# Chapter 5

# Feedback dissipativity through the dissipativity equality

## 5.1 Introduction

This chapter deals with the problem of feedback dissipativity. Sufficient conditions under which a class of non-affine discrete-time control systems can be rendered dissipative are derived. Four methodologies in order to render nonlinear single-input single-output discrete-time systems dissipative are given. The feedback dissipativity methodologies proposed will be applied to several examples in order to solve the feedback passivity or passivation problem. Although this dissertation is focused on feedback dissipativity by means of static state feedback, in one of the methodologies proposed in this chapter, the feedback dissipativity control is of dynamic type.

Dissipative and passive systems present highly desirable properties which may simplify the system analysis and control design. This fact impels to transform a system which is not dissipative or passive into a dissipative or passive one. The action of rendering a system dissipative (respectively, passive) by means of a static state feedback is known as *feedback dissipativity* (respectively, feedback passivity or *passivation*). Systems which can be rendered dissipative (respectively, passive) are regarded as feedback dissipative (respectively, feedback passive) systems. The problem of the establishment of conditions for a nonlinear discrete-time system to be rendered passive or dissipative via a state feedback has not been solved yet. This problem has only been solved for lossless systems *(Byrnes and Lin, 1994)* [14]. Therefore, the main contribution of this chapter is to solve the feedback dissipativity problem in nonlinear discrete-time systems.

For the linear case, the problem of feedback passivity and feedback dissipativity for $(Q,S,R)$-dissipative systems has been solved in the framework of the positive real control problem *(de Souza and Xie, 1992)* [164], *(Souza et al., 1993)* [165] and the $(Q,S,R)$-dissipative control problem *(Tan et al., 1999)* [170], *(Tan et al., 2000)* [171] in connection with $H_\infty$ design. These approaches are oriented to achieve the asymptotic stability of the system, and also treat uncertain linear discrete-time systems. This line is not considered in this dissertation, since we are interested in achieving feedback dissipativity of nonlinear discrete-time systems with general supply functions.

In this chapter, four procedures for dealing with the feedback dissipativity problem are posed. The proposed feedback dissipativity approaches are based on the establishment of the input $u$ which satisfies the fundamental dissipativity inequality; it is the idea underlying in *(Sira-Ramírez, 1998)* [159] for the continuous-time case. Four different ways of obtaining the control $u$ which achieves feedback dissipativity are presented. The first one proposes an implicit solution for the feedback dissipativity problem. The main disadvantage of this procedure is that finding a solution of $u$ for each iteration can be difficult or impossible in some systems. Due to the discrete-time nature of the system, an iterative method can be proposed. Our iterative-like solution to the discrete-time feedback dissipativity problem is by means of the speed-gradient (SG) algorithm, see *(Fradkov et al., 1999)* [41]. The problem of passivation for continuous-time nonlinear systems is proposed to be solved by means of the SG algorithm in *(Fradkov, 1991)* [35], *(Fradkov et al., 1995)* [37] and *(Seron et al., 1995)* [153]. The discrete-time version of the SG methodology is given in *(Fradkov and Pogromsky, 1998)* [40]. For the application of the SG algorithm for feedback dissipativity purposes, the definitions of quasi-$(V,s)$-dissipativity and feedback quasi-dissipativity are introduced. The SG-based feedback dissipativity methodology gives rise to two kinds of feedback laws which render the system quasi-$(V,s)$-dissipativity: a dynamic controller and a static one.

The third approach in order to solve the feedback dissipativity problem is of approximate type and is also based on the proposal of a control $u$ which renders the system $(V,s)$-dissipative. In this case, dissipativity is seen as a "perturbation" of the storage energy invariance or the system losslessness situations. Therefore, two approximate methods are derived for which the errors of the approximation are bounded, and sufficient conditions are given under which the approximation made is valid.

This chapter is organized as follows. Section 5.2 gives an implicit solution for the feedback dissipativity problem through the fundamental dissipativity equality. In Section 5.3, the feedback dissipativity problem is formulated through the SG algorithm. Section 5.4.1 solves the feedback dissipativity in two steps: first, the storage energy function of the system is rendered invariant, second, the control which makes the system dissipative is obtained from an approximation of the fundamental dissipativity equality. Section 5.4.2 presents another methodology in order to render a system dissipative by means of a static feedback, it also consists of two steps: first, the system is rendered lossless, second, the control which makes the system dissipative is proposed in order to satisfy an approximation of the fundamental dissipativity inequality. The feedback dissipativity methodologies are illustrated by means of two examples: a discrete-time model of the DC-to-DC buck converter proposed in *(Navarro-López et al., 2002)* [119] and an academic nonlinear discrete-time system. For the examples, the feedback passivity or passivation problem is treated. The validity of the passifying controls given will be analyzed for each example. Conclusions and suggestions for further research are presented in the last section.

Some of the results of this chapter have been extracted from *(Navarro-López et al., 2001)* [118], *(Navarro-López et al., 2002)* [119].

## 5.2 Feedback dissipativity through the dissipativity inequality: An implicit solution

Let nonlinear single-input single-output discrete-time systems of the form

$$x(k+1) = f(x(k),u(k)), \quad x \in \mathscr{X} \subset \Re^n, \quad u \in \mathscr{U} \subset \Re \tag{5.1}$$

$$y(k) = h(x(k),u(k)), \quad y \in \mathscr{Y} \subset \Re \tag{5.2}$$

where $f : \mathscr{X} \times \mathscr{U} \to \mathscr{X}$ and $h : \mathscr{X} \times \mathscr{U} \to \mathscr{Y}$ are smooth maps. $k \in Z_+ := \{0,1,2,\ldots\}$. All considerations will be restricted to an open set of $\mathscr{X} \times \mathscr{U}$ containing $(\bar{x},\bar{u})$, having $\bar{x}$ as an isolated fixed point of $f(x,\bar{u})$, with $\bar{u}$ a constant, i.e., $f(\bar{x},\bar{u}) = \bar{x}$. We consider a positive definite $C^2$ function $V : \mathscr{X} \to \Re$, with $V(0) = 0$, associated with the system (5.1)-(5.2) and addressed as the *storage function*. A second $C^2$ function is also considered, called the *supply function*, denoted by $s(y,u)$, with $s : \mathscr{Y} \times \mathscr{U} \to \Re$. Consider a dissipation rate function $\phi$ as given in Definition 4.1.

The class of dissipative systems treated in this section are $(V,s)$-dissipative and $(V,s)$-lossless systems. All the definitions given are based on Definition 4.2 presented in Chapter 4.

### 5.2.1 Definition of the problem

Let $\alpha : \mathscr{X} \times \mathscr{U} \to \mathscr{U}$ be a $\mathscr{C}^1$ function with $\alpha(0,0) = 0$. A nonlinear static state feedback control law is denoted by the expression $u = \alpha(x,v)$, where $v \in \mathscr{U}$. Consider a locally regular feedback control law $u = \alpha(x,v)$, as it was defined in Definition 2.10. The system

$$x(k+1) = f(x(k),\alpha(x(k),v(k)))$$

is referred by the *feedback transformed* system, which may be also denoted by $x(k+1) = \overline{f}(x(k),v(k))$. In addition, $\overline{h}(x,v)$ denotes the function $h(x,\alpha(x,v))$.

**Definition 5.1** *Consider the system (5.1)-(5.2) and two scalar functions $V(x)$ and $s(y,v)$ as a storage function and a supply function, respectively. The system is said to be* feedback dissipative *(resp., feedback stricly dissipative) with the functions V and s, if there exists a* regular *static state feedback control law of the form $u = \alpha(x,v)$, with v as the new input, such that the feedback transformed system is $(V,s)$-dissipative (resp., strictly $(V,s)$-dissipative).*

The existence of a feedback control law of the form $u = \alpha(x,v)$ for which the system is rendered $(V,s)$-dissipative must be assessed from the existence of solutions, for the control input $u$, of the following equation,

$$V(f(x,u)) - V(x) = s(h(x,u),v) - \phi(x,u) \tag{5.3}$$

with $V$ and $s$ considered as a storage function and a supply function, respectively.

The following theorem states sufficient conditions under which feedback dissipativity is possible.

**Theorem 5.1** (Navarro-López et al., 2002) *[119] Consider the system (5.1)-(5.2) and two scalar functions $V(x)$ and $s(y,v)$ as a storage function and a supply function, respectively. Let $\phi(x,u)$ be a given dissipation rate function. Let $(x_0,u_0,v_0) \in A = \mathscr{X} \times \mathscr{U} \times \mathscr{U}$ with A an open set. Suppose that the following two conditions are satisfied:*

1. *$\exists(x_0,u_0,v_0)$ such that equality (5.3) holds, i.e.,*

$$V(f(x_0,u_0)) - V(x_0) = s(h(x_0,u_0),v_0) - \phi(x_0,u_0) \tag{5.4}$$

2.
$$\left\{ \frac{\partial}{\partial u}\left[V(f(x,u)) - s(h(x,u),v) + \phi(x,u)\right] \right\}_{(x_0,u_0,v_0)} \neq 0 \tag{5.5}$$

*Then, there exists a* unique *static state feedback control law of the form $u = \alpha(x,v)$ defined in a neighbourhood of $(x_0,v_0)$ and valued in a neighbourhood of $u_0$ such that the feedback transformed system $x(k+1) = \overline{f}(x(k),v(k))$, $y(k) = \overline{h}(x(k),v(k))$ is $(V,s)$-dissipative.*

**Proof.** Let $\mathscr{X} \times \mathscr{U} \times \mathscr{U}$ an open set. Consider the following $\mathscr{C}^1$ function $F : \mathscr{X} \times \mathscr{U} \times \mathscr{U} \to \Re$ defined by

$$F(x,v,u) = V(f(x,u)) - V(x) - [s(h(x,u),v) - \phi(x,u)] \tag{5.6}$$

From condition (5.4), we have that $F(x_0,v_0,u_0) = 0$. Condition (5.5) states that $\frac{\partial F}{\partial u}$ is non-singular at $(x_0,v_0,u_0)$. Then, by the *implicit function theorem* there exist open neighbourhoods $\widetilde{\mathscr{X}} \subset \mathscr{X}$ of $x_0$, $\widetilde{\mathscr{U}_1} \subset \mathscr{U}$ of $v_0$ and $\widetilde{\mathscr{U}_2} \subset \mathscr{U}$ of $u_0$ and a unique map $G : \widetilde{\mathscr{X}} \times \widetilde{\mathscr{U}_1} \to \widetilde{\mathscr{U}_2}$ such that

$$F(x,v,G(x,v)) = 0, \ \forall (x,v) \in \widetilde{\mathscr{X}} \times \widetilde{\mathscr{U}_1}$$

Thus, the implicit function theorem provides sufficient conditions which guarantee the existence of a local feedback control law $u = G(x,v)$ for the nonlinear equation (5.3) to be satisfied; in other words, the existence of a control which renders system (5.1)-(5.2) $(V,s)$-dissipative with $s(y,v)$, and $v$ as the new input is guaranteed.

∎

**Remark 5.1** *As it happened in the continuous-time setting (see Section 3.4), the result given for the discrete-time domain is also local. The feedback control law which achieves dissipativity is locally valid; control $u$ satisfies (5.3) for all $(x,v)$ sufficiently close to $(x_0,v_0)$. In the sequel, when referring the feedback dissipativity problem we will implicitly refer to the local feedback dissipativity problem.*

**Remark 5.2** *If condition (5.5) is satisfied for $(x_0,u_0,v_0)$, it also holds in a neighbourhood of $(x_0,u_0,v_0)$, i.e., there exists an open neighbourhood of $(x_0,u_0,v_0)$, $\widetilde{\mathscr{W}} \subset \mathscr{X} \times \mathscr{U} \times \mathscr{U}$, where the following assertion is valid,*

$$\frac{\partial}{\partial u}\left[V(f(x,u)) + s(h(x,u),v) - \phi(x,u)\right] \neq 0, \ \forall (x,u,v) \in \widetilde{\mathscr{W}} \tag{5.7}$$

*Condition (5.7) in addition to condition (5.4) will be regarded as the* feedback dissipativity conditions *in the discrete-time setting.*

**Definition 5.2** *A system of the form (5.1)-(5.2) is said to be feedback passive if it is feedback dissipative with $s = yv$.*

**Remark 5.3** *Particularizing Theorem 5.1 for $\phi(x,u) = 0, \ \forall (x,u) \in \mathscr{X} \times \mathscr{U}$, note that conditions for rendering a system $(V,s)$-lossless via static state feedback are obtained.*

In the next two sections, two examples will be presented in order to illustrate the feedback dissipativity approach proposed. The special case of the feedback passivity problem will be studied. An analysis of the solutions given for the proposed passifying control will be presented.

### 5.2.2 Example 1. Passivation of a discrete-time model of the DC-to-DC buck converter

The feedback dissipativity scheme proposed will be applied to the passivation of a discrete-time model for the DC-to-DC buck converter, it is obtained from a normalized averaged model of the original continuous-time one, see *(Navarro-López et al., 2002)* [119]. This discrete-time model will be also used in Chapters 6, 7 and 8, and it is presented as follows. Although the model is linear, it is appropriate to illustrate the proposed feedback dissipativity methodology. It is an example for which the energy concepts introduced have a physical interpretation.

#### 5.2.2.1 The discrete-time model

The DC-to-DC buck converter is a well known physical system employed in power electronics. The simplified scheme of this converter is shown in Figure 5.1. This circuit is used to produce a constant (DC) voltage in the load $R$ lower than the power supply voltage ($V_{in}$). This goal is achieved by means of the adequate commutation of the switch. We can describe in a simple way how this circuit operates: when the switch position is 1, the inductor and the capacitor store energy, whereas if the switch position is 0, there is an exchange of energy between the inductor, the capacitor and the load. This circuit can be modelled by the following system of ordinary differential equations (ODE), *(Kassakian et al., 1991)* [70]:

$$\frac{d}{d\tau}\begin{pmatrix} i \\ v \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{pmatrix}\begin{pmatrix} i \\ v \end{pmatrix} + u\begin{pmatrix} \frac{V_{in}}{L} \\ 0 \end{pmatrix} \tag{5.8}$$

where $i$ is the current flowing through the inductor, $v$ the voltage across the capacitor; $u$ represents the switch position, which can take two values, $u \in \{0,1\}$, and $V_{in}$ is the power supply.

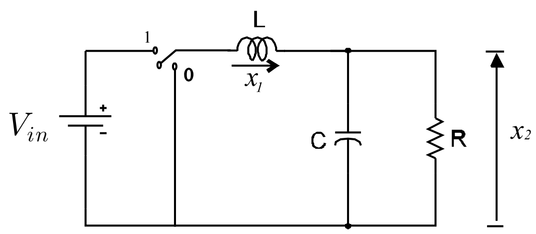


**Figure 5.1:** The DC-to-DC buck converter.

In order to have a normalized model, let us consider the following coordinate transformation:

$$x_1 = \frac{1}{V_{in}}\sqrt{\frac{L}{C}}i, \quad x_2 = \frac{v}{V_{in}}, \quad t = \frac{\tau}{\sqrt{LC}}, \quad \gamma = \frac{\sqrt{L}}{R\sqrt{C}}$$

which yields to,

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -\gamma \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + u\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad u \in \{0,1\} \tag{5.9}$$

The parameter $\gamma$ acts as the normalized load.

Perturbation theory may be used to obtain the continuous-time averaged model, see *(Khalil, 1996)* [76],

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -\gamma \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \hat{u} \\ 0 \end{pmatrix} = Ax(t) + B\hat{u}(t) \tag{5.10}$$

where now $\hat{u} \in [0,1]$. Note that in the original model (5.8), $u$ denotes the switch position, while in the average description (5.10), the control $\hat{u}$ denotes the duty cycle of a PWM control, then $\hat{u} = \frac{t_{on}}{T}$, with $t_{on}$ the time the switch is in 1 position, and $T$ the time between two $0 \to 1$ changes of the switch position; $T$ is regarded as the commutation period. See Figure 5.2.

The equilibrium points of (5.10) are

$$\begin{aligned} \bar{x}_1 &= \gamma\hat{\bar{u}} \\ \bar{x}_2 &= \hat{\bar{u}} \end{aligned} \tag{5.11}$$

with $\hat{\bar{u}} \in [0,1]$ a constant.

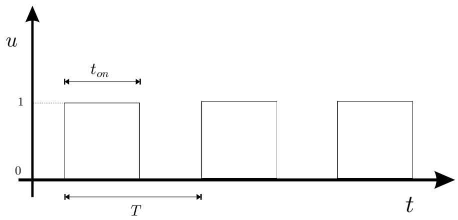


**Figure 5.2:** Switch behaviour in a commutation period.

In order to obtain a discrete-time model, let $T$ be the sampling period and let us solve equation (5.10) with $t_0 = kT$, $t_f = (k+1)T$, $x(t_0) = x_k$, $x = (x_1, x_2)^T$, assuming $\hat{u}(t) = \hat{u}(kT)$ to be constant for $t \in [kT, (k+1)T)$, as it is usual in discrete-time systems. Then, it is obtained, see *(Franklin et al., 1990)* [42],

$$x(k+1) = e^{AT}x(k) + (e^{AT} - I)A^{-1}B\hat{u}(k) \tag{5.12}$$

Let

$$e^{AT} = e^{\left[T\begin{pmatrix} 0 & -1 \\ 1 & -\gamma \end{pmatrix}\right]} = \begin{pmatrix} a & -b \\ b & c \end{pmatrix}$$

therefore, using (5.12) one yields to

$$\left[\begin{array}{c} x_1(k+1) \\ x_2(k+1) \end{array}\right] = \left[\begin{array}{cc} a & -b \\ b & c \end{array}\right]\left[\begin{array}{c} x_1(k) \\ x_2(k) \end{array}\right] + \left[\begin{array}{c} \gamma(-a+1)+b \\ -\gamma b - c + 1 \end{array}\right]\hat{u}(k) \tag{5.13}$$

which is a discrete-time model of the DC-to-DC buck converter. The sampling period $T$ is considered to be the commutation period.

