Chapter 4

Towards the characterization of general discrete-time dissipative systems

4.1 Introduction

The aim of this chapter is to make an advance in the study of the properties of nonlinear discrete-time dissipative and lossless systems. The existing approaches in the literature will be used, basically, the ones based on the state-space dynamical represetation and the use of the storage and the supply functions.

Dissipativity and losslessness properties of multiple-input multiple-output discrete-time systems which are nonlinear in the states and the control input will be examined.

Necessary conditions for the characterization of a class of dissipative and lossless systems, regarded as (V,s)-dissipative and (V,s)-lossless systems will be given. Necessary and sufficient conditions for a class of dissipative and lossless nonlinear discrete-time systems will be proposed. The kind of dissipativity and losslessness treated will be regarded as *QSS* (Quadratic Storage Supply) dissipativity and *QSS*-losslessness, respectively, referring to (V,s)-dissipative and (V,s)-lossless systems whose storage (V) and supply (s) functions are such that $V(f(x,u))$ and $s(h(x,u),u)$ are quadratic in u.

Most of the conditions existing in the literature, addressed as KYP conditions, either for dissipative, passive or lossless cases, are strictly contained in the proposed dissipativity properties.

This chapter is organized as follows. Section 4.2 explains the kind of formalization for dissipativity to be adopted along the chapter. Section 4.3 is endeavoured to the study of dissipativity properties, whereas Section 4.4 is devoted to the losslessness problem. Conclusions and further research are presented in the last section.

4.2 Formalization of dissipativity-related concepts

There are two important representations of dynamical systems: the input-output description via an operator on a function space, and the state-space description.

The input-output representation requires minimal knowledge of the physical system laws and its internal interconnections. It can be used for discrete-time and continuous-time dynamics in the same framework and is based on functional analysis.

The state-space representation deals with the internal description of the system and its physical behaviour and interconnections are described by means of a model which generally takes the form of a differential or a difference equation, for the continuous-time or the discrete-time case, respectively. As it is well known, in this description, two different parts can be distinguished: a dynamical part, usually regarded as the state equation, which describes the evolution of the states under the influence of the inputs, and a memoryless part relating the output to the state and the input.

Dissipativity and passivity can be formalized from these two different points of view: considering the input-output description of the system via an operator on a function space or via the state-space or internal dynamical representation. The former endows the frequency-domain characterization of dissipativity; for the discrete-time case, see for example *(Wu and Desoer, 1970)* [182], *(Popov, 1973)* [141], *(Goodwin and Sin, 1984)* [45], *(Albertos, 1993)* [1]. The latter interprets dissipativity by means of an energy balance equation; for the discrete-time case, see for example *(Byrnes and Lin, 1994)* [14], *(Sengör, 1995)* [151].

In this dissertation, the state-space or internal description approach will be used. Therefore, the basis of our results will be the definitions given in Chapter 2.

Let the system

$$x(k+1) = f(x(k),u(k)),\ \ x \in \mathscr{X},\ u \in \mathscr{U} \tag{4.1}$$

$$y(k) = h(x(k),u(k)),\ \ y \in \mathscr{Y} \tag{4.2}$$

where $f : \mathscr{X} \times \mathscr{U} \to \mathscr{X}$, and $h : \mathscr{X} \times \mathscr{U} \to \mathscr{Y}$ are smooth maps with $\mathscr{X}$ an open subset of $\Re^n$, and $\mathscr{U}$, $\mathscr{Y}$ open subsets of $\Re^m$. $k \in Z_+ := \{0,1,2,\ldots\}$. All considerations will be restricted to an open set of $\mathscr{X} \times \mathscr{U}$ containing $(\overline{x},\overline{u})$, having $\overline{x}$ as an isolated fixed point of $f(x,\overline{u})$, i.e., $f(\overline{x},\overline{u}) = \overline{x}$. We consider a positive definite C^2 function $V : \mathscr{X} \to \Re$, $V(0) = 0$, associated with the system (4.1)-(4.2) and addressed as the *storage function*. A second C^2 function is also considered, called the *supply function*, denoted by $s(y,u)$, with $s : \mathscr{Y} \times \mathscr{U} \to \Re$.

