && (2.128) \\ \mathcal{L}^i &= (\sqrt{\pi_{1i}} I, \ldots, \sqrt{\pi_{mi}} I)^T && (2.129) \\ \mathcal{X} &= diag\{X_1, \ldots, X_m\} && (2.130) \end{aligned}$$

with the following control action $u_k = F_i x_k$ where $F_i = M_i X_i^{-1}$

2.6.2 Stability with partly unknown transition probabilities

There exists the possibility that several mode transitions are not known in a MJLS. These conditions were developed to guarantee the stability of the system (L. Zhang & Boukas, 2009), despite this feature. The next notation will be used in this section:

$$\begin{aligned} \mathcal{I}_k^i &= \{j \mid \pi_{ji} \quad is \quad known\} && (2.131) \\ \mathcal{I}_{uk}^i &= \{j \mid \pi_{ji} \quad is \quad unknown\} && (2.132) \end{aligned}$$

Obviously $\mathcal{I}^i = \mathcal{I}_k^i + \mathcal{I}_{uk}^i$

A discrete MJLS is stable if satisfies the next theorem:

Theorem 2.19 *(L. Zhang & Boukas, 2009) Consider unforced system* (2.124) *with partly unknown transition probabilities. The corresponding system is stochastically stable, if there exist matrices $P_i > 0$, $i \in \mathcal{I}$, such that:*

$$
\begin{aligned}
A_i^T \mathbb{P}_k^i A_i + \pi_k^i P_i &< 0 && (2.133)\\
A_i^T P_j A_i + P_i &< 0 \quad \forall j \in \mathcal{I}_{uk}^i && (2.134)
\end{aligned}
$$

being $\mathbb{P}_k^i = \sum_{j \in \mathcal{I}_k^i} \pi_{ji} P_j$

Stabilization with partly unknown transition probabilities At the current section, the stochastic systems have been dealt with a free response, now, the systems will have a closed-loop proportional controller, for a system with partly unknown transition probabilities.

A discrete MJLS is stabilizable if it satisfies the next theorem:

Theorem 2.20 *(L. Zhang & Boukas, 2009) Consider a closed-loop discrete system* (2.124) *with partly unknown transition probabilities. The corresponding system is stochastically stable, if there exist matrices $X_i > 0$, $M_i \quad \forall i \in \mathcal{I}$, such that:*

$$
\begin{aligned}
\begin{pmatrix} -\mathcal{X}_k^i & \mathcal{L}_k^i Q_i \\ * & -\pi_k^i X_i \end{pmatrix} &< 0 && (2.135)\\
\begin{pmatrix} -X_j & Q_i \\ * & -X_i \end{pmatrix} &< 0 \quad \forall j \in \mathcal{I}_{uk}^i && (2.136)
\end{aligned}
$$

being:

$$
\begin{aligned}
Q_i &= A_i X_i + B_i M_i && (2.137)\\
\mathcal{L}_k^i &= \left(\sqrt{\pi_{k_1^i i}} I, \ldots, \sqrt{\pi_{k_m^i i}} I\right)^T && (2.138)\\
\mathcal{X}_k^i &= diag\{X_{k_1^i}, \ldots, X_{k_m^i}\}, \quad \forall j \in \mathcal{I}_k^i && (2.139)
\end{aligned}
$$

with the following control action $u_k = F_i x_k$ where $F_i = M_i X_i^{-1}$

There are also results in which the stability is proven in an interval where each transition probability lies, see for instance (Xiong et al., 2005; Kao, Xie, & Wang, 2014), apart from the mere known/unknown setup discussed above.

Last, MJLS have received interest in the last years, and consequently more researchers have made improvements in this matter, in fields like stability and stabilization (Yan, Song, & Park, 2017), predictive control (J. Lu, Li, & Xi, 2013) or stochastically stable model predictive control (Patrinos, Sopasakis, Sarimveis, & Bemporad, 2014; Mejías, 2016). Nevertheless, predictive control on such models is "easier" than that on Takagi-Sugeno ones, because Markov-Jump systems do not have "intermediate" models whereas TS do.

2.7 Model Predictive Control

The model predictive control (MPC) is a term that includes a set of algorithms that share a similar features (Mayne, Rawlings, Rao, & Scokaert, 2000) (Lee, 2011)(P. Scokaert & Mayne, 1998). In such algorithms, there are always the following elements:

- **Prediction Horizon**: It denotes the number of steps to be predicted.
- **Prediction Model**: Used to predict the future states.
- **Cost Index**: It denotes the objective function in order to be minimized.
- **Optimizer**: An algorithm employed to achieve the optimal control actions, so that the cost index can be minimized.
- **Receding Horizon**: It means that at each sample time, only the first control input is applied.

Such as is discussed in (Goodwin, Graebe, & Salgado, 2001): the essential idea in Model Predictive Control (MPC) algorithms is to formulate controller design as an on-line receding horizon optimization problem, which is solved (usually by quadratic programming methods) subject to certain constraints.

The MPC is a control algorithm based on solving online a optimal control problem, the receding horizon approach can be summarized in the following steps:

1. At time k and for the current state x_k, solve, on-line, an open-loop optimal control problem over some future interval taking into account of the current and future constraints.

2. Apply the first step values in the optimal control sequence.

3. Repeat the procedure at time $(k+1)$ using the current state x_{k+1}.

Given the following system:

$$x_{k+1} = f(x_k, u_k) \tag{2.140}$$

the MPC at the instant k is computed by solving the next constrained optimal control problem:

$$\mathcal{P}_N(x): \quad V_N^0(x) = \min V_N(x, U) \tag{2.141}$$

being:

$$U = u_k, u_{k+1}, \ldots, u_{k+N-1} \tag{2.142}$$

$$V_N(x, U) = \sum_{k=0}^{N-1} L(x_k, u_k) + F(x_N) \tag{2.143}$$

subject to:

$$u_k \in \mathbb{U} \quad k = 1, \ldots, N-1 \tag{2.144}$$

$$x_k \in \mathbb{X} \quad k = 1, \ldots, N-1 \tag{2.145}$$

$$x_N \in \mathbb{X}_f \tag{2.146}$$

where, the set of constrains $\mathbb{U} \in \mathbb{R}^m$, the feasible set $\mathbb{X} \in \mathbb{R}^n$ and the terminal set $\mathbb{X}_f \in \mathbb{R}^n$.

2.7.1 Quadratic Cost index

We have described the MPC algorithm in a rather general nonlinear setting. However, it might reasonably be expected that further insights can be obtained if one specializes the algorithm to cover the linear case with quadratic cost function. Again we will use a state space set-up.

Let us therefore assume that the system is described by the following linear time invariant model (Goodwin, Seron, & De Doná, 2006):

$$x_{k+1} = Ax_k + Bu_k \tag{2.147}$$

$$y_k = Cx_k \tag{2.148}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ and $y_k \in \mathbb{R}^n$.

By assuming that (A, B, C) are stabilizable and detectable, and assuming in the problem of tracking a constant set-point y_s,that is, we wish to regulate, to zero, the error $e_k = y_k - y_s$.

Given knowledge of the current state measurement $x(0)$, our aim is to find the M-move control sequence $u_0, u_1, \ldots, u_{M-1}$ that minimizes the finite horizon performance index (Goodwin et al., 2006):

$$J_0 = [x_N - x_s]^T P[x_N - x_s] + \sum_{k=0}^{N-1} e_k^T Q e_k + \sum_{k=0}^{M-1} [u_k - u_s]^T R[u_k - u_s] \tag{2.149}$$

being $P > 0$, $Q > 0$ and $R > 0$; the terminal, sequence and inputs weighing matrices. In the above expression, u_s and x_s denote the steady state values:

$$u_s = -[C(I - A)^{-1}B]^{-1}y_s \tag{2.150}$$

$$x_s = (I - A)^{-1}Bu_s \tag{2.151}$$

Finally, the prediction model to compute an optimization problem can be formulated as:

$$X = \Gamma U + \Theta x_0 \tag{2.152}$$

where:

$$X = \begin{pmatrix} x_1 & x_2 & \dots & x_N \end{pmatrix}^T \tag{2.153}$$

$$U = \begin{pmatrix} u_0 & u_1 & \dots & u_{M-1} \end{pmatrix}^T \tag{2.154}$$

$$\Theta = \begin{pmatrix} A \\ A^2 \\ \vdots \\ A^N \end{pmatrix} \tag{2.155}$$

$$\Gamma = \begin{pmatrix} B & 0 & \dots & 0 & 0 \\ AB & B & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A^{M-1}B & A^{M-2}B & \dots & AB & B \\ A^M B & A^{M-1}B & \dots & A^2B & AB \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A^{N-1}B & A^{N-2}B & \dots & \dots & A^{N-M}B \end{pmatrix} \tag{2.156}$$

being N the prediction horizon, and M the control horizon, thus $M \leq N$

2.7.2 Bellman Theorem

The Bellman principle reduces an infinite period of optimization to only two periods, thus, the optimal control input is more easy to achieve i.e. if a generical control input is optimal in two consecutive time instants, also it will be for all the other periods (R. E. Bellman & Dreyfus, 2015).

It writes the value of a decision problem at a certain point in time in terms of the payoff from some initial choices and the value of the remaining decision problem that results from those initial choices. This breaks a dynamic optimization problem into simpler subproblems, as indicated by the Bellman theorem (R. Bellman, 1956):

Theorem 2.21 *The Bellman optimality condition states that: If* $u_y = u_t^{opt}, t \in [t_o, t_f]$, *is the optimal solution for the above problem,then* u_t^{opt} *is also the optimal solution over the (sub)interval* $[t_o + \delta t, t_f]$,*where* $t_o < t_o + \delta t < t_f$.

2.7.3 Non-lineal predictive control

Although, most of the industrial processes are not linear, the linear predictive control is the most widely extended. Mainly, because of two reasons(Camacho & Bordons, 2013): the simplicity to solve the optimization problem by Quadratic Programming (QP) and the adequacy of the linear models within a bounded region.

But some cases linear models are not accurate enough, either because the linearization is lax or region is too broad. In these cases, it is necessary to use a non-lineal models (E. S. Pérez, 2011).

The main handicap to use these models, is that QP cannot be used to solve the optimization problem, and is necessary to use a Sequential Quadratic Programming (SQP), what solve a sequence of optimization subproblems, each of which, optimizes a quadratic index subject to linear constraints that change in each iteration.

This technique has several issues (Biegler, 2000):

- In the original method is necessary the second order derivative of nonlinear functions, that sometimes can't be calculated
- The global convergence of the algorithm is not always guaranteed
- At each sample time, several QP problems must be solved, what may imply a high computational cost.
- It is possible that the intermediate solutions provided by the algorithm do not satisfy the original constraints.

Nonlinear predictive control is currently an active field of research. There does not exist a generic guaranteed methodology for a general case. Some approaches listed in (Camacho & Bordons, 2013) are:

- Extended linear MPC: Originally introduced for the DMC (Hernandez & Arkun, 1991). A term that does not depends on the future inputs, so that it allows solving the problem by means of QP.
- Local models: It lies in linearizing the nonlinear model around different operating points, successively in each optimization (Kouvaritakis, Cannon, & Rossiter, 1999)(Townsend & Irwin, 2001).

- Suboptimal nonlinear MPC: It was proposed by (P. O. Scokaert, Mayne, & Rawlings, 1999). The idea is instead of minimizing the cost index, to seek a cost index so that satisfies the restrictions, for decreasing the value of this index latter.
- Using short horizon: The idea is compute only the first control action in the sequence (the unique that is going to be applied), and use the linearized model to optimize the other control actions. It is dealt with in (Kouvaritakis & Cannon, 2001).
- Control sequence decomposition: This methodology divides the control actions in two groups, a control base sequence and a free increment of this ones. Then, the response is separated due to each one, by using for the first one a nonlinear model and other lineal for the second one, it allows optimizing the free response by means of a QP. Generally is necessary to use a iterative process that sums two sequences, being such sum a new base sequence, it is done until the free sequence is zero.
- Feedback linearization: It consists in a cancellation of the nonlinear part of the system, which implies to have a quadratic cost index, but restrictions nonlinear, which are approximated by linear ones. equations(Botto, Van Den Boom, Krijgsman, & Da Costa, 1999).
- MPC based on Volterra models: When the nonlinear model of the system is a second order Volterra model may be used a iterative process, but very quick (Pearson & Ogunnaike, 2002), what, by means of the computing of several QP's can be obtained the optimal. A special case of Volterra models are Hammerstein and Wiener models, that allow solving the optimization by the cancellation of non linear part.
- Neural Networks: Because of, neural networks are universal approximations, that may also be used to optimize cost function offline (Arahal, Berenguel, & Camacho, 1998).

Nowadays, current developments in MPC range from *Contractive Sets* (Ariño, Pérez, Sala, & Bedate, 2014), to *Hammerstein models* (Khani & Haeri, 2015) or *LMIs* (Xia, Yang, Shi, & Fu, 2010), as well as applications in (Ariño, Pérez, & Sala, 2010), and (Q. Lu, Shi, Lam, & Zhao, 2015). Nonlinear optimisation is pursued in (Biegler, 2010; Diehl, Ferreau, & Haverbeke, 2009; Andersson, Åkesson, & Diehl, 2012).

2.7.4 Fuzzy Model Predictive Control

The first references to fuzzy model predictive control with a certain entity are found in (De Oliveira & Lemos, 1995) where the fuzzy models are introduced in the model predictive control due to the capacity of these models to be identified online. A multi-step prediction is implemented and with this prediction a fuzzy controller is set. As can be observed, this type of implementation falls far short of the ideas of linear model predictive control and therefore the development of the thesis. In the same line there are works such as (Maeda, Shimakawa, & Murakami, 1995) and (Wong, Shah, Bourke, & Fisher, 2000).

Other interesting applications of the model predictive control to Takagi-Sugeno fuzzy systems are found in (Roubos, Mollov, Babuška, & Verbruggen, 1999), in this article the authors treat predictive control with all its basic ingredients. The optimization problem is obtained with the linearized model in a plausible prediction of the model. These ideas are similar to the implementation proposed in Chapter 6 of the thesis, without considering feasibility issues, terminal maximal polyhedral set and stability of a suboptimal solution treated in Chapter 6. In the same line there are the works of (Li, Li, & Xi, 2004) (Abonyi, Nagy, & Szeifert, 2001) (Sivakumar, Manic, Nerthiga, Akila, & Balu, 2010) (Boumehraz & Benmahammed, 2005).

The work (Kavsek-Biasizzo, Skrjanc, & Matko, 1997) and (T. Zhang, Feng, & Lu, 2007) computes a linear MPC by freezing the memberships at a particular instant and assuming, that, these ones will be constant in the future; it might work in practice, but it lacks theoretical justification in fast transients. The work (Y. Lu & Arkun, 2000) presents an interesting approach in which a sequence of quadratic cost bounds and state-feedback gains solves (sub optimally) the MPC problem. The great advantage is its computational tractability; however, it is well-known that even for the linear case, under constraints, the optimal value function is not quadratic in the state, so the approach is conservative. Recent works, such as (Q. Lu et al., 2015), discusses networked interval type-2 systems, but their results are still based on the 1-step equation discussed above, so they are not solving a multi-step problem such as most MPC literature understands. Another interesting result is found in (García-Nieto, Salcedo, Martínez, & Reynoso-Meza, 2010). It proposed to model an N-step fuzzy system so that the membership functions are constant at each step, but the model matrices are modified. In this way a simple predictive control is achieved to be solved, although it is true that it is not able to correctly represent the TS fuzzy system.

Finally, the work of (Bedate, 2015) has served as the basis for the development of the thesis. In this work, the prediction for a fuzzy system in all its complexity is presented. The controller dependents on the values of membership function at the prediction instant and an state feedback PDC control law. The cost index is dealt with an approximation to its real value, while the constraints have to be satisfied for all possible values of the membership functions. Stability is achieved by setting a contractive function, that decreases with each step of the system. The present thesis takes the idea of exploring to the maximum the constraints for any possible value of the membership functions, but also the control action depends on the membership function of the prediction instant and of all previous ones up to current time. The proposed cost index is minimized for the worse case; as its value decreases on time, it can be stated as Lyapunov function in order to prove stability.

2.8 Invariant Set theory

The invariant set theory has been applied since the 90s, with a successful outcomes by (Blanchini, 1999) (Kerrigan, 2000), in the control engineering field.

In this section, we are going to outline the main concepts in this theory.

First of all, let us consider a polytopic region Ω, which will be expressed as:

$$\Omega = \{x \in \mathbb{R}^n | Tx + s \leq 0\} \tag{2.157}$$

The definition of invariant set was first presented by (Blanchini, 1999), and it can be expressed as:

Definition 2.6 *Invariant set (Blanchini, 1999) (Kerrigan, 2000): The set $\Omega \subset \mathbb{R}^n$ is positively invariant for the system $x_{k+1} = f(x_k)$ if and only if for all $x_k \in \Omega \Rightarrow f(x_k) \in \Omega$*

Another interesting definition is the closed loop invariant set, i.e. when a system is controlled by a control action u_k:

Definition 2.7 *Control invariant set (Blanchini, 1999) (Kerrigan, 2000): The set $\Omega \subset \mathbb{R}^n$ is positively invariant for the system $x_{k+1} = f(x_k, u_k)$ if and only if for all $x_k \in \Omega \Rightarrow \exists u_k \in \mathbb{U}\ :\ f(x_k) \in \Omega$*

Robust invariant Sets:

Analogous to previous lines, but now for a systems subject to disturbances ω_k, where $\omega_k \in \mathbb{W}$, the definition of a Robust positively invariant set is:

Definition 2.8 *Robust invariant set (Blanchini, 1999) (Kerrigan, 2000): The set $\Omega \subset \mathbb{R}^n$ is robust positively invariant for the system $x_{k+1} = f(x_k, \omega_k)$ if and only if $\forall x_0 \in \Omega$ for all $\omega_k \in \mathbb{W}$, so that the system evolution satisfies $x_k \in \Omega, \forall k \in \mathbb{N}$.*

In other words, Ω is robust invariant, if and only if, the next condition holds:

$$x_k \in \Omega \Rightarrow x_{k+1} \in \Omega,\ \forall w_k \in \mathbb{W} \tag{2.158}$$

The same for a robust control invariant set, which can be defined as:

Definition 2.9 *Robust control invariant set(Blanchini, 1999) (Kerrigan, 2000): The set $\Omega \subset \mathbb{R}^n$ is robust invariant for the system $x_{k+1} = f(x_k, u_k, \omega_k)$ if and only if $\forall x_0 \in \Omega$ and $\forall w_k \in \mathbb{W}$, exists $u_k \in \mathbb{U}$, so that the system evolution satisfies $x_k \in \Omega, \forall k \in \mathbb{N}$.*

In other words, Ω is robust invariant, if and only if:

$$x_k \in \Omega \Rightarrow \exists u_k \in \mathbb{U}\ :\ x_{k+1} \in \Omega, \quad \forall w_k \in \mathbb{W} \tag{2.159}$$

Note that, actually, invariant sets are quite related to Lyapunov level sets: if V is a Lyapunov function, its level sets $\{V(x) \leq \gamma\}$ are invariant. Also, robust invariant sets are, too, related to the concept of *inescapable* sets conceived as level sets of a Lyapunov-like function (Salcedo, Martínez, & García-Nieto, 2008; Sala & Pitarch, 2016). Such level sets are, however, in most cases, conservative (as there are larger invariant sets than the, for instance, ellipsoidal bounds arising from quadratic Lyapunov functions).

Nevertheless, exact computation of the maximal invariant sets, in a modelling region can be carried out in the linear case and an asymptotically exact inner approximation for the Takagi-Sugeno shape independent approach, via the so-called one-step sets discussed below.

2.8.1 One-step Set

The One-step set is another of the interesting concepts employed in the set theory field, and necessary to achieve the invariant set.

The definition of invariant one-step set was presented by (Blanchini, 1991)(Kerrigan, 2000) like that:

Definition 2.10 *The closed-loop one-step set $\mathcal{Q}(\Omega)$ (Blanchini, 1991) (Kerrigan, 2000): The $\mathcal{Q}(\Omega)$ is the set of states in $\mathbb{R}^n$ for which an admissible control input exists that will guarantee that the system will be lead to Ω in one step.*

what is the same:

$$\mathcal{Q}(\Omega) = \{x_k \in \mathbb{R}^n | \exists u_k \in \mathbb{U} : f(x_k, u_k) \in \Omega\} \tag{2.160}$$

Similar for a system subject to disturbances ω_k, with $\omega_k \in \mathbb{W}$:

Definition 2.11 *The robust closed-loop one-step set $\mathcal{Q}(\Omega)$ (Blanchini, 1991) (Kerrigan, 2000): The $\mathcal{Q}(\Omega)$ is the set of states in $\mathbb{R}^n$ for which an admissible control input exists which will guarantee that the system will be lead to Ω in one step, for all allowable disturbances $\omega_k \in \mathbb{W}$.*

$$\mathcal{Q}(\Omega) = \{x_k \in \mathbb{R}^n | \exists u_k \in \mathbb{U} : f(x_k, u_k, w_k) \in \Omega, \forall w_k \in \mathbb{W}\} \tag{2.161}$$

Other definition used in the current thesis is the robust controllable i-step set $\mathbb{K}_i(\Omega, \mathbb{T})$, which is defined as:

Definition 2.12 *Given an arbitrary target $\mathbb{T} \subset \Omega$ the set $\mathbb{K}_i(\Omega, \mathbb{T})$ for the system $x_{k+1} = f(x_k, u_k, w_k)$ is defined as:*

$$\mathbb{K}_i(\Omega, \mathbb{T}) = \{x_0 \in \mathbb{R}^n | u_k \in \mathbb{U}, \\ x_k \in \Omega_{k=0}^{i-1}, \quad x_i \in \mathbb{T}, \quad \forall w_k \in \mathbb{W}_{k=0}^{i-1}\} \quad (2.162)$$

From the above definition, $\mathbb{K}_i(\Omega, \mathbb{T})$ is the largest subset of Ω, where the states lies into Ω up to the *i-1* step, and moreover in the i step, states get into $\mathbb{T}$ which $\mathbb{T} \in \Omega$. The $\mathbb{K}_{i+1}(\Omega, \mathbb{T})$ can be computed as follows:

$$\mathbb{K}_{i+1}(\Omega, \mathbb{T}) = \mathcal{Q}(\mathbb{K}_i) \cap \Omega \quad (2.163)$$

And finally, in order to obtain the invariant set, the algorithm 1 is applied. It was developed by (Blanchini, 1999) and enhanced by (Kerrigan, 2000):

Algorithm 1 Calculation of the closed-loop N-step invariant set $\mathbb{K}_N(\Omega, \mathbb{T})$

1. Make $i = 0$ and $\mathbb{K}_0(\Omega, \mathbb{T}) = \mathbb{T}$
2. While $i < N$:
   (a) $\mathbb{K}_{i+1}(\Omega, \mathbb{T}) = \mathcal{Q}\left(\mathbb{K}_i(\Omega, \mathbb{T})\right) \cap \Omega$
   (b) If $\mathbb{K}_{i+1}(\Omega, \mathbb{T}) = \mathbb{K}_i(\Omega, \mathbb{T})$, end algorithm and $\mathbb{K}_N(\Omega, \mathbb{T}) = \mathbb{K}_\infty(\Omega, \mathbb{T}) = \mathbb{K}_i(\Omega, \mathbb{T})$.
   (c) i=i+1

2.8.2 Approximated One-Step Set for TS Fuzzy Systems

So far, the robust sets for linear systems have been dealt with in a generic way, but now, the goal is to formulate previous statements in more detail for non-linear systems, so that, it is necessary to employ a TS fuzzy systems with local models. This methodology is going to be presented in the following lines (Bedate, 2015).

Given a TS system with a PDC controller $u = \sum_{i=1}^{r} F_i x$. Then, by following the results in Section 2.5.7 the d-Polya expanded closed-loop

TS system is (Ariño, Perez, Bedate, & Sala, 2013):

$$x_{k+1} = \sum_{\mathbf{i}\in\mathbb{I}_d^+} n_{\mathbf{i}}\mu_{\mathbf{i}}(\widetilde{G}_{\mathbf{i}}x_k + \widetilde{B}_{\mathbf{i}_w}w_k) \tag{2.164}$$

being:

$$\widetilde{G}_{\mathbf{i}} = \frac{1}{n_{\mathbf{i}}}\sum_{\mathbf{j}\in\mathcal{P}(\mathbf{i})}(A_{j_1} + B_{j_1}F_{j_2}) \qquad \widetilde{B}_{\mathbf{i}_w} = \frac{1}{n_{\mathbf{i}}}\sum_{\mathbf{j}\in\mathcal{P}(\mathbf{i})} B_{j_1 w} \tag{2.165}$$

with $\sum_{\mathbf{i}\in\mathbb{I}_d^+} n_{\mathbf{i}}\mu_{\mathbf{i}} = 1$

In order to improve the approximation of one-step fuzzy set, Polya expansion is applied (Ariño et al., 2013), note that, a similar notation is formulated in the section 2.5.7:

$$x_{k+1} = \left(\sum_{i=1}^{r}\mu_i\right)^{d-2} \cdot x_{k+1} = \sum_{\mathbf{i}\in\mathbb{I}_d}\mu_{\mathbf{i}}(G_{i_1 i_2}x_k + B_{i_1\omega}\omega_k) \tag{2.166}$$

From the previous equation:

$$\begin{aligned}\sum_{\mathbf{i}\in\mathbb{I}_d}\mu_{\mathbf{i}}\left(G_{i_1 i_2}x_k + B_{i_1\omega}\omega_k\right) = \\ \sum_{\mathbf{i}\in\mathbb{I}_d^+}\sum_{\mathbf{j}\in\mathcal{P}(\mathbf{i})}\mu_{\mathbf{i}}\left(G_{j_1 j_2}x_k + B_{j_1\omega}\omega_k\right) = \\ \sum_{\mathbf{i}\in\mathbb{I}_d^+}\mu_{\mathbf{i}}\sum_{\mathbf{j}\in\mathcal{P}(\mathbf{i})}\left(G_{j_1 j_2}x_k + B_{j_1\omega}\omega_k\right) = \\ \sum_{\mathbf{i}\in\mathbb{I}_d^+} n_{\mathbf{i}}\mu_{\mathbf{i}}\sum_{\mathbf{j}\in\mathcal{P}(\mathbf{i})}\left(\frac{G_{j_1 j_2}x_k + B_{j_1\omega}\omega_k}{n_{\mathbf{i}}}\right) = \\ \sum_{\mathbf{i}\in\mathbb{I}_d^+} n_{\mathbf{i}}\mu_{\mathbf{i}}\left(\widetilde{G}_{\mathbf{i}}x_k + \widetilde{B}_{\mathbf{i}\omega}\omega_k\right)\end{aligned} \tag{2.167}$$

Taking into account the above information, the one-step set will be formulated as:

$$\widetilde{\mathcal{Q}}(\Omega) = \{x_k \in \mathbb{R}^n | T\left(\widetilde{G}_{\mathbf{i}}x + B_{\mathbf{i}\omega}\omega_j\right) + s \leq 0, \quad \forall j = 1, \ldots, n_{\mathbb{W}}, \forall \mathbf{i} \in \mathbb{I}_d^+\} \tag{2.168}$$

where ω_j are the vertices of $\mathbb{W}$ and $n_{\mathbb{W}}$ is the number of vertices.

The process to compute the invariant set is very similar to the discussed one in Section 2.8.1 by means of Algorithm 1.

2.9 Conclusions

In the current chapter, some concepts have been presented in order to understand the main contributions of the thesis, this background has included several topics such as Fuzzy Control, Predictive Control, Markov-Jump systems' Control and Set Theory.

At the beginning, stability formulation and the LMIs have been shown, because these ones are two of the most important math tools in the control engineering, especially in fuzzy control.

After, TS fuzzy control systems have been explained together with techniques and methodologies to deal with these systems. In such section, the main concepts introduced have been Stability/Stabilization, PDC controller, Guaranteed Cost, Constraints on inputs and/or outputs, and relaxation methodologies mainly based on copositive programming, all these focused on TS systems.

Then, the stability and control of Markov-Jump linear systems has been deal with, as well as, the formulation of these systems. Moreover, there are presented the policies to study the stability and stabilization in the stochastic control field.

Afterwards, the predictive control is summarised, which is one of the most important issues discussed in this thesis. There are listed the main features of this control, such as cost index , Bellman theorem, Riccati equation, receding horizon, and non-linear predictive control.

At the end, Invariant Sets and their properties are explained, as well as techniques to achieve a polyhedral invariant set, both for linear or fuzzy systems.

Chapter 3

Improved stability for Takagi-Sugeno systems by applying Polya's Theorem with multi-indices

The present chapter researches the inherent tensor-product structure of the Takagi-Sugeno models obtained by means of non-lineal sector methodology (Tanaka & Wang, 2004), so that, it is possible to obtain more relaxed stability conditions, that, if these ones were achieved for a generic Takagi-Sugeno model (Sala & Ariño, 2007a). It occurs because a fraction of the modeling information of these systems is usually lost, when the stability conditions are applied, such as was formulated in (Ariño & Sala, 2007). The current work extends the previous contribution to get a asymptotically exact conditions by use of Polya theorem for TS systems expressed in tensor-product structure.

A preliminary version of the contents of this chapter appears in (Querol et al., 2014).

3.1 Introduction

Currently, Takagi-Sugeno (TS) fuzzy systems are widely employed for modelling non-linear processes. It is because, there exists a systematic modelling methodology denoted as sector non-linearity (Tanaka & Wang, 2004). By means of this technique, the obtained Takagi-Sugeno model is equivalent to the non-linear in a compact bounded region.

The TS systems modelled by this technique or similar techniques consist in an interpolation of local models (Tanaka & Wang, 2004). The validity of these models is given by a set of rules, each one depends on one or several membership functions, μ_{ij}, what assessment the level of compliance for the rule. If each model depends on several functions, the product of them is necessary to obtain the interpolation.

In most of previous works, connection among the local models has been omitted, as well as, the product of membership functions μ_{ij}. These works assume a single membership function for each one of the local models. Both descriptions of the system are equivalent, the second approach simplifies the computing to obtain the controller by means of LMIs, but the system loses information about the model structure which produces that the controller might be too much conservative. It was widely discussed in (Ariño & Sala, 2007), where, there is proved that, a better PDC design can be achieved by maintaining the product structure in the membership functions μ_{ij}. Currently, it has been noted that the stability conditions presented were conservatives, i.e. sufficient but not necessary. This one has led to the suggestion, such as it has been done for another kind of models (Sala & Ariño, 2007a), that, it is possible to obtain stability conditions which are asymptotically necessary, as well as sufficient.

The structure of this chapter is divided into the next four sections: at the beginning there are the **Preliminaries**, where, stability conditions and relaxation ones based on Polya theorem are presented. Next section is **Notation and Contributions**, there are discussed the notation as well as the main contributions of this chapter, in order to compute new relax conditions based on Polya expansion. Following section contains several **Examples**, where, the potential and the improvement achieved by means of the current relaxation methodology are tested and compared. Finally, **Conclusions**, where, the results of the current work are assessed and discussed.

3.2 Preliminares

As has been mentioned previously, a TS system basically consists of several models linked with a set of rules, such as can be read in the next expression:

$$\begin{aligned} &\textbf{IF} \quad \mu_{1,i_1}(z_1(t)) \quad and \ldots and \quad \mu_{p,i_p}(z_p(t)) \\ &\qquad\qquad \textbf{THEN} \begin{cases} \dot{x}(t) = A_{\mathbf{i}} x(t) + B_{\mathbf{i}} u(t) \\ y(t) = C_{\mathbf{i}} x(t) \end{cases} \end{aligned} \tag{3.1}$$

where $\mathbf{i} = (i_1, i_2, \ldots, i_p)$.

Usually, for brevity the notation is simplified, so that, the indices of vector $\mathbf{i}$ are handled as a single one. In this chapter, these indices are not simplified, because, all the available information will be employed on the development of new stability conditions. Thus, the fuzzy system can be written as:

$$\dot{x} = \sum_{i_1=1}^{r_1} \cdots \sum_{i_p=1}^{r_p} \mu_{1,i_1} \ldots \mu_{p,i_p} (A_{\mathbf{i}} x + B_{\mathbf{i}} u) \quad \mathbf{i} = (i_1, i_2, \ldots, i_p) \tag{3.2}$$

For brevity, there are defined $\mu_{\mathbf{i}} = \mu_{1,i_1} \ldots \mu_{p,i_p}$ and $\sum_{\mathbf{i} \in \mathbb{I}} = \sum_{i_1=1}^{r_1} \cdots \cdots \sum_{i_p=1}^{r_p}$ where $\mathbb{I}$ denotes the set for all the possible values of $\mathbf{i}$.

Here, we explain a method in order to design a PDC (Parallel Distributed Controller) (Tanaka & Wang, 2004). So, the control action is formulated as:

$$u = \sum_{\mathbf{i} \in \mathbb{I}} \mu_{\mathbf{i}} F_{\mathbf{i}} x \tag{3.3}$$

Therefore, the system behavior in closed-loop can be expressed with a double sum, just as it is displayed in the next expression:

$$\dot{x} = \sum_{\mathbf{i} \in \mathbb{I}} \sum_{\mathbf{j} \in \mathbb{I}} \mu_{\mathbf{i}} \mu_{\mathbf{j}} (A_{\mathbf{i}} + B_{\mathbf{i}} F_{\mathbf{j}}) x \tag{3.4}$$

$$\mathbf{i} = (i_1, i_2, \ldots, i_p) \quad \mathbf{j} = (j_1, j_2, \ldots, j_p) \tag{3.5}$$

$$\mu_{\mathbf{i}} = \mu_{1,i_1}, \ldots, \mu_{p,i_p} \tag{3.6}$$

Stability study and stabilization problem of the TS systems, have been widely dealt with for the last ten years by several researchers like (Tanaka & Wang, 2004)(Guerra & Vermeiren, 2004)(Tuan et al., 2001)(Fang, Liu, Kau, Hong, & Lee, 2006), and more recently by (Ariño, Pérez, Sala, & Bedate, 2014). As intermediate outcome, it can be employed, in order to check the stability, the stabilization, or several capabilities, such as decay-rate, disturbance rejection. It is enough with assessing, if a second order matrix polynomial, formed by membership functions, is negative definite.

$$\sum_{\mathbf{i}} \sum_{\mathbf{j}} \mu_{\mathbf{i}} \mu_{\mathbf{j}} x^T Q_{\mathbf{ij}} x < 0, \quad \forall x \neq 0 \tag{3.7}$$

Generally, this problem is carried out with the membership functions simplified, because, it is assumed that given **i**, **j**, **k** and **l** then $\mu_{\mathbf{i}}\mu_{\mathbf{j}} \neq \mu_{\mathbf{k}}\mu_{\mathbf{l}}$, what is not always right, such as is indicated as follows:

$$\mu_{(1,1)} = \mu_{11}\mu_{21}, \quad \mu_{(2,2)} = \mu_{12}\mu_{22},$$

$$\mu_{(1,2)} = \mu_{11}\mu_{22}, \quad \mu_{(2,1)} = \mu_{12}\mu_{21}$$

Note that $\mu_{(1,1)}\mu_{(2,2)} = \mu_{(1,2)}\mu_{(2,1)}$, this idea was developed by (Ariño & Sala, 2007), where some stability conditions are obtained exploiting these equalities. The problem of the conditions obtained in such work, it is that these were not asymptotically exact because the Polya Relaxation in (Sala & Ariño, 2007b), discussed on Section 2.5.7 of this thesis, was not yet published. This issue is tackled in this work.

Following, in order to explain this relaxation sequel, there are presented the contributions of (Sala & Ariño, 2007a), as well as (Powers & Reznick, 2001), where Polya's theorem application is presented to relax inequalities with the next structure:

$$\sum_i \sum_j \mu_i \mu_j x^T Q_{ij} x < 0, \quad \forall x \neq 0 \tag{3.8}$$

by keeping the membership functions in a simplex i.e. $\mu_i > 0$, $\sum_i \mu_i = 1$.

So, the equation (3.8) can be expressed as:

$$\left(\sum_{l=1}^{r} \mu_l \right)^{d-2} \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j x^T Q_{ij} x < 0 \tag{3.9}$$

$$\sum_{i_1=1}^{r} \sum_{i_2=1}^{r} \cdots \sum_{i_d=1}^{r} \mu_{i_1} \mu_{i_2} \ldots \mu_{i_d} x^T Q_{i_1 i_2} x < 0, \quad \forall x \neq 0 \tag{3.10}$$

This expression can be arranged like it is discussed in (3.13), by doing a notation change, such as is discussed in the following lines:

$$\mathbb{I}_{d,r} = \{\mathbf{i} = (i_1, i_2, \ldots, i_d) \in \mathbb{N}^p | 1 \leq i_j \leq r \quad \forall j = 1, \ldots, d\} \tag{3.11}$$

another way to refer to k_{th} element in the index $\mathbf{i}$ is $\mathbf{i}(k) = i_k$.

$$\mathbb{I}^+_{d,r} = \{\mathbf{i} \in \mathbb{I}_{d,r} | i_k \leq i_{k+1}, \quad k = 1, \ldots, d-1\} \tag{3.12}$$

Note that, $\mathbb{I}^+_{d,r}$ contains all the indices of different monomials belonging to the polynomial (3.10), thus, this polynomial can be expressed using only these monomials, i.e.

$$\sum_{\mathbf{i}\in\mathbb{I}_{d,r}} \mu_{\mathbf{i}} x^T Q_{\mathbf{i}} x = \sum_{\mathbf{i}\in\mathbb{I}^+_{d,r}} \mu_{\mathbf{i}} x^T \widetilde{Q}_{\mathbf{i}} x < 0, \quad \forall x \neq 0 \tag{3.13}$$

where $Q_{\mathbf{i}} = Q_{i_1,i_2}$.

Therefore, the coefficients value of polynomial $\widetilde{Q}_{\mathbf{i}}$ are straightforward to get from (3.14), such as is itemized in (Sala & Ariño, 2007a).

$$\widetilde{Q}_{\mathbf{i}} = \sum_{\mathbf{j}\in\mathcal{P}(\mathbf{i})} Q_{\mathbf{j}} \quad \mathbf{i} \in \mathbb{I}^+_{d,r} \tag{3.14}$$

taking into account that $\mathcal{P}(\mathbf{i})$ denotes all the possible permutations of the vector $\mathbf{i}$.

3.3 Notation and main work

At this section, the notation is developed, which will be employed to carry out the relaxation of the inequalities by means of Polya theorem (Sala & Ariño, 2007a).

The notation (3.7) is similar to the employed one in (Tanaka & Wang, 2004), which consists of arranging the indices for rules, but, when Polya is applied (Sala & Ariño, 2007a), the computation of the polynomial coefficients of (3.7) is a complex problem. This issue makes interesting to think up a notation that collects the antecedents of each rule, to ease the coefficients collecting.

Therefore, $Q_{\bar{\mathbf{i}}}$ is defined so that this one is equivalent to $Q_{\mathbf{i},\mathbf{j}}$, presented in (3.7). Such as is written in the following example:

$$Q_{\mathbf{ij}} = Q_{(i_1,i_2,\ldots,i_p),(j_1,j_2,\ldots,j_p)} \tag{3.15}$$

$$Q_{\bar{\mathbf{i}}} = Q_{(i_1,j_1),\ldots,(i_p,j_p)} \tag{3.16}$$

In order to explore all the possible values of $Q_{\bar{\mathbf{i}}}$, the next set will be defined:

$$\mathbb{D}_{p,d} = \{\bar{\mathbf{i}} = (\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_k, \ldots, \mathbf{i}_p) | \mathbf{i}_k \in \mathbb{I}_{d,r_k} \forall k = 1, 2, \ldots, p\} \tag{3.17}$$

According to the new indices arrangement, stated in (3.15) and (3.16), as well as, set definition (3.17); it is possible to formulate the condition (3.7) as follows:

$$\Xi(t) = \sum_{\bar{\mathbf{i}} \in \mathbb{D}_{p,2}} \mu_{\bar{\mathbf{i}}} x^T Q_{\bar{\mathbf{i}}} x \tag{3.18}$$

Let membership functions be $\mu_{\bar{\mathbf{i}}} > 0$ and $\sum_{\bar{\mathbf{i}} \in \mathbb{D}_{p,1}} \mu_{\bar{\mathbf{i}}} = 1$. In the same manner as was done in (3.9), it is possible to multiply the conditions (3.18) by $(\sum_{\bar{\mathbf{j}} \in \mathbb{D}_{p,1}} \mu_{\bar{\mathbf{j}}})^{d-2} = 1$.

$$\Xi(t) = \left(\sum_{\bar{\mathbf{j}} \in \mathbb{D}_{p,1}} \mu_{\bar{\mathbf{j}}} \right)^{d-2} \sum_{\bar{\mathbf{i}} \in \mathbb{D}_{p,2}} \mu_{\bar{\mathbf{i}}} x^T Q_{\bar{\mathbf{i}}} x \tag{3.19}$$

$$\Xi(t) = \sum_{\bar{\mathbf{i}} \in \mathbb{D}_{p,d}} \mu_{\bar{\mathbf{i}}} x^T Q_{\bar{\mathbf{i}}} x \tag{3.20}$$

At this point, like in (3.13), a polynomial defined by the membership functions can be formulated as the following sum of monomials:

$$\sum_{\bar{\mathbf{i}} \in \mathbb{D}_{p,d}} \mu_{\bar{\mathbf{i}}} Q_{\bar{\mathbf{i}}} = \sum_{\bar{\mathbf{i}} \in \mathbb{D}^+_{p,d}} \mu_{\bar{\mathbf{i}}} \widetilde{Q}_{\bar{\mathbf{i}}} \tag{3.21}$$

The set $\mathbb{D}^+_{p,d}$ is defined in (3.22), so that monomials of membership functions are not repeated in the sum.

$$\mathbb{D}^+_{p,d} = \{\bar{\mathbf{i}} = (\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_k, \ldots, \mathbf{i}_p) | \mathbf{i}_k \in \mathbb{I}^+_{d,r_k} \forall k = 1, 2, \ldots, p\} \tag{3.22}$$

The coefficients $\widetilde{Q}_{\bar{\mathbf{i}}}$ are obtained by joining all the equal products of $\mu_{\bar{\mathbf{i}}}$, such as is displayed in the next expression, where $\widetilde{Q}_{\bar{\mathbf{i}}}$ is:

$$\widetilde{Q}_{\bar{\mathbf{i}}} = \sum_{\bar{\mathbf{j}} \in \overline{\mathcal{P}}(\bar{\mathbf{i}})} Q_{\bar{\mathbf{j}}} \quad \bar{\mathbf{i}} \in \mathbb{D}^+_{p,d} \tag{3.23}$$

By defining $\overline{\mathcal{P}}(\bar{\mathbf{i}})$ as all the possible permutations of each one of sub-indices $\mathbf{i}_k$ of $\bar{\mathbf{i}}$

$$\overline{\mathcal{P}}(\bar{\mathbf{i}}) = \{\bar{\mathbf{j}} \quad | \quad \mathbf{j}_k \in \mathcal{P}(\mathbf{i}_k), \quad k = 1 \ldots p\} \tag{3.24}$$

Clearly, considering that, the value of all the monomials $\mu_{\bar{\mathbf{i}}}$ is positive, the polynomial is negative if all the coefficients $\widetilde{Q}_{\mathbf{i}}$ are negative too. This condition is, indeed, sufficient but not necessary. The Polya theorem (see Theorem 2.2 on page 20) proves that, as the value of d increases, this condition is each time less conservative, and when d tends to infinity, these coefficients have to be negative.

