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VARIEDADES DE PRYM DE CURVAS BIELIPTICAS

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Certifico que la presente memoria
ha sido realizada bajo mi dirección
por Juan Carlos Naranjo del Val
y que constituye su tesis doctoral.
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Indice

Introducción

El contenido de la presente tesis se halla resumido en la última parte de esta introducción. Con el fin de facilitar su lectura y situarlo en su contexto hacemos un breve recuento de los tópicos necesarios sobre variedades abelianas, jacobianas y variedades de Prym.

Variedades abelianas

Dado un toro complejo $A = \mathbf{C}^g/\Lambda$ se tiene una identificación natural $\Lambda = H_1(A,\mathbf{Z})$. Así una forma bilineal antisimétrica

$$E : \Lambda \times \Lambda \longrightarrow \mathbf{Z}$$

corresponde a un elemento ξ de $\mathrm{Hom}(\Lambda^2 H_1(A,\mathbf{Z}),\mathbf{Z}) = H^2(A,\mathbf{Z})$. Fijemos una tal E. Sean $e_1,\ldots,e_{2g} \in \mathbf{C}^g$ generadores de Λ. Escribiendo las coordenadas de los vectores e_i en columna obtenemos una matriz Ω llamada matriz de períodos. Sea B la matriz que representa a la forma bilineal E en esta misma base. Las relaciones bilineales de Riemann

$$\Omega B^t \Omega = 0$$

$$i\Omega B^t \overline{\Omega} > 0$$

son equivalentes a la existencia de un haz invertible amplio $\mathcal{L}$ sobre A cuya clase de Chern sea ξ. En tal caso A puede sumergirse en un espacio proyectivo y es una variedad algebraica. El Teorema de Riemann-Roch afirma que

$$h^0(A,\mathcal{L}) = \sqrt{\det B}.$$

Una ***variedad abeliana principalmente polarizada*** (vapp) es un par formado por un toro complejo A junto con un haz invertible amplio $\mathcal{L}$ sobre A tal que $h^0(A,\mathcal{L}) = 1$. Es decir la matriz antisimétrica B que representa a la clase de Chern de $\mathcal{L}$ es unimodular. Por álgebra lineal existe entonces una base de Λ en la cual la matriz de E toma la forma

$$B = \begin{pmatrix} 0 & \mathrm{Id}_g \\ -\mathrm{Id}_g & 0 \end{pmatrix}.$$

Cambiando la base de $\mathbf{C}^g$ de forma adecuada se obtiene que la matriz de períodos se puede expresar de la forma (Id,τ) y las relaciones bilineales de Riemann se traducen en las condiciones

$$\tau = {}^t\tau$$

$$\mathrm{Im}(\tau) > 0.$$

Llamamos ***semiespacio de Siegel*** al conjunto de las matrices complejas simétricas $g \times g$ con parte imaginaria definida positiva, y lo denotamos por $\mathcal{H}_g$. Claramente es exhaustiva la aplicación de $\mathcal{H}_g$ en el espacio $\mathcal{A}_g$ de las clases de isomorfismos de vapp dada por

$$\tau \longmapsto \mathbf{C}^g/(\mathrm{Id},\tau)$$

(indicamos también con (Id, τ) la red de $\mathbf{C}^g$ que generan las columnas de esta matriz). El grupo simpléctico $Sp(\mathbf{Z}, 2g)$ actúa sobre $\mathcal{H}_g$ de forma compatible con esta aplicación y el cociente $\mathcal{H}_g/Sp(\mathbf{Z}, 2g)$ es isomorfo a $\mathcal{A}_g$. Si τ es una matriz del semiespacio de Siegel llamaremos $(A_\tau, \mathcal{L}_\tau)$ a la vapp correspondiente.

Sobre el producto $\mathbf{C}^g \times \mathcal{H}_g$ se define la ***función theta de Riemann***

$$\theta(z, \tau) = \sum_{n \in \mathbf{Z}^g} e^{i\pi({}^t n\tau n + 2\cdot{}^t nz)}.$$

Fijada τ, $\theta(z, \tau) = 0$ define un divisor de $\mathbf{C}^g$ invariante por las traslaciones con elementos de (Id, τ) determinando en A_τ un divisor simétrico Θ_τ que verifica $\mathcal{L}_\tau \cong \mathcal{O}_{A_\tau}(\Theta_\tau)$.

Jacobianas

Sea C una curva algebraica proyectiva irreducible no singular sobre $\mathbf{C}$ con género geométrico g. Sea $\omega_1, \dots, \omega_g$ una base de $H^0(C, \omega_C)$, es decir g formas diferenciales holomorfas sobre C linealmente independientes. Sea $\gamma_1, \dots \gamma_{2g}$ una base de $H_1(C, \mathbf{Z})$. La ***matriz de períodos*** Ω asociada a la curva con estas elecciones es la matriz compleja g×2g cuya columna j-ésima es

$$^t(\int_{\gamma_j} \omega_1, \dots, \int_{\gamma_j} \omega_g).$$

El Teorema de Torelli (cf. [To]) afirma que la curva C está determinada por la matriz de períodos Ω. Este enunciado se reformula de manera natural en el contexto de las variedades abelianas. En efecto, si ω y ω' son dos formas diferenciales holomorfas sobre la curva entonces se verifican las relaciones

$$\int_C \omega \wedge \omega' = 0$$
$$i \int_C \omega \wedge \overline{\omega} > 0 \qquad \text{si } \omega \neq 0.$$

Utilizando la relación entre el cup producto en cohomología y el producto de intersección en homología las propiedades anteriores implican que la matriz Ω verifica las relaciones bilineales de Riemann con respecto a la matriz antisimétrica unimodular entera B tal que ${}^tB^{-1}$ es igual a la matriz de los productos de intersección $(\gamma_i \cdot \gamma_j)$. En consecuencia el cociente de $\mathbf{C}^g$ por la red generada por las columnas de la matriz de períodos de C es una vapp de dimensión g a la cual llamamos *jacobiana* de C y la denotamos por JC.

Simbolizaremos con Θ_C a un divisor representando la polarización principal y le llamamos *divisor theta*.

Otra definición de JC, independiente de coordenadas es la siguiente. La integración sobre ciclos define una inyección de $H_1(C,\mathbf{Z})$ en el dual de $H^0(C,\omega_C)$, y

$$\mathrm{JC} = H^0(C,\omega_C)^*/H_1(C,\mathbf{Z}),$$

La forma bilineal es

$$H_1(C,\mathbf{Z}) \times H_1(C,\mathbf{Z}) \longrightarrow H_2(C,\mathbf{Z}) \cong \mathbf{Z},$$

con el isomorfismo dado por la clase de orientación de la estructura analítica de C.

Indicando por $\mathcal{M}_g$ el espacio de moduli de las curvas de género g se obtiene así una aplicación

$$\begin{aligned} J : \mathcal{M}_g &\longrightarrow \mathcal{A}_g \\ C &\longmapsto JC. \end{aligned}$$

En este lenguaje el Teorema de Torelli se expresa diciendo que J es inyectiva (la aplicación J recibe el nombre de *morfismo de Torelli*).

De este resultado se conocen muchas demostraciones, la mayor parte de ellas se basan en el estudio de la geometría del divisor theta. Utilizando construcciones geométricas elementales se reconstruye a partir de (JC,Θ_C) la curva C. Observemos primero que C vive de forma natural en su propia jacobiana. En efecto, fijado $p_0 \in C$, asociamos a un punto p de la curva la aplicación lineal

$$H^0(C,\omega_C) \longrightarrow \mathbf{C}$$

que integra cada forma holomorfa entre p_0 y p. Escrito en coordenadas, la imagen de p es

$$\left(\int_{p_0}^{p} \omega_1, \ldots, \int_{p_0}^{p} \omega_g\right).$$

La definición depende del camino escogido entre p_0 y p para realizar la intregación. Haciendo cociente por $H_1(C,\mathbf{Z})$ la indeterminación desaparece, obteniéndose un punto de la jacobiana.

Consideremos la curva sumergida en su jacobiana de este modo. Observemos que todos los espacios tangentes en puntos de la jacobiana se identifican canónicamente vía traslación con $T_{JC}(0)$, el cual se identifica también canónicamente con $H^0(C,\omega_C)^*$. Tomando el espacio tangente en cada punto de la curva se obtiene la "derivada" de la inmersión $C \hookrightarrow JC$

$$C \longrightarrow \mathbb{P}(H^0(C,\omega_C)^*)$$
$$p \longrightarrow (\omega_1(p) : \cdots : \omega_g(p))$$

que coincide con la aplicación canónica φ_{ω_C}. Así, cuando la curva no es hiperelíptica, también aparece sumergida en el proyectivizado del espacio tangente a la jacobiana en el origen.

El divisor theta se obtiene ahora geométricamente a partir de C merced al Teorema de parametrización de Riemann que afirma que la imagen de la aplicación suma

$$C^{g-1} \longrightarrow JC$$

es un trasladado del divisor theta.

Esta parametrización es un punto esencial en el estudio geométrico del divisor theta. Permite, por ejemplo, describir sus singularidades obteniéndose que la dimensión de SingΘ_C es g-3 si C es hiperelíptica, y g-4 si no lo es.

Indiquemos a modo de ejemplo dos de las demostraciones del Teorema de Torelli. La primera es debida a A.Andreotti ([A]). Estudia la aplicación de Gauss que asocia a cada punto no singular de Θ_C el espacio tangente de Θ_C en ese punto, lo que nos da un subespacio vectorial de codimensión 1 de $T_{JC}(0)$ y, por tanto, un punto de $\mathbb{P}(H^0(C,\omega_C))$. Tomando la adherencia del grafo de esta aplicación, obtenemos un morfismo finito. Su lugar discriminante es, si la curva es no hiperelíptica, la hipersuperficie dual de la imagen canónica de C (la cual determina la curva C por geometría proyectiva elemental).

La segunda es debida a Enriques, Petri, Andreotti, Mayer y Green. En un punto doble de Θ_C, el proyectivizado del cono tangente al divisor trasladado al origen es una cuádrica de rango menor o igual que cuatro conteniendo a la imagen de la aplicación canónica. Se prueba que, si C no es especial, la intersección de estas cuádricas coincide con $\varphi_{\omega_C}(C)$.

Variedades de Prym

Una variedad de Prym es una vapp de un tipo más general que una jacobiana. Para definirla tomemos un recubrimiento irreducible de grado dos no ramificado de una curva de género g:

$$\pi : \hat{C} \longrightarrow C.$$

Fijada C, tales recubrimientos están parametrizados por los puntos de orden dos de JC.

La involución ι del cambio de hoja en $\tilde{C}$ induce sendas involuciones ι^* de $H^0(\tilde{C},\omega_{\tilde{C}})^*$ y $H_1(\tilde{C},\mathbf{Z})$ respectivamente. Tomando los subespacios antiinvariantes respectivos se obtiene una variedad abeliana $P(\tilde{C},C) := (H^0(\tilde{C},\omega_{\tilde{C}})^*)^-/H_1(\tilde{C},\mathbf{Z})^- \hookrightarrow \mathrm{J}\tilde{C}$ a la cual llamamos ***variedad de Prym*** asociada al recubrimiento. Dicho de otro modo ι define una involución ι^* de la jacobiana de $\tilde{C}$ y $P(\tilde{C},C)$ es la componente neutra del núcleo del morfismo Id$+\iota^*$. Otra definición es la siguiente: el recubrimiento π induce una aplicación norma entre jacobianas

$$\mathrm{Nm}_\pi : \mathrm{J}\tilde{C} \longrightarrow \mathrm{JC}$$

cuya definición, interpretando las jacobianas como grupos de clases de divisores (Teoremas de Abel y de Jacobi) es: $\mathrm{Nm}_\pi(\sum n_i\tilde{p}_i) = \sum n_i\pi(\tilde{p}_i)$. El núcleo tiene dos componentes y la variedad de Prym es la que contiene al origen. Vistos como elementos de $J\tilde{C}=\mathrm{Pic}^0(\tilde{C})$, los puntos de $P(\tilde{C},C)$ corresponden a las clases de los divisores de la forma

$$\sum_{j=1}^{2k} n_j(p_j - \iota(p_j)).$$

Por otro lado la polarización de J$\tilde{C}$ restringe a dos veces una polarización principal Ξ en $P(\tilde{C},C)$. En el lenguaje de las formas bilineales se tiene que la forma bilineal sobre $H_1(\tilde{C},\mathbf{Z})$ restringe a $H_1(\tilde{C},\mathbf{Z})^-$ y la matriz reducida de esta restricción es

$$\begin{pmatrix} 0 & 2\mathrm{Id}_g \\ -2\mathrm{Id}_g & 0 \end{pmatrix}.$$

Se obtiene así una vapp. Su dimensión p es la diferencia entre las dimensiones de ambas jacobianas. Por la fórmula de Riemann-Hurwitz se obtiene por tanto que p=g-1.

Análogamente a lo que sucede en el caso de las jacobianas la curva $\tilde{C}$ vive dentro de la variedad de Prym: Fijado un punto $p_0 \in \tilde{C}$ definimos la inmersión

$$\tilde{C} \longrightarrow P(\tilde{C},C)$$

$$p \longrightarrow p - \iota(p) + p_0 - \iota(p_0).$$

Derivando esta aplicación, los paralelismos se mantienen: se obtiene un morfismo de $\tilde{C}$ en el proyectivizado del espacio tangente a $P(\tilde{C},C)$ en el origen, el cual se identifica canónicamente con $(H^0(\tilde{C},\omega_{\tilde{C}})^*)^- = H^0(C,\omega_C(\eta))^*$, donde $\eta \in \mathrm{Pic}^0(\tilde{C})$ es el punto de orden dos que define el recubrimiento. Este morfismo coincide con el asociado a la serie lineal $|\,\omega_C(\eta)\,|$ al que se le suele llamar ***semicanónico***. Si la curva no admite series lineales del tipo g^1_4, es decir no es ***tetragonal***, el morfismo semicanónico es una inmersión.

La teoría de las variedades de Prym juega un papel destacado en varios campos de la geometría algebraica, de los cuales los más destacados son los siguientes:

a) ***El problema de Schottky.*** **Es el problema de caracterizar las jacobianas entre las vapp. Numerosas aproximaciones de carácter muy variado y éxito diverso aparecen en la literatura. Los Teoremas de Matsusaka y Shiota, la conjetura de la trisecante, etc. dan nombre a otros tantos enfoques de la cuestión. El punto de vista original es debido a Schottky y Jung, y es en el que aparecen las variedades de Prym. Encuentran relaciones entre las "thetanullwierte" (que son las coordenadas proyectivas naturales de cierto recubrimiento de $\mathcal{A}_g$) que, presumiblemente, definen el lugar geométrico $\overline{J(\mathcal{M}_g)}$ en $\mathcal{A}_g$. De forma muy simplificada se podría decir que las jacobianas cumplen las ecuaciones de Schottky-Jung debido a que "se les puede asociar" variedades de Prym.** Por otro lado la existencia o no de vapp no jacobianas verificando estas ecuaciones es un problema cuya respuesta se desconoce. El mayor avance en esta dirección es debido a B.v.Geemen (cf. [Ge]) que probó que $\overline{J(\mathcal{M}_g)}$ es una componente irreducible del lugar definido por dichas relaciones.

b) *Estudio geométrico de las variedades abelianas.* Las curvas C y $\tilde{C}$ y su geometría se utiliza para obtener propiedades de la variedad de Prym. Por ejemplo el divisor Ξ admite una descripción del tipo dado por el Teorema de parametrización de Riemann. Explícitamente, escribiendo

$$\Xi^{*} = \{\tilde{\zeta} \in \mathrm{Pic}^{2g-2}(\tilde{C}) \mid \mathrm{Nm}_{\pi}(\tilde{\zeta}) = \omega_C \text{ y } h^0(\tilde{C},\tilde{\zeta}) \text{ es par y positivo}\}.$$

existe $\tilde{\zeta}_0 \in \mathrm{Pic}^{2g-2}(\tilde{C})$ tal que $\Xi^{*} - \tilde{\zeta}_0$ coincide con Ξ.

Se obtienen también descripciones explícitas del lugar singular de Ξ, que han permitido probar que su dimensión es siempre mayor o igual que p-6. También se obtiene una clasificación de los recubrimientos para cuyas variedades de Prym se cumple dim Sing$\Xi \geq p-4$.

La importancia de estos hechos radica en que habitualmente es muy difícil realizar cálculos sobre la geometría del divisor theta de una vapp general. Con gran diferencia las jacobianas son las vapp mejor entendidas, proporcionando en $\mathcal{A}_g$ un subespacio de dimensión 3g-3 (notemos que dim$\mathcal{A}_g$ =dim$\mathcal{H}_g$ = $\frac{1}{2}g(g+1)$). Como comentaremos más adelante las variedades de Prym parametrizan un subespacio de dimensión 3g conteniendo en su adherencia a las jacobianas. Así toda vapp de dimensión menor o igual que tres es una jacobiana, mientras que una vapp general de dimensión menor o igual que cinco es una variedad de Prym.

c) *Jacobianas intermedias de sólidos y problemas de racionalidad.* Las jacobianas intermedias de ciertos sólidos admiten una representación como variedades de Prym. De hecho este fenómeno se extiende a los llamados fibrados en cuádricas que son aquellas variedades que admiten un morfismo sobre el plano proyectivo con fibras isomorfas a cuádricas de dimensión constante impar. El ejemplo más conocido es el de la hipersuperficie cúbica no singular del espacio proyectivo de dimensión cuatro. Sea X un tal sólido y fijemos una recta l contenida en X. Para cada 2-plano π conteniendo a l se tiene

$$\pi \cap X = l \cup Q$$

donde Q es una cónica. Estos 2-planos están parametrizados por un espacio proyectivo de dimensión dos. Los 2-planos para los cuales la cónica Q degenera describen una quíntica plana C. Para cada punto de esta quíntica obtenemos dos rectas contenidas en X por lo que C viene dotada de forma natural de un recubrimiento doble

$$\pi : \tilde{C} \longrightarrow C.$$

Se tiene un isomorfismo de vapp

$$JX \cong P(\tilde{C}, C).$$

Este hecho lleva a una demostración de que X no es racional, resultado que fue demostrado en primer lugar por H.Clemens y P.Griffiths (cf. [C-G]). Se prueba primero que si X fuera racional se su jacobiana intermedia se expresaría como suma directa de jacobianas de curvas lisas. Por otro lado $(\tilde{C}, C)$ no pertenece al conjunto de los recubrimientos para los cuales $\dim \operatorname{Sing}\Xi \geq p - 4$. Por tanto JX no es una jacobiana.

El problema de Torelli para las variedades de Prym

Ya hemos comentado que los recubrimientos dobles irreducibles no ramificados de una curva lisa C de género g están en biyección con el conjunto de cardinal $2^{2g} - 1$ de los puntos de orden dos de JC. Llamamos $\mathcal{R}_g$ al espacio de moduli de los pares (C, η) con $\eta \in JC - \{0\}$ y $2\eta = 0$. La aplicación olvido de $\mathcal{R}_g$ a $\mathcal{M}_g$ es un recubrimiento no ramificado de $2^{2g} - 1$ hojas. En particular $\dim \mathcal{R}_g = 3g - 3$. A la aplicación

$$P : \mathcal{R}_g \longrightarrow \mathcal{A}_{g-1}$$

que asocia a un par (C, η) la variedad de Prym (con su polarización natural) del correspondiente recubrimiento se le llama *aplicación de Prym*. Es el análogo a la aplicación de Torelli. Como en el caso de las jacobianas se plantea el "problema de Torelli": ¿Es inyectiva la aplicación P?.

Comparemos las dimensiones de $\mathcal{R}_g = 3g-3$ y de $\mathcal{A}_{g-1} = \frac{1}{2}g(g-1)$:

g	$3g-3$	$\frac{1}{2}g(g-1)$
2	3	1
3	6	3
4	9	6
5	12	10
6	15	15
7	18	21
$\vdots$	$\vdots$	$\vdots$

Obviamente la aplicación no puede ser inyectiva si $g \leq 5$. Analicemos primero los casos correspondientes a géneros bajos. Es conocido desde Wirtinger (cf. [Wi]) que P es genéricamente exhaustiva si $g \leq 6$ y es genéricamente finita si $g = 6$. Sin embargo se desprende del análisis de ciertos ejemplos concretos que la aplicación de Prym no es nunca exhaustiva. Ello es un reflejo de que P no es propia. En [Be1], Beauville extiende la aplicación de Prym a determinados recubrimientos de curvas estables llamados *admisibles*, compactificando de este modo la aplicación de Prym. La idea es la siguiente: por la propiedad universal de extensión de la compactificación de Satake $\mathcal{A}^S_{g-1}$ de $\mathcal{A}_{g-1}$, P se extiende a una aplicación

$$\mathrm{P}^S : \mathcal{R}^S_g \longrightarrow \mathcal{A}^S_{g-1},$$

donde $\mathcal{R}^S_g$ es la compactificación estable de $\mathcal{R}_g$. Son admisibles los elementos de $\bar{\mathcal{R}}_g := (\bar{\mathrm{P}}^S)^{-1}(\mathcal{A}_{g-1})$ y denotamos por $\bar{\mathrm{P}}$ a la restricción de P^S a $\bar{\mathcal{R}}_g$. Explícitamente, un morfismo finito de grado dos $\pi : \tilde{C} \longrightarrow C$ entre curvas estables con involución de cambio de hoja ι es admisible si y solo si se verifican las dos siguientes condiciones:

a) los puntos fijos de ι son singulares y la involución de cambio de hoja no intercambia las ramas de la singularidad.

b) el número de nodos no fijos por ι coincide con el de componentes irreducibles no fijas por la involución.

Algunos de estos recubrimientos ya habían sido estudiados por Wirtinger, en concreto aquéllos en los que ninguna componente es fija por ι. La variedad de Prym de estos recubrimientos es isomorfa a la jacobiana de la curva base. En consecuencia las jacobianas están contenidas en la imagen de $\bar{\mathrm{P}}$.

El caso g=6 aparece como especialmente atractivo al producirse la igualdad de dimensiones. En [D-S], R.Donagi y R.Smith demuestran que el grado de P es 27 y prueban que la fibra genérica tiene la misma estructura que las 27 rectas contenidas en la superficie cúbica de $\mathbf{P}^3$.

La anterior discusión nos muestra que el enunciado del problema de Torelli en este contexto requiere la restricción $g \geq 7$. Digamos ya que la respuesta al problema es "genéricamente afirmativa", es decir la aplicación de Prym es genéricamente inyectiva. Este hecho fue probado simultáneamente por R.Friedman y R.Smith (cf. [F-S]) y por V. Kanev (cf. [K]). Ambas demostraciones se basan en el estudio de la aplicación P en fibras especiales: variedades de Prym reducibles en un caso e isomorfas a jacobianas en el otro.

En [We1], G.Welters da un método constructivo para recuperar, siempre con la hipótesis de genericidad, el recubrimiento a partir de la geometría de la vapp. Para recuperar la superficie $\tilde{C} + \tilde{C}$ en $P(\tilde{C},C)$, estudia la subvariedad de los puntos que trasladan el lugar singular del divisor theta dentro del propio divisor. Este método de reconocer datos geométricos intrínsecamente a partir de la variedad de Prym es uno de los principales recursos que utilizamos en este trabajo.

Recientemente O.Debarre ([De1]) ha dado otra demostración constructiva de la inyectividad genérica. En este caso utiliza los proyectivizados de los conos tangentes al divisor Ξ en los puntos dobles. Sus trasladados al origen son cuádricas de $\mathbf{P}(T_{P(\tilde{C},C)}(0))$ conteniendo la imagen de la aplicación semicanónica. Cuando la curva es general la intersección coincide con dicha curva, se imita por tanto una de las líneas de demostración del Teorema de Torelli para las jacobianas.

La conjetura tetragonal.

La aplicación de Prym no es inyectiva para ninguna g. Este resultado fue probado por A.Beauville (cf.[Be2]) para $g \leq 9$ utilizando la construcción de Recillas ([Re]). Esta construcción asocia a un elemento $(\tilde{C},C) \in \mathcal{R}_g$ y a una serie lineal g^1_3 sobre C una curva tetragonal Γ tal que $J\Gamma \cong P(\tilde{C},C)$.

La no inyectividad para todo g es consecuencia de la construcción tetragonal de Donagi (cf. [Do]). Moviéndose en el mismo orden de ideas, esta construcción generaliza la de Recillas y la contiene como un caso degenerado. Pasamos a describirla: fijemos una terna $(\tilde{C},C,g^1_4)$, donde $\pi : \tilde{C} \longrightarrow C$ es un recubrimiento no ramificado de curvas lisas irreducibles y la serie lineal g^1_4 está definida sobre C. Consideremos la variedad $\tilde{X}$ definida por el diagrama de pull-back siguiente:

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \tilde{C}^{(4)} \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \pi^{(4)}} \\ \mathbf{P}^1 & \xrightarrow{g^1_4} & C^{(4)}. \end{array}$$

La variedad $\tilde{X}$ descompone en la reunión disjunta de dos curvas conexas $\tilde{X}_1$ y $\tilde{X}_2$ invariantes por la involución natural en $\tilde{C}^{(4)}$. Sean X_1 y X_2 las respectivas curvas cociente por esta acción. Los recubrimientos $(\tilde{X}_i, X_i)$ son admisibles y se verifica que

$$P(\tilde{X}_1, X_1) \cong P(\tilde{X}_2, X_2) \cong P(\tilde{C}, C).$$

Además este proceso es reversible: las curvas X_i vienen dotadas de sendas series lineales g^1_4. Cuando alguna de las curvas, por ejemplo X_1 es lisa (la admisibilidad obliga a $\tilde{X}_1$ a serlo también) podemos iniciar la construcción con la terna $(\tilde{X}_1, X_1, g^1_4)$ obteniéndose las otras dos.

De aquí se deduce la no inyectividad. Al existir curvas con una infinidad de series lineales g^1_4 incluso aparecen fibras de P de dimensión positiva.

Por tanto la complejidad de la aplicación de Prym es notoriamente mayor a al de Torelli. R.Donagi enuncia la *conjetura tetragonal*: si dos recubrimientos lisos aparecen en la misma fibra de la aplicación de Prym, se pasa del uno al otro aplicando sucesivamente, cambiando la serie g^1_4 en cada paso, la construcción tetragonal.

En [De2] O.Debarre estudia la fibra de la aplicación de Prym sobre $P(\tilde{C}, C)$ con C tetragonal *general* de género $g \geq 13$. La genericidad de la curva implica la existencia de una única serie lineal g^1_4 sobre ella. La fibra está formada en este caso por los tres recubrimientos tetragonalmente relacionados confirmando de esta manera la conjetura en este caso. Es esencial en la demostración eludir aquellas curvas con una infinidad de series lineales tetragonales, a saber: las curvas hiperelípticas, las trigonales y las llamadas *bielípticas*. Estas últimas son las que admiten una presentación como recubrimiento doble de una curva elíptica.

A pesar de este resultado positivo, la conjetura, tal como está enunciada, no es cierta (cf. [De3], [Do2]). Ello se deduce del siguiente estudio del espacio de moduli $\mathcal{R}_{B,g}$ de los recubrimientos dobles no ramificados de las curvas bielípticas. Este espacio descompone en la reunión de $[\frac{g-1}{2}]+2$ componentes irreducibles las cuales notamos por $\mathcal{R}_{B,g,t}$, $t = 0,\ldots,[\frac{g-1}{2}]$ y $\mathcal{R}'_{B,g}$. Cuando se aplica una construcción tetragonal a un elemento de $\mathcal{R}_{B,g,0}$ se obtiene un recubrimiento isomorfo al inicial junto con otro no liso. Sea $\mathcal{T}$ el conjunto de recubrimientos admisibles no lisos así obtenidos (con las notaciones de (2.10) se tiene $\mathcal{H}_{g,0} \subset \mathcal{T} \subset \mathcal{H}'_{g,0}$). Por otro lado una construcción tetragonal aplicada a un elemento de $\mathcal{R}'_{B,g}$ proporciona dos elementos de $\mathcal{T}$. En consecuencia

$$P(\mathcal{R}'_{B,g}) \subset P(\mathcal{T}) = P(\mathcal{R}_{B,g,0}),$$

pero no hay forma de pasar mediante construcciones tetragonales de una componente a otra.

Este contraejemplo es más de forma que de fondo. Si la construcción tetragonal fuera siempre reversible (no sólo cuando los recubrimientos obtenidos son lisos) el contraejemplo no sería tal. Es natural esperar que la construcción se pueda extender a los recubrimientos admisibles eliminando así la obstrucción que impide transitar tetragonalmente de $\mathcal{R}'_{B,g}$ a $\mathcal{R}_{B,g,0}$. Estaremos en tal caso ante una *conjetura tetragonal extendida*.

Resultados obtenidos en la presente tesis

En esta memoria se estudia la fibra de la aplicación de Prym sobre $P(\tilde{C},C)$ donde C es una curva bielíptica *general*. Se pretende con ello extender el estudio realizado por O.Debarre sobre las curvas tetragonales generales a este caso. De las tres familias de curvas con infinidad de series tetragonales que antes mencionamos, las bielípticas presentan, a priori, el aspecto más rico: sus series lineales g^1_4 no tienen punto base, con lo que la construcción tetragonal "no degenera" y, por otro lado, es el único de los tres casos en el que la variedad de Prym no es una jacobiana. El hecho de que la discrepancia señalada con la conjetura en su formulación standard ("conjetura standard") nazca en el campo bielíptico añade un mayor interés a este caso.

En el primer capítulo delimitamos las fronteras de la conjetura standard en el campo bielíptico. Cuando $(\tilde{C},C)$ es un elemento general de $\mathcal{R}_{B,g,t}$ con $t \geq 1$ la conjetura standard se verifica, es decir, cualquier otro recubrimiento $\tilde{D} \longrightarrow D$ no ramificado de curvas lisas con $P(\tilde{C},C) \cong P(\tilde{D},D)$ se obtiene a partir de $(\tilde{C},C)$ aplicando sucesivamente la construcción tetragonal (Teoremas (5.11), (5.16), (6.11) y (6.24)).