The fixed points of (5.13) are (5.11). It is considered that

$$\bar{x}_1 \in (0, \gamma\bar{x}_2], \;\; \bar{x}_2 \in (0, 1]$$

Let $x_1 \in \mathscr{X}_1$ and $x_2 \in \mathscr{X}_2$, with $\mathscr{X}_1$, $\mathscr{X}_2$ closed sets containing $\bar{x}_1$ and $\bar{x}_2$, respectively, i.e.,

$$\begin{array}{c} x_1 \in \mathscr{X}_1 = [0, \gamma\rho] \\ x_2 \in \mathscr{X}_2 = [0, \rho] \end{array}$$

with $\rho > 1$ for $\mathscr{X}_1$ and $\mathscr{X}_2$ to contain $\bar{x}_1 = \gamma\bar{x}_2$ and $\bar{x}_2$, respectively.

The physical parameters to consider are the following ones: $V_{in} = 407V$, $L = 1mH$, $C = 80\mu F$, $R = 10\Omega$, $f_c = 10kHz$, $T = 0.3535533906$, with $f_c$ the switch commutation frequency and $T$ the sampling period. Then, the constants of the model are:

$$a = 0.9406416964, b = 0.3254699438, c = 0.8255706942, \gamma = 0.3535533906 \tag{5.14}$$

#### 5.2.2.2 Application of the passifying methodology

The feedback dissipativity scheme proposed in Section 5.2 will be applied to the passivation of (5.13) with constants $a$, $b$, $c$ and $\gamma$ given in (5.14). The energy associated to the system will be used as storage function. The system energy associated to (5.8) can be obtained from the sum of the stored energy in the inductor and in the capacitor:

$$E_s = E_{i_L} + E_{v_C} = \frac{1}{2}Li^2(t) + \frac{1}{2}Cv^2(t)$$

which takes the following form for the normalized model (5.9),

$$E_s = \frac{\eta}{2}\left(x_1^2 + x_2^2\right) \tag{5.15}$$

where $\eta = V_{in}^2 C$.

The feedback dissipativity methodology is then applied with $s = yv$ and $V = E_s$; the output of the system is considered to be $y = x_2 + \hat{u} = h(x, \hat{u})$. First of all, a function $\phi$ must be proposed. This function will be chosen in order to collect the positive terms appearing in $V(x(k+1))$. Consequently, a possibility for $\phi$ is the following one,

$$\begin{aligned} \phi(x, \hat{u}) \;\; = \;\; & \eta\mu\Big\{ x_1^2(a^2+b^2) + x_2^2(b^2+c^2) + \hat{u}^2\left[\gamma^2(-a+1)^2 + b^2(1+\gamma^2) + \right. \\ & \left. + (-c+1)^2\right]\Big\} \end{aligned} \tag{5.16}$$

with $\mu$ a positive constant.

The control which passifies the system will be obtained from the equation

$$V(f(x,\hat{u})) - V(x) = h(x,\hat{u})v - \phi(x,\hat{u}), \tag{5.17}$$

with $v$ the new control input and $v \in [0,1]$. Equation (5.17) results in a second-order equation of the form

$$a_{\hat{u}}\hat{u}^2 + b_{\hat{u}}(x_1,x_2,v)\hat{u} + c_{\hat{u}}(x_1,x_2,v) = 0 \tag{5.18}$$

with

$$\begin{aligned}
a_{\hat{u}} &= \left(\frac{1}{2}+\mu\right)\left[\gamma^2(-a+1)^2 + b^2(\gamma^2+1) + (-c+1)^2\right] + \gamma(bc-ab)\\
b_{\hat{u}}(x_1,x_2,v) &= [\gamma(-a+1)+b](ax_1-bx_2) + [-\gamma b - c + 1](bx_1+cx_2) - \frac{1}{\eta}v\\
c_{\hat{u}}(x_1,x_2,v) &= \left(\frac{1}{2}+\mu\right)\left[x_1^2(a^2+b^2) + x_2^2(b^2+c^2)\right] + (bc-ab)x_1x_2 -\\
&\quad -\frac{1}{\eta}x_2v - \frac{1}{2}(x_1^2+x_2^2)
\end{aligned}$$

Conditions (5.4) and (5.5) considered for the passivation case are met for this example if,

$$a_{\hat{u}}\hat{u}^2 + b_{\hat{u}}(x_1,x_2,v)\hat{u} + c_{\hat{u}}(x_1,x_2,v) = 0, \tag{5.19}$$

$$2a_{\hat{u}}\hat{u} + b_{\hat{u}}(x_1,x_2,v) \neq 0 \tag{5.20}$$

for some $(x_1,x_2,\hat{u},v)$. If conditions (5.19)-(5.20) are satisfied for some $(x_1,x_2,\hat{u},v)$, a passifying control $\hat{u}$ exists. This $\hat{u}$ can be obtained from the explicit solution of (5.18), then it is necessary to assure that $b_{\hat{u}}^2 - 4a_{\hat{u}}c_{\hat{u}} \geq 0$, which will be achieved by means of the value of $\mu$. The existence of control $\hat{u}$ will be analyzed in Section 5.2.2.3.

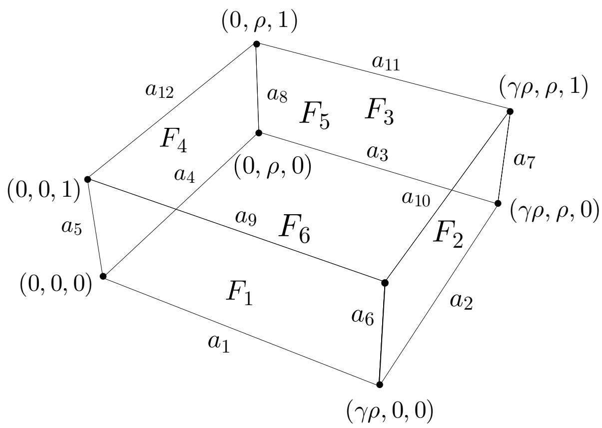


**Figure 5.3:** Domain for which $\varphi(x_1,x_2,v) = b_{\hat{u}}^2 - 4a_{\hat{u}}c_{\hat{u}}$ is defined: $[0,\gamma\rho]\times[0,\rho]\times[0,1]$.

### 5.2.2.3 Existence of solutions for the passifying control

Two solutions for the passifying control $\hat{u}$ defined in (5.18) are obtained:

$$\hat{u}_1(x_1,x_2,v) = \frac{-b_{\hat{u}} + \sqrt{b_{\hat{u}}^2 - 4a_{\hat{u}}c_{\hat{u}}}}{2a_{\hat{u}}}, \qquad \hat{u}_2(x_1,x_2,v) = \frac{-b_{\hat{u}} - \sqrt{b_{\hat{u}}^2 - 4a_{\hat{u}}c_{\hat{u}}}}{2a_{\hat{u}}} \tag{5.21}$$

For the existence of these controls in $\Re$, it is necessary to assure that $\varphi(x_1,x_2,v) = b_{\hat{u}}^2 - 4a_{\hat{u}}c_{\hat{u}} \geq 0$ when $x_1 \in \mathscr{X}_1 = [0,\gamma\rho]$, $x_2 \in \mathscr{X}_2 = [0,\rho]$ and $v \in [0,1]$ with $\rho > 1$. This goal will be achieved through the value of $\mu$. Constant $\mu$ represents the damping injection to the system. The smaller $\mu$ is, the smaller the dissipation rate is and the slower the convergence to the equilibrium point is. The smaller $\mu$ is, the greater the overshooting is. The meaning of $\mu$ will be appreciated in Chapter 6 when the feedback dissipativity method will be used for stabilization purposes.

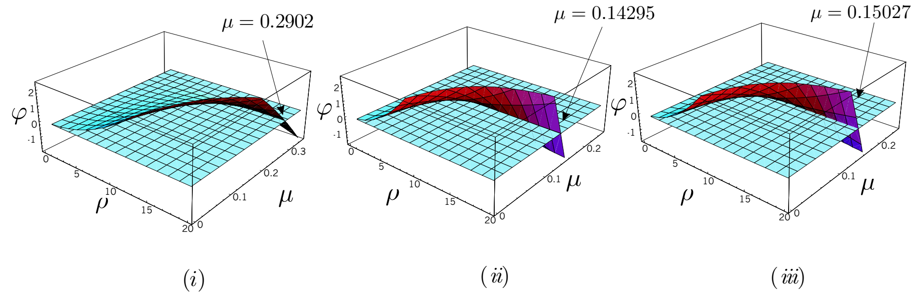


**Figure 5.4:** Representation of $\varphi(x_1,x_2,v) = b_{\hat{u}}^2 - 4a_{\hat{u}}c_{\hat{u}}$ with $x_1 = \gamma\rho$, $x_2 = \rho$ varying $\rho \in (1,20]$ and $\mu \in [0,0.25]$ at vertices (i) $(\gamma\rho,0,0)$ (ii) $(\gamma\rho,\rho,0)$ (iii) $(0,\rho,0)$.

Now, appropriate values for $\mu$, with $\mu > 0$, in order to passify the system (5.13) with $y = x_2 + \hat{u}$ by means of controls $\hat{u}_1$ and $\hat{u}_2$ are proposed. $\rho$ is considered to have a maximum value for our analysis in order to obtain $\mu$: $\rho \in (1,20]$. The main idea of the procedure to follow is that we will find an upper bound for $\mu$ for which the function $\varphi$ evaluated at all the candidates for local minima in the parallelepiped $[0,\gamma\rho] \times [0,\rho] \times [0,1]$ is positive. If $\varphi$ is positive for all the local minima, it is positive for all the values of the states and the control $v$ in $[0,\gamma\rho] \times [0,\rho] \times [0,1]$. We are interested in the worst case, i.e., the minimum value for $\mu$ which makes the function $\varphi$ positive at all the candidates for local minimum, in other words, the value for $\mu$ which makes $\varphi$ positive at the local minimum giving the lowest value of $\varphi$. The local minima of $\varphi$ must be searched among the stationary points or critical points, the local minima on the faces, along the edges and at the vertices of the parallelepiped represented in Figure 5.3. Let analyze these minima in two groups. On the one hand, the critical points of function $\varphi$ will be analyzed; on the other hand, the candidates for minima on the faces, along the edges and at the vertices of the parallelepiped will be studied.

1. **Stationary point** $(x_1,x_2,v) = (0,0,v)$. Constant $\mu$ will be designed for the point $(x_1 = 0, x_2 = 0)$ , with $v$ fixed, to be a relative minimum of $\varphi(x_1,x_2,v)$. This will be achieved by means of the second-derivative test due to the fact that the partial derivatives of $\varphi$ with respect to $x_1$ and $x_2$ at $(x_1 = 0, x_2 = 0)$ with v=0 are zero. Then, the Hessian matrix of function $\varphi$ evaluated at $x = 0, v = 0$ will be assured to be positive definite. Straightforward calculations yield to the following Hessian matrix of $\varphi$ at $x_1 = 0, x_2 = 0$ with $v = 0$:

$$H = \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix}$$

with

$$\begin{aligned} h_{11} &= -1.1892(-0.2902+\mu)(0.6972+\mu) \\ h_{12} &= -0.034764+0.02247\mu \\ h_{22} &= -9.45237(-0.150227+\mu)(0.42705+\mu) \end{aligned}$$

The real symmetric matrix $H$ is positive definite if all eigenvalues are positive. The values of $\mu$ below which $eig(H)$ are positive are: $\mu_1 = 0.137817$ and $\mu_2 = 0.296853$ for each eigenvalue, respectively. Then, the first upper bound of $\mu$ to consider is:

$$\mu < \overline{\mu}_1 = \min(\mu_1, \mu_2) = \mu_1$$

2. **Other candidates for minima**. Analyzing function $\varphi$ on the faces and along the edges of the parallelepiped, it can be concluded that the minima values of $\mu$ are given by the vertices $v_1 = (\gamma\rho, 0, 0)$, $v_2 = (\gamma\rho, \rho, 0)$, $v_3 = (0, \rho, 0)$. For the faces $F_1$, $F_2$, $F_3$ and $F_4$ the worst cases are given along the edges $a_1$, $a_2$, $a_3$, and $a_4$, respectively, i.e., for the case $v = 0$. For face $F_5$ the worst case for $\mu$ is given along $a_{10}$, and for $F_6$ is along $a_2$. The value for $\mu$ which makes $\varphi > 0$ along edge $a_1$ is the value of $\mu$ achieved for the maximum value of $x_1$, i.e., at vertex $v_1$, along edge $a_2$ for the maximum value of $x_2$, i.e., at vertex $v_2$, along $a_3$ for the maximum value of $x_1$ at vertex $v_2$, along edge $a_4$ for the maximum value of $x_2$ at vertex $v_3$ and along edge $a_{10}$ for the maximum value of $x_2$ at vertex $v_4 = (\gamma\rho, \rho, 1)$. From the analysis of the behaviour of $\varphi$ along the edges, it appears that the parameter $\mu$ depends on the constant $\rho$ which defines the intervals of the states, in addition, the higher $\rho$ is, the lower $\mu$ is obtained; since we are interested in the lowest value of $\mu$, then $\rho = 20$ has been considered.

   Now, the behaviour of $\varphi$ depending on $\mu$ and $\rho$ at the mentioned vertices will be analyzed. Due to the fact that along edge $a_7$ the lowest value for which $\varphi$ is positive is given at vertex $v_2$, the vertices to be analized will be $v_1$, $v_2$ and $v_3$. Function $\varphi$ evaluated at vertices $v_1$, $v_2$ and $v_3$ takes the following form:

$$\varphi(x_1, x_2, v)\big|_{v_1, v_2, v_3} = -p_1(-p_2+\mu)(p_3+\mu)\rho^2 \tag{5.22}$$

   with

   (a) For $v_1$: $p_1 = 0.0743241,\ p_2 = 0.2902,\ p_3 = 0.697272$
   (b) For $v_2$: $p_1 = 0.546943,\ p_2 = 0.142953,\ p_3 = 0.422947$
   (c) For $v_3$: $p_1 = 0.472618,\ p_2 = 0.150227,\ p_3 = 0.427051$

In Figure 5.4, the behaviour of $\varphi$ at these vertices varying $\rho$ and $\mu$ can be appreciated. Therefore, the minimum value of $\mu$ for which $\varphi$ is positive is $\overline{\mu}_2 = 0.14295$, that is, the value for which $\varphi$ is zero at vertex $v_2$.

Considering $\overline{\mu}_1 = 0.137817$, it is obtained that:

$$0 < \mu < \min(\overline{\mu}_1, \overline{\mu}_2) = \overline{\mu}_1$$

Therefore, it can be seen that function $\varphi$ is positive for any value of $x_1 \in [0, \gamma\rho]$, $x_2 \in [0, \rho]$, $v \in [0, 1]$, any value of $\rho > 1$, if $\mu < \overline{\mu}_1$ (see Figure 5.5). To conclude with, the system (5.13) can be rendered $(V, s)$-passive by means of controls $\hat{u}_1$ and $\hat{u}_2$ as defined in (5.21) as the solutions of (5.18), considering $s = yv$, $y = x_2 + \hat{u}$, $V = E_s$ and $\phi$ as in (5.16) if $\mu \in (0, 0.137817)$.

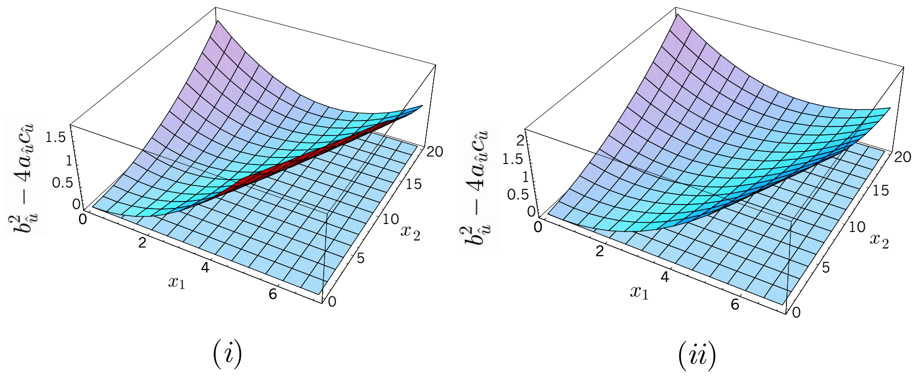


**Figure 5.5:** Representation of $\varphi(x_1,x_2,v) = b_{\hat{u}}^2 - 4a_{\hat{u}}c_{\hat{u}}$ varying $x_1 \in [0,\gamma\rho]$ and $x_2 \in [0,\rho]$, with $\rho = 20$ and $\mu = 0.135$ for (i) $v = 0$ (ii) $v = 1$.

**Remark 5.4** *In Figure 5.5, the graphics have been given for variations of the states with $\rho = 20$. It can be checked that any value of $\rho > 0$ also results in a function $\varphi > 0$.*

**Remark 5.5** *Note that the passivation method depends of the form of $\phi$ and the constant $\mu$ appearing in it.*

### 5.2.3 Example 2. Passivation of a nonlinear model

Consider the following system extracted from *(Sira-Ramírez, 1991)* [156],

$$\begin{aligned} x_1(k+1) &= \left[x_1^2(k) + x_2^2(k) + u(k)\right] \cos\left[x_2(k)\right] \\ x_2(k+1) &= \left[x_1^2(k) + x_2^2(k) + u(k)\right] \sin\left[x_2(k)\right] \\ y(k) &= x_1^2(k) + x_2^2(k) + u(k) \end{aligned} \tag{5.23}$$

Let $x_1$, $x_2$, $u \in [-\rho,\rho]$, $v \in [0,1]$, with $\rho > 0$.