As it is noted above, efficient computerization of coefficients in the polynomial (3.21) is not a easy problem. In order to simplify this issue, it is necessary to have in mind that $Q_{\bar{\mathbf{i}}} = Q_{(i_1(1),i_1(2),\ldots,i_1(d))\ldots(i_p(1),i_p(2),\ldots,i_p(d))}$ takes the same value for a different set of indices $\bar{\mathbf{i}}$, $\bar{\mathbf{j}}$ where $i_k(1) = j_k(1)$ and $i_k(2) = j_k(2)$, since this one is obtained from a double sum i.e. $Q_{\bar{\mathbf{i}}} = Q_{(i_1(1),i_1(2))\ldots(i_p(1),i_p(2))}$. This consideration shows that some of the polynomial coefficients are repeated, therefore, it is not necessary to calculate all the sum about $\overline{\mathcal{P}}(\bar{\mathbf{i}})$, such as, it has been defined in (3.24) to obtain $\widetilde{Q}_{\mathbf{i}}$.

In fact, it is possible to formulate $\widetilde{Q}_{\bar{\mathbf{i}}} = \sum_{\bar{\mathbf{j}}\in\mathbb{D}_{p,2}} C_{\bar{\mathbf{i}}}(\bar{\mathbf{j}})Q_{\bar{\mathbf{j}}}$, where $C_{\bar{\mathbf{i}}}$ is the number of times that $Q_{\bar{\mathbf{j}}}$ element appears in the sum. This value can be calculated as the number of permutations in the index $\bar{\mathbf{i}}$, when the $\bar{\mathbf{j}}$ values have been deleted. In the case when the $\bar{\mathbf{j}}$ values are not in $\bar{\mathbf{i}}$, the $C_{\bar{\mathbf{i}}}(\bar{\mathbf{j}})$ is 0. For instance, for the indices $\bar{\mathbf{i}} = (1,1,1,2)(1,1,2,2)$ and $\bar{\mathbf{j}} = (1,2)(1,2)$, the index formed after the values deleting will be $\bar{\mathbf{i}} - \bar{\mathbf{j}} = (1,1)(1,2)$. To calculate the number of times that this value appears in the sum, the number of possible permutations of each element is computed and multiplied. It is observed that, for $(1,1)$ there is a single permutation and for $(1,2)$ there are two possible permutations. Thus, it is clear that $C_{(1,1,1,2)(1,1,2,2)}((1,1)(1,2)) = 2$.

In order to obtain systematically the value of $C_{\bar{\mathbf{i}}}(\bar{\mathbf{j}})$, similar method is employed in (Ding, 2010), the following combinatorial expression may be used:

$$C_{\mathbf{i}_1,\ldots,\mathbf{i}_p}(\mathbf{j}_1,\ldots,\mathbf{j}_p) = \prod_{k=1}^{p} \frac{(d-2)!}{M(\mathbf{i}_k,\mathbf{j}_k)} \tag{3.25}$$

being

$$M(\mathbf{i},\mathbf{j}) = \prod_{l\in\mathbf{i}} (m(\mathbf{i},l) - m(\mathbf{j},l))! \tag{3.26}$$

Where the function $m(\mathbf{i},l)$ denotes the scalar l multiplicity in a vector $\mathbf{i}$, i.e. if $\mathbf{i} = (1,1,2,3)$ then $m(\mathbf{i},1) = 2$, $m(\mathbf{i},2) = 1$ and $m(\mathbf{i},3) = 1$.

3.4 Examples

3.4.1 Example 1

Let's set out a example which consists of a system with two membership functions and two antecedents. With current notation, the system equations are written as:

$$\dot{x} = \sum_{\bar{\mathbf{i}} \in \mathbb{D}_{p,1}} \mu_{\bar{\mathbf{i}}}(A_{\bar{\mathbf{i}}} x + B_{\bar{\mathbf{i}}} u) \tag{3.27}$$

being $A_{\bar{\mathbf{i}}} = A_{\mathbf{i}_1(1),\mathbf{i}_2(1),\ldots,\mathbf{i}_p(1)}$, $B_{\bar{\mathbf{i}}} = B_{\mathbf{i}_1(1),\mathbf{i}_2(1),\ldots,\mathbf{i}_p(1)}$ and $\mu_{\bar{\mathbf{i}}} = \mu_{\mathbf{i}_1(1)}\mu_{\mathbf{i}_2(1)}\cdots$ $\ldots\mu_{\mathbf{i}_p(1)}$. The system is controlled by a closed loop PDC: $u = \sum_{\bar{\mathbf{j}} \in \mathbb{D}_{p,1}} \mu_{\bar{\mathbf{j}}} F_{\bar{\mathbf{j}}} x$

$$\dot{x} = \sum_{\bar{\mathbf{i}} \in \mathbb{D}_{p,1}} \mu_{\bar{\mathbf{i}}}(A_{\bar{\mathbf{i}}} x + B_{\bar{\mathbf{i}}} \sum_{\bar{\mathbf{j}} \in \mathbb{D}_{p,1}} \mu_{\bar{\mathbf{j}}} F_{\bar{\mathbf{j}}} x) \tag{3.28}$$

the indices $\bar{\mathbf{i}}$ and $\bar{\mathbf{j}}$ can be grouped in a new index in the set $\mathbb{D}_{p,2}$

$$\dot{x} = \sum_{\bar{\mathbf{i}} \in \mathbb{D}_{p,2}} \mu_{\bar{\mathbf{i}}}(A_{\bar{\mathbf{i}}(1)} + B_{\bar{\mathbf{i}}(1)} F_{\bar{\mathbf{i}}(2)}) x \tag{3.29}$$

where $\bar{\mathbf{i}}(1) = (\mathbf{i}_1(1), \ldots, \mathbf{i}_p(1))$

In order to guarantee Lyapunov stability in the nonlinear system, it is necessary to check that, for a positive definite P matrix, the next condition is satisfied:

$$x^T P \dot{x} + \dot{x}^T P x < 0 \tag{3.30}$$

hence, in order to satisfy previous condition (3.30) for a fuzzy system, a sufficient condition (shape-independent) is the following set of inequalities:

$$\widetilde{Q}_{\bar{\mathbf{i}}} > 0 \quad \forall\, \bar{\mathbf{i}} \in \mathbb{D}^+_{p,d} \tag{3.31}$$

where, $\widetilde{Q}_{\bar{\mathbf{i}}}$ is formulated in (3.23) and associated matrices $Q_{\bar{\mathbf{i}}}$ are defined as:

$$Q_{\bar{\mathbf{i}}} = -X_{\bar{\mathbf{i}}(1)} A^T_{\bar{\mathbf{i}}(1)} - A_{\bar{\mathbf{i}}(1)} X + B_{\bar{\mathbf{i}}(1)} M_{\bar{\mathbf{i}}(2)} + M^T_{\bar{\mathbf{i}}(2)} B^T_{\bar{\mathbf{i}}(1)} \tag{3.32}$$

being

$$X > 0 \quad X = P^{-1} \tag{3.33}$$

$$M_{\bar{\mathbf{i}}(2)} = F_{\bar{\mathbf{i}}(2)} P \tag{3.34}$$

If sum expansion is not applied ($d = 2$), for a system with three membership functions and two antecedents for each one, it occurs that the coefficients $Q_{\bar{\mathbf{i}}}$ that form $\widetilde{Q}_{\bar{\mathbf{i}}}$ for $\bar{\mathbf{i}} = (1,2)(1,2)(1,2)$ are:

$$\begin{aligned} \widetilde{Q}_{(1,2)(1,2)(1,2)} = Q_{(1,2)(1,2)(1,2)} + Q_{(1,2)(1,2)(2,1)} \\ + Q_{(1,2)(2,1)(1,2)} + Q_{(1,2)(2,1)(2,1)} \\ + Q_{(2,1)(1,2)(1,2)} + Q_{(2,1)(1,2)(2,1)} \\ + Q_{(2,1)(2,1)(1,2)} + Q_{(2,1)(2,1)(2,1)} \end{aligned} \tag{3.35}$$

Following with the previously presented case, we can see what happens if the sum is expanded up to $d = 4$, for a system with two membership functions and two antecedents each one. The coefficients $Q_{\bar{\mathbf{i}}}$ belonging to $\widetilde{Q}_{\bar{\mathbf{i}}}$ for the case $\bar{\mathbf{i}} = (1,1,1,2)(1,1,1,2)$ are:

$$\begin{aligned} &\widetilde{Q}_{(1,1,1,2)(1,1,1,2)} = \\ &C_{(1,1,1,2)(1,1,1,2)}((1,1)(1,1))Q_{(1,1)(1,1)} \\ &+ C_{(1,1,1,2)(1,1,1,2)}((1,1)(1,2))Q_{(1,1)(1,2)} \\ &+ C_{(1,1,1,2)(1,1,1,2)}((1,1)(2,1))Q_{(1,1)(2,1)} \\ &+ C_{(1,1,1,2)(1,1,1,2)}((1,2)(1,1))Q_{(1,2)(1,1)} \\ &+ C_{(1,1,1,2)(1,1,1,2)}((1,2)(1,2))Q_{(1,2)(1,2)} \\ &+ C_{(1,1,1,2)(1,1,1,2)}((1,2)(2,1))Q_{(1,2)(2,1)} \\ &+ C_{(1,1,1,2)(1,1,1,2)}((2,1)(1,1))Q_{(2,1)(1,1)} \\ &+ C_{(1,1,1,2)(1,1,1,2)}((2,1)(1,2))Q_{(2,1)(1,2)} \\ &+ C_{(1,1,1,2)(1,1,1,2)}((2,1)(2,1))Q_{(2,1)(2,1)} \end{aligned} \tag{3.36}$$

$$C_{(1,1,1,2)(1,1,1,2)}((1,1)(1,1)) = \frac{2!}{1!1!}\frac{2!}{1!1!} = 4 \tag{3.37}$$

$$C_{(1,1,1,2)(1,1,1,2)}((1,1)(1,2)) = \frac{2!}{1!1!}\frac{2!}{2!0!} = 2 \tag{3.38}$$

$$C_{(1,1,1,2)(1,1,1,2)}((1,1)(2,1)) = \frac{2!}{1!1!}\frac{2!}{2!0!} = 2 \tag{3.39}$$

$$C_{(1,1,1,2)(1,1,1,2)}((1,2)(1,1)) = \frac{2!}{2!0!}\frac{2!}{1!1!} = 2 \tag{3.40}$$

$$C_{(1,1,1,2)(1,1,1,2)}((1,2)(1,2)) = \frac{2!}{2!0!}\frac{2!}{2!0!} = 1 \tag{3.41}$$

$$C_{(1,1,1,2)(1,1,1,2)}((1,2)(2,1)) = \frac{2!}{2!0!}\frac{2!}{2!0!} = 1 \tag{3.42}$$

$$C_{(1,1,1,2)(1,1,1,2)}((2,1)(1,1)) = \frac{2!}{2!0!}\frac{2!}{1!1!} = 2 \tag{3.43}$$

$$C_{(1,1,1,2)(1,1,1,2)}((2,1)(1,2)) = \frac{2!}{2!0!}\frac{2!}{2!0!} = 1 \tag{3.44}$$

$$C_{(1,1,1,2)(1,1,1,2)}((2,1)(2,1)) = \frac{2!}{2!0!}\frac{2!}{2!0!} = 1 \tag{3.45}$$

3.4.2 Example 2

In this section, an illustrative example is proposed, this one shows a TS model formed by 3 membership functions that can take 2 values, therefore, we have a model with 8 interpolated linear models. To assess and compare the current work contribution, two variables a y b have been employed. It allows solving the problem, knowing for which values of these parameters is possible to prove the stabilization of the system with a PDC (Tanaka & Wang, 2004), with different methodologies and sum expansions d. The tests have been carried out by means of two methodologies: (Sala & Ariño, 2007a) and the proposed in the current work.

$$\begin{aligned} A_{1,1,1} = A_1 &= \begin{pmatrix} 0.5 & -0.05 \\ 0 & -5 \end{pmatrix} \\ B_{1,1,1} = B_1 &= \begin{pmatrix} a + 0.01 \\ 0.1 \end{pmatrix} \end{aligned} \tag{3.46}$$

$$\begin{aligned} A_{1,1,2} = A_2 &= \begin{pmatrix} -10 & 0 \\ 0 & -10 \end{pmatrix} \\ B_{1,1,2} = B_2 &= \begin{pmatrix} 1 \\ 0.2 \end{pmatrix} \end{aligned} \tag{3.47}$$

$$\begin{aligned} A_{1,2,1} = A_3 &= \begin{pmatrix} -1 & 0.1 \\ 0 & -2 \end{pmatrix} \\ B_{1,2,1} = B_3 &= \begin{pmatrix} 1 \\ 0.4 \end{pmatrix} \end{aligned} \tag{3.48}$$

$$A_{1,2,2} = A_4 = \begin{pmatrix} b & -0.01 \\ 0 & -3 \end{pmatrix}$$
$$B_{1,2,2} = B_4 = \begin{pmatrix} 1 \\ 0.05 \end{pmatrix} \tag{3.49}$$

$$A_{2,1,1} = A_5 = \begin{pmatrix} -0.7 & 0.2 \\ 0 & -1 \end{pmatrix}$$
$$B_{2,1,1} = B_5 = \begin{pmatrix} 1 \\ 0.1 \end{pmatrix} \tag{3.50}$$

$$A_{2,1,2} = A_6 = \begin{pmatrix} 2 & -0.01 \\ 0 & -2 \end{pmatrix}$$
$$B_{2,1,2} = B_6 = \begin{pmatrix} 1 \\ 0.6 \end{pmatrix} \tag{3.51}$$

$$A_{2,2,1} = A_7 = \begin{pmatrix} -0.5 & 0.1 \\ 0 & -1 \end{pmatrix}$$
$$B_{2,2,1} = B_7 = \begin{pmatrix} 1 \\ 0.3 \end{pmatrix} \tag{3.52}$$

$$A_{2,2,2} = A_8 = \begin{pmatrix} b & -0.05 \\ 0 & -3 \end{pmatrix}$$
$$B_{2,2,2} = B_8 = \begin{pmatrix} 1 \\ 0.2 \end{pmatrix} \tag{3.53}$$

So, if the following conditions are satisfied, for all "flattened" vertices $A_1, \ldots, A_8$, the system becomes stable:

$$Q_{ij} = -XA_i - A_i^T X + B_i M_j + M_j^T B_i^T \tag{3.54}$$
$$X > 0 \tag{3.55}$$
$$M_j = F_j X^{-1} \tag{3.56}$$

In the figure 3.1, with the methodology developed by (Sala & Ariño, 2007a), there are displayed the parameters a and b for which is possible to achieve a PDC controller, in order that the system becomes stable, note that, in this particular case, obtained results have been the same ones, regardless of d value. On the other hand, in the figure 3.2, there are the values for the a and b parameters for which the system is stable with a specific PDC, employing the methodology developed in the current work, in section 3.3. In both cases have been expanded the membership sums degree up to values of $d = 2, 3, 4, 5, 6$, so comparing the figures the improvement is significant.

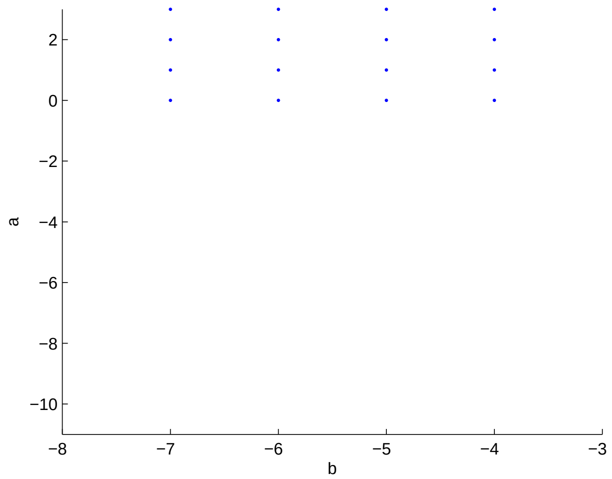

Figure 3.1: With Classic Polya (Sala & Ariño, 2007a), from $d = 2$ to $d = 6$ [no improvement occurs].

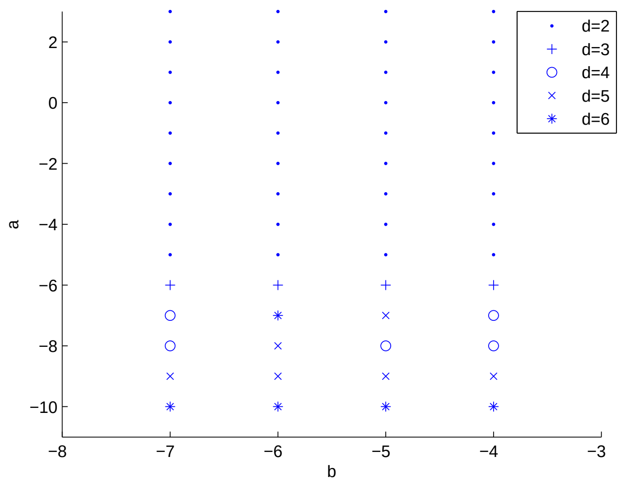

Figure 3.2: With Multi-indices Polya

3.4.3 Example 3

This example is similar to the previous one, but it will deal with a TS system with two membership functions, one has two antecedents and the other one three. The local models of the system are:

$$\begin{aligned} A_{1,1} = A_1 &= \begin{pmatrix} 0.5 & -0.05 \\ 0 & -5 \end{pmatrix} \\ B_{1,1} = B_1 &= \begin{pmatrix} a + 0.01 \\ 0.1 \end{pmatrix} \end{aligned} \tag{3.57}$$

$$\begin{aligned} A_{1,2} = A_2 &= \begin{pmatrix} -10 & 0 \\ 0 & -10 \end{pmatrix} \\ B_{1,2} = B_2 &= \begin{pmatrix} 1 \\ 0.2 \end{pmatrix} \end{aligned} \tag{3.58}$$

$$\begin{aligned} A_{1,3} = A_3 &= \begin{pmatrix} -1 & 0.1 \\ 0 & -2 \end{pmatrix} \\ B_{1,3} = B_3 &= \begin{pmatrix} 1 \\ 0.4 \end{pmatrix} \end{aligned} \tag{3.59}$$

$$\begin{aligned} A_{2,1} = A_4 &= \begin{pmatrix} b & -0.01 \\ 0 & -3 \end{pmatrix} \\ B_{2,1} = B_4 &= \begin{pmatrix} 1 \\ 0.05 \end{pmatrix} \end{aligned} \tag{3.60}$$

$$\begin{aligned} A_{2,2} = A_5 &= \begin{pmatrix} -0.7 & 0.2 \\ 0 & -1 \end{pmatrix} \\ B_{2,2} = B_5 &= \begin{pmatrix} 1 \\ 0.1 \end{pmatrix} \end{aligned} \tag{3.61}$$

$$\begin{aligned} A_{2,3} = A_6 &= \begin{pmatrix} 2 & -0.01 \\ 0 & -2 \end{pmatrix} \\ B_{2,3} = B_6 &= \begin{pmatrix} 1 \\ 0.6 \end{pmatrix} \end{aligned} \tag{3.62}$$

Note that, in the figure 3.3, obtained results have been the same ones, regardless of d value, so without our proposals no improvement appears.

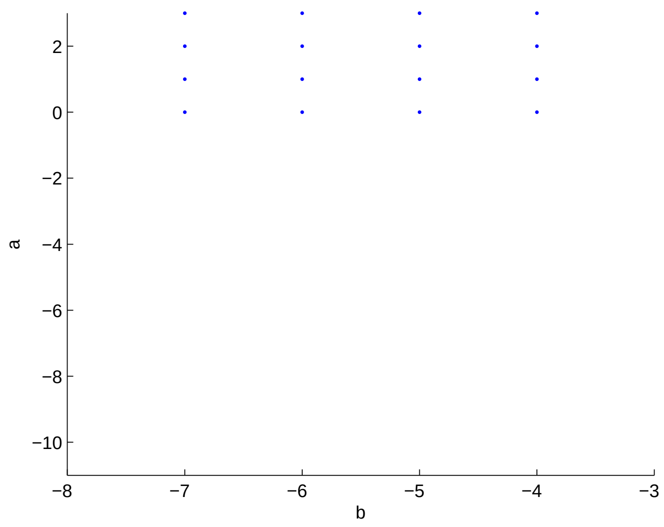

Figure 3.3: With Classic Polya (Sala & Ariño, 2007a), from $d = 2$ to $d = 6$ [no improvement appears]

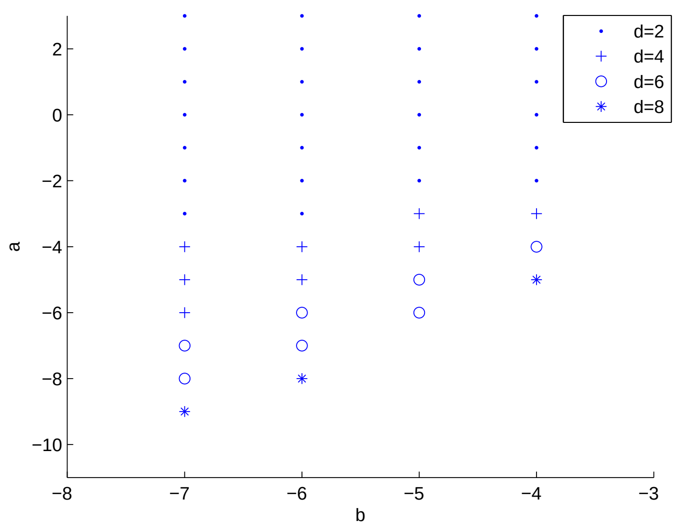

Figure 3.4: With Multi-indices Polya

Also in this example, the loss of information, caused by not using the multi-index model, produces a stability condition which is more conservative (Figure 3.3) with respect to the multi-index solution appearing on Figure 3.4.

3.5 Conclusions

In the present chapter, a methodology based on Polya theorem is developed (Sala & Ariño, 2007a). It allows expanding sums of membership functions in the TS systems, so that, the stability conditions become more relaxed, avoiding the loss of information from flattening a tensor-product expression while also introducing a convenient notation for handling such tensor-product considerations in the realm of Polya theorem.

Chapter 4

Guaranteed Cost Control for Discrete Stochastic Fuzzy Systems via LMIs

This chapter seeks to present a methodology to carry out a guaranteed cost control for a Markov-Jump Nonlinear Systems (MJNLSs), where the states and input actions can be constrained to quadratic functions. The nonlinear system will be modeled by Takagi-Sugeno methodology (TS). The PDC controller (Guerra & Vermeiren, 2004) (Tanaka & Wang, 2004) is applied in the current chapter, which allows performing an optimal or suboptimal controller with guaranteed cost, regardless membership functions, when moreover the states and the inputs can be bounded by several constraints.

4.1 Introduction

A guaranteed cost control for a Fuzzy TS system with stochastic transitions among modes is going to be dealt with in the current chapter, where the jump transitions are based on Markov matrix, which contains the transition probabilities among several TS systems. Other approaches about the stochastic control field can be found in (L. Zhang & Boukas, 2009) (Sheng, Gao, Zhang, & Chen, 2015) (Hernández-Mejías, Sala, Ariño, & Querol, 2015) (Mejías, 2016).

In order to compute a suboptimal control, since the control actions are suited for any membership function, several LMI conditions have been developed, so that the guaranteed cost can be minimized taking into account the requirements of the stochastic fuzzy system. To deal with the nonlinear conditions, methodologies like Polya's relaxation can be applied to obtain the PDC controllers and the feasible quadratic area

(Sala & Ariño, 2007a).

4.2 Preliminaries and Notation

At the current chapter, the Markov jump Takagi-Sugeno system will be formulated as:

$$x_{k+1} = \sum_{i=1}^{r_{\theta(k)}} \mu_{i,\theta(k)}(x_k) \left(A_{i,\theta(k)} x_k + B_{i,\theta(k)} u_k \right) \tag{4.1}$$

Where $\theta(k)$ is the active mode at the k step. Each one of the s stochastic modes are arranged in $\mathcal{I} = \{1, \ldots, s\}$, being $A_{i,\theta(k)} \in \mathbb{R}^n$ and $B_{i,\theta(k)} \in \mathbb{R}^{n \times m}$ the matrices of fuzzy system. Moreover, $r_{\theta(k)}$ denotes the max-number of TS models for the stochastic mode $\theta(k)$, being $\mu_{i,\theta(k)}$ each one of the membership functions, $\sum_{i=1}^{r_{\theta(k)}} \mu_{i,\theta(k)} = 1$, for stochastic mode $\theta(k) \in \mathcal{I}$.

Taking the transition probability matrix $\mathbb{P} = \pi_{ji}$ which $p(\theta(k+1) = j|\theta(k) = i) = \pi_{ji}$, and by using the Markov chain definition we can calculate each future possibility by means of:

$$p(\theta(t+k) = j|\theta(t) = i) = e_j^T \mathbb{P}^k e_i \tag{4.2}$$

being e_i a s-dimensional vector, where the ith element is equal to one, and the rest of elements are zeros.

Moreover in this chapter we assume that there exists a diagnoser such that, at time k, the mode $\theta(k)$ is known, as well as, the plant state x_k and the Markov chain probability matrix does not change over time.

The fuzzy summations for a particular mode will be denoted with:

$$\widetilde{\Xi}_{k,l} := \sum_{i=1}^{r_l} \mu_{i,l}(x_k) \Xi_{i,l} \tag{4.3}$$

Taking the previous system (4.1), PDC control action presented in (Guerra & Vermeiren, 2004) can be applied for each stochastic mode:

$$u_k = \widetilde{F}_{k,\theta(k)} \left(\widetilde{P}_{k,\theta(k)} \right)^{-1} x_k \quad \theta(k) \in \mathcal{I} \tag{4.4}$$

where $\widetilde{F}_{k,\theta(k)} = \sum_{i=1}^{r_{\theta(k)}} \mu_{i,\theta(k)}(x_k) F_{i,\theta(k)}$ and $\widetilde{P}_{k,\theta(k)} = \sum_{i=1}^{r_{\theta(k)}} \mu_{i,\theta(k)}(x_k) P_{i,\theta(k)}$, being $\widetilde{F}_{k,\theta(k)} \in \mathbb{N}^{m\times n}$ and $\widetilde{P}_{k,\theta(k)} \in \mathbb{N}^{n}$ matrices, for all $\theta(k) \in \mathcal{I}$ and $i \in \{1,\ldots,r_{\theta(k)}\}$

Thus, the equation (4.1) converts to:

$$x_{k+1} = \left(\widetilde{A}_{k,\theta(k)} + \widetilde{B}_{k,\theta(k)}\widetilde{F}_{k,\theta(k)}\left(\widetilde{P}_{k,\theta(k)}\right)^{-1}\right) x_k \tag{4.5}$$

for brevity, the active mode at k instant may be denoted as θ, so the next notation can be used further on:

$$x_{k+1} = \left(\widetilde{A}_{k,\theta} + \widetilde{B}_{k,\theta}\widetilde{F}_{k,\theta}\left(\widetilde{P}_{k,\theta}\right)^{-1}\right) x_k \tag{4.6}$$

4.3 Guaranteed cost fuzzy stochastic control

Firstly, the Lyapunov candidate equation will be presented, akin to (Guerra & Vermeiren, 2004), but having in mind that the considered system is fuzzy and stochastic, these properties have been employed to formulate the following equation:

$$V_k = x_k^T \left(\widetilde{P}_{k,\theta}\right)^{-1} x_k \tag{4.7}$$

where θ is the mode at the instant k. So that, the estimated Lyapunov equation at the instant $k+1$ may be formulated as:

$$\mathbb{E}(V_{k+1}) = \sum_{l=1}^{s} \pi_{l\theta}\left(x_{k+1}^T\left(\widetilde{P}_{k+1,l}\right)^{-1} x_{k+1}\right) \tag{4.8}$$

The objective function minimized in this chapter is a quadratic cost index, which is expressed as follows:

$$L_k = \left(x_k^T Q x_k + u_k^T R u_k\right) \tag{4.9}$$

where $Q \geq 0$ and $R \geq 0$ are the weighing matrices, with appropriate dimensions, for the states and the inputs respectively. The main idea of this chapter is to minimize the cost index. Note that, the expected infinite-time cost may be formulated as:

$$J_\infty = \mathbb{E}\left(\sum_{k=0}^{\infty}\left(x_k^T Q x_k + u_k^T R u_k\right)\right) \tag{4.10}$$

To approach this issue, note that previously, similar considerations were had in mind in (Tanaka & Wang, 2004) for continuous systems, where those were formulated in order to achieve a fuzzy guaranteed cost. So that, matrices $P_{i,\theta}$ must be able to bound the value of the infinite horizon J_∞.

This bounding may be obtained by constraining the per-stage weighting with the next condition (4.11). So according to Bellman theorem (R. Bellman, 1956), in order to optimize the control actions, the decrement in one step by the cost index can be formulated as follows:

$$\mathbb{E}\left(x_{k+1}^T\left(\widetilde{P}_{k+1,\theta_+}\right)^{-1}x_{k+1}-x_k^T\left(\widetilde{P}_{k,\theta}\right)^{-1}x_k\right)<-\mathbb{E}\left(x_k^TQx_k+u_k^TRu_k\right) \tag{4.11}$$

being θ_+ the future mode at the instant $k+1$.

Indeed, if (4.11) holds, summing from $k=0$ to $k=\infty$, the final controller will be stabilising taking $x_\infty=0$; e.i. the cost index (4.10) is bounded by $x_0^T\left(\widetilde{P}_{0,\theta(0)}\right)^{-1}x_0$, like it is expressed in the following equation:

$$\mathbb{E}\left(\sum_{k=0}^{\infty}\left(x_k^TQx_k+u_k^TRu_k\right)\right)\leq\left(x_0^T\left(\widetilde{P}_{0,\theta(0)}\right)^{-1}x_0\right) \tag{4.12}$$

Because at the k-instant, the mode is known, the previous equation (4.11) can be simplified to the following one:

$$\mathbb{E}\left(x_{k+1}^T\left(\widetilde{P}_{k+1,\theta_+)}\right)^{-1}x_{k+1}\right)-x_k^T\left(\widetilde{P}_{k,\theta}\right)^{-1}x_k<-\left(x_k^TQx_k+u_k^TRu_k\right) \tag{4.13}$$

So, the x_{k+1} is a *deterministic* value, because the x_k and the current mode $\theta(k)$ are known, $x_{k+1}=\sum_{i=1}^r\sum_{j=1}^r\mu_i\mu_j(A_{i,\theta(k)}+B_{i,\theta(k)}F_{j,\theta(k)})x_k$, thus the previous equation can be written as:

$$x_{k+1}^T\mathbb{E}\left(\left(\widetilde{P}_{k+1,\theta_+}\right)^{-1}\right)x_{k+1}-x_k^T\left(\widetilde{P}_{k,\theta}\right)^{-1}x_k<-\left(x_k^TQx_k+u_k^TRu_k\right) \tag{4.14}$$

For the estimated matrix $\mathbb{E}\left(\left(\widetilde{P}_{k+1,\theta_+}\right)\right)$, we will take $\mathbb{E}\left(\left(\widetilde{P}_{k+1,\theta_+}\right)^{-1}\right)$ $=\sum_{\theta_+\in\mathcal{I}}p(\theta_+|\theta)\left(\widetilde{P}_{k+1,\theta_+}\right)^{-1}$, so that expected matrix arises as

$\sum_{\theta_+ \in \mathcal{I}} \pi_{\theta_+\theta} \left(\widetilde{P}_{k+1,\theta_+}\right)^{-1}$, which, moreover, allows guaranteeing stochastic stability.

The equation (4.14) is based on the previous expression (4.11). Finally, this equation for a MJNLSs guaranteed cost control is formulated as:

$$x_{k+1}^T \sum_{\theta_+ \in \mathcal{I}} \pi_{\theta_+\theta} \left(\widetilde{P}_{k+1,\theta_+}\right)^{-1} x_{k+1} - x_k^T \left(\widetilde{P}_{k,\theta}\right)^{-1} x_k + x_k^T Q x_k + \\ + x_k^T \left(\widetilde{P}_{k,\theta}\right)^{-T} \left(\widetilde{F}_{k,\theta}\right)^T R \widetilde{F}_{k,\theta} \left(\widetilde{P}_{k,\theta}\right)^{-1} x_k < 0 \quad (4.15)$$

By developing the above equation (4.15), we obtain:

$$x_{k+1}^T \left(\sum_{\theta_+ = 1}^{s} \pi_{\theta_+\theta} \left(\widetilde{P}_{k+1,\theta_+}\right)^{-1} \right) x_{k+1} - x_k^T \left(\widetilde{P}_{k,\theta}\right)^{-1} x_k \\ + x_k^T Q x_k + x_k^T \left(\widetilde{P}_{k,\theta}\right)^{-T} \left(\widetilde{F}_{k,\theta}\right)^T R \widetilde{F}_{k,\theta} \left(\widetilde{P}_{k,\theta}\right)^{-1} x_k < 0 \quad (4.16)$$

Note that if the condition (4.16) holds for all k instants, the equations (4.11) and (4.12) are satisfied too. A guaranteed cost control is dealt with in theorem 4.1, where membership functions can be different among stochastic modes. So, it offers a solution to minimize estimated cost index (4.10).

Theorem 4.1 *If there exist matrices $P_{i,\theta} > 0$ and $F_{i,\theta}$ $\forall \theta \in \mathcal{I}$, $i \in \{1, \ldots, r_\theta\}$, so that the system* (4.6) *satisfies the following conditions* (4.17)*, the cost index J_∞ is bounded by V_0, defined by Lyapunov equation* (4.7) *with $k = 0$, i.e. $J_\infty < V_0$:*

$$\mathcal{R} = \sum_{i=1}^{r_\theta} \sum_{j=1}^{r_\theta} \sum_{l_1=1}^{r_1} \cdots \sum_{l_s=1}^{r_s} \mu_{i,\theta} \mu_{j,\theta} \mu_{l_1,1}^{+} \ldots \mu_{l_s,s}^{+} \mathcal{R}_{ijl_1 \ldots l_s}^{\theta} > 0 \quad \forall \theta \in \mathcal{I} \quad (4.17)$$

with

$$\mathcal{R}^{\theta}_{ijl_1\dots l_s} = \begin{pmatrix} P_{i,\theta} & (*) & \dots & (*) & (*) & (*) \\ A_{i,\theta}P_{j,\theta} + B_{i,\theta}F_{j,\theta} & \pi^{-1}_{1\theta}P_{l_1,1} & 0 & 0 & 0 & 0 \\ \vdots & 0 & \ddots & 0 & 0 & 0 \\ A_{i,\theta}P_{j,\theta} + B_{i,\theta}F_{j,\theta} & 0 & 0 & \pi^{-1}_{s\theta}P_{l_s,s} & 0 & 0 \\ P_{i,\theta} & 0 & 0 & 0 & Q^{-1} & 0 \\ F_{i,\theta} & 0 & 0 & 0 & 0 & R^{-1} \end{pmatrix} \tag{4.18}$$

where $\mu_{i,\theta} = \mu_{i,\theta}(x_k)$ *and* $\mu^{+}_{i,\theta} = \mu_{i,\theta}(x_{k+1})$. *Moreover, note that* (4.17) *polynomial may be relaxed by Polya's method (Sala & Ariño, 2007a) (Scherer & Hol, 2006) or other techniques.*

Proof:

Now, considering the Markov modes set $\mathcal{I} = \{1, \dots, s\}$; for brevity the model at k instant will be denoted as θ, and for the instant $k+1$, it will be called θ_+. So, taking the equation (4.16) for a stochastic fuzzy system:

$$x^T_{k+1}\left(\sum_{\theta_+=1}^{s}\pi_{\theta_+\theta}\left(\widetilde{P}_{k+1,\theta_+}\right)^{-1}\right)x_{k+1} - x^T_k\left(\widetilde{P}_{k,\theta}\right)^{-1}x_k + x^T_kQx_k + \\ x^T_k\left(\widetilde{P}_{k,\theta}\right)^{-T}\left(\widetilde{F}_{k,\theta}\right)^{T}R\widetilde{F}_{k,\theta}\left(\widetilde{P}_{k,\theta}\right)^{-1}x_k < 0 \tag{4.19}$$

the x_{k+1} values can be formulated in the equation (4.19), and following, in order to avoid the quadratic dependence, the Schur complement is applied:

$$\begin{pmatrix} \left(\widetilde{P}_{k,\theta}\right)^{-1} & (*) & \dots & (*) & (*) & (*) \\ \widetilde{A}_{k,\theta} + \widetilde{B}_{k,\theta}\widetilde{F}_{k,\theta}\left(\widetilde{P}_{k,\theta}\right)^{-1} & \pi^{-1}_{1\theta}\widetilde{P}_{k+1,1} & 0 & 0 & 0 & 0 \\ \vdots & 0 & \ddots & 0 & 0 & 0 \\ \widetilde{A}_{k,\theta} + \widetilde{B}_{k,\theta}\widetilde{F}_{k,\theta}\left(\widetilde{P}_{k,\theta}\right)^{-1} & 0 & 0 & \pi^{-1}_{s\theta}\widetilde{P}_{k+1,s} & 0 & 0 \\ I & 0 & 0 & 0 & Q^{-1} & 0 \\ \widetilde{F}_{k,\theta}\left(\widetilde{P}_{k,\theta}\right)^{-1} & 0 & 0 & 0 & 0 & R^{-1} \end{pmatrix} > 0 \tag{4.20}$$

Such as is done in (do Valle Costa et al., 2006), by means of χ^{θ} matrix, a congruence transformation is carried out in (4.20). So, the condition

$\mathcal{R} > 0$ arises as:

$$\chi^\theta = Diag(\widetilde{P}_{k,\theta}, I, I, \ldots, I, I) \tag{4.21}$$

$$\mathcal{R} = \begin{pmatrix} \widetilde{P}_{k,\theta} & (*) & \ldots & (*) & (*) & (*) \\ \widetilde{A}_{k,\theta}\widetilde{P}_{k,\theta} + \widetilde{B}_{\mu,\theta}\widetilde{F}_{\mu,\theta} & \pi_{1\theta}^{-1}\widetilde{P}_{k+1,1} & 0 & 0 & 0 & 0 \\ \vdots & 0 & \ddots & 0 & 0 & 0 \\ \widetilde{A}_{k,\theta}\widetilde{P}_{k,\theta} + \widetilde{B}_{k,\theta}\widetilde{F}_{k,\theta} & 0 & 0 & \pi_{s\theta}^{-1}\widetilde{P}_{k+1,s} & 0 & 0 \\ \widetilde{P}_{k,\theta} & 0 & 0 & 0 & Q^{-1} & 0 \\ \widetilde{F}_{k,\theta} & 0 & 0 & 0 & 0 & R^{-1} \end{pmatrix} \tag{4.22}$$

Finally, the previous equation can be formulated as:

$$\mathcal{R} = \sum_{i=1}^{r_\theta}\sum_{j=1}^{r_\theta}\sum_{l_1=1}^{r_1}\cdots\sum_{l_s=1}^{r_s} \mu_{i,\theta}\mu_{j,\theta}\mu_{l_1,1}^{+}\cdots\mu_{l_s,s}^{+}\mathcal{R}_{ijl_1\ldots l_s}^{\theta} > 0 \quad \forall\ \theta \in \mathcal{I} \tag{4.23}$$

$$\mathcal{R}_{ijl_1\ldots l_s}^{\theta} = \begin{pmatrix} P_{i,\theta} & (*) & \ldots & (*) & (*) & (*) \\ A_{i,\theta}P_{j,\theta} + B_{i,\theta}F_{j,\theta} & \pi_{1\theta}^{-1}P_{l_1,1} & 0 & 0 & 0 & 0 \\ \vdots & 0 & \ddots & 0 & 0 & 0 \\ A_{i,\theta}P_{j,\theta} + B_{i,\theta}F_{j,\theta} & 0 & 0 & \pi_{s\theta}^{-1}P_{l_s,s} & 0 & 0 \\ P_{i,\theta} & 0 & 0 & 0 & Q^{-1} & 0 \\ F_{i,\theta} & 0 & 0 & 0 & 0 & R^{-1} \end{pmatrix} \tag{4.24}$$

