Chapter 3

Dissipativity-related results in the continuous-time case

3.1 Introduction

The purpose of this chapter is to show some results obtained for the continuous-time case relating dissipativity, feedback dissipativity and stabilization of dissipative systems. These results will be extended for the discrete-time case in the following chapters of this dissertation, and they are based on the existing results given in the literature for the affine-in-the-input nonlinear continuous-time case.

The KYP properties will be proposed for a class of multiple-input multiple-output dissipative systems. This characterization will be extended to the case of discrete-time systems in Chapter 4.

The feedback dissipativity problem will be treated for the case of single-input single-output non-affine nonlinear systems. As it was pointed out in Section 2.4.1, feedback dissipativity will be based on a different definition for dissipative systems, what will be regarded as (V, s)-dissipative systems. The main feature of this class of dissipativity is that it is based on the establishment of an a priori storage function V.

Using the dissipativity definitions and the feedback dissipativity results, the stability implications of a class of dissipative systems will be revisited. These conclusions will be used to suitably extend the ESDI controller design method to the general nonlinear case.

A dissipativity-oriented form for nonlinear systems which is of the Generalized Hamiltonian type is also proposed. Feedback dissipative systems are shown to be easily characterized in terms of such a form and some notes are given relating the use of this form for stabilization purposes.

This chapter is organized as follows. Section 3.2 presents the systems to be treated and some notes on the nomenclature used along the chapter. A characterization for a class of dissipative systems is given in Section 3.3. Section 3.4 revisits the definitions about dissipative systems and feedback dissipative systems, and deals with the proposal of a

solution to the feedback dissipativity problem. Section 3.5, in addition to the results given in Section 3.4, sets the stage for the extension of the ESDI controller design methodology to the non-affine nonlinear systems case. Section 3.6 presents the extension of the ESDI design method. Section 3.7 deals with the derivation of a Generalized Hamiltonian-type form for nonlinear systems. Feedback dissipative systems are shown to exhibit special features if they are written in the form proposed, and some notes on the stabilization problem through this form are also pointed out. The conclusions and suggestions for further research are presented in the last section.

Most of the results presented in this chapter are extracted from *(Sira-Ramírez and Navarro-López, 2000)* [161].

3.2 Generalities

Let a nonlinear system of the form

$$\dot{x} = f(x,u), \ x \in \mathscr{X}, \ u \in \mathscr{U} \tag{3.1}$$

$$y = h(x,u), \ y \in \mathscr{Y} \tag{3.2}$$

where $f : \mathscr{X} \times \mathscr{U} \to \mathscr{X}$ and $h : \mathscr{X} \times \mathscr{U} \to \mathscr{Y}$ are smooth mappings with $f(0,0) = 0$, and $\mathscr{X}$ an open subset of $\mathfrak{R}^n$, $\mathscr{U}$, $\mathscr{Y}$ open subsets of $\mathfrak{R}^m$. The open set $\mathscr{X} \times \mathscr{U}$ is such that contains the origin $(0,0)$.

Along this chapter different dissipativity-related concepts will be treated. The formalism for dissipativity to use is the dissipativity definition given in Chapter 2 as Definition 2.4. This definition implies the use of a storage function V, a supply function s and a dissipation function ϕ. The storage function V is considered as a positive definite, C^2 function, $V : \mathscr{X} \to \mathfrak{R}$ whose row gradient, with respect to x, is denoted by $\partial V / \partial x$, and $V(0) = 0$. The supply function is another $\mathscr{C}^2$ function, with $s : \mathscr{Y} \times \mathscr{U} \to \mathfrak{R}$. The dissipation function is a $\mathscr{C}^1$ function $\phi : \mathscr{X} \times \mathscr{U} \to \mathfrak{R}$ with $\phi(\cdot, u)$ positive for each $u \in \mathscr{U}$, and $\phi(0,0) = 0$.

With some abuse of notation, we use Lie derivatives to briefly express time derivatives of scalar functions of the state. Thus, for a fixed u

$$\dot{V} = \frac{\partial V}{\partial x} f(x,u) = L_{f(x,u)} V(x) \tag{3.3}$$

Similarly, we write

$$\frac{\partial \dot{V}}{\partial u} = \frac{\partial}{\partial u} \left[\frac{\partial V}{\partial x} f(x,u) \right] = \frac{\partial V}{\partial x} \left(\frac{\partial f}{\partial u} \right) = L_{\frac{\partial f}{\partial u}} V(x) \tag{3.4}$$

3.3 Towards the KYP property in the general continuous-time case

Proposition 3.1 (Sira-Ramírez and Navarro-López, 2000) *[161] Let a system of the form (3.1)-(3.2) be dissipative, then*

$$L_{f(x,0)} V(x) \ \leq \ s(h(x,0),0) \tag{3.5}$$

$$L_{\frac{\partial f}{\partial u}} V(x) \ = \ \frac{\partial}{\partial u} s(h(x,u),u) - \frac{\partial}{\partial u} \phi(x,u) \tag{3.6}$$

where ϕ is defined by equation (2.7).

Proof. The first relation directly follows from the infinitesimal version of the definition of dissipativity (2.7). Indeed, using the fact that $\dot{V} = L_{f(x,u)}V(x) \leq s(y,u)$ and letting $u = 0$. Relation (3.6) follows by taking partial derivatives with respect to u in (2.7).

