Universitat Autònoma de Barcelona

Topological Attractors Of Quasi-periodically Forced One-dimensional Maps

Zhaoyang Dong

Topological Attractors Of Quasi-periodically Forced One-dimensional Maps

Zhaoyang Dong

Director: Prof. Lluís Alsedà i Soler

October, 2018
Departament de Matemàtiques
Universitat Autònoma de Barcelona

Contents

Chapter 1

Introduction

This memoir is concerned with topological attractors of some quasi-periodically forced one-dimensional maps. The main aim of our study is to understand the states of the attractors by analyzing the mechanisms which rule the dynamics of the maps. Concretely we investigate two types of quasi-periodically forced one-dimensional families. The first type consists of two different quasi-periodically forced increasing systems. We present rigorous proofs for the states of their attractors. The second type of systems that we consider are those quasi-periodically forced S-unimodal systems. We propose the mechanism for their changes of periodicity according to the forced terms, which is based on elaborate analysis of the S-unimodal maps and is substantiated by numerical evidence.

Attractors are one of the main subjects of dynamical systems and chaos theory. They are invariant subsets of state spaces that the asymptotic motion of the points in their neighbourhood follows them. Hence the long-term evolution behaviours of dynamical systems are mainly represented by their attractors, and the knowledges of attractors are the key for the understanding of the whole system. There are many important and impressive results on attractors, such as the famous Lorenz's attractor [51], Hénon's attractor [36], Feigenbaum's attractor [22, 23]. They are all important strange attractors, and each of them stands for an important achievement in the field of dynamical systems. Here by strangeness it means that they own some fractal features in the geometry aspect. A chaotic attractor is the one which is "sensitive dependence on initial conditions", usually measured by a positive Lyapunov exponent. In early literature, a strange attractor often refer to the chaotic attractor, since all known attractors chaotic are with the strange feature together.

The motivation of our research is the problem of strange nonchaotic attractors. In 1984 Grebogi, Ott, Pelikan and Yorke [29] found examples of strange nonchaotic attractors in quasi-periodically forced skew product systems. The existence of strange nonchaotic attractor(SNA for short) interested mathematician and physicist who work in this field greatly. In recent decades, a lot of works were devoted to find and to study SNAs. A noticeable work among them is given by Keller, who in [44] present an elegant mathematical proof on the existence of SNA in a type of quasi-periodically forced systems. However, the concrete mechanism for the birth of SNAs still remains unclear today. In this memoir we study some quasi-periodically forced one-dimensional maps. The one-dimensional maps that we choose are either simple or well-studied already, we hope this can help us to learn the essence of their mechanism from the very beginning.

The main content of this memoir consist of three chapters. The basic notions and background are introduced in the second chapter. We start from presenting the elementary concepts of dynamical systems, orbits, ω-limit sets and attractors. Next we make a short summary on ergodic theory and Lyapunov exponent, which is the major measurement of the chaoticity of attractors. These materials are all standard, we include them so that this memoir are more self-contained.

The third section is devoted to an introduction of some fundamental and important issues about SNAs, which is the most important one of this chapter. SNAs typically occur in quasi-periodically forced dynamical systems. quasi-periodically forced system is a special form of the skew products that we present in the first subsection. We point out there that, in this memoir, we focus only on quasi-periodically forced one-dimensional maps, which are on cylinder $\mathbb{S}^1 \times \mathbb{R}$ of the form

$$\begin{cases} \theta_{n+1} = \theta_n + \omega \pmod 1, \\ x_{n+1} = f(\theta_n, x_n), \end{cases} \tag{1.1}$$

where $\theta \in \mathbb{S}^1$, ω is a fixed irrational real number and $\mathbb{R}$ represents the forced space which is called fibre of the skew product. The SNA problem on such systems consists of their attractors, with the chaoticity and the strangeness of the attractors.

In current literature, the terminology of SNA is still not uniform, even ambiguous. Hence in the second subsection, we summarize the paper [1] of Alsedà and Costa for their discussion on the definition of SNA. The two important examples of Grebogi, Ott, Pelikan and Yorke [29] and Keller [44] are presented in the third subsection, these examples help us to obtain an intuitive perspective on such attractors. The strangeness is the most confusing and difficult part on confirmation of an SNA, the last subsection is devoted to some arguments involved this problem. They are the results of Stark [74] on the regularity of invariant graphs in quasi-periodically forced one-dimensional systems, and the elaborate discussions of fractalization mechanism by Jorba and Tatjer [41].

Our research are presented in the next two chapters. In the third chapter we first explore shortly the general topological properties of the pinched closed invariant sets, which are of particular important for SNAs. By pinched it means that the intersection of this set with some fibre contains only one point. We prove that, the ω-limit set of pinched points is the unique minimal set in a pinched closed invariant set, and any continuous graph contained in a pinched closed invariant set must be invariant. A system itself is called pinched, if there is one fibre who is mapped to one point. The invariant set of a pinched system is certainly pinched. Theorem A, our first main result shows that, in a pinched system the ω-limit set of pinched points is the only minimal set which must be contained in every invariant sets. This implies the orbit of pinched points plays the crucial role in a pinched system, since all the other orbits can only go around it.

The special role of the pinched orbit in a pinched system is clearly presented in the two models we study next in this chapter. They are families with two parameters which are forced monotone increasing maps, with some concave or convex structures on fibre maps. We prove rigorously the states of their attractors, which are given by the main theorems in each of the last two sections. The important features on them are: for the non-pinched systems, the invariant curves as attractors exhibit the same type of bifurcations like the unforced one-dimensional maps, which implies that they have totally analogous dynamics only with the fixed points of one-dimensional maps being replaced by invariant curves; for the pinched systems, however, the corresponding bifurcations are totally disrupted by the pinched orbit in most of cases. In a pinched system, if the points is taken away to the position where the bifurcation should occur in the unforced one-dimensional map, then there is no any bifurcation in the forced system. Only when the pinched orbit is fixed steadily, that is, it does not change with a parameter, the bifurcation can happen. When such a bifurcation happens, it gives birth of an SNA.

The last chapter is devoted to the quasi-periodically forced S-unimodal maps. In this chapter we propose the mechanism of the change of the periodicity of their attractors, which works with respect to a parameter who controls the forcing term when the forced S-unimodal map is fixed. Precisely, with the increasing of forcing term, it happens a series of merging or collision between separated parts of a periodic attractor in a certain order generally. Thus the periodicity of the attractor changes accordingly in a particular way.

This mechanism of forced systems is based on the topological structures of the forced S-uniform maps. The similar merging and collision are also reported for S-unimodal maps in physical context for decades, but no mathematical research about them is known to us. In the first section we introduce these phenomena of S-unimodal maps for an intuitive impression. Such merging and collision are thought to be the reverse of some bifurcations of periodic orbits, we prove this is true later in this chapter. The facts that we present in the proof reveal the topological structures that rule the change of periodicity of the forced S-unimodal systems.

In the second section, we reveal the block structures of attractors for S-unimodal maps with the restrictive intervals defined by our own. We show that, for a periodic orbit of an S-unimodal maps, there is a set of intervals link with this orbit. These intervals are restrictive means that their union is an invariant set. Furthermore, the intersection of all restrictive interval is a nested invariant set who contains the attractor of this map. The nested restrictive intervals also have block structures which we present using the extension patterns. With restrictive intervals, we also give the criteria on the classification of the three types of topological attractors of S-unimodal maps.

With the results of the second section, we prove Theorem D of the reverse bifurcations of S-unimodal maps in the third section. It shows a reverse bifurcation happens when the innermost restrictive intervals becomes non-restrictive. It is just corresponding to the bifurcation which gives birth of the periodic orbit that these innermost restrictive linked. Moreover, in view of the transition between these pair of bifurcations, the n-fold map (n is the period of the restrictive intervals) on each of restrictive intervals is a full family. This fact explains the self-similarity structures of the transition of S-unimodal families.

The final section is devoted to the mechanism of forced S-unimodal maps, the main conclusion Claim E of this chapter. It is reasonable to see that, if the forced S-unimodal map is fixed, the bifurcations in S-unimodal families hardly happens. While with respect to the forced term, the bifurcations are just in general according to the order of the block structures of that forced S-unimodal map fixed. This mechanism is supported clearly by the numerical simulations.

Acknowledgements

First I would like to thank my tutor, Prof. Lluís Alsedà, for your enormous patience, endless tolerance and so many kind helps. There won't be this memoir without your instructions, suggestions and criticisms, form which I learned a lot.

I would like to all the people who helped me in these years, Leopoldo, Victor, Set, with many others. Also staff and coordinators of department, Dolors, Loli, Beatriz,...

Finally and forever, my family. My parents, Zhaoyan, and Qin Jiayi. For your supports on all these years.

Chapter 2

Preliminary and background

In this chapter we introduce the basic definitions and notions that we need throughout this thesis. The reader with a good background on dynamical systems, particularly on strange nonchaotic attractors, can skip this chapter and come back only when some of these are needed. This introduction is tried to keep the "minimum amount", starting with some fundamental and essential concepts in the study of dynamical systems. The purpose of these preliminaries is to make this work as self-contained as possible, so that it allows the reader to follow the thesis without consulting other books or articles frequently. The issue of Strange Nonchaotic Attractors(SNAs for short) is the motivation of this memoir, hence the contents are all close related on it.

More precisely, the first section covers basic definitions and notions about general dynamical system, including orbits, limit sets and attractors. In the second section we introduce some of basic measure theory and the concept of Lyapunov exponent, which are usually used as measurement of the chaoticity. The third section is devoted to the problem of SNAs, where we present the notions on quasi-periodically forced skew product systems, a short discussion on the definition of SNA, and the typical examples including the famous Keller's model. Its last subsection is focused on the strangeness of attractors. The noticeable works by Stark [74] and by Jorba and Tatjer [41] on fractalization mechanism are summarized.

2.1 Basic definition on dynamical systems

The contents of this section are some of the most fundamental concepts of dynamical systems. We start from the definition of a dynamical system, followed by the notion of the orbit of a point, which is the natural object for the study of dynamical systems. The efforts to understand the orbits lead to the concepts of the periodic orbit, the ω-limit set and the attractor of a system step by step.

2.1.1 dynamical Systems

A dynamical system is one whose state changes with time t under some deterministic law. All the possible states of the system form a set X, which is called the *state space*, or *phase space*. Mathematically, a state space can be a metric, topological space or manifold and so on. Any point of X is called a *state*.

According to the time variable t, there are two main types of dynamical systems: those for which the time variables are continuous ($t \in \mathbb{R}$), and those for which they are discrete ($t \in \mathbb{Z}$). A continuous dynamical system is usually described by a differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = F(x) \quad (t \in \mathbb{R})\,, \tag{2.1}$$

while discrete one is in the form of the iteration of a map, i. e.,

$$x_{t+1} = F(x_t) \quad (t \in \mathbb{Z}). \tag{2.2}$$

In each of the above situations, x represents the state of the systems and takes value in the state space X. The differential equation in (2.1), or the map F in (2.2) represents the law which determines uniquely the evolution behaviour of the system.

Moreover, in order to form a dynamical system, either the solutions of differential equation or the iteration of a map must have some group structure, that is, it must satisfy the following definition (see, for instance [43, 48]).

Definition 2.1.1. A *dynamical system* is a triple $(\mathrm{X}, \mathrm{T}, \mathrm{F})$, where

(1) X is a set of states (normally with some special structure);

(2) T is a set of times, such that $0 \in \mathrm{T}$ and for any $s, t \in \mathrm{T}$, $s + t \in \mathrm{T}$;

(3) $\mathrm{F}\colon \mathrm{X} \times \mathrm{T} \longrightarrow \mathrm{X}$ is a function satisfying the group property $\mathrm{F}^0 = \mathrm{Id}$, $\mathrm{F}^s \circ \mathrm{F}^t = \mathrm{F}^{s+t}$.

In the above definition, if the time variable t belongs only to the set of nonnegative real number or the nonnegative integer number, and the solution or the iteration of a map satisfies the corresponding property of item (3), the system (X, T, F) is called a *semi-dynamical system*. In this case, F needs not be invertible. Notice that semi-dynamical systems are often called dynamical systems in the literature by abuse of notation. In this memoir, we only work with discrete semi-dynamical systems, that is, with $\mathrm{T} = \mathbb{Z}^+ \cup \{0\}$. So we use (X, F) to denote such a system, and call it dynamical system whenever there is no possible ambiguity.

2.1.2 Orbits and ω-limit

The basic goal of the study of dynamical systems is to understand how the states of all points change with time under the action of given mathematical laws, that is, their *orbits* or *trajectories*.

Definition 2.1.2. The *(forward) orbit*, or *trajectory*, of a point $x \in \mathrm{X}$ is the set

$$\mathrm{Orb}(x) = \mathcal{O}^+(x) = \{F^n(x) : n \in \mathbb{Z}^+ \cup \{0\}\}.$$

In general, the orbit of a point can be a very complicated set. However, there may exist some simple ones among all the orbits, which play an important role in the study of the whole system.

Definition 2.1.3. A point x is a *periodic point of period* n if there is an $n \in \mathbb{N}$ such that $f^n(x) = x$, and $f^m(x) \neq x$ for each $m < n$. If $n = 1$, then x is called a *fixed point.* Moreover, if a periodic point has a neighbourhood such that all the points in this neighbourhood eventually approach to its orbit, then it is called an *attracting periodic point.* On the contrary, if all the points in some of its neighbourhood leave this neighbourhood under the action of f^n, that is, x is the only point which stays always inside this neighborhood, then it is *repelling.*

If a point x is a periodic point of period n, let the number $\lambda(x) = \frac{d}{dx} f^n(x)$. If $|\lambda(x)| < 1$, then x is an attracting one. On the other hand, if $|\lambda(x)| > 1$ then x is repelling. Whenever $|\lambda(x)| \neq 1$, the orbit of x is called *hyperbolic.*

It is clear that if x is a periodic point of period n, then $\mathrm{Card}(\mathrm{Orb}(x)) = n$. Since $\{f^n(x)\}$ is a repeating sequence of these n points, the behaviour of x is well-understood. For those points which are not periodic, it is useful to understand their limiting behaviour. This gives rise to the notion of ω-limit set.

Definition 2.1.4. Let (X, F) be a dynamical system, and $x \in \mathrm{X}$. The *ω-limit set* of x, denoted by $\omega(x)$, is the set of the limit points of $\mathrm{Orb}(x)$, that is, $\omega(x) = \{y :$ there is a subsequence $\{n_j\}$ of $\{n\}$ such that $f^{n_j}(x) \to y$ as $n_j \to \infty\}$.

ω-limit sets have particular significance in the study of dynamical systems, they are very important invariant sets in the systems. A set $A \subseteq X$ is *(forward) invariant* in a system F if $F(A) \subseteq A$. particularly, if $F(A) = A$, then A is called *strongly invariant*. Normally, we only clarify a strongly invariant set in time of necessary.

Obviously, any orbits will stay in an invariant set eventually since they enter it. Thus it forms a special system of it own. This display the special role of such set.

Remark 2.1.1. It is well-known (see for instance [67]) that the ω-limit set of a point has the following properties.

(i) For any $x \in$ X, $\omega(x) = \cap_{n \geq 0} \mathrm{Cl}(\mathrm{Orb}(f^n(x)))$;

(ii) For all $y \in \mathrm{Orb}(x)$, $\omega(y) = \omega(x)$;

(iii) For any point x, $\omega(x)$ is closed and forward invariant, that is, $f(\omega(x)) \subset \omega(x)$;

(iv) Moreover, if $\mathrm{Orb}(x)$ is contained in some compact subset of X, then $\omega(x)$ is nonempty, compact and (totally) invariant, that is, $f(\omega(x)) = \omega(x)$;

(v) If $D \subset X$ is closed and forward invariant and $x \in D$, then $\omega(x) \subset D$. In particular, if $y \in \omega(x)$, then $\omega(y) \subset \omega(x)$.

$\diamond$

Obviously, if x is periodic, then $\omega(x) = \mathrm{Orb}(x)$.

2.1.3 Definition of attractors

In the study of dynamical systems, the main attention is devoted to the eventual behaviour of most of points in the state space. The ω-limit set can give us this information about a point, but it is not enough for the whole system. For this reason we need some more global notion.

Notice that the orbit of a fixed or a periodic point has only finite points, so its dynamics is simple and well-understood. Moreover, if such a point is attracting, then its orbit plays an important role in the study of dynamical systems. Geometrically, we can say that the asymptotic motion of these points in the neighbourhood follows the orbit of this periodic point in state space. Thus, the dynamics of all these points are well-understood through the dynamics of this attracting periodic point. Generalizing this idea, we can see that, an invariant subset of the state space with such properties also plays the same role in the study of dynamical systems. This leads to the concept of attractors. For a concrete mathematical meaning of attractor, we adopt the definition given by Milnor in [58]. We should point out that, there are many other definitions of attractors in the literature. Milnor also makes a survey of some of them in his article.

Definition 2.1.5 (Milnor). Let (X, F) be a dynamical system where X is a smooth, compact manifold endowed with a measure μ equivalent to Lebesgue measure when it is restricted to any coordinate neighbourhood. A closed subset $A \subset X$ is called an *attractor* if it satisfies the following two conditions:

(1) The set $\varrho(A) := \{x : \omega(x) \subset A\}$ has strictly positive measure;

(2) there is no strictly smaller closed set $A' \subset A$ so that $\varrho(A')$ coincides with $\varrho(A)$ up to a set of measure zero.

The set $\varrho(A)$ is called *realm of attraction of* A, it can be defined for every subset of X. When it is open, it is called *basin of attraction of* A.

Remark 2.1.2. In the above definition, the first condition assures that there is some positive possibility, such that a randomly chosen point will be attracted to A, so the realm is visible in this sense. The second is a minimality condition, that is, every part of A should play an essential role. $\diamond$

In [58], Milnor also pointed out some properties of the attractor based on this definition, and proved the following result about the existence of attractor. With this result, it is often convenient to ensure that there is an attractor in a system.

Theorem 2.1.1. *Let S be a compact subset of X such that $\mu(S) > 0$ and $f(S) \subset S$, then S contains at least one attractor.*

In this spirit, in the topological space X, the (topological) attractor refers to a set with dynamical structure similar to the metric one.

Definition 2.1.6. A closed invariant set $A \subseteq X$ is called a *topological attractor* of f if
(i) $rl(A)$ is a set of second Baire category;
(ii) for any proper closed invariant subset $A' \subset A$, the set $rl(A) \setminus rl(A')$ is of second Baire category as well.

Here $rl(A) = \{x : \omega(x) \subseteq a\}$ is its"realm of attraction".

One can see from the definition of the attractors, the long-term evolution behaviours of dynamical systems are mainly represented by their attractors. Hence the knowledge of the dynamical behaviours of the attractors is the key for the understanding of the whole system, and the study of the attractors is one of the main subjects of dynamical systems and chaos theory. The improvement on this study is always closely related with the development of the whole field. In the passed half century, many impressive discoveries were presented. These results not only interested a great number of scientists, they even aroused the public's enormous enthusiasm of this field because of the beauty of their wonderful forms.

In 1963, Lorentz published his famous and historical paper "Deterministic Nonperiodic Flow" [51], and reported the Lorentz's attractor obtained from a very rough model simulating the atmospheric motion. In this paper, he rediscovered the phenomenon of "sensitive dependence on initial conditions" described by Poincaré in his book [63] more than half century ago. Just like Poincaré pointed out, the systems with this character always exhibit complicated behaviours, and predication is impossible. It is notable that such phenomena were found to exist in many systems with very simple mathematical forms, see for instance, May [54], Li and Yorke [49], where the term "chaos" was coined. Here in this context, when we refer to chaotic attractors, we mean that there is sensitive dependence on initial conditions. The chaoticity is given in terms of positive Lyapunov exponents.

Later than Lorentz, in 1971 Ruelle and Takens [70] suggested that the turbulent motion of a fluid could be explained in terms of strange attractors. For them, a strange attractor is a chaotic attractor. While in this memoir, by strangeness we mean that the attractors have nonelementary geometrical properties, such as noninteger fractal dimension, or nowhere differentiability. But at the early time in this field, almost all the important chaotic attractors own the features of strangeness in geometry structure. So in the early literature, when people spoke of "strange attractor", sometimes they spoke of the strangeness in geometry, sometimes they referred to the chaotic behaviour of the attractor, and sometimes both. As the study of this phenomenon was proceeded further, people realized that a chaotic attractor may not be strange or fractal in its geometry form. There is a simple example of a chaotic attractor which is a nicely smooth manifold (see [58, Appendix 3]). On the other hand, a strange attractor needs not be chaotic either. The earliest literature we found about this is the works of V.M. Millionščikov [56, 57] and R. E. Vinograd [76], in which there are some constructions of continuous flows containing such attractors. The term of Strange Nonchaotic Attractor(SNA for short) was introduced and coined by Grebogi, Ott, Pelikan and Yorke in 1984. In their paper [29], they gave out models of two and three dimensional systems, and proved both numerically and theoretically that the attractors are nonchaotic with geometry structure of nowhere differentiability. So far, the Strange Nonchaotic Attractors are reported to typically appear in the quasi-periodically forced skew product systems. In the remaining of this memoir, we will focus our study on this particular class of dynamical systems, so we give an special introduction of the SNAs and quasi-periodically forced systems with the third section. Before we this, we need some knowledges on ergodic theory and Lyapunov exponent.

2.2 Ergodic theory and Lyapunov exponents

When one investigates a complicated system, a more global point of view is usually very helpful. Ergodic theory provides the measure-theoretic approach to reveal the statistical properties of the systems, which is very necessary for the understanding of general behaviour of those complicated systems. We summarize the basic materials on it first in the second subsection. One of the important application of ergodic theory is that, it can be used to simplify the calculation of Lyapunov exponents. Lyapunov exponent is another important notion in the study of dynamical system, which is often used to indicate the degree of chaoticity. We discuss it at the last of this section. All these materials are quite standard, so we just present them briefly.

At the beginning of the study of dynamical systems, people tried to use strictly analytical tools to find the precise solutions, so that they can get the information of the orbits of the points in the system. But such methods can only be used to study particular solutions, and often work only locally. Whereas a more global picture of the system is often needed, particularly, for the study of long-term asymptotic behaviour and of its qualitative aspects. It is in the late nineteenth century, Henri Jules Poincaré introduced geometrical and topological methods into the study of dynamical systems (see [63]), which began the history of the modern theory of dynamical systems. The geometrical and topological approaches do not rely on explicit calculation of solutions, they are the tools to make the system visualized. The attractor in the preceding section is just a notion from this point of view.

In addition to the qualitative study of a dynamical system, the measure-theoretic approaches are also very utilized to overcome the difficulties that arise in using strict analysis, especially for those very complicated systems. People have some complicated systems could even behave with some probabilistic characters. Around 1900, Gibbs suggested looking at what happens to subsets of state space, for instance, the probability that a subset is in another subset of the space at a given time t, and the average time that the subset spends inside of the other one, so one can discover statistical properties of dynamical systems. Such questions motivate the type of study undertaken in ergodic theory. In general, ergodic theory is the study of transformations and flows from the viewpoint of recurrence properties, mixing properties, and other global dynamical properties connected with asymptotic behaviour. In this section, we introduce some essential notions of ergodic theory. All the materials here in this section are very standard in ergodic theory, and come from the textbooks of Walters [77] or Mañe [53].

We start with some basic definitions of measure theory.

Definition 2.2.1. Let X be a set, a σ-algebra $\mathcal{B}$ over X is a nonempty collection of subsets of X, which is closed under complementation and countable unions of its members. That is, the following properties hold:

(1) If $B \in \mathcal{B}$, $X \setminus B \in \mathcal{B}$.

(2) If B_n is a sequence of elements of $\mathcal{B}$, then $\bigcup B_{i \in \mathbb{N}} \in \mathcal{B}$.

From the definition, it follows that X and the empty set are in $\mathcal{B}$, and that the σ-algebra is also closed under countable intersections. Moreover, the intersection of σ-algebras is also a σ-algebra. One can also see that, for any collection of subsets of X, there is always a σ-algebra containing it, namely, the power set of X. Thus, if $\mathcal{S}$ is a collection of subsets of X, by taking the intersection of all σ-algebras containing $\mathcal{S}$, we obtain the smallest such σ-algebra, which is called the *σ-algebra generated by $\mathcal{S}$*.

The main use of σ-algebras is in the definition of measures on X. We give this definition as below.

Definition 2.2.2. If $\mathcal{B}$ is a collection of subsets of X which forms a σ-algebra, then the pair $(X, \mathcal{B})$ is called a *measurable space*. A measure on the measurable space $(X, \mathcal{B})$ is a set function $\mu : \mathcal{B} \to \mathbb{R}^+$ satisfying the following conditions:

(1) $\mu(\varnothing) = 0$,

(2) if $B_1, B_2, \ldots, B_m, \ldots$ is a countable collection of pairwise disjoint elements of $\mathcal{B}$, $\mu(\bigcup_{i=1}^{\infty} B_i) = \Sigma_{i=1}^{\infty} \mu(B_i)$.

The triple $(X, \mathcal{B}, \mu)$ is then called a *measure space*.

In addition, if $\mu(X) = 1$, then we say that μ is a *probability measure*. Most of the times, we work with probability measures on finite dimensional topological spaces equipped with a *Borel* σ-algebra, that is, the σ-algebra generated by the topology.

Definition 2.2.3. Let $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$ be two measurable spaces. A transformation $T : (X_1, \mathcal{B}_1) \longrightarrow (X_2, \mathcal{B}_2)$ is a *measurable transformation* if $T^{-1}(B) \in \mathcal{B}_1$ whenever $B \in \mathcal{B}_2$. We denote it briefly by $T : X_1 \longrightarrow X_2$ as long as the σ-algebras are clear for us.

Moreover, a measurable transformation $T : (X_1, \mathcal{B}_1, \mu_1) \longrightarrow (X_2, \mathcal{B}_2, \mu_2)$ is *measure-preserving* or *invariant* if $\mu_1(T^{-1}(B)) = \mu_2(B)$ for every $B \in \mathcal{B}_2$.

In the study of dynamical systems and ergodic theory, we are mainly interested in self-transformations of probability measure spaces. That is, $(X_2, \mathcal{B}_2, \mu_2) = (X_1, \mathcal{B}_1, \mu_1)$. According to this situation, we have the following definition.

Definition 2.2.4. A measure μ is called T-invariant for a transformation $T : (X, \mathcal{B}, \mu) \longrightarrow (X, \mathcal{B}, \mu)$ if $\mu(T^{-1}(B)) = \mu(B)$ whenever $B \in \mathcal{B}$.

The set of all invariant probability measures on X is denoted by $\mathcal{M}_{inv}(T)$, or $\mathcal{M}_{inv}$ briefly. The Krylov-Bogolubov Theorem guarantees that there always exists an invariant measure for a compact topological space with a continuous map on it, so $\mathcal{M}_{inv}$ is non-empty. Moreover, the set $\mathcal{M}_{inv}$ is convex and compact with the weak* topology.

It may be difficult to check directly from the definition whether a measure is invariant or not, since we usually do not have explicit knowledge of all members of $\mathcal{B}$. The following proposition is a useful tool to help us to simplify this work.

Proposition 2.2.1 (Characterization of invariant measures). *A measure $\mu \in \mathcal{M}_{inv}(T)$ if and only if $\int_M g \circ T \mathrm{d}\mu = \int_M g \mathrm{d}\mu$ for all $g \in L^1(X, \mathcal{B}, \mu)$.*

In a dynamical system, we usually have a space X with some structure on it, and a transformation T of X which preserves this structure, for instance, a topological space and a continuous map on it. To apply the measure theoretic methods to the study of such system, we need an invariant measure for the transformation T which acts "nicely" with respect to the structure on X. Ergodic measure is just such a concept.

Definition 2.2.5. An invariant probability measure $\mu \in \mathcal{M}_{inv}$ is called *ergodic* if whenever $T^{-1}B = B$ for some $B \in \mathcal{B}$, then either $\mu(B) = 0$ or $\mu(B) = 1$. We denote the set of ergodic measures by $\mathcal{M}_{erg} = \{\mu \in \mathcal{M}_{inv} : \mu \text{ is ergodic}\}$.

Remark 2.2.1. Ergodicity is a concept of irreducibility for the given system from the measure theory point of view. Since, if $T^{-1}(B) = B$ for some $B \in \mathcal{B}$, then $T^{-1}(X \backslash B) = X \backslash B$, and we can study the action of T on B and $X \backslash B$ separately. If $0 < \mu(B) < 1$, this would simplify the study of the given system into two proper separated parts. We can see that this cannot happen to a system with ergodic measure. $\diamond$

We have known that the Krylov-Bogolubov Theorem guarantees that $\mathcal{M}_{inv}$ is nonempty for a continuous map on a compact topological space, while the ergodic measures are precisely those extremal points of $\mathcal{M}_{inv}$, so $\mathcal{M}_{erg}$ is also non-empty (refer to [77, 64] for details). Moreover, this fact makes the following definition allowable.

Definition 2.2.6. If a transformation T has a unique invariant measure, then this measure must be ergodic. Such a transformation T is called *uniquely ergodic*.

A classic example of this is the irrational rotation on $\mathbb{S}^1$ with the Haar-Lebesgue measure.

Analogous to Proposition 2.2.1, ergodicity can be characterized in terms of properties of functions too. We are going to see that this characterization is very useful in application of ergodic theorem.

Proposition 2.2.2 (**Characterization of ergodic measures**). *The following conditions are equivalent:*

(1) μ *is* T*-ergodic;*

(2) If a measurable function f *is* T*-invariant, that is,* $f \circ T = f$*, then* f *is constant almost everywhere;*

(3) If a measurable function f *is* T*-invariant almost everywhere, then* f *is constant almost everywhere.*

The best known and major result in ergodic theory is the Birkhoff Ergodic Theorem proved by Birkhoff in 1931.

Theorem 2.2.3 (**Birkhoff Ergodic Theorem**). *Let* $T : X \to X$ *be a transformation on a measurable space* $(X, \mathcal{B})$*, and* $m \in \mathcal{M}_{inv}(T)$*. For any* $f \in L^1(X, \mathcal{B}, m)$*, there is an integrable function* f^* *such that*

$$\frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) \longrightarrow f^*(x), \quad \text{as } n \to +\infty,$$

for almost all points $x \in X$ (with respect to m). Moreover, f^* is T-invariant, and if $m(X) < \infty$,

$$\int_X f^* \mathrm{d}m = \int_X f \mathrm{d}m.$$

Remark 2.2.2. By Proposition 2.2.2, if the invariant measure m is also ergodic, then f^* must be constant m-almost everywhere. So if $m(X) < \infty$, then $f^* = \frac{1}{m(X)} \int_X f \mathrm{d}m$ m-almost everywhere. Consequently, if m is ergodic, the "temporal averages" of f,

$$\frac{1}{n} \sum_{i=0}^{n-1} f(T^i x)$$

and the "spatial averages"

$$\frac{1}{m(X)} \int_X f \mathrm{d}m$$

coincide. ◇

Another important ergodic theorem is the so-called Subadditive Ergodic Theorem. It is not only utilized in the proofs of a number significant ergodic theorems, but also very pregnant in many applications.

Theorem 2.2.4 (Subadditive Ergodic Theorem). *Let $T : X \to X$ be a transformation on a measurable space $(X, \mathcal{B})$, and $m \in \mathcal{M}_{inv}(T)$. If $\{\varphi_n\}$ is a sequence of integrable functions satisfying*

$$\varphi_{n+m}(x) \leq \varphi_n(x) + \varphi_m(T^n x),$$

then for m-almost every x,

$$\lim_{n\to\infty} \frac{1}{n} \varphi_n(x) = \overline{\varphi}(x),$$

where the function $\overline{\varphi}$ is T-invariant and integrable. Moreover, if m is ergodic, $\overline{\varphi}$ is constant m-almost everywhere.

A notion which is closely related with ergodic theorems is the Lyapunov (characteristic) exponent. The definition of Lyapunov exponents goes back to the dissertation of Lyapunov in 1892 (see [67]). It gives the averaged rate of exponential divergence (or convergence, according as it is positive or negative) from perturbed initial conditions. Nowadays, it is commonly used as the measure of the degree of chaoticity of a system. The precise definition is as follows:

Definition 2.2.7. Let $f : M \to M$ be a differentiable map on a manifold of dimension k. For each $(x, v) \in M \times T_xM$ the *Lyapunov exponent* of (x, v) is defined as

$$\lambda(x, v) := \lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n v\|,$$

where $\|\cdot\|$ is a norm on the tangent space induced by a Riemannian metric on M.

Note that the above definition of Lyapunov exponent measures the exponential divergence rate of nearby trajectories in the v-direction. In general, the rates of separation can be different for different orientations of initial separation vectors. Hence, there is a whole spectrum of Lyapunov exponents, the number of them is equal to the number of dimensions of the phase space. This fact is just what the famous Oseledets' Multiplicative Ergodic Theorem shows. This theorem is a very important theorem on the existence and properties of Lyapunov exponents, which is given by Oseledets in 1968 [61]. One can also refer to [69] for this theorem.

Theorem 2.2.5 (**Oseledets' Multiplicative Ergodic Theorem**). *Let $(M, \mathcal{B})$ be a measurable space, where M is k-dimensional, $f : M \longrightarrow M$ a measurable transformation and μ an f-invariant probability measure. Then for μ-almost every $x \in M$ there exists $s(x) \leq n$ and numbers*

$$-\infty \leq \lambda_1(x) \leq \ldots \leq \lambda_{s(x)}(x)$$

with a sequence of subspaces

$$\{0\} = V_0(x) \subsetneq V_1(x) \subsetneq \ldots \subsetneq V_{s(x)}(x) = T_xM$$

such that

(1) $\lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n v\| = \lambda_i(x)$ *for every* $v \in V_i(x)\backslash V_{i-1}(x)$;

(2) $D_x f V_i(x) = V_i(f(x))$;

(3) $\lambda_i(x)$, $s(x)$ *and* $V_i(x)$ *are* μ*-measurable;*

(4) if we denote $E_i(x) = V_i(x)\backslash V_{i-1}(x)$ *and* $k_i(x) = \dim E_i(x)$ *for every* $i = 1, \ldots, s(x)$, *then*

$$\lim_{n\to\infty} \frac{1}{n} \log |\det D_x f^k| = \sum_{i=0}^{s(x)} k_i(x)\lambda_i(x)$$

and

$$T_xM = E_1(x) \oplus \ldots \oplus E_{s(x)}(x).$$

Moreover, if the measure μ *is* f*-ergodic, then* $\lambda_i(x)$*,* $s(x)$ *and* $V_i(x)$ *do not depend on* x*.*

Remark 2.2.3. We should point out that it is useful to use an axiomatic definition of Lyapunov exponents for the study their properties and the proof of this theorem. For this concept, see [6]. $\diamond$

It is common to just refer to the largest one of these exponents, i.e. to the *Maximal Lyapunov exponent*, because it determines the predictability of a dynamical system. The maximal Lyapunov exponent at a point $x \in M$ is normally given by

$$\lambda_{max}(x) := \limsup_{n\to\infty} \frac{1}{n} \log \|D_x f^n\|,$$

where $\|\cdot\|$ is a matrix norm compatible with the vector norm defined in the tangent space induced by the Riemannian metric on M. A positive maximal Lyapunov exponent is usually taken as an indication that the system is chaotic.