■

4.4 Constrained fuzzy stochastic control

We are going to take the following quadratic constraints for the states and inputs:

$$||x|| \le \gamma \quad ||u|| \le \phi \tag{4.25}$$

From the condition $x_0^T \left(P_\mu^{\theta_0}\right)^{-1} x_0 \le \delta^{-1}$, it is possible offer a methodology to reduce the suboptimal guaranteed cost for a certain starting point x_0. So δ^{-1} value is the max-estimated guaranteed cost of equation (4.10). At corollary 4.1, this methodology is formulated:

Corollary 4.1 *If there exist matrices $P_{i,\theta} > 0$ and $F_{i,\theta}$ $\forall\theta \in \mathcal{I}$, $i \in \{1,\dots,r_\theta\}$, so that the system* (4.6) *satisfies the following conditions* (4.26) *for a specific starting point x_0 and θ_0; the estimated cost index* (4.10) *is bounded by δ^{-1} parameter, so that $J_\infty < V_0$, and restrictions* (4.25) *are not violated:*

$$
\begin{aligned}
\max\delta \quad s.t.& \\
& P_{i,\theta_0} - \delta x_0 x_0^T \geq 0 \quad \forall\, i \\
& \gamma\delta - P_{i,\theta} \geq 0 \quad \forall\, i,\theta \in \mathcal{I} \\
& \begin{pmatrix} P_{i,\theta} & (F_{i,\theta})^T \\ F_{i,\theta} & \delta\phi \end{pmatrix} \geq 0 \quad \forall\, i,\theta \in \mathcal{I} \\
& \mathcal{R} = \sum_{i=1}^{r_\theta}\sum_{j=1}^{r_\theta}\sum_{l_1=1}^{r_1}\cdots\sum_{l_s=1}^{r_s} \mu_{i,\theta}\mu_{j,\theta}\mu^+_{l_1,1}\dots\mu^+_{l_s,s}\mathcal{R}^\theta_{ijl_1\dots l_s} > 0 \quad \forall\, \theta \in \mathcal{I}
\end{aligned}
\tag{4.26}
$$

being

$$
\mathcal{R}^\theta_{ijl_1\dots l_s} = \begin{pmatrix}
P_{i,\theta} & (*) & \dots & (*) & (*) & (*) \\
A_{i,\theta}P_{j,\theta} + B_{i,\theta}F_{j,\theta} & \pi^{-1}_{1\theta}P_{l_1,1} & 0 & 0 & 0 & 0 \\
\vdots & 0 & \ddots & 0 & 0 & 0 \\
A_{i,\theta}P_{j,\theta} + B_{i,\theta}F_{j,\theta} & 0 & 0 & \pi^{-1}_{s\theta}P_{l_s,s} & 0 & 0 \\
P_{i,\theta} & 0 & 0 & 0 & Q^{-1} & 0 \\
F_{i,\theta} & 0 & 0 & 0 & 0 & R^{-1}
\end{pmatrix}
\tag{4.27}
$$

being membership functions $\mu_{i,\theta} = \mu_{i,\theta}(x_k)$, $\mu^+_{i,\theta} = \mu_{i,\theta}(x_{k+1})$. Moreover, note that, polynomial (4.26) *may be relaxed by Polya's method (Sala & Ariño, 2007a) (Scherer & Hol, 2006) or other techniques.*

Proof: Omitted as it is a juxtaposicion of the concepts in the proof of Theorem 4.1 and the bound on the control gain (Ariño et al., 2010). ∎

4.5 Example

In this example, we discuss the TS inverted pendulum system by the approach of local approximation in fuzzy partition spaces, the TS system has the same structure that the equation (4.1):

$$x_{k+1} = \sum_{i=1}^{2} \mu_{i,\theta(k)} A_{i,\theta(k)} x_k + B_{i,\theta(k)} u_k \tag{4.28}$$

The TS local models are based on (Tanaka & Wang, 2004); and the system has three modes:
Mode 1:

$$A_{1,1} = \begin{pmatrix} 0 & 1 \\ \frac{g}{4l/3-aml} & 0 \end{pmatrix} \quad B_{1,1} = \begin{pmatrix} 0 \\ \frac{-a}{4l/3-aml} \end{pmatrix} \tag{4.29}$$

$$A_{2,1} = \begin{pmatrix} 0 & 1 \\ \frac{2g}{\pi(4l/3-aml\beta^2)} & 0 \end{pmatrix} \quad B_{2,1} = \begin{pmatrix} 0 \\ \frac{-a\beta}{4l/3-aml\beta^2} \end{pmatrix} \tag{4.30}$$

Mode 2:

$$A_{1,2} = \begin{pmatrix} 0 & 1 \\ \frac{g}{4l/3-aml} & 0 \end{pmatrix} \quad B_{1,2} = 0.5\begin{pmatrix} 0 \\ \frac{-a}{4l/3-aml} \end{pmatrix} \tag{4.31}$$

$$A_{2,2} = \begin{pmatrix} 0 & 1 \\ \frac{2g}{\pi(4l/3-aml\beta^2)} & 0 \end{pmatrix} \quad B_{2,2} = 0.5\begin{pmatrix} 0 \\ \frac{-a'\beta}{4l/3-aml\beta^2} \end{pmatrix} \tag{4.32}$$

Mode 3:

$$A_{1,2} = \begin{pmatrix} 0 & 1 \\ \frac{g}{4l/3-a'ml} & 0 \end{pmatrix} \quad B_{1,2} = 0.5\begin{pmatrix} 0 \\ \frac{-a'}{4l/3-a'ml} \end{pmatrix} \tag{4.33}$$

$$A_{2,2} = \begin{pmatrix} 0 & 1 \\ \frac{2g}{\pi(4l/3-a'ml\beta^2)} & 0 \end{pmatrix} \quad B_{2,2} = 0.5\begin{pmatrix} 0 \\ \frac{-a'\beta}{4l/3-a'ml\beta^2} \end{pmatrix} \tag{4.34}$$

being $l = 0.2$, $m = 0.1$, $M = 1$, $\beta = cos(88^o)$, $a = 1/(M+m)$ and $a' = 1/(1.25 \cdot M + m)$. With the following jump probability transition matrix $\mathbb{P}$, for Markov chain:

$$\mathbb{P} = \begin{pmatrix} 0.7 & 0.1 & 0.3 \\ 0.2 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.5 \end{pmatrix} \tag{4.35}$$

In this example, the membership functions are the same for all the modes:

$$if \quad x_1 \leq 0 \quad \mu_{1,1} = \mu_{1,2} = \frac{\pi/2 + x_1}{\pi/2} \quad \mu_{2,1} = \mu_{2,2} = 1 - \mu_{1,1} \tag{4.36}$$

$$if \quad x_1 > 0 \quad \mu_{1,1} = \mu_{1,2} = \frac{\pi/2 - x_1}{\pi/2} \quad \mu_{2,1} = \mu_{2,2} = 1 - \mu_{1,1} \tag{4.37}$$

Note that, the system is bounded, and the constraints for the control action u_k and the states x_k are:

$$-10 \leq u_k \leq 10 \tag{4.38}$$

$$\begin{pmatrix} -\pi/2 \\ -\pi/2 \end{pmatrix} \leq x_k \leq \begin{pmatrix} \pi/2 \\ \pi/2 \end{pmatrix} \tag{4.39}$$

With the next weighing matrices Q and R, for the estates and the input, respectively:

$$Q = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \quad R = 0.5 \tag{4.40}$$

Finally, PDC control actions are applied, the states trajectory and input values are displayed in figure 4.1 and 4.2[1]

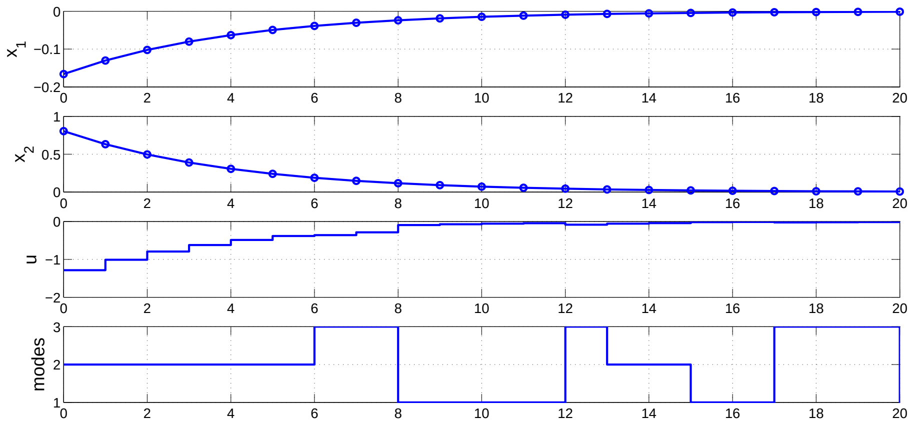

Figure 4.1: simulation 1 to 20 time instants

PDC controllers have been calculated according to the theorem 4.1 and the corollary 4.1, in order to relax the conditions, with a Polya expansion up to degree 50 of the involved fuzzy summations. The feasible

[1]During the simulation, the system has been subjected to abrupt mode changes according to the Markov matrix (4.35).

sets and the trajectory from the point $x_0 = [-0.165; 0.80]$ are displayed in the next figure:

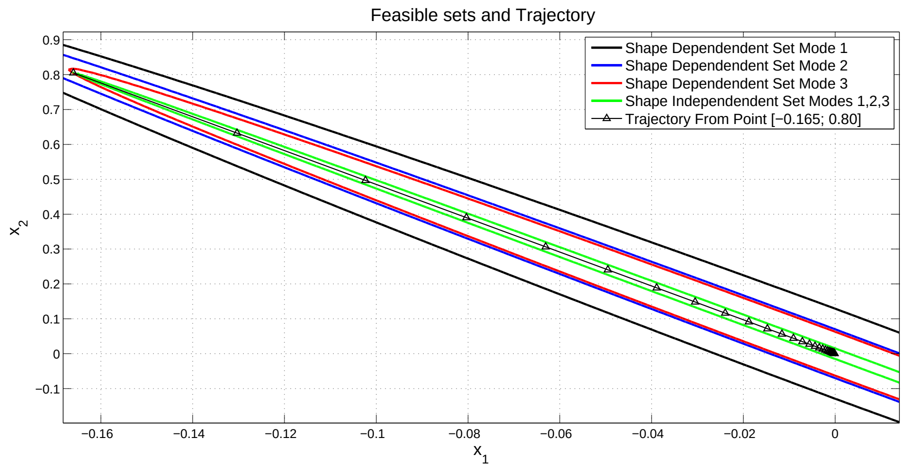

Figure 4.2: Quadratic Sets and Trajectory

The estimated guaranteed cost for the point $x_0 = [-0.165; 0.80]$ is 4.9142 and the calculated cost in this simulation has been 3.0397.

4.6 Conclusions

The present chapter presents a stochastic fuzzy guaranteed cost control, where several Markov modes and Takagi Sugeno models are combined into a single system. It attempts to show how copositive programming, Polya expansion, can be employed on several kind of systems, and not only for a TS one.

Part II

Main Contributions

Chapter 5

Asymptotically Exact Stabilization for Constrained Discrete Takagi-Sugeno Systems via Set-Invariance

Note: The contribution of this chapter is based on the following publication:

Ariño, C., Sala, A., Pérez, E., Bedate, F. and Querol, A. (2016). Asymptotically exact stabilisation for constrained discrete Takagi-Sugeno systems via set-invariance. *Fuzzy Sets and Systems*, 316, 117–138.

Given a Takagi-Sugeno system, this chapter proposes a novel methodology to obtain the state feedback controller guaranteeing the largest (membership-shape independent) possible domain-of-attraction with contraction rate performance λ, based on λ-contractive sets from polyhedral linear system literature. The resulting controller is valid for any realisation of the memberships, as usual in most TS literature. As a Polya-related complexity parameter grows, the proposal in this work obtains an asymptotically exact approximation to the largest shape-independent controllable domain of attraction. The frontier of such approximation can be understood as the level set of a polyhedral control-Lyapunov function. Convergence of a proposed iterative algorithm is asymptotically necessary and sufficient for TS system stabilisation: for a high-enough value of the complexity parameter, any conceivable shape-independent Lyapunov controller design procedure will yield a proven domain of attraction smaller or equal to the algorithm's output.

5.1 Introduction

A large class of nonlinear systems can be *exactly* expressed, locally in a *compact* region of interest (denoted as Ω in the sequel), as a fuzzy Takagi-Sugeno (TS) model, using the "sector nonlinearity" methodology (Tanaka & Wang, 2004), embedding the nonlinearity into a convex time-varying combination of "vertex" linear equations, where the convex combination's coefficients, say μ, are usually denoted as membership functions.

Once these *locally exact* fuzzy models are available, model-based stability analysis and control design for such systems can be handled via some conditions on the vertex models; conditions which involve only the vertex models and disregard the actual "shape" of the memberships are called *shape-independent* (Sala, 2009): they introduce some conservativeness, as shape-independent conditions refer to the "family" of systems sharing the same vertices, instead of the single nonlinear one which originated the TS model.

The most widespread approach to the above shape-independent stability and control design problems for TS systems are the Linear Matrix Inequality (LMI) results in literature (Tanaka, Ikeda, & Wang, 1996; Ariño & Sala, 2007; Sala et al., 2005; Seidi & Markazi, 2011; Zou & Li, 2011).

If decay-rate performance is pursued, most of the above LMI results can be understood as finding a Lyapunov function such that $V(x_{k+1}) \leq V(\lambda x_k)$, for a given value of the contraction rate λ (or optimising it via, say, bisection), see (Goh, Turan, Safonov, Papavassilopoulos, & Ly, 1994). The classes of controllers are called PDC (Tanaka & Wang, 2004) if the controller is chosen as a combination of vertex actions sharing the same membership functions as the controlled plant; or non-PDC if other functions of the memberships are used (Guerra & Vermeiren, 2004). Past and future memberships may be involved in the Lyapunov function and non-PDC controllers (Guerra, Kerkeni, Lauber, & Vermeiren, 2012; Kruszewski, Wang, & Guerra, 2008; Lendek, Guerra, & Lauber, 2015).

In most literature, once a feasible Lyapunov function is found, either quadratic $V(x) = x^T P x$ (Tanaka & Wang, 2004) or nonquadratic (Guerra & Vermeiren, 2004), the stability or control problems are considered solved, and the proven stability domain is the largest level set

$\{V(x) < V_c\}$ inside the region Ω. Actually, given $V(x)$, a slightly larger set is possible (Pitarch, Sala, Ariño, & Bedate, 2012); furthermore, the LMI solution $V(x)$ may be non-unique: so, the actual domain of attraction can be much larger than the Lyapunov level set. The developments in this chapter will be also compared to the above-cited options considering delayed/future values of membership functions in nonquadratic Lyapunov functions.

Apart from state constraints arising from the local modelling region, control action saturation is also an important issue. LMI analysis of saturated controllers needs additional restrictions forcing non-saturation on a particular level set (Tanaka & Wang, 2004) or, for instance, iterative approaches (Ariño et al., 2010), or the system states vector has to be extended in order to design an antiwindup gains (Da Silva & Tarbouriech, 2005). Determining the largest stabilisable domain of attraction in a given region Ω via LMI under constraints remains basically unsolved: there are powerful results using polynomial-fuzzy Lyapunov functions and multi-sum controllers, but changes of variable render some steps conservative in controller synthesis and, also, maximum-volume formulae do not exist for non-quadratic level sets.

In robust (polytopic) linear control, the above problem has been successfully addressed based in set-invariance ideas, originating in the 70's (Bertsekas, 1972), with later refinements (Gilbert & Tan, 1991; Kerrigan, 2000; E. Pérez, Ariño, Blasco, & Martínez, 2011; Blanchini, 1999; Kvasnica, Grieder, Baotić, & Morari, 2004). The relationship between both approaches lies in the fact that condition $V(x_{k+1}) \leq V(\lambda x_k)$ means that the level sets of the Lyapunov functions are λ-contractive, in the sense introduced in (Kerrigan, 2000).

The connection to fuzzy control systems hasn't been, however, exploited in literature to the author's knowledge. A first work in such direction appears in (Ariño, Pérez, Sala, & Bedate, 2014), and extending such results motivates the research presented in this chapter.

The goal of this chapter is studying stabilisation of *discrete-time* TS systems based on geometric set invariance considerations under affine state and control constraints, avoiding LMIs. Inspired on that idea, a prior paper (Ariño, Pérez, Sala, & Bedate, 2014) proposes using polytope-handling software to find the maximal (i.e., largest) λ-contractive set in Ω, for a given open-loop or closed-loop (being the controller fixed, *a pri-*

ori) fuzzy system, using an asymptotically exact algorithm. It is shown in such a paper that, by sheer definition, such a set will be larger than any level set obtained with a *shape-independent* Lyapunov approach. Algorithms from earlier polytopic system literature are adapted in the above-cited work to the multiple summations arising in closed-loop PDC fuzzy systems, by combining those results with the ones using Polya's theorem (Ariño & Sala, 2007). The above paper does exploit that information under state and input constraints but, however, considers only stability analysis of a *pre-existing* PDC controller.

The objective of this work is to extend the results in (Ariño, Pérez, Sala, & Bedate, 2014) to (possibly non-PDC) fuzzy controller *synthesis*, obtaining an estimate of the largest set inside a polytopic region of interest Ω in which there exists an admissible (i.e., within saturation limits) controller such that the set is made λ-contractive in closed loop. The chapter improves on current shape-independent fuzzy LMI-based literature in several key aspects:

- A Lyapunov function is not needed (although a polyhedral one is obtained as a by-product), as the argumentation is purely based on set-invariance results.
- The algorithm is asymptotically exact, so given enough computing resources it would equal or beat any shape-independent Lyapunov result, by sheer definition of the maximal λ-contractive set.
- The controller structure can also be expanded so that it may approach any continuous non-PDC controller (using polynomials in memberships, which are universal function approximators in the unit simplex (Cotter, 1989)).

Of course, it also improves over earlier robust-linear polytopic controllers using related approaches (Gilbert & Tan, 1991; Kerrigan, 2000), by the fact that the knowledge of the membership functions is actually exploited in fuzzy control systems.

There are three issues left out of the scope of this work: (a) for brevity, only a disturbance-free case is considered; extensions could be made in systems with additive disturbances adapting (Ariño et al., 2013); (b) there are other shape-*dependent* results (Sala & Ariño, 2008;

Bernal, Guerra, & Kruszewski, 2009) whose output might be less conservative than the ones in this work; (c) although the results in this work would overcome any shape-independent result with enough computational resources, there may exist LMI results which obtain acceptable controllers in practice with less computational resources than those needed to match them via the proposals in this work.

The structure of the chapter is as follows: Sections 5.2 and 5.3 state the goal of the chapter and discuss preliminary definitions and results. Section 5.4 precisely defines shape-independent sets for fuzzy control systems. Section 5.5 details an algorithm for the computation of polytopic λ-contractive sets which can be proved to *asymptotically* obtain the *maximal* shape-independent λ-contractive set. Section 5.6 presents two different procedures to compute the control action: an online optimisation and an explicit offline solution. Further discussion and comparative analysis with prior literature appears in Section 5.7. Finally, some examples appear in Section 5.8, and a conclusion section closes the chapter.

5.2 Problem statement

Consider a discrete-time nonlinear system:

$$x_{k+1} = f(x_k, u_k) \tag{5.1}$$

such that f has continuous partial derivatives, where $x_k \in \mathbb{R}^n$ represents the state vector and $u_k \in \mathbb{R}^m$ stands for the control actions at time instant k.

It is well known that such system can be equivalently expressed (*locally* in a compact region $\mathbb{X}$ of the state-space (Tanaka & Wang, 2004), denoted as modelling region), as a TS fuzzy system with r rules or local models:

$$x_{k+1} = \tilde{f}(\mu(x_k), x_k, u_k) := \sum_{i=1}^{r} \mu_i(x_k)(A_i x_k + B_i u_k) \tag{5.2}$$

where A_i, B_i are the so-called consequent model matrices and $\mu_i : \mathbb{X} \mapsto [0, 1]$ represent membership functions, grouped for convenience onto a vector of membership functions, $\mu(x) := (\mu_1(x) \ \dots \ \mu_r(x))^T$. Membership functions are defined in such a way so that, for any $x \in \mathbb{X}$, $\mu(x)$ belongs to the $(r-1)$-dimensional standard simplex $\Delta \subset \mathbb{R}^r$, defined as:

$$\Delta := \{\mu = (\mu_1, \dots, \mu_r) \in \mathbb{R}^r \mid \sum_{i=1}^{r} \mu_i = 1, \quad \mu_i \geq 0 \ \ i : 1 \dots r\} \tag{5.3}$$

Notation $\tilde{f}(\mu, x, u)$ is a shorthand for future developments; note that that $\tilde{f}$ is linear in μ and, (separately) in (x_k, u_k). Also, when memberships in several instants of time are involved, notation $h_i \in \Delta$, $h_i := \mu(x_{k+i})$ will be used.

The problem this chapter aims to solve is the determination of a fuzzy control law which stabilizes the TS system (5.2) in the largest possible subset of a polytopic region Ω, with $\Omega \subset \mathbb{X}$. Such controller design procedure must be understood as finding a set of valid initial conditions $\mathcal{C}^\lambda$ and a feedback law $u(x, \mu)$ which ensures that $x_k \in \Omega$ for all $k \geq 0$, and $\lim_{k\to\infty} x_k = 0$ if $x_0 \in \mathcal{C}^\lambda$, while fulfilling control constraints $u(x, \mu(x)) \in \mathbb{U}$ for all $x \in \Omega$, for all possible shapes of the membership $\mu(x)$ as long as $\mu(x) \in \Delta$ (shape-independent stabilisation). Coefficient λ will be related to a "contraction rate" performance measure. The formal meaning of shape-independent stabilisation with contraction rate λ will be made clear later in Section 5.4.

By assumption, modelling region $\mathbb{X}$ and input constraint set $\mathbb{U}$ will be compact, convex, polytopes, containing the origin. So, they can be defined by affine constraints, expressed as vector inequalities:

$$\mathbb{X} = \{x \in \mathbb{R}^n \mid Rx \leq l\} \tag{5.4}$$

$$\mathbb{U} = \{u \in \mathbb{R}^m \mid Su \leq s\} \tag{5.5}$$

being R, S matrices and l, s vectors with compatible dimensions, with vector inequalities to be understood as element-wise; abusing the notation, a scalar at the right-hand side of an inequality should be understood as affecting each of the rows at the left-hand side.

Actually, in most cases of practical interest Ω will be intentionally set to be equal to the modelling region $\mathbb{X}$, but the developments in this work do not necessarily require so from a theoretical point of view.

5.3 Preliminary definitions and results

Given an arbitrary set Ω, notation $\lambda\Omega$ will denote the linear scaling of the set Ω by $\lambda \geq 0$. If Ω is defined as $\Omega := \{x \in \mathbb{R}^n : M(x) \leq 0\}$, for an arbitrary vector of constraint functions $M(\cdot)$, the scaled set is $\lambda\Omega := \{x : M(\lambda^{-1}x) \leq 0\}$.

Definition 1 ((Kerrigan, 2000)) *A set $\Omega \subset \mathbb{X}$ is* control λ contractive *(given $0 \leq \lambda \leq 1$) for the system* (5.1) *if and only if, for any x in Ω there exists an admissible input such that the successor state lies in $\lambda\Omega$, i.e., if $x \in \Omega \Rightarrow \exists u \in \mathbb{U} : f(x,u) \in \lambda\Omega$.*

Obviously, u above might be non-unique, and, too, the set of feasible u depends on x, denoted as $U_\Omega(x) := \{u \in \mathbb{U} \,|\, f(x,u) \in \Omega\}$. If $\mathbb{U}$ is a polytope, and $f(x,u)$ is *affine* in control, i.e., $f(x,u) = \bar{f}(x) + \bar{g}(x)u$, then $U_\Omega(x)$ is a polytope, too. Trivially, a contracting state-feedback controller $u(x)$ can be implemented by any arbitrary selection from the set-valued map $U_\Omega(x)$; however, additional hypothesis are needed on $\mathbb{U}$, Ω and f so that there exists a *continuous* selection $u(x)$ (Michael, 1956). The scalar λ will be denoted as geometric contraction rate. Decay-rate stability requires contraction at all future time, requiring the definition of suitable Lyapunov functions:

Definition 2 *A function $V(x)$ such that $V(0) = 0$ is a (local) control Lyapunov function (CLF) ensuring geometric contraction rate λ for system* (5.1) *if there exists a set $\Omega \subset \mathbb{X}$ including the origin in which, for all $x \in \Omega \sim \{0\}$, $V(x) > 0$ and there exists $u \in \mathbb{U}$ such that $V(f(x,u)) \leq V(\lambda x)$.*

The above definition is an adaptation to the discrete-time and contraction rate setting of well-known concepts defined in, for instance, (Sontag, 1999) for continuous-time stabilization.

From the definition, it can be proved that level sets (inside Ω) of any CLF ensuring contration rate λ are control λ-contractive and, for any time $k \geq 0$, $V(x_k) \leq V(\lambda^k x_0)$. In many common cases, $V(x)$ is a homogeneous degree-q polynomial in x, then $V(\lambda x) = \lambda^q V(x)$; standard discrete decay-rate formulas $V(x_{k+1}) \leq \lambda^2 V(x_k)$ arise with, for instance, $q = 2$.

Definition 3 (Maximal control λ-contractive Set) *A set, to be denoted as $\mathcal{C}^\lambda_\infty(\Omega)$, is the* maximal *control λ-contractive set contained in a* *region Ω for the system $x_{k+1} = f(x_k, u_k)$ if and only if $\mathcal{C}_\infty^\lambda(\Omega)$ is control λ-contractive and contains all the control λ-contractive sets contained in Ω.*

Corollary 5.1 *Any level set of a local CLF in $\mathbb{X}$ ensuring contration rate λ is a subset of the maximal control λ-contractive set in $\mathbb{X}$.*

Proof: Evident, because of the above-mentioned fact that the referred level sets are control λ-contractive and all such sets are subsets of the maximal one. ∎

In the particular case of $\lambda = 1$, a control λ-contractive set is also denoted in literature (Kerrigan, 2000) as *control invariant* set, and the maximal control λ-contractive set is denoted as the *maximal control invariant set* $\mathcal{C}_\infty(\Omega)$.

Definition 4 *Given an arbitrary target set Ω, the one-step set $\mathcal{Q}(\Omega)$ is the set of states x in $\mathbb{X}$ from which the next state of system* (5.1) *can be driven to Ω with an admissible $u \in \mathbb{U}$, i.e.,*

$$\mathcal{Q}(\Omega) := \{x \in \mathbb{X} | \exists u \in \mathbb{U} : f(x,u) \in \Omega\}$$

Note that $x \in \mathcal{Q}(\Omega)$ iff $U_\Omega(x) \neq \emptyset$. Also, Definition 1 could be rewritten saying that Ω is control λ-contractive iff $\Omega \subset Q(\lambda\Omega)$.

Definition 5 ((Gilbert & Tan, 1991)) *The so-called i-step set $C_i^\lambda(\Omega)$ is recursively defined, starting with $\mathcal{C}_0^\lambda(\Omega) := \Omega$ as $\mathcal{C}_{i+1}^\lambda(\Omega) := \mathcal{Q}\left(\lambda \mathcal{C}_i^\lambda(\Omega)\right) \cap \Omega$, for $i \geq 0$.*

If there exists a finite i such that $\mathcal{C}_{i+1}^\lambda(\Omega) = \mathcal{C}_i^\lambda(\Omega)$, it can be proved (Kerrigan, 2000) that $\mathcal{C}_i^\lambda(\Omega)$ is the maximal one in Definition 3. Such set will be denoted as $\mathcal{C}_\infty^\lambda(\Omega)$. Also, in case such finite i does not exist, but there exists $\mathcal{C}_\infty^\lambda$, for any $1 \geq \lambda^* > \lambda$, there exist a finite i^* such that $\mathcal{C}_i^\lambda$ is control λ^*-contractive for all $i \geq i^*$, albeit possibly non-maximal (Blanchini, 1994, Theorem 3.2).

Efficient computational characterisation of the one-step set $\mathcal{Q}$ in Definition 4 can only be easily carried out for special cases of f; for instance, the linear case (Kerrigan, 2000). Actually, extending the idea to the TS case is the main motivation of this work.

In order to do that, we recall Polya's theorem, which is a key tool for the results presented in the Takagi-Sugeno controller synthesis in later sections.

Theorem 5.1 (Polya) *(Powers & Reznick, 2001) If a real homogeneous polynomial $F(\mu_1,\ldots,\mu_r)$ is (strictly) positive in the $(r-1)$ dimensional standard simplex Δ, then there exists a sufficiently large $d \geq 0$ such that all the coefficients of the polynomial $(\mu_1+\cdots+\mu_r)^d F(\mu_1,\ldots,\mu_r)$ are positive.*

5.4 Shape-Independent one-step and λ contractive sets for fuzzy control systems

In order to obtain the i-step sets in Definition 5, iterative computation of the one-step set in Definition 4 is needed. For a TS system, such set is:

$$\mathcal{Q}(\Omega) = \{x \in \mathbb{X} \,|\, \exists u \in \mathbb{U} : \sum_{i=1}^{r} \mu_i(x)(A_i x + B_i u) \in \Omega\} \qquad (5.6)$$

The shape of $\mathcal{Q}(\Omega)$ may be very hard to compute, due to the nonlinearities in the membership functions $\mu_i(x)$. Indeed, determining if a particular x belongs to $\mathcal{Q}(\Omega)$, for convex Ω, is computationally simple as $\tilde{f}(\mu(x), x, u)$ is, actually, an affine function[1] of u; however, the difficulty lies in determining an *explicit* expression for the boundary of $\mathcal{Q}(\Omega)$ needed for the iterations in Definition 5.

A reasonable approach, in order to deal with this drawback, is disregarding the information about the actual value of the membership functions, dealing with the Takagi-Sugeno model for *any* possible value of μ_i –assumed known to the controller, as done in most TS literature (i.e., a shape-independent analysis (Sala, 2009))–. Hence, the one-step set in Definition 4 should be *replaced* by the one below:

[1] For instance, for fixed x, $\mu(x)$, if Ω is a polytope, the problem is a linear programming feasibility one.

Definition 6 *The* shape-independent one-step set *of a TS system* (5.2) *is*

$$Q_{si}(\Omega) := \{x \in \mathbb{X} \,|\, \forall \mu \in \Delta \, \exists u \in \mathbb{U} : \sum_{i=1}^{r} \mu_i (A_i x + B_i u) \in \Omega\} \tag{5.7}$$

The definition ensures that for each $(x, \mu) \in \mathcal{Q}_{si}(\Omega) \times \Delta$ there exists a non-empty set of fuzzy (i.e., membership-dependent) control actions defined as:

$$U_\Omega(x, \mu) := \{u \in \mathbb{U} \,|\, \tilde{f}(\mu, x, u) \in \Omega\} \tag{5.8}$$

If Ω is polyhedral, the set $U_\Omega(x, \mu)$ is itself a polytope, for fixed x and μ; optimisation problems on $U_\Omega(x, \mu)$ will be discussed in Section 5.6. Unfortunately, exact computation of $\mathcal{Q}_{si}$ is still cumbersome, due to the nonlinearities involving products of μ_i with x and u.

Let us show that $\mathcal{Q}_{si}(\Omega) \subset \mathcal{Q}(\Omega)$. Indeed,

$$\mathcal{Q}(\Omega) = \{x \in \mathbb{X} \,|\, \text{for } \mu \equiv \mu(x) \, \exists u \in \mathbb{U} : \sum_{i=1}^{r} \mu_i (A_i x + B_i u) \in \Omega\} \supset \mathcal{Q}_{si}(\Omega) \tag{5.9}$$

as the conditions in the left-hand side of (5.9) involve only the single point $\mu(x)$, instead of the whole simplex in (5.7).

Any function $u(x, \mu)$, $u : \mathcal{Q}_{si}(\Omega) \times \Delta \mapsto \mathbb{U}$, so that $u(x, \mu) \in U_\Omega(x, \mu)$ would be a valid fuzzy state-feedback control law to steer any state in $\mathcal{Q}_{si}(\Omega)$ to Ω in one step applying $u(x, \mu(x))$, valid for any actual shape of $\mu(x)$. Although there might be many options, the referred controller $u(x, \mu)$ can be selected to be *continuous*, which will be important for later developments:

Lemma 5.1 *Let us assume Ω is described by $\Omega := \{x : g(x) \leq 0\}$ with g being a vector of affine functions (polytopic Ω). Then, there exists a* continuous *function $u : \mathcal{Q}_{si}(\Omega) \times \Delta \mapsto \mathbb{U}$, such that $\tilde{f}(\mu, x, u(x, \mu)) \in \Omega$.*

Proof: The proof follows an argumentation analogous to the linear case in (Artstein & Raković, 2008, Proposition 3.2). In this case, $U_\Omega(x, \mu) = \{u \in \mathbb{U} \,|\, g(\tilde{f}(\mu, x, u)) \leq 0\}$ can be understood as a *set-valued* map. Convexity of $\mathbb{U}$, plus $g \circ f$ being affine in u (for fixed μ and x), ensure $U_\Omega(x, \mu)$ is a closed convex set for all $(x, \mu) \in \mathcal{Q}_{si}(\Omega) \times \Delta$. Also, it is a set-valued map which can be proved to be continuous (because, again, $g \circ f$ is continuous). The classical Michael's *convex selection* theorem (Michael, 1956, Theorem 3.2) implies that a continuous selection $u : \mathcal{Q}_{si}(\Omega) \times \Delta \mapsto \mathbb{U}$ exists. ■

A shape-independent definition of λ-contractiveness for TS systems is now presented:

Definition 7 *Given* $0 \leq \lambda \leq 1$*, a set* $\Omega \subset \mathbb{X}$ *is* shape-independent control λ-contractive *for the system* (5.2) *if and only if, for any* (x, μ) *in* $\Omega \times \Delta$ *there exists an admissible (possible non-unique) input* $u \in \mathbb{U}$ *such that* $\tilde{f}(\mu, x, u) \in \lambda\Omega$*; equivalently, iff* $\Omega \subset \mathcal{Q}_{si}(\lambda\Omega)$*. Given a region* $\mathbb{X}$*, a shape-independent control* λ*-contractive set* Ω *is* maximal *if any other shape-independent control* λ*-contractive set in* $\mathbb{X}$ *is contained in* Ω*.*

As $Q_{si}(\lambda\Omega) \subset Q(\lambda\Omega)$, any shape-independent λ-contractive sets are also λ-contractive sets for the system (5.1) from which the TS model came from, as $\Omega \subset Q_{si}(\lambda\Omega) \subset Q(\lambda\Omega)$. So, studying shape-independent control λ-contractive sets is a way to guarantee similar contraction properties for nonlinear systems; of course this is, actually, the leitmotif of most TS fuzzy control developments.

Basically, the generic goal of shape-independent fuzzy controllers (designed with contraction objective in mind) should be approaching the above maximal shape-independent set: no algorithm can prove a larger set by definition. The results in this chapter will present a constructive procedure to approach it with increasing accuracy.

Proposition 5.1 *If* Ω *is shape-independent control* λ*-contractive for the TS system* (5.2)*, then any linear scaling* $\gamma\Omega$*, with* $0 < \gamma \leq 1$*, is shape-independent control* λ*-contractive, too.*

Proof: Considering $x \in \gamma\Omega$, with any arbitrary $\gamma \leq 1$. Then, as $\gamma\Omega \subset \Omega$, for any $(x, \mu) \in \gamma\Omega \times \Delta$ there exists u such that $\tilde{f}(\mu, \gamma^{-1}x, u) \in \lambda\Omega$, because $\gamma^{-1}x \in \Omega$. Linearity of $\tilde{f}$ in 2nd and 3rd arguments allows to state that:

$$\tilde{f}(\mu, \gamma^{-1}x, u) = \gamma^{-1}\tilde{f}(\mu, x, \gamma u) \in \lambda\Omega$$

hence, $\tilde{f}(\mu, x, \gamma u) \in \lambda(\gamma\Omega)$. So, the control $\gamma u \in \mathbb{U}$ drives $x \in \gamma\Omega$ to $\lambda(\gamma\Omega)$. ■

Hence, as shape-independent control λ-contractive sets are control λ-contractive, the following well-known result and Proposition 5.1 can be joined to induce a control Lyapunov function, if a shape-independent control λ-contractive set is found:

Proposition 5.2 ((Blanchini, 1999)) *Consider $\Omega = \{x \in \mathbb{R}^n | \max_{1\le i\le n_h} H_i x \le 1\}$. If $\gamma\Omega$ is control λ-contractive, for the TS system (5.2), for all $0 \le \gamma$ such that $\gamma\Omega \in \mathbb{X}$ then*

$$V(x) := \max_{1\le i\le n_h} (H_i x) \tag{5.10}$$

is a control Lyapunov function ensuring contraction rate λ.

The nesting of contractive sets in Proposition 5.1 allows, too, the following corollary to be stated (proof omitted for brevity):

Corollary 5.2 *A set Ω is shape-independent control λ-contractive for a TS system, if and only if, for any $x_0 \in \Omega$, for any membership sequence $(h_0, h_1, \ldots, h_{k-1}) \in \Delta^k$, there exists a control law $u(x, \mu)$, with $\mu = h_k$ at time k, such that $x_k = \tilde{f}(h_{k-1}, x_{k-1}, u(x_{k-1}, h_{k-1})) \in \lambda^k\Omega$, i.e., any initial state in it converges to the origin with a geometric contraction rate λ.*

Modifying the iterations in Definition 5, the following result can be stated:

Lemma 5.2 *For $\lambda < 1$, the maximal shape-independent control λ contractive set in a region $\Omega \subset \mathbb{X}$ would be obtained if the iteration*

$$\bar{\mathcal{C}}^{\lambda}_{i+1}(\Omega) = \mathcal{Q}_{si}\left(\lambda\bar{\mathcal{C}}^{\lambda}_{i}(\Omega)\right) \cap \Omega,$$

initialised with $\bar{\mathcal{C}}^{\lambda}_{0}(\Omega) = \Omega$, converges in a finite number of steps, i.e., $\bar{\mathcal{C}}^{\lambda}_{\infty}(\Omega) := \bar{\mathcal{C}}^{\lambda}_{i+1}(\Omega) = \bar{\mathcal{C}}^{\lambda}_{i}(\Omega)$ for some finite i. The set $\bar{\mathcal{C}}^{\lambda}_{i}(\Omega)$ will be denoted as i-step shape-independent set.

Proof: The proof comprises three steps:

1. First, the fact that $\bar{\mathcal{C}}_i^\lambda$ is shape-independent control λ-contractive if $\bar{\mathcal{C}}_i^\lambda = \bar{\mathcal{C}}_{i+1}^\lambda$. Indeed, for any i, if there exists a state $x \in \bar{\mathcal{C}}_i^\lambda$ and a membership $\mu \in \Delta$ such that $\tilde{f}(\mu, x, u)$ cannot be steered to $\lambda\bar{\mathcal{C}}_i^\lambda$ with an admissible u, then such state will *not* belong to $\bar{\mathcal{C}}_{i+1}^\lambda$. Hence, convergence will not occur until such x does not exist.

2. Second, let us prove that no point $x \in \Omega$, $x \notin \bar{\mathcal{C}}_i^\lambda$ can be steered to $\lambda\bar{\mathcal{C}}_\infty^\lambda$ for all μ: if there existed x which could be steered to $\lambda\bar{\mathcal{C}}_i^\lambda$ such point would belong to $\bar{\mathcal{C}}_{i+1}^\lambda$; again, convergence cannot happen until no such x exists.

3. Finally, as $\lambda\bar{\mathcal{C}}_\infty^\lambda$ contains the origin, and $\lambda < 1$, any stabilising trajectory should eventually enter $\lambda\bar{\mathcal{C}}_\infty^\lambda$ (Corollary 5.2). However, the above second assertion states that for states outside $\bar{\mathcal{C}}_\infty$ there exists at least one value of membership for which entering $\lambda\bar{\mathcal{C}}_\infty^\lambda$ is impossible. Hence, no larger shape-independent control λ-contractive set exists.

■

Given a nonlinear system, the set $\bar{\mathcal{C}}_\infty^\lambda$ obtained from a TS model of it is a subset of the "true" $\mathcal{C}_\infty^\lambda$ discussed in Section 5.3, due to the inherent conservatism of shape-independent TS analysis (Sala, 2009).

5.5 Inner approximation of shape-independent control λ-contractive sets for TS systems

The above shape-independent sets need choosing a particular controller parametrisation $u(x, \mu)$ in order to be computable whith available computational geometry software such as MPT (Kvasnica et al., 2004). This is the topic of this section.

The simplest approximation is choosing u not depending on memberships. Indeed, let us consider:

$$\mathcal{Q}_{si}^0(\Omega) := \{x \in \mathbb{X} \,|\, \exists u \in \mathbb{U} : A_i x + B_i u \in \Omega \quad \forall\, i = 1 \dots r\} \tag{5.11}$$

The above expression comes from plugging a membership-independent $u(x, \mu) := u(x)$ into (5.7) and considering that $\sum_{i=1}^r \mu_i(A_i x + B_i u) \in \Omega$ if and only if $A_i x + B_i u \in \Omega$ for all i. Obviously, $\mathcal{Q}_{si}^0(\Omega) \subset \mathcal{Q}_{si}(\Omega)$.

In fact, the set $\mathcal{Q}^0_{si}(\Omega)$ is the *robust* one-step set in uncertain polytopic systems literature (Pluymers, Rossiter, Suykens, & De Moor, 2005): its main drawback is its conservativeness coming from the fact that, for a given state, the control action should be the same for any value of the membership functions.