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Example 3.1 *Consider the case in which $f(x,u) = f(x) + g(x)u$ with $y = h(x)$, $s(y,u) = y^T u$ and ϕ only depending on x, then (3.5) and (3.6) take the form,*

$$L_{f(x)}V(x) \leq 0, \quad L_{g(x)}V(x) = h^T(x) = y^T \tag{3.7}$$

which is a well known form of the KYP conditions, see (2.31)-(2.32). For systems affine in the control input, with $y = h(x) + k(x)u$, conditions (3.5)-(3.6) are obtained as

$$L_{f(x)}V(x) \leq 0, \quad L_{g(x)}V(x) = (h(x) + 2k(x)u)^T = (y + k(x)u)^T \tag{3.8}$$

If functions V and s are considered in such a way that $L_{f(x,u)}V(x)$ and $s(h(x,u),u)$ are quadratic in u, then relations (3.5)-(3.6) yield necessary and sufficient conditions. In the following theorem, conditions for a class of dissipative systems such that $L_{f(x,u)}V(x)$ and $s(h(x,u),u)$ are quadratic in u are given.

Theorem 3.1 *Suppose there exists a $\mathscr{C}^2$ function V such that $L_{f(x,u)}V(x)$ is quadratic in u. Then, a continuous-time system of the form (3.1)-(3.2) is dissipative with a $\mathscr{C}^2$ supply function $s(y,u)$ satisfying $s(h(x,u),u)$ quadratic in u, and with storage function V, $V : \mathscr{X} \to \mathfrak{R}^+$, $V(0) = 0$, if and only if, there exist real functions $l(x)$ and $k(x)$, $l : \mathscr{X} \to \mathfrak{R}^q$, $k : \mathscr{X} \to \mathfrak{R}^{q \times m}$, for some integer q, such that*

$$L_{f(x,0)}V(x) = s(h(x,0),0) - l^T(x)l(x) \tag{3.9}$$

$$L_{\frac{\partial f}{\partial u}}V(x)\bigg|_{u=0} = \frac{\partial}{\partial u}s(h(x,u),u)\bigg|_{u=0} - 2l^T(x)k(x) \tag{3.10}$$

$$L_{\frac{\partial^2 f}{\partial u^2}}V(x)\bigg|_{u=0} = \frac{\partial^2}{\partial u^2}s(h(x,u),u)\bigg|_{u=0} - 2k^T(x)k(x) \tag{3.11}$$

Proof. (Necessity): If system (3.1)-(3.2) is dissipative (Definition 2.4) there exists a function ϕ satisfying equality (2.7). Since

$$L_{f(x,u)}V(x), \quad s(h(x,u),u)$$

are quadratic in u, the dissipation rate function ϕ can be written as follows (as it is proposed in *(Hill and Moylan, 1976)* [53]),

$$\phi(x,u) = [l(x) + k(x)u]^T [l(x) + k(x)u] \geq 0, \quad \forall u \in \mathscr{U} \tag{3.12}$$

for some real functions $l(x)$ and $k(x)$. Condition (3.9) is obtained from the infinitesimal version of the dissipativity equality (2.6), i.e., restricting (2.7) to $u = 0$ and considering

the derivative of V along the trajectories of system (3.1)-(3.2), and taking $\phi(x,u)$ as defined in (3.12). Conditions (3.10) and (3.11) follow from the first-order and the second-order derivative of (2.7) with respect to u, considering (3.12) and $u = 0$.

(Sufficiency): Assume there exist real functions $l(x)$, $k(x)$ which satisfy conditions (3.9)-(3.11).

Multiplying equality (3.10) by u^T from the left and adding (3.9), it is obtained

$$L_{f(x,0)}V(x) + u^T \left. L_{\frac{\partial f}{\partial u}}V(x)\right|_{u=0} = s(h(x,0),0) + u^T \left.\frac{\partial}{\partial u}s(h(x,u),u)\right|_{u=0} -$$
$$-2l^T(x)k(x)u - l^T(x)l(x) \qquad (3.13)$$

Adding to the right-hand side of (3.13) $u^T k^T(x)k(x)u - u^T k^T(x)k(x)u$ and using (3.11), one yields to

$$u^T k^T(x)k(x)u = -\frac{1}{2}u^T \left.L_{\frac{\partial^2 f}{\partial u^2}}V(x)\right|_{u=0} u + \frac{1}{2}u^T \left.\frac{\partial^2}{\partial u^2}s(h(x,u),u)\right|_{u=0} u \qquad (3.14)$$

Then, from (3.13) and (3.14),

$$L_{f(x,0)}V(x) + u^T \left.L_{\frac{\partial f}{\partial u}}V(x)\right|_{u=0} + \frac{1}{2}u^T \left.L_{\frac{\partial^2 f}{\partial u^2}}V(x)\right|_{u=0} u =$$
$$= s(h(x,0),0) + u^T \left.\frac{\partial}{\partial u}s(h(x,u),u)\right|_{u=0} + \frac{1}{2}u^T \left.\frac{\partial^2}{\partial u^2}s(h(x,u),u)\right|_{u=0} u -$$
$$-\phi(x,u) \qquad (3.15)$$

with ϕ as defined in (3.12). By claiming that $L_{f(x,u)}V(x)$ and $s(h(x,u),u)$ are quadratic in u, the second-order Taylor expansion at $u = 0$ of $L_{f(x,u)}V(x)$ and $s(h(x,u),u)$ can be considered in (3.15), and then

$$L_{f(x,u)}V(x) = s(h(x,u),u) - \phi(x,u),$$

which is the dissipativity equality (2.7).

