Chapter 7

A solution to the passivation problem through the relative degree and the zero dynamics properties

7.1 Introduction

The problem of feedback passivity for a class of multiple-input multiple-output nonlinear discrete-time systems affine in the control input is solved using the properties of the relative degree and zero dynamics of the non-passive system. This is a novel solution to such a problem in the nonlinear discrete-time setting, jointly to the results presented in Chapter 5 of the present dissertation. It is the main contribution of this chapter and can be considered as the extension to the passivity case of the approach given in *(Byrnes and Lin, 1994)* [14] where the losslessness feedback problem is given. The passivation methodology proposed in this chapter will be used for stabilization purposes. The passifying escheme wil be applied to two examples: the nonlinear model (5.23), and the linear discrete-time model for the DC-to-DC buck converter (5.13). For the linear example presented, the frequency-domain characteristics of the passified system will be analyzed. Part of the results given in this chapter have been extracted from *(Navarro-López et al., 2002)* [120], *(Navarro-López and Fossas-Colet, 2002)* [121].

The study of the properties of the relative degree and the zero dynamics of a passive system has played an important role in understanding problems such as feedback passivity or the stabilization of passive systems in the continuous-time setting, see *(Byrnes et al., 1991)* [12]. For discrete-time systems, the implications of dissipativity and passivity in the relative degree and the zero dynamics have not been studied in a deep way, they have only been studied for the losslessness case, see *(Byrnes and Lin, 1994)* [14] and *(Sengör, 1995)* [151]; only some short notes on the literature are found for the linear case concerning the properties of the relative degree of passive systems, see *(Monaco and Normand-Cyrot, 1999)* [113] or *(Byrnes and Lin, 1994)* [14].

The characteristics of the relative degree and the zero-dynamics of passive discrete-time systems will be also analyzed in this chapter. These properties give valuable information concerning the relation between the input and the output of the system. As passivity property is an input-output property, the relative degree and zero dynamics of a passive system will present distinctive features. The implications of passivity in the relative degree of linear discrete-time systems will be established, as well as for a class of nonlinear discrete-time systems which are affine in the control input. Notes on the relative degree of dissipative discrete-time systems will be given. The implications of passivity in the properties of the zero dynamics of discrete-time systems will be presented, either for the linear or the nonlinear case.

The chapter is organized as follows. Section 7.2 is devoted to the properties of the relative degree and zero dynamics of passive nonlinear discrete-time systems. In Section 7.2.1, the special properties that the relative degree of passive discrete-time systems, either linear or nonlinear, exhibits are shown. For the nonlinear case, nonlinear discrete-time systems which are affine in the control input are studied. Section 7.2.2 revisits the properties of the zero dynamics of passive systems, whether they are linear or nonlinear. A solution to the passivation problem for a class of nonlinear discrete-time systems through the relative degree and the zero dynamics properties is presented in Section 7.3. Section 7.4 shows the application of the passivation methodology proposed to the stabilization of a nonlinear system, while Section 7.5 is devoted to a linear example. Conclusions and suggestions for further research are given in the last section.

7.2 Implications of passivity in the relative degree and zero dynamics of discrete-time systems

In this section, by means of using the definitions of relative degree and zero dynamics in the discrete-time setting, the properties that the relative degree and the zero dynamics of passive discrete-time systems exhibit will be studied.

The definitions of relative degree and zero dynamics for nonlinear discrete-time systems are given in *(Monaco and Normand-Cyrot, 1987)* [105], as well as, in *(Nijmeijer and van der Schaft, 1990)* [124], where the concept of relative degree is also regarded as characteristic number.

7.2.1 Passivity implications in the relative degree of a discrete-time system

Recall the definition of a (V,s)-dissipative system, Definition 4.2 and its particular case of a V-passive system, Definition 4.3.

7.2.1.1 Relative degree of passive nonlinear discrete-time systems

In this section, the relative degree of nonlinear passive discrete-time systems which are affine in the control input will be analyzed.

Let the system,

$$x(k+1) = f(x(k)) + g(x(k))u(k) \tag{7.1}$$

$$y(k) = h(x(k)) + J(x(k))u(k) \tag{7.2}$$

where f,g,h,J are smooth maps and $f(x) \in \Re^n$, $g(x) \in \Re^{n\times m}$, $h(x) \in \Re^m$, $J(x) \in \Re^{m\times m}$, $x \in \Re^n$, $u \in \Re^m$. Let $(\overline{x},\overline{u})$ an isolated fixed point of the system. There is no loss of generality in considering $(\overline{x},\overline{u}) = (0,0)$ and $f(0) = 0$, $h(0) = 0$, they are considered in the sequel.

Assume

$$\left.\frac{\partial y(x,u)}{\partial u}\right|_{\substack{x=0\\u=0}} \neq 0 \tag{7.3}$$

Then, by the implicit function theorem, there exists $u^* : \vartheta_1 \to \mathscr{U}$ defined in a neighbourhood of $x = 0$ such that $y(x,u^*) = 0$, $\forall x \in \vartheta_1$, with $\mathscr{U}$ a neighbourhood of $u = 0$. The zero dynamics of system (7.1)-(7.2) is defined by $f^* = f(x) + g(x)u^*$ where $(x,u^*) \in \mathscr{Z}^* = \{(x,u) : x \in \vartheta_1, y(x,u) = 0\}$.