#### 5.2.3.1 Application of the feedback passivity scheme

The system (5.23) will be rendered $V$-passive with storage function $V = x_1^2 + x_2^2$ and supply function $s(y,v) = yv$. The dissipation rate function $\phi(x,u)$, chosen in order to collect the positive terms appearing in $V(x(k+1))$, is considered as follows,

$$\phi(x,u) = \mu\left[(x_1^2 + x_2^2)^2 + u^2 + x_1^2 + x_2^2\right] \tag{5.24}$$

with $\mu$ a positive constant. Then, the passifying control will be obtained from equation (5.3) and takes the following form,

$$a_u u^2 + b_u(x_1,x_2,v)u + c_u(x_1,x_2,v) = 0 \tag{5.25}$$

with

$$\begin{aligned} a_u &= 1 + \mu \\ b_u(x_1,x_2,v) &= 2(x_1^2 + x_2^2) - v \\ c_u(x_1,x_2,v) &= (x_1^2 + x_2^2)[(1+\mu)(x_1^2 + x_2^2) + (\mu - 1)] - (x_1^2 + x_2^2)v \end{aligned}$$

Conditions (5.4) and (5.5) for the passivation goal in the case of system (5.23) are met if,

$$a_u u^2 + b_u(x_1,x_2,v)u + c_u(x_1,x_2,v) = 0, \tag{5.26}$$
$$2u(1+\mu) + 2(x_1^2 + x_2^2) - v \neq 0 \tag{5.27}$$

for some $(x_1,x_2,u,v)$. The most immediate solution for control $u$ is given by the two solutions of the second-order equation (5.25),

$$u_1(x_1,x_2,v) = \frac{-b_u + \sqrt{b_u^2 - 4a_u c_u}}{2a_u}, \qquad u_2(x_1,x_2,v) = \frac{-b_u - \sqrt{b_u^2 - 4a_u c_u}}{2a_u} \tag{5.28}$$

For these controls to be real it is necessary that the radicand $b_u^2 - 4a_u c_u$ is positive, which will be achieved by means of the value of the parameter $\mu$. The admissible values for $\mu$ will be analized in the following section.

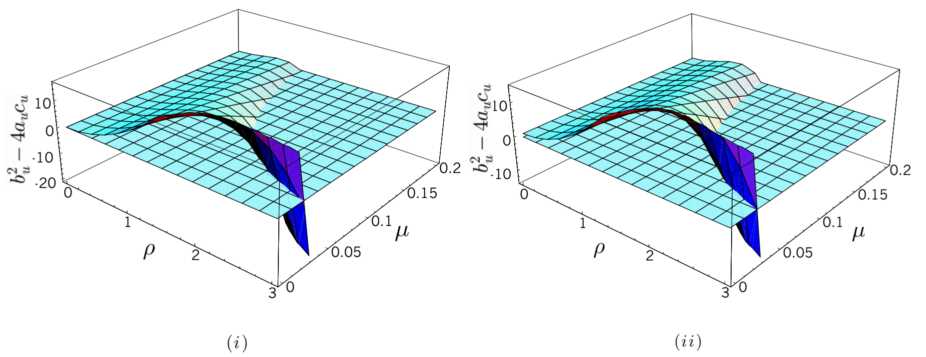


**Figure 5.6:** Function $\varphi(x_1,x_2,v) = b_u^2 - 4a_u c_u$ evaluated at the vertices of the parallelepiped defining the domain of the system (5.23) varying $\rho \in (0,3]$ and $\mu \in (0,0.2]$ (i) vertices $(-\rho,\rho,0)$, $(\rho,-\rho,0)$, $(\rho,\rho,0)$, $(-\rho,\rho,0)$ (ii) $(-\rho,\rho,1)$, $(\rho,-\rho,1)$, $(\rho,\rho,1)$, $(-\rho,\rho,1)$.

#### 5.2.3.2 Existence of solutions for the passifying control

This section illustrates that some values of $\mu$ make the radicand $\varphi(x_1,x_2,v) = b_u^2 - 4a_u c_u$ be positive and some others make it be negative. We are interested in values of $\mu$ for which $\varphi$ is positive. We will follow a similar procedure as the one presented for the buck example in order to give admissible values for $\mu$. Our goal is assuring $\varphi$ to be positive at all the candidates for local minima by means of the value of parameter $\mu$. This function is defined in the parallelepiped $[-\rho,\rho] \times [-\rho,\rho] \times [0,1]$. The candidates for local minima of $\varphi$ must be searched among the stationary points, and the local minima on the faces, along the edges and at the vertices of the mentioned parallelepiped. First, the critical points of function $\varphi$ will be analyzed; second, the candidates for local minima on the faces, along the edges and at the vertices of the parallelepiped will be studied.

1. **Relative minimum at the stationary point** $(x_1,x_2,v) = (0,0,0)$. The critical point $(0,0,0)$ will be assured to be a relative minimum of $\varphi(x_1,x_2,v)$ by means of the value of parameter $\mu$. For this goal, it is necessary to assure that the Hessian matrix

of $\varphi$ at $(0,0,0)$ is positive definite, this matrix takes the following form:

$$H = \begin{bmatrix} -8(\mu-1)(\mu+1) & 0 & 0 \\ 0 & -8(\mu-1)(\mu+1) & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$H$ is positive definite if $\mu < 1$. The value $\overline{\mu}_1 = 1$ is considered as the first candidate for upper bound of $\mu$.

2. **Local minima in the parallelepiped** $[-\rho,\rho]\times[-\rho,\rho]\times[0,1]$. Evaluating the function $\varphi(x_1,x_2,v)$ in the domain $[-\rho,\rho]\times[-\rho,\rho]\times[0,1]$, the vertices of this parallelepiped give the minimum values of $\mu$ below which function $\varphi$ is positive. The analysis of $\varphi$ at the vertices is divided in two groups: the ones with $v=0$, and the ones with $v=1$. Vertices with the same value of $v$ result in the same value for function $\varphi$. Let consider the two mentioned cases:

   (a) **Vertices** $v_1 = (-\rho,\rho,0)$, $v_2 = (\rho,-\rho,0)$, $v_3 = (\rho,\rho,0)$, $v_4 = (-\rho,\rho,0)$. Evaluating $\varphi$ at these points, function $\varphi$ takes the following form:

   $$\varphi(x_1,x_2,v)\big|_{v_i} = 8\rho^2\left\{1-\mu\left[\mu+2\rho^2(2+\mu)\right]\right\} \tag{5.29}$$

   with $i\in\{1,\dots,4\}$. The value of $\mu>0$ below which $\varphi$ is positive is calculated making $\varphi(x_1,x_2,v)=0$, that is,

   $$\overline{\mu}_2 = \frac{1}{2\rho^2+\sqrt{4\rho^4+2\rho^2+1}} \tag{5.30}$$

   From (5.30), this upper bound for $\mu$ is concluded to depend directly on the value of the extreme of the intervals the states are defined in: the higher $\rho$ is, the lower $\mu$ becomes. For instance, if $\rho = 1$ is considered, the upper bound for $\mu$ is $\overline{\mu}_2 = 0.2152504$; on the other hand, $\rho = 3$ yields in $\overline{\mu}_2 = 0.02738206$.

   (b) **Vertices** $v_5 = (-\rho,\rho,1)$, $v_6 = (\rho,-\rho,1)$, $v_7 = (\rho,\rho,1)$, $v_8 = (-\rho,\rho,1)$ . Evaluating $\varphi$ at these vertices, the following expression is obtained:

   $$\varphi(x_1,x_2,v)\big|_{v_i} = 1+8\rho^2\left\{1-\mu\left[\mu-1+2\rho^2(2+\mu)\right]\right\} \tag{5.31}$$

   with $i\in\{5,\dots,8\}$. The value of $\mu>0$ below which (5.31) is positive is calculated making $\varphi(x_1,x_2,v)=0$, that is,

   $$\overline{\mu}_3 = \frac{2\rho(1-4\rho^2)+\sqrt{64\rho^6+24\rho^2+2}}{4(\rho+2\rho^3)} \tag{5.32}$$

   From (5.32) this upper bound for $\mu$ is concluded to depend directly on the value of $\rho$. For instance, if $\rho = 1$, the upper bound for $\mu$ is $\overline{\mu}_3 = 0.2905694$; on the other hand, $\rho = 3$ yields in $\overline{\mu}_3 = 0.0285265$.

Therefore, the minimum value for $\mu$ is $\overline{\mu}_2$, achieved for the case $v=0$, i.e., at vertices $v_i$, $i\in\{1,\dots,4\}$. Considering $\rho\in(0,3]$, $\overline{\mu}_2 = 0.02738206$. See Figure 5.6 for an illustration of the behaviour of $\varphi$ at the vertices $v_i$, $i\in\{1,\dots,8\}$ varying parameters $\rho$ and $\mu$.

The upper bounds for $\mu$ obtained from the evaluation of the local minima in the parallelepiped $[-\rho,\rho]\times[-\rho,\rho]\times[0,1]$ are always less than $\overline{\mu}_1 = 1$, then, it is concluded that, for any $\rho > 0$:

$$0 < \mu \leq \overline{\mu}_2(\rho)$$

The positiveness of $\varphi$ can be appreciated in Figure 5.7 for different values of $v$, $\mu$ and $\rho$.

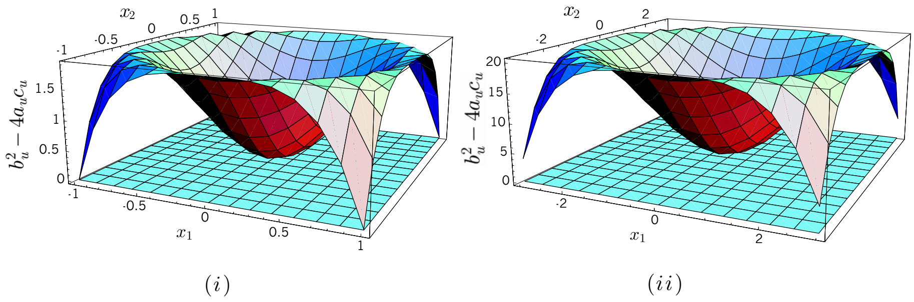


**Figure 5.7:** Function $\varphi(x_1,x_2,v) = b_u^2 - 4a_uc_u$ with different values of $\rho$ (i) $v = 0$, $\mu = 0.215$, $x_1$, $x_2 \in [-1,1]$ (ii) $v = 1$, $\mu = 0.027$, $x_1, x_2 \in [-3,3]$ .

To conclude with, system (5.23) with $x_1, x_2 \in [-\rho,\rho]$, $v \in [0,1]$, $\rho > 0$ can be rendered $V$-passive by means of controls $u_1$ and $u_2$ as defined in (5.28) as the solutions of (5.25) considering $s = yv$, $V = x_1^2 + x_2^2$ and $\phi$ as in (5.24) if $\mu \in (0,\overline{\mu}_2]$. Higher values of $\mu$ can be considered, however, in these case $\rho$ is smaller and the neighbourhood of the origin for which the control is valid gets smaller.

**Remark 5.6** *In the nonlinear example, the control $v$ is considered to vary in the interval $[0,1]$, this interval has been chosen because these values of $v$ will be considered for stabilization purposes in Chapter 6, indeed, we are only interested in $v = 0$.*

## 5.3 Feedback dissipativity of nonlinear discrete-time systems using the speed-gradient algorithm

In this section, the discrete-time version of the SG algorithm will be adapted so as to be used for achieving feedback dissipativity of nonlinear discrete-time systems. Two types of feedback dissipativiy controllers are proposed with this method: a dynamic-type one and a static-type one. In order to use the SG algorithm for feedback dissipativity purposes, the definitions of quasi-$(V,s)$-dissipativity and feedback quasi-dissipativity are proposed. The SG-based feedback dissipativity methodology will be applied to the passivation of the examples presented in the previous section. An analysis of valid values of the constants appearing in the passifying controls is given.

### 5.3.1 The speed-gradient algorithm

The speed gradient (SG) method (see *(Fradkov, 1991)* [35] or *(Fradkov et al., 1999)* [41]) is based upon the achievement of a specified control goal by means of some goal function $Q$. The discrete-time version of the SG algorithm is proposed in *(Fradkov and*

*Pogromsky, 1998)* [40]. Some of the results of this section are based on the ones presented in *(Navarro-López et al., 2001)* [118].

Let the system (5.1)-(5.2). The SG control problem is formulated as finding a control law $u(k)$ which ensures the control objective

$$Q(x(k+1)) \leq \Delta, \; when\, k > k^* \tag{5.33}$$

with some nonnegative function $Q$ and a threshold value $\Delta > 0$. Substituting (5.1) into (5.33) the following goal function is obtained, which can be seen as an infinite number of goal functions

$$Q_k(u) = Q(f_k(x(k),u)) \tag{5.34}$$

The following gradient control algorithm is proposed,

$$u(k+1) = u(k) - \gamma(k)\nabla_u Q_k(u(k)) \tag{5.35}$$

where $\gamma(k) \geq 0, \forall k$. By $\nabla_u Q_k(u(k))$, we denote the partial derivative of $Q$ with respect to $u$.

Algorithm (5.35) makes the current control correction $\Delta u(k) = u(k+1) - u(k)$ vary in the direction of decrease of the current goal function $Q_k(u)$. The variation of the control $u$ is proportional to the gradient of the speed of change of the objective function $Q$.

The SG control scheme in the discrete-time setting and its applicability conditions are based upon Theorem 2.3 proposed in *(Fradkov and Pogromsky, 1998)* [40] which is presented as follows.

**Theorem 5.2** (Fradkov and Pogromsky, 1998) *[40] Consider the gradient control algorithm (5.35). Let $Q$ a nonnegative function and $\Delta$ a positive constant. Suppose that*

**A1** *There exists $\varepsilon^* > 0$ and a vector $u^*$ such that the following inequalities are satisfied*

$$Q_k(u^*) \leq \varepsilon^* < \Delta, \; k = 0,1,2,\ldots \tag{5.36}$$

**A2** *For all admissible inputs $u$ the inequality $Q_k(u) \leq \Delta$ implies fullfilment of one of the following conditions*

$$(u^* - u)^T \nabla_u Q_k(u) \quad \leq \quad \varepsilon^* - \Delta < 0, \tag{5.37}$$

$$(u^* - u)^T \nabla_u Q_k(u) \quad \leq \quad Q_k(u^*) - Q_k(u) < 0 \tag{5.38}$$

**A3** *For any $\rho > 0$ and any $k = 0,1,2,\ldots$ there exists $\kappa(\rho) > 0$ such that the following inequalities are satisfied*

$$|\nabla_u Q_k(u)|^2 \leq \kappa(\rho) \tag{5.39}$$

*as long as $|u - u^*| \leq \rho$ and $Q_k(u) > \Delta$.*

**A4** *The gain coefficients $\gamma(k)$ in the proposed control are chosen as follows*

$$\gamma(k) = \gamma_c \delta(k) |\nabla_u Q_k(u(k))|^{-2} \tag{5.40}$$

*where (if condition (5.37) is satisfied)*

$$0 < \gamma_c < 2\,(\Delta - \varepsilon^*)\,, \; \delta(k) = \begin{cases} 1, & if\, Q_k(u(k)) \geq \Delta, \\ 0, & otherwise \end{cases}$$

*or (if condition (5.38) is satisfied)*

$$0 < \gamma_c < 2\left(1 - \frac{\varepsilon^*}{\Delta}\right), \ \delta(k) = \begin{cases} Q_k(u(k)), & if\, Q_k(u(k)) \geq \Delta, \\ 0, & otherwise \end{cases}$$

*Then, for any $u_0$ there exists a number $k^*$ such that the goal inequality $Q(x(k+1)) \leq \Delta$ is fulfilled and $u(k) = const$ for $k \geq k^*$ with $u(k)$ given by (5.35).*

■

Let us clarify assumptions A1, A2, A3 made in Theorem 5.2 in order to verify them. Assumption A1 means that there exists some control $u^*$ for which the control goal (5.33) is achievable for a given $\Delta$. On the other hand, both conditions (5.37) and (5.38) are satisfied if the control goal function $Q$ is convex in $u(k)$ *(Fradkov and Pogromsky, 1998)* [40]. The third assumption A3 supposes that the gradient $\nabla_u Q_k(u(k))$ is bounded in any bounded set of the phase space. These assumptions ensure the convergence of the control goal function to $Q \leq \Delta$ by means of a sufficiently small value of the parameter $\gamma$ and by introducing a deadzone for small values of $Q$.

Note that the control $u$ obtained by the SG algorithm is given in a dynamic form. In *(Fradkov and Pogromsky, 1998)* [40], it is established that the gradient control algorithm (5.35) is equivalent to

$$u(k) = -\gamma_{cs} \nabla_{u(k)} Q(x(k)) \tag{5.41}$$

i.e., $u(k)$ is explicitly obtained from $\nabla_u Q_k(u)$, and $\gamma_{cs}$ is a positive constant. The stability properties of the algorithm (5.41) are established by means of the value of the constant $\gamma_{cs}$; that is, $\gamma_{cs}$ is chosen in such a way to assure the asymptotic stability of the fixed point of the controlled system obtained with (5.41).

### 5.3.2 Formulation of the feedback dissipativity problem

In order to solve the feedback dissipativity problem through the SG algorithm some new dissipativity-related definitions are introduced. The ideas of *quasi-$(V,s)$-dissipativity* and feedback quasi-dissipativity are proposed in the following definitions.