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Remark 3.1 *Conditions (3.9)-(3.11) are a form of rewriting for a class of non-affine in the control input dissipative systems the conditions appeared in* (Hill and Moylan, 1976) *[53] for* (Q,R,S)*-dissipativity for nonlinear systems affine in the control input (see Theorem A.1 in Appendix A). Besides, dissipativity conditions shown in Theorem 3.1 are of the same type as the ones proposed in* (Lin, 1995) *[86], see Proposition A.1 in Appendix A, with the difference that conditions given in* (Lin, 1995) *[86] are necessary conditions for systems (3.1)-(3.2) to be passive.*

3.4 Feedback dissipativity

This section is devoted to the feedback dissipativity problem for single-input single-output continuous-time systems.

Let $\gamma : \mathscr{X} \times \mathscr{U} \to \mathscr{U}$ be a $\mathscr{C}^1$ function of its arguments. A nonlinear static state feedback control law is denoted by the expression $u = \gamma(x, v)$ with $v \in \mathscr{U} \subset \mathfrak{R}$.

As it was pointed out in Section 2.4.1 the notion of (V, s)-dissipativity will be introduced here to treat the feedback dissipativity problem in the continuous-time setting.

Definition 3.1 *Let the system (3.1)-(3.2) with $m = 1$. The system (3.1)-(3.2) with storage function V and supply function s is said to be (V,s)-dissipative (resp., strictly (V,s)-dissipative) if there exists a $\mathscr{C}^1$ function $\phi : \mathscr{X} \times \mathscr{U} \to \mathfrak{R}$ such that $\phi(\cdot, u)$ is positive (resp., strictly positive) for each $u \in \mathscr{U}$ and*

$$L_{f(x,u)}V(x) = s(h(x,u), u) - \phi(x,u), \quad \forall (x,u) \in \mathscr{X} \times \mathscr{U} \tag{3.16}$$

Consider a locally regular feedback control law $u = \gamma(x, v)$, as it was defined in Definition 2.10. By the *feedback transformed* system we mean the system $\dot{x} = f(x, \gamma(x, v))$, which may be also denoted by $\dot{x} = \overline{f}(x, v)$. We also denote by $\overline{h}(x, v)$ the function $h(x, \gamma(x, v))$.

Definition 3.2 *Consider a system of the form (3.1)-(3.2) with $m = 1$ and two scalar functions $V(x)$, $s(y, v)$ considered as a storage function and a supply function, respectively. The system is said to be feedback dissipative (resp., feedback strictly dissipative) with the functions V and s if there exists a regular static state feedback control law of the form, $u = \gamma(x, v)$, with v the new input, such that the feedback transformed system is (V,s)-dissipative (resp., strictly (V,s)-dissipative).*

The existence of a feedback control law, of the form $u = \gamma(x, v)$, for which the system is rendered (V, s)-dissipative must be assessed from the existence of solutions, for the control input u, of the following equation,

$$L_{f(x,u)}V(x) = s(h(x,u), v) - \phi(x,u) \tag{3.17}$$

The following theorem states sufficient conditions under which feedback dissipativity is possible.

Theorem 3.2 (Sira-Ramírez and Navarro-López, 2000) *[161] Consider a system of the form (3.1)-(3.2) with $m = 1$ and two scalar functions $V(x)$, $s(y, v)$ considered as a storage function and a supply function, respectively. Let $\phi(x, u)$ be a given $\mathscr{C}^1$ function $\phi : \mathscr{X} \times \mathscr{U} \to \mathfrak{R}$ such that $\phi(\cdot, u)$ is positive for each $u \in \mathscr{U}$. Let $(x_0, u_0, v_0) \in A$, with $A = \mathscr{X} \times \mathscr{U} \times \mathscr{U}$ an open set. Suppose that the following two conditions are satisfied:*

1. *There exists (x_0, u_0, v_0) for which the equality (3.17) holds true, i.e.*

$$\left. L_{f(x,u)}V(x) \right|_{(x_0,u_0)} = s(h(x_0, u_0), v_0) - \phi(x_0, u_0), \tag{3.18}$$

2.

$$\left[L_{\frac{\partial f}{\partial u}(x,u)}V(x) - \frac{\partial}{\partial u}s(h(x,u), v) + \frac{\partial}{\partial u}\phi(x,u) \right]_{(x_0,u_0,v_0)} \neq 0 \tag{3.19}$$

Then, there exists a unique *static state feedback control law of the form, $u = \gamma(x,v)$ defined in a neighbourhood of (x_0,v_0) and valued in a neighbourhood of u_0, such that the feedback transformed system $\dot{x} = \overline{f}(x,v)$, $y = \overline{h}(x,v)$ is (V,s)-dissipative.*

Proof. The proof follows directly from the *implicit function* theorem.