Lemma 7.1 (Navarro-López and Fossas-Colet, 2002) *[121] If the system (7.1)-(7.2) is V-passive and its zero dynamics is such that $\mathscr{Z}^*$ does not contain straight lines of the form $\{(\hat{x},\lambda\hat{u}),\ \lambda \in \Re\}$, with $\mathscr{Z}^*$ as defined above, then $J(x)$ is invertible.*

Proof. Let $\overline{x}$ be such that $J(\overline{x})$ is not invertible. Let us take $\overline{u}(\overline{x})$ such that $\overline{u}(\overline{x}) \in Ker\, J(\overline{x})$. Since the system is V-passive

$$V(f(x)+g(x)u) - V(x) \leq y^T u,\ \ \forall (x,u) \tag{7.4}$$

Taking into account (7.4) and the positive definiteness of V,

$$-V(x) \leq [h(x)+J(x)u(x)]^T u(x) \tag{7.5}$$

If $h^T(\overline{x})\overline{u}(\overline{x}) \neq 0$, defining

$$u_0 = -\frac{\lambda \overline{u}(\overline{x})V(\overline{x})}{h^T(\overline{x})\overline{u}(\overline{x})}, \tag{7.6}$$

and particularizing inequality (7.5) for $\overline{x}, u_0$, yields

$$-V(\overline{x}) \leq -\lambda V(\overline{x}), \forall \lambda \in \Re$$

Thus, $h^T(\overline{x})\overline{u}(\overline{x}) = 0$, in other words, $\{(\overline{x},\lambda\overline{u}(\overline{x})),\ \lambda \in \Re\} \subset \mathscr{Z}^*$. Since this is a contradiction with the hypothesis, $J(x)$ must be invertible.

■

Remark 7.1 *Note that the invertibility of $J(x)$ is equivalent to the fact that the system (7.1)-(7.2) has relative degree zero.*

7.2.1.2 Relative degree of passive linear discrete-time systems

For the nonlinear case, the relative degree has been established in a local way, whereas, in the linear case, it is possible to talk about global relative degree of the system.

The basis of our analysis will be the dissipativity conditions (2.24)-(2.26) given in Lemma 2.1 restricted to the passivity case. Let a multiple-input multiple-output linear time-invariant (LTI) discrete-time system of the form

$$x(k+1) = Ax(k)+Bu(k) \tag{7.7}$$
$$y(k) = Cx(k)+Du(k) \tag{7.8}$$

Remark 7.2 *Note that the denomination of passivity and dissipativity in the literature coincides with the denomination of V-passivity and (V,s)-dissipativity, respectively. The V-passivity and (V,s)-dissipativity denomination is preferred by us.*

Proposition 7.1 *(Monaco and Normand-Cyrot, 1999)[113] Suppose the storage function of the form $V = \frac{1}{2}x^T Px$, with P a positive definite and symmetric matrix. A system of the form (7.7)-(7.8) is V-passive, if and only if, there exists P such that*

$$A^T PA - P \leq 0 \tag{7.9}$$

$$B^T PA = C \tag{7.10}$$

$$B^T PB - (D^T + D) \leq 0 \tag{7.11}$$

Remark 7.3 *Conditions (7.9)-(7.11) are equivalent to the ones given in* (Hitz and Anderson, 1969) *[58], see (2.20)-(2.22).*

Proposition 7.2 *If the system (7.7)-(7.8) is V-passive, then it has relative degree zero.*

Proof. Having relative degree zero is equivalent to $D \neq 0$, i.e., the output depends directly on the input. From condition (7.11), with P a positive definite matrix, one concludes that $B^T PB$ is a positive definite matrix, consequently $D^T + D$ must be a positive definite matrix, and therefore $D \neq 0$.

■

Remark 7.4 *This is not the first time that the result reported in Proposition 7.2 has appeared in the literature, this fact has also been briefly pointed out in* (Byrnes and Lin, 1994) *[14] or* (Monaco and Normand-Cyrot, 1999) *[113].*

Remark 7.5 *In* (Byrnes and Lin, 1994) *[14], it is stated that it does not make sense to study passivity and losslessness of discrete-time systems having outputs independent of u. This is the case for $s(y,u) = y^T u$. Indeed, dissipative systems can have relative degree greater than zero, that is, D can be zero. For example, considering (Q,S,R)-dissipative systems, it can be concluded that ISP, VSP and FGS systems may have relative degree greater than zero. See definitions for ISP, VSP and FGS systems in Chapter 2.*

7.2.2 Zero dynamics properties of passive discrete-time systems under study

7.2.2.1 Zero dynamics of passive nonlinear discrete-time systems

Let a system of the form,

$$x(k+1) = f(x(k),u(k)),\ x \in \Re^n,\ u \in \Re^m \tag{7.12}$$

$$y(k) = h(x(k),u(k)),\ \ y \in \Re^m \tag{7.13}$$

where $f : \Re^n \times \Re^m \to \Re^n$ and $h : \Re^n \times \Re^m \to \Re^m$ are smooth maps, with $k \in \mathscr{Z}_+ := \{0,1,2,\ldots\}$. Let $\bar{x}$ an isolated fixed point of $f(x,\bar{u})$, with $\bar{u}$ a constant vector, i.e.,

$f(\bar{x},\bar{u}) = \bar{x}$. There is no loss of generality in considering $(\bar{x},\bar{u}) = (0,0)$ and $h(0,0) = 0$, from now on they both are considered.