5.5.1 Fuzzy controllers (single-sum)

Given that $\mu_i(x_k)$ are actually known, a clear improvement is defining a so-called parallel distributed control parametrisation in the form:

$$u(x) = \sum_{j=1}^{r} \mu_j(x) u_j(x) \tag{5.12}$$

which defines a different “vertex controller” $u_j(x)$ for each model. This well-known formula is, of course, the key idea behind “fuzzy” controllers since the 1990s. The closed-loop system with the parametrisation (5.12) can be written as

$$x_{k+1} = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(x_k)\mu_j(x_k)(A_i x_k + B_i u_j(x_k)) \tag{5.13}$$

Let us introduce the augmented notation

$$\bar{u}(x) = \begin{pmatrix} u_1(x) \\ \vdots \\ u_r(x) \end{pmatrix}, \quad E_j = [0_{m\times m} 0_{m\times m} \dots I_{m\times m} \dots 0_{m\times m}] \tag{5.14}$$

being E_j an $m \times (mr)$ matrix with an identity matrix in the j-th block position, for $1 \le j \le r$. In this way, we have $u_j(x_k) = E_j \bar{u}(x_k)$, so the closed-loop system can be written as the augmented-input one:

$$x_{k+1} = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(x_k)\mu_j(x_k)(A_i x_k + B_i E_j \bar{u}(x_k)) \tag{5.15}$$

where the new input is a vector of length $r \times m$. In this case, disregarding again the fact that memberships depend on state, the shape-independent one-step set of system (5.15), to be denoted as $\mathcal{Q}^1_{si}(\Omega)$, is readily expressed as:

$$\mathcal{Q}^1_{si}(\Omega) := \left\{ x \in \mathbb{X} \mid \exists \bar{u} \in \mathbb{U}^r, \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j(A_i x + B_i E_j \bar{u}) \in \Omega \quad \forall \mu \in \Delta \right\} \tag{5.16}$$

where $\bar{u}$ is understood as a length-r vector whose elements belong to $\mathbb{U}$; such elements must be the same for all membership values but might be different for different states; in an analogous way to (5.8), a set $\bar{U}_\Omega(x)$ could be suitably defined, and a continuous $\bar{u}(x) : \mathcal{Q}^1_{si}(\Omega) \mapsto \mathbb{U}^r$ can be proven to exist[2], thus justifying the chosen parametrisation (5.12), obtained from $\bar{u}(x)$ by reverting back the vertical stacking to a fuzzy summation.

Now, $\mathcal{Q}^0_{si}(\Omega) \subseteq \mathcal{Q}^1_{si}(\Omega) \subset \mathcal{Q}_{si}(\Omega)$ because forcing all u_j to be equal converts (5.16) into the particular case (5.11) and, on the other hand, the parametrisation of the underlying $u(x, \mu)$ in (5.7) is generic, not restricted to being linear as (5.12) postulates. Notation $\mathcal{Q}^1_{si}$ is used to emphasise that the candidate controller is a polynomial of degree 1 in the memberships. More general controller parametrisations will be discussed in Section 5.5.3.

5.5.2 Asymptotically exact polytopic inner approximation of $\mathcal{Q}^1_{si}(\Omega)$

On the sequel, we will assume that Ω is a polytope defined as

$$\Omega := \{x \mid R_\Omega x \leq l_\Omega\} \tag{5.17}$$

for some matrices R_Ω and vector l_Ω. As $\sum \mu_i = 1$, $\mathcal{Q}^1_{si}(\Omega)$ can be expressed as

[2]The proof in this case would be analogous to Lemma 5.1, adding the fact that the infinite intersection of closed convex sets is itself also closed and convex; details omitted for brevity.

$$\mathcal{Q}^1_{si}(\Omega) = \left\{ x \in \mathbb{X} \mid \exists \bar{u} \in \mathbb{U}^r, \ \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j \left(R_\Omega (A_i x + B_i E_j \bar{u}) - l_\Omega \right) \le 0 \quad \forall \mu \in \Delta \right\} \tag{5.18}$$

The main issue regarding $\mathcal{Q}^1_{si}(\Omega)$ in (5.16) is the fact that a double-fuzzy summation (Sala & Ariño, 2007a) appears in its expressions, so necessary and sufficient conditions for computing (5.18) cannot be stated in a convex form. So, relaxations of the double sum are needed. This kind of problems have been widely studied in the field of copositive programming and LMI control for TS systems (Sala & Ariño, 2007a; Kruszewski, Sala, Guerra, & Ariño, 2009).

The goal of this section is adapting the procedures in the referred works, based on Polya's theorem (here recalled as Theorem 5.1), to the problem of computing approximations to $\mathcal{Q}^1_{si}(\Omega)$. In order to do that, the notation for d-dimensional indices in (Ariño & Sala, 2007; Sala & Ariño, 2007a; Ariño, Pérez, Sala, & Bedate, 2014) will be used:

$$\mathbf{i} = (i_1, i_2, \dots i_d), \quad \mathbb{I}_d = \{1, \dots, r\}^d$$

$$\mathbb{I}^+_d = \{\mathbf{i} \in \mathbb{I}_d | i_s \le i_{s+1}, \ s = 1, \dots, d-1\}$$

so $\mathbb{I}^+_d$ indexes all the different monomials $\mu_{\mathbf{i}}$ of an homogeneous degree-d polynomial (taking into account commutativity). For instance, for $d = 3$ and $r = 2$ we can define $\mathbb{I}^+_3 = \{111, 112, 122, 222\}$ – with some abuse of notation shorthanding $(1, 1, 1)$ as 111, etc.

Notation $n_{\mathbf{i}}$ will denote the number of elements of $perm(\mathbf{i})$, being $perm(\mathbf{i})$ the set of permutations of an element of $\mathbb{I}^+_d$ in $\mathbb{I}_d$. In the above case $n_{111} = n_{222} = 1$, $n_{112} = n_{221} = 3$ (because $perm(111) = \{111\}$, $perm(112) = \{112, 121, 211\}$, $\dots$).

Denoting as $\mu_{\mathbf{i}} := \mu_{i_1}\mu_{i_2}\dots\mu_{i_d}$, the following identities are straightforward:

$$\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\dots\sum_{i_d=1}^{r} \mu_{i_1}\mu_{i_2}\dots\mu_{i_d} = \left(\sum_{i=1}^{r}\mu_i\right)^d = \sum_{\mathbf{i}\in\mathbb{I}_d}\mu_{\mathbf{i}} = \sum_{\mathbf{i}\in\mathbb{I}^+_d} n_{\mathbf{i}}\mu_{\mathbf{i}} = 1 \tag{5.19}$$

because $\mu_{\mathbf{i}} = \mu_{\mathbf{j}}$ if $\mathbf{j} \in perm(\mathbf{i})$. The reader is referred to (Ariño & Sala, 2007; Sala & Ariño, 2007a; Ariño, Pérez, Sala, & Bedate, 2014) for further details on the multiindex notation and relevant properties.

Continuing with the example with $d = 3$ and $r = 2$, we have

$$1 = \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2} \mu_{i_1}\mu_{i_2}\mu_{i_3} =$$

$$\underbrace{\mu_1^3}_{\mu_{111}} + \underbrace{3}_{n_{112}}\underbrace{\mu_1^2\mu_2}_{\mu_{112}} + \underbrace{3}_{n_{122}}\underbrace{\mu_1\mu_2^2}_{\mu_{122}} + \underbrace{\mu_2^3}_{\mu_{222}} = \sum_{\mathbf{i}\in\mathbb{I}_3^+} n_{\mathbf{i}}\mu_{\mathbf{i}}$$

.

In order to apply Theorem 5.1, as $\left(\sum_{i=1}^{r}\mu_i\right)^{d-2} = 1$ for any d, we rewrite equation (5.15), denoting $\bar{u}(x_k)$ with shorthand $\bar{u}_k$, as

$$x_{k+1} = \left(\sum_{i=1}^{r}\mu_i\right)^{d-2}\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j(A_i x_k + B_i E_j \bar{u}_k) =$$

$$= \sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\cdots\sum_{i_d=1}^{r}\mu_{i_1}\mu_{i_2}\ldots\mu_{i_d}(A_{i_1}x_k + B_{i_1}E_{i_2}\bar{u}_k) \quad (5.20)$$

Denoting $G_{i_1 i_2} = [A_{i_1} \quad B_{i_1}E_{i_2}]$, and reordering the terms of the summation (5.20) we get

$$x_{k+1} = \sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\cdots\sum_{i_d=1}^{r}\mu_{i_1}\mu_{i_2}\ldots\mu_{i_d}G_{i_1 i_2}\begin{bmatrix} x_k \\ \bar{u}_k \end{bmatrix} = \sum_{\mathbf{i}\in\mathbb{I}_d}\mu_{\mathbf{i}}G_{i_1 i_2}\begin{bmatrix} x_k \\ \bar{u}_k \end{bmatrix} =$$

$$= \sum_{\mathbf{i}\in\mathbb{I}_d^+}\mu_{\mathbf{i}}\left(\sum_{\mathbf{j}\in perm(\mathbf{i})}G_{j_1 j_2}\right)\begin{bmatrix} x_k \\ \bar{u}_k \end{bmatrix} = \sum_{\mathbf{i}\in\mathbb{I}_d^+}\mu_{\mathbf{i}}n_{\mathbf{i}}\widetilde{G}_{\mathbf{i}}\begin{bmatrix} x_k \\ \bar{u}_k \end{bmatrix} \quad (5.21)$$

being $\widetilde{G}_{\mathbf{i}}$ the average values of $G_{i_1 i_2}$ over all permutations of a particular ordered multidimensional index, i.e.;

$$\widetilde{G}_{\mathbf{i}} := \frac{1}{n_{\mathbf{i}}}\cdot\sum_{\mathbf{j}\in perm(\mathbf{i})}G_{j_1 j_2} \quad (5.22)$$

For instance, in the above case $d = 3$, $r = 2$, we would get:

$$\widetilde{G}_{111} = G_{11}, \quad \widetilde{G}_{112} = \frac{1}{3}(G_{11} + G_{12} + G_{21}),$$

$$\widetilde{G}_{122} = \frac{1}{3}(G_{12} + G_{21} + G_{22}), \quad \widetilde{G}_{222} = G_{22} \quad (5.23)$$

With the new equivalent expression (5.21) of the system dynamics, the one step set (of course, identical to that in (5.16), as (5.21) is a mere rewriting of (5.15) in the new notation for any $d \geq 2$), can be written as

$$\mathcal{Q}^1_{si}(\Omega) = \left\{ x \in \mathbb{X} \mid \exists \bar{u} \in \mathbb{U}^r : \sum_{\mathbf{i} \in \mathbb{I}^+_d} n_{\mathbf{i}} \mu_{\mathbf{i}} \widetilde{G}_{\mathbf{i}} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} \in \Omega \ \forall \mu \in \Delta \right\} \tag{5.24}$$

Now, if Ω were a polytope (5.17), from (5.19) we can cast the evident equivalence:

$$R_\Omega \left(\sum_{\mathbf{i} \in \mathbb{I}^+_d} n_{\mathbf{i}} \mu_{\mathbf{i}} \widetilde{G}_{\mathbf{i}} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} \right) \leq l_\Omega \ \Leftrightarrow \ R_\Omega \left(\sum_{\mathbf{i} \in \mathbb{I}^+_d} n_{\mathbf{i}} \mu_{\mathbf{i}} \widetilde{G}_{\mathbf{i}} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} \right) \leq \left(\sum_{\mathbf{i} \in \mathbb{I}^+_d} n_{\mathbf{i}} \mu_{\mathbf{i}} \right) l_\Omega \tag{5.25}$$

Now, the right-hand side inequality is actually equivalent to:

$$\sum_{\mathbf{i} \in \mathbb{I}^+_d} n_{\mathbf{i}} \mu_{\mathbf{i}} \left(R_\Omega \widetilde{G}_{\mathbf{i}} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} - l_\Omega \right) \leq 0 \tag{5.26}$$

so, as $n_{\mathbf{i}}\mu_{\mathbf{i}}$ are all non-negative, we can assert that existence of $\bar{u}$ such that

$$R_\Omega \widetilde{G}_{\mathbf{i}} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} \leq l_\Omega \tag{5.27}$$

is a sufficient condition for $x \in \mathcal{Q}^1_{si}(\Omega)$. So, it becomes clear that a sufficient condition for a given point to belong to $\mathcal{Q}^1_{si}(\Omega)$ is that it belongs to the polytopic complexity-d subset arising from the inequality in (5.27), denoted as:

$$\widetilde{\mathcal{Q}}^1_d(\Omega) := \left\{ x \in \mathbb{X} \mid \exists \bar{u} \in \mathbb{U}^r : \widetilde{G}_{\mathbf{i}} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} \in \Omega \quad \forall \mathbf{i} \in \mathbb{I}^+_d \right\} \tag{5.28}$$

because, indeed, the above argumentation ensures $\widetilde{\mathcal{Q}}^1_d(\Omega) \subset \mathcal{Q}^1_{si}(\Omega)$.

It can be proved, following the asymptotic exactness results derived from Polya argumentations (Powers & Reznick, 2001), that the polytopic set $\widetilde{\mathcal{Q}}^1_d(\Omega)$ will tend to $\mathcal{Q}^1_{si}(\Omega)$ as the complexity parameter d tends to infinity. This is done in the lemma below:

Lemma 5.3 *If x belongs to the interior of $\mathcal{Q}^1_{si}(\Omega)$, for a polytopic Ω expressed as* (5.17)*, there exists a finite d such that $x \in \widetilde{\mathcal{Q}}^1_d(\Omega)$.*

Proof: Indeed, for x in the interior of $\mathcal{Q}^1_{si}(\Omega)$, using the original definition (5.15), i.e., $d = 2$, there exists $\gamma < 0$ such that

$$\sum_{\mathbf{i}\in\mathbb{I}_2^+} n_{\mathbf{i}}\mu_{\mathbf{i}} \left(R_\Omega \widetilde{G}_{i_1 i_2} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} - l_\Omega \right) \le \gamma < 0 \tag{5.29}$$

So standard Polya-argumentations (Powers & Reznick, 2001; Sala & Ariño, 2007a) show that there is a finite d such that, expanding (5.29) in the same way as done in (5.21), i.e.,

$$\sum_{\mathbf{i}\in\mathbb{I}_d^+} n_{\mathbf{i}}\mu_{\mathbf{i}} \left(R_\Omega \widetilde{G}_{\mathbf{i}} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} - l_\Omega \right) = \sum_{\mathbf{i}\in\mathbb{I}_2^+} n_{\mathbf{i}}\mu_{\mathbf{i}} \left(R_\Omega \widetilde{G}_{i_1 i_2} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} - l_\Omega \right) \tag{5.30}$$

results in a d-th degree homogeneous polynomial in $\mu_{\mathbf{i}}$ at the left-hand side of the equation such that the polynomial coefficients will be all non-positive. Requiring non-positiveness of all such coefficients is, actually, what (5.28) states once matrices R_Ω, l_Ω defining the shape of Ω are plugged in. ∎

5.5.3 Extension to multiple-parametrization controllers.

A more flexible controller parametrisation $u_k(\mu)$ can be set up as a c-dimensional fuzzy summation, $c > 1$, as suggested in, for instance, (Ariño & Sala, 2007):

$$u(x) = u^c(x, \mu) := \sum_{\mathbf{i}\in\mathbb{I}_c^+} n_{\mathbf{i}}\mu_{\mathbf{i}}(x) u_{i_1 i_2 \ldots i_c}(x) \tag{5.31}$$

For each state, this fuzzy control parametrisation is a degree-c homogeneous polynomial in the memberships.

Lemma 5.1 ensures that, for a fixed x, there exists a control function $u(x, \mu)$ which is continuous in the memberships (in fact, so it will be in the state, too, but this will not be needed for the moment) fulfilling the required constraints on successor states. Any arbitrary continuous controller parametrization $u(x, \mu)$, in the compact region Δ can be approximated to any desired accuracy by a polynomial in μ, as (5.31) proposes (polynomials are universal function approximators (Cotter, 1989)); the idea will be later used to prove asymptotic exactness of some algorithms, via increasing the degree c.

With the above general parametrization (5.31), consider conforming $\bar{u}$ vertically stacking all $u_{i_1 i_2 \ldots i_c}(x)$ and suitably defining matrices $E_{i_1 i_2 \ldots i_c}$ so that $u_{i_1 i_2 \ldots i_c}(x) = E_{i_1 i_2 \ldots i_c}\bar{u}(x)$ in the same way as it was done in (5.14). For instance, for $c = 2$ in a system with 2 rules, $\bar{u}$ would be defined as $\bar{u} = (u_{11}^T \; u_{12}^T \; u_{22}^T)^T$, as well as suitable $E_{11} = (I \; 0 \; 0)$, $E_{12} = E_{21} = (0 \; I \; 0)$ and $E_{22} = (0 \; 0 \; I)$. Then, for $d = 3$, the closed loop would be

$$x_{k+1} = \sum_{\mathbf{i} \in \mathbb{I}_3} \mu_{\mathbf{i}}(x_k) \left(A_{i_1} x_k + B_{i_1} E_{i_2 i_3} \bar{u}(x_k) \right) \tag{5.32}$$

With the extra decision variables in $\bar{u}$, a larger polytopic approximation $\widetilde{\mathcal{Q}}_d^c$, $d > c$ of the "ideal" shape-independent one-step set $\mathcal{Q}_{si}$ will be defined below, where superscript c denotes the degree of the controller parametrisation, and subscript d denotes the total Polya complexity parameter. The definition of such $\widetilde{\mathcal{Q}}_d^c$ will be analogous to (5.28) but with different sizes of $\bar{u}$ and $\mathbf{i}$:

$$\widetilde{\mathcal{Q}}_d^c(\Omega) := \left\{ x \in \mathbb{X} \mid \exists \bar{u} \in \mathbb{U}^\rho, \rho = card(\mathbb{I}_c^+) : \widetilde{G}_{\mathbf{i}} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} \in \Omega \quad \forall \mathbf{i} \in \mathbb{I}_d^+ \right\} \tag{5.33}$$

where, actually, the expression of $\widetilde{G}_{\mathbf{i}}$ in (5.21) should be reworked in order to fit the higher dimensionality. For illustration, in the above example (5.32), in order to define $\widetilde{\mathcal{Q}}_4^2(\Omega)$ we would need $G_{ijk} = (A_i \;\; B_i E_{jk})$ and, subsequently:

$$\widetilde{G}_{1111} = G_{111}, \quad \widetilde{G}_{1112} = \frac{1}{4}(G_{111} + G_{112} + G_{121} + G_{211}),$$
$$\widetilde{G}_{1222} = \frac{1}{4}(G_{122} + G_{212} + G_{221} + G_{222}),$$
$$\widetilde{G}_{1122} = \frac{1}{6}(G_{112} + G_{121} + G_{211} + G_{221} + G_{212} + G_{122}), \quad \widetilde{G}_{2222} = G_{222}$$

For brevity, details on the construction of $\bar{u}$ and $\tilde{G}_{\mathbf{i}}$ in other cases are left to the reader. The above definition (5.33) generalises the cases of controller degrees $c = 0$, implicitly assumed in (5.11), and $c = 1$, explicitly defined in (5.12) and used in (5.28). It can be proved that $\widetilde{\mathcal{Q}}_d^c \subset \widetilde{\mathcal{Q}}_{d'}^{c'}$ when $c' \geq c$ and $d' \geq d$ (details omitted for brevity).

Algorithm 2 Computation of the λ-contractive set $\widehat{\mathcal{C}}_\infty^\lambda$

Inputs: c, d, Ω, λ.

1. Make $i = 0$, $\widehat{\mathcal{C}}_0^\lambda = \Omega$
2. Repeat :
   (a) i=i+1
   (b) $\widehat{\mathcal{C}}_i^\lambda = \widetilde{\mathcal{Q}}_d^c \left(\lambda \widehat{\mathcal{C}}_{i-1}^\lambda\right) \cap \Omega$

   Until $\widehat{\mathcal{C}}_i^\lambda = \widehat{\mathcal{C}}_{i-1}^\lambda$;
3. Set $\widehat{\mathcal{C}}_\infty^\lambda = \widehat{\mathcal{C}}_i^\lambda$; END.

5.5.4 Polytopic inner approximation of the maximal shape-independent control λ-contractive set

As the actual $\mathcal{Q}_{si}$ used in Lemma 5.2 is out of reach with finite computational resources, we will modify it by substituting $\mathcal{Q}_{si}$ by the polytopic shape-independent *approximation* $\widetilde{\mathcal{Q}}_d^c$. The result is Algorithm 2. Once restricted to polytopic sets, the computational geometry tools in the MPT toolbox (Kvasnica et al., 2004) allow implementing the above algorithm to find $\hat{\mathcal{C}}_i^\lambda$ in a few lines of MATLAB® code.

Proposition 5.3 *For any positive c, d, i, we have: $\widehat{\mathcal{C}}_i^\lambda \subset \bar{\mathcal{C}}_i^\lambda(\Omega)$; hence, if the corresponding iterations converge $\widehat{\mathcal{C}}_\infty^\lambda \subset \bar{\mathcal{C}}_\infty^\lambda(\Omega)$ (inner approximation). Also, the converged $\widehat{\mathcal{C}}_\infty^\lambda$ is shape-independent λ-contractive for the TS model* (5.2).

Proof: The first statement arises from the fact that $\widetilde{\mathcal{Q}}_d^c \subset \mathcal{Q}_{si}$ so each iteration yields a progressively smaller set. The second statement is proved from the fact that $\widehat{\mathcal{C}}_\infty^\lambda = (\widetilde{\mathcal{Q}}_d^c(\lambda\widehat{\mathcal{C}}_\infty^\lambda) \cap \Omega) \subset \mathcal{Q}_{si}(\lambda\widehat{\mathcal{C}}_\infty^\lambda)$. ■

The above proposition states that the algorithm may have obtained a *non*-maximal λ-contractive set. However, the asymptotic exactness of the Polya result allows to state the following result extending Lemma 5.3, using $int(S)$ to denote the interior of a set S:

Theorem 5.2 *Given any integer $i > 0$, for every $x \in int(\bar{\mathcal{C}}_i^\lambda(\Omega))$, there exists a pair of finite c, d such that, when Algorithm 2 is run with such complexity parameters, then $x \in int(\widehat{\mathcal{C}}_i^\lambda)$.*

Proof: Considering any arbitrary i, let us assume the polytopic S is expressed as $S = \{R_i x \leq l_i\}$ for some R_i, l_i. Then, if x belongs to the interior of $\mathcal{Q}_{si}(S)$ there exists $\gamma > 0$ and there exists a continuous $u(x, \mu)$ such that $R_i \tilde{f}(\mu, x, u(x, \mu)) - l_i \leq \gamma < 0$ for all $\mu \in \Delta$, by Lemma 5.1 and the fact that being x an interior point, inequalities defining the set must be strictly fulfilled.

Now, universal approximation of polynomials enables us tu ensure that there exists a degree-c polynomial in μ in the form (5.31), say $u^c(x, \mu)$ which, for fixed x, approximates the continuous function $u(x, \mu)$ in the compact set Δ up to a precision $\|u(x, \mu) - u^c(x, \mu)\| \leq \varepsilon$ with ε as small as needed so that $R_i(\tilde{f}(\mu, x, u(x, \mu)) - \tilde{f}(\mu, x, u^c(x, \mu))) \leq \gamma/2$. This allows us to assert that there exists a finite c such that:

$$R_i \tilde{f}(\mu, x, u^c(x, \mu)) - l_i \leq \gamma/2 < 0$$

Now, the left-hand side of the above expression can be trivially converted to an homogeneous polynomial of degree $c + 1$ on the simplex Δ. Hence, asymptotic exactness of Polya theorem (Theorem 5.1) ensures that there exists a finite complexity parameter d such that all coefficients of the degree d expansion of $R_i \tilde{f} - l_i$ are strictly negative. Hence, $x \in int(\widetilde{\mathcal{Q}}_d^c(\lambda S))$, by definition of $\widetilde{\mathcal{Q}}_d^c$.

Now, an induction argumentation is needed. Starting from $\bar{\mathcal{C}}_0^\lambda = \widehat{\mathcal{C}}_0^\lambda = \Omega$, if $x_1 \in int(\bar{\mathcal{C}}_1^\lambda)$ then there exist c_1,d_1 such that $x_1 \in int(\widetilde{\mathcal{Q}}_{d_1}^{c_1}(\Omega) \cap \Omega)$. if $x_2 \in int(\bar{\mathcal{C}}_2^\lambda)$, then there exists u, depending on x_2 and μ, in $\mathbb{U}$ such that $x_1 := \tilde{f}(\mu, x_2, u) \in int(\bar{\mathcal{C}}_1^\lambda)$ so the above c_1, d_1 ensure $\tilde{f}(\mu, x_2, u) \in int(\widetilde{\mathcal{Q}}_{d_1}^{c_1}(\Omega) \cap \Omega)$: hence, $x_2 \in \mathcal{Q}_{si}(\widetilde{Q}_{d_1}^{c_1}(\Omega) \cap \Omega)$. Now, letting $S = \widetilde{\mathcal{Q}}_{d_1}^{c_1}(\Omega) \cap \Omega$ we can assert that there exist c_2, d_2 such that $x \in int(\widehat{\mathcal{Q}}_{d_2}^{c_2}(\widetilde{\mathcal{Q}}_{d_1}^{c_1}(\Omega) \cap \Omega) \cap \Omega)$. The argumentation can follow on for any i: if $x \in int(\bar{\mathcal{C}}_i^\lambda)$ there exist a sequence $d_1, \ldots, d_i$, $c_1, \ldots, c_i$ such that x belongs to $\widehat{\mathcal{C}}_i^\lambda$. The required c and d in the theorem statement will be the maximum c and d of the respective sequences. ∎

5.6 Controller computation from λ-contractive sets

The obtained polytopic λ-contractive sets, after Algorithm 2 convergence in a finite number of iterations, induce a control Lyapunov function and associated controllers, to be discussed in this section.

Let us assume that the converged set $\widehat{\mathcal{C}}_\infty^\lambda$ is defined as a polytope $\widehat{\mathcal{C}}_\infty^\lambda = \{x \mid \max_{1\leq i\leq n_h} H_i x \leq 1\}$ for some row vectors H_i. Then, Proposition 5.2 immediately allows defining a control Lyapunov function (5.10). However, the only problem addressed up to this point is the *existence* of a continuous control law (Lemma 5.1), but not any *constructive* procedure to find it; notwithstanding, it is well known that, once a *control* Lyapunov function is available, computation of a controller is possible (Sontag, 1989).

As the set of valid control actions $U_{\lambda\widehat{\mathcal{C}}_\infty^\lambda}(x,\mu)$ defined in (5.8), is polyhedral for known x and μ (actually, μ would be the measured $\mu(x)$), optimisation of a convex cost index over $U_{\lambda\widehat{\mathcal{C}}_\infty^\lambda}(x,\mu(x))$ can be efficiently solved via convex programming. Such optimisation is a widely used choice to constructively compute the above-referred control action in the polyhedral-robust control literature referred to in the introduction; details and adaptation to the fuzzy case will be presented next. Let us discuss two possible options: *on-line* and *off-line* optimisation.

5.6.1 On-line optimisation

In on-line operation, state and membership values are known at the time of computing the control action, so the model $x_{k+1} = A(\mu(x_k))x_k + B(\mu(x_k))u_k$, affine in the control action u_k, renders:

$$x_{k+1} = M_k + N_k u_k, \qquad M_k := A(\mu(x_k))x_k, \quad N_k := B(\mu(x_k))$$

and M_k and N_k are matrices known at time k once x_k has been measured. A reasonable course of action would be proposing a cost index depending only on the current control action u_k, choosing a suitable one in the convex set $U_{\lambda\widehat{\mathcal{C}}_\infty^\lambda}(x_k,\mu(x_k)) = \{u \in \mathbb{U} \mid \max_i H_i(M_k + N_k u) \leq \lambda\}$. In this way, there would be no need to actually build up a "fuzzy" controller (5.12), as u_k can be directly optimised.

Of course, decrescence of the piecewise-linear Lyapunov function (5.10) and admissibility of u_k (i.e, $u_k \in \mathbb{U}$) need to be introduced as optimisation constraints, irrespective of the chosen cost index.

Several optimisation criteria may be chosen, for instance:

1. Achieving the fastest decay, by minimising the predicted next value of the polyhedral Lyapunov function (5.10), i.e., given x_k, selecting u_k equal to the optimal solution below

$$u_k := \arg\min_{u\in\mathbb{U}} V(x_{k+1}) = \arg\min_{u\in\mathbb{U}} \max_{i\in\{1,\ldots,n_h\}} H_i(M_k + N_k u) \quad (5.34)$$

   which is a standard linear minimax problem, which can be cast as a linear programming (LP) one,

$$u_k = \arg\min_u \delta$$
$$\text{subject to:} \quad u \in \mathbb{U},\ H_i(M_k + N_k u) \le \delta \quad \forall i \qquad (5.35)$$

   Note that, for $x_k \in \widehat{\mathcal{C}}^{\lambda}_{\infty}(\Omega)$, the minimal δ will be lower than λ because, by construction, $\widehat{\mathcal{C}}^{\lambda}_{\infty}(\Omega)$ is a set in which constraints (5.35) are feasible for $\delta = \lambda$.

2. Minimising the "control effort" subject to the contraction condition $V(x_{k+1}) = \max_i H_i(M_k + N_k u_k) \le V(\lambda x_k)$ which forces the Lyapunov function to be decreasing. If the control effort is measured in 1-norm (sum of absolute value of elements) or ∞-norm (elementwise maximum) then the problem is also an LP one; it it is measured in 2-norm, then it is a QP one. Again, feasibility is guaranteed for $x_k \in \widehat{\mathcal{C}}^{\lambda}_{\infty}(\Omega)$.

Note that, even if the controller to be found on-line does not appear to be a "fuzzy" controller, it does indeed depend on the membership values, as $M_k = A(\mu(x_k))$ and $N_k = B(\mu(x_k))$. Note, too, that the above on-line LP/QP problems may be feasible even outside the guaranteed (but conservative, shape-independent) set $\widehat{\mathcal{C}}^{\lambda}_{\infty}$ computed by Algorithm 2; however, such feasibility cannot be guaranteed by the shape-independent analysis in earlier sections.

5.6.2 Off-line optimisation

Although the above on-line optimisation solution is, actually a one-step optimisation (hence with low computational complexity as the number of decision variables is the number of inputs), an off-line computation of a controller solution can be obtained if so wished.

Indeed, analogously to (Bemporad, Morari, Dua, & Pistikopoulos, 2002), an explicit piecewise fuzzy controller can be designed under the setup in this work, as an off-line version of (5.35). Indeed, consider the augmented input $\bar{u}(x)$ defined in (5.14), either from (5.12) or, with higher-dimensional controllers (5.31), composed of vertically stacking $u_{j_1 \ldots j_c}(x)$, $\mathbf{j} \in \mathbb{I}_c^+$ in a vector of length $\rho = card(\mathbb{I}_c^+)$. Replacing in (5.35) the closed-loop fuzzy model (5.21) –or the higher-complexity versions implicitly considered in (5.33)–, we have an optimisation problem:

$$\bar{u}^*(x) := \arg\min_{\bar{u}} \delta \quad \text{subject to:}$$
$$u_{j_1 \ldots j_c} \in \mathbb{U},\ H_i \left(\sum_{\mathbf{j} \in \mathbb{I}_d^+} n_{\mathbf{j}} \mu_{\mathbf{j}}(x) \widetilde{G}_{\mathbf{j}} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} \right) \leq (\sum_{\mathbf{j} \in \mathbb{I}_d^+} n_{\mathbf{j}} \mu_{\mathbf{j}}) \delta \quad \forall i \qquad (5.36)$$

which, as written, cannot yet be solved off-line because memberships are unkown at design time. To overcome such issue, for the controller (5.31), the proposal is choosing the optimal decision variables given by the solution of

$$\bar{u}^*(x) = \arg\min_{\bar{u}} \delta$$
$$\text{subject to:} \quad u_{j_1 \ldots j_c} \in \mathbb{U},\ H_i \widetilde{G}_{\mathbf{j}} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} \leq \delta \quad \forall i\ \forall \mathbf{j} \in \mathbb{I}_d^+ \qquad (5.37)$$

because, following analogous argumentations to those leading from (5.24) to (5.28), all feasible solutions of problem (5.37) are feasible, too, in (5.36). Actually, the developments in previous section prove that problem (5.37) is feasible in the set $\widehat{\mathcal{C}}_\infty^\lambda$ resulting from Algorithm 2 (details omitted for brevity).

Explicit solution As (5.37) is a linear programming problem once x is fixed (when actually measured), the optimal state-dependent solution $\bar{u}^*(x)$ has an explicit expression, piecewise-affine in x, which can be obtained via *multi-parametric* linear programming, considering x as a parameter, in the form $\bar{u}^*(x) = \overline{F}(x)x + \bar{\sigma}(x)$ with $\overline{F}(x)$ and $\bar{\sigma}(x)$ being a piecewise constant $(m\rho)\times n$ matrix and a $(m\rho)\times 1$ vector, respectively; for details on how such solutions can be obtained with suitable software, the reader is referred to (Kvasnica et al., 2004).

Now, reverting the vertical stacking in $\bar{u}^*(x)$ to the originating multidimensional fuzzy summation, i.e., writing the controller as in (5.12) or (5.31), the optimal controller arising from (5.37), can be expressed as

$$u^*(x) = \sum_{\mathbf{i}\in\mathbb{I}_c^+} n_{\mathbf{i}}\mu_{\mathbf{i}}(x)\,(F_{\mathbf{i}}(x)x + \sigma_{\mathbf{i}}(x)) \tag{5.38}$$

where $F_{\mathbf{i}}(x)$ and $\sigma_{\mathbf{i}}(x)$ are piecewise constant $m\times n$ matrices and $m\times 1$ vectors, respectively, suitably extracted from $\overline{F}(x)$ and $\bar{\sigma}(x)$.

This formula (*piecewise-affine multi-sum PDC* controller) gives interesting theoretical insights and, as above discussed, does not require on-line optimization. The other proposed optimisation setups (control effort in 1, 2 or ∞ norm) would also give rise to piecewise fuzzy-controllers (details, almost identical, are omitted for brevity). Anyway, the drawback is that, even if likely faster in runtime execution, performance with off-line optimisation will be inferior to that with on-line one (5.34), due to the explicit use of the measured value of the membership in (5.34), instead of the setting in (5.37) where memberships do not appear. Nevertheless, proven worst-case performance bounds are identical in both alternatives.

5.7 Discussion and comparison with existing approaches

This section will compare the result with other approaches in set-invariance and Lyapunov/LMI literature, including the fuzzy control and fuzzy Lyapunov functions.

Set-invariance prior literature Let us first remind how this work generalises existing set-invariance control approaches: Contractiveness concepts are used to obtain *necessary and sufficient* constrained robust stability and stabilisation conditions for polytopic systems in (Kerrigan, 2000). The work (Ariño, Pérez, Sala, & Bedate, 2014) generalises the idea to asymptotically shape-independent necessary and sufficient stability conditions for TS systems, and the proposal here presented covers the *asymptotically shape-independent necessary and sufficient stabilisation* conditions for constrained TS systems.

Other Lyapunov/LMI approaches Omitting detail, results from Section 6 in (Ariño, Pérez, Sala, & Bedate, 2014), dealing with generic Lyapunov functions, can be adapted to the stabilisation case here, with minor modifications (changing to control Lyapunov functions), so the following can be stated:

Lemma 5.4 *If a function $V(x)$ and a controller $\bar{u}(x)$, conformed as in* (5.33) *for some controller complexity c, have been proved to exist (with whatever method) such that*

$$V(\widetilde{G}_{\mathbf{i}}\begin{bmatrix} x \\ \bar{u} \end{bmatrix}) \leq V(\gamma x) \qquad \forall \mathbf{i} \in \mathbb{I}_{d+} \tag{5.39}$$

then, Algorithm 2 with complexity parameters c, d converges in a finite number of iterations for $\gamma < \lambda$ and the resulting λ-contractive set is larger than the Lyapunov level sets.

Proof: Omitted, as it is analogous to Propositions 1, 2 and Corollary 2 (computing an explicit bound on the number of needed iterations) in (Ariño, Pérez, Sala, & Bedate, 2014), with argumentations dating to (Blanchini, 1994; Kerrigan, 2000). ∎

Theorem 5.2, combined with the above Lemma, discussing the relationship with any conceivable methods to find Lyapunov functions are the key ones in this work: the ideal shape-independent i-step sets cannot be computed, but Polya relaxations allow running the iterations in Algorithm 2 with an approximation to $\bar{\mathcal{C}}_i^{\lambda}$ which can be made as precise as wished. Then, the obtained sets with Algorithm 2 beat Lyapunov level sets in the sense of Lemma 5.4. In fact, by maximality and convexity argumentations, they beat the *union* of all feasible solutions of any Lyapunov inequality (5.39), see Figure 5.2 in a later example.

Hence, the result *closes* (in theory) the shape-independent control design for *constrained* TS systems. Note, however, that as the number of decision variables in $\bar{u}$ and the summation dimension d increases, the computational complexity of the resulting problem grows heavily so only reasonably small values of c and d in $\widetilde{\mathcal{Q}}_d^c$ can actually be searched in practice.

Generic controller parametrizations Note that the proposed piecewise multiple-sum fuzzy affine controller structure in (5.38) is more general than many *non-piecewise* fuzzy controller choices in literature and, importantly, it is a *result* (asymptotically *exact*) of the proposed *optimal* control, whereas most literature proposes a particular control structure (or a fixed piecewise partition) *a priori*.

5.7.1 Relation with Fuzzy Lyapunov functions

Some approaches in literature propose fuzzy Lyapunov functions so its level sets are $V_\gamma := \{x : V(\mu(x), x) \leq \gamma\}$. Of course, such sets are shape-dependent, as they are defined in terms of $\mu(x)$. Clearly, the largest set that can be certified to belong to V_γ without knowing the specific shape of $\mu(x)$ is:

$$V_{si,\gamma} := \bigcap_{\mu \in \Delta} \{x : V(\mu, x) \leq \gamma\} = \{x : \max_{\mu \in \Delta} V(\mu, x) \leq \gamma\} \tag{5.40}$$

Of course, the proposal in this chapter can only be compared to level sets in the above form $V_{si,\gamma}$. Importantly, the proposal here allows seamless incorporation of non-symmetric constraint sets, whereas other LMI-based approaches might have difficulties in doing it.

Future/delayed fuzzy Lyapunov approaches More powerful Lyapunov function and controller parametrisations with “future” memberships values have been proposed in the α-samples approach (Kruszewski et al., 2008), and, also, with “past” memberships ones (Guerra et al., 2012). Combinations including both past and future memberships appear in (Lendek et al., 2012, 2015). The conditions in the cited references are shape-independent, in the sense that they consider neither the relationship between the memberships in different times nor the one between memberships and states.

The remainder of the section discusses specific details about the relationship between these proposals and the set-invariance one here proposed.

In order to encompass the different fuzzy (delayed/future) Lyapunov function approaches in other literature with an unified notation, generic fuzzy Lyapunov functions will be considered in the form $V(\Upsilon, x)$, where Υ is a delay-line set of membership vectors

$$\Upsilon := \{\mu(x_{k+s}), \dots, \mu(x_{k+1}), \mu(x_k), \mu(x_{k-1}), \dots, \mu(x_{k-l})\} \tag{5.41}$$

for some chosen values of look-ahead horizon s and delay l parameters.

In order to add causality constraints (control cannot depend on future membership values), the operator $F(\cdot)$ will extract the future (non-causal) elements, i.e., $F(\Upsilon) := \{\mu(x_{k+s}), \dots, \mu(x_{k+1})\}$, and $P(\cdot)$ will contain past ones,i.e., $P(\Upsilon) := \{\mu(x_{k-1}), \dots, \mu(x_{k-l})\}$. So, under this setting causal fuzzy controllers must be in the form $u(\mu(x), P(\Upsilon), x)$. Geometric λ-contractive conditions amount to

$$V(\Upsilon, \lambda x) - V(\Upsilon_+, \tilde{f}(\mu(x), x, u(\mu(x), P(\Upsilon), x))) > 0 \quad \forall x \in \Omega \sim \{0\} \tag{5.42}$$

where Υ_+ denotes the vector of memberships evaluated one step *forward* in time, i.e., from look-ahead $s+1$ until delay $l-1$.

The above expression is shape-dependent, but we can assert the following general shape-independent stabilization conditions replacing the elements of Υ and Υ_+ by arbitrary vectors (respectively denoted as Υ_{si} and $\Upsilon_{+,si}$) lying in the unit simplex:

Lemma 5.5 *The closed-loop fuzzy system:* $x_{k+1} = \tilde{f}(\mu(x_k), x_k, u(\mu(x_k), P(\Upsilon), x_k))$ *is locally stable with contraction rate* λ *if there exist a controller* $u(\mu, P(\Upsilon_{si}), x)$ *and a Lyapunov function* $V(\Upsilon_{si}, x)$ *such that*

$$V(\Upsilon_{si}, \lambda x) - V(\Upsilon_{[+,si]}, \tilde{f}(\mu, x, u(\mu, P(\Upsilon_{si}), x))) > 0 \tag{5.43}$$

being $\Upsilon_{si} = \{h_s, \dots, h_1, \mu, h_{-1}, \cdots h_{-l}\}$, $\Upsilon_{[+,si]} = \{h_{(s+1)}, \dots, h_1, \mu, h_{-1}, \dots \dots, h_{-l+1}\}$, *holds for all* $x \in \Omega$, $x \neq 0$, *for all* $h_{s+1}, \dots, h_{-l}, \mu$ *in* Δ.