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Remark 3.2 *The result given is local. The feedback control law which achieves dissipativity is locally valid; control u satisfies (3.17) for all (x,v) sufficiently close to (x_0,v_0). In the sequel, when referring the feedback dissipativity problem we will implicitly refer to the local feedback dissipativity problem.*

Remark 3.3 *If condition (3.19) is satisfied for (x_0,u_0,v_0), it also holds in a neighbourhood of (x_0,u_0,v_0), in other words, there exists an open neighbourhood $\widetilde{\mathscr{W}} = \widetilde{\mathscr{X}} \times \widetilde{\mathscr{U}}_1 \times \widetilde{\mathscr{U}}_2$ containing (x_0,u_0,v_0) such that $\widetilde{\mathscr{W}} \subset \mathscr{X} \times \mathscr{U} \times \mathscr{U}$ where the following assertion is valid,*

$$L_{\frac{\partial f}{\partial u}(x,u)} V(x) - \frac{\partial}{\partial u}s(h(x,u),v) + \frac{\partial}{\partial u}\phi(x,u) \neq 0, \ \ \forall (x,u,v) \in \widetilde{\mathscr{W}} \qquad (3.20)$$

Condition (3.20) will be regarded as the feedback dissipativity condition *in the continuous-time setting.*

Definition 3.3 *A system of the form (3.1)-(3.2) with $m = 1$ is said to be feedback passive if it is feedback dissipative with $s = yu$.*

Example 3.2 *For systems affine in the control, of the form $\dot{x} = f(x) + g(x)u$, $y = h(x)$, $s(y,u) = yu$, and considering ϕ only depending on the state, condition (3.20) reduces to the transversality condition* (Sira-Ramírez, 1998) *[159],*

$$L_{g(x)}V(x) \neq 0,$$

which has the clear geometric interpretation of having a control vector field $g(x)$ which is locally nowhere tangent to the level sets $\{ \, x \mid V(x) = constant \, \}$. Under the validity of such a transversality condition, the existence of a feedback control law of the form, $u = \alpha(x) + \beta(x)v$, is guaranteed. This is obtained from

$$L_{f(x)}V(x) + uL_{g(x)}V(x) = h(x)v - \phi(x) \qquad (3.21)$$

and, hence

$$u = \alpha(x) + \beta(x)v = -\frac{L_{f(x)}V(x) + \phi(x)}{L_{g(x)}V(x)} + \frac{h(x)}{L_{g(x)}V(x)}v \qquad (3.22)$$

The closed-loop system

$$\dot{x} = \left(I - g(x)\frac{\partial V(x)/\partial x}{L_{g(x)}V(x)} \right) f(x) - g(x)\frac{\phi(x)}{L_{g(x)}V(x)} + g(x)\frac{h(x)}{L_{g(x)}V(x)}v \qquad (3.23)$$

satisfies,

$$
\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial x}\left[\left(I - g(x)\frac{\partial V/\partial x}{L_{g(x)}V(x)}\right)f(x) - g(x)\frac{\phi(x)}{L_{g(x)}V(x)} + g(x)\frac{h(x)}{L_{g(x)}V(x)}v\right] \\
&= h(x)v - \phi(x)
\end{aligned}
\tag{3.24}
$$

3.5 Dissipativity and stability

Dissipative systems and its particular case of passive systems present highly desirable properties for a system, namely the ones referring to its stability, which may simplify system analysis and control design. The immediate consequences of passivity are referred to the stability of the system evolving under no control actions applied to the system and the nature of the stability of the zero dynamics associated to the zero value, for an indefinite period of time, of the output function *y*. It can easily deduced that a zero-input passive system with an equilibrium point at the origin and with a differentiable positive definite storage function, which is zero at the origin, is stable in the sense of Lyapunov. Furthermore, any passive system is weakly minimum phase *(Byrnes et al., 1991)* [12], since if the system output is rendered zero by means of an adequate feedback, the remaining dynamics or *zero dynamics* is Lyapunov stable.

In this section, the implications of dissipativity in the stability of a system of the form (3.1)-(3.2) with $m = 1$ are treated. The results achieved in the previous section concerning the feedback dissipativity problem will be used.

Stability will be studied for a class of dissipative systems whose supply function holds the following property.

Definition 3.4 *The supply function $s(y,u)$ is said to satisfy the zero-input-output property if*

$$
\begin{aligned}
s(0,u) &= 0, \quad \forall u \in \mathscr{U} \\
s(y,0) &= 0, \quad \forall y \in \mathscr{Y}
\end{aligned}
\tag{3.25}
$$

A nonlinear regular static state feedback control law of the form: $u = \gamma(x,v)$, which achieves either (V,s)-*dissipativity* or *strict* (V,s)-*dissipativity* by means of static state feedback, induces an *implicit damping injection* which makes the system locally stable (resp., asymptotically stable) for certain particular values of the transformed control input. The meaning of this assertion is clarified in the following theorem.

Theorem 3.3 (Sira-Ramírez and Navarro-López, 2000) *[161] Let the system (3.1)-(3.2) with $m = 1$ and two scalar functions $V(x)$ and $s(y,v)$ as a storage function and a supply function, respectively. Let $\phi(x,u)$ be a $\mathscr{C}^1$ function $\phi : \mathscr{X} \times \mathscr{U} \to \mathfrak{R}$ such that $\phi(\cdot,u)$ is positive (resp., strictly positive) for each $u \in \mathscr{U}$. Suppose there exists a feedback control law, $u = \gamma(x,v)$, which achieves (V,s)-dissipativity (resp., strict (V,s)-dissipativity) of the feedback transformed system with s satisfying (3.25). Then, the control law $u = \gamma(x,0)$ renders the zero solution $x = 0$ of the system locally stable (resp., locally asymptotically stable).*

Proof. Suppose that $u = \gamma(x,v)$ achieves (V,s)-dissipativity (the argument is the same for strict (V,s)-dissipativity). Consider (2.7) and the time derivative of $V(x)$,

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial x} f(x, \gamma(x,v)) = s(h(x, \gamma(x,v)), v) - \phi(x, \gamma(x,v)) \\
&\leq s(h(x, \gamma(x,v)), v)
\end{aligned}$$

Since $V(x)$ is positive definite and s satisfies (3.25), and use is made of the above feedback control law with $v = 0$, then the resulting feedback transformed system is locally stable to zero. Indeed, for $v = 0$,

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial x} f(x, \gamma(x,0)) = \\
&= s(h(x, \gamma(x,0)), 0) - \phi(x, \gamma(x,0)) = -\phi(x, \gamma(x,0)) \leq 0
\end{aligned}$$

The result of the theorem follows.