Assume

$$\left.\frac{\partial h(x,u)}{\partial u}\right|_{\substack{x=0\\u=0}} \neq 0 \tag{7.14}$$

Then, by the implicit function theorem, there exists $u^* : \vartheta_1 \to \mathscr{U}$ defined in a neighbourhood of $x = 0$ such that $h(x,u^*) = 0$, $\forall x \in \vartheta_1$, with $\mathscr{U}$ a neigbourhood of $u = 0$. The zero dynamics of system (7.12)-(7.13) is defined by $f^* = f(x,u^*)$ where $(x,u^*) \in \mathscr{Z}^* = \{(x,u) : x \in \vartheta_1, h(x,u) = 0\}$.

Definition 7.1 *A system of the form (7.12)-(7.13) has a locally passive zero dynamics if there exists a $\mathscr{C}^2$ positive definite function V, locally defined on the neighbourhood ϑ_2 of $x = 0$ in $\Re^n$, s.t. $V(0) = 0$ and*

$$V(f(x,u^*)) \leq V(x), \;\; \forall x \in \vartheta_2 \tag{7.15}$$

Theorem 7.1 (Navarro-López and Fossas-Colet, 2002) *[121] Suppose system (7.12)-(7.13) satisfying (7.14) to be V-passive with a storage function V which is positive definite, and $V(0) = 0$. Then, the zero dynamics of (7.12)-(7.13) locally exists at $x = 0$ and is passive.*

Proof. By (7.14), system (7.12)-(7.13) has relative degree 0 in a neighbourhood of $x = 0$ and its zero dynamics, indeed, locally exists in a neighbourhood of $x = 0$ in $\Re^n$. As (7.12)-(7.13) is V-passive, the dissipativity relation (4.3) is met with $s(y,u) = y^T u$. Setting $u = u^*$ such that $y = h(x,u^*) = 0$, one yields to $f^*(x)$. Since the zero dynamics is restricted to $\mathscr{Z}^*$, inequality (4.3) is converted into equation (7.15).

■

Remark 7.6 *A passive zero dynamics is a Lyapunov stable dynamics, also referred as* weakly minimum phase *dynamics, denomination proposed in* (Byrnes et al., 1991) *[12].*

7.2.2.2 Zero dynamics of passive linear discrete-time systems

For the nonlinear case, it is necessary to talk about locally passive zero dynamics, whereas in the linear case it is possible to conclude properties of the zero dynamics of passive discrete-time systems in a global sense. If the system (7.7)-(7.8) is V-passive then it has relative degree zero, and consequently, its zero dynamics takes the following form

$$f^*(x(k)) = (A - BD^{-1}C)x(k) \tag{7.16}$$

Proposition 7.3 (Navarro-López et al., 2002) *[120] Let a system of the form (7.7)-(7.8) be V-passive with a storage function $V = \frac{1}{2}x^T Px$, with P a positive definite and symmetric matrix. Then, its zero dynamics is passive.*

Proof. Since the system (7.7)-(7.8) is assumed to be V-passive, there exists P a positive definite and symmetric matrix satisfying equations (7.9)-(7.11). Consider $V = \frac{1}{2}x^T Px$. The zero dynamics of the system is given by (7.16), then $V(f^*(x)) - V(x) = \frac{1}{2}x^T Mx$, where

$$M = (A - BD^{-1}C)^T P(A - BD^{-1}C) - P \tag{7.17}$$

Thus, it is needed to prove that M is negative semi-definite. Considering condition (7.10), M can be written as follows

$$M = (A^T PA - P) - C^T \left[D^{-1} + (D^{-1})^T\right] C + C^T (D^{-1})^T B^T PBD^{-1}C \tag{7.18}$$

Adding and substracting to (7.18) $C^T(D^{-1})^T(D^T + D)D^{-1}C = C^T[D^{-1} + (D^{-1})^T]C$, it is obtained

$$\begin{aligned} M &= (A^T PA - P) - C^T[D^{-1} + (D^{-1})^T]C + C^T(D^{-1})^T B^T PBD^{-1}C - \\ &\quad -C^T(D^{-1})^T(D^T + D)D^{-1}C + C^T(D^{-1})^T(D^T + D)D^{-1}C \end{aligned}$$

Since, $C^T(D^{-1})^T(D^T + D)D^{-1}C = C^T[D^{-1} + (D^{-1})^T]C$, one yields to

$$\begin{aligned} M &= (A^T PA - P) - C^T[D^{-1} + (D^{-1})^T]C + C^T[D^{-1} + (D^{-1})^T]C + \\ &\quad +C^T(D^{-1})^T[B^T PB - (D^T + D)]D^{-1}C, \end{aligned}$$

and using (7.9) and (7.11), it is concluded that M is negative semi-definite.