2.3 Quasi-periodically forced skew products and SNAs

This section is devoted an exclusive introduction of the quasi-periodically forced skew product systems. Found SNAs are usually pinched closed invariant sets in such systems, so we introduce some of their essential ingredients in the first subsection. Next we turn to the SNAs issues. A precise definition of SNAs and arguments involved are given in the second subsection, this will help us to clarify unnecessary confusion on this notion. The third subsection contains two concrete examples of SNAs by Grebogi, Ott, Pelikan and Yorke [29] and Keller [44]. In the last subsection, it covers two noticeable papers on the strangeness problem. They are on the regularity of invariant graphs by Stark [74], and the works on the fractalization mechanism by Jorba and Tatjer [41].

2.3.1 Quasi-periodically forced skew products

The essence of a forced systems is to have two or more equations which are coupled together in some way. This type of coupling is known as a *skew product*. It is written in the form

$$\begin{cases} x_{n+1} = f(x_n), \\ y_{n+1} = g(x_n, y_n), \end{cases} \tag{2.3}$$

where $x_n \in \mathrm{X}$ represents the state of the forcing (driving) system, and $y_n \in \mathrm{Y}$ represents the state of the forced (driven) system, influenced by the dynamics of the forcing system. Another way of denoting system (2.3) is by a map $F : X \times Y \longrightarrow X \times Y$, with

$$F(x, y) = (f(x), g(x, y)). \tag{2.4}$$

The space Y is called the *fibre* of the skew product, and the space X is called the *base*. f is a map of X which makes (X, f) a dynamical system in its own way, its dynamical behaviour influence the combined dynamics of x and y given by the skew product (2.3). The case that (X, f) induces a periodic influence on the skew product has been extensively studied for a long time. On the contrary, when this influence is quasi-periodic — a quasi-periodically forced system — it is in general much more poorly understood yet. Such kind of systems received more and more attention in the last three decades, because of its close relation with the occurrence of SNAs.

In this memoir, we focus only on quasi-periodically forced one-dimensional skew products. Precisely, all the bases of such systems are given by a simple one-dimensional quasi-periodically system, an irrational rotation in $\mathbb{S}^1$. For those fibres of the skew product, they are also taken as one-dimensional space X (usually $\mathbb{R}$). Hence the state space of the whole system is a cylinder $\mathbb{S}^1 \times \mathbb{R}$ if $X = \mathbb{R}$, and a system is of the form

$$\begin{cases} \theta_{n+1} = \theta_n + \omega \pmod 1, \\ x_{n+1} = f(\theta_n, x_n), \end{cases} \tag{2.5}$$

where $\theta \in \mathbb{S}^1$, ω is a fixed irrational real number and X represents the forced space. Physically, this class of dynamical systems models physical systems subject to external quasi-periodic perturbations of frequency ω.

In such a quasi-periodically forced dynamical system, due to the irrational rotation of the forcing system, there cannot be any fixed or periodic points. The simplest invariant closed subset, and hence the attractor, can only be the graph of a map from the base $\mathbb{S}^1$ to the fibre X, that is, the graph of a map $\varphi : \mathbb{S}^1 \longrightarrow X$. So it can be denoted by $\mathcal{A} = \{(\theta, \varphi(\theta) : \theta \in \mathbb{S}^1\}$. Moreover, this graph $\mathcal{A}$ must be *invariant* under the action of system (2.5), that is, $\varphi(\theta + \omega \pmod 1) = f(\theta, \varphi(\theta))$ for any $\theta \in \mathbb{S}^1$. For the existence of such a graph as an attractor, a simple condition can be obtained from the Hirsch-Pugh-Shub stability theory [38, 39] (see also [67]). We may also call it an *invariant curve* when we focus on its geometric structure.

With respect to the ergodicity of the skew product dynamical system (2.3), it can be shown that the invariant and ergodic measures of the whole system are related with the ones for the system defined on the base space. This is given by the next proposition, the reader can see [10] for its proof.

Proposition 2.3.1. *In a skew product dynamical system $F : X \times Y \longrightarrow X \times Y$ given by $F(x, y) = (f(x), g(x, y))$, if m is F-invariant, then $m \circ \pi^{-1} \in \mathcal{M}_{inv}(f)$, where $\pi : X \times Y \longrightarrow X$ is the projection over the first component. Furthermore, if m is F-ergodic then $m \circ \pi^{-1}$ is f-ergodic.*

Remark 2.3.1. In most of the literature of the study of quasi-periodically forced dynamical systems (2.5), people use the notion of the horizontal and vertical Lyapunov exponents, which correspond the two component directions, that is, the θ and the x directions respectively. They claim that the horizontal one is zero a.e. for any invariant measure due to the constant rotation on $\mathbb{S}^1$. While for every point $z = (\theta, x) \in M$, the vertical one, $\lambda_V(z) = \lambda(z, (0, 1)^t)$, is given by

$$\lambda_V(z) = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\partial x_n}{\partial x} \right| .$$

Many authors believe that they are indeed the Lyapunov exponents of such systems, but it may not be the case. In fact, it is not an easy problem like it looks, and we should be careful on this assertion. For a discussion of the problem on Lyapunov exponents, see [1] and references therein. ◇

2.3.2 A definition of Strange Nonchaotic Attractor

The existence of Strange Nonchaotic Attractors interested mathematicians and physicists working on dynamical systems greatly. Following the work of Grebogi et al., there have been lots of papers related with SNAs. In those papers, some authors try to report the existence of SNAs in some models by numerical experiments or by theoretical proof (see [19, 7, 26]), some try to explain the mechanisms of the creation of SNAs (see, for instance, [25, 35, 42, 45, 46, 60]). But this problem is still far from being solved now. Most of the reports of the existence of SNAs are only based on rough numerical experiments and are proved to be smooth curves by further research recently. Moreover, since there is no a common precisely formulated definition of an SNA, even the question of what an SNA is is still a problem. The authors use their own intuitive idea on what an SNA is. To clarify this notion, we first give a brief introduction of the definition of SNAs in this subsection.

Here we adopt the definition given by Sara Costa in her master thesis [10](see also [1]). This definition is the most common one in the literature, but is not the only one. There are several other definitions known in the literature, for instance, using Hausdorff dimension to define the strangeness in [11, 66]. For the details and the relation of these definitions, we refer the reader to Sara's thesis, in which she investigated this problem comprehensively and made an integrated survey about it.

Definition of nonchaoticity: Let $\mathcal{A}$ be an attractor and $\varrho(\mathcal{A})$ be its realm of attraction. The attractor $\mathcal{A}$ is said to be *nonchaotic* if the set of points in the realm of attraction whose maximal Lyapunov exponent is positive, that is,

$$\left\{ x \in \varrho(\mathcal{A}) : \limsup_{n\to\infty} \frac{1}{n} \log \parallel D_x f^n \parallel > 0 \right\}$$

has zero Lebesgue measure.

Remark 2.3.2. The reason that we use such a condition is to guarantee the nonchaoticity is Lebesgue observable. Notice that we use lim sup to compute the maximal Lyapunov exponent, it must exist for every point $x \in X$. Then the observability of the nonchaoticity is guaranteed by the fact that $\varrho(\mathcal{A})$ has positive Lebesgue measure from the definition of the attractor. ◇

Definition of Strangeness: For a quasi-periodically forced system of the form (2.5), an attractor, which is given by the invariant graph of a map φ from the forcing space to the forced space, is called to be strange if it is neither finite nor piecewise differentiable.

2.3.3 Examples of Strange Nonchaotic Attractors

To date, the model given by Grebogi et al. is still the most classic one, and the works on the rigorous proofs of the existence of SNA are still very few. The best results that we known on the proof of such problem is given by Keller [44] and Bezhaeva and Oseledets [7] separately, based on a generalized model of Grebogi et al.. In this subsection, we have a brief look at the model of Grebogi et al. and Keller to make the abstract notion more clear.

Example 2.3.3 (Grebogi et al.). Consider the dynamical system

$$\begin{cases} \theta_{n+1} = \theta_n + \omega \pmod 1, \\ x_{n+1} = f(\theta_n, x_n) = 2\sigma\cos(2\pi\theta)\tanh(x) \end{cases} \tag{2.6}$$

where $\omega \in \mathbb{R}\setminus\mathbb{Q}$. The authors take $\omega = \frac{\sqrt{5}-1}{2}$, the golden mean, for numerical computations. ◇

There are two Lyapunov exponents in this system. The authors claim that the horizontal one (according to the θ direction) is always zero, and the vertical one (according to the x direction) is

$$h = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \left| \frac{\partial}{\partial x} f(\theta_k, x_k) \right|.$$

Obviously, the θ-axis, i.e., $x = 0$ is invariant under the map of system (2.6). Whether it is an attractor or not is determined by its stability. Note that two orbits on $x = 0$ maintain a constant separation, thus if $h > 0$ for the $x = 0$ orbits, the nearby points can only diverge from the θ-axis, so it is unstable and cannot be an attractor.

To calculate h for the $x = 0$ orbit, applying the Birkhoff Ergodic Theorem for m-a.e. θ and $x = 0$, the authors obtain that the vertical Lyapunov exponent is

$$\lambda_V(\theta, 0) = \int_{\mathbb{S}^1} \log \left| \frac{\partial}{\partial x} f(\theta, 0) \right| d\theta = \log |\sigma|.$$

Consider for the parameter $|\sigma| > 1$, $x = 0$ is not an attractor. On the other hand, since $|x_n| \leq 2|\sigma| < \infty$ for every $n \in \mathbb{N}$, there must exist an attractor because the orbit of any point is in this compact subset of the state space. Moreover, the measure on the attractor generated by an orbit is uniform in θ, because of the ergodicity in θ. The authors note that in this system, for any point $p = (1/4, x)$ or $p = (3/4, x)$, there must be $f(p) = 0$. So the attractor must contain the points $(1/4+\omega, 0)$ and $(3/4+\omega \pmod 1, 0)$ and must not contain any other point in $\theta = 1/4+\omega$ and $\theta = 3/4+\omega \pmod 1$. Consequently, every $(1/4 + k\omega \pmod 1, 0)$, $(3/4 + k\omega \pmod 1, 0)$ for $k \in \mathbb{N}$ belong to the attractor. Hence there is a subset of points which is dense both in the attractor and in $x = 0$, which is not an attractor itself.

The authors draw the picture of the attractor for $\sigma = 1.5$ (see Figure 2.1) and calculate the vertical Lyapunov exponent by numerical method. It is seen from the picture of the attractor that there are points off $x = 0$ as expected, and the vertical Lyapunov exponentis $h \approx -1.059$. So the attractor is an example of Strange Nonchaotic Attractor.

To verify that h must be negative so that the attractor is indeed nonchaotic, the authors consider the points that $x \neq 0$. Note that the function tanh is increasing in $\mathbb{R}$, concave in $(0, \infty)$ and convex in $(-\infty, 0)$, so

$$0 \leq \frac{\mathrm{d}}{\mathrm{d}x} \tanh x \leq \frac{\tanh x}{x},$$

for every $x \neq 0$. That is,

$$\left| \frac{\partial}{\partial x} f(\theta, x) \right| \leq \left| \frac{f(\theta, x)}{x} \right|$$

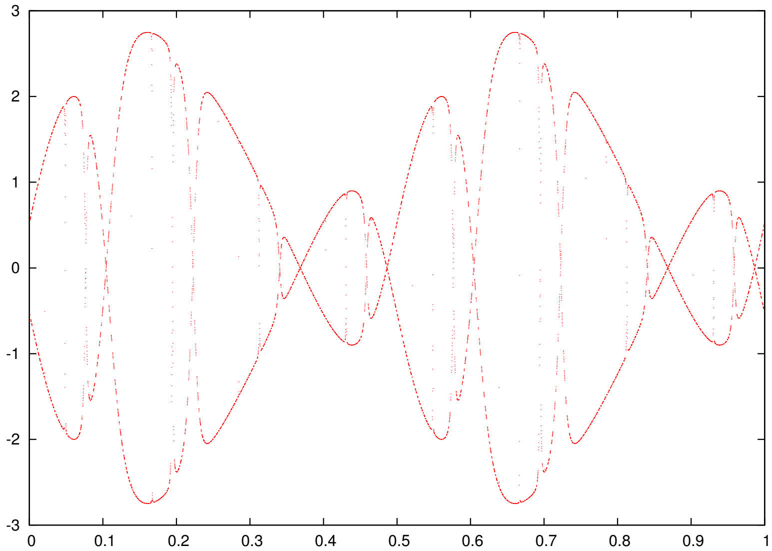


Figure 2.1: SNA of the model of Grebogi et al..

for all $\theta \in \mathbb{S}^1$ and every $x \neq 0$. In particular,

$$\left|\frac{\partial}{\partial x} f(\theta_n, x_n)\right| \leq \left|\frac{f(\theta_n, x_n)}{x_n}\right| = \left|\frac{x_{n+1}}{x_n}\right|.$$

From this inequality, the authors obtain

$$\begin{aligned}
\lambda_V(\theta, x) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log\left|\frac{\partial}{\partial x} f(\theta_k, x_k)\right| \\
&\leq \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log\left|\frac{x_{k+1}}{x_k}\right| \\
&= \lim_{n\to\infty} \frac{1}{n} (\log|x_n| - \log|x_0|) \\
&\leq \lim_{n\to\infty} \frac{1}{n} \log 2|\sigma| = 0
\end{aligned}$$

for every $\theta \in \mathbb{S}^1$ and $x \neq 0$. So the vertical Lyapunov exponent $\lambda_V(\theta, x)$ is nonpositive for every $x \neq 0$ and m-a.e. $\theta \in \mathbb{S}^1$. Since in the attractor, those points who are on $x = 0$ form only a zero measure subset, this assertion of SNA is valid.

Example 2.3.4 (**Keller**)**.** Later in 1996, Keller gave out an elegant analytical proof for the existence of SNA in a generalized model. The model that he considered is

$$\begin{cases} \theta_{n+1} = \theta_n + \omega \pmod 1, \\ x_{n+1} = f(x_n) g(\theta_n) \end{cases} \tag{2.7}$$

where $\omega \in \mathbb{R}\backslash\mathbb{Q}$ as before; $f : [0, \infty) \longrightarrow [0, \infty)$ is $\mathcal{C}^1$, bounded, increasing and strictly concave, moreover $f(0) = 0$; and $g : \mathbb{S}^1 \longrightarrow [0, \infty)$ is continuous. Thus it corresponds to the model of Grebogi et al. by changing the quasi-periodically forced map defined on $\mathbb{S}^1 \times [0, \infty)$ and replacing $\cos(2\pi\theta)$ by $|\cos(2\pi\theta)|$. ◇

Keller shows the existence of an SNA by studying the properties of the invariant graph of the given model (2.7). He successfully constructs a decreasing sequence of continuous functions and proves that the attractor of this model is the graph of a map $\psi : \mathbb{S}^1 \longrightarrow [0, \infty)$ to which that sequence of functions converges. His complete theorem is as follows:

Theorem 2.3.2 (**Keller**). *Let us consider the two-dimensional discrete dynamical system $T : \mathbb{S}^1 \times [0, \infty) \longrightarrow \mathbb{S}^1 \times [0, \infty)$ given by*

$$T(\theta, x) = (\theta + \omega, f(x) \cdot g(\theta))$$

where $\omega \in \mathbb{R}\backslash\mathbb{Q}$; $f : [0, \infty) \longrightarrow [0, \infty)$, bounded, increasing, strictly concave and $f(0) = 0$; and $g : \mathbb{S}^1 \longrightarrow [0, \infty)$ is continuous. Then there is an upper semi-continuous function $\varphi : \mathbb{S}^1 \longrightarrow [0, \infty)$ with an invariant graph such that:

(1) $\lim_{n\to\infty}(1/n)\sum_{k=0}^{n-1} |x_k - \varphi(\theta_k)| = 0$ *for m-a.e. $\theta \in \mathbb{S}^1$ and all $x > 0$, where m is the Lebesgue measure on $\mathbb{S}^1$. In particular, the Lebesgue measure on $\mathbb{S}^1$ "lifted" to the graph of φ is a SRB (Sinai-Ruelle-Bowen) measure for T, i.e.,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \nu(T^k(\theta, x)) = \int_{\mathbb{S}^1} \nu(\theta, \varphi(\theta)) \mathrm{d}\theta$$

for all $\nu \in \mathcal{C}(\mathbb{S}^1 \times [0, \infty))$ and for a.e. $(\theta, x) \in \mathbb{S}^1 \times [0, \infty)$.

Define

$$\lambda_\varphi = \int_{\mathbb{S}^1} \log g(\theta) \mathrm{d}\theta + \int_{\mathbb{S}^1} \log f'(\varphi(\theta)) \mathrm{d}\theta,$$

and consider the parameter

$$\sigma := f'(0) \exp\left(\int_{\mathbb{S}^1} \log g(\theta) \mathrm{d}\theta\right).$$

(2) *If* $\sigma \leq 1$*, then* $\varphi \equiv 0$ *and* $\lambda(\theta, x) = \lambda_\varphi = \log \sigma$ *for* m*-a.e.* $\theta \in \mathbb{S}^1$ *and each* $x \geq 0$*.*

(3) *If* $\sigma > 1$*, then* $\lambda(\theta, x) = \lambda_\varphi < 0$ *for* m*-a.e.* $\theta \in \mathbb{S}^1$ *and all* $x > 0$*. The set* $\{\theta : \varphi(\theta) > 0\}$ *has full Lebesgue measure. Furthermore,*

(3.1) *if* $g(\hat{\theta}) = 0$ *for at least one* $\hat{\theta} \in \mathbb{S}^1$*, then the set* $\{\theta : \varphi(\theta) > 0\}$ *is at the same time meagre and* φ *is* m*-a.e. discontinuous.*

(3.2) *if* $g(\theta) > 0$ *for all* $\theta \in \mathbb{S}^1$*, then* $\varphi(\theta) > 0$ *for all* $\theta \in \mathbb{S}^1$*. In this case,* φ *is continuous, and if* g *is* $\mathcal{C}^1$*, then so is* φ*.*

(4) *If* $\sigma \neq 1$*, then* $|x_n - \varphi(\theta_n)| \to 0$ *exponentially fast for* m*-a.e.* $\theta \in \mathbb{S}^1$ *and each* $x > 0$*.*

The idea of the proof is the following: taking a horizontal line which is higher than the upper bound of $f \cdot g$ and iterating it, the monotonicity of the model ensures that there exists a limit of this decreasing sequence of continuous graphs, which is an upper semicontinuous graph. The uniqueness of this semicontinuous graph and its attraction for all upper points are the consequence of the concavity of f. This semicontinuous graph, when $x = 0$ is attracting with the nonpositive vertical Lyapunov exponent, is just the invariant graph $x = 0$. Otherwise, it must be $x > 0$ almost everywhere.

The interesting case is in (3.1). The upper semi-continuous function φ in the theorem is m-a.e. discontinuous, while the closure of the graph of φ contains the circle $\{(\theta, x) : x = 0\}$, and it is the ω-set of m-a.e. points in the state space, so it is a strange attractor. Moreover, the vertical Lyapunov exponent of the points in this graph is $\lambda_\varphi < 0$, this shows that the closure of this graph is an SNA.

2.3.4 Regularity, fractalization and strangeness

In this subsection we summarize related results on the strangeness of invariant curves in quasi-periodically forced systems, which are from two papers by Stark [74] and Jorba and Tatjer [41]. The works of Stark give some sufficient conditions on the smoothness of invariant curves in skew product systems according the Lyapunov exponents. Jorba and Tatjer point out that, even a smooth invariant curve may get extremely wrinkled during the process that it goes close to the critical value 0 of its Lyapunov exponent. This process is called as fractalization, it reveals the difficulties on judging the strangeness of the curve in such situation. At last, we mention the most reliable and easy ways on deciding the continuity of an invariant curve.

There have been a lot of studies of these objects for several kinds of quasi-periodically forced dynamical systems (see, for instance, [16, 17, 18, 35, 42, 46, 68] and references therein). In these papers, the authors tried to report the existence of SNAs and to characterize them through several kinds of properties of topological, spectral and dimensional nature as well as other ones. However, until now, rigorous mathematical results are still scarce. Statements on the existence of an SNA in some papers are just based on very rough numerical evidences, and turned to be wrong in later research. For this reason, it is necessary and helpful to obtain a better understanding on the properties of the attractors in the quasi-periodically forced skew product systems first. Here we only mention the results related close to our subsequent investigations, and omit all the details on the proofs. Both the papers are elaborate and technological, which cover more results we present here. The interested reader can refer to them directly.

Regularity of invariant curves In [74] Stark study the regularity of invariant curves in general skew product systems. Particularly for quasi-periodically forced one-dimensional systems, he proves (more than) that a continuous invariant graph with a negative vertical Lyapunov exponent is as smooth as the fibre function of the system. This is the theorem below.

In a quasi-periodically forced dynamical system of the form

$$\begin{cases} \theta_{n+1} = \theta_n + \omega \pmod 1, \\ x_{n+1} = f(\theta_n, x_n), \end{cases} \tag{2.8}$$

with $\theta \in \mathbb{S}^1$, ω is irrational, and $x \in X$. We have,

Theorem 2.3.3 (Stark). *Suppose that Φ is a continuous invariant graph of (2.8) such that its largest Lyapunov exponent in the x direction is negative, then Φ is as smooth as f. In particular, if X is one-dimensional and Φ is a continuous invariant graph with a negative Lyapunov exponent in the x direction, then Φ is as smooth as f.*

Fractalization of continuous curves Now we know that the negative vertical Lyapunov exponent can guarantee the continuity of a curve. However, a smooth curve may also be a highly oscillating one. If the oscillation degree is extremely high, then it may be very difficult to detect its smoothness by only the numerical methods. Jorba and Tatjer prove that this fractalization mechanism does exist on some so-called nonreducible attracting invariant graphs for some quasi-periodically forced one-dimensional systems. Notice that fractalization is usually regard as one of the process which results in SNAs in physical contexts.

Precisely, in some quasi-periodically forced one-dimensional systems, there is a continuous change of the attracting invariant curves, which makes them more and more wrinkled as the the system parameter varies. When this happens, the attracting invariant graphs, which are the attractors of the systems, may look strange with only usual numerical methods. But in reality, it can continue being smooth as long as its vertical Lyapunov exponent is negative. This procedure of fractalization is described mathematically as below.

Consider a family of one-dimensional quasi-periodically forced dynamical systems F_μ of the form

$$\begin{cases} \theta_{n+1} = \theta_n + \omega \pmod 1, \\ x_{n+1} = f_\mu(\theta_n, x_n), \end{cases} \tag{2.9}$$

with $x \in \mathbb{R}$, $\theta \in \mathbb{S}^1$, ω an irrational number, and f_μ a smooth function of both x and θ. Here $\mu \in \mathbb{R}$ is a parameter, on which f_μ depends continuously. Moreover, the function f_0 does not depend on θ, that is, $f_0(\theta, x) = g(x)$ for some smooth function $g : \mathbb{R} \to \mathbb{R}$.

For a given value of $\mu = \mu_0$, assume that system (2.9) has an invariant curve $x = u_{\mu_0}(\theta)$ and the curve is of class $\mathcal{C}^r$ for some $r \geq 0$. Without loss of generality, it can be taken $\mu_0 = 0$. Then the invariant curve $u_0(\theta)$ must satisfy the functional equation $F(u_0, 0) = 0$, where $F : \mathcal{C}^r(\mathbb{S}^1, \mathbb{R}) \times \mathbb{R} \to \mathcal{C}^r(\mathbb{S}^1, \mathbb{R})$ is given by

$$F(u, \mu)(\theta) = f_\mu(u(\theta), \theta) - u(\theta + \omega), \tag{2.10}$$

for any $(u, \mu) \in \mathcal{C}^r(\mathbb{S}^1, \mathbb{R}) \times \mathbb{R}$. Next, use the Implicit Function Theorem to study the continuation of this curve with respect to the parameter μ. That is, look for a regular function $\mu \mapsto u_\mu$, which is defined for $|\mu|$ small enough, such that $F(u_\mu, \mu) = 0$.

On the Banach space $\mathcal{C}^r(\mathbb{S}^1, \mathbb{R})$ endowed with the standard $\mathcal{C}^r$ norm, it is not difficult to see that such an F is differentiable, and the function $D_u F(u, \mu)v \in \mathcal{C}^r(\mathbb{S}^1, \mathbb{R})$ is given by

$$[D_u F(u, \mu)v](\theta) = D_x f_\mu(u(\theta), \theta)v(\theta) - v(\theta + \omega) \tag{2.11}$$

for any $(u, \mu) \in \mathcal{C}^r(\mathbb{S}^1, \mathbb{R}) \times \mathbb{R}$, and any $v \in \mathcal{C}^r(\mathbb{S}^1, \mathbb{R})$. It is immediate to verify that $D_u F(u, \mu)v$ is a bounded operator.

Assume that an invariant curve x or $u_0(\theta)$ is of class $\mathcal{C}^r$, with $r \geq 0$, its linearized normal behaviour is described by the following linear skew product system

$$\begin{cases} \theta_{n+1} = \theta_n + \omega, \\ x_{n+1} = a(\theta)x_n. \end{cases} \tag{2.12}$$

where $a(\theta) = D_x f_0(u_0(\theta), \theta)$ is of class $\mathcal{C}^r$ too, $x \in \mathbb{R}$ and $\theta \in \mathbb{S}^1$. Moreover, assume that the invariant curve is not degenerate, in the sense that the function $a(\theta)$ is not identically zero. For the invariant curves, it will turn out that there is important effect on their behaviours according to whether the linear system (2.12) can be reduced to a form with a constant coefficient or not. That is, whether the system verifies the property of reducibility. The definition of reducibility is given by the following:

Definition 2.3.1. The system (2.12) is called *reducible* if and only if there exists a change of variable $x = c(\theta)y$ (which may be complex), continuous with respect to θ, such that (2.12) becomes

$$\begin{cases} \theta_{n+1} = \theta_n + \omega, \\ x_{n+1} = b x_n, \end{cases} \tag{2.13}$$

where b does not depend on θ.

Jorba and Tatjer proved that, under suitable conditions, the reducibility of (2.12) is equivalent to the fact that $a(\theta)$ has no zeros. Then the fractalization mechanism for nonreducible invariant curves is defined by:

Definition 2.3.2. A curve is undergoing a fractalization mechanism if its $\mathcal{C}^1$ norm – taken on any closed nontrivial interval for θ – goes to infinity much faster that its $\mathcal{C}^0$ norm, that is,

$$\limsup_{\alpha \to \alpha_0} \frac{\|x'_\alpha\|_{I,\infty}}{\|x_\alpha\|_\infty} = +\infty,$$

where $\|\cdot\|_{I,\infty}$ denotes the sup norm on a nontrivial closed interval I.

In a family of the form

$$\begin{cases} \theta_{n+1} = \theta_n + \omega, \\ x_{n+1} = \alpha a(\theta) x_n + b(\theta), \end{cases} \tag{2.14}$$

where $x \in \mathbb{R}$, $\theta \in \mathbb{S}^1$, ω is an irrational number as usual, $a(\theta)$ and $b(\theta)$ are $\mathcal{C}^r$ functions, and α is a real positive parameter. Clearly, the linearized normal behaviour around it is described by

$$\begin{cases} \theta_{n+1} = \theta_n + \omega, \\ x_{n+1} = \alpha a(\theta) x_n. \end{cases} \tag{2.15}$$

If an invariant curve of system (2.14) exists, its vertical Lyapunov exponent over this curve is given by

$$\Lambda = \ln \alpha + \int_{\mathbb{S}^1} \ln |a(\theta)| \mathrm{d}\theta. \tag{2.16}$$

If the above integral exists, then set

$$\alpha_0 = \exp\left(- \int_{\mathbb{S}^1} \ln |a(\theta)| \mathrm{d}\theta \right). \tag{2.17}$$

For the values of $\alpha < \alpha_0$, the vertical Lyapunov exponent is negative. Particularly, this implies that this invariant curve is globally attracting, hence it must be unique. We also know that there is a continuous change of this curve with respect to the parameter α when $\alpha < \alpha_0$. Let $x_\alpha(\theta)$ denote the solution of (2.14) for $\alpha < \alpha_0$ and a given continuous function $b(\theta)$. The next theorem describes the fractalization process of a nonreducible system.

Theorem 2.3.4. *Assume that $a(\theta), b(\theta) \in \mathcal{C}^1(\mathbb{S}^1, \mathbb{R})$ and that (2.15) is not reducible. Then,*

(1) if

$$\limsup_{\alpha \to \alpha_0^-} \|x_\alpha\|_\infty < +\infty,$$

and $b \in D_1$, where D_1 is some residual set, we have

$$\limsup_{\alpha \to \alpha_0^-} \|x'_\alpha\|_{I,\infty} = +\infty,$$

for any nontrivial closed interval $I \subset \mathbb{S}^1$;

(2) if

$$\limsup_{\alpha \to \alpha_0^-} \|x_\alpha\|_\infty = +\infty,$$

then for any nontrivial closed interval $I \subset \mathbb{S}^1$, we have

$$\limsup_{\alpha \to \alpha_0^-} \|x_\alpha\|_{I,\infty} = +\infty, \quad \text{and} \quad \limsup_{\alpha \to \alpha_0^-} \frac{\|x'_\alpha\|_{I,\infty}}{\|x_\alpha\|_\infty} = +\infty.$$

Remark 2.3.5. If in Theorem 2.3.4 the system (2.15) is reducible, the situation is different. In this case, if ω is Diophantine and a, b are $\mathcal{C}^r$ for r large enough, then

(i) If $\limsup_{\alpha \to \alpha_0^-} \|x_\alpha\|_\infty < +\infty$, then $\limsup_{\alpha \to \alpha_0^-} \|x'_\alpha\|_\infty < +\infty$.

(ii) If $\limsup_{\alpha \to \alpha_0^-} \|x_\alpha\|_\infty = +\infty$, then $\limsup_{\alpha \to \alpha_0^-} \dfrac{\|x'_\alpha\|_\infty}{\|x_\alpha\|_\infty} < +\infty$.

◇

Supplementary Finally we supplement two methods which can be used to prove the continuity and strangeness respectively. They are simple, traditional, and easy to use, which we will use subsequently.

It is well-known that a continuous graph is both upper and lower semicontinuous. Moreover, the limit of a decreasing sequence of continuous graphs is upper semicontinuous, and the limit of an increasing sequence of continuous graphs is lower semicontinuous. Hence, if a curve is proved to be the limits of both a decreasing and an increasing sequences of continuous graphs, it is proved that it is continuous.

For the strangeness, we known that, if two invariant graphs intersect in a quasi-periodically forced system, they must intersect in a dense set of $\theta \in \mathbb{S}^1$ by the irrational rotation on the base. This means that, when this happens, at least one of them cannot be continuous. This is the most reliable way for the proof of strangeness so far.

Chapter 3

Pinched invariant sets and quasi-periodically forced increasing Systems

In this chapter we expose the crucial role of pinched orbits on the dynamics of pinched quasi-periodically forced systems. The arguments reveal that there are notable differences between the pinched and non-pinched systems, which is shown clearly with examples that we analyze in detail.

The examples that we choose are the most possible simple and pure one among all the quasi-periodically forced one-dimensional maps: the monotone increasing systems. They help us to learn the most basic and essential nature of the quasi-periodically system without any other interference, which may cause too many extraneous confusions on the issues of SNAs. As a consequence, the role of pinched orbits help us unartificially to discover where a strange nonchaotic tractor can occur in a system.

More precise, we consider two types of forced monotone increasing systems, and demonstrate their dynamics via detailed proofs. Both of the models are generalized from Keller's by extending f to $\mathbb{R}$. One of them is set as convex in $\mathbb{R}^-$, the other is kept to be concave in all $\mathbb{R}$. Moreover, 0 is not fixed at original, which can be set at any point by a parameter. In non-pinched case, the dynamics of both models are essentially same with the unforced one-dimensional maps. There are bifurcations of smooth invariant curves which are born by the same type of bifurcation in the unforced systems. While in pinched case, the pinched condition either totally destroyed corresponding bifurcations, or move them to particularly place with SNAs.

This chapter consist of four sections. The properties of the pinched invariant sets and their role in the pinched systems are discussed in the first one. Where we show that a continuous curve contained in a closed invariant subset of quasi-periodically forced system must be invariant, and also be the ω-limit set of all pinched points. Particularly, for the system which is pinched, that is, there is at least a fibre which is mapped to one point, the ω-limit set of all pinched points is the only minimal subset which is contained in any invariant set of the system. This result reveals the crucial role of the pinched orbits in pinched systems, which is the first main theorem of this chapter.

The other sections are devoted to discussions of two generalized Keller's models. In the section section, we develop some general properties from the common setting of the two models. They are the basic tools for the investigation of both of the models. The last two sections are devoted to the dynamics of the two models respectively. With two main theorems we give the complete descriptions of each model with respect to two parameters.

3.1 Pinched invariant sets and pinched systems

In this first section we discuss some fundamental properties of the pinched invariant sets in quasi-periodically forced systems, and consider the role of pinched orbits in the pinched systems. We think that it can clarify some natural of the dynamics of quasi-periodically forced systems, and be helpful to the investigation of individual examples.

In Keller's model, there are big differences between cases of pinched and non-pinched systems, and the pinched invariant set which is not a continuous curve displays complicated dynamics. We also know from literature that it is possible that there exists pinched invariant set in a non-pinched system. In examples by Heagy and Hammel [35] and also Bjerklöv [8], they show two different invariant curves do intersect in a non-pinched system, which must be at a dense orbit of some $\theta \in \mathbb{S}^1$ at same time. So we start our study of the quasi-periodically forced systems with the most basic topological structures of pinched invariant sets. This is the topic of the first subsection, where we prove a theorem which says that: if there exists a continuous graph inside a pinched compact invariant set, this graph must be invariant and in fact is the ω-limit set of all pinched points. Certainly, it must be the only one which is continuous.

The second subsection is devoted to the pinched systems. We use the theorem above to show that, in a pinched system the unique ω-limit set of all those pinched orbits is the only minimal subsystem, and must be contained in any invariant set. This fact implies that the pinched orbit is the key to understand the dynamics of a pinched system, because all other points have to go around it.

3.1.1 Continuous graphs in pinched invariant sets

First we give the necessary definitions of our terminology and review some known results on pinched invariant sets. Next, after a simple Lemma 3.1.2, we prove our theorem.

The notions of ω-limit set and invariant set are standard. The *ω-limit set* of a point p, denoted by $\omega(p)$, is the set of the limit points of the orbit of this point $\mathrm{Orb}(p)$. If a subset $\mathcal{A}$ of the system (3.1) satisfies $F(\mathcal{A}) \subseteq \mathcal{A}$, then it is called an *invariant subset*.

A map $F\colon \mathbb{S}^1 \times X \to \mathbb{S}^1 \times X$:

$$F(\theta, x) = (\, R(\theta), G(\theta, x)\,), \tag{3.1}$$

gives a quasi-periodically forced dynamical system. Here $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is the unit circle, and X denotes some interval of $\mathbb{R}$. The function $R\colon \mathbb{S}^1 \to \mathbb{S}^1$ denotes an irrational rotation of the circle $\mathbb{S}^1$ by a fixed angle ω. Duo to it, there cannot be any fixed or periodic points. If an invariant set is compact, then the projection of the set to $\mathbb{S}^1$ is the whole circle. The simplest invariant closed subset can only be the graph of a map from $\mathbb{S}^1$ to X. If denote the map by Φ, its graph is the set $\mathcal{A} = \{(\theta, \Phi(\theta) : \theta \in \mathbb{S}^1\}$. To be invariant under the action of system (3.1), the graph need to satisfy the invariant equation $\Phi(R(\theta)) = G(\theta, \Phi(\theta))$ for any $\theta \in \mathbb{S}^1$. We abuse the notation and call Φ an *invariant graph*, or an *invariant curve*.