**Definition 5.3** *The system (5.1)-(5.2) with storage function $V(x)$ and supply function $s(y,u)$ is said to be quasi-$(V,s)$-dissipative (resp., strictly quasi-$(V,s)$-dissipative) if there exists a dissipation rate (resp., strict dissipation rate) function $\phi$ and a constant $\overline{\Delta} \geq 0$ such that*

$$V(f(x,u)) - V(x) = s(h(x,u),u) - \phi(x,u) + \overline{\Delta}, \quad \forall (x,u) \in \mathscr{X} \times \mathscr{U} \tag{5.42}$$

**Definition 5.4** *The system (5.1)-(5.2) is said to be quasi-$V$-passive if it is quasi-$(V,s)$-dissipative with respect to the supply rate $s(y,u) = yu$.*

**Remark 5.7** *The denomination of quasi-dissipative systems has already been used in the literature for continuous-time systems, see for example* (Polushin, 1995) *[139]. In Polushin's definition of quasi-dissipativity, no dissipation rate function is used.*

**Definition 5.5** *Consider the system (5.1)-(5.2) and two scalar functions $V(x)$ and $s(y,v)$ as a storage function and a supply function, respectively. The system is said to be* feedback quasi-dissipative *(resp., feedback stricly quasi-dissipative) with the functions $V$ and $s$, if there exists a* regular *static state feedback control law of the form $u = \alpha(x,v)$, with $v$ as the new input, such that the feedback transformed system is quasi-$(V,s)$-dissipative (resp., strictly quasi-$(V,s)$-dissipative).*

**Remark 5.8** *Note that the sufficient conditions (5.4)-(5.5) under which feedback dissipativity is possible are the same for the case of feedback quasi-dissipativity, with the difference that for the feedback quasi-dissipativity case, the constant $\overline{\Delta}$ appears in the right-hand side of equality (5.4).*

Now, the discrete-time version of the SG algorithm will be applied to solve the feedback quasi-dissipativity problem. Therefore, the following control goal $Q$ is proposed,

$$Q = Q_d(x,u,v) = V(f(x,u)) - V(x) - s(y,v) + \phi(x,u) \tag{5.43}$$

It can be noticed that when the SG algorithm is applied in order to achieve feedback dissipativity, feedback quasi-dissipativity with $\overline{\Delta} \leq \Delta$ is achieved.

As the SG-algorithm establishes, the feedback dissipativity goal is achieved if the control goal function $Q_d$ is rendered to a value less than $\Delta$, if $Q_d \neq 0$ quasi-$(V,s)$-dissipativity is achieved instead of $(V,s)$-dissipativity. If the control goal function $Q_d$ is rendered to zero then $(V,s)$-dissipativity of the transformed system is achieved.

The SG algorithm is applied for $\Delta$ having a value very near to zero and $\varepsilon^* <<< \Delta$, as well as supposing condition (5.38) of Theorem 5.2 to be satisfied. The dissipation rate function $\phi$ is chosen in such a way to ensure the control goal function $Q_d(x,u,v) = V(f(x,u)) - V(x) - s(y,v) + \phi(x,u)$ to be nonnegative. Then, the control $u$ which achieves feedback quasi-dissipativity can be calculated in two ways:

1. The dynamic controller solution using Theorem 5.2 with control (5.35):

$$\begin{aligned} u(k+1) &= u(k) - \gamma(k)\nabla_u Q_{d,k}(u(k)) \\ \gamma(k) &= \gamma_c \delta(k) |\nabla_u Q_{d,k}(u(k))|^{-2} \\ 0 &< \gamma_c < 2 \\ \delta(k) &= \begin{cases} Q_{d,k}(u(k)), & if\, Q_{d,k}(u(k)) \geq \Delta \\ 0, & otherwise \end{cases} \end{aligned} \tag{5.44}$$

2. The static solution given by control (5.41):

$$u(k) = -\gamma_{cs} \nabla_{u(k)} Q_d(x(k)) \tag{5.45}$$

with $\gamma_{cs}$ a positive constant such that ensures the asymptotic stability of the fixed point of the controlled system. A way to assure the asymptotic stability of the fixed point of the controlled system is by assuring that the linearized controlled system around the fixed point has all poles with modulus less than 1.

**Remark 5.9** *The feedback dissipativity process is strongly dependent on the election of $\phi$ and constants $\gamma_c$, $\gamma_{cs}$.*

**Remark 5.10** *As for the application of the method $\Delta$ is considered to be very near to zero, it can be said that the achievement of quasi-$(V,s)$-dissipativity implies the achievement of $(V,s)$-dissipativity.*

**Remark 5.11** *Sufficient conditions for feedback dissipativity or feedback $(V,s)$-dissipativity by means of the SG algorithm are the conditions of the application of the SG method, i.e., conditions A1, A2 and A3 shown in Theorem 5.2 for a control goal function $Q = Q_d$.*

The feedback dissipativity methodology proposed, in its two versions, will be applied to the passivation of two examples, the same examples analyzed in Section 5.2. An analysis of the constants appearing in the controls proposed will be made. The conditions under which the SG algorithm can be applied will be also verified.

### 5.3.3 Example 1. The discretized buck converter model

Our goal is to passify the system (5.13) having as storage function $V = \frac{\eta}{2}(x_1^2 + x_2^2)$ and as supply function $s(y,v) = yv$, with $y = x_2 + \hat{u}$. Function $\phi(x,\hat{u})$ is chosen as defined in (5.16) with $\mu$ a positive constant. Indeed, our goal is to render the sytem quasi-$V$-passive. Let $x_1 \in [0, \gamma_b \rho]$, $x_2 \in [0,\rho]$, $\hat{u}, v \in [0,1]$, and $\rho > 1$, with $\gamma_b$ the normalized load.

Taking into account (5.43), the control goal function for the example takes the following form

$$Q_d(x_1,x_2,\hat{u},v) = a_{\hat{u}}\hat{u}^2 + b_{\hat{u}}(x_1,x_2,v)\hat{u} + c_{\hat{u}}(x_1,x_2,v), \tag{5.46}$$

where

$$\begin{aligned}
a_{\hat{u}} &= \eta\left[\left(\frac{1}{2}+\mu\right)[\gamma_b^2(-a+1)^2 + b^2(\gamma_b^2+1) + (-c+1)^2] + \gamma_b(bc-ab)\right] \\
b_{\hat{u}}(x_1,x_2,v) &= \eta\left\{\left[\gamma_b(-a+1)+b\right](ax_1 - bx_2) + \left(-\gamma_b b - c + 1\right)(bx_1 + cx_2)\right\} - v \\
c_{\hat{u}}(x_1,x_2,v) &= \eta\left\{\left(\frac{1}{2}+\mu\right)[x_1^2(a^2+b^2) + x_2^2(b^2+c^2)] + (bc-ab)x_1x_2 - \right. \\
&\quad \left. -\frac{1}{2}(x_1^2+x_2^2)\right\} - x_2 v
\end{aligned}$$

For the application of the SG algorithm for feedback dissipativity purposes, it is necessary to assure that the function $Q_d$ is positive, which will be achieved by means of choosing the adequate value of the parameter $\mu$. In addition, the applicability conditions of the SG method must be checked.

First of all, let examine the positiveness of $Q_d$. A way of assuring the positiveness of $Q_d$ is by means of assuring that $Q_d$ is a positive function for all the candidates for local minima. The procedure to follow is the one used in Section 5.2.2.2. The candidates for local minima are the critical points of $Q_d$ and the local minima existing in the domain of definition of $Q_d(x_1,x_2,\hat{u},v)$, i.e., $[0,\gamma_b\rho] \times [0,\rho] \times [0,1] \times [0,1]$. Examining $Q_d$, it can be noticed that $Q_d$ is positive for a value of $\mu$ and ahead, then, the worst case to search is the greatest value of $\mu$, among all the cases examined, for which $Q_d$ becomes positive.

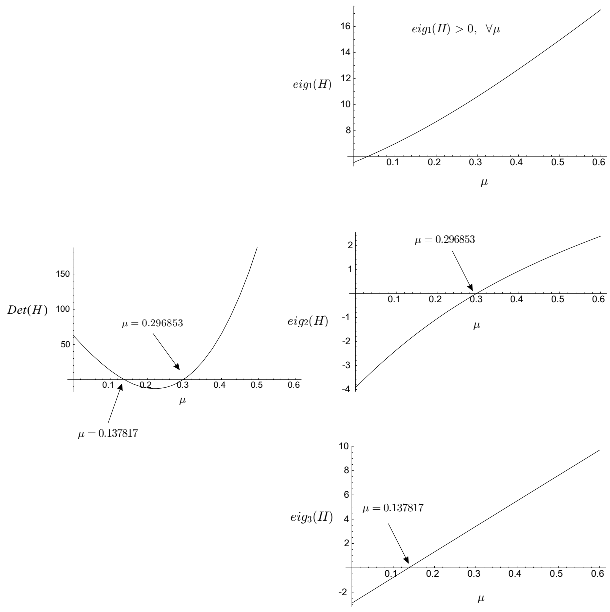


**Figure 5.8:** Values of $\mu$ for which the eigenvalues and the determinant of the Hessian matrix of $Q_d$ evaluated at $x = 0$, $\hat{u} = 0$ with $v = 0$ varying $\mu$ become positive.

1. **Assuring the critical point** $(x_1, x_2, \hat{u}, v) = (0, 0, 0, v)$ **to be a relative minimum**. Now, $\mu$ will be designed to assure that $Q_d$ has a relative minimum at $x = 0, \hat{u} = 0$. Let $\hat{u} = \alpha(x, v)$ and $\alpha(0, 0) = 0$. Therefore, the relative minimum of $Q_d$ at $x = 0$, $\hat{u} = 0$ will be analyzed for $v = 0$. Let $x\hat{u} = (x,\, \hat{u})^T$. The gradient of $Q_d$ takes the form,
$$\frac{\partial Q_d}{\partial x\hat{u}} = \left( \frac{\partial Q_d}{\partial x_1}, \frac{\partial Q_d}{\partial x_2}, \frac{\partial Q_d}{\partial \hat{u}} \right) \tag{5.47}$$
with
$$\begin{aligned} \frac{\partial Q_d}{\partial x_1} &= \eta \left[ ab_1\hat{u} + bb_2\hat{u} + 2(a^2+b^2)\left(\frac{1}{2}+\mu\right)x_1 + (bc-ab)x_2 - x_1 \right] \\ \frac{\partial Q_d}{\partial x_2} &= \eta \left[ -bb_1\hat{u} + cb_2\hat{u} + 2(b^2+c^2)\left(\frac{1}{2}+\mu\right)x_2 + (bc-ab)x_1 - x_2 \right] \\ \frac{\partial Q_d}{\partial u} &= 2a_{\hat{u}}\hat{u} + b_{\hat{u}} \end{aligned}$$
with
$$b_1 = \left[\gamma_b(-a+1) + b\right],\ b_2 = \left(-\gamma_b b - c + 1\right)$$
The gradient of $Q_d$ at $x_1 = x_2 = \hat{u} = 0$ is zero. For this stationary point to be a relative minimum of $Q_d$, a sufficient condition is that the Hessian matrix of $Q_d$ at $x_1 = x_2 = \hat{u} = 0$ $(H)$ is positive definite, it is obtained,
$$H = \begin{bmatrix} 2\eta(a^2+b^2)\left(\frac{1}{2}+\mu\right) - \eta & \eta(bc-ab) & \eta(ab_1+bb_2) \\ \eta(bc-ab) & 2\eta(b^2+c^2)\left(\frac{1}{2}+\mu\right) - \eta & \eta(cb_2-bb_1) \\ \eta(ab_1+bb_2) & \eta(cb_2-bb_1) & 2a_{\hat{u}} \end{bmatrix} \tag{5.48}$$

The real symmetric matrix $H$ is positive definite if all eigenvalues are positive. The eigenvalues of $H$ are studied in function of $\mu$ and a lower bound for $\mu$ is obtained, that is, the eigenvalues of (5.48) become greater than zero when $\mu > \overline{\mu}_1 = 0.296853$ , and therefore, with $\mu > \overline{\mu}_1$ the determinant of $H$ is positive. The constant $\mu$ is varied for values greater than zero and the determinant and the eigenvalues of $H$ are depicted in Figure 5.8. Consequently, the Hessian matrix of $Q_d$ is positive definite and the stationary point $(x = 0, \hat{u} = 0, v)$ is a relative minimum of $Q_d$ for $\mu > \overline{\mu}_1$.

2. **Studying $Q_d$ at other candidates for local minima**. The domain of definition of function $Q_d(x_1, x_2, \hat{u}, v)$ is $[0, \gamma_b \rho] \times [0, \rho] \times [0, 1] \times [0, 1]$, then, in this domain, a lower bound of $\mu$ will be found for which $Q_d$ is positive at all the candidates for local minima. Analyzing the domain of $Q_d$ in a similar way as it was made in Section 5.2.2.2, the conclusion is that the worst case is presented for $(x_1, x_2, \hat{u}, v) = (0, \rho, 1, 1)$. The value of $\mu$ in all cases depends on $\rho$, the smaller $\rho$ is, the higher $\mu$ is, then, our interest is concentrated in analyzing $\mu$ for the minimum value of $\rho$, i.e., $\rho = 1$. In the case of $(x_1, x_2, \hat{u}, v) = (0, \rho, 1, 1)$, the critical value of $\mu$ is given by the following expression:

$$\overline{\mu}_2 = \frac{1.40803(0.107028 + \rho)(1.20324 + \rho)}{1.98829 + 10.4359\rho^2} \tag{5.49}$$

Evaluating (5.49) at $\rho = 1$, $\overline{\mu}_2 = 0.276418$ is obtained. Then, for values of $\mu$ greater than $\overline{\mu}_2$, $Q_d$ is assured to be positive in all the candidates for local minima, and, consequently, $Q_d$ is positive for all $x_1$, $x_2$, $u$, $v$ in the sets where $Q_d$ is defined.

Then, two values of $\mu$ are obtained as critical ones, and the greatest one must be chosen in order to assure that $Q_d$ is a positive function, then,

$$\mu > \max(\overline{\mu}_1, \overline{\mu}_2) = \overline{\mu}_1$$

The positiveness of $Q_d$ for values of $\mu$ greater than $\overline{\mu}_1$ can be appreciated in Figure 5.9.

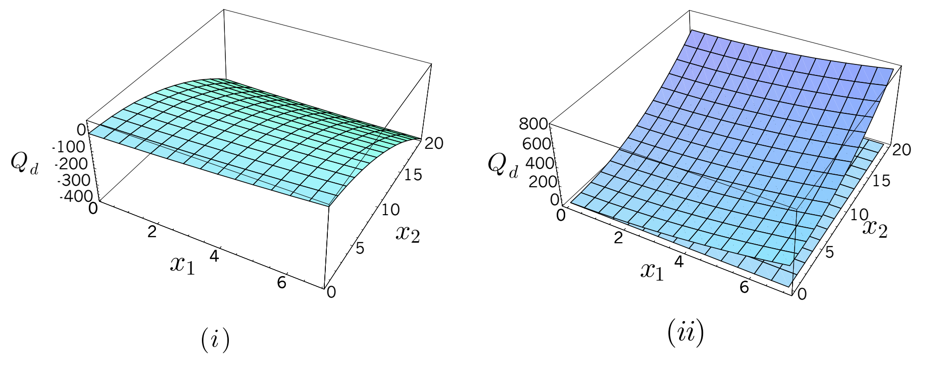


**Figure 5.9:** $Q_d$ is positive depending on the value of $\mu$. Representation of $Q_d$ with $x_1 \in [0, \gamma_b \rho]$, $x_2 \in [0, \rho]$, $\rho = 20$ (i) with $\hat{u} = 1$, $v = 0.5$, $\mu = 0.05$ (ii) with $\hat{u} = 1$, $v = 0.5$, $\mu = 0.3$.

An upper bound for $\mu$ must be also given. The maximum value of $\mu$ depends on the maximum value of $Q_d$. Since the variables of $Q_d$ vary in compact sets and $Q_d$ is continuous, this function is bounded and reaches its maximum and minimum value in some point of the compact (uniform continuity *(Marsden and Hoffman, 1998)* [98]), then $\mu$ has an upper bound. On the other hand, the maximum values of $Q_d$ in $x_1 \in [0, \gamma_b \rho]$, $x_2 \in [0, \rho]$, $\rho > 1$, $\hat{u} \in [0,1]$ and $v \in [0,1]$ are given for the extremes of the intervals of variations of the variables $x_1$, $x_2$, $\hat{u}$, $v$. Therefore, the upper bound of $\mu$ is given by the neighbourhood the system is defined in, which is established by the value of $\rho$. Consequently, it can be concluded that for any $\rho > 1$, there exists $\overline{\mu}_{max}(\rho) > 0$ such that,

$$\overline{\mu}_1 < \mu < \overline{\mu}_{max}(\rho) \tag{5.50}$$

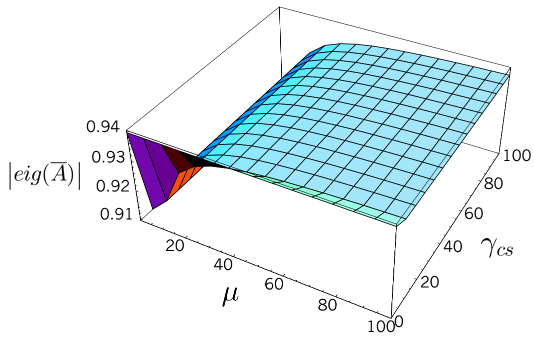


**Figure 5.10:** Modulus of the eigenvalues of $\overline{A}$ of the buck converter system passified by control (5.53) varying $\mu \in [0.296853, 100]$ and $\gamma_{cs} \in (0, 100]$.