Definition 4.1 *A $\mathscr{C}^2$ function $\phi : \mathscr{X} \times \mathscr{U} \to \Re$, such that $\phi(\cdot,u)$ is positive (respectively, strictly positive) for each $u \in \mathscr{U}$, with $\phi(0,0) = 0$ is regarded as a dissipation rate (resp., strict dissipation rate) function in the sense proposed in Hill and Moylan [55].*

The dissipativity definition in the discrete-time nonlinear setting given in Byrnes and Lin [14] will be rewritten in the following way.

Definition 4.2 *The system (4.1)-(4.2) with storage function $V(x)$ and supply function $s(y,u)$ is said to be (V,s)-dissipative (resp., strictly (V,s)-dissipative) if there exists a*

dissipation rate (resp., strict dissipation rate) function ϕ such that

$$V(f(x,u)) - V(x) = s(h(x,u),u) - \phi(x,u), \quad \forall (x,u) \in \mathscr{X} \times \mathscr{U} \tag{4.3}$$

Definition 4.3 *The system (4.1)-(4.2) is said to be V-passive if it is (V,s)-dissipative with respect to the supply rate $s(y,u) = y^T u$.*

4.3 Dissipativity in the discrete-time domain

The properties that a system has to meet in order to be passive or dissipative are usually known as the KYP conditions. The KYP conditions for passivity were originally established in the discrete-time domain by Hitz and Anderson in *(Hitz and Anderson, 1969)* [58] for the linear case as the Positive Discrete Real Lemma. As it was pointed out in Chapter 2, the KYP lemma was first proposed for linear systems, finding the relation between passivity and its frequency stability properties. In the sequel, when the KYP denomination is used, we will refer to the conditions that any dissipative or passive system fulfils.

In the literature, the KYP conditions have not been established in a general way for dissipative or passive nonlinear discrete-time systems of the form (4.1)-(4.2). There exist necessary and sufficient conditions for characterizing passive nonlinear discrete-time systems of the affine form $x(k+1) = f(x(k)) + g(x(k))u(k)$, $y(k) = h(x(k)) + J(x(k))u(k)$, presumed the stored energy function V satisfies $V(f(x)+g(x)u)$ is quadratic in u *(Byrnes and Lin, 1993)* [13]. Necessary conditions for systems of the form (4.1)-(4.2) to be passive are stated in *(Lin, 1995)* [86]. Moreover, the necessary and sufficient conditions for dissipativity in the discrete-time setting appearing in *(Sengör, 1995)* [151], *(Göknar and Sengör, 1998)* [44] are proposed for affine-in-control system structures as well. For single-input multiple-output nonlinear systems which are non-affine in the control input, the KYP conditions for passivity and losslessness are proposed in *(Monaco and Normand-Cyrot, 1997, 1999)* [112, 113], where the authors phocus on systems which can be expanded by exponential Lie series; in this dissertation, this kind of passivity and losslessness characterization will not be used.

It must be pointed out that, in the literature, the KYP denomination is usually used for the set of properties for a passive or a dissipative system characterization, even if they are only necessary conditions. In this sense, the following conditions are proposed, which are fulfilled by any nonlinear discrete-time dissipative system of the form (4.1)-(4.2). The corresponding result for passive systems is obtained taking $s = y^T u$.

Proposition 4.1 (Navarro-López et al., 2002) *[119] Let a discrete-time system of the form (4.1)-(4.2) be (V,s)-dissipative, then*

$$V(f(x,0)) - V(x) \leq s(h(x,0),0) \tag{4.4}$$

$$\frac{\partial}{\partial u} V(f(x,u)) = \frac{\partial}{\partial u_2} s(h(x,u_1),u_2)\Big|_{\substack{u_1 = u \\ u_2 = u}} + \frac{\partial}{\partial y} s(y,u) \frac{\partial}{\partial u} h(x,u) - \frac{\partial}{\partial u} \phi(x,u) \tag{4.5}$$

with ϕ a dissipation rate function.

Proof. Inequality (4.4) directly follows from (4.3), taking $u = 0$. Equality (4.5) is obtained by taking partial derivatives with respect to u in equation (4.3).

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Remark 4.1 *If we consider the dissipation rate function ϕ only depending on the state, equation (4.5) takes the form*

$$\frac{\partial}{\partial u}V(f(x,u)) = \frac{\partial}{\partial u_2}s(h(x,u_1),u_2)\Big|_{\substack{u_1=u\\u_2=u}} + \frac{\partial}{\partial y}s(y,u)\frac{\partial}{\partial u}h(x,u) \tag{4.6}$$

where we can clearly identify two terms: a term resulting from the general form of the supply function and a term resulting from the feed-forward presence of the input in the output equation.