$\blacksquare$

3.6 The energy shaping plus damping injection method

A fundamental control problem is that of stabilization of the system trajectories around a desired equilibrium. To this purpose, the ESDI control design methodology has been developed on the basis of modifying the stored energy of the system, to take into account the desired equilibrium, and the addition, through state or output feedback, of the required dissipation in order to enhance the dissipation structure of the underlying stabilization error system. As it was pointed out in Section 2.5.2 there exist two main approaches in treating the ESDI methodology.

In this section, the approach for the ESDI methodology given for nonlinear continuous-time systems which are affine in the control input in *(Sira-Ramírez, 1998)* [159] is extended to the case of non-affine nonlinear systems of the form (3.1)-(3.2). This stabilization approach will make use of the feedback dissipativity solution given in Section 3.4, and the stability comments concluded in Section 3.5.

Theorem 3.4 (Sira-Ramírez and Navarro-López, 2000) *[161] Let the system (3.1)-(3.2) with $m = 1$ be a nonlinear system which may be rendered (V,s)-dissipative with the functions $V(x)$ and $s(y,v)$ considered as a storage function and a supply function, respectively, by means of a regular nonlinear static state feedback control law of the form $u = \gamma(x,v)$, with s satisfying (3.25), i.e., let $\gamma(x,v) = \overline{u}$ be the regular static state feedback control representing a state dependent solution, parameterized by v, of the equality*

$$\frac{\partial V}{\partial x} f(x, \overline{u}) = s(h(x, \overline{u}), v) - \phi(x, \overline{u}) \tag{3.26}$$

Then, the error vector $e = x - \xi$, with ξ defined as

$$\dot{\xi} = f(x, u) - f(x - \xi, \gamma(x - \xi, 0)) + R(x - \xi) \left[\left. \frac{\partial V(e)}{\partial e^T} \right|_{e = x - \xi} \right] \tag{3.27}$$

with $R(e)$ being an $n \times n$ positive semi-definite matrix, such that

$$\frac{\partial V(e)}{\partial e}R(e)\frac{\partial V(e)}{\partial e^T} + \phi(e,u) > 0 \tag{3.28}$$

for each $u \in \mathscr{U}$, is locally asymptotically stable *to zero. Moreover, the asymptotic stability result holds even if $R(e)$ is such that,*

$$\frac{\partial V(e)}{\partial e}R(e)\frac{\partial V(e)}{\partial e^T} + \phi(e,u) \geq 0 \tag{3.29}$$

for each $u \in \mathscr{U}$, and

$$\left\{ e \mid \frac{\partial V(e)}{\partial e}f(e,\gamma(e,0)) = 0 \right\} \cap \left\{ e \mid \frac{\partial V(e)}{\partial e}R(e)\frac{\partial V}{\partial e^T} = 0 \right\} = \{0\} \tag{3.30}$$

Proof. Consider the modified stored energy function $V(x - \xi)$. The time derivative of the modified energy function is given by

$$\dot{V}(x - \xi) = \frac{\partial V(e)}{\partial e}\bigg|_{e=x-\xi}\left[f(x,u) - \dot{\xi}\right] \tag{3.31}$$

According to the dynamics assigned to the auxiliary variable ξ, it follows that the time derivative of the function $V(e)$ takes the following form

$$\begin{aligned}
\dot{V}(e) &= \frac{\partial V(e)}{\partial e}\left[f(e,\gamma(e,0)) - R(e)\frac{\partial V(e)}{\partial e^T}\right] = \\
&= \frac{\partial V(e)}{\partial e}f(e,\gamma(e,0)) - \frac{\partial V(e)}{\partial e}R(e)\frac{\partial V(e)}{\partial e^T} \leq \\
&\leq -\left[\phi(e,\gamma(e,0)) + \frac{\partial V(e)}{\partial e}R(e)\frac{\partial V(e)}{\partial e^T}\right] < 0
\end{aligned} \tag{3.32}$$

The main result of the theorem follows from fundamental results of Lyapunov stability theory.