In the above assertion, with a slight abuse of notation, $P(\Upsilon_{si})$ should be understood as the operator extracting "past" elements $\{h_{-1}, \dots, h_{-l}\}$.

On the following, shorthand notation $\Upsilon_{si} \in \Delta$ should, too, be understood as each element of Υ_{si} belonging to Δ.

Proof: Direct because (5.43) implies (5.42) (it is a particular choice of memberships). ∎

Conditions (5.43) are a contraction rate version analogous to the ones proved in LMI settings such as, for instance, (Guerra et al., 2012; Lendek et al., 2015). The relationship of such conditions with the geometric setting in this work is proven in the theorem below:

Theorem 5.3 *If condition* (5.43) *holds for all* Υ_{si}, $\Upsilon_{+,si} \in \Delta$*, then the level sets in* Ω *of:*

$$V_{si}(x) = \min_{P(\Upsilon_{si})\in\Delta} \max_{F(\Upsilon_{si})\in\Delta, \mu\in\Delta} V(\Upsilon_{si}, x) \tag{5.44}$$

are shape-independent control λ*-contractive.*

Proof: As (5.43) hold for any possible value of Υ_+, they do for the particular values of $\{h_{s+1}, \ldots, h_1\} = \{\bar{h}^+_{s+1}, \ldots, \bar{h}^+_1\}$ given by

$$\{\bar{h}^+_{s+1}, \ldots, \bar{h}^+_1\} = \arg \max_{F(\Upsilon_{+,si})\in\Delta} V(\Upsilon_{+,si}, \tilde{f}(\mu, x, u(\mu, P(\Upsilon_{si}), x)) \tag{5.45}$$

So, we can assert

$$\begin{aligned} 0 < {} & V(\bar{h}^+_s, \ldots, \bar{h}^+_1, \mu, P(\Upsilon_{si}), \lambda x) \\ & - V(\bar{h}^+_{s+1}, \ldots, \bar{h}^+_1, \mu, P(\Upsilon_{[+,si]}), \tilde{f}(\mu, x, u(\mu, P(\Upsilon_{si}), x))) \end{aligned} \tag{5.46}$$

Then, denoting

$$\{\bar{h}_s, \ldots, \bar{h}_1, \bar{h}\} = \arg \max_{h, h_1 \ldots, h_s\in\Delta} V(h_s, \ldots, h_1, h, P(\Upsilon_{si}), \lambda x) \tag{5.47}$$

for any $P(\Upsilon_{[si]})$ in Δ, we have:

$$\begin{aligned} 0 < {} & V(\bar{h}_s, \ldots, \bar{h}_1, \bar{h}, P(\Upsilon_{si}), \lambda x) \\ & - V(\bar{h}^+_{s+1}, \ldots, \bar{h}^+_1, \mu, P(\Upsilon_{[+,si]}), \tilde{f}(\mu, x, u(\mu, P(\Upsilon_{si}), x))) \end{aligned} \tag{5.48}$$

Denote now:

$$\{\underline{h}_{-1}, \ldots, \underline{h}_{-l}\} = \arg \min_{P(\Upsilon_{si})\in\Delta} \left(V(\bar{h}_s, \ldots, \bar{h}_1, \bar{h}, P(\Upsilon_{si}), \lambda x)\right) \tag{5.49}$$

so, as the above (5.48) holds for any $P(\Upsilon_{[si]})$ in Δ, it does for $\{h_{-1},\dots h_{-l}\}=\{\underline{h}_{-1},\dots,\underline{h}_{-l}\}$, i.e.,

$$0<V(\bar{h}_s,\dots,\bar{h}_1,\bar{h},\underline{h}_{-1},\dots,\underline{h}_{-l},\lambda x)- \\ V(\bar{h}^+_{s+1},\dots,\bar{h}^+_1,\mu,\underline{h}_{-1},\dots,\underline{h}_{-l+1},\tilde{f}(\mu,x,u(\mu,\underline{h}_{-1},\dots,\underline{h}_{-l},x))) \tag{5.50}$$

and, at last, denoting

$$\{\underline{h}^+,\underline{h}^+_1,\dots,\underline{h}^+_{-l+1}\}= \\ \arg\min_{h,h_1,\dots,h_{-l+1}\in\Delta} V(\bar{h}^+_{s+1},\dots,\bar{h}^+_1,h,\dots,h_{-l+1},\tilde{f}(\mu,x,u(\mu,\underline{h}_{-1},\dots,\underline{h}_{-l},x))) \tag{5.51}$$

we have, for all $\mu\in\Delta$:

$$0<V(\bar{h}_s,\dots,\bar{h}_1,\bar{h},\underline{h}_{-1},\dots,\underline{h}_{-l},\lambda x)- \\ V(\bar{h}^+_{s+1},\dots,\bar{h}^+_1,\underline{h}^+,\underline{h}^+_{-1},\dots,\underline{h}^+_{-l+1},\tilde{f}(\mu,x,u(\mu,\underline{h}_{-1},\dots,\underline{h}_{-l},x))) \tag{5.52}$$

In the last inequality, the only still "free" variable ranging in the unit simplex is the membership at the current instant. All other past of future ones have been replaced by suitable maximisers or minimisers. In particular,

$$V_{si}(\lambda x)= \\ \min_{P(\Upsilon_{si})\in\Delta}\ \max_{F(\Upsilon_{si})\in\Delta,\mu\in\Delta} V(\Upsilon_{si},\lambda x)=V(\bar{h}_s,\dots,\bar{h}_1,\bar{h},\underline{h}_{-1},\dots,\underline{h}_{-l},\lambda x) \tag{5.53}$$

and, using the resulting controller $u(\mu,\underline{h}_{-1},\dots,\underline{h}_{-l},x)$ in (5.52), such that $x_{k+1}=\tilde{f}(\mu,x,u(\mu,\underline{h}_{-1},\dots,\underline{h}_{-l},x))$, we have:

$$V_{si}(x_{k+1})= \\ V(\bar{h}^+_{s+1},\dots,\bar{h}^+_1,\underline{h}^+,\underline{h}^+_{-1},\dots,\underline{h}^+_{-l+1},\tilde{f}(\mu,x,u(\mu,\underline{h}_{-1},\dots,\underline{h}_{-l},x))) \tag{5.54}$$

Hence, expression (5.52), proves that there exists a control action such that $V_{si}(\lambda x)-V_{si}(x_{k+1})>0$. ∎

Theorem 5.3 extends the concept of shape-independent level sets (5.40) to the case of past and future memberships (indeed, (5.40) is a particular case of level sets of (5.44)). The importance of the theorem is twofold:

1. by asymptotical exactness, if any proposal in literature proves a sufficient condition for (5.43), then, any point in the interior of the level sets of $V_{si}(x)$ in $\mathbb{X}$ will be found by the proposed algorithm with a high-enough value of the complexity parameters.
2. V_{si} is a "standard" Lyapunov function: intuition is "reconciled" with the results, in the sense that Lyapunov functions involving past and future memberships are transformed to standard ones depending only on the current state (at least in the shape-independent case). Also, even if past memberships appear in the controller in (5.42), the controller actually used to prove contractiveness in the above proof is independent of the past "measured" memberships (as required by the λ-contractiveness definition): the arguments $\underline{h}_{-1}, \ldots, \underline{h}_{-l}$ in the controller $u(\mu, \underline{h}_{-1}, \ldots, \underline{h}_{-l}, x)$ are actually a function of x, as (5.49) shows.

Comparison with shape-dependent options Note that supposedly "future" values of μ and x are, in fact, predictions based on current x, $\mu(x)$. As stability conditions hold for any future memberships, the following shape-dependent Lyapunov function is, too, proven if (5.43) holds from any LMI literature result[3]:

$$V_{sd}(x_k) := \min_{h_{-1},\ldots,h_{-l}\in\Delta} V(h_{-l}, \ldots, h_{-1}, \mu(x_k), \mu(x_{k+1}), \ldots, \mu(x_{k+s}), x_k) \tag{5.55}$$

Obviously, given some scalar γ, the level sets $V_{sd,\gamma} = \{V_{sd}(x) \leq \gamma\}$ will be larger than those $V_{si,\gamma} = \{V_{si}(x) \leq \gamma\}$ with V_{si} from (5.44), as $V_{sd}(x) \leq V_{si}(x)$. Note, however, that comparing the largest level sets V_{sd,γ_1^*} and V_{si,γ_2^*} in region Ω, forcefully $\gamma_2^* \geq \gamma_1^*$, as $V_{si,\gamma_1^*} \subset V_{sd,\gamma_1^*}$. So, the proven domain of attraction with Lyapunov functions (5.55) or (5.44) will be "different": there might be some states proven to contract to the origin by V_{sd} but not V_{si} and vice-versa (with no clear inclusion in either sense).

[3] As "past" is irrelevant for stability, minimisation on past memberships can be carried out for larger level sets, details omitted for brevity. Such minimisation appears, then, in (5.55).

5.8 Examples

Consider a TS system $x_{k+1} = \sum_{i=1}^{2} \mu_i(A_i x_k + B_i u_k)$ with model matrices:

$$A_1 = \begin{pmatrix} 0.95 & 0.3 \\ 0.7 & 1.1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0.1 & 0.7 \\ 0.2 & 0.4 \end{pmatrix} \tag{5.56}$$

$$B_1 = \begin{pmatrix} 0.4 \\ 0.5 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 0.1 \\ 2 \end{pmatrix} \tag{5.57}$$

Subject to the constraints in inputs and states:

$$-10 \le u_k \le 10, \qquad \begin{pmatrix} -10 \\ -10 \end{pmatrix} \le x_k \le \begin{pmatrix} 10 \\ 10 \end{pmatrix} \tag{5.58}$$

For the sake of illustration, even if results are valid for any membership shape, some system trajectories will be later simulated using as membership functions:

$$\mu_1(x) = (10 - (1\ 0)x)/20, \quad \mu_2(x) = 1 - \mu_1(x) \tag{5.59}$$

5.8.1 Comparison with fuzzy-delayed Lyapunov function

With the above plant, a comparative study with LMIs in (Lendek et al., 2012, Corollary 1) will be made first.

LMI settings The cited proposal uses a delayed Lyapunov function and a non-PDC controller, respectively given by:

$$V(x_k, x_{k-1}) = x_k^T \left(\sum_{i=1}^{r} \mu_i(x_{k-1}) P_i \right)^{-1} x_k \tag{5.60}$$

$$u_k = \left(\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(x_{k-1}) \mu_j(x_k) F_{ij} \right) \left(\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(x_{k-1}) \mu_j(x_k) H_{ij} \right)^{-1} x_k \tag{5.61}$$

As discussed in Section 5.7.1, even if the above Lyapunov function is a "fuzzy" one, the (non-fuzzy) Lyapunov function (5.44) particularised for (5.60), i.e.,

$$V_{si}(x_k) = \min_{\mu \in \Delta} x^T \left(\sum_{i=1}^{r} \mu_i P_i \right)^{-1} x \qquad (5.62)$$

is also proved and, evidently, the level sets $V_{\gamma,x_{k-1}} = \{x_k : V(x_k, x_{k-1}) \leq \gamma\}$ for whichever value of $\mu_i(x_{k-1})$ will be smaller than those in the form $V_{si,\gamma} = \{x : V_{si}(x) \leq \gamma\}$ for the same level. In fact, the latter level set is the convex hull of the union of the ellipsoids $\mathcal{E}_i := \{x : x^T P_i^{-1} x \leq \gamma\}$, see (Hu & Lin, 2003).

In order for the comparison to be fair, apart from the LMIs in (Lendek et al., 2012, Cor. 1), also appearing in (Lendek et al., 2015), extra LMIs have been added to to force that the level set for $\gamma = 1$ of each of the ellipsoid lies inside the constraint region $\mathbb{X}$, and to account for the control saturation. Details are omitted for brevity as they follow standard S-procedure argumentations (and use of (Lendek et al., 2012, Property 2)). In order to obtain a "large" domain of attraction exploiting the non-quadratic and convex-hull ideas, ellipsoids $\mathcal{E}_i$ were forced to contain the point $\gamma \cdot (\cos\phi_i, \sin\phi_i)^T$ for $\phi_1 = \pi/4$ and $\phi_2 = 3\pi/4$, respectively, and the scalar $\gamma \geq 0$ was maximised. In this way, ellipsoids were expanded in orthogonal directions.

Set-invariance settings In the λ-contractiveness approach presented here, we set $\lambda = 0.9999$ (mere stabilisation), and we test a non-fuzzy controller ($c = 0$, $d = 1$), and a PDC control parametrisation ($c = 1$) with a Polya complexity parameter $d = 6$.

Results As a result, Figure 5.1 is obtained. The figure depicts the convex hull of the two ellipsoids arising from the LMI solution(curved line), the maximal set with the non-fuzzy controller (blue region) and the inner approximation to the maximal controllable set with single-sum controllers (union of blue and red region). Increasing d didn't visually appear to generate more stabilisable points. The set proposed by our algorithm is larger than the proposed solution in the compared work.

Computation time The LMI solution from (Lendek et al., 2012) in Figure 5.1, using YALMIP 3.2010.0611 and SEDUMI 1.3, took 1.56 seconds; with MPT Toolbox 2.6.3 (Kvasnica et al., 2004), obtaining the

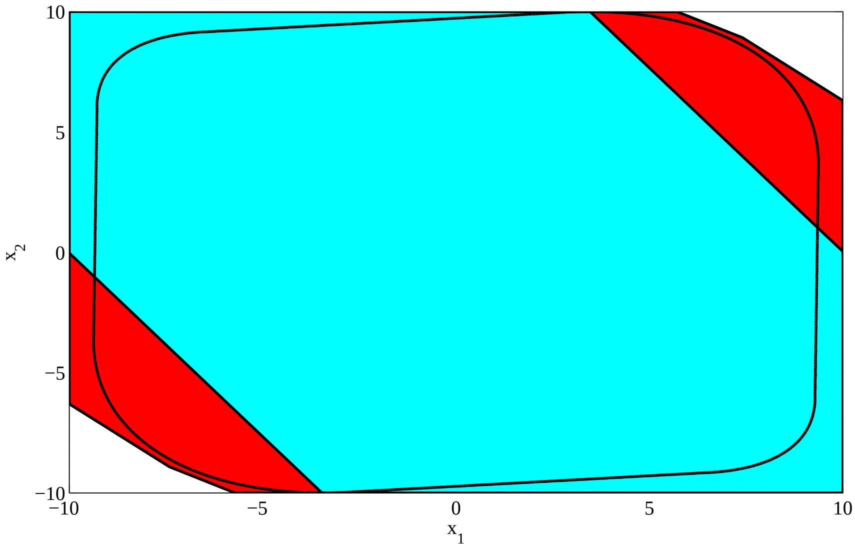

Figure 5.1: Comparative analysis with delayed-membership non-PDC control (single solution).

blue robust-polytopic region with the algorithms in (Kerrigan, 2000) took 16 iterations and 2.03 seconds; the Polya-6 red region in the above figure took 4 iterations and 0.222 seconds; the used computer was an Intel I5 2.56GHz computer with 6 Gb of RAM with Matlab 2010. The more general controller parametrisation allows to prove stability with less iterations (optimal controllers are faster): surprisingly, the more complex setting took less time to compute.

Union of all possible LMI solutions As the solution of (Lendek et al., 2012, Cor. 1) might be not unique[4], several solutions were crafted by forcing one of the ellipsoids $\mathcal{E}_i$ (i randomly chosen) to contain the largest possible ellipsoid in the form $\mathcal{E}^*_\gamma = \{x : 100x_1^2 + x_2^2 \leq \gamma\}$ rotating $\mathcal{E}^*_\gamma$ repeated times, in order to explore whether there exists a solution of the LMIs "stretching out" as much as possible in every direction.

Figure 5.2 presents the multitude of solutions for different runs of

[4]In our approach, on-line controllers might be non-unique, but the maximal set is indeed unique (such fact can be proved by convexity argumentations).

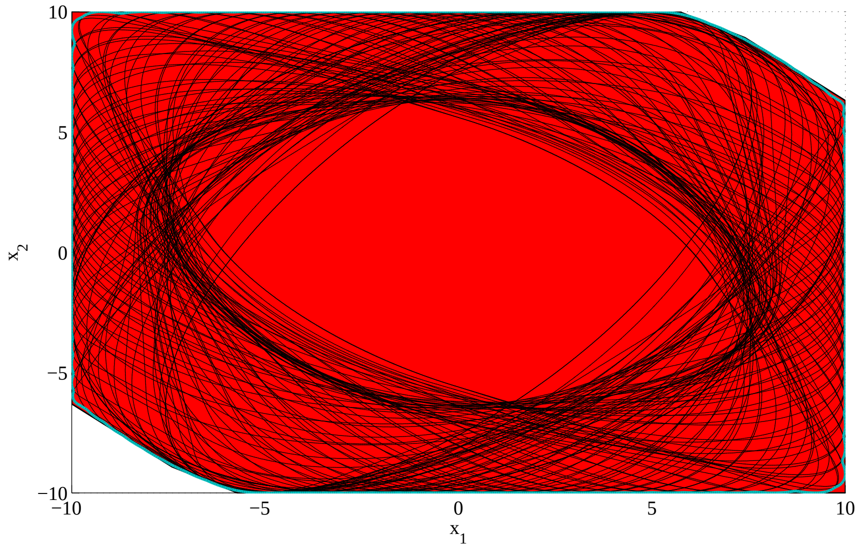

Figure 5.2: Comparative analysis with delayed-membership non-PDC control (all solutions); the cyan line depicts the union of all such solutions.

the LMIs[5], with the union emphasised in blue color. All of the solutions lie inside the converged invariant set produced by our algorithm. So, there exists a controller with piecewise-PDC structure which outperforms (larger domain of attraction) those in (Lendek et al., 2012, Cor. 1), not only individually but also outperforming the *union* of all feasible solutions (which might involve a *different* controller each, so a controller for such union set is not found in the cited work) with a *single* controller.

5.8.2 Off-line piecewise controller

In this subsection, a contraction rate $\lambda = 0.98$ has been chosen as the specification for speed of convergence, so the obtained sets are slightly smaller. The piecewise-PDC fuzzy controller (linear in memberships and affine in the state) in (5.38) has been computed for complexity

[5]Importantly, note that the LMIs in the compared work provide only *one* solution, as in Figure 5.1: computing the *union* of all LMI solutions requires a theoretically infinite number of LMIs with the settings in prior literature, whereas our proposal takes 0.2 seconds to compute a set which is larger than such union.

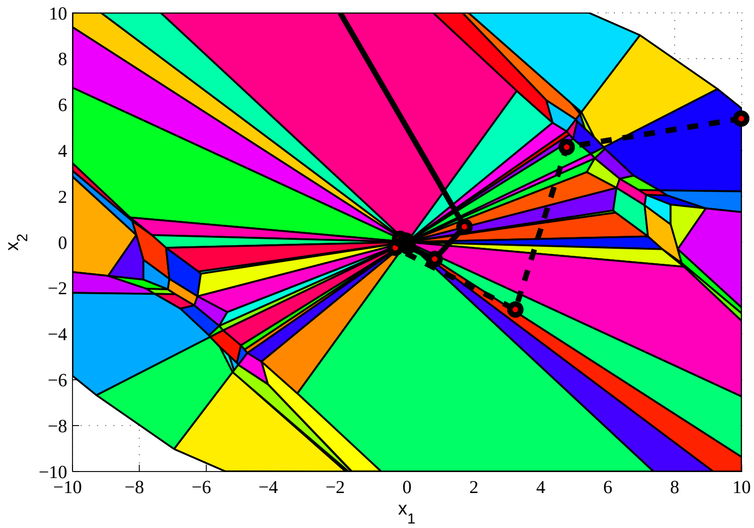

Figure 5.3: Piecewise state-space tesellation and trajectories for different simulations of the piecewise-affine controller (5.38).

parameters $c = 1$, $d = 6$, searching for the fastest decay. As previously discussed, this aims to achieve a faster on-line execution in exchange for a larger computation time in the design phase.

The optimal control problem (5.37) in the single polytope given by the maximal contractive set has a piecewise solution with a tessellation of 90 polytopic regions[6], depicted in Figure 5.3. In this example, the computation of the explicit piecewise-PDC-affine optimal feedback law took 2.35 seconds, instead of the 0.222 that took computing "only the *set* in which a controller exists".

Two trajectories are simulated with the piecewise controller. As guaranteed by the algorithms, feasible sequences of control and states can be obtained without violating any constraints.

[6]Note that the resulting regions were not known *a priori*, contrarily to other piecewise results, say (Johansson et al., 1999), in which regions are fixed at start.

5.9 Conclusions

This chapter presents an extension of the control λ-contractive set computations in robust control literature to fuzzy Takagi-Sugeno models under state and input constraints (possibly non-symmetric). Based on Polya's asymptotically exact theorems, the obtained closed-loop controllable sets will approach the maximal shape-independent control λ-contractive set: if some complexity parameters are high enough, the obtained sets and controllers improve over any conceivable (shape independent) Lyapunov-based controller design technique for TS systems. An implementation requiring on-line one-step optimisation is proposed; as an alternative, by using explicit multi-parametric software tools, a shape-independent piecewise-affine-multidimensional-PDC controller exists whose explicit expression can be obtained off-line, achieving the same worst-case performance. Comparative analysis with delayed-fuzzy Lyapunov functions show that all their shape-independent solutions lie inside the sets produced by the new algorithm for the same complexity parameter values.

Chapter 6

Model Predictive Control for Discrete Fuzzy Systems via Iterative Quadratic Programming

Note: The contribution of this chapter is based on the following publication:

Ariño, C., Pérez, E., Querol, A., and Sala, A. (2014). Model predictive control for discrete fuzzy systems via iterative quadratic programming. *In Fuzzy systems (FUZZ-IEEE), 2014 IEEE international conference on* (pp. 2288–2293).

Takagi-Sugeno fuzzy models are exact representations of nonlinear systems in a compact region. Guaranteed-cost linear matrix inequalities produce controllers which minimize a shape-independent bound on a quadratic cost; however, the controller has a fixed structure (possibly suboptimal), say a Parallel Distributed Compensator (PDC), and does not allow input saturation. By posing the problem as a Model Predictive Control one, the ideas of terminal set, terminal controller and feasible set can be used in order to improve the performance of usual guaranteed-cost controllers for Takagi-Sugeno systems via Quadratic Programming. A Polya-based approach has been introduced in order to (conservatively) transform the invariant set problem into a polytopic one, as well as computing the controller feasibility region. The optimal controller is computed iteratively.

6.1 Introduction

Takagi-Sugeno (TS) fuzzy models are exact representations of nonlinear systems in a compact region (modelling region, Ω) if well-known systematic sector-nonlinearity methodologies (Tanaka & Wang, 2004) are used.

Techniques based on Linear Matrix Inequalities (LMI) have allowed to obtain a wide range of fuzzy controllers following different specifications (stability, decay, $\mathcal{H}_\infty$, ...). Many of them result in closed-loop expressed as multi-dimensional fuzzy summations. In particular, guaranteed-cost ones (Wu, 2004) are those which generalize to TS models, with some conservativeness, the usual optimization of infinite-cost quadratic indices in linear quadratic regulator (LQR) control.

However, as fuzzy models are usually valid only locally in the compact region Ω, performance guarantees are usually stated only on level sets of the obtained Lyapunov functions included in the modelling region (Pitarch, Ariño, Bedate, & Sala, 2010). So, implicitly, the actual fuzzy control problem *should* incorporate state constraints arising from the local modelling setup. Such constraints are usually enforced via Lyapunov level sets but the actual valid initial condition region might be quite larger than that arising from the level sets (Ariño, Pérez, Sala, & Bedate, 2014). Also, in realistic applications, there is always control saturation which is not easy to handle in LMI framework: most conditions actually require the control action to *avoid* saturation in the outermost Lyapunov level set or, if that is not the case (Cao & Lin, 2003), either cannot prove improvement with respect to non-saturating laws or require iteration/Bilinear Matrix Inequalities (BMI) (Ariño et al., 2010).

In a linear case, state and input constraints are handled with on-line finite-horizon optimization in model predictive control (Goodwin & De Doná, 2005) (MPC). Stability and infinite-horizon optimality of receding-horizon predictive laws is ensured for all initial states in a so-called *feasible set* if a so-called *terminal controller* can be found which does not hit any constraint in future time for all initial states in a *terminal set*. These are well known concepts in the linear MPC framework (Goodwin & De Doná, 2005) which, however, are much harder to deal with in nonlinear systems.

The objective of this chapter is adapting the above considerations to TS fuzzy systems. As there are some causes of conservatism (in particular shape-independence and fuzzy summation issues (Sala, 2009)), subsets of the actual invariant and feasible sets are computed for a PDC terminal controller. Also, as future optimal trajectories are unknown, an iterative procedure is reported in order to converge to the optimal one for the original nonlinear system (actually suboptimal because convergence is sought only in the finite-horizon segment). The result, albeit sub-optimal (because the terminal controller is conservative because of shape-independence and Polya-like fuzzy summation issues), improves over the terminal (plain PDC) controller both in achieved cost and in the enlarged feasible zone.

The structure of the chapter is as follows: next section discusses preliminary notation and the concrete problem statement above outlined. Section 6.3 discusses the proposed setup for adapting MPC to fuzzy systems, first considering the terminal cost, and later the terminal set, feasible set, plus an iterative algorithm to compute the optimal transient system trajectories (Section 6.3.4). Section 6.4 and 3.4.3 propose examples in which the main concepts are illustrated. A conclusion section closes the chapter.

6.2 Preliminaries and Problem statement

Consider a nonlinear discrete-time system to be controlled, given by a model:

$$x_{k+1} = f_x(x_k) + f_u(x_k)u_k \tag{6.1}$$

such that $f_x(0) = 0$. This system can be expressed *locally* in a compact region of interest Ω containing the origin as a TS (Takagi & Sugeno, 1985) fuzzy system with r rules or local models in the form:

$$x_{k+1} = \sum_{i=1}^{r} \mu_i(x_k)(A_i x_k + B_i u_k) \tag{6.2}$$

where $x_k \in \mathbb{R}^n$ represent the states, $u_k \in \mathbb{R}^m$ the control actions and μ_i represent membership functions such that:

$$\sum_{i=1}^{r} \mu_i(x_k) = 1, \quad \mu_i(x_k) \geq 0 \ \forall x \ i : 1 \ldots r \tag{6.3}$$

If a fuzzy PDC state-feedback controller were used,

$$u_k = -\sum_{i=1}^{r} \mu_i K_i x_k \tag{6.4}$$

the closed loop has the form:

$$x_{k+1} = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j ((A_i - B_i K_j)x_k) \tag{6.5}$$

Note that the dependence of the membership functions on x_k has been omitted for brevity.

Let us also consider in our problem formulation some input and state constraints. When these constraints are linear they can be defined by the appropriated matrices R and S, and vectors l, s such that:

$$\mathbb{X} = \{x \in \mathbb{R}^n \mid Rx + l \leq 0\} \tag{6.6}$$

$$\mathbb{U} = \{u \in \mathbb{R}^m \mid Su + s \leq 0\} \tag{6.7}$$

6.2.1 Problem statement

In literature, guaranteed cost control is used to synthesize PDC controllers in the form (6.4) without taking into account the state and input constraints.

The objective of this chapter is using such controllers as *terminal* controllers in predictive-control-like strategies for fuzzy TS systems in order to (partly) overcome the conservativeness arising from:

- The worst-case (membership independent) approach.
- The limited choice of Lyapunov functions.
- Ensuring the satisfaction of the above defined constraints in the largest possible initial condition region.

In summary, even if terminal controllers are conservative, results (guaranteed cost bounds) will improve due to the addition of a finite-horizon segment with less conservative assumptions.

6.3 Fuzzy Model Predictive Control

MPC can be defined as a Constrained online optimization based on a model prediction. The essential parts of a MPC are:

- A model that will be able to describe the behavior of the future states.

- An objective function that represents the performance of the controlled system.

- An optimizer, that minimizes the objective function subject to the proper constraints.

- The receding horizon strategy, that implies that the optimizer has to solve the problem at each step.

The model that will be used on the MPC formulation is the following TS one:

$$x_{k+1} = \sum_{i=1}^{r} \mu_i(\tilde{x}_k)(A_i x_k + B_i u_k) \qquad (6.8)$$

Note that the main difference between (6.2) and (6.8) is that the membership functions depend on a new variable $\tilde{x}_k$. This variable is an estimation of the optimal states at time k, at the prior iteration in an algorithm to be later introduced. The introduction of that variable simplifies the problem significantly, because the non-linear dependence of the TS model can be evaluated at the beginning and then it will not be introduced into the optimization problem. The obvious drawback of that simplification is that many times $\tilde{x}_k$ may be quite different from the predicted optimal x_k, so this motivates the mentioned iterative approach.

6.3.1 The objective function

Ideally, the proposed objective function would be a quadratic performance function of the states and the inputs such as:

$$J_\infty = \sum_{k=0}^{\infty} \left(x_k^T H x_k + u_k^T F u_k \right) \tag{6.9}$$

However, the main drawback of the function (6.9) is that it is not numerically tractable (except in the well-known linear time-invariant case) because of the infinite-horizon objective. In order to avoid this problem, a finite horizon function is usually introduced in MPC:

$$J_N = x_N^T P x_N + \sum_{k=0}^{N-1} \left(x_k^T H x_k + u_k^T F u_k \right) \tag{6.10}$$

For performance and stability reasons, it will be interesting that our proposed finite horizon performance function J_N bounds the optimal infinite-horizon one J_∞ ($J_\infty \leq J_N$) while making the gap between both functions as small as possible.

To do so, analogous considerations as those in (Tanaka et al., 1999) for continuous systems have been done for the discrete TS case in this chapter. This way, a matrix P must be found which bounds the term of the infinite horizon $J_{N\to\infty}$

$$J_{N\to\infty} = \sum_{k=N}^{\infty} \left(x_k^T H x_k + u_k^T F u_k \right) \tag{6.11}$$

such that

$$J_{N\to\infty} \leq x_N^T P x_N \tag{6.12}$$

This bounding can be achieved by constraining the per-stage weighting with the condition (6.13)

$$x_{k+1}^T P x_{k+1} - x_k^T P x_k < -(x_k^T H x_k + u_k^T F u_k) \tag{6.13}$$

Indeed, if (6.13) holds, summing from $k = N$ to $k = \infty$ and assuming the resulting controller will be stabilising so $x_\infty = 0$, the cost index (6.11) can be bounded by $x_N^T P x_N$

$$\sum_{k=N}^{\infty} (x_k^T H x_k + u_k^T F u_k) < x_N^T P x_N \tag{6.14}$$

Using the Schur complement, following well-known argumentations (Boyd et al., 1994), a controller for which (6.13) holds can be found if there exist matrices M_i, $X > 0$ such that

$$\Gamma_{ij} = \begin{pmatrix} X & XA_i^T - M_j^T B_i^T & X & M_j^T \\ A_i X - B_i M_j & X & 0 & 0 \\ X & 0 & H^{-1} & 0 \\ M_j & 0 & 0 & F^{-1} \end{pmatrix} \tag{6.15}$$

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j \Gamma_{ij} > 0 \tag{6.16}$$

where $P = X^{-1}$ and the controller is defined as a PDC (6.4) with

$$K_i = M_i X^{-1} \tag{6.17}$$

The worst-case bound of the cost function (6.11) is minimized if the eigenvalues of X are maximized, that is

$$\min -\lambda$$

subject to (6.16) and $X > \lambda$.

Note that (6.16) is a fuzzy summation which can be, conservatively, expressed as an LMI following any of the relaxations in (Tanaka & Wang, 2004; Tuan et al., 2001; Sala & Ariño, 2007a), see Chapter 3.

6.3.2 Terminal Set

Many times, the PDC controller (6.17) can not be applied in the whole state space definition $\mathbb{X}$ defined in (6.6), because some inputs defined with this PDC controller will not verify the input constraints defined in (6.7). Also, maybe even if they do in a particular instant, the future optimal trajectory may exit Ω or even without exiting, it might violate the input bounds.

Then it is mandatory to obtain a set from with this controller can be applied and the system will be stable and optimal, and future states must also belong to that set. Following predictive-control argumentations, the invariant set of this controller has to be computed. It can be done following the algorithm presented in (Ariño et al., 2013). This algorithm is based on (Gilbert & Tan, 1991), adapted to TS Fuzzy models applying the Polya theorem and is also presented here as Algorithm 3.

Algorithm 3 Calculation of the closed-loop N-step invariant set $\mathbb{K}_N(\Omega, \mathbb{T})$

1. Make $i = 0$ and $\mathbb{K}_0(\Omega, \mathbb{T}) = \mathbb{T}$
2. While $i < N$:
   (a) $\mathbb{K}_{i+1}(\Omega, \mathbb{T}) = \mathcal{Q}\left(\mathbb{K}_i(\Omega, \mathbb{T})\right) \cap \Omega$
   (b) If $\mathbb{K}_{i+1}(\Omega, \mathbb{T}) = \mathbb{K}_i(\Omega, \mathbb{T})$, end algorithm and $\mathbb{K}_N(\Omega, \mathbb{T}) = \mathbb{K}_\infty(\Omega, \mathbb{T}) = \mathbb{K}_i(\Omega, \mathbb{T})$.
   (c) i=i+1

Where $\mathbb{T}$ is a target set; Ω is a generic set in the states space; and $\mathbb{K}_i(\Omega, \mathbb{T})$ denotes the subset of Ω that steers the system to $\mathbb{T}$ in at most i steps.

The algorithm needs to compute iteratively the one-step set $\mathcal{Q}(\Omega) = \{x \in \Omega | x_{k+1} \in \Omega\}$. This set, in a general case, is a complicated one arising from the non-linear dynamics embedded in the TS models.

In order to avoid this problem, an approximation of $\mathcal{Q}$ can be done using the Polya expanded TS model

$$x_{k+1} = \left(\sum_{i=1}^{r} \mu_i(x)\right)^{d-2} \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(x)\mu_j(x)G_{ij}x_k \qquad (6.18)$$

where $G_{ij} = A_i - B_iK_j$. Note that this model is equivalent to (6.5) as $\sum_{i=1}^{r} \mu_i(x) = 1$.

The Polya-expanded model in (6.18) is a d degree vector polynomial of μ_i and a suitable matrix $\tilde{G}_{\mathbf{i}}$ can be found such that

$$x_{k+1} = \sum_{\mathbf{i} \in \mathbb{I}_d^+} n_{\mathbf{i}} \mu_{\mathbf{i}}(x) \tilde{G}_{\mathbf{i}} x_k \qquad (6.19)$$

where each $\mu_{\mathbf{i}}$ represents one of the possible monomials $\prod \mu_j$ of degree d; $\mathbb{I}_d^+$ is the set of all the different monomials of degree d; and $n_{\mathbf{i}}$ is the number of times that this monomial appears in (6.18). For further details, the reader is referred to (Ariño et al., 2013).

With this notation, the one-step set can be expressed as

$$\mathcal{Q}(\Omega) = \{x \in \mathbb{R}^n | \sum_{\mathbf{i} \in \mathbb{I}_d^+} n_{\mathbf{i}} \mu_{\mathbf{i}}(x) \tilde{G}_{\mathbf{i}} x \in \Omega\} \tag{6.20}$$

Due to $\mu_{\mathbf{i}}$ and $n_{\mathbf{i}}$ being positive, a sufficient condition for a point x to belong to (6.20), can be given by ensuring that

$$\mathcal{Q}(\Omega) \supset \tilde{\mathcal{Q}}_d(\Omega) = \{x \in \mathbb{R}^n | \tilde{G}_{\mathbf{i}} x \in \Omega\} \tag{6.21}$$

Note that $\tilde{\mathcal{Q}}_d(\Omega)$ is a polytopic subset of the one-step set. Furthermore, it can be proved that as d increases, $\tilde{\mathcal{Q}}_d(\Omega)$ asymptotically approaches to the maximal shape-independent subset of $\mathcal{Q}(\Omega)$ (Ariño, Pérez, Sala, & Bedate, 2014) or chapter 5. Now, this set can be used in Algorithm 3 in order to obtain an inner approximation of the Invariant Set which can be used as the terminal set in the MPC problem. We define this set as $\mathbb{Z} = \mathbb{K}_\infty(\Omega, \Omega)$ which is a polytope. Hence, it can be represented as

$$\mathbb{Z} = \{x \in \mathbb{R}^n | Zx \leq z\} \tag{6.22}$$

for a suitable choice of matrix Z and vector z.

6.3.3 Optimization Problem

Once the terminal cost P and the terminal set $\mathbb{Z}$ are obtained, given a known initial state x_0 and a first guess of the future optimal trajectory $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_{N-1})$, the following Quadratic Programming (QP) optimization problem $\mathcal{P}_N(x_0, \tilde{\mathbf{x}})$ can be stated:

$$\begin{aligned}
&\mathcal{P}_N(x_0, \tilde{\mathbf{x}}) : \text{find } u_0, \ldots, u_{N-1} \text{ which minimize} \\
&J_N^{OPT}(x) = x_N^T P x_N + \sum_{k=0}^{N-1} \left(x_k^T H x_k + u_k^T F u_k \right) \qquad (6.23) \\
&\text{subject to:} \\
&\qquad u_k \in \mathbb{U} \text{ for } k = 0, \ldots, N-1 \\
&\qquad x_{k+1} = \sum_{i=1}^{r} \mu_i(\tilde{x}_k)(A_i x_k + B_i u_k) \in \mathbb{X} \qquad (6.24) \\
&\qquad\qquad \text{for } k = 0, \ldots, N \\
&\qquad x_N \in \mathbb{Z} \subset \mathbb{X}
\end{aligned}$$

$$\Gamma = \begin{pmatrix} B(\tilde{x}_0) & 0 & \dots & 0 \\ A(\tilde{x}_1)B(\tilde{x}_0) & B(\tilde{x}_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (\prod_{i=1}^{N-1} A(\tilde{x}_i))B(\tilde{x}_0) & (\prod_{i=2}^{N-1} A(\tilde{x}_i))B(\tilde{x}_1) & \dots & B(\tilde{x}_{N-1}) \end{pmatrix} \tag{6.28}$$

At this point it is useful to remark that $\tilde{x}_k$ for $k = 1 \dots N-1$ have to be already known in order to avoid the nonlinearities of the model's memberships and express this problem as a QP. In the next section, an iterative procedure will be presented to obtain this state estimates.

Let us show that, indeed, the problem is a standard QP one. First, note that the matrices below are known at the time of the computation

$$A(\tilde{x}_k) = \sum_{i=1}^{r} \mu_i(\tilde{x}_k)A_i, \quad B(\tilde{x}_k) = \sum_{i=1}^{r} \mu_i(\tilde{x}_k)B_i \tag{6.25}$$

With these matrices the prediction model can be expressed as:

$$\mathbf{x} = \Theta x_0 + \Gamma \mathbf{u} \tag{6.26}$$

where $\mathbf{x} = (x_1^T \dots x_N^T)^T$, $\mathbf{u} = (u_0^T \dots u_{N-1}^T)^T$, Θ is defined in (6.27) and Γ in (6.28) on top of next page.

$$\Theta = \begin{pmatrix} A(\tilde{x}_0) \\ A(\tilde{x}_1)A(\tilde{x}_0) \\ \vdots \\ A(\tilde{x}_{N-1})\dots A(\tilde{x}_0) \end{pmatrix} \tag{6.27}$$

As Θ and Γ are easily computable matrices, following standard MPC procedures (see (Goodwin & De Doná, 2005) for details), the optimization problem can be expressed as the following quadratic program on the vector of future controls $\mathbf{u}$:

$$\mathcal{P}_N(x_0, \tilde{\mathbf{x}}) : \text{minimize } \frac{1}{2}\mathbf{u}^T\mathbf{H}\mathbf{u} + x_0^T\mathbf{F}\mathbf{u} \tag{6.29}$$

subject to:

$$\Phi\mathbf{u} \leq \Delta - \Lambda x_0 \tag{6.30}$$

where

$$\Phi = \begin{pmatrix} \mathbf{R}\Gamma \\ \mathbf{S} \end{pmatrix}, \quad \Delta = \begin{pmatrix} \mathbf{l} \\ \mathbf{s} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \mathbf{R}\Theta \\ 0 \end{pmatrix} \tag{6.31}$$

$$\mathbf{R} = \text{diag}(R, \ldots, R, Z), \quad \mathbf{l} = \text{diag}(l, \ldots, l, z) \tag{6.32}$$

$$\mathbf{S} = \text{diag}(S, \ldots, S), \quad \mathbf{s} = \text{diag}(s, \ldots, s) \tag{6.33}$$

$$\mathbf{H} = \Gamma^T[\text{diag}(H, \ldots, H, P)]\Gamma + \text{diag}(F, \ldots, F) \tag{6.34}$$

$$\mathbf{F} = \Theta^T[\text{diag}(H, \ldots, H, P)]\Gamma \tag{6.35}$$

6.3.4 Iterative computation of the state trajectory estimate

As previously stated, in the proposed optimization problem an state estimate $\tilde{x}_k$ is needed in intermediate steps. For a good prediction of the trajectories, it is needed that this estimate is as close as possible to the real future optimal state, $\tilde{x}_k \approx x_k^{OPT}$. However, as these future trajectories are unknown until the actual control action is computed, an iterative setup is needed in order to compute the optimal control action as well as the optimal trajectory.

To this end, Algorithm 4 below is presented. It has been implemented with end conditions considering some (application dependent) limitations on the available time ϵ_t for the computation and desired precision in the solution ϵ_x.