Seguidamente estudiamos la aplicación P restringida a cada una de las componentes restantes $\mathcal{R}_{B,g,0}$ y $\mathcal{R}'_{B,g}$. En ambas situaciones la aplicación es inyectiva (Teoremas (7.6) y

(7.22)). Un estudio comparativo de ambos casos permite incluso dar una inyección explícita de $\mathcal{R}'_{B,g}$ en $\mathcal{R}_{B,g,0}$. Como consecuencia se prueba que si $(\tilde{D}, D) \in \mathcal{R}_g$ pertenece a la fibra de $P(\tilde{C}, C)$ con $(\tilde{C}, C)$ un elemento general de $\mathcal{R}'_{B,g} \cup \mathcal{R}_{B,g,0}$ entonces $(\tilde{D}, D)$ también pertenece a $\mathcal{R}'_{B,g} \cup \mathcal{R}_{B,g,0}$ y, si no coincide con $(\tilde{C}, C)$, está "relacionado tetragonalmente" con él a través de elementos admisibles de T. Las técnicas utilizadas en el análisis de cada componente son diferentes debido a que para valores de t inferiores a cuatro las propiedades especiales de las curvas de género bajo inciden en la geometría de las variedades de Prym correspondientes.

El siguiente paso en el estudio de la fibra consiste en pasar al terreno admisible. Ello exige extender la construcción tetragonal a este contexto. En el §12 se esboza como hacerlo siguiendo las ideas de A.Beauville (cf. [Be2]). Se sustituye el producto simétrico de las curvas por las variedades de divisores de Cartier efectivos de grado dado. Consiguientemente tiene sentido plantear la conjetura tetragonal extendida eliminando las hipótesis de lisitud en la conjetura inicial. Los resultados del capítulo I nos dicen ahora que el contraejemplo citado en la sección anterior no contradice la conjetura tetragonal extendida.

En el capítulo II damos un contraejemplo propiamente dicho a la conjetura tetragonal extendida. La construcción allí dada asocia al par formado por un elemento general de $\mathcal{R}_{B,g,4}$ y una serie lineal g^1_4 fijada sobre la curva base cuatro recubrimientos con curva base no tetragonal y con la misma variedad de Prym que el recubrimiento inicial. En la construcción interviene de forma crucial el hecho de que las jacobianas de las curvas de género cinco admiten una presentación distinguida como variedad de Prym.

Por último, en el tercer capítulo se prueba que este contraejemplo es el único que aparece en el caso de las curvas bielípticas generales (Teorema (13.1)). Es decir los elementos de la fibra estudiada se obtienen utilizando la construcción tetragonal o bien la construcción dada en el capítulo II. En el §17 damos, a modo de resumen, la descripción de la fibra estudiada.

En pocas palabras el contenido de esta memoria es el siguiente:

a) tras extender la construcción tetragonal a los recubrimientos admisibles eliminando así la discrepancia antes mencionada, se construye un contraejemplo a la conjetura extendida y

b) se describe completamente la fibra de P sobre $P(\tilde{C}, C)$ con C bielíptica general, interviniendo en ello solamente la construcción tetragonal y la construcción que da lugar al contraejemplo de a), es decir: no hay más contraejemplos en el caso bielíptico general.

Agradezco al Dr. Welters la generosidad con la que me ha ofrecido su ayuda y su tiempo durante estos años, tanto en el periodo de formación previo a la elaboración de este trabajo como durante la creación y redacción del mismo.

Quiero expresar también mi agradecimiento a los compañeros del Departament d'Algebra i Geometria por su soporte. Con ellos he compartido matemáticas, docencia y cafés.

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1. Notation.

Throughout this paper we work over the field of the complex numbers. The constant g will be greater than or equal to 10. By a curve we shall mean a projective connected curve with at most double ordinary singularities. If C is a curve we shall denote by $g(C)$ the arithmetic genus of C. For a subspace F of $\mathcal{R}_g$, the symbol $\bar{F}$ denotes the closure of F in $\bar{\mathcal{R}}_g$.

For D, D' two divisors on a smooth curve C, the expression $D \equiv D'$ will indicate that they are linearly equivalent. We shall denote by $\mathrm{Pic}^d(C)$ the set of linear equivalence classes of degree d divisors on C. Usually we shall not make difference between a divisor and its linear equivalence class in $\mathrm{Pic}^d(C)$. For two non-negative integers r,d we shall consider the algebraic subsets of $\mathrm{Pic}^d(C)$:

$$W^r_d(C) = \{\zeta \in \mathrm{Pic}^d(C) \mid h^0(\zeta) \geq r+1\}.$$

Let $\pi : \tilde{C} \longrightarrow C$ be a double cover of a smooth curve, either unramified or ramified exactly at the points $\tilde{Q}_1, \ldots, \tilde{Q}_k \in \tilde{C}$. Let Δ be the discriminant divisor, that is to say

$$\Delta = \sum_{i=1}^{k} \pi(\tilde{Q}_i).$$

Once C is given, the morphism π and the curve $\tilde{C}$ are determined by Δ and a unique element $\xi \in \mathrm{Pic}(C)$ satisfying $2\xi \equiv \Delta$. In fact $\tilde{C} \cong \mathrm{Spec}(\mathcal{O}_C \oplus \mathcal{O}_C(-\xi))$ where the $\mathcal{O}_C$-algebra structure is determined by the map $\mathcal{O}_C(-\xi) \otimes \mathcal{O}_C(-\xi) \cong \mathcal{O}_C(-\Delta) \longrightarrow \mathcal{O}_C$ given by the multiplication with an equation for Δ. The class ξ verifies $\pi^*(\xi) \equiv \sum_{i=1}^{k} \tilde{Q}_i$. We will refer to ξ and Δ as the class and the discriminant divisor respectively attached to the covering.

A curve C is said to be hyperelliptic if it can be represented as a double covering of the projective line.

Let D, D_1 and D_2 be curves. The notation

$$D = D_1 \cup_k D_2$$

means that $D = D_1 \cup D_2$ and $\# D_1 \cap D_2 = k$.

The symbols $[\;]$ and $\sim$ will mean rational cohomology class and algebraic equivalence respectively.

If A is an abelian variety and n is a positive integer, the group of the elements $x \in A$ such that $nx = 0$ will be written by ${}_nA$. For a polarized abelian variety A the symbol L_A denotes an invertible sheaf defining the polarization, we call λ_A the isogeny $A \longrightarrow \hat{A}$ induced by L_A (cf. [Mu2]) and we denote by $H(L_A)$ its kernel. We shall denote by Ξ_A an effective divisor such that $\mathcal{O}_A(\Xi_A) \cong L_A$. When speaking of the Jacobian of a smooth curve X we shall use L_X and Θ_X instead of L_{JX} and Ξ_{JX}.

If (A, L_A) and (B, L_B) are two polarized abelian varieties, the divisor $\Xi_A \times B + A \times \Xi_B$ gives on $A \times B$ a polarization whose attached invertible sheaf is written $L_A \boxtimes L_B$. We shall set

$$\zeta_A = [\Xi_A]^{a-1}/(a-1)!.$$

If X is a subvariety of an abelian variety A we define

$$I(X) := \{x \in A \mid a + X \subset X\}.$$

This is a closed algebraic subgroup of A.

Let $(\tilde{C}, C) \in \mathcal{R}_g$ and P its associated Prym variety (cf. Introduction). There is a natural model $(P^\bullet, \Xi^\bullet)$ of (P, Ξ) in $\mathrm{Pic}^{2g-2}(\tilde{C})$ described as follows ([Mu1])

$$P^\bullet = \{\tilde{\zeta} \in \mathrm{Pic}^{2g-2}(\tilde{C}) \mid \mathrm{Nm}_\pi(\tilde{\zeta}) \equiv K_C, \quad h^0(\tilde{\zeta}) \text{ even }\}$$
$$\Xi^\bullet = \{\tilde{\zeta} \in P^\bullet \mid h^0(\tilde{\zeta}) \geq 2\}.$$

In these terms, the singular locus of Ξ is described (loc. cit.) as:

$$\mathrm{Sing}\Xi^\bullet = \mathrm{Sing}^\pi_{st}\Xi^\bullet \cup \mathrm{Sing}^\pi_{ex}\Xi^\bullet \quad \text{where}$$
$$\mathrm{Sing}^\pi_{st}\Xi^\bullet = \{\tilde{\zeta} \in P^\bullet \mid h^0(\tilde{\zeta}) \geq 4\} \qquad \text{and}$$
$$\mathrm{Sing}^\pi_{ex}\Xi^\bullet = \{\tilde{\zeta} \in P^\bullet \mid \tilde{\zeta} = \pi^*(\zeta) + \tilde{\zeta}_0, \quad h^0(\tilde{\zeta}_0) \geq 1, h^0(\zeta) \geq 2\}.$$

The singularities of the first kind are called stable and the singularities of the second kind are called exceptional. These definitions depend on π.

Let $\pi : \tilde{D} \longrightarrow D$ be a double cover of curves and let ι be the involution on $\tilde{D}$ which interchanges the sheets of the cover. We say that $(\tilde{D}, D)$ is allowable (cf. [Be1]) if the only fixed points of ι are nodes where the two branches are not exchanged, and the number of nodes exchanged under ι equals the number $2c_e = 2c_e(\tilde{D}, D)$ of irreducible components exchanged under ι. The Prym variety is defined in the same way as in the smooth case. We denote by $P(\tilde{D}, D)$ (instead of $\tilde{P}(\tilde{D}, D)$) this abelian variety and we reserve the symbol $\tilde{P}$ for modular statements. Moreover the natural models described above still make sense. According to [Be1], (4.11) the Prym variety of an allowable double covering with $c_e = 0$ coincides with the Prym variety of its stable reduction. We shall assume (except in §12) that we are in the stable case.

If $\mathrm{Sing}(\tilde{D})$ is strictly contained in $\{\tilde{x} \in \tilde{D} \mid \iota(\tilde{x}) = \tilde{x}\}$ then the kernel of the norm map has a unique component, which is an abelian variety. We use also the notation $P(\tilde{D}, D)$ to refer to this abelian variety. In this case we define

$$P(\tilde{D}, D)^{*} = \mathrm{Nm}_{\pi}^{-1}(\omega_D) \subset \mathrm{Pic}^{2g-2}(\tilde{D})$$
$$\Xi^{*} = \{\tilde{\zeta} \in P(\tilde{D}, D)^{*} \mid h^0(\tilde{\zeta}) \geq 1\}$$

where ω_D is the dualizing sheaf. The codimension of Ξ^{*} is, at least, 2.

Let $\pi : \tilde{D} \longrightarrow D$ a double cover of curves and let ι be the attached involution on $\tilde{D}$. Assume that $\mathrm{Sing}(\tilde{D}) \subset \{\tilde{x} \in \tilde{D} \mid \iota(\tilde{x}) = \tilde{x}\}$, $\pi(\mathrm{Sing}\tilde{D}) = \mathrm{Sing}D$ and that all the components of $\tilde{D}$ are invariant by the covering involution. Let $\tilde{s}_1, \ldots, \tilde{s}_k \in \mathrm{Sing}(\tilde{D})$ and $s_1 = \pi(\tilde{s}_1), \ldots, s_k = \pi(\tilde{s}_k)$. Let $\tilde{f} : \tilde{N} \longrightarrow \tilde{D}$ and $f : N \longrightarrow D$ be the partial desingularizations of $\tilde{D}$ and D in $\tilde{s}_1, \ldots, \tilde{s}_k$ and $s_1, \ldots, s_k$ respectively. Then one has a finite morphism of homogeneous spaces:

$$\tilde{f}^0 : P(\tilde{D}, D)^{*} \longrightarrow P(\tilde{N}, N)^{*}$$
$$\tilde{L} \longrightarrow \tilde{f}^{*}(\tilde{L})\Big(-\sum_{\tilde{f}(\tilde{s}) \in \{\tilde{s}_1, \ldots, \tilde{s}_k\}} \tilde{s}\Big).$$

Analogously

$$\tilde{f}^{*} : P(\tilde{D}, D) \longrightarrow P(\tilde{N}, N)$$
$$\tilde{L} \longrightarrow \tilde{f}^{*}(\tilde{L})$$

is an isogeny (cf. [Sh2], p.110).

Capítulo I. The fibre of P over a generic element of $P(\mathcal{R}_{B,g})$.

2.Summary of known results

The following facts mostly are taken from [De 3].

Let $\mathcal{B}_g$ be the moduli space for the bi-elliptic curves of genus g and let $\mathcal{R}_{B,g}$ be the moduli space for unramified double coverings of bi-elliptic curves, obtained by the pull-back diagram:

$$\begin{array}{ccc} \mathcal{R}_{B,g} & \longrightarrow & \mathcal{R}_g \\ \downarrow & & \downarrow \\ \mathcal{B}_g & \longrightarrow & \mathcal{M}_g . \end{array}$$

Let us fix an element $(\tilde{C}, C) \in \mathcal{R}_{B,g}$ and let $\varepsilon : C \longrightarrow E$ be a morphism of degree two on a smooth elliptic curve E (ε is unique up to automorphisms of E if $g \geq 6$). The Galois group of $\tilde{C}$ over E may be identified with either $\mathbf{Z}/2\mathbf{Z}$ or $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. We shall denote by $\mathcal{R}'_{B,g}$ the subset of the elements with Galois group $\mathbf{Z}/2\mathbf{Z}$.

(2.1).- If $\mathrm{Gal}_E(\tilde{C}) \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, we write for the elements of the group: $Id, \iota, \iota_1, \iota_2$, where ι is the involution which interchanges the sheets of the double cover π. Let $C_1 = \tilde{C}/\langle\iota_1\rangle, C_2 = \tilde{C}/\langle\iota_2\rangle$ be the quotient curves.

One has a commutative diagram:

(2.2)

$$\begin{array}{ccccc} & & \tilde{C} & & \\ & {}^{\pi}\swarrow & \downarrow^{\pi_1} & \searrow^{\pi_2} & \\ C & & C_1 & & C_2 \\ & {}_{\varepsilon}\searrow & \downarrow^{\varepsilon_1} & \swarrow_{\varepsilon_2} & \\ & & E & & \end{array}$$

where $\pi_1, \pi_2, \varepsilon_1$ and ε_2 are the obvious morphisms. We shall always assume that $g(C_1) \leq g(C_2)$. It is easy to check the equality:

$$g(C_1) + g(C_2) = g + 1;$$

so if $g(C_1) = t+1$, we obtain $g(C_2) = g - t$ and $t + 1 \leq g - t$ implies that $t \in \{0, \ldots, [\frac{g-1}{2}]\}$. Let $\mathcal{R}_{B,g,t}$ be the subset of $\mathcal{R}_{B,g}$ consisting of the elements $(\tilde{C}, C)$ with $\mathrm{Gal}_E(\tilde{C}) \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and $g(C_1) = t+1, \quad g(C_2) = g - t$.

One finds that $\mathcal{R}'_{B,g}, \mathcal{R}_{B,g,0}, \ldots, \mathcal{R}_{B,g,[\frac{g-1}{2}]}$ are the irreducible components of $\mathcal{R}_{B,g}$ and that each one has dimension $2g - 2$.

(2.3).- Let $(\tilde{C}, C) \in \mathcal{R}_{B,g,t}$. In relation to the diagram (2.2) we fix the following notation:

i) τ, τ_1 and τ_2 are the involutions of C, C_1 and C_2 associated to $\varepsilon, \varepsilon_1$ and ε_2 respectively

ii) $\bar{P}_1, \ldots, \bar{P}_{2g-2} \in E$ are the discriminant points of ε and $\Delta = \sum_{i=1}^{2g-2} \bar{P}_i$ is the discriminant divisor. We write $P_1, \ldots, P_{2g-2}$ for the corresponding ramification points of ε.

iii) $\xi \in \mathrm{Pic}^{g-1}(E)$ is the class associated to ε. Hence $2\xi \equiv \Delta$.

iv) $\eta \in_2 JC$ is the class associated to π.

We may assume that $\bar{P}_1, \ldots, \bar{P}_{2t}$ are the discriminant points of ε_1 and that $\bar{P}_{2t+1}, \ldots, \bar{P}_{2g-2}$ are those of ε_2. We shall denote by $\Delta_1, \Delta_2, \xi_1$ and ξ_2 the discriminant divisors and the classes associated to ε_1 and ε_2 respectively. So:

$$\Delta = \Delta_1 + \Delta_2 \qquad \text{and} \qquad 2\xi_1 \equiv \Delta_1, \qquad 2\xi_2 \equiv \Delta_2.$$

(2.4).- It is easy to check the following facts:

i) $\xi = \xi_1 + \xi_2$.

ii) $\eta \equiv P_1 + \cdots + P_{2t} - \varepsilon^*(\xi_1) \equiv P_{2t+1} + \cdots + P_{2g-2} - \varepsilon^*(\xi_2)$.

iii) $\tilde{C} \cong C_1 \times_E C_2$.

iv) $\iota_1 \circ \iota_2 = \iota_2 \circ \iota_1 = \iota, \qquad \iota \circ \iota_1 = \iota_1 \circ \iota = \iota_2, \qquad \iota \circ \iota_2 = \iota_2 \circ \iota = \iota_1$.

v) The involutions ι_1 and ι_2 are liftings to $\tilde{C}$ of the involution τ of C. Analogously ι and ι_2 both lift τ_1 and ι and ι_1 both lift τ_2.

vi) $\varepsilon_1^*(\Delta_2)$ and $\varepsilon_1^*(\xi_2)$ (resp. $\varepsilon_2^*(\Delta_1)$ and $\varepsilon_2^*(\xi_1)$) are the discriminant divisor and the class associated to π_1 (resp. π_2).

(2.5).- We keep the assumption $(\tilde{C}, C) \in \mathcal{R}_{B,g,t}$ and we write $P = P(\tilde{C}, C)$. We have the description:

$$\Xi^* = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in W_t^0(C_1), \quad \zeta_2 \in W_{g-t-1}^0(C_2), \\ Nm_{\varepsilon_1}(\zeta_1) + Nm_{\varepsilon_2}(\zeta_2) = \xi\}$$

(2.6).- For $g \geq 7$ define the following subvarieties of Ξ^*:

$$V = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in W_t^1(C_1), \quad \zeta_2 \in W_{g-t-1}^1(C_2), \\ Nm_{\varepsilon_1}(\zeta_1) = \xi_1, \quad Nm_{\varepsilon_2}(\zeta_2) = \xi_2\}$$

$$W_a = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) + \pi^*(\epsilon^*(\zeta)) \mid \quad \zeta_1 \in W^0_{t-2+a}(C_1), \quad \zeta_2 \in W^0_{g-t-a-3}(C_2),$$
$$\zeta \in \mathrm{Pic}^2(E), \quad Nm_{e_1}(\zeta_1) + Nm_{e_2}(\zeta_2) + 2\zeta = \xi\}$$

where $a \in \{0, 2, -2\}$. Then $\mathrm{Sing}\Xi^* \supset V \cup W_{-2} \cup W_0 \cup W_2$ with equality if Δ does not belong to the image of the addition map $|\xi| \times |\xi| \longrightarrow |2\xi|$ (this happens if $(\tilde{C}, C)$ is general). Otherwise a finite number of new isolated singularities could appear.

(2.7).- The following table contains relevant information to be used in the sequel:

t	0	1	2	3	≥ 4
V	$\emptyset$	$\emptyset$	$\emptyset$	dim $g-7$	irred. dim $g-7$
W_{-2}	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	irred. dim $g-5$
W_0	$\emptyset$	$\emptyset$	irred. dim $g-5$	irred. dim $g-5$	irred. dim $g-5$
W_2	irred. dim $g-5$	irred. dim $g-5$	irred. dim $g-5$	irred. dim $g-5$	irred. dim $g-5$

As we shall see in (3.4), when $t = 3$ and $(\tilde{C}, C)$ is general V has two components. The singularity corresponding to an element of each one of these varieties is stable for V, exceptional for W_0 and stable and exceptional for W_{-2} and W_2.

By using (2.4.v) the reader can observe that all these varieties are fixed by the reflection with respect to $K_{\tilde{C}}$ (i.e.: fixed by ι).

(2.8).- Consider now the abelian varieties $P_1 := P(C_1, E) = Ker(Nm_{e_1})$ (if $t \geq 1$) and $P_2 := P(C_2, E) = Ker(Nm_{e_2})$. We define the morphisms:

$$\varphi : P_1 \times P_2 \longrightarrow P$$

by sending (ζ_1, ζ_2) to $\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2)$, if $t \geq 1$, and

$$\psi : P_2 \longrightarrow P$$

by sending ζ_2 to $\pi_2^*(\zeta_2)$ if $t = 0$. Then φ and ψ are isogenies and:

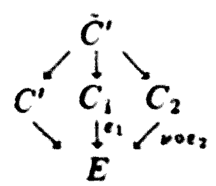
$$Ker(\varphi) = \{(\epsilon_1^*(\bar{a}), \epsilon_2^*(\bar{a})) \mid \bar{a} \in_2 JE\}$$

$$Ker(\psi) = \{0, \epsilon_2^*(\xi_1)\}$$

(2.9).- **Remark.** The definitions of $\iota, \tau, \bar{P}_1, \ldots, \bar{P}_{2g-2}, \Delta, \xi$ and η given in (2.3) make still sense if $(\tilde{C}, C) \in \mathcal{R}'_{B,g}$ and we will use them throughout.

(2.10).- Now we want to apply the tetragonal construction to an element $(\tilde{C}, C) \in \mathcal{R}_{B,g}$. Assuming first that $(\tilde{C}, C) \in \mathcal{R}_{B,g,t}$ and keeping the notation of (2.3), fix a linear series g_2^1 on E inducing an involution ν. Applying the tetragonal construction to $(\tilde{C}, C)$ with respect to $\epsilon^*(g_2^1)$ one obtains two elements $(\tilde{C}', C')$ and $(\tilde{C}'', C'')$ of $\bar{\mathcal{R}}_g$ (cf Introduction) verifying:

a) In terms of the data introduced in (2.1), one of the coverings, say $(\tilde{C}', C')$, can be described by the new set of data:



Note that $(\tilde{C}', C') \cong (\tilde{C}, C)$ if $t = 0$.

If $\nu(\bar{P}_i) \neq \bar{P}_j$ for $1 \leq i \leq 2t < j \leq 2g-2$, then $(\tilde{C}', C') \in \mathcal{R}_{B,g,t}$. In any case $(\tilde{C}', C') \in \bar{\mathcal{R}}_{B,g,t}$.

b) We now consider the second covering $(\tilde{C}'', C'')$. For $2 \leq t \leq [\frac{g-1}{2}]$ we define $\mathcal{H}'_{g,t} = \{(\tilde{\Gamma}, \Gamma) \in \bar{\mathcal{R}}_g \mid \Gamma = \Gamma_1 \cup_4 \Gamma_2$ with Γ_1, Γ_2 curves of genus $t-1, g-t-2$ respectively$\}$. (Notice that $t - 1 \leq g - t - 2$, since $t \leq [\frac{g-1}{2}]$). We call $\mathcal{H}_{g,t}$ the subspace defined by the additional condition of Γ_1, Γ_2 being irreducible and smooth. Then the second cover $(\tilde{C}'', C'')$ is an element of $\mathcal{H}'_{g,t}$ such that the components of C'' are hyperelliptic curves. If moreover $\nu(\bar{P}_i) \neq \bar{P}_j$ for $1 \leq i \leq 2t < j \leq 2g-2$, then $(\tilde{C}'', C'') \in \mathcal{H}_{g,t}$.

For $t = 1$ we put

$$\mathcal{H}'_{g,1} = \{(\tilde{\Gamma}, \Gamma) \in \bar{\mathcal{R}}_g / \tilde{\Gamma} = \mathbf{P}^1 \cup_4 \Gamma_2 \text{ and } \Gamma_2 \text{ is a hyperelliptic curve}\}.$$

Again the additional condition of Γ_2 being irreducible and smooth defines a subspace $\mathcal{H}_{g,1}$. Then $(\tilde{C}'', C'') \in \mathcal{H}'_{g,1}$. When ν verifies the same condition as above, then $(\tilde{C}'', C'') \in \mathcal{H}_{g,1}$.

Finally we define for $t = 0$

$$\mathcal{H}'_{g,0} = \{(\tilde{\Gamma}, \Gamma) \in \bar{\mathcal{R}}_g / \Gamma \text{ is obtained from a hyperelliptic curve by identifying two pairs of points}\}$$

By imposing that the hyperelliptic curve being irreducible and smooth, and each pair being non-hyperelliptic we define a subspace $\mathcal{H}_{g,0}$. Then $(\tilde{C}'', C'') \in \mathcal{H}'_{g,0}$. If ν is general then $(\tilde{C}'', C'') \in \mathcal{H}_{g,\epsilon}$

By applying the tetragonal construction to an element of $\mathcal{R}'_{B,g}$ we obtain two elements of $\mathcal{H}'_{g,0}$. Once again if the linear series g^1_2 is general, then they belong to $\mathcal{H}_{g,0}$.

The spaces $\mathcal{H}_{g,t}$ are irreducible and dense in $\mathcal{H}'_{g,t}$, $t = 0, \ldots, [\frac{g-1}{2}]$. We have also

$$\begin{aligned} \dim \mathcal{H}_{g,t} &= 3g - 7 && \text{for } t \geq 2, \\ \dim \mathcal{H}_{g,1} &= 2g - 2 && \text{and} \\ \dim \mathcal{H}_{g,0} &= 2g - 1. \end{aligned}$$

Notice that our definition of $\mathcal{H}_{g,0}$ differs a bit of that of [De3]. This change is necessary in order to have the next property.

(2.11).- Any element of $\mathcal{H}_{g,0}$ can be obtained by means of the tetragonal construction from an element of $\mathcal{R}_{B,g,0}$. In fact, this is a consequence of the construction that will be given in §12. On the other hand, notice that $P(\mathcal{R}'_{B,g}) \subset P(\mathcal{H}_{g,0})$. Hence $P(\mathcal{R}'_{B,g}) \subset P(\mathcal{R}_{B,g,0})$.

(2.12).- **Remark.** From (2.11) we deduce that for each element of $\mathcal{R}'_{B,g}$ there exists an element of $\mathcal{R}_{B,g,0}$ component giving the same Prym variety. This implies that the tetragonal conjecture as stated initially is not true. This suggests the need for extending the tetragonal construction to the case of allowable double covers and for extending the tetragonal conjecture to the proper map P.

(2.13).- Although we have no direct description of $\mathrm{Sing}\Xi^*$ when we are in $\mathcal{R}'_{B,g}$ we deduce from Remark (2.12) and Table (2.7) that $\mathrm{Sing}\Xi^*$ has a unique component of dimension $g-5$ and possibly a finite number of isolated singularities.

(2.14).- Finally we recall two lemmas borrowed from [Mu1] and [De2]. First we need a definition. Let $\pi : \tilde{C} \longrightarrow C$ be a double cover of a smooth curve. We shall say that an effective divisor on $\tilde{C}$ is π-simple if it does not contain inverse images of effective divisors of C. Let $\zeta \in \mathrm{Pic}(C)$ be the class attached to π. With this notation one has:

(2.15).- Lemma ([Mu1],p.338). *If $\mathcal{L}$ is an invertible sheaf on C and $\tilde{D}$ is an effective π-simple divisor on $\tilde{C}$ there exists an exact sequence:*

$$0 \longrightarrow \mathcal{L} \longrightarrow \pi_*(\pi^*(\mathcal{L}) \otimes_{\mathcal{O}_{\tilde{C}}} \mathcal{O}_{\tilde{C}}(\tilde{D})) \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_C} \mathcal{O}_C(Nm_\pi(\tilde{D}) - \zeta) \longrightarrow 0.$$

(2.16).- Lemma (Debarre,[De2],p.550). Let $\pi : \tilde{C} \longrightarrow C$ be an allowable double cover of a stable curve C, $\tilde{\mathcal{L}}$ an invertible sheaf on $\tilde{C}$ and D a reduced element of $\mid K_C \otimes (Nm_\pi(\tilde{\mathcal{L}}))^{-1} \mid$ with non-singular support. Suppose that $h^0(\tilde{\mathcal{L}} \otimes_{\mathcal{O}_{\tilde{C}}} \mathcal{O}_{\tilde{C}}(\tilde{D})) \geq 1$ for all effective divisors $\tilde{D}$ such that $Nm_\pi(\tilde{D}) = D$. Then $h^0(\tilde{\mathcal{L}}) \geq 1$.

3.Some properties of bi-elliptic curves

This section deals with properties of bi-elliptic curves that will be used later on. In a first reading it may be skipped and kept for reference purposes.

Let $\varepsilon : C \longrightarrow E$ be a (2:1) morphism of smooth curves where E is an elliptic curve. We denote by $\bar{\Delta}$ and ξ the discriminant divisor and the class determining ε. By Riemann-Hurwitz:

$$\deg\bar{\Delta} = 2g - 2, \qquad \deg\xi = g - 1.$$

Let $\tau : C \longrightarrow C$ be the involution which interchanges the points of each fibre.

(3.1).-Lemma Let $\bar{A}$, B be effective divisors on E and C respectively. Assume that B is ε-simple (cf (2.14)). Then:

$$\deg(\bar{A}) + \deg(B) < g(C) - 1 \quad \Rightarrow \quad h^0(\varepsilon^*(\bar{A}) + B) = h^0(\bar{A}).$$

Proof: By applying (2.15) we obtain an exact sequence:

$$0 \longrightarrow \mathcal{O}_E(\bar{A}) \longrightarrow \varepsilon_*(\mathcal{O}_C(\varepsilon^*(\bar{A}) + B)) \longrightarrow \mathcal{O}_E(\bar{A} + Nm_\varepsilon(B) - \xi) \longrightarrow 0.$$

Thus:

$$h^0(\varepsilon^*(\bar{A}) + B) \leq h^0(\bar{A}) + h^0(\bar{A} + Nm_\varepsilon(B) - \xi) = h^0(\bar{A}).$$

On the other side, one has an injection $\varphi : \mid \bar{A} \mid \longrightarrow \mid \varepsilon^*(\bar{A}) + B \mid$ given by $\varphi(\bar{R}) = \varepsilon^*(\bar{R}) + B$, so $h^0(\varepsilon^*(\bar{A}) + B) = h^0(\bar{A})$. ∎

Note that B is fixed in the linear series $\mid \varepsilon^*(\bar{A}) + B \mid$ and that φ is a bijection. In particular, if $B = 0$:

$$\deg(\bar{A}) < g(C) - 1 \quad \Rightarrow \quad h^0(\varepsilon^*(\bar{A})) = h^0(\bar{A})$$

and

$$\mid \varepsilon^*(\bar{A}) \mid = \{\varepsilon^*(\bar{R}) \mid \bar{R} \in \mid \bar{A} \mid\} = \varepsilon^*(\mid \bar{A} \mid).$$

(3.2).- If $g(C) \geq 5$, then C is not trigonal (cf [Te]).

