# Appendix A.

# THE DQ TRANSFORMATION

## A.1. Definition

The constant-power dq transformation for a three-phase system is:

$$\mathbf{T}_{dq} = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos(\theta_r) & \cos(\theta_r - \frac{2\pi}{3}) & \cos(\theta_r + \frac{2\pi}{3}) \\ -\sin(\theta_r) & -\sin(\theta_r - \frac{2\pi}{3}) & -\sin(\theta_r + \frac{2\pi}{3}) \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad \text{(A.1)}$$

where $\theta_r$ is the rotating coordinate angle defined as:

$$\theta_r = \int_0^t \omega \, dt + \theta_0 , \quad \text{(A.2)}$$

$\omega$: coordinate angular frequency.

$\theta_0$: initial position angle of the coordinate axes.

When $\omega$ is constant, such as in applications with direct connection to the utility, this transformation can be expressed as follows

$$\mathbf{T}_{dq} = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos(\omega t + \theta_0) & \cos(\omega t + \theta_0 - \frac{2\pi}{3}) & \cos(\omega t + \theta_0 + \frac{2\pi}{3}) \\ -\sin(\omega t + \theta_0) & -\sin(\omega t + \theta_0 - \frac{2\pi}{3}) & -\sin(\omega t + \theta_0 + \frac{2\pi}{3}) \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}. \quad \text{(A.3)}$$

## A.2. Power Conservative Transformation

The dq transformation is an orthogonal transformation since it accomplishes the propriety $\mathbf{T}_{dq}^{-1} = \mathbf{T}_{dq}^{\mathrm{T}}$. The internal product remains invariable under such kind of transformation. Thus, given $\mathbf{x}_{1r} = \mathbf{T}_{dq}\mathbf{x}_1$ and $\mathbf{x}_{2r} = \mathbf{T}_{dq}\mathbf{x}_2$, the internal product $< \mathbf{x}_{1r}, \mathbf{x}_{2r} >$ is:

$$< \mathbf{x}_{1r}, \mathbf{x}_{2r} > = < \mathbf{T}_{dq}\mathbf{x}_1, \mathbf{T}_{dq}\mathbf{x}_2 > = < \mathbf{x}_1, \mathbf{T}_{dq}^{\mathrm{T}}\mathbf{T}_{dq}\mathbf{x}_2 > = < \mathbf{x}_1, \mathbf{T}_{dq}^{-1}\mathbf{T}_{dq}\mathbf{x}_2 > = < \mathbf{x}_1, \mathbf{x}_2 > . \quad \text{(A.4)}$$

Similarly, the instantaneous power of a set of three-phase voltages and currents is:

$$p = v_a\, i_a + v_b\, i_b + v_c\, i_c = \begin{bmatrix} v_a & v_b & v_c \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} = \mathbf{v}_{ph}^{\mathrm{T}}\, \mathbf{i}_{ph}\, , \quad \text{(A.5)}$$

and applying the dq transformation to the variables:

$$p = \mathbf{v}_{ph}^{\mathrm{T}}\, \mathbf{i}_{ph} = \mathbf{v}_{ph}^{\mathrm{T}}\left(\mathbf{T}_{dq}^{-1}\, \mathbf{T}_{dq}\right) \mathbf{i}_{ph} = \left(\mathbf{v}_{ph}^{\mathrm{T}}\mathbf{T}_{dq}^{-1}\right)\left(\mathbf{T}_{dq}\mathbf{i}_{ph}\right) = \left(\mathbf{v}_{ph}^{\mathrm{T}}\mathbf{T}_{dq}^{\mathrm{T}}\right)\left(\mathbf{T}_{dq}\mathbf{i}_{ph}\right) = \mathbf{v}_{phr}^{\mathrm{T}}\, \mathbf{i}_{phr}\, . \quad \text{(A.6)}$$

In addition, when the sum of the three currents is zero, the homopolar current component ($i_o$) is also zero. Therefore, calculation of the instantaneous power is simplified as follows:

$$p = \mathbf{v}_{phr}^{\mathrm{T}}\, \mathbf{i}_{phr} = \begin{bmatrix} v_d & v_q & v_o \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_o \end{bmatrix} = v_d\, i_d + v_q\, i_q + v_o\, i_o = v_d\, i_d + v_q\, i_q\, . \quad \text{(A.7)}$$