In order to apply the SG algorithm it is necessary to verify assumptions A1, A2 and A3. From the fact that for $\mu > 0.296853$ it follows that $Q_d > 0$ and $Q_d$ has a relative minimum at $x = 0, \hat{u} = 0$, and due to the increasing nature of $Q_d$ in the domain considered, the conditions appearing in A1 and A2 are met. On the one hand, the condition to verify in A1 is met due to the fact that $Q_d$ is positive with a relative minimum at $(x = 0, \hat{u} = 0, v)$, then we can always find a constant $0 < \varepsilon^* < \Delta$ and a control $u^*$ for which $Q_d$ is smaller and then the control goal (5.33) is achieved, indeed, this $u^*$ can be $\hat{u}^* = u^* = 0$ considering $x_1, x_2, v$ fixed near to zero. On the other hand, condition (5.38) of assumption A2 can be checked to be met considering $\hat{u}^* = u^* = 0$. This condition takes the following form:

$$-(2a_{\hat{u}}\hat{u}^2 + b_{\hat{u}}\hat{u}) \leq -(a_{\hat{u}}\hat{u}^2 + b_{\hat{u}}\hat{u}) < 0 \tag{5.51}$$

It can be noticed that (5.51) is always met. For the condition of assumption A3 that $\nabla_u Q(u) = 2a_{\hat{u}}\hat{u} + b_{\hat{u}}$ must be bounded, we can apply the property of uniform continuity in a set *(Marsden and Hoffman, 1998)* [98]. Then, the SG algorithm can be applied.

Two passifying control schemes can be proposed, they are presented as follows:

1. A passifying dynamic control scheme given by (5.44):

$$\begin{aligned}
\hat{u}(k+1) &= \hat{u}(k) - \gamma(k)\left[2a_{\hat{u}}\hat{u}(k) + b_{\hat{u}}(x_1(k), x_2(k), v(k))\right] \\
\gamma(k) &= \gamma_c \delta(k)\left|2a_{\hat{u}}\hat{u}(k) + b_{\hat{u}}(x_1(k), x_2(k), v(k))\right|^{-2} \\
0 &< \gamma_c < 2 \\
\delta(k) &= \begin{cases} Q_{d,k}(\hat{u}(k)), & if\, Q_{d,k}(\hat{u}(k)) \geq \Delta, \\ 0, & otherwise \end{cases}
\end{aligned} \tag{5.52}$$

with $\mu > 0.296853$.

2. A passifying static control given by (5.45):

$$\hat{u}(k) = -\frac{\gamma_{cs} b_{\hat{u}}(x_1(k), x_2(k), v(k))}{1 + 2\gamma_{cs} a_{\hat{u}}} \tag{5.53}$$

with $\gamma_{cs}$ a positive constant chosen in such a way to ensure the fixed point of the controlled system to be asymptotically stable. By means of $\gamma_{cs}$, the poles of the system will be assured to have modulus less than 1. If control (5.53) is applied to (5.13), a linear system is obtained with the following form:

$$\begin{aligned}
&\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \\
&= \begin{bmatrix} a - b_1 c_1 \eta (b_1 a + b_2 b) & -b + b_1 c_1 \eta (b_1 b - b_2 c) \\ b - b_2 c_1 \eta (b_1 a + b_2 b) & c + b_2 c_1 \eta (b_1 b - b_2 c) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \\
&+ \begin{bmatrix} b_1 c_1 \\ b_2 c_1 \end{bmatrix} v(k) = \\
&= \overline{A} x(k) + \overline{B} v(k) \\
y(k) &= [-c_1 \eta (b_1 a + b_2 b), \, -c_1 \eta (b_2 c - b_1 b)] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + c_1 v(k) = \\
&= \overline{C} x(k) + \overline{D} v(k)
\end{aligned} \tag{5.54}$$

with

$$b_1 = \left[\gamma_b(-a+1) + b\right], \; b_2 = \left(-\gamma_b b - c + 1\right), \; c_1 = \frac{\gamma_{cs}}{1 + 2\gamma_{cs} a_{\hat{u}}}$$

The matrix $\overline{A}$ of the quasi-$V$-passified system (5.54) depends on the constants $\mu$ and $\gamma_{cs}$, then, these constants will be chosen in order to have the eigenvalues of $\overline{A}$ with modulus less than 1. The eigenvalues of $\overline{A}$ are complex conjugate. The modulus of these eigenvalues can be evaluated for different values of $\mu$ and $\gamma_{cs}$ with $\mu > 0.296853$ in order to have $Q_d$ positive. As Figure 5.10 depicts, the modulus of the eigenvalues of $\overline{A}$ are less than one with $\mu > 0.296853$ and $\gamma_{cs} > 0$.

In Chapter 6, the passifying scheme (5.53) will be used for stabilization purposes. The control scheme (5.52) will not be used due to the fact that this dissertation concentrates more attention to the feedback dissipativity problem using a static feedback law.

### 5.3.4 Example 2. A nonlinear example

The feedback dissipativity methodology proposed in Section 5.3.2 will be applied to the passivation of an academic example: a nonlinear discrete-time system whose fixed point $(0,0)$ is unstable in open loop. The system to study is the one proposed in Section 5.2.3. Our goal is to render system (5.23) quasi-$V$-passive with a storage function $V = x_1^2 + x_2^2$ and with a supply function $s(y,v) = yv$, $y = x_1^2 + x_2^2 + u$. Function $\phi(x,u)$ is considered as in (5.24) with $\mu > 0$. Suppose $x_1, x_2 \in [-\rho_x, \rho_x]$, $u \in [-\rho_u, \rho_u]$, $v \in [-\rho_v, \rho_v]$, with $\rho_x$, $\rho_u$, $\rho_v$ positive constants.

Taking into account (5.43), we have that for our example the control goal function takes the following form

$$Q_d(x_1, x_2, u, v) = u^2(1+\mu) + 2u(x_1^2 + x_2^2) + (x_1^2 + x_2^2)[(1+\mu)(x_1^2 + x_2^2) + \mu - 1] - yv \tag{5.55}$$

First of all, it is necessary to assure that the function $Q_d$ is positive, which will be achieved by means of the value of the parameter $\mu$. The procedure to follow in order to achieve this goal will be the same one as the one given for the buck example: function $Q_d$ will be assured to be positive in all the candidates for local minima in the domain of definition of $Q_d$ by means of the value of $\mu$. Two groups of local minima will be analyzed: the critical points of $Q_d$ and other candidates for minima in the domain $[-\rho_x, \rho_x] \times [-\rho_x, \rho_x] \times [-\rho_u, \rho_u] \times [-\rho_v, \rho_v]$. Examining $Q_d$, it can be noticed that $Q_d$ becomes positive for $\mu$ greater than a value $\overline{\mu}$, then, the worst case to search is the greatest value of $\mu$, among all the cases examined, for which $Q_d$ is positive.

1. **Assuring the critical point** $(x_1, x_2, u, v) = (0,0,0,v)$ **to be a relative minimum**. We will follow the same procedure as in the buck converter example where it was imposed for $Q_d$ to have a relative minimum at $x = 0, u = 0$. Let $u = \alpha(x, v)$ and $\alpha(0,0) = 0$. Therefore, the relative minimum of $Q_d$ at $x = 0$, $u = 0$ will be analyzed for $v = 0$. Consider $xu = (x,\ u)^T$. The gradient of $Q_d$ takes the form,

$$\frac{\partial Q_d}{\partial xu} = \left( \frac{\partial Q_d}{\partial x_1}, \frac{\partial Q_d}{\partial x_2}, \frac{\partial Q_d}{\partial u} \right) \tag{5.56}$$

with

$$\begin{aligned} \frac{\partial Q_d}{\partial x_1} &= 4ux_1 + 4\left(x_1^3 + x_1 x_2^2\right)(1+\mu) + 2(\mu - 1)x_1 \\ \frac{\partial Q_d}{\partial x_2} &= 4ux_2 + 4\left(x_2^3 + x_2 x_1^2\right)(1+\mu) + 2(\mu - 1)x_2 \\ \frac{\partial Q_d}{\partial u} &= 2u(1+\mu) + 2\left(x_1^2 + x_2^2\right) \end{aligned}$$

For $x_1 = x_2 = u = 0$ the gradient of $Q_d$ is zero. Now, the Hessian matrix of $Q_d$ will be evaluated for this stationary point. The Hessian matrix of $Q_d$ at $\overline{xu} = (x_1 = 0, x_2 = 0,\ u = 0)^T$ results in,

$$\left.\frac{\partial^2 Q_d}{\partial xu^2}\right|_{\overline{xu}} = \begin{pmatrix} 2(\mu-1) & 0 & 0 \\ 0 & 2(\mu-1) & 0 \\ 0 & 0 & 2(1+\mu) \end{pmatrix} \tag{5.57}$$

Therefore, for (5.57) to be positive definite it is necessary that all the eigenvalues are positive, i.e., $\mu$ must be greater than 1. Consequently, for $\mu > 1$, $(x_1, x_2, u, v) = (0,0,0,v)$ is a relative minimum of $Q_d$. In addition, for these values of $\mu$, the function $Q_d$ is positive. This fact is illustrated in Figure 5.11, where it can be seen that for a value $\mu < 1$, although function $Q_d$ is positive, the stationary point $\overline{xu}$ is not a relative minimum of the control goal function, whereas for $\mu > 1$ it is. Besides, for $v = 0$, $\mu < 1$ the function $Q_d$ yields negative for some values of $x$. To conclude with, the first lower bound for $\mu$ to considered is $\overline{\mu}_1 = 1$.

2. **Studying $Q_d$ at other candidates for local minima**. The domain of definition of function $Q_d(x_1, x_2, u, v)$ is $[-\rho_x, \rho_x] \times [-\rho_x, \rho_x] \times [-\rho_u, \rho_u] \times [-\rho_v, \rho_v]$, then, in this domain, a lower bound of $\mu$ will be found for which $Q_d$ is positive at all the candidates for local minima. For the sake of simplicity, $\rho_x = \rho_u = \rho_v = \rho$. Analyzing the domain of $Q_d$ and the behaviour of the function in it, the conclusion is that the worst case of $\mu$ is presented for the following $(x_1, x_2, u, v)$ points:

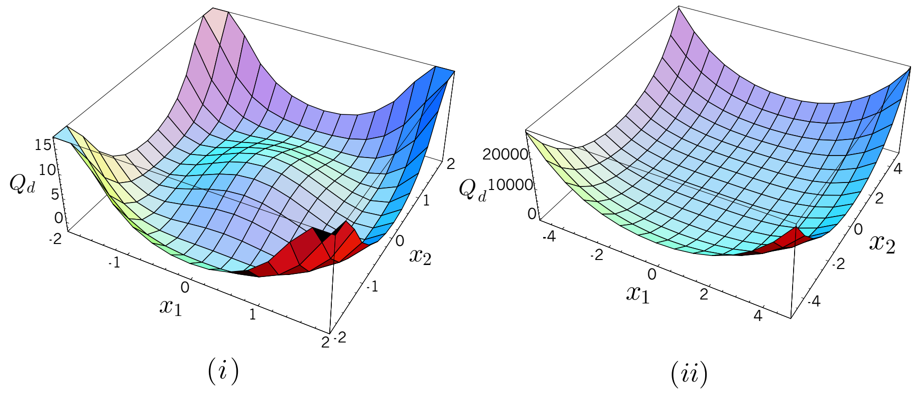


**Figure 5.11:** Representation of $Q_d$ (i) with $x_1,x_2 \in [-2,2]$, $v=0$, $\mu=0.06$, $u=-2$ (ii) with $x_1,x_2 \in [-5,5]$, $v=0$, $\mu=10$, $u=2$.

$p_1=(-\rho,\rho,0,0)$, $p_2=(-\rho,-\rho,0,0)$, $p_3=(\rho,\rho,0,0)$, $p_4=(\rho,-\rho,0,0)$, $p_5=(0,\rho,0,0)$, $p_6=(\rho,0,0,0)$, $p_7=(0,-\rho,0,0)$, $p_8=(-\rho,0,0,0)$. The value of $\mu$ in all cases depends on $\rho$, the smaller $\rho$ is, the higher $\mu$ is, then, our interest is concentrated in analyzing $\mu$ for the lower bound of $\rho$, i.e., $\rho=0$. In the cases of points $p_1$, $p_2$, $p_3$ and $p_4$, the critical value of $\mu$ is given by the following expression:

$$\overline{\mu}_2=\frac{1-2\rho^2}{1+2\rho^2} \tag{5.58}$$

For $p_5$, $p_6$, $p_7$ and $p_8$ the critical value for $\mu$ is given as follows,

$$\overline{\mu}_3=\frac{1-\rho^2}{1+\rho^2} \tag{5.59}$$

If $\mu$ is greater than $\overline{\mu}_2$ and $\overline{\mu}_3$, function $Q_d$ becomes positive. Considering the lower bound of $\rho$, i.e., $\rho=0$, $\overline{\mu}_2=\overline{\mu}_3=1$. This lower bound of $\mu$ coincides with $\overline{\mu}_1$. Then, in order to have $Q_d(x_1,x_2,u,v)>0$ it is necessary that

$$\mu>1$$

As it was done in the buck example, an upper bound for $\mu$ can be also given. The maximum value of $\mu$ depends on the maximum value of $Q_d$. Since the variables of $Q_d$ vary in a compact set and $Q_d$ is continuous, this function is bounded, then $\mu$ has an upper bound. The maximum value of $\mu$ depends on the extreme values of the variables $x_1$, $x_2$, $u$, for a fixed $v$, i.e., it depends upon constants $\rho_x$, $\rho_u$, $\rho_v$. Consequently, for any $\rho_x>0$, $\rho_u>0$, $\rho_v>0$, there exist $\overline{\mu}(\rho_x,\rho_u,\rho_v)>0$ such that,

$$1<\mu<\overline{\mu}(\rho_x,\rho_u,\rho_v) \tag{5.60}$$

In order to apply the SG algorithm it is necessary to verify assumptions A1, A2 and A3. From the fact that for $\mu>1$ it follows that $Q_d$ is a positive function with a relative minimum at $(x=0,u=0,v)$, the condition appearing in A1 is met, for $u^*=0$ and $x_1$, $x_2$,

$v$ fixed near to zero. The condition which is met in assumption A2 is (5.38), i.e., it takes the form:

$$-\left[2u^2(1+\mu)+2u(x_1^2+x_2^2)-uv\right] \leq -\left[u^2(1+\mu)+2u(x_1^2+x_2^2)-uv\right] < 0 \qquad (5.61)$$

Relation (5.61) is always met. For the condition of assumption A3 that $\nabla_u Q_d(u) = 2u(1+\mu)+2\left(x_1^2+x_2^2\right)-v$ must be bounded, the property of uniform continuity in a set *(Marsden and Hoffman, 1998)* [98] is applied. Then, the SG algorithm can be considered in its two versions:

1. The dynamic passifying control:

$$\begin{aligned}
u(k+1) &= u(k)-\gamma(k)\left\{2u(k)\left(1+\mu\right)+2\left[x_1^2(k)+x_2^2(k)\right]-v(k)\right\} \\
\gamma(k) &= \gamma_c\delta(k)\left|2u(k)\left(1+\mu\right)+2\left[x_1^2(k)+x_2^2(k)\right]-v(k)\right|^{-2} \\
0 &< \gamma_c < 2 \\
\delta(k) &= \begin{cases} Q_{d,k}(u(k)), & if\, Q_{d,k}(u(k)) \geq \Delta, \\ 0, & otherwise \end{cases}
\end{aligned} \qquad (5.62)$$

   with $\mu > 1$.

2. The static passifying control:

$$u(k) = -\frac{\gamma_{cs}\left[2x_1^2(k)+2x_2^2(k)-v(k)\right]}{1+2\gamma_{cs}(1+\mu)} \qquad (5.63)$$

   with $\mu > 1$ and $\gamma_{cs} > 0$ such that assure the asymptotic stability of the fixed point of the controlled system. Applying the control (5.63) to the system (5.23), the following quasi-$(V,s)$-passive system is obtained:

$$\begin{aligned}
x_1(k+1) &= \left\{x_1^2(k)+x_2^2(k)-\frac{\gamma_{cs}}{1+2\gamma_{cs}(1+\mu)}\left[2x_1^2(k)+x_2^2(k)\right]\right\}\cos[x_2(k)]+ \\
&\quad +\frac{\gamma_{cs}}{1+2\gamma_{cs}(1+\mu)}\cos[x_2(k)]v(k) = \\
&= \overline{f}_1[x_1(k),x_2(k)]+\overline{g}_1[x_1(k),x_2(k)]v(k) = \mathscr{F}_1[x(k),v(k)] \\
x_2(k+1) &= \left\{x_1^2(k)+x_2^2(k)-\frac{\gamma_{cs}}{1+2\gamma_{cs}(1+\mu)}\left[2x_1^2(k)+x_2^2(k)\right]\right\}\sin[x_2(k)]+ \\
&\quad +\frac{\gamma_{cs}}{1+2\gamma_{cs}(1+\mu)}\sin[x_2(k)]v(k) = \\
&= \overline{f}_2[x_1(k),x_2(k)]+\overline{g}_2[x_1(k),x_2(k)]v(k) = \mathscr{F}_2[x(k),v(k)] \\
y(k) &= \left\{x_1^2(k)+x_2^2(k)-\frac{\gamma_{cs}}{1+2\gamma_{cs}(1+\mu)}\left[2x_1^2(k)+x_2^2(k)\right]\right\}+ \\
&\quad +\frac{\gamma_{cs}}{1+2\gamma_{cs}(1+\mu)}v(k) = \\
&= \overline{h}[x_1(k),x_2(k)]+\overline{J}[x_1(k),x_2(k)]v(k) = \mathscr{H}[x(k),v(k)]
\end{aligned} \qquad (5.64)$$

   A way to ensure the local asymptotic stability of the fixed point $(\overline{x},\overline{v})$ of system (5.64) is by assuring that the linearized system around the fixed point has poles with modulus less than 1, i.e., the eigenvalues of the matrix,

$$\overline{A} = \left. \begin{pmatrix} \frac{\partial \mathscr{F}_1(x,v)}{\partial x_1} & \frac{\partial \mathscr{F}_1(x,v)}{\partial x_2} \\ \frac{\partial \mathscr{F}_2(x,v)}{\partial x_1} & \frac{\partial \mathscr{F}_2(x,v)}{\partial x_2} \end{pmatrix} \right|_{(x=\overline{x}, v=\overline{v})} \tag{5.65}$$

has modulus less than 1. For the fixed point $\overline{x} = 0$, $\overline{v} = 0$, the eigenvalues of (5.65) are zero, then the fixed point $(0,0)$ is locally asymptotically stable for any $\mu > 1$ and $\gamma_{cs} > 0$.