Then, the zero dynamics of a V-passive LTI discrete-time system is Lyapunov stable or as it has been defined, it is passive.

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7.3 A solution to the passivation problem

The passivation problem will be studied for a class of nonlinear discrete-time systems linear in the input of the form (7.1)-(7.2).

Consider Definition 4.4 of QSS-dissipativity given in Chapter 4. The following definition is introduced as a particular case of Definition 4.4.

Definition 7.2 *A system of the form (7.1)-(7.2) is said to be QS (*Quadratic Storage*) passive if it is V-passive with a storage function V such that $V(f(x) + g(x)u)$ is quadratic in u.*

This chapter is devoted to render a system of the form (7.1)-(7.2) QS-passive. The characterization for QS-passive systems to use will be the restriction of QSS-dissipativity conditions (4.7)-(4.9) to $s(y,u) = y^T u$ and the dynamics (7.1)-(7.2). Then, conditions (4.7)-(4.9) take the following form.

Theorem 7.2 (Byrnes and Lin, 1993) *[13] Let V be a storage function such that $V(f(x,u))$ is quadratic in u. Then, a discrete-time nonlinear system of the form (7.1)-(7.2) is QS-passive with V, if and only if, there exist real functions $l(x)$, $m(x)$ and $k(x)$, all of appropriate dimensions such that*

$$V(f(x)) - V(x) = -l^T(x)l(x) - m^T(x)m(x) \tag{7.19}$$

$$\left.\frac{\partial V(\alpha)}{\partial \alpha}\right|_{\alpha=f(x)} g(x) + 2l^T(x)k(x) = h^T(x) \tag{7.20}$$

$$g^T(x)\left.\frac{\partial^2 V(\alpha)}{\partial \alpha^2}\right|_{\alpha=f(x)} g(x) = J^T + J(x) - 2k^T(x)k(x) \tag{7.21}$$

■

Remark 7.7 *The denomination of QS-passivity has been introduced in this dissertation. It is considered to be appropriated to use in Theorem 7.2.*

Let $\alpha(x)$ and $\beta(x)$ be smooth functions, with $\alpha(0) = 0$ and $\beta(x)$ invertible $\forall x$. A nonlinear static state feedback control law is denoted by

$$u = \alpha(x) + \beta(x)w \tag{7.22}$$

Theorem 7.3 (Navarro-López and Fossas-Colet, 2002) *[121] Suppose $h(0) = 0$, and there exists a $\mathscr{C}^2$ storage function V, which is positive definite, $V(0) = 0$ and $V(f(x) + g(x)u)$ is quadratic in u, $\forall f$, $\forall g$. Then, system (7.1)-(7.2) is locally feedback equivalent to a QS-passive system with V as storage function if and only if (7.1)-(7.2) has locally relative degree 0 and its zero dynamics is locally passive in a neighbourhood $\mathscr{X}$ of $x = 0$.*

Proof. (Necessity): Assume that there is a static state control law of the form (7.22) which renders the system (7.1)-(7.2) *QS*-passive system in a neighbourhood of $x = 0$. Then the feedback transformed system

$$\begin{aligned} x(k+1) &= \overline{f}(x(k)) + \overline{g}(x(k))w(k) \\ y(k) &= \overline{h}(x(k)) + \overline{J}(x(k))w(k) \end{aligned}$$

is locally *QS*-passive, with $\overline{f}(x) = f(x) + g(x)\alpha(x)$, $\overline{g}(x) = g(x)\beta(x)$, $\overline{h}(x) = h(x) + J(x)\alpha(x)$, $\overline{J}(x) = J(x)\beta(x)$. Therefore, the feedback transformed system has locally relative degree zero and passive zero dynamics, see Theorem 7.1, Lemma 7.1 and Remark 7.1. Consequently, the system (7.1)-(7.2) has locally relative degree zero and passive zero dynamics.

(Sufficiency): It will be shown that if system (7.1)-(7.2) has locally relative degree 0 and passive zero dynamics, it is feedback equivalent to a *QS*-passive system, i.e., there exists a control $u = \alpha(x) + \beta(x)w$, such that the feedback system $x(k+1) = \overline{f}(x(k)) + \overline{g}(x(k))w(k)$ fulfils

$$V(\overline{f}(x) + \overline{g}(x)w) - V(x) \le (\overline{h}(x) + \overline{J}(x)w)^T w$$