Let $\mathcal{A} \subset \mathbb{S}^1 \times X$ be a compact invariant set for F, the *lower* and *upper boundaries* of $\mathcal{A}$ are respectively the functions $\mathcal{A}^-$, $\mathcal{A}^+ : \mathbb{S}^1 \to X$ given by

$$\mathcal{A}^-(\theta) = \inf\{x \in X\colon (\theta, x) \in \mathcal{A}\},$$

$$\mathcal{A}^+(\theta) = \sup\{x \in X\colon (\theta, x) \in \mathcal{A}\}.$$

Since $\mathcal{A}$ is compact in $\mathbb{S}^1 \times X$, it is bounded and closed, so these functions are well-defined, and $(\theta, \mathcal{A}^-(\theta)) \in \mathcal{A}$, $(\theta, \mathcal{A}^+(\theta)) \in \mathcal{A}$. The *pinched compact invariant subset* is a compact F-invariant set $\mathcal{A}$ such that $\mathcal{A} \cap \{\theta_0 \times \mathbb{R}\}$ contains only one point for at least one $\theta_0 \in \mathbb{S}^1$, that is, $\mathcal{A}^+(\theta_0) = \mathcal{A}^-(\theta_0)$ for such $\theta_0 \in \mathbb{S}^1$. We call those points $(\theta_0, \mathcal{A}(\theta_0))$ the *pinched points*, and denote the set of all the pinched points by $P(\mathcal{A})$. Due to the continuity of F and the invariance of $\mathcal{A}$, it is easily to see that $P(\mathcal{A})$ is also an F-invariant set.

There are already some known properties of the topological structures the compact invariant sets in the literature (see [21] and reference therein), which we summarize as the proposition below.

Proposition 3.1.1. *Suppose $\mathcal{A}$ is a compact F-invariant set, its upper and lower boundaries $\mathcal{A}^+$ and $\mathcal{A}^-$ are upper and lower semicontinuous graphs respectively. If it is pinched, then $\mathcal{A}^+(\theta) = \mathcal{A}^-(\theta)$ for a residual set of $\theta \in \mathbb{S}^1$. Moreover, both $\mathcal{A}^+$ and $\mathcal{A}^-$ are continuous at all these $\theta \in \mathbb{S}^1$.*

In fact, we can say a little more about the continuity of the pinched compact invariant set, derived from the property below.

Lemma 3.1.2. *For a point $p = (\theta, x)$ in the quasi-periodically forced dynamical system* (3.1)*, if its orbit* $\mathrm{Orb}(p) \subset \Psi$ *where Ψ is a continuous graph, then Ψ is an invariant graph and $\Psi = \omega(p)$.*

Proof. We only need to prove that $\Psi = \omega(p)$. Because the F-invariance of Ψ just follows the F-invariance of $\omega(p)$.

First we prove that $\Psi \subset \omega(p)$, that is, taking any point $q = (\theta_q, \Psi(\theta_q)) \in \Psi$, we have $q \in \omega(p)$. Denote $F^n(p)$ by (θ_n, x_n). Since the set $\{F^n(p)\} \subset \Psi$ by assumption, we have $x_n = \Psi(\theta_n)$ for all $n \geq 0$. Notice $F(\theta, x) = (\, R(\theta), G(\theta, x)\,)$ with R an irrational rotation on $\mathbb{S}^1$, this implies that $\{\theta_n\}$ is a dense set in $\mathbb{S}^1$. So we can choose a subsequence (θ_{n_j}) of (θ_n) such that its limit is θ_q. Thus, $(x_{n_j} = \Psi(\theta_{n_j}))$ must converge to $\Psi(\theta_q)$ by the continuity of Ψ. This shows q is a limit point of $\mathrm{Orb}(p)$, that is $q \in \omega(p)$.

Next we show that $\omega(p) \subset \Psi$. As a continuous graph, Ψ is a closed subset in $\mathbb{S}^1 \times X$. Thus $\mathrm{Orb}(p) \subset \Psi$ implies any limit point of $\mathrm{Orb}(p)$ must also belong to this closed set Ψ. □

Next we prove our theorem which show the property of continuous invariant graph in a pinched compact invariant set.

Theorem 3.1.3. *Suppose $\mathcal{A}$ is a pinched compact F-invariant subset of system* (3.1)*, and let $p = (\theta_p, x_p) \in P(\mathcal{A})$ be any pinched point. If there exists a continuous graph $\Psi \subset \mathcal{A}$, then Ψ must be invariant and $\Psi = \omega(p)$.*

Proof. $\mathcal{A}$ is pinched means that $\mathcal{A}^+(\theta_p) = \mathcal{A}^-(\theta_p) = x_p$ for any $p \in P(\mathcal{A})$. Since $\Psi \subset \mathcal{A}$, we have $\mathcal{A}^+(\theta_p) \geq \Psi(\theta_p) \geq \mathcal{A}^-(\theta_p)$ by the definition of $\mathcal{A}^+$ and $\mathcal{A}^-$. Notice that $\mathcal{A}^+(\theta_p) = \mathcal{A}^-(\theta_p)$ since p is a pinched point, so $p \in \Psi$ because $x_p = \Psi(\theta_p)$. Thus we have proved that $P(\mathcal{A}) \subset \Psi$. This implies $\mathrm{Orb}(p) \subset \Psi$ since $\mathrm{Orb}(p) \subset P(\mathcal{A})$ for any pinched point q. Now the results follow directly from Lemma 3.1.2 above. □

This theorem demonstrates a mechanism for the creation of the geometric strangeness in a pinched invariant set. In fact, there are two possibilities: one is that, if a pinched compact invariant set is not the ω-limit set of the pinched points itself, then there must be some strange geometry of this invariant set (at least one of its boundaries cannot be continuous, like Keller's model); the other one is more difficult to analyze and is still an open problem, which is the case that the ω-limits set of the pinched points is just this pinched compact invariant set. One cannot ignore the possibility that it is not a continuous graph itself, although we have still not seen an explicit example of such strange set. Below we summarize some useful results on this question as a remark.

Remark 3.1.4. In [3], this ω-limits set of the pinched points is treated as a pseudo curve, which must be a curve if it contains any piece of curve. Stark [73] proves that, if the projection of the set of pinched points is with full measure on $\mathbb{S}^1$, and the normal Lyapunov exponent of the unique invariant measure supported on this pinched set is negative, then this pinched set must be a smooth curve provided F is a C^1 map. In general, it is still not known if any ω-limit sets of the pinched points are all curves. Particularly, its shape may be extremely wrinkled when the Lyapunov exponent approaches to critical value zero. There are some works which claim such curves are not continuous based on the numerical looking, but are reported as mistakes (see, for example [12, 27, 62]). A detailed theoretical discussion on this fractalization mechanism is given in [41], which we have summarized in the last section of the previous chapter. Finally, some examples for the situation of the Lyapunov exponent at critical value zero can be found in [37]. $\diamond$

3.1.2 Orbits of pinched points in pinched systems

For the bifurcation mechanism which can result some pinched invariant set, the *pinched systems* are natural and important candidate. A dynamical system given by the map $F\colon \mathbb{S}^1 \times X \to \mathbb{S}^1 \times X$ in (3.1) is pinched if there is at least one fibre who is mapped to a single point. That is, if there is some $\theta_0 \in \mathbb{S}^1$ such that $G(\theta_0, x) = c$ is a constant for all $x \in X$. The point $(R(\theta_0), c)$, who is the image of a pinched fibre, is called a *(system) pinched point*, and we reserve the notation p for (system) pinched points in the following of this chapter.

Obviously, in a pinched system, the compact invariant set must also be pinched. Furthermore, in such system, all the interesting dynamics can only happen around the pinched points. This is the first main result of this chapter, which we prove in Theorem A that, $\omega(p)$ is the only minimal invariant subset in the system which is a subset for any compact invariant set. This distinctive feature reveals the crucial role of the pinched orbits in those pinched systems.

First we show that, if the pinched points remain in the pinched system, all the points have to come back arbitrarily close to them.

Lemma 3.1.5. *In a pinched system, $p \in \omega(x)$ for any point x who doesn't go to infinity.*

Proof. Denote p by (θ_p, x_p) and $F^n(x)$ by (θ_n, x_n). Because R is an irrational rotation on $\mathbb{S}^1$, this implies that $\{\theta_n\}$ is a dense set in $\mathbb{S}^1$. Choose a subsequence (θ_{n_j}) of (θ_n) such that the limit of (θ_{n_j}) is just $R^{-1}(\theta_p)$ and (x_{n_j}) is bounded. If we cannot do it, this implies that $(F^n(x))$ goes to infinity. Otherwise, because F is continuous and maps the whole fibre over $R^{-1}(\theta_p)$ to p, it must be that p is the limit point of $(F^{n_j+1}(x))$. □

Corollary 3.1.6. *It is easy to see that:*

(1) if there is some system pinched point p who goes to infinity, then all the points in the system also go to infinity;

(2) the ω-limit set for all the pinched points is unique.

Proof. The first claim is trivial from the above lemma, in such case the ω-limit of any point is empty. Now we consider the case that the pinched points do not go to infinity. If p_1 and p_2 all two different pinched points, Lemma 3.1.5 gives both $p_1 \in \omega(p_2)$ and $p_2 \in \omega(p_1)$. By the property of the ω-limit sets, if $y \in \omega(x)$, then $\omega(y) \subset \omega(x)$ for any points x and y. Thus $\omega(p_1) \subset \omega(p_2)$ and $\omega(p_2) \subset \omega(p_1)$, we have $\omega(p_1) = \omega(p_2)$. □

Thus we can use $\omega(p)$ to denote the ω-limit set in a pinched system, it doesn't matter which one the point p is. Now, it is almost trivial for us to get the proposition below.

Theorem A. *In the pinched quasi-periodically forced systems, if $\omega(p)$ is not empty, then it is the only minimal compact invariant set in the whole system, and is a subset for any compact invariant set.*

Proof. Let $\mathcal{A}$ be a compact invariant set and $x \in \mathcal{A}$, $p \in \omega(x)$ means $\omega(p) \subset \omega(x) \subset \mathcal{A}$. To see that $\omega(p)$ is minimal, we need to prove it is the ω-limit set of all its points. Taken any $x \in \omega(p)$, the property of the ω-limit set gives $\omega(x) \subset \omega(p)$. We have also $\omega(p) \subset \omega(x)$ because $p \in \omega(x)$ by Lemma 3.1.5. Hence $\omega(x) = \omega(p)$ as required. □

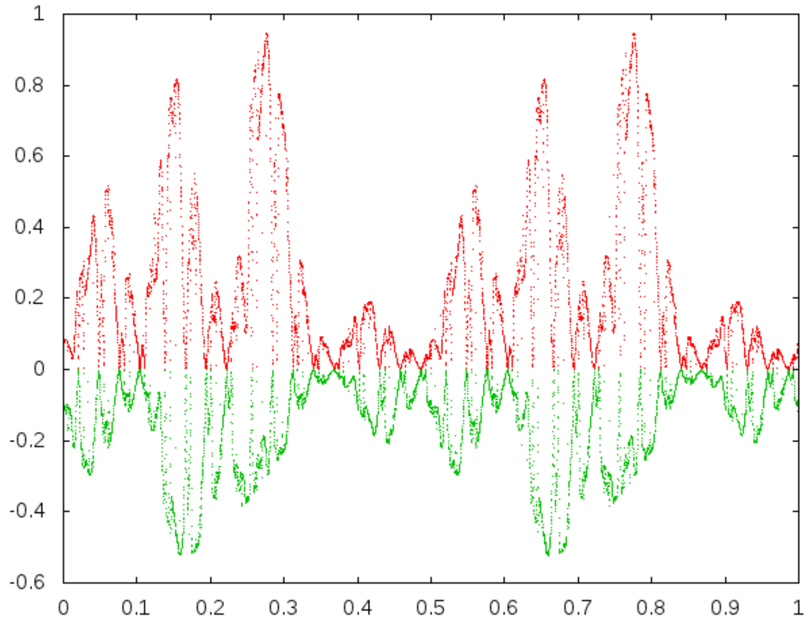


Figure 3.1: Period doubling bifurcation at $x = 0$, obtained with $G(\theta, x) = 2.2\,|\cos(2\pi\theta)|\,x(x-1)$ and ω equals the golden mean.

Remark 3.1.7. Notice that, for each pinched compact invariant set, the ω-limit set of its pinched points is also the unique minimal set inside. The arguments above work the same for this case. Therefore, the uniquely possible continuous graph in a pinched invariant set can only be ω-limit set of pinched points. $\diamond$

Theorem A is simple but instructive, it reveals the special significance of orbits of pinched points on the dynamics of pinched systems. Naturally we can hope that a forced system owns the similar dynamics with the unforced interval map when the perturbation given by forced term is relative small. This is really the case for the non-pinched systems, in which there can exist several disjoint compact invariant sets, each corresponding to the different fixed or periodic points of the interval maps. But it is not true generally for pinched systems. Theorem A implies any two compact invariant sets cannot be disjoint, hence all interesting dynamics can only be in a piece which is around the ω-limit set of the pinched points.

We will discuss some monotone interval families extensively in the following sections, it illustrates more clear this special significance of the pinched systems and the difference with the non-pinched ones. Before that, we give a simple example below to show that, how the position of the pinched orbits in a pinched system affects the general dynamical behaviour of the system.

Example 3.1.8. Consider the system

$$F(\theta, x) = (\,\theta + \omega \text{ mod}(1),\; 2.2\,|\cos(2\pi\theta)|\, x(x-1)\,),$$

a pinched system whose attractor is plotted in Figure 3.1.

In [4], the authors study the pinched quasi-periodically forced unimodal map

$$F(\theta, x) = (\,\theta + \omega \text{ mod}(1),\; \mu x(1-x) \cdot g(\theta)\,),$$

with $g = 0$ at some θ. They propose a question that, if there are some periodic invariant graphs in such system system, just like the unimodal maps can go into the period doubling cascade. Notice that $x = 0$ is invariant and contains all the pinched points, hence the dynamics of the system happens only around it. But $x = 0$ is not the place where the period doubling takes place for the unimodal map $\mu x(1-x)$, which should be at the other fixed point.

If we replace $\mu x(1-x)$ by $\mu x(x-1)$, then Figure 3.1 shows clearly that the period doubling occurs. That is because, for this new model, $x = 0$ is just the right place for the first period doubling. In fact, it is equivalent that there is a change of variable, so that the fixed point $1 - 1/\mu$ of the common logistic map $\mu x(1-x)$ is move to $x = 0$ of this new model $\mu x(x-1)$. ◇

3.2 Notations and tools

In this section we develop some properties of the quasi-periodically forced monotonic systems. They are derived from the monotonicity, and from the concavity and convexity respectively. These properties are basic tools of our arguments of the two models in the next two sections. Moreover, they work in any systems of monotone increasing fibre maps with concave or convex structures. This implies that, although the fibres of our systems are set as $\mathbb{R}$, but our results in fact are true for any systems with fibre maps are interval maps who have the similar structure of monotonicity with concavity or convexity.Such systems maybe with fibre maps on a finite interval of X, or even just locally with these structures around some attractors.

There are three subsections in this section. In the first one we give the notations of transfer operators, they are very useful tools in the study of the quasi-periodically forced systems. Next we obtain some properties of these operators due to the monotonicity in the second subsection. At last, we discuss the concepts of concavity and convexity, and prove some contraction results of the fibre maps with such structures. Notice that, to derive these results, it is not necessary to ask for the domain to be $\mathbb{S}^1 \times \mathbb{R}$. They are also true on any locally situation of $\mathbb{S}^1 \times I$ with $I \subset \mathbb{R}$.

3.2.1 Transfer operators

If we consider that, let the system (3.1) act on functions from $\mathbb{S}^1$ to $\mathbb{R}$, we get a functional version of the system (3.1). Then an invariant function is a fixed point in this functional version of system. For the skew products (3.1), an invariant function is a function $\varphi\colon \mathbb{S}^1 \to \mathbb{R}$ that satisfies the following invariance equation

$$\varphi(R(\theta)) = G(\theta, \varphi(\theta)).$$

Recall that we abuse terminology and refer an invariant graph φ to a graph of an invariant function φ. Thus its graph is kept to be fixed under the action of system (3.1). An easy example is the function $\varphi = 0$ in Keller's model. This idea leads to the important tools for the study of invariant graphs, which are the *transfer operators* defined as follows.

Let $\mathcal{P}$ be the space of all functions (not necessarily continuous) from $\mathbb{S}^1$ to $\mathbb{R}$, $\psi \in \mathcal{P}$. The *(forward) transfer operator* $\mathcal{T}\colon \mathcal{P} \to \mathcal{P}$ of the system (3.1) is defined as:

$$(\mathcal{T}\psi)(\theta) = G(R^{-1}(\theta), \psi(R^{-1}(\theta))).$$

Until the end of this chapter, we only consider transfer operators for system given by

$$F(\theta, x) = (\,\theta + \omega \ \mathrm{mod}(1),\ \lambda f(x+a) \cdot g(\theta)\,), \tag{3.2}$$

here we have $R(\theta) = \theta + \omega \pmod 1$, and $G(\theta, x) = \lambda f(x+a) \cdot g(\theta)\,)$. In the map $G\colon \mathbb{S}^1 \times \mathbb{R} \to \mathbb{R}$, λ and a are used as real parameters. Assume that $g\colon \mathbb{S}^1 \to \mathbb{R}$ is continuous and $g(\theta) \geq 0$, and the real function $f(x+a)$ is strictly increasing for any fixed a, which satisfies $f(0) = 0$.

Then the transfer operator of system (3.2) is given by

$$(\mathcal{T}\psi)(\theta) = \lambda f_a(\psi(\theta - \omega)) \cdot g(\theta - \omega). \tag{3.3}$$

We save the notation $\mathcal{T}$ for operator given above of such system (3.2) from now on, and call it just as transfer operator for short. Notice that, the graph of $\mathcal{T}\psi$ is the image of the graph of ψ under F, and φ is invariant if and only if $\mathcal{T}\varphi = \varphi$.

When considering the backward iterates or the preimages of a function ψ for the system (3.2), we need the *backward transfer operator* $\mathcal{R}$ (only defined when $g(\theta) > 0$ for all $\theta \in \mathbb{S}^1$):

$$(\mathcal{R}\psi)(\theta) = f_a^{-1}\left(\frac{\psi(\theta+\omega)}{\lambda g(\theta)}\right). \tag{3.4}$$

Notice that this operator is well-defined if $\lambda g(\theta) \neq 0$ for all $\theta \in \mathbb{S}^1$, since we require f to be strictly increasing in (3.2).

3.2.2 Some facts due to monotonicity

Next, we show some simple facts coming from the monotonicity of f and $g(\theta) \geq 0$ for the system (3.2). They will be frequently used later in the next two sections when we deal with the dynamics of the models.

Observation 3.2.1. *The fibre map $\lambda f(x+a)g(\theta)$ is also monotone increasing for all $\theta \in \mathbb{S}^1$. That is, taking (θ_0, x_0) and (θ_0, y_0) for any $\theta_0 \in \mathbb{S}^1$ with $x_0 \geq y_0$, then*

$$\lambda f_a(x_0)g(\theta_0) \geq \lambda f_a(y_0)g(\theta_0).$$

Observation 3.2.2. *If $a \geq 0$ and $x_k \geq -a$ for some $k \in \mathbb{N}$, the monotonicity implies $x_{k+n} \geq 0$ for all $n \geq 1$.*

Proof. First note that, if $x_k \geq 0$ for some $k \in \mathbb{N}$, we have $f(x_k+a) \geq f(a) \geq f(0) = 0$, which implies $x_{k+1} = \lambda f(x_k+a)g(\theta_n) \geq 0$. Hence $x_{k+n} \geq 0$ for all $n \geq 1$. Moreover, if $x_k \geq -a$, then $x_{k+1} \geq 0$ since $f(x_k+a) \geq f(0) = 0$. □

Remark 3.2.3. The above observations imply that, if $a \geq 0$ in the system (3.2), then $\mathbb{S}^1 \times \mathbb{R}^+$ is invariant, and any points with initial value $x \geq -a$ enter this invariant region after one iterate. It is the same case for $\mathbb{S}^1 \times \mathbb{R}^-$ if $a \leq 0$. ◇

In terms of transfer operators, the monotonicity of fibre maps in system (3.2) gives the following lemma.

Lemma 3.2.4. *Let $\psi, \varphi \in \mathcal{P}$, and $\psi \leq \varphi$. Then*

(1) $\mathcal{T}^n\psi \leq \mathcal{T}^n\varphi$ for all $n \geq 1$.

(2) $\mathcal{R}^n\psi \leq \mathcal{R}^n\varphi$, for all $n \geq 1$, whenever $\mathcal{R}$ is well-defined.

(3) particularly, if $\mathcal{T}\psi \leq \psi$ ($\mathcal{T}\psi \geq \psi$), then $\mathcal{T}^{n+1}\psi \leq \mathcal{T}^n\psi$ ($\mathcal{T}^{n+1}\psi \geq \mathcal{T}^n\psi$) for all $n \geq 1$ respectively. The same is for the backward transfer operator $\mathcal{R}$.

Proof. Here $\mathcal{T}\psi \leq \mathcal{T}\varphi$ follows clearly from monotonicity of all the fibre maps in the above observation, then $\mathcal{T}^n\psi \leq \mathcal{T}^n\varphi$ is the result of induction. The same is the case of backward transfer operator $\mathcal{R}$. □

Remark 3.2.5. Particularly, the above lemma shows that, if $\mathcal{T}\psi \leq \psi$, we have $\mathcal{T}^{n+1}\psi \leq \mathcal{T}^n\psi$ for all $n \geq 1$, which results a decreasing sequence of graphs $\{\mathcal{T}^n\psi\}$. It is commonly known that, if the pointwise limit of such a decreasing or an increasing sequence of continuous function exits, this limit is an upper or a lower semicontinuous function. Hence, if a function is both the limit of a decreasing and an increasing sequence at the same time, this function must be a continuous one. ◇

3.2.3 Contraction due to concavity

If we assume that the function f in system (3.2) are also with some concave or convex structures, then we can obtain a property derived from these structures. This property gives the relation of the contraction of an interval map with the concavity or convexity, which is the reason of the existence and the attraction of the invariant graph in our model. We treat this issue in a different but more general way other than Keller. Our treatments follow the arguments in [5], which extend their result to more general settings.

First, let us introduce the notion of α-concavity and β-convexity.

Definition 3.2.6. Let f be a continuous real-valued function on a closed interval I of the real line and let $\alpha \geq 0$. The function f will be called *α-concave* if the function f_α, given by

$$f_\alpha(x) = f(x) + \alpha x^2$$

is concave.

The following properties of an α-concave function f follow immediately from the definition:

(1) f is concave;

(2) if $\alpha > 0$ then f is strictly concave;

(3) if $0 \leq \gamma \leq \alpha$ then f is γ-concave.

Similarly, we define a β-convex function as follows:

Definition 3.2.7. Let f be a continuous real-valued function on a closed interval I of the real line and let $\beta \geq 0$. The function f will be called *β-convex* if the function f_β, given by

$$f_\beta(x) = f(x) - \beta x^2$$

is convex.

A β-concave function f also satisfies:

(1) f is convex;

(2) if $\beta > 0$ then f is strictly convex;

(3) if $0 \leq \gamma \leq \beta$ then f is γ-convex.

Now we start to prove an inequality satisfied by α-concave functions $f(x+a)$ in (3.2) with $a \geq 0$.

If $f(x)$ is α-concave for $x \geq 0$, so is $f(x+a)$ since $x + a \geq a \geq 0$ for $x \geq 0$. Thus given two points $uv > 0$, we define

$$\kappa(u, v) := \frac{|v - u|}{\min\{|u|, |v|\}}.$$

Lemma 3.2.8. *Assume that f in (3.2) is α-concave for $x \geq 0$, and $a \geq 0$. Let $0 < x < y$, then,*

$$\frac{\kappa(f(x+a), f(y+a))}{\kappa(x, y)} \leq \frac{f(y+a)}{f(y+a) + \alpha(y+a)^2}.$$

Proof. Since $y > x > 0 \geq -a$ and $f_\alpha(x+a) = f(x+a) + \alpha(x+a)^2$ is concave for $x \geq -a$, we have

$$\frac{f_\alpha(x+a) - f_\alpha(-a+a)}{x - (-a)} \geq \frac{f_\alpha(y+a) - f_\alpha(-a+a)}{y - (-a)}.$$

That is

$$\begin{aligned}\frac{f_\alpha(x+a) - f_\alpha(-a+a)}{x - (-a)} &= \frac{f_\alpha(x+a) - 0}{x+a} \geq \frac{f_\alpha(y+a) - f_\alpha(-a+a)}{y - (-a)} \\ &= \frac{f_\alpha(y+a) - 0}{y+a}.\end{aligned}$$

Thus

$$\frac{f(x+a)}{x+a} + \alpha(x+a) \geq \frac{f(y+a)}{y+a} + \alpha(y+a),$$

which implies

$$\frac{f(x+a)}{x+a} \geq \frac{f(y+a)}{y+a} + \alpha(y-x). \tag{3.5}$$

so

$$f(x+a) \geq \frac{x+a}{y+a} f(y+a) + \alpha(y-x)(x+a).$$

which gives,

$$\begin{aligned} f(y+a) - f(x+a) &\leq \frac{y-x}{y+a} f(y+a) - \alpha(x+a)(y-x) \\ &= (y-x)\left(\frac{f(y+a)}{y+a} - \alpha(x+a)\right). \end{aligned}$$

Now we have

$$\frac{f(y+a)-f(x+a)}{f(x+a)} \cdot \frac{x}{y-x} \leq \frac{x}{f(x+a)}\left(\frac{f(y+a)}{y+a} - \alpha(x+a)\right).$$

By (3.5)

$$\frac{f(x+a)}{x} > \frac{f(x+a)}{x+a} \geq \frac{f(y+a)}{y+a} + \alpha(y-x),$$

we get

$$\begin{aligned} \frac{\kappa(f(x+a), f(y+a))}{\kappa(x,y)} &= \frac{f(y+a)-f(x+a)}{f(x+a)} \cdot \frac{x}{y-x} \leq \frac{\frac{f(y+a)}{y+a} - \alpha(x+a)}{\frac{f(y+a)}{y+a} + \alpha(y-x)} \\ &= 1 - \frac{\alpha(y+a)}{\frac{f(y+a)}{y+a} + \alpha(y-x)} \leq 1 - \frac{\alpha(y+a)}{\frac{f(y+a)}{y+a} + \alpha(y+a)} \\ &= \frac{\frac{f(y+a)}{y+a}}{\frac{f(y+a)}{y+a} + \alpha(y+a)} = \frac{f(y+a)}{f(y+a) + \alpha(y+a)^2}. \end{aligned}$$

□

Corollary 3.2.9. *For two initial points (θ_0, x_0) and (θ_0, y_0) in the same fibre, if $0 < x_0 < y_0$ and $g(\theta_k) \neq 0$ for all $0 \leq k \leq n$, then $0 < x_k < y_k$ for all $0 \leq k \leq n+1$. Furthermore, for all $0 \leq k \leq n$,*

$$\frac{\kappa(x_{k+1}, y_{k+1})}{\kappa(x_k, y_k)} = \frac{\kappa(f(x_k+a), f(y_k+a))}{\kappa(x_k, y_k)} \leq \frac{f(y_k+a)}{f(y_k+a) + \alpha(y_k+a)^2}. \tag{3.6}$$

Proof. $0 < x_k < y_k$ for $0 \leq k \leq n+1$ follows directly from Observation 3.2.1, while the strict inequality is due to $g(\theta_k) \neq 0$ for all $0 \leq k \leq n$. The inequality of (3.6) is just the result of Lemma 3.2.8 above, while the equality follows from $\kappa(x_{k+1}, y_{k+1}) = \kappa(f(x_k + a), f(y_k + a))$, which is because

$$\frac{|x_{k+1} - y_{k+1}|}{\min\{|x_{k+1}|, |y_{k+1}|\}} = \frac{\lambda g(\theta_k)|f(x_k + a) - f(y_k + a)|}{\lambda g(\theta_k)|f(x_k + a)|} = \frac{|f(x_k + a) - f(y_k + a)|}{|f(x_k + a)|}.$$

□

Remark 3.2.10. Analogously, for f is β-concave and $a \leq 0$, we have also that, if $y < x < 0$ then

$$\frac{\kappa(x_{k+1}, y_{k+1})}{\kappa(x_k, y_k)} = \frac{\kappa(f(x_k - a), f(y_k - a))}{\kappa(x_k, y_k)} \leq \frac{f(y_k - a)}{f(y_k - a) + \beta(y_k - a)^2}. \quad (3.7)$$

The proof goes literally same except for the sign, so we omit it here. ⋄

3.3 First monotonic increasing model

Both this and the next section are devoted to the discussions of quasi-periodically forced monotonic increasing systems. These two models can be viewed as generalization of the Keller's model to the whole cylinder. We take the interval maps to be monotone increasing with some concave or convex structure on it. Such settings are very typical for one dimensional systems, and the arguments can suit also the situation that the particular domain is locally increasing. More precisely, the families in both sections are given by the maps $F\colon \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$, which are in the same form of (3.2) given in the previous section.

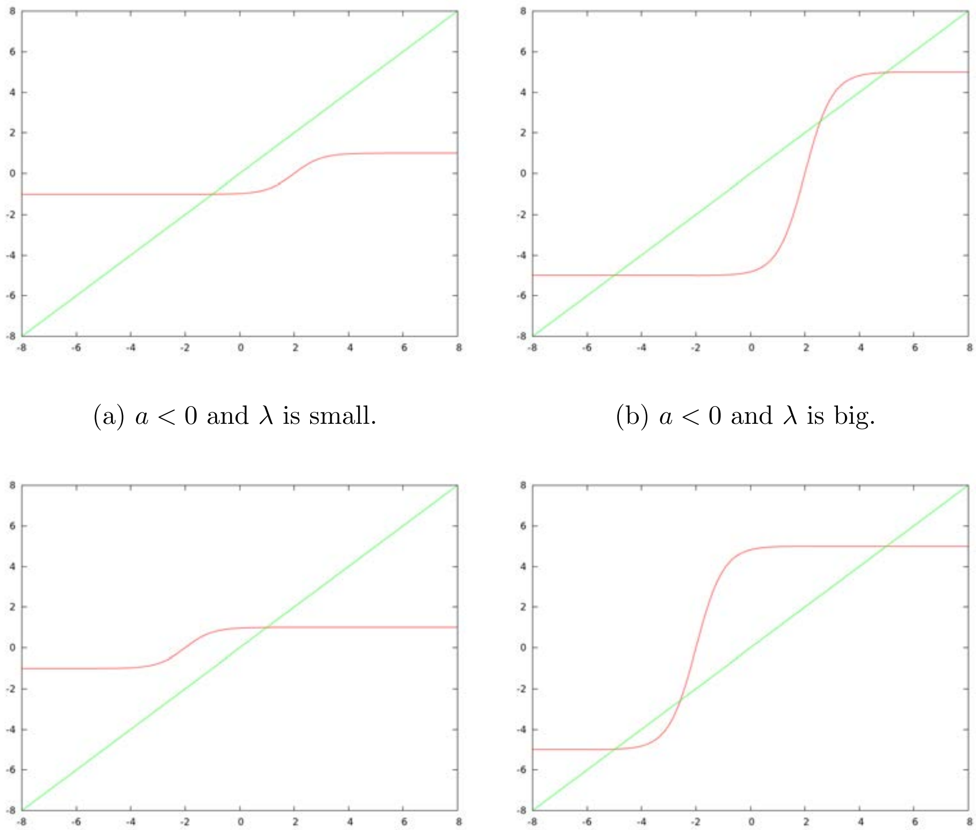
The first family is generalized from the Keller's model in a symmetric-like fashion, the upper part of the fibre maps is kept to be concave like the Keller's, the lower is set to be convex. There is quite several cases of the dynamics of this family according to different situations of parameters, so we devote the first subsection to describe them. Their proofs are left to the second subsection.

3.3.1 The model and its dynamics

Now we study our first family, which generalizes the Keller's model to the whole cylinder $\mathbb{S}^1 \times \mathbb{R}$ in the following way:

$$F(\theta, x) = (\,\theta + \omega \bmod(1),\, \lambda f(x + a) \cdot g(\theta)\,), \quad (3.8)$$

with f a real function satisfies



(a) $a < 0$ and λ is small. (b) $a < 0$ and λ is big.

(c) $a > 0$ and λ is small. (d) $a > 0$ and λ is big.

Figure 3.2: Graphs of $\lambda f(x + a)$ for different cases.

(1) f is continuous and bounded;

(2) f is strictly increasing on $\mathbb{R}$ and $f(0) = 0$;

(3) f is strictly concave for $x > 0$ and is strictly convex for $x < 0$.

Thus the fibre maps are all the same with Keller for $x \geq 0$, but extended to the negative part with a convex curve. We let $\lambda > 0$ so that each fibre map is strictly increasing. The reason of setting the fibre maps to be $f(x + a)$ is that, we hope to study the dynamics of any maps with this type of shape, particularly not only fixed at 0. With the change of a, the fixed points and zeros of $f(x + a)$ change too, this makes $x = 0$ no longer invariant as long as $a \neq 0$.

The family of this kind of interval maps $f(x+a)$ is a typical example of the saddle-node bifurcation of the one dimensional systems. We plot their pictures for λ big and small, $a < 0$ and $a > 0$ respectively in Figure 4.2, they correspond to the situation before and after the saddle-node bifurcation. The dynamics of $f(x+a)$ in this case can be easily understood via graph analysis. Notice that, the cases for $a < 0$ and $a > 0$ are in fact essentially the same, except the direction that those points go are opposite. We do no plot the graph of $a = 0$, which $x = 0$ is always fixed, where a pitchfork bifurcation occurs with the increasing of λ, which is different with $a \neq 0$.

The dynamical behaviours of the system (3.8) for the cases of $a = 0$ and of $a \neq 0$ turn out also different in the same way as the interval family. We first describe each case of them respectively.

Case $a = 0$ (1) $x = 0$ is invariant of the system for any λ. Moreover, if λ is relatively small, $x = 0$ is the only invariant graph which attracts all other points of the system;

(2) With the increasing of λ, two invariant graphs other than $x = 0$ will bifurcate from above and below independently at individual critical value of λ for each own, provided the upper and lower parts are not symmetric. These two new invariant graphs attract the points with positive and negative initial x values correspondingly;

(3) If $g(\theta) > 0$ for all $\theta \in \mathbb{S}^1$, all the invariant graphs are as smooth as g; if $g(\theta_0) = 0$ for some $\theta_0 \in \mathbb{S}^1$, the invariant graphs bifurcate out from $x = 0$ are SNAs.

Remark 3.3.1. This case is a rather special case for families (3.8), since at $x = 0$, $f(x)$ is fixed. It is clearly a directly generalization of Keller's model, hence its dynamical behaviour can be easily understood via Keller's model without proof. Precisely, the parts of $x \geq 0$ and $x \leq 0$ are two subsystems respectively. The upper one is exactly Keller's model; while for the lower one, the only difference with the upper is convexity instead, but dynamics and the proof goes analogously due to the symmetric structure. It should be noticed that, here by symmetry we don't mean the strictly geometrically symmetric of the two parts, we only refer that the general structure are symmetric in the topological sense. Hence the bifurcations of these two subsystems may happen at two different values of λ individually. $\diamond$

Case $a \neq 0$ In this case, $x = 0$ is no longer invariant. The consequence of this fact is that, the bifurcations go entirely distinct for the pinched and non-pinched cases. Their complete dynamical behaviours are given by our main theorem of this section below.