## 5.4 Feedback dissipativity seen as a perturbation of the energy invariance and losslessness situations

This section deals with the problem of feedback dissipativity for a class of discrete-time systems from the viewpoint proposed in Section 5.2. The feedback dissipativity problem is treated by means of the fundamental dissipativity inequality.

Two feedback dissipativity methodologies are proposed. They both are of approximate type, due to the fact that they both are based upon the first-order Taylor expansion formula at $u$ of $V(f(x,u))$ and $s(h(x,u),v)$. The underlying idea is that dissipativity is seen as a “perturbation” of the storage energy invariance or the system losslessness situations, in the sense that the control which makes the system dissipative ($u$) is based on the control that makes the storage energy function $V$ invariant or on the control that renders the system lossless ($u^*$), and $u$ is locally valid in a neighbourhood of $u^*$. The approaches are local since the $(V,s)$-dissipativity of the feedback transformed system is assured in a compact subset of $\mathscr{X} \times \mathscr{U}$ containing the fixed point of the system. The orbits of the feedback transformed system are assured not to leave this compact by means of the stability properties of the class of dissipativity treated.

The second method which renders a system $(V,s)$-dissipative by means of rendering it $(V,s)$-lossless can be also seen as a new way of treating the feedback losslessness problem. The local feedback losslessness methodology proposed is derived from the feedback dissipativity one proposed in Section 5.2.

### 5.4.1 Feedback dissipativity seen as a perturbation of the energy invariance situation

#### 5.4.1.1 Description of the methodology

Consider the system (5.1)-(5.2). Suppose there exists a control $u^* : \mathscr{X} \to \mathscr{U}$ such that

$$V(f(x,u^*)) - V(x) = 0, \;\; \forall (x,u^*) \in \mathscr{X} \times \mathscr{U} \tag{5.66}$$

with $V$ a storage function associated to the system, as defined in Section 5.2.

Let a function $\delta u^*(x,u^*,v)$ such that $\delta u^* : \mathscr{X} \times \mathscr{U} \times \mathscr{U} \to \mathscr{U}$. Define the following state dependent input coordinate transformation,

$$u = u^*(x) + \delta u^*(x,u^*,v) \tag{5.67}$$

where,

$$\delta u^*(x,u^*,v) = \frac{s(h(x,u^*),v) - \phi(x,u^*)}{\left\{ \frac{\partial}{\partial u}[V(f(x,u)) + \phi(x,u) - s(h(x,u),v)] \right\}_{u=u^*}}, \tag{5.68}$$

with $v$ the new input to the system, $s$ the supply function, and $\phi$ acting as a smooth dissipation rate function.

**Proposition 5.1** *Let $V(x)$ be a storage function. Let $(x_0,u_0^*) \in A = \mathscr{X} \times \mathscr{U}$ with $A$ an open set. Suppose that the following two conditions are satisfied:*

1. $\exists(x_0,u_0^*)$ *such that equality (5.66) holds, i.e.,*

$$V(f(x_0,u_0^*)) - V(x_0) = 0, \tag{5.69}$$

2. 

$$\left.\frac{\partial}{\partial u^*}V(f(x,u^*))\right|_{(x_0,u_0^*)} \neq 0 \tag{5.70}$$

*Then, there exists a unique static state feedback control law of the form $u^* = \alpha(x)$ defined in a neighbourhood of $x_0$, $\widetilde{\mathscr{X}} \subset \mathscr{X}$, and valued in a neighbourhood of $u_0^*$, $\widetilde{\mathscr{U}} \subset \mathscr{U}$, such that equation (5.66) is satisfied with $u^* = \alpha(x)$, for all $x \in \widetilde{\mathscr{X}}$.*

**Proof.** The proof follows from the implicit function theorem.

■

**Remark 5.12** *If condition (5.70) is satisfied for $(x_0,u_0^*)$, it also holds in a neighbourhood $\widetilde{\mathscr{W}}$ of $(x_0,u_0^*)$, with $\widetilde{\mathscr{W}} \subset \mathscr{X} \times \mathscr{U}$, where the following assertion is valid,*

$$\frac{\partial}{\partial u^*}V(f(x,u^*)) \neq 0, \ \ \forall (x,u^*) \in \widetilde{\mathscr{W}} \tag{5.71}$$

**Proposition 5.2** *Let $V(x)$ and $s(y,v)$ smooth storage and supply functions. Suppose conditions (5.69)-(5.70) are satisfied. Let $\overline{x}$ an isolated fixed point of $f(x,\overline{u})$, with $\overline{u}$ a constant. Let $\widetilde{\mathscr{X}} \subset \mathscr{X}$ and $\widetilde{\mathscr{U}}$, $\widetilde{\mathscr{V}}$, $\widetilde{\mathscr{P}} \subset \mathscr{U}$ be compact sets containing $\overline{x}$ and $\overline{u}$, respectively. Then, the system (5.1)-(5.2) is locally feedback dissipative with the functions $V$ and $s$, by means of a feedback of the form (5.67), with $|\delta u^*| \leq \rho$, $\rho > 0$ small enough, $u^* : \widetilde{\mathscr{X}} \to \widetilde{\mathscr{U}}$ obtained from (5.66), $\delta u^* : \widetilde{\mathscr{X}} \times \widetilde{\mathscr{U}} \times \widetilde{\mathscr{V}} \to \widetilde{\mathscr{P}}$ given by (5.68) and $u^* + \delta u^*$ defined in a neighbourhood of $u^*$ if there exists a smooth dissipation rate function $\phi(x,u)$ for which*

$$\left| \frac{s(h(x,u^*),v) - \phi(x,u^*) - \min(R_V + R_\phi) + \max R_s}{\left\{ \frac{\partial}{\partial u}[V(f(x,u)) + \phi(x,u) - s(h(x,u),v)] \right\}_{u=u^*}} \right| \leq \rho \tag{5.72}$$

*with $R_V$, $R_\phi$ and $R_s$ the remainder of the Taylor expansion of $V(f(x,u^* + \delta u^*))$, $\phi(x,u^* + \delta u^*)$ and $s(h(x,u^* + \delta u^*),v)$ at $u^*$, respectively.*

**Proof.** Let consider control $u = u^*(x) + \delta u^*(x,u^*,v)$ is applied to the system (5.1)-(5.2), then

$$\begin{aligned} x(k+1) &= f(x(k),u^*(k)+\delta u^*(k)) \\ y(k) &= h(x(k),u^*(k)+\delta u^*(k)), \end{aligned} \tag{5.73}$$

with $u^*(x)$ such a control that makes the system orbits lie on the level surfaces of $V$ and $|\delta u^*| \leq \rho$, $\rho > 0$. Control $\delta u^*$ is proposed in such a way that makes the system (5.73) be $(V,s)$-dissipative, i.e.,

$$V(f(x,u^*+\delta u^*)) - V(x) \leq s(h(x,u^*+\delta u^*),v) \tag{5.74}$$

with $v$ the new input. Inequality (5.74) can be transformed into an equality as follows, see Definition 4.2,

$$V(f(x,u^*+\delta u^*)) - V(x) = s(h(x,u^*+\delta u^*),v) - \phi(x,u^*+\delta u^*), \tag{5.75}$$

with $\phi$ a smooth dissipation rate function.

Considering the first-order Taylor approximation at $u^*$ of (5.75), it is obtained

$$\begin{aligned} V(f(x,u^*)) \quad &+ \quad \frac{\partial}{\partial u}V(f(x,u))\Big|_{u=u^*}\delta u^* + R_V - V(x) = \\ &= \quad s(h(x,u^*),v) + \frac{\partial}{\partial u}s(h(x,u),v)\Big|_{u=u^*}\delta u^* + R_s - \\ &\quad - \left\{\phi(x,u^*) + \frac{\partial}{\partial u}\phi(x,u)\Big|_{u=u^*}\delta u^* + R_\phi\right\}, \end{aligned} \tag{5.76}$$

Taking into account that for $u^* = \alpha(x)$, $V(f(x,\alpha(x))) - V(x) = 0$, $\forall x \in \widetilde{\mathscr{X}}$, then the control $\delta u^*$ which renders the system $(V,s)$- dissipative is obtained from (5.76) and takes the form,

$$\delta u^* = \frac{s(h(x,u^*),v) - \phi(x,u^*) - (R_V + R_\phi - R_s)}{\left\{\frac{\partial}{\partial u}\left[V(f(x,u)) + \phi(x,u) - s(h(x,u),v)\right]\right\}_{u=u^*}} \tag{5.77}$$

First of all, for the existence of control $\delta u^*$ defined as in (5.77), it is necessary that,

$$\left\{\frac{\partial}{\partial u}\left[V(f(x,u)) + \phi(x,u) - s(h(x,u),v)\right]\right\}_{u=u^*} \neq 0, \;\; \forall (x,u^*,v) \in \widetilde{\mathscr{W}} \subset \mathscr{X} \times \mathscr{U} \times \mathscr{U} \tag{5.78}$$

If condition (5.70) holds, condition (5.78) will be satisfied for an open neighbourhood $\widetilde{\mathscr{W}} \subset \mathscr{X} \times \mathscr{U} \times \mathscr{U}$ (see Remark 5.12).

Besides, control $\delta u^*$ is needed to be bounded and small enough in order to have $u^* + \delta u^*$ defined in a neighbourhood of $u^*$, i.e., $u^* + \delta u^* \in [u^* - \rho, u^* + \rho]$, $\rho > 0$. Therefore, from (5.77),

$$\begin{aligned} |\delta u^*| \quad &= \quad \left| \frac{s(h(x,u^*),v) - \phi(x,u^*) - (R_V + R_\phi - R_s)}{\left\{\frac{\partial}{\partial u}\left[V(f(x,u)) + \phi(x,u) - s(h(x,u),v)\right]\right\}_{u=u^*}} \right| \leq \\ &\leq \quad \left| \frac{s(h(x,u^*),v) - \phi(x,u^*) - \min(R_V + R_\phi) + \max R_s}{\left\{\frac{\partial}{\partial u}\left[V(f(x,u)) + \phi(x,u) - s(h(x,u),v)\right]\right\}_{u=u^*}} \right| \leq \rho \end{aligned}$$

Indeed, function $|\delta u^*|$ is bounded if $s$, $\phi$, the residues $R_V$, $R_\phi$ and

$$\frac{\partial}{\partial u}[V(f(x,u)) + \phi(x,u) - s(h(x,u),v)]$$

are bounded. Due to the fact that functions $V$, $s$, $\phi$ are defined in compact sets, they are bounded. It is the same for the residues $R_V$, $R_\phi$: they depend on the 2nd-order derivative of $V(f(x,u))$, $\phi(x,u)$ and $s(h(x,u),v)$, and since they are smooth functions defined on a compact, their derivatives are bounded.

To conclude with, if condition (5.72) is satisfied then $u^* + \delta u^*$ with $u^*$ obtained from equation (5.66) and $\delta u^*$ given by (5.68) is valid and the system (5.1)-(5.2) is rendered locally $(V,s)$-dissipative. Note that in addition to condition (5.72) which assures that $\delta u^*$ is bounded, control $\delta u^*$ must be also small enough, $\rho$ must be small enough, in other words, control $u^* + \delta u^*$ must be defined in a neighbourhood around $u^*$ small enough.

■

**Remark 5.13** *The orbits of the feedback transformed system are assured not to leave the compact $\widetilde{\mathscr{X}}$, where $(V,s)$-dissipativity (strictly $(V,s)$-dissipativity) is achieved, if they start in $\widetilde{\mathscr{X}}$. The idea is the following one. If for a $(x^*,v^*) \in \widetilde{\mathscr{X}} \times \widetilde{\mathscr{V}}$ of the feedback transformed system $x(k+1) = \overline{f}(x,v)$, $y = \overline{h}(x,v)$, we have that $s(\overline{h}(x^*,v^*),v^*) = 0$, and $(x^*,v^*)$ is closed enough to the fixed point $(\overline{x},\overline{u})$, then there exists a neighbourhood $\mathscr{W} \subset \widetilde{\mathscr{X}} \times \widetilde{\mathscr{V}}$ of $(x^*,v^*)$ containing $(\overline{x},\overline{u})$ where $s(\overline{h}(x,v),v) = 0, \forall (x,v) \in \mathscr{W}$, and $V(x(k+1)) - V(x(k)) \leq 0$ (resp., $V(x(k+1)) - V(x(k)) < 0$), then there is a region where the fixed point is stable (resp., asymptotically stable). If $v^* \neq 0$, the function $v$ should be bounded. The bounds of $v$ will depend on the compacts where $(V,s)$-dissipativity is assured. For more details on the stability properties of $(V,s)$-dissipative systems, see Chapter 6.*

**Remark 5.14** *Condition (5.78) for the existence of $\delta u^*$ has the same form as the feedback dissipativity condition (5.5), see Section 5.2.*

**Remark 5.15** *For the validity of this method, it is necessary to check how good the first-order Taylor approximations at $u^*$ used for $V(f(x,u^* + \delta u^*))$, $\phi(x,u^* + \delta u^*)$ and $s(h(x,u^* + \delta u^*),v)$ are. Then, the validity of the method can be also tested by means of the boundedness of*

$$|V_1 - V_2| = |R_V| \tag{5.79}$$

$$|\phi_1 - \phi_2| = |R_\phi| \tag{5.80}$$

$$|s_1 - s_2| = |R_s| \tag{5.81}$$

*with,*

- $V_1 = V(f(x,u^* + \delta u^*))$, $V_2 = V(f(x,u^*)) + \frac{\partial}{\partial u} V(f(x,u))\Big|_{u=u^*} \delta u^*$.
- $\phi_1 = \phi(x,u^* + \delta u^*)$ , $\phi_2 = \phi(x,u^*) + \frac{\partial}{\partial u}\phi(x,u)\Big|_{u=u^*} \delta u^*$.

- $s_1 = s(h(x,u^* + \delta u^*),v)$ , $s_2 = s(h(x,u^*),v) + \frac{\partial}{\partial u} s(h(x,u),v)\Big|_{u=u^*} \delta u^*$.

*This corresponds to the study of the error of the approximation made. This will be checked in the example treated in Section 5.4.1.2.*

**Remark 5.16** *If $\delta u^*(k) \to 0$ as $k \to \infty$ then the remainders $R_V$, $R_s$ and $R_\phi$ tend to zero* (Marsden and Hoffman, 1998) *[98], and consequently, the approximation of the feedback dissipativity equality made is valid.*

**Remark 5.17** *The control proposed in order to render a nonlinear discrete-time system $(V,s)$-dissipative consists of two controls: a control which makes the storage energy function invariant ($u^*$), and the control which alters the energy invariance situation to make the system $(V,s)$-dissipative ($\delta u^*$). Control $\delta u^*$ is defined by means of $u^*$, however, the form of $u^*$ must be proposed. The computation of $u^*$ from equality (5.66) can be carried out in many different ways. On account of the discrete-time nature of the system, iterative methods may be used. The iterative-like method used in Section 5.3 in order to render a system quasi-$(V,s)$-dissipative, i.e., the speed-gradient algorithm in its discrete version, can not be used here at a first sight; if a control goal function $Q$ of the form $Q = V(f(x,u^*)) - V(x)$ was proposed (which is the one we are interested in), function $Q$ would not be assured to be non-negative. The way proposed in order to obtain $u^*$ will be the one shown in Section 5.2, consisting in computing the explicit expression of $u^*$ from equation (5.66).*

#### 5.4.1.2 Example. A nonlinear example

The feedback dissipativity methodology shown above will be applied to the passivation of an academic nonlinear discrete-time system whose fixed point $(0,0)$ is unstable in open loop. This system was also considered in Sections 5.2, 5.3 and has the form presented in (5.23). System (5.23) is aimed to be rendered $V$-passive with storage function $V = x_1^2 + x_2^2$ and supply function $s(y,v) = yv$, $y = x_1^2 + x_2^2 + u$. The dissipation rate function $\phi(x,u)$ is chosen as in (5.24). Suppose the variables of the system are defined in the following sets: $x_1 \in [-\varepsilon_{x_1}, \varepsilon_{x_1}]$, $x_2 \in [-\varepsilon_{x_2}, \varepsilon_{x_2}]$, $u, u^*, v \in [-\varepsilon_u, \varepsilon_u]$, $\delta u^* \in [-\rho, \rho]$, with $\varepsilon_{x_1}$, $\varepsilon_{x_2}$, $\varepsilon_u$, $\rho$ positive constants.

Now, the existence and validity of controls $u^*(x)$ and $\delta u^*(x, u^*, v)$ will be analyzed.

