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MODULI SPACES OF VECTOR BUNDLES ON ALGEBRAIC VARIETIES

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CERTIFICA

que la present memòria ha estat realitzada sota la seva direcció per Laura Costa i Farràs, i que constitueix la tesi d'aquesta per a aspirar al grau de Doctora en Matemàtiques.

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Signat: Rosa Maria Miró-Roig.

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Introduction

This thesis seeks to contribute to a deeper understanding of the moduli spaces $M_{X,H}(r; c_1, \cdots, c_{min\{r,n\}})$ of rank r, H-stable vector bundles E on an n-dimensional variety X, with fixed Chern classes $c_i(E) = c_i \in H^{2i}(X, \mathbb{Z})$, displaying new and interesting geometric properties of $M_{X,H}(r; c_1, \cdots, c_{min\{r,n\}})$ which nicely reflect the general philosophy that moduli spaces inherit a lot of geometrical properties of the underlying variety X.

More precisely, we consider a smooth, irreducible, n-dimensional, projective variety X defined over an algebraically closed field k of characteristic zero, H an ample divisor on X, $r \geq 2$ an integer and $c_i \in H^{2i}(X, \mathbb{Z})$ for $i = 1, \cdots, min\{r, n\}$. We denote by $M_{X,H}(r; c_1, \cdots, c_{min\{r,n\}})$ the moduli space of rank r, vector bundles E on X, H-stable, in the sense of Mumford-Takemoto, with fixed Chern classes $c_i(E) = c_i$ for $i = 1, \cdots, min\{r, n\}$.

The main questions and problems we have considered are:

(1) Let X be a smooth, irreducible, rational surface, H an ample divisor on X and $0 << c_2 \in \mathbb{Z}$. Is the moduli space $M_{X,H}(2; c_1, c_2)$ rational?

More generally,

(2) Let X be a smooth, irreducible, rational surface, H an ample divisor on X and $0 << c_2 \in \mathbb{Z}$. Is the moduli space $M_{X,H}(r; c_1, c_2)$ rational?

(3) Let X be an algebraic $K3$ surface and H an ample divisor on X. Determine invariants (r, c_1, c_2, l) for which the moduli space $M_{X,H}(r; c_1, c_2)$ and the punctual Hilbert scheme $Hilb^l(X)$ are birational.

(4) What can be said about the geometry of moduli spaces $M_{X,H}(2; c_1, c_2)$ if X is a variety of arbitrary dimension? Is $M_{X,H}(2; c_1, c_2)$ connected, smooth, rational and irreducible?

Questions (**1**) and (**2**) were formulated in [Sch90]; Problem 21, [Sch85]; Problem 2 and [OV88]; Problem 2 and question (**3**) was stated by Nakashima in [Nak97]. The aim of the fourth question is to show that on varieties X of higher dimension, provided that we choose the ample divisor H on X closely related to c_2, $M_{X,H}(2; c_1, c_2)$ inherits a lot of the geometric properties of X.

Moduli spaces for semistable (resp. stable) torsion free sheaves (resp. vector bundles) on smooth, irreducible, algebraic, projective varieties were constructed in the 1970's. Once the existence of the moduli space is established, the question arises as what can be said about its local and global structure. Many authors have studied its structure, from the point of view of algebraic geometry, of topology and of differential geometry; giving very pleasant connections between these areas. In this work, we will take an algebraic-geometric point of view.

Many subtle and interesting results have been proved regarding the moduli spaces $M_{X,H}(r; c_1, \cdots, c_{min\{r,n\}})$ when the underlying variety X has dimension $n = 2$, and almost nothing is known if the variety has dimension n greater or equal than three. Let us briefly recall some of the main results. To this end, we will denote by $\overline{M}_{X,H}(r; c_1, \cdots, c_{min\{r,n\}})$ the moduli space of rank r, torsion free sheaves E on X of dimension n, semistable with respect to H, in the sense of Gieseker-Maruyama, with fixed Chern classes $c_i(E) = c_i$, $i = 1, \cdots, min\{r, n\}$. Notice that $M_{X,H}(r; c_1, \cdots, c_{min\{r,n\}})$ is an open subset of $\overline{M}_{X,H}(r; c_1, \cdots, c_{min\{r,n\}})$.

In the 1980's, Donaldson proved that if X is a smooth, irreducible, projective surface then, the moduli space $M_{X,H}(2; 0, c_2)$ is generically smooth of the expected dimension $4c_2 - 4\chi(O_X) + p_g(X) + 1$, provided c_2 is large enough ([Don86]). As a consequence, he obtained some spectacular new results on the classification of $\mathbb{C}^\infty$ four manifolds. Since then, many interesting results have been proved. For instance, it is well known that $\overline{M}_{X,H}(r; c_1, c_2)$ (resp. $M_{X,H}(r; c_1, c_2)$) is a projective (resp. quasi-projective) variety and for c_2 sufficiently large, it is non-empty ([Sor97], [Mar77] and [Mar78]), generically smooth, irreducible, normal ([Don86], [Zuo91], [GL96] and [OGr96]) of the expected dimension $2rc_2-(r-1)c_1^2-r^2\chi(O_X)+p_g(X)+1$.

As a consequence of Mukai's work ([Muk84]) we have that, if X is a smooth, irreducible, K3 surface then $\overline{M}_{X,H}(2;c_1,c_2)$ has Kodaira dimension 0 and, very recently, Li has proved that if X is a minimal surface of general type with a reduced canonical divisor then $\overline{M}_{X,H}(2;c_1,c_2)$ is of general type ([Li94]). This shows that the geometry of the surface and of the moduli spaces of sheaves on the surface are intimately linked.

We turn now our attention to the study of the rationality of the moduli space $M_{X,H}(r;c_1,c_2)$. For $X = \mathbb{P}^2$, Maruyama (resp. Ellingsrud and Stromme) proved that if $c_1^2-4c_2 \not\equiv 0 \pmod 8$, then the moduli space $M_{\mathbb{P}^2,O_{\mathbb{P}^2}(1)}(2;c_1,c_2)$ of $O_{\mathbb{P}^2}(1)$-stable, rank 2 vector bundles on $\mathbb{P}^2$ with Chern classes c_1 and c_2 is rational ([Mar85] and [ES87]). Later on, Maeda proved that the rationality of the moduli space $M_{\mathbb{P}^2,O_{\mathbb{P}^2}(1)}(2;c_1,c_2)$ holds for all $(c_1,c_2) \in \mathbb{Z}^2$ provided $M_{\mathbb{P}^2,O_{\mathbb{P}^2}(1)}(2;c_1,c_2)$ is non-empty ([Mae90]). Maeda's result together with the remark that there is no counterexample known to the fact that the moduli space $M_{X,H}(2;c_1,c_2)$ is always rational provided the underlying surface is rational, gives rise to the first question considered in this work, which we have reformulated here:

QUESTION (1): Let X be a smooth, irreducible, rational surface. Fix $c_1 \in Pic(X)$ and $0 << c_2 \in \mathbb{Z}$. Is there an ample divisor H on X such that $M_{X,H}(2;c_1,c_2)$ is rational?

We give an affirmative answer to this question. Furthermore, we prove that if X is a minimal rational surface or a Fano surface then for any ample divisor H on X, the moduli space $M_{X,H}(2;c_1,c_2)$ is rational provided it is non-empty. For all other rational surfaces, $M_{X,H}(2;c_1,c_2)$ is rational provided $c_2 >> 0$ and $(K_X+F)H<0$, being F a fiber of the ruling and K_X the canonical divisor of X.

As main tools we use the theory of prioritary sheaves introduced by Lazlo and Hirschowitz and developed by Walter, the theory of moduli spaces of stable vector bundles on blown up surfaces (see [Nak93]), and the birational properties of moduli spaces of rank two, stable vector bundles on algebraic surfaces. It is clear that the definition of stability depends on the choice of the ample divisor. Hence, it is natural to ask for the changes of $M_{X,H}(r;c_1,c_2)$ when H varies. It turns out that the ample cone of X has a chamber structure such that $M_{X,H}(r;c_1,c_2)$ only depends on the chamber of H and, in general, $M_{X,H}(r;c_1,c_2)$ changes when H crosses a wall between two chambers.

In this setup, we have proved that if X is a smooth, irreducible, anticanonical, rational surface, then the moduli spaces $M_{X,H}(2;c_1,c_2)$ and $M_{X,H'}(2;c_1,c_2)$ are birationaly equivalent, whenever non-empty and c_2 is bigger than some constant, depending on X, which we give explicitly. In case X is a smooth, rational surface, not necessarily anticanonical, we have proved that if $c_2 >> 0$ then the moduli spaces $M_{X,H}(r;c_1,c_2)$ and $M_{X,H'}(r;c_1,c_2)$ are birational, provided $H(K_X+F)<0$ and $H'(K_X+F)<0$. Again, F denotes the ruling of the rational surface $X \to \mathbb{P}^1$ and K_X the canonical divisor of X.

All these results imply that for many purposes we can fix the ample divisor H and this is what we usually do to study the birational geometry of moduli spaces $M_{X,H}(r;c_1,c_2)$.

In order to solve question **(1)**, we have established two criterions of rationality for moduli spaces $M_{X,H}(2;c_1,c_2)$ of rank two, H-stable, vector bundles E on a smooth, irreducible, rational surface X, with fixed Chern classes $c_1(E)=c_1$ and $c_2(E)=c_2$. The first one works for anticanonical rational surfaces, i.e., rational surfaces with effective anticanonical divisor. The second one works for arbitrary rational surfaces. Then, using either these criterions or constructing irreducible families of simple, prioritary torsion free sheaves (resp. stable vector bundles) over a big enough rational basis, we have completely solved question **(1)**.

We have extended the results concerning the rationality of the moduli spaces $M_{X,H}(2;c_1,c_2)$ of rank two, H-stable vector bundles, to arbitrary rank, partially answering question **(2)**. The affirmative answer to question **(1)** together with our contribution to question **(2)** strongly support the fact that the moduli space $M_{X,H}(r;c_1,c_2)$ is rational whenever X is rational.

Regarding question (**3**) we have extended results of Zuo ([Zuo91b]) and Nakashima ([Nak93b]) from rank two to arbitrary rank. It is well known that if X is an algebraic $K3$ surface, then the Hilbert scheme $Hilb^l(X)$ of zero dimensional subschemes of X of length l has a symplectic structure. On the other hand, in 1984, Mukai showed that the moduli space of simple sheaves has also a symplectic structure. Hence, it seems natural to look for a closer relation between Hilbert schemes and moduli spaces. Using Serre's correspondence and elementary transformations we have determined invariants (r, c_1, c_2, l) for which there is a birational map between the Hilbert scheme $Hilb^l(X)$ and the moduli space $M_{X,H}(r; c_1, c_2)$ of rank r, H-stable, vector bundles E on a smooth, irreducible, algebraic $K3$ surface, with fixed Chern classes $c_i(E) = c_i$.

For moduli spaces of vector bundles on higher dimensional varieties X, no general results are known and, as we will stress, the situation changes drastically. Results like the smoothness and irreducibility of moduli spaces of vector bundles on algebraic surfaces, turn to be false for moduli spaces of vector bundles on higher dimensional algebraic varieties. The existence of moduli spaces of stable vector bundles on a higher dimensional variety which are neither irreducible nor smooth is rather common. Indeed, in [Ein88] (resp. [AO95]), Ein (resp. Ancona and Ottaviani) proved that the minimal number of irreducible components of the moduli space of rank two (resp. rank 3) stable vector bundles on $\mathbb{P}^3$ (resp. $\mathbb{P}^5$) with fixed c_1 and c_2 going to infinity, grows to infinity. See [BM97] for a generalization of Ein's result to arbitrary projective 3-folds and [MO97] for examples of singular moduli spaces of vector bundles on $\mathbb{P}^{2n+1}$ with $c_2 >> 0$.

However, our contributions to question (**4**) show that for a $\mathbb{P}^d$-bundle, X, over a smooth curve C, and for a suitable choice of an ample divisor L on X, the moduli space $M_{X,L}(2; c_1, c_2)$ of rank 2, L-stable vector bundles E on X is a smooth, irreducible, projective variety. Namely, we prove that the moduli space is a $\mathbb{P}^N$-bundle over $Pic^0(C) \times Pic^0(C)$. If, in addition, X is a rational normal scroll, then the moduli space $M_{X,H}(2; c_1, c_2)$ is also rational. Once again these results reflect that a lot of the geometrical properties of the moduli space are related to the corresponding ones of the underlying variety.

Nevertheless, we want to point out that our result strongly depends on the polarization we have fixed. We will see how the moduli space $M_{X,L}(2; c_1, c_2)$ changes when the ample divisor L varies and we will show, by means of examples, that our results turn to be false for other ample divisors.

As byproduct of the methods we use throughout this work, we also compute the Picard group of some of the moduli spaces we deal with. We use the Picard group to see that, in general, given two ample divisors H and H' on X, the birational map between $M_{X,H}(r; c_1, c_2)$ and $M_{X,H'}(r; c_1, c_2)$ is not an isomorphism.

Part of the results of this thesis will appear in:

- L. Costa and R.M. Miró-Roig, *On the rationality of moduli spaces of vector bundles on Fano surfaces*, Journal Pure Appl. Algebra, to appear, (**[CM97]**).
- L. Costa, *K3 surfaces: moduli spaces and Hilbert schemes*, Collectanea Mathematica, to appear, (**[Cos98]**).

The contents of this Thesis is the following: **Chapter 1** is devoted to provide the reader with the general background that we will need in the sequel. In the first two sections, we have collected the main definitions and results concerning coherent sheaves and moduli spaces, at least, those we will need through this work.

In section 1.3, we review some facts on walls and chambers that we will use to understand and describe how the moduli space $M_{X,L}(2; c_1, c_2)$ changes when the ample divisor L varies.

Finally, in section 1.4 we overhaul the classification, up to isomorphism, of smooth, irreducible, rational surfaces and for the sake of completeness we prove some facts on the cohomology of line bundles on smooth, rational surfaces, which we have not found explicitly in the literature.

The aim of **Chapter 2** is to establish the criterions of rationality for moduli spaces $M_{X,H}(2; c_1, c_2)$ of rank two, H-stable vector bundles on a smooth, irreducible, rational surface X that will be used as one of our tools for answering **Question (1)**.

In section 2.1, using the theory of chambers and walls introduced in section 1.3, we will prove that if X is a smooth, irreducible, anticanonical, rational surface and

$4c_2 - c_1^2 > 2 - \frac{3K_X^2}{2}$ then, for any two ample divisors H_1 and H_2 on X, the moduli spaces $M_{X,H_1}(2; c_1, c_2)$ and $M_{X,H_2}(2; c_1, c_2)$ are birationaly equivalent, whenever non-empty (Theorem 2.1.10). As an important consequence, we will obtain the first criterion of rationality (Criterion 2.1.13) valid for moduli spaces $M_{X,H}(2; c_1, c_2)$ of rank two, H-stable vector bundles E, on a smooth, irreducible, anticanonical, rational surface, with fixed Chern classes $c_i(E) = c_i$.

Along section 2.2 we will show, by means of a family of examples that the birational map between $M_{X,H}(r; c_1, c_2)$ and $M_{X,H'}(r; c_1, c_2)$ is not an isomorphism.

In section 2.3, we generalize Theorem 2.1.10 and Criterion 2.1.13 to arbitrary smooth, rational surfaces. As a main tool, we will use prioritary sheaves which were introduced on $\mathbb{P}^2$ by Hirschowitz-Laszlo ([HL93]) and on birationaly ruled surfaces by Walter ([Wal93]). Using the fact that the moduli space of simple prioritary sheaves is smooth and irreducible (Theorem 2.3.5) and the fact that, under some conditions on H, H-stable vector bundles are prioritary (Lemma 2.3.2), we obtain Theorem 2.3.6 which extends Theorem 2.1.10 to arbitrary rational surfaces and the second criterion of rationality for moduli spaces $M_{X,H}(2; c_1, c_2)$ of rank two, H-stable, vector bundles E on a smooth, irreducible, rational surface X with fixed Chern classes $c_1(E) = c_1$ and $c_2(E) = c_2$ (Criterion 2.3.7) which generalizes Criterion 2.1.13 to arbitrary rational surfaces.

Let $\pi : X \to \mathbb{P}^1$ be a smooth, algebraic, rational surface and H an ample divisor on X such that $H(K_X + F) < 0$, being F the ruling of π and K_X the canonical divisor of X. In **Chapter 3** we prove that the moduli space $M_{X,H}(2; c_1, c_2)$ of rank two, H-stable, vector bundles E on a smooth, irreducible, rational surface X, with fixed Chern classes $c_1(E) = c_1 \in Pic(X)$ and $0 << c_2(E) = c_2 \in \mathbb{Z}$ is a smooth, irreducible, rational, quasi-projective variety (Theorem 3.3.7) which solves **Question (1)**.

According to the classification, up to isomorphism, of smooth, projective, rational surfaces (Theorem 1.4.1) and the methods we use, we have divided this Chapter in three sections.

In section 3.1, we deal with the case of minimal rational surfaces. For any smooth, irreducible, minimal, rational surface X and for any ample divisor H on X, we prove the rationality of the non-empty moduli spaces $M_{X,H}(2; c_1, c_2)$ of rank two, H-stable, vector bundles E on X with Chern classes $c_i(E) = c_i$ (Theorem 3.1.8). We prove this result using either Criterion 2.1.13 or constructing suitable families of rank two, H-stable (resp. simple, prioritary) vector bundles (resp. torsion free sheaves), over a big enough rational base.

In section 3.2, X is a Fano surface and for any ample divisor H on X, we prove the rationality of the moduli space $M_{X,H}(2; c_1, c_2)$ of rank two, H-stable, vector bundles E on X with Chern classes $c_i(E) = c_i$ (Theorem 3.2.7). We want to stress that in this section, we strongly use the fact that X is an anticanonical rational surface. So, we are able to use Criterion 2.1.13 and this is in fact what we will do (together with the construction of families of simple, prioritary vector bundles over a big, rational base), in order to prove the rationality of the moduli space $M_{X,H}(2; c_1, c_2)$.

Finally, in section 3.3, we prove the rationality of the moduli space $M_{X,H}(2; c_1, c_2)$ for the remaining rational surfaces, i.e. for non-minimal, rational surface obtained blowing up at least 8 points of a Hirzebruch surface. As a main tool, we use Criterion 2.3.7. In the case where this second criterion cannot be applied, we construct families of rank two, simple, prioritary, torsion free sheaves, over a rational base. Then, using the fact that for any ample divisor H on X such that $(K_X + F)H < 0$, H-stable vector bundles are prioritary (Lemma 2.3.2), we will deduce the rationality of the moduli space $M_{X,H}(2; c_1, c_2)$ (Theorem 3.3.6). We remark that the assumption $H(K_X + F) < 0$ is only used in this last section.

In **Chapter 4** we study moduli spaces $M_{X,H}(r; c_1, c_2)$ of rank r, H-stable vector bundles on either minimal rational surfaces or on algebraic K3 surfaces.

In section 4.1 we will come back to the delicate problem concerning the rationality of the moduli space $M_{X,H}(r; c_1, c_2)$ when X is a minimal, rational surface. First of all, by means of constructing a family of simple prioritary sheaves over a big enough rational base, we will prove the rationality of some moduli spaces $M_{X_e,H}(r; c_1, c_2)$ of rank r, H-stable vector bundles E on a smooth, Hirzeburch surface X_e, with fixed Chern classes $c_1(E) = c_1$ and $c_2(E) = c_2$ (Theorem 4.1.13). As a consequence, we will obtain the rationality of some moduli spaces $M_{\mathbb{P}^2, O_{\mathbb{P}^2}(1)}(r; c_1, c_2)$ of rank r,

$O_{\mathbb{P}^2}(1)$-stable vector bundles on the projective plane $\mathbb{P}^2$, with fixed Chern classes $(c_1, c_2) \in \mathbb{Z}^2$ (Theorem 4.1.14).

We want emphasize that Theorem 4.1.13 and Theorem 4.1.14 constitute an important contribution to the problem whether moduli spaces $M_{X,H}(r; c_1, c_2)$ are always rational provided X is rational. They round off the series of works developped by Göttsche ([Got96]), Katsylo ([Kat92]), Yoshioka ([Yos96]) and Li ([Li97]).

In section 4.2, we will turn our attention to studying moduli spaces $M_{X,H}(r; c_1, c_2)$ of rank r, H-stable vector bundles E on a smooth, algebraic K3 surface X, with fixed Chern classes $c_1(E) \in Pic(X)$ and $c_2(E) \in \mathbb{Z}$. We determine invariants $((r, c_1, c_2), l) \in (\mathbb{Z} \times Pic(X) \times \mathbb{Z}) \times \mathbb{Z}$ for which there is a birational map ϕ between the moduli space $M_{X,H}(r; c_1, c_2)$ and the Hilbert scheme $Hilb^l(X)$ of 0-dimensional subschemes of X (Theorem 4.2.1). The pullback of the symplectic structure on $Hilb^l(X)$ ([Bea78]) via the birational map ϕ, gives a symplectic structure on $M_{X,H}(r; c_1, c_2)$, which coincides with the symplectic structure constructed on $M_{X,H}(r; c_1, c_2)$ by Mukai in [Muk84].

In **Chapter 5** we deal with moduli spaces $M_{X,L}(2; c_1, c_2)$ of rank two, L-stable vector bundles E, on $\mathbb{P}^d$-bundles of arbitrary dimension, with fixed Chern classes $c_i(E) = c_i$, $i = 1, 2$.

To begin section 5.1 we recall some basic facts on $\mathbb{P}^d$-bundles over a smooth, projective curve of genus $g \geq 0$, in order to supply the reader with the background that we will use in forthcoming sections. We will end this section with a key Proposition that will allow us to guarantee the existence of a section of a suitable twist of a rank two vector bundle, whose scheme of zeros has codimension greater or equal than two (Proposition 5.1.13).

In section 5.2 we prove our main results of moduli spaces of rank two, vector bundles on normal scrolls $X = \mathbb{P}(\mathcal{E}) \to C$, which are defined over a smooth, projective curve C of genus $g \geq 0$ (i.e. on $\mathbb{P}^d$-bundles). Namely, we state that the moduli space $M_{X,L}(2; c_1, c_2)$ of rank two, L-stable vector bundles E on X, with fixed Chern classes $c_i(E) = c_i$ is a smooth, irreducible, projective variety and we compute its dimension (Theorem 5.2.4, Theorem 5.2.8 and Theorem 5.2.12). If, in addition, X is a rational, normal scroll, we obtain that $M_{X,L}(2; c_1, c_2)$ is, as well, a rational variety

(Corollary 5.2.5 and Corollary 5.2.9).

The proof of this results will allow us to compute the Kodaira dimension (Corollary 5.2.14) and to describe the Picard group (Corollary 5.2.15) of these moduli spaces. Notice that, once again, the moduli space captures a lot of information of the underlying variety.

The main idea is to construct non-trivial rank two vector bundles as an extension of two line bundles. Despite what happens on other projective varieties, for instance on Fano manifolds and on projective spaces, where any extension of two line bundles splits, on normal scrolls X, there are huge families of rank two, L-stable vector bundles, given by a non-trivial extension of two line bundles, being L a suitable ample divisor on X.

Section 5.3 is devoted to illustrate, by means of an example, how moduli spaces $M_{X,L}(2; c_1, c_2)$ vary when the ample divisor L crosses different walls (Theorem 5.3.2). Roughly speaking, we will see that there is a "extremal" wall W such that if L crosses it, in some direction, then the moduli space $M_{X,L}(2; c_1, c_2)$ is empty, meanwhile, if L sits in a chamber close to W, then the moduli space can be nicely described.

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Notation and conventions

Throughout this work k will be an algebraically closed field of characteristic zero.

For any smooth, irreducible, projective variety X of dimension n, we denote by $Pic(X)$ the group of divisors modulo linear equivalence, by $Num(X)$ the group of divisors modulo numerical equivalence, and by C_X the cone in $Num(X) \otimes \mathbb{R}$ generated by all ample divisors. A polarization on X is an element in C_X. We will identify $H^{2n}(X, \mathbb{Z})$ with $\mathbb{Z}$ and we will denote by $K = K_X$ the canonical divisor of X.

We will use the words vector bundles and locally free sheaves indistinctly and we will talk about ample divisors instead of ample line bundles.

For any coherent sheaf E on a smooth, projective algebraic variety X, we will often write H^iE (resp. h^iE) to denote $H^i(X, E)$ (resp. $dim_k H^i(X, E)$) and we will denote by $c_i(E) \in H^{2i}(X, \mathbb{Z})$ the i-th Chern class of E. The dual of E is written as $E^* := \mathcal{H}om(E, O_X)$. We will say that E is simple if $Hom(E, E) \cong k$. Given two coherent sheaves E and F, we will often write $ext^i(E, F)$ to denote $dim_k Ext^i(E, F)$.

The symbol " $\square$ " will stand for "end of proof ".

Chapter 1

Generalities on moduli spaces and surfaces

As we pointed out in the introduction, this first chapter essentially does not contain new results. However, considering that most of the material concerning moduli spaces on vector bundles is scattered through the literature, we have thought convenient to recall here the main definitions and results, at least those which we will need through this work, in order to provide a general background on the subject.

The lay out of this Chapter is as follows: We start in section one, by collecting some well known facts on Chern classes of coherent sheaves (resp. vector bundles), as well as by reviewing Serre's duality and Hirzebruch-Riemann-Roch's Theorem. We also introduce the notion of H-stability (resp. G-stability with respect to H) in the sense of Mumford-Takemoto (resp. Gieseker-Maruyama) of vector bundles (resp. torsion free sheaves) on a smooth, projective variety X, being H an ample divisor on X.

In section two, we first go over the concept of moduli space from a general point of view. As a general reference on moduli spaces the reader can look up, for example, [MFK92] or [New78]. Then, we restrict our attention to moduli spaces of vector bundles on smooth, projective varieties and, in particular, on smooth, projective surfaces. We will gather the results on moduli spaces of vector bundles that are important to our study. For more information on moduli spaces of vector bundles on smooth, projective varieties, the reader can consult, among others, [Mar77], [Mar78], [GL96] or [OGr96], being the main source [HL97].

The dependence of the different notions of stability on the fixed ample divisor H on the variety X is clear. Among others, Friedman and Qin have studied the problem from various angles and revealed interesting phenomena. Since in Chapter 2 and in Chapter 5 we go back to the problem of how the moduli space changes when the polarization varies, in section three we recall the basic facts on walls and chambers used in Chapters 2 and 5. The main source is [Qin93].

Finally, in section four we summarize some facts on the classification, up to isomorphism, of smooth, rational surfaces and we prove some easy results about cohomology groups of line bundles on smooth, rational surfaces which we have not found explicitly in the literature.

1.1 Coherent sheaves and stability

In this first section, we give a general account of basic properties of coherent sheaves and vector bundles on smooth, projective varieties and we recall the different notions of stability needed in the sequel.

1.1.1. *Let E be a coherent sheaf of rank $r \geq 0$ on a non-singular projective variety of dimension n and let L be a line bundle on X. Then,*

$$c_k(E \otimes L) = \sum_{i=0}^{k} \binom{r-i}{k-i} c_i(E) c_1(L)^{k-i}.$$

1.1.2. *Let E be a rank two vector bundle on a non-singular projective variety of dimension n. Then,*

$$E^* \cong E(-c_1)$$

being $c_1 = c_1(E)$.

1.1.3. *Let E be a coherent sheaf of rank $r > 0$ on a non-singular projective variety X of dimension n. Let c_i be the Chern classes of E. The discriminant of E is* *defined by*

$$\Delta(E) = \Delta(r; c_1, c_2) := \frac{1}{r}(c_2 - \frac{r-1}{2r}c_1^2).$$

Notice that for any line bundle L on X, $\Delta(E) = \Delta(E \otimes L)$.

Let us recall the well known Hirzebruch-Riemann-Roch's Theorem.

1.1.4. Hirzebruch-Riemann-Roch's Theorem: *Let E be a torsion free sheaf of rank $r \geq 0$ on a non-singular projective variety X of dimension n. Let $td(T)$ be the Todd class of the tangent bundle of X. Then,*

$$\chi(E) := \sum_{i=0}^{n} (-1)^i dim H^i(X, E) = deg(ch(E).td(T))_n$$

where $(\ \ )_n$ denotes the component of degree n in $A(X) \otimes \mathbb{Q}$ and $ch(E)$ is the Chern character of E.

In particular, if X is a smooth, projective surface we have

1.1.5. *Let E be a torsion free sheaf of rank $r \geq 0$ on a non-singular projective surface X. Let c_1 and c_2 be the Chern classes of E. Then,*

$$\chi(r; c_1, c_2) := \chi(E) = \sum_{i=0}^{2} dim H^i(X, E) = r(1 + p_a(X)) - \frac{c_1 K}{2} + \frac{c_1^2 - 2c_2}{2}.$$

Throughout the thesis we will very often use the well known Serre's duality Theorem which we recall now.

1.1.6. Serre's duality *Let X be a smooth, algebraic variety of dimension n, with canonical line bundle K_X and F_1, F_2 two torsion free sheaves on X. Then, $Ext^i(F_1, F_2)$ is canonically isomorphic to the dual of $Ext^{n-i}(F_2, F_1 \otimes K_X)$. If, in addition, F_1 is locally free then, $H^i(X, F_1)$ is canonically isomorphic to the dual of $H^{n-i}(X, F_1^* \otimes K_X)$.*

The main objects of our study are moduli spaces of vector bundles on smooth, projective varieties. There are very few classification problems for which a fine (resp. coarse) moduli space exists (see Definition 1.2.2 and Definition 1.2.1). To get a moduli space of vector bundles we must somehow restrict the class of vector bundles that we consider. What kind of vector bundles should we consider? The answer to this question is stable vector bundles. Now we will recall the different notions of stability that we will use later on.

Definition 1.1.7. *Let X be a smooth projective variety of dimension n and H an ample divisor on X. For a torsion free sheaf F on X one sets*

$$\mu_H(F) := \frac{c_1(F)H^{n-1}}{rk(F)}.$$

The sheaf F is H-semistable if

$$\mu_H(E) \leq \mu_H(F)$$

for all non-zero subsheaves $E \subset F$ with $rk(E) < rk(F)$. If strict inequality holds, then F is H-stable.

Definition 1.1.8. *Let X be a smooth projective variety of dimension n and H an ample divisor on X. For a torsion free sheaf F on X one sets*

$$P_F(m) := \frac{\chi(F \otimes O_X(mH))}{rk(F)}.$$

The sheaf F is G-semistable with respect to H if

$$P_E(m) \leq P_F(m) \ \ for \ \ m >> 0$$

and all non-zero subsheaves $E \subset F$ with $rk(E) < rk(F)$. If strict inequality holds, then F is G-stable with respect to H.

Using Hirzebruch-Riemann-Roch Theorem (1.1.4) one easily checks the following implications

$$L - stable \Rightarrow G - stable \Rightarrow G - semistable \Rightarrow L - semistable.$$

Remark 1.1.9. It easy follows from the definition of stability and 1.1.1 that a rank r vector bundle E on X is L-stable if, and only if, for any $D \in Pic(X)$, $E \otimes O_X(D)$ is L-stable.

1.1.10. *Let X be a smooth algebraic surface with canonical line bundle K and let V be a rank 2 vector bundle on X. It holds*

(1) If V is H-stable and $c_1(V)H < 0$ then $H^0V = 0$.

(2) If V is H-stable, $\chi(V) > 0$ and $c_1(V^* \otimes K)H < 0$ then $H^0V \neq 0$.

Remark 1.1.11. We want to emphasize that both notions of stability depend on the ample divisor we fix on the underlying variety X. We deeply study this problem in Chapter 2 and we will go back to it in Chapter 5.

1.2 Moduli spaces of vector bundles

Moduli spaces are one of the fundamental constructions of Algebraic Geometry and they arise in connection with classification problems. Roughly speaking, a moduli space for a collection of algebraic objects A and an equivalence relation $\sim$ is a "space" (in some sense of the word) which parameterizes equivalence classes of objects in a "continuous way", i.e., it takes into account how the equivalence classes of objects change in one or more parameter families. In our setting, the objects are algebraic and therefore we want an algebraic structure on our classification space $A/\sim$. Moreover, we want our moduli space to be unique up to isomorphism.

In this section, we will gather the results of moduli spaces of torsion free sheaves (resp. vector bundles) on a smooth, projective variety that are important to our study; all of them are well known to the experts. Let us start with a formal definition of a moduli space.

Let *(Sch/k)* be the category of Noetherian schemes over an algebraically closed field k. Suppose we are given a contravariant functor

$$F : (Sch/k) \longrightarrow (Sets).$$

Definition 1.2.1. *A fine moduli space of the functor F is a scheme $\mathcal{M}$ such that $Mor(.,\mathcal{M})$ represents F, i.e. F and $Mor(.,\mathcal{M})$ are isomorphic functors.*

If a fine moduli space exists, it is unique up to isomorphism. Unfortunately, there are very few classification problems for which a fine moduli space exists and it is necessary to find some weaker conditions, which nevertheless determine a unique algebraic structure on $A/\sim$. This leads to the following definition.

Definition 1.2.2. *A coarse moduli space of the functor F is a scheme $\mathcal{M}$ with the following properties.*

i) There is a natural transformation $\Psi : F \to Mor(.,\mathcal{M})$, such that

$$\Psi(Spec(k)) : F(Spec(k)) \xrightarrow{\cong} Mor(Spec(k),\mathcal{M})$$

is a bijection.

ii) For any other scheme $\mathcal{N}$ with natural transformation $\Phi : F \to Mor(.,\mathcal{N})$ there is a unique morphism of schemes $f : \mathcal{M} \to \mathcal{N}$ such that the diagram of natural transformations

$$\begin{array}{ccc} F & \xrightarrow{\Psi} & Mor(.,\mathcal{M}) \\ & {\scriptstyle\Phi}\searrow \quad \swarrow{\scriptstyle f_*} & \\ & Mor(.,\mathcal{N}) & \end{array}$$

commutes.

Again, if a coarse moduli space exists, it is unique up to isomorphism. A fine moduli space of the functor F is always a coarse moduli space of this functor but, in general, not vice versa. In fact, there is *a priori* no reason why the map

$$\Psi(S) : F(S) \to Mor(S,\mathcal{M})$$

should be bijective for varieties S other than $\{pt\}$. General facts on moduli spaces can be found, for instance, in [MFK92] or [New78].

From now on, we will restrict our attention to moduli spaces of vector bundles on smooth, projective varieties. The first step in the classification of vector bundles is to determine which cohomology classes on a projective variety can be viewed as Chern classes of vector bundles. On curves the answer is known. On surfaces the existence of vector bundles was settled by Schwarzenberger and it remains open on higher dimensional varieties. The next step aims at a deeper understanding of the set of all vector bundles with a fixed rank and Chern classes. This naturally leads to the concept of moduli spaces which we will shortly recall.

We fix a smooth, projective scheme X of dimension n, an ample divisor H on X and a numerical polynomial $P \in \mathbb{Q}[x]$. We define a T-family of vector bundles on X as a vector bundle V over $X \times T$ flat over T. Two T-families V, W are equivalent, which is denoted $V \sim W$, if there is a line bundle $\mathcal{L}_T$ on T such that $V \otimes \pi_2^* \mathcal{L}_T \cong W$, being $\pi_2 : X \times T \to T$ the natural projection. Note that if a vector bundle V on $X \times T$ is flat over T and T is irreducible, then the Hilbert polynomial $\chi(V(n)|_{X \times \{t\}})$ does not depend on $t \in T$ ([Har77]; Theorem III.9.9 and Corollary III.9.10).

We now consider the following contravariant functor

$$F_{X,P} : (Sch/k) \longrightarrow (Sets)$$

where

$$F_{X,P}(T) = \left\{ T\text{-families of vector bundles on } X \text{ with Hilbert polynomial P} \right\} \Big/ \sim$$

and if $f : T' \to T$ is a morphism in (Sch/k), then $F_{X,P}(f)$ is the map obtained by pulling-back sheaves via $f_X := id_X \times f$, i.e.

$$\begin{aligned} F_{X,P}(f) : \ F_{X,P}(T) &\longrightarrow F_{X,P}(T') \\ [E] &\longrightarrow [f_X^* E]. \end{aligned}$$

Definition 1.2.3. *A fine moduli space of vector bundles on X with Hilbert polynomial $P \in \mathbb{Q}[x]$ is a scheme $M_X(P)$ together with a family (Poincaré bundle) of vector* *bundles $\mathcal{U}$ on $M_X(P) \times X$ such that the contravariant functor $F_{X,P}$ is represented by $(M_X(P), \mathcal{U})$*

As we pointed out before, there are very few classification problems for which a fine moduli space exists. To get at least a coarse moduli space we must restrict our attention to stable vector bundles. A T-family V of vector bundles on X is H-stable if $V_t = V|_{X\times\{t\}}$ is H-stable for every geometric point $t \in T$.

We now consider the following contravariant functor

$$F^s_{(X,H),P} : (Sch/k) \longrightarrow (Sets)$$

where

$$F^s_{(X,H),P}(T) = \left\{ T\text{-families of } H\text{-stable vector bundles on } X \text{ with Hilbert polynomial P} \right\} \Big/ \sim$$

and if $f : T' \to T$ is a morphism in (Sch/k), then $F^s_{(X,H),P}(f)$ is the map obtained by pulling-back sheaves via $f_X := id_X \times f$, i.e.

$$\begin{array}{rccc} F^s_{(X,H),P}(f): & F^s_{(X,H),P}(T) & \longrightarrow & F^s_{(X,H),P}(T') \\ & [E] & \longrightarrow & [f_X^* E]. \end{array}$$

Theorem 1.2.4. *The functor $F^s_{(X,H),P}$ has a coarse moduli scheme $M_{X,H}(P)$ which is a separated scheme and locally of finite type over k. This means*

(1) There is a natural transformation

$$\Psi : F^s_{(X,H),P} \longrightarrow Hom(., M_{X,H}(P)),$$

which is bijective for any reduced point x_0.

(2) For every scheme $\mathcal{N}$ and every natural transformation $\Phi : F^s_{(X,H),P} \to Hom(., \mathcal{N})$ there is a unique morphism $f : M_{X,H}(P) \to \mathcal{N}$ for which the diagram

$$\begin{array}{ccc} F^s_{(X,H),P} & \xrightarrow{\quad\Psi\quad} & Mor(.,M_{X,H}(P)) \\ & \searrow^{\Phi} \quad\quad \swarrow_{f_*} & \\ & Mor(.,\mathcal{N}) & \end{array}$$

commutes.

In addition,

(i) There is a natural map $\lambda : M_{X,H}(P) \longrightarrow Pic(X)$ such that if a geometric point v corresponds to a vector bundle V on X, then $\lambda(v)$ corresponds to the line bundle $det(V)$.

(ii) $M_{X,H}(P)$ decomposes into a disjoint union of schemes

$$M_{X,H}(r; c_1, \cdots, c_{min\{r,n\}}),$$

where $n = dimX$ and $M_{X,H}(r; c_1, \cdots, c_{min\{r,n\}})$ is the moduli scheme of H-stable rank r vector bundles with Chern classes $(c_1, \cdots, c_{min\{r,n\}})$ up to numerical equivalence.

Proof. See [Mar77]; Theorem 5.6. □

Remark 1.2.5. The moduli space $M_{X,H}(r; c_1, \cdots, c_{min\{r,n\}})$ is contained in a projective variety $\overline{M}_{X,H}(r; c_1, \cdots, c_{min\{r,n\}})$ with a natural moduli interpretation. It is the closure of $M_{X,H}(r; c_1, \cdots, c_{min\{r,n\}})$ in the moduli space $\overline{M}_{X,H}(r; c_1, \cdots, c_{min\{r,n\}})$ of G-semistable with respect to H, torsion free sheaves on X. See [Mar78]; Corollary 5.9.1 for the details.

It is one of the deepest problems in algebraic geometry to determine when the moduli space of H-semistable (resp. H-stable) vector bundles is non-empty. If the underlying variety is a curve C of genus $g \geq 2$, it is well known that the moduli space of H-stable vector bundles on C of rank r and fixed determinant bundle is smooth of dimension $(r^2 - 1)(g - 1)$ ([HL97]; Corollary 4.5.5).

If the underlying variety X has dimension greater or equal to three, there are no general results which guarantee the non-emptiness of the moduli space of H-stable (resp. H-semistable) vector bundles on X. Finally, if the underlying variety is a smooth, projective surface X, then the existence conditions are well known whenever X is $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ and, in general, it is known that the moduli space $M_{X,H}(r; c_1, c_2)$ of H-stable, rank r, vector bundles E on X with Chern classes $c_1(E) = c_1$ and $c_2(E) = c_2$ is non-empty provided $\Delta(r; c_1, c_2) >> 0$ (see for instance [GL96], [HL93], [Mar77], [Mar78], [LQ96b] and [OGr96]) and empty if $\Delta(r; c_1, c_2) < 0$ (Bogomolov's inequality).

Recently, Sorger in [Sor97] has given an explicit lower bound for $\Delta(r; c_1, c_2)$ in order to assure that $M_{X,H}(r; c_1, c_2)$ is non-empty. Namely, he has proved that if $\Delta(r; c_1, c_2) > C(X, H, c_1, \sqrt{c_2})$ being $C(X, H, c_1, \sqrt{c_2})$ a real constant which only depends on X, H, c_1 and $\sqrt{c_2}$, then the moduli space $M_{X,H}(r; c_1, c_2)$ of H-stable, rank r, vector bundles E on X, with Chern classes $c_1(E) = c_1$ and $c_2(E) = c_2$ is non-empty.

We now turn to the question under which conditions the functor $F^s_{(X,H),P}$ is represented by $M_{X,H}(P)$ or, equivalently, whether we have a universal family on $M_X := M_{X,H}(r; c_1, \cdots, c_{min\{r,n\}})$ i.e. a M_X-family of vector bundles E on X such that if $t \in M_X$ is a generic point, then E_t on $X \times \{t\} \cong X$ corresponds to t. The existence of such a universal family (Poincaré sheaf) is guaranteed by the following criterion.

For any $c_1 \in Num(X)$ we denote by $a(c_1)$ the greatest integer such that c_1 is divisible by $a(c_1)$ in $Num(X)$ if $c_1 \neq 0$ and $a(c_1) = 0$ if $c_1 = 0$.

Theorem 1.2.6. *Let X be a smooth, algebraic surface, $c_1 \in Num(X)$ and $c_2 \in \mathbb{Z}$. If $gcd(r, a(c_1), \chi(r; c_1, c_2)) = 1$, then $M_{X,H}(r; c_1, c_2)$ has a universal family.*

Proof. See [Dre91]; Theoreme D. □

To state our next result we need to fix some more notations.

Let $H^l := Hilb^l(X)$ be the Hilbert scheme of zero-dimensional subschemes of length l on X and let $\mathcal{I}_{\mathcal{Z}_l}$ be the ideal sheaf of the universal subscheme $\mathcal{Z}_l$ in $X \times H^l$. Let π and p_X be the projections of $X \times H^l$ to H^l and X respectively. For any $D, c_1 \in Pic(X)$, we define $G_1 := p_X^*(-D)$ and $G_2 := \mathcal{I}_{\mathcal{Z}_l} \otimes p_X^*(D + c_1)$. We put

$$\mathcal{E}_{D,c_1} := Ext^1_\pi(G_2, G_1)$$

where $Ext^1_\pi(G_2, .)$ is the right derived functor of $Hom_\pi(G_2, .) := \pi_* \mathcal{H}om(G_2, .)$.

Lemma 1.2.7. *Let U (resp. $\mathcal{F}$) be the family of rank 2 torsion free sheaves (resp. vector bundles) E on X given by a non-trivial extension*

$$0 \to O_X(-D) \to E \to O_X(D + c_1) \otimes I_Z \to 0$$

where $Z \subset X$ is a 0-cycle of length l. Assume that $\pm(2D + c_1)$ and $2D + c_1 + K$ are non effective divisors. Then, U (resp. $\mathcal{F}$) is an irreducible, rational, projective (resp. quasi-projective) variety.

Proof. We apply [Rot79]; Corollary 11.44 to the functor $F = \pi_*$ and to the functor $G = \mathcal{H}om(G_2, .)$ and we get the long exact sequence

$$0 \to R^1\pi_*(\mathcal{H}om(G_2, G_1)) \to R^1(\pi_*\mathcal{H}om(G_2, .))(G_1) \to \pi_*(R^1\mathcal{H}om(G_2, G_1))$$

$$\to R^2\pi_*(\mathcal{H}om(G_2, G_1)) \to R^2(\pi_*\mathcal{H}om(G_2, .))(G_1) \to \cdots .$$

By base-change Theorem, we can see, arguing as in [Got96b]; Lemma 3.2 (see also [HS80]), that

$$\mathcal{E}_{D,c_1}$$

is a locally free sheaf of rank $r = dim Ext^1(O_X(D + c_1) \otimes I_Z, O_X(-D))$ and there is a natural bijective morphism

$$\psi : \mathbb{P}(\mathcal{E}_{D,c_1}) \longrightarrow U.$$

Therefore, U (resp. $\mathcal{F}$) is an irreducible, rational, projective (resp. quasi-projective) variety. $\square$

Remark 1.2.8. The conclusions of Lemma 1.2.7 are true if instead of assuming that $\pm(2D+c_1)$ and $2D+c_1+K$ are non effective divisors, we assume that $-(2D+c_1)$ and $2D+c_1+K$ are non effective divisors and $H^0E(D)=1$ for a generic E of U (resp. $\mathcal{F}$).

Once the existence of the moduli space is established, the question arises as what can be said about its local and global structure. In spite of the great progress made during the last decades in the problem of moduli spaces of vector bundles on smooth projective varieties (essentially in the framework of the Geometric Invariant Theory by Mumford) a lot of problems remain open and very little is known for varieties of arbitrary dimension. We refer to [HL97]; section 4.5 for general facts on the infinitesimal structure of $M_{X,H}(r;c_1,\cdots,c_{min\{r,n\}})$ and here, we only recall the results which are basic for us.

Theorem 1.2.9. *Let X be a smooth, projective variety of dimension n and F a H-stable, rank r vector bundle on X with Chern classes $c_i(F)=c_i$. Then the Zariski tangent space of the moduli space $M_{X,H}(r;c_1,\cdots,c_{min\{r,n\}})$ at $[F]$ is canonically given by*

$$T_{[F]}M_{X,H}(r;c_1,\cdots,c_{min\{r,n\}})\cong Ext^1(F,F).$$

If $Ext^2(E,E)=0$ then $M_{X,H}(r;c_1,\cdots,c_{min\{r,n\}})$ is smooth at $[F]$. In general, we have the following bound

$$ext^1(F,F)\geq dim_{[F]}M_{X,H}(r;c_1,\cdots,c_{min\{r,n\}})\geq ext^1(F,F)-ext^2(F,F).$$

Proof. See for instance [HL97]; Corollary 4.5.2. □

If F is a locally free sheaf on X, then the trace map $tr:\mathcal{E}nd(X)\to O_X$ induces maps $tr^i: Ext^i(F,F)\longrightarrow H^iO_X$. We denote the kernel of tr^i by $Ext^i(F,F)_0$. If we fix $L\in Pic(X)$ and we denote by $\mathcal{M}_{X,H}(r;L,c_2,\cdots,c_{min\{r,n\}})$ the moduli space of rank r, H-stable vector bundles E with fixed determinant $det(E)=L\in Pic(X)$ and $c_i(E)=c_i\in H^{2i}(X,\mathbb{Z})$ for $2\leq i\leq min\{r,n\}$, then we have the analogous result.

Theorem 1.2.10. *Let X be a smooth, projective variety of dimension n and F a H-stable, rank r vector bundle on X with fixed $det(F) = L \in Pic(X)$ and $c_i(F) = c_i$ for $2 \leq i \leq min\{r,n\}$. Then the Zariski tangent space of $\mathcal{M}_{X,H}(r;L,c_2,\cdots,c_{min\{r,n\}})$ at $[F]$ is canonically given by $T_{[F]}\mathcal{M}_{X,H}(r;L,c_2,\cdots,c_{min\{r,n\}}) \cong Ext^1(F,F)_0$. If $Ext^2(E,E)_0 = 0$ then the moduli space $\mathcal{M}_{X,H}(r;L,c_2,\cdots,c_{min\{r,n\}})$ is smooth at $[F]$. In general, we have the following bound*

$$ext^1(F,F)_0 \geq dim_{[F]}\mathcal{M}_{X,H}(r;L,c_2,\cdots,c_{min\{r,n\}}) \geq ext^1(F,F)_0 - ext^2(F,F)_0.$$

Proof. See for instance [HL97]; Theorem 4.5.4 . □

In case X is a smooth surface, we can make the dimension bound more explicit. Indeed, for any H-stable vector bundle F on X with $det(F) = L$ and $c_2(F) = c_2$, we have

$$\begin{aligned} ext^1(F,F)_0 - ext^2(F,F)_0 &= \chi(O_X) - \textstyle\sum_{i=0}^{2}(-1)^i ext^i(F,F) \\ &= 2rc_2(F) - (r-1)det(F)^2 - (r^2-1)\chi(O_X) \end{aligned}$$

where the last equality follows from Hirzebruch-Riemann-Roch's Theorem. The number

$$2rc_2 - (r-1)L^2 - (r^2-1)\chi(O_X)$$

is called the expected dimension of $\mathcal{M}_{X,H}(r;L,c_2)$.

Remark 1.2.11. For any smooth, projective, rational surface X we have the isomorphism $Num(X) \cong Pic(X)$. Hence, for any rank r vector bundle E on X, we can identify $c_1 = c_1(E) \in Num(X)$ with $L = det(E) \in Pic(X)$. Therefore, there is no difference between $M_{X,H}(r;c_1,c_2)$ and $\mathcal{M}_{X,H}(r;L,c_2)$ and we will write $M_{X,H}(r;c_1,c_2)$ instead of $\mathcal{M}_{X,H}(r;c_1,c_2)$.

Convention 1.2.12. Given a smooth, projective variety X of diemension n, an ample divisor H on X, $r \in \mathbb{Z}$, $c_i \in H^{2i}(X,\mathbb{Z})$ and $L \in Pic(X)$, we will denote $M_{X,H}(r;c_1,\cdots,c_{min\{r,n\}})$ by $M_H(r;c_1,\cdots,c_{min\{r,n\}})$ and $\mathcal{M}_{X,H}(r;L,\cdots,c_{min\{r,n\}})$ by $\mathcal{M}_H(r;L,\cdots,c_{min\{r,n\}})$ if there is no confusion.

For small values of the discriminant $\Delta(r;c_1,c_2)$, moduli spaces $\overline{M}_H(r;c_1,c_2)$ (resp. $M_H(r;c_1,c_2)$) of H-semistable torsion free sheaves (resp. H-stable vector bundles) on a smooth projective surface can look rather wild: their dimension need not be the expected one, they need not be neither irreducible nor reducible, let alone non-singular (see for instance [GL96], [OGr96], [Mar78], [Mes96]). This changes if the discriminant increases: moduli spaces become irreducible, normal, of the expected dimension and the codimension of the locus of points which are singular increases. We have summarized more precisely this ideas in the next Theorem.

Theorem 1.2.13. *Let X be a smooth algebraic surface, H an ample divisor on X and $L \in Pic(X)$. For all $c_2 >> 0$, the moduli space $\overline{\mathcal{M}}_H(r;L,c_2)$ of G-semistable with respect to H, rank r torsion free sheaves on X (resp. $\mathcal{M}_H(r;L,c_2)$ of H-stable, rank r vector bundles on X), is a generically smooth, irreducible, projective (resp. quasi-projective) variety of the expected dimension $2rc_2-(r-1)L^2-(r^2-1)\chi(O_X)$.*

Proof. See [GL96], [OGr96] and [OGr96b]. □

Remark 1.2.14. For smooth, projective, anticanonical, rational surfaces (i.e. rational surfaces X whose anticanonical divisor $-K_X$ is effective) and for the rank two case, we can omit the hypothesis $c_2 >> 0$. The irreducibility and smoothness of $M_H(2;c_1,c_2)$ holds whenever $M_H(2;c_1,c_2)$ is non-empty. Indeed, assume $M_H(2;c_1,c_2) \neq \emptyset$. Since $-K_X$ is effective, for any vector bundle $E \in M_H(2;c_1,c_2)$, we have $Ext^2(E,E)_0 = 0$. Hence, $M_H(2;c_1,c_2)$ is smooth at $[E]$ and

$$dim_{[E]}M_H(2;c_1,c_2) = 4c_2 - c_1^2 - 3.$$

The irreducibility of $M_H(2;c_1,c_2)$ follows from [Bal87]; Theorem 2.2 and Theorem 2.1.10 below. □

To end with generalities on moduli spaces of vector bundles we recall the following nice result due to Nakashima that will be used in Chapter 3. Let X be a smooth, projective surface, H an ample divisor on X, $L \in Pic(X)$ and c_2 an integer.

We consider $\pi : \tilde{X} \longrightarrow X$ the blow up of X at l distinct points p_i, $1 \leq i \leq l$, and we denote by E_i be the exceptional divisors. For $n >> 0$, $H_n := n\pi^*H - \sum_{i=1}^{l} E_i$ is an ample divisor on $\tilde{X}$ and we have

Theorem 1.2.15. *For sufficiently large n, there exists an open immersion*

$$\varphi : \mathcal{M}_H(r; L, c_2) \hookrightarrow \mathcal{M}_{H_n}(r; \pi^*L, c_2)$$

defined by $\varphi(V) = \pi^*(V)$ on closed points.

Proof. See [Nak93]; Theorem 1. □

1.3 Walls and chamber structures

Let X be a smooth, irreducible, projective variety of dimension n. In [Qin93], Qin considered the problem: What is the difference between $\mathcal{M}_{X,L_1}(r; c_1, c_2)$ and $\mathcal{M}_{X,L_2}(r; c_1, c_2)$ where L_1 and L_2 are two different polarizations ?.

It turns out that the ample cone of X has a chamber structure such that $\mathcal{M}_{X,L}(r; c_1, c_2)$ only depends on the chamber of L and the change of $\mathcal{M}_{X,L}(r; c_1, c_2)$ when L passes through the wall between chambers can be somehow controlled.

In this section we will recall the basic results about walls and chambers due to Qin ([Qin93]). In Chapters 2 and 5, we will use these notions in order to study when two moduli spaces $\mathcal{M}_{X,L_1}(r; c_1, c_2)$ and $\mathcal{M}_{X,L_2}(r; c_1, c_2)$ are birational. Moreover, an accurate study of these structures will allow us to give a very useful criterion of rationality for moduli spaces of rank two vector bundles on rational surfaces.

Definition 1.3.1. *Let L_1, L_2 be two polarizations on a smooth, irreducible, projective variety of dimension n. We define $L_1 \overset{s}{\geq} L_2$ if every rank two vector bundle with c_1 and c_2 as its first and second Chern classes is L_1-stable whenever is L_2-stable. We define $L_1 \overset{s}{=} L_2$ if both $L_1 \overset{s}{\geq} L_2$ and $L_2 \overset{s}{\geq} L_1$.*

Remark 1.3.2. Notice that for fixed $c_1 \in Pic(X)$ and $c_2 \in H^4(X, \mathbb{Z})$ if $L_1 \overset{s}{=} L_2$, then the moduli spaces $\mathcal{M}_{X,L_1}(2; c_1, c_2)$ and $\mathcal{M}_{X,L_2}(2; c_1, c_2)$ can be naturally identified.

Definition 1.3.3. *(i) Let $S \in A^{n-2}_{num}(X)$ and $\xi \in Num(X) \otimes \mathbb{R}$. We define*

$$W^{(\xi,S)} := C_X \cap \{x \in Num(X) \otimes \mathbb{R} | x\xi S = 0\}.$$

(ii) Define $\mathcal{W}(c_1, c_2)$ as the set whose elements consist of $W^{(\xi,S)}$, where S is a complete intersection surface in X, and ξ is the numerical equivalence class of a divisor G on X such that $O_X(G + c_1)$ is divisible by 2 in $Pic(X)$, and that

$$G^2S < 0; \qquad c_2 + \frac{G^2 - c_1^2}{4} = [Z]$$

for some locally complete intersection codimension-two cycle Z in X.
(iii) A wall of type (c_1, c_2) is an element in $\mathcal{W}(c_1, c_2)$. A chamber of type (c_1, c_2) is a connected component of $C_X \setminus \mathcal{W}(c_1, c_2)$. A $\mathbb{Z}$-chamber of type (c_1, c_2) is the intersection of $Num(X)$ with some chamber of type (c_1, c_2).
(iv) A face of type (c_1, c_2) is $\mathcal{F} = W^{(\xi,S)} \cap \overline{\mathcal{C}}$, where $W^{(\xi,S)}$ is a wall of type (c_1, c_2) and $\mathcal{C}$ is a chamber of type (c_1, c_2).

We say that a wall $W^{(\xi,S)}$ of type (c_1, c_2) separates two polarizations L and L' if, and only if, $\xi SL < 0 < \xi SL'$.

Notice that when $dim(X) = 2$, S disappears in the definition of wall and we will denote by W^ξ a wall of type (c_1, c_2) defined by ξ instead of $W^{(\xi,S)}$.

Remark 1.3.4. In [Qin93]; Corollary 2.2.2 and Remark 2.2.6, Qin proves that the moduli space $\mathcal{M}_{X,L}(2; c_1, c_2)$ only depends on the chamber of L and that the study of moduli spaces of rank two vector bundles stable with respect to a polarization lying on walls may be reduced to the study of moduli spaces of rank two vector bundles stable with respect to a polarization lying on $\mathbb{Z}$-chambers.

We will denote by $\mathcal{M}_{\mathcal{C}}(2; c_1, c_2)$ (resp. $\mathcal{M}_{\mathcal{F}}(2; c_1, c_2)$) the moduli space $\mathcal{M}_L(2; c_1, c_2)$ where L is a polarization lying in the chamber $\mathcal{C}$ (resp. face $\mathcal{F}$).

Proposition 1.3.5. *If $dim(X) = 2$, then the set of walls of type (c_1, c_2) is locally finite.*

Proof. See [Qin93]; Proposition 2.1.6. □

If $dim(X) > 2$, Proposition 1.3.5 will no longer hold. This will limit the application of the theory of walls and chambers structures in higher-dimensional cases. Nevertheless, this theory is quite satisfactory in dimension two and an accurate description of these structures will allow us to give a criterion of rationality of moduli spaces of rank two vector bundles on rational surfaces.

Let us now focus our attention on the case where X is an algebraic surface. We will introduce some extra notations and Remarks about the theory of chambers and walls that will be very useful for us in Chapter 2.

Definition 1.3.6. Let ξ be a numerical equivalence class defining a wall of type (c_1, c_2). We define $E_\xi(c_1, c_2)$ as the quasi-projective variety parameterizing rank 2 vector bundles E on X given by an extension

$$0 \to O_X(G) \to E \to O_X(c_1 - G) \otimes I_Z \to 0$$

where G is a divisor with $2G - c_1 \equiv \xi$ and Z is a locally complete intersection 0-cycle of length $c_2 + (\xi^2 - c_1^2)/4$. Moreover, we require that E is not given by the trivial extension when $\xi^2 = c_1^2 - 4c_2$.

We define $D(\xi) := dimE_\xi(c_1, c_2)$ and we put

$$d_\xi(c_1, c_2) := d(\xi) = D(\xi) - (4c_2 - c_1^2 - 3\chi(O_X)).$$

In other words, $d(\xi)$ is the difference between the dimension of $E_\xi(c_1, c_2)$ and the expected dimension of a non-empty moduli space $\mathcal{M}_L(2; c_1, c_2)$.

Remark 1.3.7. By [Qin93]; Theorem 1.2.5, if L_1 and L_2 are two ample divisors on X and E is a rank 2 vector bundle on X which is L_1-stable but L_2-unstable, then we have $E \in E_\xi(c_1, c_2)$ where ξ defines a non-empty wall of type (c_1, c_2) separating L_1 and L_2 (i.e. $\xi L_1 < 0 < \xi L_2$; moreover, we can consider the ample divisor $L := (\xi L_2)L_1 - (\xi L_1)L_2$ on X and we have $L\xi = 0$).

We end this section with a remark that is essential in the study of the birational equivalence between moduli spaces.

Remark 1.3.8. If $d(\xi) < 0$ for any ξ which defines a non-empty wall of type (c_1, c_2), then any two moduli spaces $\mathcal{M}_{L_1}(2; c_1, c_2)$ and $\mathcal{M}_{L_2}(2; c_1, c_2)$ are birational whenever non-empty and any polarization L is trivial of type (c_1, c_2) provided $\mathcal{M}_L(2, c_1, c_2)$ is non-empty (see introduction of Chapter two for the precise definition).

1.4 Background on surfaces

In this section we will gather all relevant results on smooth, irreducible, rational surfaces and cohomology of divisors on rational surfaces which we will need in the sequel.

First of all, we will recall the classification, up to isomorphism, of smooth, irreducible, projective, rational surfaces.