Remark 4.2 *The necessary conditions for passive systems of the form (4.1)-(4.2) given in* (Lin, 1995) *[86] are different from the ones given in (4.4) and (4.5). In Lin's work, the passivity inequality is used and no dissipation rate function ϕ is introduced.*

If functions $V(f(x,u))$ and $s(h(x,u),u)$ are quadratic in u, then relations (4.4)-(4.5) yield necessary and sufficient conditions as Theorem 4.1 shows. Taking into account these conditions, our characterization is restricted to a class of dissipative systems for which the storage function V is such that $V(f(x,u))$ is quadratic in u, as well as the supply function s is such that $s(h(x,u),u)$ is quadratic in u. Therefore, for clearness' sake, the following definition is introduced.

Definition 4.4 *A system of the form (4.1)-(4.2) is said to be QSS (*Quadratic Storage Supply*) dissipative if it is (V,s)-dissipative with a storage function V and a supply function s such that $V(f(x,u))$ and $s(h(x,u),u)$ are quadratic in u.*

Theorem 4.1 (Navarro-López and Fossas, 2002) *[121] Let V and s be storage and supply functions such that $V(f(x,u))$ and $s(h(x,u),u)$ are quadratic in u. Then, a discrete-time nonlinear system of the form (4.1)-(4.2) is QSS-dissipative with V and s, if and only if, there exist real functions $l(x)$, $m(x)$ and $k(x)$, all of appropriate dimensions such that*

$$V(f(x,0)) - V(x) = s(h(x,0),0) - l^T(x)l(x) - m^T(x)m(x) \tag{4.7}$$

$$\frac{\partial V(z)}{\partial z}\Big|_{z=f(x,0)}\frac{\partial f(x,u)}{\partial u}\Big|_{u=0} + 2l^T(x)k(x) = \frac{\partial}{\partial u}s(h(x,u),u)\Big|_{u=0} \tag{4.8}$$

$$\left(\frac{\partial f(x,u)}{\partial u}\right)^T\Big|_{u=0}\frac{\partial^2 V(z)}{\partial z^2}\Big|_{z=f(x,0)}\frac{\partial f(x,u)}{\partial u}\Big|_{u=0} + \frac{\partial V(z)}{\partial z}\Big|_{z=f(x,0)}\frac{\partial^2 f(x,u)}{\partial u^2}\Big|_{u=0} = \frac{\partial^2}{\partial u^2}s(h(x,u),u)\Big|_{u=0} - 2k^T(x)k(x) \tag{4.9}$$

Proof. (Necessity): Consider Definitions 4.2 and 4.4. If system (4.1)-(4.2) is *QSS*-dissipative, there exists a positive function ϕ satisfying (4.3). Since $V(f(x,u))$ and $s(h(x,u),u)$ are quadratic in u, the dissipation rate function ϕ can be written as follows

$$\phi(x,u) = [l(x)+k(x)u]^T[l(x)+k(x)u] + m^T(x)m(x) \geq 0, \ \ \forall u \in \mathscr{U} \tag{4.10}$$

for some real functions $m(x)$, $l(x)$ and $k(x)$. Condition (4.7) is obtained restricting (4.3) to $u=0$, and taking $\phi(x,u)$ as defined in (4.10). Conditions (4.8) and (4.9) follow from the first-order derivative and the second-order derivative of (4.3) with respect to u, considering (4.10) and $u=0$.

(Sufficiency): Assume there exist real functions $m(x), l(x), k(x)$ which satisfy conditions (4.7)-(4.9). Multiplying equality (4.8) by u^T from the left and adding (4.7), it is obtained

$$\begin{aligned} V(f(x,0)) - V(x) + u^T \left.\frac{\partial}{\partial u} V(f(x,u))\right|_{u=0} &= s(h(x,0),0) + u^T \left.\frac{\partial}{\partial u} s(h(x,u),u)\right|_{u=0} - \\ &\quad -2l^T(x)k(x)u - l^T(x)l(x) - m^T(x)m(x) \end{aligned} \tag{4.11}$$