## A.3. Transformation of a State-Space-Formulated System

### A.3.1. General Application

A three-phase system in the state-space representation can be transformed into the dq components as in the following. The standard state-space notation is:

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}\, , \quad \text{(A.8)}$$

and the relationship with the transformed variables is:

$$\mathbf{x}_r = \mathbf{T}_{dq}\,\mathbf{x} \quad \text{and} \quad \mathbf{x} = \mathbf{T}_{dq}^{-1}\,\mathbf{x}_r = \mathbf{T}_{dq}^{\mathrm{T}}\,\mathbf{x}_r\, . \quad \text{(A.9)}$$

Substituting **x** from (A.9) into (A.8):

$$\frac{d}{dt}\mathbf{x}_r = \mathbf{A}_r\,\mathbf{x}_r + \mathbf{B}_r\,\mathbf{u}_r\,, \tag{A.10}$$

in which:

$$\mathbf{A}_r = \mathbf{T}_{dq}\mathbf{A}\,\mathbf{T}_{dq}^{-1} - \mathbf{T}_{dq}\frac{d}{dt}\mathbf{T}_{dq}^{-1}, \quad \text{and} \quad \mathbf{B}_r = \mathbf{T}_{dq}\mathbf{B}\mathbf{T}_{dq}^{-1}. \tag{A.11}$$

Solving for the term $\mathbf{T}_{dq}\frac{d}{dt}\mathbf{T}_{dq}^{-1}$ with the definition of $\mathbf{T}_{dq}$ given in (A.3):

$$\mathbf{T}_{dq}\frac{d}{dt}\mathbf{T}_{dq}^{-1} = \mathbf{T}_{dq}\frac{d}{dt}\mathbf{T}_{dq}^{\mathrm{T}} = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{A.12}$$

The new matrices of the transformed system are:

$$\mathbf{A}_r = \mathbf{T}_{dq}\mathbf{A}\mathbf{T}_{dq}^{\mathrm{T}} + \begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B}_r = \mathbf{T}_{dq}\mathbf{B}\mathbf{T}_{dq}^{\mathrm{T}}\,. \tag{A.13}$$

### A.3.2. Application to the Three-Level System

The three-level system formulated in (2.22) is:

$$\frac{d}{dt}\begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & 0 & 0 \\ 0 & -\frac{R}{L} & 0 \\ 0 & 0 & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} + \begin{bmatrix} -\frac{1}{L} & 0 & 0 \\ 0 & -\frac{1}{L} & 0 \\ 0 & 0 & -\frac{1}{L} \end{bmatrix}\begin{bmatrix} e_a - v_{a0} \\ e_b - v_{b0} \\ e_c - v_{c0} \end{bmatrix} + \begin{bmatrix} -\frac{1}{L} \\ -\frac{1}{L} \\ -\frac{1}{L} \end{bmatrix} v_{N0}\,, \tag{A.14}$$

in which the matrices can be identified as follows:

$$\mathbf{x} = \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -\frac{R}{L} & 0 & 0 \\ 0 & -\frac{R}{L} & 0 \\ 0 & 0 & -\frac{R}{L} \end{bmatrix}, \quad \mathbf{B}_1 = \mathbf{B}_2 = \begin{bmatrix} -\frac{1}{L} & 0 & 0 \\ 0 & -\frac{1}{L} & 0 \\ 0 & 0 & -\frac{1}{L} \end{bmatrix},$$

$$\mathbf{u}_1 = \begin{bmatrix} e_a - v_{a0} \\ e_b - v_{b0} \\ e_c - v_{c0} \end{bmatrix}, \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} v_{N0} \\ v_{N0} \\ v_{N0} \end{bmatrix}. \tag{A.15}$$