Since the system relative degree is zero, $J(x)$ is invertible in a neighbourhood $\mathscr{X}$ of $x = 0$, then

$$J^{-1}(x) = \left(\frac{\partial}{\partial u} y(x,u)\right)^{-1}$$

is well defined $\forall x \in \mathscr{X}$. It is chosen

$$u(k) = u^*(k) + J^{-1}(x)v(k) \tag{7.23}$$

with u^* such that $y(x,u^*) = 0$, i.e., $u^* = -J^{-1}(x)h(x)$. System (7.1)-(7.2) with (7.23) yields to

$$\begin{aligned} x(k+1) &= f^*(x(k)) + g^*(x(k))v(k), \\ y(k) &= v(x(k)), \end{aligned} \tag{7.24}$$

where $f^*(x) = f(x) - g(x)J^{-1}(x)h(x)$ represents the zero dynamics of the original system and $g^*(x) = g(x)J^{-1}(x)$. Now, a new input control and a new output are defined

$$y(k) = v(k) := \overline{h}(x(k)) + \overline{J}(x(k))w(k) \tag{7.25}$$

Then, the new system dynamics is given by

$$\begin{aligned} x(k+1) &= f^*(x(k)) + g^*(x(k))\overline{h}(x(k)) + g^*(x(k))\overline{J}(x)w(k) \\ y(k) &= \overline{h}(x(k)) + \overline{J}(x)w(k) \end{aligned} \tag{7.26}$$

It is defined,

$$\overline{J}(x) = \left(\frac{1}{2} g^{*T} \left. \frac{\partial^2 V}{\partial z^2} \right|_{z=f^*(x)} g^*(x) \right)^{-1} \tag{7.27}$$

$$\overline{h}(x) = -\overline{J}(x) \left(\left. \frac{\partial V}{\partial z} \right|_{z=f^*(x)} g^*(x) \right)^T \tag{7.28}$$

System (7.26) with (7.27) and (7.28) will be shown to be QS-passive with a $\mathscr{C}^2$ storage function V. Since $V(f^*(x) + g^*(x)u)$ is quadratic in u, using the Taylor expansion formula, it follows that

$$\begin{aligned} V(f^*(x) + g^*(x)\overline{h}(x)) &= V(f^*(x)) + \left. \frac{\partial V}{\partial z} \right|_{z=f^*(x)} g^*(x)\overline{h}(x) + \\ &+ \frac{1}{2}\overline{h}^T(x) g^{*T}(x) \left. \frac{\partial^2 V}{\partial z^2} \right|_{z=f^*(x)} g^*(x)\overline{h}(x) \end{aligned} \tag{7.29}$$

Substracting $V(x)$ in both sides of (7.29) and considering that the zero dynamics of (7.1)-(7.2) is locally passive,

$$\begin{aligned} V(f^*(x) + g^*(x)\overline{h}(x)) - V(x) &= -\left[l(x) + k(x)\overline{h}(x) \right]^T \left[l(x) + k(x)\overline{h}(x) \right] - \\ &- m^T(x)m(x) + \left. \frac{\partial V}{\partial z} \right|_{z=f^*(x)} g^*(x)\overline{h}(x) + \\ &+ \frac{1}{2}\overline{h}^T(x) g^{*T}(x) \left. \frac{\partial^2 V}{\partial z^2} \right|_{z=f^*(x)} g^*(x)\overline{h}(x) \end{aligned} \tag{7.30}$$

Differentiating both sides of (7.30) with respect to $\overline{h}(x)$, and multiplying the result by $\overline{J}(x)$, in addition to use (7.27) and (7.28), the passivity condition (7.20) for system (7.26) follows, that is

$$\left. \frac{\partial V}{\partial z} \right|_{z=f^*(x)+g^*(x)\overline{h}(x)} g^*(x)\overline{J}(x) = \overline{h}^T(x) - 2\left[l^T(x) + \overline{h}^T(x)k^T(x) \right] k(x)\overline{J}(x) \tag{7.31}$$

Taking the second-order derivative with respect to $\overline{h}(x)$ in both sides of (7.30) and multiplying both sides of the result from the left by $\overline{J}^T(x)$ and from the right by $\overline{J}(x)$, using (7.27) and supposing $J(x)$ to be symmetric, one yields to

$$\left[g^*(x)\overline{J}(x)\right]^T \left.\frac{\partial^2 V}{\partial \alpha^2}\right|_{\alpha=f^*+g^*\overline{h}} \left[g^*(x)\overline{J}(x)\right] = \overline{J}^T(x) + \overline{J}(x) - 2\overline{J}^T(x)k^T(x)k(x)\overline{J}(x)$$

which is the passivity condition (7.21) for system (7.26). For the passivity conditions of system (7.26), the equivalent of function $l(x)$ is $l(x)+k(x)\overline{h}(x)$, and the equivalent of function $k(x)$ is $k(x)\overline{J}(x)$.

Besides, using (7.27) and (7.28) on (7.29)

$$\begin{aligned} V(f^*(x)+g^*(x)\overline{h}(x)) &= V(f^*(x)) - \overline{h}^T \left(\overline{J}(x)^{-1}\right)^T(x)\overline{h}(x) + \overline{h}^T\overline{J}(x)^{-1}(x)\overline{h}(x) = \\ &= V(f^*(x)) \end{aligned}$$

Taking into account that the original system has passive zero dynamics, i.e., $V(f^*(x)) \le V(x)$, it can be written

$$V(f^*(x)) = V(x) - \left[l(x)+k(x)\overline{h}(x)\right]^T \left[l(x)+k(x)\overline{h}(x)\right] - m^T(x)m(x)$$

then,

$$V(f^*(x)+g^*(x)\overline{h}(x)) - V(x) = -\left[l(x)+k(x)\overline{h}(x)\right]^T \left[l(x)+k(x)\overline{h}(x)\right] - m^T(x)m(x)$$

which is the passivity condition (7.19) for the system (7.26). In conclusion, system (7.1)-(7.2) with the passifying control law (7.25) is QS-passive.