Adding to the right-hand side term of (4.11) $u^T k^T k u \ -u^T k^T k u$, and using (4.9), one yields

$$\begin{aligned} & V(f(x,0)) + u^T \left.\frac{\partial}{\partial u} V(f(x,u))\right|_{u=0} + \frac{1}{2} u^T \left.\frac{\partial^2}{\partial u^2} V(f(x,u))\right|_{u=0} u - V(x) = \\ = \ & s(h(x,0),0) + u^T \left.\frac{\partial}{\partial u} s(h(x,u),u)\right|_{u=0} + \frac{1}{2} u^T \left.\frac{\partial^2}{\partial u^2} s(h(x,u),u)\right|_{u=0} u - \phi(x,u) \end{aligned} \tag{4.12}$$

with $\phi(x,u)$ given in (4.10). By claiming that $V(f(x,u))$ and $s(h(x,u),u)$ are quadratic in u, the second-order Taylor expansion at $u=0$ of $V(f(x,u))$ and $s(h(x,u),u)$ can be considered in (4.12), and (4.3) is obtained. In other words, the system is *QSS*-dissipative with supply function $s(y,u)$.

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Remark 4.3 *As it is recommended in* (Sengör, 1995) *[151], a new function $m(x)$ has been considered, in comparison to the passivity conditions given in* (Byrnes and Lin, 1993) *[13] for nonlinear discrete-time systems affine in the control input.*

Necessary and sufficient KYP conditions appeared in the literature for dissipative multiple-input multiple-output discrete-time systems are strictly contained in Theorem 4.1, either for the linear or the nonlinear case. Passivity conditions for nonlinear discrete-time systems which are affine in the control input appearing in *(Byrnes and Lin, 1993)* [13] (see Theorem 2.4) are obtained taking $s(y,u) = y^T u$ and $m(x) = 0$. In order to obtain dissipativity conditions for nonlinear affine in the control input systems presented in *(Sengör, 1995)* [151] (see Theorem A.8), the left-hand side of equality (4.7) would be $V(f(x)-x)$ and in (4.8) and (4.9), $z = f(x)-x$ should be considered with $V(x) = B^T(x) + x^T C(x)x$, with B and C matrices of appropriate dimensions, and $s(y,u) = y^T Q y + 2y^T S u +$

$u^T Ru$, with Q, S, R constant matrices, Q and R symmetric. For the linear case: passivity conditions appearing in *(Hitz and Anderson, 1969)* [58] (see Theorem 2.2) are obtained taking $m(x) = 0$, $V(x) = \frac{1}{2}x^T Px$, $s(y,u) = y^T u$, and dissipativity conditions presented in *(Goodwin and Sin, 1984)* [45] (see Lemma 2.1) are obtained by taking $m(x) = 0$, $s(y,u) = y^T Qy + 2y^T Su + u^T Ru$, $V(x) = \frac{1}{2}x^T Px$, with P a real symmetric positive definite matrix. As it can be seen in these results, the supply functions considered as functions of the input and the output are affine in u. Furthermore, all these systems are also affine in the input; this results in supply functions quadratic in the input when the variables are the states and the input.

4.4 Losslessness in the discrete-time domain

This section deals with the properties of lossless systems in the nonlinear discrete-time setting. Lossless systems are considered as a special case of dissipative ones. Losslessness is treated independently from dissipativity due to two main reasons. On the one hand, some differences in the proofs of the concluded properties are found. On the other hand, lossless systems are usually studied separate from dissipative ones in the discrete-time setting, since there are more losslessness-related studies than dissipativity-related ones in the discrete-time domain.

Definition 4.5 *The system (4.1)-(4.2) is said to be (V,s)-lossless if it is (V,s)-dissipative with $\phi(x,u) = 0, \forall (x,u) \in \mathscr{X} \times \mathscr{U}$.*

Conditions (4.4) and (4.5) may be rewritten for the losslessness case in the following way.