Theorem 1.4.1. *Let X be a smooth, minimal, rational surface. Then, X is either isomorphic to $\mathbb{P}^2$ or to a Hirzebruch surface X_e with $e \neq 1$.*

Proof. See [Bea78]; Theorem V.10. □

Remark 1.4.2. Recall that $\mathbb{P}^2$ with one point blown up is isomorphic to the Hirzebruch surface X_e with $e = 1$.

Remark 1.4.3. It follows from Theorem 1.4.1 and Remark 1.4.2 that every smooth, irreducible, projective, rational surface is either isomorphic to $\mathbb{P}^2$, or to a Hirzebruch surface X_e or to the blow up of a Hirzebruch surface at a finite set of $s > 0$ points. Indeed, we identify the blow up of $\mathbb{P}^2$ at $s \geq 1$ points with the Hirzebruch surface $X_{e=1}$ with $s - 1$ points blown up.

Going ahead with this summary of smooth, rational surfaces, we introduce the notion of anticanonical, rational surface and of Fano surface.

Definition 1.4.4. *A smooth, irreducible, rational surface X is said to be anticanonical if its anticanonical divisor $-K_X$ is effective and X is said to be a Fano surface if $-K_X$ is ample.*

Smooth projective rational surfaces X whose anticanonical divisor $-K_X$ is effective constitute an interesting class of surfaces with Kodaira dimension $\kappa(X) \leq 0$. For instance, they include all Del Pezzo surfaces, all blowing up of relatively minimal models of rational surfaces at 8 or fewer points, and all smooth complete toric surfaces, but also include surfaces for which there is an effective but highly non-reduced anticanonical divisor.

The following Theorem gives us the classification, up to isomorphism, of smooth Fano surfaces.

Theorem 1.4.5. *Let X be a smooth Fano surface. Set $d = K_X \cdot K_X$. We have*

(i) $1 \leq d \leq 9$.

(ii)
- *If $d = 9$ then $X \cong \mathbb{P}^2$.*
- *If $d = 8$ then $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ or X is the blow up of $\mathbb{P}^2$ in a point.*
- *If $1 \leq d \leq 7$ then X is the blow up of $\mathbb{P}^2$ in $9 - d$ different points.*

Proof. See [Bea78]. □

Remark 1.4.6. It easily follows from the definition and Theorem 1.4.1 that Fano surfaces and smooth, minimal, rational surfaces are anticanonical and the only minimal Fano surfaces are $\mathbb{P}^2$ and the quadric surface $X \cong \mathbb{P}^1 \times \mathbb{P}^1$. Furthermore, if X is obtained by blowing up a Hirzebruch surface X_e at $s \geq 8$ points, the question whether X is anticanonical strongly depends on the position of the points that we blow up.

Our aim is to study moduli spaces of vector bundles on rational surfaces. According to the methods we will use, we distinguish the following cases:

- Minimal rational surfaces $\begin{cases} X = \mathbb{P}^2. \\ X = X_e \text{ with } e \neq 1. \end{cases}$

- Non-minimal rational surfaces $\begin{cases} \text{Fano surfaces.} \\ \text{Blow up of } X_e \text{ at s points.} \end{cases}$

From now, until the end of the section we will describe the Picard group, intersection product of divisors on rational surfaces and we will prove some technical lemma on divisors that will be very useful for us in Chapters 2, 3 and 4.

Hirzebruch surfaces X_e.

For any integer $e \geq 0$, let $X_e \cong \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-e))$ be a non singular, Hirzebruch surface; i.e., a rational, ruled surface defined by the vector bundle on $\mathbb{P}^1$

$$\mathcal{E} = O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-e).$$

We denote by C_0 and F the standard basis of $Pic(X_e) \cong \mathbb{Z} \oplus \mathbb{Z}$ such that $C_0^2 = -e$. They correspond to sections and π-fibers respectively of the natural projection map $\pi : X_e \to \mathbb{P}^1$. ([Har77]; V, Proposition 2.3). We have $C_0^2 = -e$, $F^2 = 0$, $C_0F = 1$ and the canonical divisor

$$K_{X_e} = -2C_0 - (e+2)F.$$

So $K_{X_e}^2 = 8$ and $-K_{X_e}$ is effective.

Remark 1.4.7. It is well known that a divisor $L = aC_0 + bF$ on X_e is ample if, and only if, is very ample, if and only if, $a > 0$ and $b > ae$, and that $D = a'C_0 + b'F$ is effective if and only if $a' \geq 0$ and $b' \geq 0$ ([Har77]; V, Corollary 2.18).

In the following Lemma we will compute the cohomology groups of line bundles on a smooth Hirzebruch surface X_e.

Lemma 1.4.8. *We consider the line bundle $O_{X_e}(aH+bF)$ on a smooth, Hirzebruch surface X_e and $\pi : X_e = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ the natural projection. We have*

$$H^i(X_e, O_{X_e}(aH+bF)) = \begin{cases} 0 & \text{if } a=-1 \\ H^i(\mathbb{P}^1, S^a(\mathcal{E}) \otimes O_{\mathbb{P}^1}(b)) & \text{if } a \geq 0 \\ H^{2-i}(\mathbb{P}^1, S^{-2-a}(\mathcal{E}) \otimes O_{\mathbb{P}^1}(-e-b-2)) & \text{if } a \leq -2 \end{cases}$$

being $S^a(\mathcal{E})$ the a-th symmetric power of $\mathcal{E}$.

Proof. By the projection formula we have

$$R^i\pi_* O_{X_e}(aH+bF) = R^i\pi_* O_{\mathbb{P}(\mathcal{E})}(a) \otimes O_{\mathbb{P}^1}(b)$$

being $R^1\pi_* O_{\mathbb{P}(\mathcal{E})}(a) = 0$ for $a > -2$. Moreover, using the Base Change Theorem we get $R^i\pi_* O_{\mathbb{P}(\mathcal{E})}(a) = 0$ for $i \geq 2$.

Since, $R^i\pi_* O_{X_e}(aH+bF) = 0$ for $i > 0$ and $a > -2$, by the degeneration of the Leray Spectral sequence

$$H^i(\mathbb{P}^1, R^j\pi_* O_{\mathbb{P}(\mathcal{E})}(aH+bF)) \Rightarrow H^{i+j}(\mathbb{P}(\mathcal{E}), O_{\mathbb{P}(\mathcal{E})}(aH+bF))$$

we obtain

$$H^i(X_e, O_{X_e}(aH+bF)) = H^i(\mathbb{P}^1, \pi_* O_{X_e}(aH+bF)) \quad \text{for all } a > -2$$

with $\pi_* O_{X_e}(aH+bF) = S^a(\mathcal{E}) \otimes O_{\mathbb{P}^1}(b)$ if $a \geq 0$ and 0 otherwise. The case $a \leq -2$ follows from the case $a \geq 0$ and Serre's duality. Hence, the Lemma is proved. □

We also have the following easy Lemma that will be of crucial importance in forthcoming chapters.

Lemma 1.4.9. *Let X_e be a smooth, Hirzebruch surface. Then, for any ample divisor H on X_e and any non-zero effective divisor C on X_e, $CH > 0$. In particular, $(K_{X_e} + F)H < 0$.* □

Despite we know that the blow up of $\mathbb{P}^2$ at s points can be naturally identified with the blow up of $X_{e=1}$ at $s-1$ points, in order to fix the notations that we will use later on, we prefer to describe this two blow ups separately.

Blowing up points in $\mathbb{P}^2$

Let $Z = \{P_1, \cdots, P_s\}$ be a set of s distinct points in $\mathbb{P}^2 = \mathbb{P}^2_k$, let $X := Bl_Z(\mathbb{P}^2)$ be the rational surface obtained from $\mathbb{P}^2$ by blowing up the points of Z and let $\pi : X := Bl_Z(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$ be the blow up. Let $E_1, \cdots, E_s$ be the divisor classes in $Pic(X)$ which contain the exceptional lines corresponding to the blow ups of the points $P_1, \cdots, P_s$, respectively and E_0 the divisor class in $Pic(X)$ which contains the transform of a line in $\mathbb{P}^2$ which misses all the points of Z. Then

$$Pic(X) \cong \mathbb{Z}^{s+1} \cong < E_0, E_1, \cdots, E_s >,$$

with $E_0^2 = 1 = -E_1^2 = \cdots = -E_s^2$ and $E_iE_j = 0$ if $i \neq j$.

Once we have obtained X, there may be other such morphisms $X \longrightarrow \mathbb{P}^2$ and any such morphism factors into a sequence of blowings-up at points giving rise, as before, to a basis of $Pic(X)$. Such a basis, arising from a morphism $X \longrightarrow \mathbb{P}^2$, is called an *exceptional configuration.*

The canonical divisor is given by

$$K_X = -3E_0 + \sum_{i=1}^{s} E_i.$$

Notice that for $s \leq 8$, the anticanonical divisor $-K_X$ is effective and hence X is an anticanonical rational surface. Moreover, by Theorem 1.4.5, X is also a Fano surface.

Remark 1.4.10. On any smooth, rational surface X obtained by blowing up $\mathbb{P}^2$ at $s \geq 1$ different points, there exist ample divisors L on X such that

$$L(K_X + E_0 - E_1) < 0.$$

Indeed, we consider on X the ample divisors (see [Kuc96])

$$L_t = tE_0 - \sum_{i=1}^{s} E_i, \quad t >> 0 \quad \text{and}$$

$$L_m = 3(mE_0 - (m-1)E_1) - \sum_{i=2}^{s} E_i, \quad m >> 0.$$

They verify

$$L_t(K_X + E_0 - E_1) = -2t + s - 1 < 0 \quad \text{and}$$

$$L_m(K_X + E_0 - E_1) = -6m + s - 1 < 0$$

which proves what we want.

As we pointed out in Remark 1.4.2, $\mathbb{P}^2$ with one point blown up is isomorphic to the Hirzebruch surface $X_{e=1}$. Let us describe the corresponding isomorphism between the Picard groups according the notation just introduced.

Remark 1.4.11. Let X be the rational surface obtained by blowing up one point of $\mathbb{P}^2$ and $X_{e=1}$ a Hirzebruch surface. Then, we have the following isomorphism

$$\begin{array}{ccc} Pic(X_{e=1}) & \longrightarrow & Pic(X) \\ C_0 & \longmapsto & E_1 \\ F & \longmapsto & E_0 - E_1 \end{array}$$

between the corresponding Picard groups.

The following Lemma will be very useful to compute the cohomology groups of line bundles on a smooth, rational, surface X obtained blowing up $\mathbb{P}^2$ at $s \geq 1$ points.

Lemma 1.4.12. *Let X be a smooth, irreducible, rational surface obtained blowing up s different points of $\mathbb{P}^2$ and $D = aE_0 - \sum_{i=1}^{s} b_i E_i$ a divisor on X. The following conditions hold* *(a) If $DE_i \leq 0$, then $H^0O_X(D+nE_i) = H^0O_X(D)$ for all $n \geq 1$;*

(b) If $a < 0$, then $H^0O_X(D) = 0$.

Proof. (a) Assume $n \geq 1$ and consider the long exact cohomology sequence

$$0 \longrightarrow H^0O_X(D+(n-1)E_i) \longrightarrow H^0O_X(D+nE_i) \longrightarrow H^0O_{E_i}(D+nE_i) \longrightarrow \cdots$$

associated to the exact sequence

$$0 \longrightarrow O_X(D+(n-1)E_i) \longrightarrow O_X(D+nE_i) \longrightarrow O_{E_i}(D+nE_i) \longrightarrow 0.$$

Since by assumption $DE_i \leq 0$, we have $E_i(D+nE_i) \leq -n < 0$ which implies that $H^0O_{E_i}(D+nE_i) = 0$. Hence,

$$H^0O_X(D+nE_i) = H^0O_X(D+(n-1)E_i)$$

and iterating the process we deduce $H^0O_X(D+nE_i) = H^0O_X(D)$ for all $n \geq 1$.

(b) If $a < 0$, then $DL_t = at - \sum_{i=1}^{s} b_i < 0$ for $t >> 0$, which implies that D is not effective or, equivalently, $H^0O_X(D) = 0$ (recall that for $t >> 0$, $L_t = tE_0 - \sum_{i=1}^{s} E_i$ is an ample divisor on X). □

Blowing up points in Hirzebruch surfaces.

For any $e \geq 0$, we consider $X_e \cong \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-e))$ a non singular Hirzebruch surface. As before, we denote by C_0 and F the standard basis of $Pic(X_e) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Let $Z = \{P_1, \cdots, P_s\}$ be a set of s distinct points in X_e and consider the rational surface $X := Bl_Z(X_e)$ obtained from X_e by blowing up the points of Z. Consider the blow up $\pi : X \longrightarrow X_e$ and $\pi^* : Pic(X_e) \longrightarrow Pic(X)$ the associated map between the Picard groups. Let $E_1, \cdots, E_s$ be the divisor classes in $Pic(X)$ which contain the exceptional lines corresponding to the blow ups of the points $P_1, \cdots, P_s$, respectively. It is well known that the Picard group of X is generated by π^*C_0, π^*F, E_1,...,E_s, with $(\pi^*C_0)^2 = -e$, $\pi^*C_0\pi^*F = 1$, $(\pi^*F)^2 = 0$, $\pi^*C_0E_i = \pi^*FE_i = 0$, for $1 \leq i \leq s$, $1 = -E_1^2 = \cdots = -E_s^2$, and $E_iE_j = 0$ if $i \neq j$.

The canonical divisor has the form

$$K_X = \pi^*K_{X_e} + \sum_{i=1}^{s} E_i = -2\pi^*C_0 - (e+2)\pi^*F + \sum_{i=1}^{s} E_i$$

(See [Har77]; V, Propositions 3.2 and 3.3). For simplicity we will write C_0 and F instead of π^*C_0 and π^*F. Therefore, we have

$$Pic(X) \cong \mathbb{Z}^{s+2} \cong < C_0, F, E_1, \cdots, E_s > \quad \text{and}$$

$$K_X = -2C_0 - (e+2)F + \textstyle\sum_{i=1}^{s} E_i.$$

Remark 1.4.13. Let X be a smooth rational surface obtained by blowing up a Hirzebruch surface X_e at $s \geq 1$ different points. Then there exist ample divisors L on X such that $L(K_X + F) < 0$.

Indeed, let $\tilde{L}_1 = C_0 + (e+s)F$ and $\tilde{L}_2 = C_0 + nF$ with $n >> 0$ be two ample divisors on X_e (Remark 1.4.7). By [Kuc96],

$$L_1 := 3C_0 + 3(e+s)F - \textstyle\sum_{i=1}^{s} E_i \quad \text{and}$$

$$L_2 := 3C_0 + 3nF - \textstyle\sum_{i=1}^{s} E_i$$

are two ample divisors on X and they verify

$$L_1(K_X + F) = 6e - 3(e+1) - 6(e+s) + s < 0 \quad \text{and}$$

$$L_2(K_X + F) = 6e - 3(e+1) - 6n + s < 0$$

which proves what we want.

Now, we are going to give two lemmas on cohomology groups of line bundles that we will use in Chapter 3.

Lemma 1.4.14. *Let X be a smooth, irreducible, rational surface obtained blowing up s different points of a Hirzebruch surface X_e and $D = aC_0 + bF + \sum_{i=1}^{s} b_i E_i$ a divisor on X. The following is satisfied*

(a) If $a < 0$ or $b < 0$, then $H^0 O_X(D) = 0$;

(b) If $DE_i \leq 0$, then $H^0\mathcal{O}_X(D+\alpha E_i) = H^0\mathcal{O}_X(D)$ for all $\alpha \geq 0$.

Proof. (a) Assume that $a < 0$. By [Kuc96], the divisor $L = 3C_0 + 3tF - \sum_{i=1}^{s} E_i$ with $t >> 0$ on X is ample. We have

$$DL = -3ae + 3b + 3at + \sum_{i=1}^{s} b_i < 0.$$

So D is not effective or, equivalently, $H^0\mathcal{O}_X(D) = 0$. If $b < 0$, we consider the ample divisor $H = 3nC_0 + 3(ne+1)F - \sum_{i=1}^{s} E_i$ with $n >> 0$ on X (see [Kuc96]) and we get

$$DH = 3nb + 3a + \sum_{i=1}^{s} b_i < 0.$$

Hence, D is not effective or, equivalently, $H^0\mathcal{O}_X(D) = 0$ which proves (a).

(b) For all $\alpha > 0$ we have $E_i(D+\alpha E_i) < 0$. Therefore, for all $\alpha > 0$ we obtain $H^0\mathcal{O}_{E_i}(D+\alpha E_i) = 0$. Thus, from the exact cohomology sequence

$$0 \longrightarrow H^0\mathcal{O}_X(D+(\alpha-1)E_i) \longrightarrow H^0\mathcal{O}_X(D+\alpha E_i) \longrightarrow H^0\mathcal{O}_{E_i}(D+\alpha E_i) \longrightarrow \cdots$$

associated to the exact sequence

$$0 \longrightarrow \mathcal{O}_X(D+(\alpha-1)E_i) \longrightarrow \mathcal{O}_X(D+\alpha E_i) \longrightarrow \mathcal{O}_{E_i}(D+\alpha E_i) \longrightarrow 0$$

we deduce that $H^0\mathcal{O}_X(D+(\alpha-1)E_i) = H^0\mathcal{O}_X(D+\alpha E_i)$ and iterating the process we deduce that $H^0\mathcal{O}_X(D+\alpha E_i) = H^0\mathcal{O}_X(D)$, which proves what was stated. □

Lemma 1.4.15. *Let X be a smooth, irreducible, rational surface obtained blowing up one point p of a Hirzebruch surface X_e and $D = aC_0 + bF - 2E$ with $a, b \geq 0$ a divisor on X. If $a > 0$ or $b > 0$ then $h^0\mathcal{O}_X(D) \leq max\{0, h^0\mathcal{O}_X(aC_0+bF) - 3\}$.*

Proof. Since $a > 0$ or $b > 0$ we can think of D as the strict transformation of a curve $aC_0 + bF$ on X_e which has p as a double point. Hence, the dimension of $H^0\mathcal{O}_X(D)$ coincides with the dimension of the linear system of curves $aC_0 + bF$ on X_e which has a node at p. Therefore, $h^0\mathcal{O}_X(D) \leq max\{0, h^0\mathcal{O}_X(aC_0+bF) - 3\}$, which proves what we want. □.

We will end this Chapter with the following easy but very useful Lemma.

Lemma 1.4.16. *Let X be a smooth, projective surface and $Z \subset X$ a generic 0-cycle in the Hilbert scheme $Hilb^l(X)$. If $l \geq h^0 O_X(D)$, then $H^0 I_Z(D) = 0$.*

Proof. It easily follows after a straightforward computation. □

Chapter 2

Criterion of rationality for moduli spaces on surfaces

The aim of this chapter is to supply criterion of rationality for moduli spaces $M_L(2;c_1,c_2)$ of L-stable, rank 2, vector bundles over a smooth, irreducible, rational surface X, which will allow us to give, in the forthcoming chapters, an affirmative answer to the following question (see [Sch90]; Problem 21, [Sch85]; Problem 2, [OV88]; Problem 2):

QUESTION: Let X be a smooth, rational, projective surface. Fix a polarization L, $c_1 \in Pic(X)$ and $0 << c_2 \in \mathbb{Z}$. Is $M_L(2;c_1,c_2)$ rational?

As a main tool we use the birational properties of the moduli spaces $M_L(2;c_1,c_2)$ of rank 2, L-stable vector bundles on algebraic surfaces. In [Qin91], [Qin91b] and [Qin93], Qin studies the change of $M_L(2;c_1,c_2)$ when L varies. It turns out that the ample cone of X has a chamber structure such that $M_L(2;c_1,c_2)$ only depends on the chamber of L and, in general, $M_L(2;c_1,c_2)$ changes when L passes through the wall between two chambers (see [Qin93] and [Qin91]).

We say that an irreducible component M of a moduli space $M_L(2;c_1,c_2)$ is trivial if for any polarization H, there exists a sheaf in M which is also H-stable. A polarization L is trivial of type (c_1,c_2) if every irreducible component of the moduli space $M_L(2;c_1,c_2)$ is trivial. In [Qin91], Qin states the following conjecture:

CONJECTURE ([Qin91]): Trivial polarizations of type $(c_1,c_2) \in Pic(X) \times \mathbb{Z}$

exist when $4c_2 - c_1^2$ is larger than some constant $c = c(X)$ depending on X.

The first goal of this chapter is to prove Qin's conjecture for smooth, projective, anticanonical, rational surfaces, i.e. smooth, irreducible, rational surfaces X whose anticanonical divisor $-K_X$ is effective. To be more precise, we prove that if X is an anticanonical rational surface then any polarization L is trivial of type (c_1, c_2) provided $M_L(2; c_1, c_2)$ is non-empty and $4c_2 - c_1^2 > 2 - 3K_X^2/2$; and for any two ample divisors L_1 and L_2 the moduli spaces $M_{L_1}(2; c_1, c_2)$ and $M_{L_2}(2; c_1, c_2)$ are birational whenever non-empty and $4c_2 - c_1^2 > 2 - 3K_X^2/2$ (Theorem 2.1.10). Therefore, for many purposes we can fix the polarization L and this is what we will always do when we want to study the rationality of the moduli space $M_L(2; c_1, c_2)$.

The second objective of this chapter is to give criterion of rationality for moduli spaces $M_L(2; c_1, c_2)$ of rank two, L-stable, vector bundles over smooth, irreducible, rational surfaces.

The techniques we use, strongly depend on whether $-K_X$ is effective or not. According to this fact, we distinguish two cases: in section 1, X is an anticanonical rational surface and in section 3, X is a non-anticanonical rational surface.

We start section 1, computing the invariant $d(\xi)$ (see Definition 1.3.6 and Corollary 2.1.4) and proving Theorem 2.1.10, which fully solves Qin's conjecture for smooth, projective, anticanonical, rational surfaces X. Moreover, we give explicitly the constant $c = c(X)$ which only depends on X. As an application, we give sufficient conditions on $c_1 \in Pic(X)$ and $c_2 \in \mathbb{Z}$ in order to assure, for any polarization L, the rationality of the moduli space $M_L(2; c_1, c_2)$ of rank two, L-stable, vector bundles with Chern classes (c_1, c_2) on a smooth, projective, anticanonical, rational surface X, i.e. the first criterion of rationality (Criterion 2.1.13). In section 2, by means of an example, we will see that if L_1 and L_2 are two polarizations on a smooth, rational, anticanonical surface, lying in different chambers, then the birational map between $M_{L_1}(2; c_1, c_2)$ and $M_{L_2}(2; c_1, c_2)$ is not, in general, an isomorphism. In section 3 we will turn our attention to the question whether two moduli spaces $M_{L_1}(2; c_1, c_2)$ and $M_{L_2}(2; c_1, c_2)$ are birational when the underlying variety X is a smooth, irreducible, non-anticanonical, rational surface (Theorem 2.3.6). As an application we will give the second criterion of rationality (Criterion 2.3.7). The proof of the results of section 3 relies on Walter's results on the stack of prioritary sheaves.

The results of section 1 of this chapter will appear in [CM98].

2.1 Anticanonical rational surfaces

Throughout this section X will be a smooth, irreducible, projective, anticanonical, rational surface, i.e. a smooth, irreducible, rational surface whose anticanonical divisor $-K_X$ is effective. In particular, $p_a(X) = p_g(X) = q(X) = 0$. From now on, we fix $c_1 \in Pic(X)$ and we assume that the numerical equivalence class ξ on X determines a non-empty wall of type (c_1, c_2) and G is any divisor such that $\xi \equiv 2G - c_1$ (see Definition 1.3.3). The first goal of this section is to compute $D(\xi)$ and $d(\xi)$, i.e. the dimension of $E_\xi(c_1, c_2)$ and the difference between the dimension of $E_\xi(c_1, c_2)$ and the expected dimension of a non-empty moduli space $M_L(2; c_1, c_2)$ (see Definition 1.3.6).

Lemma 2.1.1. *With the above notation it holds*

(1) $H^0\mathcal{O}_X(2G - c_1) = 0$ *and* $H^0\mathcal{O}_X(-2G + c_1) = 0$.

(2) $H^0\mathcal{O}_X(K_X - (2G - c_1)) = 0$ *and* $H^0\mathcal{O}_X(K_X + (2G - c_1)) = 0$.

Proof. (1) Since ξ defines a non-empty wall we have $\xi L_1 > 0 > \xi L_2$ for some ample divisors L_1 and L_2. Thus $2G - c_1$ and $c_1 - 2G$ are not effective divisors and we get $H^0\mathcal{O}_X(2G - c_1) = H^0\mathcal{O}_X(-2G + c_1) = 0$.

(2) Since X is a smooth, irreducible, projective, anticanonical surface, the divisor $-K_X$ is effective and $-K_X L \geq 0$ for any ample divisor L. If $K_X + 2G - c_1$ or $K_X - 2G + c_1$ are effective, then $(K_X + 2G - c_1)L \geq 0$ or $(K_X - 2G + c_1)L \geq 0$ for any ample divisor L; i.e., $(2G - c_1)L \geq -K_X L$ or $(-2G + c_1)L \geq -K_X L$ for any ample divisor L. However, the inequalities $0 > (2G - c_1)L_2$ and $0 > (-2G + c_1)L_1$ give us $-K_X L_1 < 0$ and $-K_X L_2 < 0$ which contradicts the fact that $-K_X$ is effective. □

Remark 2.1.2. Assume that ξ determines a non-empty wall of type (c_1, c_2). Then,

(1) $-h^1O_X(-\xi) = \chi(O_X(-\xi)) = \xi(\xi + K_X)/2 + 1$ and

(2) $-h^1O_X(\xi) = \chi(O_X(\xi)) = \xi(\xi - K_X)/2 + 1.$

Proof. It easily follows from Lemma 2.1.1 and the Riemann-Roch's Theorem (1.1.5). In particular, we have

(1) $\xi(\xi + K_X)/2 + 1 \leq 0$ and

(2) $\xi(\xi - K_X)/2 + 1 \leq 0.$ □

Lemma 2.1.3. *Let $Z \subset X$ be a locally complete intersection 0-cycle of length $l(Z) = c_2 + (\xi^2 - c_1^2)/4$. Then, we have*

$$dimExt^1(I_Z, O_X(2G - c_1)) = \frac{4c_2 - c_1^2 - \xi^2}{4} + \frac{\xi K_X}{2} - 1.$$

Proof. We apply the functor $Hom(., O_X(2G - c_1))$ to the exact sequence

$$0 \longrightarrow I_Z \longrightarrow O_X \longrightarrow O_Z \longrightarrow 0$$

and we get the exact sequence

$$0 \longrightarrow H^1O_X(2G - c_1) \longrightarrow Ext^1(I_Z, O_X(2G - c_1)) \longrightarrow H^0O_Z$$

$$\longrightarrow H^2O_X(2G - c_1) \longrightarrow Ext^2(I_Z, O_X(2G - c_1)) \longrightarrow 0.$$

Using Lemma 2.1.1, Serre's duality and Riemann-Roch's Theorem, we obtain $H^2O_X(2G - c_1) = 0$ and

$$h^1O_X(2G - c_1) = -\chi(O_X(2G - c_1)) = -1 - \frac{\xi^2 - \xi K_X}{2}.$$

Therefore, we have

$$\begin{aligned} dimExt^1(I_Z, O_X(2G - c_1)) &= h^0O_Z + h^1O_X(2G - c_1) \\ &= \tfrac{4c_2 - c_1^2 - \xi^2}{4} + \tfrac{\xi K_X}{2} - 1. \end{aligned}$$ □

We are now in position to calculate $d(\xi)$.

Corollary 2.1.4. *With the above notation, we have*

(1) $D(\xi) = 3(4c_2 - c_1^2)/4 + \xi^2/4 + (\xi K_X)/2 - 2$

(2) $d(\xi) = (c_1^2 - 4c_2)/4 + \xi^2/4 + (\xi K_X)/2 + 1.$

Proof. (1) Any vector bundle E in $E_\xi(c_1, c_2)$ sits in a non-trivial extension

$$0 \longrightarrow O_X(G) \longrightarrow E \longrightarrow O_X(c_1 - G) \otimes I_Z \longrightarrow 0$$

where G is a divisor with $2G - c_1 \equiv \xi$ and $Z \subset X$ is a locally complete intersection 0-cycle of length $c_2 + (\xi^2 - c_1^2)/4$ (Definition 1.3.6). Moreover, the invertible sheaf $O_X(G)$ and the 0-cycle Z are uniquely determined by E. Therefore,

$$\begin{aligned} dimE_\xi(c_1, c_2) &= \#moduliO_X(G) + \#moduli(Z) \\ &\quad + dimExt^1(I_Z, O_X(2G - c_1)) - h^0E(-G). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &h^0E(-G) = h^0O_X = 1, \\ &\#moduliO_X(G) = q(X) = 0, \\ &\#moduli(Z) = 2length(Z) = \tfrac{4c_2+\xi^2-c_1^2}{2}, \\ &dimExt^1(I_Z, O_X(2G - c_1)) = \tfrac{4c_2-c_1^2-\xi^2}{4} + \tfrac{\xi K_X}{2} - 1 \end{aligned}$$

where the last equality follows from Lemma 2.1.3. Thus, we obtain

$$D(\xi) = 3\frac{4c_2 - c_1^2}{4} + \frac{\xi^2}{4} + \frac{\xi K_X}{2} - 2.$$

(2) By definition we have

$$\begin{aligned} d(\xi) := d_\xi(c_1, c_2) &= D(\xi) - (4c_2 - c_1^2 - 3\chi(O_X)) \\ &= \tfrac{c_1^2-4c_2}{4} + \tfrac{\xi^2}{4} + \tfrac{\xi K_X}{2} + 1. \end{aligned}$$

□

Remark 2.1.5. Notice that for any numerical equivalence class ξ which defines a non-empty wall of type (c_1, c_2) we have $d(\xi) \leq 0$. In fact, it follows from Remark 2.1.2 and the inequality $c_1^2 - 4c_2 \leq \xi^2$. $\square$

Now, we will give a technical result to be used later on.

Proposition 2.1.6. *Let X be a smooth, projective, anticanonical, rational surface, L a polarization and ξ a numerical equivalence class defining a wall of type (c_1, c_2). Assume $d(\xi) = 0$. It holds*

(a) If $E \in M_L(2; c_1, c_2)$ then $\chi(E(-\frac{c_1+\xi}{2})) = 1$.

(b) If $E \in M_L(2; c_1, c_2)$ and $4c_2 - c_1^2 > 2 - \frac{3K_X^2}{2}$ then $h^0E(-\frac{c_1+\xi}{2}) > 0$.

Remark 2.1.7. We point out that $E(-\frac{c_1+\xi}{2})$ has sense because $\xi + c_1$ is divisible by 2 in $Pic(X)$ (see Definition 1.3.3).

Proof. First of all, notice that, by Remark 2.1.5, the hypothesis $d(\xi) = 0$ is equivalent to $\xi^2 = c_1^2 - 4c_2$ and $\xi^2 + \xi K_X + 2 = 0$.

(a) Applying 1.1.1 and 1.1.5 we easily see that

$$c_1(E(-\frac{c_1+\xi}{2})) = -\xi, \quad c_2(E(-\frac{c_1+\xi}{2})) = 0, \quad \text{and} \quad \chi(E(-\frac{c_1+\xi}{2})) = 1.$$

(b) First we prove that the divisor $-(2K_X + \xi)$ is effective. Indeed, since ξ is not effective and $-K_X$ is effective we have $h^0O_X(\xi + 3K_X) = 0$ and by Serre's duality $h^2O_X(-\xi - 2K_X) = 0$. Therefore, applying Riemann-Roch's Theorem we get

$$\begin{aligned} h^0O_X(-\xi - 2K_X) - h^1O_X(-\xi - 2K_X) &= \chi(O_X(-\xi - 2K_X)) \\ &= \tfrac{(-\xi-2K_X)(-\xi-3K_X)}{2} + 1 \\ &= 2(4c_2 - c_1^2 - 2) + 3K_X^2 > 0 \end{aligned}$$

which gives us $h^0 O_X(-\xi - 2K_X) > 0$ or, equivalently, $-(\xi + 2K_X)$ is effective. Hence, $-(2K_X + \xi)L \geq 0$ for any ample divisor L on X or, equivalently,

$$c_1((E(-\frac{c_1+\xi}{2}))^* \otimes K_X)L = (2K_X + \xi)L \leq 0.$$

If the last inequality is strict we obtain (Fact 1.1.10)

$$h^0 E(-\frac{c_1+\xi}{2}) > 0.$$

If $c_1((E(-\frac{c_1+\xi}{2}))^* \otimes K_X)L = 0$ we get

$$h^0 E(-\frac{c_1+\xi}{2}) > 0 \quad \textit{or} \quad h^2 E(-\frac{c_1+\xi}{2}) > 0$$

and we will prove that the last inequality is not possible. Indeed, by Serre duality,

$$0 < h^2 E(-\frac{c_1+\xi}{2}) = h^0 E^*(\frac{c_1+\xi}{2} + K_X).$$

A non-zero section $\sigma \in H^0 E^*(\frac{c_1+\xi}{2} + K_X)$ defines an injection

$$O_X(-\frac{c_1+\xi}{2} - K_X) \hookrightarrow E^* \cong E(-c_1),$$

or, equivalently,

$$O_X(\frac{c_1-\xi}{2} - K_X) \hookrightarrow E.$$

From the L-stability of E we have

$$(\frac{c_1-\xi}{2} - K_X)L < \frac{c_1 L}{2}$$

which contradicts the fact that $(2K_X + \xi)L = 0$. □

Remark 2.1.8. Notice that if the moduli space $M_L(2; c_1, c_2)$ is non-empty, by Bogomolov's inequality, $c_1^2 - 4c_2 < 0$, the condition $4c_2 - c_1^2 > 2 - 3K_X^2/2$ is automatically satisfied whenever the underlying surface is a Hirzebruch surface or a Fano surface.

The following Proposition will be the key point for proving the first main result of this section.

Proposition 2.1.9. *Let X be a smooth, projective, anticanonical, rational surface, L a polarization and ξ a numerical class defining a wall of type (c_1, c_2). Assume $d(\xi) = 0$. If $\xi L \geq 0$ and $4c_2 - c_1^2 > 2 - 3K_X^2/2$ then $M_L(2; c_1, c_2) = \emptyset$.*

Proof. Assume $M_L(2; c_1, c_2) \neq \emptyset$. For any $E \in M_L(2; c_1, c_2)$ we apply Proposition 2.1.6 and we take a nonzero section $s \in H^0E(-\frac{c_1+\xi}{2}) \neq 0$. It defines an injection

$$O_X(\frac{c_1 + \xi}{2}) \hookrightarrow E.$$

Since E is L-stable, we have

$$(\frac{c_1 + \xi}{2})L < \frac{c_1 L}{2}$$

i.e.; $\xi L < 0$, which contradicts the hypothesis $\xi L \geq 0$. □

In the next Theorem, we prove that for any two ample divisors L_1 and L_2 on a smooth, irreducible, projective, anticanonical, rational surface X, the moduli spaces $M_{L_1}(2; c_1, c_2)$ and $M_{L_2}(2; c_1, c_2)$ with $4c_2 - c_1^2 > 2 - 3K_X^2/2$ are birational whenever non-empty.

Theorem 2.1.10. *Let X be a smooth, projective, anticanonical, rational surface, $c_1 \in Pic(X)$ and $c_2 \in \mathbb{Z}$. Assume $4c_2 - c_1^2 > 2 - \frac{3K_X^2}{2}$. We have*

(a) Any polarization L is trivial of type (c_1, c_2) provided $M_L(2; c_1, c_2)$ is non-empty.

(b) For any two ample divisors L_1 and L_2 on X the moduli spaces $M_{L_1}(2; c_1, c_2)$ and $M_{L_2}(2; c_1, c_2)$ are birational whenever non-empty.

Proof. By Remark 1.3.4 we may assume that L_1 and L_2 lie in chambers. Let $\mathcal{C}_1$ be the chamber containing L_1 and $\mathcal{C}_2$ the chamber containing L_2. If $\mathcal{C}_1 = \mathcal{C}_2$, then the moduli spaces can be naturally identified (see [Qin93]; Proposition 2.2.2). Assume $\mathcal{C}_1 \neq \mathcal{C}_2$. Since the set of walls of type (c_1, c_2) is locally finite (Proposition 1.3.5), we can choose finitely many ample divisors

$$L_1 = L^{(1)}, L^{(2)}, ..., L^{(r-1)}, L^{(r)} = L_2$$

on the line segment connecting L_1 and L_2 in such a way that we have

(1) $L^{(i)}$ lies in some chamber for all $i = 1, ..., r$ and

(2) $L^{(i)}$ and $L^{(i+1)}$ are separated by a single wall for all $i = 1, ..., r-1$.

So, without loss of generality, we may suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ share a common wall W of type (c_1, c_2). Take $\xi \in Num(X)$ such that $W^\xi = W$. Since $d(\xi) = 0$ and $L\xi \geq 0$ implies $M_L(2; c_1, c_2) = \emptyset$ (Proposition 2.1.9) and the moduli spaces $M_{L_1}(2; c_1, c_2)$ and $M_{L_2}(2; c_1, c_2)$ are non-empty we deduce $d(\xi) \neq 0$ and, hence, $d(\xi) < 0$ (Remark 2.1.5). Therefore, we have ([Qin93]; Theorem 1.3.3)

$$M_{L_1}(c_1, c_2) = (M_{L_2}(c_1, c_2) \setminus \sqcup_\eta E_{-\eta}(c_1, c_2)) \sqcup (\sqcup_\eta E_\eta(c_1, c_2))$$

where η satisfies $\eta L < 0$ for some $L \in \mathcal{C}_1$ and runs over all numerical equivalence classes which define the common wall $W = W^\xi$. Moreover, $d(\eta) < 0$ (Remark 2.1.5 and Proposition 2.1.9) and we conclude that $M_{L_1}(2; c_1, c_2)$ and $M_{L_2}(2; c_1, c_2)$ are birationally equivalent. □

As an application we have

Corollary 2.1.11. *Let X be a smooth Fano surface, $c_1 \in Pic(X)$ and $c_2 \in \mathbb{Z}$. We have*

(a) Any polarization L is trivial of type (c_1, c_2) provided $M_L(2; c_1, c_2)$ is non-empty.

(b) For any two ample divisors L_1 and L_2 on X the moduli spaces $M_{L_1}(2; c_1, c_2)$ and $M_{L_2}(2; c_1, c_2)$ are birational whenever non-empty.

Proof. Any smooth Fano surface X is rational and anticanonical (see Remark 1.4.6). So, the result follows from Theorem 2.1.10 because if the moduli space $M_L(2; c_1, c_2)$ is non-empty then by Bogomolov's inequality we have $4c_2 - c_1^2 > 0 > 2 - \frac{3K_X^2}{2}$. □

Remark 2.1.12. We will see in the next section that, in general, the birational map between $M_{L_1}(2; c_1, c_2)$ and $M_{L_2}(2; c_1, c_2)$ is not an isomorphism.

We are going to finish this section proving a very useful criterion of rationality for moduli spaces $M_L(2; c_1, c_2)$ of rank two, L-stable, vector bundles over a smooth, irreducible, projective, anticanonical, rational surfaces.

First criterion of rationality 2.1.13. *Let X be a smooth, projective, anticanonical, rational surface, $c_1 \in Pic(X)$ and $c_2 \in \mathbb{Z}$. Assume $4c_2 - c_1^2 > 2 - \frac{3K_X^2}{2}$ and that there exists a numerical equivalence class ξ which defines a non-empty wall of type (c_1, c_2) such that $d(\xi) = 0$ (i.e. $\xi^2 = c_1^2 - 4c_2$ and $\xi^2 + \xi K_X + 2 = 0$). Then, the following holds*

(1) There exists an ample divisor $\tilde{L}$ on X such that the moduli space $M_{\tilde{L}}(2; c_1, c_2)$ is a smooth, irreducible, rational projective variety of dimension $4c_2 - c_1^2 - 3$ and $Pic(M_{\tilde{L}}(2; c_1, c_2)) \cong \mathbb{Z}$ whenever non-empty.

(2) For any ample divisor L on X, the moduli space $M_L(2; c_1, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - c_1^2 - 3$, whenever non-empty.

Proof. By Theorem 2.1.10 (b), it is enough to see that if $\tilde{L}$ is an ample divisor such that $\xi\tilde{L} < 0$ and $\tilde{L} \in \mathcal{C}$ with $W^{\xi} \cap \overline{\mathcal{C}} \neq \emptyset$ then $M_{\tilde{L}}(2; c_1, c_2) \cong \mathbb{P}^{4c_2 - c_1^2 - 3}$. For such $\tilde{L}$ and $\mathcal{C}$ we have ([Qin93]; Proposition 1.3.1)

$$M_{\tilde{L}}(2; c_1, c_2) = M_{\mathcal{F}}(2; c_1, c_2) \sqcup (\sqcup_{\mu} E_{\mu}(c_1, c_2))$$

where $\mathcal{F}$ is the face of $\mathcal{C}$ contained in W^{ξ}, $\mu L < 0$ for some $L \in \mathcal{C}$ and μ runs over all numerical equivalence classes which define the wall W^{ξ}. For any $L' \in \mathcal{F}$, $L'\xi = 0$. So, since $d(\xi) = 0$, $M_{\mathcal{F}}(2; c_1, c_2) = \emptyset$ (Proposition 2.1.9). Moreover, $W^{\mu} = W^{\eta}$ if, and only if, $\mu = \lambda\eta$, for some $\lambda \in \mathbb{R}$. Therefore, we conclude

$$M_{\tilde{L}}(2; c_1, c_2) \cong E_{\xi}(c_1, c_2).$$

Let us see that $E_{\xi}(c_1, c_2) \cong \mathbb{P}^{4c_2 - c_1^2 - 3}$. By definition, for any E in $E_{\xi}(c_1, c_2)$, we have the exact sequence (see Definition 1.3.6)

$$0 \longrightarrow O_X(G) \longrightarrow E \longrightarrow O_X(c_1 - G) \otimes I_Z \longrightarrow 0$$

where G is a divisor with $2G - c_1 \equiv \xi$ and Z is a locally complete intersection 0-cycle with $l(Z) = c_2 + \frac{\xi^2 - c_1^2}{4}$. By hypothesis $d(\xi) = 0$. Thus, $\xi^2 = c_1^2 - 4c_2$ and $Z = \emptyset$. Therefore, E is given by a non-trivial extension

$$0 \longrightarrow O_X(G) \longrightarrow E \longrightarrow O_X(c_1 - G) \longrightarrow 0$$

where $G \equiv \frac{\xi + c_1}{2}$ i.e.; $E \in \mathbb{P}(H^1 O_X(\xi))$. Finally, using Riemann-Roch's Theorem, we get $\mathbb{P}(H^1 O_X(\xi)) \cong \mathbb{P}^{4c_2 - c_1^2 - 3}$. $\square$

Remark 2.1.14. We want to stress that the ample divisors $\tilde{L}$ on X such that $M_{\tilde{L}}(2; c_1, c_2)$ is a projective variety strongly depends on c_2 (Criterion 2.1.13). Indeed, in [OGr96]; Theorem A, O'Grady proves that if we fix an ample divisor L on X and $c_2 \geq n(X, L)$ being $n(X, L)$ a numerical function which depends on X and L, then the dimension of complete subvarieties of $M_L(2; c_1, c_2)$ is strictly less than the expected dimension $4c_2 - c_1^2 - 3$.

Remark 2.1.15. The results that we have proved in this section also work for smooth, irreducible, projective surfaces with anticanonical divisor $-K_X$ numerically effective (a divisor is said to be numerically effective if its intersection number with any effective divisor is nonnegative) and arithmetic genus $p_a = 0$. See [CP92] for a complete classification of smooth projective surfaces with anticanonical divisor numerically effective.

2.2 Examples

The aim of this section is to illustrate by means of a family of examples that if L_1 and L_2 are two polarization on a smooth, irreducible, projective, anticanonical, rational surface lying in different chambers, then the birational map between $M_{L_1}(2; c_1, c_2)$ and $M_{L_2}(2; c_1, c_2)$ is not, in general, an isomorphism. More precisely, in the following example we will see that two moduli spaces $M_{L_1}(2; c_1, c_2)$ and $M_{L_2}(2; c_1, c_2)$ are birational but, $Pic(M_{L_1}(2; c_1, c_2)) \neq Pic(M_{L_2}(2; c_1, c_2))$, which in particular proves that $M_{L_1}(2; c_1, c_2)$ and $M_{L_2}(2; c_1, c_2)$ are not isomorphic. To this end, we start recalling some basic facts on Picard groups.

Proposition 2.2.1. *Let Z be an irreducible subset in a smooth variety Y. It holds*

a) If Z has codimension greater or equal than 2 then,

$$Pic(Y) = Pic(Y \setminus Z).$$

b) If Z has codimension 1, then there exists an exact sequence

$$0 \longrightarrow G_Z \longrightarrow Pic(Y) \longrightarrow Pic(Y \setminus Z) \longrightarrow 0$$

where G_Z is the cyclic subgroup of $Pic(Y)$ generated by $O_Y(Z)$.

Proof. See [Har77]; II, Proposition 6.5. □

Let us start with the precise family of examples.

Let X_e be a smooth, irreducible Hirzebruch surface with $e \geq 2$, $1 < c_2$ an integer, $c_1 = C_0 \in Pic(X_e)$ and $\xi \in Num(X_e)$ defining a non-empty wall of type (C_0, c_2). It follows from Remark 2.1.2, Corollary 2.1.4 and Remark 2.1.5 that $d(\xi) \leq 0$ and

i) $d(\xi) = 0$ if, and only if, $\xi^2 = c_1^2 - 4c_2$ and $\xi^2 + \xi K_{X_e} + 2 = 0$;

ii) $d(\xi) = -1$ if, and only if, $\xi^2 = c_1^2 - 4c_2 + 4$ and $\xi^2 + \xi K_{X_e} + 2 = 0$ or, $\xi^2 = c_1^2 - 4c_2$ and $\xi^2 + \xi K_{X_e} + 4 = 0$.

In the two following Lemmas we will determine all possible numerical equivalence classes ξ, defining a non-empty wall of type (C_0, c_2) with $d(\xi) = 0$ (Lemma 2.2.2) or $d(\xi) = -1$ (Lemma 2.2.3).

Lemma 2.2.2. *With the above notations we consider the numerical equivalence class $\xi_0 = C_0 - 2c_2F$. ξ_0 defines a non-empty wall of type (C_0, c_2) and $d(\xi_0) = 0$. Moreover, ξ_0 is the only numerical equivalence class defining a non-empty wall of type (C_0, c_2) such that $d(\xi_0) = 0$.*

Proof. Since $\xi_0 + c_1 = 2C_0 - 2c_2F$, $d(\xi_0) = 0$ and $c_1^2 - 4c_2 = {\xi_0}^2 < 0$, we only have to check (see Definition 1.3.3 and Remark 1.3.7) that there exist ample divisors L and L' on X_e such that

$$\xi_0 L < 0 < \xi_0 L'.$$

We take the ample divisors $L = C_0 + (e+1)F$ and $L' = C_0 + (e + 2c_2 + 1)F$ on X_e (see Remark 1.4.7). We have

$$L\xi_0 = -e - 2c_2 + (e+1) = -2c_2 + 1 < 0 \quad \text{and}$$

$$L'\xi_0 = -e - 2c_2 + e + 2c_2 + 1 = 1 > 0$$

which proves that ξ_0 defines a non-empty wall of type (C_0, c_2).

Assume that there exists a numerical equivalence class, say $\xi = aC_0 + bF$, defining a non-empty wall of type (C_0, c_2) such that $d(\xi) = 0$. Since $d(\xi_0) = d(\xi) = 0$ we have

$$\xi^2 = c_1^2 - 4c_2 = \xi_0^2 \text{ and } \xi^2 + \xi K_{X_e} + 2 = \xi_0^2 + \xi_0 K_{X_e} + 2 = 0$$

which implies that $\xi K_{X_e} = \xi_0 K_{X_e}$, or equivalently, $2b = ae - 2a - e + 2 - 4c_2$, which together with the equality

$$-ae^2 + 2ab = \xi^2 = c_1^2 - 4c_2 = -e - 4c_2$$

gives us

$$(1-a)(2a + e + 4c_2) = 0.$$

Hence, $a = 1$ or $2a + e + 4c_2 = 0$. If $2a = -e - 4c_2$, then $b = -\frac{e^2}{4} - ec_2 + 1$ and ξ does not define a non-empty wall of type (C_0, c_2) because $\xi L < 0$ for any ample divisor L on X_e. Indeed, since $a < 0$ and $b < 0$, for any ample divisor $L = \alpha C_0 + \beta F$ on X_e we get

$$\xi L = (aC_0 + bF)(\alpha C_0 + \beta F) = -a(\alpha e - \beta) + \alpha b < 0$$

where the last inequality follows from the fact that since L is ample we have $\beta > \alpha e$ (see Remark 1.4.7). If $a = 1$, then $b = -2c_2$ and $\xi = \xi_0$. Therefore, ξ_0 is the only numerical equivalence class ξ defining a non-empty wall of type (C_0, c_2) such that $d(\xi) = 0$. $\square$

Lemma 2.2.3. *With the above notations we consider the numerical equivalence class* $\xi_1 = C_0 - 2(c_2 - 1)F$. ξ_1 *defines a non-empty wall of type* (C_0, c_2) *and* $d(\xi_1) = -1$. *Moreover,* ξ_1 *is the only numerical equivalence class defining a non-empty wall of type* (C_0, c_2) *such that* $d(\xi_1) = -1$.

Proof. Since $\xi_1+c_1 = 2C_0-2(c_2-1)F$, $c_1^2-4c_2 = {\xi_1}^2-4 < 0$ and $\xi_1^2+\xi_1 K_{X_e}+2 = 0$, we have $d(\xi_1) = -1$. Let us see that there exist ample divisors L and L' on X_e such that

$$\xi_1 L < 0 < \xi_1 L'.$$

We take the ample divisors $L = C_0 + (e+1)F$ and $L' = C_0 + (e+2c_2+1)F$ on X_e (Remark 1.4.7). We have

$$L\xi_1 = -e - 2(c_2-1) + (e+1) = -2c_2 + 3 < 0 \quad \text{and}$$

$$L'\xi_1 = -e - 2(c_2-1) + e + 2c_2 + 1 = 3 > 0$$

which proves that ξ_1 defines a non-empty wall of type (C_0, c_2).

Assume that there exists a numerical equivalence class, say $\xi = aC_0+bF$, defining a non-empty wall of type (C_0, c_2) such that $d(\xi) = -1$. Since $d(\xi) = -1$, we have two possibilities

(a) $\xi^2 = c_1^2 - 4c_2$ and $\xi^2 + \xi K_{X_e} + 4 = 0$ or

(b) $\xi^2 = c_1^2 - 4c_2 + 4$ and $\xi^2 + \xi K_{X_e} + 2 = 0$.

(a) If $\xi^2 = c_1^2 - 4c_2$ and $\xi^2 + \xi K_{X_e} + 4 = 0$, we have

$$-e - 4c_2 + 4 = c_1^2 - 4c_2 + 4 = -\xi K_{X_e} = 2a - ae + 2b.$$

The equality

$$-ea^2 + 2ab = \xi^2 = c_1^2 - 4c_2 = -e - 4c_2$$

together with $2b = ae - 2a - e - 4c_2 + 4$, gives us

$$(e + 4c_2)(1-a) - 2a(a-2) = 0$$

which implies that $a \leq -1$ and $b = \frac{a(e-2)}{2} - \frac{e}{2} - 2c_2 + 2 < 0$. Arguing as in Lemma 2.2.2 we see that ξ does not define a non-empty wall of type (C_0, c_2).

(b) If $\xi^2 = c_1^2 - 4c_2 + 4$ and $\xi^2 + \xi K_{X_e} + 2 = 0$ we have $-\xi K_{X_e} = -e - 4c_2 + 6$, i.e.

$$2b = ae - 2a - e - 4c_2 + 6.$$

This equality together with $\xi^2 = c_1^2 - 4c_2 + 4$ gives us

$$(1 - a)(4 - 4c_2 - e - 2a) = 0.$$

Therefore, $a = 1$ or $4 - 4c_2 - e - 2a = 0$. If $2a = 4 - 4c_2 - e$, then $b = 1 + e - ec_2 - \frac{e^2}{4}$ and again ξ does not define a non-empty wall of type (C_0, c_2). If $a = 1$ then $b = -2(c_2 - 1)$ and $\xi = \xi_1$, which proves what was stated. $\square$

By Remark 2.1.5, for all numerically equivalence class ξ defining a non-empty wall of type (C_0, c_2) we have $d(\xi) \leq 0$. Hence, using the above two Lemmas we obtain

Corollary 2.2.4. *For all numerical equivalence class $\xi \notin \{\xi_0, \xi_1\}$ defining a non-empty wall of type (C_0, c_2), we have $d(\xi) \leq -2$.*

Now we will describe how the Picard group of the moduli space $M_L(2; c_1, c_2)$ changes when L crosses different walls. To this end we need to recall the following result due to Qin ([Qin93]; Theorem 1.3.3). Given L_1 and L_2 two polarizations lying on chambers C_1 and C_2, sharing a common wall, we have

$$(2.1) \quad M_{L_1}(2; c_1, c_2) = (M_{L_2}(2; c_1, c_2) \setminus \sqcup_\xi E_{-\xi}(c_1, c_2)) \sqcup (\sqcup_\xi E_\xi(c_1, c_2)),$$

where ξ satisfies $\xi L_1 < 0$ and runs over all numerical equivalence classes which define the common wall W.

Let L_1 and L_2 be two polarizations sharing a common wall $W = W^\xi$. Assume $\xi L_2 > 0$ and $\xi L_1 < 0$. We distinguish 3 cases:

- If $\xi \notin \{\xi_0, \xi_1\}$, then $d(\xi) \leq -2$ (Corollary 2.2.4). Hence, applying (2.1) and Proposition 2.2.1 we obtain

$$\begin{aligned} Pic(M_{L_1}(2; c_1, c_2)) &\cong Pic(M_{L_1}(2; c_1, c_2) \setminus \sqcup_\xi E_\xi(c_1, c_2)) \\ &\cong Pic(M_{L_2}(2; c_1, c_2) \setminus \sqcup_\xi E_{-\xi}(c_1, c_2)) \\ &\cong Pic(M_{L_2}(2; c_1, c_2)). \end{aligned}$$

- If $\xi = \xi_0$, then $\xi_0 L_2 > 0$ and by Proposition 2.1.9 we have

$$M_{L_2}(2; c_1, c_2) = \emptyset.$$

Moreover, L_1 lies on a chamber $\mathcal{C}$ with $\overline{\mathcal{C}} \cap W^{\xi_0} \neq \emptyset$ and from the proof of Criterion 2.1.13 we deduce

$$Pic(M_{L_1}(2; c_1, c_2)) \cong Pic(\mathbb{P}^{4c_2 - c_1^2 - 3}) \cong \mathbb{Z}.$$

- If $\xi = \xi_1$, then $\xi_1 L_2 > 0$ and $d(\xi_1) = -1$. By (2.1) and Proposition 2.2.1 we have an exact sequence

$$0 \longrightarrow G_{E_{\xi_1}} \longrightarrow Pic(M_{L_1}(2; c_1, c_2)) \longrightarrow Pic(M_{L_2}(2; c_1, c_2)) \longrightarrow 0$$

being $G_{E_{\xi_1}}$ the cyclic subgroup of $Pic(M_{L_1}(2; c_1, c_2))$ generated by the line bundle $O_{M_{L_1}(2;c_1,c_2)}(E_{\xi_1})$. Let $\mathcal{C}_1$ be the chamber such that $\overline{\mathcal{C}}_1 \cap W^{\xi^0} \neq \emptyset$ and $L\xi_0 < 0$ for any $L \in \mathcal{C}_1$. Fix $L_2' \in \mathcal{C}_1$. By Proposition 1.3.5 we can choose finitely many ample divisors

$$L_2 = L_2^{(1)}, L_2^{(2)}, ..., L_2^{(r-1)}, L_2^{(r)} = L_2'$$

on the line segment connecting L_2 and L_2' in such a way that we have

(1) $L_2^{(i)}$ lies in some chamber for all $i = 1, ..., r$ and

(2) $L_2^{(i)}$ and $L_2^{(i+1)}$ are separated by a single wall W^{ξ^i} for all $i = 1, ..., r-1$, such that $d(\xi^i) \leq -2$.

Hence, we have

$$Pic(M_{L_2}(2; c_1, c_2)) \cong Pic(M_{L_2^{(i)}}(2; c_1, c_2)) \cong Pic(M_{L_2'}(2; c_1, c_2)) \cong \mathbb{Z}$$

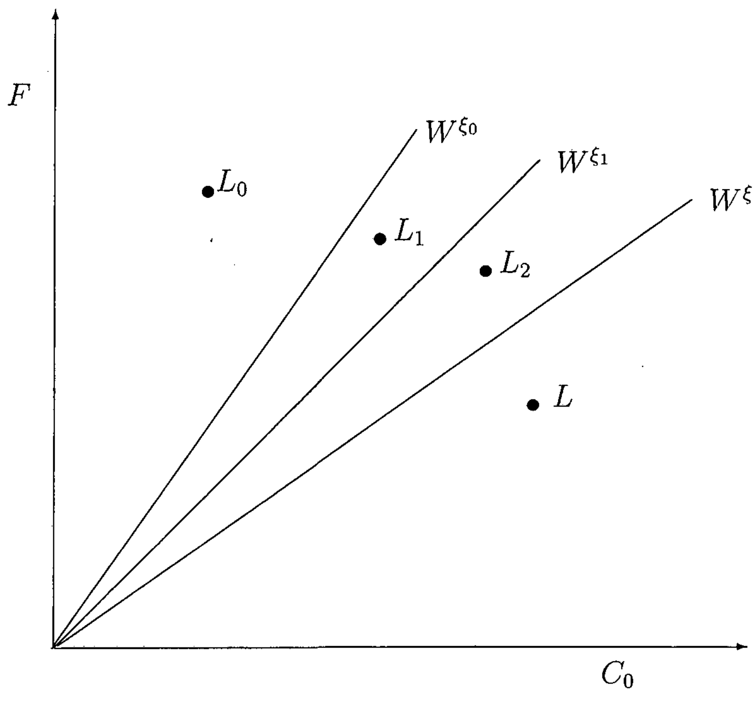
where the last isomorphism follows from the second case we have studied. Therefore, we obtain the exact sequence

$$0 \longrightarrow G_{E_{\xi_1}} \longrightarrow Pic(M_{L_1}(2; c_1, c_2)) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

We have seen that indeed, there are moduli spaces with different Picard group, which in particular implies that they are not isomorphic; but since X_e is an anticanonical rational surface, by Criterion 2.1.13 they are birational whenever non-empty (for instance when $c_2 >> 0$).

We denote by $\mathcal{C}_0$ the chamber such that $\overline{\mathcal{C}}_0 \cap W^{\xi_0} \neq \emptyset$ and for any $L \in \mathcal{C}_0$, $L\xi_0 \geq 0$, $\mathcal{C}_1$ the chamber with $\overline{\mathcal{C}}_1 \cap W^{\xi_0} \neq \emptyset$ and for any $L \in \mathcal{C}_1$, $L\xi_0 < 0$, $\mathcal{C}_2$ the chamber such that $\overline{\mathcal{C}}_2 \cap W^{\xi_1} \neq \emptyset$ and for any $L \in \mathcal{C}_2$, $L\xi_1 < 0$ and $\mathcal{C}$ a chamber, different from $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_2$, such that for all $L \in \mathcal{C}$, we have $L\xi_1 < 0$.

Given a polarization $L = aC_0 + bF$, we can represent L as a point of coordinates (a, b) in the plane. The following picture gives us an idea of the situation just described



Let us summarize what has been seen in the example

- If $L_0 \in \mathcal{C}_0$, then $M_{L_0}(2; c_1, c_2) = \emptyset$ and $Pic(M_{L_0}(2; c_1, c_2)) = 0$.

- If $L_1 \in \mathcal{C}_1$, then $M_{L_1}(2;c_1,c_2) \cong \mathbb{P}^{4c_2-c_1^2-3}$ and $Pic(M_{L_1}(2;c_1,c_2)) = \mathbb{Z}$.
- If $L_2 \in \mathcal{C}_2$, then $Pic(M_{L_2}(2;c_1,c_2))$ sits in the exact sequence

$$0 \longrightarrow G_{E_{\xi_1}} \longrightarrow Pic(M_{L_2}(2;c_1,c_2)) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

- If $L \in \mathcal{C}$, we have $Pic(M_L(2;c_1,c_2)) \cong Pic(M_{L_2}(2;c_1,c_2))$.

In forthcoming chapters, we will give more examples where the moduli spaces are birational but not isomorphic (see for instance, Chapter 5, section 3).