In order to obtain control $u^*$, equation (5.66), for our example, takes the following form,

$$a_{u^*}(u^*)^2 + b_{u^*}(x_1,x_2)u^* + c_{u^*}(x_1,x_2) = 0 \tag{5.82}$$

with

$$\begin{aligned} a_{u^*} &= 1 \\ b_{u^*}(x_1,x_2) &= 2(x_1^2 + x_2^2) \\ c_{u^*}(x_1,x_2) &= (x_1^2 + x_2^2)^2 - (x_1^2 + x_2^2) \end{aligned}$$

Sufficient conditions for the existence of control $u^*$ are (5.69) and (5.70) which are met for this example if,

$$a_{u^*}(u^*)^2 + b_{u^*}(x_1,x_2)u^* + c_{u^*}(x_1,x_2) = 0, \tag{5.83}$$

$$2a_{u^*}u^* + b_{u^*}(x_1,x_2) \neq 0 \tag{5.84}$$

for some $(x_1, x_2, u^*)$. If conditions (5.83)-(5.84) are satisfied for some $(x_1, x_2, u^*)$, a control $u^*$ satisfying (5.66) exists. This $u^*$ can be obtained from the explicit solution of (5.82), then it is necessary to assure that $\varphi(x_1, x_2) = b_{u^*}^2 - 4a_{u^*}c_{u^*} \geq 0$, which will be always achieved. This function is always positive and has a minimum in $x = 0$. Then, two solutions for control $u^*$ are obtained,

$$u_1^*(x_1, x_2) = \frac{-b_{u^*} + \sqrt{b_{u^*}^2 - 4a_{u^*}c_{u^*}}}{2a_{u^*}}, \; u_2^*(x_1, x_2) = \frac{-b_{u^*} - \sqrt{b_{u^*}^2 - 4a_{u^*}c_{u^*}}}{2a_{u^*}} \tag{5.85}$$

Concerning the computation of control $\delta u^*$, condition (5.72) is needed to be assured. In other words, it is necessary to verify that all the terms appearing in (5.72) are bounded. On the one hand, we can use the property of uniform continuity for the functions $s(h(x,u^*),v)$, $\phi(x,u^*)$, $\beta(x,u^*,v) = \left\{ \frac{\partial}{\partial u}\left[V(f(x,u)) + \phi(x,u) - s(h(x,u),v)\right] \right\}_{u=u^*}$, indeed, function $\phi(x,u)$ is a convex function with a minimum in $x = 0, u = 0$ for all $\mu > 0$. Function $\beta(x,u^*,v)$ was analyzed in Section 5.3 when the SG algorithm was applied to passify the system (5.23), notice that,

$$\beta(x,u^*,v) = \frac{\partial}{\partial u} Q_d(x,u,v)\bigg|_{u=u^*}$$

with $Q_d$ as defined in (5.55). To conclude with, as functions $s(h(x,u^*),v)$, $\phi(x,u^*)$, $\beta(x,u^*,v)$ are continuous and their variables vary in a compact set, we can say that they are bounded.

Now, it is necessary to conclude that the remainders $R_V$, $R_s$ and $R_\phi$ are also bounded. The expressions for these errors are presented as follows,

$$\begin{aligned}
R_V &= \frac{\partial^2}{\partial u^2} V(f(x,u))\bigg|_{u=u^*} \frac{(\delta u^*)^2}{2} = (\delta u^*)^2 = V_1 - V_2 \\
R_s &= \frac{\partial^2}{\partial u^2} s(h(x,u),v)\bigg|_{u=u^*} \frac{(\delta u^*)^2}{2} = 0 = s_1 - s_2 \\
R_\phi &= \frac{\partial^2}{\partial u^2} \phi(x,u)\bigg|_{u=u^*} \frac{(\delta u^*)^2}{2} = \mu(\delta u^*)^2 = \phi_1 - \phi_2
\end{aligned}$$

with,

$$
\begin{aligned}
V_1 &= V(f(x,u^*+\delta u^*)) = \\
&= (u^*+\delta u^*)^2 + 2(u^*+\delta u^*)(x_1^2+x_2^2) + (x_1^2+x_2^2)^2, \\
V_2 &= V(f(x,u^*)) + \frac{\partial}{\partial u} V(f(x,u))\Big|_{u=u^*} \delta u^* = \\
&= (u^*)^2 + 2u^*(x_1^2+x_2^2) + (x_1^2+x_2^2)^2 + 2\left(u^*+x_1^2+x_2^2\right)\delta u^* \\
s_1 &= s(h(x,u^*+\delta u^*),v) = \\
&= (x_1^2+x_2^2+u^*+\delta u^*)v \\
s_2 &= s(h(x,u^*),v) + \frac{\partial}{\partial u} s(y,u)\Big|_{u=u^*} \delta u^* = \\
&= (x_1^2+x_2^2+u^*)v + v\delta u^* \\
\phi_1 &= \phi(x,u^*+\delta u^*) = \\
&= \mu\left[(x_1^2+x_2^2)^2 + (u^*+\delta u^*)^2 + x_1^2+x_2^2\right] \\
\phi_2 &= \phi(x,u^*) + \frac{\partial}{\partial u}\phi(x,u)\Big|_{u=u^*} \delta u^* = \\
&= \mu\left[(x_1^2+x_2^2)^2 + (u^*)^2 + x_1^2+x_2^2\right] + 2u^*\mu\delta u^*
\end{aligned}
$$

The remainders $R_V$, $R_s$ and $R_\phi$ represent how much

$$
\begin{aligned}
V_1 &= V(f(x,u^*+\delta u^*)), \\
s_1 &= s(h(x,u^*+\delta u^*),v), \\
\phi_1 &= \phi(x,u^*+\delta u^*)
\end{aligned}
$$

differ from their approximations of first order at $u^*$ ($V_2$, $s_2$ and $\phi_2$, respectively). It can be concluded that the error of the approximation depends on $\delta u^*$. If $\delta u^*$ is bounded then $R_V$ and $R_\phi$ are bounded in the sets where $x$, $u^*$ and $v$ are defined. The errors of the approximation tend to zero as long as $\delta u^*$ does.

In addition, an upper bound for $\mu$ is needed to be given. The maximum value for $\mu$ for control $\delta u^*$ to be valid is such that the relation (5.72) is met, i.e.,

$$
\left| \frac{(x_1^2+x_2^2+u^*)v - \mu\left[(x_1^2+x_2^2)^2+(u^*)^2+x_1^2+x_2^2\right] - (1+\mu)\min[(\delta u^*)^2]}{2u^*(1+\mu)+2\left(x_1^2+x_2^2\right)-v} \right| \le \rho \quad (5.86)
$$

The value for $\mu$ given by relation (5.86) will depend on the maximum and minimum values of the states and the controls, i.e., on the constants $\varepsilon_{x_1}$, $\varepsilon_{x_2}$, $\varepsilon_u$, $\rho$. Then, it can be concluded that for any $\rho > 0$, $\varepsilon_{x_1} > 0$, $\varepsilon_{x_2} > 0$, $\varepsilon_u > 0$ there exists $\overline{\mu} > 0$ such that,

$$
0 < \mu < \overline{\mu}(\rho, \varepsilon_{x_1}, \varepsilon_{x_2}, \varepsilon_u) \quad (5.87)
$$

Therefore, control $\delta u^*$ is given by expression (5.68), i.e.,

$$
\delta u^*(x_1,x_2,u^*,v) = \frac{(x_1^2+x_2^2+u^*)v - \mu\left[(x_1^2+x_2^2)^2+(u^*)^2+x_1^2+x_2^2\right]}{2u^*(1+\mu)+2\left(x_1^2+x_2^2\right)-v} \quad (5.88)
$$

with $\mu > 0$ and satisfying (5.86), and control $u^*$ as defined in (5.85). For the existence of control $\delta u^*$, it is also needed to assure that,

$$
2u^*(1+\mu) + \left(2x_1^2+2x_2^2\right) - v \neq 0 \quad (5.89)
$$

The control which renders the system $V$-passive is given by $u^* + \delta u^*$ as defined above.

**Remark 5.18** *For stabilization purposes, $x_1 = x_2 = v = 0$ will be considered, then $u^*$ and $\delta u^*$ will tend to zero, and consequently, the errors of the approximations used will be zero in the steady situation, see Chapter 6.*

### 5.4.2 Feedback dissipativity seen as a perturbation of the losslessness situation

The methodology proposed in this section in order to render a system $(V,s)$-dissipative by means of a static feedback follows the same approach as the one presented in Section 5.4.1, it is based on the fundamental dissipativity inequality, indeed, on an approximation of the dissipativity inequality. It consists of two steps: first, the system is rendered lossless; second, the control which makes the system $(V,s)$-dissipative it will be a "perturbation" of the one which makes it lossless.

The feedback losslessness methodology proposed is derived from the feedback dissipativity one proposed in Section 5.2, see Definition 5.1 for feedback dissipativity and Theorem 5.1 for feedback dissipativity conditions.

#### 5.4.2.1 Description of the methodology

Before proposing the control which renders the system (5.1)-(5.2) $(V,s)$-dissipative, the feedback losslessness problem is defined in terms of the dissipativity inequality. This method can be also seen as an alternative way of treating the feedback losslessness problem to the one proposed in *(Byrnes and Lin, 1994)* [14].

**Definition 5.6** *Consider a system of the form (5.1)-(5.2) and two scalar functions $V(x)$ and $s(y,v)$ considered as a storage function and a supply function, respectively. The system is said to be* feedback lossless *with the functions V and s, if there exists a* regular *static state feedback control law of the form, $u = \alpha(x,v)$, with $v$ the new input, such that the feedback transformed system is $(V,s)$-lossless.*

The existence of a feedback control law of the form $u^* = \alpha(x,v)$ for which the system is rendered lossless must be assessed from the existence of solutions, for the control input $u^*$, of the following equation,

$$V(f(x,u^*)) - V(x) = s(h(x,u^*),v) \tag{5.90}$$

with $V$ and $s$ considered as a storage function and a supply function, respectively, as defined in Section 5.2.

The following proposition states sufficient conditions under which local feedback losslessness is possible.

**Proposition 5.3** *Consider a system of the form (5.1)-(5.2) and two scalar functions $V(x)$ and $s(y,v)$ considered as a storage function and a supply function, respectively. Let $(x_0,u_0^*,v_0) \in A = \mathscr{X} \times \mathscr{U} \times \mathscr{U}$ with $A$ an open set. Suppose that the following two conditions are satisfied:*

*1.* $\exists(x_0,u_0^*,v_0)$ *such that equality (5.90) holds, i.e.*

$$V(f(x_0,u_0^*)) - V(x_0) - s(h(x_0,u_0^*),v_0) = 0, \tag{5.91}$$

*2.*

$$\left\{\frac{\partial}{\partial u^*}[V(f(x,u^*)) - s(h(x,u^*),v)]\right\}_{(x_0,u_0^*,v_0)} \neq 0 \tag{5.92}$$

*Then, there exists a unique static state feedback control law of the form* $u^* = \alpha(x,v)$ *defined in a neighbourhood of* $(x_0,v_0)$ *and valued in a neighbourhood of* $u_0^*$ *such that the feedback transformed system* $x(k+1) = \overline{f}(x(k),v(k)),\ \ y(k) = \overline{h}(x(k),v(k))$ *is* $(V,s)$*-lossless.*

**Proof.** The proof follows from the implicit function theorem.

■

**Remark 5.19** *If condition (5.92) is satisfied for* $(x_0,u_0^*,v_0)$*, it also holds in a neighbourhood* $\widetilde{\mathscr{W}}$ *of* $(x_0,u_0^*,v_0)$*, with* $\widetilde{\mathscr{W}} \subset \mathscr{X} \times \mathscr{U} \times \mathscr{U}$*, where the following assertion is valid,*

$$\frac{\partial}{\partial u^*}[V(f(x,u^*)) - s(h(x,u^*),v)] \neq 0,\ \ \forall (x,u^*,v) \in \widetilde{\mathscr{W}} \tag{5.93}$$

Consider the system (5.1)-(5.2). Suppose there exists a regular static state feedback $u^* : \mathscr{X} \times \mathscr{U} \to \mathscr{U}$ which renders the system $(V,s)$-lossless. Let a function $\delta u^*(x,u^*,v)$ such that $\delta u^* : \mathscr{X} \times \mathscr{U} \times \mathscr{U} \to \mathscr{U}$. Define the following state dependent input coordinate transformation,

$$u = u^*(x,v) + \delta u^*(x,u^*,v) \tag{5.94}$$

with,

$$\delta u^*(x,u^*,v) = -\mu\left\{\frac{\partial}{\partial u}[V(f(x,u)) - s(h(x,u),v)]\right\}_{u=u^*} \tag{5.95}$$

where $\mu$ is a positive constant.

**Proposition 5.4** *Let* $V(x)$ *and* $s(y,v)$ *smooth storage and supply functions. Suppose conditions (5.91)-(5.92) are satisfied. Let* $\overline{x}$ *an isolated fixed point of* $f(x,\overline{u})$*, with* $\overline{u}$ *a constant. Let* $\widetilde{\mathscr{X}} \subset \mathscr{X}$ *and* $\widetilde{\mathscr{U}}, \widetilde{\mathscr{V}}, \widetilde{\mathscr{P}} \subset \mathscr{U}$ *be compact sets containing* $\overline{x}$ *and* $\overline{u}$*, respectively. Then, the system (5.1)-(5.2) is locally feedback dissipative with the functions V and s by means of a feedback of the form (5.94), with* $u^* : \widetilde{\mathscr{X}} \times \widetilde{\mathscr{V}} \to \widetilde{\mathscr{U}}$ *obtained from (5.90),* $\delta u^* : \widetilde{\mathscr{X}} \times \widetilde{\mathscr{U}} \times \widetilde{\mathscr{V}} \to \widetilde{\mathscr{P}}$ *given by (5.95), and* $u^* + \delta u^*$ *defined in a neighbourhood of* $u^*$ *if there exists a positive constant* $\mu$*, for which the following conditions are satisfied*

*1.*

$$|\max R_V - \min R_s| \leq \mu\left\{\frac{\partial}{\partial u}[V(f(x,u)) - s(h(x,u),v)]\right\}^2_{u=u^*} \tag{5.96}$$

2.
$$\mu \left| \left\{ \frac{\partial}{\partial u} [V(f(x,u)) - s(h(x,u),v)] \right\}_{u=u^*} \right| \leq \rho \tag{5.97}$$

*with $\rho$ a positive constant small enough, and $R_V$, $R_s$ the remainder of the Taylor expansion of $V(f(x,u^*+\delta u^*))$ and $s(h(x,u^*+\delta u^*),v)$ at $u^*$, respectively.*

**Proof.** Let consider control $u = u^* + \delta u^*$ is applied to the system (5.1)-(5.2), then

$$\begin{aligned} x(k+1) &= f(x(k), u^*(k) + \delta u^*(k)) \\ y(k) &= h(x(k), u^*(k) + \delta u^*(k)), \end{aligned} \tag{5.98}$$

with $u^*$ such a control that makes the system $(V,s)$-lossless with $V$ and $s$ as storage energy and supply functions, respectively. Control $\delta u^*$ is proposed in such a way that makes the system (5.98) be $(V,s)$-dissipative, with $v$ the new input, i.e.,

$$V(f(x,u^*+\delta u^*)) - V(x) \leq s(h(x,u^*+\delta u^*),v) \tag{5.99}$$

Considering the first-order Taylor approximation at $u^*$ of $V(f(x,u^*+\delta u^*))$ and $s(h(x,u^*+\delta u^*),v)$, (5.99) is rewritten as follows

$$\begin{aligned} V(f(x,u^*)) \quad + & \quad \frac{\partial}{\partial u} V(f(x,u)) \bigg|_{u=u^*} \delta u^* + R_V - V(x) \leq \\ \leq & \quad s(h(x,u^*),v) + \frac{\partial}{\partial u} s(h(x,u),v) \bigg|_{u=u^*} \delta u^* + R_s \end{aligned} \tag{5.100}$$

Taking into account that for $u^* = \alpha(x,v)$, $V(f(x,\alpha(x,v))) - V(x) - s(h(x,\alpha(x,v)),v) = 0$, $\forall (x,v) \in \widetilde{\mathscr{X}} \times \widetilde{\mathscr{U}}$, relation (5.100) takes the form,

$$\left\{ \frac{\partial}{\partial u} [V(f(x,u)) - s(h(x,u),v)] \right\}_{u=u^*} \delta u^* - R_s + R_V \leq 0$$

Besides, using (5.95),

$$\begin{aligned} & \left\{ \frac{\partial}{\partial u} [V(f(x,u)) - s(h(x,u),v)] \right\}_{u=u^*} \delta u^* - R_s + R_V \leq \\ \leq & \left\{ \frac{\partial}{\partial u} [V(f(x,u)) - s(h(x,u),v)] \right\}_{u=u^*} \delta u^* - \min R_s + \max R_V = \\ = & -\mu \left\{ \frac{\partial}{\partial u} [V(f(x,u)) - s(h(x,u),v)] \right\}^2_{u=u^*} - \min R_s + \max R_V \leq 0 \end{aligned}$$

which is assured by means of condition (5.96).

Control $\delta u^*$ is also needed to be bounded and small enough in order to have $u^* + \delta u^*$ defined in a neighbourhood of $u^*$, i.e., let $\rho$ be a positive constant, $u^* + \delta u^* \in [u^* - \rho, u^* + \rho]$. This holds if

$$|\delta u^*| = \mu \left| \left\{ \frac{\partial}{\partial u} [V(f(x,u)) - s(h(x,u),v)] \right\}_{u=u^*} \right| \leq \rho$$

which is what condition (5.97) proposes with $\rho$ small enough. Indeed, due to the fact that the smooth functions $V$, $s$ are defined in compact sets, it is the same for their derivatives and consequently, they are bounded. $|\delta u^*|$ can be as small as wanted, by means of constant $\mu$. Constant $\mu$ is chosen in order to achieve the applicability conditions (5.96)-(5.97).