(3.3).- If $g(C) \geq 4$, then C is not hyperelliptic. To see this, take $D \geq 0$ a divisor of degree two on C. By (3.1) $h^0(D) = 1$.

(3.4).- Assume that C is general, of genus 4. Then $W_3^1(C)$ has two different points. In fact the canonical model of C is the complete intersection of a cubic and a quadric in $\mathbf{P}^3$. The rulings of the quadric cut on C the linear series g_3^1 of this curve. Let $D \geq 0$ be a degree 3 divisor on C such that $h^0(D) = 2$. If $|D|$ were the unique g_3^1 then:

$$D \equiv \tau(D).$$

We write $Q_1, \dots, Q_6 \in C$ for the ramification points of ε. For each $i \in \{1, \dots, 6\}$ we find points $x_i, y_i \in C$ such that $D \equiv Q_i + x_i + y_i$, hence:

$$x_i + y_i \equiv \tau(x_i) + \tau(y_i).$$

By (3.3) this is an equality. If $y_i = \tau(x_i)$ then $D \equiv Q_i + \varepsilon^*(\varepsilon(x_i))$ and by (3.1) $h^0(D) = 1$. We deduce that $x_i = \tau(x_i)$, $y_i = \tau(y_i)$. Now, by taking norms, there appear linear equivalences between divisors on E having support on discriminant points. This contradicts the generality of C.

(3.5).- Assume that C is general, of genus 3. Then C is not hyperelliptic. Indeed, by using that C can be hyperelliptic in at most one way we can imitate the proof of (3.4).

(3.6).- We consider the following subvarieties of $\mathrm{Pic}^{g(C)-1}(C)$:

$$Z' = \{\zeta \in \mathrm{Sing}\Theta^* \mid \quad Nm_\varepsilon(\zeta) = \xi\}$$
$$Z'' = \{\varepsilon^*(x+y) + \zeta' \mid \quad x, y \in E, \quad \zeta' \in W^0_{g(C)-5}\} \qquad \text{if} \quad g(C) \geq 5$$
$$A = \{\varepsilon^*(x) + \zeta' \mid \quad x \in E, \quad \zeta' \in W^0_{g(C)-3}\} \supset Z'' \qquad \text{if} \quad g(C) \geq 3.$$

Remarks: i) If $g(C) \geq 3$, then A is irreducible of dimension $g(C) - 2$.

ii) If $g(C) \geq 6$, then Z' and Z'' are irreducible of dimension $g(C) - 4$ and $\mathrm{Sing}\Theta^* = Z' \cup Z''$ ([We2],Prop.3.6). If $g(C) = 5$, then the equality holds but Z' is not always irreducible (in fact by [Te] there is a bijection between its components and the bi-elliptic structures on C).

(3.7).- In order to avoid annoying notation for the varieties V and W_a (where $a \in 2, 0, -2$) described in (2.6) we use the definitions of (3.6) applied to C_1 and C_2. In these terms the descriptions of (2.5) read:

$$V = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \quad \zeta_1 \in Z_1', \quad \zeta_2 \in Z_2'\}$$
$$W_{-2} = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \quad \zeta_1 \in Z_1'', \quad \zeta_2 \in \Theta_2^*, \quad Nm_{\varepsilon_1}(\zeta_1) + Nm_{\varepsilon_2}(\zeta_2) = \xi\}$$
$$W_0 = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \quad \zeta_1 \in A_1, \quad \zeta_2 \in A_2, \quad Nm_{\varepsilon_1}(\zeta_1) + Nm_{\varepsilon_2}(\zeta_2) = \xi\}$$
$$W_2 = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \quad \zeta_1 \in \Theta_1^*, \quad \zeta_2 \in Z_2'', \quad Nm_{\varepsilon_1}(\zeta_1) + Nm_{\varepsilon_2}(\zeta_2) = \xi\}.$$

(3.8).- Lemma([De3],Lemma 5.2.10). Assume $g(C) \geq 6$ and fix $\bar{\lambda} \in \mathrm{Pic}^{g(C)-1}(E)$. Then $\{\zeta \in Z'' \mid Nm_\varepsilon(\zeta) = \bar{\lambda}\}$ is irreducible of dimension $g(C) - 5$.

In particular $Z' \cap Z''$ is irreducible.

The following facts will be used throughout.

(3.9).-**Proposition. One has the following equalities:**

i) If $g(C) \geq 3$ then:

$$\{a \in \mathrm{JC} \mid a + A \subseteq A\} = \{\varepsilon^*(\bar{a}) \mid \bar{a} \in \mathrm{Pic}(E)\}.$$

ii) If $g(C) \geq 5$ then:

$$\begin{aligned}&\{a \in \mathrm{JC} \mid a + Z'' \subset A\} = \{a \in \mathrm{JC} \mid a + Z'' \subset \Theta^*\} = \\ &=\{a \in \mathrm{JC} \mid a + Z' \cap Z'' \subset A\} = \{a \in \mathrm{JC} \mid a + Z' \cap Z'' \subset \Theta^*\} = \\ &=\{\varepsilon^*(\bar{x}) - r - s \mid \bar{x} \in E, r, s \in C\}.\end{aligned}$$

iii) If $g(C) \geq 5$ then:

$$\begin{aligned}&\{a \in \mathrm{JC} \mid a + Z'' \subset Z''\} = \{a \in \mathrm{JC} \mid a + Z' \cap Z'' \subset Z''\} = \\ &=\{\varepsilon^*(\bar{a}) \mid \bar{a} \in \mathrm{Pic}^0(E)\}.\end{aligned}$$

PROOF: i). The inclusion of the second member in the first one is clear. Let $a \in JC$ such that $a + A \subset A$. In particular, for all $\bar{x} \in E$ and $D \in C^{(g-3)}$ one has $h^0(a + \varepsilon^*(\bar{x}) + D) > 0$. Then:

$$h^0(a + \varepsilon^*(\bar{x})) > 0.$$

Indeed, otherwise by Riemann-Roch

$$h^0(K_C - \varepsilon^*(\bar{x}) - a) = h^0(\varepsilon^*(\xi - \bar{x}) - a) = g(C) - 3.$$

By taking points $x_1, \ldots, x_{g(C)-3}$ such that x_{i+1} is not a base point of the linear series $| \varepsilon^*(\xi - \bar{x}) - a - x_1 - \cdots - x_i |$, one finds an effective divisor $D = \sum_{i=1}^{g(C)-3} x_i$ verifying $h^0(\varepsilon^*(\xi - \bar{x}) - a - D) = 0$. So $h^0(a + \varepsilon^*(\bar{x}) + D) = 0$, which is a contradiction.

Now we may write $a \equiv D - \varepsilon^*(\bar{x})$ where D is an effective divisor of degree two verifying

$$h^0(D + \varepsilon^*(\bar{\alpha})) > 0 \text{ for all } \bar{\alpha} \in \mathrm{Pic}^0(E).$$

By applying (2.15) we conclude that $D \in \mathrm{Im}(\varepsilon^*)$, thereby proving i).

ii). All the equalities are an easy consecuence of the following one:

$$\{a \in JC \mid a + Z' \cap Z'' \subset \Theta^*\} = \{\varepsilon^*(\bar{x}) - r - s \mid \bar{x} \in E, \quad r, s \in C\}.$$

This fact was proved by Debarre in [De5]. We give here an sketch of the proof. We only prove the inclusion of the left hand side member in the right hand side member. Write $a \equiv D - \varepsilon^*(\bar{A})$, where $\bar{A} \in \mathrm{Pic}^r(E)$ and D is effective. If we assume that D is ε-simple then $2r \leq g + 1$. In fact it is not necessary to consider the case the case $2r = g + 1$. It suffices to obtain a contradiction if $r \geq 2$.

Suppose that $2r \leq g - 2$. For a generic element $\bar{B} \in \mathrm{Pic}^r(F)$ there exists $D' \geq 0$ such that:

- $D + D'$ is ε-simple.
- $2\bar{B} + Nm_\varepsilon(D') \equiv \xi$.

Then $\varepsilon^*(\bar{B}) + D' \in Z' \cap Z''$. By applying (2.15)

$$\begin{aligned} 0 &< h^0(a + \varepsilon^*(\bar{B}) + D') = h^0(D + D' + \varepsilon^*(\bar{B} - \bar{A})) \\ &\leq h^0(\bar{B} - \bar{A}) + h^0(Nm_\varepsilon(D + D') + \bar{B} - \bar{A} - \xi) \\ &= h^0(\bar{B} - \bar{A}) + h^0(Nm_\varepsilon(D) - \bar{A} - \bar{B}) \end{aligned}$$

which is a contradiction because $\bar{B}$ is generic. The cases $2r = g - 1, g$ are similar.

Part iii) follows from ii). ∎

4.A key lemma

Let $f : \tilde{N} \longrightarrow N$ be a (2:1) morphism of smooth curves with ramification divisor $\sum_{i=1}^{k} \tilde{Q}_i$. We denote by σ the involution of $\tilde{N}$ attached to f.

Let $\tilde{\mathrm{L}}$ be a line bundle on $\tilde{N}$ with $\sigma^*(\tilde{\mathrm{L}}) \cong \tilde{\mathrm{L}}$. Choose an isomorphism φ normalized in such a way that:

$$\sigma^*(\varphi) \circ \varphi = \mathrm{Id}_{\tilde{L}}.$$

Writing $\tilde{\mathrm{L}}[\tilde{x}]$ for the pointwise fibre of $\tilde{\mathrm{L}}$ over $\tilde{x} \in \tilde{N}$, one obtains isomorphisms:

$$\varphi[\tilde{Q}_i] : \tilde{\mathrm{L}}[\tilde{Q}_i] \longrightarrow \sigma^*(\tilde{\mathrm{L}})[\tilde{Q}_i] = \tilde{\mathrm{L}}[\sigma(\tilde{Q}_i)] = \tilde{\mathrm{L}}[\tilde{Q}_i] \qquad i \in \{1, \dots, k\}$$

given by multiplication with constants λ_i with $\lambda_i^2 = 1$. We attach to $\tilde{\mathrm{L}}$ a vector $v(\tilde{\mathrm{L}}) = (\lambda_1, \dots, \lambda_k) \in (\mu_2)^k$ which depends on the choice of φ. The ambiguity disappears when we pass to the quotient modulo μ_2 by the natural action. Then we have an homomorphism of groups:

$$v : \mathrm{Ker}(\sigma^* - 1) \longrightarrow \frac{(\mu_2)^k}{\mu_2}.$$

We use the notation $v(\tilde{D})$ and $v(\tilde{\mathcal{L}})$ for $\tilde{D}$ a divisor and $\tilde{\mathcal{L}}$ an invertible sheaf on $\tilde{N}$.

(4.1).- **Proposition**. There exists a line bundle L on N such that $f^*(\mathrm{L}) \cong \tilde{\mathrm{L}}$ iff $v(\tilde{\mathrm{L}}) = \overline{(1, \dots, 1)}$.

PROOF: It suffices to use [G],Th.1,p.17. ∎

(4.2).- **Proposition**. Let $\tilde{\mathcal{L}}$ be an invertible sheaf on $\tilde{N}$ such that $\sigma^*(\tilde{\mathcal{L}}) \cong \tilde{\mathcal{L}}$. Then there exists a divisor $\tilde{D}$ on $\tilde{N}$ with $0 \leq \tilde{D} \leq \sum_{i=1}^{k} \tilde{Q}_i$ and an invertible sheaf $\mathcal{L}$ on N such that

$$f^*(\mathcal{L}) \cong \tilde{\mathcal{L}} \otimes \mathcal{O}_{\tilde{N}}(-\tilde{D})$$

PROOF: By using the exact sequence:

$$0 \longrightarrow \mathcal{O}_{\tilde{C}}(-\tilde{Q}_i) \longrightarrow \mathcal{O}_{\tilde{C}} \longrightarrow \mathcal{O}_{\tilde{Q}_i} \longrightarrow 0$$

and by observing that $\mathcal{O}_{\tilde{C}}(-\tilde{Q}_i) \notin Im(f^*)$ (hence by (4.1) $v(-\tilde{Q}_i) \neq \overline{(1, \dots, 1)}$) one has $v(-\tilde{Q}_i) = \overline{(1, \dots, -1, \dots, 1)}$. Then, by tensoring $\tilde{\mathcal{L}}$ with suitable sheaves $\mathcal{O}_{\tilde{C}}(-\tilde{Q}_i)$ we can make all the coordinates of the corresponding vector be equal. ∎

Let $(\tilde{C}, C) \in \mathcal{R}_{B,g}$. We keep the notations of §2. In particular $\eta \in_2 JC$ is the class determining $\pi : \tilde{C} \longrightarrow C$.

(4.3).- **Corollary**. One has $(\tilde{C}, C) \in \mathcal{R}'_{B,g}$ iff $\tau^*(\eta) \neq \eta$.

PROOF: By (2.4.ii), $\tau^*(\eta) = \eta$ when $(\tilde{C}, C) \notin \mathcal{R}'_{B,g}$. Conversely suppose $\tau^*(\eta) = \eta$. Applying (4.2) we may write:

$$\eta = D - \varepsilon^*(\bar{A}) \qquad \text{with} \qquad 0 \leq D \leq \sum_{i=1}^{2g-2} P_i.$$

Let C_1 (resp. C_2) be the double cover on E given by the class of $\bar{A}$ (resp. $\xi - \bar{A}$) and the discriminant divisor $Nm_\varepsilon(D)$ (resp. $\bar{\Delta} - Nm_\varepsilon(D)$). Observe that:

$$\varepsilon^*(Nm_\varepsilon(\eta)) = 2\eta = 0.$$

So due to the injectivity of ε^*:

$$Nm_\varepsilon(\eta) = 0 \qquad \text{and} \qquad 2\bar{A} \equiv Nm_\varepsilon(D).$$

Then $\tilde{C} \cong C_1 \times_E C_2$ and $C \cong \tilde{C}/(\iota_1 \circ \iota_2)$, ι_1 and ι_2 being the involutions of $\tilde{C}$ attached to the projections on C_1 and C_2 respectively. Hence $(\tilde{C}, C) \in \mathcal{R}_{B,g,t}$ for some t. ∎

(4.4).- **Lemma**. Assume $t > 0$. We consider the commutative diagram:

$$\begin{array}{ccc} JC_1 & \xrightarrow{\pi_1^*} & J\tilde{C} \\ \varepsilon_1^* \uparrow & & \uparrow \pi_2^* \\ JE & \xrightarrow[\varepsilon_2^*]{} & JC_2 \end{array}$$

Then:

$$\pi_1^*(JC_1) \cap \pi_2^*(JC_2) = \{\pi^*(\varepsilon^*(\alpha)) \mid \alpha \in \mathrm{Pic}^0(E)\}$$

PROOF: Fix $\tilde{\beta} \in \mathrm{Imag}(\pi_1^*) \cap \mathrm{Imag}(\pi_2^*)$ and $\beta_1 \in \mathrm{J}C_1$, $\beta_2 \in \mathrm{J}C_2$ such that $\tilde{\beta} = \pi_1^*(\beta_1) = \pi_2^*(\beta_2)$. By the assumption $t > 0$, the morphisms π_1 and π_2 are ramified, hence π_1^* and π_2^* are injective. Thus we need to find an element $\beta \in \mathrm{Pic}^0(E)$ such that $\beta_1 = \pi_1^*(\beta)$ and $\beta_2 = \pi_2^*(\beta)$. To see this observe that $\iota_2^*(\tilde{\beta}) = \tilde{\beta}$. On the other side from (2.4.v) one has:

$$\iota_2^*(\tilde{\beta}) = \iota_2^*(\pi_1^*(\beta_1)) = \pi_1^*(\tau_1^*(\beta_1))$$

therefore

$$\pi_1^*(\beta_1) = \tilde{\beta} = \pi_1^*(\tau_1^*(\beta_1))$$

and

$$\beta_1 = \tau_1^*(\beta_1).$$

In a similar way we get $\beta_2 = \tau_2^*(\beta_2)$. By applying (4.2), there exist divisors D_1 on C_1, D_2 on C_2 and classes $\bar{\alpha}_1, \bar{\alpha}_2 \in \mathrm{Pic}^0(E)$ such that:

$$\beta_1 \equiv \varepsilon_1^*(\bar{\alpha}_1) - D_1, \quad \beta_2 \equiv \varepsilon_2^*(\bar{\alpha}_2) - D_2 \tag{4.5}$$

where $0 \leq D_i \leq$ ramification divisor of ε_i, $i = 1, 2$.

Hence:

$$\pi^*(\varepsilon^*(\bar{\alpha}_1 - \bar{\alpha}_2)) \equiv \pi_1^*(D_1) - \pi_2^*(D_2)$$

Let R_1 and R_2 be the effective divisors on C such that

$$\pi^*(R_1) = \pi_1^*(D_1), \quad \pi^*(R_2) = \pi_2^*(D_2)$$

thus

$$0 \leq R_1 \leq \sum_{i=1}^{2t} P_i, \quad 0 \leq R_2 \leq \sum_{i=2t+1}^{2g-2} P_i.$$

From

$$\pi^*(\varepsilon^*(\bar{\alpha}_1 - \bar{\alpha}_2)) \equiv \pi^*(R_1 - R_2),$$

two possibilities appear:

$$\text{either} \quad \text{i)} \quad \varepsilon^*(\bar{\alpha}_1 - \bar{\alpha}_2) \equiv R_1 - R_2$$

$$\text{or} \quad \text{ii)} \quad \varepsilon^*(\bar{\alpha}_1 - \bar{\alpha}_2) \equiv R_1 - R_2 + \eta.$$

We first suppose i). From (4.1) we have $v(R_1 - R_2) = \overline{(1, \ldots, 1)}$, i.e.: $v(R_1) = v(R_2)$. By applying the proof of (4.2) we can compute these vectors:

$$v(R_1) = \overline{(\lambda_1, \ldots, \lambda_{2t}, 1, \ldots, 1)} \quad \text{with} \quad \lambda_i = -1 \quad \text{iff} \quad P_i \in \mathrm{Supp}(R_1)$$

$$v(R_2) = \overline{(1, \ldots, 1, \lambda_{2t+1}, \ldots, \lambda_{2g-2})} \quad \text{with} \quad \lambda_i = -1 \quad \text{iff} \quad P_i \in \mathrm{Supp}(R_2)$$

We conclude that $\lambda_1 = \cdots = \lambda_{2t} = \lambda_{2t+1} = \cdots = \lambda_{2g-2}$, that is to say, either $R_1 = R_2 = 0$ or $R_1 = \sum_{i=1}^{2t} P_i$, $R_2 = \sum_{i=2t+1}^{2g-2} P_i$. If $R_1 = R_2 = 0$, then $D_1 = D_2 = 0$ and we finish by taking $\beta = \bar{\alpha}_1 = \bar{\alpha}_2$. Similarly, if $R_1 = \sum_{i=1}^{2t} P_i$, $R_2 = \sum_{i=2t+1}^{2g-2} P_i$ we get $D_1 \equiv \varepsilon_1^*(\xi_1)$ and $D_2 \equiv \varepsilon_2^*(\xi_2)$ (see (2.3)). By replacing in (4.5):

$$\beta_1 = \varepsilon_1^*(\bar{\alpha}_1 - \xi_1), \quad \beta_2 = \varepsilon_2^*(\bar{\alpha}_2 - \xi_2).$$

On the other side, by (2.4.ii):

$$\varepsilon^*(\bar{\alpha}_1 - \bar{\alpha}_2) \equiv \sum_{i=1}^{2t} P_i - \sum_{i=2t+1}^{2g-2} P_i \equiv \varepsilon^*(\xi_1 - \xi_2)$$

and one finally obtains $\beta = \bar{\alpha}_1 - \xi_1 = \bar{\alpha}_2 - \xi_2$.

In the case ii) we can imitate the above proof by replacing η by the expression of (2.4 ii). ∎

5.The components $\mathcal{R}_{B,g,t}$ for $t \geq 3$

In the first half of this paragraph $(\tilde{C}, C)$ is an element of $\mathcal{R}_{B,g,t}$ with $t \geq 4$ and $P = P(\tilde{C}, C)$. We keep the notations of §§1 and 2. In particular $g \geq 10$.

In order to describe the fibre of the Prym map over P we shall use ideas from [We1] and [De2]. Via geometric constructions we recover essential information that almost gives the initial data. To make the construction intrinsical we will need to recognize some components of SingΞ^*. For instance we can recognize V by its dimension. Our first goal will be to recover from P other components.

Recalling the descriptions of (3.7) one has:

(5.1).- **Proposition**. The variety $W_{-2} \cap W_2$ is irreducible of dimension $g-9$ and one has the equality:

$$W_{-2} \cap W_2 = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in Z_1'', \zeta_2 \in Z_2'', Nm_{\epsilon_1}(\zeta_1) + Nm_{\epsilon_2}(\zeta_2) = \xi\}.$$

PROOF: We check first the equality. Clearly the second member is contained in the first one. To see the opposite inclusion we apply (4.4). Indeed, suppose that

$$\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) = \pi_1^*(\zeta_1') + \pi_2^*(\zeta_2') \tag{5.2}$$

where

$$\zeta_1 \in \Theta_1^*, \quad \zeta_2 \in Z_2'', \quad \zeta_1' \in Z_1'', \quad \zeta_2' \in \Theta_2^*$$
$$\text{and} \quad Nm_{\epsilon_1}(\zeta_1) + Nm_{\epsilon_2}(\zeta_2) = Nm_{\epsilon_1}(\zeta_1') + Nm_{\epsilon_2}(\zeta_2') = \xi.$$

Then:

$$\pi_1^*(\zeta_1 - \zeta_1') = \pi_2^*(\zeta_2 - \zeta_2').$$

By (4.4) there exists $\alpha \in \mathrm{Pic}^0(E)$ such that:

$$\zeta_1 - \zeta_1' = \varepsilon_1^*(\bar{\alpha})$$
$$\zeta_2' - \zeta_2 = \varepsilon_2^*(\bar{\alpha}).$$

In particular $\zeta_1 = \varepsilon_1^*(\bar{\alpha}) + \zeta_1'$ and replacing this in (5.2) we are done.

Consider now the morphism:

$$\Psi : Z_1'' \times Z_2'' \longrightarrow \mathrm{Pic}^{2g-2}(\tilde{C})$$
$$(\zeta_1, \zeta_2) \longrightarrow \pi_1^*(\zeta_1) + \pi_2^*(\zeta_2)$$

Let us define $T = \{(\zeta_1, \zeta_2) \in Z_1'' \times Z_2'' \mid Nm_{e_1}(\zeta_1) + Nm_{e_2}(\zeta_2) = \xi\}$. Clearly $\Psi(T) = W_{-2} \cap W_2$. Since each fibre of the induced map $T \longrightarrow W_{-2} \cap W_2$ is isomorphic to E (use (4.4)) it suffices to prove that T is irreducible of dimension $g-8$. To see this look at the first projection: $T \longrightarrow Z_1''$. Clearly Z_1'' is irreducible and by (3.8) the fibres are irreducible of dimension $g-t-5$ (note that $g \geq 10$, $t \geq 4$ and $t+1 \leq g-t$ imply $g-t \geq 6$). Thus T is irreducible and $\dim T = \dim Z_1'' + g - t - 5 = t - 3 + g - t - 5 = g - 8$. ∎

(5.3).- **Proposition**. The varieties $W_0 \cap W_{-2}$ and $W_0 \cap W_2$ are both irreducible of dimension $g-7$ and they are described as follows:

$$W_0 \cap W_{-2} = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in Z_1'', \zeta_2 \in A_2, Nm_{e_1}(\zeta_1) + Nm_{e_2}(\zeta_2) = \xi\}$$
$$W_0 \cap W_2 = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in A_1, \zeta_2 \in Z_2'', Nm_{e_1}(\zeta_1) + Nm_{e_2}(\zeta_2) = \xi\}.$$

PROOF: By symmetry only one variety has to be considered, for instance $W_0 \cap W_2$. Imitating the proof of (5.1) one finds the equality. The irreducibility and dimension may be obtained as in loc. cit. replacing Ψ by the morphism:

$$\Psi' : A_1 \times Z_2'' \longrightarrow \mathrm{Pic}^{2g-2}(\tilde{C})$$
$$(\zeta_1, \zeta_2) \longrightarrow \pi_1^*(\zeta_1) + \pi_2^*(\zeta_2)$$

and T by $T' = \{(\zeta_1, \zeta_2) \in A_1 \times Z_2'' \mid Nm_{e_1}(\zeta_1) + Nm_{e_2}(\zeta_2) = \xi\}$. ∎

(5.4).- **Remark**. The second statement of Proposition (5.3) still holds true if $t \geq 2$.

(5.5).- We put

$$\Lambda_a = \{x \in P \mid x + W_0 \cap W_a \subset W_0\}$$

where $a = 2, -2$.

Due to (5.1) and (5.3) we know W_0 among the components of $\mathrm{Sing}\Xi^*$ of dimension $g-5$. Therefore we recognize also $\{(W_0 \cap W_2), (W_0 \cap W_{-2})\}$. Combining both facts $\{\Lambda_{-2}, \Lambda_2\}$ is intrinsically recovered from P. Our next aim is to compute Λ_{-2} and Λ_2.

(5.6).- **Proposition.** One has the equalities:

i) $\Lambda_{-2} = \{\pi_1^*(\varepsilon_1^*(\bar{x}) - r - s) \mid \bar{x} \in E, \quad r, s \in C_1, \quad 2\bar{x} \equiv \varepsilon_1(r) + \varepsilon_1(s)\}$

ii) $\Lambda_2 = \{\pi_2^*(\varepsilon_2^*(\bar{x}) - r - s) \mid \bar{x} \in E, \quad r, s \in C_2, \quad 2\bar{x} \equiv \varepsilon_2(r) + \varepsilon_2(s)\}$.

PROOF: We only prove the second one, the first one being equivalent. Looking at (5.3) it is easy to check that the second member of this equality is contained in the first one (by (2.8) its elements belong to P). We show the opposite inclusion. Fix $\tilde{a} \in \Lambda_2$. By using (2.8) we may write

$$\tilde{a} = \pi_1^*(a_1) + \pi_2^*(a_2) \qquad \text{with } Nm_{\varepsilon_1}(a_1) = Nm_{\varepsilon_2}(a_2) = 0.$$

Let $\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \in W_0 \cap W_2$ where $\zeta_1 \in A_1$, $\zeta_2 \in Z_2''$ and $Nm_{\varepsilon_1}(\zeta_1) + Nm_{\varepsilon_2}(\zeta_2) = \bar{\xi}$ (cf (5.3)). Applying Lemma (4.4) there exist elements $\zeta_1' \in A_1$, $\zeta_2' \in A_2$ and $\alpha \in \mathrm{Pic}^0(E)$ such that:

$$a_1 + \zeta_1 = \zeta_1' + \varepsilon_1^*(\alpha) \in A_1$$
$$a_2 + \zeta_2 = \zeta_2' - \varepsilon_2^*(\alpha) \in A_2.$$

Therefore $a_1 + A_1 \subset A_1$ and $a_2 + Z_2'' \subset A_2$. Then by using (3.9.i) and (3.9.ii) we finish the proof. ∎

(5.7).- **Proposition.** Assume $t \geq 4$. The sets $\Lambda_{-2} \cap 2\Lambda_{-2}$ and $\Lambda_2 \cap 2\Lambda_2$ are two symmetric irreducible curves. Their normalizations are C_1 and C_2 respectively, and τ_1 and τ_2 are the involutions induced by the (-1) map of P.

PROOF: We first observe that:

$$2\Lambda_{-2} = \{\pi_1^*(x + y - \tau_1(x) - \tau_1(y)) \mid \quad x, y \in C_1\}$$
$$2\Lambda_2 = \{\pi_2^*(x + y - \tau_2(x) - \tau_2(y)) \mid \quad x, y \in C_2\}.$$

Now, it suffices to consider the set $\Lambda_{-2} \cap 2\Lambda_{-2}$. One has:

$$\Lambda_{-2} \cap 2\Lambda_{-2} = \{\pi_1^*(x - \tau_1(x)) \mid \quad x \in C_1\}.$$

Indeed, since τ_1 has fixed points, $\pi_1^*(x - \tau_1(x)) \in 2\Lambda_{-2}$ for all $x \in C_1$. Moreover:

$$\pi_1^*(x - \tau_1(x)) = \pi_1^*(\varepsilon_1^*(\varepsilon_1(x)) - 2\tau_1(x)) \in \Lambda_{-2}.$$

So the right hand side member of the equality is contained in the left hand side member. To see the opposite inclusion, take $\bar{x} \in \bar{E}$ and $r, s \in C_1$ such that $2\bar{x} \equiv \epsilon_1(r) + \epsilon_2(s)$ and suppose that $\pi_1^*(\epsilon_1^*(\bar{x}) - r - s) \in 2\Lambda_{-2}$. We obtain a linear equivalence:

$$\pi_1^*(\epsilon_1^*(\bar{x}) - r - s) \equiv \pi_1^*(y + z - \tau_1(y) - \tau_1(z))$$

where $y, z \in C_1$. Since π_1^* is injective:

$$(5.8) \qquad \epsilon_1^*(\bar{x}) + \tau_1(y) + \tau_1(z) \equiv y + z + r + s.$$

By assumption $t \geq 4$ and then (3.1) implies that $h^0(\epsilon_1^*(\bar{x}) + \tau_1(y) + \tau_1(z)) = 1$ iff $\tau_1(z) \neq y$. If $y = \tau_1(z)$ the initial element belongs to the right hand side member trivially. Thus we can assume that (5.8) is an equality of divisors and then either $y = \tau_1(z)$ or $y = \tau_1(y)$ or $z = \tau_1(z)$. In any case the inclusion follows.