Theorem B. *If $a \neq 0$ in family (3.8), the dynamics of the system is the following:*

(1) In all the values of a and λ, there is an invariant graph Φ_λ in the system, which is attracting and continuous. If $a > 0$, $\Phi_\lambda > 0$; otherwise $\Phi_\lambda < 0$.

(2) If there exists $g(\theta_0) = 0$ for some $\theta_0 \in \mathbb{S}^1$, Φ_λ is the only *invariant graph who attracts all points of the whole system;*

(3) If $g(\theta) > 0$ for all $\theta \in \mathbb{S}^1$, then there is a critical value of λ, denoted by λ_0, such that

(a) If $\lambda < \lambda_0$, Φ_λ is the only invariant graph which is the attractor for the whole system $\mathbb{S}^1 \times \mathbb{R}$;

(b) If $\lambda > \lambda_0$, there exist another two continuous invariant graphs other than Φ_λ, one of them is repelling and the other one is attracting. Denote this new attracting graph by Ψ_λ, $\Psi_\lambda < -a$ if $\Phi_\lambda > 0$ ($a > 0$); $\Psi_\lambda > -a$ if $\Phi_\lambda < 0$ ($a < 0$). In any cases, the repelling graph lies between Φ_λ ($x = -a$ in fact) and Ψ_λ.

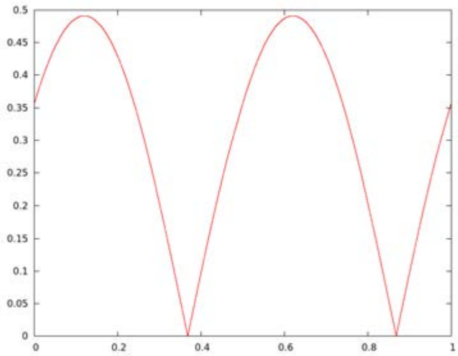
Observation 3.3.2. *We notice the following facts which lead to a simpler dynamical explanation of the general situation:*

(1) The existence of one attracting invariant graph, Φ_λ namely, depends, neither on if the system is pinched or not, nor on λ is small or big. But the bifurcations do.

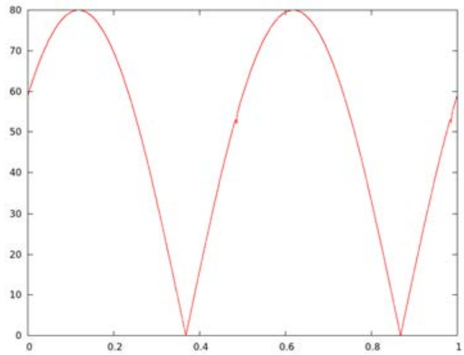
(2) For the non-pinched case, there is a bifurcation occurs at the opposite side of Φ_λ with the increasing of λ. The type of this bifurcation is exactly the saddle-node bifurcation just corresponding the unforced one dimensional interval maps.

(3) In the pinched case instead, unlike $a = 0$, there is ***no any*** *bifurcation. Moreover, since there is no other curves bifurcate out, there is no the strange curve also.*

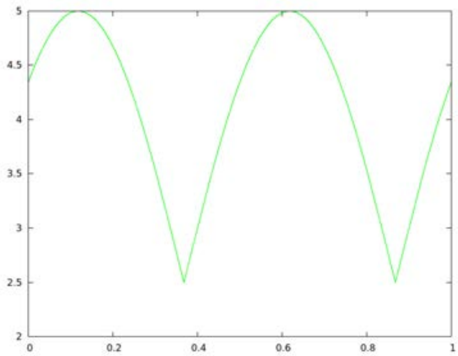
Recall that the unforced interval map f of this type has been well understood already, as shown by the graphs of the maps of $f(x + a)$ plotted in Figure 4.2. Comparing with the unforced interval systems, we can summarize briefly the dynamics of this models as follows. Viewing the invariant graphs of the forced systems as the fixed points of the unforced interval maps, the dynamics for $a = 0$ is the same as the interval maps. The particular case is the invariant graph may be a strange one when the forced systems are



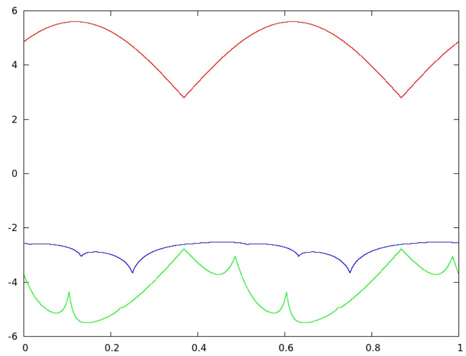
(a) The attractor obtained by model with $\lambda g(\theta)f(x + a) = 0.5\,|\cos(2\pi\theta)|\tanh(x+2)$.



(b) The attractor obtained by model with $\lambda g(\theta)f(x + a) = 80\,|\cos(2\pi\theta)|\tanh(x+2)$.



(c) The attractor obtained by model with $\lambda g(\theta)f(x + a) = 2.5(1 + |\cos(2\pi\theta)|)\tanh(x+2)$. No repellor founded in system.



(d) The two attractors and repellor obtained by model with $\lambda g(\theta)f(x + a) = 2.8(1 + |\cos(2\pi\theta)|)\tanh(x+2)$. The top and bottom curves are attractors, the middle one is repellor.

Figure 3.3: Graphs of some cases of the first family.

pinched. When $a \neq 0$, the dynamics of family (3.8) is also exactly the same with the unforced $f(x+a)$, but the bifurcation is destroyed by the pinched condition, the reason we will see clearly in the proof at the next subsection.

To summarize this model, we plot in Figure 3.3 the invariant graphs for each of the cases with a concrete model of this type. The figures correspond to the pinched and non-pinched system, with a fixed positive a and λ values small and big respectively. We can see that from their pictures at the bottom row, the dynamics of the non-pinched cases are exactly similar to the unforced interval maps, except that the fixed points are replaced by invariant graphs. While for the pinched system whose pictures at top row, even though the λ value is taken to be very large, because the pinched points are attracted by the invariant graph, all points follow with them to this invariant graph too.

3.3.2 Proof of Theorem B

Theorem B consists of three assertions, we prove them one by one with the propositions given below in this subsection for the case $a > 0$. For $a < 0$, the proof goes analogously, the only difference is the change of sign. More precise, first we prove the existence of the unique invariant graph Φ_λ and its attraction inside the invariant region $\mathbb{S}^1 \times \mathbb{R}^+$. Next we show that, for both the pinched case and small enough λ, all the points of the system enter eventually this region of $\mathbb{S}^1 \times \mathbb{R}^+$, thus this graph Φ_λ is also the unique attracting invariant graph of the system in these two situations. Finally, we try to prove there exists bifurcation with the increasing of λ in the non-pinched case.

Existence of Φ_λ as attractor

One way for this proof is to just follow Keller. Taking an upper bound M of f, the transfer operator on the function space acting on $x = M$ produce a decreasing sequence of continuous curves. Meanwhile the same action on $x = 0$ gives an increasing sequence. This two sequences can be seen have the same limit Φ_λ, which is then continuous. In this memoir we don't involve the detailed investigation of the regularity of the attracting curves, or the estimation of the convergent speed of the points to the curve. Since the discussion of these issues is in fact exactly the same as Keller for this model, we refer interested reader to [44]. Here we prove the existence of Φ_λ as attractor by a direct method, using the contraction from the concavity.

The proposition below is derived from Lemma 3.2.9, which requires all the fibre maps to be α-concave defined on a closed interval. Since the fibre maps for our model are bounded, we can apply this lemma after the second iterate, then each f_θ are limited in a closed interval. Moreover, the concavity condition implies that, at least the left one-sided derivative of f exists at each point of x and is monotonically decreasing. Hence value at the upper bound is the minimum of the left one-sided derivative in this interval, then we can choose a small enough α such that f is α-concave in this domain. Now Lemma 3.2.9 gives directly the following.

Lemma 3.3.3. *Take (θ_0, x_0) and (θ_0, y_0) for any $\theta_0 \in \mathbb{S}^1$, if both $x_0 > -a$ and $y_0 > -a$, then*

$$\lim_{n\to\infty} |x_n - y_n| = 0.$$

Proof. First note that, Observation 3.2.2 shows if $-a < x_0 < 0$, then $0 \leq x_1$, and then $x_2 > 0$ if $g(\theta_1) > 0$. Thus we can start the following arguments from x_1 or x_2 if necessary.

If there is any n such that $g(\theta_n) = 0$, the result is true trivially because $x_{n+1} = y_{n+1} = 0$. Otherwise, $g(\theta_n) > 0$ for all $n \in \mathbb{N}$, then we have

$$|x_n - y_n| = \min\{x_n, y_n\}\kappa(x_n, y_n) = \min\{x_n, y_n\}\kappa(x_0, y_0)\prod_{k=0}^{n-1}\frac{\kappa(x_{k+1}, y_{k+1})}{\kappa(x_k, y_k)}.$$

Corollary 3.2.9 tells us that, for each $\frac{\kappa(x_{k+1},y_{k+1})}{\kappa(x_k,y_k)}$, it is smaller than a constant number which is smaller than 1. Hence, the products in the above equality goes to 0 as $n \to \infty$. □

Proposition 3.3.4. *There is a unique invariant graph Φ_λ in the region $\mathbb{S}^1 \times \mathbb{R}^+$. It is continuous and attracts all the points in the region $\mathbb{S}^1 \times [-a, +\infty)$.*

Proof. The above lemma says that, for all the initial values (θ_0, x_0) with $x_0 > -a$, there is only one point left eventually for each fibre after iterations. Thus we obtain a function by corresponding each $\theta \in \mathbb{S}^1$ with the point left in that fibre, this function is the Φ_λ that we claim. By definition Φ_λ must be unique and attract all the points belong to $\mathbb{S}^1 \times \mathbb{R}^+$ ($\mathbb{S}^1 \times [-a, +\infty)$ in fact). In view of the transfer operator, Φ_λ is also the only possible limit for $\mathcal{T}^n(x = -a)$ and $\mathcal{T}^n(x = c)$ with any constant c larger than the upper bound of f. These two sequences of graphs are monotone decreasing and increasing respectively, so their common limits Φ_λ must be invariant and continuous. □

Remark 3.3.5. We point out two facts here.

(1) Notice first that, by the proof of this proposition, the existence and the properties of Φ_λ don't depend on whether the system (3.8) is pinched or not.

(2) It can also be seen that f is not necessarily bounded. In fact, if there is big enough $c > 0$ which satisfies that $\mathcal{T}(x = c) \leq c$, the arguments of our proof still work. This is indeed the case for a large number of unbounded maps who are increasing and concave.

$\diamond$

Bifurcation problem

The essential difference between the second and third parts of our theorem is that, whether there exists a bifurcation at the negative part. There is a bifurcation that could occur at the negative part if and only if the system (3.8) is non-pinched.

In the following, we prove first that, in the pinched system, Φ_λ is the only invariant graph who attracts all the points in the system, so there is no any bifurcation possible. The non-pinched case is considered after that.

Pinched case: bifurcation destroyed

Proposition 3.3.6. *If there is some $\theta_0 \in \mathbb{S}^1$ such that $g(\theta_0) = 0$, then all points $(\theta, x) \in \mathbb{S}^1 \times \mathbb{R}$ will eventually enter $\mathbb{S}^1 \times [-a, +\infty)$, hence Φ_λ is the only invariant attracting graph for the whole system.*

Proof. We just prove that all the points must enter the region $\mathbb{S}^1 \times [-a, +\infty)$. Thus, by Proposition 3.3.4, all the points are eventually attracted by Φ_λ.

Let M denote any upper bound of $|f|$, since g is continuous and $g(\theta_0) = 0$, there must be an interval $J = (\theta_0 - \delta, \theta_0 + \delta)$ such that, for any $\theta \in J$, $g(\theta) \leq \frac{a}{2\lambda M}$. Then we have $-a < \frac{-a}{2} < \lambda f(x+a)g(\theta) < \frac{a}{2}$ for any $x \in \mathbb{R}$.

Due to the ergodicity of the irrational orbit, all the orbits eventually enter $J \times \mathbb{R}$, and then enter $\mathbb{S}^1 \times [-a, +\infty)$ at the next iterate. $\square$

Non-pinched case: saddle-node bifurcation For the one dimensional family $\lambda f(x + a)$ with $a > 0$, the dynamics of its negative part is a typical example of the saddle-node bifurcation. When the forced system is non-pinched, this saddle-node bifurcation is exact what happens at the negative part too. That is, for λ small enough, there is no invariant graph at the negative part, and all points enter $\mathbb{S}^1 \times \mathbb{R}^+$; with the increasing of λ over a critical value, there occur two new invariant graphs in the system, one is attracting, the other is repelling.

The proof of these results goes in the following way: first we show that there is no invariant graph in $\mathbb{S}^1 \times \mathbb{R}^-$ for small enough λ; next we prove that there are indeed two invariant graphs in this part if λ is big enough; and then we prove the facts that, if there are two invariant graphs at $\mathbb{S}^1 \times \mathbb{R}^-$ then this is the case for all bigger λ; similarly the case for no invariant graph and smaller value of λ; these facts together with the properties of invariant graphs in monotone system indicate the bifurcation can only happen in one λ value.

We give some notations that we need for the proof first. Recall that non-pinched case means $g(\theta) > 0$ for all $\theta \in \mathbb{S}^1$. Since $\mathbb{S}^1$ is compact, we denote that $m_g = \min\{g(\theta)\}, M_g = \max\{g(\theta)\}$. Clearly $M_g \geq m_g > 0$. Let l be some lower bound of f, thus $l \leq f(x)$ for all $x \in \mathbb{R}$, and $l < 0$.

Now we begin our proof. First we show that, for the same reason of the pinched case, if λ is small enough, all points enter $\mathbb{S}^1 \times \mathbb{R}^+$, then Φ_λ is the only invariant graph for the whole system.

Proposition 3.3.7. *If $\lambda < \frac{-a}{M_g l}$, all initial points go eventually into $\mathbb{S}^1 \times [-a, +\infty)$, thus Φ_λ is the only invariant curve and the attractor for the whole system.*

Proof. We only need to show that the x value for all the points will go larger than $-a$ under some iterates. Let $\lambda < \frac{-a}{M_g l}$, which is equivalent to $\lambda l M_g > -a$. By definition of l and M_g, $\lambda f(x+a)g(\theta) \geq \lambda l M_g > -a$ as required. □

Next, if λ is big enough, we prove that there are two other invariant graphs in the region of $\mathbb{S}^1 \times (-\infty, -a)$, thus there are two attracting invariant graph and one repelling one in the system simultaneously.

Proposition 3.3.8. *For any $c > a$, let $\lambda \geq \frac{-c}{m_g f(-c+a)}$, then there exist an invariant curve Ψ_λ with $\Psi_\lambda \leq -c$, which attracts all the points of $(\theta, x) \in \mathbb{S}^1 \times (-\infty, -c)$. Moreover, there is a repelling invariant graph Γ_λ, which lies in $\mathbb{S}^1 \times (-c, -a)$.*

Proof. Notice that $\lambda \geq \frac{-c}{m_g f(-c+a)}$ is equivalent to $\lambda m_g f(-c+a) \leq -c$. Take any points on $x = -c$, they all have $\lambda f(-c+a)g(\theta) \leq \lambda m_g f(-c+a) \leq -c$. It implies that, in terms of transfer operators, we have $\mathcal{T}(-c) \leq -c$. Hence $\mathcal{T}^{n+1}(-c) \leq \mathcal{T}^n(-c)$ for all $n \geq 0$. We define $\Psi_\lambda = \lim_{n\to+\infty} \mathcal{T}^n(-c)$. Moreover, similar to Lemma 3.2.8, now the convexity gives the contraction which makes Ψ_λ an attracting invariant graph. On the other hand, Ψ_λ must also be the limit of $(T^n(l))$, which is monotone increasing. This means that Ψ_λ is continuous just like the case of Φ_λ for $x > 0$.

Using the backward transfer operator to the graph $\mathcal{T}(-c)$ and $x = -c$, we have $\mathcal{R}(\mathcal{T}(-c)) = -c \geq \mathcal{T}(-c)$. Hence $\mathcal{R}^2(\mathcal{T}(-c)) = \mathcal{R}(-c) \geq -c = \mathcal{R}(\mathcal{T}(-c))$, which follows $\mathcal{R}^{n+1}(-c) \geq \mathcal{R}^n(-c)$ for all $n \geq 0$. Similarly, we also have $\mathcal{R}^{n+1}(-a) \leq \mathcal{R}^n(-a)$ for all $n \geq 0$. The limit of these two sequences is just the repelling invariant graph. □

Finally, we show that, these two new invariant graphs come from a bifurcation at some critical value of λ. This can be derived from the fact that, in such kinds of systems, the invariant graphs also have some monotonicity with respect to the parameters.

Proposition 3.3.9. *If the invariant graphs Ψ_{λ_0} and Γ_{λ_0} exit for some parameter value λ_0, then the invariant graphs Ψ_λ and Γ_λ also exit for any $\lambda > \lambda_0$. Moreover, $\Psi_\lambda < \Psi_{\lambda_0}$ and $\Gamma_\lambda > \Gamma_{\lambda_0}$.*

Proof. Ψ_{λ_0} is invariant under the value λ_0 means the invariant equation below is satisfied,

$$(\mathcal{T}_{\lambda_0}\Psi_{\lambda_0})(\theta+\omega) = \lambda_0 f(\Psi_{\lambda_0}(\theta)+a)g(\theta) = \Psi_{\lambda_0}(\theta+\omega).$$

We know that $\Psi_{\lambda_0} < -a$, this implies $f(\Psi_{\lambda_0}(\theta)+a) < 0$. Now for any $\lambda > \lambda_0$, the the system is given by function $\lambda g(\theta)f(x+a)$, the transfer operator acts on the curve Ψ_{λ_0} gives

$$(\mathcal{T}_\lambda\Psi_{\lambda_0})(\theta+\omega) = \lambda f(\Psi_{\lambda_0}(\theta)+a)g(\theta) < \lambda_0 f(\Psi_{\lambda_0}(\theta)+a)g(\theta) = \Psi_{\lambda_0}(\theta+\omega).$$

That is, $\mathcal{T}_\lambda\Psi_{\lambda_0} < \Psi_{\lambda_0}$. Hence $\mathcal{T}_\lambda^{n+1}\Psi_{\lambda_0} \leq \mathcal{T}_\lambda^{n}\Psi_{\lambda_0}$ for all $n \geq 0$. Now define $\Psi_\lambda = \lim_{n\to+\infty}\mathcal{T}_\lambda^n\Psi_{\lambda_0}$, according to the proof of the above proposition, this limit Ψ_λ is just the invariant graph under the value λ.

Analogously, under the action of the backward operator $\mathcal{R}$, it is easy to show $\mathcal{R}_\lambda\Gamma_{\lambda_0} > \Gamma_{\lambda_0}$. Define $\Gamma_\lambda = \lim_{n\to+\infty}\mathcal{R}_\lambda^n\Gamma_{\lambda_0}$ then the statement on invariant graphs Γ_λ follows also. □

Now we come to the last claim of the bifurcation at a critical value with respect to parameter λ.

Proposition 3.3.10. *For any fixed a, there is a value λ_0 such that: when $\lambda < \lambda_0$, all the points of system are attracted by the only invariant graph which is $\Phi_\lambda > 0$; when $\lambda > \lambda_0$, there are two other invariant graphs occurring in the region $\mathbb{S}^1 \times (-\infty, -a)$. The lower one is attracting and is decreasing with respect to the increasing of λ; the upper one is repelling and is increasing with respect to the increasing of λ.*

Proof. The monotonicity of the invariant graphs with respect to λ is proved in the above proposition. Proposition 3.3.9 also gives that, for any $\lambda \in (\lambda_0, +\infty)$, there also exist two invariant graphs provided λ_0 admits these two. Thus we can define a lower bound of all values of λ which admit two invariant graphs at $\mathbb{S}^1 \times (-\infty, -a)$. Denote this value by λ_l, we have two invariant graphs at $\mathbb{S}^1 \times (-\infty, -a)$ for any $\lambda \in (\lambda_l, +\infty)$.

Meanwhile, the values of λ for which all the points in $\mathbb{S}^1 \times (-\infty, -a)$ go eventually up to $x = -a$ (and then to $\mathbb{S}^1 \times \mathbb{R}^+$) also form an interval $(\lambda_h, 0)$. Next, we show that the values of these two bounds can only be $\lambda_l = \lambda_h$, which indicate our claim.

Notice that, the proof of Proposition 3.3.9 shows clearly that the monotonicity of the invariant graph with respect to the parameter λ is valid for any invariant graph at $\mathbb{S}^1 \times (-\infty, -a)$. It depends on only the forward or the backward transfer operator to be used, not on which invariant graph is. Precisely, if φ_{λ_0} is any invariant graph at $\mathbb{S}^1 \times (-\infty, -a)$ for some λ_0, then $\mathcal{T}_\lambda \varphi_{\lambda_0} < \varphi_{\lambda_0}$ and $\mathcal{R}_\lambda \varphi_{\lambda_0} > \varphi_{\lambda_0}$ for any $\lambda > \lambda_0$. Therefore, whenever there is a value λ_0 such that there exists an invariant graph in this region, then Proposition 3.3.9 indicate there must be two invariant graphs for all $\lambda > \lambda_0$. Hence $[\lambda_l, \lambda_h]$ can only consist of a single point, which is the critical value of bifurcation. □

3.4 The second model and its dynamics

In this final section we make a short discussion of the second model, which is also a generalized Keller's model. The main task is to understand the transition with respect to the parameter a and λ. It can be seen that the approach of the proof is in the same spirit of the previous one. With the tools and steps we have developed in the second section, basing on the graph analysis of the unforced interval maps intuitively, it is enough to obtain a good understanding of the models. To avoid duplication, here we adopt templet of different situations of the previous model to give the proof where there occur the same cases. Then our focus is put more on the dynamical behaviour and the distinction between the pinched and non-pinched systems of this family.

We first give out the concrete conditions and a general description of the dynamics of this family, then check the dynamics of the corresponding unforced interval maps shortly, and explain why the dynamics of the forced systems behave as we describe in the last.

The family that we consider is also in the form of (3.2). We remove the restriction of f being bounded, and preserve the monotonicity also with a concave or convex structure on it. Precisely, in (3.2) the maps f is a real function which satisfies:

(1) f is continuous;

(2) f is strictly increasing on $\mathbb{R}$ and $f(0) = 0$;

(3) f is strictly concave or strictly convex for all $x \in \mathbb{R}$.

We take only the model of f being convex as examples to exhibit our treatments. The arguments for the concave case is in fact totally analogous, just as the difference for their dynamics is only in appearance. That is, for the invariant graphs of f convex, the attracting one is below the repelling; while for f concave, the attracting one is above the repelling. Any points which is not on the invariant graphs either go toward the attracting one monotonically, or leave the repelling one to infinity monotonically, which is the same way as the unforced interval maps f. We plot in Figure 3.5 and Figure 3.6 the attractors and repellors of some concrete examples of this kinds of models, which correspond to the non-pinched and pinched one respectively. Theorem C gives the details of the dynamics of such families.

Theorem C. *Let λ be fixed, we can see the dynamics are as follows.*

(1) There is an $a_0 \geq 0$ such that, if $a > a_0$, there is no any invariant graph and all points of the system go to infinity.

(2) If $a < a_0$ there exist two invariant graphs. The upper one is repelling such that all points up to it go to infinity; the lower one is attracting who attracts all points below the upper invariant graph.

Moreover, if we let a increase from negative, we have the following.

(3) When $a < 0$, the upper one is upper than $x = 0$, the other is lower than $x = 0$.

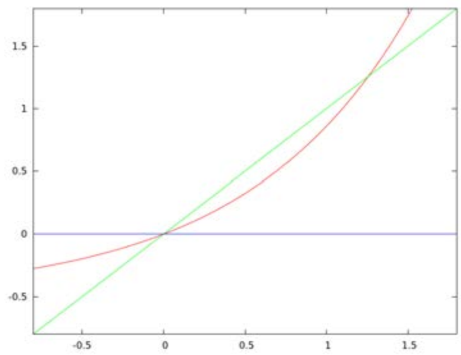
(4) With the increasing of a, the upper invariant graph goes down and the lower one goes up. At $a = 0$, one of them becomes $x = 0$.

(a) If the system is non-pinched, both the invariant graphs go up or below $x = 0$ with the increasing of a, and finally merge into one at the value $a_0 \geq 0$.

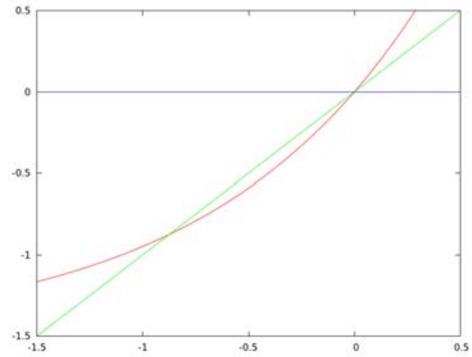
(b) If the system is pinched, and

i. if λ is relative small such that the lower invariant graph is $x = 0$ at $a = 0$, both invariant graphs are upper than $x = 0$ with the increasing of a, and finally they merge into one at the value $a_0 > 0$;

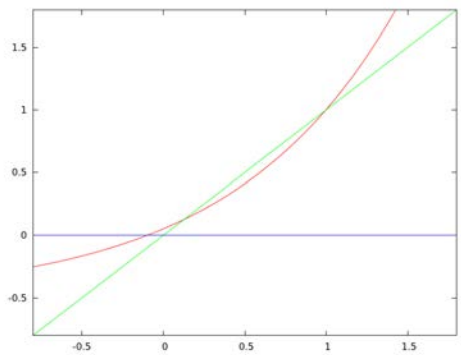
ii. if λ is big such that the upper one is $x = 0$ at $a = 0$, then $x = 0$ is repelling and the lower one is attracting. This lower one is strange and nonchaotic who intersects $x = 0$ in a dense set, that is, it is an SNA. Moreover, the critical value is $a_0 = 0$.



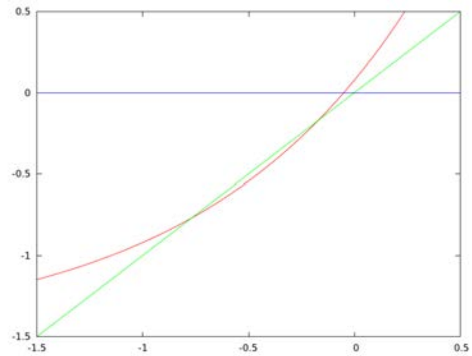
(a) $a = 0$ with λ small.



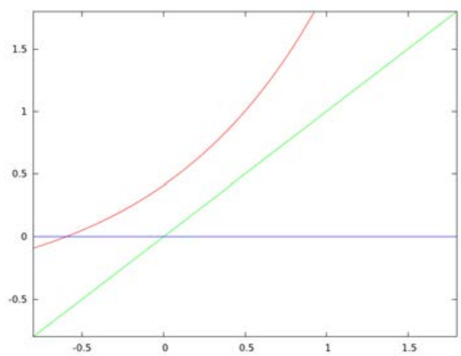
(b) $a = 0$ with λ big.



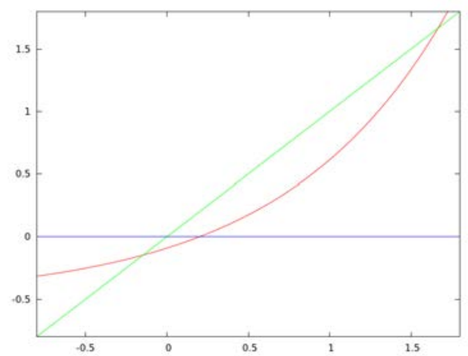
(c) $a > 0$ small with λ small.



(d) $a > 0$ small with λ big.



(e) $a > 0$ big enough.



(f) $a < 0$.

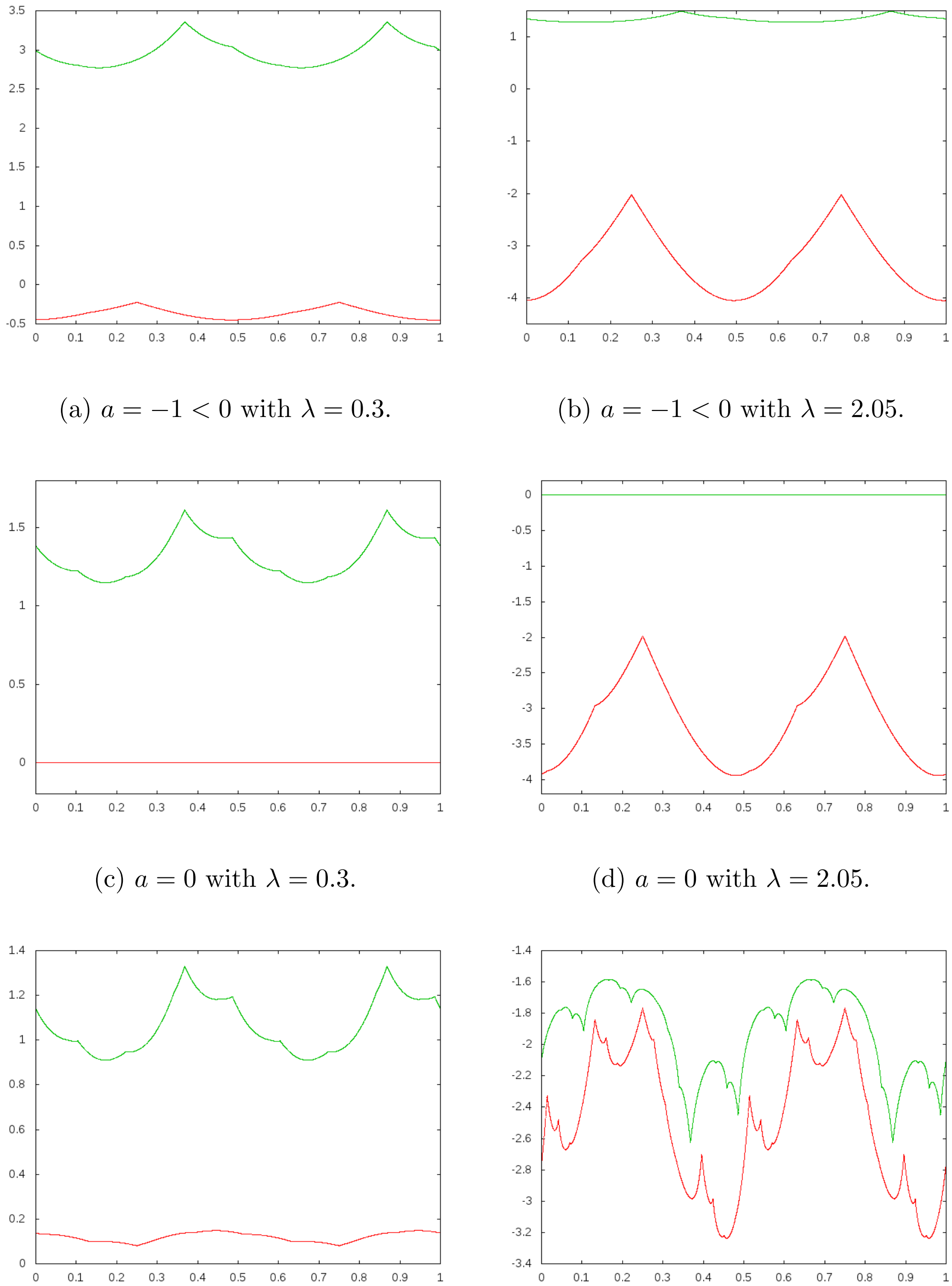
Figure 3.4: Graphs of $\lambda f(x + a)$ for different cases.

Comparing with the dynamics of the unforced one-dimensional maps $\lambda f(x+a)$, which are all well-known already, it can be seen clearly the common and the specific features between the forced and unforced systems. The one-dimensional maps admit a saddle-node bifurcation with the increasing of the parameter a. The above results demonstrate that this is exactly what happens for those non-pinched systems, only the fixed points are replaced by continuous invariant graphs now. But for the pinched ones, all the bifurcations which take place at critical value of $a_0 < 0$ in the unforced systems are moved to $a_0 = 0$ in the last case of the above theorem.

Unforced interval maps First we have a short look at the unforced interval maps, whose dynamics is the base for the analyzing of the corresponding forced one. These convex increasing interval maps are rather simple and well-understood already, which are usually used as typical models for the saddle-node bifurcation. More precisely, they have no any fixed point before a critical bifurcation value, and all the points go to $+\infty$ eventually. After the bifurcation at critical value, there are two fixed points occurring in the system. The left (small) one is attracting, and the right (big) one is repelling. The attracting fixed point attracts all the points which are smaller than the repelling one, while other points larger than the repelling one all go eventually to $+\infty$.

In Figure 3.4, we plot all the possible cases for such maps in form of $\lambda f(x+a)$ with $f(0) = 0$, which is not in exactly the bifurcations. Fix λ and let a increase from negative value, the behaviour of such maps go in the following way. When $a < 0$, for any points $x \leq 0$, $\lambda f(x+a) < 0$ and then remain to be negative for all iterations. Hence the attracting fixed point is negative, the repelling one can only be positive so that they locate at the two sides of zero. With the increasing of a, the left fixed points moves to the right continuously, while the right one to left. When a reaches 0, $\lambda f(0+a) = 0$, hence 0 is one of the fixed point. If $x = 0$ is attracting or repelling, depends on the value of λ: if λ is big enough, it is repelling, and the attracting one is at its left side, so is negative; for smaller λ, $x = 0$ is attracting and the repelling ones is positive (see Figure 3.4(b) and Figure 3.4(a)). Keeping increasing a to $a > 0$, then both of the fixed points go to the same side of 0, which are positive for λ small or negative for λ big. At the last, they merge into one at the bifurcation point, then go to $+\infty$ with all other points as shown by Figure 3.4(e).

Non-pinched case With the behaviours of the unforced interval maps in mind, now we analyze the forced system with fibre maps $\lambda g(\theta) f(x+a)$. In



(a) $a = -1 < 0$ with $\lambda = 0.3$.

(b) $a = -1 < 0$ with $\lambda = 2.05$.

(c) $a = 0$ with $\lambda = 0.3$.

(d) $a = 0$ with $\lambda = 2.05$.

(e) $a = 0.1 > 0$ small with $\lambda = 0.3$.

(f) $a = 1 > 0$ small with $\lambda = 2.05$.

Figure 3.5: Graphs of different cases for the non-pinched system $\lambda g(\theta) f(x + a) = \lambda |1 + \cos(2\pi\theta)| (\exp(x + a) - 1)$. The red one is attracting and the green one is repelling.

Figure 3.5, we take $\lambda g(\theta)f(x+a) = \lambda\,|1+\cos(2\pi\theta)|\,(\exp(x+a)-1)$ as an example, and plot the attractors and repellors for different cases of this model. The attractors are obtained by forward iterations with very small initial value, and are plotted in red. The repellors are by backward iteration with very big initial value and are plotted in green. From top to bottom, each row corresponds to a negative, zero and positive value of a before all points go to infinity. Concerning for the values of λ, it is taken as 0.3 for the left column, and 2.05 for the right.

First we point out two basic facts of such system, from which the results for each cases are derived.