■

Remark 7.8 *The passifying control proposed above is the same that the control which renders a system lossless in* (Byrnes and Lin, 1994) *[14].*

Summing up, the proposed passifying control scheme has the following form, with u as passifying control and w as the new control input to the system:

$$\begin{aligned} u(x,v) &= u^*(x) + J^{-1}(x)v \\ u^*(x) &= -J^{-1}(x)h(x) \\ v(x,w) &= y = \overline{h}(x) + \overline{J}(x)w \\ \overline{h}(x) &= -\overline{J}(x)\left(\left.\frac{\partial V}{\partial z}\right|_{z=f^*(x)} g^*(x)\right)^T \\ \overline{J}(x) &= \left(\frac{1}{2}g^{*T}(x)\left.\frac{\partial^2 V}{\partial z^2}\right|_{z=f^*(x)} g^*(x)\right)^{-1} \\ g^*(x) &= g(x)J^{-1}(x) \\ f^*(x) &= f(x) - g(x)J^{-1}(x)h(x) \end{aligned} \tag{7.32}$$

7.4 A nonlinear example

Let the system (5.23) with a modified output,

$$\begin{aligned} x_1(k+1) &= \left[x_1^2(k)+x_2^2(k)+u(k)\right]\cos[x_2(k)] \\ x_2(k+1) &= \left[x_1^2(k)+x_2^2(k)+u(k)\right]\sin[x_2(k)] \\ y(k) &= [x_1^2(k)+x_2^2(k)]+J[x(k)]u(k) \end{aligned} \tag{7.33}$$

In this section, the feedback passivity methodology presented in Section 7.3 will be applied to system (7.33) in order to stabilize it to the system fixed point $(0,0)$. The passifying control scheme given in (7.32) is then used. The control which renders the output of the system (7.33) to zero is,

$$u^*(x) = -J^{-1}(x)(x_1^2+x_2^2) \tag{7.34}$$

Control u^* substituted in (7.33) gives the zero dynamics of the system,

$$\begin{aligned} x_1^*(k+1) &= f_1^*(x(k)) = \left\{x_1^2(k)+x_2^2(k)-J^{-1}(x)\left[x_1^2(k)+x_2^2(k)\right]\right\}\cos[x_2(k)] \\ x_2^*(k+1) &= f_2^*(x(k)) = \left\{x_1^2(k)+x_2^2(k)-J^{-1}(x)\left[x_1^2(k)+x_2^2(k)\right]\right\}\sin[x_2(k)] \end{aligned}$$

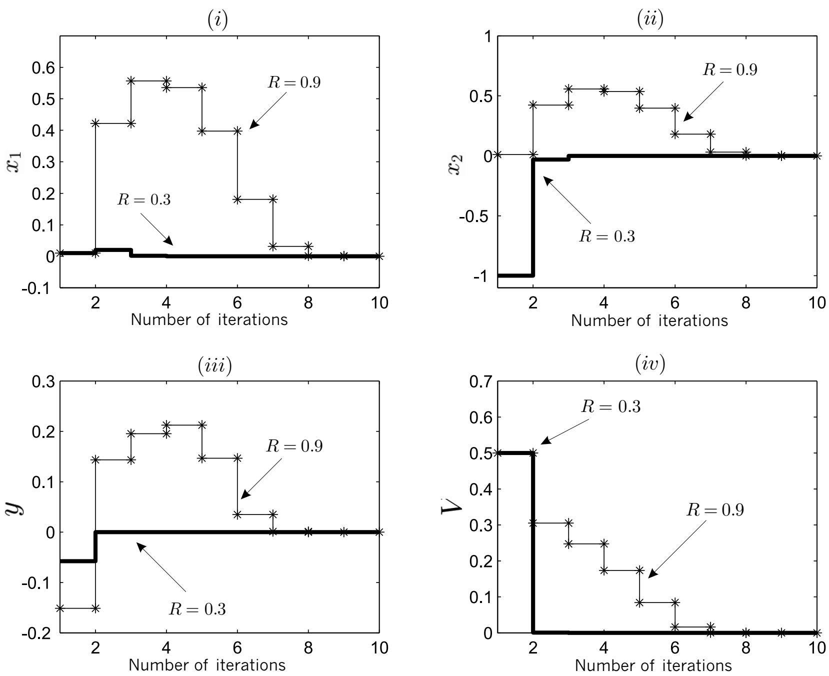

Figure 7.1: Stabilized system response for (7.33) with initial conditions $x_0 = (0.01,-1)^T$ for $R = 0.3$ and $R = 0.9$ (i) x_1 (ii) x_2 (iii) output y (iv) storage energy function V.