2.3 Non-anticanonical rational surfaces

The main tool we will use to prove the birational equivalence between moduli spaces of rank 2 vector bundles on non-anticanonical, rational surfaces, stable with respect to different polarizations will be prioritary sheaves. Prioritary sheaves were introduced on $\mathbb{P}^2$ (resp. on birationaly ruled surfaces) by Hirschowitz-Laszlo (resp. Walter) as a generalization of semistable sheaves. (The reader can see [HL93] and [Wal93] for more information on prioritary sheaves). Let us start recalling the precise definition of prioritary sheaf.

Definition 2.3.1. *Let $\pi : X \longrightarrow \mathbb{P}^1$ be a birationaly ruled surface and we consider $F \in Num(X)$ the numerical class of a fiber of π. A coherent sheaf E on X is said to be prioritary if it is torsion free and if $Ext^2(E, E(-F)) = 0$.*

The following Lemma allows us to use prioritary sheaves in order to deduce results on moduli spaces of stable vector bundles. We want to stress that this result is due to Walter (see [Wal93]) but, because of its importance, we prefer to reproduce here its proof.

Lemma 2.3.2. *Let $\pi : X \longrightarrow \mathbb{P}^1$ be a birationaly ruled surface, $F \in Num(X)$ the numerical class of a fiber of π and H an ample divisor on X with $H(K_X + F) < 0$. Then any H-semistable, torsion free sheaf E is prioritary.*

Proof. If E is a H-semistable torsion free sheaf on X, then any nonzero torsion-free quotient Q of E would have H-slope satisfying $\mu_H(Q) \geq \mu_H(E)$, while any nonzero subsheaf S of E would have H-slope satisfying $\mu_H(S) \leq \mu_H(E)$. So if E were not prioritary, there would exist a nonzero homomorphism

$$\phi \in Hom(E, E(K_X + F)) \cong Ext^2(E, E(-F))^*.$$

The image of ϕ would then satisfy

$$\mu_H(E) \leq \mu_H(im(\phi)) \leq \mu_H(E(K_X + F)) = \mu_H(E) + H(K_X + F),$$

contradicting $H(K_X + F) < 0$. □

Remark 2.3.3. Since every rational surface is a birationaly ruled surface, the above results, together with the forthcoming facts on prioritary sheaves can, and will, be applied to studying the moduli spaces we deal with.

Remark 2.3.4. See Lemma 1.4.9 (resp. Remarks 1.4.10 and 1.4.13) for the existence of ample divisors L on a Hirzebruch surface X_e (resp. on the blow-up X of a Hirzebruch surface) such that the condition $L(K_{X_e} + F) < 0$ (resp. $L(K_X + F) < 0$) is satisfied.

For a given $1 \leq r \in \mathbb{Z}$, $c_1 \in Pic(X)$, and $c_2 \in \mathbb{Z}$, we will denote by $Prior(r; c_1, c_2)$ the stack of prioritary sheaves E on X of rank r and Chern classes c_1 and c_2, and by $Spl(r; c_1, c_2)$ the moduli space of simple prioritary torsion free sheaves E on X of rank r and Chern classes c_1 and c_2. In [Wal93]; Proposition 2, Walter proves that the stack $Prior(r; c_1, c_2)$ of prioritary sheaves is irreducible and smooth. It follows from Lemma 2.3.2 that for any ample divisor H on X verifying $(K_X + F)H < 0$, the moduli space $M_H(r; c_1, c_2)$ (resp. $Spl(r; c_1, c_2)$) of H-stable vector bundles (resp. simple prioritary sheaves) on X is an open substack of $Prior(r; c_1, c_2)$. We would like to mention that the reader unfamiliar with algebraic stacks only has to know that algebraic stacks are some sort of generalization of schemes and that there always exists a moduli stack of coherent sheaves with fixed rank and Chern classes. Moreover, there are notions of smoothness and irreducibility.

In the following Theorem we have summarized more precisely the results we will need later on.

Theorem 2.3.5. *Let $\pi : X \longrightarrow \mathbb{P}^1$ be a birationaly ruled surface and $F \in Num(X)$ the numerical class of a fiber of π. Suppose $2 \leq r \in \mathbb{Z}$, $c_1 \in Pic(X)$ and $c_2 \in \mathbb{Z}$ are given. Then, the stack $Prior(r; c_1, c_2)$ is smooth and irreducible. Let H be an ample divisor on X such that $(K_X + F)H < 0$. For any integer c_2 such that $\Delta(r; c_1, c_2) >> 0$ the moduli space $M_H(r; c_1, c_2)$ is a non-empty, smooth, irreducible, quasi-projective variety of the expected dimension $2rc_2 - (r-1)c_1^2 - r^2 + 1$.*

Proof. See [Wal93]; Theorem 1 and Proposition 2. □

Now we will prove that for any two polarizations L_1 and L_2 verifying the inequality $L_i(K_X + F) < 0$, $i = 1, 2$, the moduli spaces $M_{L_1}(r; c_1, c_2)$ and $M_{L_2}(r; c_1, c_2)$ are birationaly equivalent whenever non-empty. This result can be considered as generalization of Theorem 2.1.10 to arbitrary rational surfaces.

Theorem 2.3.6. *Let $\pi : X \longrightarrow \mathbb{P}^1$ be a birationaly ruled surface, $F \in Num(X)$ the numerical class of a fiber of π, $c_1 \in Pic(X)$ and $c_2 \in \mathbb{Z}$ such that $\Delta(r; c_1, c_2) >> 0$. Then, for any two ample divisors L_1 and L_2 on X with $L_i(K_X + F) < 0$, $i = 1, 2$, the moduli spaces $M_{L_1}(r; c_1, c_2)$ and $M_{L_2}(r; c_1, c_2)$ are birationaly equivalent.*

Proof. By Theorem 2.3.5 and Lemma 2.3.2, the moduli spaces $M_{L_1}(r; c_1, c_2)$ and $M_{L_2}(r; c_1, c_2)$ are non-empty open substacks of $Prior(r; c_1, c_2)$ and the result follows from the smoothness and irreducibility of $Prior(r; c_1, c_2)$. □

In the next Proposition we will give a criterion of rationality for moduli spaces of rank two vector bundles on arbitrary rational surfaces. This second criterion can be viewed as a generalization of Criterion 2.1.13 to arbitrary rational surfaces.

Second criterion of rationality 2.3.7. *Let* $\pi : X \longrightarrow \mathbb{P}^1$ *be a birationaly ruled surface,* $F \in Num(X)$ *the numerical class of a fiber of* π, $c_1 \in Pic(X)$ *and* $c_2 \in \mathbb{Z}$. *Assume* $4c_2 - c_1^2 > 2 - \frac{3K_X^2}{2}$ *and that there exists a numerical equivalence class* ξ *which defines a non-empty wall of type* (c_1, c_2) *and that it satisfies*

$$(2.2) \qquad \xi^2 = c_1^2 - 4c_2, \quad \xi^2 + \xi K_X + 2 = 0,$$

$$(2.3) \qquad H^0O_X(\xi + 3K_X) = H^0O_X(\xi + K_X + F) = H^0O_X(K_X + F - \xi) = 0.$$

Then, the following holds

(1) There exists an ample divisor $\tilde{L}$ *on* X *such that the moduli space* $M_{\tilde{L}}(2; c_1, c_2)$ *is a smooth, irreducible, rational projective variety of dimension* $4c_2 - c_1^2 - 3$ *and* $Pic(M_{\tilde{L}}(2; c_1, c_2)) \cong \mathbb{Z}$ *whenever non-empty.*

(2) For $c_2 >> 0$ *and any ample divisor* L *on* X *such that* $L(K_X + F) < 0$, *the moduli space* $M_L(2; c_1, c_2)$ *is a smooth, irreducible, rational, quasi-projective variety of dimension* $4c_2 - c_1^2 - 3$.

Proof. (1) Let L be any ample divisor on X, $c_2 \in \mathbb{Z}$ with $4c_2 - c_1^2 > 2 - \frac{3K_X^2}{2}$ and ξ a numerical equivalence class verifying (2.2) and (2.3).

Claim 1: For any $E \in M_L(2; c_1, c_2)$

$$h^0E(-\frac{c_1 + \xi}{2}) > 0.$$

Proof of Claim 1: Applying 1.1.1 and 1.1.5 we easily see that

$$c_1(E(-\frac{c_1 + \xi}{2})) = -\xi, \quad c_2(E(-\frac{c_1 + \xi}{2})) = 0 \quad \text{and} \quad \chi(E(-\frac{c_1 + \xi}{2})) = 1.$$

By hypothesis, $H^0O_X(\xi + 3K_X) = 0$ and by Serre's duality $H^2O_X(-\xi - 2K_X) = 0$. Therefore, applying Riemann-Roch's Theorem we get

$$\begin{aligned} h^0O_X(-\xi - 2K_X) - h^1O_X(-\xi - 2K_X) &= \chi(O_X(-\xi - 2K_X)) \\ &= \tfrac{(-\xi - 2K_X)(-\xi - 3K_X)}{2} + 1 \\ &= 2(4c_2 - c_1^2 - 2) + 3K_X^2 > 0 \end{aligned}$$

which gives us $h^0O_X(-\xi-2K_X)>0$ or, equivalently, $-(\xi+2K_X)$ is effective. Hence, $-(2K_X+\xi)L\geq 0$ for any ample divisor L on X or, equivalently,

$$c_1((E(-\frac{c_1+\xi}{2}))^*\otimes K_X)L=(2K_X+\xi)L\leq 0.$$

If the last inequality is strict we obtain (Fact 1.1.10)

$$h^0E(-\frac{c_1+\xi}{2})>0.$$

If $c_1((E(-\frac{c_1+\xi}{2}))^*\otimes K_X)L=0$ we get

$$h^0E(-\frac{c_1+\xi}{2})>0 \quad \text{or} \quad h^2E(-\frac{c_1+\xi}{2})>0$$

and we will prove that the last inequality is not possible. Indeed, by Serre's duality,

$$0<h^2E(-\frac{c_1+\xi}{2})=h^0E^*(\frac{c_1+\xi}{2}+K_X).$$

A non-zero section $\sigma\in H^0E^*(\frac{c_1+\xi}{2}+K_X)$ defines an injection

$$O_X(\frac{c_1-\xi}{2}-K_X)\hookrightarrow E$$

and from the L-stability of E we have

$$(\frac{c_1-\xi}{2}-K_X)L<\frac{c_1L}{2}$$

which contradicts the fact $(2K_X+\xi)L=0$. Therefore, $h^0E(-\frac{c_1+\xi}{2})>0$, which proves Claim 1.

Claim 2: If $\xi L\geq 0$ then, $M_L(2;c_1,c_2)=\emptyset$.

Proof of Claim 2: Assume $M_L(2;c_1,c_2)\neq\emptyset$. For any $E\in M_L(2;c_1,c_2)$, we can take a nonzero section $s\in H^0E(-\frac{c_1+\xi}{2})$. It defines an injection

$$O_X(\frac{c_1+\xi}{2})\hookrightarrow E.$$

Since E is L-stable, we have

$$(\frac{c_1+\xi}{2})L<\frac{c_1L}{2}$$

i.e.; $\xi L < 0$ which contradicts the hypothesis $\xi L \geq 0$. Hence, $M_L(2; c_1, c_2) = \emptyset$ which proves Claim 2.

Let $\tilde{L}$ be an ample divisor on X such that $\xi \tilde{L} < 0$ and $\tilde{L} \in \mathcal{C}$ with $W^\xi \cap \overline{\mathcal{C}} \neq \emptyset$. Let us see that

$$M_{\tilde{L}}(2; c_1, c_2) \cong \mathbb{P}^{4c_2 - c_1^2 - 3}.$$

For such $\tilde{L}$ and $\mathcal{C}$ we have ([Qin93]; Proposition 1.3.1)

$$M_{\tilde{L}}(2; c_1, c_2) = M_{\mathcal{F}}(2; c_1, c_2) \sqcup (\sqcup_\mu E_\mu(c_1, c_2))$$

where $\mathcal{F}$ is the face of $\mathcal{C}$ contained in W^ξ, $\mu \tilde{L} < 0$ for some $\tilde{L} \in \mathcal{C}$ and μ runs over all numerical equivalence classes which define the wall W^ξ. For any $L' \in \mathcal{F}$, $L'\xi = 0$. So, by Claim 2, $M_{\mathcal{F}}(2; c_1, c_2) = \emptyset$. Moreover, $W^\mu = W^\eta$ if, and only if, $\mu = \lambda \eta$, for some $\lambda \in \mathbb{R}$. Therefore, we conclude

$$M_{\tilde{L}}(2; c_1, c_2) \cong E_\xi(c_1, c_2).$$

By definition, any $E \in E_\xi(c_1, c_2)$, sits in an exact sequence

$$0 \longrightarrow O_X(G) \longrightarrow E \longrightarrow O_X(c_1 - G) \otimes I_Z \longrightarrow 0$$

where G is a divisor with $2G - c_1 \equiv \xi$ and Z is a locally complete intersection 0-cycle with $l(Z) = c_2 + \frac{\xi^2 - c_1^2}{4}$. By hypothesis $\xi^2 = c_1^2 - 4c_2$ (see (2.2)). Therefore, $Z = \emptyset$ and

$$\begin{aligned} M_{\tilde{L}}(2; c_1, c_2) \cong E_\xi(c_1, c_2) &\cong \mathbb{P}(Ext^1(O_X(c_1 - G), O_X(G))) \\ &\cong \mathbb{P}(H^1 O_X(\xi)) \cong \mathbb{P}^{4c_2 - c_1^2 - 3} \end{aligned}$$

where the last isomorphism follows from the hypothesis (2.3), the fact that since ξ defines a non-empty wall of type (c_1, c_2),

$$h^2 O_X(\xi) = h^0 O_X(K_X - \xi) \leq h^0 O_X(K_X - \xi + F) = 0$$

and we have $h^0 O_X(\xi) = h^2 O_X(\xi) = 0$, and Riemann-Roch's Theorem.

Therefore, the moduli space $M_{\tilde{L}}(2; c_1, c_2)$ is a smooth, irreducible, rational, projective variety of dimension $4c_2 - c_1^2 - 3$ and

$$Pic(M_{\tilde{L}}(2; c_1, c_2)) \cong \mathbb{Z}$$

whenever the moduli space is non-empty.

(2) Let L be an ample divisor on X with $L(K_X+F)<0$. By Theorem 2.3.5, we only need to prove that the moduli space $M_L(2;c_1,c_2)$ is rational. It follows from the proof of (1) that there exists an ample divisor $\tilde{L}$ on X such that

$$M_{\tilde{L}}(2;c_1,c_2)\cong E_\xi(c_1,c_2).$$

Claim 3: Any $E\in E_\xi(c_1,c_2)$ is a prioritary sheaf.

Proof of Claim 3: Since E is a rank two vector bundle, we only need to check that $Ext^2(E,E(-F))=0$. By assumption, $\xi^2=c_1^2-4c_2$, so every $E\in E_\xi(c_1,c_2)$ is given by a non-trivial extension

$$(2.4)\qquad 0\longrightarrow O_X(\frac{c_1+\xi}{2})\longrightarrow E\longrightarrow O_X(\frac{c_1-\xi}{2})\longrightarrow 0.$$

By Serre's duality, we have $dimExt^2(E,E(-F))=dimHom(E,E(K_X+F))$. Applying the functor $Hom(.,E(K_X+F))$ to the exact sequence (2.4) we get the long exact sequence

$$0\longrightarrow Hom(O_X(\tfrac{c_1-\xi}{2}),E(K_X+F))\longrightarrow Hom(E,E(K_X+F))$$
$$\longrightarrow Hom(O_X(\tfrac{c_1+\xi}{2}),E(K_X+F))\longrightarrow\cdots.$$

We consider the long exact cohomology sequence

$$0\longrightarrow H^0O_X(F+K_X+\xi)\longrightarrow H^0E(\frac{\xi-c_1}{2}+K_X+F)\longrightarrow H^0O_X(F+K_X)\longrightarrow\cdots$$

associated to the exact sequence (2.4). Since $F+K_X$ is not an effective divisor and by assumption $H^0O_X(F+K_X+\xi)=0$, we get

$$Hom(O_X(\frac{c_1-\xi}{2}),E(K_X+F))=H^0E(\frac{\xi-c_1}{2}+K_X+F)=0.$$

Using the long exact cohomology sequence

$$0\longrightarrow H^0O_X(F+K_X)\longrightarrow H^0E(-\frac{\xi+c_1}{2}+K_X+F)\longrightarrow H^0O_X(F+K_X-\xi)\longrightarrow\cdots$$

associated to the exact sequence (2.4) and the hypothesis $H^0O_X(F+K_X-\xi)=0$, we obtain

$$Hom(O_X(\frac{c_1+\xi}{2}), E(K_X+F)) = H^0E(-\frac{\xi+c_1}{2}+K_X+F)=0$$

which proves that $Hom(E,E(K_X+F))=0$. Therefore, E is a prioritary sheaf and Claim 3 is proved.

It follows from Claim 3 that

$$M_{\tilde{L}}(2;c_1,c_2)\cong E_\xi(c_1,c_2)\cong \mathbb{P}^{4c_2-c_1^2-3}\subset Prior(2;c_1,c_2).$$

Since $\Delta(2;c_1,c_2)>>0$, the moduli spaces $M_{\tilde{L}}(2;c_1,c_2)$ and $M_L(2;c_1,c_2)$ are smooth and irreducible. It follows from Claim 3 (resp. Lemma 2.3.2) that $M_{\tilde{L}}(2;c_1,c_2)$ (resp. $M_L(2;c_1,c_2)$) is an open substack of $Prior(2;c_1,c_2)$. By Theorem 2.3.5, $Prior(2;c_1,c_2)$ is smooth and irreducible and we have proved that $M_{\tilde{L}}(2;c_1,c_2)$ is rational. Therefore $M_L(2;c_1,c_2)$ is rational, which proves what we want. □

In the next Chapter, we will use Criterion 2.1.13 and Criterion 2.3.7 in order to prove the rationality of some moduli spaces.

Chapter 3

Rank 2 vector bundles on surfaces

Moduli spaces of vector bundles with fixed determinant on smooth, algebraic curves are unirational and very often even rational. For moduli spaces of vector bundles on smooth, algebraic surfaces the situation differs drastically and, from the point of view of birational geometry, discloses highly interesting features. The general philosophy is that the geometry of a smooth surface X and of the moduli spaces of vector bundles on X are intimately related. For example, based on Li's work [Li94], one should expect that the moduli space of stable vector bundles over a surface of general type is also of general type. Similarly, moduli spaces associated to rational surfaces are expected to be rational.

There is at present no counterexample known to the question whether the moduli spaces are always rational provided the underlying surface X is rational. For $X = \mathbb{P}^2$, Maruyama (resp. Ellingsrud and Stromme) proved that if $c_1^2 - 4c_2 \not\equiv 0 \pmod 8$, then the moduli space $M_{\mathbb{P}^2, O_{\mathbb{P}^2}(1)}(2; c_1, c_2)$ of $O_{\mathbb{P}^2}(1)$-stable, rank 2 vector bundles on $\mathbb{P}^2$ with Chern classes c_1 and c_2 is rational ([Mar85] and [ES87]). Later on, Maeda proved that the rationality of the moduli space $M_{\mathbb{P}^2, O_{\mathbb{P}^2}(1)}(2; c_1, c_2)$ holds for all $(c_1, c_2) \in \mathbb{Z}^2$ provided $M_{\mathbb{P}^2, O_{\mathbb{P}^2}(1)}(2; c_1, c_2)$ is non-empty ([Mae90]). In recent papers [Li97], [LQ96], [LQ96b], [Got96], [Kat92] and [Yos96], the question of the rationality of the moduli space $M_L(r; c_1, c_2)$ of rank $r \geq 2$, L-stable vector bundles on $\mathbb{P}^2$ and other rational surfaces was investigated. All these results give rise to the following question (see [Sch90]; Problem 21, [Sch85]; Problem 2, [OV88]; Problem 2):

QUESTION: Let X be a smooth, rational, projective surface. Fix a polarization L, $c_1 \in Pic(X)$ and $0 << c_2 \in \mathbb{Z}$. Is $M_L(r; c_1, c_2)$ rational?

The goal of this chapter is to give for $r = 2$ an affirmative answer to the above question. More precisely, we will prove that if X is a smooth, rational, projective surface and we fix an ample divisor L, $c_1 \in Pic(X)$ and $0 << c_2 \in \mathbb{Z}$, then the moduli space $M_{X,L}(2; c_1, c_2)$ is rational (Theorem 3.3.7). When there is no confusion, we will write $M_L(2; c_1, c_2)$ instead of $M_{X,L}(2; c_1, c_2)$.

Next we outline the ideas used to prove the main Theorem of this chapter (Theorem 3.3.7). According to the classification, up to isomorphism, of smooth, projective, rational surfaces and Remark 1.4.6, we have divided this chapter in three sections. In the first one, we will prove the rationality of the moduli space $M_{X,L}(2; c_1, c_2)$ (Theorem 3.1.8) of L-stable, rank two, vector bundles E with Chern classes $c_1(E) = c_1$ and $c_2(E) = c_2$ over a smooth, minimal, rational surface X, i.e. X is either $\mathbb{P}^2$ or a smooth Hirzebruch surface X_e (Theorem 1.4.1). In section 2, we will study the rationality of the moduli space $M_{X,L}(2; c_1, c_2)$ of rank 2, L-stable vector bundles over a smooth Fano surface X. We want to point out that in this case we will strongly use the fact that X is an anticanonical rational surface and, therefore, we will study this case separately from the case of other non-minimal rational surfaces, which will be studied along the last section of this chapter.

In all sections, we analyze separately all possible values of the first Chern class and we prove the rationality using either the Criterion 2.1.13 or the Criterion 2.3.7; or constructing suitable families of rank two stable vector bundles (resp. prioritary torsion free sheaves) over a big enough rational base.

3.1 Moduli spaces of vector bundles on minimal rational surfaces

The goal of this section is to prove the rationality of the moduli spaces $M_L(2; c_1, c_2)$ of rank two, L-stable vector bundles E on smooth, minimal, rational surfaces X with fixed Chern classes $c_1 \in Pic(X)$ and $c_2 \in \mathbb{Z}$. According to Theorem 1.4.1, if X is a minimal rational surface, then X is either isomorphic to $\mathbb{P}^2$ or a Hirzebruch surface X_e with $e \neq 1$.

As we pointed out in the introduction, the case $X \cong \mathbb{P}^2$ has been studied by several authors (see [ES87], [Mae90] and [Mar85]) and it is summarized in the following Theorem

Theorem 3.1.1. *For any pair of integers (c_1, c_2), the moduli space $M_{O_{\mathbb{P}^2}(1)}(2; c_1, c_2)$ of rank two, $O_{\mathbb{P}^2}(1)$-stable vector bundles E on $\mathbb{P}^2$ with fixed Chern classes $c_1(E) = c_1$ and $c_2(E) = c_2$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - c_1^2 - 3$, provided it is non-empty.*

Proof. See [Mar75]; Proposition 4.15 for the smoothness. The irreducibility follows from [Bar77]; 4.3 Corollary 1 (resp. [Hul79]; Theorem 2.1) if c_1 is even (resp. odd) and the rationality was proved by Maeda in [Mae90]. □

Hence, we will study the rationality of the moduli space $M_L(2; c_1, c_2)$ of rank two, L-stable vector bundles E on a Hirzebruch surface X_e with fixed Chern classes $c_1 \in Pic(X_e)$ and $c_2 \in \mathbb{Z}$.

Remark 3.1.2. We will prove the rationality of $M_L(2; c_1, c_2)$ distinguishing different cases, according to the value of $c_1 \in Pic(X_e)$. Since a rank 2 vector bundle E on X_e is L-stable if, and only if, $E \otimes O_{X_e}(G)$ is L-stable for any divisor $G \in Pic(X_e)$, we may assume, without loss of generality, that $c_1(E)$ is one of the following: 0, $C_0 + \alpha F$ with $\alpha \in \{0, 1\}$ or F.

Let us start with the case $c_1 = C_0 + \alpha F$ with $\alpha \in \{0, 1\}$.

Proposition 3.1.3. *Let X_e be a smooth, Hirzebruch surface, $c_2 \in \mathbb{Z}$ and $\alpha \in \{0, 1\}$. Then, the following is satisfied*

(1) There exists an ample divisor $\tilde{L}$ on X_e such that $M_{\tilde{L}}(2; C_0 + \alpha F, c_2)$ is a smooth, irreducible, rational, projective variety of dimension $4c_2 + e - 2\alpha - 3$ and $Pic(M_{\tilde{L}}(2; C_0 + \alpha F, c_2)) \cong \mathbb{Z}$ whenever non-empty.

(2) For any ample divisor L on X_e, the moduli space $M_L(2; C_0 + \alpha F, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of the expected dimension $4c_2 + e - 2\alpha - 3$, whenever non-empty.

Proof. First of all, notice that by [Nak93]; Theorem 1.5 we have: $c_2 \geq 1$. Now, we will apply Criterion 2.1.13. To this end, we take the numerical equivalence class, say $\xi = C_0 - (2c_2 - \alpha)F$.

Claim: ξ defines a non-empty wall of type $(C_0 + \alpha F, c_2)$ and $d(\xi) = 0$.

Proof of the Claim: Notice that $\xi + c_1 = 2C_0 - 2(c_2 - \alpha)F$ and

$$\xi^2 = 2\alpha - e - 4c_2 = c_1^2 - 4c_2 < 0.$$

Moreover, we have (see Corollary 2.1.4)

$$\begin{aligned} d(\xi) &= \tfrac{c_1^2-4c_2}{4} + \tfrac{\xi^2}{4} + \tfrac{\xi K_{X_e}}{2} + 1 \\ &= \tfrac{\xi^2}{2} + \tfrac{\xi K_{X_e}}{2} + 1 \\ &= \alpha - \tfrac{e+4c_2}{2} + \tfrac{(C_0-(2c_2-\alpha)F)(-2C_0-(e+2)F)}{2} + 1 = 0. \end{aligned}$$

Hence, we only have to check (see Definition 1.3.3 and Remark 1.3.7) that there exist ample divisors L and L' on X_e such that

$$\xi L \leq 0 < \xi L'.$$

We take the ample divisors $L = C_0 + (e+1)F$ and $L' = C_0 + (e + 2c_2 + 1)F$ (Remark 1.4.7) on X_e. We have

$$L\xi = -e - 2c_2 + \alpha + e + 1 = \alpha - 2c_2 + 1 \leq 0$$

$$L'\xi = -e - 2c_2 + \alpha + e + 2c_2 + 1 = \alpha + 1 > 0.$$

Since a smooth, Hirzebruch surface is an anticanonical, rational surface we can apply Criterion 2.1.13 (see Remark 2.1.8) and this leads us to prove the proposition.

□

Before studying the case $c_1 = 0$, we need a low bound for c_2 which is given by the following result.

Lemma 3.1.4. *Let X_e be a smooth, Hirzebruch surface and L an ample divisor on X_e. If E is a rank two, L-stable, vector bundle on X_e with Chern classes $(0, c_2)$, then $c_2 \geq 2$.*

Proof. It follows from Bogomolov's inequality, $4c_2 - c_1^2 > 0$, that $c_2 > 0$. Assume that $c_2 = 1$. By Riemann-Roch's Theorem we have

$$\chi(E) = 1.$$

Since E is L-stable and $c_1(E) = 0$, we get $h^2E = h^0E(K_{X_e}) = 0$ and $h^0E \geq 1$. On the other hand, a non-zero section defines an injection

$$O_{X_e} \hookrightarrow E$$

which contradicts the L-stability of E. Therefore, $c_2 \geq 2$ which proves what was stated. □

Now we will deal with the case $c_1 = 0$.

Proposition 3.1.5. *Let X_e be a smooth, Hirzebruch surface, $c_2 \in \mathbb{Z}$ and L any ample divisor on X_e. Then, the moduli space $M_L(2; 0, c_2)$ is either empty or a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - 3$.*

Proof. Since $c_2 \geq 2$ (Lemma 3.1.4), the result follows from Theorem 3.1.1 and [Art90]; Theorem 1.7 and Corollary 3.4. □

In order to study the last case, which corresponds to $c_1 = F$, we will distinguish two cases according to the parity of c_2. Let us start with the odd case.

Proposition 3.1.6. *Let X_e be a smooth, Hirzebruch surface, $\alpha \in \{1, 3\}$ and L an ample divisor on X_e. Then, the moduli space $M_L(2; F, 4m + \alpha)$ is either empty or a smooth, irreducible, rational, quasi-projective variety of dimension $4(4m + \alpha) - 3$.*

Proof. Let L be an ample divisor on X_e such that the moduli space $M_L(2; F, 4m+\alpha)$ is non-empty. We consider X the smooth, rational surface obtained by blowing up one point of X_e and the divisor on X

$$L_n := n\pi^*L - E_1$$

where $\pi : X \longrightarrow X_e$ is the blow up and E_1 is the exceptional divisor. For n sufficiently large, L_n is an ample divisor on X and there is an open immersion (see Theroem 1.2.15)

$$M_{X_e,L}(2; F, 4m+\alpha) \hookrightarrow M_{X,L_n}(2; F, 4m+\alpha).$$

Furthermore, by Remark 1.2.14, the moduli space $M_{X_e,L}(2; F, 4m+\alpha)$ is a smooth, irreducible, quasi-projective variety of dimension $4(4m+\alpha)-3$ and the moduli space $M_{X,L_n}(2; F, 4m+\alpha)$ is a smooth, irreducible, quasi-projective variety of the same dimension. Therefore, we only need to check that $M_{X,L_n}(2; F, 4m+\alpha)$ is rational. To this end, we consider the irreducible family $\mathcal{F}$ of rank two torsion free sheaves E on X given by a non-trivial extension

$$(3.1) \qquad \epsilon : 0 \longrightarrow O_X(-D) \longrightarrow E \longrightarrow O_X(D+F) \otimes I_Z \longrightarrow 0$$

where Z is a locally complete intersection 0-cycle of length $|Z| = 6m + \frac{3(\alpha-1)}{2}$ verifying $H^0 I_Z(2D+F) = 0$ and

$$D = C_0 + bF - cE_1 = \begin{cases} C_0 + (m+n-1)F & \text{if } e = 2n \quad \text{and} \quad \alpha = 1 \\ C_0 + (m+n)F - E_1 & \text{if } e = 2n \quad \text{and} \quad \alpha = 3 \\ C_0 + (m+n)F - E_1 & \text{if } e = 2n+1 \quad \text{and} \quad \alpha = 1 \\ C_0 + (m+n)F & \text{if } e = 2n+1 \quad \text{and} \quad \alpha = 3. \end{cases}$$

Let us see that such a 0-cycle Z exists. Indeed, applying Lemma 1.4.8 we get

$$(3.2) \qquad \begin{aligned} h^0 O_X(2C_0 + (2b+1)F) &= h^0 O_{\mathbb{P}^1}(2b+1-2e) + h^0 O_{\mathbb{P}^1}(2b+1-e) \\ &\quad + h^0 O_{\mathbb{P}^1}(2b+1) \\ &= 6b + 6 - 3e. \end{aligned}$$

If $\alpha = 1$ and $e = 2n$ or $\alpha = 3$ and $e = 2n+1$, then

$$h^0 O_X(2D+F) = h^0 O_X(2C_0 + (2b+1)F) = 6m + \frac{3(\alpha-1)}{2} = |Z|.$$

So, for a generic $Z \in Hilb^{|Z|}(X)$ we have $H^0 I_Z(2D+F) = 0$ (see Lemma 1.4.16).

Assume $\alpha = 3$ and $e = 2n$ or $\alpha = 1$ and $e = 2n+1$. By Lemma 1.4.15 we have

$$\begin{aligned} h^0 O_X(2D+F) &= h^0 O_X(2C_0 + (2b+1)F - 2E_1) \\ &\leq h^0 O_X(2C_0 + (2b+1)F) - 3 \\ &= 6m + \tfrac{3(\alpha-1)}{2} \end{aligned}$$

where the last equality follows from (3.2). Hence, since $|Z| = 6m + \frac{3(\alpha-1)}{2}$, for a generic $Z \in Hilb^{|Z|}(X)$ we have $H^0 I_Z(2D+F) = 0$ (see Lemma 1.4.16).

Let us show:

(a) $h^0 E(D) = 1$.

(b) $dim\mathcal{F} = 4(4m+\alpha) - 3$.

(c) Any $E \in \mathcal{F}$ is a simple prioritary sheaf with Chern classes $c_1(E) = F$ and $c_2(E) = 4m + \alpha$.

(*a*) We have the long exact cohomology sequence

$$0 \longrightarrow H^0 O_X \longrightarrow H^0 E(D) \longrightarrow H^0 I_Z(2D+F) \longrightarrow \cdots$$

associated to the exact sequence (3.1). By hypothesis, we have

$$(3.3) \qquad H^0 I_Z(2D+F) = 0.$$

Therefore, $h^0 E(D) = 1$ which proves (*a*).

(*b*) By construction we have

$$(3.4) \quad \begin{aligned} dim\mathcal{F} &= \#moduli(Z) + dimExt^1(I_Z(D+F), O_X(-D)) - h^0 E(D) \\ &= 2length(Z) + dimExt^1(I_Z(D+F), O_X(-D)) - h^0 E(D) \\ &= 2length(Z) + dimExt^1(I_Z(D+F), O_X(-D)) - 1 \end{aligned}$$

where the last equality follows from (a).

By Serre's duality (see 1.1.6) we have

$$\begin{aligned} dimExt^1(I_Z(D+F), O_X(-D)) &= dimExt^1(O_X(-D), I_Z(D+F+K_X)) \\ &= h^1 I_Z(2D+F+K_X). \end{aligned}$$

Since Z is a 0-cycle, $h^2 I_Z(2D+F+K_X) = h^2 O_X(2D+F+K_X)$ and using again Serre's duality and Lemma 1.4.14; (a) we get

$$\begin{aligned} H^2 O_X(2D+F+K_X) &= H^0 O_X(-2D-F)^* \\ &= H^0 O_X(-2C_0-(2b+1)F+2cE_1)^* = 0. \end{aligned}$$

Using Lemma 1.4.8 and Lemma 1.4.14; (b) we obtain

$$\begin{aligned} h^0 O_X(2D+F+K_X) &= h^0 O_X((2b-1-e)F-(2c-1)E_1) \\ &\leq h^0 O_X((2b-1-e)F) \\ &= h^0 O_{\mathbb{P}^1}(2b-1-e) = 2b-e. \end{aligned}$$

Hence, since $|Z| = 6m + \frac{3(\alpha-1)}{2} > h^0 O_X(2D+F+K_X)$, using Lemma 1.4.16, for a generic $Z \in Hilb^{|Z|}(X)$ we have

$$H^0 I_Z(2D+F+K_X) = 0. \tag{3.5}$$

Therefore, putting this results together we get

$$\begin{aligned} h^1 I_Z(2D+F+K_X) &= -\chi(I_Z(2D+F+K_X)) \\ &= -\chi(O_X(2D+F+K_X)) + |Z|. \end{aligned}$$

By Riemann-Roch's Theorem (see 1.1.5) we have

$$\begin{aligned} \chi(O_X(2D+F+K_X)) &= \frac{(2D+F+K_X)(2D+F)}{2} + 1 \\ &= \frac{(2C_0+(2b+1)F-2cE_1)((2b-e-1)F-(2c-1)E_1)}{2} + 1 \\ &= 2b-e-2c^2+c. \end{aligned}$$

Finally, we substitute in (3.4) and we get

$$\begin{aligned} dim\mathcal{F} &= 2(6m + \tfrac{3(\alpha-1)}{2}) - 2b + e + 2c^2 - c + 6m + \tfrac{3(\alpha-1)}{2} - 1 \\ &= 4(4m+\alpha) - 3 \end{aligned}$$

which proves (b).

(c) It is easy to check that any $E \in \mathcal{F}$ is a rank two torsion free sheaf with Chern classes $c_1(E) = F$ and $c_2(E) = 4m + \alpha$. Let us see that E is a prioritary sheaf. Since E is torsion free, we only need to check that $Ext^2(E, E(-F)) = 0$ (see Definition 2.3.1). Applying the functor $Hom(., E(-F))$ to the exact sequence (3.1), we get the long exact sequence

$$\cdots \longrightarrow Ext^2(I_Z(D+F), E(-F)) \longrightarrow Ext^2(E, E(-F)) \longrightarrow$$
$$Ext^2(O_X(-D), E(-F)) \longrightarrow 0.$$

Claim 1: $Ext^2(O_X(-D), E(-F)) = 0$.

Proof of Claim 1: We consider the exact cohomology sequence

$$\cdots \longrightarrow H^2O_X(-F) \longrightarrow H^2E(D-F) \longrightarrow H^2I_Z(2D) \longrightarrow 0$$

associated to the exact sequence (3.1). By Serre's duality and Lemma 1.4.14 we have

$$H^2O_X(-F) = H^0O_X(F+K_X)^* = H^0O_X(-2C_0 - (e+1)F + E_1)^* = 0,$$
$$H^2I_Z(2D) = H^2O_X(2D) = H^0O_X(-2D+K_X)^*.$$

Since the divisor $-2D + K_X = -4C_0 - (2b+e+2)F + (2c+1)E_1$ is non effective (Lemma 1.4.14; (a)) we get

$$H^0O_X(-2D+K_X)^* = 0,$$

and hence, $Ext^2(O_X(-D), E(-F)) = H^2E(D-F) = 0$ which proves Claim 1.

Claim 2: $Ext^2(I_Z(D+F), E(-F)) = 0$.

Proof of Claim 2: Applying the functor $Hom(I_Z(D+2F), .)$ to the exact sequence (3.1), we get the long exact sequence

$$\cdots \longrightarrow Ext^2(I_Z(D+2F), O_X(-D)) \longrightarrow Ext^2(I_Z(D+2F), E) \longrightarrow$$
$$Ext^2(I_Z(D+2F), I_Z(D+F)) \longrightarrow 0.$$

By Serre's duality we have

$$\begin{aligned} Ext^2(I_Z(D+2F), O_X(-D)) &= Hom(O_X(-D), I_Z(D+2F+K_X))^* \\ &= H^0 I_Z(2D+2F+K_X)^* = 0 \end{aligned}$$

where the last group vanishes for a generic $Z \in Hilb^{|Z|}(X)$ due to the fact that

$$|Z| > h^0 O_X(2D+2F+K_X).$$

Using again Serre's duality we get

$$\begin{aligned} dimExt^2(I_Z(D+2F), I_Z(D+F)) &= dimHom(I_Z(D+F), I_Z(D+2F+K)) \\ &\leq dimHom(I_Z(D+F), O_X(D+2F+K)) \\ &= h^0 O_X(F+K_X) = 0. \end{aligned}$$

Therefore, $Ext^2(I_Z(D+2F), E) = 0$ which proves Claim 2.

It easily follows from Claim 1 and Claim 2 that E is a prioritary sheaf.

Let us see that E is simple, i.e., $dimHom(E, E) = 1$. We always have the inequality $1 \leq dimHom(E, E)$. Let us see the other one.

Applying the functor $Hom(., E)$ to the exact sequence (3.1), we get the long exact sequence

$$0 \longrightarrow Hom(I_Z(D+F), E) \longrightarrow Hom(E, E) \longrightarrow Hom(O_X(-D), E) \longrightarrow \cdots.$$

From (a) we have

$$dimHom(O_X(-D),E) = h^0E(D) = 1.$$

Hence, we only need to check that $Hom(I_Z(D+F),E) = 0$. To this end, we apply the functor $Hom(I_Z(D+F),.)$ to the exact sequence (3.1) and we obtain the long exact sequence

$$0 \longrightarrow Hom(I_Z(D+F),O_X(-D)) \longrightarrow Hom(I_Z(D+F),E) \longrightarrow$$
$$Hom(I_Z(D+F),I_Z(D+F)) \xrightarrow{\delta} Ext^1(I_Z(D+F),O_X(-D)) \longrightarrow \cdots .$$

Applying Serre's duality and Lemma 1.4.14; (a) we get

$$\begin{aligned} Hom(I_Z(D+F),O_X(-D)) &= Ext^2(O_X(-D),I_Z(D+F+K_X))^* \\ &= H^2I_Z(2D+F+K_X)^* \\ &= H^2O_X(2D+F+K_X)^* \\ &= H^0O_X(-2D-F) \\ &= H^0O_X(-2C_0-(2b+1)F+2cE_1) = 0. \end{aligned}$$

Since the extension ϵ given in (3.1) is non-trivial, the map

$$Hom(I_Z(D+F),I_Z(D+F)) \cong k \xrightarrow{\delta} Ext^1(I_Z(D+F),O_X(-D))$$

defined by $\delta(1) = \epsilon$, is an injection. Hence, $Hom(I_Z(D+F),E) = 0$ and E is simple.

We have a morphism

$$\phi : \mathcal{F} \longrightarrow Spl(2; F, 4m+\alpha)$$

from $\mathcal{F}$ to the moduli space $Spl(2; F, 4m+\alpha)$ of simple prioritary sheaves, which is an injection. Indeed, assume that there are two non-trivial extensions

$$0 \longrightarrow O_X(-D) \xrightarrow{\alpha_1} E \xrightarrow{\alpha_2} O_X(D+F) \otimes I_Z \longrightarrow 0;$$
$$0 \longrightarrow O_X(-D) \xrightarrow{\beta_1} E \xrightarrow{\beta_2} O_X(D+F) \otimes I_{Z'} \longrightarrow 0.$$

From (3.3), we have

$$Hom(O_X(-D), I_Z(D+F)) = H^0 I_Z(2D+F) = 0,$$

$$Hom(O_X(-D), I_{Z'}(D+F)) = H^0 I_{Z'}(2D+F) = 0.$$

Thus, $\beta_2 \circ \alpha_1 = \alpha_2 \circ \beta_1 = 0$. So, there exists $\gamma \in Aut(O_X(-D)) \cong k$ such that $\beta_1 = \alpha_1 \circ \gamma$. Therefore, $Z = Z'$ and ϕ is an injection.

Now, let us see that $Spl(2; F, 4m+\alpha)$ is rational. In fact, since the moduli space $Spl(2; F, 4m+\alpha)$ of simple prioritary sheaves is smooth and irreducible (Theorem 2.3.5), its rationality follows from the fact that ϕ is an injection, Remark 1.2.8, which states that $\mathcal{F}$ is rational and the fact that $dim\mathcal{F} = dimSpl(2; F, 4m+\alpha)$.

By Lemma 1.4.9, $L(K_{X_e} + F) < 0$. Thus, $L_n(K_X + F) < 0$ for $n >> 0$ and the moduli space $M_{X,L_n}(2; F, 4m+\alpha)$ is an open subscheme of the moduli space $Spl(2; F, 4m+\alpha)$ of simple prioritary sheaves (Lemma 2.3.2). Therefore, the moduli space $M_{X,L_n}(2; F, 4m+\alpha)$ is also rational and, as we pointed out at the beginning of the proof, this implies that the moduli space $M_{X_e,L}(2; F, 4m+\alpha)$ is rational, which proves what we want. □

Now we will deal with the remaining case: $c_1 = F$ and $c_2 \in \mathbb{Z}$ even.

Proposition 3.1.7. *Let X_e be a smooth, Hirzebruch surface and L an ample divisor on X_e. Then, the moduli space $M_L(2; F, 2n)$ is either empty or a smooth, irreducible, rational, quasi-projective variety of dimension $4(2n)-3$.*

Proof. Assume that the moduli space $M_L(2; F, 2n)$ is non-empty. Then, from Bogomolov's inequality we get

$$4(2n) > 2 - \frac{3K_{X_e}^2}{2}.$$

Therefore, since X_e is an anticanonical rational surface, we can apply Theorem 2.1.10 and Remark 1.2.14 and we only need to check the rationality of $M_L(2; F, 2n)$ for a suitable ample divisor L on X_e. We take $L = C_0 + (2e^2+n)F$ (see Remark 1.4.7).

We consider the irreducible family $\mathcal{F}_n$ of rank 2 vector bundles E on X_e given by a non trivial extension

$$(3.6) \qquad 0 \to O_{X_e}(-D) \to E \to O_{X_e}(D+F) \otimes I_Z \to 0$$

where $D = (n-1)F$ and Z is a locally complete intersection 0-cycle of length $2n$ such that $H^0 I_Z(2D+F) = 0$.

Notice that since $|Z| = 2n$, $2D + F = (2n-1)F$ and $h^0 O_X((2n-1)F) = 2n$ (see Lemma 1.4.8), the condition

$$H^0 I_Z(2D+F) = 0$$

is satisfied for all generic $Z \in Hilb^{2n}(X)$ (see Lemma 1.4.16). By [Mir93]; Proposition 1.3, $\mathcal{F}_n$ is non-empty.

Let us show:

(a) $h^0 E(D) = 1$.

(b) $dim \mathcal{F}_n = 4(2n) - 3$.

(c) There is an injection $\mathcal{F}_n \hookrightarrow M_L(2; F, 2n)$.

(a) It follows from the exact cohomology sequence associated to the exact sequence (3.6) and the fact that $H^0 I_Z(2D+F) = 0$.

(b) By definition we have

$$\begin{aligned} dim \mathcal{F}_n &= \#moduli(Z) + dimExt^1(I_Z(D+F), O_{X_e}(-D)) - h^0 E(D) \\ &= 2length(Z) + dimExt^1(I_Z, O_{X_e}(-2D-F)) - h^0 E(D) \\ &= 2length(Z) + dimExt^1(I_Z, O_{X_e}(-2D-F)) - 1 \end{aligned}$$

where the last equality follows from (a).

Applying the functor $Hom(., O_{X_e})$ to the exact sequence

$$0 \longrightarrow I_Z(2D+F) \longrightarrow O_{X_e}(2D+F) \longrightarrow O_Z(2D+F) \longrightarrow 0$$

we get the long exact sequence

$$0 \to H^1 O_{X_e}(-2D-F) \to Ext^1(I_Z, O_{X_e}(-2D-F)) \to$$

$$H^0 O_Z \to H^2 O_{X_e}(-2D-F) \to Ext^2(I_Z, O_{X_e}(-2D-F)) \to 0.$$

But the divisor $-2D-F$ is not effective, so we obtain

$$\begin{aligned} dim Ext^1(I_Z, O_{X_e}(-2D-F)) &= dim Ext^2(I_Z, O_{X_e}(-2D-F)) \\ &\quad + h^0 O_Z - \chi(O_{X_e}(-2D-F)). \end{aligned}$$

By Serre's duality, we have

$$\begin{aligned} Ext^2(I_Z, O_{X_e}(-2D-F)) &= Hom(O_{X_e}(-2D-F), I_Z(K_{X_e}))^* \\ &= H^0 I_Z(2D+F+K_{X_e})^* \\ &= H^0 I_Z(-2C_0+(2n-e-3)F)^* = 0 \end{aligned}$$

and applying Riemann-Roch's Theorem, we get

$$\begin{aligned} \chi(O_{X_e}(-2D-F)) &= \tfrac{(-2D-F)(-2D-F-K_{X_e})}{2} + 1 \\ &= -\tfrac{(2C_0+(e+3-2n)F)((2n-1)F)}{2} + 1 \\ &= 2-2n. \end{aligned}$$

Therefore,

$$dim Ext^1(I_Z, O_{X_e}(-2D-F)) = length(Z) - (2-2n) = 4n-2$$

and

$$dim \mathcal{F}_n = 2(2n) + (4n-2) - 1 = 4(2n) - 3$$

which proves (b).

(c) Let $E \in \mathcal{F}_n$ be a rank two vector bundle on X_e given by a non-trivial extension

$$0 \to O_{X_e}(-D) \to E \to O_{X_e}(D+F) \otimes I_Z \to 0$$

where $Z \subset X_e$ is a locally complete intersection 0-cycle of length $2n$ such that $H^0 I_Z(2D+F) = 0$.

It is easy to check that $c_1(E) = F$ and $c_2(E) = 2n$. Let us see that E is L-stable; i.e., for any rank 1 subbundle $O_{X_e}(G)$ of E we have $c_1(O_{X_e}(G))L < \frac{1}{2}$ or, equivalently,

$$c_1(O_{X_e}(G))L < \frac{c_1(E)L}{2}.$$

Indeed, since E sits in an extension

$$0 \to O_{X_e}(-(n-1)F) \to E \to O_{X_e}(nF) \otimes I_Z \to 0$$

we have

$$(1) \quad O_{X_e}(G) \hookrightarrow O_{X_e}(-(n-1)F) \quad \text{or}$$

$$(2) \quad O_{X_e}(G) \hookrightarrow O_{X_e}(nF) \otimes I_Z.$$

In the first case, $-G-(n-1)F$ is an effective divisor. Since L is an ample divisor we have $(-G-(n-1)F)L \geq 0$ and

$$c_1(O_{X_e}(G))L = GL \leq -(n-1)FL = -(n-1) < \frac{1}{2} = \frac{c_1(E)L}{2}.$$

If $O_{X_e}(G) \hookrightarrow O_{X_e}(nF) \otimes I_Z$ then $nF - G$ is an effective divisor. On the other hand, we have

$$\begin{aligned} H^0 O_{X_e}(G+(n-1)F) &\subset H^0 I_Z((2n-1)F) \\ &= H^0 I_Z(2D+F) = 0. \end{aligned}$$

So $G+(n-1)F$ is not an effective divisor and writing $G = \alpha C_0 + \beta F$, we have either $\beta + n - 1 < 0$ or $\alpha < 0$ (see Remark 1.4.7).

Assume that $\beta + n - 1 < 0$ (in particular $\beta < 0$). Since $nF - G$ is an effective divisor it must be $\alpha \leq 0$ (Remark 1.4.7) and we have

$$\begin{aligned} c_1(O_{X_e}(G))L = GL &= -\alpha e + \alpha(2e^2 + n) + \beta \\ &= \alpha(2e^2 - e + n) + \beta \\ &< \tfrac{1}{2} = \tfrac{c_1(E)L}{2}. \end{aligned}$$

Assume that $\alpha < 0$ and $\beta + n - 1 \geq 0$. Using again the fact that $nF - G$ is an effective divisor and hence $\beta \leq n$, we obtain

$$\begin{aligned} c_1(O_{X_e}(G))L = GL &= -\alpha e + \alpha(2e^2 + n) + \beta \\ &\leq -\alpha e + \alpha(2e^2 + n) + n \\ &= n(\alpha + 1) + e\alpha(2e - 1) \\ &< \tfrac{1}{2} = \tfrac{c_1(E)L}{2}, \end{aligned}$$

which proves the L-stability of E. Thus, we have a morphism

$$\phi : \mathcal{F}_n \longrightarrow M_L(2; F, 2n)$$

which is an injection because by assumption $H^0 I_Z(2D + F) = 0$.

Finally, since the moduli space $M_L(2; F, 2n)$ is smooth and irreducible (Remark 1.2.14), its rationality easily follows from (c), Remark 1.2.8 and the fact that $dim \mathcal{F}_n = dim M_L(2; F, 2n)$. □.

Putting this results together we get the main result of this section.

Theorem 3.1.8. *Let X be a smooth, irreducible, projective, minimal, rational surface, $c_1 \in Pic(X)$ and $c_2 \in \mathbb{Z}$. Then, for any polarization L on X, the moduli space $M_L(2; c_1, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - c_1^2 - 3$, whenever non-empty.*

Proof. It follows from Theorem 1.4.1, Propositions 3.1.3, 3.1.5, 3.1.6, 3.1.7 and Theorem 3.1.1. □

3.2 Moduli spaces of vector bundles on Fano surfaces

In this section, we will prove the rationality of moduli spaces $M_L(2; c_1, c_2)$ of rank two, L-stable, vector bundles on a Fano surface X. According to Theorem 1.4.5, X is isomorphic to either $\mathbb{P}^2$, or the quadric surface in $\mathbb{P}^3$ or the blow up of $\mathbb{P}^2$ at s different points with $1 \leq s \leq 8$. Since the cases $X = \mathbb{P}^2$ and $X = X_e$ with $e = 0$ (i.e X is the quadric surface in $\mathbb{P}^3$) have been studied in the first section of this Chapter, now we will deal with the remaining cases. So, throughout this section, X is the blow up of $\mathbb{P}^2$ at s different points with $1 \leq s \leq 8$.

Remark 3.2.1. Since a rank 2 vector bundle E on X is H-stable if, and only if, $E \otimes O_X(G)$ is H-stable for any divisor $G \in Pic(X)$, we may assume, without loss of generality, that $c_1(E)$ is one of the following: 0, E_0, E_i with $i = 1, \cdots, s$, $\sum_{j=2}^{\rho} E_{i_j}$ with $2 \leq \rho \leq s$ or $E_0 + \sum_{j=1}^{\rho} E_{i_j}$ with $1 \leq \rho \leq s$; and $c_2(E) \geq 1$([Mir93]; Theorem 2.1 and Theorem 2.2). For simplicity, we will write $\sum_{i=1}^{\rho} E_i$ instead of $\sum_{j=1}^{\rho} E_{i_j}$.

We will discuss separately all possible values of c_1 (Remark 3.2.1). Let us start with the case $c_1 = 0$.

Proposition 3.2.2. *Let X be a Fano surface obtained blowing up s, $1 \leq s \leq 8$, different points of $\mathbb{P}^2$ and L any ample divisor on X. Then the moduli space $M_L(2; 0, c_2)$ is either empty or a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - 3$.*

Proof. For $t >> 0$, the divisor $H = tE_0 - \sum_{i=1}^{s} E_i$ is ample on X and there is an open immersion (Theorem 1.2.15)

$$M_{\mathbb{P}^2, O_{\mathbb{P}^2}(1)}(2; 0, c_2) \hookrightarrow M_{X,H}(2; 0, c_2).$$

Furthermore, $M_{\mathbb{P}^2, O_{\mathbb{P}^2}(1)}(2; 0, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - 3$ (Theorem 3.1.1) and $M_{X,H}(2; 0, c_2)$ is a smooth, irreducible, quasi-projective variety of dimension $4c_2 - 3$. Hence, $M_{X,H}(2; 0, c_2)$ is rational and the moduli space $M_L(2;0,c_2)$ is empty or a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2-3$ (Corollary 2.1.11 and Remark 1.2.14). $\square$

In the next two propositions, we will consider the case $c_1 = E_0$ and the case $c_1 = \sum_{j=1}^{\rho} E_{i_j}$ and using Criterion 2.1.13 we will prove the rationality of the moduli space $M_L(2;c_1,c_2)$.

Proposition 3.2.3. *Let X be a Fano surface obtained blowing up s, $1 \le s \le 8$, different points of $\mathbb{P}^2$. Then, the following holds*

(1) There exists an ample divisor $\tilde{L}$ on X such that the moduli space $M_{\tilde{L}}(2;E_0,c_2)$ is a smooth, irreducible, rational, projective variety of dimension $4c_2-4$ and $Pic(M_{\tilde{L}}(2;E_0,c_2)) \cong \mathbb{Z}$ whenever non-empty.

(2) For any ample divisor L on X, the moduli space $M_L(2;E_0,c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2-4$, whenever non-empty.

Proof. We take the numerical equivalence class, say $\xi = (1-2c_2)E_0 + 2c_2E_1$. For all $c_2 > 0$, $\xi \equiv (1-2c_2)E_0 + 2c_2E_1$ defines a non-empty wall of type (E_0,c_2) and $d(\xi) = 0$ (see Definition 1.3.3 and Corollary 2.1.4). Indeed, since

$$c_1^2 - 4c_2 = 1 - 4c_2 = \xi^2 < 0,$$

$$\xi + c_1 = (2-2c_2)E_0 + 2c_2E_1$$

and

$$\begin{aligned} d(\xi) &= \tfrac{c_1^2-4c_2}{4} + \tfrac{\xi^2}{4} + \tfrac{\xi K_X}{2} + 1 \\ &= \tfrac{\xi^2}{2} + \tfrac{\xi K_X}{2} + 1 \\ &= \tfrac{1-4c_2}{2} + \tfrac{((1-2c_2)E_0+2c_2E_1)(-3E_0+\sum_{i=1}^{s} E_i)}{2} + 1 \\ &= 0, \end{aligned}$$

we only have to check that there exist ample divisors L_t and L_n' on X such that

$$\xi L_t < 0 < \xi L_n'.$$

It is well known that the divisor $L_t = tE_0 - \sum_{i=1}^{s} E_i$ is ample for $t \geq 3$ and we have

$$\begin{aligned} L_t\xi &= (tE_0 - \textstyle\sum_{i=1}^{s} E_i)((1-2c_2)E_0 + 2c_2E_1) \\ &= t(1-2c_2) + 2c_2 < 0. \end{aligned}$$

On the other hand, we take the divisor

$$L'_n = n((4c_2+1)E_0 - (4c_2-1)E_1) - \sum_{j=2}^{s} E_j.$$

For $n >> 0$, L'_n is ample (see [Kuc96]) and we have

$$\xi L'_n = ((1-2c_2)E_0 + 2c_2E_1)(n(4c_2+1)E_0 - n(4c_2-1)E_1) = n > 0$$

which proves what we want.

If the moduli space $M_L(2; E_0, c_2)$ is non-empty, using Bogomolov's inequality $c_1^2 - 4c_2 < 0$ and the fact that $K_X^2 = 9 - s > 0$, we obtain

$$4c_2 - c_1^2 > 2 - \frac{3K_X^2}{2}.$$

Thus, we can apply Criterion 2.1.13 and we deduce what was stated. □

Now we will deal with the case $c_1 = \sum_{j=1}^{\rho} E_{i_j}$.

Proposition 3.2.4. *Let X be a Fano surface obtained blowing up s, $1 \leq s \leq 8$, different points of $\mathbb{P}^2$. Then, the following is satisfied*

(1) There exists an ample divisor $\tilde{L}$ on X such that $M_{\tilde{L}}(2; \sum_{j=1}^{\rho} E_{i_j}, c_2)$ is a smooth, irreducible, rational, projective variety of dimension $4n + \rho - 3$ and $Pic(M_{\tilde{L}}(2; \sum_{j=1}^{\rho} E_{i_j}, c_2)) \cong \mathbb{Z}$ whenever non-empty.

(2) For any ample divisor L on X, the moduli space $M_L(2; \sum_{j=1}^{\rho} E_{i_j}, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4n + \rho - 3$, whenever non-empty.

Proof. We take the numerical equivalence class, say

$$\xi = -2c_2E_0 + (2c_2+1)E_{i_1} - E_{i_2} - \dots - E_{i_\rho}, \quad \text{with} \quad 1 \le \rho \le s.$$

For all $c_2 > 0$, $\xi \equiv -2c_2E_0 + (2c_2+1)E_{i_1} - E_{i_2} - \dots - E_{i_\rho}$ defines a non-empty wall of type $(\sum_{j=1}^{\rho} E_{i_j}, c_2)$ and $d(\xi) = 0$ (see Definition 1.3.3 and Corollary 2.1.4). Indeed, since

$$c_1^2 - 4c_2 = -\rho - 4c_2 = \xi^2 < 0,$$

$$\xi + c_1 = 2(-c_2E_0 + (c_2+1)E_{i_1})$$

and

$$\begin{aligned} d(\xi) &= \tfrac{c_1^2-4c_2}{4} + \tfrac{\xi^2}{4} + \tfrac{\xi K_X}{2} + 1 \\ &= \tfrac{\xi^2}{2} + \tfrac{\xi K_X}{2} + 1 \\ &= -\tfrac{\rho+4c_2}{2} + \tfrac{(-2c_2E_0+(2c_2+1)E_{i_1} - \sum_{j=2}^{\rho} E_{i_j})(-3E_0+\sum_{i=1}^{s} E_i)}{2} + 1 \\ &= 0 \end{aligned}$$

we only have to check that there exist ample divisors L_t and L_n' on X such that

$$\xi L_t < 0 < \xi L_n'.$$

We take the divisors $L_t = tE_0 - \sum_{i=1}^{s} E_i$ and $L_n' = n(4c_2E_0 - (4c_2-1)E_{i_1}) - \sum_{j=2}^{s} E_{i_j}$ on X. For $t >> 0$ (resp. $n >> 0$) the divisor L_t (resp. L_n') is ample (see [Kuc96]). Moreover, for $t >> 0$ and $n >> 0$, we have

$$L_t\xi = -2c_2t + (2c_2+1) - (\rho - 1) < 0$$

$$L_n'\xi = -2c_2n(4c_2) + n(4c_2-1)(2c_2+1) - (\rho-1) = n(2c_2-1) - (\rho-1) > 0,$$

which proves that ξ defines a non-empty wall of type $(\sum_{j=1}^{\rho} E_{i_j}, c_2)$ (see Remark 1.3.7).

If the moduli space $M_L(2; \sum_{j=1}^{\rho} E_{i_j}, c_2)$ is non-empty, using Bogomolov's inequality, $c_1^2 - 4c_2 < 0$, and the fact that $K_X^2 = 9 - s > 0$, we obtain

$$4c_2 - c_1^2 > 2 - \frac{3K_X^2}{2}.$$

Thus, we can apply again Criterion 2.1.13 and we deduce what we want. □

For the remaining values of c_1, there is not a numerical equivalence class ξ satisfying the hypothesis of Criterion 2.1.13. In these cases, we will prove the rationality constructing a suitable family over a big enough rational variety. Let us start with the case $c_1 = E_0 + \sum_{i=1}^{\rho} E_i$.