Remark 3.4.1. (1) Like the previous model, there exists monotonicity with respect to both a and λ in this system. Particularly with respect to the increasing of a, the attracting invariant graph is strictly increasing, and the repelling one is strictly decreasing. This is exactly what happens for the fixed points of the unforced systems.

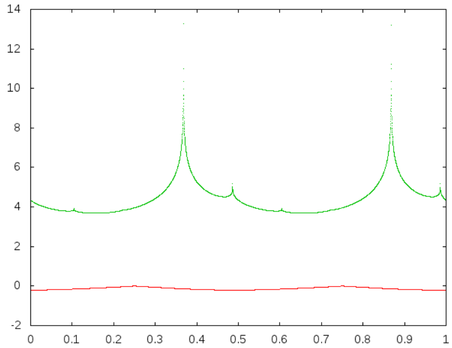
The arguments of the monotonicity are the same as those in the previous model with respect to λ. For example, when we fix the value of λ and let a vary as a parameter, then if $a_1 < a_2$, $f(x+a_1) < f(x+a_2)$ for any x. In term of transfer operator, $\mathcal{T}_{a_1}\psi < \mathcal{T}_{a_2}\psi$ for any curve ψ. Since the x values of any points is bigger in system of parameter a_2 than a_1. Thus the attracting invariant graph must be increasing with the increasing of a. It is because . Similarly, the points go to infinity for some a_0 must also go to infinity for any $a > a_0$, which implies the repelling fixed curve goes down as a increasing. Therefore the two invariant graphs go closer with the increasing of a.

(2) The inverse function of a monotone increasing convex function is a monotone increasing concave one, and vice versa. So the backward iteration acts just like the forward one on a monotone increasing concave system. Notice that an attracting invariant graph of the backward iteration is the repelling one of the forward iteration, and vice versa.

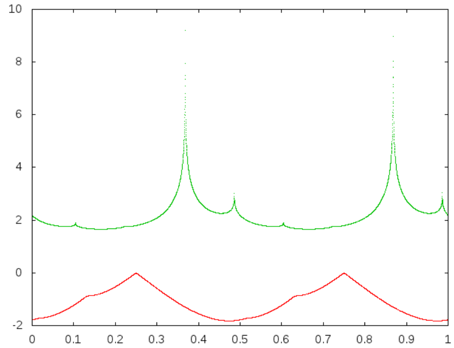
$\diamond$

Now let us fix λ and consider the dynamics of this second model with respect to the increasing of parameter a from negative.

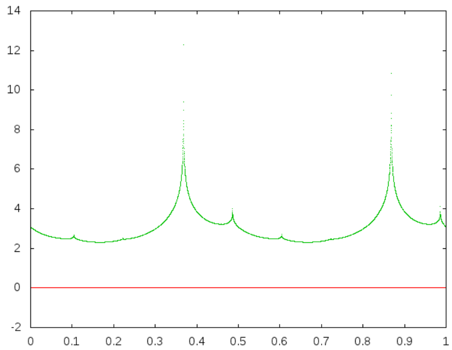
(1) $a < 0$ implies $\lambda g(\theta)f(0+a) < 0$ for any $(\theta, 0)$, hence all the points (θ, x) with $x \le 0$ always remain at the negative part. Notice that all the fibre maps have the shapes shown in Figure 3.4(f). So the negative part is exactly the case of the negative part of the previous model when $a < 0$,



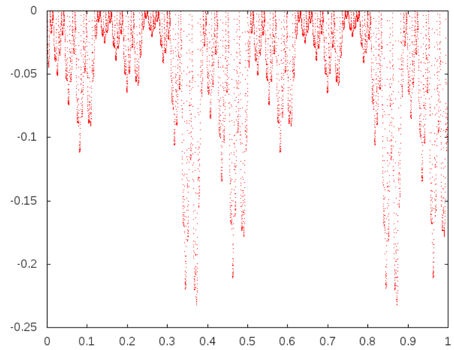
(a) $a = -1 < 0$ with $\lambda = 0.3$.



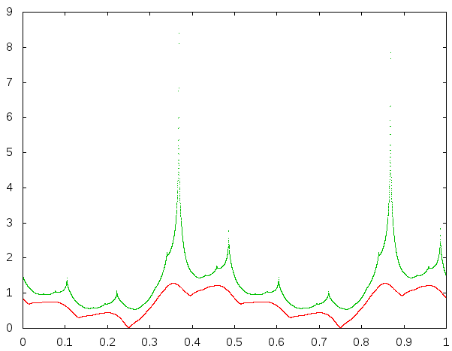
(b) $a = -1 < 0$ with $\lambda = 2.05$.



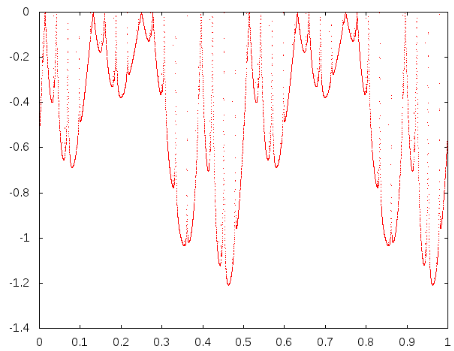
(c) $a = 0$ with $\lambda = 0.3$.



(d) $a = 0$ with $\lambda = 2.05$.



(e) $a = 0.9 > 0$ small with $\lambda = 0.3$.



(f) $a = 0$ small with $\lambda = 2.5$.

Figure 3.6: Graphs of different cases for the pinched system $\lambda g(\theta) f(x+a) = \lambda \left|\cos(2\pi\theta)\right| (\exp(x+a) - 1)$. The red one is attracting and the green one is repelling.

which says there is a unique attracting graph in this region. For the positive part, we consider the backward iteration. This is the case of the positive part of the previous when $a > 0$, so the unique attracting invariant graph under backward iteration gives the repelling one for the our original one.

(2) When $a = 0$, since $f(0 + a) = f(0) = 0$, we have $x = 0$ is an invariant graph. Whether it is attracting or repelling depends on the value of λ now That is, it is attracting for smaller λ and repelling for big (refer to Figure 3.4(a) and Figure 3.4(b)). The negative part is the same with the first model at $a = 0$, the positive part under backward iteration can also be viewed as the positive part in that model.

(3) When $a > 0$, due to the monotonicity with respect to a, the invariant graphs move continuously, so we have:

   (a) when λ is big such that $x = 0$ is repelling, the repelling invariant graph moves down so that both invariant graphs are at the negative part.

   (b) when λ is small such that $x = 0$ is attracting, the attracting invariant graph moves up so that both invariant graphs are all at the positive part.

(4) There is a critical value a_0 so that, if $a > a_0$, all the points go to infinity and there is no any invariant graph now. This is also derived by the monotonicity with respect to a, the arguments are the same in Proposition 3.3.10.

Generally, the dynamics of non-pinched systems of this type model are generic the same as the unforced interval maps also. Comparing the invariant graphs in Figure 3.5 and the fixed points in Figure 3.4, this same character is exhibited clearly.

pinched case For the pinched case, the basic analysis goes the same way as the non-pinched case, except for, we need to pay special attention to where the pinched points go. Since we have proved in the second section that, all points who don't go infinity must follow their orbit in a pinched system.

Taken $\lambda g(\theta) f(x+a) = \lambda\,|\cos(2\pi\theta)|\,(\exp(x+a) - 1)$ as example now. We plot the attractors and repellors in Figure 3.6 for a reference. The arrangement for the parameter value is same with Figure 3.5, except for the last one

Figure 3.6(f).

For the analysis of the general models of this type of family, most of situations are same with the non-pinched systems. For example, when $a < 0$, we can consider the forward iteration of the negative part and the backward iteration of the positive one residual, it gives the existence and uniqueness of one invariant graph at each part. The only difference with the non-pinched case is that, some fibres are pinched to 0 now. Hence the lower and attracting invariant graph has value 0 at those pinched points. Meanwhile, the repelling invariant graph at the positive part forms the boundary for points with two different dynamics, which are points who is attracted by the attracting invariant graph and follow the pinched orbit, and points who go to infinity. Notice that, all the preimages of the pinched points are fibres, so the pictures of Figure 3.6 show there are places of this repelling invariant graph which is at infinity, thus this repelling invariant graph cannot be continuous everywhere.

We don't repeat other similar situations, and discuss only the difference below. When $a = 0$, $x = 0$ is also invariant, it changes from attracting to repelling with the increasing of λ. When it becomes repelling, the attracting invariant graph is below it. Moreover, this attracting graph must intersect $x = 0$ at a dense set since the system is pinched. So there is an SNA in the system, which is exactly the case of the negative part of the first model. We plot in Figure 3.6(f) with a little large λ for a better view.

The difference with the non-pinched case is then when a goes larger than 0. For the value of λ with the SNA in the system, unlike the case λ is small such that $x = 0$ is attracting, for which the attracting graph increases with a as the same way (refer to the fiber maps in Figure 3.4(c)), and the system keeps two invariant graphs until the bifurcation value; if $x = 0$ is repelling with an SNA, the repelling invariant graph decreases with respect to the increasing of a, this implies that the repelling invariant graph should go lower than $x = 0$, thus all points on $x = 0$ go to infinity now. So there is no any invariant graph in the system, since all the other points have to follow the pinched points to infinity too. This is the same as the case of $a > 0$ of the previous model, all the points of the negative part are taken away to positive part by the pinched points.

It is interesting to notice that, for the values of λ such that there exist SNAs when $a = 0$, we have just shown that 0 is only the critical value for all these λ at which the bifurcations happen. Thus we obtain an example of an SNA created via a saddle-node bifurcation. Moreover, for the non-pinched system, the bifurcations all occur at some different values $a > 0$ corresponding the values of λ, now they are all fixed at $a = 0$. Meanwhile, if we fix $a = 0$ and let λ be the changing parameters, this example is also an example of a transcritical bifurcation which gives the birth of an SNA. Then we know that it is possible for the birth of SNA for all those typical bifurcations corresponding to the one dimensional systems.

Chapter 4

Attractors of quasi-periodically forced S-unimodal maps

The objects of this chapter are the transition of attractors in quasi-periodically forced S-unimodal systems, and the reverse bifurcations in form of invariant sets. We explain the reason of the reverse bifurcations, and present the general mechanism of the transition with respect to the increasing of forced term parameter for a fixed S-unimodal map. This mechanism is well verified with numerical evidences.

In fact, corresponding to every bifurcation of periodic orbit in the well-known S-unimodal families, there exists a reverse bifurcation of the attractor in form of chaotic intervals. The theories of bifurcations of periodic orbits are classical results on S-unimodal families. However, their reverse bifurcations can only be found in physical context up to now. The first main result of this chapter is Theorem D, by which we prove mathematically the reverse bifurcations. It also display clearly the correspondence of each pair of the bifurcations, which is exposed by the block structures of the unforced S-unimodal maps. The structures of restrictive intervals turn out to be the key factor for the transitions of both S-unimodal families and the quasi-periodically forced systems.

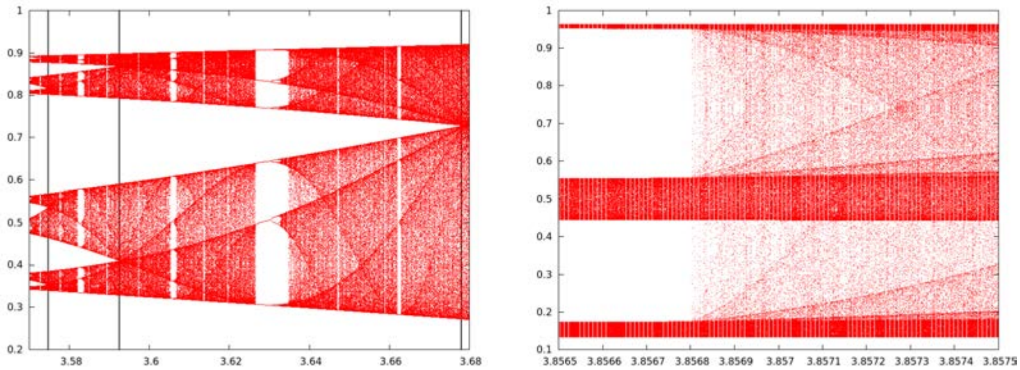
This chapter has four sections. In the first one we give a short introduction of the reverse bifurcation phenomena reported in physical context. The second section is devoted to the formal definition and properties of restrictive intervals, their representation via extension pattern, and the characterization of the dynamics of attractors by these intervals. We exhibit that, for a fixed S-unimodal map, the intersection of all respective intervals is a nested invariant set which contains the attractor, and the state of attractor can be specified by case of these restrictive intervals. With these knowledge, in the third section we discuss the reverse bifurcations of S-unimodal maps. We can also obtain an integrated perspectives on the transition and self-similarity for full S-unimodal families, by viewing each pair of corresponding bifurcations. The final main result of this chapter is given in the last section, where we claim the general mechanism of bifurcations of attractors in quasi-periodically forced S-unimodal maps. It displays how the periodicity changes with the increasing of the perturbation to a fixed S-unimodal map.

4.1 Introduction of reverse bifurcations

In this section we exhibit the phenomena of reverse bifurcations for S-unimodal maps. What follows is a brief introduction to the idea of their treatment and the concepts involved. Such reverse bifurcations of S-unimodal maps are reported for decades, but few mathematical treatments are known yet. Their mechanism is the base of the understanding of the corresponding forced system. So we get acquainted with these phenomena intuitively first, and then develop a series of discussions on this issue in the following sections.

Unimodal maps are continuous interval maps $f : [a, b] \to [a, b]$ who have only one extreme c on $[a, b]$ and map the endpoints to one of them. S-unimodal maps are unimodal maps with negative Schwarzian derivative in every point except for c. A prototype of S-unimodal family is given by the popular logistic family $f_\mu(x) = \mu x(1-x)$ where μ is a parameter. The classification of topological attractors of S-unimodal maps and the bifurcations of periodic orbits are all very classical results in the field of dynamical systems on intervals. It is known that (see [32, 52]), for an S-unimodal map, there exists only one topological attractor which belongs to one of the following three types: an attracting periodic orbit, a solenoidal (Feigenbaum-like) attractor, or a cycle of intervals. In the transition of an S-unimodal family like f_μ, attractors of these three types alternate in a certain order. The solenoidal attractors are not generic in the sense that they occur only for a zero measure set of parameter value μ. For the other two types, it is well-known that a new attracting periodic orbit comes from a bifurcation. The two generic types of such bifurcations are period-doubling and saddle-node respectively, the detailed theories of them can refer to [31] or any standard textbook. The period-doubling cascade to chaos is one of the most important result in chaotic dynamical systems. Relative to the developed theory of bifurcations for periodic orbits, the situation of cycles of intervals can only be seen in physical context. In this chapter we demonstrate that there are reverse bifurcations of these cycles of intervals, each corresponds a classical bifurcation of periodic orbit.

When the attractor is a cycle of n intervals, each of the interval is a



(a) Attractor diagram of $f(x) = \mu x(1-x)$ for $c \in [3.57, 3.68]$. The attractor changes from 8 bands to 4, 4 bands to 2, and 2 bands into 1 at about 3.5747, 3.5924 and 3.678 respectively, where we marked by vertical lines.

(b) Attractor diagram of $f(x) = \mu x(1-x)$ for $c \in [3.8565, 3.8575]$. At the place a little larger than 3.8658, the attractor changes from 3 bands directly into a one piece band. It is the reserve of saddle-node bifurcation.

Figure 4.1: Examples of band merging by reverse bifurcations.

chaotic subsystem under f^n with sensitive dependence on initial conditions, hence it appears as n bands. There are two types of reverse bifurcations of such attractors. In 1980, Lorenz published a paper [51], where he found a series of procedure which he thought to be reverse of the period-doubling cascade in a one-parameter unimodal family. The phenomena are: after the period-doubling cascade to the Feigenbaum attractor, there exists a series of μ_i values for $i = \dots n, n-1, \dots 1$, such that 2^i bands merge into 2^{i-1} bands at μ_i, with two adjacent bands getting in touch and becoming one. See Figure 4.1(a) for these reverse bifurcation of the logistic family.

Another type of bands merging was found by Grebogi et al [30], which happens when the unstable period 3 orbit born from a saddle-node bifurcation just touches the chaotic three-bands attractor. It results in a bifurcation from a chaotic attractor in three narrow bands to a chaotic attractor filling a entire interval. This bifurcation is called an interior crisis in physical context, which occurs at a precise parameter value that marks a discontinuous jump in size of the chaotic attractor, see Figure 4.1(b) for an example. Such jumps or explosions are typical and common in nonlinear dynamics, for example, phenomenon in the forced Duffing oscillator by Ueda [75], and also the quasi-periodically forced S-unimodal maps we will study later in this chapter. It is observed that these explosions always involve collisions between attractors and unstable periodic motions or their insets, which are basin boundaries.

In this chapter we show mathematically their precise mechanisms and the reason for the S-unimodal maps, and then explore those quasi-periodically forced ones.

In fact, the reasons of mechanism for these two types of reverse bifurcations are totally consentaneous for the S-unimodal maps. The crucial concept to understand the mechanism is the restrictive intervals of a periodic orbit, which we extend the original notion from Guckenheimer [32]. Brief saying, if $\{p_1, p_2, \dots, p_n\}$ is a periodic orbit of period n for a unimodal map, then associated to this orbit, there exists a unique set of K intervals $[p_i, q_i]$ (or $[q_i, p_i]$), for $0 \leq i < K$ with $K = n$ or $2n$. These intervals $\{[p_i, q_i]\}$ satisfies the following properties: $f^K(p_i) = f^K(q_i)$; if $i \neq j$ and $p_i \neq p_j$, $[p_j, q_j] \cap [p_i, q_i] = \emptyset$; and $f([p_i, q_i]) = [p_{i+1}, q_{i+1}]$ except for the central one. By the central interval we mean the only one of $\{[p_i, q_i]\}$ which has $f(p_i) = f(q_i)$. These intervals $\{[p_i, q_i]\}$ are restrictive if $f^K([p_i, q_i]) \subseteq [p_{i+1}, q_{i+1}]$ for all $0 \leq i < K$. For any unimodal map f, the union of a set of restrictive intervals is forward invariant, and the intersection of all its restrictive intervals forms a nested set. If f is in addition S-unimodal, then this nested invariant set contains the only attractor of f. Moreover, the situation of restrictive intervals also characterize the feature of the attractor of this S-unimodal map f: if the restrictive intervals are infinitely many, then the attractor is solenoidal; if the restrictive intervals are finitely many, and there is no other periodic point inside the innermost one, the attractor is a periodic orbit; otherwise, the attractor is a cycle of intervals when there are other periodic points in the innermost one.

When a bifurcation happens for S-unimodal maps, it gives birth of a new attracting periodic orbit, and together a repelling one in the case of a saddle-node bifurcation, or the original attracting orbit changes to be repelling in the period-doubling case. Simultaneously, intervals linked to new orbit(s) also come into being inside the old innermost restrictive intervals. The new attracting orbit becomes attractor, contained inside those intervals of the repelling orbit of the bifurcation. After that a series of new bifurcations happen in order, while all new born attractors are still contained inside these restrictive intervals, since they are still restrictive. This lasts to the final moment before they becomes non-restrictive, at which time the attractor comes to a cycle of intervals which fill in these intervals, so it certainly touches the unstable orbit whose points are the endpoints of the intervals. Then the reverse bifurcation happens, the innermost restrictive intervals becomes the one before the bifurcation, and the attractor is seen as bands each inside one of them. This gives the mechanism of the reverse bifurcations. Under this framework, there is no difference between the mechanism of reverse bifurcations for both period-doubling and saddle-node ones. Hence we do not distinguish them particularly, and normally call them as bands merging for what happens at both of these reverse bifurcations.

For the attractors of quasi-periodically forced S-unimodal maps, if we fix an S-unimodal map and let a control parameter of the forcing term increase, we can see that there are also the reverse bifurcations of the bands merging. This also happens like the unforced maps, when the attractor goes beyond the boundary of the invariant region and spreads to the larger invariant region outside. What caused by it is the same combinatorial change as the unforced S-unimodal maps. With help of numerical evidences, this progress of a series of reverse bifurcations can been seen coincident with the order of the structure of the restrictive intervals of the fixed S-unimodal map. This gives a general mechanism on the change of the periodicity of the attractors for the forced systems.

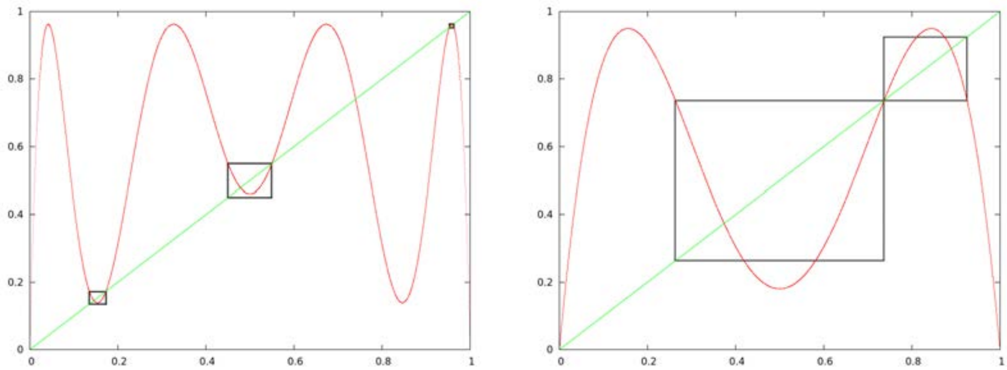
4.2 Restrictive intervals and structures of attractors

In this section we introduce the basic terminologies and definitions of this chapter. The key concept is restrictive intervals, which are sets whose union is forward invariant with one of their endpoint from a periodic orbit. They are the major tool for characterizing the periodicity and states of topological attractors. The content consists of three parts, each is given in one subsection. The first is to give the definition and basic properties of restrictive intervals. Next we introduce a combinatorial tool, extension pattern, which is a natural and convenient tool to describe the dynamics structure of restrictive intervals. Finally, we introduce the topological attractors of S-unimodal maps, and characterize them by restrictive intervals.

4.2.1 Definition and properties

This subsection is devoted to set up the concept of restrictive intervals of unimodal maps and to exhibit their properties. It can be seen finally that the intersection of all the restrictive intervals is a forward invariant nested set.

To understand the restrictive intervals, we need to learn a series of facts first. We start by construction to show that, to every periodic orbit of a unimodal map, there exists a set of intervals associated naturally. These intervals are selected preimages of the central interval (see definition below),



(a) Intervals of a periodic orbit of period 3. The red line is graph of f^3 with $f = 3.85x(1-x)$ (in green).

(b) Intervals of $2n$ type of a repelling fixed point. The red line is graph of f^2 with $f = 3.8x(1-x)$ (in green).

Figure 4.2: Examples of periodic intervals associated with periodic orbits.

and are uniquely determined by the periodic orbit. The construction of these intervals will show us that, it is equivalent for these intervals that they are restrictive, periodic and invariant. We list all those equivalent conditions by Proposition 4.2.4. Thereafter we give our definition of the restrictive intervals based on this proposition. Due to the invariance of these intervals, the periodicity of any orbit inside them must be a multiple of the periodicity of these intervals, the details are given by Proposition 4.2.7. Notice that, the invariance also implies that all these series restrictive intervals form a nested set, with the critical point inside their intersection. This are the crucial facts that we study the attractors of S-unimodal maps with restrictive intervals .

First we introduce some definitions and notations. By saying “an interval” we mean a “closed interval”. The notation $I = [a, b]$ means that a and b are the endpoints of I but does not necessarily mean that $a < b$ unless this is specified. This is also the case when we discuss other kinds of intervals. The n-fold iterates of a map f is denoted by f^n. Assume $f^n(p) = p$, one says then that p is periodic with *period* n. Moreover, if n is the smallest integer for which this is true, then we say that it is the *prime period*. In most of situations, when we say “period” we refer to the prime period. If $n = 1$, p is called a fixed point. The set $\mathcal{O}(x) = \{f^n(x)\}_{n=0}^{\infty}$ is called the *orbit* of $x \in I$. The limit set of $\mathcal{O}(x)$ is denoted by $\omega(x)$.

A continuous interval map $f : I \to I$ is called *unimodal* if it has only one extreme c on I, and if it maps the endpoints of I to one of them. Moreover, we assume f is strictly monotone on each of intervals $[a, c]$ and $[c, b]$.

For simplicity reason, we often prove our results for the case that c is the maximum. It is easy to see that the other case of c being minimum can be treated analogously, since they can be transformed into each other just by a simple change of coordinates. Points p and p' are called *"c-symmetric"* if $f(p) = f(p')$, and *"c-symmetric" interval* refers to an interval whose endpoints are just "c-symmetric". For two points $p, q \in I$, if $p \in (q, q')$, we say p is *"closer" to* c than q. If p is a periodic point of period n, we say p is *central* if it is closer to c than any other periodic points of its orbit, and call $[p, p']$ a *center interval.* Thus in the central interval, there is no any other periodic points of the orbit of p. Like the case of periodic points, we are also interested in whether the central intervals can exhibit some periodicity. For this we first need to show that, to each orbit of periodic points, there are a series of specific intervals linked. The details of these intervals are just given by the proof of the lemma below.

Lemma 4.2.1. *Given p_0 the central point of a periodic orbit of period n and $p_i = f^i(p_0)$, there is a set of intervals $J_i = [p_i, q_i]$ for $0 \leq i \leq K$ with $K = n$ or $2n$ such that, each $f|_{J_i}$ is monotone and $f(J_i) = J_{i+1}$ for $1 \leq i < K$, where $J_K = J_0 = [p_0, p_0']$.*

Proof. Assume that $p_0 \neq c$, otherwise we just take this set of intervals to be degenerated, which is exact the periodic orbit itself. Now we construct the set of intervals $J_i = [p_i, q_i]$ as follows.

If $n = 1$ and $f(c) \in [p_0, p_0']$, we are done with $J_0 = [p_0, p_0']$. Otherwise, let $p_2 = p_1 = p_0$, $q_2 = p_0'$, choose q_1 such that $f(q_1) = q_2$ and q_1 is at the same side of c as p_0 (see Figure 4.2(b)). This is for the case $n = 1$.

Now assume that $n > 1$. Let $q_n = q_0 = p_0'$ first, then choose q_i such that $f(q_i) = q_{i+1}$ and $f|_{[p_i, q_i]}$ is monotone for $0 < i < n$. This means that q_i is one of the preimage of q_{i+1} which belongs to the same side of c as p_i. So the construction can be completed if the preimages of q_{i+1} exist.

The existence of the preimages of points q_{i+1} is equivalent to $q_{i+1} \leq f(c)$, we know that $p_1 = f(p_0) < f(c)$, so we just need to prove $q_i < p_1$ for all $1 < i \leq n$. First we start from q_n. By assumption, $q_n = p_0'$, thus $q_n < p_1$ since $p_1 \notin (p_0, p_0')$ by the definition of the central point, hence we have the q_{n-1} required now. Using induction next, assume that we obtain q_k for $1 < k < n$ now. Notice that $p_i = f^i(p_0) < p_1$ for all $i \neq 1$, since p_0 is central. Thus we can obtain $q_k < p_1$. Otherwise, $q_k > p_1$ and $p_k < p_1$ implies $p_1 \in (p_k, q_k)$, and then $f^{n-k}(p_1) = p_{n-k+1} \in (p_n, q_n) = (p_0, q_0)$, which is a contradiction to p_0 being central.

When we get q_1 and if $q_1 > p_1$, the claimed intervals are all constructed completely, hence we stop with $K = n$ (see Figure 4.2(a)). If $q_1 < p_1$, we continue the same construction and recode the subindex by replacing n

with $2n$, which means that q_1 is reindexed as q_{n+1}. The same arguments above guarantee the construction process until we get the new q_1 reindexed, and finally $q_1 > p_1$ for this time. Notice that if $K = 2n$, then $p_{n+k} = p_k$, q_{n+k} and q_k are at different side of p_k (see the periodic orbit of period 3 in Figure 4.4(b)). □

Thus if $K = n$, $J_i \cap J_j = \emptyset$ for any $0 \leq i < j < K$; and for $K = 2n$, $J_i \cap J_j = \emptyset$ when $|i-j| \neq n$, and $J_i \cap J_j = \{p_i = p_j\}$ when $|i-j| = n$. In fact, these intervals linked to periodic orbit can expose many useful dynamical structures of both the orbit and the system, one is as the corollary below.

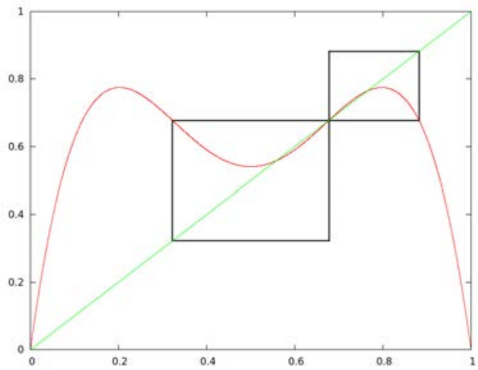
Corollary 4.2.2. *For p a periodic point of period n of a unimodal map f, if $(f^n)'(p)$ exits, then: $(f^n)'(p) = 0$ if and only if $c = f^i(p)$ for some $0 \leq i < n$; $(f^n)'(p) > 0$ if and only if $K = n$; and $(f^n)'(p) < 0$ if and only if $K = 2n$.*

Proof. By the chain rule, $(f^n)'(p) = \prod_{i=0}^{n-1} f'(f^i(p))$. Hence it is 0 if and only if the only extreme c belongs to this orbit. Now for the other two cases we assume that c is not in the orbit of p, and then we can just prove the "if" part to complete all the statements.

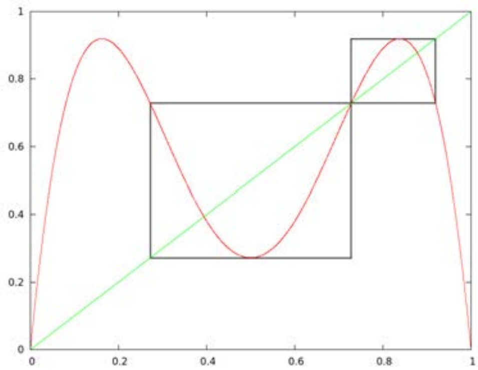
Notice that $(f^n)'(p)$ is a constant for all the points of the orbit of p, so we consider only the largest point of the orbit, who is denoted by p_1 like the previous lemma. If $K = n$, we have $[p_0, p_0'] = (f^{n-1})([p_1, q_1]) = [p_0, q_0]$, so $(f^n)([p_1, q_1]) = f([p_0, p_0']) \subseteq [p_1, b]$. This implies that, for any $x \in (p_1, p_1 + \delta)$ with $\delta > 0$, $f^n(x) > p_1 = f^n(p_1)$, hence $(f^n)'(p) > 0$. The same arguments show that, if $K = 2n$, $(f^n)([p_1, q_1]) \subseteq [a, p_1]$, so $(f^n)'(p) < 0$ (note for K, $(f^K)'(p) = (f^{2n})'(p) > 0$). □

Remark 4.2.3. For a family of unimodal maps who is continuous with respect to a parameter, it is common that the orbit of a periodic point also moves continuously with respect to the parameter. The arguments above implies that, the series of intervals linked to an orbit of periodic points changes its style whenever the central point moves cross the extreme c. ◇

Now, we discuss when a central interval can be periodic. An interval $J \subseteq I$ is called to be *periodic* with period n if $\mathrm{Int}(f^k(J)) \cap \mathrm{Int}(f^j(J)) = \emptyset$ $(0 \leq k < j \leq n-1)$ and $f^n(J) \subseteq J$. In such a case the set $\mathrm{Orb}(J) := \cup_{k=0}^{n-1} f^k(J)$ is called a *cycle* of intervals. Any cycle of intervals is necessarily forward invariant. A set I is *forward invariant* if $f(I) \subseteq I$. Thus a point p stays eventually inside a forward invariant set after it enters. For the S-unimodal maps, we will see in the fourth section that, the cycles of intervals containing the critical point c are of particular interest for their attractors. A simple and direct condition which ensures that a central interval is periodic is that, the central point p is restrictive, which means that $f^n(c) \in [p, p']$ by



(a) Restrictive intervals and graph of f^2 (red line) for $f = 3.1x(1-x)$.



(b) Restrictive intervals and graph of f^2 (red line) for $f = 3.6785x(1-x)$.

Figure 4.3: Examples of restrictive intervals of a repelling fixed point of $2n$ type.

the original idea of Guckenheimer in [32]. We will give our new definition of this concept after the following proposition, which gives out some clear and handy equivalent conditions for a central interval being periodic. These conditions are easily obtained by the construction of the series intervals of the periodic orbit.

Proposition 4.2.4. *let p_0 be the central periodic point of period n and p_i, J_i, K be defined as Lemma 4.2.1 above, then the following conditions are equivalent:*

(1) $f^K(c) \in [p_0, p_0']$;

(2) $f(c) \leq q_1$;

(3) $J_0 \ldots J_{K-1}$ *are all periodic;*

(4) each $f^K|_{J_i}$ *is unimodal for* $0 \leq i < K$;

(5) $\cup_{i=0}^{K-1} J_i$ *is invariant.*

Proof. By Lemma 4.2.1, we know that $f(J_i) = J_{i+1}$ for $1 \leq i < K$ and they are all monotone. This implies that, we only need to verify whether $[p_1, f(c)] = f(J_0) \subseteq J_1 = [p_1, q_1]$ or not. All of the above follow straightforward by it. $\square$

Now we give our formal definition of the restrictive central points and a few remarks on it. By *intervals of the periodic orbit of* p, we mean the set of those intervals constructed as Lemma 4.2.1 for this orbit of p. We also denote the point q_1 in the lemma by $p(R)$, which is the most right point of all those intervals. If p is a central periodic point of period n, then both the points and intervals of the orbit of p are called to be *restrictive* if $f^K(c) \in [p, p']$. Notice that the period of such intervals is K with $K = n$ or $2n$, not necessarily the same as the points orbit. If $K = 2n$, we call p a periodic point of *period-doubling (or* $2n$*) type*. Thus a periodic point of $2n$ type is itself of periodic point of prime period n, but the set of intervals of its orbit has $2n$ elements. In Figure 4.2(a), we plot the restrictive intervals of a periodic orbit of period 3. While the example in Figure 4.2(b), the repelling fixed point is of $2n$ type, but the intervals of its orbit is not restrictive. Examples of restrictive intervals of this type can be seen in Figure 4.3.

Remark 4.2.5. Notice that the $2n$ type is not viewed as restrictive by Guckenheimer in [32]. Furthermore, he requires the restrictive central points to be repelling, but we do not. Our definition has better consistency for our later arguments. In fact, the reader who is familiar with the renormalization theory of interval maps knows that, some authors use this name refer to the commonly called renormalization intervals. Although a little similar, but our definition are different with the renormalization intervals either. Their definition requires also the endpoints being repelling in fact. Here we will not dwell on the details of this field, which considers the forward iterations of these intervals under renormalization operator. We only discuss the topological dynamics aspect exposed by these intervals. ◇

Some other useful and simple facts about the intervals of periodic orbit can be easily concluded. First, due to the construction, it can be seen that $f^K(q_i) = f^{K-n+i}(p_0') = p_i = f^K(p_i)$, thus we can view q_i and p_i as being "$f^i(c)$ central symmetric" for f^K (see Figure 4.2). Second, if $K = 2n$, then $f^n(c) \in f^n([p_0, p_0']) \subseteq J_n$. Notice that $J_n = [q_n, p_0]$ and q_n, p_0' are at different side of p_0. This is the case for all the points in the orbit of a central periodic point of $2n$ type. There are two associated intervals which are at each side of every point of its orbit, and this point is the common endpoint of both of two intervals. Therefore, when there are new periodic points occur in the restrictive intervals, it must be simultaneously in two intervals at both sides (see Figure 4.3(a)). It is for this reason that we call them as period-doubling type.