To prove the second part of the theorem, consider the set,

$$\left\{ e \in \mathfrak{R}^n \mid \dot{V}(e) = \frac{\partial V(e)}{\partial e}\left[f(e,\gamma(e,0)) - R(e)\frac{\partial V(e)}{\partial e}\right] = 0 \right\} \tag{3.33}$$

This set clearly coincides with the set,

$$\left\{ e \in \mathfrak{R}^n \mid \frac{\partial V(e)}{\partial e}f(e,\gamma(e,0)) = \frac{\partial V(e)}{\partial e}R(e)\frac{\partial V(e)}{\partial e^T} \right\}$$

The left hand side of the last equality, according to the Theorem 3.3, is a negative function, while, from the definition of $R(e)$, the right hand side is a positive function of e. The equality can only be valid over the set of values of e where both expressions are zero. This implies that the set is given by

$$\left\{ e \in \mathfrak{R}^n \mid \frac{\partial V(e)}{\partial e}f(e,\gamma(e,0)) = 0 \right\} \cap \left\{ e \in \mathfrak{R}^n \mid \frac{\partial V(e)}{\partial e}R(e)\frac{\partial V}{\partial e^T} = 0 \right\} \tag{3.34}$$

It follows, according to the assumption in the theorem, that the invariance set, $\{\, e \mid \dot{V}(e) = 0\,\}$, is constituted just by the singleton represented by $\{\, e \in \Re^n \mid e = 0\,\}$. The asymptotic stability to zero of the trajectories of the error, $e(t) = x(t) - \xi(t)$, follows as a consequence of LaSalle's invariance theorem.

$\blacksquare$

3.7 A dissipativity-oriented form for nonlinear systems

It will be shown that, under very mild conditions, nonlinear systems enjoy a form which is of Generalized Hamiltonian-type form treated extensively in *(Crouch and van der Schaft, 1987)* [23]. This form will present special features for dissipative systems, and besides it will be used for stabilization purposes.

Theorem 3.5 (Sira-Ramírez and Navarro-López, 2000) *[161] Let the following condition be satisfied over all (x,u) in the open set $\mathscr{X} \times \mathscr{U}$,*

$$L_{\frac{\partial f}{\partial u}} V(x) \neq 0 \tag{3.35}$$

Then, the system (3.1)-(3.2) with $m = 1$ can be written in the form,

$$\dot{x} = [J(x,u) + S(x,u)]\frac{\partial V}{\partial x^T}, \quad y = h(x,u) \tag{3.36}$$

with

$$J(x,u) + J^T(x,u) = 0, \quad S(x,u) = S^T(x,u), \ \ \forall (x,u) \in \mathscr{X} \times \mathscr{U}$$

Proof. Consider the following string of equalities

$$
\begin{aligned}
\dot{x} \ = \ & f(x,u) = \\[2mm]
= \ & f(x,u) + \frac{\partial f(x,u)}{\partial u}\left[\frac{\partial V/\partial x}{L_{\frac{\partial f}{\partial u}} V(x)}f(x,u)\right] - \frac{\partial f(x,u)}{\partial u}\left[\frac{\partial V/\partial x}{L_{\frac{\partial f}{\partial u}} V(x)}f(x,u)\right] = \\[2mm]
= \ & \left[I - \frac{\partial f(x,u)}{\partial u}\left(\frac{\partial V/\partial x}{L_{\frac{\partial f}{\partial u}} V(x)}\right)\right]f(x,u) + \frac{\partial f(x,u)}{\partial u}\left[\frac{\partial V/\partial x}{L_{\frac{\partial f}{\partial u}} V(x)}f(x,u)\right] = \\[2mm]
= \ & \frac{1}{L_{\frac{\partial f}{\partial u}} V(x)}\left[f(x,u)\frac{\partial f^T(x,u)}{\partial u} - \frac{\partial f(x,u)}{\partial u}f^T(x,u)\right]\frac{\partial V}{\partial x^T} + \\[2mm]
& + \frac{\partial f(x,u)}{\partial u}\left[\frac{\partial V/\partial x}{L_{\frac{\partial f}{\partial u}} V(x)}f(x,u)\right] \tag{3.37}
\end{aligned}
$$

On the other hand, the term

$$\frac{\partial f(x,u)}{\partial u}\left[\frac{\partial V/\partial x}{L_{\frac{\partial f}{\partial u}} V(x)}f(x,u)\right] = \frac{1}{L_{\frac{\partial f}{\partial u}} V(x)}\left[\frac{\partial f(x,u)}{\partial u}f^T(x,u)\right]\frac{\partial V}{\partial x^T}$$

can be decomposed as the sum of two further terms; one containing a symmetric matrix and a second one having a skew-symmetric matrix.

$$\frac{1}{L_{\frac{\partial f}{\partial u}}V(x)}\left[\frac{\partial f(x,u)}{\partial u}f^T(x,u)\right]\frac{\partial V}{\partial x^T} =$$

$$= \frac{1}{2L_{\frac{\partial f}{\partial u}}V(x)}\left\{\left[\frac{\partial f(x,u)}{\partial u}f^T(x,u)+f(x,u)\frac{\partial f^T(x,u)}{\partial u}\right]+\right.$$

$$\left.+\left[\frac{\partial f(x,u)}{\partial u}f^T(x,u)-f(x,u)\frac{\partial f^T(x,u)}{\partial u}\right]\right\}\frac{\partial V}{\partial x^T} \tag{3.38}$$

Combining this last expression with those in (3.37), one obtains,

$$\dot{x} = \frac{1}{2L_{\frac{\partial f}{\partial u}}V(x)}\left[f(x,u)\frac{\partial f^T(x,u)}{\partial u}-\frac{\partial f(x,u)}{\partial u}f^T(x,u)\right]\frac{\partial V}{\partial x^T} +$$

$$+\frac{1}{2L_{\frac{\partial f}{\partial u}}V(x)}\left[f(x,u)\frac{\partial f^T(x,u)}{\partial u}+\frac{\partial f(x,u)}{\partial u}f^T(x,u)\right]\frac{\partial V}{\partial x^T} =$$

$$= [J(x,u)+S(x,u)]\frac{\partial V}{\partial x^T} \tag{3.39}$$

therefore, one has

$$J(x,u) = \frac{1}{2L_{\frac{\partial f}{\partial u}}V(x)}\left[f(x,u)\frac{\partial f^T(x,u)}{\partial u}-\frac{\partial f(x,u)}{\partial u}f^T(x,u)\right]$$

$$S(x,u) = \frac{1}{2L_{\frac{\partial f}{\partial u}}V(x)}\left[f(x,u)\frac{\partial f^T(x,u)}{\partial u}+\frac{\partial f(x,u)}{\partial u}f^T(x,u)\right] \tag{3.40}$$