Proposition 4.2 (Navarro-López et al., 2002) *[119] Let a discrete-time system of the form (4.1)-(4.2) be (V,s)-lossless, then*

$$V(f(x,0)) - V(x) = s(h(x,0),0) \tag{4.13}$$

$$\frac{\partial}{\partial u}V(f(x,u)) = \frac{\partial}{\partial u_2}s(h(x,u_1),u_2)\Big|_{\substack{u_1 = u \\ u_2 = u}} + \frac{\partial}{\partial y}s(y,u)\frac{\partial}{\partial u}h(x,u) \tag{4.14}$$

If functions $V(f(x,u))$ and $s(h(x,u),u)$ are quadratic in u, then equalities (4.13)-(4.14) yield necessary and sufficient conditions as Theorem 4.2 shows. The denomination of *QSS*-losslessness will be used in the same sense as *QSS*-dissipativity was used above.

Definition 4.6 *A system of the form (4.1)-(4.2) is said to be QSS-lossless if it is (V,s)-lossless with a storage function V and a supply function s such that $V(f(x,u))$ and $s(h(x,u),u)$ are quadratic in u.*

Theorem 4.2 (Navarro-López et al., 2002) *[119] Let V and s be storage and supply functions such that $V(f(x,u))$ and $s(h(x,u),u)$ are quadratic in u. Then, a system of the*

form (4.1)-(4.2) is QSS-lossless with V and s if and only if

$$V(f(x,0))-V(x) = s(h(x,0),0) \tag{4.15}$$

$$\left.\frac{\partial V(z)}{\partial z}\right|_{z=f(x,0)} \left.\frac{\partial f(x,u)}{\partial u}\right|_{u=0} = \left.\frac{\partial}{\partial u}s(h(x,u),u)\right|_{u=0} \tag{4.16}$$

$$\left(\frac{\partial f(x,u)}{\partial u}\right)^T\Bigg|_{u=0} \left.\frac{\partial^2 V(z)}{\partial z^2}\right|_{z=f(x,0)} \left.\frac{\partial f(x,u)}{\partial u}\right|_{u=0} + \left.\frac{\partial V(z)}{\partial z}\right|_{z=f(x,0)} \left.\frac{\partial^2 f(x,u)}{\partial u^2}\right|_{u=0} = \left.\frac{\partial^2}{\partial u^2}s(h(x,u),u)\right|_{u=0} \tag{4.17}$$

Proof. The proof follows the one of *Theorem 2.6* in *(Byrnes and Lin, 1994)* [14].

(Necessity): If system (4.1)-(4.2) is *QSS*-lossless with supply function $s(y,u)$, and a storage function $V : \mathscr{X} \to \Re^+$, then

$$V(f(x,u))-V(x) = s(h(x,u),u), \quad \forall (x,u) \in \mathscr{X} \times \mathscr{U} \tag{4.18}$$

Condition (4.15) is obtained substituting $u = 0$ in (4.18). Conditions (4.16) and (4.17) follow from the first-order derivative and the second-order derivative of (4.18) with respect to u and taking $u = 0$.

(Sufficiency): Since V is a $\mathscr{C}^2$ function and $V(f(x,u))$ is quadratic in u, it is concluded that

$$V(f(x,u)) = A(x) + B(x)u + u^T C(x)u, \quad \forall (x,u) \in \mathscr{X} \times \mathscr{U} \tag{4.19}$$

and applying Taylor's expansion formula to $V(f(x,u))$ at $u = 0$, one yields to

$$\begin{aligned}
A(x) &= V(f(x,0)) \\
B(x) &= \left.\frac{\partial V(f(x,u))}{\partial u}\right|_{u=0} = \left.\frac{\partial V(z)}{\partial z}\right|_{z=f(x,0)} \left.\frac{\partial f(x,u)}{\partial u}\right|_{u=0} \\
C(x) &= \frac{1}{2}\left.\frac{\partial^2 V(f(x,u))}{\partial u^2}\right|_{u=0} = \\
&= \frac{1}{2}\left[\left(\frac{\partial f(x,u)}{\partial u}\right)^T\Bigg|_{u=0} \left.\frac{\partial^2 V(z)}{\partial z^2}\right|_{z=f(x,0)} \left.\frac{\partial f(x,u)}{\partial u}\right|_{u=0} + \right. \\
&\quad \left. + \left.\frac{\partial V(z)}{\partial z}\right|_{z=f(x,0)} \left.\frac{\partial^2 f(x,u)}{\partial u^2}\right|_{u=0}\right]
\end{aligned}$$