Proposition 3.2.5. *Let X be a Fano surface obtained blowing up s, $1 \leq s \leq 8$, different points of $\mathbb{P}^2$ and L any ample divisor on X. Then, the moduli space $M_L(2; E_0+\sum_{i=1}^{\rho} E_i, c_2)$, $2 \leq \rho \leq s$, is either empty or a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 + \rho - 4$.*

Proof. By Corollary 2.1.11 we only need to check the rationality of the moduli space $M_L(2; E_0 + \sum_{i=1}^{\rho} E_i, c_2)$ for a suitable ample divisor L on X and by Theorem 1.2.15 we can assume $s = \rho$. To this end, we take an ample divisor L on X such that $L(K_X + (E_0 - E_1)) < 0$ (see Remark 1.4.10).

We consider $c_2 = 2n + \beta$ with $\beta \in \{0, 1\}$ and the irreducible family $\mathcal{F}_{n,\beta}$ of rank 2 vector bundles E on X given by a non-trivial extension

$$(3.7) \quad \epsilon: \quad 0 \to O_X(-D) \to E \to O_X(D + E_0 + \sum_{i=1}^{\rho} E_i) \otimes I_Z \to 0$$

where $D = (n + 1 - \beta)E_0 - (n + 2 - \beta)E_1 - (2 - \beta)E_2$ and Z is a locally complete intersection 0-cycle of length $2n - 2 + 3\beta$ verifying $H^0 I_Z(2D + E_0 + \sum_{i=1}^{\rho} E_i) = 0$.

Applying Lemma 1.4.12; (a) and using the fact that there is no plane curve of degree $2n + 3$ with a point of multiplicity $2n + 3$ and a point of multiplicity three we get

$$h^0 O_X(2D + E_0 + \sum_{i=1}^{\rho} E_i) = \begin{cases} 0 & \text{if } \beta = 0 \\ 2n & \text{if } \beta = 1 \end{cases}$$

and we infer the existence of such a 0-cycle Z from the inequality

$$|Z| \geq h^0 O_X(2D + E_0 + \sum_{i=1}^{\rho} E_i) \quad .$$

(see Lemma 1.4.16). By [Mir93]; Proposition 1.3, $\mathcal{F}_{n,\beta}$ is non-empty. Let us show

(a) $h^0E(D) = 1$.

(b) $dim\mathcal{F}_{n,\beta} = 4c_2 + \rho - 4$.

(c) Any $E \in \mathcal{F}_{n,\beta}$ is a simple prioritary vector bundle and it has Chern classes $(E_0 + \sum_{i=1}^{\rho} E_i, c_2)$.

(a) From the exact cohomology sequence

$$0 \to H^0O_X \to H^0E(D) \to H^0I_Z(2D + E_0 + \sum_{i=1}^{\rho} E_i) \to 0$$

associated to the exact sequence (3.7) and the fact $H^0I_Z(2D + E_0 + \sum_{i=1}^{\rho} E_i) = 0$ we get

$$h^0E(D) = h^0O_X + h^0I_Z(2D + E_0 + \sum_{i=1}^{\rho} E_i) = 1$$

which proves (a).

(b) By definition we have

$$\begin{aligned} dim\mathcal{F}_{n,\beta} &= \#moduli(Z) + dimExt^1(I_Z(D + E_0 + \textstyle\sum_{i=1}^{\rho} E_i), O_X(-D)) - h^0E(D) \\ &= 2length(Z) + dimExt^1(I_Z, O_X(-2D - E_0 - \textstyle\sum_{i=1}^{\rho} E_i)) - 1 \end{aligned}$$

where the last equality follows from (a). Applying the functor $Hom(., O_X)$ to the exact sequence

$$0 \longrightarrow I_Z(2D+E_0+\sum_{i=1}^{\rho} E_i) \longrightarrow O_X(2D+E_0+\sum_{i=1}^{\rho} E_i) \longrightarrow O_Z(2D+E_0+\sum_{i=1}^{\rho} E_i) \longrightarrow 0$$

we get the long exact cohomology sequence

$$0 \to H^1O_X(-2D - E_0 - \textstyle\sum_{i=1}^{\rho} E_i) \to Ext^1(I_Z, O_X(-2D - E_0 - \sum_{i=1}^{\rho} E_i)) \to$$

$$H^0O_Z \to H^2O_X(-2D - E_0 - \textstyle\sum_{i=1}^{\rho} E_i) \to Ext^2(I_Z, O_X(-2D - E_0 - \sum_{i=1}^{\rho} E_i)) \to 0.$$

Since the divisor $-2D - E_0 - \sum_{i=1}^{\rho} E_i$ is not effective (Lemma 1.4.12; (b)), we have

$$h^1O_X(-2D-E_0-\sum_{i=1}^{\rho} E_i)-h^2O_X(-2D-E_0-\sum_{i=1}^{\rho} E_i) = -\chi(O_X(-2D-E_0-\sum_{i=1}^{\rho} E_i))$$

and hence, we obtain

$$dimExt^1(I_Z, O_X(-2D - E_0 - \sum_{i=1}^{\rho} E_i)) = dimExt^2(I_Z, O_X(-2D - E_0 - \sum_{i=1}^{\rho} E_i)) + h^0O_Z - \chi(O_X(-2D - E_0 - \sum_{i=1}^{\rho} E_i)).$$

By Riemann-Roch's Theorem (see 1.1.5) we have

$$\begin{aligned}\chi(O_X(-2D - E_0 - \sum_{i=1}^{\rho} E_i)) &= \frac{(-(2D+E_0+\sum_{i=1}^{\rho} E_i))(-(2D+E_0+\sum_{i=1}^{\rho} E_i)+3E_0-\sum_{i=1}^{\rho} E_i)}{2} + 1 \\ &= -\frac{(-(2n+3-2\beta)E_0+(2n+3-2\beta)E_1+(3-2\beta)E_2-\sum_{i=3}^{\rho} E_i)((2n-2\beta)E_0)}{2} \\ &+\frac{(-(2n+3-2\beta)E_0+(2n+3-2\beta)E_1+(3-2\beta)E_2-\sum_{i=3}^{\rho} E_i)((2n+2-2\beta)E_1)}{2} \\ &+\frac{(-(2n+3-2\beta)E_0+(2n+3-2\beta)E_1+(3-2\beta)E_2-\sum_{i=3}^{\rho} E_i)((2-2\beta)E_2)}{2} \\ &-\frac{(-(2n+3-2\beta)E_0+(2n+3-2\beta)E_1+(3-2\beta)E_2-\sum_{i=3}^{\rho} E_i)(2\sum_{i=3}^{\rho} E_i)}{2} + 1 \\ &= \frac{(2n+3-2\beta)(2n-2\beta)-(2n+3-2\beta)(2n+2-2\beta)-(3-2\beta)(2-2\beta)-2(\rho-2)}{2} + 1 \\ &= -2n + 2\beta - \rho - (3-2\beta)(1-\beta),\end{aligned}$$

and using Serre's duality, we get

$$\begin{aligned}Ext^2(I_Z, O_X(-2D - E_0 - \sum_{i=1}^{\rho} E_i)) &= Hom(O_X(-2D - E_0 - \sum_{i=1}^{\rho} E_i), I_Z(K_X))^* \\ &= H^0I_Z(2D + E_0 + \sum_{i=1}^{\rho} E_i + K_X)^* = 0\end{aligned}$$

where the last equality follows from the fact that

$$2D + E_0 + \sum_{i=1}^{\rho} E_i + K_X = (2n-2\beta)E_0 - (2n+2-2\beta)E_1 - (2-2\beta)E_2 + 2\sum_{i=3}^{\rho} E_i$$

is a non effective divisor. Indeed, it follows from Lemma 1.4.12; (a) and the fact that there is no plane curve of degree $(2n-2\beta)$ with a point of multiplicity $2n+2-2\beta$.

Putting all this together we obtain

$$\begin{aligned}dimExt^1(I_Z, O_X(-2D - E_0 - \sum_{i=1}^{\rho} E_i)) &= length(Z) - (-2n + 2\beta - \rho - (3-2\beta)(1-\beta)) \\ &= 4n + \beta + \rho - 2 + (3-2\beta)(1-\beta)\end{aligned}$$

and

$$\begin{aligned} dim\mathcal{F}_{n,\beta} &= 2(2n-2+3\beta)+(4n+\beta+\rho-2+(3-2\beta)(1-\beta))-1 \\ &= 8n-2(1-\beta)(\beta+2)+\rho \\ &= 4c_2+\rho-4 \end{aligned}$$

which proves (b).

(c) It is easy to see that for any $E \in \mathcal{F}_{n,\beta}$, $c_1(E) = E_0 + \sum_{i=1}^{\rho} E_i$ and $c_2(E) = c_2$. Let us see that E is a simple prioritary sheaf. Since E is torsion free, in order to see that E is a prioritary sheaf, we only need to see that $Ext^2(E, E(-(E_0 - E_1))) = 0$ (see Definition 2.3.1 and Remark 1.4.11).

Applying the functor $Hom(., E(-E_0 + E_1))$ to the exact sequence (3.7), we get the long exact cohomology sequence

$$\cdots \longrightarrow Ext^2(I_Z(D+E_0+\textstyle\sum_{i=1}^{\rho} E_i), E(-E_0+E_1)) \longrightarrow$$

$$Ext^2(E, E(-E_0+E_1)) \longrightarrow Ext^2(O_X(-D), E(-E_0+E_1)) \longrightarrow 0$$

Claim 1: $Ext^2(O_X(-D), E(-E_0+E_1)) = H^2E(D-E_0+E_1) = 0.$

Proof of Claim 1: We consider the exact cohomology sequence

$$\longrightarrow H^2O_X(-E_0+E_1) \longrightarrow H^2E(D-E_0+E_1) \longrightarrow H^2I_Z(2D+2E_1+\sum_{i=2}^{\rho} E_i) \longrightarrow 0$$

associated to the exact sequence (3.7). Using Serre's duality and the fact that Z is a zero dimensional subscheme we get

$$\begin{aligned} H^2I_Z(2D+2E_1+\textstyle\sum_{i=2}^{\rho} E_i) &= H^2O_X(2D+2E_1+\textstyle\sum_{i=2}^{\rho} E_i) \\ &= H^0O_X(-2D-2E_1-\textstyle\sum_{i=2}^{\rho} E_i+K_X)^* = 0 \end{aligned}$$

where the last equality follows from the fact that the divisor

$$-2D-2E_1-\sum_{i=2}^{\rho} E_i+K_X = -(2n+5-2\beta)E_0+(2n+3-2\beta)E_1+(4-2\beta)E_2$$

is non effective (Lemma 1.4.12; (b)). Therefore, since

$$H^2\mathcal{O}_X(-E_0+E_1)=H^0\mathcal{O}_X(E_0-E_1+K_X)^*=0$$

we obtain

$$Ext^2(\mathcal{O}_X(-D),E(-E_0+E_1))=H^2E(D-E_0+E_1)=0,$$

which proves Claim 1.

Claim 2: $Ext^2(I_Z(D+E_0+\sum_{i=1}^{\rho}E_i),E(-E_0+E_1))=0.$

Proof of Claim 2. Applying the functor $Hom(I_Z(D+E_0+\sum_{i=1}^{\rho}E_i),.)$ to the exact sequence (3.7), we get the long exact sequence

$$\longrightarrow Ext^2(I_Z,\mathcal{O}_X(-2D-2E_0-\textstyle\sum_{i=2}^{\rho}E_i))\longrightarrow Ext^2(I_Z(D+2E_0+\sum_{i=2}^{\rho}E_i),E)$$
$$\longrightarrow Ext^2(I_Z,I_Z(-E_0+E_1))\longrightarrow 0.$$

Since Z is a 0-dimensional subscheme, by Serre's duality we get

$$\begin{aligned} dim Ext^2(I_Z,I_Z(-E_0+E_1)) &= dim Hom(I_Z(-E_0+E_1),I_Z(K_X))\\ &\le dim Hom(I_Z(-E_0+E_1),\mathcal{O}_X(K_X))\\ &= h^0\mathcal{O}_X(E_0-E_1+K_X)=0 \end{aligned}$$

where the last equality follows from Lemma 1.4.12; (b). Finally, using once again Serre's duality we obtain

$$\begin{aligned} Ext^2(I_Z,\mathcal{O}_X(-2D-2E_0-\textstyle\sum_{i=2}^{\rho}E_i)) &= Hom(\mathcal{O}_X,I_Z(2D-E_0+E_1+2\textstyle\sum_{i=2}^{\rho}E_i))^*\\ &= H^0I_Z(2D-E_0+E_1+2\textstyle\sum_{i=2}^{\rho}E_i)^*=0 \end{aligned}$$

where the last equality follows from the fact that the divisor

$$2D-E_0+E_1+2\sum_{i=2}^{\rho}E_i=(2n+1-2\beta)E_0-(2n+3-2\beta)E_1-(2-2\beta)E_2+2\sum_{i=3}^{\rho}E_i$$

is not effective. Indeed, it follows from Lemma 1.4.12; (a) and the fact that there is no plane curve of degree $2n+1-2\beta$ with a point of multiplicity $2n+3-2\beta$. Therefore,

$$Ext^2(I_Z(D+E_0+\sum_{i=1}^{\rho}E_i), E(-E_0+E_1)) = 0$$

which proves Claim 2.

It easily follows from Claim 1 and Claim 2 that E is a prioritary sheaf.

Now we will see that E is a simple sheaf, i.e., $dimHom(E,E)=1$.

Applying the functor $Hom(.,E)$ to the exact sequence (3.7) we get the long exact sequence

$$0 \longrightarrow Hom(I_Z(D+E_0+\sum_{i=1}^{\rho}E_i),E) \longrightarrow Hom(E,E) \longrightarrow Hom(O_X(-D),E) \longrightarrow \cdots .$$

From (a) we have

$$dimHom(O_X(-D),E) = h^0E(D) = 1.$$

Hence, we only have to see that $Hom(I_Z(D+E_0+\sum_{i=1}^{\rho}E_i),E)=0$. To this end, we apply the functor $Hom(I_Z(D+E_0+\sum_{i=1}^{\rho}E_i),.)$ to the exact sequence (3.7) and we get the long exact sequence

$$0 \longrightarrow Hom(I_Z(D+E_0+\sum_{i=1}^{\rho}E_i),O_X(-D)) \longrightarrow Hom(I_Z(D+E_0+\sum_{i=1}^{\rho}E_i),E) \longrightarrow$$

$$Hom(I_Z(D+E_0+\sum_{i=1}^{\rho}E_i),I_Z(D+E_0+\sum_{i=1}^{\rho}E_i)) \xrightarrow{\delta} Ext^1(I_Z(D+E_0+\sum_{i=1}^{\rho}E_i),O_X(-D)) \longrightarrow .$$

Since E is given by a non-trivial extension ϵ, the map

$$\begin{array}{cccc} \delta : Hom(I_Z,I_Z)\cong k & \longrightarrow & Ext^1(I_Z(D+E_0+\sum_{i=1}^{\rho}E_i),O_X(-D)) \\ 1 & \longrightarrow & \epsilon \end{array}$$

is an injection, which together with

$$Hom(I_Z(D+E_0+\sum_{i=1}^{\rho}E_i),O_X(-D)) = H^0O_X(-2D-E_0-\sum_{i=1}^{\rho}E_i) = 0$$

gives us

$$Hom(I_Z(D + E_0 + \sum_{i=1}^{\rho} E_i), E) = 0$$

which proves that E is simple and hence (c).

It follows from (c) that there is an open injection

$$\phi : \mathcal{F}_{n,\beta} \longrightarrow Spl(2; E_0 + \sum_{i=1}^{\rho} E_i, c_2)$$

that maps the irreducible family $\mathcal{F}_{n,\beta}$ into the moduli space $Spl(2; E_0 + \sum_{i=1}^{\rho} E_i, c_2)$ of simple prioritary sheaves. Indeed, assume that there are two non-trivial extensions

$$0 \rightarrow O_X(-D) \xrightarrow{\alpha_1} E \xrightarrow{\alpha_2} O_X(D + E_0 + \sum_{i=1}^{\rho} E_i) \otimes I_Z \rightarrow 0;$$

$$0 \rightarrow O_X(-D) \xrightarrow{\lambda_1} E \xrightarrow{\lambda_2} O_X(D + E_0 + \sum_{i=1}^{\rho} E_i) \otimes I_{Z'} \rightarrow 0.$$

Since $Hom(O_X(-D), I_Z(D + E_0 + \sum_{i=1}^{\rho} E_i) = H^0 I_Z(2D + E_0 + \sum_{i=1}^{\rho} E_i) = 0$ we have $\lambda_2 \circ \alpha_1 = \alpha_2 \circ \lambda_1 = 0$. So, there exists $\mu \in Aut(O_X(-D)) \cong k$ such that $\alpha_1 = \lambda_1 \circ \mu$. Therefore, $Z = Z'$ and ϕ is an injection.

Since the moduli space $Spl(2; E_0 + \sum_{i=1}^{\rho} E_i, c_2)$ of simple prioritary sheaves is smooth and irreducible (Theorem 2.3.5) of dimension $4c_2 + \rho - 4$, its rationality easily follows from Remark 1.2.8 and the fact that $dim\mathcal{F}_{n,\beta} = dimSpl(2; E_0 + \sum_{i=1}^{\rho} E_i, c_2)$. Moreover, since $L(K_X + E_0 - E_1) < 0$, the moduli space $M_L(2; E_0 + \sum_{i=1}^{\rho} E_i, c_2)$ is an open dense subset of $Spl(2; E_0 + \sum_{i=1}^{\rho} E_i, c_2)$ (Lemma 2.3.2). Therefore, the moduli space $M_L(2; E_0 + \sum_{i=1}^{\rho} E_i, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 + \rho - 4$, whenever non-empty. □

Notice that in the previous Proposition we have studied the rationality of the moduli space $M_L(2; E_0 + \sum_{i=1}^{\rho} E_i, c_2)$, assuming that $\rho \geq 2$. Hence, it remains the case $c_1 = E_0 - E_1$, that will be treated in the next Proposition.

Proposition 3.2.6. *Let X be a Fano surface obtained blowing up s, $1 \leq s \leq 8$, different points of $\mathbb{P}^2$ and L any ample divisor on X. Then, the moduli space $M_L(2; E_0+E_1, c_2)$, is either empty or a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - 3$.*

Proof. By Remark 1.2.14 and Corollary 2.1.11 we only need to check the rationality of $M_L(2; E_0 + E_1, c_2)$ for a suitable ample divisor L on X and by Theorem 1.2.15 we can assume that $s = 1$, i.e., that X is obtained by blowing up one point of $\mathbb{P}^2$.

Since $X \cong X_e$ with $e = 1$, the Proposition follows from Remark 1.4.11 and Propositions 3.1.6 and 3.1.7. □

The results of this section will appear in [CM98] and they can be summarized in the following Theorem.

Theorem 3.2.7. *Let X be a Fano surface, $c_1 \in Pic(X)$ and $c_2 \in \mathbb{Z}$. Then, for any polarization L on X, the moduli space $M_L(2; c_1, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - c_1^2 - 3$, whenever non-empty.*

Proof. It follows from Theorem 1.4.5, Propositions 3.2.2-6 and Theorem 3.1.8. □

3.3 Moduli spaces of vector bundles on non-minimal rational surfaces

In this section we prove the rationality of the moduli space $M_L(2; c_1, c_2)$ of rank two, L-stable vector bundles E with Chern classes $c_1(E) = c_1$ and $c_2(E) = c_2$, over a smooth, non-minimal, rational surface X, i.e. the underlying surface X of the moduli spaces we deal with is a smooth, irreducible, Hirzebruch surface X_e, blown up at s different points. Since in section two we have studied moduli spaces of vector bundles on Fano surfaces, in this section we can assume that X is obtained blowing up $s \geq 8$ different points of X_e.

Remark 3.3.1. Since a rank 2 vector bundle E on X is H-stable if, and only if, $E \otimes O_X(G)$ is H-stable for any divisor $G \in Pic(X)$, we may assume, without loss of generality, that $c_1(E)$ is one of the following: 0, $\sum_{j=1}^{\rho} E_{i_j}$ with $1 \leq \rho \leq s$, C_0, F, $C_0 + F$, $C_0 + \sum_{j=1}^{\rho} E_{i_j}$ with $1 \leq \rho \leq s$, $F + \sum_{j=1}^{\rho} E_{i_j}$ with $1 \leq \rho \leq s$ or $C_0 + F + \sum_{j=1}^{\rho} E_{i_j}$ with $1 \leq \rho \leq s$. For simplicity, we will write $\sum_{i=1}^{\rho} E_i$ instead of $\sum_{j=1}^{\rho} E_{i_j}$.

We will analyze separately different values of c_1 (Remark 3.3.1). Let us start with the case

$$c_1 \in \{0, C_0, F, C_0 + F\} \subset Pic(X).$$

Proposition 3.3.2. *Let X be a smooth, rational surface obtained blowing up s different points of X_e, H any ample divisor on X with $H(K_X + F) < 0$ and $c_1 \in \{0, C_0, F, C_0 + F\} \subset Pic(X)$. For $c_2 >> 0$, the moduli space $M_H(2; c_1, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - c_1^2 - 3$.*

Proof. Let L be an ample divisor on X_e. Since $c_2 >> 0$, the moduli space $M_{X_e,L}(2; c_1, c_2)$ is non-empty. We consider on X the divisor

$$L_n = n\pi^* L - \sum_{i=1}^{s} E_i.$$

Since $L(K_{X_e} + F) < 0$ (Lemma 1.4.9) we also have $L_n(K_X + F) < 0$ for n sufficiently large. Moreover, for $n >> 0$, L_n is an ample divisor on X and there is an open immersion (see Theorem 1.2.15)

$$M_{X_e,L}(2; c_1, c_2) \hookrightarrow M_{X,L_n}(2; c_1, c_2).$$

Furthermore, for $c_2 >> 0$ the moduli space $M_{X_e,L}(2; c_1, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - c_1^2 - 3$ (Theorem 3.1.8) and $M_{X,L_n}(2; c_1, c_2)$ is a smooth, irreducible, quasi-projective variety of the same dimension (Theorem 2.3.5). Hence, $M_{X,L_n}(2; c_1, c_2)$ is rational and the moduli space $M_{X,H}(2; c_1, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - c_1^2 - 3$ (Theorem 2.3.6). □

Now using Criterion 2.3.7, we will deal with the cases $c_1 = C_0 + \sum_{i=1}^{\rho} E_i$ and $c_1 = C_0 + F + \sum_{i=1}^{\rho} E_i$ respectively.

Proposition 3.3.3. *Let X be a smooth, rational surface obtained blowing up s different points of X_e and $c_2 \in \mathbb{Z}$ with $4c_2 > max\{2+\alpha-\frac{3K_X^2}{2}, 2(s+\alpha)\}$ and $\alpha \in \{0,1\}$. Then, the following holds*

(1) *There exists an ample divisor $\tilde{L}$ on X such that $M_{\tilde{L}}(2; C_0+\alpha F+\sum_{i=1}^{\rho} E_i, c_2)$ is a smooth, irreducible, rational projective variety of dimension $4c_2+e-2\alpha+\rho-3$ and $Pic(M_{\tilde{L}}(2; C_0 + \alpha F + \sum_{i=1}^{\rho} E_i, c_2)) \cong \mathbb{Z}$ whenever non-empty.*

(2) *For $c_2 >> 0$ and any ample divisor L on X such that $L(K_X + F) < 0$, the moduli space $M_L(2; C_0 + \alpha F + \sum_{i=1}^{\rho} E_i, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 + e - 2\alpha + \rho - 3$.*

Proof. We take the numerical equivalence class, say $\xi = C_0 - (2c_2 - \alpha)F - \sum_{i=1}^{\rho} E_i$.
Claim: For all $c_2 > \frac{s+\alpha}{2}$, $\xi \equiv C_0 - (2c_2 - \alpha)F - \sum_{i=1}^{\rho} E_i$ defines a non-empty wall of type $(C_0 + \alpha F + \sum_{i=1}^{\rho} E_i, c_2)$ and it satisfies

(1) $\xi^2 = c_1^2 - 4c_2, \quad \xi^2 + \xi K_X + 2 = 0,$

(2) $H^0 O_X(\xi + 3K_X) = H^0 O_X(\xi + K_X + F) = H^0 O_X(K_X + F - \xi) = 0.$

Proof of the Claim: (1) Notice that since

$$\xi + c_1 = 2C_0 - (2c_2 - 2\alpha)F,$$

$$\xi^2 = 2\alpha - e - 4c_2 - \rho = c_1^2 - 4c_2$$

and

$$\begin{aligned} \xi^2 + \xi K_X + 2 &= (C_0 - (2c_2 - \alpha)F - \textstyle\sum_{i=1}^{\rho} E_i)(-2C_0 - (e+2)F + \sum_{i=1}^{s} E_i) \\ &\quad +2\alpha - e - 4c_2 - \rho + 2 \\ &= 2\alpha - e - 4c_2 - \rho + (e - 2 + 4c_2 - 2\alpha + \rho) + 2 = 0 \end{aligned}$$

we only need to check (see Definition 1.3.3 and Remark 1.3.7) that there exist ample divisors L_1 and L_2 on X such that

$$\xi L_1 < 0 < \xi L_2.$$

In fact, if we take the ample divisors on X

$$L_1 = 3C_0 + 3(e+s)F - \sum_{i=1}^{s} E_i \quad \text{and}$$

$$L_2 = 3C_0 + 3(2c_2 + e + \rho)F - \sum_{i=1}^{s} E_i$$

(see [Kuc96] for the ampleness of L_1 and L_2) then we get

$$\xi L_1 = -3e + 3e + 3s - 6c_2 + 3\alpha - \rho < 0 \quad \text{and}$$

$$\xi L_2 = -3e + 6c_2 + 3e + 3\rho - 6c_2 + 3\alpha - \rho = 2\rho + 3\alpha > 0,$$

where the first inequality follows from the assumption $2c_2 > s + \alpha$. This finishes the proof of (1).

(2) It is easily seen that

$$\xi + 3K_X = -5C_0 - (2c_2 + 3e + 6 - \alpha)F + 2\sum_{i=1}^{\rho} E_i + 3\sum_{i=\rho+1}^{s} E_i,$$

$$\xi + K_X + F = -C_0 - (2c_2 + e + 1 - \alpha)F + \sum_{i=\rho+1}^{s} E_i,$$

$$K_X + F - \xi = -3C_0 + (2c_2 - e - 1 - \alpha)F + 2\sum_{i=1}^{\rho} E_i + \sum_{i=\rho+1}^{s} E_i.$$

Therefore, by Lemma 1.4.14; (a) we obtain

$$H^0 O_X(\xi + 3K_X) = H^0 O_X(\xi + K_X + F) = H^0 O_X(K_X + F - \xi) = 0$$

which proves (2) and the Claim.

Since by hypothesis $4c_2 > 2 + \alpha - \frac{3K_X^2}{2}$, we also have $4c_2 - c_1^2 > 2 - \frac{3K_X^2}{2}$ ($c_1^2 = 2\alpha - e - \rho$). Thus, we can apply Criterion 2.3.7 and we get what we want. □

For the remaining values of c_1 there is no numerical equivalence class ξ verifying the hypothesis of Criterion 2.3.7. In these cases, we will prove the rationality of the moduli space $M_L(2; c_1, c_2)$ constructing a suitable family of simple prioritary vector bundles (see Definition 2.3.1 and Lemma 2.3.2) over a big enough rational base. We will start with the case $c_1 = \sum_{i=1}^{\rho} E_i$.

Proposition 3.3.4. *Let X be a smooth, rational surface obtained blowing up s different points of X_e and L any ample divisor on X with $L(K_X+F)<0$. For $c_2>>0$, the moduli space $M_L(2;\sum_{i=1}^{\rho}E_i,c_2)$, $1\leq\rho\leq s$, is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2+\rho-3$.*

Proof. By Theorem 2.3.5 we only need to check the rationality of $M_L(2;\sum_{i=1}^{\rho}E_i,c_2)$ and by Theorem 1.2.15 we can assume $s=\rho$.

We write $c_2=2n+\beta$ with $\beta\in\{0,1\}$ and we consider the irreducible family $\mathcal{F}_{n,\beta}$ of rank 2 vector bundles E on X given by a non-trivial extension

$$(3.8)\qquad \epsilon:\quad 0\rightarrow O_X(-D)\rightarrow E\rightarrow O_X(D+\sum_{i=1}^{\rho}E_i)\otimes I_Z\rightarrow 0$$

where $D=nF-(1-\beta)E_1$ and Z is a locally complete intersection 0-cycle of length $2n+\beta$ such that $H^0I_Z(2D+\sum_{i=1}^{\rho}E_i)=0$.

Let us see that such a 0-cycle Z exists. To this end, we call $p_1,\cdots,p_\rho$ the ρ points of X_e we blow up and we distinguish two cases: $\beta=0$ and $\beta=1$.

If $\beta=1$, then $2D+\sum_{i=1}^{\rho}E_i=2nF+\sum_{i=1}^{\rho}E_i$ and $|Z|=2n+1$. Since $H^0I_Z(2nF+\sum_{i=1}^{\rho}E_i)=H^0I_Z(2nF)$, it is enough to take Z to be $2n+1$ points sitting on $2n+1$ different fibers.

If $\beta=0$, then $2D+\sum_{i=1}^{\rho}E_i=2nF-E_1+\sum_{i=2}^{\rho}E_i$ and $|Z|=2n$. Since $H^0I_Z(2nF-E_1+\sum_{i=1}^{\rho}E_i)=H^0I_Z(2nF-E_1)$, it is enough to take Z to be $2n$ points such that $Z\cup\{p_1\}$ sits on $2n+1$ different fibers.

By [Mir93]; Proposition 1.3, $\mathcal{F}_{n,\beta}$ is non-empty. Indeed, the divisor

$$2D+\sum_{i=1}^{\rho}E_i+K_X=-2C_0+(2n-e-2)F+2\beta E_1+2\sum_{i=2}^{\rho}E_i$$

is not effective (Lemma 1.4.14; (a)).

Let us show:

(a) $h^0E(D)=1$.

(b) $dim\mathcal{F}_{n,\beta}=4c_2+\rho-3$.

(c) Any $E \in \mathcal{F}_{n,\beta}$ is a simple prioritary vector bundle and it has Chern classes $(\sum_{i=1}^{\rho} E_i, c_2)$.

(a) From the exact cohomology sequence

$$0 \to H^0 O_X \to H^0 E(D) \to H^0 I_Z(2D + \sum_{i=1}^{\rho} E_i) \to 0$$

associated to the exact sequence (3.8) we get

$$h^0 E(D) = h^0 O_X + h^0 I_Z(2D + \sum_{i=1}^{\rho} E_i) = 1$$

where the last equality follows from the assumption $H^0 I_Z(2D + \sum_{i=1}^{\rho} E_i) = 0$ and we have proved (a).

(b) By definition we have

$$\begin{aligned} dim\mathcal{F}_{n,\beta} &= \#moduli(Z) + dimExt^1(I_Z(D + \textstyle\sum_{i=1}^{\rho} E_i), O_X(-D)) - h^0 E(D) \\ &= 2length(Z) + dimExt^1(I_Z, O_X(-2D - \textstyle\sum_{i=1}^{\rho} E_i)) - h^0 E(D) \\ &= 2length(Z) + dimExt^1(I_Z, O_X(-2D - \textstyle\sum_{i=1}^{\rho} E_i)) - 1 \end{aligned}$$

where the last equality follows from (a). If we apply the functor $Hom(., O_X)$ to the exact sequence

$$0 \longrightarrow I_Z(2D + \sum_{i=1}^{\rho} E_i) \longrightarrow O_X(2D + \sum_{i=1}^{\rho} E_i) \longrightarrow O_Z(2D + \sum_{i=1}^{\rho} E_i) \longrightarrow 0$$

we get the long exact sequence

$$\begin{aligned} &0 \to H^1 O_X(-2D - \textstyle\sum_{i=1}^{\rho} E_i) \to Ext^1(I_Z, O_X(-2D - \textstyle\sum_{i=1}^{\rho} E_i)) \to \\ &H^0 O_Z \to H^2 O_X(-2D - \textstyle\sum_{i=1}^{\rho} E_i) \to Ext^2(I_Z, O_X(-2D - \textstyle\sum_{i=1}^{\rho} E_i)) \to 0. \end{aligned}$$

But the divisor $-2D - \sum_{i=1}^{\rho} E_i = -2nF + (1-2\beta)E_1 - \sum_{i=2}^{\rho} E_i$ is not effective (see Lemma 1.4.14; (a)), so we obtain

$$\begin{aligned} dimExt^1(I_Z, O_X(-2D - \textstyle\sum_{i=1}^{\rho} E_i)) &= dimExt^2(I_Z, O_X(-2D - \textstyle\sum_{i=1}^{\rho} E_i)) \\ &\quad + h^0 O_Z - \chi(O_X(-2D - \textstyle\sum_{i=1}^{\rho} E_i)). \end{aligned}$$

By Riemann-Roch's Theorem (1.1.5) we have

$$\begin{aligned}\chi(O_X(-2D-\textstyle\sum_{i=1}^{\rho}E_i)) &= \frac{(-2D-\sum_{i=1}^{\rho}E_i)(-2D-\sum_{i=1}^{\rho}E_i+2C_0+(e+2)F-\sum_{i=1}^{\rho}E_i)}{2}+1\\ &= \frac{(-2nF+(1-2\beta)E_1-\sum_{i=2}^{\rho}E_i)(2C_0+(e+2-2n)F-2\beta E_1-2\sum_{i=2}^{\rho}E_i)}{2}+1\\ &= -2n+\beta-2\beta^2+2-\rho\end{aligned}$$

and using Serre's duality, we get

$$\begin{aligned}Ext^2(I_Z,O_X(-2D-\textstyle\sum_{i=1}^{\rho}E_i)) &= Hom(O_X(-2D-\textstyle\sum_{i=1}^{\rho}E_i),I_Z(K_X))^*\\ &= H^0I_Z(2D+\textstyle\sum_{i=1}^{\rho}E_i+K_X)^*=0.\end{aligned}$$

The last equality follows from the fact that the divisor

$$2D+\sum_{i=1}^{\rho}E_i+K_X=-2C_0+(2n-e-2)F+2\beta E_1+2\sum_{i=2}^{\rho}E_i$$

is non effective (see Lemma 1.4.14; (a)). Putting this results together we obtain

$$\begin{aligned}dimExt^1(I_Z,O_X(-2D-\textstyle\sum_{i=1}^{\rho}E_i)) &= length(Z)-(-2n+\beta-2\beta^2+2-\rho)\\ &= 4n+2\beta^2-2+\rho\end{aligned}$$

and

$$\begin{aligned}dim\mathcal{F}_{n,\beta} &= 2(2n+\beta)+(4n+\rho+2\beta^2-2)-1\\ &= 8n+2\beta+2\beta^2+\rho-3=4(2n+\beta)+\rho-3\end{aligned}$$

which proves (b).

(c) It is easy to see that for any $E\in\mathcal{F}_{n,\beta}$, $c_1(E)=\sum_{i=1}^{\rho}E_i$ and $c_2(E)=2n+\beta=c_2$. Let us see that E is a prioritary sheaf. Since E is a torsion free sheaf, we only have to check that $Ext^2(E,E(-F))=0$ (see Definition 2.3.1). Applying the functor $Hom(.,E(-F))$ to the exact sequence (3.8) we get the long exact sequence

$$\cdots\longrightarrow Ext^2(I_Z(D+\textstyle\sum_{i=1}^{\rho}E_i),E(-F))\longrightarrow$$

$$Ext^2(E,E(-F))\longrightarrow Ext^2(O_X(-D),E(-F))\longrightarrow 0.$$

Claim 1: $Ext^2(O_X(-D), E(-F)) = 0$.

Proof of Claim 1: We consider the exact cohomology sequence

$$\cdots \longrightarrow H^2O_X(-F) \longrightarrow H^2E(D-F) \longrightarrow H^2I_Z(2D-F+\sum_{i=1}^{\rho}E_i) \longrightarrow 0$$

associated to the exact sequence (3.8). Since Z is a 0-dimensional subscheme, using Lemma 1.4.14 and Serre's duality we get

$$H^2O_X(-F) = H^0O_X(K_X+F)^* = 0 \quad \text{and}$$

$$H^2I_Z(2D-F+\textstyle\sum_{i=1}^{\rho}E_i) = H^0O_X(-2D+F-\sum_{i=1}^{\rho}E_i+K_X)^* = 0$$

which proves that $H^2E(D-F) = 0$ or, equivalently, $Ext^2(O_X(-D), E(-F)) = 0$, and this proves Claim 1.

Claim 2: $Ext^2(I_Z(D+\sum_{i=1}^{\rho}E_i), E(-F)) = 0$.

Proof of Claim 2: Applying the functor $Hom(I_Z(D+\sum_{i=1}^{\rho}E_i), .)$ to the exact sequence (3.8) we get the long exact sequence

$$\longrightarrow Ext^2(I_Z, O_X(-2D-F-\textstyle\sum_{i=1}^{\rho}E_i)) \longrightarrow Ext^2(I_Z(D+\sum_{i=1}^{\rho}E_i), E(-F))$$

$$\longrightarrow Ext^2(I_Z, I_Z(-F)) \longrightarrow 0.$$

Using once more Serre's duality we obtain

$$\begin{aligned} Ext^2(I_Z, O_X(-2D-F-\textstyle\sum_{i=1}^{\rho}E_i)) &= Hom(O_X, I_Z(2D+F+\textstyle\sum_{i=1}^{\rho}E_i+K_X))^* \\ &= H^0I_Z(2D+F+\textstyle\sum_{i=1}^{\rho}E_i+K_X)^* = 0 \end{aligned}$$

where the last equality follows from the fact that the divisor

$$2D+F+\sum_{i=1}^{\rho}E_i+K_X = -2C_0+(2n-e-1)F+2\beta E_1+2\sum_{i=2}^{\rho}E_i$$

is non effective (Lemma 1.4.14; (a)). Finally, we have

$$\begin{aligned} dimExt^2(I_Z, I_Z(-F)) &\leq dimExt^2(O_X, I_Z(-F)) \\ &= h^0O_X(K_X+F) = 0, \end{aligned}$$

which gives us $Ext^2(I_Z(D+\sum_{i=1}^{\rho} E_i), E(-F)) = 0$ and this proves Claim 2.

It easily follows from Claim 1 and Claim 2 that E is a prioritary sheaf.

Let us see that E is a simple vector bundle, i.e., $dimHom(E,E) = 1$. We always have $1 \leq dimHom(E,E)$. Let us prove the other inequality. Applying the functor $Hom(.,E)$ to the exact sequence (3.8) we get the long exact sequence

$$0 \longrightarrow Hom(I_Z(D+\sum_{i=1}^{\rho} E_i), E) \longrightarrow Hom(E,E) \longrightarrow Hom(O_X(-D),E) \longrightarrow \cdots .$$

By (a) we have

$$dimHom(O_X(-D),E) = h^0E(D) = 1.$$

Hence, we only have to check that

$$Hom(I_Z(D+\sum_{i=1}^{\rho} E_i), E) = 0.$$

To this end, we consider the long exact sequence

$$0 \longrightarrow Hom(I_Z(D+\textstyle\sum_{i=1}^{\rho} E_i), O_X(-D)) \longrightarrow Hom(I_Z(D+\sum_{i=1}^{\rho} E_i), E) \longrightarrow$$

$$Hom(I_Z(D+\textstyle\sum_{i=1}^{\rho} E_i), I_Z(D+\sum_{i=1}^{\rho} E_i)) \stackrel{\delta}{\longrightarrow} Ext^1(I_Z(D+\sum_{i=1}^{\rho} E_i), O_X(-D)) \longrightarrow$$

obtained applying the functor $Hom(I_Z(D+\sum_{i=1}^{\rho} E_i), .)$ to the exact sequence (3.8). By Lemma 1.4.14; (a) we have

$$Hom(I_Z(D+\sum_{i=1}^{\rho} E_i), O_X(-D)) = H^0O_X(-2D-\sum_{i=1}^{\rho} E_i) = 0.$$

On the other hand, since E is given by a non-trivial extension ϵ, the map

$$\begin{array}{rcl} \delta : Hom(I_Z, I_Z) \cong k & \longrightarrow & Ext^1(I_Z(D+\sum_{i=1}^{\rho} E_i), O_X(-D)) \\ 1 & \longrightarrow & \epsilon \end{array}$$

is an injection. Therefore, $Hom(I_Z(D+\sum_{i=1}^{\rho} E_i), E) = 0$ and E is a simple vector bundle, which proves (c).

It follows from (c) that there is an open injection

$$\phi : \mathcal{F}_{n,\beta} \longrightarrow Spl(2; \sum_{i=1}^{\rho} E_i, c_2)$$

from the irreducible family $\mathcal{F}_{n,\beta}$ to the moduli space $Spl(2; \sum_{i=1}^{\rho} E_i, c_2)$ of simple prioritary sheaves. Indeed, assume that there are two non-trivial extensions

$$0 \to O_X(-D) \xrightarrow{\alpha_1} E \xrightarrow{\alpha_2} O_X(D + \sum_{i=1}^{\rho} E_i) \otimes I_Z \to 0;$$

$$0 \to O_X(-D) \xrightarrow{\lambda_1} E \xrightarrow{\lambda_2} O_X(D + \sum_{i=1}^{\rho} E_i) \otimes I_{Z'} \to 0.$$

Since by assumption

$$Hom(O_X(-D), I_Z(D + \sum_{i=1}^{\rho} E_i) = H^0 I_Z(2D + \sum_{i=1}^{\rho} E_i) = 0$$

we have $\lambda_2 \circ \alpha_1 = \alpha_2 \circ \lambda_1 = 0$. So, there exists $\mu \in Aut(O_X(-D)) \cong k$ such that $\alpha_1 = \lambda_1 \circ \mu$. Therefore, $Z = Z'$ and ϕ is an injection.

Now, let us see that $Spl(2; \sum_{i=1}^{\rho} E_i, c_2)$ is rational. In fact, since the moduli space $Spl(2; \sum_{i=1}^{\rho} E_i, c_2)$ of simple prioritary sheaves is smooth and irreducible (Theorem 2.3.5), its rationality follows from the fact that ϕ is an injection, Remark 1.2.8 and the fact that $dim\mathcal{F}_{n,\beta} = dimSpl(2; \sum_{i=1}^{\rho} E_i, c_2)$.

Since $L(K_X + F) < 0$, the moduli space $M_L(2; \sum_{i=1}^{\rho} E_i, c_2)$ is an open dense subset of the moduli space $Spl(2; \sum_{i=1}^{\rho} E_i, c_2)$ of simple prioritary sheaves (Lemma 2.3.2). Therefore, the moduli space $M_L(2; \sum_{i=1}^{\rho} E_i, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 + \rho - 3$ which proves what we want. □

Now we will deal with the last case, namely, $c_1 = F + \sum_{i=1}^{\rho} E_i$.

Proposition 3.3.5. *Let X be a smooth, rational surface obtained blowing up s different points of X_e and L any ample divisor on X with $L(K_X+F) < 0$. For $c_2 >> 0$, the moduli space $M_L(2; F + \sum_{i=1}^{\rho} E_i, c_2)$, $1 \leq \rho \leq s$, is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 + \rho - 3$.*

Proof. By Theorem 2.3.5 we only need to check the rationality of the moduli space $M_L(2; F+\sum_{i=1}^{\rho} E_i, c_2)$ and by Theorem 1.2.15 we can assume $s=\rho$.

We write $c_2 = 2n+\beta$ with $\beta \in \{0,1\}$ and we consider the irreducible family $\mathcal{F}_{n,\beta}$ of rank 2 vector bundles E on X given by a non-trivial extension

$$(3.9) \quad \epsilon: \quad 0 \to O_X(-D) \to E \to O_X(D+F+\sum_{i=1}^{\rho} E_i) \otimes I_Z \to 0$$

where $D=(n+\beta-1)F-\beta E_1$ and Z is a locally complete intersection 0-cycle of length $2n+\beta$ such that $H^0 I_Z(2D+F+\sum_{i=1}^{\rho} E_i)=0$.

Arguing as in Proposition 3.3.4 we can show:

(a) $h^0E(D)=1$.

(b) $dim\mathcal{F}_{n,\beta} = 4c_2+\rho-3$.

(c) Any $E \in \mathcal{F}_{n,\beta}$ is a simple prioritary vector bundle and it has Chern classes $(F+\sum_{i=1}^{\rho} E_i, c_2)$.

Once more, from (c) we can deduce, using Theorem 2.3.5 and Lemma 2.3.2, that the moduli space $M_L(2; F+\sum_{i=1}^{\rho} E_i, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2+\rho-3$. □

Gathering this results we obtain the main result of this section

Theorem 3.3.6. *Let X be a smooth, rational surface obtained blowing up s different points of X_e and L any ample divisor on X such that $L(K_X+F)<0$. For any integer $c_2 >> 0$, the moduli space $M_L(2;c_1,c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2-c_1^2-3$.*

Proof. It follows from Propositions 3.3.2-5 . □

Finally, we are ready to state the main result of this chapter.

Theorem 3.3.7. *Let X be a smooth, algebraic, rational surface. For any divisor $c_1 \in Pic(X)$, $0 << c_2 \in \mathbb{Z}$ and any polarization L on X such that $L(K_X + F) < 0$, being $F \in Pic(X)$ the ruling of $\pi : X \to \mathbb{P}^1$, the moduli space $M_L(2; c_1, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - c_1^2 - 3$.*

Proof. It is a consequence of Theorems 3.1.8, 3.2.7 and 3.3.6 and the classification of smooth, rational, algebraic surfaces (Theorem 1.4.1). □

Final Remark 3.3.8. Let X be a smooth, irreducible, rational surface. We have seen that the moduli space $M_{X,L}(2; c_1, c_2)$ of rank two, L-stable vector bundles E on X, with fixed Chern classes $c_1(E) = c_1$ and $c_2(E) = c_2 >> 0$ is rational. Therefore, we are led to pose the following question

QUESTION: Fix a smooth, irreducible, projective surface, an ample divisor L on X, $0 << c_2 \in \mathbb{Z}$ and $c_1 \in Pic(X)$. Assume that the moduli space $M_{X,L}(2; c_1, c_2)$ is rational. Is X rational?

The answer is affirmative provided that $p_g(X) = 0$ or $q(X) = 0$. To be more precise we have

Proposition 3.3.9. *Let X be a smooth, irreducible, projective surface satisfying $p_g(X) \neq 0$ or $q(X) \neq 0$. Fix an ample divisor L on X, $c_1 \in Pic(X)$ and an integer $c_2 >> 0$. Then, the moduli space $M_{X,L}(2; c_1, c_2)$ is a non rational, quasi-projective variety.*

Proof. If $p_g(X) \neq 0$, then there exists a nontrivial regular 2-form on X. Hence, by [Zuo92]; Theorem 1 the moduli space $M_{X,L}(2; c_1, c_2)$ has also a non-zero regular 2-form. Indeed, Zuo shows that for $c_2 >> 0$, there is a natural injective map $H^0(X, \Omega^2) \hookrightarrow H^0(M_{X,L}(2; c_1, c_2), \Omega^2)$. Therefore, the moduli space $M_{X,L}(2; c_1, c_2)$ is an irrational variety.

In case that $q(X) \neq 0$, by [HL97]; Theorem 11.1.4 the moduli space $M_{X,L}(2; c_1, c_2)$ is not unirational and hence not rational. Alternatively, by [Li97b]; Theorem 0.2,

$b_1(M_{X,L}(2;c_1,c_2)) = b_1(X)$, being $b_1(Y)$ the first Betti number of a variety Y. Hence, the result follows from the fact that for any smooth, projective surface X, $b_1(X) = 2q(X)$ and the fact that the first Betti number is a birational invariant of the variety. Indeed, since $q(X) \neq 0$, we have

$$b_1(M_{X,L}(2;c_1,c_2)) \neq 0$$

which proves that $M_{X,L}(2;c_1,c_2)$ is not rational. □

Chapter 4

Rank r vector bundles on surfaces

In this chapter we take up the study of moduli spaces $M_{X,L}(r; c_1, c_2)$ of rank r vector bundles E, L-stable with fixed Chern classes $c_1(E) = c_1 \in Pic(X)$ and $c_2(E) = c_2 \in \mathbb{Z}$ on X where in section 1, X is a smooth, minimal, rational surface and in section 2, X is a smooth, algebraic $K3$ surface.

In the first section of this chapter, we will turn our attention to the delicate problem concerning the rationality of the moduli space $M_{X,L}(r; c_1, c_2)$ when X is a smooth, minimal, rational surface. At present, there is no counterexample known to the question whether moduli spaces $M_{X,L}(r; c_1, c_2)$ are always rational provided X is rational. For $r = 2$, we have seen in chapter 3 that the answer is affirmative. In recent papers, [Li97], [LQ96] and [LQ96b] the rationality of the moduli space $M_{\mathbb{P}^2, O_{\mathbb{P}^2}(1)}(3; c_1, c_2)$ of rank 3, $O_{\mathbb{P}^2}(1)$-stable vector bundles on $\mathbb{P}^2$ was investigated and not much is known about the rationality when the rank is greater than 3 (see [Got96], [Kat92] and [Yos96]). The aim of the first section of this chapter is to prove the rationality of many moduli spaces $M_{X,L}(r; c_1, c_2)$ provided X is a smooth, minimal, rational surface (see Theorem 4.1.13 and Theorem 4.1.14). The proof relies on Walter's results on the stack of prioritary sheaves.

In the second section of this chapter, we will study the moduli space $\mathcal{M}_{X,H}(r; c_1, c_2)$ of rank r, H-stable vector bundles E on a smooth, algebraic $K3$ surface X with fixed Chern classes $c_1(E) = c_1 \in Pic(X)$ and $c_2(E) = c_2 \in \mathbb{Z}$.

In 1984, Mukai showed that the moduli space of simple sheaves on a $K3$ surface

X has a symplectic structure (see [Muk84]). On the other hand, it is well known that the Hilbert scheme $Hilb^l(X)$ of 0-dimensional subschemes of X has also a symplectic structure (see [Bea83]). So, it seems natural to look for a closer relation between the moduli space $\mathcal{M}_{X,H}(r; c_1, c_2)$ and the Hilbert scheme $Hilb^l(X)$. The goal of section 2 is to determine invariants $(r; c_1, c_2) \in \mathbb{Z} \times Pic(X) \times \mathbb{Z}$ and $l \in \mathbb{Z}$ for which the moduli space $\mathcal{M}_{X,L}(r; c_1, c_2)$ and the Hilbert scheme $Hilb^l(X)$ of 0-dimensional subschemes of X of length l are birationaly equivalent, partially answering the question proposed by Nakashima in [Nak97].

The results of section 2 of this chapter will appear in [Cos98] and generalize previous results of Zuo in [Zuo91b] and Nakashima in [Nak93b] and [Nak97].

4.1 Moduli spaces of rank r vector bundles on rational surfaces

Keeping the notations introduced in chapter 1 we denote by $M_{X,L}(r; c_1, c_2)$ the moduli space of rank r, vector bundles E on X, L-stable (in the sense of Mumford-Takemoto) with Chern classes $c_1(E) = c_1 \in Pic(X)$ and $c_2(E) = c_2 \in \mathbb{Z}$. We will write $M_L(r; c_1, c_2)$ when there is no confusion.

Recall that if X is a smooth, minimal, rational surface, then X is either isomorphic to a Hirzebruch surface X_e with $e \neq 1$ or $\mathbb{P}^2$ (see Theorem 1.4.1). The goal of this section is to prove the rationality of the moduli space $M_{X_e,H}(r; c_1, c_2)$ of rank r vector bundles E on a Hirzebruch surface X_e, H-stable, with fixed Chern classes $c_1(E) = c_1 \in Pic(X_e)$, $c_2(E) = c_2 \in \mathbb{Z}$ and $\Delta(r; c_1, c_2) >> 0$, provided one of the following conditions is verified

- $c_1F = 1$ or $r-1$ $(mod \quad r)$;
- $c_1F = r-2$ $(mod \quad r)$ and $c_2 - \frac{c_1^2}{2} - \frac{c_1K}{2} - (r-1) = 0$ $(mod \quad 2)$;
- $c_1F = 2$ $(mod \quad r)$ and $c_2 + c_1C_0 - \frac{c_1^2}{2} + \frac{c_1K}{2} + 1 = 0$ $(mod \quad 2)$

being F the fiber of $\pi : X_e \to \mathbb{P}^1$ and $K = K_{X_e}$ the canonical divisor of X_e.

As a corollary we will deduce the rationality of the moduli space $M_{\mathbb{P}^2,O_{\mathbb{P}^2}(1)}(r;c_1,c_2)$ of rank r, $O_{\mathbb{P}^2}(1)$-stable vector bundles E on the projective plane with $\Delta(r;c_1,c_2)$ sufficiently large, provided one of the following conditions is verified

- $c_1 = 1$ or $r-1$ $(mod \quad r)$;
- $c_1 = r-2$ $(mod \quad r)$ and $c_2 - \frac{c_1^2}{2} + \frac{3c_1}{2} - (r-1) = 0$ $(mod \quad 2)$;
- $c_1 = 2$ $(mod \quad r)$ and $c_2 + c_1 - \frac{c_1^2}{2} - \frac{3c_1}{2} + 1 = 0$ $(mod \quad 2)$.

We will end the section with some comments and remarks.

As we pointed out in the introduction of this chapter, the main tool we will use to prove the rationality of our moduli spaces will be prioritary sheaves (see Definition 2.3.1) and some of their basic properties (see Theorem 2.3.5).

We want to stress that Lemma 2.3.2 and Theorem 2.3.5 enable us to reduce our problem of proving the rationality of $M_H(r;c_1,c_2)$ to the construction of a family of simple prioritary vector bundles over a big enough rational base. To this end, we will distinguish different cases according to the value of c_1F. In each case we will construct our irreducible family of rank r vector bundles on X_e by means of extensions of vector bundles on X_e of lower rank. The first case that we will study is the following

Theorem 4.1.1. *Let X_e be a smooth, Hirzebruch surface, H an ample divisor on X_e, $2 \leq r \in \mathbb{Z}$, $d \in \mathbb{Z}$ and $c_1 = C_0 + dF \in Pic(X_e)$. For any integer c_2 such that $\Delta(r;c_1,c_2) >> 0$, the moduli space $M_H(r;c_1,c_2)$ is a non-empty, smooth, irreducible, rational, quasi-projective variety of dimension $2rc_2 - (r-1)c_1^2 - (r^2-1)$.*

Remark 4.1.2. Theorem 4.1.1 generalizes to arbitrary rank Proposition 3.1.3 proved in chapter 3 of this thesis.

Remark 4.1.3. For any ample divisor H on X_e we have $(K+F)H < 0$ (Lemma 1.4.9). Hence, H-stable torsion free sheaves are prioritary (see Lemma 2.3.2) and for $\Delta(r;c_1,c_2) >> 0$, the moduli space $M_H(r;c_1,c_2)$ (resp. $Spl(r;c_1,c_2)$) of rank r vector bundles (resp. torsion free sheaves) H-stable (resp. simple prioritary) with Chern classes (c_1,c_2) is irreducible and non-empty of the expected dimension $2rc_2-(r-1)c_1^2-(r^2-1)$ (Theorem 2.3.5). In addition, $M_H(r;c_1,c_2)$ is smooth. Therefore, to prove Theorem 4.1.1 we only need to check the rationality of the moduli space $M_H(r;c_1,c_2)$.

Let us start with a technical lemma which will be very useful for us.

Lemma 4.1.4. *Let X_e be a smooth, Hirzebruch surface, c_2, d and $r \geq 2$ integers such that $d-c_2-\frac{c_2-\alpha}{r-1}-1<0$, being $\alpha = c_2 \ (mod\ r-1)$, $0<\alpha\leq r-1$. Consider the family $\mathcal{H}_\alpha$ of vector bundles $E_{\alpha+1}$ on X_e given by an extension*

$$(4.1) \quad \epsilon: \quad 0 \to O_{X_e}(C_0+(d-c_2)F) \to E_{\alpha+1} \to O_{X_e}((1+\frac{c_2-\alpha}{r-1})F)^\alpha \to 0$$

where $\epsilon=(e_1,\cdots,e_\alpha)$ with $e_i \in Ext^1(O_{X_e}((1+\frac{c_2-\alpha}{r-1})F), O_{X_e}(C_0+(d-c_2)F))$, $1\leq i\leq\alpha$, are k-linearly independent. We have

(a) $E_{\alpha+1}$ is a rank $(\alpha+1)$ simple, prioritary vector bundle on X_e with Chern classes

$$(c_1', c_2') = (C_0 + (d+\alpha-c_2+\frac{\alpha(c_2-\alpha)}{r-1})F, \frac{\alpha(c_2+r-1-\alpha)}{r-1}).$$

(b) $\mathcal{H}_\alpha$ is a non-empty, irreducible, rational family of the expected dimension

$$2(\alpha+1)c_2'-\alpha {c_1'}^2-(\alpha+1)^2+1.$$

In addition, if the moduli space $M_H(\alpha+1;c_1',c_2')$ is non-empty, then it is a rational, smooth, irreducible, quasi-projective variety of dimension

$$2(\alpha+1)c_2'-\alpha {c_1'}^2-(\alpha+1)^2+1.$$

Proof. (a) It easily follows from the construction that $E_{\alpha+1}$ is a rank $(\alpha+1)$ vector bundle on X_e with Chern classes

$$c_1' = c_1(E_{\alpha+1}) = C_0+(d+\alpha-c_2+\tfrac{\alpha(c_2-\alpha)}{r-1})F \quad \text{and}$$

$$c_2' = c_2(E_{\alpha+1}) = \alpha+\tfrac{\alpha(c_2-\alpha)}{r-1}.$$

Let us show that $E_{\alpha+1}$ is a prioritary sheaf. Since $E_{\alpha+1}$ is a torsion free sheaf, we only have to check that $Ext^2(E_{\alpha+1}, E_{\alpha+1}(-F)) = 0$ (Definition 2.3.1). Applying the functor $Hom(E_{\alpha+1}, .)$ to the exact sequence (4.1) twisted by $O_{X_e}(-F)$, we get the long exact sequence

$$\cdots \to Ext^2(E_{\alpha+1}, O_{X_e}(C_0 + (d - c_2 - 1)F)) \to Ext^2(E_{\alpha+1}, E_{\alpha+1}(-F))$$
$$\to Ext^2(E_{\alpha+1}, O_{X_e}(\tfrac{c_2-\alpha}{r-1}F)^{\alpha}) \to 0.$$

By Serre's duality and using again the exact sequence (4.1), we get

$$Ext^2(E_{\alpha+1}, O_{X_e}(C_0 + (d - c_2 - 1)F)) = H^0E_{\alpha+1}(-C_0 - (d - c_2 - 1)F + K)^* = 0,$$
$$Ext^2(E_{\alpha+1}, O_{X_e}(\tfrac{c_2-\alpha}{r-1}F)) = H^0E_{\alpha+1}(K - \tfrac{c_2-\alpha}{r-1}F)^* = 0,$$

where $K = K_{X_e}$ is the canonical divisor of X_e. Thus, $Ext^2(E_{\alpha+1}, E_{\alpha+1}(-F)) = 0$ and $E_{\alpha+1}$ is a prioritary vector bundle.