Example 4.2.6. In Figure 4.3(a), we plot the picture of f^2 with $f = \mu x(1-x)$ for $x \in [0, 1]$ and $\mu = 3.1$. f has a fixed point $p = 1 - 1/\mu \approx 0.67742$, hence its central interval is $[p', p] = [1/\mu, 1 - 1/\mu]$. Although p is fixed for f, its central interval $J_0 = [p', p]$ is not restrictive under f, since $f(x) \geq p$ for all $x \in [p', p]$. However, if we consider for map f^2, then it is restrictive. Moreover, both this interval J_0 and its preimage J_1 of f, are invariant under f^2.

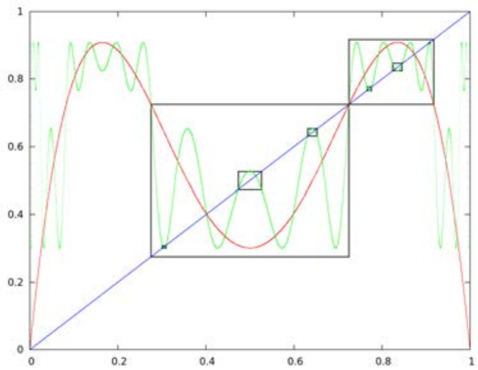
If we consider this example from the family $f_\mu(x) = \mu x(1 - x)$ point of view, the following facts are worth noting. When $\mu > 1$, there occurs a second fixed point $p = 1 - 1/\mu$ other than $x = 0$. This fixed point p is restrictive of n type for $\mu \in (1, 2)$ with $p < f_\mu(c) < c < p'$; and becomes $2n$ type when $\mu > 2$, for which $p' < c < p < f_\mu(c)$. Then it keeps to be restrictive (of $2n$ type) until $\mu_1 \approx 3.6785735$, which is the root of $f_\mu^2(1/2) = 1/\mu$, equivalently, of $\mu^3 - 2\mu^2 - 4\mu - 8 = 0$.

It is clear that, when $2 < \mu < 3.6785735$, for any $q \in J_0$, it must be $f^{2k+i}(q) \in J_i$ for $k \in \mathbb{N}$ and $i = 0, 1$. Particularly, when $2 < \mu < 3$, we know that all points inside these two intervals converge to p oscillatingly from its two sides. At $\mu = 3$, a period-doubling bifurcation takes place with p becoming repelling and two new attracting periodic point of period two occurring at both its sides (refer to Figure 4.3(a)). Thus for $3 < \mu < 3.6785735$, the attractors all stay in $J_0 \cup J_1$ with p repelling. Figure 4.2(b) shows that, at $\mu = 3.6785$, the period-doubling intervals of p is coming to the end of being restrictive. In fact, a reverse bifurcation just happen when these intervals becomes non-restrictive, the attractor merge into one piece instead of two as shown in Figure 4.1(a). This is an example of the correspondence of the bifurcations, whose mechanism is universal for all the restrictive intervals of period-doubling type. $\diamond$

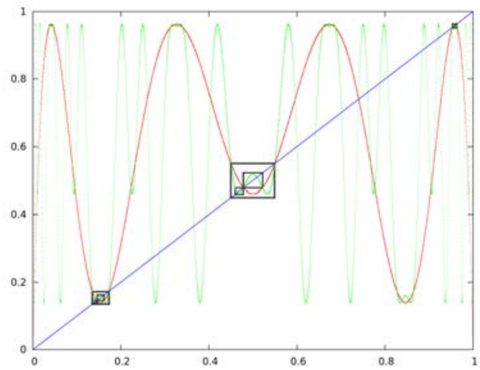
Finally, let us show the properties on the periodicity of the periodic points in restrictive intervals. Since J_i is invariant under f^K for each $0 \leq i < K$, denote $f^K|_{J_i}$ by G_i we have $f^{mK}|_{J_i} = G_i^m$. This implies that, if $s \in J_i$ is a periodic point of period mK of f, then it is a periodic point of period m of G_i (see Figure 4.4).

At the other hand, given p_0 a periodic point of period n with $\{J_i\}$ the set of intervals of its orbit constructed as Lemma 4.2.1. If the central interval J_0 is restrictive, then for any point $p \in J_0$, it must be $f^{n+k}(p) \in J_k$ due to the invariance of the union of these intervals. Hence, if p is also periodic, its period must be a multiple of n. In this case, each J_k can be viewed as a block who contains some points of the orbit of p with the same number. The same arguments also work if this point p is replaced by a subinterval $J \subseteq J_0$. Hence we have the following proposition.

Proposition 4.2.7. *All the restrictive intervals of a unimodal map f form a nested set of invariant intervals. Particularly, if p is a restrictive central*



(a) Nested restrictive intervals for $f(x) = 3.635x(1-x)$. The red line is graph of f^2, The green one is of f^6.



(b) Nested restrictive intervals for $f(x) = 3.85x(1-x)$. The red line is graph of f^3, The green one is of f^6.

Figure 4.4: Examples of nested restrictive intervals.

periodic point, and the set of intervals of its orbit are $\{P_i\}$ *for* $0 \leq i \leq K_p$, *then*

(1) if r *is a central periodic point of period* n *who is closer to* c *than* p, *that is,* $r \in (p, p')$, *then* $n = k \times K_p$ *for some* $k \in \mathbb{Z}^+$, *and* $f^{mK_p+j}(r) \in P_j$ *for all* $0 \leq j < K_p$, $m \geq 0$;

(2) if r *is also restrictive and the periodic intervals of its orbit are* R_i *for* $0 \leq i \leq K_r$, *then* $R_{mK_p+j} \subset P_j$ *for all* $0 \leq j < K_p$, $m \geq 0$;

(3) for any central periodic point s, *if* s *is closer to* c *than* p *and the restrictive central point* r *is closer to* c *than* s, *its orbit* $\mathcal{O}(s) \subset \cup_{i=0}^{K_p} P_i \setminus \cup_{i=0}^{K_r} R_i$.

The proof of this proposition is straightforward from the invariance of the union of restrictive intervals. In Figure 4.4 we plot two examples of restrictive periodic intervals of period 6, they can show this proposition intuitively. Note that, Figure 4.4(a) shows three intervals inside each of two intervals of a period-doubling type fixed point; while in Figure 4.4(b), it is two period-doubling intervals contained in each of a periodic three interval. Clearly, although there are orbits of period 6 in both of the inner restrictive intervals, but the dynamics of these two period orbits are not same. Next we introduce a concept of combinatorial dynamics, extension pattern, which can naturally specify the dynamical structures of these restrictive intervals and periodic orbits.

4.2.2 Extension patterns

This subsection is devoted to extension pattern, a concept from combinatorial dynamics, which is the most natural and convenient language for the dynamics and structures of restrictive intervals and those periodic orbits inside. The reader interested more on combinatorial dynamics can refer to [2] for a comprehensive discussion. Here we introduce only the knowledge fitting our purpose: first the general definitions and then a special property for restrictive intervals of unimodal maps. With this property, we define an operation on the patterns of them, which is the naturally description of the dynamics of attractors of S-unimodal maps.

The combinatorial structure we wish to study can be set up as follows. Given $f\colon I \to I$ a continuous map of a closed interval I to itself and $\mathcal{P}$ a f-invariant set (i.e., $f(\mathcal{P}) \subset \mathcal{P}$) with finite elements which are intervals (may be degenerated, that is, a single point), label the elements of $\mathcal{P}$

$$p_1 < p_2 < \cdots < p_n$$

(where $p_i < p_{i+1}$ means the right endpoint p_i is no larger than the left of p_{i+1}). Then the action of f on $\mathcal{P}$ can be codified in the map

$$\theta : \{1, \ldots, n\} \to \{1, \ldots, n\}$$

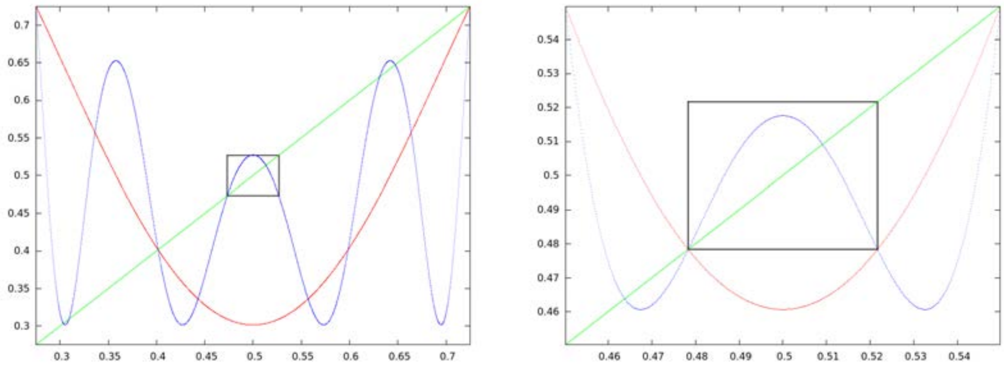
defined by

$$f(p_i) = p_\theta(i) \quad i = 1, \ldots, n.$$

The map θ encodes the combinatorial structure of each orbit and the way these orbits intertwine. To stress the combinatorial role of θ, we refer to any map of $\{1, \ldots, n\}$ to itself as a *combinatorial pattern* on n elements, or a *pattern* for short. The *degree* of θ, denoted by $|\theta|$, is the number n. We say that the map f exhibits the combinatorial pattern θ on $\mathcal{P}$, and call $\mathcal{P}$ a *representative* of θ in f. A given finite invariant set $\mathcal{P}$ represents a unique combinatorial pattern θ, but a given combinatorial pattern θ may have many representatives in f. To make the situation clear, we may denote a pattern θ and a representative $\mathcal{P}$ by $(\mathcal{P}, \theta)$, and may also call it a pattern for short.

Let $\mathcal{P} = \{1, \ldots, n\}$. A *block* in $\mathcal{P}$ is defined as a set of the form $B = \{i \in \mathcal{P} | a \le i \le b\}$, where $a \le b \in \mathcal{P}$. By a *block structure* for a pattern θ represented by $\mathcal{P}$, we mean a partition $\mathcal{B} = \{B_1, \ldots, B_k\}$ of $\mathcal{P}$ into disjoint blocks such that if $x, y \in \mathcal{P}$ belong to the same block, their images under θ belong to a single block. We number the blocks B_j so that $x \in B_i, y \in B_j$ and $i < j$ implies that $x < y$; then there is a unique pattern γ defined by

$$\theta[B_j] \subset B_{\gamma(j)}, \quad i = 1, \ldots, k.$$



(a) Enlarged central restrictive intervals for $f(x) = 3.635x(1-x)$. The red line is graph of f^2, The blue one is of f^6.

(b) Enlarged central restrictive intervals for $f(x) = 3.85x(1-x)$. The red line is graph of f^3, The blue one is of f^6.

Figure 4.5: Structures of nested restrictive intervals inside the central ones.

We call γ a *reduction* of θ, θ an *extension* of γ, and refer to $\mathcal{B}$ as a *block structure for θ over γ*.

Particularly, if $\mathcal{P}$ is a single periodic orbit, the combinatorial pattern θ represented by $\mathcal{P}$ is a cyclic permutation. Let $(\mathcal{P}, \theta)$ be a cycle of period n and let $(\mathcal{B}, \gamma)$ be a pattern of period m. Let $\mathcal{P} = p_1 < p_2 < \cdots < p_n$ and $\mathcal{B} = i_1, i_2, \ldots, i_m$. Then $(\mathcal{P}, \theta)$ has a block structure over $\mathcal{B}$ provided that, $n = sm$, $\mathcal{P} = P_1 \cup P_2 \cup \ldots \cup P_m$ with $P_i = \{p_{(i-1)s+1}, p_{(i-1)s+2}, \ldots, p_{(i-1)s+s}\}$ for all $i = 1, 2, \ldots, m$; $\theta(P_{i_j}) = P_{i_{j+1}}$ for all $j = 1, 2, \ldots, m-1$ and $\theta(P_{i_m}) = P_{i_1}$. Thus each of the sets P_i is a block of $\mathcal{P}$. We can view each block as a "fat" point and $\mathcal{P}$ as a "fat" cycle with the pattern $(\mathcal{B}, \gamma)$.

Example 4.2.8. Both the periodic orbits of period six in Figure 4.4 have typical block structures linked to their restrictive intervals. They are of maps $f(x) = \mu x(1-x)$ for two different values μ respectively, and their pattens and block structures are not the same either.

The map $f(x) = 3.635x(1-x)$ in Figure 4.4(a) has two orbits of period six. We can see that, both these orbits belong to the two restrictive intervals which is the $2n$ type of the repelling fixed point, and each of these two intervals contains three points of each orbit. That is, for these two period six orbits, each of these three points form a block over the pattern of two restrictive intervals.

In the case of the periodic orbit of period six for the map $f(x) = 3.85x(1-x)$ in Figure 4.4(b), the six points in the plotted boxes belong to three blocks respectively, the block are given by restrictive intervals of a period three orbit who contain this period six orbit. ◇

Remark 4.2.9. Here we'd like to point out that, the two orbits of $\mu = 3.635$ come together continuously from a saddle-node bifurcation at a smaller μ. They exist thereafter for all value of μ larger than this bifurcation value, even after they become unstable. For instance, in Figure 4.4(b) for $\mu = 3.85$, it can been seen that, outside the boxes we plot, there are also two other orbits which given by the intersection of f^6 and the diagonal. They are just the continuous successors of these orbits of $\mu = 3.635$. However, this cannot be shown in the bifurcation diagram for the logistic family, whose attractor must be contained in all restrictive intervals. For this case $\mu = 3.635$, refer to the our discussion of Example 4.2.6 and Figure 4.1(a), the period 2 intervals created by the first period-doubling bifurcation are still restrictive, hence new restrictive intervals and orbits brought by bifurcations can only lie in these two intervals.

While for $\mu = 3.85$, the orbit inside the boxes originate from a period-doubling bifurcation (see Figure 4.5(b)) of a period three orbit, which starts the big period 3 window of the bifurcation diagram for the logistic maps. Thus the structure of extension pattern represented by this orbit is over the three blocks given by restrictive intervals. ◇

The examples above demonstrate concretely the extension pattern over the blocks of restrictive intervals. This block structure is exact the combinatorial aspect of Proposition 4.2.7, and therefore is valid for all orbits (points or restrictive intervals) inside any restrictive intervals. Moreover, for unimodal maps there is also a special property of this structure. Briefly saying, the extension pattern over blocks of restrictive intervals is uniquely determined by the pattern represented by the blocks, and the pattern of the orbit in the central block. For simplicity reason, when the unimodal map f is known, we abuse the notation of a pattern and an orbit on which f exhibits this pattern. More precisely, if p is a periodic point, we use $\mathcal{O}(p)$ to denote also the pattern represented by this orbit, and call it the pattern of (the orbit of) p. If p is a restrictive central point, we denote by $\mathcal{O}([p, p'])$ the pattern represented by the orbit of $f^i([p, p'])$. Particularly, let $G = f^K|_{[p,p']}$ with K the period of this interval $[p, p']$, then we know this restriction function G itself is also a unimodal map on $[p, p']$. If q is also a central restrictive point of f with $q \in [p, p']$, then q is a periodic point of G on $[p, p']$. We denote by $\mathcal{O}_{[p,p']}(q)$ and $\mathcal{O}_{[p,p']}([q, q'])$ respectively the patterns which G exhibits on the orbit and the restrictive intervals of q. Thus we have the following property.

Proposition 4.2.10. *Given p and q two central periodic points of a unimodal map f with $q \in [p, p']$. If p is restrictive, then $\mathcal{O}(q)$, the pattern of point*

q, is uniquely decided by its reduction $\mathcal{O}([p,p'])$ and the pattern $\mathcal{O}_{[p,p']}(q)$ of the central block. Similarly, if q is also restrictive, $\mathcal{O}([q,q'])$ is also uniquely decided by $\mathcal{O}([p,p'])$ and $\mathcal{O}_{[p,p']}([q,q'])$.

Proof. This proposition is a direct consequence of the monotonicity of unimodal map on each side of c. Recall the construction of the orbit of intervals of p, $[p,p']$ is set as $J_K = J_0$, and for $0 < i < K$, $f(J_i) = J_{i+1}$ with f strictly monotone on all J_i. Hence, when the pattern $\mathcal{O}_{[p,p']}(q)$ on $[p,p']$ is known, every pattern on J_i is uniquely given by the preimages of the representative $\mathcal{O}_{[p,p']}(q)$. The same is for $\mathcal{O}_{[p,p']}([q,q'])$ with those preimages of q'. □

To illustrate this proposition, in Figure 4.5 we plot the pictures of the central restrictive intervals of Examples 4.2.8 in Figure 4.4. Denote the central restrictive intervals by $[p,p']$, in the pictures we plot the graphs of f^6 and f^K with K the period of $[p,p']$ under f, which is 3 for Figure 4.5(a) and 2 for Figure 4.5(b) respectively. Clearly, both $G = f^K|_{[p,p']}$ are unimodal, and $f^6|_{[p,p']} = G^i$ where $i = 6/K$. In each case, there is $q \in [p,p']$, which is central for both maps G and f. The pattern of q that G exhibits is $\mathcal{O}_{[p,p']}(q)$, which is represented by the orbit of those i points inside $[p,p']$, shown by Figure 4.5. $\mathcal{O}(q)$ is the pattern of q that f exhibits, represented by the orbit of period 6 in Figure 4.4. And $\mathcal{O}([p,p'])$, represented by the orbit of interval $[p,p']$ in f, is the reduction pattern of $\mathcal{O}(q)$, whose elements are blocks for $\mathcal{O}(q)$. Such block structure is shown by boxes in Figure 4.4. If we embed the central blocks in Figure 4.5 back into Figure 4.4 and take its preimages, then what obtain is exact the same boxes plotted here.

The significance of this proposition is that, we can define an operation on the patterns of central points and intervals with their block structures, which we write as

$$\mathcal{O}(q) = \mathcal{O}([p,p']) \ltimes \mathcal{O}_{[p,p']}(q),$$

for $q \in [p,p']$ and p restrictive. Concerning the nested sets of series of restrictive central points, an easy induction yields that: if $p_{i+1} \in [p_i, p_i']$ for restrictive central points p_i of $0 \le i \le k$, then

$$\mathcal{O}(p_{k+1}) = \mathcal{O}([p_0,p_0']) \ltimes \mathcal{O}_{[p_0,p_0']}([p_1,p_1']) \ltimes \ldots \ltimes \mathcal{O}_{[p_k,p_k']}(p_{k+1}).$$

In this case, we call $\mathcal{O}([p_i,p_i'])$, the pattern of the orbit of intervals $[p_i,p_i']$, *the layer of p_i*. For different layers of a given unimodal map, the one who has block structure over another is said as an upper layer over its reduction. We point out that, if a upper layer has only one element in the central block of a lower one, then these two layers have the same combinatorial pattern in the sense of the original definition. For instance, this is the case of the repelling and attracting orbits of a saddle-node bifurcation when they are just separated. But we regard them as different patterns to emphasis the fact that they are representatives with different structures, particularly, that the attracting one has a upper layer over the repelling.

This operation provide us a convenient tool to describe the dynamics of the topological attractors of S-unimodal maps, because an S-unimodal map can have only one attractor who must be contained inside the innermost layer.

4.2.3 Criteria for topological attractors of S-unimodal maps

In this subsection we demonstrate how the structures of topological attractors of S-unimodal maps can be characterized by restrictive intervals. Simply saying, an attractor is a limit set of many typical points of the system. Now it is well-known that, an S-unimodal map f has at most one topological attractor such that points of a residual subset of the system tend to it. Moreover, this attractor can only be one of the following three types: a periodic orbit, a solenoidal set, or a finite cycle of intervals that f is topologically transitive on their union. Here we give a brief introduction of this classification first, and then show that each type of the attractors just corresponds to a kind of situation of the central intervals: a finite number of restrictive central intervals without any periodic point within the innermost one; infinite many of restrictive central intervals; and a finite number of restrictive central intervals with all periodic points within the innermost one being non-restrictive.

We review the classic results on topological attractors of S-unimodal maps first. By attractor we mean only the topological attractor, which a set in the topological space I with dynamical structure similar to the metric one in the sense of Milnor. More precisely, a closed invariant set $A \subseteq I$ is called a *topological attractor* of f if
(i) $rl(A)$ is a set of second Baire category;
(ii) for any proper closed invariant subset $A' \subset A$, the set $rl(A) \setminus rl(A')$ is of second Baire category as well.
Here $rl(A) = \{x : \omega(x) \subseteq a\}$ is its"realm of attraction".

A unimodal map is called *S-unimodal* if it is three times differentiable with negative Schwarzian derivative outside c:

$$Sf = \frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2 \leq 0.$$

A primary reason for working with negative Schwarzian derivative is *Singer's theorem*. To state this theorem we recall some of definitions. A periodic point x of period n is (one-sided) *stable* if there is a non-trivial interval U of x with $f^n(y) \to x$ for all $y \in U$. A necessary condition for x to be stable is that $|Df^n(x)| \leq 1$. A sufficient condition for x to be stable is that $|Df^n(x)| < 1$. We denote the derivative of f by either f' or Df as convenient.

Theorem 4.2.11 (Singer[72]). *Let $f : I \to I$ has negative Schwarzian derivative. For every stable periodic point x of period n, there is an $i < n$ and a critical point c or endpoint of I such that $y \in [c, f^i(x)]$ implies $f^{kn}(y) \to f^i(x)$ as $k \to \infty$.*

The proof of Singer's theorem is based upon several facts about Schwarzian derivatives which we list here and use later:

(1) If Sf is negative, then Sf^n is negative for all $n > 0$.

(2) If Sf is negative, then $|f'|$ has no positive local minimum. If $J = [a, b]$ is an interval on which f is monotone and $x \in J$, then $|f'(x)| \geq \min(|f'(a)|, |f'(b)|)$.

(3) If $x_1 < x_2 < x_3$ are consecutive fixed points of $G = f^n$ and $[x_1, x_3]$ contains no critical point of G, then $G'(x_2) > 1$.

Due to this theorem, except for the fixed endpoint of I, an S-unimodal map can have at most one stable orbit, which attracts the critical point c if it exist. However, if the fixed endpoint is stable, either it is the only one of f which attracts all points of I, or there is some other point between it and c, say p, who is fixed and repelling. For the latter case, if p is restrictive, then all points of $I \setminus [p, p']$ are attracted to this endpoint, while $[p, p']$ is invariant with f a unimodal map on it. In this case what we really need to study is just $f|_{[p,p']}$. If p is not restrictive, then almost all points go outside $[p, p']$ and are attracted by this stable fixed point. This is not an interesting case for us. So without loss of generality, we assume that, for an S-unimodal map, if this fixed endpoint is no longer a stable one which attracts all points of I, then it must change to be unstable. This implies that there can be at most one stable periodic orbit, which must attracts the critical point c in this case. In fact, with this assumption, there also can be only one attractor in such system, which is given by the following theorem.

Theorem 4.2.12. *Let $f : I \to I$ be an S-unimodal map. Then there exists a set $A \subseteq I$ of second Baire category so that for each $x \in A$ the set $\omega(x)$ has to be of one of the following three types:*

(1) $\omega(x)$ *is a stable periodic orbit;*

(2) $\omega(x) = \omega(c)$ *with* $\omega(c)$ *a minimal, solenoidal set of zero Lebesgue measure;*

(3) $\omega(x)$ *is equal to a finite union of intervals containing* c *and* f *acts as a topologically transitive map on this union of intervals.*

Remark 4.2.13. The above theorem first dated back to Guckenheimer, who prove in [32] that there is sensitive dependence on initial conditions in the cycle of intervals, while it must not for the case of stable periodic orbits. Later it was proved in increasing integrity (see [52] for transitivity) and generality, refer to [55] for more details. Now we have known that, the negative Schwarzian derivative is just one of the sufficient conditions such that a family of unimodal maps have topological attractors of these types. One can obtain the same kind of results by estimating distortion of cross-ratios, thus the assumption of negative Schwarzian derivative is rather restrictive and turns out to be unnecessary. Here in this work we choose such families just as typical examples to study the changes of their attractors, but note that our results also work for any unimodal family whose topological attractors are of the three types as the above theorem, for instance, the real analytic unimodal families. ◇

Now we give the detailed definitions of the other two types besides the stable periodic orbit. Clearly, for any finite union of intervals of the third type, these intervals must be periodic as we have defined in the second section, so we call this type of attractors as *cycle of intervals* too. More precise, such an attractor of map f is given by $A = \cup_i^K [f^K(c_i), f^{2K}(c_i)]$, where c_i is the extreme of f^K inside J_i in Lemma 4.2.1 with those J_i's the smallest restrictive intervals of f. In this case, $f|_A$ in chaotic in the sense of Devaney. An invariant set S is called a *solenoidal attractor* or a *Feigenbaum-like attractor* if it has the following structure:

$$S = \cap_{n=1}^{\infty} \cup_{k=0}^{p_n - 1} f^k(I_n),$$

where each I_n is a periodic interval of period p_n, with $I_1 \supset I_2 \ldots$ and $p_n \to \infty$. Notice that a solenoidal attractor is a minimal invariant set, which means that it is the limit set for any point of it.

Concerning on a family of unimodal maps, a natural and fundamental problem is to describe the set of parameters corresponding to these three different types. In the quadratic case we have known the following (see [47, 28]).

- for the case of periodic orbits, the parameters form a set dense in parameter space, which consists of countably infinitely many nontrivial intervals. Moving the parameter inside one connected component of this set, we see the period-doubling scenario, with universal scaling in parameter space.
- for the case of solenoidal attractors, the parameters set is a completely disconnected set of Lebesgue measure zero.
- for the case of cycles of intervals, the parameters set is a completely disconnected set of positive Lebesgue measure.

So any nontrivial parameter interval contains maps with stable periodic orbits, and we cannot find cycles of intervals in a whole connected components of the parameter space. But close some parameter with such attractor, we are likely to find also interval attractors. Recall that this is a general property of a set of positive Lebesgue measure: almost all points of the set are so-called Lebesgue density points, where measure accumulates. We will discuss this a little bit when later we review the intermittency with saddle-node bifurcations in the next section.

Next we show simple classification criteria for these three types of attractors. Both the definition and the above theorem are given in terms of the limit sets, which is not so easy to use in practice. However, these three types of attractors can be characterized exactly by the three cases of the sets of central periodic points of the maps, which we will use this fact as criteria of the classification.

Theorem 4.2.14. *The attractor of an S-unimodal map is contained in the nested set of all the restrictive intervals, and the sets of central periodic points and the attractors of S-unimodal maps correspond with each other as follows:*

(1) if there are infinite many of restrictive central periodic points of f, then its attractor is a solenoid;

(2) if there are a finite number of restrictive central periodic points of f, and no any other (central) periodic point within the smallest central interval, then the orbit of the restrictive central periodic points closest to c is stable;

(3) if there are a finite number of restrictive central periodic points of f, with other periodic points (hence not restrictive) inside the smallest central interval, then its attractor is a cycle of intervals, who is contained inside the cycle of the smallest central interval.

Proof. We discuss these three cases one by one as follows.

(1) The case that infinite many of restrictive central periodic points means the solenoidal or Feigenbaum-like attractor, is just from the definition of solenoid and Proposition 4.2.7.

(2) Now for the case that there are a finite number of restrictive central periodic points of f without more closer periodic point inside. The equivalence of this case to the existence of a stable periodic orbit follows easily by the simple lemma below.

Lemma 4.2.15. *If $f : [a, b] \to [a, b]$ $(a < b)$ is a unimodal map with $f(a) = f(b) = a$, then it has only one periodic point a if and only if $f(x) < x$ for all $x \in (a, b]$.*

Proof. $f(x) < x$ for all $x \in (a, b]$ means trivially a is the only fixed point of f. On the other hand, if there is some $x_0 \in (a, b)$ such that $f(x_0) \geq x_0$ then, either x_0 is another fixed point when $f(x_0) = x_0$, or there must be some other fixed point at (x_0, b) for $f(x_0) > x_0$, which is because $f(b) = a < b$. □

Now suppose that p is the closest restrictive central periodic point among these finite ones of f, and whose period is n. We have $f^n : [p, p'] \to [p, p']$ is unimodal by Proposition 4.2.4. Then there is no any other periodic point inside $(p, p']$ means that p is the only periodic point of $f^n|_{[p,p']}$. So p attracts all the points in this interval due to the lemma above, its orbit must be a stable one.

On the other hand, if there is a stable orbit of f, Singer's Theorem says that, it must be some point p of this orbit such that any $y \in [p, c]$ is attracted to p under f^n, where n is the period of p. We prove only for the case of $p < c$, $p > c$ can be treated similarly. Of course, we can let p be the central one, and clearly its attracted interval is at least $[p, p']$, which means that there is no any fixed or periodic points of f^n in $[p, p']$. Moreover, by the above lemma, we have that p is restrictive since $p < f^n(c) < c < p'$.

(3) This last case is just the classic result of Guckenheimer, who proved in [32] that:

Theorem 4.2.16 (**Guckenheimer [32]**)**.** *Suppose an S-unimodal map f has no stable periodic orbit. Then f has sensitivity to initial conditions if and only if there is an integer N such that $n \geq N$ implies f^n does not have a restrictive central point.*

We will not repeat the prove here, but just point out why his condition is qualified by ours, although the definition of restrictive central points is not exactly same. We have shown that, f has a stable periodic orbit can only happen in the case above. The arguments of above case also implies that all the restrictive central points which are not the closest one must all be repelling, required by the definition of Guckenheimer. Now taken the period of the closest restrictive central periodic point p as required N of this theorem, then all the other possible restrictive central points have period no more than N, because $[p', p]$ is inside all of their central intervals, and hence be multiple of their periods.

□

Finally, this criteria of topological attractors by restrictive intervals implies that we can present the dynamics of those attractors with their patterns, using the operation we defined in the previous section. Namely, for the cases that we interested, periodic orbit and cycle of intervals, if the finite set of restrictive central periodic points of an S-unimodal map f is $\{p_0, p_1, \ldots, p_k\}$ with $p_{i+1} \in [p_i, p'_i]$ for $0 \leq i < k$, then

$$\mathcal{O}(p_k) = \mathcal{O}([p_0, p'_0]) \ltimes \mathcal{O}_{[p_0,p'_0]}([p_1, p'_1]) \ltimes \ldots \ltimes \mathcal{O}_{[p_{k-1},p'_{k-1}]}([p_k, p'_k]) \ltimes \mathcal{O}_{[p_k,p'_k]}(p_k)$$

gives the pattern of the stable periodic orbit of p_k. If instead, the attractor is cycle of intervals inside these restrictive intervals, then its pattern is

$$\mathcal{O}([p_k]) = \mathcal{O}([p_0, p'_0]) \ltimes \mathcal{O}_{[p_0,p'_0]}([p_1, p'_1]) \ltimes \ldots \ltimes \mathcal{O}_{[p_{k-1},p'_{k-1}]}([p_k, p'_k]).$$

For a solenoidal attractor, we allow to apply this operation infinite times, thus its pattern can be written as

$$\mathcal{O}(A) = \mathcal{O}([p_0, p'_0]) \ltimes \mathcal{O}_{[p_0,p'_0]}([p_1, p'_1]) \ltimes \ldots \ltimes \mathcal{O}_{[p_{k-1},p'_{k-1}]}([p_k, p'_k]) \ltimes \ldots.$$

Notice that, if an attracting p_k moves continuously cross c with a parameter of maps varying, its pattern actually changes. In these two cases, $\mathcal{O}_{[p_{k-1},p'_{k-1}]}([p_k, p'_k])$ presents a n type and a $2n$ type patterns respectively. However, we will keep this single notation for these two cases, since we will not need distinguish them in our subsequent arguments when we use this notation.

4.3 Bifurcations and transition of full family

With knowledge of the previous section, we give Theorem D, the first main theorem of this chapter, in the first subsection. It explains the mechanism of reverse bifurcation in an S-unimodal family. The reverse bifurcation together with classic bifurcations of periodic orbits, are exactly at two sides of a full family. This provide us a complete perspective of the transition of S-unimodal maps. The second subsection is then devoted to a short review of the transition, and the self-similarity during it.

4.3.1 Reverse bifurcations as bands merging

This subsection is devoted to bifurcations of S-unimodal maps, particularly the reverse bifurcations of attractors of those cycles of intervals. As we have shown with particular examples in the first section of this chapter, such bifurcations are in forms of bands merging. We will exposes precisely the mechanism of their changes with size and periodicity. The key factor for the mechanism is just those restrictive intervals, whose destruction yields bands merging. On the other hand, the birth of the restrictive intervals is certainly related to the occurrence of their corresponding periodic orbit, hence they naturally have a definite relation.

To call them as "reverse" bifurcations, we actually discuss the problem in a setting of the one-parameter continuous family of maps, for instance, the logistic family of $f_\mu(x) = \mu x(1-x)$ with $\mu \in [0, 4]$. Although it is not necessary indeed for the understanding of each single reverse bifurcation independently, this setting is very helpful to learn not only comprehensively the general mechanism of bifurcations, but also its role in the overall transition process. Therefore we try to deal with them from this point of view. Furthermore, the process from the birth to the end of a set of restrictive intervals reveals the common feature of self-similarity of an S-unimodal family, which we will explore briefly in the next section.

With regard to the content of this section, it is arranged as follows. First we make a brief review of theories on the bifurcations of periodic orbits. The knowledge of them are common known in any popular textbook, so we do not go into much details. A special issue is the review on the intermittency phenomena linked with those saddle-node bifurcations. Such discussions are necessary for clarification of the mathematical nature with the numerical appearance of the attractors. After that, we start to discuss the reserve bifurcations themselves. Their mechanism can be presented clearly and concisely with notation of extension patterns. We take some examples to illustrate the situation much clear at the end of this section.

Bifurcations of periodic orbits

Generally, the qualitative changes of dynamics with respect to parameters are known as *bifurcations*. Here for us, this problem is given by how the state and periodicity of the attractor change as the parameter μ of S-unimodal family f_μ varying. Roughly speaking, when there is a qualitative change at μ, one says that μ is a bifurcation value of the parameter. Concerning those bifurcations which involve periodic orbits, assume $f_\mu^n(p) = p$ and that p is periodic with prime period n, the behaviors of orbits near p are usually decided by the number $\lambda(p) = \frac{d}{dx} f_\mu^n(p)$. If $|\lambda(p)| < 1$, then f^n is a contraction in some neighborhood of p. Hence, for x close enough to p, $f^{ni}(x) \to p$ as $i \to \infty$. On the other hand, if $|\lambda(p)| > 1$ there is some neighborhood of p such that p is the only point which stays always inside this neighborhood. Moreover, the implicit function theorem implies that when $\lambda(p) \neq 1$ there is a periodic point $p(\mu)$ of prime period n depending smoothly on μ.

Thus the bifurcations of a periodic orbit for S-unimodal maps are of two sorts. The number of periodic orbits of a given prime period n can only change at a value of μ for which there is a periodic point p of period n with $\lambda(p) = 1$. The stability of a periodic orbit only changes when $|\lambda(p)| = 1$. Bifurcations take place "generically" in the two cases $\lambda(p) = \pm 1$ are the well-known saddle-node type and the period-doubling one correspondingly. The detailed analytic forms of their sufficient conditions and the proofs can be found in any common textbook. We refer to [15, 31] for discussions focused on S-unimodal families.