Algorithm 4 Iterative computation of the state estimate

1. Obtain initial estimate $\tilde{\mathbf{x}}$ from previous sampling step.
2. Solve the program $\mathcal{P}_N(x_0, \tilde{\mathbf{x}})$ obtaining u_k
3. $\tilde{x}_0^* = x_0$, $\tilde{x}_{k+1}^* = \sum_{i=1} \mu_i(\tilde{x}_k^*)(A_i\tilde{x}_k^* + B_iu_k)$ for $k = 0 \ldots N-2$
4. If $|\tilde{\mathbf{x}}^* - \tilde{\mathbf{x}}| > \epsilon_x|\tilde{\mathbf{x}}|$ and $t - t_0 < \epsilon_t$ go to step 2 with $\tilde{\mathbf{x}} := \tilde{\mathbf{x}}^*$

In Algorithm 4, the initial state sequence estimate $\tilde{\mathbf{x}}$ can be seeded to, for instance, all elements equal to x_0 (which is implicitly done in (T. Zhang et al., 2007), by “freezing” the memberships to the initial value $\mu(x_0)$).

6.3.5 Feasible region

At this point, it is important to know the set of states where the proposed problem $\mathcal{P}_N(x_0, \tilde{\mathbf{x}})$ has a solution, given an horizon N. Otherwise, Algorithm 4 may be infeasible. This feasible set, can be computed as the set of states that can reach the terminal set in N steps while holding the imposed constraints in inputs and states. Of course, the larger the horizon, the larger the resulting set would be.

A possible way to compute this feasible set is applying Algorithm 3 with horizon N and $\mathbb{T} = \mathbb{Z}$, where $\mathbb{Z}$ is the terminal set previously computed in Section 6.3.2. Now, as the input is not determined by a given "optimal" controller (only the existance of a valid input is needed), the one-step set is redefined as

$$\mathcal{Q}(\Omega) = \left\{ x \in \mathbb{R}^n | \exists u \in \mathbb{U}, \sum_{i=1}^{r} \mu_i(\tilde{x}_k)(A_i x + B_i u) \in \Omega \right\} \tag{6.36}$$

As the values of $\tilde{x}$ are uncertain, an inner approximation of this set is here proposed, which is shape-independent, i.e., valid for any possible value of μ_i:

$$\tilde{\mathcal{Q}}(\Omega) = \{ x \in \mathbb{R}^n | \exists u \in \mathbb{U}, \, A_i x + B_i u \in \Omega, \; \forall i = 1 \ldots r \} \tag{6.37}$$

and the standard algorithm is applied with the above set for a number of steps equal to the finite-horizon N.

6.3.6 Receding Horizon Optimization and Stability

The optimal controller obtained by solving problem (6.29) is implemented, as usual in MPC, in a receding-horizon strategy in which only the first action u_0 is applied and, then, a new state is measured and everything is recomputed.

Given the fact that the terminal cost verifies (6.13), using the results in (Goodwin & De Doná, 2005), assuming Algorithm 4 has converged to the optimal trajectory, then stability of the receding horizon implementation can be ensured; also, some contractive-set constraints (de Oliveira Kothare & Morari, 2000) can be additionally enforced to ensure stability even if Algorithm 4 has not converged.

This contractiveness restriction can be introduced in problem (6.29) constraints using a Lyapunov condition. In our proposal, this Lyapunov function will be a piecewise linear function, formulated in chapter 5, equation (5.10), in the form:

$$V(x) = \max_i H_i x \tag{6.38}$$

that can be easily introduced in the quadratic programming problem adding the following linear constraints

$$H_i x < V(x_0) \tag{6.39}$$

The proposed optimization problem is implemented in a receding horizon schema. Therefore at each step the quadratic program $\mathcal{P}_N(x_0, \tilde{\mathbf{x}})$ has to be solve at least once. In order to perform this optimization a value of $\tilde{\mathbf{x}}$ has to be computed. To this end, Algorithm 5 has been presented. It has been implemented with some limitations on the available time ϵ_t for the computation and precision in the solution ϵ_x.

Algorithm 5 Receding Horizon schema

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1. read x_0 and the time (t_0) from sensors
2. If $t_0 = 0 \rightarrow \tilde{x}_k = x_0$ for $k = 0 \dots N-1$ and go step 4
3. $\tilde{x}_0 = x_0$, $\tilde{x}_{k+1} = \sum_{i=1} \mu_i(\tilde{x}_k)(A_i \tilde{x}_k + B_i u_{k+1})$ for $k = 0 \dots N-2$
4. solve the program $\mathcal{P}_N(x_0, \tilde{\mathbf{x}})$ obtaining u_k
5. $\tilde{x}_0^* = x_0$, $\tilde{x}_{k+1}^* = \sum_{i=1} \mu_i(\tilde{x}_k^*)(A_i \tilde{x}_k^* + B_i u_k)$ for $k = 0 \dots N-2$
6. If $|\tilde{\mathbf{x}}^* - \tilde{\mathbf{x}}| > \epsilon_x$ and $t - t_0 < \epsilon_t$ go to step 4 with $\tilde{\mathbf{x}} := \tilde{\mathbf{x}}^*$
7. The action u_0 is applied to the system.

6.4 Example

This example will illustrate the proposed MPC methodology for a TS system

$$x_{k+1} = \sum_{i=1}^{r} \mu_i(x_k)(A_i x_k + B_i u_k) \tag{6.40}$$

with the local models and membership functions defined as (6.41)-(6.43)

$$A_1 = \begin{pmatrix} -0.9 & 0.3 \\ 0 & 0.4 \end{pmatrix} A_2 = \begin{pmatrix} 0.8 & 0.6 \\ -0.5 & 0.2 \end{pmatrix} \tag{6.41}$$

$$B_1 = \begin{pmatrix} 0.4 \\ 1.1 \end{pmatrix} B_2 = \begin{pmatrix} 0.9 \\ 0.3 \end{pmatrix} \tag{6.42}$$

$$\mu_1 = \frac{10 - x_1(k)}{20} \quad \mu_2 = 1 - \mu_1 \tag{6.43}$$

The system will be constrained in the input and states as given by

$$-1 \leq u_k \leq 1 \quad -10 \leq x_k \leq 10 \tag{6.44}$$

where state restrictions are understood as component-wise.

First, a terminal state weighting P and terminal PDC controller $u_k = \sum_{i=1}^{r} \mu_i K_i x_k$ are computed as discussed in section 6.3.1 with weighting matrices H and F being chosen as:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F = 1 \tag{6.45}$$

The obtained PDC controller gains K_i are

$$K_1 = (-0.3519 \quad 0.3136), \ K_2 = (0.3898 \quad 0.5664) \tag{6.46}$$

and the resulting terminal weighting P matrix is:

$$P = \begin{pmatrix} 8.5967 & -0.1159 \\ -0.1159 & 5.5136 \end{pmatrix} \tag{6.47}$$

Next, the terminal set is obtained following Algorithm 3 with constraints (6.48) and a Polya complexity index $d = 50$ in the computation of the inner approximation of the one-step set (6.21), with the state constraints arising from the use of the terminal controller, i.e.:

$$\begin{gathered} \begin{pmatrix} -10 \\ -10 \end{pmatrix} \leq x_k \leq \begin{pmatrix} 10 \\ 10 \end{pmatrix} \\ -1 \leq K_1 x_k \leq 1 \quad -1 \leq K_2 x_k \leq 1 \end{gathered} \tag{6.48}$$

The obtained terminal set is illustrated in Figure 6.1.

At this point all the required elements for stating the QP fuzzy predictive control problem are already available. However, it is also interesting to obtain the set of states for which the optimization problem will we feasible, i.e. the feasible set, as described in Section 6.3.5. Choosing an horizon of $N = 6$ the shape-independent feasible set presented in grey in Figure 6.1 is found.

Finally, in order to evaluate the MPC controller performance, the closed-loop trajectory from an arbitrarily chosen point $x_0 = (-1\ 8)^T$ is also shown in Figure 6.1. Algorithm 4 needs 2 iterations to find an state estimate of relative precision of $\epsilon_x = 0.1\%$. In Algorithm 4, initial $\tilde{\mathbf{x}}$ has been seeded with all elements equal to x_0, as previously discussed.

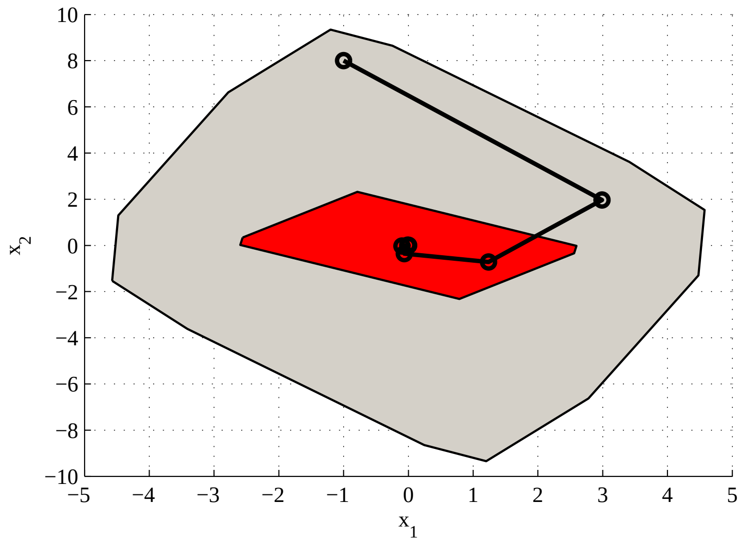

Figure 6.1: Terminal Set (red), Feasible Set (grey), and state trajectory

Additionally, time responses of the states and the control action are shown in figures 6.2 and 6.3 respectively, showing a fast convergence to the equilibrium point.

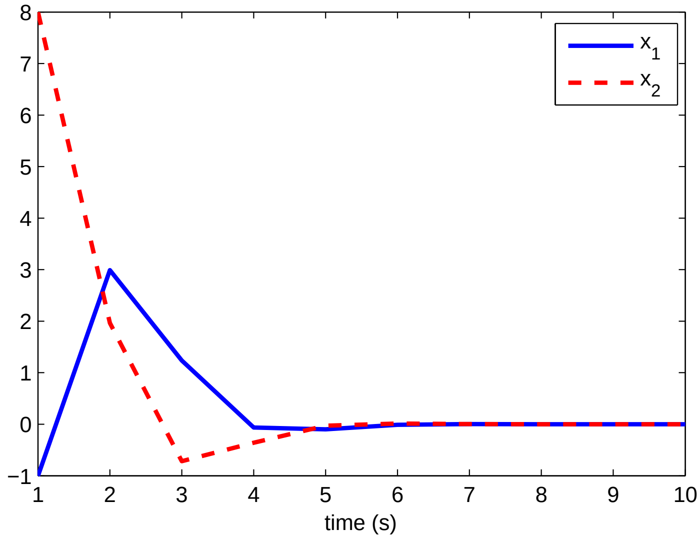

Figure 6.2: States time response.

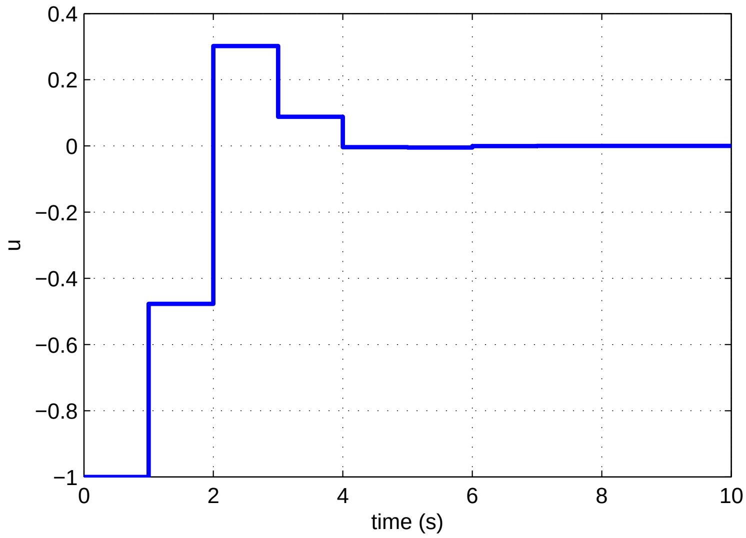

Figure 6.3: Control action time response.

The total time of a single iteration of algorithm 4 is 0.11*sec.* with a non optimized implementation, the time employed by solving the quadratic problem (6.29)-(6.30) is 0.025*sec.* As two iterations are needed, total computation time for each control action is 0.22 s[1].

Comparative analysis and discussion:

The use of the MPC approach allows improving over the performance index from the shape-independent LMI PDC controller in two ways:

First, by allowing a larger feasible zone, in which input constraints may be hit for several time steps. The standard literature controller would only be valid in its invariant set (and, actually, published fuzzy guaranteed-cost literature would only consider a Lyapunov level set inside it).

Second, even inside the terminal set, a few steps of actual optimization will beat in most cases the worst-case cost with a fixed PDC controller structure. For instance, in the example, the state $\psi = (-2, 0)$ inside the terminal set yields a computed cost of 0.1411 with the terminal controller (note that the cost bound proved with the LMIs is $\psi^T P \psi = 34.387$ as it is a shape-independent worst-case estimation –almost 250 times higher than the actual cost–), whereas the actual cost index computed with the predictive iterative controller reduces it to 0.0916 (i.e, a 35% reduction). Random trials with states in the terminal set result in a reduction between 0% and 93% over the LMI-based PDC controller. Note also that the larger the prediction horizon the less relevant the role of the terminal cost and terminal controller is, as usual in dynamic-programming based optimal control setups.

6.5 Conclusions

This chapter presents an application of predictive-control ideas to fuzzy control. The MPC algorithm follows an standard structure in which a fuzzy PDC terminal controller and terminal state weighting are calculated by LMIs. An algorithm for obtaining an inner approximation of the terminal set for this controller with a Polya-based approach is also introduced. As future memberships are unknown, an iterative quadratic

[1]Calculated by a Intel i5-7200U, RAM: 8GB, Windows 10, software: Matlab 2013 and quadprog.m function

programming procedure is proposed. Stability guarantees are also discussed.

Chapter 7

Shape-Independent Model Predictive Control for Takagi-Sugeno Fuzzy Systems

Note: The contribution of this chapter is based on the future publication:

Ariño, C., Querol, A. Sala, A.. Shape-Independent Model Predictive Control for Takagi-Sugeno Fuzzy Systems. *Submitted, under review.*

Predictive control of TS fuzzy systems has been addressed in prior literature with some unclear simplifying assumptions or heuristic approaches. This chapter presents a formulation of the model predictive control of TS systems so that results are valid for any membership value (shape-independent) with a suitable account of causality (control can depend on current and past memberships and state), and a family of progressively better controllers can be obtained by increasing Polya-related complexity parameters.

7.1 Introduction

Takagi-Sugeno (TS) systems are a widely-used tool to exactly model nonlinear systems via the so-called sector-nonlinearity approach (Tanaka & Wang, 2004); models are valid on a compact modelling region Ω. Subsequently, stability analysis and control design tasks can be carried out on the TS models. Control techniques for TS fuzzy systems based on LMIs have been deeply developed in recent years in literature, see (Guerra et al., 2015) and references therein. Of course, the TS+LMI approach is conservative with respect to an "ideal" nonlinear control approach (Sala, 2009), but it allows solving the problems via convex optimisation tools, derived from linear systems and related to the so-called linear parameter varying (LPV) approach (Mohammadpour & Scherer, 2012). As the model is only valid on a compact set, the results are also limited to this modelling region, usually the largest Lyapunov level set in Ω (slightly larger sets can be actually proven, see (Pitarch, Sala, & Ariño, 2014)) or an inescapable set in disturbance-rejection problems (Sala & Pitarch, 2016).

In realistic applications, there is always control saturation, which is not easy to handle with LMI: most conditions conservatively require the control action to avoid saturation in the outermost Lyapunov level set (Tanaka & Wang, 2004), others allow saturation (Cao & Lin, 2003) but they cannot prove improvement with respect to non-saturating laws or require iteration/Bilinear Matrix Inequalities, see (Ariño et al., 2010). Note also that if the operation point is not at the center of actuator range, constraints are non-symmetric; however, in an LMI setup constraints must be forced to be symmetric in the vast majority of cases.

A recent alternative to the LMI approach in TS systems is the geometric polytope manipulation approach in (Ariño, Pérez, Sala, & Bedate, 2014) (stability analysis) and (Ariño et al., 2017) (controller design, Chapter 5), extending well-known results in the robust-polytopic control field (Kerrigan, 2000). These results are important due to two features: first, the natural consideration of non-symmetric state and input constraints and, second, the key fact that, asymptotically, no shape-independent controller can prove a larger domain of attraction so they close the control synthesis problem with stability and decay rate performance (only in theory, as severe computational issues hinder practical implementation of Polya-based asymptotic solutions (Sala & Ariño, 2007a)).

Apart from sheer stability, optimality of a quadratic index for TS systems is a problem that has been also addressed in many works in literature in the so-called guaranteed cost setup (Tanaka & Wang, 2004; Ariño et al., 2010; Tanaka, Ohtake, & Wang, 2009; Guan & Chen, 2004); the "guaranteed cost" terminology stems from the fact that the controllers in such references only prove an "upper bound" to the actual cost. The above-referred works use a 1-step version of the Bellmann equation, i.e., $V(x_k) \geq L(x_k, u_k) + V(x_{k+1})$, being L some quadratic "step cost", to prove that V is a cost bound via some LMIs.

Obviously, the natural multi-step extension to the Bellmann equation is $V(x_k) \geq \sum_{j=1}^{N-1} L(x_k, u_k) + V(x_{k+N})$; such optimisations are the key point behind model predictive control (MPC) approaches (Camacho & Bordons, 2013). Linear MPC under quadratic stage and terminal cost are routinely solved via very efficient quadratic programming (QP, convex) optimisation and, also, there are well-known stability and optimality guarantees (Goodwin et al., 2006) emanating from classical LQR theory. In the uncertain case, the so-called minimax predictive control is addressed in the well-known references (P. Scokaert & Mayne, 1998; Löfberg, 2003; Kerrigan, 2000); state and input constraints are naturally handled in the resulting constrained optimisation setups (QP, LMI). These "uncertain" MPC approaches can, of course, apply to TS systems but, they are conservative in the same way as "robust linear" controllers are, as they do not exploit the fact that membership functions are measurable in on-line operation so control can depend on them. On the other hand, the "ideally" least-conservative framework would be directly applying nonlinear MPC (Grüne & Pannek, 2011; Diehl et al., 2002) to the original nonlinear system from which the TS model was originated; the main drawback is the fact that computational cost of nonlinear MPC is high (convexity cannot be guaranteed) and, in many cases, convergence of the optimisation iterations is, if they actually converge, to a possibly local minimum.

So, fuzzy MPC stands in the "middle ground" between the well-developed linear case and the elusive nonlinear one. However, quite a few of the fuzzy MPC earlier literature works have significant shortcomings. For instance, the widely-cited work (Sousa, Babuška, & Verbruggen, 1997) resorts to clustering but, actually, it carries out nonlinear MPC on the resulting identified fuzzy model, via nonconvex branch-and-bound optimisation. The work (Kavsek-Biasizzo et al., 1997) computes a linear MPC by "freezing" the memberships at a particular instant and assuming they will be constant in the future; this might work in practice, but it lacks theoretical justification in fast transients. The work (Y. Lu & Arkun, 2000) presents an interesting approach in which a sequence of quadratic cost bounds and state-feedback gains solves (suboptimally) the MPC problem. The great advantage is its computational tractability; however, it is well-known that even for the linear case, under constraints, the optimal value function is not quadratic in the state, so the approach is conservative. Recent works such as (Q. Lu et al., 2015) discusses networked interval type-2 systems, but their results are still based on the 1-step equation discussed above so they are not solving a “proper” multi-step problem as most MPC literature understands. For brevity in this introduction, further discussion and comparative analysis with related results[1] are presented in Section 7.6.

The objective of this chapter is adapting the linear MPC for TS fuzzy models in a rigorous way, suitably posing the problem and proposing linear matrix inequality conditions for the resulting optimisation. The key idea is introducing the shape-independent concept to the model predictive control problem formulation on TS fuzzy systems. This concept implies that the MPC is valid not only for the actual non-linear system originating a TS model, but also for any other possible realization of the membership functions. This is a source of conservatism with respect ot pure nonlinear-MPC but, in exchange, systematic LMI conditions can be posed. Importantly, the proposed controllers make use of the fact that future memberships will be measurable at the moment of computing the future control action, absent in other literature proposals.

The structure of the chapter is as follows: Section 7.2 presents preliminary results, notation and problem statement; Section 7.3 presents the main contribution on shape-independent fuzzy-MPC, prediction model and constraints; this section is followed by a stability analysis in Section 7.4, where terminal cost and controller are discussed in depth. Two brief sections present a shape-dependent MPC variation and a comparative analysis and discussion. Section 7.7 develops an example, and a conclusion section closes the chapter.

7.2 Preliminaries

Consider a nonlinear discrete-time system, given by:

$$x_{k+1} = f_x(x_k) + g(x_k)u_k \tag{7.1}$$

where $x_k \in \mathbb{R}^n$ represents the state vector, $u_k \in \mathbb{R}^m$ the control actions and $f_x(0) = 0$. The above system can be expressed in compact regions

[1]There are other related works on LMI-based suboptimal MPC for Wiener and Hammerstein models (Khani & Haeri, 2015) or for input-output LPV systems (uncertain impulse-response coefficients) (Abbas, Tóth, Meskin, Mohammadpour, & Hanema, 2016) anyway, these non-TS representations are intentionally out of the scope of the present manuscript, as well as other multi-agent/cooperative versions of MPC (Killian, Mayer, Schirrer, & Kozek, 2016).

$\mathbb{X}$, $\mathbb{U}$ containing the origin as a Takagi-Sugeno fuzzy model with r rules in the form

$$x_{k+1} = \tilde{f}(\mu, x_k, u_k) := \sum_{i=1}^{r} \mu_i(x_k)(A_i x_k + B_i u_k) \tag{7.2}$$

In this representation $\mu_i(x)$ are the membership functions which, for later manipulations, they will be grouped in the vector of memberships

$$\mu(x) := \begin{pmatrix} \mu_1(x) & \mu_2(x) & \dots & \mu_r(x) \end{pmatrix}^T \tag{7.3}$$

which belongs to the standard simplex:

$$\Delta := \left\{ \mu \in \mathbb{R}^r \mid \sum_{i=1}^{r} \mu_i = 1,\ \mu_i \geq 0 \right\} \tag{7.4}$$

The regions where system (7.1) is modelled will be assumed to be polyhedral

$$\mathbb{X} = \{x \in \mathbb{R}^n \mid Rx + r \leq 0\}, \quad \mathbb{U} = \{u \in \mathbb{R}^m \mid Su + s \leq 0\} \tag{7.5}$$

where inequalities in vectors are understood to hold element-wise.

As discussed in the introduction, well-known LMIs have been developed to synthesise state-feedback controllers for systems in the TS form (7.2). In particular, this chapter will root on the so-called guaranteed-cost results (Tanaka & Wang, 2004; Ariño et al., 2010) and will propose predictive controllers which improve on the obtained cost bounds.

7.2.1 Fuzzy predictive control: problem statement

Generic Predictive control problems (Camacho & Bordons, 2013; Goodwin et al., 2006) are based on solving finite-horizon constrained optimisation problems. Let us discuss how the predictive control problem can be formally stated in a TS framework.

First, in a TS fuzzy system, let us define a membership-dependent cost index:

$$J_\infty := \sum_{k=0}^{\infty} L(\mu(x_k), x_k, u_k) \tag{7.6}$$

In this way, some non-quadratic costs can be embedded into the framework of this chapter. Then, the fuzzy version of the finite-time problems in predictive control requires defining a N-step cost as:

$$J_N(\boldsymbol{\mu}, \mathbf{u}, x_0) := V(\mu(x_N), x_N) + \sum_{k=0}^{N-1} L(\mu(x_k), x_k, u_k) \tag{7.7}$$

where, for convenience, memberships are arranged as a matrix:

$$\boldsymbol{\mu} := \begin{pmatrix} \mu_1(x_0) & \mu_1(x_1) & \dots & \mu_1(x_N) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_r(x_0) & \mu_r(x_1) & \dots & \mu_r(x_N) \end{pmatrix} \in \mathbb{R}^{r \times (N+1)} \tag{7.8}$$

and $\boldsymbol{\mu} \in \Delta^{N+1}$ will indicate that each column belongs to Δ. The term $V(\mu(x_N), x_N)$ will be denoted as *terminal cost* and the term $L(\mu(x_k), x_k, u_k)$ as *stage* cost; $\mathbf{u}$ denotes the set of control actions $u_0, \dots, u_{N-1}$.

Generic nonlinear predictive control (NMPC, (Camacho & Bordons, 2013)) would, subsequently, try to obtain a sequence of future inputs u_j^*, $j = 0, \dots, N-1$ so that $J_N(\boldsymbol{\mu}, \mathbf{u}, x_0)$ is minimised (understanding $\boldsymbol{\mu}$ to be obtained from the simulated states under $\mathbf{u}^*$ from an initial condition x_0). The basic problem of the above generic NMPC approach is the fact that the optimisation problem is, in general, non-convex and without guaranteed convergence or execution time bound in real-time implementation, and also stability guarantees are missing.

Disregarding μ, system (7.2) can be considered as an uncertain polytopic one, so such early linear minimax results cited in the introduction can be applied to them to design robust controllers, but in such a case the knowledge of the membership functions in on-line operation is not exploited.

The objective of this chapter is providing a Polya-based asymptotically exact solution to a shape-independent version of the above predictive control problem:

$$J^*(x_0) := \min_{U} \max_{\boldsymbol{\mu} \in \Delta^{N+1}} J_N(\boldsymbol{\mu}, g(\boldsymbol{\mu}, U), x_0) \tag{7.9}$$

under suitable constraints, where $\mathbf{u} := g(\boldsymbol{\mu}, U)$ will denote a causal controller parametrisation, detailed in Section 7.3. Causality refers to the fact that the actual shape of the membership functions is unknown at design time but it *will* be known when implemented, and *future* memberships will, too, be eventually known. The proposed solution will be in LMI form.

7.2.2 Homogeneous polynomial notation

In order to solve fuzzy control problems, homogeneous polynomials in memberships are widely used (for instance, in the asymptotically exact solutions in (Sala & Ariño, 2007a), in (Ding, 2010), etc.) Considering a degree $d \in \mathbb{N}$, $d \geq 1$, and state x, all monomials of degree d in the memberships can be expressed as

$$\mu^{\alpha}(x) := \mu_1^{\alpha_1}(x)\mu_2^{\alpha_2}(x)\ldots\mu_r^{\alpha_r}(x) \tag{7.10}$$

being $\alpha \in \mathbb{N}^r$ a vector of natural numbers (understood including zero) such that $|\alpha| := \sum_{i=1}^{r} \alpha_i = d$. In the sequel, the following notation will be used to denote a degree-d homogeneous polynomial in memberships, say Ξ_{μ}^{d}, which will be expressed as:

$$\Xi_{\mu(x)}^{d} := \sum_{|\alpha|=d} \mu^{\alpha}(x) n_{\alpha} \Xi_{\alpha} \tag{7.11}$$

where $n_{\alpha}\Xi_{\alpha}$ are the polynomial coefficients, factorised as Ξ_{α} and a combinatorial number n_{α}, defined to be the multiset permutation of α in the set $|\alpha| = d$, i.e.:

$$n_{\alpha} := \begin{pmatrix} |\alpha| \\ \alpha \end{pmatrix} = \frac{(\alpha_1 + \cdots + \alpha_r)!}{\alpha_1! \ldots \alpha_r!} \tag{7.12}$$

It can be proved that:

$$\sum_{|\alpha|=d} \mu^{\alpha} n_{\alpha} = 1 \tag{7.13}$$

The above expressions follow, with some shorthand changes, the notation in (Ding, 2010). In 7.A, some results on products of homogeneous polynomials and Polya-theorem, later used, are recalled. State dependence $\mu^{\alpha}(x)$ will be shorthanded to μ^{α} if clear from the context.

Remark 1 *In order to use this notation in TS models, we will understand*

$$\sum_{|\xi|=1} \mu^{\xi} A_{\xi} := \sum_{i=1}^{r} \mu_i A_i \tag{7.14}$$

where $\xi \in \mathbb{N}^r$ is, forcedly, a vector with a single element equal to 1, being the rest zero, and we understand A_{ξ} to be equal to A_i, being i the unique index such that $\xi_i = 1$. As $n_{\xi} = 1$, we omit writing it at the left-hand side of (7.14).

Homogeneous polynomials in delayed instants The appearence of memberships of states from x_0 to x_N in the problem statement motivates extending the fuzzy summation notation to encompass different instants.

This chapter will discuss homogeneous polynomials in membership functions, evaluated at several instants of time, arranged as (7.8), so $\boldsymbol{\mu}_{ik} := \mu_i(x_k)$. The degree of the homogeneous polynomial at different time instants may differ: the homogeneous polynomial in memberships at time k will, by assumption, have degree d_k. In order to compactly handle such situation, we will introduce a degree vector $d \in \mathbb{N}^{1\times(N+1)}$, conformed with said elements d_k.

Considering now (7.11), as a different vector α will be needed for different instants, the definition of α will be generalised to being a matrix, using its k-th column to index a particular monomial at instant k: considering a matrix of natural numbers $\alpha \in \mathbb{N}^{r\times(N+1)}$, notation $\alpha_{\underline{k}}$ for $k = 0, \ldots, N$, will denote the k-th column of a matrix α, considering the first one to be indexed by zero, in order to be consistent with the idea that the degrees correspond to $\mu(x_0)$. Also, in this matrix case, $|\alpha|$ will denote the vector of dimension $1 \times (N + 1)$ formed by the column-wise sums, i.e., whose element at position j is $|\alpha_{\underline{k}}| = \sum_{i=1}^{r} \alpha_{ik}$.

Now, notation $\boldsymbol{\mu}^\alpha$ will represent a monomial in the membership functions (at different instants of time), given by:

$$\boldsymbol{\mu}^\alpha := \Pi_{k=0}^{N} \mu^{\alpha_{\underline{k}}}(x_k) = \Pi_{i=1}^{r} \Pi_{k=0}^{N} \mu_i^{\alpha_{ik}}(x_k) \tag{7.15}$$

and n_α is

$$n_\alpha := \prod_{j=0}^{N} n_{\alpha_{\underline{k}}} = \prod_{j=0}^{N} \frac{|\alpha_{\underline{k}}|!}{\prod_{i=1}^{r} \alpha_{ik}!} \tag{7.16}$$

It can be proved in a straightforward way that $\sum n_\alpha \boldsymbol{\mu}^\alpha = 1$.

The generalisation of (7.11) to the multiple-instant setting will be denoted by:

$$\Xi_{\boldsymbol{\mu}}^{d} := \sum_{|\alpha|=d} \boldsymbol{\mu}^\alpha n_\alpha \Xi_\alpha \tag{7.17}$$

where d will now be a degree *vector*, $d \in \mathbb{N}^{N+1}$ indicating the degree at each instant of time of the overall fuzzy summation.

Example 7.1 *Consider a prediction horizon $N = 3$ and two rules, $r = 2$. Example membership monomials for some degree matrices α are, say:*

$$\alpha = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix} \Rightarrow \boldsymbol{\mu}^\alpha = (\mu_1(x_0))^3(\mu_2(x_0))^4, \quad n_\alpha = \tfrac{7!}{3!4!}\tfrac{0!}{0!0!}\tfrac{0!}{0!0!} = 35$$
$$|\alpha| = (7,0,0), \ |\alpha_{\underline{1}}| = 7, \ |\alpha_{\underline{2}}| = |\alpha_{\underline{3}}| = 0$$

$$\alpha = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \Rightarrow \boldsymbol{\mu}^\alpha = (\mu_1(x_0))^2\mu_1(x_1)(\mu_2(x_2))^4, \quad n_\alpha = 1$$
$$|\alpha| = (2,1,4), \ |\alpha_{\underline{1}}| = 2, \ |\alpha_{\underline{2}}| = 1, \ |\alpha_{\underline{3}}| = 4$$

7.3 Shape-independent predictive control

Let us now completely state the shape-independent problem (7.9) rooting from (7.7) and its associated constraints.

At the moment of computing u_k in (7.7), memberships $\mu(x_0), \ldots \mu(x_k)$ will be known; hence, we will search over controller depending on these memberships, i.e., $u_k := g_k(\mu(x_0), \ldots, \mu(x_k), U_k)$, where U_k is a vector of numeric parameters. Juxtaposing all controls, defining the set of control laws $\mathbf{u} = \{u_1, u_2, \ldots, u_{N-1}\}$, and the set of numeric decision variables $U := \{U_0, \ldots, U_{N-1}\}$, where U_k are those associated to control u_k at instant k, we will express them with the shorthand $\mathbf{u} := g(\boldsymbol{\mu}, U)$. A homogeneous polynomial structure for g and U will be detailed shortly below.

Under the above causality constraints, then, the shape-independent fuzzy MPC problem would amount to computing:

$$J^*(x_0) := \min_U \max_{\boldsymbol{\mu}\in\Delta^{N+1}} J_N(\boldsymbol{\mu}, g(\boldsymbol{\mu}, U), x_0) \tag{7.18}$$

subject to, for $k = 0, \ldots N-1$,

$$x_{k+1} = \tilde{f}(\mu, x_k, g_k(\mu(x_0), \ldots, \mu(x_k), U_k)) \tag{7.19}$$
$$g_k(\mu(x_0), \ldots, \mu(x_k), U_k) \in \mathbb{U} \tag{7.20}$$
$$x_k \in \mathbb{X} \tag{7.21}$$
$$x_N \in \mathbb{T} \tag{7.22}$$

where $\mathbb{T}$ is a so-called terminal set (see Section 7.4 for details on how to obtain it to guarantee stability).

Causal fuzzy controller parametrisation As future values of membership functions are unknown, the control action cannot depend on them, as discussed in earlier sections. So, u_0 will be chosen to be an homogeneous polynomial in $\mu(x_0)$ with, say, degree c_{00}, u_1 will be a polynomial in $\mu(x_0)$ and $\mu(x_1)$, with degrees c_{10} and c_{11} in $\mu(x_0)$ and $\mu(x_1)$, respectively, and so on with $u_2, \ldots, u_{N-1}$.

We will introduce notacion $c^{[0]} = (c_{00}, 0, 0, 0)$, $c^{[1]} = (c_{10}, c_{11}, 0, 0)$, ... so c_{kl} indicates the degree of u_k in $\mu(x_l)$, for $l \leq k$, $0 \leq k \leq N$. Based on the above discussion, we will consider a control action to be applied to the TS system (7.2) to be given by:

$$u_k = \sum_{|\alpha|=c^{[k]}} \boldsymbol{\mu}^\alpha n_\alpha u_{\alpha,k} \tag{7.23}$$

where $u_{\alpha,k}$ are control decision variables conforming U_k, and $c^{[k]} \in \mathbb{N}^N$ is a user-defined degree vector, with the above-discussed causality constraints (i.e., elements $k+1$ to N equal to zero).

7.3.1 Prediction model

1-step closed-loop model With the above control law (7.23), at time k, the closed-loop successor state would be:

$$x_{k+1} = \sum_{i=1}^{r} \sum_{|\alpha|=c^{[k]}} \boldsymbol{\mu}^\alpha \cdot \mu_i \cdot n_\alpha (A_i x_k + B_i u_{\alpha,k}) \tag{7.24}$$

We will vectorise all decision variables in $u_{\alpha,k}$, i.e., stacking them in a column vector U_k defined as:

$$U_k := vec_{|\alpha|=c^{[k]}}(u_{\alpha,k}) \tag{7.25}$$

using any arbitrary enumeration ordering for α, for instance lexicographic. Conversely, we will invert the vectorisation using notation $u_{\alpha,k} = E_{\alpha,k} U_k$ being $E_{\alpha,k}$ the matrix that selects the suitable vector elements, according to the chosen order in the vectorisation operation. With this notation, the input at instant k can be written as:

$$u_k = \sum_{|\alpha|=c^{[k]}} n_\alpha \boldsymbol{\mu}^\alpha E_{\alpha,k} U_k \tag{7.26}$$

Hence,

$$x_{k+1} = \sum_{i=1}^{r} \sum_{|\alpha|=c^{[k]}} \boldsymbol{\mu}^{\alpha} \mu_i(x_k) n_\alpha (A_i x_k + B_i E_{\alpha,k} U_k) \tag{7.27}$$

With suitable manipulations, using Corollary 7.1 and, if so wished Polya expansion (Corollary 7.2), we can express the closed-loop model (details omitted for brevity) as:

$$x_{k+1} = \sum_{|\gamma|=q^{[k]}} \boldsymbol{\mu}^{\gamma} n_\gamma G_{\gamma,k} \begin{pmatrix} x_k \\ U_k \end{pmatrix} \tag{7.28}$$

where, $q^{[k]} \geq c^{[k]} + e_k$, and

$$G_{\gamma,k} := \frac{1}{n_\gamma} \sum_{\alpha,\xi \in \mathcal{S}_\gamma^{c^{[k]} e_k}} n_\alpha n_{\gamma-\alpha-\xi} \begin{pmatrix} A_\xi & B_\xi E_{\alpha,k} \end{pmatrix} \tag{7.29}$$

with $\mathcal{S}_\gamma^{c^{[k]} e_k}$ defined according to (7.82) as

$$\mathcal{S}_\gamma^{c^{[k]} e_k} = \{\alpha, \xi \in \mathbb{N}^{r\times(N+1)} \mid |\alpha| = c^{[k]},\ |\xi| = e_k, \gamma - \alpha - \xi \geq 0\} \tag{7.30}$$

and e_k denotes the vector whose elements are all zero except the k-th one, equal to 1. Abusing the notation introduced in Remark 1, A_ξ, when indexed by a matrix ξ such that $|\xi| = e_k$, should be understood as the consequent matrix A_i, being i the row number at k-th column of the single element of ξ equal to 1.

Note also that, if we choose $q^{[k]} = c^{[k]} + e_k$ then no Polya expansion is carried out; for any other larger choice of elements of $q^{[k]}$ the expression of the Polya expansion in Corollary 7.2 is implicitly considered in (7.28).

Example 7.2 *For instance, in a TS model with $r = 2$, $N = 3$, let us consider predicting x_2 from x_1 with $c^{[1]} = (2, 2, 0)$, which respects causality (u_1 cannot depend on $\mu(x_2)$). As $e_1 = (0, 1, 0)$, the above expression (7.28) with no Polya expansion would entail $q^{[1]} = (2, 3, 0)$. Let us, for instance, show the element*

$$G_{\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix},1} = \frac{1}{6} \left(4 \begin{bmatrix} A_1 & B_1 E_{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix},1} \end{bmatrix} + 2 \begin{bmatrix} A_2 & B_2 E_{\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix},1} \end{bmatrix} \right)$$

being E suitable selection matrices $(0, \ldots, 0, I, 0, \ldots, 0)$.

Prediction along the full horizon In MPC, from an initial state x_0, predictions of x_1, x_2, $\dots x_N$ must be made. So, the above 1-step ahead prediction must be nested, and expression (7.28) generalised.

For instance, given x_0, we could predict x_2 with an homogeneous controller of degree vector $c^{[0]} := (c_{00}\ 0\dots 0)$ at instant 0, and a controller which, at instant 1, can depend on memberships at instants 0 and 1, with degrees $c^{[1]} := (c_{10}\ c_{11}\ 0\dots 0)$, the prediction of x_2, without any Polya expansion, would be:

$$\begin{aligned} x_2 &= \sum_{i=1}^{r} \mu_i(x_1) \left(A_i x_1 + B_i \left(\sum_{|\beta|=c^{[1]}} \boldsymbol{\mu}^{\beta} n_\beta E_{\beta,1} U_1 \right)\right) \\ &= \sum_{i=1}^{r} \mu_i(x_1) \left(A_i \left(\sum_{j=1}^{r} \mu_j(x_0) \left(A_j x_0 + B_j \left(\sum_{|\alpha|=c^{[0]}} \boldsymbol{\mu}^{\alpha} n_\alpha E_{\alpha,0} U_0 \right)\right)\right) \right. \\ &\qquad \left. + B_i \left(\sum_{|\beta|=c^{[1]}} \boldsymbol{\mu}^{\beta} n_\beta E_{\beta,1} U_1 \right)\right) \end{aligned} \tag{7.31}$$

Following similar steps, predictions of x_3, $\dots x_N$ can be crafted as follows:

In expression (7.31) and in those of larger horizon, products of model matrices at different instants appear in convolution-like formulae. In order to set a compact notation for such products, let us consider the product of matrices from instant j to k. The auxiliary degree vector $\mathbf{1}_{jk}$, used in (7.34) and (7.35) below, will be defined as the vector whose components j to k are equal to 1, being the rest equal to zero. Thus, matrices ξ such that $|\xi| = \mathbf{1}_{jk}$ are those in which a single element is equal to one in each of the columns from j to k. Now, for such ξ, we will define:

$$A_\xi := \prod_{l=j}^{k} A_{\underline{\xi_l}} \tag{7.32}$$

Example 7.3 *Considering an index matrix* $|\xi| = \mathbf{1}_{23}$*, in a 2-rule model, we would have, say:*

$$A_{\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}} = A_2 A_1, \qquad A_{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}} = A_2^2$$

Note that in the model predictions, the above matrix would be multiplied by $\mu_2(x_3)\mu_1(x_2)$ (left) and $\mu_2(x_3)\mu_2(x_2)$ (right).