Next we will see that $E_{\alpha+1}$ is simple, i.e., $dimHom(E_{\alpha+1}, E_{\alpha+1}) = 1$. Applying the functor $Hom(., E_{\alpha+1})$ to the exact sequence (4.1) we get the following long exact sequence

$$0 \to Hom(O_{X_e}(\tfrac{c_2-\alpha+r-1}{r-1}F)^{\alpha}, E_{\alpha+1}) \to Hom(E_{\alpha+1}, E_{\alpha+1})$$
$$\to Hom(O_{X_e}(C_0 + (d - c_2)F), E_{\alpha+1}) \to \cdots.$$

Since $H^0O_{X_e}(-C_0 - (d - c_2 - 1 - \frac{c_2-\alpha}{r-1})F) = 0$, from the exact sequence (4.1) we deduce

$$dimHom(O_{X_e}(C_0 + (d - c_2)F), E_{\alpha+1}) = h^0E_{\alpha+1}(-C_0 - (d - c_2)F) = 1.$$

Consider the long exact cohomology sequence

$$0 \to H^0O_{X_e}(C_0 + (d - c_2 - \tfrac{c_2-\alpha+r-1}{r-1})F) \to H^0E_{\alpha+1}(-\tfrac{c_2-\alpha+r-1}{r-1}F)$$
$$\to H^0O_{X_e}^{\alpha} \xrightarrow{\delta} H^1O_{X_e}(C_0 + (d - c_2 - \tfrac{c_2-\alpha+r-1}{r-1})F) \to \cdots$$

associated to the exact sequence

$$0 \to O_{X_e}(C_0+(d-c_2-\frac{c_2-\alpha+r-1}{r-1})F) \to E_{\alpha+1}(-\frac{c_2-\alpha+r-1}{r-1}F) \to O_{X_e}^{\alpha} \to 0.$$

Since $(O = O_{X_e})$

$$H^1O(C_0+(d-c_2-\frac{c_2-\alpha+r-1}{r-1})F) = Ext^1(O(\frac{c_2-\alpha+r-1}{r-1}F), O(C_0+(d-c_2)F)),$$

the map

$$\delta : H^0O_{X_e}^{\alpha} \longrightarrow H^1O_{X_e}(C_0+(d-c_2-\tfrac{c_2-\alpha+r-1}{r-1})F)$$

given by

$$\delta((0, \cdots, \overset{i)}{1}, \cdots, 0)) = e_i \quad \text{for} \quad 1 \leq i \leq \alpha,$$

is an injection. By hypothesis $d - c_2 - \frac{c_2-\alpha}{r-1} - 1 < 0$, hence the divisor on X_e, $C_0 + (d - c_2 - \frac{c_2-\alpha+r-1}{r-1})F$ is not effective (Remark 1.4.7) and

$$H^0O_{X_e}(C_0+(d-c_2-\frac{c_2-\alpha+r-1}{r-1})F) = 0.$$

The latest, together with the fact that δ is an injection gives us

$$H^0E_{\alpha+1}(-\frac{c_2-\alpha+r-1}{r-1}F) = 0.$$

Therefore, $dimHom(E_{\alpha+1}, E_{\alpha+1}) = 1$ which proves (a).

(b) Let us compute the dimension of $\mathcal{H}_{\alpha}$. We set

$$\begin{aligned} s &= dimExt^1(O_{X_e}((1+\tfrac{c_2-\alpha}{r-1})F), O_{X_e}(C_0+(d-c_2)F)) \\ &= h^1O_{X_e}(C_0+(d-c_2-1-\tfrac{c_2-\alpha}{r-1})F). \end{aligned}$$

We have

$$dim\mathcal{H}_{\alpha} = dimGrass(\alpha, s) = \alpha(s-\alpha)$$

being $Grass(\alpha, s)$ the Grassmann variety of α-dimensional linear subspaces of

$$Ext^1(O_{X_e}((1+\frac{c_2-\alpha}{r-1})F), O_{X_e}(C_0+(d-c_2)F)).$$

Since $d-c_2-\frac{c_2-\alpha}{r-1}-1<0$, $H^0O_{X_e}(C_0+(d-c_2-1-\frac{c_2-\alpha}{r-1})F)=0$ (see Remark 1.4.7) and by Serre's duality

$$\begin{aligned} H^2O_{X_e}(C_0+(d-c_2-1-\tfrac{c_2-\alpha}{r-1})F) &= H^0O_{X_e}(K-C_0-\tfrac{(d-1)(r-1)-rc_2+\alpha}{r-1}F)^* \\ &= H^0O_{X_e}(-3C_0-\tfrac{(e+1+d)(r-1)-rc_2+\alpha}{r-1}F)^*=0. \end{aligned}$$

Therefore, applying Riemann-Roch's Theorem we obtain

$$\begin{aligned} s &= -\chi(O_{X_e}(C_0+(d-c_2-\tfrac{c_2+r-\alpha-1}{r-1})F)) \\ &= -\frac{(C_0+(d-c_2-\frac{c_2+r-\alpha-1}{r-1})F)(3C_0+(e+2+d-c_2-\frac{c_2+r-\alpha-1}{r-1})F)}{2}-1 \\ &= e-2-2(d-c_2-\tfrac{c_2+r-\alpha-1}{r-1}). \end{aligned}$$

Therefore,

$$\begin{aligned} dim\mathcal{H}_\alpha=\alpha s-\alpha^2 &= \alpha(e-2-2(d-c_2-\tfrac{c_2+r-\alpha-1}{r-1}))-\alpha^2 \\ &= 2(\alpha+1)\alpha(\tfrac{c_2+r-\alpha-1}{r-1})-\alpha(-e+2(d-c_2)) \\ &\quad -\tfrac{2\alpha^2}{r-1}(c_2+r-\alpha-1)-(\alpha+1)^2+1 \\ &= 2(\alpha+1)c_2'-\alpha c_1'^2-(\alpha+1)^2+1 \\ &= dimSpl(\alpha+1;c_1',c_2') \end{aligned}$$

where the last equality follows from Remark 4.1.3.

Let us see that the induced map

$$\phi:\mathcal{H}_\alpha\longrightarrow Spl(\alpha+1;c_1',c_2')$$

is an open embedding. Assume that there are two nontrivial extensions

$$0\to O_{X_e}(C_0+(d-c_2)F)\xrightarrow{\beta_1}E_{\alpha+1}\xrightarrow{\beta_2}O_{X_e}((1+\frac{c_2-\alpha}{r-1})F)^\alpha\to 0;$$

$$0\to O_{X_e}(C_0+(d-c_2)F)\xrightarrow{\gamma_1}E_{\alpha+1}\xrightarrow{\gamma_2}O_{X_e}((1+\frac{c_2-\alpha}{r-1})F)^\alpha\to 0.$$

Since we have $(O = O_{X_e})$

$$Hom(O(C_0+(d-c_2)F), O(\frac{r-1+c_2-\alpha}{r-1}F)) = H^0O(-C_0+\frac{(1-d)(r-1)+rc_2-\alpha}{r-1}F) = 0,$$

we get $\beta_2 \circ \gamma_1 = \gamma_2 \circ \beta_1 = 0$. Thus, there exists $\lambda \in Aut(O_{X_e}(C_0 + (d - c_2)F)) \cong k$ such that $\gamma_1 = \beta_1 \circ \lambda$ and hence ϕ is an open embedding.

Since ϕ is an open embedding and $\mathcal{H}_\alpha$ is a non-empty, irreducible, rational family with dimension equal to the dimension of $Spl(\alpha + 1; c_1', c_2')$, we can also state the rationality of $Spl(\alpha + 1; c_1', c_2')$. Finally, since $M_H(\alpha + 1; c_1', c_2')$ is an open dense subset of $Spl(\alpha + 1; c_1', c_2')$, we have the rationality of $M_H(\alpha + 1; c_1', c_2')$ whenever it is non-empty. □

Remark 4.1.5. With the above notation, if the moduli space $M_H(\alpha + 1; c_1', c_2')$ is non-empty (namely, in case that $\Delta(\alpha+1; c_1', c_2') >> 0$), then a generic vector bundle $E \in M_H(\alpha + 1; c_1', c_2')$ sits in an exact sequence of the following type

$$(4.2) \qquad 0 \to O_{X_e}(C_0 + (d - c_2)F) \to E \to O_{X_e}((1 + \frac{c_2 - \alpha}{r-1})F)^\alpha \to 0.$$

From now on, we will call $\mathcal{H}_\alpha^o$ the open subset of $M_H(\alpha+1; c_1', c_2')$ parameterizing H-stable, vector bundles given by extensions of type (4.2).

Proof of Theorem 4.1.1. We write $c_2 = \lambda(r-1)+\alpha$ with $0 < \alpha \leq r-1$. We consider the irreducible family $\mathcal{F}$ of vector bundles E on X_e given by an extension

$$(4.3) \qquad \epsilon : 0 \to E_{\alpha+1} \to E \to O_{X_e}(\frac{c_2 - \alpha}{r-1}F)^{r-\alpha-1} \to 0$$

where $\epsilon = (e_1, \cdots, e_{r-\alpha-1})$ with $e_i \in Ext^1(O_{X_e}(\frac{c_2-\alpha}{r-1}F), E_{\alpha+1})$, $1 \leq i \leq r - \alpha - 1$, are k-linearly independent and $E_{\alpha+1} \in \mathcal{H}_\alpha^o$, being $\mathcal{H}_\alpha^0$ the family of vector bundles described in Remark 4.1.5.

Let us show:

(a) $dimHom(E_{\alpha+1}, E) = 1$.

(b) $dim\mathcal{F} \geq 2rc_2 - (r-1)c_1^2 - (r^2-1)$.

(c) Any $E \in \mathcal{F}$ is a rank r, simple, prioritary vector bundle on X_e with Chern classes $c_1 = C_0 + dF$ and c_2.

(a) Applying the functor $Hom(E_{\alpha+1}, .)$ to the exact sequence (4.3), we get the long exact sequence

$$0 \to Hom(E_{\alpha+1}, E_{\alpha+1}) \to Hom(E_{\alpha+1}, E) \to Hom(E_{\alpha+1}, O_{X_e}(\frac{c_2-\alpha}{r-1}F)^{r-\alpha-1}) \to \cdots .$$

Since $E_{\alpha+1}$ is simple (i.e. $Hom(E_{\alpha+1}, E_{\alpha+1}) \cong k$), we only need to check that

$$Hom(E_{\alpha+1}, O_{X_e}(\frac{c_2-\alpha}{r-1}F)^{r-\alpha-1}) = 0.$$

Since $E_{\alpha+1} \in \mathcal{H}^o_\alpha$, applying the functor $Hom(., O_{X_e}(\frac{c_2-\alpha}{r-1}F))$ to the exact sequence

$$(4.4) \qquad 0 \to O_{X_e}(C_0 + (d-c_2)F) \to E_{\alpha+1} \to O_{X_e}((1+\frac{c_2-\alpha}{r-1})F)^\alpha \to 0$$

we obtain the long exact sequence

$$0 \to Hom(O_{X_e}((1+\tfrac{c_2-\alpha}{r-1})F)^\alpha, O_{X_e}(\tfrac{c_2-\alpha}{r-1}F)) \to Hom(E_{\alpha+1}, O_{X_e}(\tfrac{c_2-\alpha}{r-1}F))$$
$$\to Hom(O_{X_e}(C_0 + (d-c_2)F), O_{X_e}(\tfrac{c_2-\alpha}{r-1}F)) \to \cdots .$$

Since

$$Hom(O_{X_e}(\frac{c_2+r-\alpha-1}{r-1}F)^\alpha, O_{X_e}(\frac{c_2-\alpha}{r-1}F)) = H^0O_{X_e}(-F)^\alpha = 0$$

and

$$Hom(O_{X_e}(C_0+(d-c_2)F), O_{X_e}(\frac{c_2-\alpha}{r-1}F)) = H^0O_{X_e}(-C_0+(c_2-d+\frac{c_2-\alpha}{r-1})F) = 0,$$

we get $Hom(E_{\alpha+1}, O_{X_e}(\frac{c_2-\alpha}{r-1}F)^{r-\alpha-1}) = 0$ which proves (a).

(b) Set $n = n(\alpha) := dimExt^1(O_{X_e}(\frac{c_2-\alpha}{r-1}F), E_{\alpha+1})$. Since $dimHom(E_{\alpha+1}, E) = 1$, we have

$$dim\mathcal{F} = dim\mathcal{H}^o_\alpha + dimGrass(r-\alpha-1, n)$$

being $Grass(r-\alpha-1,n)$ the Grassmann variety of linear subspaces of dimension $(r-\alpha-1)$ of $Ext^1(O_{X_e}(\frac{c_2-\alpha}{r-1}F),E_{\alpha+1})$.

Notice that

$$n = h^1E_{\alpha+1}(\frac{\alpha-c_2}{r-1}F) \geq -\chi(E_{\alpha+1}(\frac{\alpha-c_2}{r-1}F)).$$

Since

$$c_1(E_{\alpha+1}) = C_0 + (d+\alpha-c_2+\tfrac{\alpha(c_2-\alpha)}{r-1})F \quad \text{and}$$

$$c_2(E_{\alpha+1}) = \tfrac{\alpha(c_2+r-1-\alpha)}{r-1},$$

using 1.1.1 we deduce

$$c_1E_{\alpha+1}(\tfrac{\alpha-c_2}{r-1}F) = C_0 + (d+\alpha-c_2+\tfrac{\alpha-c_2}{r-1})F,$$

$$c_2E_{\alpha+1}(\tfrac{\alpha-c_2}{r-1}F) = \tfrac{\alpha(c_2+r-1-\alpha)}{r-1} + \tfrac{\alpha(\alpha-c_2)}{r-1} = \alpha$$

and by Riemann-Roch's Theorem 1.1.5 we obtain

$$\begin{aligned}\chi(E_{\alpha+1}(\tfrac{\alpha-c_2}{r-1}F)) &= \alpha+1-\tfrac{c_1E_{\alpha+1}(\frac{\alpha-c_2}{r-1}F)K}{2}+\tfrac{(c_1E_{\alpha+1}(\frac{\alpha-c_2}{r-1}F))^2}{2}-c_2E_{\alpha+1}(\tfrac{\alpha-c_2}{r-1}F)\\ &= \alpha+1-\tfrac{c_1E_{\alpha+1}K}{2}+\tfrac{(\alpha+1)(\alpha-c_2)}{r-1}+\tfrac{(c_1E_{\alpha+1})^2}{2}\\ &\quad +\tfrac{(\alpha+1)(\alpha-c_2)}{r-1}c_1E_{\alpha+1}F-c_2E_{\alpha+1}-\tfrac{\alpha(\alpha-c_2)}{r-1}c_1E_{\alpha+1}F.\end{aligned}$$

Therefore, we have

$$\begin{aligned}dim\mathcal{F} &= dim\mathcal{H}^o_\alpha+(r-\alpha-1)n-(r-\alpha-1)^2\\ &= 2(\alpha+1)c_2E_{\alpha+1}-\alpha(c_1E_{\alpha+1})^2-(\alpha+1)^2+1+(r-\alpha-1)n-(r-\alpha-1)^2\\ &\geq 2(\alpha+1)c_2E_{\alpha+1}-\alpha(c_1E_{\alpha+1})^2-(\alpha+1)^2+1\\ &\quad +(r-\alpha-1)(-\alpha-1+\tfrac{c_1E_{\alpha+1}K}{2}-\tfrac{(\alpha+1)(\alpha-c_2)}{r-1})\\ &\quad +(r-\alpha-1)(-\tfrac{(c_1E_{\alpha+1})^2}{2}-\tfrac{(\alpha-c_2)}{r-1}c_1E_{\alpha+1}F+c_2E_{\alpha+1})-(r-\alpha-1)^2\\ &= 2rc_2E-(r-1)c_1E^2-(r^2-1)\end{aligned}$$

where the last equality follows after a straightforward computation using

$$\begin{aligned} c_1E &= c_1E_{\alpha+1} + \tfrac{(r-\alpha-1)(c_2-\alpha)}{r-1}F \\ &= C_0 + (d+\alpha-c_2+\tfrac{\alpha(c_2-\alpha)}{r-1})F + \tfrac{(r-\alpha-1)(c_2-\alpha)}{r-1}F = c_1 \quad \text{and} \\ c_2E &= c_2E_{\alpha+1} + \tfrac{(r-\alpha-1)(c_2-\alpha)}{r-1}c_1E_{\alpha+1}F \\ &= \tfrac{(r-\alpha-1)(c_2-\alpha)}{r-1}(C_0+(d+\alpha-c_2+\tfrac{\alpha(c_2-\alpha)}{r-1})F)F + \tfrac{\alpha(c_2+r-1-\alpha)}{r-1} = c_2. \end{aligned}$$

(c) It easily follows from the construction that E is a rank r vector bundle on X_e with Chern classes $c_1E = c_1$ and $c_2E = c_2$. Let us see that E is a prioritary sheaf. Since E is a rank r torsion free sheaf, we only have to check that $Ext^2(E, E(-F)) = 0$ (see Definition 2.3.1). Applying the functor $Hom(., E(-F))$ to the exact sequence (4.3) we get the long exact sequence

$$Ext^2(O_{X_e}(\frac{c_2-\alpha}{r-1}F)^{r-\alpha-1}, E(-F)) \to Ext^2(E, E(-F)) \to Ext^2(E_{\alpha+1}, E(-F)) \to 0.$$

First of all, we will see that $Ext^2(O_{X_e}(\frac{c_2-\alpha}{r-1}F)^{r-\alpha-1}, E(-F)) = 0$. To this end, consider the long exact sequence

$$\cdots \to H^2E_{\alpha+1}(-(\frac{c_2-\alpha}{r-1}+1)F) \to H^2E(-(\frac{c_2-\alpha}{r-1}+1)F) \to H^2O_{X_e}(-F)^{r-\alpha-1} \to 0$$

associated to the exact sequence

$$0 \to E_{\alpha+1}(-(\frac{c_2-\alpha}{r-1}+1)F) \to E(-(\frac{c_2-\alpha}{r-1}+1)F) \to O_{X_e}(-F)^{r-\alpha-1} \to 0.$$

Since $H^2O_{X_e} = 0$ and $(O = O_{X_e})$

$$H^2O(C_0 + (d - c_2 - \tfrac{c_2-\alpha}{r-1} - 1)F) = H^0O(-C_0 - (d - c_2 - \tfrac{c_2-\alpha}{r-1} - 1)F + K)^* = 0;$$

using the exact sequence (4.4) one can see that $H^2E_{\alpha+1}(-(\frac{c_2-\alpha}{r-1}+1)F) = 0$. Hence, since $H^2O_{X_e}(-F) = 0$ we get

$$Ext^2(O_{X_e}(\frac{c_2-\alpha}{r-1}F)^{r-\alpha-1}, E(-F)) = H^2E(-(\frac{c_2-\alpha}{r-1}+1)F)^{r-\alpha-1} = 0.$$

Now we will see that $Ext^2(E_{\alpha+1}, E(-F)) = 0$. To this end, we apply the functor $Hom(., E(-F))$ to the exact sequence (4.4) and we get the long exact sequence

$$\cdots \to Ext^2(O_{X_e}(\tfrac{c_2+r-\alpha-1}{r-1}F)^\alpha, E(-F)) \to Ext^2(E_{\alpha+1}, E(-F))$$
$$\to Ext^2(O_{X_e}(C_0 + (d-c_2)F), E(-F)) \to 0.$$

Using once more the exact sequences (4.3) and (4.4) we obtain

$$\begin{aligned} dimExt^2(O_{X_e}(\tfrac{c_2+r-\alpha-1}{r-1}F)^\alpha, E(-F)) &= \alpha h^2E(-(\tfrac{c_2-\alpha}{r-1}+2)F) \\ &\leq \alpha h^2E_{\alpha+1}(-(\tfrac{c_2-\alpha}{r-1}+2)F) = 0, \\ dimExt^2(O_{X_e}(C_0+(d-c_2)F), E(-F)) &= h^2E(-C_0-(d-c_2+1)F) \\ &\leq h^2E_{\alpha+1}(-C_0-(d-c_2+1)F) = 0. \end{aligned}$$

Gathering this results we get $Ext^2(E, E(-F)) = 0$. This proves that E is a prioritary vector bundle.

Let us see that E is simple. Applying the functor $Hom(., E)$ to the exact sequence (4.3), we get the long exact sequence

$$0 \to Hom(O_{X_e}(\frac{c_2-\alpha}{r-1}F)^{r-\alpha-1}, E) \to Hom(E,E) \to Hom(E_{\alpha+1}, E) \to \cdots.$$

In (a) we have proved that $dimHom(E_{\alpha+1}, E) = 1$, therefore we only need to check that $Hom(O_{X_e}(\frac{c_2-\alpha}{r-1}F), E) = H^0E(-\frac{c_2-\alpha}{r-1}F) = 0$. To this end, we consider the long exact cohomology sequence

$$0 \to H^0E_{\alpha+1}(-\tfrac{c_2-\alpha}{r-1}F) \to H^0E(-\tfrac{c_2-\alpha}{r-1}F) \to$$
$$H^0O_{X_e}^{r-\alpha-1} \xrightarrow{\delta} H^1E_{\alpha+1}(-\tfrac{c_2-\alpha}{r-1}F) \to \cdots.$$

Claim:

$$H^0E_{\alpha+1}(-\frac{c_2-\alpha}{r-1}F) = 0.$$

Proof of the Claim: Take the ample divisor $H = C_0 + (e+1)F$ on X_e (see Remark 1.4.7). By hypothesis

$$\Delta(r; c_1, c_2) = \frac{1}{r}(c_2 - \frac{r-1}{2r}c_1^2) >> 0,$$

so we have

$$\begin{aligned} \Delta(\alpha+1; c_1E_{\alpha+1}, c_2E_{\alpha+1}) &= \tfrac{1}{\alpha+1}(\tfrac{\alpha(c_2+r-1-\alpha)}{r-1}) \\ &\quad -\tfrac{\alpha}{2(\alpha+1)^2}(C_0 + (d+\alpha-c_2+\tfrac{\alpha(c_2-\alpha)}{r-1})F)^2 \\ &= \tfrac{1}{\alpha+1}(\tfrac{\alpha(c_2+r-1-\alpha)}{r-1}) \\ &\quad -\tfrac{\alpha}{2(\alpha+1)^2}(2d-e+2\alpha-2c_2+\tfrac{2\alpha(c_2-\alpha)}{r-1}) \\ &= \tfrac{1}{\alpha+1}(\tfrac{\alpha r}{(\alpha+1)(r-1)}c_2+\alpha) \\ &\quad -\tfrac{\alpha}{2(\alpha+1)^2}(-e+2d+2\alpha+\tfrac{2\alpha}{r-1}) >> 0 \end{aligned}$$

and hence, the moduli space $M_H(\alpha+1; c_1E_{\alpha+1}, c_2E_{\alpha+1})$ is non-empty.

Assume $H^0E_{\alpha+1}(-\frac{c_2-\alpha}{r-1}F) \neq 0$. The H-stability of any vector bundle $E_{\alpha+1} \in \mathcal{H}_\alpha^o$ implies

$$\frac{c_2-\alpha}{r-1}FH = \frac{c_2-\alpha}{r-1} < \frac{c_1E_{\alpha+1}H}{\alpha+1} = \frac{1}{\alpha+1}(\alpha - c_2 + d + 1 + \frac{\alpha c_2}{r-1} - \frac{\alpha^2}{r-1})$$

or, equivalently,

$$\frac{c_2}{r} < \frac{r-1}{r^2}(\alpha+d+1-\frac{\alpha^2}{r-1}) + \frac{\alpha(\alpha+1)}{r^2},$$

which contradicts the fact that

$$\Delta(r; c_1, c_2) = \frac{1}{r}(c_2 - \frac{r-1}{2r}c_1^2) = \frac{1}{r}(c_2 - \frac{r-1}{2r}(2d-e)) >> 0.$$

Therefore, $H^0E_{\alpha+1}(-\frac{c_2-\alpha}{r-1}F) = 0$ which proves our claim.

Since the morphism

$$\delta : H^0O_{X_e}^{r-\alpha-1} \longrightarrow H^1E_{\alpha+1}(-\tfrac{c_2-\alpha}{r-1}F)$$

given by

$$\delta((0, \cdots, \overset{i)}{1}, \cdots, 0)) = e_i \quad \text{for} \quad 1 \leq i \leq r - \alpha - 1,$$

is an injection, we have $H^0E(-\frac{c_2-\alpha}{r-1}F) = 0$ which proves that E is simple.

Let $\mathcal{A}^0_\alpha \longrightarrow \mathcal{H}^0_\alpha \times X_e$ be a Poincaré sheaf of H-stable vector bundles on X_e such that for any $h \in \mathcal{H}^0_\alpha$, $\mathcal{A}^0_\alpha|_{\{h\}\times X_e}$ is isomorphic to the bundle $E_{\alpha+1}$ corresponding to $h \in \mathcal{H}^0_\alpha$. Consider the natural projections

$$\pi : \mathcal{H}^0_\alpha \times X_e \longrightarrow \mathcal{H}^0_\alpha \quad \text{and}$$

$$p : \mathcal{H}^0_\alpha \times X_e \longrightarrow X_e.$$

Set

$$\mathcal{E}_\alpha := Ext^1_\pi(p^*O_{X_e}(\frac{c_2-\alpha}{r-1}F), \mathcal{A}^0_\alpha).$$

$\mathcal{E}_\alpha$ is a locally free sheaf of rank $n = dim Ext^1(O_{X_e}(\frac{c_2-\alpha}{r-1}F), E_{\alpha+1})$ on $\mathcal{H}^0_\alpha$ and compatible with arbitrary base change.

Define $\mathcal{B}_\alpha := Gr(r-\alpha-1, \mathcal{E}_\alpha)$. $\mathcal{B}_\alpha$ is rational as a locally free fibre bundle with a Grassmannian as fibre over the rational variety $\mathcal{H}^0_\alpha$.

It follows from (*b*) and (*c*) that there is a component W of $Spl(r; c_1, c_2)$ which is birational to $\mathcal{B}_\alpha$ and thus rational. The simple prioritary sheaves form an open substack in the irreducible substack $Prior(r; c_1, c_2)$. Thus the moduli space of simple prioritary sheaves is also irreducible and rational. Assume now that $M_H(r; c_1, c_2)$ is non-empty for an ample divisor H on X_e. Since $(K+F)H < 0$ (Lemma 1.4.9), the moduli space $M_H(r; c_1, c_2)$ is an open subscheme of the moduli space of simple prioritary sheaves (Lemma 2.3.2). Therefore, the moduli space $M_H(r; c_1, c_2)$ is rational which proves what we want. □

Remark 4.1.6. Fix $c_1 = C_0 + dF \in Pic(X_e)$ and $c_2 \in \mathbb{Z}$ such that $\Delta(r; c_1, c_2) >> 0$. It follows from the above construction that a generic $E \in Spl(r; C_0 + dF, c_2)$ sits in an extension $(O = O_{X_e})$

$$0 \to O(C_0+(d-c_2)F) \to E \to O((1+\frac{c_2-\alpha}{r-1})F)^\alpha \oplus O(\frac{c_2-\alpha}{r-1}F)^{r-\alpha-1} \to 0 \tag{4.5}$$

where $c_2 = \lambda(r-1) + \alpha$ with $0 < \alpha \leq r-1$. The same holds for a generic vector bundle $E \in M_H(r; C_0 + dF, c_2)$. Indeed, a generic $E \in Spl(r; C_0 + dF, c_2)$ belongs to an exact sequence

$$0 \to E_{\alpha+1} \to E \to O_{X_e}(\frac{c_2 - \alpha}{r-1}F)^{r-\alpha-1} \to 0$$

where $E_{\alpha+1} \in \mathcal{H}^0_\alpha$, being $\mathcal{H}^0_\alpha$ the family of vector bundles described in Remark 4.1.5. So, $E_{\alpha+1}$ lies in an exact sequence

$$0 \to O_{X_e}(C_0 + (d - c_2)F) \to E_{\alpha+1} \to O_{X_e}((1 + \frac{c_2 - \alpha}{r-1})F)^\alpha \to 0.$$

Therefore, we have the following commutative diagram ($O = O_{X_e}$)

$$\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
 & O(C_0 + (d-c_2)F) & = & O(C_0 + (d-c_2)F) & & & \\
 & \downarrow & & \downarrow & & & \\
0 \to & E_{\alpha+1} & \to & E & \to & O(\frac{c_2-\alpha}{r-1}F)^\beta \to 0 \\
 & \downarrow & & \downarrow & & \| \\
0 \to & O(\frac{r-1+c_2-\alpha}{r-1}F)^\alpha & \to & O(\frac{r-1+c_2-\alpha}{r-1}F)^\alpha \oplus O(\frac{c_2-\alpha}{r-1}F)^\beta & \to & O(\frac{c_2-\alpha}{r-1}F)^\beta \to 0 \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & &
\end{array}$$

where $\beta := r - \alpha - 1$. This proves what we want.

Since duality preserves H-stability, we get, as by-product of Theorem 4.1.1, the following result.

Theorem 4.1.7. *Let X_e be a smooth, Hirzebruch surface, H an ample divisor on X_e, $2 \leq r \in \mathbb{Z}$, $d \in \mathbb{Z}$ and $c_1 = (r-1)C_0 + dF \in Pic(X_e)$. For any integer c_2 such that $\Delta(r; c_1, c_2) >> 0$, the moduli space $M_H(r; c_1, c_2)$ is a non-empty, smooth, irreducible, rational, quasi-projective variety of dimension $2rc_2 - (r-1)c_1^2 - (r^2-1)$.*

Proof. Since a rank r torsion free sheaf E on X_e is H-stable if, and only if, for any $L \in Pic(X)$, $E \otimes O_X(L)$ is H-stable if, and only if, its dual E^* is H-stable, we have an isomorphism

$$M_H(r; c_1, c_2) \longrightarrow M_H(r; \tilde{c_1}, \tilde{c_2})$$

which sends $E \in M_H(r; c_1, c_2)$ to $(E \otimes O_{X_e}(-C_0))^* \in M_H(r; \tilde{c_1}, \tilde{c_2})$ where

$$\tilde{c_1} = C_0 - dF \quad \text{and}$$

$$\tilde{c_2} = c_2 + \tfrac{(r-1)(r-2)}{2} e - (r-1)d.$$

Let H be an ample divisor on X_e such that the moduli space $M_H(r; c_1, c_2)$ is non-empty. Since by hypothesis

$$\begin{aligned} \Delta(r; c_1, c_2) &= \tfrac{1}{r}(c_2 - \tfrac{r-1}{2r} c_1^2) \\ &= \tfrac{1}{r}(c_2 - \tfrac{(r-1)^2}{r} d + \tfrac{(r-1)^3}{2r} e) >> 0, \end{aligned}$$

we have

$$\begin{aligned} \Delta(r; \tilde{c_1}, \tilde{c_2}) &= \tfrac{1}{r}(c_2 + \tfrac{(r-1)(r-2)}{2} e - (r-1)d - \tfrac{r-1}{2r}(C_0 - dF)^2) \\ &= \tfrac{1}{r}(c_2 + \tfrac{(r-1)(r-2)}{2} e + \tfrac{r-1}{2r} e - \tfrac{(r-1)^2}{r} d) >> 0. \end{aligned}$$

Therefore, it follows from Theorem 4.1.1 that the moduli space $M_H(r; \tilde{c_1}, \tilde{c_2})$ is a smooth, irreducible, rational, quasi-projective variety of the expected dimension $2r\tilde{c_2} - (r-1)\tilde{c_1}^2 - (r^2-1)$ i.e. of dimension $2rc_2 - (r-1)c_1^2 - (r^2-1)$. Therefore, the same holds for the moduli space $M_H(r; c_1, c_2)$. □

Remark 4.1.8. Fix $c_1 = (r-1)C_0 + dF \in Pic(X_e)$ and an integer c_2 such that $\Delta(r; c_1, c_2) >> 0$. Using the notations introduced in the proof of Theorem 4.1.7 and Remark 4.1.6 we can state that a generic $E \in Spl(r; (r-1)C_0 + dF, c_2)$ sits in an extension

$$(4.6) \quad 0 \to O_{X_e}(C_0 + (\tfrac{\alpha - \tilde{c}_2}{r-1} - 1)F)^{\alpha} \oplus O_{X_e}(C_0 + \tfrac{\alpha - \tilde{c}_2}{r-1} F)^{r-\alpha-1} \to$$

$$E \to O_{X_e}((\tilde{c}_2 + d)F) \to 0$$

where $\tilde{c}_2 = \lambda(r-1) + \alpha$ with $0 < \alpha \leq r-1$. The same holds for a generic vector bundle $E \in M_H(r;(r-1)C_0 + dF, c_2)$.

From now on we denote by $\mathcal{G}^o_H(r;(r-1)C_0 + dF, c_2)$ the open subset of the moduli space $M_H(r;(r-1)C_0 + dF, c_2)$ parameterizing rank r, H-stable, vector bundles given by an extension of type (4.6).

Next we will deal with the case $c_1F = r-2$.

Theorem 4.1.9. *Let X_e be a smooth, Hirzebruch surface, H an ample divisor on X_e, $2 \leq r \in \mathbb{Z}$, $d \in \mathbb{Z}$, $c_1 = (r-2)C_0 + dF \in Pic(X_e)$ and an integer c_2 such that $c_2 - \frac{c_1^2}{2} - \frac{c_1K}{2} - (r-1) = 0$ (mod 2). For $\Delta(r;c_1,c_2) >> 0$ the moduli space $M_H(r;c_1,c_2)$ is a non-empty, smooth, irreducible, rational, quasi-projective variety of dimension $2rc_2 - (r-1)c_1^2 - (r^2-1)$.*

Remark 4.1.10. We want to stress that once we fix c_1, the equation

$$c_2 - \frac{c_1^2}{2} - \frac{c_1K}{2} - (r-1) = 0 \pmod 2$$

only determines the parity of c_2.

Proof of Theorem 4.1.9. By Remark 4.1.3, we only need to check the rationality of the moduli space $M_H(r;c_1,c_2)$.

To this end, we take $D = bF \in Pic(X_e)$ with

$$2b = c_2 - \frac{c_1^2}{2} - \frac{c_1K}{2} - (r-1). \tag{4.7}$$

In order to simplify the notation we set

$$d' := b - d \quad \text{and}$$

$$c_2' := c_2 - (r-2)(e+d) + \tfrac{(r-1)(r-2)}{2}e.$$

We consider the ample divisor $L = C_0 + (e+1)F$ on X_e and the irreducible, rational family

$$\mathcal{G}_L^o := \mathcal{G}_L^o(r-1; (r-2)C_0 + (d-b)F, c_2 - (r-2)b)$$

of rank $(r-1)$, L-stable vector bundles W on X_e with fixed Chern classes

$$(c_1W, c_2W) = ((r-2)C_0 + (d-b)F, c_2 - (r-2)b)$$

such that any $W \in \mathcal{G}_L^o$ sits in an exact sequence of the following type

$$(4.8) \quad 0 \to O_{X_e}(C_0 + (\tfrac{\alpha - c_2'}{r-2} - 1)F)^{\alpha} \oplus O_{X_e}(C_0 + \tfrac{\alpha - c_2'}{r-2}F)^{r-\alpha-2} \to$$
$$W \to O_{X_e}((c_2' - d')F) \to 0$$

where $c_2' = \lambda(r-2) + \alpha$ with $0 < \alpha \leq r-2$. $\mathcal{G}_L^o$ is non-empty. Indeed, by Remark 4.1.8 it is enough to check that the hypothesis $\Delta(r; c_1, c_2) >> 0$ implies that $\Delta(W) = \frac{1}{r-1}(c_2W - \frac{r-2}{2(r-1)}c_1W^2) >> 0$. In fact,

$$\Delta(r; c_1, c_2) = \frac{1}{r}(c_2 + \frac{(r-1)(r-2)^2}{2r}e - \frac{(r-1)(r-2)}{r}d) >> 0$$

implies that

$$\begin{aligned}
\Delta(W) &= \tfrac{1}{r-1}(c_2W - \tfrac{r-2}{2(r-1)}c_1W^2) \\
&= \tfrac{1}{r-1}(c_2 - (r-2)b - \tfrac{r-2}{2(r-1)}((r-2)C_0 + (d-b)F)^2) \\
&= \tfrac{1}{r-1}(c_2 - \tfrac{r-2}{r-1}b + \tfrac{(r-2)^3}{2(r-1)}e - \tfrac{(r-2)^2}{r-1}d) \\
&= \tfrac{1}{r-1}(\tfrac{r}{2(r-1)}c_2 + \tfrac{r-2}{r-1}(\tfrac{c_1^2}{4} + \tfrac{c_1K}{4}) + \tfrac{r-2}{2} + \tfrac{(r-2)^3}{2(r-1)}e - \tfrac{(r-2)^2}{r-1}d) >> 0
\end{aligned}$$

where the last equality follows from (4.7). Hence, $M_L(r-1; c_1W, c_2W)$ is non-empty ([Sor97]).

Let $\mathcal{F}$ be the irreducible family of rank r vector bundles E on X_e given by a non-trivial extension

$$(4.9) \qquad \epsilon: \quad 0 \to W \to E \to O_{X_e}(D) \to 0$$

where $W \in \mathcal{G}_L^o$, being $\mathcal{G}_L^o$ the family of L-stable vector bundles and $D = bF$ fixed above.

Let us show:

(a) $dimHom(W, E) = 1$.

(b) $dim\mathcal{F} \geq 2rc_2 - (r-1)c_1^2 - (r^2-1)$.

(c) E is a rank r, simple, prioritary vector bundle on X_e with Chern classes $c_1 = (r-2)C_0 + dF$ and c_2.

(a) Applying the functor $Hom(W, .)$ to the exact sequence (4.9), we get the long exact sequence

$$0 \to Hom(W, W) \to Hom(W, E) \to Hom(W, O_{X_e}(D)) \to \cdots .$$

Since W is simple (i.e. $Hom(W, W) \cong k$), we only need to check that

$$Hom(W, O_{X_e}(D)) = 0.$$

Applying the functor $Hom(., O_{X_e}(D))$ to the exact sequence (4.8) we obtain the long exact sequence

$$0 \to Hom(O_{X_e}((c_2' - d')F), O_{X_e}(D)) \to Hom(W, O_{X_e}(D)) \to$$

$$Hom(O_{X_e}(C_0 + (\tfrac{\alpha - c_2'}{r-2} - 1)F)^\alpha, O_{X_e}(D)) \oplus Hom(O_{X_e}(C_0 + \tfrac{\alpha - c_2'}{r-2}F)^{r-\alpha-2}, O_{X_e}(D)) \to .$$

Since

$$Hom(O_{X_e}(C_0 + (\tfrac{\alpha - c_2'}{r-2} - 1)F), O_{X_e}(D)) = H^0O_{X_e}(-C_0 + (b - \tfrac{\alpha - c_2'}{r-2} + 1)F) = 0,$$

$$Hom(O_{X_e}(C_0 + \tfrac{\alpha - c_2'}{r-2}F), O_{X_e}(D)) = H^0O_{X_e}(-C_0 + (b - \tfrac{\alpha - c_2'}{r-2})F) = 0,$$

we only have to see that $Hom(O_{X_e}((c_2' - d')F), O_{X_e}(D)) = H^0O_{X_e}((b - c_2' + d')F) = 0$.

To this end, notice that by (4.7)

$$\begin{aligned} b - c_2' + d' &= 2b - c_2' - d \\ &= c_2 - \tfrac{c_1^2}{2} - \tfrac{c_1 K}{2} - (r-1) - d - c_2 + (r-2)(e+d) \\ &\quad - \tfrac{(r-1)(r-2)}{2} e \\ &= \tfrac{(r-2)^2}{2} e - (r-2)d - e(r-2) + d + \tfrac{(e+2)(r-2)}{2} \\ &\quad -(r-1) - d + (r-2)d + (r-2)e - \tfrac{(r-1)(r-2)}{2} e \\ &= -1. \end{aligned}$$

Therefore $H^0 O_{X_e}((b - c_2' + d')F) = H^0 O_{X_e}(-F) = 0$ which proves (a).

(b) By definition we have

$$\begin{aligned} dim\mathcal{F} &= dim\mathcal{G}_L^o + dimExt^1(O_{X_e}(D), W) - dimHom(W, E) \\ &= dim\mathcal{G}_L^o + h^1 W(-D) - 1 \\ &\geq dim\mathcal{G}_L^o - \chi(W(-D)) - 1. \end{aligned}$$

From the exact sequence (4.9) we get

$$c_1 W(-D) = c_1 - rD \quad and \quad c_2 W(-D) = c_2 - (r-1)c_1 D.$$

Applying Riemann-Roch's Theorem 1.1.5, we obtain

$$\begin{aligned} \chi(W(-D)) &= (r-1) - \tfrac{(c_1 - rD)K}{2} + \tfrac{(c_1 - rD)^2}{2} - c_2 + (r-1)Dc_1 \\ &= (r-1) - \tfrac{c_1 K}{2} + \tfrac{rDK}{2} + \tfrac{c_1^2}{2} - c_2 - Dc_1. \end{aligned}$$

Therefore,

$$\begin{aligned}
dim\mathcal{F} &\geq 2(r-1)c_2W - (r-2)c_1W^2 - (r-1)^2 + 1 - (r-1) \\
&\quad + \tfrac{c_1K}{2} - \tfrac{rDK}{2} - \tfrac{c_1^2}{2} + c_2 + Dc_1 - 1 \\
&= 2(r-1)(c_2 - c_1D) - (r-2)(c_1-D)^2 - (r-1)^2 - (r-1) \\
&\quad + \tfrac{c_1K}{2} - \tfrac{rDK}{2} - \tfrac{c_1^2}{2} + c_2 + Dc_1 \\
&= (2r-1)c_2 - \tfrac{(2r-3)c_1^2}{2} - c_1D - r^2 + r + \tfrac{c_1K}{2} - \tfrac{rDK}{2} \\
&= 2rc_2 - (r-1)c_1^2 - r^2 + 1
\end{aligned}$$

where the last equality follows from (4.7).

(c) It easily follows from the construction that E is a rank r vector bundle on X_e with Chern classes $c_1E = c_1 = (r-2)C_0 + dF$ and $c_2E = c_2$. Let us see that E is a prioritary sheaf. Since E is a rank r torsion free sheaf, we only have to check that $Ext^2(E, E(-F)) = Hom(E, E(K+F))^* = 0$ (see Definition 2.3.1). Applying the functor $Hom(E, .)$ to the exact sequence

$$0 \to W(K+F) \to E(K+F) \to O_{X_e}(D+K+F) \to 0$$

we get the long exact sequence

$$0 \to Hom(E, W(K+F)) \to Hom(E, E(K+F)) \to Hom(E, O_{X_e}(D+K+F)) \to \cdots.$$

Let us see $Hom(E, O_{X_e}(D+K+F)) = H^2E(-D-F)^* = 0$. Using the exact sequence (4.9) we get the long exact sequence

$$\cdots \to H^2W(-D-F) \to H^2E(-D-F) \to H^2O_{X_e}(-F) \to 0.$$

Using the exact sequence (4.8) and Serre's duality we have

$$H^2W(-D-F) = 0 \quad \text{and}$$

$$H^2O_{X_e}(-F) = H^0O_{X_e}(K+F)^* = 0.$$

Hence, we obtain $H^2E(-D-F)=0$.

Now we will see that $Hom(E,W(K+F))=0$. To this end, we apply the functor $Hom(.,W(K+F))$ to the exact sequence (4.9) and we get the long exact sequence

$$0\to Hom(O_{X_e}(D),W(K+F))\to Hom(E,W(K+F))\to Hom(W,W(K+F))\to\cdots.$$

Since W is a prioritary vector bundle, $Hom(W,W(K+F))=0$ and using once again the exact sequence (4.8) we get

$$Hom(O_{X_e}(D),W(K+F))=H^0W(K+F-D)=0.$$

Therefore, $Hom(E,W(K+F))=0$ and $Hom(E,E(K+F))=0$ which proves that E is a prioritary sheaf.

Let us see that E is simple, i.e. $Hom(E,E)\cong k$. Applying the functor $Hom(.,E)$ to the exact sequence (4.9), we get the long exact sequence

$$0\to Hom(O_{X_e}(D),E)\to Hom(E,E)\to Hom(W,E)\to\cdots.$$

By (a) we have $dimHom(W,E)=1$. Since we always have $dimHom(E,E)\geq 1$, we only need to check that

$$Hom(O_{X_e}(D),E)=0.$$

To this end, we apply the functor $Hom(O_{X_e}(D),.)$ to the exact sequence (4.9) and we get the long exact sequence

$$0\to Hom(O_{X_e}(D),W)\to Hom(O_{X_e}(D),E)\to$$

$$Hom(O_{X_e}(D),O_{X_e}(D))\xrightarrow{\delta}Ext^1(O_{X_e}(D),W)\to\cdots.$$

Since $W\in\mathcal{G}_L^o$ is L-stable, if $Hom(O_{X_e}(D),W)=H^0W(-D)\neq 0$ we have

$$DL=b<\frac{c_1WL}{r-1}=\frac{(c_1-D)L}{r-1}$$

which is equivalent to

$$rDL=rb=\frac{r}{2}(c_2-\frac{c_1^2}{2}-\frac{c_1K}{2}-(r-1))<c_1L=r-2+d$$

or, equivalently,

$$\frac{c_2}{r} < \frac{c_1^2}{2r} + \frac{r-2}{2r}e - \frac{d-1}{r} + \frac{2(r-2+d)}{r^2}$$

and this clearly contradicts the hypothesis $\Delta(r; c_1, c_2) = \frac{1}{r}(c_2 - \frac{r-1}{2r}c_1^2) >> 0$.

Therefore, $Hom(O_{X_e}(D), W) = 0$. Since the extension (4.9) is non-trivial, the map

$$\begin{array}{ccc} \delta : H^0 O_{X_e} & \longrightarrow & Ext^1(O_{X_e}(D), W) \\ 1 & \longrightarrow & \epsilon \end{array}$$

is an injection and hence, $Hom(O_{X_e}(D), E) = 0$ which proves that E is simple.

By Theorem 4.1.7 and Remark 4.1.8, W sits in a smooth, irreducible, rational, quasi-projective variety of dimension $2(r-1)c_2(W) - (r-2)c_1(W)^2 - (r-1)^2 + 1$. Set $a(c_1W)$ the greatest integer such that c_1W is divisible by $a(c_1W)$ in $Pic(X_e)$ if $c_1W \neq 0$ and $a(c_1W) = 0$ if $c_1W = 0$. Since

$$gcd(r-1, a(c_1W), \chi(r-1; c_1W, c_2W)) = 1,$$

there exists an open dense subset $\mathcal{U} \subset M_L(r-1; c_1(W), c_2(W))$ and a Poincaré sheaf (Theorem 1.2.6)

$$\begin{array}{c} \mathcal{W} \\ \downarrow \\ \mathcal{U} \times X_e \end{array}$$

of L-stable rank $r-1$ vector bundles on X_e with Chern classes $(c_1(W), c_2(W))$, parameterized by $\mathcal{U}$ such that for all $u \in \mathcal{U}$,

$$\mathcal{W}|_{\{u\} \times X_e}$$

is isomorphic to the vector bundle W corresponding to $u \in \mathcal{U}$.

Consider the projections

$$\pi_1 : \mathcal{U} \times X_e \longrightarrow \mathcal{U} \quad \text{and}$$

$$\pi_2 : \mathcal{U} \times X_e \longrightarrow X_e.$$

Let $\vartheta := \pi_2^* O_{X_e}(D)$ and consider the relative extension sheaf

$$\mathcal{E} := Ext^1_{\pi_1}(\vartheta, \mathcal{W})$$

which is a coherent sheaf over $\mathcal{U}$.

For any element $[W] \in \mathcal{U}$, $\mathcal{E}|_{[W]} = Ext^1(O_{X_e}(D), W)$ has constant dimension. Hence, $\mathcal{E}$ is a locally free sheaf over $\mathcal{U}$ and compatible with arbitrary base change.

Define $\mathcal{B} := \mathbb{P}(\mathcal{E})$. $\mathcal{B}$ is a projective bundle over $\mathcal{U}$. Let $\gamma : \mathcal{B} \to \mathcal{U}$ be the natural projection and we consider the morphism

$$p := \gamma \times id_{X_e} : \mathcal{B} \times X_e \longrightarrow \mathcal{U} \times X_e.$$

It follows from [Lan83]; Corollary 4.5 that over $\mathcal{B} \times X_e$ there is a tautological extension

$$0 \longrightarrow p^*(\vartheta_1) \longrightarrow \mathcal{V} \longrightarrow p^*(\vartheta_2) \otimes O_{\mathcal{B}}(-1) \longrightarrow 0\ .$$

such that for each $t \in \mathcal{B}$ the restriction to $\{t\} \times X_e$ is isomorphic to the extension corresponding to t, i.e.

$$0 \longrightarrow W \longrightarrow E \longrightarrow O_{X_e}(D) \longrightarrow 0.$$

Moreover, there is a natural bijective morphism

$$\mathcal{B} := \mathbb{P}(\mathcal{E}) \longrightarrow \mathcal{F}.$$

Therefore, the family $\mathcal{F}$ which parameterizes all such $E's$ is a smooth, irreducible, rational variety of dimension $2rc_2 - (r-1)c_1^2 - (r^2-1)$. On the other hand, we have a morphism

$$\psi : \mathcal{F} \cong \mathcal{B} \longrightarrow Spl(r; c_1, c_2),$$

$$\psi(t) := \mathcal{V}|_{\{t\}\times X_e} \cong E$$

from $\mathcal{F}$ to the moduli space of simple prioritary sheaves, which is an injection. Indeed, assume that there are two non-trivial extensions

$$0 \longrightarrow W \xrightarrow{\alpha_1} E \xrightarrow{\alpha_2} O_{X_e}(D) \longrightarrow 0;$$

$$0 \longrightarrow W' \xrightarrow{\beta_1} E \xrightarrow{\beta_2} O_{X_e}(D) \longrightarrow 0.$$

From the corresponding exact sequences of W and W' we deduce that

$$Hom(W', O_{X_e}(D)) = Hom(W, O_{X_e}(D)) = 0.$$

Thus, $\beta_2 \circ \alpha_1 = \alpha_2 \circ \beta_1 = 0$. So, there exists $\gamma \in Aut(W) \simeq k$ (W is simple) such that $\beta_1 = \alpha_1 \circ \gamma$. Therefore, $W \cong W'$ and ψ is an injection.

Since $dim\mathcal{F} = dimSpl(r; c_1, c_2)$, we conclude that the moduli space of simple prioritary sheaves is also rational. Since $(K + F)H < 0$ (Lemma 1.4.9), the moduli space $M_H(r; c_1, c_2)$ is an open subscheme of the moduli space of simple prioritary sheaves. Therefore, the moduli space $M_H(r; c_1, c_2)$ is rational which proves what was stated. □

Since H-stability is preserved by duality, we get as a consequence of Theorem 4.1.9 the following result.

Theorem 4.1.11. *Let X_e be a smooth, Hirzebruch surface, H an ample divisor on X_e, an integer c_2 such that $c_2 + c_1C_0 - \frac{c_1^2}{2} + \frac{c_1K}{2} + 1 = 0$ (mod 2), $2 \leq r \in \mathbb{Z}$, $d \in \mathbb{Z}$ and $c_1 = 2C_0 + dF \in Pic(X_e)$. For $\Delta(r; c_1, c_2) >> 0$ the moduli space $M_H(r; c_1, c_2)$ is a non-empty, smooth, irreducible, rational, quasi-projective variety of dimension $2rc_2 - (r-1)c_1^2 - (r^2 - 1)$.*

Remark 4.1.12. We want to stress that once we have fixed c_1, the condition

$$c_2 + c_1C_0 - \frac{c_1^2}{2} + \frac{c_1K}{2} + 1 = 0 \quad (\text{mod } 2)$$

is equivalent to fix the parity of c_2.

Proof of Theorem 4.1.11. Since a rank r torsion free sheaf E on X_e is H-stable if, and only if, for any $L \in Pic(X_e)$, $E \otimes O_X(L)$ is H-stable if, and only if, its dual E^* is H-stable, we have an isomorphism

$$M_H(r; c_1, c_2) \longrightarrow M_H(r; \tilde{c_1}, \tilde{c_2})$$

which sends $E \in M_H(r; c_1, c_2)$ to $E^* \otimes O_{X_e}(C_0) \in M_H(r; \tilde{c_1}, \tilde{c_2})$ where

$$\tilde{c_1} = (r-2)C_0 - dF \quad \text{and}$$

$$\tilde{c_2} = c_2 - \tfrac{(r-1)(r-4)}{2}e - (r-1)d.$$

Hence, using Theorem 4.1.9 and arguing as in Theorem 4.1.7 we get the desired result.

Now we are in a position to prove the main result of this section.

Theorem 4.1.13. *Let X_e be a smooth, Hirzebruch surface, F a fiber of the ruling, H an ample divisor on X_e, $2 \leq r \in \mathbb{Z}$, $c_1 \in Pic(X_e)$ and $c_2 \in \mathbb{Z}$. Assume that one of the following conditions is satisfied*

i) $c_1F = 1$ *or* $r-1$ $(mod\ \ r)$;

ii) $c_1F = r-2$ $(mod\ \ r)$ *and* $c_2 - \frac{c_1^2}{2} - \frac{c_1K}{2} - (r-1) = 0$ $(mod\ \ 2)$;

iii) $c_1F = 2$ $(mod\ \ r)$ *and* $c_2 + c_1C_0 - \frac{c_1^2}{2} + \frac{c_1K}{2} + 1 = 0$ $(mod\ \ 2)$.

If $\Delta(r; c_1, c_2) >> 0$, then the moduli space $M_H(r; c_1, c_2)$ is a non-empty, smooth, irreducible, rational, quasi-projective variety of dimension $2rc_2 - (r-1)c_1^2 - (r^2-1)$.

Proof. Since a rank r torsion free sheaf E on X_e is H-stable (resp. simple, prioritary) if, and only if, for any $L \in Pic(X_e)$, $E \otimes O_{X_e}(L)$ is H-stable (resp. simple, prioritary) we may assume, without loss of generality, that $c_1(E) = aC_0 + dF$ with $0 \leq a < r$. Hence, Theorem 4.1.13 easily follows from Theorems 4.1.1, 4.1.7, 4.1.9 and 4.1.11. □

To close this section, we will apply Theorem 4.1.13 to study the rationality of the moduli space $M_{\mathbb{P}^2, O_{\mathbb{P}^2}(1)}(r; c_1, c_2)$ of rank r, $O_{\mathbb{P}^2}(1)$-stable vector bundles on $\mathbb{P}^2$ with given Chern classes c_1 and c_2.

In chapter 3 we have reviewed what is known about the rationality of the moduli space $M_{\mathbb{P}^2, O_{\mathbb{P}^2}(1)}(2; c_1, c_2)$ of rank two, $O_{\mathbb{P}^2}(1)$-stable, vector bundles on $\mathbb{P}^2$ with

Chern classes c_1 and c_2. However not much is known about rationality when the rank is greater than two, in general. Recent progress has been made for the rank 3 case. Indeed, it follows from [LQ96b], [LQ96] and [Li97] that $M_{\mathbb{P}^2,O_{\mathbb{P}^2}(1)}(3;1,c_2)$ with $c_2 \geq 1$ and $M_{\mathbb{P}^2,O_{\mathbb{P}^2}(1)}(3;2,c_2)$ with $c_2 \geq 1$ are rational, and it follows from [Got96] that the moduli space $M_{\mathbb{P}^2,O_{\mathbb{P}^2}(1)}(3;0,c_2)$ with $c_2 = 1$ $(mod\ 3)$ is rational. As far as we know, the only contributions to the rationality when the rank is greater than 3 are due to Göttsche, Katsylo and Yoshioka. We easily deduce from [Got96] that $M_{\mathbb{P}^2,O_{\mathbb{P}^2}(1)}(r;0,c_2)$ is rational when $c_2 = 1$ $(mod\ r)$. In [Kat92], Katsylo generalizes Göttsche result and he proves that $M_{\mathbb{P}^2,O_{\mathbb{P}^2}(1)}(r;0,c_2)$ is rational provided $gcd(r,c_2) = 1,2,3$ or 4. In [Yos96], Yoshioka proves that $M_{\mathbb{P}^2,O_{\mathbb{P}^2}(1)}(r;c_1,c_2)$ is rational provided $gcd(r,3c_1) = 1$.

Our main result in this direction is the following

Theorem 4.1.14. *Take $c_1, c_2 \in \mathbb{Z}$ and $2 \leq r \in \mathbb{Z}$. Assume that one of the following conditions holds*

i) $c_1 = 1$ *or* $r-1$ $(mod\ \ r)$*;*

ii) $c_1 = r-2$ $(mod\ \ r)$ *and* $c_2 - \frac{c_1^2}{2} + \frac{3c_1}{2} - (r-1) = 0$ $(mod\ \ 2)$*;*

iii) $c_1 = 2$ $(mod\ \ r)$ *and* $c_2 + c_1 - \frac{c_1^2}{2} - \frac{3c_1}{2} + 1 = 0$ $(mod\ \ 2)$*.*

If $\Delta(r;c_1,c_2) >> 0$, then the moduli space $M_{\mathbb{P}^2,O_{\mathbb{P}^2}(1)}(r;c_1,c_2)$ is a non-empty, smooth, irreducible, rational, quasi-projective variety of dimension

$$2rc_2 - (r-1)c_1^2 - (r^2-1).$$

Proof. By Theorem 2.3.5 we only need to check the rationality of the moduli space $M_{\mathbb{P}^2,O_{\mathbb{P}^2}(1)}(r;c_1,c_2)$. We consider $X = X_{e=1}$ a smooth, Hirzebruch surface with invariant $e = 1$. It is well known that X is obtained by blowing up $\mathbb{P}^2$ at one point (see Remark 1.4.2). Let l be a line generating $Pic(\mathbb{P}^2)$ and $\pi : X \longrightarrow \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at one point not contained in l. We consider on X the ample divisor

$$H_n = n\pi^*l - C_0 = n(C_0+F) - C_0$$

with $n >> 0$. Under our assumptions, by Theorem 4.1.13, the moduli space $M_{X,H_n}(r; c_1(C_0+F), c_2) \cong M_{X,H_n}(r; c_1\pi^*l, c_2)$ is an irreducible, smooth, rational, quasi-projective variety of dimension $2rc_2 - (r-1)c_1^2 - (r^2-1)$.

Applying Theorem 1.2.15, we get an open immersion

$$\varphi : M_{\mathbb{P}^2, O_{\mathbb{P}^2}(1)}(r; c_1, c_2) \hookrightarrow M_{X,H_n}(r; c_1\pi^*l; c_2)$$

defined by $\varphi(V) = \pi^*(V)$ on closed points, between two smooth moduli spaces of the same dimension. Therefore, the moduli space $M_{\mathbb{P}^2, O_{\mathbb{P}^2}(1)}(r; c_1, c_2)$ is rational. □

We want to point out that part (*i*) of Theorem 4.1.14 extends to arbitrary rank Li's Theorem ([Li97]; Theorem 0.3). Moreover, Theorem 4.1.14 works for arbitrary rank r while Yoshioka's Theorem does not apply when $r = 0$ (*mod* 3).

Final Remarks: Let X be a smooth rational surface obtained by blowing up $s > 0$ different points of a smooth, Hirzebruch surface X_e (resp. $\mathbb{P}^2$). Denote by E_i, $1 \leq i \leq s$, the exceptional divisors. Fix $c_1 = aC_0 + dF - \sum_{i=1}^{\rho} \beta_i E_i$, with $\beta_i \in \{0,1\}$ (*mod r*), $c_2 \in \mathbb{Z}$ and H an ample divisor on X . Using either Nakashima's Theorem or arguing as in the proof of Theorem 4.1.13 , we can prove that if $\Delta(r; c_1, c_2) >> 0$, then the moduli space $M_H(r; c_1, c_2)$ is a non-empty, irreducible, smooth, rational, quasi-projective variety of dimension $2rc_2 - (r-1)c_1^2 - (r^2-1)$ provided one of the following conditions is satisfied

i) $c_1F = 1$ or $r - 1$ (*mod* r);

ii) $c_1F = r - 2$ (*mod* r) and $c_2 - \frac{c_1^2}{2} - \frac{c_1K}{2} - (r-1) = 0$ (*mod* 2);

iii) $c_1F = 2$ (*mod* r) and $c_2 + c_1C_0 - \frac{c_1^2}{2} + \frac{c_1K}{2} + 1 = 0$ (*mod* 2).

The results of this section together with the fact that there is at present no counterexample known to the question whether moduli spaces are always rational provided X is rational, strongly support the following

Conjecture: The moduli space $M_H(r; c_1, c_2)$ of rank r, H-stable vector bunles on a smooth surface X, with Chern classes c_1 and c_2 is rational provided X is rational.

Notice that once more an affirmative answer to this question will show that the moduli space of vector bundles inherits a lot of geometric properties from the underlying surface.

4.2 Moduli spaces of rank r vector bundles on K3-surfaces

Let X be a smooth, algebraic, $K3$ surface defined over an algebraically closed field k of characteristic 0, i.e., X is a smooth, algebraic surface with trivial canonical line bundle $K_X \simeq O_X$ and vanishing irregularity $q(X) = 0$. Fix an ample divisor H on X, $L \in Pic(X)$ and $c_2 \in \mathbb{Z}$. As usual, let $\mathcal{M}_H(r; L, c_2)$ be the moduli space of rank r, H-stable vector bundles E on X with $det(E) = L$ and $c_2(E) = c_2$. The goal of this section is to determine invariants $(r; L, c_2) \in \mathbb{Z} \times Pic(X) \times \mathbb{Z}$ and $l \in \mathbb{Z}$ for which the moduli space $\mathcal{M}_H(r; L, c_2)$ is birational to the Hilbert scheme $Hilb^l(X)$ of 0-dimensional subschemes of X of length l.

In 1984, Mukai ([Muk84]; Corollary 0.2) proved that the moduli space of simple sheaves on X has a symplectic structure. On the other hand, it is well known that the Hilbert scheme $Hilb^l(X)$ of 0-dimensional subschemes of X with length l has also a symplectic structure (see [Muk84]; Example 0.4 and [Bea83]). Hence, it seems natural to look for a closer relation between Hilbert schemes $Hilb^l(X)$ and moduli spaces $\mathcal{M}_H(r; L, c_2)$. In [Nak97], Nakashima proposes the following

Problem: To determine for arbitrary $K3$ surfaces X, all invariants $(r; L, c_2)$ for which $\mathcal{M}_H(r; L, c_2)$ are birational to some $Hilb^l(X)$.