Briefly, for the period-doubling bifurcation, it happens at the place that an originally existed periodic orbit of period n loses its stability with a new periodic orbit of period $2n$ occurs around it. When the periodic point $p(\mu)$ of prime period n depending smoothly on the parameter μ, it takes place at μ_0 where $\lambda(p(\mu_0)) = -1$. The orbit lose its stability if $\lambda(p(\mu)) < -1$ with parameter μ varying. From the restrictive point of view, we know by Corollary 4.2.2 that, with the central point of an attracting periodic orbit moving to another side of c, the derivative λ of this orbit change from positive to negative, and the original n type restrictive intervals linked to the orbit change to be $2n$ type. When finally this orbit loses its stability, a new attracting orbit occurs with every linked $2n$ type restrictive intervals containing each point of the new orbit inside. For typical S-unimodal families, for example, a quadratic family, the period-doubling scenario can be seen clearly in its transition diagram of attractors, recall the parameters of this scenario are inside each connected component of the dense set of parameter space, which consists of countably infinitely many nontrivial intervals.

For the case of saddle-node bifurcations, it takes place when the graph of $f^n_{\mu_0}$ for some n touches the diagonal tangentially at some μ_0, so there occurs a periodic orbit of period n. At one side of μ_0, this intersection does not happen, the orbit does not exist. As μ varies to the other side of μ_0, the graph of f^n_μ meets the diagonal at two points, the new orbit splits in two with one attracting and the other repelling. In the transition diagram, this attracting one can be seen as the attractor provided its period n is rather small. Unlike a period-doubling bifurcation with which the new orbit splits out of an existing attracting orbit, the new orbit of a saddle-node bifurcation is observed occurring from a chaotic attractor of cycle of intervals. The dynamical behaviors on such cycles of intervals is known as (type I) intermittency introduced by Pomeau and Manneville in [65].

Intermittency is regarded as a route to chaos in context of physics. The observed phenomena before a saddle-node bifurcation are as follows. There appears to be a chaotic orbit in the system, see Figure 4.6(a) for example. Examining the chaotic orbit for parameter close to the bifurcation values, the character of its transition is that: the orbit appears to be a period orbit for long stretches of time as the same period of the orbit born by the saddle-node bifurcation. But after that, there is a short burst (the "intermittent burst") of chaotic-like behavior, followed by another long stretch of almost period behavior, followed again by a chaotic burst, and so on. The average duration of the long stretches between the intermittent bursts becomes longer and longer, and approaches infinity with the pure periodic orbit appears at the bifurcation value.

Remark 4.3.1. Notice that, if one only looks at the picture of transition diagram, it seems that the parameters for intermittency occurring occupy quite a large nontrivial interval. But we know that, the parameters set of cycles of intervals is a completely disconnected set for a quadratic family, so cycles of intervals cannot appear in a whole connected components of the parameter space. In fact, a long stable cycle is indistinguishable from non-periodic motion, and are discernible due to both its high period and short occurrence. Hence, whether or not an attractor in this case is truly non-periodic is difficult to judge only by numerical simulation. One should be aware that, any nontrivial parameter interval do contain maps with stable periodic orbits. But close a parameter value with chaotic attractor, we are likely to find also interval attractors. Such values are almost all Lebesgue density points with measure accumulates. We will meet this similar situation when we deal with reverse bifurcations. ◇

The reverse bifurcations

The reverse bifurcations happen when a set of restrictive intervals become non-restrictive, the performance is that some intervals of the attractor merge into a larger size one, as we show in the Introduction. Here we reveal the concrete mechanism of band merging, which is given in form of their extension patterns. It is simple, but maybe abstract, so we illustrate two examples to make them more intuitive. The periodic endpoints of restrictive intervals of these two examples are born by a period-doubling and a saddle-node bifurcation respectively, the examples display clearly that each bifurcation of this type is exact the reserve procedure of the bifurcation of the corresponding periodic endpoints of its intervals.

For a reverse bifurcation, its parameter value is a so-called Misiurewicz point. Before we exclusively investigate reverse bifurcations, we make a short exposition on the necessary known results on the dynamics at such critical values. To be precise, we deal with a family of one-parameter S-unimodal maps $f_\mu(x)$ on the interval $I = [a, b]$, and assume that f_μ is also continuous with respect to the parameter μ. Denote by p_{μ_0} the restrictive central points of period n for some f_{μ_0}. For a value μ_0 such that $f_{\mu_0}^{K_p}(c) = p'_{\mu_0}$ with $K_p = n$ or $2n$ the period of its restrictive interval,. we study what happens locally when the parameter μ passes through μ_0 (in a small enough region). Here $f_{\mu_0}^{K_p+1}(c) = f(p_{\mu_0})$, so μ_0 is a *Misiurewicz point* which means the critical point c is preperiodic (i.e., it becomes periodic after finitely many iterations but is not periodic itself), and the map f_{μ_0} belongs to the set of uncountably many Misiurewicz maps in generic one-parameter families. In [59] Misiurewicz proved that Misiurewicz maps admit absolutely continuous invariant measures. Now it is known that the parameters of such maps are Lebesgue density points of parameters corresponding to absolutely continuous invariant measures.

Moreover, for each side of such μ_0, there are two different types of bifurcations. Where μ comes to μ_0 with the intervals keeping restrictive, it is the homoclinic bifurcation. A good introduction on it is in the popular textbook [15] of Devaney. It shows that this critical value μ_0 is an accumulation point of infinitely saddle-node and period-doubling bifurcation, and $f_{\mu_0}^{K_p}|_{[p'_{\mu_0}, p_{\mu_0}]}$ is conjugate with the shift map of the sequence space on two symbols, hence chaotic in the sense of Devaney. Particularly, every attractor of such μ_0 is a cycle of intervals which is exactly the cycle of $[p'_{\mu_0}, p_{\mu_0}]$ itself.

A reverse bifurcation occurs at the other side of such critical value μ_0, where $f_\mu^{K_p}(c) \notin [p'_\mu, p_\mu]$ for values of μ . By general theory, an unstable periodic orbit (in fact any hyperbolic set) persists and moves smoothly under small perturbations of the map. But if $f_\mu^{K_p}(c) \notin [p'_\mu, p_\mu]$, the intervals of periodic orbit of p_μ is not restrictive, this implies the attractor is not limited inside them any longer. While we know that the attractor has to be contained in the innermost set of restrictive intervals, hence it must be inside the originally second innermost restrictive intervals now. This yields the mechanism of the change of attractors of the reverse bifurcations.

Theorem D. *Suppose that μ_0 is a critical value such that $f_{\mu_0}^{K_p}(c) = p'_{\mu_0}$ for a central periodic point p_{μ_0}, and q_{μ_0} is the restrictive central period point who is the second closest to c of f_{μ_0}, that is, with p_{μ_0} the only restrictive central point in $[q'_{\mu_0}, q_{\mu_0}]$. For value (in Lebesgue measure sense) of μ arbitrarily close to μ_0 with $f_\mu^n(c) \notin [p'_\mu, p_\mu]$, the attractor changes from a cycle of intervals inside periodic intervals $[p'_{\mu_0}, p_{\mu_0}]$ of period K_p, to a cycle of intervals of period K_q contained in the restrictive intervals of orbit $[q'_\mu, q_\mu]$. Precisely for its patterns, as $\mu_0 \to \mu$,*

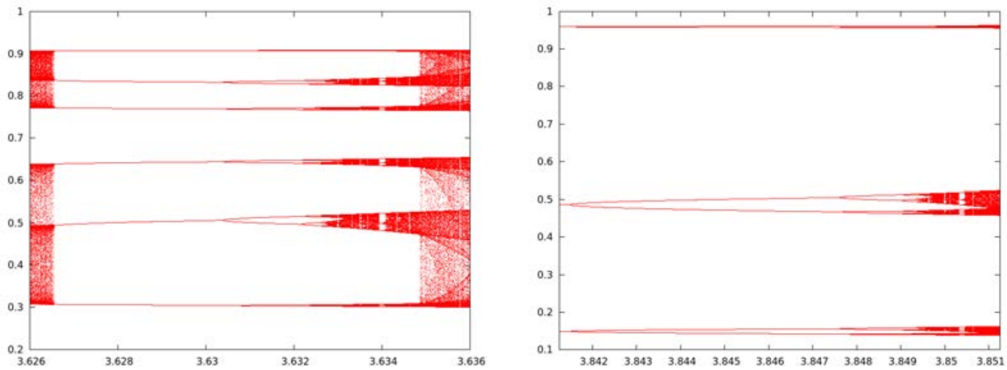
$$\mathcal{O}([p_{\mu_0}]) = \mathcal{O}([q'_{\mu_0}, q_{\mu_0}]) \ltimes \mathcal{O}_{[q'_{\mu_0}, q_{\mu_0}]}([p'_{\mu_0}, p_{\mu_0}]) \to \mathcal{O}([q_{\mu_0}]) = \mathcal{O}([q'_{\mu_0}, q_{\mu_0}]).$$

Proof. We show that (in Lebesgue measure sense), for those parameters μ arbitrarily close to μ_0 with $f_\mu^{K_p}(c) \notin [p'_\mu, p_\mu]$, the cycle of intervals $[q'_{\mu_0}, q_{\mu_0}]$ is the innermost restrictive one of f_μ, and there exist non-restrictive central point inside $[q'_{\mu_0}, q_{\mu_0}]$. Therefore, our assertion follows by Theorem 4.2.14.

Consider $p_\mu(R)$ the most right endpoints of the set of restrictive intervals of p_μ. By Proposition 4.2.4, $f_{\mu_0}^n(c) = p'_{\mu_0}$ is equivalent to $f_{\mu_0}(c) = p_{\mu_0}(R)$. Meanwhile, it is $f_{\mu_0}(c) = p_{\mu_0}(R) > s_{\mu_0}(R)$ for any non-restrictive central point s_{μ_0}, and $f_{\mu_0}(c) < q_{\mu_0}(R)$ since q_{μ_0} is restrictive by assumption. That is, $s_\mu(R) < p_\mu(R) = f_\mu(c) < q_\mu(R)$.

Now for μ arbitrarily close to μ_0 with $f_\mu^n(c) \notin [p'_\mu, p_\mu]$, the above inequality changes to be $s_\mu(R) \le p_\mu(R) < f_\mu(c) \le q_\mu(R)$ due to continuity of $f(\mu, x)$. It means that q_μ is still a restrictive central period point, and p_μ becomes non-restrictive like any other non-restrictive central points of f_{μ_0}. So q_μ is closest restrictive central period point now, with all other central period points inside (q_μ, q'_μ) being non-restrictive.

By "in Lebesgue measure sense", we mean the following. It is exact same with what we introduce for intermittency, in a real system, parameters corresponding to cycles of intervals cannot appear in any whole connected components of the parameter space. In fact, in any connected interval of μ_0, there are infinitely many saddle-node and period-doubling bifurcations of periodic orbits, discernible with high periods and extremely short life. These infinitely many tiny windows opening and then closing, and all locate in restrictive intervals $[q'_\mu, q_\mu]$ which continuously changed from $[q'_{\mu_0}, q_{\mu_0}]$ with the same pattern. The closed window gives attractor exactly of bands with



(a) Reverse bifurcation of saddle-node type with a period 6 cycle.

(b) Reverse bifurcation of period-doubling type with a period 6 cycle.

Figure 4.6: Examples of nested restrictive intervals.

this pattern. The opening one cannot be distinguished hence also look like it. μ_0 is Lebesgue density points with measure accumulates, which means that one can observe chaotic interval attractors in positive possibility, thus those nearby attractors all look like chaotic bands in any transition diagram by numerical simulation (see figures of examples below). □

Maybe the mechanism above given by patterns of restrictive intervals is lack of intuition, because it is hard to know the related restrictive intervals in case of being just limited at a given parameter. The situation becomes much clear if one date back to the birth of these intervals in the transition of family. Recall any set of restrictive intervals must be linked to a periodic orbit, and hence its state is certainly decided by the bifurcation which gives birth of its orbit. The examples we display next can clearly illustrate the relation of the birth from bifurcations of periodic orbit and the destruction showing reverse way in form of chaotic bands. Moreover, this is very helpful for the understanding of the overall structure of family transition.

We give two examples of reverse bifurcations of the logistic family $f_\mu(x) = \mu x(1-x)$, which correspond to the two generic bifurcations of periodic orbits, the saddle-node and the period-doubling respectively. Apparently, it looks like they exhibit two different behaviors, so they are thought as two sorts of behaviors in physical context. We will explain their detailed mechanism, then expose briefly that their different appearances are natural from the point of view of their mechanisms of bifurcations one by one.

Example 4.3.2. Our first example is shown in Figure 4.6(a), which is a reverse bifurcation of a cycle of intervals of period 6, occurring at the value about $\mu = 3.6348$.

In Figure 4.6(a), the attractor appears as a disjoint 6-bands, with each three in two groups. It suddenly becomes to be a 2-bands, because the three disjoint bands of each group merge directly into one. This is an example of a saddle-node type, because the restrictive intervals of those six bands originally due to a saddle-node bifurcation at about $\mu = 3.6265$. Notice that before this bifurcation of periodic orbit, the attractor is a 2-bands. With this bifurcation, a periodic orbit of period 6 occurs, in each original band there presents 3 points of the new orbit.

From the restrictive intervals point of view, this saddle-node bifurcation takes place inside a set of restrictive intervals with 2 intervals, which contains the 2-bands in Figure 4.6(a). After this bifurcation, there are new set of restrictive intervals linked every new periodic orbits now. Figure 4.4(a) display the situation for one of such moment, where we plot in boxes the old 2 restrictive intervals (due to the repelling fixed point, refer Example 4.2.6) and the new 6 restrictive intervals of the repelling periodic orbit from this bifurcation.

This orbit of period 6 cannot be seen in transition diagram above since it is repelling, but it persists after arising from the saddle-node bifurcation, so does its restrictive intervals until the place that the reverse bifurcation happens. During this process, a series of complicated dynamical transition presents in order, starting from the period-doubling bifurcation of the attracting orbit from its same bifurcation, ending with chaotic bands completely coincident with its restrictive intervals. Notice that, the reverse bifurcation make it go back to exactly a 2-bands just like that before they are born, because the innermost restrictive intervals come back to those before the saddle-node bifurcation which brings them.

A reverse bifurcation for restrictive intervals of an orbit from a saddle-node bifurcation is common refer as "crisis" in physics, because it always present a sudden change of a jump in size of a chaotic attractor, with several bands merge into one piece. There is also evidence in the diagram that the density of attractor points in the large attractor near the crisis concentrates in the original bands, and gradually spreads out, indicating another form of intermittency. These are thought as its character. Another example of this same type can see Figure 4.1(b). $\diamond$

Example 4.3.3. Our next example in Figure 4.6(b) is also a case of period 6. Differently, it is a reverse of period-doubling one.

The reverse bifurcation takes place at about $\mu = 3.8415$, the periodic orbit of the restrictive intervals occurs at about $\mu = 3.851$, from the period-doubling bifurcations of a period 3 orbit. In this case, the original attracting orbit of period 3 comes from the saddle-node bifurcation of period 3 which starts the big period 3 window. The type of its intervals changes to be $2n$ since it moves to the other side of the repelling orbit, whose intervals keeps to be restrictive of period 3 (see Figure 4.4(b)). Also after a series of complicated dynamical transition in order, the reverse bifurcation comes when the chaotic bands attractor is completely coincident with its restrictive intervals. Since for the restrictive intervals of this orbit, it is every pairs of intervals bearing at each side of it 3 points, the outer series of restrictive intervals is the one of the repelling period 3 orbit. Hence when the critical value is reached, the pairs inside each outer intervals merge into one, the periodicity of the chaotic intervals attractor changes from 6 to 3.

This reverse bifurcation of the restrictive intervals of period-doubling type is usually called by band merging. Apparently, it is more "smooth" than the type of above example. This is only because that, the pairs of the inner restrictive intervals are not totally disjoint, but with a common endpoint instead. Therefore, as the attractor spreads inside and occupies the intervals finally, every pairs meet naturally and continuously. $\diamond$

Remark 4.3.4. In the literature, phenomena similar as these reverse bifurcations are considered to be are typical and common in nonlinear dynamics, which are popularly thought by apparent collisions between attractors and unstable periodic motions. In the S-unimodal case, that is the chaotic three-band attractor touches the unstable periodic orbit. Notice that, these orbits are all repelling, hence the points nearby moves away from them. The possible way for an attractor touches a repelling orbit is that, it hits some preimage of a point of the orbit. The c-symmetric points are exactly the preimages of the central points in S-unimodal case. Form the phase space point of view, when the repelling orbit together with its preimage confines the region of an attractor, its behaviors also character the features of the attractor to a certain extent. naturally with the attractor spreading out of such limited region, its dynamics changes correspondingly, just like the size and periodicity in this case of S-unimodal maps. $\diamond$

4.3.2 Self-similarity in transition of S-unimodal family

In this subsection we make a brief descriptive exposition of the transition of a generic S-unimodal family. From the restrictive intervals point of view, this transition naturally exposes naturally the noticeable self-similarity during it.

For S-unimodal maps, the transition of a family is also an important issue besides the classification of their behaviors, that is, how the dynamical be-

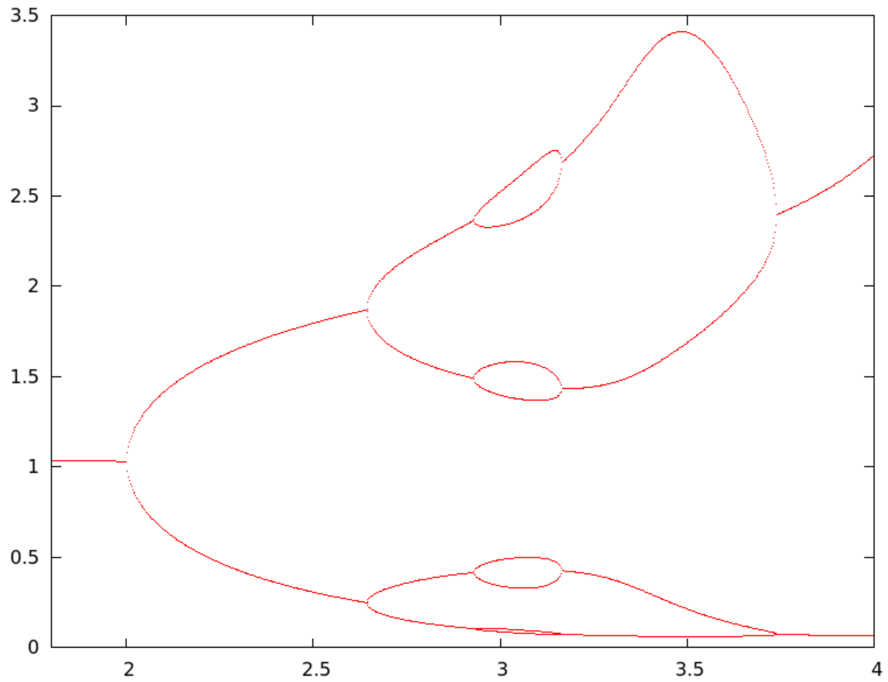


Figure 4.7: Example of not generic family with period-halving.

haviors change with respect to a parameter. The works of Guckenheimer [31] and Devaney [14] are devoted to the systematic genealogy of periodic orbits in the transition, which display that there exists particular regularity on the order of the occurrence and also coexistence for periodic orbits of unimodal maps. People also notice the obvious self-similarity in the transition diagram. In [13], Derrida Gervois and Pomeau exhibited this internal similarity with a composition law of MSS (Metropolis-Stein-Stein) sequences, the same as the popular kneading sequences nowadays. Below we examine these dynamical behaviors from the restrictive intervals point of view, with their convenient patterns expression. It is easily to see that the life cycle of every existed set of restrictive intervals gives a full family, so self-similarity is naturally an inevitable consequence, the same as order of the occurrence.

For simplicity, we only consider a generic family as Devaney in [14]. We assume that $f_\mu(c)$ goes from a to b with the increasing of μ. Furthermore, any periodic orbit must always last until $f_\mu(c) = b$ from its occurrence at a bifurcation, and must remain unstable since it changes to be so. Hence, the situation like period-halving in Figure 4.7 cannot occur.

Self-similarity of full families

With an eye on the life cycles of restrictive intervals, we consider from their creations with bifurcations of periodic orbits, to their destructions at reverse bifurcations. This is called an “window” in the transition diagram. It turns out that every such period is a unit of similarity, because of a significant theorem of the full families. More precise, given a family of one-parameter S-unimodal maps $f_\mu(x)$ on the interval $I = [a, b]$ such that f_μ is continuous with respect to the parameter μ too, a *full family* is the one that its extreme $f_\mu(c)$ goes continuously from one endpoint a to the other b with μ. A prototype is the logistic family $f_\mu(x) = \mu x(1-x)$ for $\mu \in [0, 4]$. For a full family of S-unimodal maps, a theorem by Guckenheimer [32]) says that, for any unimodal map g, there exists a map f_{μ_0} of this family such that f_{μ_0} and g have the same kneading sequence. This means that all the possible combinatorial dynamics of unimodal maps will occur in a full family. Moreover, the results of Guckenheimer in [32] imply that, there also must be some map of a full family which is topologically conjugate to any map of $\mu x(1-x)$ for $\mu \in [0, 4]$.

For our purpose, we may relax a little the restriction of $f_\mu(c)$ moving from a to b on the definition of a full family, and ask $f_\mu(c)$ for going from any $r \in [a, c)$ to b instead. Doing so, we only miss at most one trivial case: a is the only attracting point for all $x \in [a, b]$. Corresponding to $f_\mu(x) = \mu x(1-x)$, it is the case of $\mu \in (1, 4]$. We regard any of such family as a full family also. Notice that, the transition of a full family of S-unimodal maps can be viewed as inside a biggest restrictive interval, their common domain of definition I. Moreover, if $[p_\mu, q_\mu]$ is a restrictive interval of period K of f_μ, the maps $f_\mu^K|_{[p_\mu, q_\mu]}$ form certainly a full family from the creation of $[p_\mu, q_\mu]$ to its destruction with reverse bifurcation. Recall any new restrictive interval occurs with a (only) stable orbit as their endpoints, and finishes when $f_\mu^K(c)$ touches the other endpoint. This fact reveals the self-similarity in the transition of a full S-unimodal family, because each restrictive interval has the same transition with the whole full family.

Structures of transition

Below we make some short arguments on the structures o the transition diagram for the full S-unimodal families, by considering the pattens inside a window.

For any window, we have known that all those pattens of attractors existed in it are extension patterns who have some block structure over its pattern. So we can try to arrange the parameter space of a full family according to the patterns of restrictive intervals. More precise, if p_μ is a central restrictive periodic point of f_μ, we denote by $\{\mathcal{O}(p_\mu)\}$ the set of all the patterns of the attractors who have a block structure over $\mathcal{O}([p_\mu, p'_\mu])$. That is,

$$\{\mathcal{O}(p_\mu)\} := \{\text{patterns with form denoted by } \mathcal{O}([p_\mu, p'_\mu]) \ltimes *\},$$

here $*$ is any admissible patten for an S-unimodal map.

The following two facts are keys for understanding the structure of a family and transition of topological attractors:

- similarity: for every restrictive central periodic point p, $\{\mathcal{O}(p)\}$ is one-to-one with $\{\mathcal{O}(I)\}$.
- For sets of $\{\mathcal{O}(p)\} \neq \{\mathcal{O}(q)\}$ of two central restrictive points p and q, either there is an inclusion relation of $\{\mathcal{O}(p)\} \subset \{\mathcal{O}(q)\}$ or $\{\mathcal{O}(p)\} \supset \{\mathcal{O}(q)\}$; or they are disjoint ($\{\mathcal{O}(p)\} \bigcap \{\mathcal{O}(q)\} = \emptyset$) with an order relation of $\{\mathcal{O}(p)\} < \{\mathcal{O}(q)\}$ or $\{\mathcal{O}(p)\} > \{\mathcal{O}(q)\}$. Here $<$ and $>$ refer to the forcing relation when $\{\mathcal{O}(p)\}$ and $\{\mathcal{O}(q)\}$ are sets of patterns. That is, if a window does not embed into another one as a part of it, then they are completely independent.

  This is because, if $p \in [q, q']$, then $[p, p'] \subset [q, q']$, and so $p(R) < q(R) = f(c)$ at the end of $[q, q']$ being restrictive.

Using the recursive method on the above two rules, we can know more exactly the general structures. In the general interval $I = [a, b]$, if it is $\mu \in [0, 1]$ such that $f_0(c) = a$ and $f_1(c) = b$. there are two series of parameters μ_i^s and μ_i^e for $i \geq 0$ with $0 = \mu_0^s < \mu_1^s \ldots < \mu_i^s < \ldots < \mu_i^e \ldots < \mu_1^e < \mu_0^e = 1$, such that a new central interval $[p_i, p_i{}']$ occurs at μ_i^s and comes to the end at μ_i^e. That is, when $\mu \in (\mu_i^s, \mu_i^e]$, the attractors all have some block structure over $\mathcal{O}([p_i, p_i{}'])$. Correspondingly, we have the following results on the attractors:

- for $\mu \in (\mu_i^s, \mu_{i+1}^s]$, the attractor is a period orbit of period 2^i;
- for $\mu \in (\mu_i^s, \mu_i^e]$, the period of a periodic attractor (orbit or cycle of intervals) is $s \cdot 2^i$ for some $s \in \mathbb{Z}^+$;
- if the period of a periodic attractor (orbit or cycle of intervals) is $s \cdot 2^i$ for some prime number s, then it can only occur at $\mu \in (\mu_{i+1}^e, \mu_i^e]$.

Notice that, any periodic orbit with prime number period can only occur after the central interval of the repelling fixed point becomes non-restrictive, so the reverse bifurcation terminates the period 2 window. For the logistic family $f_\mu(x) = \mu x(1 - x)$, it is after $\mu_1 \approx 3.6785735$, just as we discussed after Example 4.2.6. More generally, this is also the case for all periodic orbits who are not a period-doubling type, for example, the periodic orbit of period 4 with points $p_2 < p_3 < p_0 < p_1$.

The final thing is, this is the same structure in every window of restrictive intervals $\mu \in [\mu_0, \mu_1]$. We just need replace $\mu \in [0, 1]$ by $[\mu_0, \mu_1]$, and change all those period with 2^i to be with $2^i n$ for n the period of the pattern of the window (refer to Figure 4.6). Although, there are infinitely countably many windows inside every window, and there are also infinitely countably many windows between any two independent windows, we still get a clear perspective of the general structure.

4.4 Attractors of Quasi-periodically forced S-unimodal maps

In this final section we investigate the change of periodicity of attractors in quasi-periodically forced S-unimodal systems. We illustrate with clear numerical evidences that, with respect to the increasing of forcing term, generally the periodicity reduces according to the block structures of the extension pattern for the unforced S-unimodal map.

More precise, a quasi-periodically forced S-unimodal system on cylinder $\mathbb{S}^1 \times \mathbb{R}$ is given by map $F\colon \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$ of the form:

$$\begin{cases} \theta_{n+1} &= \theta_n + \omega \pmod 1, \\ x_{n+1} &= \psi(\theta_n, x_n), \end{cases} \tag{4.1}$$

where $(\theta, x) \in \mathbb{S}^1 \times \mathbb{R}$. Here $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is the unit circle and $\omega \in \mathbb{R} \setminus \mathbb{Q}$ is a fixed irrational number. So the function $R\colon \mathbb{S}^1 \to \mathbb{S}^1$ denotes an irrational rotation of the circle $\mathbb{S}^1$ by the fixed angle ω. Furthermore, we assume $\psi(\theta_n, x_n)$ is a continuous function on both x and θ. Such a system (4.1) is in the form of *skew product*: the map $R\colon \mathbb{S}^1 \to \mathbb{S}^1$ is called the *base map* and $\{\theta\} \times \mathbb{R}$ is called the *fiber over* θ. For a fixed θ, the map

$$\psi(\theta, \cdot)\colon \{\theta\} \times \mathbb{R} \to \{R(\theta)\} \times \mathbb{R}$$

is a map from the fiber over θ into the fiber over $R(\theta)$. Since the map that will be applied at time n depends on the irrational rotation $R^n(\theta)$, the system (4.1) is called *quasi-periodically forced.*

For a quasi-periodically forced S-unimodal system, the fibre map $\psi_\theta(x)$ is a function of S-unimodal map $f(x)$ perturbed by some function of θ, for instance, $f(x) \cdot g(\theta)$ or $f(x) + g(\theta)$. For the $f(x) \cdot g(\theta)$ case, we assume that $g(\theta) \geq 0$ so that the S-unimodal structure can be maintained in the forced system. Moreover, we suppose that $\psi_\theta(x) = f(x)$ for all $\theta \in \mathbb{S}^1$ if $\epsilon = 0$. We present the general mechanism of the change of periodicity with respect to the increasing of ϵ with fixed f.

Claim E. *For this type of systems, let ϵ increase from 0, we can see a process of the patterns of attractors for the forced systems change as follows. If the attractor of the fibre map $f(x)$ is contained in restrictive intervals of pattern*

$$\mathcal{O}([p_k]) = \mathcal{O}([p_0, p_0']) \ltimes \mathcal{O}_{[p_0,p_0']}([p_1, p_1']) \ltimes \ldots \ltimes \mathcal{O}_{[p_{k-1},p_{k-1}']}([p_k, p_k']),$$

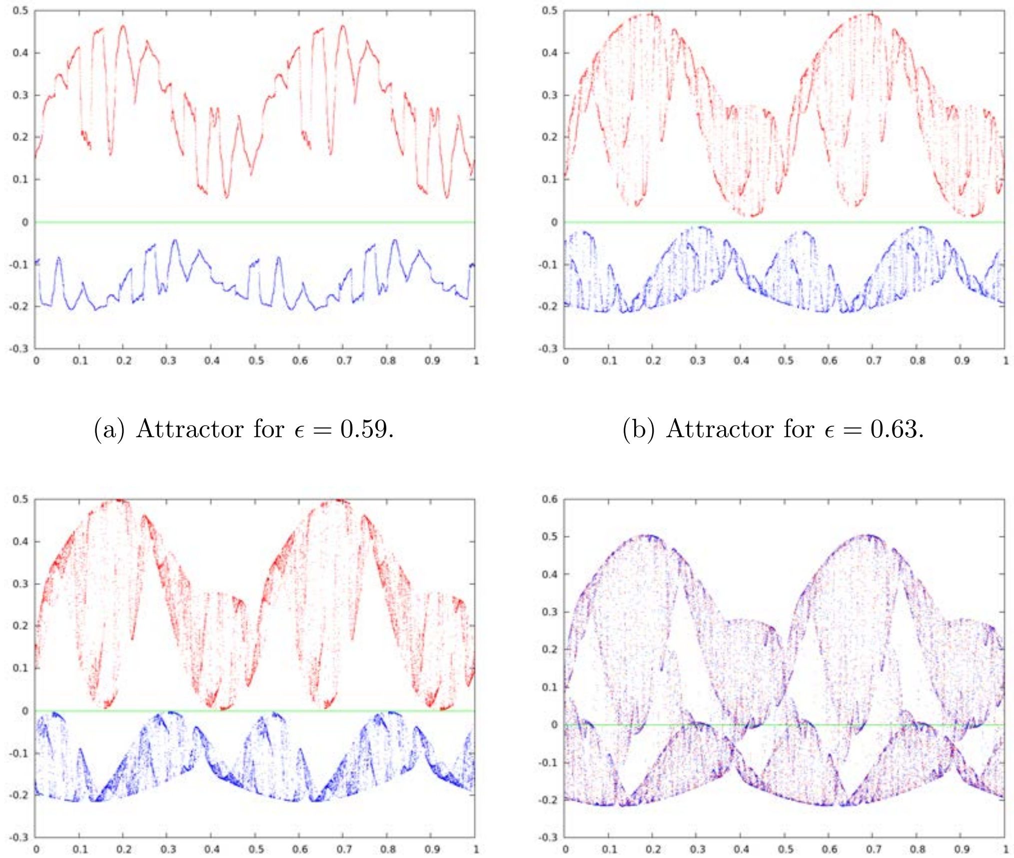
the attractors of forced systems become step by step (maybe not monotonically) stripes on the cylinder with patterns

$$\mathcal{O}([p_k]) \to \mathcal{O}([p_{k-1}]) \to \ldots \to \mathcal{O}([p_1]) \to \mathcal{O}([p_0]).$$

That is, the attractor merges into stripes with bigger size and less periodicity, each time it goes beyond the old region containing them, just like these restrictive intervals of the fibre map $f(x)$. Hence their changes are according to the block structure of $f(x)$.

Such systems are studied extensively with numerical methods, and many phenomena are reported already, which can be seen in an exclusive summary book [24] and references therein. It is found that, similar phenomena like bangs merging, interior crisis are all found as in S-unimodal families also occur for the attractors of such systems. Different with S-unimodal family, the successive period-doubling bifurcations of periodic orbit, which occur with the increasing of the maximum of S-unimodal maps, cannot always continue in any forced systems. We will make simple and informal analysis of the reason of these bifurcations, we can see that this process according to the extension pattern is certainly reasonable.

We have known that, due to the irrational rotation of the base map, there cannot exist any fixed or periodic points in a quasi-periodically forced system. As a closed invariant subset, the projection of any attractor to the base space has to be the whole circle $\mathbb{S}^1$. The simplest invariant closed subset can only be the graph of a map from $\mathbb{S}^1$ to X. More generally, the attractors of quasi-periodically S-unimodal maps are in forms of strips introduced in [21, 3]. That is, let $\pi : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1$ the projection from $\mathbb{S}^1 \times \mathbb{R}$ to the base circle $\mathbb{S}^1$, a *strip* in $\mathbb{S}^1 \times \mathbb{R}$ is a closed set $A \subset \mathbb{S}^1 \times X$ such that $\pi(A) = \mathbb{S}^1$ and $\pi^{-1}(\theta) \cap A$ is a closed interval (perhaps degenerate to a point) for a residual set of $\theta \in \mathbb{S}^1$. Thus an attractor of system (4.1) is in a form of union of n strips $A = A_1 \cup A_2 \cup \ldots \cup A_n$, such that $A_i \neq A_j$ if $i \neq j$ and $F(A_i) = A_{i+1 \pmod 1}$ for $1 \leq i \leq n$. Moreover, A is also transitive, which is the closure of a dense



(a) Attractor for $\epsilon = 0.59$.

(b) Attractor for $\epsilon = 0.63$.

(c) Attractor for $\epsilon = 0.64$.

(d) Attractor for $\epsilon = 0.65$.

Figure 4.8: Attractors of system (4.2) for different parameter values.

orbit. We call n the period of this attractor A and each A_i a periodic strip of the attractor. Notice that, $A_i \neq A_j$ for $i \neq j$ does not mean that $A_i \cap A_j = \emptyset$, this intersection may be a nowhere dense set.

To analyze the mechanism we claim, first let us look at a particular example, whose transition is shown briefly in Figure 4.8.