With the above notations, the classical convolution formula in linear MPC $x_k = A^k x_0 + \sum_{j=0}^{k-1} A^{k-(j+1)} B u_j$ is extended to the TS case as follows:

Theorem 7.1 *For any degree vector $q \in \mathbb{N}^{N+1}$, for $q_j \geq \max_i(c_{jj} + 1, c_{ij})$, $j = 0, \ldots, N-1$, the prediction of x_{k+1} as a function of x_0 and future controls in the form* (7.26) *can be expressed as:*

$$x_{k+1} = \sum_{|\gamma|=q} \boldsymbol{\mu}^{\gamma} n_{\gamma} \Xi_{\gamma} \tag{7.33}$$

being

$$\Xi_{\gamma} \;:=\; \frac{1}{n_{\gamma}} \left(\sum_{\xi \in \mathcal{S}_{\gamma}^{\mathbf{1}0k}} n_{\gamma-\xi} A_{\xi} x_0 + \sum_{j=0}^{k} \sum_{\alpha,\xi \in \mathcal{D}_{\gamma}^{j,k}} n_{\gamma-\alpha-\xi} n_{\alpha} A_{\xi} B_{\beta} E_{\alpha,j} U_j \right) \tag{7.34}$$

where $\mathcal{S}_{\gamma}^{\mathbf{1}0k}$ is built as defined in (7.79)*, and:*

$$\mathcal{D}_{\gamma}^{j,k} := \{\alpha, \xi, \beta \in \mathbb{N}^{r \times (N+1)}, \,|\, |\alpha| = c^{[j]}, |\xi| = \mathbf{1}_{(j+1)k}, |\beta| = e_j, \gamma - \alpha - \xi - \beta \geq 0\} \tag{7.35}$$

Proof: Carried out by exhaustively repeating the analogous operations to (7.31) for x_3, etc., omitted for brevity because of the tedious nature of the developments. ∎

Finally, the k-step prediction can be written as

$$x_{k+1} = \sum_{|\gamma|=q} \boldsymbol{\mu}^{\gamma} n_{\gamma} G_{\gamma,k} \begin{pmatrix} x_0 \\ U \end{pmatrix} \tag{7.36}$$

been $U = (U_0^T, \ldots, U_{N-1}^T)^T$ and $G_{\gamma,k}$ is formulated by extracting out x_0 and U from Ξ_{γ} in (7.34), in order to express (7.33) in matrix form (7.36).

Hence, juxtaposing $G_{\gamma,k}$ in column form, resulting in a matrix to be denoted as G_γ, the full prediction from $k = 1$ up to $k = N$ can be expressed as follows:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \sum_{|\gamma|=q} \boldsymbol{\mu}^\gamma n_\gamma G_\gamma \begin{pmatrix} x_0 \\ U \end{pmatrix} \tag{7.37}$$

7.3.2 Constraints on decision variables

This section discusses how to enforce the constraints on states and inputs. Indeed, from prediction model (7.37), future states depend on x_0 and U; thus, constraints (7.20)–(7.22) must be formulated as constraints on decision variables U.

Input constraints Carrying out a Polya expansion, if so wished of (7.26), the input at instant k can expressed, taking any degree vector $h \in \mathbb{R}^N$, so that $h_j \geq \max_i(c_{jj}, c_{ij}),\ j = 0, \ldots, N-1$, as:

$$u_k = \sum_{|\alpha|=c^{[k]}} n_\alpha \boldsymbol{\mu}^\alpha E_{\alpha,k} U_k = \sum_{|\gamma|=h} n_\gamma \boldsymbol{\mu}^\gamma W_{\gamma,k} U \tag{7.38}$$

where $W_{\gamma,k}$, from (7.84), is defined as:

$$W_{\gamma,k} := \frac{1}{n_\gamma} \sum_{\alpha \,\in\, \mathcal{S}_\gamma^{c^{[k]}}} n_{\gamma-\alpha} n_\alpha \left(0 \ \ldots \ 0 \ E_{\alpha,k} \ 0 \ \ldots \ 0\right) \tag{7.39}$$

At this point, as (7.38) is an homogeneous polynomial, in order to enforce $u_k \in \mathbb{U}\ k = 0 \ldots N$, as required by constraint (7.20), we formulate the constraint on the polynomial coefficients, in terms of the decision variables U, using matrix S and vector s in (7.5) as:

$$S \cdot W_{\gamma,k} \cdot U + s \leq 0 \quad \forall\ |\gamma| = h,\ k = 0, \ldots, N-1 \tag{7.40}$$

indeed, if the above holds, as $\sum n_\alpha \boldsymbol{\mu}^\alpha = 1$, we have

$$S u_k + s = \sum_{|\gamma|=h} n_\gamma \boldsymbol{\mu}^\gamma (S W_{\gamma,k} U + s) \leq 0$$

Finally, juxtaposing $W_{\gamma,k}$ in column form, as a matrix to be denoted as W_γ, the full prediction inputs from $k=1$ to $k=N-1$ can be expressed as follows:

$$\mathbf{u} = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} = \sum_{|\gamma|=h} n_\gamma \boldsymbol{\mu}^\gamma W_\gamma U \tag{7.41}$$

Future state constraints In a similar way (details omitted for brevity), the condition $x_k \in \mathbb{X}$, for $k = 1 \dots N-1$, required in (7.21), is expressed in term of the decision variables U and the current state values x_0, for any Polya expansion of degree h such that $h_j \geq \max_i(c_{jj}+1, c_{ij}),\ j = 0, \dots, N-1$, as the sufficient condition

$$R \cdot \frac{1}{n_\gamma} \sum_{\alpha\,\in\,\mathcal{S}_\gamma^q} n_{\gamma-\alpha} n_\alpha G_{\alpha,k} \begin{pmatrix} x_0 \\ U \end{pmatrix} + r \leq 0 \quad \forall\ |\gamma| = h,\ k = 0, \dots, N-2 \tag{7.42}$$

Finally, for stability reasons to be later discussed, the terminal set must be defined as:

$$\mathbb{T} := \{x : x_N^T \left(P_\mu^c\right)^{-1} x_N \leq \lambda^{-1}\ \forall \mu \in \Delta\} \tag{7.43}$$

being P_μ^c a homogeneous polynomial of degree c, following notation (7.17). Taking the prediction model (7.28) for x_N:

$$x_N = \sum_{|\alpha|=q} \boldsymbol{\mu}^\alpha n_\alpha G_{\alpha,N-1} \begin{pmatrix} x_0 \\ U \end{pmatrix} \tag{7.44}$$

and applying well-kwown Schur-complement manipulations, we have that $x_N \in \mathbb{T}$, required in (7.22), is equivalent to the matrix inequality:

$$\begin{pmatrix} \lambda^{-1} & \sum_{|\alpha|=q} \boldsymbol{\mu}^\alpha n_\alpha \left(G_{\alpha,N-1} \begin{pmatrix} x_0 \\ U \end{pmatrix} \right)^T \\ \sum_{|\alpha|=q} \boldsymbol{\mu}^\alpha n_\alpha G_{\alpha,N-1} \begin{pmatrix} x_0 \\ U \end{pmatrix} & \sum_{|\sigma|=l} \boldsymbol{\mu}^\sigma n_\sigma P_{\sigma_{\underline{N}}} \end{pmatrix} \geq 0$$

$$l = (0, 0, \dots, 0, c) \tag{7.45}$$

If a Polya expansion of the polynomials above is done up to degree vector $h \geq q,\ h \geq l$, the following (asymptotically exact) sufficient conditions for $x_N \in \mathbb{T}$ in LMI form are obtained, requiring the coefficients of the referred expansion to be positive semidefinite:

$$\sum_{\substack{\alpha \in \mathcal{S}_\gamma^q \\ \sigma \in \mathcal{S}_\gamma^l}} \begin{pmatrix} n_\gamma \lambda^{-1} & n_{\gamma-\alpha} n_\alpha \left(G_{\alpha,N-1} \begin{pmatrix} x_0 \\ U \end{pmatrix} \right)^T \\ n_{\gamma-\alpha} n_\alpha G_{\alpha,N-1} \begin{pmatrix} x_0 \\ U \end{pmatrix} & n_{\gamma-\sigma} n_\sigma P_{\sigma_{\underline{N}}} \end{pmatrix} \geq 0$$
$$\forall\ |\gamma| = h,\ l = (0, 0, \ldots, 0, c),\ h \geq q,\ h \geq l \quad (7.46)$$

7.3.3 LMI formulation of the model predictive control problem

Considering the cost index in (7.7), in order to cast the MPC problem as LMI, the k-step cost $L(\mu(x_k), x_k, u_k)$ is defined as the following expression, quadratic in the state and inputs, but involving homogeneous polynomials of degree d in the involved matrices:

$$L(\mu(x_k), x_k, u_k) = x_k^T \left(H^d_{\mu(x_k)} \right)^{-1} x_k + u_k^T \left(F^d_{\mu(x_k)} \right)^{-1} u_k \qquad (7.47)$$

and the terminal cost $V(\mu(x_N), x_N)$ is defined as:

$$V(\mu(x_N), x_N) = x_N^T \left(P^c_{\mu(x_N)} \right)^{-1} x_N \qquad (7.48)$$

We are now in conditions of presenting the main result of this chapter:

Theorem 7.2 *Given x_0 and the degree-q prediction model* (7.37)*, the cost index J^* in* (7.9) *fulfills $J^*(x_0) \leq \delta^*$ if, for a degree vector $h \in \mathbb{N}^{N+1}$, $h \geq q$, $h \geq f$, $h \geq l$ –see* (7.54) *below for definitions of f and l–, the following LMI optimisation is feasible:*

$$\delta^* := \min_U \delta \tag{7.49}$$

subject to:

$$\sum_{\substack{\alpha\,\in\,\mathcal{S}^q_\gamma\\ \sigma\,\in\,\mathcal{S}^f_\gamma}} \begin{pmatrix} n_\gamma\delta & \begin{pmatrix} x_0^T & U^T \end{pmatrix} n_{\gamma-\alpha}n_\alpha G_\alpha^T & U^T n_{\gamma-\alpha}n_\alpha W_\alpha^T \\ * & n_{\gamma-\sigma}n_\sigma diag(H_{\sigma_{\underline{1}}},\ldots,H_{\sigma_{\underline{N-1}}},P_{\sigma_{\underline{N}}}) & 0 \\ * & 0 & n_{\gamma-\sigma}n_\sigma diag(F_{\sigma_{\underline{0}}},\ldots,F_{\sigma_{\underline{N-1}}}) \end{pmatrix} > 0 \tag{7.50}$$

$$R\frac{1}{n_\gamma}\sum_{\alpha\,\in\,\mathcal{S}^q_\gamma} n_{\gamma-\alpha}n_\alpha G_{\alpha,k}\begin{pmatrix} x_0 \\ U\end{pmatrix} + r \le 0 \quad k = 0,\ldots,N-2 \tag{7.51}$$

$$\sum_{\substack{\alpha\,\in\,\mathcal{S}^q_\gamma\\ \sigma\,\in\,\mathcal{S}^l_\gamma}} \begin{pmatrix} n_\gamma\lambda^{-1} & n_{\gamma-\alpha}n_\alpha\left(G_{\alpha,N-1}\begin{pmatrix} x_0 \\ U\end{pmatrix}\right)^T \\ n_{\gamma-\alpha}n_\alpha G_{\alpha,N-1}\begin{pmatrix} x_0 \\ U\end{pmatrix} & n_{\gamma-\sigma}n_\sigma P_{\sigma_{\underline{N}}} \end{pmatrix} \ge 0 \tag{7.52}$$

$$S\frac{1}{n_\gamma}\sum_{\alpha\,\in\,\mathcal{S}^q_\gamma} n_{\gamma-\alpha}n_\alpha W_{\alpha,k}U + s \le 0 \quad k = 0,\ldots,N-1 \tag{7.53}$$

$$\forall\ |\gamma| = h,\ f = (d,d,\ldots,d,c),\ l = (0,0,\ldots,0,c) \tag{7.54}$$

Proof: Considering the index J_N in (7.7) and (7.9), let us prove that

$$\delta - J_N(\boldsymbol{\mu},\mathbf{u},x_0) > 0,\ \forall\boldsymbol{\mu}\in\Delta^{N+1} \tag{7.55}$$

if the LMI conditions in the theorem hold, once u is parametrised as (7.26). Indeed, replacing in J_N the prediction model (linear in x_0 and U), a quadratic expression in control decision variables U arises. In order to remove the quadratic dependence on the decision variables U in (7.55), the Schur complement is applied, resulting in the condition:

$$\begin{pmatrix} \delta & \begin{pmatrix} x_0^T & U^T \end{pmatrix}\sum_{|\alpha|=q}\boldsymbol{\mu}^\alpha n_\alpha G_\alpha^T & U^T\sum_{|\alpha|=q}\boldsymbol{\mu}^\alpha n_\alpha W_\alpha^T \\ * & \sum_{|\sigma|=f}\boldsymbol{\mu}^\sigma n_\sigma diag(H_{\sigma_{\underline{1}}},\ldots,H_{\sigma_{\underline{N-1}}},P_{\sigma_{\underline{N}}}) & 0 \\ * & 0 & \sum_{|\sigma|=f}\boldsymbol{\mu}^\sigma n_\sigma diag(F_{\sigma_{\underline{0}}},\ldots,F_{\sigma_{\underline{N-1}}}) \end{pmatrix} > 0 \tag{7.56}$$

According to Definition 7.1 and Proposition 7.2, the degree of the polynomial (7.56) can be extended to any complexity parameter h, being $h \ge f$ and $h \ge q$. Thus, the Polya expansion of (7.56) yields:

$$\sum_{|\gamma|=h} \boldsymbol{\mu}^{\gamma} \begin{pmatrix} n_\gamma \delta & \begin{pmatrix} x_0^T & U^T \end{pmatrix} \sum_{\alpha \in \mathcal{S}_\gamma^q} n_{\gamma-\alpha} n_\alpha G_\alpha^T & U^T \sum_{\alpha \in \mathcal{S}_\gamma^q} n_{\gamma-\alpha} n_\alpha W_\alpha^T \\ * & \sum_{\sigma \in \mathcal{S}_\gamma^f} n_{\gamma-\sigma} n_\sigma diag(H_{\sigma_{\underline{1}}}, \ldots, H_{\sigma_{\underline{N-1}}}, P_{\sigma_{\underline{N}}}) & 0 \\ * & 0 & \sum_{\sigma \in \mathcal{S}_\gamma^f} n_{\gamma-\sigma} n_\sigma diag(F_{\sigma_{\underline{0}}}, \ldots, F_{\sigma_{\underline{N-1}}}) \end{pmatrix} > 0 \tag{7.57}$$

As all μ_k are positive, the inequality (7.57) will hold if the inequalities (7.50) hold; henceforth, δ will be an upper bound of the cost index J_N. Note that it is also needed that future states belong to $\mathbb{X}$, future control actions must lie in $\mathbb{U}$ and the state at instant N must be driven inside the terminal set. In order to ensure that, the constraints (7.51)–(7.53) have been added to the problem as sufficient conditions to constraints (7.20)–(7.22) in Section 7.3.2. Note that (7.51) and (7.53) are element-wise (scalar) constraints instead of full matrix inequalities. ■

Feasible set of initial conditions For fixed horizon, terminal cost and terminal sets, the set of x_0 yielding feasible LMIs (7.50)–(7.54) in Theorem 7.2 (if x_0 were now considered to be a decision variable) is a convex LMI set. Such a set will be denoted as *shape-independent feasible set.*

Note that, as the terminal set is an intersection of ellipsoids, the feasible set will not be polytopic, so considerations in Section 6.3.5 do not apply. Also, the terminal controller (see next section) is neither the PDC one in (Ariño, Pérez, Sala, & Bedate, 2014), (used in Section 6.3.2 of this thesis) nor the one achieving contraction in the largest possible set in Section 5.5.4. Thus, these earlier polyhedral approaches do not apply to the setting under study now.

7.4 Stability

As in classical predictive control, in order to prove stability, the terminal set and terminal costs must be computed assuming there is a so-called terminal controller which guarantees that the infinite cost (7.6) is bounded by the terminal cost. Let us discuss the details.

7.4.1 Terminal Controller

Theorem 7.3 *The system* (7.2) *is stabilizable and the cost index* J_∞ *in* (7.6)*, with step cost* (7.47)*, is bounded by* $V(\mu(x_0), x_0) = x_0^T \left(P^c_{\mu(x_0)}\right)^{-1} x_0$*, being* $c \in \mathbb{N}$ *a degree parameter, if there exist matrices* P_α*, and* K_α*, for all* $|\alpha| = c$*, such that the following LMIs are feasible:*

$$\sum_{\substack{\alpha,\xi \in \mathcal{S}^{c1}_\gamma \\ \sigma \in \mathcal{S}^d_\gamma}} \begin{pmatrix} n_\alpha n_{\gamma-\alpha} P_\alpha & n_\alpha n_{\gamma-\alpha-\xi}(A_\xi P_\alpha - B_\xi K_\alpha)^T & n_\alpha n_{\gamma-\alpha} K_\alpha^T & n_\alpha n_{\gamma-\alpha} P_\alpha \\ * & n_\gamma P_\theta & 0 & 0 \\ * & 0 & n_\sigma n_{\gamma-\sigma} F_\sigma & 0 \\ * & 0 & 0 & n_\sigma n_{\gamma-\sigma} H_\sigma \end{pmatrix} > 0 \tag{7.58}$$

for all $\gamma \in \mathbb{N}^r$ *such that* $|\gamma| = q$*, with any arbitrarily chosen* $q \geq max(c+1, d)$*, and all* $\theta \in \mathbb{N}^r$ *such that* $|\theta| = c$*, being* S^c_γ*,* S^d_γ *and* $\mathcal{S}^{c1}_\gamma$ *defined in* (7.79) *and* (7.82)*, and the control action is:*

$$u(x) = K^c_{\mu(x)} (P^c_{\mu(x)})^{-1} x \tag{7.59}$$

Optimal controller It is straightforward to check that adding the condition

$$\delta I \;<\; \frac{1}{n_\gamma} \sum_{\alpha \in \mathcal{S}^c_\gamma} n_{\gamma-\alpha} n_\alpha P_\alpha \tag{7.60}$$

and minimising δ would guarantee that $J_\infty \leq \delta^* x_0^T x_0$, being δ^* the optimal solution.

Proof: Let us prove that $J_\infty \leq V(x_0)$, when using controller $u_k = K^c_\mu (P^c_\mu)^{-1} x_k$.

As widely known, the theorem will be proved if the one-step Bellmann equation below holds:

$$x_k^T (P^c_{\mu(x_k)})^{-1} x_k - x_{k+1}^T (P^c_{\mu(x_{k+1})})^{-1} x_{k+1} - \left(x_k^T (H^d_{\mu(x_k)})^{-1} x_k + u_k^T (F^d_{\mu(x_k)})^{-1} u_k\right) > 0 \tag{7.61}$$

Considering that $x_{k+1} = \sum_{i=1}^{r} \mu_i \left(A_i + B_i K^c_{\mu(x_k)} \left(P^c_{\mu(x_k)} \right)^{-1} \right) x_k$, the previous inequality is positive, for any x_k, if the next matrix is positive definite:

$$\begin{aligned} &\left(P^c_{\mu(x_k)}\right)^{-1} \\ &-\left(\sum_{i=1}^{r} \mu_i \left(A_i + B_i K^c_{\mu(x_k)} \left(P^c_{\mu(x_k)}\right)^{-1}\right)\right)^T \left(P^c_{\mu(x_{k+1})}\right)^{-1} \left(\sum_{i=1}^{r} \mu_i \left(A_i + B_i K^c_{\mu(x_k)} \left(P^c_{\mu(x_k)}\right)^{-1}\right)\right) \\ &-\left(H^d_{\mu(x_k)}\right)^{-1} - \left(K^c_{\mu(x_k)} \left(P^c_{\mu(x_k)}\right)^{-1}\right)^T \left(F^d_{\mu(x_k)}\right)^{-1} K^c_{\mu(x_k)} \left(P^c_{\mu(x_k)}\right)^{-1} > 0 \end{aligned} \tag{7.62}$$

In order to remove the dependence on $(P^c_{\mu(x_k)})^{-1}$, we need to multiply the expression by a congruence with matrix $P^c_{\mu(x_k)}$

$$\begin{aligned} &P^c_{\mu(x_k)} \\ &-\left(\sum_{i=1}^{r} \mu_i \left(A_i P^c_{\mu(x_k)} + B_i K^c_{\mu(x_k)}\right)\right)^T \left(P^c_{\mu(x_{k+1})}\right)^{-1} \left(\sum_{i=1}^{r} \mu_i \left(A_i P^c_{\mu(x_k)} + B_i K^c_{\mu(x_k)}\right)\right) \\ &-\left(P^c_{\mu(x_k)}\right)^T \left(H^d_{\mu(x_k)}\right)^{-1} \left(P^c_{\mu(x_k)}\right) - \left(K^c_{\mu(x_k)}\right)^T \left(F^d_{\mu(x_k)}\right)^{-1} \left(K^c_{\mu(x_k)}\right) > 0 \end{aligned} \tag{7.63}$$

and by applying the Schur complement,

$$\begin{pmatrix} \sum\limits_{|\alpha|=c} n_\alpha \mu^\alpha P_\alpha & \sum\limits_{|\alpha|=c} \sum\limits_{i=1}^{r} n_\alpha \mu^\alpha \mu_i (A_i P_\alpha - B_i K_\alpha)^T & \sum\limits_{|\alpha|=c} n_\alpha \mu^\alpha K_\alpha^T & \sum\limits_{|\alpha|=c} n_\alpha \mu^\alpha P_\alpha \\ * & \sum\limits_{|\theta|=c} n_\theta \mu^\theta (x_{k+1}) P_\theta & 0 & 0 \\ * & 0 & \sum\limits_{|\sigma|=d} n_\sigma \mu^\sigma F_\sigma & 0 \\ * & 0 & 0 & \sum\limits_{|\sigma|=d} n_\sigma \mu^\sigma H_\sigma \end{pmatrix} > 0 \tag{7.64}$$

being $\mu^\alpha = \mu^\alpha(x_k)$. Subsequently, A degree q Polya expansion (7.84) to (7.64) can be done. The coefficients of resulting expanded polynomial are $L_{\gamma,\theta}$ which correspond to powers $\mu^{\gamma}(x_k)$ and $\mu^{\theta}(x_{k+1})$. As condition (7.58) states that all coefficients are positive definite, then the polynomial $\sum_{\gamma,\theta}\mu(x_k)^{\gamma}\mu(x_{k+1})^{\theta}n_{\theta}L_{\gamma,\theta}$ is positive (Theorem 7.6). ■

7.4.2 Computation of terminal sets.

Consider the terminal set defined in (7.43). The developments below will discuss how to obtain a bound on λ such that $\mathbb{T}\subset\mathbb{X}$ and the control action (7.59) is admissible, i.e., $u\in\mathbb{U}$, for all $x\in\mathbb{T}$. Note that, as $\mathbb{T}$ is a subset of every Lyapunov level set[2], once $\mathbb{T}$ is entered, the state trajectory will never leave $\mathbb{T}$ in the future under the terminal control law computed with LMIs in Theorem 7.3.

Theorem 7.4 *For a TS fuzzy system* (7.2)*, with input* (7.59)*, the terminal set* $\mathbb{T}$ *verifies* $\mathbb{T}\subset\mathbb{X}$ *and* $u(x)\in\mathbb{U}$ *for all* $x\in\mathbb{T}$*, if* λ *is the minimum positive scalar such that, for all* $|\gamma|=q$*, being* $q\geq c$*,*

$$\lambda \geq \frac{1}{n_\gamma}\frac{1}{r_i^2}R_i\left(\sum_{\alpha\,\in\,\mathcal{S}_\gamma^c} n_{\gamma-\alpha}n_\alpha P_\alpha\right)R_i^T \tag{7.65}$$

$$\sum_{\alpha\,\in\,\mathcal{S}_\gamma^c} n_{\gamma-\alpha}n_\alpha\begin{pmatrix} P_\alpha & K_\alpha^T S_j^T \\ S_j K_\alpha & \lambda s_j^2\end{pmatrix}\geq 0 \tag{7.66}$$

where R_i *denotes the* i*-th row in matrix* R *and* S_j *the* j*-th row of* S*, and* i *ranges from one to up to the number of rows of* R*,* j *ufrom one to the number of rows of* S*.*

Proof: As the shape-independent levels sets of the Lyapunov function from Theorem 7.3 are symmetric, because $V(\mu,x)=V(\mu,-x)$, in order for such a level set to lie inside the (possibly non-symmetric $\mathbb{X}$), the referred level set must belong to the symmetric set $\mathbb{X}_{sym}:=$

[2]Basically, as μ is unknown at design time, only the shape-independent level set $\{\max_{\mu\in\Delta}V(\mu,x)\leq\lambda\}$ can be proven to belong to any “true” shape-dependent level set $\{V(\mu(x),x)\leq\lambda\}$, motivating the definition (7.43), see the discussion in (Ariño et al., 2017, Section 7).

$\{x \mid |R_i \frac{1}{-r_i} x| \leq 1 \; \forall i\} \subset \mathbb{X}$. According to (Boyd et al., 1994), it is well known that the set $x^T \left(P_\mu^c\right)^{-1} x \leq \lambda^{-1}$ is contained in $\mathbb{X}$ if

$$\frac{1}{r_i^2} R_i \left(P_\mu^c\right) R_i^T \leq \lambda \;\; \forall i \tag{7.67}$$

Substituting $P_\mu^c = \sum_{|\gamma|=c} \mu^\gamma P_\gamma$ by an arbitrary Polya expansion $P_\mu^c = \mathcal{E}(P_\mu^c, q)$ where $q \geq c$

$$\lambda \geq \frac{1}{r_i^2} R_i \left(\sum_{|\gamma|=h} \mu^\gamma \sum_{\alpha \,\in\, \mathcal{S}_\gamma^c} n_{\gamma-\alpha} n_\alpha P_\alpha \right) R_i^T =$$

$$= \sum_{|\gamma|=q} n_\gamma \mu^\gamma \frac{1}{n_\gamma r_i^2} R_i \left(\sum_{\alpha \in \mathcal{S}_\gamma^c} n_{\gamma-\alpha} n_\alpha P_\alpha \right) R_i^T \quad \forall i \tag{7.68}$$

So, from Theorem 7.6, a sufficient condition for the above is that each of the homogeneous polynomial coefficients fulfills the inequality, which is what (7.65) states.

On the other hand, the control action must belong to set $\mathbb{U}$ for all states inside the sought level set. Again, as done with the state constraints, symmetry of the level set and linearity in the state of the terminal controller needs to enforce $u \in \mathbb{U}_{sym}$, where:

$\mathbb{U}_{sym} := \left\{ u \middle| \left| \frac{S_j}{-s_j} u \right| \leq 1 \; \forall j \right\} \subset \mathbb{U}.$

If $x^T (P_\mu^c)^{-1} x \leq (\lambda)^{-1}$, and the condition below holds,

$$\frac{1}{s_j^2} u^T S_j^T S_j u = \underbrace{\frac{1}{s_j^2} x^T (P_\mu^c)^{-T} (K_\mu^c)^T S_j^T S_j K_\mu^c (P_\mu^c)^{-1} x \leq x^T (P_\mu^c)^{-1} x \lambda} \leq 1$$

then the control action will be admissible for all x in $\mathbb{T}$. Concentrating on the inequality over the braces, if congruence with matrix P_μ^c is applied and, later on, a Schur complement, we get the equivalent condition:

$$\begin{pmatrix} P_\mu^c & (K_\mu^c)^T S_j^T \\ S_j K_\mu^c & \lambda s_j^2 \end{pmatrix} \geq 0 \quad \forall \, j \tag{7.69}$$

Finally, an arbitrary Polya expansion can be done

$$\sum_{|\gamma|=q} \mu^\gamma \begin{pmatrix} \sum_{\alpha \in \mathcal{S}_\gamma^c} n_{\gamma-\alpha} n_\alpha P_\alpha & \sum_{\alpha \in \mathcal{S}_\gamma^c} n_{\gamma-\alpha} n_\alpha K_\alpha^T S_j^T \\ \sum_{\alpha \in \mathcal{S}_\gamma^c} n_{\gamma-\alpha} n_\alpha S_j K_\alpha & n_\gamma \lambda s_j^2 \end{pmatrix} \geq 0 \quad \forall\, j \tag{7.70}$$

This inequality will hold if all the coefficients of the polynomial are positive, which amounts to (7.66). ∎

7.4.3 Stability: main result

The second main result of the chapter is the following:

Theorem 7.5 *Given x_0, if $F_\mu^d \geq 0$ and $H_\mu^d > 0$ for any $\mu \in \Delta$, and control actions* (7.26) *are applied once the solution to the optimisation in Theorem 7.2 is obtained, from instant $k = 0$ to $k = N - 1$, and the terminal controller is applied from instants N onwards, then, the system will reach the origin asymptotically and the cost* (7.6) *will be bounded by δ^*.*

Proof: Note that, the theorem 7.2 includes the condition (7.52), which forces that control actions (7.26) steer the system to terminal set (7.43) within N steps. If once the system is inside the terminal set the terminal controller is applied, the system will reach the origin asymptotically with it. With regard to a bound on cost (7.6), the terminal cost bounds the infinite cost (Theorem 7.3) once $x_N \in \mathbb{T}$ and the total cost (including the transient until $\mathbb{T}$ is reached) is bounded by δ^* discussed in the statement and proof of Theorem 7.2. ∎

7.5 Shape-dependent solution (known $\mu(x_0)$).

Note that the MPC problem stated in Section 7.3 is fully shape independent. Thus, any x_0 in the shape-independent feasible set discussed on page 160 would be guaranteed feasible for any TS system, whatever the value of $\mu(x_0)$ happened to be. However, in actual implementation this would be suboptimal, given the fact that $\mu(x_0)$ would be known so the solution needs not to be valid for "all" possible $\mu(x_0)$ but only for the currently measured value. So, if matrix $\boldsymbol{\mu}$ contained only the unknown memberships from instants 1 to N, the shape-dependent MPC problem would be stated as:

$$J^*(x_0) := \min_{U} \max_{\boldsymbol{\mu}\in\Delta^N} J_N(\boldsymbol{\mu}, g(\boldsymbol{\mu}, U), \mu(x_0), x_0) \tag{7.71}$$

subject to the same constraints as in the original problem. This entails some minor modifications to the previously-presented setting in Theorem 7.2: the part of the prediction model G_α computing x_1 from x_0 is a single linear model $x_1 = A(\mu(x_0))x_0 + B(\mu(x_0))u_0$, thus, the degree of all polynomials involving $\mu(x_0)$ can be set to zero, because all "vertices" are the same. For brevity, details on these modifications are left to the reader.

Note that the feasible set of x_0 in problem (7.71) (and the associated constraints) would not be a convex LMI set as $\mu(x_0)$ may be any arbitrarily complex nonlinearity, see the example on Section 7.7. This set will be denoted as *shape-dependent feasible set*.

7.6 Discussion and comparative analysis

As discussed in the introduction, other references have dealt with predictive control in a fuzzy context. Some comparative discussion with a few references appears in the introduction. We deferred to this section a comparison with other more recent LMI-based results on similar problems.

Regarding comparison with (Yang, Feng, & Zhang, 2014), their approach pursues similar goals to ours. However, their determination of the terminal set and controller is conservative, in the sense that, first, their proposed controller is a PDC one (see their equation 4) and, second, they pose conditions *for all* i, j, l in their Theorems 1 and 2, where i affects the current process and Lyapunov function vertex model, j affects the controller vertex and l is the next-instant Lyapunov function vertex. In that way, say, controller vertex 1 proves stability and cost bounds for every i and k so their results cannot be better than those from a robust linear controller. Our terminal controller even for the same level of Polya complexity parameter will, hence, prove a better cost bound. Note, however, that they consider *uncertainty* in the TS vertex models as well as persistent disturbances so such conservative steps are clearly justified and needed to synthesise the terminal controller; on the other hand, our approach purposely does not consider uncertain models.

Some of the literature uses the name “model predictive control” to discuss strategies in which an optimisation problem is solved once x_k is measured, see (Xia et al., 2010), (Zhao, Gao, & Chen, 2010), (T. Zhang et al., 2007). However, what we understand as “predictive control” includes a multi-step prediction model, whereas one-step optimisation is more akin to the so-called “guaranteed-cost” literature: thus, our objective is not comparing to these “one-step MPC” setups, in principle.

In (Xia et al., 2010), a state-dependent guaranteed-cost control is proposed so that, once x_k is measured, a cost bound γ can be guaranteed solving some LMIs ensuring that control action does not saturate. As states get closer to the origin, better cost bounds can be found by increasing the controller gain. In fact, when our terminal set is eventually reached, the same bound would be obtained as LMIs are equivalent. Our improvement lies in the fact that we allow full saturation until the terminal set is reached.

7.7 Example

Consider a 2-rule TS system (7.2) with the local models and membership functions defined as:

$$\begin{gathered} A_1 = \begin{pmatrix} -0.9 & 0.3 \\ 0 & 0.4 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0.8 & 0.6 \\ -0.5 & 0.2 \end{pmatrix} \\ B_1 = \begin{pmatrix} 0.4 \\ 1.1 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0.9 \\ 0.3 \end{pmatrix} \end{gathered} \tag{7.72}$$

$$\mu_1(x_k) = \frac{\sin(0.5\pi x_k(2)) + 1}{2} \qquad \mu_2(x_k) = 1 - \mu_1(x_k) \tag{7.73}$$

Input and state constraints are defined as:

$$-1 \leq u_k \leq 1 \qquad \begin{pmatrix} -5 \\ -5 \end{pmatrix} \leq x_k \leq \begin{pmatrix} 5 \\ 5 \end{pmatrix} \tag{7.74}$$

For simplicity, non-fuzzy weighing matrices H and F will be employed, with $d = 0$:

$$H = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad F = 2 \tag{7.75}$$

Terminal set and controller The terminal controller has been computed with degree $c = 1$, with LMIs (7.58), (7.65) and (7.66) expanded to degree $q = 20$.

The terminal set $\mathbb{T}$ (shape-independent) is drawn with black line on Figure 7.1; it is an intersection of ellipsoids. For curiosity, the shape-dependent set that the terminal controller would actually guarantee as valid is plotted in red (for the chosen memberships); however, such red set cannot be used in the optimisation as $\mu(x_N)$ is not yet known at $t = 0$. The shape-dependent outline has been obtained by the contour function in Matlab, using the actual Lyapunov function replacing the memberships by its explicit expressions (7.73).

Feasible sets The shape-independent predictive control problem (i.e., valid for any $\mu(x_0)$) has been solved with horizon $N = 4$, and the chosen controller degree parameters are:

$$c^{[0]} = (1,0,0,0,0), c^{[1]} = (1,1,0,0,0), c^{[2]} = (1,1,1,0,0), c^{[3]} = (1,1,1,1,0)$$

Furthermore, homogeneous polynomials have been expanded, to exploit Polya's theorem, to degree $h = (4,4,4,4,1)$. The feasible set of this shape-independent MPC problem has also been computed; it is a convex LMI set, presented in Figure 7.1 with a purple[3] line.

Additionally, the shape-dependent solution outlined in Section 7.5 has, too, been computed, with controller complexities:

$$c^{[0]} = (0,0,0,0,0), c^{[1]} = (0,1,0,0,0), c^{[2]} = (0,1,1,0,0), c^{[3]} = (0,1,1,1,0)$$

and Polya expansion to degree vector $h = (0,4,4,4,1)$. Note that the initial zero in $c^{[j]}$ indicates that polynomials in the shape-dependent solution do not depend on $\mu(x_0)$ as it is directly replaced in the model matrices, as discussed in the above-referred section. The shape-dependent feasible set was computed and presented in green in Figure 7.1. As it depends on the actual values of the membership functions it is clearly non-convex (for the chosen memberships) and has been approximately computed determining feasibility of the MPC optimization problem point by point in a dense grid.

[3]Fixing any arbitrary direction and determining the point in the feasible set at a largest distance from the origin in such direction is an LMI problem; this enables a reasonably easy "ray-tracing" computation of the boundary (details omitted for brevity).

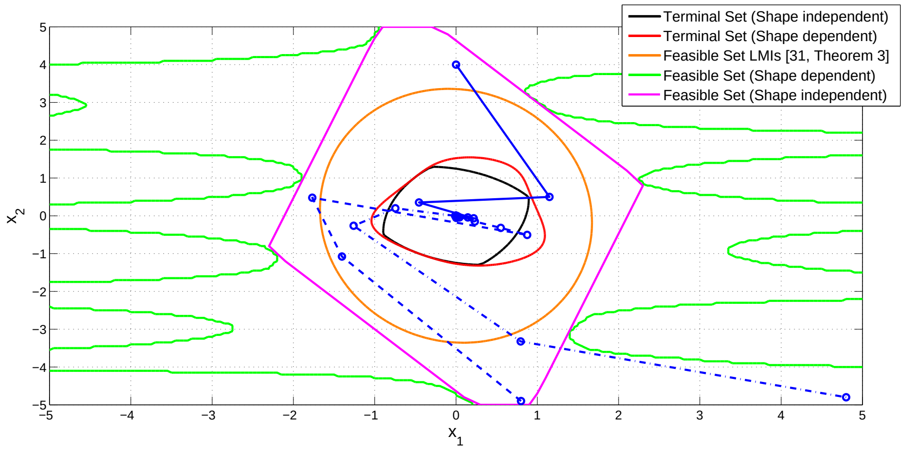

Figure 7.1: Terminal and feasible sets, plus three simulated trajectories of the MPC controller (in blue).

Simulation For illustration, trajectories (using the less-conservative shape-dependent solution) from $x_0 = (0, 4)$ (solid blue), $x_0 = (0.8, -4.9)$ (dashed blue) and $x_0 = (4.8, -4.8)$ (dash-dot blue) are shown in the phase plane in Figure 7.1 and in the time-domain, jointly with the control action, in Figure 7.2. Note that the last initial conditions lie outside the guaranteed shape-independent feasible set but are, anyway, feasible for the particular membership values at this point.

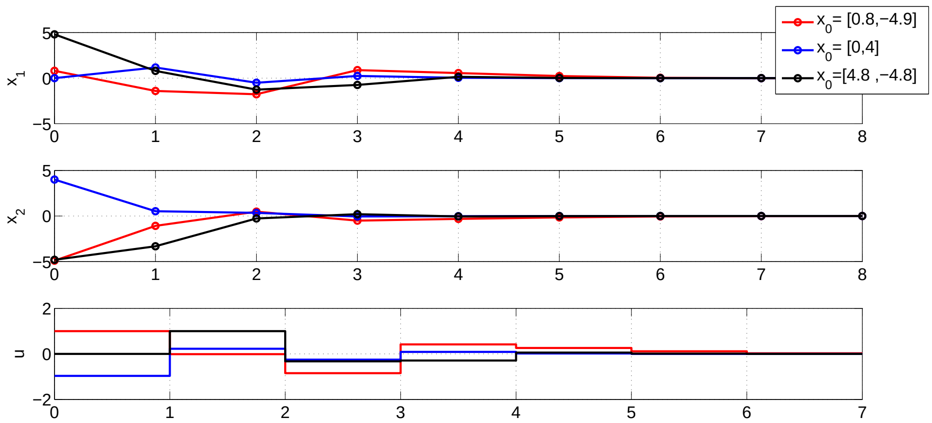

Figure 7.2: Time response of the trajectories on figure 7.1.

Comparative analysis Let us compare our feasible set with that obtained in (Yang et al., 2014, Theorem 3) with the same horizon $N = 4$, depicted in orange, in the above figure. Importantly, the cited work tries to maximise the volume of an ellipsoidal feasible set via *logdet* convex optimisation subject to LMI constraints. Conservatism of the proposed conditions there makes our approach to yield significantly larger feasible sets both in the shape-independent case (purple) and in the shape-dependent one (green).

Table 7.1: Cost comparison with other literature for $x_0 = (0, 4)$.

	Cost bound:
Theorem 7.2	10.89
(Xia et al., 2010, Thm 3.1)	13.63
(Yang et al., 2014, Theorem 3)	45.72

Considering $x_0 = (0, 4)$, the comparison of the guaranteed cost bounds with different approaches is summarised on Table 7.1. Note that the main goal of the paper (Yang et al., 2014) is not to minimize the cost, but to maximise the feasible area; hence, a large penalty with respect to (Xia et al., 2010) is incurred. Nevertheless, our approach beats both (Xia et al., 2010) in cost and (Yang et al., 2014) in size of the feasible set. In fact, a suboptimal lower-gain terminal controller in (Xia et al., 2010) was needed to ensure feasibility of x_0 with $N = 4$ (enlarging the terminal set) and an horizon of $N = 7$ was needed in the implementation of (Yang et al., 2014) for x_0 to be feasible.

7.8 Conclusions

This chapter has presented a generalisation of the predictive control approach to Takagi-Sugeno systems. The approach obtains better results than prior literature both in achieved cost and in the feasible region, due to a suitable control action parametrisation as a function of future memberships and the use of Polya relaxations. Of course, many guaranteed-cost results in prior literature are particular cases considering an horizon of a single sample; in fact, such results are used to build the terminal controller and terminal cost needed in the developments.