From (4.15)-(4.17), (4.19) takes the form

$$V(f(x,u)) = V(x) + s(h(x,0),0) + \left.\frac{\partial}{\partial u}s(h(x,u),u)\right|_{u=0} u + \frac{1}{2}u^T \left.\frac{\partial^2}{\partial u^2}s(h(x,u),u)\right|_{u=0} u \tag{4.20}$$

By claiming that $s(y,u)$ is quadratic in u, the second-order Taylor expansion at $u = 0$ of $s(h(x,u),u)$ can be considered in (4.20) and it is obtained

$$V(f(x,u))-V(x) = s(h(x,u),u)$$

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Remark 4.4 *Conditions (4.15)-(4.17) can be derived from the QSS-dissipativity conditions (4.7)-(4.9), with $\phi(x,u) = 0$, i.e., with $l(x) = k(x) = m(x) = 0$.*

Necessary and sufficient conditions existing in the literature for lossless multiple-input multiple-output discrete-time systems of the form

$$\begin{aligned} x(k+1) &= f(x(k)) + g(x(k))u(k) \\ y(k) &= h(x(k)) + J(x(k))u(k) \end{aligned}$$

are strictly contained in Theorem 4.2. For example, conditions appearing in *(Byrnes and Lin, 1994)* [14](see Theorem 2.3) are obtained taking $s(y,u) = y^T u$. Losslessness conditions presented in *(Sengör, 1995)* [151] (see Theorem A.7) are obtained considering the left-hand side of equality (4.15) as $V(f(x) - x)$ and in (4.16) and (4.17) $z = f(x) - x$ with $V = B^T(x) + x^T C(x)x$, where B and C are matrices of appropriate dimensions, and $s(y,u) = y^T Qy + 2y^T Su + u^T Ru$, where Q,R,S are constant matrices of appropriate dimensions, and Q, R are symmetric.

Remark 4.5 *In* (Byrnes and Lin, 1994) *[14], it is stated that it is not possible to study passivity and losslessness of discrete-time systems having outputs independent of u. Notice that for general supply functions, lossless systems can have outputs independent of the input. This is also pointed out in* (Sengör, 1995) *[151]. We delay for now a more complete discussion of this point.*

4.5 Conclusions and future work

Some properties of multiple-input multiple-output nonlinear discrete-time dissipative and lossless systems have been treated. On the one hand, necessary conditions for the characterization of what is regarded as (V,s)-dissipative and (V,s)-lossless systems have been proposed. On the other hand, necessary and sufficient conditions for a class of dissipative and lossless nonlinear systems have been given, actually, the dissipative and lossless systems treated are those whose storage and supply functions satisfy $V(f(x,u))$ and $s(h(x,u),u)$ are quadratic in u. This class of dissipativity and losslessness has been regarded as *QSS*-dissipativity and *QSS*-losslessness, respectively. *QSS*-dissipativity conditions will be used to solve the passivation problem in nonlinear discrete-time systems affine in the control input in Chapter 7.

The results obtained are compared with the ones already existing in the literature for the dissipative, passive and lossless cases. The results presented in this chapter follow the idea of the already existing characterizations of dissipative, passive and lossless systems. Our contribution is the extension of the established dissipativity-related properties to multiple-input multiple-output nonlinear discrete-time systems which are non-affine in the control input, i.e., systems of the form $x(k+1) = f(x(k),u(k))$, with associated outputs $y(k) = h(x(k),u(k))$ for a class of dissipativity called *QSS*-dissipativity. The conditions contributed gather all the existing KYP conditions in the discrete-time setting, with the exclusion of the ones presented in *(Monaco and Normand-Cyrot, 1997, 1999)* [112, 113].

New ways of treating dissipativity concepts are needed to be explored. Concerning the results presented here, it is desirable to give dissipativity conditions without the restriction of $V(f(x,u))$ and $s(h(x,u),u)$ to be quadratic in u.

The domains of the states and the inputs in most control systems are compact sets. Thus, from the *Stone-Weierstrass Theorem*, functions $V(f(x,u))$ and $s(h(x,u),u)$ can be uniformly approximated by polynomials. The work presented here covers a first step in this direction since we have considered *QSS*-dissipative, respectively *QSS*-lossless, systems, in which $V(f(x,u))$ and $s(h(x,u),u)$ have been approximated by polynomials of second order in u. A way of extending our results may be by means of the use of higher order polynomial approximations and studying the conditions that dissipative systems must fulfil with this kind of storage and supply functions.