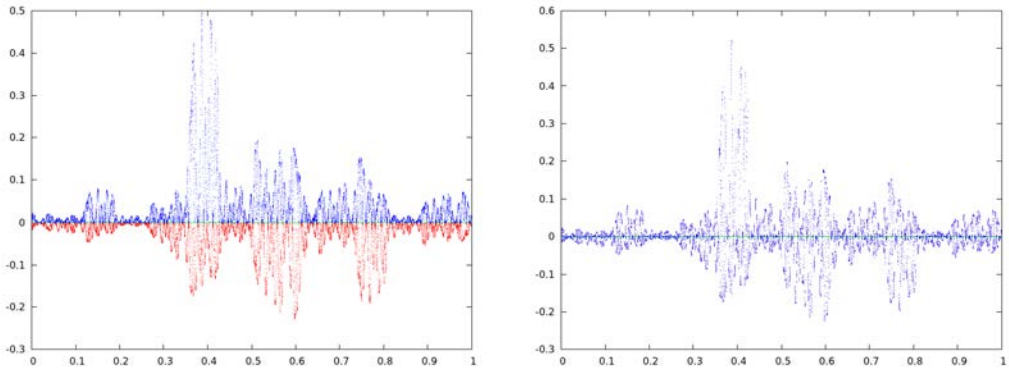
Example 4.4.1. Let the system given by the following map:

$$\begin{cases} \theta_{n+1} &= \theta_n + \omega \pmod 1, \\ x_{n+1} &= (1 + \epsilon |\cos(2\pi\theta_n)|)\, 2.1 x_n (x_n - 0.5). \end{cases} \tag{4.2}$$

In this example, the forcing term is $1 + \epsilon|\cos(2\pi\theta_n)|$ with parameter ϵ. We choose specially the forced map to be $2.1x_n(x_n - 0.5)$, for the reason that $x = 0$ is always an invariant graph, with a preimage $x = 0.5$ in this system.

In fact, it is equivalent to a change of variables of the common logistic maps $\mu x(1 - x)$, so that its second fixed points $x = 1 - 1/\mu$ is moved and fixed to $x = 0$ in new system now. Refer to map $2.1x_n(x_n - 0.5)$, now the region of $\mathbb{S}^1 \times [0, 0.5]$ for (4.2) is equivalent to the central interval of its fixed point. Notice that, we have $x_{n+1} < 0$ whenever $0 < x_n < 0.5$ and $x_{n+1} > 0$ if $x_n < 0$. Moreover, there must be a graph ψ in the negative part $\mathbb{S}^1 \times (-\infty, 0)$ who is one of the preimage of the graph $x = 0.5$. When this region given by ψ and $x = 0.5$ is invariant, and with $x = 0$ repelling, any attractor inside it must be periodic strips whose period is a multiple of 2. This is displayed clearly in the pictures of the attractors for ϵ from 0.59 to 0.63, where we plot them using different colors for the even and odd iterates. Certainly, with the top of the attractor is more close to $x = 0.5$, the lower bound of its upper strip becomes more close to $x = 0$. When this happen, the attractor looks more and more complicated, their pictures illustrate the fractalization progress proved by Jorba and Tatjer in [41]. However, notice that the periodicity never changes as long as the top does not touch $x = 0.5$.

The change occurs when the attractor finally goes beyond $x = 0.5$, which terminates the invariance of the region given by ψ and $x = 0.5$. At the same time, the periodicity reduces accordingly. We notice that, in the case that there is some $x_n > 0.5$, it must be $x_{n+1} > 0$ too, hence the old periodicity must be destroyed for such orbit. Furthermore, there is also some other points of the attractor with the second variable x to be 0, this implies the attractor must intersect with the graph $x = 0$ at a dense set on it. What follows is, the two separated before bands bot cross over $x = 0$, and so mix at the neighborhood of $x = 0$, then all other region everywhere as in Figure 4.8(d).



(a) Attractor for $\epsilon = 1.236$.

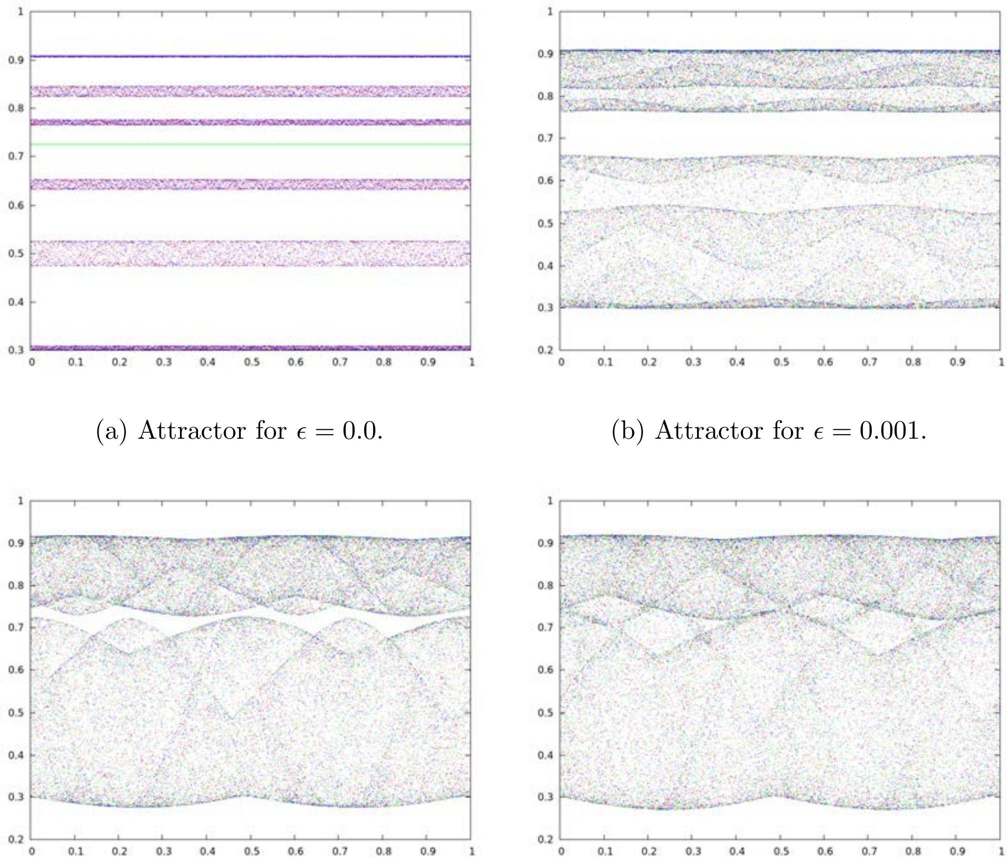
(b) Attractor for $\epsilon = 1.237$.

Figure 4.9: Attractors of $|1 + \epsilon \cos(2\pi\theta_n))|\, 3.3x(x - 0.5)$ for two parameters.

Another example with $x_{n+1} = |1 + \epsilon \cos(2\pi\theta_n))|\, 3.3x_n(x_n - 0.5)$ also show this critical point of $x = 0.5$ clearly. This system is pinched when $\epsilon \geq 1$, we can see in Figure 4.9(a) that the attractor is a periodic SNA of period 2, whose upper bound is very close but still lower than $x = 0.5$. The two parts of upper and lower semicontinuous graphs are not mixed although they intersect at $x = 0$ densely. While the two parts do merge into one piece when the upper boundary goes forward over $x = 0.5$ as in Figure 4.9(b). ◇

Remark 4.4.2. This example is a modified version of the so-called “HH” model introduced by Heagy and Hammel in [35], where they tried to find an SNA by intersection of two periodic curves. They observed that the two curves mixing into one piece immediately after their collision, and related this phenomenon with the bands merging introduced by Lorenz([51]). After them, this kind of bands merging were extensively found between couples of adjacent curves which are corresponding to period-doubling points of unforced logistic maps. A band merging is widely thought as the sign of terminal of period-doubling cascade of periodic invariant curves in the forced systems. ◇

With “HH” model, all these analyses have to be carried out on the base of numerical simulations, since both the invariant curves and the preimages as boundary of invariant region cannot be expressed analytically. Unfortunately, this is the case for almost all the systems that we meet. Even in this artificial example that we choose the model particularly, we can either not get the formulas for any other invariant periodic curves than $x = 0$. Even though



(a) Attractor for $\epsilon = 0.0$.

(b) Attractor for $\epsilon = 0.001$.

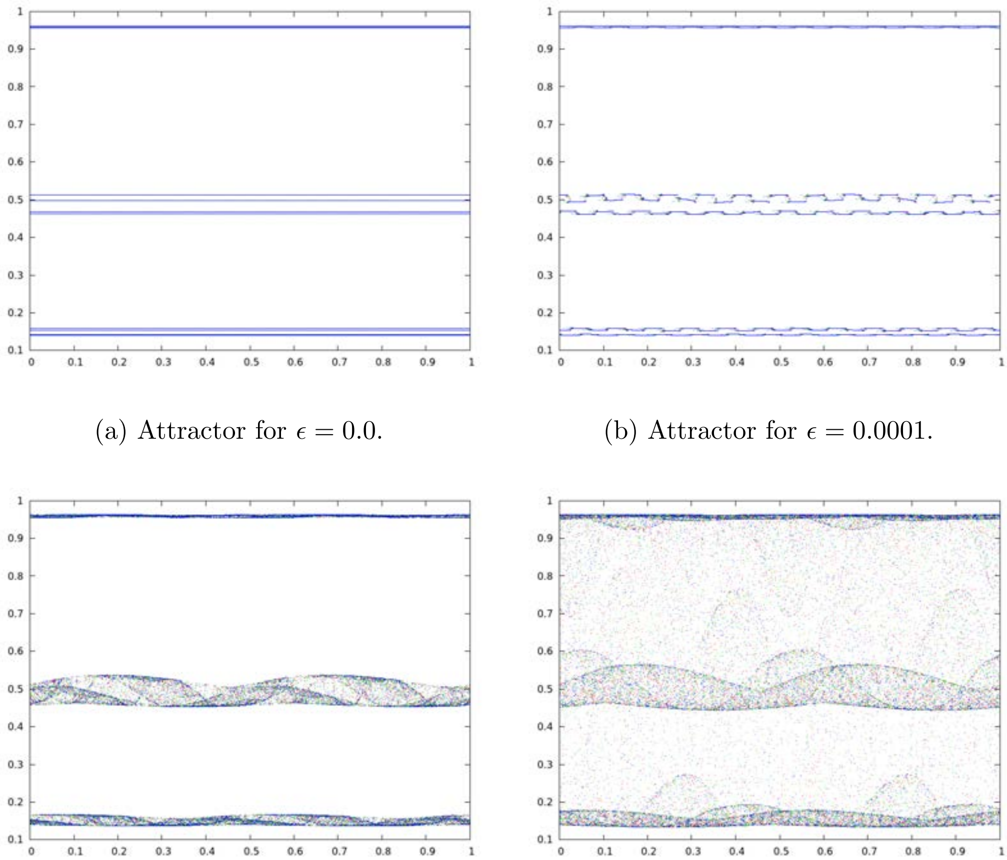
(c) Attractor for $\epsilon = 0.01$.

(d) Attractor for $\epsilon = 0.012$.

Figure 4.10: Attractors of system (4.3) for $\mu = 3.635$ with different ϵ values.

this example still has thrown light on the periodicity problem of the forced systems.

Let us consider the forced systems starting with the parameter of the forced term being $\epsilon = 0$, thus each fibre map is in fact unforced which is just the original S-unimodal map itself. In this case, each periodic point corresponds exactly a periodic curve in the forced system with the same periodicity and stability, and the pattern structures of all the restrictive intervals are also preserved by strips. If we increase ϵ gradually, by the Implicit Function Theorem, each of those repelling periodic invariant curves persists and moves smoothly as hyperbolic set, particularly those ones who correspond to the endpoints of restrictive intervals. Meanwhile, the attractor also changes gradually with respect to the perturbation, usually only in its



(a) Attractor for $\epsilon = 0.0$.

(b) Attractor for $\epsilon = 0.0001$.

(c) Attractor for $\epsilon = 0.001$.

(d) Attractor for $\epsilon = 0.002$.

Figure 4.11: Attractors of system (4.3) for $\mu = 3.85$ with different parameter values.

shape and size, before it goes beyond the invariant region from originally a forward invariant set of restrictive intervals.[1]

A reverse bifurcation happen only when the attractor breaks through the old invariant region. Due to the continuity of the systems, such breakthrough can only lead the attractors to the very outside layer. This means what we can observe for combinatorial dynamics is the same as the unforced S-unimodal maps, with a reduction of periodicity to the degree of the current pattern. That is, if the pattern of the attractor before bifurcation corresponds to some $\mathcal{O}([p_0])$ of the unforced map, then it becomes $\mathcal{O}([p_1])$ where the

[1] In rare occasion, there may be bifurcation causing a increase of its periodicity. We shortly explore the non-monotonicity problem in the final remark.

central points $p_0 \in (p_1, p_1')$ with no other central restrictive points inside.

More precisely, if the original blocks are of period-doubling type, then the bands merging of halving the period occurs; for the case of saddle-node type, it is the interior crisis with several disjoint bands suddenly merging into one. Notice in this latter case, there are infinitely many repelling periodic curves in a blank region between any two such periodic invariant strips, who correspond those non-restrictive periodic points in such interval of the one-dimensional S-unimodal map. Therefore, these blank regions are in general transitional areas for transitive orbits. This mechanism works for blocks of each layer, and hence the attractor goes all the way out to the final biggest invariant region, and comes into one piece. Figure 4.10 and 4.11 of the example below illustrate intuitively this process of the mechanism we present.

Example 4.4.3. We show the transition of attractors of systems:

$$\begin{cases} \theta_{n+1} & = \theta_n + \omega \quad (\text{mod } 1), \\ x_{n+1} & = (1 + \epsilon \cos(2\pi\theta_n))\, \mu x_n(1 - x_n). \end{cases} \tag{4.3}$$

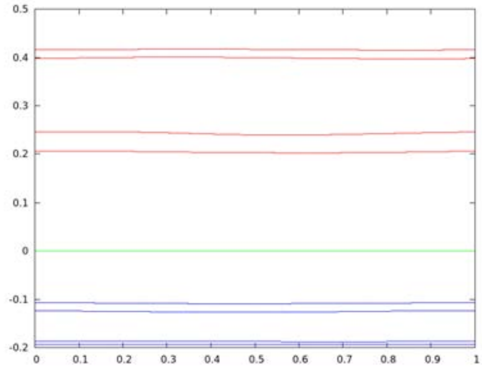
In Figure 4.10 and Figure 4.11 we plot the pictures of some attractors for $\mu = 3.635$ and $\mu = 3.85$ respectively. The corresponding unforced one-dimensional logistic maps have been discussed in Example 4.2.8, and their block structures of restrictive intervals can refer to Figure 4.4.

As we have known, for $\mu = 3.635$, the unforced map has a extension pattern with two layers, the period 2 blocks form the lower one, over each with a period 3 pattern. This is shown clearly too in Figure 4.10(a) with ϵ set at 0, which in addition illustrates that the attractor is not a periodic orbit of points, but a cycle of intervals instead. So we start here with an attractor of six period strips. With increasing of ϵ, quickly the 3 strips inside each block merge into one piece, the period of the attractor becomes 2, and then these two bands also merge into one at last.

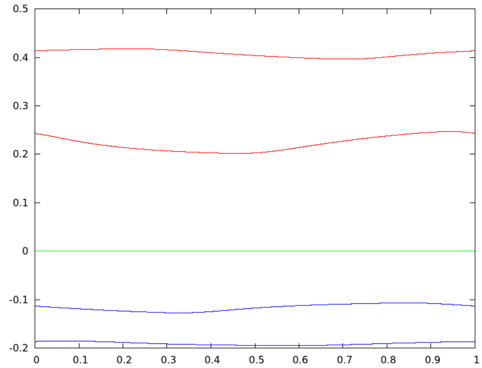
For $\mu = 3.85$, the outer two layers are period-doubling patterns over each period 3 block this time. From Figure 4.11(a) for $\epsilon = 0$, we see it is with curves of period 12, so there are still two times period-doubling bifurcations over the inside layers that we know. The other pictures there exhibit each of the reverse bifurcations in order, three times of reversed period-doubling, followed by the final merging of the three strips left. ◇

Finally, we give some remarks on some involved issues.

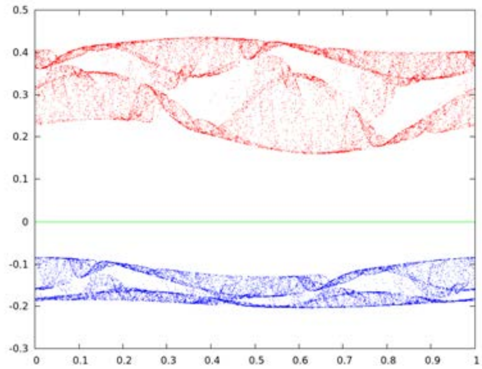
Remark 4.4.4. We have three points, which are either not so crucial or rather trivial to the above mechanism, but still are worth mentioning.



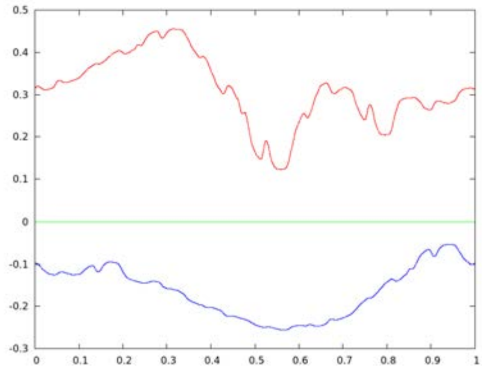
(a) Attractor for $\epsilon = 0.001$.



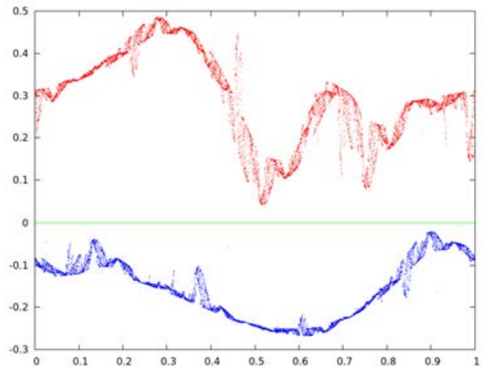
(b) Attractor for $\epsilon = 0.005$.



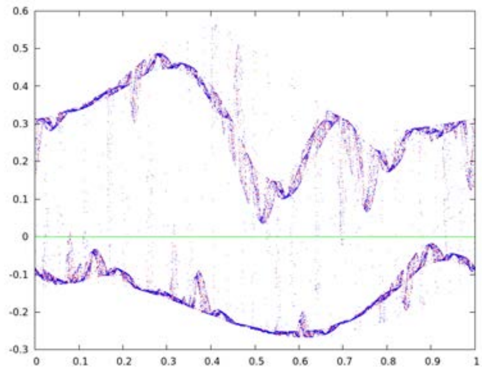
(c) Attractor for $\epsilon = 0.05$.



(d) Attractor for $\epsilon = 0.36$.



(e) Attractor for $\epsilon = 0.377$.



(f) Attractor for $\epsilon = 0.378$.

Figure 4.12: Attractors of system with fibre maps $x_{n+1} = (1 + \epsilon\cos(2\pi\theta_n))\,3.3x_n(x_n - 0.5)$.

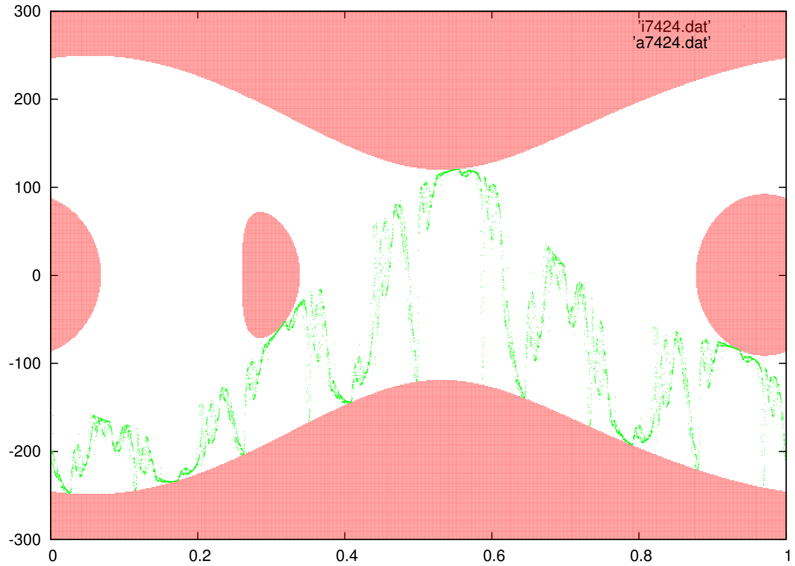


Figure 4.13: Attractor touch boundary of basin of attraction inside invariant region.

(1) Neither the size nor the chaoticity of the attractor monotonically increases with respective to the increasing of the parameter of forcing term completely. The periodicity does not decrease all the way in every system either. For any fixed S-unimodal map, it is only the general tendency that, we can expect bigger size, more chaoticity and less period of the attractor with larger parameter of perturbation. However, occasionally, backward behaviors do exist.

The precise dynamics depends on both the S-unimodal map and the perturbation, particularly on their actions at the position of the attractor. The attractors in Figure 4.12 are given by fibre maps of $x_{n+1} = (1+\epsilon\cos(2\pi\theta_n))\, 3.3x_n(x_n - 0.5)$, which start at periodic curves of period 8 from four times period-doubling bifurcation. Notice the situation of Figure 4.12(c) and Figure 4.12(d). The attractor consists of complicated strips just after a reverse bifurcation of bands merging at $\epsilon = 0.05$, which is normally accompanied with fractalization before the merging. However, during a long time of transition to $\epsilon = 0.36$, the two bands of attractor become very simple and smooth curves. We observe that, in case of Figure 4.12(c), the attractor is more close to the repelling invariant graph $x = 0$, so there are more oscillations on the two sides around contraction region. While for the latter, the attractor is pushed a little farther from $x = 0$, thus there should be more contractions on it. We also point out that, sometimes the increasing of forced parameter even can cause a period-doubling bifurcation on the curves of some attractors, but this cannot stop the next bands merging at large enough perturbations, which takes the attractor back to the same combinatorial type. So the mechanism that the pattern decreases according to the block structures is still valid in general.

(2) Any fixed forcing term, whatever how small it is, terminates the period-doubling cascade at some moment. The period-doubling cascade requires smaller and smaller spaces for happening, which tends to infinitely small at relative firmed place. This cannot be satisfied by a fixed forcing term, who in general only make a fixed amplitude at a certain position.

(3) We have shown that, whenever the attractor touches directly and then goes beyond the boundary of invariant blocks, it leaves the original limit of invariance. However, it is not the only way for the attractor to break through. Another observed possibility exists. This happens for the case of Figure 4.13. Inside the invariant region, there are some other points rather than the attractor go out. If such points meet the attractor, then they also take way all points inside. However, when this happens, the points only break the limit of blocks of one layer, and still cause a mixing of bands within the outside one, which means that the mechanism we present also works.

$\diamond$

Bibliography

[1] Ll. Alsedà and S. Costa, *On the definition of strange nonchaotic attractor*, Fund. Math. **206** (2009), 23–39.

[2] Ll. Alsedà J. Llibre and M. Misiurewicz, *Combinatorial Dynamics and Entropy in Dimension One*, Second Edition, World Scientific (Advanced Series in Nonlinear Dynamics, vol. 5), Singapore, 2000.

[3] Ll. Alsedà, F. Mañosas and L. Morales, *Forcing and entropy of strip patterns of quasiperiodic skew products in the cylinder*, J. Math. Anal. Appl. **429** (2015), no. 1, 542–561.

[4] Ll. Alsedà and M. Misiurewicz, *Attractors for unimodal quasiperiodically forced maps*, J. Difference Equ. Appl. **14** (2008), 1175–1196.

[5] Ll. Alsedà and M. Misiurewicz, *Skew Product Attractors and concavity*, Proc. Amer. Math. Soc. **143** (2015), no. 2, 703–716.

[6] L. Barreira and Y. B. Pesin. *Lyapunov exponents and smooth ergodic theory*, volume 23 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2002.

[7] Z. I. Bezhaeva and V. I Oseledets. On an example of a strange nonchaotic attractor. *Funktisional. Anal. i Prilozhen.*, 30(4):1-9, 95, 1996.

[8] K. Bjerklöv, *SNA's in the quasi-periodic quadratic family*, Comm. Math. Phys. **286** (2009), 137–161.

[9] P. Collet and J-P. Eckmann, *Iterated Maps of the Interval as Dynamical Systems*, Progress in Physics, Vol. 1, Birkhäuser: Boston, 1980.

[10] S. Costa Romero. *Strange Nonchaotic Attractor: a definition.* Master thesis, Departament de Matemàtiques, Universitat Autònoma de Barcelona, 2005.

[11] S. Datta, T. Jäger, G. Keller, and R. Ramaswamy. ON the dynamics of the critical Harper map. *Nonlinearity*, 17(6):2315-2323, 2004.

[12] J. H. P. Dawes and M. C. Freeland, *The 0-1 test for chaos and. strange nonchaotic attractors*, preprint (2008), http://people.bath.ac.uk/jhpd20/. publications/sna.pdf .

[13] B. Derrida, A. Gervois and Y. Pomeau, *Iteration of endomorphisms on the real axis and representation of numbers*, Annales de l'I.H.P. Physique théorique, section A, Tome 29, no.3, (1978), 305–356.

[14] R. L. Devaney, *Genealogy of periodic points of maps of the interval*, Trans. Amer. Math. Soc., **(265)** (1981), 137–146.

[15] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Second Edition, Westview Press, 2003.

[16] M. Ding, G. Grebogi and E. Ott. Evolution of attractors in quasiperiodically forced systems: From quasiperiodic to strange nonchaotic to chaotic. *Phys. Rev. A*, 39(5):2593-2598, 1989.

[17] M. Ding, G. Grebogi and E. Ott. Dimensions of strange nonchaotic attractors. *Phys. Lett. A*, 137(4-5):167-172, 1989.

[18] W. L. Ditto, M. L. Spano, H. T. Savage, S. N. Rauseo, J. F. Heagy and E. Ott.Experimental observation of a strange nonchaotic attractor. *Phys. Rev. Lett.*, 65(5):533-536, 1990.

[19] W. X. Ding, H. Deutsch, A. Dinklage, and C. Wilke. Observation of a strange nonchaotic attractor in a neon glow discharge. *Phys. Rev. E*, 55(3):3769-3772, 1997.

[20] J.-P. Eckmann and D. Ruelle. Ergodic theory of chaos and stange attractors. *Rev. Mod. Phys.*, 57(3):617-656, 1985.

[21] R. Fabbri, T. H. Jäger, R. Johnson, and G. Keller, *A Sharkovskii-type theorem for minimally forced interval maps*, Topol. Methods Nonlinear Anal. **26** (2005), no. 1, 163–188.

[22] M. J. Feigenbaum. Quantitative universality for a class of nonlinear transformations. *J. Stat. Phys.*, 19:25-52, 1978.

[23] M. J. Feigenbaum. The universal metric properties of non-linear transformations. *J. Stat. Phys.*, 21:669-706, 1979.

[24] U. Feudel, S. Kuznetsov and A. Pikovsky, *Strange Nonchaotic Attractors: Dynamics between Order and Chaos in Quasiperiodically Forced Systems*, World Scientific (World Scientific Series on Nonlinear Science, Series A, Vol. 56), Singapore, 2006.

[25] P. Glendinning. Intermittency and strange nonchaotic attractors in quasi-periodically forced circle maps. *Phys. Lett. A*, 244(6):545-552, 1998.

[26] P. Glendinning. Global attractors of pinched skew products. *Dyn. Syst.*, 17:287-294, 2002.

[27] G. A. Gottwald and I. Melbourne, *On the Implementation of the 0â1 Test for Chaos*, SIAM J. Appl. Dyn. Syst. **8(1)** (2009), 129–145.

[28] J. Graczyk and G. ÌŚwiaÌ§tek, *Smooth unimodal maps in the 1990s*, Ergod. Th. & Dyn. Sys., **(19)** (1999), 263–287.

[29] C. Grebogi, E. Ott, S. Pelikan, and J. A. Yorke, Strange attractors that are not chaotic. *Phys. D*, **13** (1984), 261–268.

[30] C. Grebogi, E. Ott and J. A. Yorke, *Crises: Sudden Changes in Chaotic Attractors and Chaotic Transients*, *Physica D*, **7** (1983), 181–200.

[31] J. Guckenheimer, *On the bifurcation of maps of the interval*, Invent. Math., **(39)** (1977), 165–178.

[32] J. Guckenheimer, *Sensitive dependence to initial conditions for one-dimensional maps*, Comm. Math. Phys., **(70)** (1979), 133–160.

[33] A. Haro and R. de la Llave. Spectral theory of transfer operators (II): Vector bundle maps over rotations. In progress, 2005.

[34] A. Haro and C. Simó. To be or not to be a SNA: That is the question. Preprint, 2005.

[35] J. F. Heagy and S. M. Hammel, *The birth of strange nonchaotic attractors.* Phys. D, **70(1-2)** (1994), 140–153.

[36] M. Hénon. A two-dimensional mapping with a strange attractor. *Comm. Math. Phys.*, 50:69-77, 1976.

[37] T. H. Jäger, *The creation of strange non-chaotic attractors in non-smooth saddle-node bifurcations.* Mem. Amer. Math. Soc. **201** (2009), no.945.

[38] M. W. Hirsch and C. C. Pugh. Stable manifolds and hyperbolic sets. *Global Analysis*. Amer. Math. Soc. Proc. Symp. Pure Math., 14:133-164, American Mathematical Society, 1983.

[39] M. W. Hirsch, C. C. Pugh and M. Shub. *Invariant manifolds*. Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin, 1977.

[40] À. Jorba. Numerical computation of the normal behaviour of invariant curves of n-dimensional maps. *Nonlinearity*, 14(5):943-976, 2001.

[41] À. Jorba and J. C. Tatjer, *A mechanism for the fractalization of invariant curves in quasi-periodically forced 1-D maps*, Discrete Contin. Dyn. Syst. Ser. B **10** (2008), no. 2-3, 537–567.

[42] T. Kapitaniak, E. Ponce, and J. Wojewoda. Route to chaos via strange nonchaotic attractors. *J. Phys. A*, 23(8):L383-L387, 1990.

[43] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995.

[44] G. Keller. *A note on strange nonchaotic attractors*, Fund. Math. **151** (1996), 139–148.

[45] Sang-Yoon Kim, and W. Lim. Universal mechanism for the intermittent route to strange nonchaotic attractors in quasiperiodically forced systems. *J. Phys. A*, 37(25):16477-6489, 2004.

[46] Sang-Yoon Kim, W. Lim, and E. Ott. Mechanism for the intermittent route to strange nonchaotic attractors. *Phys. Rev. E*, 67,056203, 2003.

[47] O. S. Kozlovski, *Structural Stability in One-Dimensional Dynamics*, Ph.D. Thesis, Amsterdam University, Amsterdam, 1997.

[48] P. Kůrka. *Topological and symbolic dynamics*. Cours Spécialisés 11, Société Mathématique de France, Paris, 2003

[49] T. Y. Li and J. A. Yorke. Period three implies chaos. *Amer. Math. Monthly*, 82(10):985-992, 1975.

[50] E. N. Lorenz. Deterministic Nonperiodic Flow. *J. Atmos. Sci.*, Vol.20, NO.2:130-141, 1963.

[51] E. N. Lorenz, *Noisy periodicity and reverse bifurcation*, Ann. New York Acad. Sci. **357(1)** (1980), Issue. 1, 282-291.

[52] M. Y. Lyubich, *Non-existence of wandering intervals and structure of topological attractors of one dimensionsl dynamical systems. 1. The case of negative Schwarzian derivative*, Ergod. Th. & Dyn. Sys. **9** (1989), no. 4, 751–758.

[53] R. Mañé. *Ergodic theory and differentiable dynamics.* Springer-Verlag, Berlin, 1987. Translated from the Portuguese by Silvio Levy.

[54] R. M. May. Simple mathematical models with very complicated dynamics. *Nature.* Vol.261, No.5560:459-467, 1976(June).

[55] W. de Melo, and S. van Strien, *One-dimensional Dynamics*, Springer-Verlag, 1993.

[56] V. M. Millionščikov. Proof of the existence of irregular systems of linear differential equations with almost periodic coefficients. *Differ. Uravn.*, 4(3):391-396, 1968.

[57] V. M. Millionščikov. Proof of the existence of irregular systems of linear differential equations with quasi periodic coefficients. *Differ. Uravn.*, 5(11):1979-1983, 1969.

[58] J. Milnor. On the concept of attractor. *Comm. Math. Phys.*, 99(2):177-195, 1985. (Erratum: Comm. Math. Phys., 102(3):517-519, 1985.)

[59] M. Misiurewicz, *Absolutely continuous measures for certain maps of an interval*, Inst. Hautes Études Sci. Publ. Math., **(53)** (1981), 17–51.

[60] T. Nishikawa and K. Kaneko. Fractalization of torus as a strange non-chaotic attractor. *Phys. Rev. E*, 54(6):6114-6124, 1990.

[61] V. I. Oseledets. A multiplicative ergodic theorem. Characteristic Lyapunov exponents of dynamical systems. *Trudy Moskov. Mat. Obšč.*, 19:179-210, 1968.

[62] A. S. Pikovsky and U. Feudel, *Characterization of strange nonchaotic attractors*, CHAOS, **5** n.1 (1995), 253–260.

[63] H. Poincaré. *Science and method.* Dover Publications Inc., New York, 1952. Translated by Francis Maitland. With a preface by Bertrand Russell.

[64] M. Pollicott. *Lectures on ergodic theory and Pesin theory on compact manifolds*, volume 180 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1993.

[65] Y. Pomeau and P. Manneville, *Intermittent Transition to Turbulence in Dissipative Dynamical Systems*, Comm. Math. Phys., **(74)** (1980), 189–197.

[66] A. Prasad, S. S. Negi and R. Ramaswamy. Strange nonchaotic attractors. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 11(2):291-309, 2001.

[67] C. Robinson. *Dynamical systems: Stability, symbolic dynamics, and chaos*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, second edition, 1999.

[68] F. J. Romeiras, A. Bondeson, E. Ott, T. M. Antonsen and G. Grebogi. Quasiperiodically forced dyanmical systems with strange nonchaotic attrctors. *Phys. D*, 26:277-294, 1987.

[69] D. Ruelle. Ergodic theory of differentiable dynamical systems. *Publ. Math. I.H.E.S.*, 50:27-58, 1979.

[70] D. Ruelle and F. Takens. On the nature of turbulence. *Comm. Math. Phys.*, 20:167-192, 1971.

[71] A. N. Sharkovskii. Coexistence of cycles of a continuous map of the line into itself. *Ukr. Mat. Z.*, 16:61-71, 1964.

[72] D. Singer, *Stable orbits and bifurcations of maps of the interval*, S.I.A.M. J. Appl. Math. **35** (1978), 260-267.

[73] J. Stark. *Invariant graphs for forced systems*, Phys. D, **109(1-2)** 1997, 163–179. Physics and dynamics between chaos, order, and noise (Berlin, 1996).

[74] J. Stark. Regularity of invariant graphs for forced systems. *Ergod. Th. & Dynam. Sys.*, 19:155-199, 1999.

[75] Y. Ueda, *Explosion of strange attractors exhibited by Duffing's equation.* In *Nonlinear Dynamics*, R. H. G. Helleman (ed.), pp. 422-434. New York Academy of Sciences: New York, 1980.

[76] R. E. Vinograd. A problem suggested by N. P. Erugin. *Differ. Uravn.*, 11(4):632-638, 1975.

[77] P. Walters. *An introduction to ergodic theory*, volume 79 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.