∎

Remark 3.4 *$J(x,u)$ is an $n \times n$-skew-symmetric matrix and $J(x,u)\frac{\partial V}{\partial x^T}$ represents the energy conservative part of the system. The term $S(x,u)\frac{\partial V}{\partial x^T}$ groups the dissipative and non-dissipative terms for the system dynamics. For stability purposes, we would like to convert the terms appearing in $S(x,u)\frac{\partial V}{\partial x^T}$ into dissipative ones for some energy function, which may be different from the initial one V.*

According to the last theorem, any nonlinear system for which the condition (3.35) is satisfied, can be written in the above form. If the system is feedback dissipative (respectively, feedback strictly dissipative) with s satisfying the zero-input-output property, one can more specifically characterize the resulting closed-loop matrix, $S(x,\gamma(x,v)) = \overline{S}(x,v)$, as a symmetric negative semi-definite (respectively, negative definite) matrix. This is the topic of the next theorem where we also denote $J(x,\gamma(x,v))$ as $\overline{J}(x,v)$.

Theorem 3.6 *Let the system (3.1)-(3.2) with $m = 1$. Let $\dot{x} = \overline{f}(x,v)$, $y = \overline{h}(x,v)$ be a system which has been rendered (V,s)-dissipative (resp., strictly (V,s)-dissipative) by means of a regular static state feedback with s satisfying (3.25). Suppose that the following condition is satisfied*

$$L_{\frac{\partial \overline{f}}{\partial v}}V(x) \neq 0 \tag{3.41}$$

Then, there exists a neighbourhood $\widetilde{\mathscr{W}} = \widetilde{\mathscr{X}} \times \widetilde{\mathscr{U}} \subset \mathscr{W} = \mathscr{X} \times \mathscr{U}$ where the feedback transformed system can be written as

$$\dot{x} = \left[\overline{J}(x,v) + \overline{S}(x,v) \right] \frac{\partial V}{\partial x^T} \tag{3.42}$$

with

$$\overline{J}(x,v) + \overline{J}^T(x,v) = 0, \ \ \overline{S}(x,v) = \overline{S}^T(x,v) \leq 0, \ \ (\text{resp.}, \ < 0) \tag{3.43}$$

Proof. The fact that the feedback transformed system can be written in the Generalized Hamiltonian-type form follows from the assumption (3.41) and from the result of the previous Theorem 3.5. To prove the fact that $\overline{S}(x,v)$ is negative semi-definite or negative definite, consider the time derivative of the storage function $V(x)$, and assume, for simplicity, that the feedback transformed system is (V,s)-dissipative (the argument is the same when the feedback transformed system is strictly (V,s)-dissipative),

$$
\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial x^T} \overline{S}(x,v) \frac{\partial V}{\partial x} = \frac{\partial V}{\partial x^T} \overline{f}(x,v) = \\
&= s(h(x, \gamma(x,v)), v) - \phi(x, \gamma(x,v))
\end{aligned}
\tag{3.44}
$$

Letting $v = 0$ and considering (3.25), one has

$$\frac{\partial V}{\partial x^T} \overline{S}(x,0) \frac{\partial V}{\partial x} = -\phi(x,0) \leq 0 \ \ \forall x \in \mathscr{X} \tag{3.45}$$

Therefore, there exists a neighborhood $\widetilde{\mathscr{U}}$ of the origin in $\mathscr{U}$, and an open subset, $\widetilde{\mathscr{X}}$ of $\mathscr{X}$, i.e. an open neighborhood $\widetilde{\mathscr{W}} = \widetilde{\mathscr{X}} \times \widetilde{\mathscr{U}} \subset \mathscr{W} = \mathscr{X} \times \mathscr{U}$, where the following assertion is valid,

$$\frac{\partial V}{\partial x^T} \overline{S}(x,v) \frac{\partial V}{\partial x} \leq 0 \ \ \forall (x,v) \in \widetilde{\mathscr{W}} \tag{3.46}$$

∎

Dynamics (3.36) for a feedback dissipative system can be used for stabilization purposes by means of the ESDI methodology as it was proposed in Section 3.6. Here, a note on this topic will be given. The ESDI will be based on the proposal of an auxiliary dynamics based on the new expression for the system which has been rendered dissipative, using matrices J and S as the ones proposed above.