Now, taking the morphism

$$\begin{aligned} \varphi_1 : C_1 &\longrightarrow \Lambda_{-2} \cap 2\Lambda_{-2} \\ x &\longrightarrow \pi_1^*(x - \tau_1(x)) \end{aligned}$$

the statement follows by observing that φ_1 is birational (C_1 is not hyperelliptic by (3.3)) and that $\varphi_1(\tau_1(x)) = -\varphi_1(x)$ ∎

(5.9).- Let $\pi' : \tilde{D} \longrightarrow D$ be an unramified double cover of smooth curves such that $P(\tilde{D}, D) \cong P$. Since the singular locus of the theta divisor of P has dimension $g - 5 = \dim P - 4$, D is either trigonal or bi-elliptic (cf. [Mu1],p.344). If D is trigonal P is the Jacobian of a curve (cf [Re]). Then, by [Sh1] C has to be either hyperelliptic or trigonal, which contradicts either (3.2) or (3.3). Thus D is bi-elliptic.

Moreover, looking at the table (2.7) plus the observation of (2.13) we deduce that $(\tilde{D}, D) \in \mathcal{R}_{B,g,s}$ with $s \geq 4$. Let D_1 and D_2 be the bi-elliptic curves of genus $s + 1$ and $g - s$ attached to $(\tilde{D}, D)$ in the usual way (cf (2.1)). Since as we have seen in (5.7) (C_1, τ_1) and (C_2, τ_2) can be recovered from P, one has isomorphisms $\varphi_i : D_i \longrightarrow C_i \quad i = 1,2$ commuting with the corresponding involutions. In particular the base elliptic curve is the same and $s = t$. Summarizing, if the diagram attached to $(\tilde{D}, D)$ is:

$$\begin{array}{ccccc} & & \tilde{D} & & \\ & {}^{\pi'}\swarrow & \downarrow{\scriptstyle \pi_1'} & \searrow{}^{\pi_2'} & \\ D & & D_1 & & D_2 \\ & {}_{\epsilon'}\searrow & \downarrow{\scriptstyle \epsilon_1'} & \swarrow{}_{\epsilon_2'} & \\ & & E & & \end{array}$$

there exist $\Phi_i \in \mathrm{Aut}(E)$ $(i = 1,2)$ such that

$$\begin{array}{ccc} D_i & \xrightarrow{\varphi_i} & C_i \\ {\scriptstyle \epsilon'_i}\downarrow & & \downarrow{\scriptstyle \epsilon_i} \\ E & \xrightarrow[\Phi_i]{} & E. \end{array}$$

Thus we obtain a diagram

$$\begin{array}{ccccc} & & \tilde{D} & & \\ & \swarrow & \downarrow & \searrow & \\ D & & C_1 & & C_2 \\ & \searrow & \downarrow {\scriptstyle \Phi_1 \circ \epsilon_1} & \swarrow {\scriptstyle \Phi_2^{-1} \circ \epsilon_2} & \\ & & E & & \end{array}$$

Composing with a suitable automorphism of E we get

(5.10)

$$\begin{array}{ccccc} & & \tilde{D} & & \\ & \swarrow & \downarrow & \searrow & \\ D & & C_1 & & C_2 \\ & \searrow & \downarrow {\scriptstyle \epsilon_1} & \swarrow {\scriptstyle \Phi \circ \epsilon_2} & \\ & & E & & \end{array}$$

where $\Phi \in \mathrm{Aut}(E)$ and $\Phi(\bar{P}_i) \neq \bar{P}_j$, for all $1 \leq i \leq 2t < j \leq 2g-2$.

(5.11).-**Theorem.** Let $(\tilde{C}, C)$ be a general element of $\mathcal{R}_{B,g,t}$ with $t \geq 4$ and $g \geq 10$. Let $(\tilde{D}, D) \in \mathcal{R}_g$ such that $P(\tilde{D}, D) \cong P(\tilde{C}, C)$. Then $(\tilde{D}, D) \in \mathcal{R}_{B,g,t}$ and $(\tilde{C}, C)$ and $(\tilde{D}, D)$ are tetragonally related.

PROOF: By (5.9) it only remains to see that each diagram (5.10) can be obtained by applying successively the tetragonal construction starting from the initial element $(\tilde{C}, C)$. By (2.10) it suffices to see the following fact:

Lemma. Assume that E is general. Then the set:

$$\Gamma = \{\Phi \in \mathrm{Aut}(E) \mid \Phi(\bar{P}_i) \neq \bar{P}_j, \quad \text{for} \quad 1 \leq i \leq 2t < j \leq 2g-2\}$$

is generated multiplicatively by the elements of Γ that correspond to the linear series g^1_2 of E.

PROOF: Let $\Phi \in \Gamma$. Take a point $\bar{r} \in E$ and put $\bar{s} = \Phi(\bar{r})$. Let $\tilde{\Phi}$ be the associated isomorphism of JE. In other words:

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & E \\ {\scriptstyle \iota_1}\downarrow & & \downarrow{\scriptstyle \iota_2} \\ JE & \xrightarrow[\tilde{\Phi}]{} & JE \end{array}$$

where t_1, t_2 are the embeddings of E in its Jacobian via translations by $\bar{r}$ and $\bar{s}$ respectively. Since E is general and $\dot{\Phi}(0) = 0$ one has $\dot{\Phi} = \pm Id$. Then either $\Phi(\bar{x}) \equiv \bar{x} - \bar{r} + \bar{s}$ for all $\bar{x} \in E$ or $\Phi(\bar{x}) \equiv \bar{r} + \bar{s} - \bar{x}$ for all $\bar{x} \in E$. In the second case Φ is the automorphism determined by the linear series $| \bar{r} + \bar{s} |$. Now assume that $\Phi(\bar{x}) \equiv \bar{x} - \bar{r} + \bar{s}$. Let us consider the automorphisms Φ_1 and Φ_2 of E given by the linear series $| 2\bar{r} |$ and $| \bar{r} + \bar{s} |$. Then:

$$\Phi_1(\bar{x}) \equiv 2\bar{r} - \bar{x}$$

and

$$\Phi_2(\Phi_1(\bar{x})) \equiv \bar{r} + \bar{s} - \Phi_1(\bar{x}) \equiv \bar{r} + \bar{s} - 2\bar{r} + \bar{x} \equiv \bar{x} - \bar{r} + \bar{s}.$$

So: $\Phi_2 \circ \Phi_1 = \Phi$. Moreover for a general $\bar{r} \in E$ one has $\Phi_1 \in \Gamma$. Since the linear series $| \bar{x} + \Phi(\bar{x}) |$ varies with $\bar{x}$ when Φ is not an automorphism associated to a g^1_2 we obtain $\Phi_2 \in \Gamma$ also for a general $\bar{r} \in E$. This ends the proof of the lemma. ∎

The rest of the section will be devoted to proving the analogue of the Theorem (5.11) for the component $\mathcal{R}_{B,g,3}$. Let $(\tilde{C}, C)$ be a general element of this component. There are two components of dimension $g-5$ in $\mathrm{Sing}\Xi^*$: W_0 and W_2 (cf. (2.7)).

(5.12).- **Lemma.** One has the equalities (cf. §1 and (2.8) for notations, part iii will not be needed here, but later one):

$$\begin{aligned} &i) \quad I(W_2) = \pi_1^*(P_1) \qquad \text{for } t \geq 1 \\ &ii) \quad I(W_0) = \pi^*(\epsilon^*({}_2JE)) \qquad \text{for } t \geq 2 \\ &iii) \quad I(W_{-2}) = \pi_2^*(P_2) \qquad \text{for } t \geq 4. \end{aligned}$$

Proof: We show first the equality i). To prove the inclusion of $\pi_1^*(P_1)$ in the left hand side member we consider $\pi_1^*(\beta) \in \pi_1^*(P_1)$ and we take an element $\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \in W_2$ where $\zeta_1 \in \Theta_2^*$, $\zeta_2 \in Z_2''$ and $Nm_{\epsilon_1}(\zeta_1) + Nm_{\epsilon_2}(\zeta_2) = \xi$. Since the map

$$\mathrm{Pic}^0(E) \times C_1^{(3)} \longrightarrow \mathrm{Pic}^3(C_1)$$

is surjective, we may write

$$\beta + \zeta_1 \equiv \zeta_1' + \epsilon_1^*(\bar{\rho}), \qquad \text{where} \quad \zeta_1' \in \Theta_1^*, \quad \rho \in \mathrm{Pic}^0(E).$$

Then

$$\pi_1^*(\beta) + \pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) = \pi_1^*(\zeta_1') + \pi_2^*(\zeta_2 + \epsilon_2^*(\bar{\rho})) \in W_2.$$

To see the opposite inclusion take $\bar{a} = \pi_1^*(a_1) + \pi_2^*(a_2) \in P$ with $a_1 \in P_1, a_2 \in P_2$ and such that $\bar{a} + W_2 \subset W_2$. By applying Lemma (4.4) as before (see for instance the proof of (5.6)) we get $a_2 + Z_2'' \subset Z_2''$. By (3.9.iii) there exists $\bar{a} \in_2 JE$ such that $a_2 = \varepsilon_2^*(\bar{a})$. Therefore $\hat{a} \in \pi_1^*(P_1)$.

In ii), the inclusion of the right hand side member in the left hand side member is obvious. Take now $\hat{a} = \pi_1^*(a_1)+\pi_2^*(a_2)$ with $a_1 \in P_1$ and $a_2 \in P_2$. Assume that $\hat{a}+W_0 \subset W_0$. Again as a consequence of Lemma (4.4) one has $a_1 + A_1 \subset A_1$ and $a_2 + A_2 \subset A_2$. By using (3.9.i) we obtain that $\hat{a} \in \pi^*(\varepsilon^*(JE))$. This ends the proof of the inclusion $I(W_0) \subset \pi^*(\varepsilon^*({}_2JE))$.

Part iii) is analogous to part i). ∎

By dimension count of $I(W_i)$, $i = 0,2$, we may distinguish W_0 from W_2.

(5.13).- **Proposition.** One has:

$$\bigcup_{\zeta \in W_0} ((W_0)_{-\zeta} \cap \pi_1^*(P_1)) = \{\pi_1^*(\varepsilon_1^*(\bar{x}) - r - s) \mid \bar{x} \in E; r,s \in C_1 \text{ and } 2\bar{x} \equiv \varepsilon_1(r) + \varepsilon_1(s)\}.$$

PROOF: Let $\zeta = \pi_1^*(\varepsilon_1^*(\bar{z}) + r) + \pi_2^*(\zeta_2)$, where $\bar{z} \in E$, $r \in C_1$ $\zeta_2 \in A_2$ and such that $\mathrm{Nm}_{\varepsilon_1}(\varepsilon_1^*(\bar{z}) + r)+\mathrm{Nm}_{\varepsilon_2}(\zeta_2) = \xi$. We take $a_1 \in P_1$ such that

$$\pi_1^*(a_1) + \zeta \in W_0.$$

By Lemma (4.4) this implies that

$$a_1 + \varepsilon_1^*(\bar{z}) + r \in A_1.$$

Hence $a_1 = r' - r + \varepsilon_1^*(\bar{a})$ where $\bar{a} \in JE$, $r,r' \in C_1$. By replacing $\bar{a}$ by $\bar{x} - \varepsilon_1(r')$ for some $\bar{x} \in E$ we get

(5.14) $\quad (W_0)_{-\zeta} \cap \pi_1^*(P_1) \subset \{\pi_1^*(\varepsilon_1^*(\bar{x}) - r - s) \mid \bar{x} \in E, s \in C_1 \text{ and } 2\bar{x} \equiv \varepsilon_1(r) + \varepsilon_1(s)\}.$

The inclusion of the right hand side member in the left hand side member in (5.14) is trivial. The equality in (5.14) clearly implies the equality we wanted to prove. ∎

(5.16).-**Theorem.** Let $(\check{C},C)$ be generic element of $\mathcal{R}_{B,g,3}$ with $g \geq 10$ and let $(\check{D},D) \in \mathcal{R}_g$ such that $P(\check{D},D) \cong P(\check{C},C)$. Then $(\check{D},D) \in \mathcal{R}_{B,g,3}$ and $(\check{C},C)$ and $(\check{D},D)$ are tetragonally related.

PROOF: First we observe that the methods used in the first part of this section (i.e.: for $(\check{C},C) \in \mathcal{R}_{B,g,t}$, $t \geq 4$) in order to recover the set of data (C_2,τ_2) are still valid (cf. (5.4),(5.6.ii) and (5.7)). On the other side we have seen in (5.13) how to recognize intrinsically in P the set

$$\{\pi_1^*(\varepsilon_1^*(\bar{x}) - r - s) \mid \bar{x} \in E, r,s \in C_1 \text{ and } 2\bar{x} \equiv \varepsilon_1(r) + \varepsilon_1(s)\}.$$

Since it coincides with the set obtained in (5.6.i) we can also imitate the process given in (5.7) to obtain the set of data (C_1,τ_1). Then the proof continues as in (5.11). ∎

6.The components $\mathcal{R}_{B,g,2}$ and $\mathcal{R}_{B,g,1}$

In this paragraph we wish to prove the analogue of Theorem (5.11) for the components $\mathcal{R}_{B,g,2}$ and $\mathcal{R}_{B,g,1}$. We use essentially the same ideas. Only the way to recover (C_1, τ_1) needs a new point of view, we shall consider the study of some intersections $\Xi^* \cap \Xi^*_{\bar{a}}$. We keep the assumptions and notations of §1 and §2.

Let us denote by $(\tilde{C}, C)$ a general element of $\mathcal{R}_{B,g,2}$. From (2.6) and (2.7) we may suppose that:

$$\mathrm{Sing}\Xi^* = W_0 \cup W_2.$$

(6.1).- Due to (5.12) we can make a difference between both components.

(6.2).- **Remark**. We imitate §5 (see (5.4), (5.5) and (5.7)) and obtain from P^* the curve

$$\Lambda_2 \cap 2\Lambda_2 = \{\pi_2^*(x - \tau_2(x)) \mid x \in C_2\}.$$

By normalizing we recover (C_2, τ_2).

Now we aim at describing a subvariety of $\pi_1^*(P_1)$ that determines the curve C_1.

(6.3).- **Proposition**. One has the following equalities:
i) If $\bar{a} = \pi_2^*(x - \tau_2(x))$, where $x \in C_2$, then

$$\Xi^* \cap \Xi^*_{\bar{a}} = F \cup X(\bar{a})$$

where

$$X(\bar{a}) = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in \Theta_1^*, \zeta_2 \in \Theta_2^*, h^0(\zeta_2 - x) > 0 \text{ and } Nm_{\epsilon_1}(\zeta_1) + Nm_{\epsilon_2}(\zeta_2) = \xi\}$$

is the moving part of this algebraic system and F is the fix part (see below for a description of F).

ii) Let $N = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in \Theta_1^*, Nm_{\epsilon_1}(\zeta_1) = \xi_1, \zeta_2 \in Z_2'\}$. Then:

$$\bigcap_{\bar{a} \in \Lambda_2 \cap 2\Lambda_2 - \{0\}} X(\bar{a}) = W_0 \cup W_2 \cup N,$$

and N is the union of the irreducible components distinct from $W_0 \cup W_2$.

iii) If $\tilde{a} = \pi_1^*(a_1)$, where $a_1 \in P_1 - \{0\}$, then:

$$N \cap \Xi_{\tilde{a}}^* = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in \Theta_1^* \cap (\Theta_1^*)_{a_1}, Nm_{e_1}(\zeta_1) = \xi_1, \zeta_2 \in Z_2'\}$$
$$\cup \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in \Theta_1^*, Nm_{e_1}(\zeta_1) = \xi_1, \zeta_2 \in Z_2' \cap Z_2''\}.$$

PROOF: i). Let $\zeta = \pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \in \Xi^* \cap \Xi_{\tilde{a}}^*$ with $\tilde{a} = \pi_2^*(x - \tau_2(x))$. By applying Lemma (4.4) we find elements $\zeta_1' \in \Theta_1^*, \zeta_2' \in \Theta_2^*$ and $\bar{\rho} \in \mathrm{Pic}^0(E)$ such that:

$$\text{(6.4)} \qquad \zeta_1 \equiv \zeta_1' + \varepsilon_1^*(\bar{\rho})$$
$$\tau_2(x) - x + \zeta_2 \equiv \zeta_2' - \varepsilon_2^*(\bar{\rho}).$$

Suppose first that $\bar{\rho} = 0$. Then

$$\zeta_2 \in \Theta_2^* \cap (\Theta_2^*)_{x - \tau_2(x)} = \{\zeta_2 \in \Theta_2^* \mid h^0(\zeta_2 - x) > 0\}$$
$$\cup \{\zeta_2 \in \Theta_2^* \mid h^0(\zeta_2 + \tau_2(x)) \geq 2\}.$$

If ζ_2 belongs to the second set, by Riemann-Roch one has

$$h^0(K_{C_2} - \zeta_2 - \tau_2(x)) > 0.$$

Define $\lambda = \xi_2 - Nm_{e_2}(\zeta_2)$, $\beta_1 = \zeta_1 - \varepsilon_1^*(\lambda)$ and $\beta_2 = \zeta_2 + \varepsilon_2^*(\lambda)$. Then

$$h^0(\beta_1) = h^0(\zeta_1 - \varepsilon_1^*(\xi_2 - Nm_{e_2}(\zeta_2))) = h^0(-\tau_1^*(\zeta_1) + \varepsilon_1^*(\xi_1)) = h^0(\tau_1(\zeta_1)) > 0$$
$$h^0(\beta_2 - x) = h^0(K_{C_2} - \tau_2^*(\zeta_2) - x) = h^0(K_{C_2} - \zeta_2 - \tau_2(x)) > 0.$$

Therefore $\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) = \pi_1^*(\beta_1) + \pi_2^*(\beta_2) \in X(\tilde{a})$.

On the other hand if $\rho \neq 0$ then (cf. [De4], p.9)

$$\zeta_1 \in \Theta_1^* \cap (\Theta_1^*)_{\varepsilon_1^*(\bar{\rho})} = A_1 \cup \{\zeta_1 \in \Theta_1^* \mid Nm_{e_1}(\zeta_1) = \xi_1 + \bar{\rho}\}.$$

If $Nm_{e_1}(\zeta_1) = \xi_1 + \bar{\rho}$ then $\bar{\rho} = Nm_{e_1}(\zeta_1) - \xi_1 = \xi_2 - Nm_{e_2}(\zeta_2)$ and by replacing in (6.4) one has

$$\tau_2^*(\zeta_2) + x - \tau_2(x) \equiv K_{C_2} - \zeta_2'.$$

Thus $\zeta_2 \in \Theta_2^* \cap (\Theta_2^*)_{x - \tau_2(x)}$ and proceeding as above we conclude that $\zeta_2 \in X(\tilde{a})$. We have proved the inclusion $\Xi^* \cap \Xi_{\tilde{a}}^* \subset F \cup X(\tilde{a})$, where

$$F = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in A_1, \zeta_2 \in \Theta_2^*, Nm_{e_1}(\zeta_1) + Nm_{e_2}(\zeta_2) = \xi\}.$$

The inclusion of $X(\bar{a})$ in the left hand side member is trivial. Take now $\tilde{\zeta} = \pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \in F$. Since the map

$$\begin{aligned} \mathrm{Pic}^0(E) \times C_2^{(g-3)} &\longrightarrow \mathrm{Pic}^{g-3}(C_2) \\ (\bar{a}, D) &\longrightarrow \varepsilon_2^*(\bar{a}) + D \end{aligned}$$

is surjective we can write

$$x - \tau_2(x) + \zeta_2 \equiv D + \varepsilon_2^*(\bar{a})$$

and then $\pi_2^*(x - \tau_2(x)) + \tilde{\zeta} \equiv \pi_1^*(\zeta_1 + \varepsilon_1^*(\bar{a})) + \pi_2^*(D) \in \Xi^*$.

The reader may observe that F and $X(\bar{a})$ have pure dimension $g-3$ and that $\dim(F \cap X(\bar{a})) = g-4$ for all $\bar{a}$. This concludes the proof of i).

ii) The inclusion

$$W_0 \cup W_2 \cup N \subset \bigcap_{\bar{a} \in \Lambda_2 \cap 2\Lambda_2 - \{0\}} X(\bar{a})$$

is left to the reader.

To see the opposite inclusion let $\tilde{\zeta} = \pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \in X(\bar{a})$ for all $\bar{a} \in \Lambda_2 \cap 2\Lambda_2 - \{0\}$. Then for all $x \in C_2$ there exist $\zeta_1' \in \Theta_1^*, \zeta_2' \in \Theta_2^*$ and $\rho \in \mathrm{Pic}^0(E)$ such that

$$\begin{aligned} & h^0(\zeta_2' - x) > 0 \\ & \zeta_1 \equiv \zeta_1' + \varepsilon_1^*(\rho) \\ & \zeta_2 \equiv \zeta_2' - \varepsilon_2^*(\rho). \end{aligned} \tag{6.5}$$

Let T be an irreducible component of the fibre of the map

$$\begin{aligned} \mathrm{Pic}^0(E) \times C_2 \times C_2^{(g-4)} &\longrightarrow \mathrm{Pic}^{g-3}(C_2) \\ (\rho, x, D) &\longrightarrow x + D - \varepsilon_2^*(\rho) \end{aligned}$$

over ζ_2. In these terms the conditions (6.5) say that we may (and will) assume that the projection from T to C_2 is surjective. Suppose that the projection $T \longrightarrow \mathrm{Pic}^0(E)$ is constant and let ρ_0 be the image. Then for all $x \in C_2$ we find an effective divisor D such that:

$$\zeta_2 \equiv x + D - \varepsilon_2^*(\rho_0).$$

Therefore $h^0(\zeta_2 + \varepsilon_2^*(\rho_0) - x) > 0$ for all $x \in C_2$ and hence $\zeta_2 \in \mathrm{Sing}\Theta_2^* = Z_2' \cup Z_2''$. So $\tilde{\zeta}$ belongs to $W_2 \cup N$.

If $T \longrightarrow \mathrm{Pic}^0(E)$ is surjective we find that

$$h^0(\zeta_2 + \varepsilon_2^*(\tilde{\rho})) > 0$$

for all $\tilde{\rho} \in \mathrm{Pic}^0(E)$. Hence $\zeta_2 \in A_2$. Now it is not hard to deduce that $\tilde{\zeta} \in W_0 \cup W_2$.

From the descriptions it is clear that N has not components contained in $W_0 \cup W_2$. This finishes the proof of ii).

iii) The inclusion of the right hand side member in the left hand side member is left to the reader. To see the opposite inclusion let $\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2)$ such that $\zeta_1 \in \Theta_1^*$, $Nm_{\varepsilon_1}(\zeta_1) = \xi_1$, $\zeta_2 \in Z_2'$ and suppose that

$$\pi_1^*(-a_1 + \zeta_1) + \pi_2^*(\zeta_2) \in \Xi^*.$$

Again there exist $\zeta_1' \in \Theta_1^*$, $\zeta_2' \in \Theta_2^*$ and $\rho \in \mathrm{Pic}^0(E)$ with

$$\begin{aligned} -a_1 + \zeta_1 &\equiv \zeta_1' + \varepsilon_1^*(\rho) \\ \zeta_2 &\equiv \zeta_2' - \varepsilon_2^*(\rho). \end{aligned}$$

If $\rho = 0$ then $\zeta_1 \in \Theta_1^* \cap (\Theta_1^*)_{a_1}$. On the other hand $\rho \neq 0$ implies that

$$\zeta_2 \in \Theta_2^* \cap (\Theta_2^*)_{-\varepsilon_2^*(\rho)} = A_2 \cup \{\zeta_2 \in \Theta_2^* \mid Nm_{\varepsilon_2}(\zeta_2) = \xi_2 - \rho\}.$$

Since $\zeta_2 \in Z_2'$, only $\zeta_2 \in A_2$ is possible and then $\zeta_2 \in Z_2' \cap Z_2''$. ∎

(6.6).- We shall define for $\tilde{a} = \pi_1^*(a_1), a_1 \in P_1 - \{0\}$

$$\mathcal{N}(\tilde{a}) = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in \Theta_1^* \cap (\Theta_1^*)_{a_1}, Nm_{\varepsilon_1}(\zeta_1) = \xi_1, \zeta_2 \in Z_2'\}.$$

This set is recovered from $\mathcal{N} \cap \Xi_{\tilde{a}}^*$ as the union of the components not contained in W_2. Our next goal is to distinguish points in $\pi_1^*(P_1)$ looking at the number of components of $\mathcal{N}(\tilde{a})$. We will see below that the set $\Theta_1^* \cap (\Theta_1^*)_{a_1} \cap Nm_{\varepsilon_1}^{-1}(\xi_1)$ is finite. The cardinal of this set coincides with the number of irreducible components of $\mathcal{N}(\tilde{a})$.

(6.7).- Let D be the ample divisor induced by Θ_1 on the abelian surface P_1. By Riemann-Roch

$$h^0(D) = \frac{D^2}{2} \qquad \text{and} \qquad h^0(D)^2 = \deg(\lambda_D).$$

By using [Mu1],p.330 we obtain $\deg(\lambda_D) = 4$ and therefore $D^2 = 4$.

(6.8).- **Let $x \in C_1$ and let $a_1 = x - \tau_1(x) \in P_1$. One has**

$$\Theta_1^* \cap (\Theta_1^*)_{a_1} = (x + C_1) \cup \{\zeta_1 \in \Theta_1^* \mid h^0(\zeta_1 + \tau_1(x)) = 2\}.$$

The first component meets $Nm_{\varepsilon_1}^{-1}(\bar{\xi}_1)$ at the points $x + y$ and $x + \tau_1(y)$ where $\varepsilon_1(y) = \bar{\xi}_1 - \varepsilon_1(x)$. On the other hand if the divisor $r + s$ verifies $h^0(r + s + \tau_1(x)) = 2$ and $\varepsilon_1(r) + \varepsilon_1(s) \equiv \bar{\xi}_1$ then by Riemann-Roch

$$0 < h^0(K_{C_1} - r - s - \tau_1(x)) = h^0(\tau_1(r) + \tau_1(s) - \tau_1(x)) = h^0(r + s - x).$$

Hence the second component intersects $Nm_{\varepsilon_1}^{-1}(\bar{\xi}_1)$ at the same points. That is to say, in this case $N(\bar{a})$ has at most two components.

(6.9).- Let Σ be the curve given by the pull-back diagram:

$$\begin{array}{ccc} \Sigma & \longrightarrow & C_1^{(2)} \\ \downarrow & & \downarrow{\scriptstyle \varepsilon_1^{(2)}} \\ |\bar{\xi}_1| & \longrightarrow & E^{(2)} \end{array}$$

the horizontal arrows being inclusions. Since C_1 is general it is easy to obtain (cf. §11) that Σ is a smooth curve of genus 3, the quotient $\Sigma/\tau_1^{(2)}$ is an elliptic curve E_1 and E_1 is not isomorphic to E.

We shall denote by Σ_0 the image of the map

$$\begin{array}{ccc} \Sigma & \longrightarrow & P_1 \\ x + y & \longrightarrow & x + y - \tau_1(x) - \tau_1(y). \end{array}$$

(6.10).- **Proposition**. One has:

$$\{\bar{a} \in \pi_1^*(P_1) \mid \text{ number comp. } N(\bar{a}) < 4\} = \Pi \cup \pi_1^*(\Sigma_0)$$

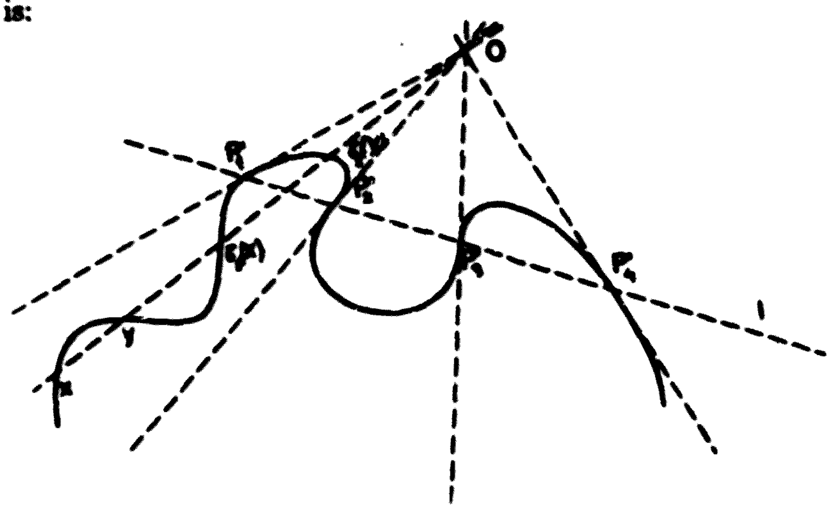
where $\Pi = \{\pi_1^*(x - \tau_1(x)) \mid x \in C_1\}$.

PROOF: By (6.6) we must study te cardinal of the set $\Theta_1^* \cap (\Theta_1^*)_{a_1} \cap Nm_{\varepsilon_1}^{-1}(\bar{\xi}_1)$ when $a_1 \in P_1$. From (6.8) we have the inclusion of Π in the left hand side member. To see the rest of the statement we shall need the following properties of the quartic plane curve C_1:

- The lines determined by the divisors $\varepsilon_1^*(\bar{x})$ with $x \in E$ all pass through a common point $O \in \mathbf{P}^2$, where $O \notin C_1$. (In fact $O = \mathbf{P}(H^0(E, \mathcal{O}_E(\bar{\xi}_1))^{\perp}) \subset \mathbf{P}H^0(C_1, K_{C_1})^*$).

- The ramification points $P_1', \ldots, P_4'$ of ε_1 belong to a line l and $O \notin l$.
- If $x, y \in C_1$ verify $\varepsilon_1(x) + \varepsilon_1(y) \equiv \bar{\xi}_1$ then $O \in \overline{xy}$.

The picture is:



In fact the involution τ_1 admits the following description: let $x \in C_1$ and take $x' \in l \cap \overline{Ox}$, then $| O, x'; x, \tau_1(x) | = -1$.