Applying (A.9) and (A.13), the vectors and matrices become:

$$\mathbf{x}_r = \begin{bmatrix} i_d \\ i_q \\ i_o \end{bmatrix}, \quad \mathbf{A}_r = \begin{bmatrix} -\frac{R}{L} & \omega & 0 \\ -\omega & -\frac{R}{L} & 0 \\ 0 & 0 & -\frac{R}{L} \end{bmatrix}, \quad \mathbf{B}_{1r} = \mathbf{B}_{2r} = \begin{bmatrix} -\frac{1}{L} & 0 & 0 \\ 0 & -\frac{1}{L} & 0 \\ 0 & 0 & -\frac{1}{L} \end{bmatrix},$$

$$\mathbf{u}_{1r} = \begin{bmatrix} e_d - v_d \\ e_q - v_q \\ e_o - v_o \end{bmatrix}, \quad \text{and} \quad \mathbf{u}_{2r} = \begin{bmatrix} 0 \\ 0 \\ \sqrt{3}\, v_{N0} \end{bmatrix}. \tag{A.16}$$

As a result, the new state-space representation of the system is as follows:

$$\frac{d}{dt}\begin{bmatrix} i_d \\ i_q \\ i_o \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & \omega & 0 \\ -\omega & -\frac{R}{L} & 0 \\ 0 & 0 & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} i_d \\ i_q \\ i_o \end{bmatrix} + \begin{bmatrix} -\frac{1}{L} & 0 & 0 \\ 0 & -\frac{1}{L} & 0 \\ 0 & 0 & -\frac{1}{L} \end{bmatrix}\begin{bmatrix} e_d - v_d \\ e_q - v_q \\ e_o - v_o \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{\sqrt{3}}{L} \end{bmatrix} v_{N0}. \tag{A.17}$$

# Appendix B.

# THE TWO-DIMENTIONAL SPACE-VECTOR TRANSFORMATION

## B.1. Definition

The two-dimensional SV transformation is defined as follows:

$$\vec{x} = x_a + x_b e^{j\frac{2\pi}{3}} + x_c e^{-j\frac{2\pi}{3}}, \tag{B.1}$$

in which the variables $x_a$, $x_b$ and $x_c$ are a set of three-phase components, either voltages or currents.

This transformation is sometimes multiplied by the coefficient 2/3 in order normalize the length of the resulting vector. The coefficient $\sqrt{2/3}$ is also used.

## B.2. Influence of Harmonics in the Vector

A balanced set of three-phase components with addition of a generic *h* order harmonic and a common component *f(t)* is given as:

$$\begin{cases} x_a = \hat{X}_1 \cos(\theta + \varphi_1) + \hat{X}_h \cos(h\theta + \varphi_h) + f(t), \\ x_b = \hat{X}_1 \cos(\theta - \frac{2\pi}{3} + \varphi_1) + \hat{X}_h \cos\left[ h(\theta - \frac{2\pi}{3}) + \varphi_h \right] + f(t), \text{ and} \\ x_c = \hat{X}_1 \cos(\theta + \frac{2\pi}{3} + \varphi_1) + \hat{X}_h \cos\left[ h(\theta + \frac{2\pi}{3}) + \varphi_h \right] + f(t), \end{cases} \tag{B.2}$$

which can be also expressed as follows:

$$\begin{cases} x_a = \hat{X}_1 \dfrac{e^{j(\theta+\varphi_1)} + e^{-j(\theta+\varphi_1)}}{2} + \hat{X}_h \dfrac{e^{j(h\theta+\varphi_h)} + e^{-j(h\theta+\varphi_h)}}{2} + f(t), \\ x_b = \hat{X}_1 \dfrac{e^{j\left(\theta-\frac{2\pi}{3}+\varphi_1\right)} + e^{-j\left(\theta-\frac{2\pi}{3}+\varphi_1\right)}}{2} + \hat{X}_h \dfrac{e^{j\left[h\left(\theta-\frac{2\pi}{3}\right)+\varphi_h\right]} + e^{-j\left[h\left(\theta-\frac{2\pi}{3}\right)+\varphi_h\right]}}{2} + f(t), \text{ and} \\ x_c = \hat{X}_1 \dfrac{e^{j\left(\theta+\frac{2\pi}{3}+\varphi_1\right)} + e^{-j\left(\theta+\frac{2\pi}{3}+\varphi_1\right)}}{2} + \hat{X}_h \dfrac{e^{j\left[h\left(\theta+\frac{2\pi}{3}\right)+\varphi_h\right]} + e^{-j\left[h\left(\theta+\frac{2\pi}{3}\right)+\varphi_h\right]}}{2} + f(t). \end{cases} \tag{B.3}$$