Summing up, if conditions (5.96)-(5.97) are satisfied then $u$ given by (5.94), with $u^*$ obtained from (5.90) and $\delta u^*$ given by (5.95), renders the system (5.1)-(5.2) locally $(V,s)$-dissipative. In addition, the orbits of the feedback transformed system are assured not to leave the compact where $(V,s)$-dissipativity (strictly $(V,s)$-dissipativity) is achieved. For this purpose, $v$ will be established from the relation $s(\overline{h}(x,v),v) = 0$, assuring that $V(x(k+1)) - V(x(k)) \leq 0$ $(< 0)$. See Remark 5.13.

■

**Remark 5.20** *For the validity of this method, it is necessary to check how good the first-order Taylor approximations at $u^*$ used for $V(f(x,u^*+\delta u^*))$ and $s(h(x,u^*+\delta u^*),v)$ are. Then, the validity of the method can be also tested by means of the boundedness of*

$$|V_1 - V_2| = |R_V| \tag{5.101}$$
$$|s_1 - s_2| = |R_s| \tag{5.102}$$

*with,*

- $V_1 = V(f(x,u^*+\delta u^*))$, $V_2 = V(f(x,u^*)) + \frac{\partial}{\partial u}V(f(x,u))\Big|_{u=u^*}\delta u^*$.
- $s_1 = s(h(x,u^*+\delta u^*),v)$ , $s_2 = s(h(x,u^*),v) + \frac{\partial}{\partial u}s(h(x,u),v)\Big|_{u=u^*}\delta u^*$.

*This corresponds to the study of the error of the approximation made.*

**Remark 5.21** *As it happens with the feedback dissipativity methodology proposed in Section 5.4.1, not only is it necessary that $\delta u^*$ is bounded, but also it must be small enough, depending upon the value of $\rho$. This will be achieved by means of the parameter $\mu$.*

**Remark 5.22** *If $\delta u^*(k) \to 0$ as $k \to \infty$ then the remainders $R_V$ and $R_s$ tend to zero* (Marsden and Hoffman, 1998) *[98], and consequently, $\delta u^*$ given in (5.95) is valid in order to achieve feedback dissipativity. This happens when the feedback dissipativity methodology is used for stabilization purposes (see Chapter 6).*

**Remark 5.23** *As it was pointed out in Remark 5.17, methods for the computation of control $u^*$ must be proposed. The way in order to obtain $u^*$ will be the one shown in Section 5.2, consisting in computing the explicit expression of $u^*$ from equation (5.90).*

#### 5.4.2.2 Example. A nonlinear example

The feedback dissipativity methodology presented will be applied to the passivation of the system (5.23). System (5.23) is aimed to be rendered $V$-passive with storage function

$V = x_1^2 + x_2^2$ and supply function $s = yv$, using the output $y = x_1^2 + x_2^2 + u$. Let $x_1 \in [-\varepsilon_{x_1}, \varepsilon_{x_1}]$, $x_2 \in [-\varepsilon_{x_2}, \varepsilon_{x_2}]$, $u, u^* \in [-\varepsilon_u, \varepsilon_u]$, $\delta u^* \in [-\rho, \rho]$, $v \in [-\varepsilon_v, \varepsilon_v]$ with $\varepsilon_{x_1}$, $\varepsilon_{x_2}$, $\varepsilon_u$, $\rho$, $\varepsilon_v$ positive constants.

Now, the existence and validity of controls $u^*$ and $\delta u^*$ will be analyzed.

In order to obtain control $u^*$, equation (5.90), for the example, is calculated and takes the following form,

$$a_{u^*}(u^*)^2 + b_{u^*}(x_1, x_2, v)u^* + c_{u^*}(x_1, x_2, v) = 0 \tag{5.103}$$

with,

$$\begin{aligned} a_{u^*} &= 1 \\ b_{u^*}(x_1, x_2, v) &= 2(x_1^2 + x_2^2) - v \\ c_{u^*}(x_1, x_2, v) &= (x_1^2 + x_2^2)^2 - (x_1^2 + x_2^2) - (x_1^2 + x_2^2)v \end{aligned}$$

Sufficient feedback losslessness conditions (5.91) and (5.92), or conditions for the existence of $u^*$ are met for (5.23) if,

$$a_{u^*}(u^*)^2 + b_{u^*}(x_1, x_2, v)u^* + c_{u^*}(x_1, x_2, v) = 0, \tag{5.104}$$

$$2u^* + 2(x_1^2 + x_2^2) - v \neq 0 \tag{5.105}$$

for some $(x_1, x_2, u^*, v)$. If conditions (5.104)-(5.105) are satisfied for some $(x_1, x_2, u^*, v)$, a control $u^*$ satisfying (5.103) exists. This $u^*$ can be obtained from the explicit solution of (5.103), as it was made in Section 5.4.1.2, obtaining,

$$u_1^*(x_1, x_2, v) = \frac{-b_{u^*} + \sqrt{b_{u^*}^2 - 4a_{u^*}c_{u^*}}}{2a_{u^*}}, \; u_2^*(x_1, x_2, v) = \frac{-b_{u^*} - \sqrt{b_{u^*}^2 - 4a_{u^*}c_{u^*}}}{2a_{u^*}} \tag{5.106}$$

Then, it is necessary to assure that $\varphi(x_1, x_2, v) = b_{u^*}^2 - 4a_{u^*}c_{u^*} \geq 0$, which will be always achieved since this function is always positive.

Concerning the computation of control $\delta u^*$, conditions (5.96)-(5.97) must be verified. They will be achieved by means of choosing an appropriate value of $\mu$.

Referring (5.96), as it was calculated in Section 5.4.1.2, it is known that

$$R_V = (\delta u^*)^2, \;\; R_s = 0$$

and

$$\left\{ \frac{\partial}{\partial u} \left[ V(f(x,u)) - s(h(x,u), v) \right] \right\}_{u=u^*} = 2(u^* + x_1^2 + x_2^2) - v$$

Therefore, condition (5.96) takes the form,

$$\max \left[ (\delta u^*)^2 \right] \leq \mu \left[ 2(u^* + x_1^2 + x_2^2) - v \right]^2 \tag{5.107}$$

Taking into account (5.95), inequality (5.107) yields to,

$$\begin{aligned}
\max\left\{\mu^2\left[2(u^*+x_1^2+x_2^2)-v\right]^2\right\} &\leq \mu\left[2(u^*+x_1^2+x_2^2)-v\right]^2 \\
\mu^2\max\left\{\left[2(u^*+x_1^2+x_2^2)-v\right]^2\right\} &\leq \mu\left[2(u^*+x_1^2+x_2^2)-v\right]^2 \\
\mu &\leq \frac{\left[2(u^*+x_1^2+x_2^2)-v\right]^2}{\max\left\{\left[2(u^*+x_1^2+x_2^2)-v\right]^2\right\}}
\end{aligned} \tag{5.108}$$

The worst case to consider is that $\mu$ is lower than the minimum value of the right-hand side of inequality (5.108), then

$$\mu \leq \frac{\min\left\{\left[2(u^*+x_1^2+x_2^2)-v\right]^2\right\}}{\max\left\{\left[2(u^*+x_1^2+x_2^2)-v\right]^2\right\}} \tag{5.109}$$

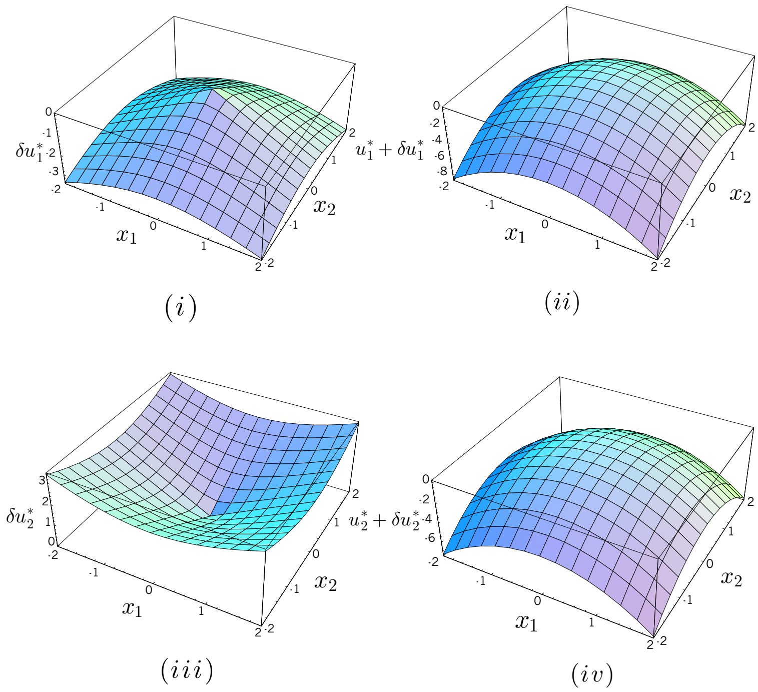


**Figure 5.12:** Control $u^*+\delta u^*$ given by (5.106) and (5.117) using $\mu=0.6$, $v=0$ (i) $\delta u^*=\delta u_1^*$ for $u_1^*$ (ii) $\delta u_1^*+u_1^*$ (iii) $\delta u^*=\delta u_2^*$ for $u_2^*$ (iv) $\delta u_2^*+u_2^*$.

On the other hand, condition (5.97) takes the following form,

$$\begin{aligned}
\mu\left|2(u^*+x_1^2+x_2^2)-v\right| &\leq \rho \\
\mu &\leq \frac{\rho}{\left|2(u^*+x_1^2+x_2^2)-v\right|}
\end{aligned} \tag{5.110}$$

for some positive constant $\rho$. The worst case to be considered in (5.110) is that $\mu$ must be lower than the minimum value of the term appearing in the right-hand side of inequality (5.110), therefore,

$$\mu \leq \frac{\rho}{\max\left|2(u^* + x_1^2 + x_2^2) - v\right|} \tag{5.111}$$

It can be noticed that function $2(u^* + x_1^2 + x_2^2) - v$ is bounded, and it can be as small as $u^*$, $x_1$, $x_2$, $v$ are.

Then, an upper bound of constant $\mu$ can be established from the minimum value given by (5.109) and (5.111), and will depend on the bounds of the states and the controls.

It is also necessary to give a bound for $v$ in order to ensure that the orbits of the feedback transformed system will remain in the compacts where the feedback dissipativity is considered. An option is proposing $v$ in such a way to have $s(\overline{h}(x,v),v) = (x_1^2 + x_2^2 + u^* + \delta u^*)v = 0$. Control $u^*$ is approximated by its linearization at $x_1 = x_2 = v = 0$, obtaining $u^* = \frac{1}{2}v$, and using this in $s(\overline{y},v) = 0$, it is obtained that $v = 0$ or,

$$v = -2(1-2\mu)(x_1^2 + x_2^2) \tag{5.112}$$

Control $v$ will be bounded by means of $\mu$, $x_1$ and $x_2$. Considering (5.111) and the fact that $\rho$ must be small enough, it can be concluded that the denominator appearing in (5.111) will be greater than $\rho$ and, consequently, $\mu$ will be less than one. Then, from (5.112), one yields to,

$$|v| < 2(\varepsilon_{x_1}^2 + \varepsilon_{x_2}^2) \tag{5.113}$$

A value for $\varepsilon_v$ is proposed as $\varepsilon_v = 2(\varepsilon_{x_1}^2 + \varepsilon_{x_2}^2)$, and relations (5.109) and (5.111) yield to,

$$\mu \leq \left[\frac{-\varepsilon_u}{2(\varepsilon_{x_1}^2 + \varepsilon_{x_2}^2) + \varepsilon_u}\right]^2 = \overline{\mu}_1 \tag{5.114}$$

$$\mu \leq \frac{\rho}{4(\varepsilon_{x_1}^2 + \varepsilon_{x_2}^2) + 2\varepsilon_u} = \overline{\mu}_2 \tag{5.115}$$

Finally, an upper bound for $\mu$ is proposed as follows,

$$0 < \mu \leq \min(\overline{\mu}_1, \overline{\mu}_2) \tag{5.116}$$

This bound of $\mu$ ensures conditions (5.96)-(5.97) to hold and control $v$ to be bounded.

**Remark 5.24** *As it was expected, $v$ depends on the sets where the states are defined, and there is a clear relation between the sets where the controls are defined ($\varepsilon_v$, $\varepsilon_u$, $\rho$), paremeter $\mu$ and the sets where the states are defined ($\varepsilon_{x_1}$, $\varepsilon_{x_2}$). The greater $\varepsilon_{x_1}$, $\varepsilon_{x_2}$ are, the greater the controls can be and the smaller $\mu$ must be, considered for a fixed $\rho$.*

**Remark 5.25** *The upper bound of parameter $\mu$ established by means of (5.109) and (5.111) is the worst case. Inequality (5.116) is only a sufficient condition for conditions (5.96)-(5.97) to met, and consequently, for the applicability of the passivation methodology. There can be higher values of $\mu$ for the method to be applicable.*

Under the conditions studied, control $\delta u^*$ is given by expression (5.95), i.e.,

$$\delta u^*(x_1, x_2, u^*, v) = -\mu\left[2(u^* + x_1^2 + x_2^2) - v\right] \tag{5.117}$$

with $\mu$ satisfying (5.116), and control $u^*$ as defined by equation (5.90), for which two possibilities are the controls (5.106). The control which renders the system $V$-passive is given by $u^* + \delta u^*$.

**Remark 5.26** *For the analysis of the approximations used for $V(f(x,u^* + \delta u^*))$ and $s(h(x,u^* + \delta u^*),v)$ see Section 5.4.1.2.*

**Remark 5.27** *As Chapter 6 will show, an interesting value for stabilization purposes will be $v = 0$, then for $v = 0$ the sum of controls $u_1^*$, $u_2^*$ and $\delta u^*$ for $u_1^*$ and $u_2^*$ has been obtained for a fixed value of $\mu$ and the states varying in an interval. See Figure 5.12. The concave nature of function $u^* + \delta u^*$ is obtained for different values of $v \neq 0$ and different values of $\mu$ as the ones presented in Figure 5.12.*

## 5.5 Conclusions and future work

Sufficient conditions under which a class of non-affine discrete-time control systems are locally feedback dissipative have been given. Four methodologies in order to deal with the local feedback dissipativity problem in nonlinear discrete-time systems have been proposed. They are based upon the fundamental dissipativity equality. Applicability conditions for each of these methods are given. The first one proposes the feedback dissipativity control as the implicit solution of the dissipativity equality, whereas the second one proposes an iterative-like solution by means of the speed-gradient algorithm in its discrete-time version. The third and fourth methodologies are based upon a first-order approximation of the fundamental dissipativity inequality. The first one achieves the feedback dissipativity goal by means of the storage energy invariance, whereas the second one proposes dissipativity as a "perturbation" of the system losslessness situation. Sufficient conditions under which the approximation considered is valid have been posed.

The contribution of this chapter is to give a solution to the feedback dissipativity problem for nonlinear discrete-time systems, this problem had not been solved in the literature before. Referring the use of the SG algorithm, it can be said that although the SG algorithm had been used to solve the passivation problem in the continuous-time setting, it had not ever been used for this purpose in the discrete-time domain. In addition, in the fourth proposed feedback dissipativity scheme, the feedback losslessness problem is also solved for nonlinear discrete-time systems and sufficient conditions under which feedback losslessness is possible are also given. This method can be also considered as an alternative feedback losslessness approach to the one given in *(Byrnes and Lin, 1994)* [14], where feedback losslessness is proposed by means of using the properties of the relative degree and the zero dynamics of the system.

The feedback dissipativity problem has been solved in a non-general manner since the four approaches are based on the establishment of the input $u$ which satisfies the fundamental dissipativity equality; it is therefore necessary to associate a priori functions $V$ and $\phi$ to the system, i.e., a storage function and a dissipation rate function with respect to which the feedback transformed system will be $(V,s)$-dissipative. The feedback dissipativity conditions guarantee the existence of the control $u$ which satisfies the dissipativity equation. At any rate, it can be considered as an application-oriented feedback dissipativity method, since, when dealing with physical systems, we are interested in defining our storage function as the energy of the system and proposing a desired dissipation. This fact has been shown in the buck example.

Some suggestions for future work can be proposed for each of the feedback dissipativity methods.

In the first procedure, a drawback is that finding an explicit solution of $u$ for all $k$ can be difficult or impossible in some systems; in these cases, an iterative feedback dissipativity algorithm can be proposed.

The feedback dissipativity methodology which uses the SG algorithm is also based on the establishment of the input $u$ which satisfies the fundamental dissipativity equality, which is seen as a goal function. The main problem of this procedure is in the static-type control solution in which we are compelled to obtain $u(k)$ from equation $u(k) = -\gamma(k)\nabla_{u(k)}Q(x(k))$. Finding an explicit solution of $u$ for all $k$ can be also difficult in some systems, in these cases, an iterative algorithm may be also proposed.

An alternative way of designing the parameters appearing in the feedback dissipativity controls given by the first and second methodologies may be given, as well as, a more detailed study of the influence of these controller parameters in the system response. Some notes on the influence of these parameters in the feedback dissipativity schemes have been pointed out through the examples considered.

Referring the last two methodologies, the main problem presented by them is that control $u^* + \delta u^*$ is locally valid in a neighbourhood of $u^*$. The approximations of $V(f(x,u))$, $s(h(x,u),v)$ and $\phi(x,u)$ could be improven, using a higher order Taylor approximation type. Alternative methods in order to calculate $u^*$ can be proposed. Referring control $\delta u^*$, a more complete study of the influence of the changes of parameter $\mu$ in the response of the feedback dissipative system is necessary in the two methods. A geometric interpretation of the underlying idea of these two methods would be interesting; some notes on this topic will be presented in Chapter 9.

It would be interesting to extend the results here obtained to the case of general multiple-input multiple-output nonlinear systems.

The use of the proposed feedback dissipativity schemes to solve the stabilization problem will be shown in Chapter 6.