Take now a point $x + y \in \Theta_1^* \cap (\Theta_1^*)_{a_1} \cap Nm_{\varepsilon_1}^{-1}(\bar{\xi}_1)$. The following equalities are well-known:

$$\overline{xy} = \mathbf{P}T_{\Theta_1^*}(x+y) \subset \mathbf{P}T_{JC_1}(x+y) \cong \mathbf{P}H^0(C_1, K_{C_1})^*$$
$$\overline{rs} = \mathbf{P}T_{(\Theta_1^*)_{a_1}}(x+y) \text{ where } r + s \in | x + y - a_1 | .$$

Since $\varepsilon_1(x) + \varepsilon_1(y) \equiv \varepsilon_1(r) + \varepsilon_1(s) \equiv \bar{\xi}_1$ both lines pass through O. They are equal iff the following equality of divisors holds

$$x + y + \tau_1(x) + \tau_1(y) = r + s + \tau_1(r) + \tau_1(s),$$

that is to say iff $\pi_1^*(a_1) \in \Pi \cup \pi_1^*(\Sigma_0)$.

Assume first that $\pi_1^*(a_1) \notin \Pi \cup \pi_1^*(\Sigma_0)$. In this case the curve $\Theta_1^* \cap (\Theta_1^*)_{a_1}$ is not singular at $x + y$ and it suffices to show that $O \notin \mathbf{P}T_{P_1}(0)$ in order to obtain transversality in the intersection. Indeed:

$$T_{P_1}(0) = (H^0(C_1, K_{C_1})^-)^* = H^0(E, \mathcal{O}_E(\bar{\xi}_1))^* = H^0(E, \mathcal{O}_E)^\perp \subset H^0(C_1, K_{C_1})^*.$$

On the other hand, if s_R is an equation for the ramification divisor $R = \sum_{i=1}^4 P_i'$ then the inclusion

$$H^0(E, \mathcal{O}_E) \hookrightarrow H^0(C_1, K_{C_1})$$
$$s \quad \to \varepsilon_1^*(s)s_R$$

induces an equality $\mathbf{P}H^0(E, \mathcal{O}_E) = \{R\}$. By dualizing we get $O \notin l = \mathbf{P}T_{P_1}(0)$. Observe in particular that it follows from this that the set $\Theta_1^* \cap (\Theta_1^*)_{a_1} \cap Nm_{\varepsilon_1}^{-1}(\xi_1)$ is finite. Combining (6.7) with the transversality we find

$$\pi_1^*(a_1) \notin \Pi \cup \pi_1^*(\Sigma_0) \Longrightarrow \text{ number comp. } N(\tilde{a}) = 4.$$

Finally if $a_1 \in \Sigma_0$ then $\mathbf{P}T_{\Theta_1^*}(x+y) = \mathbf{P}T_{(\Theta_1^*)_{a_1}}(x+y)$. Thus $\Theta_1^* \cap (\Theta_1^*)_{a_1}$ is singular at x+y.

Therefore $a_1 \in \Sigma_0 \Longrightarrow$ number comp. $N(\tilde{a}) < 4$. ∎

(6.11).- **Theorem.** Let $(\tilde{C}, C)$ be a generic element of $\mathcal{R}_{B,g,2}$ with $g \geq 10$ and let $(\tilde{D}, D) \in \mathcal{R}_g$ be such that $P(\tilde{D}, D) \cong P(\tilde{C}, C)$. Then $(\tilde{D}, D) \in \mathcal{R}_{B,g,2}$ and $(\tilde{D}, D)$ and $(\tilde{C}, C)$ are tetragonally related.

PROOF: In view of the proof of (5.11) it suffices to show how to recognize (C_1, τ_1) and (C_2, τ_2) from P. Observe that (6.2) says how to recover (C_2, τ_2). In particular we recover the curve E. By combining (6.1), (6.2), (6.3), (6.6) and (6.10) we recover the set $\Pi \cup \pi_1^*(\Sigma_0)$ intrinsically. By (6.9) one obtains that the normalization of $\pi_1^*(\Sigma_0)$ is an irreducible curve of genus ≤ 3. Thus if it has genus < 3 then we distinguish Π as the component of the set with normalization of genus 3. Otherwise since the quotient of Σ by the involution given by symmetry is not isomorphic to E we also recover Π. Now by normalizing the symmetric curve Π we obtain (C_1, τ_1). ∎

In the rest of this section $(\tilde{C}, C)$ will be a general element of $\mathcal{R}_{B,g,1}$. By (2.6) and (2.7) we can assume that $\mathrm{Sing}\Xi^* = W_2$ is irreducible of dimension $g-5$.

(6.12).- **Proposition.** One has the following equality:

$$\{\tilde{a} \in P \mid \tilde{a} + W_2 \subset \Xi^*\} = \{\pi_1^*(a_1) + \pi_2^*(\varepsilon_2^*(\bar{x}) - r - s) \mid$$
$$a_1 \in P_1, \bar{x} \in E, \quad r, s \in C_2, \quad 2\bar{x} \equiv \varepsilon_2(r) + \varepsilon_2(s)\}.$$

PROOF: The inclusion of the second set in the first one is clear. To see the opposite inclusion take $\tilde{a} = \pi_1^*(a_1) + \pi_2^*(a_2) \in P$ where $a_1 \in P_1$, $a_2 \in P_2$ and such that $\tilde{a} + W_2 \subset \Xi^*$.

Let $\tilde{\zeta} = \pi_1^*(x) + \pi_2^*(\zeta_2) \in W_2$, with $x \in C_1$, $\zeta_2 \in Z_2''$ and $\varepsilon_1(x) + Nm_{\varepsilon_2}(\zeta_2) = \xi$. By applying Lemma (4.4) one finds elements $x' \in C_1, \zeta_2' \in W_{g-2}^0(C_2)$ and $\bar{\rho} \in \mathrm{Pic}^0(E)$ such that

$$(6.13) \qquad \begin{aligned} a_1 + x &\equiv x' + \varepsilon_1^*(\bar{\rho}) \\ a_2 + \zeta_2 &\equiv \zeta_2' - \varepsilon_2^*(\bar{\rho}). \end{aligned}$$

Let us define the following subvariety of $C_1 \times Z_2''$

$$Y = \{(x, \zeta_2) \in C_1 \times Z_2'' \mid \varepsilon_1(x) + Nm_{\varepsilon_2}(\zeta_2) \equiv \xi\}.$$

Consider now the morphism:

$$\begin{aligned} \Psi : \mathrm{Pic}^0(E) \times C_1 \times C_2^{(g-2)} &\longrightarrow \mathrm{Pic}^1(C_1) \times \mathrm{Pic}^{g-2}(C_2) \\ (\rho, x', D) &\longrightarrow (x' + \varepsilon_1^*(\rho) - a_1, D - \varepsilon_2^*(\rho) - a_2). \end{aligned}$$

In these terms the equivalences of (6.13) read: $Y \subset \mathrm{Im}(\Psi)$. Since Y is irreducible (it suffices to apply (3.8) to the fibres of the projection map from Y to C_1) there exists an irreducible component X of $\Psi^{-1}(Y)$ such that the induced map

$$\tilde{\Psi} : X \longrightarrow Y$$

is dominant. If $q : X \longrightarrow \mathrm{Pic}^0(E)$ is the first projection we call $Y_\rho := \tilde{\Psi}(q^{-1}(\rho))$ for all $\rho \in \mathrm{Pic}^0(E)$. Two cases are possible:

either a) $Y_\rho = Y$ for some $\rho \in \mathrm{Pic}^0(E)$

or b) $Y_\rho \neq Y$ for all $\rho \in \mathrm{Pic}^0(E)$.

In case a) define

$$b_1 = a_1 - \varepsilon_1^*(\rho) \quad \text{and} \quad b_2 = a_2 + \varepsilon_2^*(\rho).$$

Then (6.13) says:

$$h^0(b_1 + x) > 0, \quad h^0(b_2 + \zeta_2) > 0 \quad \text{for all} \quad (x, \zeta_2) \in Y.$$

Hence $b_1 = 0$ and $b_2 + Z_2'' \subset \Theta_2^*$. Therefore by using (3.9.ii) we finish the proof.

In case b) we write $\lambda : Y \longrightarrow C_1 \subset \mathrm{Pic}^1(C_1)$ for the first projection. We claim that $\lambda_{|Y_\rho}$ is non-surjective for general $\rho \in \mathrm{Pic}^0(E)$. Otherwise for all $x \in C_1$ one finds an element $\zeta_2 \in Z_2''$ such that $(x, \zeta_2) \in Y_\rho$. In particular $h^0(a_1 + x - \varepsilon_1^*(\bar{\rho})) > 0$ and then $a_1 = \varepsilon_1^*(\bar{\rho})$.

Now since for a general $\bar{\rho}$, $Y_{\bar{\rho}}$ contains components of codimension 1 in Y we deduce from the claim the following fact: there exists $x_0 \in C_1$ such that $\lambda^{-1}(x_0) \subset Y_{\bar{\rho}}$. Hence (6.13) reads:

$$h^0(a_1 + x_0 - \varepsilon_1^*(\bar{\rho})) > 0 \quad \text{and} \quad h^0(a_2 + \zeta_2 + \varepsilon_2^*(\bar{\rho})) > 0$$

for all $\zeta_2 \in Z_2''$ with $Nm_{e_2}(\zeta_2) = \xi - \varepsilon_1(x_0)$. In particular

$$a_2 + \varepsilon_2^*(\rho) + \{\zeta_2 \in Z_2'' \mid Nm_{e_2}(\zeta_2) \equiv \xi - \varepsilon_1(x_0)\} \subset \Theta_2^*.$$

The proof ends by observing that

$$\{\zeta_2 \in Z_2'' \mid Nm_{e_2}(\zeta_2) = \xi - \varepsilon_1(x_0)\} = \varepsilon_2^*(\bar{\alpha}) + Z_2' \cap Z_2''$$

where $2\bar{\alpha} = \xi_1 - \varepsilon_1(x_0)$, and applying (3.9.ii). ∎

We shall denote by B the set described in (6.12).

(6.14).- **Proposition**. The abelian variety $\pi_1^*(P_1)$ acts on $B \cap 2B$ by translations on P and the quotient

$$\frac{B \cap 2B}{\pi_1^*(P_1)} \subset \frac{P}{\pi_1^*(P_1)}$$

is a symmetric curve with normalization C_2. The reflection on $P/\pi_1^*(P_1)$ induces on C_2 the involution τ_2.

PROOF: By using the standard arguments of §5 one has:

$$B \cap 2B = \{\pi_1^*(a_1) + \pi_2^*(x - \tau_2(x)) \mid a_1 \in P_1, x \in C_2\}.$$

Now the morphism

$$\lambda : C_2 \longrightarrow \frac{B \cap 2B}{\pi_1^*(P_1)}$$
$$x \longrightarrow \overline{\pi_2^*(x - \tau_2(x))}$$

is birational and verifies $\lambda(\tau_2(x)) = -\lambda(x)$. ∎

(6.15).-The reader can prove without much work the following properties:

- $P_1 \subset JC_1$ is an elliptic curve .
- The morphism

$$\mu : C_1 \longrightarrow P_1$$
$$x \longrightarrow x - \tau_1(x)$$

is a double cover with two ramification points : ing on C_1 a new bi-elliptic structure. The attached involution τ_1' is the composition of τ_1 with the hyperelliptic involution.

• We shall write Q_1 and Q_2 for the fixed points of τ_1' and P_1', P_2' for the ramification points of ε_1. With the notations of (2.1) :

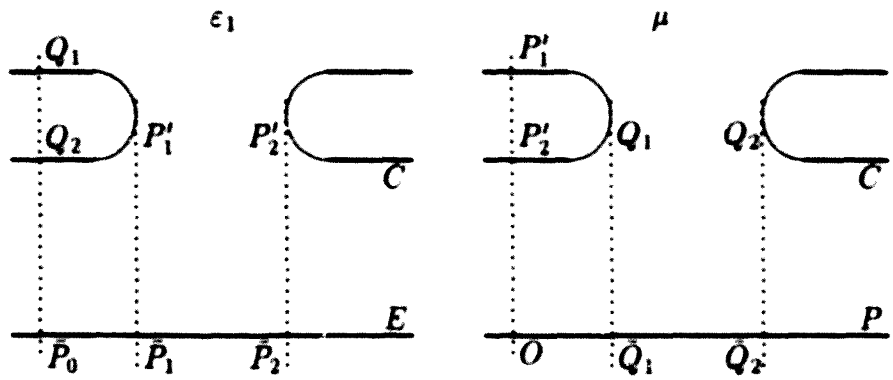
$$Q_1 + Q_2 \equiv P_1' + P_2' \equiv K_{C_1}$$

and

$$\tau_1(Q_1) = Q_2, \quad \varepsilon_1(Q_1) = \varepsilon_1(Q_2) \in |\, \xi_1 \,|$$
$$\tau_1'(P_1') = P_2', \quad \mu(P_1') = \mu(P_2') = 0.$$

We write $\bar{Q}_1 = \mu(Q_1)$ and $\bar{Q}_2 = \mu(Q_2)$.

The picture is:



where $|\, \xi_1 \,| = \{\bar{P}_0\}$.

• Note that $\bar{Q}_1 = \mu(Q_1) = Q_1 - \tau_1(Q_1) = -(Q_2 - \tau_1(Q_2)) = -\mu(Q_2) = -\bar{Q}_2$. Moreover $\mu^*(0) = P_1' + P_2' \equiv Q_1 + Q_2$.

Summarizing we obtain (composing with $\pi_1^* : P_1 \longrightarrow \pi_1^*(P_1)$) that C_1 can be represented as the double cover of $\pi_1^*(P_1)$ associated to the class of the origin (as a point of the abelian subvariety of P) and the discriminant divisor $\pi_1^*(\bar{Q}_1) + \pi_1^*(\bar{Q}_2)$. Since the class is trivially recovered from $\pi_1^*(P_1)$, we only need to find the divisor inside P. Moreover the involution τ_1 will appear when composing the canonical involution of C_1 with the involution attached to this cover.

(6.16).- Proposition. Let $\tilde{a} = \pi_1^*(x - \tau_1(x)) \neq 0$ where $x \in C_1$. Then:

$$\Xi^* \cap \Xi_{\tilde{a}}^* = F' \cup R(\tilde{a})$$

where

$$F' = \{\pi_1^*(y) + \pi_2^*(\zeta_2) \mid y \in C_1, \zeta_2 \in A_2, \varepsilon_1(y) + Nm_{e_2}(\zeta_2) \equiv \bar{\xi}\}$$
$$R(\tilde{a}) = \{\pi_1^*(x) + \pi_2^*(\zeta_2) \mid \zeta_2 \in \Theta_2^*, \varepsilon_1(x) + Nm_{e_2}(\zeta_2) \equiv \bar{\xi}\}.$$

PROOF: To see the inclusion $F' \cup R(\tilde{a}) \subset \Xi^* \cap \Xi_{\tilde{a}}^*$ compare with (6.3.i). We prove the opposite inclusion. Fix $\tilde{\zeta} = \pi_1^*(y) + \pi_2^*(\zeta_2)$ with $y \in C_1, \zeta_2 \in \Theta_2^*$ and $\varepsilon_1(y) + Nm_{e_2}(\zeta_2) \equiv \bar{\xi}$ verifying

$$-\tilde{a} + \tilde{\zeta} \in \Xi^*.$$

By lemma (4.4) one finds elements $z \in C_1, \zeta_2' \in W_{g-2}^0(C_2)$ and $\rho \in \mathrm{Pic}^0(E)$ such that

$$(6.17) \qquad y + \tau_1(x) - x \equiv z + \varepsilon_1^*(\rho)$$
$$\zeta_2 = \zeta_2' - \varepsilon_2^*(\rho).$$

We consider the possibilities:

i) $\rho = 0$

ii) $\rho \neq 0$.

If i) occurs one has

$$y + \tau_1(x) \equiv x + z.$$

The equality would lead to $y = x$ (recall that $x \neq \tau_1(x)$ due to $\tilde{a} \neq 0$) and $\tilde{\zeta} \in R(\tilde{a})$. So we only consider the case:

$$(6.18) \qquad y + \tau_1(x) \equiv K_{C_1}.$$

Defining $\alpha = \xi_1 - \varepsilon_1(y)$ one finds

$$(6.19) \qquad y + \varepsilon_1^*(\alpha) \equiv y + K_{C_1} - \varepsilon_1^*(\varepsilon_1(y)) \equiv K_{C_1} - \tau_1(y) \equiv \tau_1(K_{C_1} - y) \equiv x.$$

Moreover $Nm_{e_2}(\zeta_2) \equiv \xi - \varepsilon_1(y) \equiv \xi_2 + \alpha$. Hence

$$(6.20) \qquad h^0(\zeta_2 - \varepsilon^*(\alpha)) = h^0(K_{C_2} - \tau_2(\zeta_2)) = h^0(\tau_2(\zeta_2)) = h^0(\zeta_2) > 0.$$

Combining (6.19) and (6.20) one has:

$$\tilde{\zeta} \equiv \pi_1^*(y + \varepsilon_1^*(\tilde{a})) + \pi_2^*(\zeta_2 - \varepsilon_2^*(\tilde{a})) \equiv \pi_1^*(x) + \pi_2^*(\zeta_2 - \varepsilon_2^*(\tilde{a})) \in R(\tilde{a}).$$

From now on we assume $\bar{\rho} \neq 0$. By (6.17) we get

$$\zeta_2 \in \Theta_2^* \cap (\Theta_2^*)_{-\varepsilon_2^*(\rho)} = A_2 \cup \{\zeta_2 \in W_{g-2}^0(C_2) \mid Nm_{e_2}(\zeta_2) = \bar{\xi}_2 - \bar{\rho}\}$$

(cf.[De4],p.9). When $\zeta_2 \in A_2$, clearly $\tilde{\zeta} \in F'$. On the other hand if $Nm_{e_2}(\zeta_2) = \bar{\xi}_2 - \bar{\rho}$ we deduce

$$\varepsilon_1(y) \equiv \bar{\xi}_1 + \bar{\rho}.$$

By replacing $\bar{\rho}$ in the first linear equivalence of (6.17):

$$\tau_1(x) + K_{C_1} \equiv x + \tau_1(y) + z.$$

Writing $K_{C_1} \equiv z + z'$ we have now $\tau_1(x) + z' \equiv x + \tau_1(y)$. We deduce as before that either $x = y$ or $x + \tau_1(y) \equiv K_{C_1}$. The first possibility leads directly to $\tilde{\zeta} \in R(\tilde{a})$ and the second one implies

$$\tau_1(x) + y \equiv K_{C_1}.$$

Going back to (6.18) we end the proof as in case i). ∎

(6.21).- **Remark**. As a matter of fact the computation of $\Xi^* \cdot \Xi_{\tilde{a}}^*$ has been made by Debarre in [De5]. He finds:

$$\Xi^* \cdot \Xi_{\tilde{a}}^* = \{\tilde{\zeta} \in \Xi^* \mid h^0(\tilde{\zeta} - \pi_1^*(x)) \geq 1\} \text{ for } \tilde{a} = \pi_1^*(x - \tau_1(x))$$

and

$$\mathrm{Sing}(\Xi^* \cdot \Xi_{\tilde{a}}^*) \supset \{\tilde{\zeta} \in \Xi^* \mid h^0(\tilde{\zeta} - \pi_1^*(x)) \geq 2\}.$$

We adapt his general proof to our case. First we note that

$$F' \cup R(\tilde{a}) = \{\tilde{\zeta} \in \Xi^* \mid h^0(\tilde{\zeta} - \pi_1^*(x)) \geq 1\}.$$

Indeed, the inclusion of $F' \cup R(\tilde{a})$ in the right hand side member comes from the descriptions of (6.16). Next if $\tilde{\zeta} = \pi_1^*(y) + \pi_2^*(\zeta_2) \in \Xi^*$ with $h^0(\pi_1^*(y - x) + \pi_2^*(\zeta_2)) \geq 1$ one has either that $\pi_2^*(\zeta_2)$ is not π_1-simple (cf. (2.14)) or

$$\begin{aligned} 0 &< h^0(y - x) + h^0(y - x + Nm_{\pi_1}(\pi_2^*(\zeta_2)) - \varepsilon_1^*(\bar{\xi}_2)) = \\ &= h^0(y - x) + h^0(y - x + \varepsilon_1^*(\bar{\xi} - \varepsilon_1(y)) - \varepsilon_1^*(\bar{\xi}_2)) = \\ &= h^0(y - x) + h^0(\varepsilon_1^*(\bar{\xi}_1) - x - \tau_1(y)). \end{aligned}$$

It is easy to check that the first case implies that $\zeta_2 \in A_2$ and then $\check{\zeta} \in F'$. The second case leads to one of the following two possibilities:

$$\begin{array}{ll} i) & y = x \\ ii) & x + \tau_1(y) \equiv \epsilon_1^*(\xi_1) \equiv K_{C_1}. \end{array}$$

The recurrent argument starting at (6.18) finishes the proof of the inclusion of the right hand side member above in $F' \cup R(\check{a})$.

Next, note that $\{\check{\zeta} \in \Xi^* \mid h^0(\check{\zeta} - \pi_1^*(x)) \geq 1\}$ is a special subvariety associated to the linear system $\mid K_C - Nm_\pi(\pi_1^*(x)) \mid$ in the sense of [Be2] (cf. also [We3]). In other words, defining X by the pull-back diagram

$$\begin{array}{ccc} X & \longrightarrow & \check{C}^{(2g-4)} \\ \Big\downarrow & & \Big\downarrow \pi^{(2g-4)} \\ \mid K_C - Nm_\pi(\pi_1^*(x)) \mid & \longrightarrow & C^{(2g-4)} \end{array}$$

the image of X by the morphism

$$\begin{array}{c} \varphi : \check{C}^{(2g-4)} \longrightarrow \mathrm{Pic}^{2g-2}(\check{C}) \\ \check{D} \longrightarrow \pi_1^*(x) + \check{D} \end{array}$$

has two connected components. Only one of these components sits inside P^* and it equals $\Xi^* \cap \Xi_{\check{a}}^*$. In loc. cit., Beauville computes the cohomology class of special subvarieties. In the present case the class is $[\Xi]^2$. So $\Xi^* \cdot \Xi_{\check{a}}^*$ is reduced and the equality of the statement holds.

Suppose now that $\check{\zeta} \in \Xi^*$ verifies $h^0(\check{\zeta} - \pi_1^*(x)) \geq 2$. We can assume that $h^0(\check{\zeta}) = h^0(\check{\zeta} - \pi_1^*(x - \tau_1(x))) = 2$ (otherwise either $\check{\zeta} \in \mathrm{Sing}\Xi^*$ or $\check{\zeta} \in \mathrm{Sing}\Xi_{\check{a}}^*$, hence $\check{\zeta} \in \mathrm{Sing}(\Xi^* \cdot \Xi_{\check{a}}^*)$). It suffices to check that the tangent spaces of the two divisors at the point are equal. Taking bases for both $H^0(\check{\zeta})$ and $H^0(\check{\zeta} - \pi_1^*(x - \tau_1(x)))$ the reader can apply [Mui], p.343 for this computation. This finishes the proof of the statements in (6.21).

We remark also that, since on Ξ^* φ is one-to-one outside $\{\check{\zeta} \in \Xi^* \mid h^0(\check{\zeta} - \pi_1^*(x)) \geq 2\}$, a characterization of Welters (cf. [We3] and [Be2]) of the singularities of the special subvarieties gives the inclusion:

(6.22)

$$\begin{aligned} \mathrm{Sing}(\Xi^* \cdot \Xi_{\check{a}}^*) \subset \{\check{\zeta} \in \Xi^* \mid h^0(\check{\zeta} - \pi_1^*(x)) \geq 2\} \cup \{\pi_1^*(x) + \check{D} \text{ such that} \\ \text{for } A \geq 0 \text{ maximal with } \pi^*(A) \leq \check{D}, \quad h^0(A + \varepsilon^*(\varepsilon_1(x))) > 1\}. \end{aligned}$$

The next result allows one to find distinguished points in $\pi_1^*(P_1)$.

(6.23).- **Proposition.** Let $0 \neq \tilde{a} = \pi_1^{\circ}(x - \tau_1(x)) \in \pi_1^{*}(P_1)$. Then $\dim(\mathrm{Sing}(\Xi^{*} \cdot \Xi_{\tilde{a}}^{*}) - F')$ iff $\varepsilon_1(x) \equiv \bar{\xi}_1$.

PROOF: From (6.21) and (6.22) it suffices to see the following facts:

i) If $\varepsilon_1(x) \notin |\bar{\xi}_1|$, then $R(\tilde{a}) - F'$ intersects the second member of (6.22) in a finite number of points.

ii) If $\varepsilon_1(x) \in |\bar{\xi}_1|$, then $\dim(R(\tilde{a}) - F') \cap \{\tilde{\zeta} \in \Xi^{*} \mid h^0(\tilde{\zeta} - \pi_1^{*}(x)) \geq 2\} > 0$.

To see ii) observe that the set

$$\{\pi_1^{*}(x) + \pi_2^{*}(\zeta_2) \mid \zeta_2 \in Z_2' - Z_2' \cap Z_2'', \varepsilon_1(x) \equiv \xi_1\}$$

of dimension g-6 is contained in the above intersection.

Assume now that $\varepsilon_1(x) \notin |\xi_1|$ and take $\tilde{\zeta} = \pi_1^{*}(x) + \pi_2^{*}(\zeta_2)$ such that $\zeta_2 \notin A_2$. Then

$$h^0(\tilde{\zeta} - \pi_1^{*}(x)) = h^0(\pi_2^{*}(\zeta_2)) = h^0(\zeta_2) + h^0(\zeta_2 - \varepsilon_2^{*}(\xi_1)) = h^0(\zeta_2).$$

So $h^0(\tilde{\zeta} - \pi_1^{*}(x)) \geq 2$ implies $\zeta_2 \in \mathrm{Sing}\Theta_2^{*} = Z_2' \cup Z_2''$. Since $\zeta_2 \notin A_2 \supset Z_2''$ and $Nm_{\varepsilon_2}(\zeta_2) \neq \xi_2$ this is a contradiction.

Suppose finally that there exists a divisor $A \geq 0$ on C such that

$$h^0(\pi_2^{*}(\zeta_2) - \pi^{*}(A)) > 0 \text{ and } h^0(A + \varepsilon^{*}(\varepsilon_1(x))) \geq 2.$$

In particular $A \neq 0$. By using (3.1) the second inequality says that either A is not ε-simple or deg(A)=g-2. In the first case we conclude that we may write

$$\pi_2^{*}(\zeta_2) \equiv \pi^{*}(\varepsilon^{*}(A)) + \tilde{B}$$

where A and $\tilde{B}$ are effective divisors on E and $\tilde{C}$ respectively and A is not trivial. Then

$$0 < h^0(\pi_2^{*}(\zeta_2 - \varepsilon_2^{*}(A))) = h^0(\zeta_2 - \varepsilon_2^{*}(A)) + h^0(\zeta_2 - \varepsilon_2^{*}(A) - \varepsilon_2^{*}(\xi_1)),$$

which contradicts that $\zeta_2 \notin A_2$. On the other hand, if deg(A)=g-2 then $\pi^{*}(A) = \pi_2^{*}(\zeta_2)$. Taking norms one obtains that $2A \equiv \varepsilon^{*}(\varepsilon_2(\zeta_2))$. By (3.1) there exists an effective divisor $\bar{A}_0$ of degree g-2 on E such that $2A = \varepsilon^{*}(\bar{A}_0)$. As above A not ε-simple leads to a contradiction. If A is ε-simple, then it has support in the ramification locus of ε_1, and a finite number of possibilities appear. ∎

(6.24).- Theorem. **Let $(\tilde{C}, C)$ be a generic element of $\mathcal{R}_{B,g,1}$ and let $(\tilde{D}, D) \in \mathcal{R}_g$ such that $P(\tilde{C}, C) \cong P(\tilde{D}, D)$. Then $(\tilde{D}, D) \in \mathcal{R}_{B,g,1}$, and $(\tilde{C}, C)$ and $(\tilde{D}, D)$ are tetragonally related.**

Proof: By using the arguments of (5.9) we conclude that $(\tilde{D}, D) \in \mathcal{R}_{B,g}$. Then, from the number of irreducible components of SingΞ^* (cf. (2.7) and (2.13)) we conclude that

$$(\tilde{D}, D) \in \mathcal{R}'_{B,g} \cup \mathcal{R}_{B,g,0} \cup \mathcal{R}_{B,g,1}.$$

As we shall see (independently) in (7.1.i) (combined with (2.12)) the property

$$\dim\{\check{a} \in P \mid \check{a} + \mathrm{Sing}\Xi^* \subset \mathrm{Sing}\Xi^*\} = 1$$

(cf. (5.12.i)) does not hold for the elements of the components $\mathcal{R}'_{B,g}$ and $\mathcal{R}_{B,g,0}$. So $(\tilde{D}, D) \in \mathcal{R}_{B,g,1}$. By arguing as in (5.11) it suffices to explain how to recover (C_1, τ_1) and (C_2, τ_2) from P. The way to do it in the second case comes from Propositions (6.12) and (6.14). In the first case this is done by combining (6.15), (6.16) and (6.23). ∎

7.The components $\mathcal{R}_{B,g,0}$ and $\mathcal{R}'_{B,g}$

As we remarked in (2.12) the study of these two components should be done simultaneously. Our approach consist in looking at each component independently and then, comparing the constructions made in each case, to establish an injection from $\mathcal{R}'_{B,g}$ in $\mathcal{R}_{B,g,0}$ commuting with the Prym map.

Let $(\tilde{C},C) \in \mathcal{R}_{B,g,0}$. We keep the notations of §1 and §2. In this section we do not need the assumption of generality. Although the equality Sing$\Xi^* = W_2$ cannot be used, W_2 appears as the unique component of dimension greater than 0. Recall that $t = 0$ implies that ε_1 and π_2 are unramified. We shall denote by λ the non trivial element of $\pi^*(\varepsilon^*({}_2JE))$.