Applying (B.1) to this set of variables the following expression is obtained:

$$\vec{x} = \vec{x}_1 + \vec{x}_h, \quad \text{with } \vec{x}_1 = \frac{3\hat{X}_1}{2} e^{j(\theta+\varphi_1)} \quad \text{and}$$
$$\vec{x}_h = \frac{\hat{X}_h}{2} \left\{ \left[ 1 + 2\cos\left( (h-1)\frac{2\pi}{3} \right) \right] e^{j(h\theta+\varphi_h)} + \left[ 1 + 2\cos\left( (h+1)\frac{2\pi}{3} \right) \right] e^{-j(h\theta+\varphi_h)} \right\}. \tag{B.4}$$

It can be observed that the common component *f(t)* has disappeared in the obtained vector. As a result, in the case of transforming voltages, the reference potential of $v_a$, $v_b$ and $v_c$ does not affect the obtained vector. Therefore, the point taken as a reference for the three variables can be changed with no effect on the voltage-vector representation.

Triplen order harmonics, $h$={3, 6, 9, 12, …}, have neither influence in the vector, since $\vec{x}_h = 0$ for these order of components.

Triplen-plus-one order harmonics, or $h$={4, 7, 10, 13, …}, produce direct rotational vectors with the same frequency as the harmonic, such that:

$$\vec{x}_h = \frac{3\hat{X}_h}{2} e^{j(h\theta+\varphi_h)}, \tag{B.6}$$

while triplen-minus-one order harmonics, or $h$={2, 5, 8, 11, …}, produce reverse rotational vectors, as follows:

$$\vec{x}_h = \frac{3\hat{X}_h}{2} e^{-j(h\theta+\varphi_h)}. \tag{B.7}$$

## B.3. Equivalences in the First Sextant

Assuming no distortion in the scalar components $\hat{X}_h = 0 \;\; \forall \;\; h \neq 1$, and $\varphi_1 = 0$, the resulting vector is represented by:

$$\vec{x} = x_{a(\theta)} + x_{b(\theta)} e^{j\frac{2\pi}{3}} + x_{c(\theta)} e^{-j\frac{2\pi}{3}} = \frac{3\hat{X}_1}{2} e^{j\theta}. \tag{B.8}$$

An equivalent vector in the first sextant can be found for any vector. These equivalences are very useful for processing all of the calculations in the first sextant. The relationships are described in Table B.1.

The equivalent vector in the first sextant can be obtained just interchanging the scalar variables when applying the two-dimensional transformation. The rotational direction of the equivalent vector depends on the number of variables that have been interchanged; thus, if the reference vector is in the 3rd sextant or 5th sextant, the equivalent vector in the first sextant is also a direct rotational vector. In contrast, for 2nd, 4th and 6th sextants, the equivalent vector is a reverse rotational vector.

The equivalences given in Table B.1 have the reflexive propriety. Thus, interchanging the three-scalar components of the equivalent vector in the first sextant in the opposite direction than in Table B.1, the original vector in the corresponding sextant is obtained.

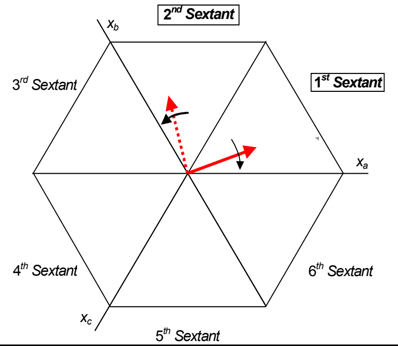
***Table B.1.*** *Equivalent vector in the first sextant*

Reference Vector in the 2nd Sextant: $\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$

Translation from the 2nd Sext. into the 1st Sext.: $\begin{cases} x_a \to x_b \\ x_b \to x_a \end{cases}$