For the rank 2 case, the first contribution to this problem is due to Zuo. He proved

Theorem ([Zuo91b]; Theorem 1) *Suppose X is an algebraic $K3$ surface and H is an ample divisor on X. Let $\mathcal{M}_H(2; 0, k(n))$ be the moduli space of H-stable rank 2 vector bundles E on X with $det(E) = 0$, $c_2(E) = k(n) := n^2H^2 + 3$, $n \in \mathbb{N}^+$ and let $Hilb^{2k(n)-3}(X)$ be the Hilbert scheme of 0-dimensional subschemes of X of length*

$2k(n)-3$. *Then there is a birational map*

$$\phi : \mathcal{M}_H(2;0,k(n)) \longrightarrow Hilb^{2k(n)-3}(X)$$

Later on, Nakashima generalized Zuo's Theorem to the triples $(r;L,c_2) = (2;L,k(n))$ where $k(n) := (n^2+n+\frac{1}{2})L^2+3$ and L is an ample divisor ([Nak93b]). In the arbitrary rank case almost nothing is known. Very recently, Nakashima has proved

Theorem ([Nak97]; Theorem 0.2; see also [OGr95]) *Let S be a $K3$ surface with (D,H) of degree one. If $c = \frac{D^2}{2}+r+1$ and $c \geq h^0(D)+1$ then $\mathcal{M}_H(r;D,c)$ is birational to the Hilbert scheme $Hilb^c(S)$ of zero dimensional cycles of length c.*

We would like to stress that the hypothesis (D,H) being of degree one is very "restrictive". The goal of this section is to prove the following Theorem

Theorem 4.2.1. *Let X be an algebraic, $K3$ surface and H an ample divisor on X. Let $\mathcal{M}_H(r;c_1,k(n))$ be the moduli space of H-stable, rank r vector bundles E on X with $det(E) = c_1 \in Pic(X)$, $c_2(E) = k(n) := \frac{c_1^2}{2}+\frac{r}{2}n^2H^2+nc_1H+(r+1)$ and let $Hilb^{l(n)}(X)$ be the Hilbert scheme of 0-dimensional subschemes of X of length $l(n)$. For $n >> 0$ there is a birational map*

$$\phi : \mathcal{M}_H(r;c_1,k(n)) \longrightarrow Hilb^{l(n)}(X)$$

where $l(n) := k(n)+\frac{r(r-1)}{2}n^2H^2+(r-1)nc_1H$.

Remark 4.2.2. Notice that when $r=2$ we recover the results of Zuo and Nakashima.

Let us start recalling the notation that we will use in this section. Until the end of this section, X will be an algebraic $K3$ surface. Given a divisor c_1 on X, an integer c_2 and an ample divisor H on X, we will denote by $\overline{\mathcal{M}}_H(r;c_1,c_2)$ the moduli space of rank r, torsion free sheaves F on X, G-semistable with respect to

H with $det(F) = c_1$ and $c_2(F) = c_2$ and by $\mathcal{M}_H(r; c_1, c_2) \subset \overline{\mathcal{M}}_H(r; c_1, c_2)$ the open subset parameterizing rank r, H-stable vector bundles F on X with $det(F) = c_1$ and $c_2(F) = c_2$.

In order to establish the birational correspondence, we will construct a suitable family of torsion free sheaves on X. To this end, let us fix $c_1 \in Pic(X)$ and an ample divisor H on X. Let n_0 be an integer such that for all $n \geq n_0$, $c_1 + rnH$ is ample. Set

$$k(n) := \tfrac{c_1^2}{2} + \tfrac{r}{2}n^2H^2 + nc_1H + (r+1);$$

$$l(n) := k(n) + \tfrac{r(r-1)}{2}n^2H^2 + (r-1)nc_1H.$$

Construction 4.2.3. *Let $\mathcal{F}$ be the irreducible family of rank r torsion free sheaves F on X, G-semistable with respect to H with $det(F) = c_1$ and $c_2(F) = k(n)$, given by a non-trivial extension*

$$0 \longrightarrow O_X^{r-1} \longrightarrow F(nH) \longrightarrow I_Z(c_1 + rnH) \longrightarrow 0$$

where Z is a 0-dimensional subscheme of X of length

$$\begin{aligned} |Z| &= c_2(F(nH)) = c_2(F) + (r-1)nc_1(F)H + \tfrac{r(r-1)}{2}n^2H^2 \\ &= k(n) + (r-1)nc_1H + \tfrac{r(r-1)}{2}n^2H^2 = l(n) \end{aligned}$$

such that $H^0 I_Z(c_1 + rnH) = 0$.

Lemma 4.2.4. *For $n >> 0$, $\mathcal{F}$ is non-empty.*

Proof. We fix $c_2' \in \mathbb{Z}$ such that $\mathcal{M}_H(r; c_1, c_2') \neq \emptyset$ ([Sor97]). It is well known that there exists an integer $n_{c_2'} \in \mathbb{Z}$ such that for all $n \geq n_{c_2'}$ and for any rank r vector bundle $E \in \mathcal{M}_H(r; c_1, c_2')$, $E(nH)$ is generated by its global sections and $\chi(E(nH)) \geq r-1$. We choose $(r-1)$ generic sections of $E(nH)$ and we get an exact sequence

$$0 \longrightarrow O_X^{r-1} \longrightarrow E(nH) \longrightarrow I_{\tilde{Z}}(c_1 + rnH) \longrightarrow 0$$

where $\tilde{Z}$ is a 0-dimensional subscheme of X of length

$$|\tilde{Z}| = c_2(E(nH)) = c_2' + \frac{r(r-1)}{2}n^2H^2 + (r-1)nc_1H.$$

Moreover, there exists an integer $l_{c_2'} \in \mathbb{Z}$ such that for all $l \geq l_{c_2'}$, if we choose l generic points $p_1, \cdots, p_l$ appropriately and a surjective map

$$\alpha : E \longrightarrow \oplus_{j=1}^{l} k_{p_j},$$

then F, the kernel of α, is a rank r, torsion free sheaf, G-semistable with respect to H, sitting into an exact sequence

$$0 \longrightarrow O_X^{r-1} \longrightarrow F(nH) \longrightarrow I_Z(c_1 + rnH) \longrightarrow 0$$

where $Z = \tilde{Z} \cup \{p_1, \cdots, p_l\}$. Indeed, since $F = Ker(\alpha) \subset E$ and $det(F) = det(E)$, the H-stability of E implies the G-semistability of F with respect to H.

For $n >> 0$ we can assume $k(n) - c_2' \geq l_{c_2'}$ and $n \geq max\{n_{c_2'}, n_0\}$. Define $l := k(n) - c_2' \geq l_{c_2'}$. As we have seen, there exists an exact sequence

$$0 \longrightarrow O_X^{r-1} \longrightarrow F(nH) \longrightarrow I_Z(c_1 + rnH) \longrightarrow 0$$

where Z is a 0-dimensional subscheme of X of length

$$\begin{aligned} |Z| = |\tilde{Z}| + l &= (c_2' + \tfrac{r(r-1)}{2}n^2H^2 + (r-1)nc_1H) + l \\ &= k(n) + \tfrac{r(r-1)}{2}n^2H^2 + (r-1)nc_1H \end{aligned}$$

and F is a rank r, torsion free sheaf, G-semistable with respect to H with $det(F) = c_1$ and $c_2(F) = k(n)$.

Since $c_1 + rnH$ is ample, by Kodaira's Vanishing Theorem $H^iO_X(c_1 + rnH) = 0$ for $i > 0$ and applying Riemann-Roch's Theorem (see 1.1.5) we get

$$h^0O_X(c_1 + rnH) = \chi(O_X(c_1 + rnH)) = \frac{c_1^2}{2} + \frac{r^2n^2H^2}{2} + rnc_1H + 2.$$

On the other hand,

$$|Z| = k(n) + \frac{r(r-1)}{2}n^2H^2 + (r-1)nc_1H$$

$$= \frac{c_1^2}{2} + \frac{r^2}{2} n^2 H^2 + rnc_1 H + (r+1).$$

Therefore, since $0 < r - 1$,

$$h^0 O_X(c_1 + rnH) - |Z| = -(r-1) < 0 \tag{4.10}$$

and hence for $l >> 0$ and l generic points,

$$H^0 I_Z(c_1 + rnH) = 0$$

(see Lemma 1.4.16). Putting this results together we get $F \in \mathcal{F}$, which proves our Lemma. □

The following two Lemmas are crucial for the definition of a birational correspondence between the Hilbert scheme $Hilb^{l(n)}(X)$ and the moduli space $\mathcal{M}_H(r; c_1, k(n))$.

Lemma 4.2.5. *With the above notation*

$$dim\mathcal{F} = 2l(n).$$

Proof. By definition

$$\begin{aligned} dim\mathcal{F} &= 2|Z| + dimGrass(r-1, Ext^1(I_Z(c_1 + rnH), O_X)) \\ &\quad -dimGrass(r-1, H^0F(nH)) \end{aligned}$$

where $Grass(s, V)$ is the Grassmann variety of s-dimensional subspaces of V and $dimGrass(s, V) = s \cdot dimV - s^2$.

Consider the exact cohomology sequence

$$0 \longrightarrow H^0 O_X^{r-1} \longrightarrow H^0 F(nH) \longrightarrow H^0 I_Z(c_1 + rnH) \longrightarrow \cdots$$

associated to the exact sequence

$$0 \longrightarrow O_X^{r-1} \longrightarrow F(nH) \longrightarrow I_Z(c_1 + rnH) \longrightarrow 0.$$

Since $H^0 I_Z(c_1 + rnH) = 0$, we obtain

$$h^0 F(nH) = h^0 O_X^{r-1} = r - 1.$$

On the other hand, the exact cohomology sequence

$$0 \longrightarrow H^0I_Z(c_1+rnH) \longrightarrow H^0O_X(c_1+rnH) \longrightarrow H^0O_Z(c_1+rnH) \longrightarrow$$
$$\longrightarrow H^1I_Z(c_1+rnH) \longrightarrow H^1O_X(c_1+rnH) \longrightarrow \cdots$$

associated to the exact sequence

$$0 \longrightarrow I_Z(c_1+rnH) \longrightarrow O_X(c_1+rnH) \longrightarrow O_Z(c_1+rnH) \longrightarrow 0,$$

together with the fact that c_1+rnH is ample and hence $H^iO_X(c_1+rnH)=0$ for $i>0$, gives us

$$\begin{aligned} dimExt^1(I_Z(c_1+rnH),O_X) &= h^1I_Z(c_1+rnH) \\ &= |Z| - h^0O_X(c_1+rnH) = r-1 \end{aligned}$$

where the last equality follows from (4.10). Putting these results together we get

$$\begin{aligned} dim\mathcal{F} &= 2l(n)+(r-1)dimExt^1(I_Z(c_1+rnH),O_X)-(r-1)^2 \\ &\quad -((r-1)h^0F(nH)-(r-1)^2) \\ &= 2l(n) \end{aligned}$$

which proves the Lemma. □

Remark 4.2.6. It follows from the definition of $l(n)$, $k(n)$ and Lemma 4.2.5 that for $n >> 0$

$$\begin{aligned} dim\mathcal{F} = dimHilb^{l(n)}(X) &= 2l(n) \\ &= 2rk(n)-(r-1)c_1^2-2(r^2-1) \\ &= dim\overline{\mathcal{M}}_H(r;c_1,k(n)). \end{aligned}$$

Lemma 4.2.7. *Any torsion free sheaf $F \in \mathcal{F}$ is simple.*

Proof. Applying the functor $Hom(F(nH),.)$ to the exact sequence

$$(4.11) \quad 0 \longrightarrow O_X^{r-1} \longrightarrow F(nH) \longrightarrow I_Z(c_1 + rnH) \longrightarrow 0$$

we get the long exact sequence

$$0 \longrightarrow Hom(F(nH), O_X^{r-1}) \longrightarrow Hom(F(nH), F(nH))$$

$$\longrightarrow Hom(F(nH), I_Z(c_1 + rnH)) \longrightarrow \cdots.$$

Since $n >> 0$, by Serre's duality we have

$$Hom(F(nH), O_X^{r-1}) = H^2 F(nH)^{r-1} = 0.$$

Therefore, it suffices to see that $dimHom(F(nH), I_Z(c_1 + rnH)) = 1$. To this end, we consider the exact cohomology sequence

$$0 \longrightarrow Hom(I_Z(c_1 + rnH), I_Z(c_1 + rnH)) \longrightarrow Hom(F(nH), I_Z(c_1 + rnH))$$

$$\longrightarrow Hom(O_X^{r-1}, I_Z(c_1 + rnH)) \longrightarrow \cdots$$

obtained applying the functor $Hom(\cdot, I_Z(c_1 + rnH))$ to the exact sequence (4.11). Since $F \in \mathcal{F}$, $H^0 I_Z(c_1 + rnH) = 0$ and we get

$$dimHom(F(nH), I_Z(c_1 + rnH)) = dimHom(I_Z(c_1 + rnH), I_Z(c_1 + rnH)) = 1$$

which proves the Lemma. □

Keeping in mind previous results, we can prove the main result of this section.

Proof of Theorem 4.2.1. For $n >> 0$, we have two natural rational morphisms

$$\pi : \mathcal{F} \longrightarrow Hilb^{l(n)}(X) \quad \text{and}$$

$$e : \mathcal{F} \longrightarrow \overline{\mathcal{M}}_H(r; c_1, k(n)).$$

The fiber $\pi^{-1}(Z)$ over $Z \in Hilb^{l(n)}(X)$ is identified with a non-empty open subset of the Grassmann variety

$$Grass(r-1, Ext^1(I_Z(c_1 + rnH), O_X))$$

and the fiber $e^{-1}(F)$ over $F \in \overline{\mathcal{M}}_H(r; c_1, k(n))$ is canonically isomorphic to a non-empty Zariski open subset of

$$Grass(r-1, H^0F(nH)).$$

Notice that from the above dimension computations, we have

$$dim(\pi^{-1}(Z)) = dim(e^{-1}(F)) = 0$$

for all generic $Z \in Hilb^{l(n)}(X)$ and for all generic $F \in \overline{\mathcal{M}}_H(r; c_1, k(n))$ respectively.

Let us see that e is an injection. Assume that there are two non trivial extensions

$$0 \longrightarrow O_X^{r-1} \xrightarrow{\alpha_1} F(nH) \xrightarrow{\alpha_2} I_Z(c_1 + rnH) \longrightarrow 0;$$

$$0 \longrightarrow O_X^{r-1} \xrightarrow{\beta_1} F(nH) \xrightarrow{\beta_2} I_{Z'}(c_1 + rnH) \longrightarrow 0$$

where Z and Z' are 0-dimensional subschemes of X of length $l(n)$.

From the fact that $H^0I_Z(c_1 + rnH) = H^0I_{Z'}(c_1 + rnH) = 0$ we get

$$Hom(O_X^{r-1}, I_Z(c_1 + rnH)) = Hom(O_X^{r-1}, I_{Z'}(c_1 + rnH)) = 0.$$

Thus, $\beta_2 \circ \alpha_1 = \alpha_2 \circ \beta_1 = 0$. So, there exists $\gamma \in Aut(F(nH)) \simeq k$ (Lemma 4.2.7) such that $\beta_1 = \gamma \circ \alpha_1$. Therefore, $Z \simeq Z'$ and hence, e is an injection.

Since $h^0F(nH) = r - 1$, π is also an injection and by Remark 4.2.6,

$$dim\mathcal{F} = dimHilb^{l(n)}(X) = dim\overline{\mathcal{M}}_H(r; c_1, k(n)).$$

Hence, e and π are birational maps. Composing, we get a birational map

$$\pi \circ e^{-1} = \psi : \overline{\mathcal{M}}_H(r; c_1, k(n)) \longrightarrow Hilb^{l(n)}(X).$$

Moreover, since $\mathcal{M}_H(r; c_1, k(n))$ is an open dense subset of $\overline{\mathcal{M}}_H(r; c_1, k(n))$, restricting ψ to $\mathcal{M}_H(r; c_1, k(n))$ we obtain the birational morphism claimed in Theorem 4.2.1. □

Remark 4.2.8. The pullback of the symplectic structure on $Hilb^{l(n)}(X)$ via the birational map ϕ of Theorem 4.2.1, gives a symplectic structure on $\mathcal{M}_H(r; c_1, k(n))$. This symplectic structure coincides with the symplectic structure given by Mukai ([Muk84]).

Remark 4.2.9. It is possible to describe explicitly the birational map

$$\phi : \mathcal{M}_H(r; c_1, k(n)) \longrightarrow Hilb^{l(n)}(X)$$

and, as an application, we check that the Hodge numbers of $\overline{\mathcal{M}}_H(r; c_1, k(n))$ and the Hilbert scheme $Hilb^{l(n)}(X)$ coincide. Furthermore, since the Hodge numbers of $Hilb^{l(n)}(X)$ can be expressed in terms of the Hodge numbers $h^{p,q}(X)$ of X (see [GS93];[Che93]), we deduce that the Hodge numbers of $\overline{\mathcal{M}}_H(r; c_1, k(n))$ can be computed in terms of $h^{p,q}(X)$.

Chapter 5

Vector bundles on higher dimensional varieties

This chapter is devoted to moduli spaces of vector bundles on $\mathbb{P}^d$-bundles. Once the existence of the moduli space is established (see [Mar77], [Mar78]), the question arises as what can be said about its local and global structure. More precisely, what does the moduli space look like as an algebraic variety ? Is it, for example, connected, irreducible, rational or smooth? How does it look as a topological space? What is its geometry?

Very little is known concerning moduli spaces $M_{X,L}(r; c_1, \cdots, c_{min\{r,n\}})$ of rank r, L-stable (in the sense of Mumford-Takemoto) vector bundles E with fixed Chern classes $c_i(E) = c_i \in H^{2i}(X, \mathbb{Z})$, on a n-dimensional, smooth, irreducible, projective variety, X, if the underlying variety has dimension greater or equal than three. Up to now, there are no general results about these moduli spaces concerning the number of connected components, dimension, smoothness, rationality, topological invariants, etc.

It was a major result in the theory of vector bundles on an algebraic surface S, the proof that, for large c_2, the moduli space $M_{S,L}(r; c_1, c_2)$ of rank r, L-stable, vector bundles on S with fixed c_1 and fixed polarization, L, is irreducible, generically smooth and of the expected dimension $2rc_2 - (r-1)c_1^2 - r^2\chi(O_S) + p_g(S) + 1$. For moduli spaces of vector bundles on a higher dimensional variety, X, the situation changes drastically. Smoothness and irreducibility turn to be false when $dim X \geq 3$. For instance, in [BM97]; Theorem 0.1, Ballico and Miró-Roig prove that under a certain technical restriction on c_1, the number of irreducible components of the moduli space $M_{X,L}(2;c_1,c_2)$ of L-stable, rank 2 vector bundles on a smooth, projective 3-fold, (X,L), with fixed c_1 and c_2L going to infinity, grows to infinity. See [MO97] for examples of singular moduli spaces of vector bundles on $\mathbb{P}^{2n+1}$ with $c_2 >> 0$.

Let $X = \mathbb{P}(\mathcal{E}) \to C$ be a $\mathbb{P}^d$-bundle over a smooth, projective curve C of genus $g \geq 0$. The goal of this chapter is to compute the dimension, prove the irreducibility and smoothness and describe the structure of the moduli space $M_L(2;c_1,c_2)$ of L-stable, rank 2 vector bundles E on X with certain Chern classes and for a suitable polarization L closely related to c_2. More precisely, we will cover the study of all moduli spaces $M_L(2;c_1,c_2)$ such that the general point $[E] \in M_L(2;c_1,c_2)$ is given as a non-trivial extension of line bundles (see Theorems 5.2.4, 5.2.8 and 5.2.12; and Remark 5.2.13). In particular, for rational normal scrolls, i.e. $\mathbb{P}^d$-bundles over $\mathbb{P}^1$, and for a certain choice of c_1, c_2 and L, we get that the moduli space $M_L(2;c_1,c_2)$ is rational (see Corollaries 5.2.5 and 5.2.9). Notice that once again, the geometry of the underlying variety and of the moduli spaces are intimately related.

Next, we outline the ideas used for proving our results and the structure of the chapter. In section 1, we first recall some basic facts on $\mathbb{P}^d$-bundles over a smooth, projective curve of genus $g \geq 0$ needed later on. We also include the key Proposition of our results. Namely, the existence of a section s of a suitable twist of a rank 2 vector bundle E on a $\mathbb{P}^d$-bundle, $X = \mathbb{P}(\mathcal{E}) \to C$, whose zero scheme, $(s)_0$, has codimension greater or equal than two (Proposition 5.1.13).

Section 2 contains our main results on moduli spaces, namely the irreducibility, smoothness and structure of the moduli space $M_L(2;c_1,c_2)$ of L-stable, rank 2 vector bundles on a $\mathbb{P}^d$-bundle $X = \mathbb{P}(\mathcal{E}) \to C$ with certain Chern classes and a suitable polarization L. Our approach is to write L-stable, rank 2 vector bundles E on X as an extension of two line bundles. A well known result for vector bundles over curves is that any vector bundle of rank $r \geq 2$, can be written as an extension of lower rank vector bundles. For higher dimensional varieties we may not to be able to get such a nice result. (For instance it is not true for $X = \mathbb{P}^n$, $n \geq 2$). However, it turns out to be true for certain L-stable, rank 2 vector bundles E on $\mathbb{P}^d$-bundles X.

In section 3, we illustrate by means of an example the changes of the moduli space $M_L(2; c_1, c_2)$ that occur when the polarization L varies (Theorem 5.3.2).

5.1 General results and preliminaries

Let us start this section reviewing some facts about $\mathbb{P}^d$-bundles over non-singular, projective curves of genus $g \geq 0$ that we will need in the sequel.

Definition 5.1.1. *Let $\mathcal{E}$ be a vector bundle over a smooth, projective curve C. The vector bundle $\mathcal{E}$ is said to be normalized if $h^0\mathcal{E} \neq 0$ but $h^0\mathcal{E}(L) = 0$ for all $L \in Pic(C)$ with $deg(L) < 0$.*

From now until the end of chapter 5, we fix a smooth, irreducible, projective curve C of genus $g \geq 0$. Let $\mathcal{E}$ be a rank $(d+1)$ vector bundle on C and consider

$$X = \mathbb{P}(\mathcal{E}) = Proj(Sym\mathcal{E}) \xrightarrow{\pi} C$$

the projectivized vector bundle associated to $\mathcal{E}$ with the natural projection π. X is a $(d+1)$-dimensional variety called a $\mathbb{P}^d$-bundle over C. When $d = 1$, we simply say that X is a ruled surface.

Notice that two vector bundles $\mathcal{E}$ and $\mathcal{E}'$ on C define the same $\mathbb{P}^d$-bundle if, and only if, there is an invertible sheaf $\mathcal{L}$ on C such that $\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{L}$.

Let $\mathcal{H} := O_{\mathbb{P}(\mathcal{E})}(1)$ be the tautological line bundle and for any point $\mathfrak{p} \in C$ we write $\mathcal{F}_{\mathfrak{p}} := \pi^* O_C(\mathfrak{p})$ and $F_{\mathfrak{p}} := \pi^*\mathfrak{p}$. Let H be the numerical equivalence class associated to the tautological line bundle $\mathcal{H}$ on X and let F be the numerical equivalence class associated to $\mathcal{F}_{\mathfrak{p}}$. We have

$$Pic(X) \cong \mathbb{Z}H \oplus \pi^* Pic(C) \quad \text{and} \quad Num(X) \cong \mathbb{Z}^2 \cong \mathbb{Z}H \oplus \mathbb{Z}F$$

with the intersection product given by

$$H^{d+1} = deg(\mathcal{E}), \quad H^d F = 1 \quad \text{and} \quad F^2 = 0.$$

The canonical divisor is

$$K_X \sim -(d+1)H + \pi^*(det(\mathcal{E}) + K_C)$$

being K_C the canonical divisor of C.

Moreover, if $D \sim aH + \pi^*\mathfrak{b}$ with $a \in \mathbb{Z}$ and $deg\mathfrak{b} = b$, then $D \equiv aH + bF$ and if, in addition, $a \geq 0$, then $\pi_* D = S^a(\mathcal{E}) \otimes O_C(\mathfrak{b})$, being $S^a(\mathcal{E})$ the a-th symmetric power of $\mathcal{E}$.

Example 5.1.2. If $\mathcal{E}$ is a rank two vector bundle on a smooth, projective curve C of genus $g \geq 0$, then X is a ruled surface with $p_a = -g$, $p_g = 0$ and irregularity $q = g$.

Example 5.1.3. Let

$$\mathcal{E} = \bigoplus_{i=0}^{d} O_{\mathbb{P}^1}(a_i)$$

be a rank $d+1$ vector bundle on $\mathbb{P}^1$ and assume $0 = a_0 \leq a_1 \leq \cdots \leq a_d$ with $a_d > 0$. Let $Y = Y(a_0, \cdots, a_d) := \mathbb{P}(\mathcal{E}) = Proj(Sym\mathcal{E}) \xrightarrow{\pi} \mathbb{P}^1$ be the projectivized vector bundle and let $O_{\mathbb{P}(\mathcal{E})}(1)$ be the tautological line bundle.

The line bundle $O_{\mathbb{P}(\mathcal{E})}(1)$ is generated by its global sections and defines a birational map

$$Y = Y(a_0, \cdots, a_d) := \mathbb{P}(\mathcal{E}) \xrightarrow{f} \mathbb{P}^N$$

with $N := d + \sum_{i=0}^{d} a_i$. We write $V(\mathcal{E})$ for the image of this map. It is a variety of dimension $d+1$ and minimal degree $c = \sum_{i=0}^{d} a_i$, called rational normal scroll. By abusing, sometimes Y is also called rational normal scroll (see [EH87] for more details). In case $d = 1$, we get the Hirzebruch surfaces described in chapter 1.

The Lemma below gives a characterization of ample divisors on a $\mathbb{P}^d$-bundle X over a smooth projective curve C.

Lemma 5.1.4. *Let $\mathcal{E}$ be a rank $d+1$ vector bundle over a smooth, projective curve C of genus $g \geq 0$ and $X = \mathbb{P}(\mathcal{E})$. If $D \equiv aH + bF$ is a divisor on X, then D is ample if, and only if,*

$$a > 0 \quad \text{and} \quad b + a\mu^-(\mathcal{E}) > 0$$

being $\mu^-(\mathcal{E}) := min\{\mu(Q) | \mathcal{E} \to Q \to 0\}$.

Proof. See [Miy85]; Theorem 3.1 □

Remark 5.1.5. Notice that if $d = 1$ and $C = \mathbb{P}^1$, we recover the ampleness criterion for divisors on Hirzebruch surfaces (see Remark 1.4.7).

Notation 5.1.6. Given $X = \mathbb{P}(\mathcal{E})$ a $\mathbb{P}^d$-bundle over C we will write

$$\gamma = \gamma(\mathcal{E}) := max\{-\mu^-(\mathcal{E}) + 1, 1\}$$

being $\mu^-(\mathcal{E}) := min\{\mu(Q) | \mathcal{E} \to Q \to 0\}$.

Remark 5.1.7. We deduce from Lemma 5.1.4 that the divisor $L \equiv H + \gamma F$ is ample. Hence, the following inequality holds for any effective divisor $D \equiv nH + mF$

$$0 \leq (nH + mF)(H + \gamma F)^d = nH^{d+1} + nd\gamma + m.$$

Now we shall compute some cohomology groups of line bundles on X needed later on, and the dimension of the irreducible family of codimension two closed subschemes Z of X which are complete intersection of type $(H, F_{\mathfrak{p}})$ being $\mathfrak{p}$ a point of C.

Lemma 5.1.8. *For any $\mathfrak{b} \in Pic(C)$, we consider the line bundle $O_X(aH + \pi^*\mathfrak{b})$ on a $\mathbb{P}^d$-bundle $\pi : X = \mathbb{P}(\mathcal{E}) \to C$ over C. We have*

$$H^i(X, O_X(aH + \pi^*\mathfrak{b})) = \begin{cases} 0 & \text{if } -d-1 < a < 0 \\ H^i(C, S^a(\mathcal{E}) \otimes O_C(\mathfrak{b})) & \text{if } a \geq 0 \\ H^{d+1-i}(C, S^{-d-1-a}(\mathcal{E}) \otimes O_C(\tilde{\mathfrak{b}})) & \text{if } a \leq -d-1 \end{cases}$$

being $\tilde{\mathfrak{b}} := -\mathfrak{b} + det(\mathcal{E}) + K_C$, K_C the canonical divisor of C and $S^a(\mathcal{E})$ the a-th symmetric power of $\mathcal{E}$.

Proof. By the projection formula we have

$$R^i\pi_*O_X(aH + \pi^*\mathfrak{b}) = R^i\pi_*O_{\mathbb{P}(\mathcal{E})}(a) \otimes O_C(\mathfrak{b})$$

being $R^i\pi_*O_{\mathbb{P}(\mathcal{E})}(a) = 0$ for $0 < i < d$ and all $a \in \mathbb{Z}$ and $R^d\pi_*O_{\mathbb{P}(\mathcal{E})}(a) = 0$ for $a > -d-1$. Moreover, using the Base Change Theorem we get $R^i\pi_*O_{\mathbb{P}(\mathcal{E})}(a) = 0$ for $i \geq d+1$.

Since, $R^i\pi_*O_X(aH + \pi^*\mathfrak{b}) = 0$ for $i > 0$ and $a > -d-1$, by the degeneration of the Leray Spectral sequence

$$H^i(C, R^j\pi_*O_{\mathbb{P}(\mathcal{E})}(aH + \pi^*\mathfrak{b})) \Rightarrow H^{i+j}(\mathbb{P}(\mathcal{E}), O_{\mathbb{P}(\mathcal{E})}(aH + \pi^*\mathfrak{b}))$$

we obtain

$$H^i(X, O_X(aH + \pi^*\mathfrak{b})) = H^i(C, \pi_*O_X(aH + \pi^*\mathfrak{b})) \quad \text{for all } a > -d-1$$

with $\pi_*O_X(aH+\pi^*\mathfrak{b}) = S^a(\mathcal{E})\otimes O_C(\mathfrak{b})$ if $a \geq 0$ and 0 otherwise. The case $a \leq -d-1$ follows from the case $a \geq 0$ and Serre's duality. Hence, the Lemma is proved. □

Lemma 5.1.9. *Let $X = \mathbb{P}(\mathcal{E})$ be a $\mathbb{P}^d$-bundle over C and let $\mathcal{L}$ be the irreducible family of codimension two closed subschemes Z of X which are complete intersection of type $(H, F_\mathfrak{p})$ being $\mathfrak{p}$ a point of C. Then, $dim\mathcal{L} = h^0\mathcal{E} + h^0O_C(\mathfrak{p}) - h^0\mathcal{E}(-\mathfrak{p}) - 2$. Moreover, if $\mathcal{E}$ is normalized then $dim\mathcal{L} = h^0\mathcal{E} + h^0O_C(\mathfrak{p}) - 2$.*

Proof. From the exact sequence

$$(5.1) \qquad 0 \longrightarrow O_X(-H - F_\mathfrak{p}) \longrightarrow O_X(-H) \oplus O_X(-F_\mathfrak{p}) \longrightarrow I_Z \longrightarrow 0$$

we deduce

$$\begin{aligned} dim\mathcal{L} &= dimHom(O_X(-H - F_\mathfrak{p}), O_X(-H) \oplus O_X(-F_\mathfrak{p})) - dimAut(O_X(-H - F_\mathfrak{p})) \\ &\quad -dimAut(O_X(-H) \oplus O_X(-F_\mathfrak{p})) + dimI_f \end{aligned}$$

where $f \in Hom(O_X(-H - F_\mathfrak{p}), O_X(-H) \oplus O_X(-F_\mathfrak{p}))$ is a general element and I_f denotes its isotropy group under the action of

$$Aut(O_X(-H - F_\mathfrak{p})) \times Aut(O_X(-H) \oplus O_X(-F_\mathfrak{p})).$$

From Lemma 5.1.8 we obtain

$$dimAut(O_X(-H) \oplus O_X(-F_\mathfrak{p})) = 2h^0 O_X + h^0 O_X(H - F_\mathfrak{p}) = 2 + h^0 \mathcal{E}(-\mathfrak{p}),$$

$$dimAut(O_X(-H - F_\mathfrak{p})) = h^0 O_X = 1,$$

$$dimHom(O_X(-H - F_\mathfrak{p}), O_X(-H) \oplus O_X(-F_\mathfrak{p})) = h^0 O_X(H) + h^0 O_X(F_\mathfrak{p}).$$

Finally, since $h^0 O_X(H) = h^0 \mathcal{E}$, $h^0 O_X(F_\mathfrak{p}) = h^0 O_C(\mathfrak{p})$ and $dimI_f = 1$ we get

$$dim\mathcal{L} = h^0 \mathcal{E} + h^0 O_C(\mathfrak{p}) - h^0 \mathcal{E}(-\mathfrak{p}) - 2.$$

If $\mathcal{E}$ is normalized $H^0 \mathcal{E}(-\mathfrak{p}) = 0$ and hence $dim\mathcal{L} = h^0\mathcal{E} + h^0 O_C(\mathfrak{p}) - 2$ which proves the Lemma. □

Let E be a rank 2 vector bundle on a $\mathbb{P}^d$-bundle X. Since $H^2(X, \mathbb{Z})$ is generated by the classes H, F and $H^4(X, \mathbb{Z})$ is generated by the classes H^2 and HF, the Chern classes $c_i(E) \in H^{2i}(X, \mathbb{Z})$, $i = 1, 2$ of E may be written as $c_1(E) \equiv aH + bF$ and $c_2(E) \equiv xH^2 + yHF$ with $a, b, x, y \in \mathbb{Z}$.

By 1.1.1 and Remark 1.1.9, we may assume, without loss of generality, that for any rank two vector bundle E on a $\mathbb{P}^d$-bundle X, $c_1(E)$ is numerically equivalent to one of the following classes: H, $H + F$, F, 0.

We also need the following result

Lemma 5.1.10. *Let I_Z be the ideal sheaf of the codimension two closed subscheme $Z \equiv aH^2 + bHF$ on a $\mathbb{P}^d$-bundle X. Then,*

$$c_1(I_Z(pH + \pi^*\mathfrak{q})) \equiv pH + qF,$$

$$c_2(I_Z(pH + \pi^*\mathfrak{q})) \equiv aH^2 + bHF \quad \text{and}$$

$$c_3(I_Z(pH + \pi^*\mathfrak{q})) \equiv -a(p - 1 - a)H^3 + (2ab - aq - pb + b)H^2F$$

being $q = deg(\mathfrak{q})$.

Proof. Let $\mathfrak{b} \in Pic(C)$ be a divisor on C with $deg(\mathfrak{b}) = b$. The exact sequence

$$0 \longrightarrow O_X(-aH - \pi^*\mathfrak{b} - H) \longrightarrow O_X(-aH - \pi^*\mathfrak{b}) \oplus O_X(-H) \longrightarrow I_Z \longrightarrow 0$$

gives us

$$c_t(O_X(-aH - \pi^*\mathfrak{b} + pH + \pi^*\mathfrak{q}))c_t(O_X(pH + \pi^*\mathfrak{q} - H))$$

$$= c_t(O_X(pH + \pi^*\mathfrak{q} - aH - \pi^*\mathfrak{b} - H))c_t(I_Z(pH + \pi^*\mathfrak{q}))$$

being $c_t(G)$ the Chern polynomial of G. The Lemma follows after a straightforward computation. □

We shall end this section with two results that will be very useful for us in the sequel.

Lemma 5.1.11. *Let E be a rank two vector bundle on $\mathbb{P}^d$. If $c_2(E) \leq 0$, then*

$$H^0(\mathbb{P}^d, E) \neq 0.$$

Remark 5.1.12. We identify the Chern classes $c_i = c_i(E) \in H^{2i}(\mathbb{P}^d, \mathbb{Z})$ of a vector bundle E on $\mathbb{P}^d$ with integers.

Proof of Lemma 5.1.11. Since $c_2(E) \leq 0$, we have $c_1^2(E) - 4c_2(E) \geq 0$. Notice that Schwarzenberger's inequality $c_1^2 - 4c_2 < 0$ for stable rank 2 vector bundles on $\mathbb{P}^2$ together with Barth's Theorem which states that the restriction of a stable rank 2 vector bundle on $\mathbb{P}^d$ to a general hyperplane is again stable (with exception of the Null-correlation bundle on $\mathbb{P}^3$) implies that E is not stable.

Since $E^* \cong E(-c_1)$, $c_1(E^*) = -c_1(E)$ and $c_2(E^*) = c_2(E)$, being E^* the dual of E, we may assume $c_1(E) \leq 0$ and $c_2(E) \leq 0$.

Let n be the least integer such that $H^0(\mathbb{P}^d, E(n)) \neq 0$. We have to show that $n \leq 0$. We take a non-zero section $s \in H^0E(n)$. Then the scheme of zeros of s represents the second Chern class of $E(n)$. Hence,

$$0 \leq c_2(E(n)) = c_2(E) + nc_1(E) + n^2.$$

Since $c_2(E) \leq 0$ it follows that $n(c_1(E)+n) \geq 0$. If $n > 0$, then

$$c_1(E) + 2n > c_1(E) + n \geq 0.$$

Let us see that in such case E is stable. To this end, we take a rank one subbundle $O_{\mathbb{P}^d}(r)$ of E. Since $h^0E(-r) \neq 0$ we have $n \leq -r$. Therefore,

$$2r \leq -2n < c_1(E)$$

and E is stable which is a contradiction.Therefore, $n \leq 0$ and the Lemma follows. □

The following proposition will be the key point for proving our main results on moduli spaces of vector bundles on $\mathbb{P}^d$-bundles. It will assure us the existence of sections vanishing in codimension ≥ 2, that will allow us to prove the irreducibility and smoothness of the moduli spaces we deal with.

Proposition 5.1.13. *Let X be a $\mathbb{P}^d$-bundle over C, c_2 an integer, $L \equiv dH + bF$ an ample divisor on X, $e \in \{0,1\}$ and E a rank two, L-stable vector bundle on X. Assume that one of the following conditions is satisfied*

(i) $c_1E \equiv H + eF$, $c_2E \equiv (c_2+e)HF$, $b = 2c_2 - H^{d+1} + e - 1$ *and*

$$c_2 > \frac{d\gamma + H^{d+1}}{2} + 1;$$

(ii) $c_1E \equiv eF$, $c_2E \equiv -H^2 + (2c_2+e)HF$, $b = c_2 - H^{d+1} + e - 1$ *and*

$$c_2 > d\gamma + H^{d+1} + 2.$$

Then, $E(-H + \pi^*\mathfrak{c}_2)$ has a non-zero section whose scheme of zeros has codimension greater or equal than two, being $\mathfrak{c}_2 \in Pic(C)$ a divisor on C with $deg(\mathfrak{c}_2) = c_2$.

Proof. (i) First of all, notice that Lemma 5.1.4 implies that $L \equiv dH + bF$ with $b = 2c_2 - H^{d+1} + e - 1$ and $2c_2 > d\gamma + H^{d+1} + 2$ is an ample divisor on X. For any L-stable rank two vector bundle E on X with Chern classes $c_1E \equiv H + eF$ and

$c_2E \equiv (c_2+e)HF$ we consider $\overline{E} := E(-H+\pi^*\mathfrak{c}_2)$ being $\mathfrak{c}_2 \in Pic(C)$ a divisor on C of degree c_2. Using 1.1.1 we obtain

$$c_1(\overline{E}) \equiv -H + (2c_2+e)F,$$

$$c_2(\overline{E}) \equiv c_2E + (H+eF)(-H+c_2F) + (-H+c_2F)^2 = 0.$$

Since, $c_2(\overline{E}) = 0$ and $F \cong \mathbb{P}^d$, from Lemma 5.1.11 we deduce $h^0(F, \overline{E}|_F) \neq 0$. Therefore, there exists an integer $a \geq 0$ such that

$$O_{\mathbb{P}^d}(a) \hookrightarrow \overline{E}|_{\mathbb{P}^d}.$$

This injection induces an injection $O_X(aH+\pi^*\mathfrak{b}') \hookrightarrow \overline{E}$ for some divisor $\mathfrak{b}'$ on C.

Take $0 \neq s \in H^0\overline{E}(-aH-\pi^*\mathfrak{b}')$ and let Y be its scheme of zeros. Let A be the maximal effective divisor contained in Y. s can be regarded as a section of $\overline{E}(-aH-\pi^*\mathfrak{b}'-A)$ and its scheme of zeros has codimension ≥ 2. Then, if

$$l'H + \pi^*\mathfrak{m}' \equiv aH + \pi^*\mathfrak{b}' + A \quad \text{with} \quad l' \geq 0,$$

$\overline{E}(-l'H-\pi^*\mathfrak{m}')$ with $l' \geq 0$ has a non-zero section whose scheme of zeros has codimension ≥ 2. Therefore, $E(-lH-\pi^*\mathfrak{m})$, being $l > 0$ and $\mathfrak{m}$ a divisor on C with $deg(\mathfrak{m}) = m$, has a non-zero section whose scheme of zeros has codimension ≥ 2. To end the proof of (i) we only need to show that $l = 1$ and $m = -c_2$.

Since E is L-stable and $O_X(lH+\pi^*\mathfrak{m}) \hookrightarrow E$ we have

$$(lH+mF)L^d = d^d(lH^{d+1}+lb+m) < \frac{c_1(E)L^d}{2} = \frac{d^d(H^{d+1}+b+e)}{2}$$

which, since $b = 2c_2 - H^{d+1} + e - 1$, is equivalent to

$$2m < -2(2l-1)c_2 - (2l-1)(e-1) + e.$$

On the other hand, since $E(-lH-\pi^*\mathfrak{m})$ has a non-zero section whose scheme of zeros has codimension ≥ 2, we get

$$\begin{aligned} 0 \leq c_2(E(-lH-\pi^*\mathfrak{m}))H^{d-1} &= ((c_2+e+2lm-m-el)HF + l(l-1)H^2)H^{d-1} \\ &= c_2 + e(1-l) + (2l-1)m + l(l-1)H^{d+1}. \end{aligned}$$

Therefore,

$$m \geq \frac{-l(l-1)H^{d+1} - c_2 + e(l-1)}{2l-1}.$$

By hypothesis $2c_2 > d\gamma + H^{d+1} + 2$. Hence, putting all these inequalities together we get

$$(5.2) \quad \tfrac{-2l(l-1)c_2}{2l-1} - \tfrac{c_2}{2l-1} + \tfrac{l(l-1)d\gamma}{2l-1} + \tfrac{2l(l-1)}{2l-1} + \tfrac{e(l-1)}{2l-1} \leq m$$
$$< -(2l-1)c_2 - \tfrac{(2l-1)(e-1)}{2} + \tfrac{e}{2}$$

which implies that

$$l^2(2c_2 + d\gamma + 2e) - l(2c_2 + d\gamma + 2e) - \frac{1}{2} < 0 \quad \text{with } l \geq 1.$$

Hence, $l = 1$ and using again (5.2) we obtain $m = -c_2$, which proves (i).

(*ii*) By Lemma 5.1.4, $L \equiv dH + bF$ with $b = c_2 - H^{d+1} + e - 1$ is an ample divisor on X. For any L-stable rank two vector bundle E on X with Chern classes $c_1E \equiv eF$ and $c_2E \equiv -H^2 + (2c_2 + e)HF$ we have $c_2(E(-H + \pi^*\mathfrak{c}_2)) = 0$ and arguing as in the case (*i*) we get that $E(-lH - \pi^*\mathfrak{m})$ with $l > 0$ and $\mathfrak{m} \in Pic(C)$ of degree m, has a non-zero section whose scheme of zeros has codimension ≥ 2. Therefore, to end the proof of (*ii*) we only need to see that $l = 1$ and $m = -c_2$.

Since E is L-stable and $O_X(lH + \pi^*\mathfrak{m}) \hookrightarrow E$ we have

$$(lH + mF)L^d = d^d(lH^{d+1} + lb + m) < \frac{c_1(E)L^d}{2} = \frac{ed^d}{2}$$

which, since $b = c_2 - H^{d+1} + e - 1$, is equivalent to $2m < -2lc_2 - 2l(e-1) + e$.

On the other hand, since $E(-lH - \pi^*\mathfrak{m})$ has a non-zero section whose scheme of zeros has codimension ≥ 2, we get

$$0 \leq c_2(E(-lH - \pi^*\mathfrak{m}))H^{d-1} = ((2c_2 + e + 2lm - el)HF + (l^2-1)H^2)H^{d-1}$$
$$= 2c_2 + e(1-l) + 2lm + (l^2-1)H^{d+1}.$$

Therefore,

$$m \geq -\frac{2c_2}{2l} - \frac{e(1-l)}{2l} - \frac{(l^2-1)H^{d+1}}{2l}.$$

By hypothesis $c_2 > d\gamma + H^{d+1} + 2$. Hence, putting all this together we get

$$l^2(c_2 + d\gamma + 2e) - (c_2 + d\gamma + e + 2) < 0 \quad \text{with } l \geq 1.$$

Hence, $l = 1$ and $m = -c_2$, which proves the Proposition. □

5.2 Moduli spaces of vector bundles on $\mathbb{P}^d$-bundles

Throughout this section X will be a $\mathbb{P}^d$-bundle over a smooth, projective curve C of genus $g \geq 0$ and we will keep the notations introduced in the first section of this chapter.

As usual, we will denote by $M_{X,L}(2; c_1, c_2)$ the moduli space of L-stable, rank two vector bundles on X, with Chern classes c_1 and c_2. If there is no possible confusion we will write $M_L(2; c_1, c_2)$ instead of $M_{X,L}(2; c_1, c_2)$. The goal of this section is to compute the dimension, prove the irreducibility and smoothness and describe the structure of moduli spaces $M_L(2; c_1, c_2)$ of L-stable, rank 2 vector bundles with certain Chern classes and for a suitable polarization L closely related to c_2. We want to stress that the polarization L that we choose, strongly depends on c_2 and our results turn to be untrue if we fix c_1, L and c_2L^{d-1} goes to infinity. Indeed, for $d = 2$ and fixed L, the minimal number of irreducible components of the moduli space $M_L(2; c_1, c_2)$ of L-stable, rank 2 vector bundles with fixed c_1 and c_2L going to infinity grows to infinity ([BM97]; Theorem 0.1).

One way to study rank 2 vector bundles over an algebraic variety X is to use extensions of line bundles. Using this idea we construct the following families.

Construction 5.2.1. For $c_1 \equiv H + eF \in Num(X)$ with $e \in \{0, 1\}$ and any integer $c_2 > \frac{H^{d+1}+d\gamma}{2} + 1$, we construct a rank 2 vector bundle E on X as a non-trivial extension

$$(5.3) \quad \epsilon: \quad 0 \longrightarrow O_X(H - \pi^*\mathfrak{c}_2') \longrightarrow E \longrightarrow O_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}) \longrightarrow 0$$

where $\mathfrak{c}_2, \mathfrak{c}_2' \in Pic(C)$ are divisors on C of degree c_2 and $\mathfrak{e} \in Pic(C)$ is a divisor of degree e. We shall call $\mathcal{F}$ the irreducible family of rank two vector bundles constructed in this way.

Proposition 5.2.2. *Let X be a $\mathbb{P}^d$-bundle over C , $\mathbb{Z} \ni c_2 > \frac{H^{d+1}+d\gamma}{2} + 1$, an ample divisor $L \equiv dH + bF$ on X with $b = 2c_2 - H^{d+1} - (1-e)$, $e \in \{0,1\}$ and $\mathfrak{c}_2, \mathfrak{c}_2', \mathfrak{e} \in Pic(C)$ divisors on C with $deg(\mathfrak{c}_2) = deg(\mathfrak{c}_2') = c_2$ and $deg(\mathfrak{e}) = e$. For any vector bundle $E \in \mathcal{F}$, we have*

(a) $H^0E(-\pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) = 0.$

(b) E is a rank two, L-stable vector bundle with Chern classes $c_1(E) \equiv H + eF$ and $c_2(E) \equiv (c_2 + e)HF$.

(c) $dim\mathcal{F} = h^1O_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) + 2g - 1.$

Proof. First of all, notice that since $b = 2c_2 - H^{d+1} - (1-e) > d\gamma$, by Lemma 5.1.4 L is an ample divisor on X.

(a) We will start proving that $H^0O_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) = 0$. By Lemma 5.1.4 $\overline{L} \equiv H + \gamma F$ is an ample divisor. If $H^0O_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) \neq 0$, since $deg(\mathfrak{c}_2) = deg(\mathfrak{c}_2') = c_2$ and $deg(\mathfrak{e}) = e$, applying Remark 5.1.7 we get

$$0 \leq (H - (2c_2 + e)F)(H + \gamma F)^d = H^{d+1} + d\gamma - 2c_2 - e$$

which contradicts the assumption $2c_2 > d\gamma + H^{d+1} + 2$. Therefore,

$$H^0O_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) = 0.$$

Now we consider the long exact cohomology sequence

$$(5.4) \quad 0 \longrightarrow H^0O_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) \longrightarrow H^0E(-\pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) \longrightarrow$$
$$H^0O_X \xrightarrow{\delta} H^1O_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) \longrightarrow H^1E(-\pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e})$$
$$\longrightarrow H^1O_X \longrightarrow H^2O_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) \longrightarrow H^2E(-\pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e})$$
$$\longrightarrow H^2O_X \longrightarrow \cdots$$

associated to the exact sequence (5.3). Since

$$H^1O_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) = Ext^1(O_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}), O_X(H - \pi^*\mathfrak{c}_2')),$$

the map

$$\delta : H^0 O_X \longrightarrow H^1 O_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2^{'} - \pi^*\mathfrak{e})$$

given by $\delta(1) = \epsilon$ is an injection. This fact, together with

$$H^0 O_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2^{'} - \pi^*\mathfrak{e}) = 0$$

gives us $H^0 E(-\pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) = 0$, which proves (a).

(b) It is easy to see that for any $E \in \mathcal{F}$, $c_1(E) \equiv H + eF$ and $c_2(E) \equiv (c_2 + e)HF$. Let us see that E is L-stable, i.e., for any rank 1 subbundle $O_X(D)$ of $E \in \mathcal{F}$ we have

$$DL^d < \frac{c_1(E)L^d}{2}.$$

For any subbundle $O_X(D)$ of E we have

$$(1) \quad O_X(D) \hookrightarrow O_X(H - \pi^*\mathfrak{c}_2^{'}) \quad \text{or}$$

$$(2) \quad O_X(D) \hookrightarrow O_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}).$$

In the first case, $D \equiv H - c_2F - C$ being C numerically equivalent to some effective divisor. Hence,

$$\begin{aligned} DL^d = (H - c_2F - C)L^d &\leq (H - c_2F)L^d \\ &= d^d(H^{d+1} + b - c_2) \\ &< \tfrac{d^d(H^{d+1}+b+e)}{2} = \tfrac{c_1(E)L^d}{2} \end{aligned}$$

where the last inequality follows from the fact that $b < 2c_2 - H^{d+1} + e$.

Assume $O_X(D) \hookrightarrow O_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e})$. From (a) we have $H^0 E(-\pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) = 0$. Therefore, $D \equiv (c_2 + e)F - C'$ being $C' \equiv nH + mF$ numerically equivalent to a non-zero effective divisor. Hence,

$$\begin{aligned} DL^d = ((c_2+e)F - C')L^d &= ((c_2+e)F - nH - mF)L^d \\ &= d^d(c_2+e-nH^{d+1}-nb-m) \\ &= d^d(c_2+e-2nc_2+n(1-e)-m) \\ &< \tfrac{c_1(E)L^d}{2} = \tfrac{d^d(2c_2+2e-1)}{2} \end{aligned}$$

if, and only if,

$$-4nc_2 + 2n(1-e) - 2m < -1.$$

Since C' is numerically equivalent to a non-zero effective divisor, by Remark 5.1.7 we have $-m \leq n(H^{d+1}+d\gamma)$ and $n > 0$ or $n = 0$ and $m > 0$. By hypothesis $c_2 > \frac{H^{d+1}+d\gamma}{2} + 1$, therefore $-4nc_2 + 2n(1-e) - 2m < -1$ and E is L-stable.

(c) Since $\mathcal{F}$ is a $\mathbb{P}^N$-bundle over $Pic^0(C) \times Pic^0(C)$ where

$$N = dimExt^1(O_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}), O_X(H - \pi^*\mathfrak{c}_2'))$$

we have

$$\begin{aligned} dim\mathcal{F} &= dimExt^1(O_X(\pi^*\mathfrak{c}_2+\pi^*\mathfrak{e}), O_X(H-\pi^*\mathfrak{c}_2')) + 2dimPic^0(C) - 1 \\ &= h^1 O_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) + 2g - 1 \end{aligned}$$

which proves (c). ☐

Remark 5.2.3. The existence of big families of indecomposable rank 2 vector bundles over $\mathbb{P}^d$-bundles of arbitrary dimension faces up to Hartshorne's conjecture on the non-existence of indecomposable rank two vector bundles on projective spaces $\mathbb{P}^n$, $n \geq 6$ ([Har74]).

Now we are ready to state one of the main theorems of this section

Theorem 5.2.4. *Let X be a $\mathbb{P}^d$-bundle over C and c_2 an integer such that $c_2 > \frac{H^{d+1}+d\gamma}{2}+1$. We fix the ample divisor $L \equiv dH+bF$ on X with $b = 2c_2-H^{d+1}-(1-e)$ and $e \in \{0,1\}$. Then the moduli space $M_L(2; H+eF, (c_2+e)HF)$ is a smooth, irreducible, projective variety of dimension $h^1O_X(H-\pi^*\mathfrak{c}_2-\pi^*\mathfrak{c}_2'-\pi^*\mathfrak{e})+2g-1$, being $\mathfrak{c}_2, \mathfrak{c}_2' \in Pic(C)$ of degree c_2 and $\mathfrak{e} \in Pic(C)$ of degree e. Namely, it is a $\mathbb{P}^N$-bundle over $Pic^0(C) \times Pic^0(C)$ being $N := h^1O_X(H-\pi^*\mathfrak{c}_2-\pi^*\mathfrak{c}_2'-\pi^*\mathfrak{e})-1$.*

Proof. Using Proposition 5.2.2 and the universal property of the moduli space $M_L(2; H+eF, (c_2+e)HF)$ we obtain a morphism

$$\phi : \mathcal{F} \longrightarrow M_L(2; H+eF, (c_2+e)HF)$$

which is an injection. In fact, assume that there are two non-trivial extensions

$$0 \longrightarrow O_X(H-\pi^*\mathfrak{c}_2') \xrightarrow{\alpha_1} E \xrightarrow{\alpha_2} O_X(\pi^*\mathfrak{c}_2+\pi^*\mathfrak{e}) \longrightarrow 0;$$

$$0 \longrightarrow O_X(H-\pi^*\bar{\mathfrak{c}}_2') \xrightarrow{\beta_1} E \xrightarrow{\beta_2} O_X(\pi^*\bar{\mathfrak{c}}_2+\pi^*\bar{\mathfrak{e}}) \longrightarrow 0.$$

Since

$$Hom(O_X(H-\pi^*\mathfrak{c}_2'), O_X(\pi^*\bar{\mathfrak{c}}_2+\pi^*\bar{\mathfrak{e}})) = H^0O_X(-H+\pi^*\mathfrak{c}_2'+\pi^*\bar{\mathfrak{c}}_2+\pi^*\bar{\mathfrak{e}}) = 0$$

$$Hom(O_X(H-\pi^*\bar{\mathfrak{c}}_2'), O_X(\pi^*\mathfrak{c}_2+\pi^*\mathfrak{e})) = H^0O_X(-H+\pi^*\bar{\mathfrak{c}}_2'+\pi^*\mathfrak{c}_2+\pi^*\mathfrak{e}) = 0$$

(see Lemma 5.1.8), we have $\beta_2 \circ \alpha_1 = \alpha_2 \circ \beta_1 = 0$. So, there exists an automorphism $\lambda \in Aut(O_X(H-\pi^*\mathfrak{c}_2')) \cong k$ such that $\beta_1 = \alpha_1 \circ \lambda$. Therefore, ϕ is an injection.

Let us see that ϕ is surjective. To this end, we take a rank two vector bundle $E \in M_L(2; H+eF, (c_2+e)HF)$. It follows from Proposition 5.1.13 that $E(-H+\pi^*\mathfrak{c}_2')$ with $\mathfrak{c}_2' \in Pic(C)$ of degree c_2, has a non-zero section s whose scheme of zeros has codimension ≥ 2. Since $c_2E(-H+\pi^*\mathfrak{c}_2') = 0$, the section s defines an exact sequence

$$0 \longrightarrow O_X(H-\pi^*\mathfrak{c}_2') \longrightarrow E \longrightarrow O_X(\pi^*\mathfrak{c}_2+\pi^*\mathfrak{e}) \longrightarrow 0$$

of type (5.3). Therefore, ϕ is surjective.

Claim: For any $E \in M_L(2; H+eF, (c_2+e)HF)$ we have

$$dimT_{[E]}M_L(2; H+eF, (c_2+e)HF) = h^1O_X(H-\pi^*\mathfrak{c}_2-\pi^*\mathfrak{c}_2'-\pi^*\mathfrak{e})+2g-1.$$

Proof of the Claim: Take $E \in M_L(2; H + eF, (c_2 + e)HF)$. By deformation theory we know that (see Theorem 1.2.9)

$$T_{[E]}M_L(2; H + eF, (c_2 + e)HF) \cong Ext^1(E, E).$$

Let us compute $dimExt^1(E, E)$. We have just seen that any rank two vector bundle $E \in M_L(2; H + eF, (c_2 + e)HF)$ sits in an extension of type (5.3). Applying the functor $Hom(., E)$ to the exact sequence (5.3) we get the long exact sequence

$$(5.5) \quad 0 \longrightarrow Hom(O_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}), E) \longrightarrow Hom(E, E) \longrightarrow$$
$$Hom(O_X(H - \pi^*\mathfrak{c}_2'), E) \longrightarrow Ext^1(O_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}), E) \longrightarrow$$
$$Ext^1(E, E) \longrightarrow Ext^1(O_X(H - \pi^*\mathfrak{c}_2'), E) \longrightarrow Ext^2(O_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}), E) \longrightarrow \cdots.$$

Since $h^1O_X = g$, $H^2O_X(H - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) = H^2O_X = 0$ (Lemma 5.1.8) and $H^0E(-\pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) = 0$ (Proposition 5.2.2; (a)) from the exact sequence (5.4) we get

$$(5.6) \quad h^1E(-\pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) = h^1O_X(H - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) + g - 1 \quad \text{and}$$
$$Ext^2(O_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}), E) = H^2E(-\pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) = 0.$$

We consider the exact cohomology sequence

$$(5.7) \quad 0 \longrightarrow H^0O_X \longrightarrow H^0E(-H + \pi^*\mathfrak{c}_2') \longrightarrow H^0O_X(-H + \pi^*\mathfrak{c}_2' + \pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e})$$
$$\longrightarrow H^1O_X \longrightarrow H^1E(-H + \pi^*\mathfrak{c}_2') \longrightarrow H^1O_X(-H + \pi^*\mathfrak{c}_2' + \pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}) \longrightarrow \cdots$$

associated to the exact sequence (5.3).

Since

$$H^0O_X(-H + \pi^*\mathfrak{c}_2' + \pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}) = H^1O_X(-H + \pi^*\mathfrak{c}_2' + \pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}) = 0$$

and $h^1O_X = g$ (Lemma 5.1.8) we get

$$h^0E(-H + \pi^*\mathfrak{c}_2') = 1 \quad \text{and} \quad h^1E(-H + \pi^*\mathfrak{c}_2') = g.$$

Therefore, from the exact sequence (5.5) we obtain

$$\begin{aligned} dimExt^1(E,E) &= h^1E(-\pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) + h^1E(-H + \pi^*\mathfrak{c}_2') - h^0E(-H + \pi^*\mathfrak{c}_2') \\ &\quad + dimHom(E,E) \\ &= h^1O_X(H - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) + 2g - 1 \end{aligned}$$

where the second equality follows from the fact that E is L-stable (and therefore simple) and (5.6), which proves our Claim.