(7.1).- **Proposition**. One has the equalities:

$$i) \quad \{\tilde{a} \in P \mid \tilde{a} + W_2 \subset W_2\} = \{0, \lambda\}$$

$$ii) \quad \{\tilde{a} \in P \mid \tilde{a} + W_2 \subset \Xi^*\} = \{\pi_2^*(\varepsilon_2^*(x) - r - s) \mid x \in E, \quad r,s \in C_2, \quad 2x \equiv \varepsilon_2(r) + \varepsilon_2(s)\}.$$

PROOF: In both equalities the inclusion of the right hand side member in the left hand side member is clear. We fix $\tilde{a} \in P$. By (2.8) we may write $\tilde{a} = \pi_2^*(a_2)$ where $a_2 \in P_2$. First we suppose that $\tilde{a} + W_2 \subset W_2$. Since

$$W_2 = \{\pi_2^*(\zeta_2) \mid \zeta_2 \in Z_2'', Nm_{\varepsilon_2}(\zeta_2) = \xi\}$$

and

$$\mathrm{Ker}(\pi_2^*) = \{0, \varepsilon_2^*(\xi_1)\}$$

we deduce that for all $\zeta_2 \in Z_2''$ with $Nm_{\varepsilon_2}(\zeta_2) = \xi$

(7.2) $$a_2 + \zeta_2 \in Z_2''.$$

By taking $\alpha = \varepsilon_2^*(\bar{\alpha})$ with $2\bar{\alpha} \equiv \xi_1$ we get

$$\alpha + \{\zeta_2 \in Z_2'' \mid Nm_{\varepsilon_2}(\zeta_2) = \xi\} = Z_2' \cap Z_2''.$$

So (7.2) says that

$$a_2 - \alpha + Z_2' \cap Z_2'' \subset Z_2'.$$

Hence, by (3.9.iii), $a_2 \in P_1 \cap \mathrm{Im}(\varepsilon_1^*)$ and we are done.

Next we assume that $\tilde{a} + W_2 \subset \Xi^*$. We obtain similarly

$$a_2 + \{\zeta_2 \in Z_2'' \mid Nm_{\epsilon_2}(\zeta_2) = \xi\} \subset \Theta_2^* \cup (\Theta_2^*)_{\epsilon_2^*(\xi_1)}.$$

Taking α as before one has:

$$a_2 - \alpha + Z_2' \cap Z_2'' \subset \Theta_2^* \cup (\Theta_2^*)_{\epsilon_2^*(\xi_1)}.$$

By using the irreducibility of $Z_2' \cap Z_2''$ (cf.(3.8)) and (3.9.ii) we end the proof. ∎

Let us denote by S the set described in (7.1.ii). Then

(7.3).- **Proposition** The set S∩2S is a symmetric curve with normalization C_2. Moreover τ_2 is the involution induced on C_2 by the (-1) map of P.

PROOF: We claim that the following equality holds

$$S \cap 2S = \{\pi_2^*(x - \tau_2(x)) \mid x \in C_2\}. \tag{7.4}$$

First we see that all the statements are a consequence of this claim. In fact, only the birationality of the map

$$\begin{aligned} \varphi : C_2 &\longrightarrow S \cap 2S \\ x &\longrightarrow \pi_2^*(x - \tau_2(x)) \end{aligned}$$

needs to be proved. Assume that $\varphi(x) = \varphi(y)$. Then $x + \tau_2(y) - \tau_2(x) - y \in \mathrm{Ker}(\pi_2^*) = \{0, \varepsilon_2^*(\xi_1)\}$. Hence:

$$2x + 2\tau_2(y) \equiv 2y + 2\tau_2(x).$$

The equality of divisors would conduce to either $x = y$ or $x = \tau_2(x)$. So we can suppose that $h^0(2x + 2\tau_2(y)) \geq 2$. Since all the linear series g_4^1 on C_2 come from g_2^1's on E one finds a divisor $A \in E^{(2)}$ such that $2x + 2\tau_2(y) = \varepsilon^*(A)$ and then we have again either $x = y$ or $x = \tau_2(x)$.

In order to prove (7.4) we observe that

$$2S = \{\pi_2^*(\zeta_2 - \tau_2(\zeta_2)) \mid \zeta_2 \in W_2^0(C_2)\}.$$

Suppose that $\pi_2^*(\varepsilon_2^*(\bar{x}) - r - s) \in 2S$ where $x \in E$, $r, s \in C_2$ and $2\bar{x} \equiv \varepsilon_2(r) + \varepsilon_2(s)$. For some points $y, z \in C_2$ one has

$$\begin{aligned} \text{either} \quad & \varepsilon_2^*(\bar{x}) - r - s \equiv y + z - \tau_2(y) - \tau_2(z) \\ \text{or} \quad & \varepsilon_2^*(\bar{x}) - r - s \equiv y + z - \tau_2(y) - \tau_2(z) + \varepsilon_2^*(\xi_1). \end{aligned}$$

Since both cases are similar we suppose that

$$\epsilon_2^*(\bar{x}) + \tau_2(y) + \tau_2(z) \equiv y + z + r + s.$$

If $h^0(\epsilon_2^*(\bar{x}) + \tau_2(y) + \tau_2(z)) = 2$, then Lemma (3.1) implies that $z = \tau_2(y)$ and one has:

$$\pi_2^*(\epsilon_2^*(\bar{x}) - r - s) \equiv \pi_2^*(y + z - \tau_2(y) - \tau_2(z)) = 0.$$

Otherwise we get an equality of divisors. The proof ends by looking at the different possibilities. The opposite inclusion is left to the reader. ∎

(7.5).- **Remark**. The data (C_2, τ_2) do not determine the initial element $(\tilde{C}, C)$. However, by recovering the class $\epsilon_2^*(\xi_1)$, the curve C_1 (hence $(\tilde{C}, C)$) may be reconstructed from our information.

(7.6).- **Theorem**. Let $(\tilde{C}, C)$ and $(\tilde{D}, D)$ be two elements of $\mathcal{R}_{B,g,0}$ verifying $P(\tilde{C}, C) \cong P(\tilde{D}, D)$. Then: $(\tilde{C}, C) \cong (\tilde{D}, D)$.

Proof: By (7.1), (7.3) and (7.5) it suffices to exhibit a way to recover $\epsilon_2^*(\xi_1)$ from P. Going back to the proof of (7.3) one finds a morphism:

$$C_2 \longrightarrow P$$

inducing a morphism:

$$j : JC_2 \longrightarrow P.$$

By construction one can factorize j into $j' \circ h$, where

$$h : JC_2 \longrightarrow \mathrm{Imag}(Id - \epsilon_2^*) \cong P_2$$

is the obvious map and $j' = \pi_{2|P_2}^*$. Then $\mathrm{Ker}(j') = \{0, \epsilon_2^*(\xi_1)\}$. Hence we obtain $\epsilon_2^*(\xi_1) \in P_2 \subset JC_2$. ∎

Let $(\tilde{C}, C) \in \mathcal{R}'_{B,g}$. We keep the notations and assumptions of §1 and §2 (see specially (2.9)). In particular $g \geq 10$. Recall that by (4.3) one has $\tau^*(\eta) \neq \eta$.

(7.7).- **Proposition**. With the above notations, $\mathrm{Sing}\Xi^*$ has a unique irreducible component of dimension $g - 5$. This component admits the description:

$$W = \{\pi^*(\epsilon^*(\bar{x} + \bar{y})) + \tilde{\zeta} \in P^* \mid \bar{x}, \bar{y} \in E, \tilde{\zeta} \in W^0_{2g-10}(\tilde{C})\}.$$

PROOF: Acording to (2.13) it only remains to check that $\dim W = g-5$ (note that by definition $W \subset \mathrm{Sing}^{\pi}_{\epsilon_\pi}\Xi^* \subset \mathrm{Sing}\Xi^*$ (cf.§1)). We consider the morphism:

$$\kappa : E^{(2)} \times \tilde{C}^{(2g-10)} \longrightarrow \mathrm{Pic}^{2g-2}(\tilde{C})$$
$$(\bar{x}+\bar{y}, \tilde{D}) \longrightarrow \tilde{D} + \pi^*(\varepsilon^*(\bar{x}+\bar{y})).$$

Let us define $T = \kappa^{-1}(P^*)$ and $\tilde{\kappa} : T \longrightarrow P^*$ the restriction of κ. The fibre of the projection of T onto $E^{(2)}$ over a point $\bar{x}+\bar{y}$ is isomorphic to a special subvariety (in the sense of [We2] and [Be2]) associated to the linear system $\mid K_C - 2\varepsilon^*(\bar{x}+\bar{y}) \mid = \mid \varepsilon^*(\xi - 2\bar{x} - 2\bar{y}) \mid$ of dimension $g-6$. Hence $\dim T = g-4$. Since $\tilde{\kappa}(T) = W$ and the generic fibre of $\tilde{\kappa}$ has dimension 1 we conclude that $\dim W = g-5$. ∎

(7.8).- **Proposition**. One has the equality:

$$\{\tilde{a} \in P \mid \tilde{a} + W \subset \Xi^*\} = \{\pi^*(\varepsilon^*(\bar{x})) - \tilde{\zeta} \in P \mid \bar{x} \in E, \tilde{\zeta} \in W^0_4(\tilde{C})\}$$

PROOF: Let $\tilde{a} \in P$ such that $\tilde{a} + W \subset \Xi^*$. We wish to see that $\tilde{a}$ is in the right hand side member. By hypothesis

$$h^0(\tilde{a} + \pi^*(\varepsilon^*(\bar{x}+\bar{y})) + \tilde{\zeta}) \geq 2$$

for all $(\bar{x}+\bar{y}, \tilde{\zeta}) \in E^{(2)} \times W^0_{2g-10}(\tilde{C})$ such that $\pi^*(\varepsilon^*(\bar{x}+\bar{y})) + \tilde{\zeta} \in P^*$. By standard facts from the theory of Prym varieties this implies that:

$$h^0(\tilde{a} + \pi^*(\varepsilon^*(\bar{x}+\bar{y})) + \tilde{\zeta}) \geq 1$$

for all $(\bar{x}+\bar{y}, \tilde{\zeta}) \in E^{(2)} \times W^0_{2g-10}(\tilde{C})$ such that $2\varepsilon^*(\bar{x}+\bar{y}) + Nm_\pi(\tilde{\zeta}) \equiv K_C$. By applying (2.16) one has:

$$h^0(\tilde{a} + \pi^*(\varepsilon^*(\bar{x}+\bar{y}))) > 0 \quad \text{for all} \quad \bar{x}, \bar{y} \in E. \tag{7.9}$$

In particular we may write $\tilde{a} \equiv \tilde{D} - \pi^*(\varepsilon^*(\bar{x}+\bar{y}))$ where $\tilde{D} \in \tilde{C}^{(8)}$ and $Nm_\pi(\tilde{D}) \equiv 2\varepsilon^*(\bar{x}+\bar{y})$. Then (7.9) reads

$$h^0(\tilde{D} + \pi^*(\varepsilon^*(\bar{\alpha}))) > 0 \quad \text{for all} \quad \bar{\alpha} \in \mathrm{Pic}^0(E).$$

Let us make the decomposition $\tilde{D} = \pi^*(\varepsilon^*(\bar{D}) + D) + \tilde{D}'$ where $\bar{D}, D, \tilde{D}'$ are effective divisors on E, C and $\tilde{C}$ respectively, D is ε-simple and $\tilde{D}'$ is π-simple (cf. (2.14)). Note that it suffices to see that the divisor $\bar{D}$ is not trivial.

Then by applying (2.15) twice one has:

$$0 < h^0(\tilde D + \pi^*(\varepsilon^*(\tilde a))) = h^0(\pi^*(\varepsilon^*(\bar D + \tilde a) + D) + \tilde D') \le$$
$$\le h^0(\varepsilon^*(\bar D + \tilde a) + D) + h^0(\varepsilon^*(\bar D + \tilde a) + D + Nm_\pi(\tilde D') - \eta)$$

and

$$h^0(\varepsilon^*(\bar D + \tilde a) + D) \le h^0(\bar D + \tilde a) + h^0(\bar D + \tilde a + Nm_\varepsilon(D) - \xi).$$

Since $\deg(\bar D + \tilde a + Nm_\varepsilon(D)) \le \deg(\tilde D) = 8 < g - 1 = \deg(\xi)$ we obtain

$$0 < h^0(\bar D + \tilde a) + h^0(\varepsilon^*(\bar D + \tilde a) + D + Nm_\pi(\tilde D') - \eta).$$

Suppose now that $\deg(\bar D) = 0$. Then we have

(7.10) $$0 < h^0(\varepsilon^*(\tilde a) + D + Nm_\pi(\tilde D') - \eta) \quad \text{for all } \tilde a \in \mathrm{Pic}^0(E) - \{0\}.$$

On the other side $Nm_\pi(\tilde D) = 2\varepsilon^*(\bar x + y)$ implies that

$$2D + Nm_\pi(\tilde D') \equiv 2\varepsilon^*(\bar x + \bar y).$$

So (7.10) reads

$$0 < h^0(\varepsilon^*(2\bar x + 2\bar y + \alpha) - D - \eta) \quad \text{for all } \alpha \in \mathrm{Pic}^0(E) - \{0\}.$$

Let $D' \ge 0$ such that

(7.11) $$\varepsilon^*(2\bar x + 2\bar y + \alpha) - D - D' \equiv \eta.$$

Then $2(D + D') \in |\varepsilon^*(2(2\bar x + 2\bar y + \alpha))|$. Since $\deg(2(2\bar x + 2\bar y + \alpha)) = 8 < g - 1$, we can apply (3.1). Therefore there exists an effective divisor $\bar F$ on E such that

$$2(D + D') = \varepsilon^*(\bar F).$$

In particular $D + D'$ is τ-invariant. Looking at (7.11) we conclude that $\tau^*(\eta) = \eta$, which is a contradiction. ∎

Let us denote by S' the set $\{\tilde a \in P \mid \tilde a + W \subset \Xi^-\}$.

(7.12).- Proposition. **The following inclusions hold:**

$$S' \cap 2S' \subset T' = \{\tilde{D} - \iota^*(\tilde{D}) \in J\tilde{C} \mid \tilde{D} \in W_2^0(\tilde{C}), Nm_\pi(\tilde{D}) \in \mathrm{Im}(\varepsilon^*)\} \subset S'$$

Proof: Let us define

$$U = \{\tilde{D} - \iota^*(\tilde{D}) \mid \tilde{D} \in W_4^0(\tilde{C}), Nm_\pi(\tilde{D}) \in \mathrm{Im}(\varepsilon^*)\}$$

By (7.8) one has $2S' \subset U$. So, our statements follow from the claim:

$$U \cap S' = T'.$$

The inclusion $T' \subset U \cap S'$ is clear. We prove the opposite inclusion. Let $\tilde{D} - \iota^*(\tilde{D}) \in U$ and $\bar{r}, \bar{s} \in E$ such that $Nm_\pi(\tilde{D}) = \varepsilon^*(\bar{r} + \bar{s})$. If we suppose that $\tilde{D} - \iota^*(\tilde{D}) \in S'$ then one finds elements $\tilde{D}' \in \tilde{C}^{(4)}$ and $\bar{x} \in E$ such that

$$\iota^*(\tilde{D}) + \tilde{D}' \equiv \tilde{D} + \pi^*(\varepsilon^*(\bar{x})). \tag{7.13}$$

We may write $\tilde{D} = \pi^*(A) + \tilde{B}$ where $\tilde{B} \geq 0$ is π-simple and A is effective. Looking at the degree of A we have three possibilities:

a) $\deg(A) = 2$. In this case $\tilde{D} - \iota^*(\tilde{D}) = 0 \in T'$.

b) $\deg(A) = 1$. Therefore $\deg(\tilde{B}) = 2$. By replacing in (7.13)

$$\tilde{D}' + \iota^*(\tilde{B}) \equiv \tilde{B} + \pi^*(\varepsilon^*(\bar{x})).$$

The equality of divisors would imply $\tilde{B} \leq \pi^*(\varepsilon^*(\bar{x}))$. Since $\tilde{B}$ is π-simple, $Nm_\pi(\tilde{B}) = \varepsilon^*(\bar{x})$ and then

$$\tilde{D} - \iota^*(\tilde{D}) \equiv \tilde{B} - \iota^*(\tilde{B}) \in T'.$$

We suppose now that

$$2 \leq h^0(\tilde{B} + \pi^*(\varepsilon^*(\bar{x}))).$$

By applying (2.15)

$$2 \leq h^0(\varepsilon^*(\bar{x})) + h^0(Nm_\pi(\tilde{B}) + \varepsilon^*(\bar{x}) - \eta) = 1 + h^0(Nm_\pi(\tilde{B}) + \varepsilon^*(\bar{x}) - \eta).$$

On the other side $Nm_\pi(\tilde{B}) = Nm_\pi(\tilde{D}) - 2A = \varepsilon^*(\bar{r} + \bar{s}) - 2A$. So

$$0 < h^0(Nm_\pi(\tilde{B}) + \varepsilon^*(\bar{x}) - \eta) = h^0(\varepsilon^*(\bar{r} + \bar{s} + \bar{x}) - 2A - \eta).$$

Arguing as in (7.11) we get $\tau^*(\eta) = \eta$, which is a contradiction.

c) $\deg(A) = 0$. Then $\tilde{D}$ is π-simple. We go back to (7.13). If there is an equality, then $\tilde{D} = \pi^*(\varepsilon^*(\bar{x}))$ and one has a contradiction. Otherwise, by applying (2.15)

$$2 \le h^0(\tilde{D}) + \pi^*(\varepsilon^*(\bar{x}))) \le 1 + h^0(\varepsilon^*(\bar{x}) + Nm_\pi(\tilde{D})) - \eta).$$

Since $Nm_\pi(\iota^*(\tilde{D})) = \varepsilon^*(\bar{r} + \bar{s})$ one has $h^0(\varepsilon^*(\bar{x} + \bar{r} + \bar{s}) - \eta) > 0$. Again this implies $\tau^*(\eta) = \eta$, which is a contradiction. ∎

(7.14).- Note that the remark (2.12) makes it possible to compare the variety S' with the variety S which appears in the first part of this section. So $S' \cap 2S'$ is a symmetric irreducible curve of genus g. Since T' is also a curve we conclude that $S' \cap 2S'$ is an irreducible component of T'.

In order to study the curve T' we define T as the variety given by following pull-back diagram:

$$\begin{array}{ccc} T & \longrightarrow & \tilde{C}^{(2)} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi^{(2)}} \\ E & \xrightarrow{\varepsilon^*} & C^{(2)}. \end{array}$$

It is not hard to see that the morphism

$$\begin{array}{ccc} \tilde{C}^{(2)} & \longrightarrow & P \\ \tilde{D} & \longrightarrow & \tilde{D} - \iota^*(\tilde{D}) \end{array}$$

sends T birationally to T'.

We shall denote by j the involution of T induced by $\iota^{(2)}$. The above diagram can be factorized as follows:

$$\begin{array}{ccc} T & \longrightarrow & \tilde{C}^{(2)} \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle h_1} \\ T/j & \longrightarrow & \tilde{C}^{(2)}/\iota^{(2)} \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle h_2} \\ E & \xrightarrow{\varepsilon^*} & C^{(2)} \end{array}$$

where:

i) the maps f_1 and f_2 are double covers, f_1 is ramified at the points $\pi^*(P_i) \in T$, $i = 1, \ldots, 2g-2$, where P_i are the ramification points of ε and f_2 is unramified. In particular T/j is a smooth curve.

ii) Similarly h_1, h_2 are (2:1) morphisms, the ramification locus of h_1 is the diagonal $\Delta \subset \tilde{C}^{(2)}$ and h_1 is unramified. In particular $\tilde{C}^{(2)}/\iota^{(2)}$ is an smooth surface.

(7.16).- **Lemma.**-The morphism f_2 is the double cover associated to the class $\bar{\eta} = \mathrm{Nm}_{\bullet}(\eta) \in_2 JE - \{0\}$.

PROOF: We start by noting that $\bar{\eta} = 0$ implies $\eta + \tau^*(\eta) = 0$ and then $\tau^*(\eta) = \eta$. So $\bar{\eta} \neq 0$. Next, fixing suitable elements ζ and $\tilde{\zeta}$ of $C^{(2)}$ and $\tilde{C}^{(2)}$ one has a commutative diagram:

$$\begin{array}{ccc} \tilde{C}^{(2)} & \xrightarrow{-\tilde{\zeta}} & J\tilde{C} \\ {\scriptstyle \pi^{(2)}}\downarrow & & \downarrow{\scriptstyle Nm_*} \\ C^{(2)} & \xrightarrow{-\zeta} & JC. \end{array}$$

By applying the functor Pic^0 we obtain:

$$\begin{array}{ccc} \mathrm{Pic}^0(\tilde{C}^{(2)}) & \xleftarrow{t} & \mathrm{Pic}^0(J\tilde{C}) \\ {\scriptstyle \mathrm{Pic}^0(\pi^{(2)})}\uparrow & & \uparrow{\scriptstyle (Nm_*)^*} \\ \mathrm{Pic}^0(C^{(2)}) & \xleftarrow{t} & \mathrm{Pic}^0(JC) \end{array}$$

the horizontal arrows being isomorphisms. Since π^* and Nm_π are dual to each other (cf.[Mu1]) and $\mathrm{Ker}(\pi^*) = \{0, \eta\}$ we obtain that $t(\lambda_{JC}(\eta)) \in \mathrm{Ker}(\mathrm{Pic}^0(\pi^{(2)}))$. In this way we get the class defining h_2. Now restricting this class to E we end the proof by noting that to restrict this class to E is equivalent to taking the norm of η. ∎

(7.17).- **Proposition.** T is an irreducible smooth curve of genus g.

PROOF: By (7.16) T/j is an irreducible smooth elliptic curve. As we indicate in (7.14) T' contains an irreducible curve of genus g, namely $S' \cap 2S'$. Therefore T contains an irreducible component of geometric genus g. We call this component X. Since

$$f_{1|X} : X \longrightarrow T/j$$

ramifies in at most $2g-2$ points, by applying Riemann-Hurwitz to the normalization of X we conclude that $f_{1|X}$ ramifies in exactly $2g-2$ points. Thus X is smooth and $X = T$. ∎

(7.18).- **Corollary**. The equality $T' = S' \cap 2S'$ holds. Moreover T' is symmetric and the multiplication by -1 induces on T the involution j.

(7.19).- Comparing with the construction made in the first part of the section (from (7.1) to (7.6)) we note that (T, j) play the rôle of (C_2, τ_2). There we obtained a point of $_2(JC_2)$ which allowed us to reconstruct C_1.

By translating this to the present context we can conclude that there exists an intrinsical way to recognize a certain element of ${}_2JT$. Moreover this class appears in $\mathrm{Im}(f_1^*)$.

Our next aim is to compute this point in terms of the initial data. To do this we imitate the proof of (7.6).

Let $\gamma : T \longrightarrow P$ be the composition of the normalization map with the inclusion $T' \hookrightarrow P$. The induced map between JT and P factorizes through a morphism

$$\tilde{\gamma} : (\mathrm{Id} - j^*)(JT) = \mathrm{Ker}(Nm_{f_1}) \longrightarrow P.$$

We want to find the kernel of $\tilde{\gamma}$. Previously:

(7.20).- **Lemma.** Let $\check{\zeta} \in \mathrm{Pic}^2(\check{C})$. Consider the morphism

$$T \hookrightarrow \check{C}^{(2)} \xrightarrow{-\check{\zeta}} J\check{C}$$

an the induced morphism

$$\nu : JT \longrightarrow J\check{C}.$$

Then: $\mathrm{Im}(\nu_{|\mathrm{Ker}(Nm_{f_1})}) \subset P$ and the restriction

$$\tilde{\nu} : \mathrm{Ker}(Nm_{f_1}) \longrightarrow P$$

is $\tilde{\gamma}$.

PROOF: Straightforward. ∎

Now, since the unique non zero element of $\mathrm{Ker}(\tilde{\gamma})$ appears in $\mathrm{Im}(f_1^*)$ it suffices to study the Kernel of $\nu_{|\mathrm{Im}(f_1^*)}$.

An easy computation gives the following result:

(7.21).- **Lemma.** The following diagram commutes:

$$\begin{array}{ccc} JT & \xrightarrow{\nu} & J\check{C} \\ {\scriptstyle f^*=(f_2\circ f_1)^*}\uparrow & & \uparrow{\scriptstyle (\varepsilon\circ\pi)^*} \\ JE & \xrightarrow{\cdot 2} & JE. \end{array}$$

Corollary. $\mathrm{Ker}(\tilde{\gamma}) = f^*({}_2JE)$.

PROOF: From Lemma (7.20) we have $\mathrm{Ker}(\tilde{\gamma}) = \mathrm{Ker}\,(\tilde{\nu})$. By applying Lemma (7.21) one finds

$$\mathrm{Ker}(\nu\,|_{\mathrm{Im}(f^*)}) = f^*({}_2JE) \subset \mathrm{Ker}(Nm_{f_1}).$$

Since f_2^* is surjective $\mathrm{Im}((f_2 \circ f_1)^*) = \mathrm{Im}(f_1^*)$ and hence $\mathrm{Ker}\,(\tilde{\gamma}) = f^*({}_2JE)$. ∎

(7.22).-**Theorem.** Let $(\tilde{C},C),(\tilde{D},D) \in \mathcal{R}'_{B,g}$ such that $P(\tilde{C},C) \cong P(\tilde{D},D)$. Then $(\tilde{C},C) \cong (\tilde{D},D)$.

PROOF: It suffices to show that the initial data are determined by T, j and $f^*({}_2JE)$. Indeed the non-zero element of $f^*({}_2JE)$ gives a point of ${}_2J(T/j)$ that allows us to recover the morphism $T/j \longrightarrow E$.

Now consider the pull-back diagram

$$\begin{array}{ccc} X & \longrightarrow & T^{(2)} \\ \downarrow & & \downarrow \\ E & \xrightarrow{f_1^*} & (T/j)^{(2)}. \end{array}$$

Then, the morphism

$$\begin{array}{l} \tilde{C} \longrightarrow X \\ \tilde{x} \longrightarrow (\tilde{x} + \tilde{x}') + (\tilde{x} + \iota(\tilde{x}')) \end{array}$$

where $\pi(\tilde{x}) + \pi(\tilde{x}') \in \mathrm{Im}(\varepsilon^*)$, is an isomorphism and the involution $j^{(2)}$ of $T^{(2)}$ induces on $\tilde{C}$ the involution ι. ∎

Now we point out that the constructions used to prove Theorems (7.6) and (7.22) can be compared in order to ol a an injection from $\mathcal{R}'_{B,g}$ in $\mathcal{R}_{B,g,0}$ commuting with the Prym map. We explain how this map goes.

Let $(\tilde{C}',C') \in \mathcal{R}'_{B,g}$. Suppose that $\varepsilon' : C' \longrightarrow E'$ is a bi-elliptic structure of C'.

Construct the pull-back diagram

$$\begin{array}{ccc} T & \longrightarrow & \tilde{C}'^{(2)} \\ \downarrow & & \downarrow \\ E' & \xrightarrow{(\varepsilon')^*} & C'^{(2)} \end{array}$$

The involution $\iota^{(2)}$ restricts to an involution j of T. Then T/j is an elliptic curve. We call $\varepsilon_1 : E' \longrightarrow T/j$ to the transposed map. By taking again a pull-back diagram we get

$$\begin{array}{ccc} \tilde{C} & \longrightarrow & T \\ \downarrow & & \downarrow{\scriptstyle \varepsilon_2} \\ E' & \xrightarrow{\varepsilon_1} & T/j \end{array}$$

The curve $\tilde{C}$ has two involutions attached to the projections; call ι the composition of this involutions. Then $(\tilde{C}, \tilde{C}/\iota) \in \mathcal{R}_{B,g,0}$ is the image of $(\tilde{C}', C')$.

There is a natural way to inverting the injection above: Start with an element $(\tilde{C}, C) \in \mathcal{R}_{B,g,0}$. With the notations of §2, observe that $t = 0$ implies that C_1 is also elliptic. We call $f_1 : E \longrightarrow C_1$ to the transposed morphism. Then the pull-back diagram

$$\begin{array}{ccc} \tilde{C}' & \longrightarrow & C_2^{(2)} \\ \downarrow & & \downarrow{\scriptstyle \varepsilon_2^{(2)}} \\ C_1 & \xrightarrow{f_1^t} & E^{(2)} \end{array}$$

gives an element $(\tilde{C}', C') \in \bar{\mathcal{R}}_g$, where $C' = \tilde{C}'/\iota$, ι being the restriction to $\tilde{C}'$ of the involution $\iota^{(2)}$. In general this element belongs to $\mathcal{R}'_{B,g}$ and in this case its image by the injection given above is $(\tilde{C}, C)$. In any case $(\tilde{C}', C') \in \bar{\mathcal{R}}'_{B,g}$ and C' is a double covering of a smooth curve of genus 1.

Finally we get

(7.23).- **Theorem.** Let $(\tilde{C}, C) \in \mathcal{R}_{B,g,0} \cup \mathcal{R}'_{B,g}$ and let $(\tilde{D}, D) \in \mathcal{R}_g$ such that $P(\tilde{D}, D) \cong P(\tilde{C}, C)$. Then $(\tilde{D}, D) \in \mathcal{R}_{B,g,0} \cup \mathcal{R}'_g$ and $(\tilde{C}, C)$ and $(\tilde{D}, D)$ are tetragonally related (in the general sense explained in the Introduction).

PROOF: By arguing as in (5.9) one obtains that D is bi-elliptic. The table (2.7) implies that

$$(\tilde{D}, D) \in \mathcal{R}_{B,g,1} \cup \mathcal{R}_{B,g,0} \cup \mathcal{R}'_{B,g}.$$

By comparing (6.12) with (7.1) we exclude the first possibility. If $(\tilde{C}, C)$ and $(\tilde{D}, D)$ belong to the same component, then the statement is a consequence of (7.6) and (7.22). If they belong to different components, say $(\tilde{D}, D) \in \mathcal{R}'_{B,g}$ and $(\tilde{C}, C) \in \mathcal{R}_{B,g,0}$, then the injection given above gives an element $(\tilde{D}_0, D_0) \in \mathcal{R}_{B,g,0}$ with $P(\tilde{D}, D) \cong P(\tilde{D}_0, D_0)$. By (7.16) $(\tilde{C}, C) = (\tilde{D}_0, D_0)$ and by construction $(\tilde{C}, C)$ and $(\tilde{D}, D)$ are tetragonally related through an element of $\mathcal{H}_{g,0}$. ∎

Capítulo II. A construction of non-tetragonal type

For all this part we fix a generic element $(\check{C}, C)$ of $\mathcal{R}_{B,g,4}$ and a linear series g_2^1 on the elliptic curve E (we keep the notations of §§1 and 2). The first section (§8) is devoted to the description of four allowable covers constructed from this set of data. These covers belong to the fibre of $\bar{P}$ over $P(\check{C}, C)$. The proof of this fact is given in §10.