It is chosen $J(x) = \frac{1}{x_1^2+x_2^2-R^2}$ with R a constant such that $R \in (0,1)$. Considering $V = \frac{1}{2}(x_1^2+x_2^2)$, as storage function, the system is not passive, in fact, with $u = 0$ the origin is unstable. However, the system can be rendered V-passive by means of a static state feedback control law of the form (7.23), see Theorem 7.3, due to the fact that $J(x)$ is invertible and the zero dynamics of system (7.33) is passive. Therefore, the passifying control scheme proposed (7.32) is applied to (7.33), obtaining,

$$\begin{aligned}
u(x,w) &= -J^{-1}(x)(x_1^2+x_2^2)+J^{-1}(x)[\overline{h}(x)+\overline{J}(x)w] = \\
&= J^{-1}(x)(x_1^2+x_2^2)-2\left(x_1^2+x_2^2\right)+2J(x)w \\
g^*(x) &= J^{-1}(x)\begin{pmatrix}\cos(x_2)\\ \sin(x_2)\end{pmatrix} \\
\overline{J}(x) &= \left[\frac{1}{2}g^{*T}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}g^*\right]^{-1} = 2J^2(x) \\
\overline{h}(x) &= -2J(x)\left[x_1^2+x_2^2-J^{-1}(x)(x_1^2+x_2^2)\right] \\
y(x,w) &= 2J(x)\left\{J(x)w-\left[x_1^2+x_2^2-J^{-1}(x)(x_1^2+x_2^2)\right]\right\}
\end{aligned}$$

In the passified system, it is chosen $w(k)=-y(k)$, as it is proposed in *(Lin and Byrnes, 1995)* [87]; this control locally asymptotically stabilizes a V-passive discrete-time system. In this case, $w(k)=-\left[\overline{J}(x(k))+1\right]^{-1}\overline{h}(x(k))$. The passified system response is shown in Figures 7.1 and 7.2 for two different values of the constant R: $R=0.3$, $R=0.9$. The greater R is the more oscillating the system response is. As it can be appreciated, the states and the output are stabilized to the origin $x=(0,0)^T$, $y=0$.

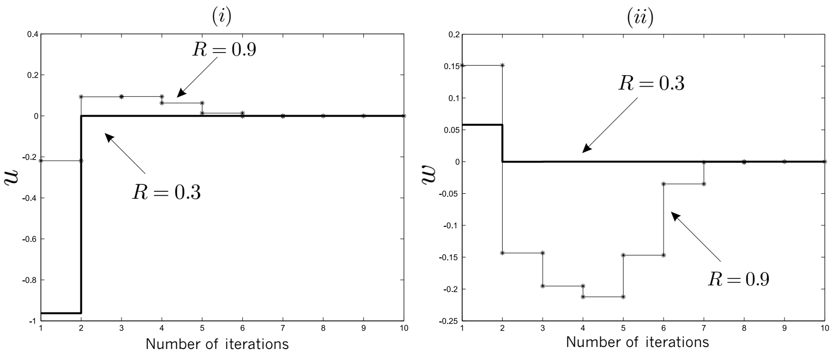

Figure 7.2: Stabilized system response for (7.33) with $x_0=(0.01,-1)^T$ for $R=0.3$ and $R=0.9$ (i) passifying control u (ii) stabilizing control w.

7.5 A linear example. Frequency-domain interpretation

In this section, the passifying control scheme (7.32) will be restricted to the LTI case, i.e., to systems of the form (7.7)-(7.8), and it will be applied to the passivation of the DC-to-DC buck converter. Passivity conditions (7.9)-(7.11) which have been obtained for a storage function of the form $V=\frac{1}{2}x^TPx$, with P a positive symmetric matrix will be taken into account. The passifying control scheme proposed applied to the linear time-invariant case takes the following form,

$$\begin{aligned} u(x,w) &= u^*(x) + D^{-1}\left[\overline{h}(x) + \overline{J}w\right] \\ u^*(x) &= -D^{-1}Cx \\ f^*(x) &= (A - BD^{-1}C)x \\ g^* &= BD^{-1} \\ \overline{J} &= 2D(B^TPB)^{-1}D^T \\ \overline{h}(x) &= -2D(B^TPB)^{-1}B^TP(A - BD^{-1}C)x \end{aligned} \tag{7.35}$$

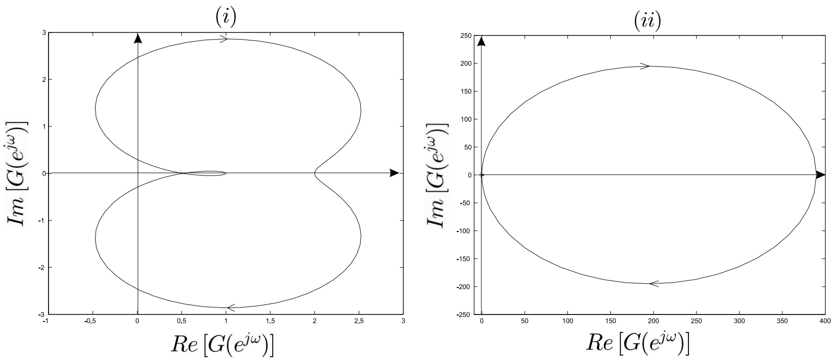

Figure 7.3: (i) Nyquist plot of the discrete non-passive buck converter (ii) Nyquist plot for the passified discrete-time buck converter.