Since $dim\mathcal{F} = h^1O_X(H - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) + 2g - 1$ (Proposition 5.2.2), it follows from the Claim that the moduli space $M_L(2; H + eF, (c_2 + e)HF)$ is smooth. Moreover, we have

$$M_L(2; H + eF, (c_2 + e)HF) \cong \mathbb{P}(Ext^1(O_X(\pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}), O_X(H - \pi^*\mathfrak{c}_2'))) \times T,$$

being $T := Pic^0(C) \times Pic^0(C)$, i.e. $M_L(2; H + eF, (c_2 + e)HF)$ is a $\mathbb{P}^N$-bundle over $Pic^0(C) \times Pic^0(C)$ where $N := h^1O_X(H - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) - 1$. Therefore, the moduli space $M_L(2; H + eF, (c_2 + e)HF)$ is a non-empty, smooth, irreducible, projective variety of dimension $h^1O_X(H - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) + 2g - 1$ which proves the Theorem. □

As a corollary we obtain the rationality of the following moduli spaces on rational normal scrolls

Corollary 5.2.5. *Let $Y := Y(a_0, \cdots, a_d)$ be a $(d+1)$-dimensional, rational, normal scroll as in Example 5.1.3 and c_2 an integer such that $c_2 > \frac{H^{d+1}+d}{2} + 1$. We fix the ample divisor $L = dH + bF$ on Y with $b = 2c_2 - H^{d+1} - (1 - e)$ and $e \in \{0, 1\}$. Then the moduli space $M_L(2; H + eF, (c_2 + e)HF)$ is a smooth, irreducible, rational, projective variety of dimension $2(d + 1)c_2 - H^{d+1} + e(d + 1) - (d + 2)$.*

Proof. It follows from Example 5.1.3, that a $(d + 1)$-dimensional, rational, normal scroll is a $\mathbb{P}^d$-bundle over a smooth, projective curve of genus $g = 0$. Thus, the result follows from Theorem 5.2.4. □

Now, we will deal with the cases $c_1 \equiv 0$ and $c_1 \equiv F$. To this end, we will consider the following family of rank two vector bundles

Construction 5.2.6. For $c_1 \equiv eF \in Num(X)$ with $e \in \{0,1\}$ and any integer $c_2 > H^{d+1} + d\gamma + 2$, we construct a rank 2 vector bundle E on X as a non-trivial extension

$$(5.8) \quad \epsilon : \quad 0 \longrightarrow O_X(H - \pi^*\mathfrak{c}_2') \longrightarrow E \longrightarrow O_X(-H + \pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}) \longrightarrow 0$$

where $\mathfrak{c}_2, \mathfrak{c}_2' \in Pic(C)$ are divisors on C of degree c_2 and $\mathfrak{e} \in Pic(C)$ is a divisor on C of degree e. We shall call $\mathcal{G}$ the irreducible family of rank two vector bundles constructed in this way.

Proposition 5.2.7. *Let X be a $\mathbb{P}^d$-bundle over C, $\mathbb{Z} \ni c_2 > H^{d+1} + d\gamma + 2$, an ample divisor $L \equiv dH + bF$ on X with $b = c_2 - H^{d+1} - (1-e)$, $e \in \{0,1\}$ and $\mathfrak{c}_2, \mathfrak{c}_2', \mathfrak{e} \in Pic(C)$ divisors on C with $deg(\mathfrak{c}_2) = deg(\mathfrak{c}_2') = c_2$ and $deg(\mathfrak{e}) = e$. For any vector bundle $E \in \mathcal{G}$, we have*

(a) $H^0E(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) = 0$.

(b) E is a rank two, L-stable vector bundle with Chern classes $c_1(E) \equiv eF$ and $c_2(E) \equiv -H^2 + (2c_2 + e)HF$.

(c) $dim\mathcal{G} = h^1O_X(2H - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) + 2g - 1$.

Proof. First of all, notice that since $b = c_2 - H^{d+1} - (1-e) > d\gamma$, L is an ample divisor on X (see Lemma 5.1.4).

(a) We will start checking that $H^0O_X(2H - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) = 0$. Since $H + \gamma F$ is numerically equivalent to an ample divisor, if $H^0O_X(2H - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) \neq 0$, we get

$$0 \leq (2H - (2c_2 + e)F)(H + \gamma F)^d = 2H^{d+1} + 2d\gamma - 2c_2 - e$$

which contradicts the assumption $c_2 > d\gamma + H^{d+1} + 2$. Therefore,

$$H^0O_X(2H - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) = 0.$$

Now we consider the long exact cohomology sequence

$$(5.9)\quad 0 \longrightarrow H^0O_X(2H - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) \longrightarrow H^0E(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) \longrightarrow$$

$$H^0O_X \xrightarrow{\delta} H^1O_X(2H - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) \longrightarrow H^1E(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e})$$

$$\longrightarrow H^1O_X \longrightarrow H^2O_X(2H - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) \longrightarrow H^2E(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e})$$

$$\longrightarrow H^2O_X \longrightarrow \cdots$$

associated to the exact sequence (5.8).

Since $H^1O_X(2H-\pi^*\mathfrak{c}_2'-\pi^*\mathfrak{c}_2-\pi^*\mathfrak{e}) = Ext^1(O_X(-H+\pi^*\mathfrak{c}_2+\pi^*\mathfrak{e}), O_X(H-\pi^*\mathfrak{c}_2'))$, the map

$$\delta : H^0O_X \longrightarrow H^1O_X(2H - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e})$$

given by $\delta(1) = \epsilon$ is an injection. This fact, together with

$$H^0O_X(2H - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) = 0$$

gives us $H^0E(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) = 0$, which proves (a).

(b) It is easy to see that for any $E \in \mathcal{G}$, $c_1(E) \equiv eF$ and $c_2(E) \equiv -H^2+(2c_2+e)HF$. Let us see that E is L-stable, i.e., for any rank 1 subbundle $O_X(D)$ of $E \in \mathcal{G}$ we have

$$DL^d < \frac{c_1(E)L^d}{2}.$$

For any subbundle $O_X(D)$ of E we get

$$(1)\quad O_X(D) \hookrightarrow O_X(H - \pi^*\mathfrak{c}_2') \quad \text{or}$$

$$(2)\quad O_X(D) \hookrightarrow O_X(-H + \pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}).$$

In the first case, $D \equiv H - c_2F - C$ being C numerically equivalent to some effective divisor. Hence,

$$\begin{aligned} DL^d = (H - c_2F - C)L^d &\le (H - c_2F)L^d \\ &= d^d(H^{d+1} + b - c_2) \\ &< \tfrac{ed^d}{2} = \tfrac{c_1(E)L^d}{2} \end{aligned}$$

where the last inequality follows from the fact that $b < c_2 - H^{d+1} + \frac{e}{2}$.

Assume $O_X(D) \hookrightarrow O_X(-H + \pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e})$. Since $H^0E(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) = 0$ (see (a)) we have $D \equiv -H + (c_2 + e)F - C'$ being $C' \equiv nH + mF$ numerically equivalent to a non-zero effective divisor. Therefore, we obtain

$$\begin{aligned} DL^d = (-H + (c_2+e)F - C')L^d &= (-H + (c_2+e)F - nH - mF)L^d \\ &= d^d(-H^{d+1} - b + c_2 + e - nH^{d+1} - nb - m) \\ &= d^d(-nc_2 + n(1-e) + 1 - m) \\ &< \tfrac{c_1(E)L^d}{2} = \tfrac{ed^d}{2} \end{aligned}$$

if, and only if,

$$-2nc_2 + (2n+2)(1-e) + e - 2m < 0.$$

Since C' is numerically equivalent to a non-zero effective divisor, we have $n > 0$ and $-m \leq n(H^{d+1} + d\gamma)$ or $n = 0$ and $m > 0$ (see Remark 5.1.7). By hypothesis $c_2 > H^{d+1} + d\gamma + 2$, therefore

$$-2nc_2 + (2n+2)(1-e) + e - 2m < 0$$

unless $e = 0$, $n = 0$ and $m = 1$. Let us see that this case cannot occur.

If $n = 0$, $e = 0$ and $m = 1$, take $\mathfrak{c}_2' - \mathfrak{p} \in Pic(C)$ a divisor on C of degree $c_2 - 1$ and a non-zero section $s \in H^0E(H - \pi^*(\mathfrak{c}_2' - \mathfrak{p}))$. We have the associated exact sequence

$$(5.10) \quad 0 \longrightarrow O_X((x-1)H + \pi^*(\mathfrak{c}_2' - \mathfrak{p}) + \pi^*\mathfrak{y}) \longrightarrow E$$
$$\longrightarrow I_Z((1-x)H - \pi^*(\mathfrak{c}_2 - \mathfrak{p}') - \pi^*\mathfrak{y}') \longrightarrow 0$$

being $xH + \pi^*\mathfrak{y}$ an effective divisor with $deg(\mathfrak{y}) = y$, $\mathfrak{y}' \in Pic(C)$ of degree y, $\mathfrak{c}_2 - \mathfrak{p}' \in Pic(C)$ a divisor on C of degree $c_2 - 1$ and

$$\begin{aligned} [Z] &\equiv c_2E((1-x)H - \pi^*(\mathfrak{c}_2' - \mathfrak{p}) - \pi^*\mathfrak{y}) \\ &\equiv ((1-x)^2 - 1)H^2 + (2c_2 + 2(1-x)(1-c_2-y))HF. \end{aligned}$$

Applying Lemma 5.1.10, we get

$$\begin{aligned} c_3(I_Z((1-x)H - \pi^*(\mathfrak{c}_2 - \mathfrak{p}' + \mathfrak{y}))) &\equiv (1-(1-x)^2)(1-x)xH^3 \\ &+(c_2+y-1)((1-x)^2-1)(4x-3)H^2F \\ &-(c_2+y-1)2x(1-x)H^2F \\ &+(4c_2((1-x)^2-1)+2xc_2)H^2F. \end{aligned}$$

Using the exact sequence (5.10) we obtain

$$\begin{aligned} c_3(E) &= c_2(I_Z((1-x)H - \pi^*(\mathfrak{c}_2-\mathfrak{p}') - \pi^*\mathfrak{y}))c_1(O_X((x-1)H + \pi^*(\mathfrak{c}_2'-\mathfrak{p}) + \pi^*\mathfrak{y})) \\ &\quad + c_3(I_Z((1-x)H - \pi^*(\mathfrak{c}_2-\mathfrak{p}') - \pi^*\mathfrak{y})) \\ &\equiv (1-(1-x)^2)(1-x)(x+1)H^3 \\ &\quad +(c_2+y-1)(((1-x)^2-1)(4x-3) - 2(1-x)(2x-1) + (1-x)^2-1)H^2F \\ &\quad +(4c_2((1-x)^2-1)+4xc_2-2c_2)H^2F \\ &=: \alpha H^3 + \beta H^2F. \end{aligned}$$

On the other hand, since E is a rank 2 vector bundle we have $c_3(E) = 0$. Therefore, $\alpha = \beta = 0$. The equality

$$\alpha = (1-(1-x)^2)(1-x)(x+1) = 0$$

implies that $x \in \{-1, 0, 1, 2\}$. Notice that since $xH + \pi^*\mathfrak{y}$ is an effective divisor, the case $x = -1$ cannot occur.

If $x = 1$ we have

$$0 = \beta = -2(c_2+y-1) - 2c_2$$

or, equivalently, $y = 1 - 2c_2$, which contradicts the fact that $xH + \pi^*\mathfrak{y}$ is an effective divisor.

If $x = 2$, using once more that $\beta = 0$ we get $y = -2c_2 + 1$ which again is a contradiction.

Finally, if $x = 0$ we get that $y = 1$ and thus $H^0E(H-\pi^*\mathfrak{c}_2') \neq 0$ which contradicts (a). Therefore, the case $n = 0$, $e = 0$ and $m = 1$ cannot occur and E is L-stable.

(c) Since $\mathcal{G}$ is a $\mathbb{P}^M$-bundle over $Pic^0(C) \times Pic^0(C)$ with

$$M = dimExt^1(O_X(-H + \pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}), O_X(H - \pi^*\mathfrak{c}_2'))$$

we have

$$\begin{aligned} dim\mathcal{G} &= dimExt^1(O_X(-H + \pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}), O_X(H - \pi^*\mathfrak{c}_2')) + 2dimPic^0(C) - 1 \\ &= h^1O_X(2H - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) + 2g - 1 \end{aligned}$$

which proves (c). □

Theorem 5.2.8. *Let X be a $\mathbb{P}^d$-bundle over C with $d > 1$ and c_2 an integer such that $c_2 > H^{d+1} + d\gamma + 2$. We fix the ample divisor $L \equiv dH + bF$ on X being $b = c_2 - H^{d+1} - (1 - e)$ and $e \in \{0, 1\}$. Then $M_L(2; eF, -H^2 + (2c_2 + e)HF)$ is a smooth, irreducible, projective variety of dimension $h^1O_X(2H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) + 2g - 1$, where $\mathfrak{c}_2, \mathfrak{c}_2', \mathfrak{e} \in Pic(C)$ are divisors on C with $deg(\mathfrak{c}_2) = deg(\mathfrak{c}_2') = c_2$ and $deg(\mathfrak{e}) = e$. Namely, it is a $\mathbb{P}^M$-bundle over $Pic^0(C) \times Pic^0(C)$ being*

$$M := h^1O_X(2H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) - 1.$$

Proof. Using Proposition 5.2.7 and the universal property of the moduli space $M_L(2; eF, -H^2 + (2c_2 + e)HF)$ we obtain a morphism

$$\phi : \mathcal{G} \longrightarrow M_L(2; eF, -H^2 + (2c_2 + e)HF)$$

which is an injection. Indeed, assume that there are two non-trivial extensions

$$0 \longrightarrow O_X(H - \pi^*\mathfrak{c}_2') \xrightarrow{\alpha_1} E \xrightarrow{\alpha_2} O_X(-H + \pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}) \longrightarrow 0;$$

$$0 \longrightarrow O_X(H - \pi^*\bar{\mathfrak{c}}_2') \xrightarrow{\beta_1} E \xrightarrow{\beta_2} O_X(-H + \pi^*\bar{\mathfrak{c}}_2 + \pi^*\bar{\mathfrak{e}}) \longrightarrow 0.$$

Since

$$Hom(O_X(H - \pi^*\mathfrak{c}_2'), O_X(-H + \pi^*\bar{\mathfrak{c}}_2 + \pi^*\bar{\mathfrak{e}})) = H^0O_X(-2H + \pi^*\bar{\mathfrak{c}}_2 + \pi^*\mathfrak{c}_2' + \pi^*\bar{\mathfrak{e}}) = 0,$$

$$Hom(O_X(H - \pi^*\bar{\mathfrak{c}}_2'), O_X(-H + \pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e})) = H^0O_X(-2H + \pi^*\mathfrak{c}_2 + \pi^*\bar{\mathfrak{c}}_2' + \pi^*\mathfrak{e}) = 0,$$

we have $\beta_2 \circ \alpha_1 = \alpha_2 \circ \beta_1 = 0$. So there exists $\lambda \in Aut(O_X(H - \pi^*\mathfrak{c}_2')) \cong k$ such that $\beta_1 = \alpha_1 \circ \lambda$. Therefore, ϕ is an injection.

Let us see that ϕ is surjective. To this end, we take a rank two vector bundle $E \in M_L(2; eF, -H^2 + (2c_2 + e)HF)$. By Proposition 5.1.13, $E(-H + \pi^*\mathfrak{c}_2')$ has a non-zero section s whose scheme of zeros has codimension greater or equal than two, being $\mathfrak{c}_2' \in Pic(C)$ of degree c_2. Since $c_2E(-H + \pi^*\mathfrak{c}_2') = 0$, the section s defines an exact sequence

$$0 \longrightarrow O_X(H - \pi^*\mathfrak{c}_2') \longrightarrow E \longrightarrow O_X(-H + \pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}) \longrightarrow 0$$

of type (5.8). Therefore, ϕ is surjective.

Claim: For any $E \in M_L(2; eF, -H^2 + (2c_2 + e)HF)$ we have

$$dim T_{[E]} M_L(2; eF, -H^2 + (2c_2 + e)HF) = h^1 O_X(2H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) + 2g - 1.$$

Proof of the Claim: Take $E \in M_L(2; eF, -H^2 + (2c_2 + e)HF)$. By deformation theory we know that

$$T_{[E]} M_L(2; eF, -H^2 + (2c_2 + e)HF) \cong Ext^1(E, E)$$

(see Theorem 1.2.9). Let us compute $dim Ext^1(E, E)$. We have just seen that any rank two vector bundle $E \in M_L(2; eF, -H^2 + (2c_2 + e)HF)$ sits in an extension of type (5.8). Applying the functor $Hom(., E)$ to the exact sequence (5.8) we get the long exact sequence

$$\begin{aligned} (5.11) \quad & 0 \longrightarrow Hom(O_X(-H + \pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}), E) \longrightarrow Hom(E, E) \longrightarrow \\ & Hom(O_X(H - \pi^*\mathfrak{c}_2'), E) \longrightarrow Ext^1(O_X(-H + \pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}), E) \\ & \longrightarrow Ext^1(E, E) \longrightarrow Ext^1(O_X(H - \pi^*\mathfrak{c}_2'), E) \\ & \longrightarrow Ext^2(O_X(-H + \pi^*\mathfrak{c}_2 + \pi^*\mathfrak{e}), E) \longrightarrow \cdots . \end{aligned}$$

Since $h^1 O_X = g$, $H^2 O_X(2H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) = H^2 O_X = 0$ (Lemma 5.1.8) and $H^0 E(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) = 0$ (Proposition 5.2.7; (a)) from the exact sequence (5.9) we get

$$(5.12) \quad h^1E(H-\pi^*\mathfrak{c}_2-\pi^*\mathfrak{e}) = h^1O_X(2H-\pi^*\mathfrak{c}_2-\pi^*\mathfrak{c}_2'-\pi^*\mathfrak{e})+g-1;$$

$$Ext^2(O_X(-H+\pi^*\mathfrak{c}_2+\pi^*\mathfrak{e}),E) = H^2E(H-\pi^*\mathfrak{c}_2-\pi^*\mathfrak{e}) = 0.$$

Consider the long exact sequence

$$(5.13) \quad 0 \longrightarrow H^0O_X \longrightarrow H^0E(-H+\pi^*\mathfrak{c}_2') \longrightarrow H^0O_X(-2H+\pi^*\mathfrak{c}_2'+\pi^*\mathfrak{c}_2+\pi^*\mathfrak{e})$$

$$\longrightarrow H^1O_X \longrightarrow H^1E(-H+\pi^*\mathfrak{c}_2') \longrightarrow H^1O_X(-2H+\pi^*\mathfrak{c}_2'+\pi^*\mathfrak{c}_2+\pi^*\mathfrak{e})$$

associated to the exact sequence (5.8).
Since $H^0O_X(-2H+\pi^*\mathfrak{c}_2'+\pi^*\mathfrak{c}_2+\pi^*\mathfrak{e}) = H^1O_X(-2H+\pi^*\mathfrak{c}_2'+\pi^*\mathfrak{c}_2+\pi^*\mathfrak{e}) = 0$ and $h^1O_X = g$ (Lemma 5.1.8) we get

$$h^0E(-H+\pi^*\mathfrak{c}_2') = 1 \quad \text{and} \quad h^1E(-H+\pi^*\mathfrak{c}_2') = g.$$

Notice that in the computation of $h^1O_X(-2H+\pi^*\mathfrak{c}_2'+\pi^*\mathfrak{c}_2+\pi^*\mathfrak{e})$ we strongly use the fact that $d > 1$.

Therefore, from the exact sequence (5.11) we obtain

$$\begin{aligned} dimExt^1(E,E) &= h^1E(H-\pi^*\mathfrak{c}_2-\pi^*\mathfrak{e})+h^1E(-H+\pi^*\mathfrak{c}_2')-h^0E(-H+\pi^*\mathfrak{c}_2') \\ &\quad +dimHom(E,E) \\ &= h^1O_X(2H-\pi^*\mathfrak{c}_2'-\pi^*\mathfrak{c}_2-\pi^*\mathfrak{e})+2g-1 \end{aligned}$$

where the second equality follows from the fact that E is L-stable (hence simple) and (5.12). This proves our Claim.

Since $dim\mathcal{G} = h^1O_X(2H-\pi^*\mathfrak{c}_2'-\pi^*\mathfrak{c}_2-\pi^*\mathfrak{e})+2g-1$ (Proposition 5.2.7; (c)), it follows from the Claim that the moduli space $M_L(2;eF,-H^2+(2c_2+e)HF)$ is smooth. Moreover, we have

$$M_L(2;eF,-H^2+(2c_2+e)HF) \cong \mathbb{P}(Ext^1(O_X(-H+\pi^*\mathfrak{c}_2+\pi^*\mathfrak{e}),O_X(H-\pi^*\mathfrak{c}_2'))) \times T,$$

being

$$T := Pic^0(C) \times Pic^0(C),$$

i.e., $M_L(2; eF, -H^2 + (2c_2 + e)HF)$ is a $\mathbb{P}^M$-bundle over T where

$$M = h^1 O_X(2H - \pi^* \mathfrak{c}_2' - \pi^* \mathfrak{c}_2 - \pi^* \mathfrak{e}) - 1.$$

Therefore, the moduli space $M_L(2; eF, -H^2 + (2c_2 + e)HF)$ is a non-empty, smooth, irreducible, projective variety of dimension $h^1 O_X(2H - \pi^* \mathfrak{c}_2' - \pi^* \mathfrak{c}_2 - \pi^* \mathfrak{e}) + 2g - 1$. $\square$

As a corollary we obtain the rationality of the following moduli spaces on rational normal scrolls

Corollary 5.2.9. *Let $Y := Y(a_0, \cdots, a_d)$ be a $(d+1)$-dimensional, rational, normal scroll as in Example 5.1.3 with $d > 1$ and $\mathbb{Z} \ni c_2 > H^{d+1} + d + 2$. We fix the ample divisor $L = dH + bF$ on Y with $b = c_2 - H^{d+1} - (1 - e)$ and $e \in \{0, 1\}$. Then the moduli space $M_L(2; eF, -H^2 + (2c_2 + e)HF)$ is a smooth, irreducible, rational, projective variety of dimension $2(ed + 1)c_2 - e(d + 2)H^{d+1} + 2(e - 1)$.*

Proof. It follows from Example 5.1.3, that a $(d + 1)$-dimensional, rational, normal scroll is a $\mathbb{P}^d$-bundle over a smooth, projective curve of genus $g = 0$. Thus, the result follows from Theorem 5.2.8. $\square$

Remark 5.2.10. Using again Bogomolov's inequality $(c_1(E)^2 - 4c_2(E))H^{d-1} < 0$, one can see that the hypothesis $2c_2 > H^{d+1} + d\gamma + 2$ (resp. $c_2 > H^{d+1} + d\gamma + 2$) when $c_1(E) \equiv H + eF$ (resp. $c_1(E) \equiv eF$) with $e \in \{0, 1\}$, is not too restrictive.

Remark 5.2.11. 1) Theorem 5.2.4 generalizes to $\mathbb{P}^d$-bundles results on moduli spaces of vector bundles on ruled surfaces obtained by Qin in [Qin92]; Proposition 3.4 and in [Qin92b] Theorem A, by Hoppe and Spindler in [HS80]; main Satz and by Brosius in [Bro83]; Propositions 3 and 8.

2) Corollary 5.2.5 generalizes to rational normal scrolls Proposition 3.1.3 on moduli spaces of vector bundles on smooth, irreducible, Hirzebruch surfaces obtained in chapter 3.

3) Once again, Theorem 5.2.4, Theorem 5.2.8, Corollary 5.2.5 and Corollary 5.2.9, reflect the general philosophy that the moduli space inherits some of the geometric properties of the underlying variety.

In the following Theorem we will generalize Theorem 5.2.4 (resp. Theorem 5.2.8) to other classes of c_2. Since the proof is essentially the same, we will skip it.

Theorem 5.2.12. *Let X be a $\mathbb{P}^d$-bundle over C with $d > 1$, integers b, a, e with $e \in \{0,1\}$ and we fix an ample divisor $L \equiv \alpha H + \beta F$ on X. Assume that the following conditions holds*

$$0 > -2a > -d-1, \quad \alpha = ad, \quad \beta = -b - aH^{d+1} + e - 1 \quad \text{and}$$

$$-ab > a^2 H^{d+1} + a^2 d\gamma + a(a+2).$$

$$(\text{resp.} \quad 0 > 1-2a > -d-1, \quad \alpha = (2a-1)d, \quad \beta = -2b - (2a-1)H^{d+1} + e - 1$$

$$\text{and} \quad -2ab > (2a-1)aH^{d+1} + (2a-1)ad\gamma + a(2a-1) + 1.)$$

Then the moduli space $M_L(2; eF, -a^2H^2 + (ae-2ab)HF)$ (resp. the moduli space $M_L(2; H+eF, a(1-a)H^2 + (b+ae-2ab)HF)$) is a smooth, irreducible, projective variety of dimension $h^1 O_X(2aH + \pi^*(\mathfrak{b}+\mathfrak{b}'-\mathfrak{e})) + 2g - 1$ (resp. of dimension $h^1 O_X((2a-1)H + \pi^*(\mathfrak{b}+\mathfrak{e})) + 2g - 1$), being $\mathfrak{b}, \mathfrak{b}', \mathfrak{e} \in Pic(C)$ of degree b and e respectively. Namely, it is a $\mathbb{P}^N$-bundle (resp. $\mathbb{P}^M$-bundle) over $Pic^0(C) \times Pic^0(C)$ being $N := h^1 O_X(2aH + \pi^*(\mathfrak{b}+\mathfrak{b}'-\mathfrak{e}))$ (resp. $M := h^1 O_X((2a-1)H + \pi^*(\mathfrak{b}+\mathfrak{e}))$).

Remark 5.2.13. We want to stress that with the above result we have covered the study of all moduli spaces $M_L(2; c_1, c_2)$ such that the general point $[E]$ of $M_L(2; c_1, c_2)$ is given as a non-trivial extension of line bundles. Indeed, the Chern classes of vector bundles E studied in Theorem 5.2.12 are the only ones which can be obtained as Chern classes of a vector bundle E constructed as a non-trivial extension of line bundles. In fact, if a rank two vector bundle E sits in a non-trivial extension

$$0 \longrightarrow O_X(aH + \pi^*\mathfrak{b}) \longrightarrow E \longrightarrow O_X(a'H + \pi^*\mathfrak{b}') \longrightarrow 0$$

where $\mathfrak{b}, \mathfrak{b}' \in Pic(C)$ are two divisors of degree b and b' respectively and $c_1(E) \equiv eF$ (resp. $c_1(E) \equiv H + eF$) with $e \in \{0, 1\}$, then

$$a + a' = 0 \quad \text{and} \quad b + b' = e$$

$$(resp. \quad a + a' = 1 \quad \text{and} \quad b + b' = e)$$

which implies that

$$c_2(E) \equiv -a^2H^2 + (ae - 2ab)HF$$

$$(\text{resp.} \quad c_2(E) \equiv a(1-a)H^2 + (b + ae - 2ab)HF).$$

We will finish this section computing the Kodaira dimension and the Picard group of moduli spaces that we have studied above.

Corollary 5.2.14. *Let X be a $\mathbb{P}^d$-bundle over C and $c_2, d \in \mathbb{Z}$ such that $d > 0$ and $c_2 > \frac{H^{d+1}+d\gamma}{2} + 1$ (resp. $c_2 > H^{d+1} + d\gamma + 2$ and $d > 1$). We fix the ample divisor $L \equiv dH + bF$ on X with $b = 2c_2 - H^{d+1} - (1-e)$ (resp. $b = c_2 - H^{d+1} - (1-e)$) and $e \in \{0, 1\}$. Then*

$$Kod(M_L(2; H + eF, (c_2 + e)HF)) = -\infty$$

$$(resp. \quad Kod(M_L(2; eF, -H^2 + (2c_2 + e)HF)) = -\infty).$$

Proof. It follows from the fact that the moduli space $M_L(2; H + eF, (c_2 + e)HF)$ (resp. $M_L(2; eF, -H^2 + (2c_2 + e)HF)$) is a $\mathbb{P}^N$-bundle (resp. $\mathbb{P}^M$-bundle) over the variety $Pic^0(C) \times Pic^0(C)$ being $N := h^1\mathcal{O}_X(H - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) - 1$ (resp. $M := h^1\mathcal{O}_X(2H - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{e}) - 1$) with $\mathfrak{c}_2, \mathfrak{c}_2' \in Pic(C)$ of degree c_2 and $\mathfrak{e} \in Pic(C)$ of degree e. □

The fact that $M_L(2; H+eF, (c_2+e)HF)$ (resp. $M_L(2; eF, -H^2+(2c_2+e)HF)$) is a $\mathbb{P}^N$-bundle (resp. $\mathbb{P}^M$-bundle) over $Pic^0(C) \times Pic^0(C)$ with natural projection Π (resp. Π') allows us to describe in the next result the Picard group of the moduli space $M_L(2; H + eF, (c_2 + e)HF)$ (resp. $M_L(2; eF, -H^2 + (2c_2 + e)HF)$).

Corollary 5.2.15. *Let X be a $\mathbb{P}^d$-bundle over C and $c_2, d \in \mathbb{Z}$ such that $d > 0$ and $c_2 > \frac{H^{d+1}+d\gamma}{2} + 1$ (resp. $c_2 > H^{d+1} + d\gamma + 2$ and $d > 1$). We fix the ample divisor $L \equiv dH + bF$ on X with $b = 2c_2 - H^{d+1} - (1-e)$ (resp. $b = c_2 - H^{d+1} - (1-e)$) and $e \in \{0,1\}$. Then*

$$Pic(M_L(2; H + eF, (c_2 + e)HF)) \cong \mathbb{Z} \oplus \Pi^* Pic(Pic^0(C) \times Pic^0(C)).$$

$$(resp. \quad Pic(M_L(2; eF, -H^2 + (2c_2 + e)HF)) \cong \mathbb{Z} \oplus \Pi'^* Pic(Pic^0(C) \times Pic^0(C)) \ ).$$

In particular, if X is a rational normal scroll, then

$$Pic(M_L(2; H + eF, (c_2 + e)HF)) \cong Pic(M_L(2; eF, -H^2 + (2c_2 + e)HF)) \cong \mathbb{Z}.$$

5.3 Change of polarizations.

In section 2 we have studied the moduli space $M_L(2; c_1, c_2)$ for fixed c_1, c_2 and a suitable polarization L on a $\mathbb{P}^d$-bundle X over a smooth, projective curve. In this section, we will illustrate some of the changes of the moduli space $M_L(2; c_1, c_2)$ that occur when the polarization L varies.

We keep the notations introduced in sections 5.1 and 5.2. In particular

$$X = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} C$$

is a $\mathbb{P}^d$-bundle over a smooth, projective curve C of genus $g \geq 0$. For technical reasons, we will also assume that $\mathcal{E}$ is normalized (see Definition 5.1.1). Under these assumptions, for any integer $n >> 0$ we have

$$(i) \quad deg(O_C(-\mathfrak{n} - \mathfrak{n}' + \mathfrak{p} + \mathfrak{p}' + K_C)) < 0,$$

$$(ii) \quad h^0\mathcal{E} + h^0 O_C(\overline{\mathfrak{p}}) \leq h^1\mathcal{E}(-\mathfrak{n} - \mathfrak{n}'),$$

$$(iii) \quad n > \tfrac{H^{d+1}+d\gamma}{2} + 3,$$

being $\mathfrak{n}, \mathfrak{n}' \in Pic(C)$ divisors on C of degree n, and $\mathfrak{p}, \mathfrak{p}', \overline{\mathfrak{p}} \in C$ points of C. Indeed, (i) and (iii) are trivial. Let us check (ii). Since $\mathcal{E}$ is a normalized vector bundle on a smooth, projective curve C, $h^0\mathcal{E}(-\mathfrak{n}-\mathfrak{n}')=0$ and by Riemann-Roch's Theorem, the inequality (ii) is equivalent to

$$h^0\mathcal{E}+h^0O_C(\overline{\mathfrak{p}})\leq 2n(d+1)+(d+1)(g-1)-deg(\mathcal{E})$$

which clearly holds for $n>>0$.

In chapter 1 we introduced the concepts of walls and chambers due to Qin. Let us discuss a precise example.

Example 5.3.1. Let X be a $\mathbb{P}^d$-bundle over C. We fix $c_1\equiv H\in Num(X)$ and for any $\mathbb{Z}\ni n>\frac{H^{d+1}+d\gamma}{2}+3$, $c_2\equiv nHF\in H^4(X,\mathbb{Z})$. We consider $S,S'\in A^{d-1}_{num}$ and $\xi\in Num(X)\otimes\mathbb{R}$ defined by

$$S:=dH^{d-1}+\beta H^{d-2}F,$$

$$S':=dH^{d-1}+(\beta-2)H^{d-2}F,$$

$$\xi:=H-2nF,$$

being $\beta=(2n-H^{d+1})(d-1)+2$. Notice that ξ is the numerical equivalence class of a divisor D on X such that $D+c_1$ is divisible by 2 in $Pic(X)$ and

$$\xi^2S=(H^2-4nHF)(dH^{d-1}+\beta H^{d-2}F)=dH^{d+1}+\beta-4nd<0,$$

$$\xi^2S'=(H^2-4nHF)(dH^{d-1}+(\beta-2)H^{d-2}F)=dH^{d+1}+\beta-2-4nd<0,$$

$$c_2+\frac{\xi^2-c_1^2}{4}=nHF+\frac{H^2-4nHF-H^2}{4}=0.$$

Therefore, both $W^{(\xi,S)}$ and $W^{(\xi,S')}$ define a wall of type (c_1,c_2). Moreover $W^{(\xi,S)}$ and $W^{(\xi,S')}$ are non-empty. In fact, take

$$\tilde{L}\equiv dH+(2n-H^{d+1}-2)F \quad \text{and} \quad \tilde{L}'\equiv dH+(2n-H^{d+1})F.$$

For $n>>0$, $\tilde{L}$ and $\tilde{L}'$ are ample divisors on X (Lemma 5.1.4) and we have

$$\tilde{L}\xi S=(dH+(2n-H^{d+1}-2)F)(dH^d+(\beta-2nd)H^{d-1}F)=0,$$

$$\tilde{L}'\xi S'=(dH+(2n-H^{d+1})F)(dH^d+(\beta-2-2nd)H^{d-1}F)=0$$

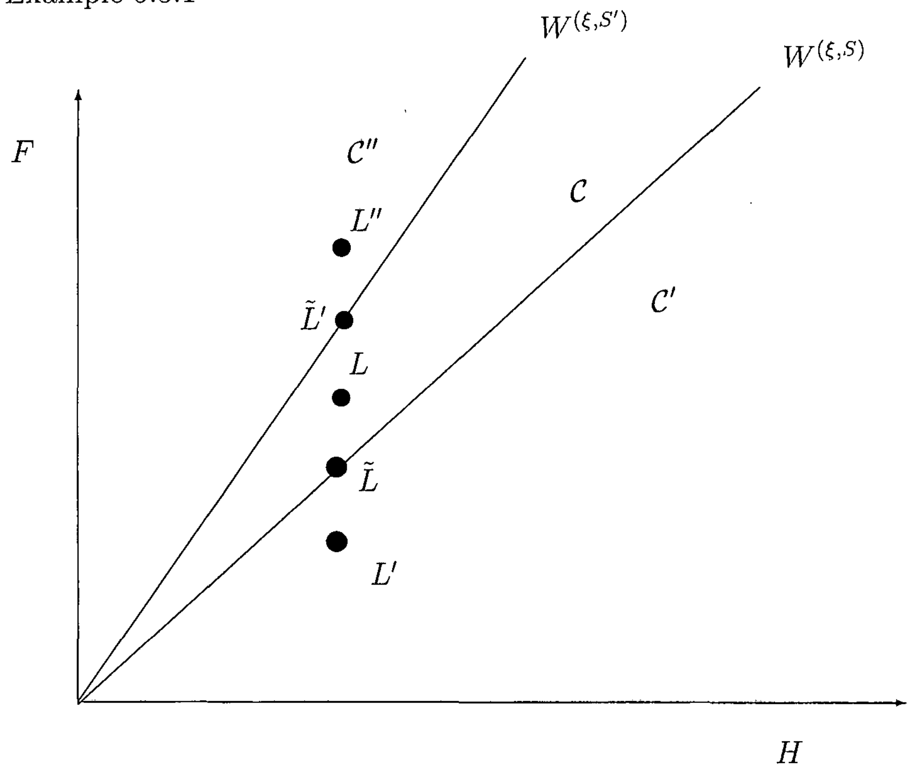
which is equivalent to $\tilde{L} \in W^{(\xi,S)}$ and $\tilde{L}' \in W^{(\xi,S')}$. Finally, if we consider the polarizations (Lemma 5.1.4)

$$L \equiv dH + (2n - H^{d+1} - 1)F, \quad L' \equiv dH + (2n - H^{d+1} - 3)F \quad \text{and}$$

$$L'' \equiv dH + (2n - H^{d+1} + 1)F,$$

since $L'\xi S < 0 < L\xi S$ and $L\xi S' < 0 < L''\xi S'$, the wall $W^{(\xi,S)}$ separates L and L' and the wall $W^{(\xi,S')}$ separates L and L''. We will denote by $\mathcal{C}$ (resp. $\mathcal{C}'$ and $\mathcal{C}''$) the chamber containing L (resp. L' and L'').

Given a polarization $L \equiv aH + bF$, we can represent the class of L as a point of coordinates (a, b) in the plane. The following picture gives as an idea of the situation described in Example 5.3.1



Now we will determine and compare the moduli spaces $M_L(2; H, nHF)$ corresponding to polarizations L lying in the chambers $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{C}''$ respectively. Keeping the notation introduced in Example 5.3.1, we have

Theorem 5.3.2. *Let X be a $\mathbb{P}^d$-bundle over C and $0 << n \in \mathbb{Z}$. The following holds*

(a) For all $\overline{L}'' \in \mathcal{C}''$, the moduli space $M_{\overline{L}''}(2; H, nHF)$ is empty.

(b) For all $\overline{L} \in \mathcal{C}$, the moduli space $M_{\overline{L}}(2; H, nHF)$ is a $\mathbb{P}^N$-bundle over the variety $Pic^0(C) \times Pic^0(C)$ being $N := h^1\mathcal{O}_X(H - \pi^*\mathfrak{n}' - \pi^*\mathfrak{n}) - 1$ with $\mathfrak{n}, \mathfrak{n}' \in Pic(C)$ divisors of degree n. In particular, it is a non-empty, smooth, irreducible, projective variety of dimension $h^1\mathcal{O}_X(H - \pi^*\mathfrak{n}' - \pi^*\mathfrak{n}) + 2g - 1$ and Kodaira dimension $-\infty$.

(c) For all $\overline{L}' \in \mathcal{C}'$, the moduli space $M_{\overline{L}'}(2; H, nHF)$ is a non-empty open subset of the moduli space $M_{\overline{L}}(2; H, nHF)$ and

$$dim(M_{\overline{L}}(2; H, nHF) \setminus M_{\overline{L}'}(2; H, nHF)) = h^0\mathcal{E} + h^0\mathcal{O}_C(\mathfrak{p}) + 2(g-1),$$

with $\mathfrak{p} \in C$ a point of C. In particular, $M_{\overline{L}'}(2; H, nHF)$ is a non-empty, smooth, irreducible, quasi-projective variety of dimension

$$h^1\mathcal{O}_X(H - \pi^*\mathfrak{n}' - \pi^*\mathfrak{n}) + 2g - 1$$

with $\mathfrak{n}, \mathfrak{n}' \in Pic(C)$ of degree n and $Kod(M_{\overline{L}'}(2; H, nHF)) = -\infty$.

Proof. (a) It follows from Proposition 5.3.3 and Remark 5.3.4.
(b) It follows from Theorem 5.2.4 and Remark 1.3.4.
(c) It follows from Proposition 5.3.5 and Remark 1.3.4. □

Let us start discussing what happens for polarizations lying on the chamber $\mathcal{C}''$.

Proposition 5.3.3. *Let X be a $\mathbb{P}^d$-bundle over C, $0 < a \in \mathbb{Z}$, n an integer such that $n > \frac{H^{d+1}+d\gamma}{2} + 3$ and $L_0 \equiv aH + bF$ an ample divisor on X such that $\frac{b}{a} \geq \frac{2n - H^{d+1}}{d}$. Then the moduli space $M_{L_0}(2; H, nHF)$ is empty.*

Remark 5.3.4. Keeping the notations introduced in Example 5.3.1 we have that $\frac{b}{a} \geq \frac{2n-H^{d+1}}{d}$ is equivalent to $\xi S' L_0 \geq 0$. Hence, $M_{L_0}(2; H, nHF)$ is empty for any polarization L_0 on the shaded area

In particular for any $L \in C''$.

Proof of Proposition 5.3.3. We consider an ample divisor $L_0 \equiv aH + bF$ with $\frac{b}{a} \geq \frac{2n-H^{d+1}}{d}$ on X (Lemma 5.1.4) and a L_0-stable, rank two vector bundle E on X with Chern classes $c_1E \equiv H$ and $c_2E \equiv nHF$. Since $c_2(E(-H+\pi^*\mathfrak{n})) = 0$ being $\mathfrak{n} \in Pic(C)$ of degree n, arguing as in Proposition 5.1.13, we get that $E(-lH - \pi^*\mathfrak{m})$ being $l > 0$ and $\mathfrak{m} \in Pic(C)$ of degree m, has a non-zero section whose scheme of zeros has codimension ≥ 2.

Since E is L_0-stable and $O_X(lH + \pi^*\mathfrak{m}) \hookrightarrow E$ we have

$$\begin{aligned}(lH+mF){L_0}^d &= a^{d-1}(alH^{d+1}+dlb+am)\\ &< \tfrac{c_1(E){L_0}^d}{2} = \tfrac{a^{d-1}(aH^{d+1}+db)}{2}\end{aligned}$$

which is equivalent to

$$2am < -a(2l-1)H^{d+1} - (2l-1)db.$$

Since by hypothesis $\frac{b}{a} \geq \frac{2n-H^{d+1}}{d}$ we get

$$m < -(2l-1)n.$$

On the other hand, since $E(-lH - \pi^*\mathfrak{m})$ has a non-zero section whose scheme of zeros has codimension ≥ 2, we get

$$\begin{aligned}0 \leq c_2(E(-lH-\pi^*\mathfrak{m}))H^{d-1} &= ((n+2lm-m)HF + l(l-1)H^2)H^{d-1}\\ &= n + (2l-1)m + l(l-1)H^{d+1}.\end{aligned}$$

Hence, $m \geq \frac{-l(l-1)H^{d+1}-n}{2l-1}$.

By hypothesis $2n > H^{d+1} + d\gamma + 6$. Therefore, putting all this inequalities together we obtain

$$(5.14)\qquad \tfrac{-2l(l-1)n}{2l-1} - \tfrac{n}{2l-1} + \tfrac{l(l-1)d\gamma}{2l-1} + \tfrac{6l(l-1)}{2l-1} \leq m < -(2l-1)n$$

which implies that

$$l^2(2n+d\gamma+6)-l(2n+d\gamma+6)<0.$$

Since, $l>0$ we get a contradiction and hence the moduli space $M_{L_0}(2;H,nHF)$ is empty, which proves what was stated. □

In the next Proposition, we compare moduli spaces corresponding to polarizations L lying in the chambers $\mathcal{C}$ and $\mathcal{C}'$ respectively.

Proposition 5.3.5. *Let X be a $\mathbb{P}^d$-bundle over C and $0 << n \in \mathbb{Z}$. We fix an ample divisor $\overline{L}' \equiv dH+bF \in \mathcal{C}'$ on X with $b=2n-H^{d+1}-3$. Then the moduli space $M_{\overline{L}'}(2;H,nHF)$ is a non-empty open subset of $M_{\overline{L}}(2;H,nHF)$ being $\overline{L} \equiv dH+(2n-H^{d+1}-1)F \in \mathcal{C}$. In particular, $M_{\overline{L}'}(2;H,nHF)$ is a smooth, irreducible, quasi-projective variety of dimension*

$$h^1\mathcal{O}_X(H-\pi^*\mathfrak{n}-\pi^*\mathfrak{n}')+2g-1=h^1\mathcal{E}(-\pi^*\mathfrak{n}-\pi^*\mathfrak{n}')+2g-1$$

being $\mathfrak{n}$ and $\mathfrak{n}'$ two divisors on C of degree n.

Proof. We consider the open subset $\mathcal{U}$ of $M_{\overline{L}}(2;H,nHF)$ defined by

$$\mathcal{U}:=\{E\in M_{\overline{L}}(2;H,nHF)|H^0E(-\pi^*(\mathfrak{n}-\mathfrak{p}'))=0\}$$

being $\mathfrak{n}-\mathfrak{p}'\in Pic(C)$ of degree $n-1$. In order to prove the Proposition it is enough to see that $\mathcal{U}$ is non-empty, that $\mathcal{U}\cong M_{\overline{L}'}(2;H,nHF)$ and that

$$dim(M_{\overline{L}}(2;H,nHF)\setminus\mathcal{U})=h^0\mathcal{E}+h^0\mathcal{O}_C(\overline{\mathfrak{p}})+2g-2<dim M_{\overline{L}}(2;H,nHF)$$

where $\overline{\mathfrak{p}}$ is a point of C.

Claim 1: $\mathcal{U}$ is a non-empty open subset of $M_{\overline{L}}(2;H,nHF)$ and

$$dim(M_{\overline{L}}(2;H,nHF)\setminus\mathcal{U})=h^0\mathcal{E}+h^0\mathcal{O}_C(\overline{\mathfrak{p}})+2g-2$$

being $\overline{\mathfrak{p}}$ a point of C.

Proof of Claim 1: For any $E \in M_{\overline{L}}(2; H, nHF) \setminus \mathcal{U}$, we take a non-zero section $s \in H^0E(-\pi^*(\mathfrak{n}-\mathfrak{p}'))$ with $deg(\mathfrak{n}-\mathfrak{p}') = n-1$ and the associated exact sequence

$$(5.15) \quad 0 \longrightarrow O_X(D + \pi^*(\mathfrak{n}-\mathfrak{p}')) \longrightarrow E \longrightarrow I_Z(H - \pi^*(\mathfrak{n}'-\mathfrak{p}) - D') \longrightarrow 0$$

where $D' \equiv D \equiv xH + yF$ are numerically equivalent to some effective divisor, $\mathfrak{n}'-\mathfrak{p} \in Pic(C)$ is a divisor on C of degree $n-1$ and $[Z]$ is a codimension two closed subscheme of X.

Since E is $\overline{L}$-stable we have

$$\begin{aligned}(D+(n-1)F)\overline{L}^d &= d^d(xH^{d+1} + x(2n - H^{d+1} - 1) + y + (n-1)) \\ &= d^d(2nx - x + y + n - 1) \\ &< \tfrac{d^d(2n-1)}{2} = \tfrac{c_1(E)\overline{L}^d}{2}\end{aligned}$$

which is equivalent to

$$4xn - 2x + 2y - 1 < 0.$$

Since D is numerically equivalent to an effective divisor, $x = 0$ and $y \geq 0$ or $x > 0$ and $-y \leq x(H^{d+1} + d\gamma)$ (see Remark 5.1.7). By hypothesis $n >> 0$, in particular $n > \frac{H^{d+1}+d\gamma}{2} + 3$ and hence the only solution is $x = y = 0$. Thus $D \equiv D' \equiv 0$ and we have the exact sequence

$$(5.16) \qquad 0 \longrightarrow O_X(\pi^*(\mathfrak{n}-\mathfrak{p}')) \longrightarrow E \longrightarrow I_Z(H - \pi^*(\mathfrak{n}'-\mathfrak{p})) \longrightarrow 0$$

where $[Z] = c_2(E(-\pi^*(\mathfrak{n}-\mathfrak{p}')))$ is a complete intersection of type $(H, F_{\overline{\mathfrak{p}}})$ being $\overline{\mathfrak{p}}$ a point of C and $\mathfrak{n}'-\mathfrak{p}, \mathfrak{n}-\mathfrak{p}' \in Pic(C)$ are two divisors on C of degree $n-1$.

Let us call $\mathcal{M}$ the irreducible family of rank 2 vector bundles given by an exact sequence of type (5.16). We have

$$\begin{aligned}dim(M_{\overline{L}}(2; H, nHF) \setminus \mathcal{U}) &= dim\mathcal{M} \\ &= dimExt^1(I_{HF}(H - \pi^*(\mathfrak{n}'-\mathfrak{p})), O_X(\pi^*(\mathfrak{n}-\mathfrak{p}'))) \\ &\quad -h^0E(-\pi^*(\mathfrak{n}-\mathfrak{p}')) + 2dimPic^0(C) + dim\mathcal{L}\end{aligned}$$

being $\mathcal{L}$ the family of codimension two closed subschemes Z of X, complete intersection of type $(H, F_{\overline{\mathfrak{p}}})$ being $\overline{\mathfrak{p}} \in Pic(C)$ a point of C.

Applying the functor $Hom(., O_X(-H + \pi^*(\mathfrak{n}' - \mathfrak{p}) + \pi^*(\mathfrak{n} - \mathfrak{p}')))$ to the exact sequence

$$0 \longrightarrow O_X(-H - F_{\overline{\mathfrak{p}}}) \longrightarrow O_X(-H) \oplus O_X(-F_{\overline{\mathfrak{p}}}) \longrightarrow I_Z \longrightarrow 0$$

we get the long exact sequence

$$0 \to Hom(I_Z, O_X(-H + \pi^*(\mathfrak{n}' - \mathfrak{p}) + \pi^*(\mathfrak{n} - \mathfrak{p}'))) \to$$

$$H^0 O_X(\pi^*(\mathfrak{n}' - \mathfrak{p}) + \pi^*(\mathfrak{n} - \mathfrak{p}')) \oplus H^0 O_X(-H + \pi^*(\mathfrak{n}' - \mathfrak{p}) + \pi^*(\mathfrak{n} - \mathfrak{p}') + F_{\overline{\mathfrak{p}}})$$

$$\to H^0 O_X(\pi^*(\mathfrak{n}' - \mathfrak{p}) + \pi^*(\mathfrak{n} - \mathfrak{p}') + F_{\overline{\mathfrak{p}}}) \to Ext^1(I_Z, O_X(-H + \pi^*(\mathfrak{n}' - \mathfrak{p}) + \pi^*(\mathfrak{n} - \mathfrak{p}')))$$

$$\to H^1 O_X(\pi^*(\mathfrak{n}' - \mathfrak{p}) + \pi^*(\mathfrak{n} - \mathfrak{p}')) \oplus H^1 O_X(-H + \pi^*(\mathfrak{n}' - \mathfrak{p}) + \pi^*(\mathfrak{n} - \mathfrak{p}') + F_{\overline{\mathfrak{p}}}) \to \cdots .$$

Once more, applying Lemma 5.1.8 and Serre's duality, together with the hypothesis $deg(O_C(-\mathfrak{n} + \mathfrak{p}' - \mathfrak{n}' + \mathfrak{p} + K_C)) < 0$ we get

$$dim Hom(I_Z, O_X(-H + \pi^*(\mathfrak{n}' - \mathfrak{p}) + \pi^*(\mathfrak{n} - \mathfrak{p}'))) = h^0 O_X(-H + \pi^*(\mathfrak{n}' - \mathfrak{p}) + \pi^*(\mathfrak{n} - \mathfrak{p}')) = 0,$$

$$\begin{aligned} h^0 O_X(\pi^*(\mathfrak{n}' - \mathfrak{p}) + \pi^*(\mathfrak{n} - \mathfrak{p}')) &= h^0 O_C(\mathfrak{n}' - \mathfrak{p} + \mathfrak{n} - \mathfrak{p}') \\ &= \chi(O_C(\mathfrak{n}' - \mathfrak{p} + \mathfrak{n} - \mathfrak{p}')) = 2n - 1 - g, \\ h^0 O_X(\pi^*(\mathfrak{n}' - \mathfrak{p}) + \pi^*(\mathfrak{n} - \mathfrak{p}') + F_{\overline{\mathfrak{p}}}) &= h^0 O_C(\mathfrak{n}' - \mathfrak{p} + \mathfrak{n} - \mathfrak{p}' + \overline{\mathfrak{p}}) \\ &= \chi(O_C(\mathfrak{n}' - \mathfrak{p} + \mathfrak{n} - \mathfrak{p}' + \overline{\mathfrak{p}})) = 2n - g, \end{aligned}$$

$$h^1 O_X(\pi^*(\mathfrak{n}' - \mathfrak{p}) + \pi^*(\mathfrak{n} - \mathfrak{p}')) = h^1 O_C(\mathfrak{n}' - \mathfrak{p} + \mathfrak{n} - \mathfrak{p}') = 0,$$

$$h^0 O_X(-H + \pi^*(\mathfrak{n}' - \mathfrak{p}) + \pi^*(\mathfrak{n} - \mathfrak{p}') + F_{\overline{\mathfrak{p}}}) = 0,$$

$$h^1 O_X(-H + \pi^*(\mathfrak{n}' - \mathfrak{p}) + \pi^*(\mathfrak{n} - \mathfrak{p}') + F_{\overline{\mathfrak{p}}}) = 0.$$

Therefore,

$$dim Ext^1(I_Z(H - \pi^*(\mathfrak{n}' - \mathfrak{p})), O_X(\pi^*(\mathfrak{n} - \mathfrak{p}'))) = 2n - g - 2n + 1 + g = 1.$$

From the exact sequence (5.16) we get

$$h^0E(-\pi^*(\mathfrak{n}-\mathfrak{p}')) = h^0O_X = 1.$$

Putting all these results together we obtain

$$dim\mathcal{M} = 1-1+2g+dim\mathcal{L} = h^0\mathcal{E}+h^0O_C(\overline{\mathfrak{p}})+2g-2$$

where the last equality follows from Lemma 5.1.9.

By hypothesis $h^0\mathcal{E}+h^0O_C(\overline{\mathfrak{p}}) \leq h^1\mathcal{E}(-\mathfrak{n}-\mathfrak{n}')$, so

$$\begin{aligned} h^0\mathcal{E}+h^0O_C(\overline{\mathfrak{p}})+2g-2 &= dim\mathcal{M} \\ &< dimM_{\overline{L}}(2;H,nHF) \\ &= h^1\mathcal{E}(-\mathfrak{n}-\mathfrak{n}')+2g-1 \end{aligned}$$

where the last equality follows from Theorem 5.2.4. Therefore, $\mathcal{U}$ is a non-empty, open dense subset of $M_{\overline{L}}(2;H,nHF)$ and

$$dim(M_{\overline{L}}(2;H,nHF)\setminus\mathcal{U}) = h^0\mathcal{E}+h^0O_C(\overline{\mathfrak{p}})+2g-2$$

which proves Claim 1.

Claim 2: Take any $E \in M_{\overline{L}}(2;H,nHF)$. E is $\overline{L}'$-stable if, and only if, $E \in \mathcal{U}$.

Proof of Claim 2: Let us see that any $E \in \mathcal{U}$ is $\overline{L}'$-stable or, equivalently, that for any rank 1 subbundle $O_X(D)$ of E we have

$$D\overline{L}'^d < \frac{c_1(E)\overline{L}'^d}{2}.$$

By Theorem 5.2.4 any $E \in \mathcal{U} \subset M_{\overline{L}}(2;H,nHF)$ sits in an exact sequence

$$0 \longrightarrow O_X(H-\pi^*\mathfrak{n}') \longrightarrow E \longrightarrow O_X(\pi^*\mathfrak{n}) \longrightarrow 0$$

where $\mathfrak{n},\mathfrak{n}' \in Pic(C)$ are two divisors of degree n.

Hence, for any subbundle $O_X(D)$ of $E \in \mathcal{U}$ we have

$$(1) \quad O_X(D) \hookrightarrow O_X(H - \pi^*\mathfrak{n}') \quad \text{or}$$

$$(2) \quad O_X(D) \hookrightarrow O_X(\pi^*\mathfrak{n}).$$

In the first case, $D \equiv H - nF - C$ being C numerically equivalent to an effective divisor. Hence,

$$\begin{aligned} D\overline{L}'^d = (H - nF - C)\overline{L}'^d & \leq (H - nF)\overline{L}'^d \\ & = d^d(H^{d+1} + b - n) \\ & < \tfrac{d^d(H^{d+1}+b)}{2} = \tfrac{c_1(E)\overline{L}'^d}{2} \end{aligned}$$

where the last inequality follows from the fact that $b < 2n - H^{d+1}$.

Assume $O_X(D) \hookrightarrow O_X(\pi^*\mathfrak{n})$. From Proposition 5.2.2; (a), $H^0E(-\pi^*\mathfrak{n}) = 0$. Therefore $D = \pi^*\mathfrak{n} - C'$ being $C' \equiv lH + mF$ numerically equivalent to a non-zero effective divisor. Hence,

$$\begin{aligned} D\overline{L}'^d = (nF - lH - mF)\overline{L}'^d & = d^d(n - lH^{d+1} - lb - m) \\ & = d^d(n - 2ln + 3l - m) \\ & < \tfrac{c_1(E)\overline{L}'^d}{2} = \tfrac{d^d(2n-3)}{2} \end{aligned}$$

if, and only if, $-4ln + 6l + 3 - 2m < 0$. Since $C' \equiv lH + mF$ is numerically equivalent to a non-zero effective divisor, $l > 0$ and $-m \leq l(H^{d+1} + d\gamma)$ or $l = 0$ and $m > 0$ (see Remark 5.1.7) . Since by hypothesis $n > \frac{H^{d+1}+d\gamma}{2} + 3$, if $l > 0$ then $-4ln + 6l + 3 - 2m < 0$. If $l = 0$, since $E \in \mathcal{U}$, $H^0E(-\pi^*(\mathfrak{n} - \mathfrak{p}')) = 0$ for any divisor $\mathfrak{n} - \mathfrak{p}'$ on C of degree $n - 1$. Therefore, we have $m > 1$ and $3 - 2m < 0$ which proves that E is $\overline{L}'$-stable.

Assume that $E \in M_{\overline{L}}(2; H, nHF) \setminus \mathcal{U}$. Let us see that E is not $\overline{L}'$-stable. Since $E \in M_{\overline{L}}(2; H, nHF) \setminus \mathcal{U}$, we have

$$O_X(\pi^*(\mathfrak{n} - \mathfrak{p}')) \hookrightarrow E$$

being $\mathfrak{n} - \mathfrak{p}' \in Pic(C)$ of degree $n-1$. If E is $\overline{L}'$-stable, we have

$$(n-1)F\overline{L}'^d = d^d(n-1) < \frac{c_1(E)\overline{L}'^d}{2} = \frac{d^d(2n-3)}{2}$$

which is a contradiction. Therefore, E is not $\overline{L}'$-stable and we have proved Claim 2.

Claim 3: Any $E \in M_{\overline{L}'}(2; H, nHF)$ sits in a non-trivial exact sequence

$$0 \longrightarrow O_X(H - \pi^*\mathfrak{n}') \longrightarrow E \longrightarrow O_X(\pi^*\mathfrak{n}) \longrightarrow 0$$

being $\mathfrak{n}, \mathfrak{n}' \in Pic(C)$ two divisors on C of degree n. In particular,

$$M_{\overline{L}'}(2; H, nHF) \subset M_{\overline{L}}(2; H, nHF).$$

Proof of Claim 3: Since $c_2(E(-H + \pi^*\mathfrak{n}')) = 0$, arguing as in Proposition 5.1.13 we get that $E(-lH - \pi^*\mathfrak{m})$ being $l > 0$ and $\mathfrak{m} \in Pic(C)$ of degree m, has a non-zero section whose scheme of zeros has codimension ≥ 2. To end the proof of Claim 3 we only need to show that $l = 1$ and $m = -n$.

Since E is $\overline{L}'$-stable and $O_X(lH + \pi^*\mathfrak{m}) \hookrightarrow E$ we have

$$(lH + mF)\overline{L}'^d = d^d(lH^{d+1} + lb + m) < \frac{c_1(E)\overline{L}'^d}{2} = \frac{d^d(H^{d+1} + b)}{2}$$

which is equivalent to $m < -(2l-1)n + \frac{3(2l-1)}{2}$.

On the other hand, since $E(-lH - \pi^*\mathfrak{m})$ has a non-zero section whose scheme of zeros has codimension ≥ 2, we get

$$\begin{aligned} 0 \leq c_2(E(-lH - \pi^*\mathfrak{m}))H^{d-1} &= ((n + 2lm - m)HF + l(l-1)H^2)H^{d-1} \\ &= n + (2l-1)m + l(l-1)H^{d+1}. \end{aligned}$$

Therefore, $m \geq \frac{-l(l-1)H^{d+1} - n}{2l-1}$.

By hypothesis $2n > H^{d+1} + d\gamma + 6$. Hence, putting these inequalities together we obtain

$$(5.17) \qquad \tfrac{-2l(l-1)n}{2l-1} - \tfrac{n}{2l-1} + \tfrac{l(l-1)d\gamma}{2l-1} + \tfrac{6l(l-1)}{2l-1} \leq m < -(2l-1)n + \tfrac{3(2l-1)}{2}$$

which implies that

$$l^2(2n+d\gamma)-l(2n+d\gamma)-\frac{3}{2}<0 \quad \text{with } l\geq 1.$$

Hence, $l=1$ and using once more (5.17) we obtain $m=-n+1$ or $m=-n$.