8. The construction.

We shall give the description of the attached coverings in three steps.

Step 1

The curve C_1 is bi-elliptic of genus 5. Since it is general it has a unique bi-elliptic structure. It is well known that (cf. [A-C-G-H], p.270, or remark ii in (3.6))

$$W_4^1(C_1) = \tilde{D}_1 \cup \varepsilon_1^*(\mathrm{Pic}^2(E))$$

where $\tilde{D}_1 = \{\zeta \in W_4^1(C_1)/\mathrm{Nm}_{\varepsilon_1}(\zeta) = \xi_1\}$ is a smooth curve of genus 7. The intersection

$$\tilde{D}_1 \cap \varepsilon_1^*(\mathrm{Pic}^2(E)) = \{\varepsilon_1^*(x+y)/x, y \in E \text{ and } 2x + 2y \equiv \xi_1\}$$

consists of four different points.

The variety $W_4^1(C_1)$ is invariant by the action $\zeta \to K_{C_1} - \zeta$ and, by passing to the quotient, we get an allowable double cover

(8.1) $$W_4^1(C_1) \longrightarrow D_1 \cup l$$

where D_1 is a smooth irreducible plane quartic and l is a line intersecting D_1 in four different points.

In fact the curve $D_1 \cup l$ is the discriminant curve of the system of quadrics containing the canonical image of C_1. Then, each point of $D_1 \cup l$ parametrices such a quadric of rank ≤ 4 (4 if the point does not belong to $D_1 \cap l$). The 2-planes contained in the quadric define two linear series g_4^1 on the curve (both linear series coincide if and only if the point belongs to $D_1 \cap l$). This is another description of the cover (8.1).

As a particular case of known results one has an isomorphism of principally polarized abelian varieties ([Ma],[Be3] and [K-K])

(8.2) $$JC_1 \cong P(W_4^1(C_1), D_1 \cup l).$$

(cf. (9.9) for a discusion of this isomorphism).

We will say that $(\tilde{D}_1, D_1)$ is "the cover associated to C_1".

Step 2

Let us consider the commutative pull-back diagram

(8.3)

$$\begin{array}{ccc} \tilde{D}_2 & \longrightarrow & C_2^{(2)} \\ \downarrow & & \downarrow \varepsilon_2^{(2)} \\ \mathbf{P}^1 & \xrightarrow{g_2^1} & E^{(2)}. \end{array}$$

The involution $\tau_2^{(2)}$ leaves invariant the curve $\tilde{D}_2$. Call D_2 the quotient curve. Then $\tilde{D}_2 \longrightarrow D_2$ is an allowable double covering except for additional ramification at four smooth points. Since C_2 is general, $\tilde{D}_2$ has, at most, one singularity (use Critetion (13.1)). The curve D_2 is hyperelliptic of genus $g-6$. For simplicity we will supose that the linear series g_2^1 is general. Then $\tilde{D}_2$ and D_2 are smooth (they are connected by (13.1)).

We will say that $(\tilde{D}_2, D_2)$ is "the cover associated to (C_2, g_2^1)".

Step 3

To construct an allowable cover $(\tilde{D}, D)$ from the pairs $(\tilde{D}_1, D_1)$ and $(\tilde{D}_2, D_2)$ we identify the ramification points of both covers (and the discriminant points correspondingly) in the following way:

Let $\eta_i \in \mathrm{Pic}^2(E)$, such that $2\eta_i \equiv \xi_1 \quad i = 1, \ldots, 4$. The classes $\varepsilon_1^*(\eta_i)$ correspond to the ramification points of $\tilde{D}_1 \longrightarrow D_1$. Note that

$$\{0, \eta_1 - \eta_2, \eta_1 - \eta_3, \eta_1 - \eta_4\} = {}_2JE.$$

On the other hand the ramification points of $\tilde{D}_2 \longrightarrow D_2$ are $\varepsilon_2^*(\bar{x}_i) \in C_1^{(2)}$ where $2\bar{x}_i \in g_2^1, \quad i = 1, \ldots, 4$. One has also

$$\{0, x_1 - x_2, x_1 - x_3, x_1 - x_4\} = {}_2JE.$$

(8.4).- Let σ be a bijection

$$\{\eta_i\}_{i=1,\ldots,4} \longrightarrow \{x_i\}_{i=1,\ldots,4}$$
$$\eta_i \longrightarrow \sigma(\eta_i)$$

such that $\eta_i - \eta_j$ and $\sigma(\eta_i) - \sigma(\bar{\eta}_j)$ coincide in ${}_2JE$. It is easy to see that four such bijections exist. We then identify $\varepsilon_1^*(\bar{\eta}_i)$ with $\varepsilon_2^*(\sigma(\bar{\eta}_i))$, $i = 1, \ldots, 4$, thus obtaining an allowable covering $(\tilde{D}, D)$. The corresponding covering map will be denoted by $p : \tilde{D} \longrightarrow D$. Moreover up to change in the indices of the x_i we may assume that $x_i = \sigma(\bar{\eta}_i)$, $i = 1, \ldots, 4$.

(8.5).-**Theorem.** There exist an isomorphism of principally polarized abelian varieties

$$P(\tilde{C}, C) \cong P(\tilde{D}, D).$$

The proof will be given in §10.

(8.6).- **Remark.** Observe that the curve D is neither tetragonal nor stable reduction of a tetragonal curve. Therefore $(\tilde{D}, D)$ and $(\tilde{C}, C)$ are not tetragonally related (cf. §12 for the definition of tetragonal relation).

9. The isogenies g_i and h_i.

In this section we keep the notations $p : \tilde{D} \longrightarrow D$, $(\tilde{D}_i, D_i)$, $i = 1,2$ to refer to the coverings constructed in §8. We put $p_i := p_{|\tilde{D}_i}$, $i = 1,2$.

For a line bundle $\tilde{L}$ on $\tilde{D}_i$ invariant by the covering involution we defined in §4 an element

$$v_i(\tilde{L}) \in \frac{(\mu_2)^4}{\mu_2}, \quad i = 1,2.$$

We shall take the ordering of the factors of $(\mu_2)^4$ for v_1 and v_2 compatible with the identifications made in Step 3 of §8.

The aim of this section is to prove the following technical result:

(9.1).- **Proposition.** There exist isogenies

$$g_i : P(\tilde{D}_i, D_i) \longrightarrow P(C_i, E) \qquad \text{and}$$
$$h_i : P(C_i, E) \longrightarrow P(\tilde{D}_i, D_i) \qquad \text{con } i = 1,2$$

verifying $h_i \circ g_i = 2$ and such that

i) $\mathrm{Ker}(g_i) = p_i^*({}_2JD_i)$

ii) $g_i({}_2P(\tilde{D}_i, D_i)) = \varepsilon_i^*({}_2JE)$

iii) $g_i^*(L_{P(C_i,E)}) \sim L^{\otimes 2}_{P(\tilde{D}_i,D_i)}$

iv) If $\tilde{\alpha}_i \in {}_2 P(\tilde{D}_i, D_i)$, then

$$v_1(\tilde{\alpha}_1) = v_2(\tilde{\alpha}_2) \text{ iff } \exists \tilde{\rho} \in {}_2 JE \text{ such that } g_i(\tilde{\alpha}_i) = \varepsilon_i^*(\tilde{\rho})$$

i') $\mathrm{Ker}(h_i) = \varepsilon_i^*({}_2JE)$

ii') $h_i({}_2P(C_i, E)) = p_i^*({}_2JD_i)$

iii') $h_i^*(L_{P(\tilde{D}_i,D_i)}) \sim L^{\otimes 2}_{P(C_i,E)}$

for $i = 1,2$, where $L_{P(\tilde{D}_i,D_i)}$ and $L_{P(C_i,E)}$ are the polarizations induced by the inclusions in the respective Jacobians.

PROOF: We first consider the case $i = 1$. The inclusion

$$\tilde{D}_1 \hookrightarrow \mathrm{Nm}_{\varepsilon_1}^{-1}(\xi_1) \cong P(C_1, E),$$

yields a morphism

$$g_1' : J\tilde{D}_1 \longrightarrow P(C_1, E).$$

We define $g_1 := (g_1')_{|P(\tilde{D}_1, D_1)}$.

It is convenient to describe the map g_1' explicitely. Let $\tilde{z} \in \tilde{D}_1$. We denote by $< \tilde{z} >$ the corresponding element of $\mathrm{Pic}^4(C_1)$. Then

$$g_1'(\sum_i n_i \tilde{z}_i) = \sum_i n_i < \tilde{z}_i >, \quad \text{where } \sum_i n_i = 0.$$

(9.2).- **Lemma.** One has $g_1'(p_1^*(JD_1)) = 0$. In particular $g_1(p_1^*({}_2JD_1)) = 0$.

PROOF: Let $\sum_i n_i z_i \in JD_1$ with $\sum_i n_i = 0$. Then

$$g_1'(p_1^*(\sum_i n_i z_1)) = g_1'(\sum_i n_i p^*(z_i)) = \sum_i n_i < p^*(z_i) >=$$

$$= \sum_i n_i \varepsilon_1^*(\bar{\xi}_1) = (\sum_i n_i) \cdot \varepsilon_1^*(\bar{\xi}_1) = 0 \blacksquare$$

On the other side in Proposition (4.7) of [C-G-T] the following result is proved: for a general bi-elliptic curve Γ the Jacobian $J\Gamma$ is isogenous to a product of an elliptic curve by a simple abelian variety A verifying $\mathrm{End}(A) \cong \mathbf{Z}$. Thus $g_1 \neq 0$ implies that g_1 is an isogeny. To study the behaviour of g_1 with respect to the points of order two we use the following result:

(9.3).- **Lemma.** One has the equality

$${}_2P(\tilde{D}_1, D_1) = p_1^*({}_2JD_1) \cup \{p_1^*(\gamma) - \tilde{z}_1 - \tilde{z}_2 \in P(\tilde{D}_1, D_1) / \gamma \in \mathrm{Pic}^1(D_1),$$
$$\tilde{z}_1, \tilde{z}_2 \in \tilde{D}_1 \text{ ramification points of } p_1\}$$

PROOF: Let $\tilde{\alpha} \in_2 P(\tilde{D}_1, D_1)$. Since it is invariant by the involution on $\tilde{D}_1$ we can apply Proposition (4.2). We get that there exists an effective divisor $\tilde{A}$ contained in the ramification divisor of $\mathbf{P}^1$ such that $\tilde{\alpha} + \tilde{A} \in p_1^*(\mathrm{Pic}(D_1))$. In particular $0 \leq \deg \tilde{A} \leq 4$ and $\deg \tilde{A}$ is even. Since the ramification divisor belongs to $p_1^*(\mathrm{Pic}(D_1))$, the cases $\deg \tilde{A} = 0, 4$ imply $\tilde{\alpha} \in p_1^*({}_2JD_1)$. When $\deg \tilde{A} = 2$ there exist two ramification points $\tilde{z}_1, \tilde{z}_2$ such that $\tilde{\alpha} + \tilde{z}_1 + \tilde{z}_2 \in p_1^*(\mathrm{Pic}^1(D_1))$ and we are done. $\blacksquare$

(9.4).- **Corollary.** Let $\tilde{z}_1, \tilde{z}_2$ be two ramification points of p_1 such that $< \tilde{z}_1 >= \varepsilon_1^*(\bar{\eta}_1)$, $< \tilde{z}_2 >= \varepsilon_1^*(\eta_2)$ and $p_1^*(\gamma) - \tilde{z}_1 - \tilde{z}_2 \in_2 P(\tilde{D}_1, D_1)$ for some $\gamma \in \mathrm{Pic}^1(D_1)$. Then

$$g_1(p_1^*(\gamma) - \tilde{z}_1 - \tilde{z}_2) = \varepsilon_1^*(\bar{\eta}_1 - \bar{\eta}_2).$$

PROOF: By using the explicit description of g_1' one has

$$g_1(p_1^*(\gamma) - \tilde{z}_1 - \tilde{z}_2) =< p_1^*(\gamma) > - \varepsilon_1^*(\bar{\eta}_1) - \varepsilon_1^*(\bar{\eta}_2) =$$

$$= \varepsilon_1^*(\bar{\xi}_1) - \varepsilon_1^*(\bar{\eta}_1) - \varepsilon_1^*(\bar{\eta}_2) = \varepsilon_1^*(\bar{\eta}_1 - \bar{\eta}_2). \blacksquare$$

Clearly this implies ii) of Proposition (9.1).

To prove i) we shall see that $\deg(g_1) = 2^6 (= \#_2 J\tilde{D}_1)$. This will be enough due to (9.2). To begin with:

(9.5).- **Lemma.** In $P(C_1, E)$ one has the equality of cohomology classes

$$[\tilde{D}_1] = \zeta_{P(C_1,E)}.$$

PROOF: One has an exact sequence

$$0 \longrightarrow {}_2JE \xrightarrow{(\varepsilon_1^*, 1)} P(C_1, E) \times JE \xrightarrow{\sigma} JC_1 \longrightarrow 0$$

$$(x, y) \longrightarrow x + \varepsilon_1^*(y)$$

and

$$\sigma^* \Theta_{C_1} \sim \Xi_{P(C_1,E)} \times JE + 2P(C_1, E) \times \{0\}$$

(cf. [Mu1],p.330). On the other hand the following equality holds in JC_1

$$[\tilde{D}_1 + E] = [W_4^1(C_1)] = 2\zeta_{C_1}$$

(cf. [A-C-G-H],p.320, Th.4.4). By applying σ^*:

$$4[\tilde{D}_1 \times \{0\}] + 4[\{0\} \times JE] = 2\sigma^*(\zeta_{C_1}) = \frac{2}{4!}\sigma^*([\Theta_{C_1}^4]) =$$

$$= \frac{2}{4!}[\Xi_{P(C_1,E)} \times JE + 2P(C_1, E) \times \{0\}]^4 =$$

$$= \frac{2}{4!}[\Xi_{P(C_1,E)}^4 \times JE] + \frac{2}{4!} \cdot 4 \cdot 2[\Xi_{P(C_1,E)}^3 \times \{0\}]$$

(in the last equality we use that $[P(C_1, E) \times \{0\}]^2 = 0$). Therefore $[\tilde{D}_1 \times \{0\}] = [\Xi_{P(C_1,E)}]^3/3! \times \{0\}$ and we are done. ∎

(9.6).- **Lemma.** The isogeny g_1 has degree 2^6.

PROOF: Taking quotient by a maximal isotropic subgroup of $H(L_{P(C_1,E)}) = \varepsilon_1^*({}_2JE)$ we get a isogeny of degree 2

$$c : P(C_1, E) \longrightarrow A$$

where A is a principally polarized abelian variety such that $c^*(L_A) \sim L_{P(C_1,E)}$. By the projection formula $c_*(\zeta_{P(C_1,E)}) = 2\zeta_A$. Thus (9.5) implies that $c_*(\tilde{D}_1)$ is twice the minimal class in A. Hence the principal polarization of $(J\tilde{D}_1)\hat{}$ induces on $\hat{A}$ twice the principal polarization, that is to say, there is a commutative diagram

(9.7)

$$\begin{array}{ccccc}
\hat{A} & \xrightarrow{\hat{c}} & P(C_1,E)\hat{} & \xrightarrow{(g_1')\hat{}} & (J\tilde{D}_1)\hat{} \\
{\scriptstyle 2\lambda_A^{-1}}\downarrow & & {\scriptstyle \mu}\downarrow & & {\scriptstyle \lambda_{J\tilde{D}_1}^{-1}}\downarrow \\
A & \xleftarrow{c} & P(C_1,E) & \xleftarrow{g_1'} & J\tilde{D}_1.
\end{array}$$

In particular

$$\deg(\mu) = \frac{\deg(2\lambda_A^{-1})}{\deg(c)^2} = \frac{2^{2\dim A}}{4} = 2^6.$$

On the other hand $g_1'(p_1^*(JD_1)) = 0$ with the fact of g_1 being an isogeny imply that $(\mathrm{Ker} g_1')^0 = p_1^*(JD_1)$. Now let us consider the diagram

(9.8)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & (\mathrm{Ker} g_1')^0 & \xrightarrow{\cong} & JD_1 & & & & \\
 & & \downarrow & & {\scriptstyle p_1^*}\downarrow & & & & \\
0 & \longrightarrow & \mathrm{Ker} g_1' & \longrightarrow & J\tilde{D}_1 & \xrightarrow{g_1'} & P(C_1,E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{Ker} g_1'/(\mathrm{Ker} g_1')^0 & \longrightarrow & J\tilde{D}_1/p_1^*(JD_1) & \xrightarrow{g_0} & P(C_1,E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Combining (9.7) and the dual diagram of (9.8) one gets a commutative diagram

$$\begin{array}{ccccc}
 & & P(C_1,E) & \xleftarrow{g_1'} & J\tilde{D}_1 \\
 & \nearrow^{\mu} & & \nearrow^{\cong} & \uparrow{\scriptstyle v} \\
P(C_1,E)^{\hat{}} & \xrightarrow{(g_1')^{\hat{}}} & J\tilde{D}_1^{\hat{}} & & P(\tilde{D}_1,D_1) \\
\| & & \uparrow & \nearrow^{\cong} & \\
P(C_1,E)^{\hat{}} & \xrightarrow{g_0} & (J\tilde{D}_1/p_1^*(JD_1))^{\hat{}} & &
\end{array}$$

where v is the inclusion map ($g_1 = g_1' \circ v$) and the commutative diagram

$$\begin{array}{ccc}
J\tilde{D}_1^{\hat{}} & \xrightarrow{\cong} & J\tilde{D}_1 \\
\uparrow & & \uparrow{\scriptstyle v} \\
(J\tilde{D}_1/p_1^*(JD_1))^{\hat{}} & \xrightarrow{\cong} & P(\tilde{D}_1,D_1)
\end{array}$$

is a consequence of the relation

$$(p_1^*)^{\hat{}} = \lambda_{D_1} \circ \mathrm{Nm}_{p_1} \circ \lambda_{\tilde{D}_1}^{-1}$$

(cf. [Mu1], p.328). Then

$$2^6 = \deg(\mu) = \deg(g_0) \cdot \deg(g_1).$$

By (9.2) we have $\deg(g_1) \geq 2^6$. Thus $\deg(g_0) = 1$ and $\deg(g_1) = 2^6$. This finishes the proof of Lemma (9.6) and hence of part i) of Proposition (9.1) . ∎

To prove iii) we use part i). One has $\mathrm{Ker} g_1 = H(L_{P(\tilde{D}_1,D_1)})$. Hence there exists an isomorphism of abelian varieties $\alpha : P(\tilde{D}_1,D_1)^{\hat{}} \longrightarrow P(\check{C}_1,C_1)$ such that $\alpha \circ \lambda_{P(\tilde{D}_1,D_1)} = g_1$.

From part ii) it then follows that

$$\alpha(\lambda_{P(\tilde{D}_1,D_1)}({}_2P(\tilde{D}_1,D_1))) = \varepsilon_1^*({}_2JE) = H(L_{P(C_1,E)}).$$

We then have the following diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & p_1^*({}_2JD_1) & \longrightarrow & {}_2P(\tilde{D}_1,D_1) & \rightarrow & H(L_{P(C_1,E)}) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & p_1^*({}_2JD_1) & \longrightarrow & P(\tilde{D}_1,D_1) & \xrightarrow{\alpha\circ\lambda_{P(\tilde{D}_1,D_1)}} & P(C_1,E) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \cdot 2} & & \downarrow{\scriptstyle \lambda_{P(C_1,E)}} & & \\
 & & & & P(\tilde{D}_1,D_1) & \underset{\cong}{\xrightarrow{\beta}} & P(C_1,E)^{\hat{}} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with β an isomorphism of abelian varieties, and

$$\hat{g}_1 \circ \lambda_{P(C_1,E)} \circ g_1 = \hat{\lambda}_{P(\tilde{D}_1,D_1)} \circ \hat{\alpha} \circ \lambda_{P(C_1,E)} \circ \alpha \circ \lambda_{P(\tilde{D}_1,D_1)} = 2\hat{\lambda}_{P(\tilde{D}_1,D_1)} \circ \hat{\alpha} \circ \beta.$$

Since $\mathrm{End}(P(C_1,E)) \cong \mathbf{Z}$, one has $\hat{\alpha} \circ \beta = Id$ and

$$g_1^*(L_{P(C_1,E)}) \sim L^{\otimes 2}_{P(\tilde{D}_1,D_1)},$$

so part iii) follows.

The isogeny $h_1 : P(C_1,E) \longrightarrow P(\tilde{D}_1,D_1)$ is defined by the condition $h_1 \circ g_1 = 2$. It is then easy to deduce i'), ii') and iii') from i), ii) and iii). All this for i=1, of course.

(9.9).- **Remark.** As announced in §8 we explain how the isomorphism (8.2) goes in this context. Consider the morphism

$$\varphi_0 : P(W_4^1(C_1), D_1 \cup l) \xrightarrow{\tilde{f}^*} P(\tilde{D}_1, D_1) \times JE \xrightarrow{(g_1, Id)} P(C_1,E) \times JE \xrightarrow{\bullet} JC_1,$$

where $\tilde{f} : \tilde{D}_1 \sqcup E \longrightarrow W_4^1(C_1)$ is the normalization map. Then one has:

Proposition([Be3],[K-K]). The morphism φ_0 verifies

i) $\mathrm{Ker}\varphi_0 =_2 P(W_4^1(C_1), D_1 \cup l)$

ii) $\varphi_0^*(\Theta_{C_1}) \sim 4\Xi_{P(W_4^1(C_1), D_1 \cup l)}$.

In particular there exists an isomorphism φ of pricipally polarized abelian varieties such that $\varphi_0 = 2\varphi$.

PROOF: The proof is left to the reader. ∎

Going back to the proof of (9.1), we consider now the case $i = 2$. The inclusion

$$\tilde{D}_2 \hookrightarrow C_2^{(2)}$$

gives a map $g_2' : J\tilde{D}_2 \longrightarrow P(C_2, E)$. That is

$$g_2'(\sum_i n_i \tilde{z}_i) = \sum_i n_i(z_{i,1} + z_{i,2}) \qquad (9.10)$$

where $\sum_i n_i = 0$ and $z_{i,1} + z_{i,2} \in C_2^{(2)}$ is the divisor corresponding to the point $\tilde{z}_i \in \tilde{D}_2$.

It is straightforward to check that

$$g_2'(p_1^*(JD_2)) = 0.$$

Let $g_2 = g'_{2|P(\tilde{D}_2,D_2)}$. As in the case $i = 1$, g_2 is an isogeny and $g_2(p_2^*({}_2JD_2))$=0.

We can reverse the construction of diagram (8.3): by using the linear series g_2^1 on D_2 given by the hyperelliptic structure and normalizing the curve obtained from the natural pull-back diagram we get

$$\begin{array}{ccc} C_2 & \longrightarrow & \tilde{D}_2^{(2)} \\ \downarrow & & \downarrow \\ \mathbf{P}^1 & \xrightarrow{g_2^1} & D_2^{(2)}. \end{array}$$

Moreover the involution of $\tilde{D}_2^{(2)}$ induces on C_2 an involution that coincides with τ_2. Imitating the construction of g_2 we get an isogeny

$$h_2 : P(C_2, E) \longrightarrow P(\tilde{D}_2, D_2)$$

verifying $h_2(\epsilon_2^*({}_2JE)) = 0$. By using the descriptions of g_2 and h_2 we obtain $h_2 \circ g_2 = 2$. Now parts i), ii), i') and ii') are obvious. Part iii) is (as in case i=1) a formal consequence of i) and ii) and the fact that End($P(C_2, E)$) = $\mathbf{Z}$. Now $h_2 \circ g_2 = 2$ and iii) give iii').

It remains only to prove iv). First of all we note that the Lemma (9.3) is still valid for ${}_2P(\tilde{D}_2, D_2)$. Let $\tilde{\alpha}_2 = p_2^*(\gamma) - \tilde{x}_i - \tilde{x}_j \in_2 P(\tilde{D}_2, D_2)$ with $\tilde{x}_i, \tilde{x}_j \in \tilde{D}_2$ ramification points given by the divisors $\epsilon_2^*(\bar{x}_i)$ and $\epsilon_2^*(\bar{x}_j)$ respectively. Then by using (9.10) one has

(9.11) $$g_2(\tilde{\alpha}_2) = \epsilon_2^*(\bar{x}_i - \bar{x}_j).$$

Let $\tilde{\alpha}_1 = p_1^*(\gamma') - \epsilon_1^*(\bar{\eta}_{i'}) - \epsilon_1^*(\bar{\eta}_{j'})$, $\gamma' \in \mathrm{Pic}^1(D_1)$. From (9.4) and (9.11) it follows

(9.12) $$\exists \bar{\rho} \text{ such that } g_1(\tilde{\alpha}_1) = \epsilon_1^*(\bar{\rho}) \text{ and } g_2(\tilde{\alpha}_2) = \epsilon_2^*(\bar{\rho}) \Leftrightarrow \bar{\eta}_{i'} - \bar{\eta}_{j'} = \bar{x}_i - \bar{x}_j.$$

Hence, by (8.4)

$$\exists \bar{\rho} \text{ such that } g_1(\tilde{\alpha}_1) = \epsilon_1^*(\bar{\rho}) \text{ and } g_2(\tilde{\alpha}_2) = \epsilon_2^*(\bar{\rho}) \Leftrightarrow$$
$$\Leftrightarrow \text{ either } \{i, j, i', j'\} = \{1, 2, 3, 4\} \text{ or } \{i, j\} = \{i', j'\} \text{ or } i = j \text{ and } i' = j'.$$

On the other hand $v_1(\tilde{\alpha}_1)$ (resp. $v_2(\tilde{\alpha}_2)$) gives -1 in the entries i' and j' (resp. i and j) when $i' \neq j'$ (resp. $i \neq j$). If $i = j$ (resp. $i' = j'$), then $v_1(\tilde{\alpha}_1) = \overline{(1,1,1,1)}$ (resp. $v_2(\tilde{\alpha}_2) = \overline{(1,1,1,1)}$). Since the classes of the vectors v_1 and v_2 are defined modulo a global sign we finally get

(9.13) $$\exists \bar{\rho} \text{ such that } g_1(\tilde{\alpha}_1) = \epsilon_1^*(\bar{\rho}) \text{ and } g_2(\tilde{\alpha}_2) = \epsilon_2^*(\bar{\rho}) \Leftrightarrow v_1(\tilde{\alpha}_1) = v_2(\tilde{\alpha}_2).$$

This ends the proof of Proposition (9.1). ∎

Observe that properties i), ii) iii), i'), ii') and iii') are independent for the cases i=1 and i=2. On the contrary (9.1.iv) depends on the way that $(\tilde{D}, D)$ has been constructed (cf. Step 3 of §8). In fact by combining (9.12) and (9.13) one finds:

(9.14).- **Remark.** Once a bijection σ

$$\{\bar{\eta}_i\}_{i=1,\ldots,4} \longrightarrow \{\bar{z}_i\}_{i=1,\ldots,4}$$
$$\bar{\eta}_i \longrightarrow \sigma(\eta_i)$$

(cf. §8 for definitions) is given, the following two facts are equivalent:

i) $\bar{\eta}_i - \bar{\eta}_j$ and $\sigma(\eta_i) - \sigma(\bar{\eta}_j)$ coincide in ${}_2JE$ for all $i, j = 1, \ldots, 4$

ii) For all $\tilde{\alpha}_1 \in_2 P(\tilde{D}_1, D_1)$ and $\tilde{\alpha}_2 \in_2 P(\tilde{D}_2, D_2)$:

$$v_1(\tilde{\alpha}_1) = v_2(\tilde{\alpha}_2) \text{ iff } \exists \bar{\rho} \in_2 JE \text{ such that } g_i(\tilde{\alpha}_i) = \varepsilon_i^*(\bar{\rho}), i = 1, 2.$$

In other words, the property we require in (8.4) and property (9.1.iv) are equivalent.