Equivalent Vector in the 1st Sextant:

$$\vec{x}_1 = x_b + x_a e^{j\frac{2\pi}{3}} + x_c e^{-j\frac{2\pi}{3}} = \frac{3\hat{X}_1}{2} e^{j(\frac{2\pi}{3} - \theta)}$$

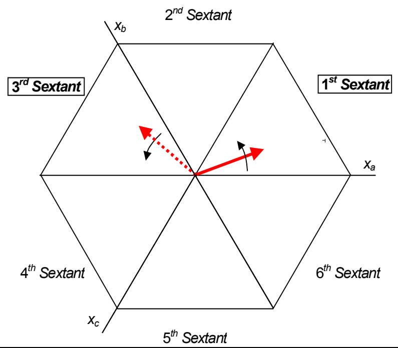



Reference Vector in the 3rd Sextant: $\frac{2\pi}{3} \le \theta \le \pi$

Translation from the 3nd Sext. into the 1st Sext.: $\begin{cases} x_a \to x_c \\ x_b \to x_a \\ x_c \to x_b \end{cases}$

Equivalent Vector in the 1st Sextant:

$$\vec{x}_1 = x_b + x_c e^{j\frac{2\pi}{3}} + x_a e^{-j\frac{2\pi}{3}} = \frac{3\hat{X}_1}{2} e^{j(\theta - \frac{2\pi}{3})}$$

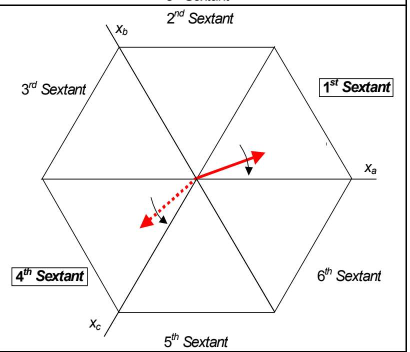



Reference Vector in the 4th Sextant: $\pi \le \theta \le \frac{4\pi}{3}$

Translation from the 4nd Sext. into the 1st Sext.: $\begin{cases} x_a \to x_c \\ x_c \to x_a \end{cases}$

Equivalent Vector in the 1st Sextant

$$\vec{x}_1 = x_c + x_b e^{j\frac{2\pi}{3}} + x_a e^{-j\frac{2\pi}{3}} = \frac{3\hat{X}_1}{2} e^{j(\frac{4\pi}{3} - \theta)}$$

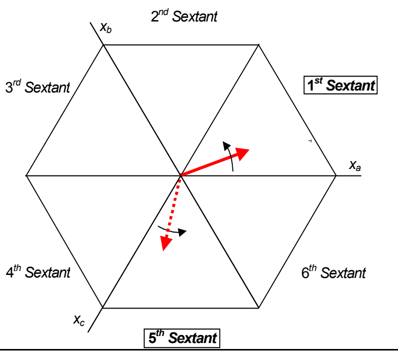



Reference Vector in the 5th Sextant: $\frac{4\pi}{3} \le \theta \le \frac{5\pi}{3}$

Translation from the 5th Sext. into the 1st Sext.: $\begin{cases} x_a \to x_b \\ x_b \to x_c \\ x_c \to x_a \end{cases}$

Equivalent Vector in the 1st Sextant:

$$\vec{x}_1 = x_c + x_a e^{j\frac{2\pi}{3}} + x_b e^{-j\frac{2\pi}{3}} = \frac{3\hat{X}_1}{2} e^{j(\theta - \frac{4\pi}{3})}$$




Reference Vector in the 6th Sextant: $\frac{4\pi}{3} \le \theta \le 2\pi$

Translation from the 6nd Sext. into the 1st Sext.: $\begin{cases} x_b \to x_c \\ x_c \to x_b \end{cases}$

Equivalent Vector in the 1st Sextant:

$$\vec{x}_1 = x_a + x_c e^{j\frac{2\pi}{3}} + x_b e^{-j\frac{2\pi}{3}} = \frac{3\hat{X}_1}{2} e^{j(2\pi - \theta)}$$