Theorem 3.7 *Let the system (3.1)-(3.2) with $m = 1$. Let $\dot{x} = \overline{f}(x,v)$, $y = \overline{h}(x,v)$ a system which has been rendered (V,s)-dissipative (resp., strictly (V,s)-dissipative) by means of a regular static state feedback with s satisfying (3.25) and is written in the following form in a neighbourhood $\widetilde{\mathscr{W}} = \widetilde{\mathscr{X}} \times \widetilde{\mathscr{U}} \subset \mathscr{W} = \mathscr{X} \times \mathscr{U}$*

$$\dot{x} = [\overline{J}(x,v) + \overline{S}(x,v)] \frac{\partial V}{\partial x^T}, \ \ y = \overline{h}(x,v) \tag{3.47}$$

with

$$\overline{J}(x,v) + \overline{J}^T(x,v) = 0, \ \ \overline{S}(x,v) = \overline{S}^T(x,v), \ \ \forall (x,v) \in \widetilde{\mathscr{W}}, \tag{3.48}$$

where $\overline{S}$ is a negative semi-definite matrix. Suppose the storage function V be of the form $V = x^T P x$, with P a positive definite matrix.

Then, the error vector $e = x - \xi$, with ξ defined as

$$\dot{\xi} = [\overline{J}(x,v) + \overline{S}(x,v)]\frac{\partial V(\xi)}{\partial \xi^T} + R(x - \xi)\left[\left.\frac{\partial V(e)}{\partial e^T}\right|_{e=x-\xi}\right], \qquad (3.49)$$

with $R(e)$ being an $n \times n$ positive definite matrix, is locally asymptotically stable *to zero.*

Proof. Consider the modified stored energy function $V(x - \xi)$. The time derivative of the modified energy function along the trajectories of (3.47) is given by

$$\dot{V}(x - \xi) = \left.\frac{\partial V(e)}{\partial e}\right|_{e=x-\xi}\left\{[\overline{J}(x,v) + \overline{S}(x,v)]\frac{\partial V(x)}{\partial x^T} - \dot{\xi}\right\} \qquad (3.50)$$

Taking into account the dynamics assigned to the auxiliary variable ξ (3.49), it follows from (3.50) that the time derivative of the function $V(e)$ is

$$\begin{aligned} \dot{V}(e) &= \frac{\partial V(e)}{\partial e}\left\{[\overline{J}(x,v) + \overline{S}(x,v)]\left[\frac{\partial V(x)}{\partial x^T} - \frac{\partial V(\xi)}{\partial \xi^T}\right] - R(e)\frac{\partial V(e)}{\partial e^T}\right\} = \\ &= \frac{\partial V(e)}{\partial e}[\overline{J}(x,v) + \overline{S}(x,v) - R(e)]\frac{\partial V(e)}{\partial e^T} \end{aligned} \qquad (3.51)$$

In an open subset $\widetilde{\mathscr{W}} = \widetilde{\mathscr{X}} \times \widetilde{\mathscr{U}}$ of $\mathscr{X} \times \mathscr{U}$, $\overline{S}(x,v)$ is negative semi-definite, then

$$\begin{aligned} \dot{V}(e) &= \frac{\partial V(e)}{\partial e}[\overline{J}(x,v) + \overline{S}(x,v) - R(e)]\frac{\partial V(e)}{\partial e^T} \leq \\ &\leq -\frac{\partial V(e)}{\partial e}R(e)\frac{\partial V(e)}{\partial e^T} < 0 \end{aligned}$$

The main result follows from fundamental results of Lyapunov stability.

∎

3.8 Conclusions and future work

In this chapter, some dissipativity-related results have been obtained for continuous-time nonlinear systems which are non-affine in the control input.

A set of necessary and sufficient conditions fulfilled for a class of multiple-input multiple-output dissipative nonlinear systems which are non-affine in the control input have been proposed. The characterization is restricted to systems for which the derivative of the storage function V along the trajectories of the system and the supply function s are quadratic in u. These conditions can be considered a form of rewriting for the non-affine case the conditions appeared in *(Hill and Moylan, 1976)* [53] given for (Q,R,S)-dissipativity for nonlinear systems affine in the control input.

The feedback dissipativity problem has also been treated for a class of single-input single-output dissipative systems regarded as (V,s)-dissipative systems. The control which achieves feedback dissipativity is obtained as the solution of the fundamental dissipativity equality, and is locally valid. This proposal can be considered as an extension to the nonlinear non-affine case of the feedback passivity problem given in *(Sira-Ramírez, 1998)* [159] for nonlinear systems which are affine in the control input.

Some stability implications have been derived for dissipative systems with supply functions having the property of what has been regarded as zero-input-output property. These stability conclusions in addition to the feedback dissipativity results have been used in order to extend the ESDI controller design methodology to the case of non-affine nonlinear systems. Besides, under mild conditions, a form of writing dissipative systems has been clearly found indicating the conservative, the dissipation, the de-stabilizing and the external energy acquisition terms. This form points to an interesting "energy managing structure", which drives the state velocity in terms of contributions of stored energy gradient projections. This way of writing dissipative systems is quite helpful in nonlinear feedback controller design tasks, as it was shown in Section 3.7, where the (J,S)-form for dissipative systems is used for stabilization purposes through the ESDI control design methodology.

The results here obtained can be extended, with some technical difficulties, to the case of general multivariable nonlinear systems. We have obtained some preliminary results for the multivariable case. They concern to the proposal of a dissipativity form as the one proposed in Section 3.7 using Lyapunov vector functions in order to solve the feedback dissipativity problem and to stabilize the system by means of the ESDI approach. These results are considered to be out of the interest of this dissertation, however, they will be revisited in the Conclusions chapter of this dissertation (Chapter 9).

The conclusions obtained in this chapter referring dissipativity, feedback dissipativity and stability implications for dissipative systems, as well as the ESDI approach will be extended to the discrete-time case in Chapters 4, 5 and 6, respectively.