10. Proof of Theorem (8.5).

The isogenies h_1 and h_2 defined in §9 and the map φ given in (2.8) allow us to construct a commutative diagram:

(10.1)

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}_2JE & \longrightarrow & \epsilon_1^*({}_2JE)\times\epsilon_2^*({}_2JE) & \longrightarrow & \pi^*(\epsilon^*({}_2JE)) & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}_2JE & \longrightarrow & P(C_1,E)\times P(C_2,E) & \xrightarrow{\varphi} & P(\tilde{C},C) & \longrightarrow & 0\\
 & & & & {\scriptstyle h_1\times h_2}\downarrow & & {\scriptstyle r}\downarrow & & \\
 & & & & P(\tilde{D}_1,D_1)\times P(\tilde{D}_2,D_2) & = & P(\tilde{D}_1,D_1)\times P(\tilde{D}_2,D_2) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

In [De3], Debarre proves that $\varphi^*(L_{P(\tilde{C},C)}) \sim L_{P(C_1,E)} \boxtimes L_{P(C_2,E)}$. From (9.1.iii') one obtains that

$$\varphi^*(r^*(L_{P(\tilde{D}_1,D_1)} \boxtimes L_{P(\tilde{D}_2,D_2)})) \sim L^{\otimes 2}_{P(C_1,E)} \boxtimes L^{\otimes 2}_{P(C_2,E)} \sim \varphi^*(L_{P(\tilde{C},C)})^{\otimes 2}.$$

The map φ is an isogeny, this implies

(10.2) $$r^*(L_{P(\tilde{D}_1,D_1)} \boxtimes L_{P(\tilde{D}_2,D_2)}) \sim L^{\otimes 2}_{P(\tilde{C},C)}.$$

Let $\tilde{f} : \tilde{D}_1 \sqcup \tilde{D}_2 \longrightarrow \tilde{D}$ be the partial desingularization at $\tilde{D}_1 \cap \tilde{D}_2$. Consider the morphism

$$f' = \lambda^{-1}_{P(\tilde{D},D)} \circ \widehat{(\tilde{f}^*)} \circ \lambda_{P(\tilde{D}_1,D_1)\times P(\tilde{D}_2,D_2)} : P(\tilde{D}_1,D_1)\times P(\tilde{D}_2,D_2) \longrightarrow P(\tilde{D},D).$$

The following holds:

(10.3) $$(f')^*(L_{P(\tilde{D},D)}) \sim L^{\otimes 2}_{P(\tilde{D}_1,D_1)} \boxtimes L^{\otimes 2}_{P(\tilde{D}_2,D_2)}.$$

Indeed, by definition

(10.4)

$$\widehat{(f')} \circ \lambda_{P(\tilde{D},D)} \circ f' = \lambda_{P(\tilde{D}_1,D_1)\times P(\tilde{D}_2,D_2)} \circ \tilde{f}^* \circ \lambda^{-1}_{P(\tilde{D},D)} \circ \widehat{(\tilde{f}^*)} \circ \lambda_{P(\tilde{D}_1,D_1)\times P(\tilde{D}_2,D_2)}.$$

On the other hand the pull-back of the polarization of $P(\tilde{D}_1, D_1) \times P(\tilde{D}_2, D_2)$ induces on $P(\tilde{D}, D)$ twice the principal polarization (cf. [Be1]). That is:

$$(\tilde{f}^*)\hat{} \circ \lambda_{P(\tilde{D}_1,D_1)\times P(\tilde{D}_2,D_2)} \circ \tilde{f}^* = 2\lambda_{P(\tilde{D},D)}. \tag{10.5}$$

Hence by composing on the right hand side in (10.4) with $\tilde{f}^*$ we obtain

$$(f')\hat{} \circ \lambda_{P(\tilde{D},D)} \circ f' \circ \tilde{f}^* = 2\lambda_{P(\tilde{D}_1,D_1)\times P(\tilde{D}_2,D_2)} \circ \tilde{f}^*.$$

Since $\tilde{f}^*$ is surjective (10.3) is proved.

Now we define the isogeny

$$\Phi : P(\tilde{C}, C) \xrightarrow{r} P(\tilde{D}_1, D_1) \times P(\tilde{D}_2, D_2) \xrightarrow{f'} P(\tilde{D}, D).$$

Note that

$$\begin{aligned} \deg(\Phi) &= \deg(r) \cdot \deg(f') \\ &= \frac{\deg(h_1) \cdot \deg(h_2)}{4} \cdot \deg(\tilde{f}^*).\deg(\lambda_{P(\tilde{D}_1,D_1)\times P(\tilde{D}_2,D_2)}) \\ &= \frac{4 \cdot 4}{4} \cdot 4 \cdot 2^6 \cdot 2^{2(g-6)} = 2^{2g-2} = \#_2 P(\tilde{C}, C). \end{aligned} \tag{10.6}$$

On the other hand by combining (10.2) and (10.3) one has

$$\Phi^*(L_{P(\tilde{D},D)}) \sim L^{\otimes 4}_{P(\tilde{C},C)}. \tag{10.7}$$

Theorem (8.5) now follows in an obvious way from (10.7) and the next

(10.8).- **Lemma.** The following equality holds: $\mathrm{Ker}\Phi =_2 P(\tilde{C}, C)$.

PROOF: By (10.6) the statement can be written alternatively $r(_2P(\tilde{C}, C)) = \mathrm{Ker}(f')$. On the other hand, by definition

$$\mathrm{Ker}(f') = \mathrm{Ker}((\tilde{f}^*)\hat{} \circ \lambda_{P(\tilde{D}_1,D_1)\times P(\tilde{D}_2,D_2)}).$$

From this fact and (10.5) one gets

$$\mathrm{Ker}(f') = \tilde{f}^*(_2P(\tilde{D}, D)).$$

Consequenctly, to establish the Lemma is equivalent to seeing that

(10.9) $$r({}_2P(\check{C},C)) = \tilde{f}^*({}_2P(\tilde{D},D)).$$

A first step to show this is to "bound" both sets. One has

(10.10)
$$p_1^*({}_2JD_1) \times p_2^*({}_2JD_2) \subset r({}_2P(\check{C},C)) \subset_2 P(\tilde{D}_1,D_1) \times_2 P(\tilde{D}_2,D_2)$$

(10.11)
$$p_1^*({}_2JD_1) \times p_2^*({}_2JD_2) \subset \tilde{f}^*({}_2P(\tilde{D},D)) \subset_2 P(\tilde{D}_1,D_1) \times_2 P(\tilde{D}_2,D_2).$$

Indeed the right hand side inclusions are obvious. On the other side the definition of r (cf (10.1)) and (9.1.ii') give

$$r({}_2P(\check{C},C)) \supset (h_1 \times h_2)({}_2P(C_1,E) \times_2 P(C_2,E)) = p_1^*({}_2JD_1) \times p_2^*({}_2JD_2).$$

This proves (10.10). Moreover (10.5) implies that

$$\tilde{f}^*({}_2P(\tilde{D},D)) = \lambda^{-1}_{P(\tilde{D}_1,D_1)\times P(\tilde{D}_2,D_2)}(\mathrm{Ker}(\hat{f}^*)) \supset$$
$$\supset \mathrm{Ker}(\lambda_{P(\tilde{D}_1,D_1)\times P(\tilde{D}_2,D_2)}) = p_1^*({}_2JD_1) \times p_2^*({}_2JD_2),$$

which gives (10.11).

By combining (10.10) and (10.11) with (9.1.i) one obtains that (10.9) and the following equality are equivalent

(10.12) $$(g_1 \times g_2)(r({}_2P(\check{C},C))) = (g_1 \times g_2)(\tilde{f}^*({}_2P(\tilde{D},D))).$$

But from the definition of r (cf. (10.1)) it is easy to see that

$$r({}_2P(\check{C},C)) = (h_1 \times h_2)(\{(\tilde{\alpha},\tilde{\beta}) \mid \exists \tilde{\rho} \in_2 {}'E \text{ such that } 2\tilde{\alpha} = \epsilon_1^*(\tilde{\rho}), \quad 2\tilde{\beta} = \epsilon_2^*(\tilde{\rho})\}).$$

Thus, since $h_i \circ g_i = 2$, $i = 1,2$:

(10.13) $$(g_1 \times g_2)(r({}_2P(\check{C},C))) = \{(\epsilon_1^*(\tilde{\rho}),\epsilon_2^*(\tilde{\rho})) \mid \tilde{\rho} \in_2 JE\}.$$

If we show that

(10.14) $$\tilde{f}^*({}_2P(\tilde{D},D)) = \{(\tilde{\alpha},\tilde{\beta}) \in_2 P(\tilde{D}_1,D_1) \times_2 P(\tilde{D}_2,D_2) \mid v_1(\tilde{\alpha}) = v_2(\tilde{\beta})\},$$

(10.12) will follow from (9.1.iv) and (10.13).

We check equality (10.14). We first prove the inclusion of the left hand side member in the right hand side member. Let $(\tilde{\alpha},\tilde{\beta}) \in \tilde{f}^*({}_2P(\tilde{D},D))$. Denote by $L(\tilde{\alpha})$ and $L(\tilde{\beta})$ the corresponding line bundles on $\tilde{D}_1$ and $\tilde{D}_2$ respectively. Then there exists a line bundle $\tilde{L} \in P(\tilde{D},D)$ such that $\tilde{L}^{\otimes 2}$ is trivial and $\tilde{f}^*(\tilde{L}) = (L(\tilde{\alpha}), L(\tilde{\beta}))$. Let $\tilde{z} \in \tilde{D}_1 \cap \tilde{D}_2$. We call $\tilde{z}_1$ (resp. $\tilde{z}_2$) the point $\tilde{z}$ when viewed as a point of $\tilde{D}_1$ (resp. $\tilde{D}_2$). Taking pointwise fibres we obtain an isomorphism $\lambda : L(\tilde{\alpha})[\tilde{z}_1] \xrightarrow{\cong} L(\tilde{\beta})[\tilde{z}_2]$ as the composition of the natural identification $L(\tilde{\alpha})[\tilde{z}_1] \xrightarrow{\cong} L[\tilde{z}] \xleftarrow{\cong} L(\tilde{\beta})[\tilde{z}_2]$.

Since $\mathrm{Nm}_p(\tilde{L}) = 0$, $\tilde{L} \otimes \iota^*(\tilde{L})$ is trivial. So $\iota^*(\tilde{L}) \cong \tilde{L}^{-1} \cong \tilde{L}$. We choose an isomorphism $\varphi : \tilde{L} \longrightarrow \iota^*(\tilde{L})$ normalized in order to have $\iota^*(\varphi) \circ \varphi = Id$. The morphism φ induces by restriction

$$\varphi_1 : L(\tilde{\alpha}) \xrightarrow{\cong} \iota^*(L(\tilde{\alpha}))$$
$$\varphi_2 : L(\tilde{\beta}) \xrightarrow{\cong} \iota^*(L(\tilde{\beta})).$$

By construction one has a commutative diagram

$$\begin{array}{ccc}
L(\tilde{\alpha})[\tilde{z}_1] & \xrightarrow[\cong]{\lambda} & L(\tilde{\beta})[\tilde{z}_2] \\
\cong \big\downarrow \varphi_1[\tilde{z}_1] & & \cong \big\downarrow \varphi_2[\tilde{z}_2] \\
\iota^*(L(\tilde{\alpha}))[\tilde{z}_1] & & \iota^*(L(\tilde{\beta}))[\tilde{z}_2] \\
\| & & \| \\
L(\tilde{\alpha})[\tilde{z}_1] & \xrightarrow[\cong]{\lambda} & L(\tilde{\beta})[\tilde{z}_2].
\end{array}$$

Thus $v_1(L(\tilde{\alpha})) = v_2(L(\tilde{\beta}))$ (see §4 for the definition of v_1) and therefore

$$\tilde{f}^*(\tilde{L}) \in \{(\tilde{\alpha},\tilde{\beta}) \in_2 P(\tilde{D}_1,D_1) \times_2 P(\tilde{D}_2,D_2) \mid v_1(\tilde{\alpha}) = v_2(\tilde{\beta})\}.$$

Now, to obtain (10.14) we prove that both sets have the same cardinality. From (9.3) (applied to both $P(\tilde{D}_1,D_1)$ and $P(\tilde{D}_2,D_2)$) one gets

$$v_1({}_2P(\tilde{D}_1,D_1)) = v_2({}_2P(\tilde{D}_2,D_2))(= \{\overline{(\lambda_1,\ldots,\lambda_4)} \in (\mu_2)^4/\mu_2 \mid \prod_{i=1}^{4} \lambda_i = 1\}).$$

Since $\mathrm{Ker}(v_i)=p_i^*({}_2JD_i)$, i=1,2 (cf. (4.1)) we conclude

$$\#\{(\tilde{\alpha},\tilde{\beta}) \in_2 P(\tilde{D}_1,D_1) \times_2 P(\tilde{D}_2,D_2) \mid v_1(\tilde{\alpha}) = v_2(\tilde{\beta})\} = \#_2P(\tilde{D}_1,D_1) \cdot \#\mathrm{Ker}(v_2) =$$
$$= \frac{1}{4} \#_2P(\tilde{D}_1,D_1) \cdot \#_2P(\tilde{D}_2,D_2) = \#\tilde{f}^*({}_2P(\tilde{D},D)).$$

This finishes the proof of the Theorem (8.5). ∎

Capítulo 3. The fibre of $\bar{P}$ over a generic element of $P(\mathcal{R}_{B,g})$

This part is devoted to studying the fibre of the extended Prym map for generic elements of $\mathcal{R}_{B,g}$. The results we obtain are summarized in Theorem (13.1). Essentially we prove that the elements described in Part II yield the unique counterexemple to the extended tetragonal conjecture that exists generically in the bi-eliptic case.

Some results on special subvarieties of divisors for ramified double coverings appear in §11. In §12 we extend the tetragonal construction to allowable covers and we apply this construction to the coverings considered in our situation. In §13 we start the proof of Theorem (13.1), leaving three cases for sections 14,15 and 16. Finally in §17 we give a complete description of the fibre of $\bar{P}$ over $P(\tilde{C}, C)$ with $(\tilde{C}, C)$ a generic element of $\mathcal{R}_{B,g}$.

11. Special subvarieties of divisors for ramified double coverings.

In this section we shall collect various results that are required for our work. They are generalizations of known results (cf. [We3], [Be2]). The proofs are not given because they are similar to those of [We3].

Let N be a projective irreducible smooth curve of genus g and let $\pi : \tilde{N} \longrightarrow N$ be a double cover ramified at the points $\tilde{R}_1, \dots, \tilde{R}_{2n}$. Let Λ a linear system on N of degree d (not necessarily complete) of dimension ≥ 1. The special subvariety determined by Λ is, by definition, the variety X_Λ given by the following pull-back diagram:

$$\begin{array}{ccc} X_\Lambda & \xrightarrow{\ i\ } & \tilde{N}^{(d)} \\ {\scriptstyle \pi^{(d)}_{|X_\Lambda} =: \pi_\Lambda} \big\downarrow & & \big\downarrow {\scriptstyle \pi^{(d)}} \\ \Lambda & \xrightarrow{\ j\ } & N^{(d)}. \end{array}$$

(11.1).- **Proposition** (Connectedness criterion). If Λ is base-point-free, then X_Λ is connected.

(11.2).- **Proposition** (Irreducibility criterion). If Λ is base-point-free and the codimension of SingX_Λ in X_Λ is greater than or equal to 2, then X_Λ is irreducible.

(11.3).- **Proposition** (Smoothness criterion). Assume that Λ is complete and base-point-free. Let $D \in \Lambda$ and let $\tilde{D} \in X_\Lambda$ such that $\pi_\Lambda(\tilde{D}) = D$. Put

$$\tilde{D} = \pi^*(A) + \tilde{B} + \tilde{R}_{i_1} + \cdots + \tilde{R}_{i_k}, \quad i_j \neq i_{j'} \text{ if } j \neq j'$$

with $A, \tilde{B}$ effective and $\tilde{B}$ simple with respect to π and not containing ramification points. Then X_Λ is smooth at $\tilde{D}$ if and only if

$$h^0(D - A - \pi(\tilde{R}_{i_1}) - \cdots - \pi(\tilde{R}_{i_k})) = h^0(D) - \deg(A) - k$$

12. The generalised tetragonal construction.

In this section we give a natural way to extend the tetragonal construction (cf. [Do], [Be2]) to allowable double covers. We follow the idea suggested by Beauville in [Be2], Remarque 4, p.364. We do not need here the hypothesis of stability on the curves. We only give sketchs of the proofs.

Let $\pi : \tilde{D} \longrightarrow D$ an allowable double covering with $c_e(\tilde{D}, D) = 0$ and ι the exchange sheets involution on $\tilde{D}$. We say that D is tetragonal if it can be represented as a four-to-one cover of the projective line. We denote by $\mathrm{Div}^4(\tilde{D})$ and $\mathrm{Div}^4(D)$ the varieties which parametrize the effective Cartier divisors of degree d on $\tilde{D}$ and D respectively. Recall that the group of Cartier divisors on $\tilde{D}$ is:

$$\mathrm{Div}(\tilde{D}) = \bigoplus_{x \in \tilde{C}_{reg}} \mathbf{Z}x + \bigoplus_{s \text{ singular}} \tilde{K}_s^*/\mathcal{O}_s^*.$$

where $\tilde{K}$ is the ring of rational functions on $\tilde{D}$. Choosing uniformizing parameters t_1 and t_2 at the preimages $\tilde{s}_1$ and $\tilde{s}_2$ of a singular point $\tilde{s}$ in the normalization of $\tilde{D}$ we can write

$$\begin{array}{l} \tilde{K}_{\tilde{s}}^*/\mathcal{O}_{\tilde{s}}^* \xrightarrow{\cong} \mathbf{C}^* \times \mathbf{Z} \times \mathbf{Z} \\ \text{class of } h \longrightarrow ((h/t_1^{\nu_1(h)})(\tilde{s}_1)/(h/t_2^{\nu_2(h)})(\tilde{s}_2), \nu_1(h), \nu_2(h)) \end{array}$$

where ν_1 and ν_2 are the corresponding valuations on $\tilde{K}$.

The four-to-one covering $\gamma : D \longrightarrow \mathbf{P}^1$ induces an inclusion $\mathbf{P}^1 \xrightarrow{\gamma^*} \mathrm{Div}^4(D)$. On the other hand there exists a well-defined norm map ([Be1],p.158):

$$\mathrm{Nm}_\pi : \mathrm{Div}^4(\tilde{D}) \longrightarrow \mathrm{Div}^4(D).$$

Let us construct the pull-back diagram

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \mathrm{Div}^4(\tilde{D}) \\ \downarrow & & \downarrow \mathrm{Nm}_\pi \\ \mathbf{P}^1 & \xrightarrow{\gamma^*} & \mathrm{Div}^4(D). \end{array}$$

By choosing an element in $\tilde{X}$ one gets morphisms $\tilde{\varphi}$ and φ making commutative the diagram:

$$\begin{array}{ccc} \mathrm{Div}^4(\tilde{D}) & \xrightarrow{\tilde{\varphi}} & J\tilde{D} \\ \mathrm{Nm}_\pi \downarrow & & \mathrm{Nm}_\pi \downarrow \\ \mathrm{Div}^4(D) & \xrightarrow{\varphi} & JD \end{array}$$

and such that $\mathrm{Nm}_\pi(\tilde\wp(\tilde X)) = 0$. Since $\mathrm{Ker}(\mathrm{Nm}_\pi)$ has two connected components, the variety $\tilde X$ is the disjoint union of two varieties $\tilde X_1$ and $\tilde X_2$. Two points $a, a' \in \tilde X$ over the same point of $\mathbf{P}^1$ belong to the same half if and only if the class of the divisor $a - a'$ can be written as $(\iota - 1)^*(\alpha_0)$ with $\alpha_0 \in \mathrm{Pic}(\tilde D)$ and $\deg(\alpha_0)$ even. In particular $\tilde X_i, i = 1,2$ are invariant by the action of $\iota^{(4)}$. Moreover $\tilde X_i$, i=1,2 are connected. This can be proven by imitating the proof of (8.8) in [We3]. Let X_i be the quotient of $\tilde X_i, i = 1,2$ by this action. Notice that X_1 and X_2 are tetragonal.

Lemma. The singularities of $\tilde X_i$ are ordinary and the covers $\tilde X_i \longrightarrow X_i, i = 1,2$ are allowable with $c_e = 0$.

SKETCH OF PROOF: Let $\tilde N_i$, N_i, $i = 1, \ldots, r$ the normalizations of the components of $\tilde D$ and D respectively. Let $\tilde Y$ be the variety obtained by the pull-back diagram

$$\begin{array}{ccc} \tilde Y & \longrightarrow & \tilde N_1^{(d_1)} \times \cdots \times \tilde N_r^{(d_r)} \\ \big\downarrow & & \big\downarrow \\ \mathbf{P}^1 & \longrightarrow & N_1^{(d_1)} \times \cdots \times N_r^{(d_r)} \end{array}$$

where the horizontal arrow below comes from the natural four-to-one map $N_1 \sqcup \cdots \sqcup N_r \longrightarrow \mathbf{P}^1$ (so $d_i = [N_i : \mathbf{P}^1]$ and $d_1 + \cdots + d_r = 4$). There is a birational morphism from $\tilde X$ to $\tilde Y$ sending a divisor $\sum_{\tilde x \text{ regular}} n_{\tilde x}\tilde x + \sum_{\tilde s \text{ singular}} (c_{\tilde s}, m_{\tilde s}, n_{\tilde s})$ to $\sum_{\tilde x \text{ regular}} n_{\tilde x}\tilde x + \sum_{\tilde s \text{ singular}} m_{\tilde s}\tilde s_1 + n_{\tilde s}\tilde s_2$. We call $\tilde Y_1$ and $\tilde Y_2$ the images of $\tilde X_1$ and $\tilde X_2$.

When r=1, criterion (11.3) gives the singularities of $\tilde Y$. Then, computations of local monodromy (cf. [D-S], p.48 for similar arguments) provide the singularities of $\tilde Y_1$ and $\tilde Y_2$. Since the birational maps $\tilde X_i \longrightarrow \tilde Y_i$, i=1,2, can be described explicitly the statement of the lemma follows easily.

When $r \geq 2$ one combine the study of the projections of $\tilde Y_i$ with (11.3) to carry out a case-by-case analysis of the singularities of $\tilde Y_1$ and $\tilde Y_2$. One obtains the proof of the lemma as above. ∎

Notice that in the case $r \geq 2$ there appear new involutions on $\tilde Y$, for instance $\iota_1^{(d_1)} \times \cdots \times 1^{(d_i)} \times \cdots \times \iota_r^{(d_r)}$ restricts to $\tilde Y$. When d_i is odd the involution exchanges the halves and one gets an isomorphism between $\tilde Y_1$ and $\tilde Y_2$.

Let $(\tilde D_0, D_0) \in \bar{\mathcal{R}}_g$ be the covering obtained by stable reduction of the pair $(\tilde D, D)$. We shall say that $(\tilde X_i, X_i)$, i=1,2 are the result of the tetragonal construction applied to $(\tilde D, D)$ (or $(\tilde D_0, D_0)$). We extend the definition to the stable reduction $(\tilde X_i', X_i')$ of the pairs $(\tilde X_i, X_i)$, i=1,2. In this way we restrict the tetragonal construction to the context of the stable curves. We shall also say that two pairs are tetragonally related when one is obtained from the other by succesive applications of the tetragonal construction.

(12.1).- Proposition. The following properties hold:

i) The tetragonal construction applied to $(\tilde{X}_1, X_1)$ (resp. $(\tilde{X}_2, X_2)$) with its inhereted tetragonal structure yields $(\tilde{X}_2, X_2)$ (resp. $(\tilde{X}_1, X_1)$) and $(\tilde{D}, D)$.

ii) $P(\tilde{X}_1, X_1) \cong P(\tilde{X}_2, X_2) \cong P(\tilde{D}, D)$.

SKETCH OF PROOF: i) Take j=1. We call $\hat{S} \subset \mathrm{Div}^4(\tilde{X}_1)$ the variety of divisors determined by the morphism $X_1 \longrightarrow \mathbf{P}^1$. Let $\tilde{x} \in \tilde{D}$ sufficiently general. There exist eight points in $\tilde{X}$ containing $\tilde{x}$, four in $\tilde{X}_1$. One gets an element in $\hat{S}$ attached to $\tilde{x}$. The study of the special points of $\tilde{D}$ leads to a case-by-case description. For example: assume $\tilde{x} \in \tilde{D}_{reg}$ with $\pi(\tilde{x}) = x$ and the Cartier divisor belonging to $\mathbf{P}^1 \subset \mathrm{Div}^4(\tilde{X}_1)$ and containing x is of the form $x + y + (r, 1, 1)_s$ where s is a singular point of C with preimage $\tilde{s}$. If $\tilde{x} + \tilde{y} + (\sqrt{r}, 1, 1)_{\tilde{s}} \in \tilde{X}_1$, then the other points of $\tilde{X}_1$ in the fibre are

$$\iota(\tilde{x}) + \iota(\tilde{y}) + (\sqrt{r}, 1, 1)_{\tilde{s}}$$
$$\tilde{x} + \iota(\tilde{y}) + (-\sqrt{r}, 1, 1)_{\tilde{s}}$$
$$\iota(\tilde{x}) + \tilde{y} + (-\sqrt{r}, 1, 1)_{\tilde{s}}$$

(recall that $\mathrm{Nm}_\pi((c, m, n)_{\tilde{s}}) = ((-1)^{m+n}c^2, m, n)_{\pi(\tilde{s})}$ and that $\iota^*((c, m, n)_{\tilde{s}}) = ((-1)^{m+n}c, m, n)_{\tilde{s}}$, cf.[Be1], p.158). Since they are not fixed points for $\iota^{(4)}$ they are all regular in $\tilde{X}_1$ and its images $[\tilde{x} + \tilde{y} + (\sqrt{r}, 1, 1)]$, $[\tilde{x} + \iota(\tilde{y}) + (-\sqrt{r}, 1, 1)_{\tilde{s}}] \in X_1$ are also regular. Therefore $2[\tilde{x} + \tilde{y} + (\sqrt{r}, 1, 1)] + 2[\tilde{x} + \iota(\tilde{y}) + (-\sqrt{r}, 1, 1)_{\tilde{s}}]$ is an element of $\mathrm{Div}^4(X_1)$ in the linear series given by $X_1 \longrightarrow \mathbf{P}^1$. Hence $2(\tilde{x} + \tilde{y} + (\sqrt{r}, 1, 1)_{\tilde{s}}) + 2(\tilde{x} + \iota(\tilde{y}) + (-\sqrt{r}, 1, 1))$ is a point in $\hat{S}$. We leave to the reader the remaining possibilities. The case of $\tilde{X}_2$ is similar.

ii) According to [Be2],p.364 the cohomology class of $\varphi(\tilde{X}_1)$ in $P(\tilde{C}, C)$ is twice the minimal class. The universal property of Prym varieties (cf. [Ma], [K-K]) gives the isomorphisms. ∎

Remark.- To simplify we have supposed:

a) D is four-to-one cover of $\mathbf{P}^1$.

b) $c_e(\tilde{D}, D) = 0$.

One can define the generalized tetragonal construction without these restrictions (although it is not needed here), that is to say: for all allowable double covers of tetragonal curves, where tetragonal curve means that there exists a line bundle L of degree four on the curve such that $h^0(L) = 2$, it has a finite base locus B and $L(-B)$ defines a finite morphism from the curve to $\mathbf{P}^1$. The definition of the construction extends verbatim and (12.1) holds. Now, in some cases, the halves $\tilde{X}_i$,i=1,2 are not connected. When B above is a point we recover the construction of Recillas (cf [Re] or [Be2]).

(12.2).- We want to apply the tetragonal construction to the elements $(\tilde{D}, D) \in \mathcal{H}'_{g,t}$, with $t \geq 1$ verifying

- $D = D_1 \cup_4 D_2$, where D_1 and D_2 are irreducible.
- There exists $\gamma : D \longrightarrow \mathbf{P}^1$ such that $\gamma_{|D_1}$ and $\gamma_{|D_2}$ have degree 2. In particular D_1 and D_2 are hyperelliptic.

Observe that the second condition is vacuous when t=1.

We keep the notations $\mathrm{Div}^4(\tilde{D})$, $\mathrm{Div}^4(D)$, $\tilde{N}_1$, $\tilde{N}_2$, N_1, N_2, $\tilde{X}$, $\tilde{X}_1$, $\tilde{X}_2$, $\tilde{Y}_1$ and $\tilde{Y}_2$. The involutions $1^{(2)} \times \iota^{(2)}$ ans $\iota^{(2)} \times 1^{(2)}$ act both on each component $\tilde{Y}_i$ and also on the varieties $\tilde{X}_i$. Then, easy local computations show that these involutions exchange the branches at the singular points of $\tilde{X}_i$, i=1,2. Hence the curves $C_1 := \tilde{X}_i/1^{(2)} \times \iota^{(2)}$ and $C_2 := \tilde{X}_i/\iota^{(2)} \times 1^{(2)}$, i=1,2 are smooth. On the other hand, let $\tilde{B}_i$, i=1,2 be the curves given by the pull-back diagram

$$\begin{array}{ccc} \tilde{B}_i & \longrightarrow & \tilde{N}_i^{(2)} \\ \downarrow & & \downarrow \\ \mathbf{P}^1 & \longrightarrow & N_i^{(2)}. \end{array}$$

We then have that C_1 (resp. C_2) is the normalization of $\tilde{B}_1$ (resp. $\tilde{B}_2$). One obtains easily that C_1 and C_2 are bi-elliptic and $g(C_1) = t+1$, $g(C_2) = g - t$. As a consequence one gets that both covers $(\tilde{X}_i, X_i)$, i=1,2 belong to $\mathcal{R}_{B,g,t}$. Moreover the curves X_1 and X_2 can be represented as double coverings of a smooth curve of genus 1 (with the notations of (13.3) this means in particular that $(\tilde{X}_i, X_i) \in \mathcal{R}'_{B,g,t}$, i=1,2).

(12.3).- Next we indicate how to apply the tetragonal construction to a covering $(\tilde{D}, D) \in \mathcal{H}'_{g,0}$ such that D is obtained from an irreducible hyperelliptic curve H by identifying two non-hyperelliptic pairs of points x_1, x_2 and y_1, y_2. The curve D is tetragonal in two different ways:

a) The curve D is the stable reduction of the curve $D' = \mathbf{P}^1 \cup H \cup \mathbf{P}^1$ where H intersects the first copy of $\mathbf{P}^1$ in two points: x_1 and x_2, the second copy in the points y_1 and y_2 and the two $\mathbf{P}^1$ are disjoint. The curve D' is clearly tetragonal. Applying the tetragonal construction we obtain a single cover (with the notations of the lemma above, this case corresponds to $r = 3, d_1 = 1, d_2 = 2$, and $d_3 = 1$, and since at least one of the d_i is odd the two halves are isomorphic). Imitating the arguments of (12.2) one shows that it belongs to $\mathcal{R}_{B,g,0}$ (observe that in this case C_1 and C_2 smooth implies that the coverings $(\tilde{X}_i, X_i)$ are smooth).

b) Let $\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2 \in \mathbf{P}^1$ be the images of x_1, x_2, y_1, y_2 by the hyperelliptic morphism. There is a unique double covering $\mathbf{P}^1 \xrightarrow{(2:1)} \mathbf{P}^1$ sending each pair $\bar{x}_1, \bar{x}_2$ and $\bar{y}_1, \bar{y}_2$ to a single point. The four-to-one covering $H \longrightarrow \mathbf{P}^1$ obtained by composing the hyperelliptic map with the (2:1) morphism above factorizes through D. In this case the tetragonal construction give two covers: one in $\mathcal{H}'_{g,0}$ and the other in $\bar{\mathcal{R}}'_{B,g}$ (compare with (2.10)) (in fact, with the notations of (13.3), this second element belongs to $\mathcal{R}''_{B,g}$). This is a consequence of case a) above and the map from $\mathcal{R}_{B,g,0}$ to $\bar{\mathcal{R}}'_{B,g}$ explained in §7.