In the first case, let $\mathfrak{n}_1 \in Pic(C)$ be a divisor on C of degree $n-1$. Since $c_2(E(-H+\pi^*\mathfrak{n}_1)) \equiv Z \equiv HF$ and $E(-H+\pi^*\mathfrak{n}_1)$ has a non-zero section whose scheme of zeros has codimension ≥ 2, we have the exact sequence

$$0 \longrightarrow O_X(H-\pi^*\mathfrak{n}_1) \longrightarrow E \longrightarrow I_Z(\pi^*\mathfrak{n}_1^{'}) \longrightarrow 0$$

where $\mathfrak{n}_1, \mathfrak{n}_1^{'} \in Pic(C)$ are two divisors on C of degree $n-1$. Hence, from Lemma 5.1.10 we have

$$c_3(E)=c_3(I_Z(\pi^*\mathfrak{n}_1^{'}))+c_2(I_Z(\pi^*\mathfrak{n}_1^{'}))c_1(O_X(H-\pi^*\mathfrak{n}_1)) \equiv 2H^2F$$

which contradicts the fact that for any rank two vector bundle E we have $c_3(E)=0$. Therefore, $m=-n$ and E sits in the exact sequence

$$0 \longrightarrow O_X(H-\pi^*\mathfrak{n}') \longrightarrow E \longrightarrow O_X(\pi^*\mathfrak{n}) \longrightarrow 0$$

where $\mathfrak{n}, \mathfrak{n}'$ are divisors on C of degree n.

Since $n > \frac{H^{d+1}+d\gamma}{2}+1$, by Proposition 5.2.2, E is $\overline{L}$-stable which proves our last Claim.

From Claims 2 and 3, we deduce

$$M_{\overline{L}'}(2;H,nHF) \cong \mathcal{U} \subset M_{\overline{L}}(2;H,nHF)$$

is a non-empty open dense subset. Indeed,

$$\begin{aligned} dim(M_{\overline{L}}(2;H,nHF)\setminus \mathcal{U}) &= h^0\mathcal{E}+h^0O_C(\overline{\mathfrak{p}})+2g-2 \\ &< dim(M_{\overline{L}}(H,nHF)) \\ &= h^1\mathcal{E}(-\pi^*\mathfrak{n}-\pi^*\mathfrak{n}')+2g-1 \end{aligned}$$

and this proves what we want. □

Final Remark: We want to finish this chapter showing, by means of two examples, that moduli spaces $M_L(2; c_1, c_2)$, we have studied, strongly depends on the fixed ample divisor L.

(1) Let X be a $\mathbb{P}^2$-bundle over a smooth, projective curve C of genus $g \geq 0$ and $L \equiv H + 2F$ an ample divisor on X. It follows from [BM97]; Theorem 0.1, that the number of irreducible components of the moduli space

$$M_L(2; H, c_2HF) \cong M_L(2; H + 2F, (c_2 + 1)HF)$$

grows to infinity when c_2 goes to infinity.

(2) Let X be a $\mathbb{P}^2$-bundle over a smooth, projective curve C of genus $g \geq 0$, an integer $k \geq 2$, $c_2 \in H^4(X, \mathbb{Z})$ and L an ample divisor on X such that $c_2L = 2k^2L^3$. It follows from [Bal98]; Theorem 0.1, that $Sing(M_L(2; 0, c_2)) \neq \emptyset$, i.e., the moduli space $M_L(2; 0, c_2)$ of rank two, L-stable, vector bundles E on X with fixed Chern classes $c_1(E) = 0$ and $c_2(E) = c_2$ is a singular variety.

This two examples show again that the moduli space $M_L(2; H, c_2HF)$ strongly depends on the fixed polarization L.

Appendix A

Resum en Català

En aquesta tesi estudiem els espais de moduli $M_{X,H}(r; c_1, \cdots, c_{min\{r,n\}})$ de fibrats vectorials E de rang r, H-estables, en una varietat projectiva X amb classes de Chern $c_i(E) \in H^{2i}(X, \mathbb{Z})$ fixades, tot mostrant noves i interessants propietats geomètriques de $M_{X,H}(r; c_1, \cdots, c_{min\{r,n\}})$ que clarament reflecteixen la filosofia general la qual sosté que els espais de moduli hereten moltes de les propietats geomètriques de la varietat base X.

De manera més precisa, considerem X una varietat projectiva, llisa i irreductible, de dimensió n, definida sobre un cos algebraicament tancat de característica zero, k, H un divisor ample en X, $r \geq 2$ un enter i $c_i \in H^{2i}(X, \mathbb{Z})$ per $i = 1, \cdots, min\{r, n\}$. Denotem per $M_{X,H}(r; c_1, \cdots, c_{min\{r,n\}})$ l'espai de moduli de fibrats E, de rang r, H-estables segons la noció de Mumford-Takemoto, amb classes de Chern $c_i(E) = c_i$ per $i = 1, \cdots, min\{r, n\}$.

Les principals qüestions i problemes que s'han considerat són:

(1) Sigui X una superfície racional, llisa, irreductible, H un divisor ample en X i $0 << c_2 \in \mathbb{Z}$. L'espai de moduli $M_{X,H}(2; c_1, c_2)$ és racional?

De forma més general,

(2) Sigui X una superfície racional, llisa, irreductible, H un divisor ample en X i $0 << c_2 \in \mathbb{Z}$. L'espai de moduli $M_{X,H}(r; c_1, c_2)$ és racional?

(3) Sigui X una superfície $K3$ algebraica i H un divisor ample en X. Determinar invariants (r, c_1, c_2, l) pels quals l'espai de moduli $M_{X,H}(r; c_1, c_2)$ és biracional a l' esquema de Hilbert $Hilb^l(X)$.

(4) Que es pot dir de la geometria de l'espai de moduli $M_{X,H}(2; c_1, c_2)$ si X és una varietat de dimensió arbitrària?. És connex, llis, irreductible i racional?

Les dues primeres qüestions varen ser formulades en [Sch90]; Problema 21, [Sch85]; Problema 2 i [OV88]; Problema 2 i la qüestió (**3**) va ser formulada per Nakashima en [Nak97]. El principal objectiu de la quarta qüestió és provar que per varietats X de dimensió més gran que dos, si triem adequadament un divisor ample H en X, estretament lligat a c_2, aleshores $M_{X,H}(2; c_1, c_2)$ reflecteix molts dels atributs geomètrics de X.

Els espais de moduli de feixos lliures de torsió (resp. de fibrats vectorials) semiestables (resp. estables) en una varietat projectiva, algebraica, llisa i irreductible, varen ser introduïts en la dècada dels setanta. Un cop s'ha establert l'existència de l'espai de moduli, la pregunta natural que sorgeix és: què es pot dir sobre la seva estructura local i global?. Diversos autors s'han dedicat a estudiar-ne l' estructura, des del punt de vista de la geometria algebraica, de la topologia i de la geometria diferencial, tot establint interessants connexions entre aquestes disciplines. En aquesta memòria, nosaltres pendrem un punt de vista algebraic i geomètric.

Al llarg dels anys, s'han provat molts resultats interessants referents als espais de moduli $M_{X,H}(r; c_1, \cdots, c_{min\{r,n\}})$ en el cas en què la varietat base X té dimensió dos, i es coneix molt poc en el cas en què la varietat base té dimensió tres o superior a tres. Permeteu-nos recordar breument algun d'aquests resultats. Per això, denotarem per $\overline{M}_{X,H}(r; c_1, \cdots, c_{min\{r,n\}})$ l'espai de moduli de feixos lliures de torsió E de rang r, semiestables respecte H segons la noció de Gieseker-Maruyama, en una varietat X de dimensió n, amb classes de Chern $c_i(E) = c_i$, $i = 1, \cdots, min\{r, n\}$ fixades. Observis que $M_{X,H}(r; c_1, \cdots, c_{min\{r,n\}})$ és un obert de $\overline{M}_{X,H}(r; c_1, \cdots, c_{min\{r,n\}})$.

En els anys vuitanta, Donaldson va provar que els espais de moduli $M_{X,H}(2; 0, c_2)$

de fibrats vectorials de rang dos, H-estables en una superfície projectiva, llisa i irreductible X, són genèricament llisos i de la dimensió esperada $4c_2 - c_1^2 - 4\chi(O_X) + p_g(X) + 1$, sempre i quan c_2 sigui prou gran ([Don86]). Com a conseqüència va obtenir espectaculars resultats sobre la classificació de varietats $\mathbb{C}^\infty$ de dimensió quatre. Des d'aleshores, han estat provats molts i interessants resultats. Per citar-ne algun, és conegut que $\overline{M}_{X,H}(r; c_1, c_2)$ (resp. $M_{X,H}(r; c_1, c_2)$) és una varietat projectiva (resp. quasi projectiva) i que per c_2 suficientment gran, no és buit ([Sor97], [Mar77] i [Mar78]), genèricament llis, irreductible i normal ([Don86], [Zuo91], [GL96] i [OGr96]) de la dimensió esperada $2rc_2 - (r-1)c_1^2 - r^2\chi(O_X) + p_g(X) + 1$.

Els espais de moduli de fibrats vectorials en corbes algebraiques llises amb determinant fixat són sempre uniracionals i molt sovint racionals. Per espais de moduli de fibrats vectorials en superfícies algebraiques, llises i irreductibles, la situació canvia dràsticament i des del punt de vista de la geometria biracional, revela interessants trets. Per exemple, com a conseqüència dels treballs de Mukai ([Muk84]) sabem que si X és una superfície K3, algebraica, llisa i irreductible, aleshores $\overline{M}_{X,H}(2; c_1, c_2)$ té dimensió de Kodaira 0. Molt recentment Li ha provat que si X és una superfície minimal de tipus general, amb divisor canònic reduït, aleshores $\overline{M}_{X,H}(2; c_1, c_2)$ és també de tipus general ([Li94]), tot mostrant que la geometria de la superfície i la de l'espai de moduli estan estretament lligades.

A continuació centrarem la nostra atenció en l'estudi de la racionalitat dels espais de moduli $M_{X,H}(r; c_1, c_2)$. Per $X = \mathbb{P}^2$, Maruyama (resp. Ellingsrud i Stromme) varen provar que si $c_1^2 - 4c_2 \not\equiv 0 \pmod 8$, aleshores l'espai de moduli $M_{\mathbb{P}^2, O_{\mathbb{P}^2}(1)}(2; c_1, c_2)$ de fibrats vectorials de rang dos en $\mathbb{P}^2$ que són $O_{\mathbb{P}^2}(1)$-estables, amb classes de Chern c_1 i c_2 és racional ([Mar85] and [ES87]). Temps més tard, Maeda va provar la racionalitat de l'espai de moduli $M_{\mathbb{P}^2, O_{\mathbb{P}^2}(1)}(2; c_1, c_2)$ per a tot parell d'enters $(c_1, c_2) \in \mathbb{Z}^2$ sempre i quan $M_{\mathbb{P}^2, O_{\mathbb{P}^2}(1)}(2; c_1, c_2)$ no sigui buit ([Mae90]). El resultat de Maeda juntament amb el fet que fins ara no es coneix cap contraexemple al fet que l'espai de moduli $M_{X,H}(2; c_1, c_2)$ és racional quan la superfície base ho és, dóna lloc a la primera qüestió considerada en aquest treball, la qual reformulem a continuació:

QÜESTIÓ (1): Sigui X una superfície racional, llisa i irreductible. Fixem $c_1 \in Pic(X)$ i $0 << c_2 \in \mathbb{Z}$. Existeix algun divisor ample H en X pel qual $M_{X,H}(2; c_1, c_2)$

és racional?

En aquest treball donem una resposta afirmativa a aquesta qüestió. A més a més, provem que si X és una superfície racional minimal o una superfície de Fano, aleshores per tot divisor ample H en X, l'espai de moduli $M_{X,H}(2; c_1, c_2)$ és racional sempre i quan no és buit. Per a totes les altres superfícies racionals, provem que $M_{X,H}(2; c_1, c_2)$ és racional sempre i quan $c_2 >> 0$ i $(K_X + F)H < 0$, sent F una fibra de $\pi : X \to \mathbb{P}^1$ i K_X el divisor canònic de X.

Com a principals ingredients utilitzem: la teoria de feixos prioritaris introduïda per Lazlo i Hirschowitz i desenvolupada més tard per Walter; la relació que hi ha entre espais de moduli de fibrats vectorials en una superfície llisa i irreductible X i els espais de moduli de fibrats vectorials en la superfície $\overline{X}$ obtinguda explotant un nombre finit de punts de X, desenvolupada per Nakashima en [Nak93b]; i les propietats biracionals dels espais de moduli de fibrats vectorials estables de rang dos en superfícies algebraiques. Si recordem la definició d'estabilitat en el sentit de Mumford-Takemoto i en el sentit de Gieseker-Maruyama

Definició: Sigui X una varietat projectiva i llisa de dimensió n i H un divisor ample en X. Per a un feix lliure de torsió F en X es defineix

$$\mu_H(F) := \frac{c_1(F)H^{n-1}}{rk(F)},$$

$$P_F(m) := \frac{\chi(F \otimes O_X(mH))}{rk(F)}.$$

El feix F és H-semiestable, en el sentit de Mumford-Takemoto, si

$$\mu_H(E) \leq \mu_H(F)$$

per a tot subfeix no nul $E \subset F$ amb $rk(E) < rk(F)$. Si es verifica la desigualtat estricta, diem que F és H-estable en el sentit de Mumford-Takemoto
i diem que el feix F és G-semiestable respecte H, en el sentit de Gieseker-Maruyama, si

$$P_E(m) \leq P_F(m) \text{ sent } m >> 0$$

per a tot subfeix no nul $E \subset F$ amb $rk(E) < rk(F)$. Si es verifica la desigualtat estricta, diem que F és G-estable en el sentit de Gieseker-Maruyama

veiem, sense cap mena de dubte, que la definició d'estabilitat depèn del divisor ample H triat. Així doncs, és natural preguntar-nos pels canvis que experimenta $M_{X,H}(r; c_1, c_2)$ quan H varia. Sabem que el con de divisors amples de X té una estructura de cambres tal que $M_{X,H}(r; c_1, c_2)$ només depèn de la cambra de H i, en general, $M_{X,H}(r; c_1, c_2)$ varia quan H travessa una paret entre dues cambres.

Es diu que una component irreductible M de $M_{X,L}(2; c_1, c_2)$ és trivial si per a qualsevol polarització H, existeix un feix en M que és H-estable. Una polarització L és trivial de tipus (c_1, c_2) si tota component de l'espai de moduli $M_{X,L}(2; c_1, c_2)$ és trivial. En [Qin91], Qin va formular la següent conjectura

CONJECTURA: Existeixen polaritzacions trivials de tipus $(c_1, c_2) \in Pic(X) \times \mathbb{Z}$ si $4c_2 - c_1^2$ és més gran que certa constant $c = c(X)$ que depèn només de X.

Nosaltres hem provat aquesta conjectura de Qin per a superfícies projectives, anticanòniques, llises i irreductibles. En altres paraules, hem vist que si X és una superfície llisa, racional i anticanònica, aleshores els espais de moduli $M_{X,H}(2; c_1, c_2)$ i $M_{X,H'}(2; c_1, c_2)$ són biracionalment equivalents, quan no són buits i la c_2 és més gran que certa constant $c = c(X)$, que només depèn de X i que nosaltres calculem explícitament. En el cas en què X és una superfície llisa, racional, no necessàriament anticanònica, hem provat que si $c_2 >> 0$, aleshores els espais de moduli $M_{X,H}(r; c_1, c_2)$ i $M_{X,H'}(r; c_1, c_2)$ són biracionals, sempre i quan $H(K_X + F) < 0$ i $H'(K_X + F) < 0$. Com abans, F denota la fibra de $\pi : X \to \mathbb{P}^1$ i K_X el divisor canònic de X.

Tots aquests resultats ens permeten, en moltes ocasions, fixar convenientment el divisor ample H en X i això és justament el que sovint fem quan volem estudiar propietats biracionals de l'espai de moduli $M_{X,H}(r; c_1, c_2)$.

Per tal de resoldre la qüestió (**1**), hem establert dos criteris de racionalitat per als espais de moduli $M_{X,H}(2; c_1, c_2)$ de fibrats vectorials E de rang dos, H-estables, en una superfície racional, llisa i irreductible, amb classes de Chern $c_1(E) = c_1$ i $c_2(E) = c_2$ fixades. El primer criteri funciona per superfícies racionals anticanòniques, en altres paraules, en superfícies racionals que tenen el divisor anticanònic efectiu. El segon, funciona per superfícies racionals arbitràries. Aleshores, utilitzant aquests criteris o construïnt famílies irreductibles de fibrats vectorials (resp. feixos simples i prioritaris) sobre una base racional suficientment gran, hem resolt completament la qüestió (**1**).

Hem extès els resultats sobre la racionalitat dels espais de moduli $M_{X,H}(2; c_1, c_2)$ de fibrats vectorials de rang dos a espais de moduli $M_{X,H}(r; c_1, c_2)$ de fibrats vectorials de rang arbitrari, tot donant una resposta parcial a la qüestió (**2**). La resposta afirmativa a la qüestió (**1**) juntament amb les noves aportacions a la qüestió (**2**) donen un ferm suport al fet que l'espai de moduli $M_{X,H}(r; c_1, c_2)$ és racional sempre i quan X sigui racional.

En relació a la qüestió (**3**), hem generalitzat a rang arbitrari els resultats de rang dos provats independentment per Zuo ([Zuo91b]) i per Nakashima ([Nak93]) que fan referència a aquesta qüestió. És ben conegut el fet que si X és una superfície algebraica K3 aleshores l'esquema de Hilbert $Hilb^l(X)$ de subesquemes zero dimensionals de X de longitud l té una estructura simplèctica (veure [Bea83]). D'altra banda, en 1984, Mukai va demostrar que l'espai de moduli de feixos simples també té una estructura simplèctica. Per tant, és natural que ens preguntem si els esquemes de Hilbert $Hilb^l(X)$ i els espais de moduli $M_{X,H}(r; c_1, c_2)$ estan relacionats. Utilitzant la correspondència de Serre i transformacions elementals hem determinat invariants (r, c_1, c_2, l) per als quals existeix una aplicació biracional entre l'esquema de Hilbert $Hilb^l(X)$ i l'espai de moduli $M_{X,H}(r; c_1, c_2)$ de fibrats vectorials E de rang r, H-estables en una superfície K3, llisa i irreductible, amb classes de Chern $c_i(E) = c_i$ fixades.

En la literatura, no és possible trobar resultats generals sobre espais de moduli de fibrats vectorials estables en varietats de dimensió més gran o igual a tres. No ens cansarem de remarcar el fet que quan es treballa amb fibrats vectorials estables en varietats de dimensió arbitrària la situació canvia dràsticament i quasi no té res a veure amb la que ens trobem quan treballem en superfícies. Resultats com ara la llisor i la irreductibilitat dels espais de moduli de fibrats vectorials estables en superfícies esdevenen falsos per espais de moduli de fibrats vectorials estables en varietats de dimensió més gran o igual a tres. És força freqüent l'existència d'espais de moduli de fibrats vectorials en varietats de dimensió més gran o igual a tres que no són ni llisos ni irreductibles. Per citar-ne algun, en [Ein88] (resp. [AO95]), Ein (resp. Ancona i Ottaviani) prova (resp. proven) que el nombre mínim de components irreductibles de l'espai de moduli de fibrats vectorials estables de rang dos (resp. rang 3) en $\mathbb{P}^3$ (resp. $\mathbb{P}^5$) amb c_1 fixada i c_2 tendint cap a infinit, creix cap a infinit. El lector pot consultar [BM97] per tenir una generalització del resultat d'Ein a varietats projectives de dimensió tres i [MO97] per veure exemples d'espais de moduli de fibrats vectorials en $\mathbb{P}^{2n+1}$ amb $c_2 >> 0$, que són singulars.

Donada C una corba projectiva, llisa i irreductible i $\mathcal{E}$ un fibrat de rank $d+1$ en C, anomenem $\mathbb{P}^d$-fibrat a la varietat X de dimensió d definida per $X = \mathbb{P}(\mathcal{E})$. Les nostres contribucions a la qüestió (**4**) proven que per a $\mathbb{P}^d$-fibrats, X, i per un divisor ample L en X convenientment escollit, l'espai de moduli $M_{X,L}(2; c_1, c_2)$ de fibrats vectorials E de rang dos, L-estables en X, amb classes de Chern fixades, és una varietat projectiva, llisa i irreductible. En altres paraules i de forma més precisa, provem que l'esmentat espai de moduli és un $\mathbb{P}^N$-fibrat sobre $Pic^0(C) \times Pic^0(C)$. Si, a més a més, X és un $\mathbb{P}^d$-fibrat racional, i.e. X és un $\mathbb{P}^d$-fibrat definit sobre $C = \mathbb{P}^1$, aleshores l'espai de moduli $M_{X,H}(2; c_1, c_2)$ és també racional. Un cop més, aquest resultats desvetllen el fet que moltes de les propietats geomètriques de l'espai de moduli estan lligades a les corresponents propietats geomètriques de la varietat base.

De tota manera, volem fer esment que el nostre resultat depèn fortament de la polarització que hem fixat. Veurem com l'espai de moduli $M_{X,L}(2; c_1, c_2)$ canvia quan el divisor ample L varia i provarem, mitjançant uns exemples, que els nostres resultats esdevenen falsos per altres divisors amples.

Com a conseqüència dels mètodes que utilitzem al llarg d'aquest treball, també calculem el grup de Picard d'alguns espais de moduli. Utilitzarem els grups de Picard per veure que, en general, donats dos divisors amples H i H' en X, l'aplicació biracional existent entre $M_{X,H}(r; c_1, c_2)$ i $M_{X,H'}(r; c_1, c_2)$ no és un isomorfisme.

Part dels resultats d'aquesta tesi apareixeran publicats en:

• L. Costa and R.M. Miró-Roig, *On the rationality of moduli spaces of vector bundles on Fano surfaces*, Journal Pure Appl. Algebra, to appear, **[CM97]**.

• L. Costa, *K3 surfaces: moduli spaces and Hilbert schemes*, Collectanea Mathematica, to appear, **[Cos98]**.

Els continguts d'aquesta tesi són els següents: **Capítol 1** està dedicat a proveir el lector dels coneixements més elementals que es faran servir al llarg del treball. En les primeres dues seccions, hem recopilat les principals definicions i els principals resultats referents a feixos coherents i a espais de moduli, com a mínim aquells que es faran servir al llarg de la memòria.

En la secció 1.3, revisem alguns fets sobre parets i cambres que més endavant utilitzarem per tal d'entendre i descriure com l'espai de moduli $M_{X,L}(2;c_1,c_2)$ canvia quan variem el divisor ample L.

Finalment, en la secció 1.4, recordem la classificació, llevat isomorfisme, de superfícies racionals, llises i irreductibles. A més a més, per tal que el treball sigui el màxim autocontingut possible, provarem alguns resultats referents a grups de cohomologia de fibrats de línia en superfícies racionals, llises i irreductibles, que nosaltres no hem trobat explícitament en la literatura.

El designi del **Capítol 2** és establir criteris de racionalitat per espais de moduli $M_{X,H}(2;c_1,c_2)$ de fibrats vectorials E de rang 2, H-estables en una superfície racional, llisa i irreductible X. Aquests criteris seran bàsics per tal de donar una resposta afirmativa a la **Qüestió (1)**.

En la secció 2.1, tot utilitzant la teoria de parets i cambres introduïda en la secció 1.3, estudiarem com canvia l'espai de moduli $M_{X,H}(2;c_1,c_2)$ quan H travessa una paret que separa dues cambres adjacents. Més concretament provarem:

Teorema 2.1.10: Sigui X una superfície racional, anticanònica, llisa i irreductible, $c_1 \in Pic(X)$ i $c_2 \in \mathbb{Z}$. Suposem que $4c_2 - c_1^2 > 2 - \frac{3K_X^2}{2}$. Aleshores tenim

(a) Tota polarització L de tipus (c_1,c_2) és trivial sempre i quan $M_{X,H}(2;c_1,c_2)$ no sigui buit.

(b) Donades dues polaritzacions L_1 i L_2 els espais de moduli $M_{X,L_1}(2;c_1,c_2)$ i $M_{X,L_2}(2;c_1,c_2)$ són biracionals si no són buits.

Aquest resultat prova la conjectura de Qin, esmentada anteriorment, en el cas en què la superfície base X és una superfície racional, anticanònica, llisa i irreductible.

Com a conseqüència important obtindrem el primer criteri de racionalitat, vàlid per a espais de moduli $M_{X,H}(2;c_1,c_2)$ de fibrats vectorials E de rang dos, H-estables en una superfície racional, anticanònica (i.e. $-K_X$ és efectiu), llisa i irreductible, amb classes de Chern $c_i(E) = c_i$ fixades.

Primer criteri de racionalitat 2.1.13: Sigui X una superfície racional, anticanònica, llisa i irreductible, $c_1 \in Pic(X)$ i $c_2 \in \mathbb{Z}$. Suposem $4c_2 - c_1^2 > 2 - \frac{3K_X^2}{2}$ i que existeix una classe d'equivalència numèrica ξ que defineix una paret no buida de tipus (c_1, c_2) i satisfà $d(\xi) = 0$ (i.e. $\xi^2 = c_1^2 - 4c_2$ i $\xi^2 + \xi K_X + 2 = 0$). Aleshores, es verifica el següent

(1) Existeix un divisor ample $\tilde{L}$ en X tal que l'espai de moduli $M_{\tilde{L}}(2;c_1,c_2)$ és buit o és una varietat projectiva, racional, llisa i irreductible de dimensió $4c_2 - c_1^2 - 3$ i $Pic(M_{\tilde{L}}(2;c_1,c_2)) \cong \mathbb{Z}$.

(2) Per qualsevol divisor ample L en X, l'espai de moduli $M_L(2;c_1,c_2)$ és buit o és és una varietat quasi projectiva, racional, llisa i irreductible de dimensió $4c_2 - c_1^2 - 3$.

L'objectiu de la secció 2.2 és el d'il.lustrar mitjançant una sèrie d'exemples, el fet que si L_1 i L_2 són dues polaritzacions en una superfície projectiva, anticanònica, llisa i irreductible, que es troben en dues cambres diferents, aleshores l'aplicació biracional existent entre els espais de moduli $M_{X,L_1}(2;c_1,c_2)$ i $M_{X,L_2}(2;c_1,c_2)$ no és, en general, un isomorfisme.

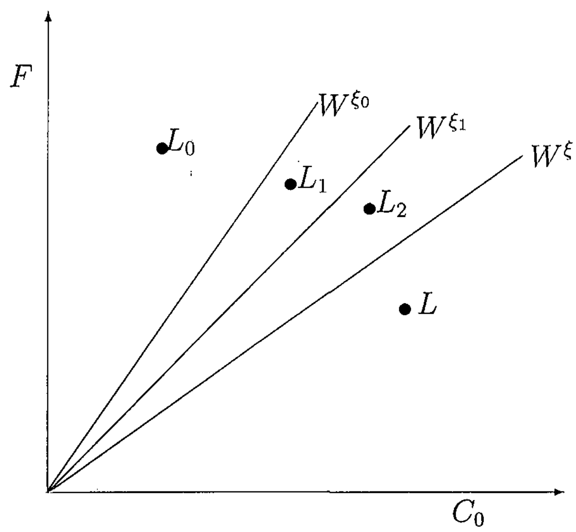
De manera més precisa, fixem X_e una superfície de Hirzebruch, llisa i irreductible amb $e \geq 1$, $1 < c_2$ un enter i $c_1 = C_0 \in Pic(X_e)$. Considerem les classes d'equivalència numèrica

$$\xi_0 = C_0 - 2c_2F \text{ i } \xi_1 = C_0 - 2(c_2 - 1)F$$

sent $< C_0, F >$ els generadors de $Pic(X_e)$ i les corresponents parets W^{ξ_0} i W^{ξ_1} definides per ξ_0 i ξ_1 respectivament.

Denotem per $\mathcal{C}_0$ la cambra tal que $\overline{\mathcal{C}}_0 \cap W^{\xi_0} \neq \emptyset$ i per a qualsevol $L \in \mathcal{C}_0$, $L\xi_0 \geq 0$, per $\mathcal{C}_1$ la cambra amb $\overline{\mathcal{C}}_1 \cap W^{\xi_0} \neq \emptyset$ i per a qualsevol $L \in \mathcal{C}_1$, $L\xi_0 < 0$, per $\mathcal{C}_2$ la cambra tal que $\overline{\mathcal{C}}_2 \cap W^{\xi_1} \neq \emptyset$ i per a qualsevol $L \in \mathcal{C}_2$, $L\xi_1 < 0$ i finalment per $\mathcal{C}$ una cambra, diferent de $\mathcal{C}_0$, $\mathcal{C}_1$ i $\mathcal{C}_2$, tal que per tot $L \in \mathcal{C}$, tenim $L\xi_1 < 0$.

Donada una polarització $L \equiv aC_0 + bF$, podem representar L com un punt de coordenades (a, b) en el pla. El següent dibuix ens descriu la situació de parets i cambres que estem considerant



Nosaltres en aquest exemple hem provat que

- Si $L_0 \in \mathcal{C}_0$, aleshores $M_{L_0}(2; c_1, c_2) = \emptyset$ i $Pic(M_{L_0}(2; c_1, c_2)) = 0$.
- Si $L_1 \in \mathcal{C}_1$, aleshores $M_{L_1}(2; c_1, c_2) \cong \mathbb{P}^{4c_2 - c_1^2 - 3}$ i $Pic(M_{L_1}(2; c_1, c_2)) = \mathbb{Z}$.
- Si $L_2 \in \mathcal{C}_2$, aleshores $Pic(M_{L_2}(2; c_1, c_2))$ es troba en la succesió exacta

$$0 \longrightarrow G_{E_{\xi_1}} \longrightarrow Pic(M_{L_2}(2; c_1, c_2)) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

- Si $L \in \mathcal{C}$, tenim $Pic(M_L(2; c_1, c_2)) \cong Pic(M_{L_2}(2; c_1, c_2))$.

En particular això ens diu que els espais de moduli són biracionals però no isomorfs, que és justament el que es pretenia veure.

En la secció 2.3, hem generalitzat el Teorema 2.1.10 i el Criteri 2.1.13 a superfícies racionals, llises i irreductibles arbitraries. Per fer possible aquesta generalització, fem servir feixos prioritaris, com a principal ingredient. Aquests van ser introduïts per primer cop en $\mathbb{P}^2$ per Hirschowitz-Laszlo ([HL93]) i en superfícies reglades en general per Walter ([Wal93]). Tot fent ús del fet que l'espai de moduli de feixos simples i prioritaris és llis i irreductible (Theorem 2.3.5) i del fet que sota certes condicions en H (que $(K_X+F)H<0$ sent K_X el divisor canònic de X i F la fibra de $\pi : X \to \mathbb{P}^1$) , els fibrats H-estables són prioritaris (Lemma 2.3.2), obtenim el Teorema 2.3.6 que extén el Teorema 2.1.10 a superfícies racionals arbitràries. En concret, obtenim

Teorema 2.3.6 Sigui $\pi : X \longrightarrow \mathbb{P}^1$ una superfície biracionalment reglada, $F \in Num(X)$ la classe d'equivalència numèrica d'una fibra de π, $c_1 \in Pic(X)$ i $c_2 \in \mathbb{Z}$ tal que $\Delta(r; c_1, c_2) >> 0$. Aleshores, donats dos divisors amples qualssevols L_1 i L_2 en X verificant $L_i(K_X + F) < 0$, $i = 1, 2$, els espais de moduli $M_{L_1}(r; c_1, c_2)$ i $M_{L_2}(r; c_1, c_2)$ són biracionalment equivalents.

També hem obtingut un segon criteri de racionalitat, aplicable en el cas en què X és una superfície racional, llisa i irreductible qualsevol, que ens garanteix la racionalitat de l'espai de moduli $M_{X,H}(2; c_1, c_2)$ de fibrats vectorials E de rang dos, H-estables en X, amb classes de Chern $c_1(E) = c_1$ i $c_2(E) = c_2$ fixades. Aquest criteri generalitza el Criteri 2.3.7 a superfícies racionals arbitràries i és formulat com segueix

Segon criteri de racionalitat 2.3.7 Sigui $\pi : X \longrightarrow \mathbb{P}^1$ una superfície biracionalment reglada, $F \in Num(X)$ la classe d'equivalència numèrica d'una fibra de π, $c_1 \in Pic(X)$ i $c_2 \in \mathbb{Z}$. Suposem que $4c_2 - c_1^2 > 2 - \frac{3K_X^2}{2}$ i que existeix una classe d'equivalència numèrica ξ que defineix una paret no buida de tipus (c_1, c_2) i que verifica

(1) $\quad \xi^2 = c_1^2 - 4c_2, \quad \xi^2 + \xi K_X + 2 = 0,$

(2) $\quad H^0 O_X(\xi + 3K_X) = H^0 O_X(\xi + K_X + F) = H^0 O_X(K_X + F - \xi) = 0.$

Aleshores,

(1) Existeix un divisor ample $\tilde{L}$ en X tal que l'espai de moduli $M_{\tilde{L}}(2; c_1, c_2)$ és buit o és una varietat projectiva, racional, llisa i irreductible de dimensió $4c_2 - c_1^2 - 3$ i $Pic(M_{\tilde{L}}(2; c_1, c_2)) \cong \mathbb{Z}$.

(2) Per $c_2 >> 0$ i qualsevol divisor ample L en X tal que $L(K_X + F) < 0$, l'espai de moduli $M_L(2; c_1, c_2)$ és una varietat quasi projectiva, racional, llisa i irreductible de dimensió $4c_2 - c_1^2 - 3$.

Sigui $\pi : X \longrightarrow \mathbb{P}^1$ una superfície llisa, biracionalment reglada i H un divisor ample en X verificant $L(K_X + F) < 0$, sent $F \in Num(X)$ la classe d'equivalència numèrica d'una fibra de π i K_X el divisor canònic de X. En el **Capítol 3** provem que l'espai de moduli $M_{X,H}(2; c_1, c_2)$ de fibrats vectorials E de rang dos, H-estables en una superfície racional, llisa i irreductible X, amb classes de Chern $c_1(E) = c_1 \in Pic(X)$ i $0 << c_2(E) = c_2 \in \mathbb{Z}$ fixades, és una varietat quasi projectiva, racional, llisa i irreductible de la dimensió esperada. En efecte, provem el següent Teorema que resolt completament la **Qüestió (1)**

Teorema 3.3.7 Sigui X una superfície racional, algebraica, llisa i irreductible. Per a qualsevol divisor $c_1 \in Pic(X)$, $0 << c_2 \in \mathbb{Z}$ i qualsevol polarització L en X tal que $L(K_X + F) < 0$, sent $F \in Pic(X)$ la classe d'equivalència numèrica d'una fibra de $\pi : X \to \mathbb{P}^1$, l'espai de moduli $M_{X,L}(2; c_1, c_2)$ és una varietat racional, quasi projectiva, llisa i irreductible de dimensió $4c_2 - c_1^2 - 3$.

Ens agradaria remarcar que un cop fixada la primera classe de Chern $c_1 \in Pic(X)$, la condició $c_2 >> 0$, o equivalentment,

$$\Delta(r; c_1, c_2) := \frac{1}{r}(c_2 - \frac{r-1}{2r}c_1^2) >> 0$$

ens garanteix que l'espai de moduli $M_{X,L}(r; c_1, c_2)$ és no buit.

Segons la classificació, llevat isomorfisme, de les superfícies racionals, llises i irreductibles,

Teorema 1.4.1 Sigui X una superfície minimal, racional, llisa i irreductible . Aleshores, X és isomorfa a $\mathbb{P}^2$ o a una superfície de Hirzebruch X_e amb $e \neq 1$.

Al llarg de la secció 3.1, centrem la nostra atenció al cas en el que la varietat base és una superfície racional minimal. Per a qualsevol superfície racional, minimal, llisa i irreductible X, i qualsevol divisor ample H en X, provem la racionalitat de tot espai de moduli $M_{X,H}(2; c_1, c_2)$ de fibrats vectorials E de rang dos, H-estables en X amb classes de Chern $c_i(E) = c_i$ fixades, que sigui no buit. Dit d'una altra manera, utilitzant el primer criteri de racionalitat (Criteri 2.1.13) o construint families de fibrats vectorials de rang dos, H-estables o construint families de feixos simples i prioritaris de rang dos, totes elles sobre una base racional de dimensió prou gran, provem el següent resultat

Teorema 3.1.8 Sigui X una superfície racional, minimal, llisa i irreductible, $c_1 \in Pic(X)$ i $c_2 \in \mathbb{Z}$. Aleshores, per a qualsevol polarització L en X, l'espai de moduli $M_{X,L}(2; c_1, c_2)$ és una varietat quasi projectiva, racional, llisa i irreductible de dimensió $4c_2 - c_1^2 - 3$, sempre i quan no sigui buit.

En la secció 3.2, X és una superfície de Fano i per a qualsevol divisor ample H en X, provem la racionalitat de l'espai de moduli $M_{X,H}(2; c_1, c_2)$ de fibrats vectorials E de rang dos, H-estables en X amb classes de Chern $c_i(E) = c_i$. Això és, provem el següent resultat

Teorema 3.2.7 Sigui X una superfície de Fano, $c_1 \in Pic(X)$ i $c_2 \in \mathbb{Z}$. Aleshores, per a qualsevol polarització L en X, l'espai de moduli $M_{X,L}(2; c_1, c_2)$ és una varietat quasi projectiva, racional, llisa i irreductible de dimensió $4c_2 - c_1^2 - 3$, sempre i quan no sigui buit.

Volem remarcar que en aquesta secció és de vital importància el fet què X és una superfície racional i anticanònica i que per tant podem utilitzar el Criteri 2.1.13 de racionalitat. És precisament això el que fem (juntament amb la construcció de famílies de fibrats vectorials de rang dos H-estables, sobre una base racional prou gran) per tal de provar la racionalitat de l'espai de moduli $M_{X,H}(2; c_1, c_2)$.

Finalment, en al secció 3.3, provem la racionalitat de $M_{X,H}(2; c_1, c_2)$ per a la resta de casos de superfícies racionals, i.e. per superfícies racionals, no-minimals, obtingudes en explotar una superfície de Hirzebruch en, com a mínim, més de vuit punts diferents, tot utilitzant el segon criteri de racionalitat (Criteri 2.3.7). En el cas en què aquest segon criteri no es pot aplicar, construïm famílies de feixos de rang dos, simples i prioritaris, sobre una base racional adequada. Aleshores, utilitzant el fet que per a qualsevol divisor ample H en X tal que $(K_X + F)H < 0$, els fibrats vectorials H-estables són prioritaris (Lemma 2.3.2), deduirem la racionalitat de l'espai de moduli $M_{X,H}(2; c_1, c_2)$ (Theorem 3.3.6). És a dir, el següent Teorema

Teorema 3.3.6 Sigui X_e una superfície de Hirzebruch i sigui X una superfície racional i llisa obtinguda en explotar X_e en s punts diferents i L un divisor ample en X tal que $L(K_X + F) < 0$. Per a qualsevol enter $c_2 >> 0$, l'espai de moduli $M_L(2; c_1, c_2)$ és una varietat quasi projectiva, racional, llisa i irreductible de dimensió $4c_2 - c_1^2 - 3$.

Volem remarcar que la hipòtesi $H(K_X + F) < 0$ tan sols es fa servir en aquesta darrera secció.

En el **Capítol 4** estudiem espais de moduli $M_{X,H}(r; c_1, c_2)$ de fibrats vectorials de rang r, H-estables en superfícies racionals, minimals o en superfícies K3 algebraiques.

Durant la secció 4.1, ens tornem a dedicar a estudiar el delicat problema de la racionalitat de l'espai de moduli $M_{X,H}(r; c_1, c_2)$ sent X una superfície racional i minimal. En primer lloc, tot construint una família de feixos simples i prioritaris sobre una base racional prou gran, provarem la racionalitat de determinats espais de moduli $M_{X_e,H}(r; c_1, c_2)$ de fibrats vectorials E de rang r, H-estables, en una superfície de Hirzeburch X_e, llisa i irreductible, amb classes de Chern $c_1(E) = c_1$ i $c_2(E) = c_2$ fixades. Permeteu-nos precisar que volem dir amb això de determinats, tot enunciant el resultat que es prova.

Teorema 4.1 Sigui X_e una superfície de Hirzebruch, llisa i irreductible, H un divisor ample en X_e, $c_1(E) = c_1 \in Pic(X_e)$ i $c_2(E) = c_2 \in \mathbb{Z}$ tal que $\Delta(r; c_1, c_2) >> 0$. Suposem que es verifica una de les següents condicions

- $c_1F = 1$ o $r - 1$ $(mod \quad r)$;
- $c_1F = r - 2$ $(mod \quad r)$ i $c_2 - \frac{c_1^2}{2} - \frac{c_1K}{2} - (r-1) = 0$ $(mod \quad 2)$;
- $c_1F = 2$ $(mod \quad r)$ i $c_2 + c_1C_0 - \frac{c_1^2}{2} + \frac{c_1K}{2} + 1 = 0$ $(mod \quad 2)$,

sent F la fibra de $\pi : X_e \to \mathbb{P}^1$. Aleshores, l'espai de moduli $M_{X_e,H}(r; c_1, c_2)$ és una varietat quasi projectiva, racional, llisa i irredcutible de la dimensió esperada $2rc_2 - (r-1)c_1^2 - (r^2-1)$.

Com a corolari deduïm la racionalitat de l'espai de moduli $M_{\mathbb{P}^2, O_{\mathbb{P}^2}(1)}(r; c_1, c_2)$ de fibrats vectorials E de rang r, $O_{\mathbb{P}^2}(1)$-estables en el pla projectiu amb discriminant $\Delta(r; c_1, c_2)$ prou gran (Teorema 4.1.14), sempre i quan es verifiqui una de les següents condicions

- $c_1 = 1$ o $r-1$ $(mod \quad r)$;
- $c_1 = r-2$ $(mod \quad r)$ i $c_2 - \frac{c_1^2}{2} + \frac{3c_1}{2} - (r-1) = 0$ $(mod \quad 2)$;
- $c_1 = 2$ $(mod \quad r)$ i $c_2 + c_1 - \frac{c_1^2}{2} - \frac{3c_1}{2} + 1 = 0$ $(mod \quad 2)$.

Volem remarcar que el Teorema 4.1.13 i el Teorema 4.1.14 constitueixen una important contribució al problema de demostrar que els espais de moduli $M_{X,H}(r; c_1, c_2)$ són sempre racionals si X és racional. Aquest resultats amplien, i en diversos sentits milloren, treballs anteriors duts a terme per Göttsche ([Got96]), Katsylo ([Kat92]), Yoshioka ([Yos96]) i Li ([Li97]).

En la secció 4.2, centrarem la nostra atenció en l'estudi dels espais de moduli $M_{X,H}(r; c_1, c_2)$ de fibrats vectorials E de rang r, H-estables en una superfície K3, algebraica, llisa i irreductible X, amb classes de Chern $c_1(E) \in Pic(X)$ i $c_2(E) \in \mathbb{Z}$ fixades. Determinarem invariants $((r, c_1, c_2), l) \in (\mathbb{Z} \times Pic(X) \times \mathbb{Z}) \times \mathbb{Z}$ per als quals existeix una aplicació biracional ϕ entre l'espai de moduli $M_{X,H}(r; c_1, c_2)$ i l'esquema de Hilbert $Hilb^l(X)$ de subesquemes zero dimensionals de X (Theorem 4.2.1). Concretament, provarem el següent resultat

Teorema 4.2.1 Sigui X una superfície algebraica $K3$ i H un divisor ample en X. Considerem l'espai de moduli $\mathcal{M}_H(r; c_1, k(n))$ de fibrats vectorials E de rang r, H-estables, en X amb classes de Chern $det(E) = c_1 \in Pic(X)$ i

$$c_2(E) = k(n) := \frac{c_1^2}{2} + \frac{r}{2}n^2H^2 + nc_1H + (r+1).$$

D'altra banda, considerem l'esquema de Hilbert $Hilb^{l(n)}(X)$ de subesquemes zero dimensionals de X de longitud $l(n) := k(n) + \frac{r(r-1)}{2}n^2H^2 + (r-1)nc_1H$. Aleshores, per $n >> 0$ hi ha una aplicació biracional

$$\phi : \mathcal{M}_H(r; c_1, k(n)) \longrightarrow Hilb^{l(n)}(X).$$

Notem que el "pullback" de l'estructura simplèctica de $Hilb^l(X)$ ([Bea78]) via l'aplicació biracional ϕ dóna una estructura simplèctica a $M_{X,H}(r; c_1, c_2)$, que coincideix amb l'estructura simplèctica de $M_{X,H}(r; c_1, c_2)$ determinada per Mukai en [Muk84].

Al llarg del **Capítol 5** estudiem espais de moduli $M_{X,L}(2; c_1, c_2)$ de fibrats vectorials E de rang dos, L-estables en $\mathbb{P}^d$-fibrats definits sobre una corba C projectiva, llisa i irreductible, de dimensió arbitraria, amb classes de Chern $c_i(E) = c_i$, $i = 1, 2$ fixades.

Iniciem la secció 5.1 recordant fets bàsics sobre $\mathbb{P}^d$-fibrats definits sobre una corba projectiva, llisa i irreductible, de gènere $g \geq 0$, amb la finalitat de proveir el lector dels coneixements que es faran servir al llarg de la resta de les seccions. En destaquem:

Lema 5.1.8 Donat $\mathfrak{b} \in Pic(C)$, considerem el fibrat de línia $O_X(aH + \pi^*\mathfrak{b})$ en un $\mathbb{P}^d$-fibrat $X = \mathbb{P}(\mathcal{E})$ sobre una corba C. Tenim

$$H^i(X, O_X(aH + \pi^*\mathfrak{b})) = \begin{cases} 0 & \text{si } -d-1 < a < 0 \\ H^i(C, S^a(\mathcal{E}) \otimes O_C(\mathfrak{b})) & \text{si } a \geq 0 \\ H^{d+1-i}(C, S^{-d-1-a}(\mathcal{E}) \otimes O_C(\tilde{\mathfrak{b}})) & \text{si } a \leq -d-1 \end{cases}$$

sent $\tilde{\mathfrak{b}} := -\mathfrak{b} + det(\mathcal{E}) + K_C$, K_C el divisor canònic de C i $S^a(\mathcal{E})$ l' a-èssima potència simetrica de $\mathcal{E}$.

Finalitzarem aquesta secció amb una Proposició clau per al desenvolupament dels posteriors resultats. Aquest resultat ens permet garantir l'existència d'una secció, σ, de cert torsat d'un fibrat vectorial de rang dos, amb lloc de zeros, σ_0, de codimensió més gran o igual que dos. El resultat és el següent

Proposició 5.1.13 Sigui X un $\mathbb{P}^d$-fibrat sobre una corba C, c_2 un enter, $L \equiv dH + bF$

un divisor ample en X, $e \in \{0,1\}$ i E un fibrat vectorial de rang dos, L-estable en X. Suposem que una de les següents condicions es verifica

(i) $c_1E \equiv H + eF$, $c_2E \equiv (c_2+e)HF$, $b = 2c_2 - H^{d+1} + e - 1$ i

$$c_2 > \frac{d\gamma + H^{d+1}}{2} + 1;$$

(ii) $c_1E \equiv eF$, $c_2E \equiv -H^2 + (2c_2+e)HF$, $b = c_2 - H^{d+1} + e - 1$ i

$$c_2 > d\gamma + H^{d+1} + 2.$$

Aleshores, $E(-H + \pi^*\mathfrak{c}_2)$ té una secció, σ, tal que el seu esquema de zeros, σ_0, té codimensió més gran o igual que dos, sent $\mathfrak{c}_2 \in Pic(C)$ un divisor de grau $deg(\mathfrak{c}_2) = c_2$.

En la secció 5.2 provem els nostres resultats principals sobre espais de moduli $M_{X,L}(2;c_1,c_2)$ de fibrats vectorials de rang dos, en $\mathbb{P}^d$-fibrats, $X = \mathbb{P}(\mathcal{E}) \to C$, definits sobre una corba C projectiva, llisa i irreductible, de gènere $g \geq 0$.

Alhora d'estudiar els espais de moduli $M_{X,L}(2;c_1,c_2)$ hem considerat diferent casos, segons la classe d'equivalència numèrica de c_1. Els resultats obtinguts, juntament amb la posterior generalització, són els següents

Teorema 5.2.4 Sigui X un $\mathbb{P}^d$-fibrat sobre C i c_2 un enter tal que $c_2 > \frac{H^{d+1}+d\gamma}{2} + 1$. Fixem un divisor ample $L \equiv dH + bF$ en X amb $b = 2c_2 - H^{d+1} - (1-e)$ i $e \in \{0,1\}$. Aleshores, l'espai de moduli $M_L(2; H+eF, (c_2+e)HF)$ és una varietat projectiva, llisa i irreductible de dimensió $h^1O_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) + 2g - 1$, sent $\mathfrak{c}_2, \mathfrak{c}_2' \in Pic(C)$ de grau c_2 i $\mathfrak{e} \in Pic(C)$ de grau e. En efecte , és un $\mathbb{P}^N$-fibrat sobre $Pic^0(C) \times Pic^0(C)$ sent

$$N := h^1O_X(H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) - 1.$$

Teorema 5.2.8 Sigui X un $\mathbb{P}^d$-fibrat sobre C amb $d > 1$ i c_2 un enter tal que $c_2 > H^{d+1} + d\gamma + 2$. Fixem un divisor ample $L \equiv dH + bF$ en X sent $b = c_2 - H^{d+1} - (1-e)$ i $e \in \{0,1\}$. Aleshores, $M_L(2; eF, -H^2 + (2c_2+e)HF)$ és una varietat projectiva, llisa i irreductible de dimensió $h^1O_X(2H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}_2' - \pi^*\mathfrak{e}) + 2g - 1$, on $\mathfrak{c}_2, \mathfrak{c}_2', \mathfrak{e} \in Pic(C)$

són divisors en C amb $deg(\mathfrak{c}_2) = deg(\mathfrak{c}'_2) = c_2$ i $deg(\mathfrak{e}) = e$. En efecte, és un $\mathbb{P}^M$-fibrat sobre $Pic^0(C) \times Pic^0(C)$ sent

$$M := h^1 O_X(2H - \pi^*\mathfrak{c}_2 - \pi^*\mathfrak{c}'_2 - \pi^*\mathfrak{e}) - 1.$$

Com a conseqüència dels resultats anteriors veiem que si X és un $\mathbb{P}^d$-fibrat, normal i racional, i.e. $C \cong \mathbb{P}^1$, aleshores, l'espai de moduli $M_{X,L}(2; c_1, c_2)$ és també racional.

Arribats a aquest punt, ens agradaria remarcar que amb el següent resultat, el qual generalitza els anteriors, queden estudiats tots els espais de moduli $M_{X,L}(2; c_1, c_2)$ de fibrats vectorials E de rang dos en un $\mathbb{P}^d$-fibrat, X, tals que un punt genèric $[E] \in M_{X,L}(2; c_1, c_2)$ està donat per una extensió no trivial de dos fibrats de línia.

Teorema 5.2.12 Sigui X un $\mathbb{P}^d$-fibrat sobre C amb $d > 1$. Considerem enters b, a, e amb $e \in \{0, 1\}$ i fixem un divisor ample $L \equiv \alpha H + \beta F$ en X. Suposem que es verifiquen les següents condicions

$$0 > -2a > -d - 1, \quad \alpha = ad, \quad \beta = -b - aH^{d+1} + e - 1 \quad \text{i}$$

$$-ab > a^2 H^{d+1} + a^2 d\gamma + a(a + 2).$$

$$(resp. \quad 0 > 1 - 2a > -d - 1, \quad \alpha = (2a - 1)d, \quad \beta = -2b - (2a - 1)H^{d+1} + e - 1$$

$$\text{i} \quad -2ab > (2a - 1)aH^{d+1} + (2a - 1)ad\gamma + a(2a - 1) + 1.)$$

Aleshores, l'espai de moduli $M_L(2; eF, -a^2H^2 + (ae - 2ab)HF)$ (resp. l'espai de moduli $M_L(2; H + eF, a(1-a)H^2 + (b + ae - 2ab)HF)$) és una varietat projectiva, llisa i irreductible de dimensió $h^1 O_X(2aH + \pi^*(\mathfrak{b} + \mathfrak{b}' - \mathfrak{e})) + 2g - 1$ (resp. de dimensió $h^1 O_X((2a - 1)H + \pi^*(\mathfrak{b} + \mathfrak{e})) + 2g - 1$), sent $\mathfrak{b}, \mathfrak{b}', \mathfrak{e} \in Pic(C)$ de grau b i e respectivament. És a dir, és un $\mathbb{P}^N$-fibrat (resp. $\mathbb{P}^M$-fibrat) sobre $Pic^0(C) \times Pic^0(C)$ sent $N := h^1 O_X(2aH + \pi^*(\mathfrak{b} + \mathfrak{b}' - \mathfrak{e}))$ (resp. $M := h^1 O_X((2a - 1)H + \pi^*(\mathfrak{b} + \mathfrak{e}))$).

La prova de tots aquests resultats ens permet calcular la dimensió de Kodaira (Corolari 5.2.14) i descriure el grup de Picard (Corollary 5.2.15) d'aquests espais de moduli. Així, doncs, tenim els següents resultats

Teorema 5.2.14 Sigui X un $\mathbb{P}^d$-fibrat sobre una corba C i $c_2, d \in \mathbb{Z}$ tal que $d > 0$ i $c_2 > \frac{H^{d+1}+d\gamma}{2} + 1$ (resp. $c_2 > H^{d+1} + d\gamma + 2$ i $d > 1$). Fixem un divisor ample $L \equiv dH + bF$ en X amb $b = 2c_2 - H^{d+1} - (1-e)$ (resp. $b = c_2 - H^{d+1} - (1-e)$) i $e \in \{0, 1\}$. Aleshores,

$$Kod(M_L(2; H + eF, (c_2 + e)HF)) = -\infty$$

$$(resp.\quad Kod(M_L(2; eF, -H^2 + (2c_2 + e)HF)) = -\infty).$$

Teorema 5.2.15 Sigui X un $\mathbb{P}^d$-fibrat sobre una corba C i $c_2, d \in \mathbb{Z}$ tal que $d > 0$ i $c_2 > \frac{H^{d+1}+d\gamma}{2} + 1$ (resp. $c_2 > H^{d+1} + d\gamma + 2$ i $d > 1$). Fixem un divisor ample $L \equiv dH + bF$ en X amb $b = 2c_2 - H^{d+1} - (1-e)$ (resp. $b = c_2 - H^{d+1} - (1-e)$) i $e \in \{0, 1\}$. Aleshores,

$$Pic(M_L(2; H + eF, (c_2 + e)HF)) \cong \mathbb{Z} \oplus \Pi^* Pic(Pic^0(C) \times Pic^0(C)).$$

$$(resp.\quad Pic(M_L(2; eF, -H^2 + (2c_2 + e)HF)) \cong \mathbb{Z} \oplus \Pi'^* Pic(Pic^0(C) \times Pic^0(C))\ ).$$

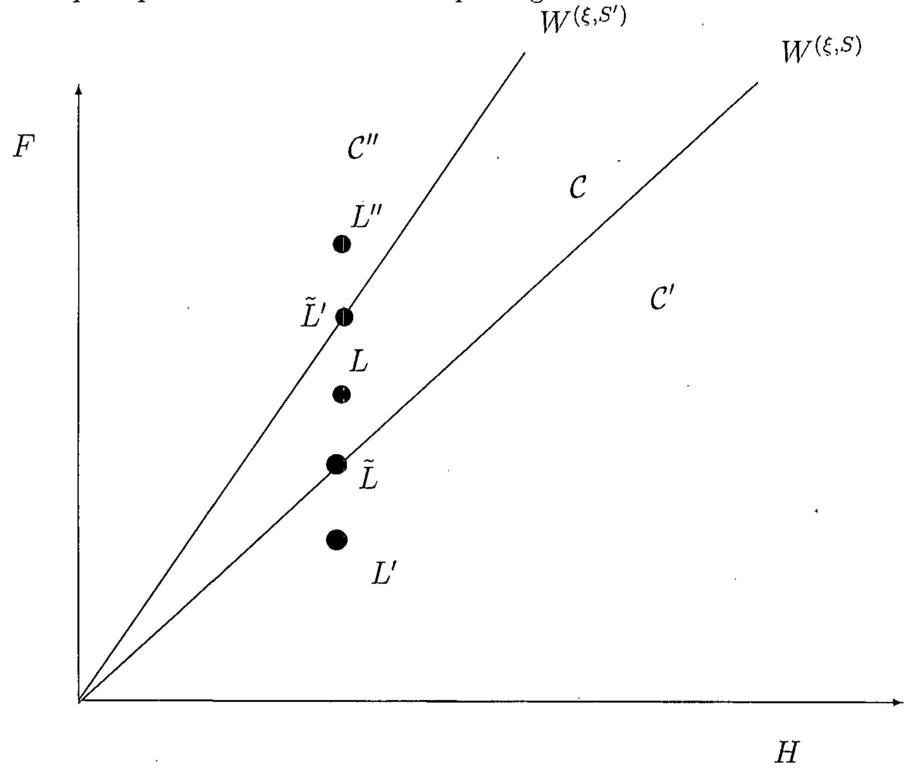
En particular, si X és un $\mathbb{P}^d$-fibrat racional, aleshores

$$Pic(M_L(2; H + eF, (c_2 + e)HF)) \cong Pic(M_L(2; eF, -H^2 + (2c_2 + e)HF)) \cong \mathbb{Z}.$$

Notem que, un cop més, els espais de moduli capturen molta informació de la varietat base.

La clau per obtenir aquests resultats és la construcció de fibrats vectorials no trivials de rang dos com a extensió de dos fibrats de línia. En contra del que passa en altres varietats projectives, com per exemple en les varietats de Fano o en els espais projectius, on tota extensió escindeix i per tant no és possible aquesta construcció, en $\mathbb{P}^d$-fibrats és possible construir bones famílies de fibrats vectorials de rang dos, L-estables, donats per una extensió no trivial de dos fibrats de linia, essent L un divisor ample en X triat adequadament.

En la secció 5.3 il.lustrem, mitjançant una col.lecció d'exemples, com canvia l'espai de moduli $M_{X,L}(2; c_1, c_2)$ quan el divisor ample L travessa diferents parets. De manera planera, el següent resultat ens diu que hi ha una paret "crítica" W tal que si L la travessa, aleshores l'espai de moduli $M_{X,L}(2; c_1, c_2)$ és buit, i per contra, si L està en certa cambra propera a W, aleshores l'espai de moduli queda elegantment descrit. Per tal d'enunciar aquest resultat, ens cal introduir més notació, si més no, ens cal fer una descripció geomètrica de la situació de la qual partim en el Teorema. Donada una polarització $L \equiv aH + bF$, podem representar la classe de L com a un punt del pla de coordenades (a, b). El següent esquema ens dona una idea de les parets i cambres que apareixen en el resultat que segueix a continuació.



Teorema 5.2.12 Sigui X un $\mathbb{P}^d$-fibrat sobre una corba C i $0 << n \in \mathbb{Z}$. Es verifica el següent

(a) Per a tot $\overline{L}'' \in \mathcal{C}''$, l'espai de moduli $M_{\overline{L}''}(2; H, nHF)$ és buit.

(b) Per a tot $\overline{L} \in \mathcal{C}$, l'espai de moduli $M_{\overline{L}}(2; H, nHF)$ és un $\mathbb{P}^N$-fibrat sobre $Pic^0(C) \times Pic^0(C)$ sent $N := h^1 O_X(H - \pi^*\mathfrak{n}' - \pi^*\mathfrak{n}) - 1$ amb $\mathfrak{n}, \mathfrak{n}' \in Pic(C)$

divisors de grau n. En particular, és una varietat projectiva, llisa i irreductible de dimensió $h^1O_X(H - \pi^*\mathfrak{n}' - \pi^*\mathfrak{n}) + 2g - 1$ i té dimensió de Kodaira $-\infty$.

(c) Per a tot $\overline{L}' \in \mathcal{C}'$, l'espai de moduli $M_{\overline{L}'}(2; H, nHF)$ és un obert no buit de l'espai de moduli $M_{\overline{L}}(2; H, nHF)$ i

$$dim(M_{\overline{L}}(2; H, nHF) \setminus M_{\overline{L}'}(2; H, nHF)) = h^0\mathcal{E} + h^0O_C(\mathfrak{p}) + 2(g-1),$$

amb $\mathfrak{p} \in C$ un punt de C. En particular, $M_{\overline{L}'}(2; H, nHF)$ és una varietat quasi-projectiva, llisa i irreductible de dimensió

$$h^1O_X(H - \pi^*\mathfrak{n}' - \pi^*\mathfrak{n}) + 2g - 1$$

amb $\mathfrak{n}, \mathfrak{n}' \in Pic(C)$ de grau n i $Kod(M_{\overline{L}'}(2; H, nHF)) = -\infty$.

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