Now, the control scheme (7.35) will be applied to the exact discretization of the buck converter with sampling period $T = 0.3535533906$ presented in Chapter 5 as (5.13), considering the output $y = x_2 + \hat{u}$ and $V = \frac{1}{2}xPx^T$, with

$$P = \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix}$$

and η a positive constant depending on the physical parameters of the system. The model to consider takes the form (7.7)-(7.8), where

$$\begin{aligned} A &= \begin{pmatrix} a & -b \\ b & c \end{pmatrix} & B &= \begin{bmatrix} (-a+1)\gamma + b \\ -b\gamma - c + 1 \end{bmatrix} \\ C &= (0, 1) & D &= 1 \end{aligned}$$

The constants of the model are

$$a = 0.9406416964, b = 0.3254699438, c = 0.8255706942, \gamma = 0.3535533906$$

The corresponding transfer function of the above state-space representation takes the following form,

$$G(z) = \frac{z^2 - 1.707z + 0.9394}{z^2 - 1.766z + 0.8825} \tag{7.36}$$

which corresponds to a non-passive system, as its Nyquist diagram shows, see Figure 7.3. Although the original system is not V-passive, it can be rendered V-passive by means of a static state feedback law $\hat{u} = u$ given in (7.35), due to the fact that:

1. System (7.36) has relative degree zero.
2. The zero dynamics of (7.36), which takes the form

$$f^*(x) = (A - BD^{-1}C)x = \begin{pmatrix} 0.9406 & -0.6719 \\ 0.3255 & 0.7662 \end{pmatrix} x,$$

   is passive since all the eigenvalues of $A_0 = (A - BD^{-1}C)$ have modulus less than one.

Then, the control scheme (7.35) is applied to (5.13), and the following system is obtained,

$$\begin{aligned} x(k+1) &= A_p x(k) + B_p w(k) \\ y(k) &= C_p x(k) + D_p w(k), \end{aligned} \tag{7.37}$$

with

$$\begin{aligned} A_p &= \begin{pmatrix} -0.9953 & 0.3785 \\ -0.0062 & 0.9462 \end{pmatrix} & B_p &= \begin{pmatrix} 0.4232 \\ 0.0725 \end{pmatrix} \\ C_p &= (-5.5879, 3.032) & D_p &= 1.2215 \end{aligned}$$

The associated transfer function of the state-space representation (7.37) takes the form,

$$G_p = \frac{1.221z^2 - 2.085z + 1.147}{z^2 + 0.04915z - 0.9394}, \tag{7.38}$$

which is a positive real transfer function, as its Nyquist diagram illustrates, see Figure 7.3.

7.6 Conclusions and future work

In this chapter, the properties of the relative degree and the zero dynamics of a class of V-passive systems have been related to its feedback passivity property, and a passivation methodology has been proposed for a class of multiple-input multiple-output nonlinear discrete-time systems affine in the control input. The class of systems for which feedback passivity has been solved are regarded as QS-passive systems, i.e., V-passive systems for which $V(f(x) + g(x)u)$ is quadratic in u. The passivation methodology proposed have been validated by means of several examples. Furthermore, the relative degree and the zero dynamics of V-passive discrete-time systems have been revisited. Notes on the relative degree of dissipative discrete-time systems have also been given.

The contribution of this chapter is to solve the feedback passivity problem for a class of nonlinear discrete-time systems which, as far as we know, has not been solved for the nonlinear discrete-time case in the literature before. The nonlinear discrete-time systems treated are those which are linear in the control input and for which $V(x(k+1))$ is quadratic in u. The results here presented are an extension to the passivity case of the ones given in *(Byrnes and Lin, 1994)* [14] where the losslessness feedback problem is reported. It is also a contribution, concluding the properties of the relative degree of V-passive nonlinear discrete-time systems affine in the control input. The way of proving the properties of the zero dynamics of V-passive linear and nonlinear discrete-time systems is a contribution, as well.

There is a great variety of feedback passivity-related problems remaining unsolved in the discrete-time setting. New ways of treating dissipativity and passivity concepts are needed to be explored. Concerning the results presented here, it is desirable to study the feedback passivity problem without the restriction of $V(x(k+1))$ to be quadratic in u. In order to compass this goal, it is necessary to propose new dissipativity and passivity characterizations. This may lead to extend the feedback passivity method proposed to systems which are non-affine in the control input.

The passivation approach given presents several problems. First, non-minimum phase systems can not be passified via this method, a solution would be finding a fictitious output for which the system has a Lyapunov stable zero dynamics. Indeed, the system is made V-passive with respect to a new transformed output. This output can be considered as a fictitious output, without meaning, but used for the passivation or the stabilization of the system.

Feedback dissipativity would be interesting to be solved from the viewpoint of using the relative degree and the zero dynamics properties of the system. The solution of the feedback dissipativity problem will allow to study systems whose output does not depend on the input. For this purpose, a complete analysis of the relative degree and the zero dynamics of dissipative systems would be required, or proposing new approaches to treat dissipativity-related concepts.