7.A Appendix

Proposition 7.1 *Consider the summations*

$$M_{\boldsymbol{\mu}}^{d} = \sum_{|\alpha|=d} \boldsymbol{\mu}^{\alpha} n_{\alpha} M_{\alpha}, \quad L_{\boldsymbol{\mu}}^{c} = \sum_{|\beta|=c} \boldsymbol{\mu}^{\beta} n_{\beta} L_{\beta} \tag{7.76}$$

then the expression

$$M_{\boldsymbol{\mu}}^{d} L_{\boldsymbol{\mu}}^{c} = \left(\sum_{|\alpha|=d} \boldsymbol{\mu}^{\alpha} n_{\alpha} M_{\alpha} \right) \left(\sum_{|\beta|=c} \boldsymbol{\mu}^{\beta} n_{\beta} L_{\beta} \right) \tag{7.77}$$

can be expressed as an homogeneous polynomial of degree $d + c$*,*

$$\Xi_{\boldsymbol{\mu}}^{d+c} = \sum_{|\gamma|=d+c} \boldsymbol{\mu}^{\gamma} \left(\sum_{\alpha \, \in \, S_{\gamma}^{d}} n_{\alpha} n_{\gamma-\alpha} M_{\alpha} L_{\gamma-\alpha} \right) \tag{7.78}$$

where

$$\mathcal{S}_{\gamma}^{d} := \{\alpha \in \mathbb{N}^{r\times(N+1)} | \; |\alpha| = d, \;\; \alpha \leq \gamma\} \tag{7.79}$$

Proof: See paper (Ding, 2010). ∎

Corollary 7.1 *the previous proposition can be used recursively for the product of three or more summations. With three polynomials, we would have:*

$$M_{\boldsymbol{\mu}}^{d} = \sum_{|\alpha|=d} \boldsymbol{\mu}^{\alpha} n_{\alpha} M_{\alpha}, \quad L_{\boldsymbol{\mu}}^{c} = \sum_{|\beta|=c} \boldsymbol{\mu}^{\beta} n_{\beta} L_{\beta}, \quad P_{\boldsymbol{\mu}}^{l} = \sum_{|\xi|=l} \boldsymbol{\mu}^{\xi} n_{\alpha} P_{\xi}, \tag{7.80}$$

the product of the three polynomials can be expressed as an homogeneous polynomial of degree $d + c + l$*,*

$$M_{\boldsymbol{\mu}}^{d} L_{\boldsymbol{\mu}}^{c} P_{\boldsymbol{\mu}}^{l} = \sum_{|\gamma|=d+c+l} \boldsymbol{\mu}^{\gamma} \left(\sum_{\alpha,\xi \in \mathcal{S}_{\gamma}^{dl}} n_{\alpha} n_{\gamma-\alpha-\xi} n_{\xi} M_{\alpha} L_{\gamma-\alpha-\xi} P_{\xi} \right) \tag{7.81}$$

where the set $\mathcal{S}_{\gamma}^{dl}$ *is defined as:*

$$\mathcal{S}_{\gamma}^{dl} := \{\alpha, \xi \in \mathbb{N}^{r\times(N+1)} | |\alpha| = d, \;, \; |\xi| = l, \; \gamma - \alpha - \xi \geq 0\} \tag{7.82}$$

Obviously, if any of d*,* c*,* l *were equal to 1, the corresponding combinatorial number* n_{α}*,* n_{β}*,* n_{ξ} *would be omitted, as in Remark 1.*

Definition 7.1 (Polya expansion) *The degree q expansion of an homogeneous polynomial $M^d_{\boldsymbol{\mu}}$, denoted by $\mathcal{E}(q, M^d_{\boldsymbol{\mu}})$, with degree $q \geq d$, is defined as the degree-q homogeneous polynomial:*

$$\mathcal{E}(q, M^d_{\boldsymbol{\mu}}) := \left(\sum_{|\beta|=(q-d)} \boldsymbol{\mu}^\beta n_\beta \right) \cdot M^d_{\boldsymbol{\mu}} \tag{7.83}$$

Evidently from (7.13), $\mathcal{E}(q, M^d_{\boldsymbol{\mu}}) = M^d_{\boldsymbol{\mu}}$.

Corollary 7.2 *Using Corollary 7.1, we can assert that*

$$\mathcal{E}(q, M^d_{\boldsymbol{\mu}}) = \sum_{|\gamma|=q} \boldsymbol{\mu}^\gamma \sum_{\alpha \,\in\, \mathcal{S}^d_\gamma} n_\alpha n_{\gamma-\alpha} M_\alpha \tag{7.84}$$

where $\mathcal{S}^d_\gamma$ was defined in (7.79).

Theorem 7.6 (Polya theorem (Sala & Ariño, 2007a)) *The summation $M^d_{\boldsymbol{\mu}}$ is positive for all $\boldsymbol{\mu} \in \Delta$ if there exists q such that all coefficients of the expanded degree-q polynomial $\mathcal{E}(q, M^d_{\boldsymbol{\mu}})$ are positive. Furthermore, if $M^d_{\boldsymbol{\mu}} > \epsilon > 0$ for all $\mu \in \Delta$ there exists a finite q such that $\mathcal{E}(q, M^d_{\boldsymbol{\mu}})$ has all its coefficients positive.*

Proposition 7.2 *The summation over a single instant k, $H^q_{\mu(x_k)}$ in* (7.11)*, can be represented as:*

$$H^q_{\mu(x_k)} = \sum_{|\alpha|=d} \boldsymbol{\mu}^\alpha n_\alpha H_{\alpha_{\underline{k}}}$$

where $d_k = q$ and the other values of the vector d are non-negative.

Proof: From the fact that $\sum_{|\alpha|=d_i} n_\alpha \mu(x_i)^\alpha = 1$ for all i and non-negative natural d_i, we have

$$H^{d_k}_{\mu(x_k)} = \left(\prod_{\substack{i=0 \\ i \neq k}}^{N} \sum_{|\alpha|=d_i} n_\alpha \mu(x_i)^\alpha \right) \sum_{|\alpha|=d_k} \mu(x_k)^\alpha n_\alpha H_\alpha = \sum_{|\alpha|=d} \boldsymbol{\mu}^\alpha n_\alpha H_{\alpha_{\underline{k}}}$$

∎

Part III

Conclusions and Bibliography

Chapter 8

Conclusions

The main issues dealt with in the current thesis are stability and predictive control for non linear systems, all these ones employing copositive programming, as well as preliminary results needed in order to pursue the mentioned main goal.

Let us discuss the developments, chapter by chapter:

In Chapter 3, a methodology to relax stability conditions in fuzzy systems has been developed, it lies in the fact that the antecedents are usually known, especially in the modeling step and they have tensor-product structure.

In Chapter 4, LMI methods were presented, but now, for a system with stochastic modes as well as fuzzy models, in order to obtain guaranteed cost control for them.

Furthermore, these methodologies allows guaranteeing a specific cost index with quadratic constraints on the inputs and the states.

The above chapters are preliminary ideas developed in the early stages of the thesis, so they are grouped in the same part as the state of the art. The main contributions now follow.

In Chapter 5 a new approach for the stability was presented for nonlinear discrete-time TS systems, based on invariant set theory instead of LMIs, i.e. usual LMI conditions are replaced by other ones according to set theory. So, outcomes are more relaxed. The obtained Lyapunov function is polyhedral.

The chapter 6 discussed an iterative methodology to approach predictive control for TS systems, rooting on sequential quadratic programming ideas from mainstream nonlinear predictive control.

Last, chapter 7 presented a minimax predictive control approach for TS systems, developed by means of a generic copositive programming, and suitable prediction models. The basic idea is dealing with suitable Polya relaxation with the different values of the membership at different times, and how to incorporate onto the prediction models the knowledge that such memberships will be known at the moment of computing the control law.

Perspectives and future work

The notion of copositiveness is central to current fuzzy control developments, not only in the predictive control. Nevertheless, it usually exacerbates computing requirements, and this is even more exaggerated if it must be applied at several time instants. Thus, even if the theoretical solution to fuzzy MPC in a shape-independent setup is asymptotically closed in Chapter 7 with an elegant Polya-based setup, we reckon that the iterative approaches in Chapter 6 might be more suited to applications. Indeed, if converged, the shape-dependent iterative solution might provide better cost figures; however, the iterative approach might not converge in complex cases, whereas, if computational resources suffice, theoretical guarantees and cost bounds can be achieved with the developments in Chapter 7.

From this thesis, future lines of research can be focused on:

- The stability problem such as is formulated in the paper (Kruszewski et al., 2008), can be adapted to the prediction model presented in the thesis. In this paper the stability is proven by a function that decreases in a period of time, and not for every step of time. So the prediction model and Polya relaxations can improve the results in this line of research.
- Programming a Matlab Copositive-Predictive Toolbox, so that the contributions presented in the present work can be employed by other researchers.
- From a theoretical point of view, the extension of the fuzzy predictive control to the fuzzy+Markov approach would be of interest,

generalising the preliminary exploration of the non-predictive setups on Chapter 4.

- Also, incorporation of sum-of-squares argumentations to some results would, perhaps, allow some further improvements; the most direct approach would be improving the terminal controllers as using SOS in the finite-horizon optimisation seems to bring severe difficulties.

- Convergence issues in the iterative approach might be also worth studying, as well as devising strategies to diminish the computational cost of the proposals, even if heuristic or approximate.

Chapter 9
Bibliography

References

Abbas, H. S., Tóth, R., Meskin, N., Mohammadpour, J., & Hanema, J. (2016). A robust MPC for input-output LPV models. *IEEE Transactions on Automatic Control*, *61*(12), 4183–4188.

Abonyi, J., Nagy, L., & Szeifert, F. (2001). Fuzzy model-based predictive control by instantaneous linearization. *Fuzzy Sets and Systems*, *120*(1), 109–122.

Andersson, J., Åkesson, J., & Diehl, M. (2012). Casadi: A symbolic package for automatic differentiation and optimal control. In *Recent advances in algorithmic differentiation* (pp. 297–307). Springer.

Arahal, M., Berenguel, M., & Camacho, E. (1998). Neural identification applied to predictive control of a solar plant. *Control Engineering Practice*, *6*(3), 333–344.

Ariño, C., Perez, E., Bedate, F., & Sala, A. (2013). Robust polytopic invariant sets for discrete fuzzy control systems. In *Fuzzy systems (FUZZ), 2013 IEEE international conference on* (pp. 1–7).

Ariño, C., Pérez, E., Querol, A., & Sala, A. (2014). Model predictive control for discrete fuzzy systems via iterative quadratic programming. In *Fuzzy systems (FUZZ-IEEE), 2014 IEEE international conference on* (pp. 2288–2293).

Ariño, C., Pérez, E., & Sala, A. (2010). Guaranteed cost control analysis and iterative design for constrained Takagi–Sugeno systems. *Engineering Applications of Artificial Intelligence*, *23*(8), 1420–1427.

Ariño, C., Pérez, E., Sala, A., & Bedate, F. (2014). Polytopic invariant and contractive sets for closed-loop discrete fuzzy systems. *Journal* *of the Franklin Institute*, *351*(7), 3559–3576.

Ariño, C., Querol, A., & Sala, A. (2017, submitted). Shape-independent model predictive control for Takagi-Sugeno fuzzy systems.

Ariño, C., & Sala, A. (2007). Relaxed LMI conditions for closed-loop fuzzy systems with tensor-product structure. *Engineering Applications of Artificial Intelligence*, *20*(8), 1036–1046.

Ariño, C., Sala, A., Pérez, E., Bedate, F., & Querol, A. (2017). Asymptotically exact stabilisation for constrained discrete takagi–sugeno systems via set-invariance. *Fuzzy Sets and Systems*, *316*, 117–138.

Ariño, C., & Sala, A. (2007). Design of multiple-parameterisation PDC controllers via relaxed conditions for multi-dimensional fuzzy summations. In *IEEE international conference on fuzzy systems* (pp. 1–6).

Artstein, Z., & Raković, S. V. (2008). Feedback and invariance under uncertainty via set-iterates. *Automatica*, *44*(2), 520–525.

Audet, C., Hansen, P., Jaumard, B., & Savard, G. (2000). A branch and cut algorithm for nonconvex quadratically constrained quadratic programming. *Mathematical Programming*, *87*(1), 131–152.

Bedate, F. (2015). *Control predictivo de modelos borrosos Takagi-Sugeno mediante funciones de Lyapunov contractivas* (Unpublished doctoral dissertation). Universitat Jaume I.

Bellman, R. (1956). Dynamic programming and Lagrange multipliers. *Proceedings of the National Academy of Sciences*, *42*(10), 767–769.

Bellman, R. E., & Dreyfus, S. E. (2015). *Applied dynamic programming.* Princeton university press.

Bemporad, A., Morari, M., Dua, V., & Pistikopoulos, E. N. (2002). The explicit linear quadratic regulator for constrained systems. *Automatica*, *38*(1), 3 - 20.

Bernal, M., Guerra, T. M., & Kruszewski, A. (2009). A membership-function-dependent approach for stability analysis and controller synthesis of Takagi–Sugeno models. *Fuzzy sets and systems*, *160*(19), 2776–2795.

Bernal, M., Sala, A., Jaadari, A., & Guerra, T.-M. (2011). Stability analysis of polynomial fuzzy models via polynomial fuzzy Lyapunov functions. *Fuzzy Sets and Systems*, *185*(1), 5–14.

Bertsekas, D. (1972). Infinite time reachability of state-space regions by using feedback control. *Automatic Control, IEEE Transactions on*, *17*(5), 604–613.

Biegler, L. T. (2000). Efficient solution of dynamic optimization and NMPC problems. In *Nonlinear model predictive control* (pp. 219–243). Springer.

Biegler, L. T. (2010). *Nonlinear programming: concepts, algorithms, and applications to chemical processes*. SIAM.

Blanchini, F. (1991). Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions. In *Decision and control, 1991. Proceedings of the 30th IEEE conference on* (pp. 1755–1760).

Blanchini, F. (1994). Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions. *Automatic Control, IEEE Transactions on*, *39*(2), 428–433.

Blanchini, F. (1999). Set invariance in control. *Automatica*, *35*(11), 1747–1767.

Botto, M. A., Van Den Boom, T. J., Krijgsman, A., & Da Costa, J. S. (1999). Predictive control based on neural network models with i/o feedback linearization. *International Journal of Control*, *72*(17), 1538–1554.

Boukas, E.-K. (2007). *Stochastic switching systems: analysis and design*. Springer.

Boulkroune, A., Tadjine, M., M'Saad, M., & Farza, M. (2008). How to design a fuzzy adaptive controller based on observers for uncertain affine nonlinear systems. *Fuzzy Sets and Systems*, *159*(8), 926–948.

Boumehraz, M., & Benmahammed, K. (2005). A switching controller for nonlinear systems via fuzzy models. *Adaptive and Natural Computing Algorithms*, 120–123.

Boyd, S. P., & El Ghaoui, L. (1993). Method of centers for minimizing generalized eigenvalues. *Linear algebra and its applications*, *188*, 63–111.

Boyd, S. P., El Ghaoui, L., Feron, E., & Balakrishnan, V. (1994). *Linear matrix inequalities in system and control theory* (Vol. 15). SIAM.

Boyd, S. P., & Vandenberghe, L. (2004). *Convex optimization*. Cambridge university press.

Bundfuss, S., & Dür, M. (2008). Algorithmic copositivity detection by simplicial partition. *Linear Algebra and its Applications*, *428*(7), 1511–1523.

Camacho, E. F., & Bordons, C. (2013). *Model predictive control*. Springer.

Campos, V. C., Souza, F. O., Tôrres, L. A., & Palhares, R. M. (2013). New stability conditions based on piecewise fuzzy Lyapunov functions and tensor product transformations. *IEEE Transactions on Fuzzy Systems*, *21*(4), 748–760.

Cao, Y.-Y., & Lin, Z. (2003). Robust stability analysis and fuzzy-scheduling control for nonlinear systems subject to actuator saturation. *Fuzzy Systems, IEEE Transactions on*, *11*(1), 57–67.

Chesi, G. (2010). LMI techniques for optimization over polynomials in control: a survey. *IEEE Transactions on Automatic Control*, *55*(11), 2500–2510.

Cotter, N. E. (1989). The Stone-Weierstrass theorem and its application to neural networks. *IEEE transactions on neural networks*, *1*(4), 290–295.

Cottle, R. W. (1974). Manifestations of the Schur complement. *Linear Algebra and its Applications*, *8*(3), 189–211.

Daafouz, J., & Bernussou, J. (2001). Parameter dependent Lyapunov functions for discrete time systems with time varying parametric uncertainties. *Systems & control letters*, *43*(5), 355–359.

Daafouz, J., Riedinger, P., & Iung, C. (2002). Stability analysis and control synthesis for switched systems: a switched Lyapunov function approach. *Automatic Control, IEEE Transactions on*, *47*(11), 1883–1887.

Da Silva, J. G., & Tarbouriech, S. (2005). Antiwindup design with guaranteed regions of stability: an lmi-based approach. *IEEE Transactions on Automatic Control*, *50*(1), 106–111.

De Oliveira, J. V., & Lemos, J. M. (1995). Long-range predictive adaptive fuzzy relational control. *Fuzzy Sets and Systems*, *70*(2-3), 337–357.

de Oliveira, M. C., Bernussou, J., & Geromel, J. C. (1999). A new discrete-time robust stability condition. *Systems & control letters*, *37*(4), 261–265.

de Oliveira Kothare, S. L., & Morari, M. (2000). Contractive model predictive control for constrained nonlinear systems. *Automatic Control, IEEE Transactions on*, *45*(6), 1053–1071.

Diehl, M., Bock, H. G., Schlöder, J. P., Findeisen, R., Nagy, Z., & Allgöwer, F. (2002). Real-time optimization and nonlinear model predictive control of processes governed by differential-algebraic equations. *Journal of Process Control*, *12*(4), 577–585.

Diehl, M., Ferreau, H. J., & Haverbeke, N. (2009). Efficient numerical methods for nonlinear mpc and moving horizon estimation. In *Nonlinear model predictive control* (pp. 391–417). Springer.

Ding, B. (2010). Homogeneous polynomially nonquadratic stabilization of discrete-time Takagi–Sugeno systems via nonparallel distributed compensation law. *Fuzzy Systems, IEEE Transactions on*, *18*(5), 994–1000.

do Valle Costa, O. L., Fragoso, M. D., & Marques, R. P. (2006). *Discrete-time Markov jump linear systems*. Springer.

Dür, M. (2010). Copositive programming–a survey. In *Recent advances in optimization and its applications in engineering* (pp. 3–20). Springer.

Fang, C.-H., Liu, Y.-S., Kau, S.-W., Hong, L., & Lee, C.-H. (2006). A new LMI-based approach to relaxed quadratic stabilization of TS fuzzy control systems. *Fuzzy Systems, IEEE Transactions on*, *14*(3), 386–397.

Feng, G. (2006). A survey on analysis and design of model-based fuzzy control systems. *IEEE Transactions on Fuzzy systems*, *14*(5), 676–697.

Gahinet, P., Nemirovskii, A., Laub, A. J., & Chilali, M. (1994). The LMI control toolbox. In *IEEE conference on decision and control* (Vol. 2, pp. 2038–2038).

García-Nieto, S., Salcedo, J., Martínez, M., & Reynoso-Meza, G. (2010). Iterative discrete forward-backward fuzzy predictive control. In *Fuzzy systems (FUZZ), 2010 IEEE international conference on* (pp. 1–7).

Gilbert, E. G., & Tan, K. T. (1991). Linear systems with state and control constraints: The theory and application of maximal output admissible sets. *Automatic Control, IEEE Transactions on*, *36*(9), 1008–1020.

Goh, K., Turan, L., Safonov, M., Papavassilopoulos, G., & Ly, J. (1994, June). Biaffine matrix inequality properties and computational methods. In *American control conference, 1994* (Vol. 1, p. 850-855 vol.1).

González, T., Sala, A., Bernal, M., & Robles, R. (2015). Invariant sets of nonlinear models via piecewise affine Takagi-Sugeno models. In *Fuzzy systems (FUZZ-IEEE), 2015 IEEE international conference on* (pp. 1–6).

Gonzalez, T., Sala, A., Bernal, M., & Robles, R. (2017). Piecewise Takagi-Sugeno asymptotically exact estimation of the domain of attraction of nonlinear systems. *Journal of the Franklin Institute*, *354*(3), 1514 - 1541.

Goodwin, G. C., & De Doná, J. A. (2005). *Constrained control and estimation: And optimization approach.* Springer.

Goodwin, G. C., Graebe, S. F., & Salgado, M. E. (2001). *Control system design* (Vol. 240). Prentice Hall New Jersey.

Goodwin, G. C., Seron, M. M., & De Doná, J. A. (2006). *Constrained control and estimation: an optimisation approach.* Springer Science & Business Media.

Grüne, L., & Pannek, J. (2011). Nonlinear model predictive control. In *Nonlinear model predictive control* (pp. 43–66). Springer.

Guan, X.-P., & Chen, C.-L. (2004). Delay-dependent guaranteed cost control for TS fuzzy systems with time delays. *IEEE Transactions on Fuzzy Systems*, *12*(2), 236–249.

Guerra, T. M., Kerkeni, H., Lauber, J., & Vermeiren, L. (2012, Feb). An efficient Lyapunov function for discrete T–S models: Observer design. *Fuzzy Systems, IEEE Transactions on*, *20*(1), 187-192.

Guerra, T. M., Kruszewski, A., Vermeiren, L., & Tirmant, H. (2006). Conditions of output stabilization for nonlinear models in the Takagi–Sugeno's form. *Fuzzy Sets and Systems*, *157*(9), 1248–1259.

Guerra, T. M., Sala, A., & Tanaka, K. (2015). Fuzzy control turns 50: 10 years later. *Fuzzy sets and systems*, *281*, 168–182.

Guerra, T. M., & Vermeiren, L. (2004). LMI-based relaxed non-quadratic stabilization conditions for nonlinear systems in the Takagi–Sugeno's form. *Automatica*, *40*(5), 823–829.

Hernandez, E., & Arkun, Y. (1991). A nonlinear DMC controller: some modeling and robustness considerations. In *American control conference, 1991* (pp. 2355–2360).

Hernández-Mejías, M. A., Sala, A., Ariño, C., & Querol, A. (2015). Reliable controllable sets for constrained markov-jump linear systems. *International Journal of Robust and Nonlinear Control*.

Hu, T., & Lin, Z. (2003). Composite quadratic Lyapunov functions for constrained control systems. *Automatic Control, IEEE Transactions on*, *48*(3), 440–450.

Jarvis-Wloszek, Z. W. (2003). *Lyapunov based analysis and controller synthesis for polynomial systems using sum-of-squares optimization* (Unpublished doctoral dissertation). University of California, Berkeley.

Johansson, M., Rantzer, A., & Arzen, K. (1999). Piecewise quadratic stability of fuzzy systems. *Fuzzy Systems, IEEE Transactions on*, *7*(6), 713–722.

Kalman, R., & Bertram, J. (1960). Control system analysis and design via the second method of lyapunov. *Trans. ASME*, *1*(82), 394–400.

Kao, Y., Xie, J., & Wang, C. (2014). Stabilization of singular Markovian jump systems with generally uncertain transition rates. *IEEE Transactions on Automatic Control*, *59*(9), 2604–2610.

Kavsek-Biasizzo, K., Skrjanc, I., & Matko, D. (1997). Fuzzy predictive control of highly nonlinear pH process. *Computers & chemical engineering*, *21*, 613–618.

Kerrigan, E. C. (2000). *Robust constraint satisfaction: Invariant sets and predictive control* (Unpublished doctoral dissertation). PhD thesis, Cambridge.

Khalil, H. K. (1996). *Noninear systems*. Prentice-Hall, New Jersey.

Khani, F., & Haeri, M. (2015). Robust model predictive control of nonlinear processes represented by Wiener or Hammerstein models. *Chemical Engineering Science*, *129*, 223–231.

Killian, M., Mayer, B., Schirrer, A., & Kozek, M. (2016). Cooperative fuzzy model-predictive control. *IEEE Transactions on Fuzzy Systems*, *24*(2), 471–482.

Kouvaritakis, B., & Cannon, M. (2001). *Non-linear predictive control: theory and practice* (No. 61). Iet.

Kouvaritakis, B., Cannon, M., & Rossiter, J. (1999). Non-linear model based predictive control. *International Journal of Control*, *72*(10), 919–928.

Kruszewski, A., Sala, A., Guerra, T. M., & Ariño, C. (2009). A triangulation approach to asymptotically exact conditions for fuzzy summations. *IEEE Transactions on Fuzzy Systems*, *17*(5), 985–994.

Kruszewski, A., Wang, R., & Guerra, T. M. (2008). Nonquadratic stabilization conditions for a class of uncertain nonlinear discrete time TS fuzzy models: a new approach. *Automatic Control, IEEE Transactions on*, *53*(2), 606–611.

Kvasnica, M., Grieder, P., Baotić, M., & Morari, M. (2004). Multiparametric toolbox (MPT). In *Hybrid systems: computation and control* (pp. 448–462). Springer.

Labiod, S., Boucherit, M. S., & Guerra, T. M. (2005). Adaptive fuzzy control of a class of MIMO nonlinear systems. *Fuzzy sets and systems*, *151*(1), 59–77.

Lee, J. H. (2011). Model predictive control: review of the three decades of development. *International Journal of Control, Automation and Systems*, *9*(3), 415–424.

Lendek, Z., Guerra, T. M., Babuska, R., & De Schutter, B. (2011). *Stability analysis and nonlinear observer design using Takagi-Sugeno fuzzy models.* Springer.

Lendek, Z., Guerra, T. M., & Lauber, J. (2012). Construction of extended Lyapunov functions and control laws for discrete-time TS systems. In *Fuzzy systems (FUZZ-IEEE), 2012 IEEE international conference on* (pp. 1–6).

Lendek, Z., Guerra, T.-M., & Lauber, J. (2015). Controller design for TS models using delayed nonquadratic Lyapunov functions. *Cybernetics, IEEE Transactions on*, *45*(3), 453–464.

Li, N., Li, S.-Y., & Xi, Y.-G. (2004). Multi-model predictive control based on the Takagi–Sugeno fuzzy models: a case study. *Information Sciences*, *165*(3), 247–263.

Löfberg, J. (2003). *Minimax approaches to robust model predictive control* (Vol. 812). Linköping University Electronic Press.

Löfberg, J. (2004). YALMIP: A toolbox for modeling and optimization in MATLAB. In *Computer aided control systems design, 2004 IEEE international symposium on* (pp. 284–289).

Lu, J., Li, D., & Xi, Y. (2013). Constrained model predictive control synthesis for uncertain discrete-time Markovian jump linear systems. *Control Theory & Applications, IET*, *7*(5), 707–719.

Lu, Q., Shi, P., Lam, H.-K., & Zhao, Y. (2015). Interval type-2 fuzzy model predictive control of nonlinear networked control systems. *IEEE Transactions on Fuzzy Systems*, *23*(6), 2317–2328.

Lu, Y., & Arkun, Y. (2000). Quasi-min-max MPC algorithms for LPV systems. *Automatica*, *36*(4), 527–540.

Maeda, M., Shimakawa, M., & Murakami, S. (1995). Predictive fuzzy control of an autonomous mobile robot with forecast learning function. *Fuzzy Sets and Systems*, *72*(1), 51–60.

Mayne, D. Q., Rawlings, J. B., Rao, C. V., & Scokaert, P. O. (2000). Constrained model predictive control: Stability and optimality. *Automatica*, *36*(6), 789–814.

Mejías, M. A. H. (2016). *Control predictivo basado en escenarios para sistemas lineales con saltos Markovianos* (Unpublished doctoral dissertation).

Michael, E. (1956). Continuous selections I. *The Annals of Mathematics*, 361–382.

Mohammadpour, J., & Scherer, C. W. (2012). *Control of linear parameter varying systems with applications*. Springer Science & Business Media.

Ogata, K. (1996). *Sistemas de control tiempo discreto* (Segunda Edición ed.). Prentice-Hall Internacional.

Pardalos, P. M., & Schnitger, G. (1988). Checking local optimality in constrained quadratic programming is np-hard. *Operations Research Letters*, *7*(1), 33–35.

Patrinos, P., Sopasakis, P., Sarimveis, H., & Bemporad, A. (2014). Stochastic model predictive control for constrained discrete-time Markovian switching systems. *Automatica*, *50*(10), 2504–2514.

Pearson, R., & Ogunnaike, B. A. (2002). *Identification and control using Volterra models*. Springer.

Pérez, E., Ariño, C., Blasco, F., & Martínez, M. (2011). Maximal closed loop admissible set for linear systems with non-convex polyhedral constraints. *Journal of Process Control*, *21*(4), 529–537.

Pérez, E. S. (2011). *Control predictivo sujeto a restricciones poliédricas no convexas: Solución explícita y estabilidad* (Unpublished doctoral dissertation). Universidad Politécnica de Valencia.

Pitarch, J. L., Ariño, C., Bedate, F., & Sala, A. (2010). Local fuzzy modeling: Maximising the basin of attraction. In *Fuzzy systems (fuzz), 2010 IEEE international conference on* (pp. 1–7).

Pitarch, J. L., Sala, A., Ariño, C., & Bedate, F. (2012). Domain of attraction estimation for nonlinear systems with fuzzy polynomial models. *Rev. Iberoam. Automatica e Informatica Industr.*, *9*(2), 152–161.

Pitarch, J. L., Sala, A., & Ariño, C. V. (2014). Closed-form estimates of the domain of attraction for nonlinear systems via fuzzy-polynomial models. *IEEE transactions on cybernetics*, *44*(4), 526–538.

Pluymers, B., Rossiter, J. A., Suykens, J. A. K., & De Moor, B. (2005). The efficient computation of polyhedral invariant sets for linear systems with polytopic uncertainty. In *Proceedings of the american control conference* (Vol. 2, p. 804-809).

Pólya, G. (1928). Über positive Darstellung von Polynomen. *Vierteljschr. Naturforsch. Ges. Zürich*, *73*, 141–145.

Powers, V., & Reznick, B. (2001). A new bound for Pólya's theorem with applications to polynomials positive on polyhedra. *Journal of Pure and Applied Algebra*, *164*(1), 221–229.

Prajna, S., Papachristodoulou, A., & Parrilo, P. A. (2002). Introducing sostools: A general purpose sum of squares programming solver. In *Decision and control, 2002, proceedings of the 41st ieee conference on* (Vol. 1, pp. 741–746).

Prajna, S., Papachristodoulou, A., & Parrilo, P. A. (2004). SOSTOOLS: sum of squares optimization toolbox for MATLAB–users guide. *Control and Dynamical Systems, California Institute of Technology, Pasadena, CA*, *91125*.

Prajna, S., Papachristodoulou, A., & Wu, F. (2004). Nonlinear control synthesis by sum of squares optimization: A Lyapunov-based approach. In *Control conference, 2004. 5th Asian* (Vol. 1, pp. 157–165).

Qiu, J., Feng, G., & Gao, H. (2012). Observer-based piecewise affine output feedback controller synthesis of continuous-time T–S fuzzy affine dynamic systems using quantized measurements. *IEEE Transactions on Fuzzy Systems*, *20*(6), 1046–1062.

Querol, A., Ariño, C. V., Hernández-Mejías, M. A., & Sala, A. (2014). Mejora de la estabilidad en sistemas Takagi-Sugeno mediante la aplicación del teorema de Polya con multiíndices. In *2014 Actas de las XXXV Jornadas de Automatica.*

Reznick, B. (1995). Uniform denominators in Hilbert's seventeenth problem. *Mathematische Zeitschrift*, *220*(1), 75–97.

Robles, R., Sala, A., Bernal, M., & González, T. (2016). Optimal-performance Takagi-Sugeno models via the LMI null space. *IFAC-PapersOnLine*, *49*(5), 13–18.

Robles, R., Sala, A., Bernal, M., & Gonzalez, T. (2016). Subspace-based Takagi-Sugeno modeling for improved LMI performance. *IEEE Transactions on Fuzzy Systems*. doi: 10.1109/TFUZZ.2016.2574927

Roubos, J. A., Mollov, S., Babuška, R., & Verbruggen, H. B. (1999). Fuzzy model-based predictive control using Takagi–Sugeno models. *International Journal of Approximate Reasoning*, *22*(1-2), 3–30.

Sala, A. (2009). On the conservativeness of fuzzy and fuzzy-polynomial control of nonlinear systems. *Annual Reviews in Control*, *33*(1), 48–58.

Sala, A., & Ariño, C. (2007a). Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: Applications of Polya's theorem. *Fuzzy Sets and Systems*, *158*(24), 2671–2686.

Sala, A., & Ariño, C. (2007b). Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: Applications of Polya's theorem. *Fuzzy Sets and Systems*, *158*(24), 2671–2686.

Sala, A., & Ariño, C. (2008). Relaxed stability and performance LMI conditions for Takagi–Sugeno fuzzy systems with polynomial constraints on membership function shapes. *Fuzzy Systems, IEEE Transactions on*, *16*(5), 1328–1336.

Sala, A., & Ariño, C. (2009). Polynomial fuzzy models for nonlinear control: A Taylor series approach. *IEEE Transactions on Fuzzy Systems*, *17*(6), 1284–1295.

Sala, A., Guerra, T. M., & Babuska, R. (2005). Perspectives of fuzzy systems and control. *Fuzzy Sets and Systems*, *156*(3), 432-444.

Sala, A., & Pitarch, J. L. (2016). Optimisation of transient and ultimate inescapable sets with polynomial boundaries for nonlinear systems. *Automatica*, *73*, 82–87.

Salcedo, J. V., Martínez, M., & García-Nieto, S. (2008). Stabilization conditions of fuzzy systems under persistent perturbations and their application in nonlinear systems. *Engineering Applications of Artificial Intelligence*, *21*(8), 1264–1276.

Scherer, C. W., & Hol, C. W. (2006). Matrix sum-of-squares relaxations for robust semi-definite programs. *Mathematical programming*, *107*(1-2), 189–211.

Scokaert, P., & Mayne, D. (1998). Min-max feedback model predictive control for constrained linear systems. *Automatic Control, IEEE Transactions on*, *43*(8), 1136–1142.

Scokaert, P. O., Mayne, D. Q., & Rawlings, J. B. (1999). Suboptimal model predictive control (feasibility implies stability). *Automatic Control, IEEE Transactions on*, *44*(3), 648–654.

Seidi, M., & Markazi, A. H. D. (2011). Performance-oriented parallel distributed compensation. *Journal of the Franklin Institute*, *348*(7), 1231-1244.

Sheng, L., Gao, M., Zhang, W., & Chen, B.-S. (2015). Infinite horizon H_∞ control for nonlinear stochastic Markov jump systems with (x, u, v)-dependent noise via fuzzy approach. *Fuzzy Sets and Systems*, *273*, 105–123.

Sivakumar, R., Manic, K. S., Nerthiga, V., Akila, R., & Balu, K. (2010). Application of fuzzy model predictive control in multivariable control of distillation column. *International Journal of Chemical Engineering and Applications*, *1*(1), 38.

Skelton, R. E., Iwasaki, T., & Grigoriadis, D. E. (1997). *A unified algebraic approach to control design*. CRC Press.

Slotine, J.-J. E., Li, W., et al. (1991). *Applied nonlinear control* (Vol. 199) (No. 1). Prentice-Hall Englewood Cliffs, NJ.

Sontag, E. D. (1989). A universal construction of Artstein's theorem on nonlinear stabilization. *Systems & control letters*, *13*(2), 117–123.

Sontag, E. D. (1999). Control-Lyapunov functions. In *Open problems in mathematical systems and control theory* (pp. 211–216). Springer.

Sousa, J., Babuška, R., & Verbruggen, H. (1997). Fuzzy predictive control applied to an air-conditioning system. *Control Engineering Practice*, *5*(10), 1395–1406.

Sturm, J. F. (1999). Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization methods and software*, *11*(1-4), 625–653.

Sugeno, M., & Kang, G. (1986). Fuzzy modelling and control of multilayer incinerator. *Fuzzy sets and systems*, *18*(3), 329–345.

Takagi, T., & Sugeno, M. (1985). Fuzzy identification of systems and its applications to modeling and control. *Systems, Man and Cybernetics, IEEE Transactions on*(1), 116–132.

Tanaka, K., Ikeda, T., & Wang, H. O. (1996). Robust stabilization of a class of uncertain nonlinear systems via fuzzy control: Quadratic stabilizability, H^∞ control theory, and linear matrix inequalities. *IEEE Transactions on Fuzzy Systems*, *4*(1), 1-13.

Tanaka, K., Ikeda, T., & Wang, H. O. (1998). Fuzzy regulators and fuzzy observers: relaxed stability conditions and LMI-based designs. *Fuzzy Systems, IEEE Transactions on*, *6*(2), 250–265.

Tanaka, K., Nishimura, M., & Wang, H. O. (1998). Multi-objective fuzzy control of high rise/high speed elevators using LMIs. In *American control conference, 1998. proceedings of the 1998* (Vol. 6, pp. 3450–3454).

Tanaka, K., Ohtake, H., & Wang, H. O. (2009). Guaranteed cost control of polynomial fuzzy systems via a sum of squares approach. *Systems, Man, and Cybernetics, Part B: Cybernetics, IEEE Transactions on*, *39*(2), 561–567.

Tanaka, K., & Sugeno, M. (1990). Stability analysis of fuzzy systems using Lyapunov's direct method. In *Proc. NAFIPS* (Vol. 90, pp. 133–136).

Tanaka, K., & Sugeno, M. (1992). Stability analysis and design of fuzzy control systems. *Fuzzy sets and systems*, *45*(2), 135–156.

Tanaka, K., Taniguchi, T., & Wang, H. O. (1998a). Fuzzy control based on quadratic performance function-a linear matrix inequality approach. In *Decision and control, 1998. proceedings of the 37th IEEE conference on* (Vol. 3, pp. 2914–2919).

Tanaka, K., Taniguchi, T., & Wang, H. O. (1998b). Model-based fuzzy control of TORA system: fuzzy regulator and fuzzy observer design via LMIs that represent decay rate, disturbance rejection, robustness, optimality. In *Fuzzy systems proceedings, 1998. IEEE world congress on computational intelligence* (Vol. 1, pp. 313–318).

Tanaka, K., Taniguchi, T., & Wang, H. O. (1999). Robust and optimal fuzzy control: a linear matrix inequality approach. In *1999 international federation of automatic control world congress* (pp. 213–218).

Tanaka, K., & Wang, H. O. (2004). *Fuzzy control systems design and analysis: a linear matrix inequality approach.* John Wiley & Sons.

Tanaka, K., Yoshida, H., Ohtake, H., & Wang, H. O. (2009). A sum-of-squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems. *IEEE Transactions on Fuzzy systems*, *17*(4), 911–922.

Tong, S., Huo, B., & Li, Y. (2014). Observer-based adaptive decentralized fuzzy fault-tolerant control of nonlinear large-scale systems with actuator failures. *IEEE Transactions on Fuzzy Systems*, *22*(1), 1–15.

Townsend, S., & Irwin, G. W. (2001). Nonlinear model based predictive control using multiple local models. *IEE Control Engineering series*, 223–244.

Tuan, H. D., Apkarian, P., Narikiyo, T., & Kanota, M. (2004). New fuzzy control model and dynamic output feedback parallel distributed compensation. *IEEE Transactions on Fuzzy Systems*, *12*(1), 13–21.

Tuan, H. D., Apkarian, P., Narikiyo, T., & Yamamoto, Y. (2001). Parameterized linear matrix inequality techniques in fuzzy control system design. *IEEE Transactions on fuzzy systems*, *9*(2), 324–332.

Wang, H. O., Tanaka, K., & Griffin, M. (1995). Parallel distributed compensation of nonlinear systems by Takagi-Sugeno fuzzy model. In *Fuzzy systems, 1995. IEEE international conference on fuzzy systems* (Vol. 2, pp. 531–538).

Wong, C., Shah, S. L., Bourke, M. M., & Fisher, D. G. (2000). Adaptive fuzzy relational predictive control. *fuzzy Sets and systems*, *115*(2), 247–260.

Wu, H.-N. (2004). Reliable LQ fuzzy control for nonlinear discrete-time systems via LMIs. *Systems, Man, and Cybernetics, Part B: Cybernetics, IEEE Transactions on*, *34*(2), 1270–1275.

Xia, Y., Yang, H., Shi, P., & Fu, M. (2010). Constrained infinite-horizon model predictive control for fuzzy-discrete-time systems. *IEEE Transactions on Fuzzy Systems*, *18*(2), 429–436.

Xiaodong, L., & Qingling, Z. (2003). New approaches to H_∞ controller designs based on fuzzy observers for TS fuzzy systems via LMI. *Automatica*, *39*(9), 1571–1582.

Xie, L., & de Souza, C. E. (1992). Robust H_∞ control for linear systems with norm-bounded time-varying uncertainty. *Automatic Control, IEEE Transactions on*, *37*(8), 1188–1191.

Xie, X., Ma, H., Zhao, Y., Ding, D.-W., & Wang, Y. (2013). Control synthesis of discrete-time T–S fuzzy systems based on a novel non-PDC control scheme. *IEEE Transactions on Fuzzy Systems*, *21*(1), 147–157.

Xie, X., Yue, D., Zhang, H., & Xue, Y. (2016). Control synthesis of discrete-time T–S fuzzy systems via a multi-instant homogenous polynomial approach. *IEEE transactions on cybernetics*, *46*(3), 630–640.

Xiong, J., Lam, J., Gao, H., & Ho, D. W. (2005). On robust stabilization of Markovian jump systems with uncertain switching probabilities. *Automatica*, *41*(5), 897–903.

Yan, Z., Song, Y., & Park, J. H. (2017). Finite-time stability and stabilization for stochastic Markov jump systems with mode-dependent time delays. *ISA Transactions*.

Yang, W., Feng, G., & Zhang, T. (2014). Robust model predictive control for discrete-time Takagi–Sugeno fuzzy systems with structured uncertainties and persistent disturbances. *IEEE Transactions on Fuzzy Systems*, *22*(5), 1213–1228.

Zhang, L., & Boukas, E.-K. (2009). Stability and stabilization of Markovian jump linear systems with partly unknown transition probabilities. *Automatica*, *45*(2), 463–468.

Zhang, T., Feng, G., & Lu, J. (2007). Fuzzy constrained min-max model predictive control based on piecewise Lyapunov functions. *IEEE Transactions on Fuzzy Systems*, *15*(4), 686–698.

Zhao, Y., Gao, H., & Chen, T. (2010). Fuzzy constrained predictive control of non-linear systems with packet dropouts. *IET control theory & applications*, *4*(9), 1665–1677.

Zou, T., & Li, S. (2011). Stabilization via extended nonquadratic boundedness for constrained nonlinear systems in Takagi-Sugeno's form. *Journal of the Franklin Institute*, *348*(10), 2849-2862.