UNIVERSITAT DE BARCELONA

VARIEDADES DE PRYM DE CURVAS BIELIPTICAS

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13. The Main Theorem.

In this section we state the central Theorem of Part III and we reduce the proof to three cases (cf. (13.14)).

(13.1).- **Theorem.** Let $(\tilde{C},C)$ be a generic element of $\mathcal{R}_{B,g}$ and let $(\tilde{D},D) \in \bar{\mathcal{R}}_g$ such that $P(\tilde{C},C) \cong P(\tilde{D},D)$. Then one (and only one) of the following two facts occurs:

i) $(\tilde{C},C)$ and $(\tilde{D},D)$ are tetragonally related.

ii) $(\tilde{C},C) \in \mathcal{R}_{E,g,4}$ and $(\tilde{D},D)$ is obtained from $(\tilde{C},C)$ as in the construction of §8.

The aim of this section is to prove Proposition (13.14) which is a first step in the proof of the theorem.

Let $(\tilde{C},C)$ be a generic element of $\mathcal{R}_{B,g}$. Let $(\tilde{D},D) \in \bar{\mathcal{R}}_g$ be such that $P(\tilde{D},D) \cong P(\tilde{C},C)$. In particular the theta divisor of $P(\tilde{D},D)$ is singular in codimension 3 and $P(\tilde{D},D)$ is not the Jacobian of a curve (cf. [Sh1] and (3.2), (3.3)). Then, [Be1], Th. 5.4 implies that $c_e(\tilde{D},D) = 0$. On the other hand in Th. (4.10) of loc. cit. there is a list of the coverings with $c_e = 0$ and dimension of the singular locus of the theta divisor equal to g-5. Since $P(\tilde{C},C)$ is not a Jacobian and $g \geq 10$ this list becomes shorter: one has that the pair $(\tilde{D},D)$ verifies at least one of the following possibilities:

(13.2)

a) D is a double cover of a stable curve of genus 1,

b) $(\tilde{D},D) \in \mathcal{H}'_{g,0}$,

c) $(\tilde{D},D) \in \mathcal{H}'_{g,1}$,

d) $(\tilde{D},D) \in \mathcal{H}'_{g,t}$ where $2 \leq t \leq [\frac{g-1}{2}]$

(cf. (2.10) for definitions).

(13.3).- **Remark.** We shall use the notations

$$\mathcal{R}'_{B,g,t} = \{(\tilde{\Gamma},\Gamma) \in \bar{\mathcal{R}}_{B,g,t} \mid \Gamma \text{ verifies } (13.2.a)\}, \qquad t = 0,\dots,[\tfrac{g-1}{2}]$$
$$\mathcal{R}''_{B,g} = (\mathcal{R}'_{B,g})' = \{(\tilde{\Gamma},\Gamma) \in \bar{\mathcal{R}}'_{B,g} \mid \Gamma \text{ verifies } (13.2.a)\}.$$

The spaces $\mathcal{H}'_{g,t}$, $\mathcal{R}'_{B,g,t}$ for $t = 0,\dots,[\frac{g-1}{2}]$ and $\mathcal{R}''_{B,g}$ are not closed in $\bar{\mathcal{R}}_g$. In fact all the inclusions

$$\mathcal{H}_{g,t} \subset \mathcal{H}'_{g,t} \subset \bar{\mathcal{H}}_{g,t} \text{ for } t = 0,\dots,[\tfrac{g-1}{2}]$$
$$\mathcal{R}_{B,g,t} \subset \mathcal{R}'_{B,g,t} \subset \bar{\mathcal{R}}_{B,g,t} \text{ for } t = 0,\dots,[\tfrac{g-1}{2}]$$
$$\mathcal{R}'_{B,g} \subset \mathcal{R}''_{B,g} \subset \bar{\mathcal{R}}'_{B,g}$$

are strict (cf. (2.10) for the definitions of $\mathcal{H}_{g,t}$ and $\mathcal{H}'_{g,t}$). In all three cases the first space is open dense in the third space. Recall that the respective dimensions have been given in (2.2) and (2.10).

We first treat the possibility (13.2.b).

(13.4).- **Proposition.** Let $(\tilde{C},C)$ be a generic element of $\mathcal{R}_{B,g}$. Let $(\tilde{D},D)\in\mathcal{H}'_{g,0}$ be such that $P(\tilde{D},D)\cong P(\tilde{C},C)$. Then $(\tilde{C},C)\in\mathcal{R}_{B,g,0}\cup\mathcal{R}'_{B,g}$, and $(\tilde{C},C)$ and $(\tilde{D},D)$ are tetragonally related.

PROOF: Let H be a hyperelliptic curve such that D is constructed from H by identifying two pairs of points. If any of the pairs is hyperelliptic, then D is obtained from a hyperelliptic curve by identifying a pair of points. By (4.10) in [Bel] $P(\tilde{D},D)$ is a Jacobian and we get a contradiction. Assume first that H is irreducible. By (12.3), the tetragonal construction gives a cover $(\tilde{C}',C')\in\mathcal{R}_{B,g,0}$ tetragonally related with $(\tilde{D},D)$. Then by (7.23) and (7.6) either $(\tilde{C}',C')=(C,C)$ or $(\tilde{C},C)$ is tetragonally related with $(\tilde{C}',C')$ (and hence with $(\tilde{D},D)$). Now we want to show that the genericity of $(\tilde{C},C)$ is enough to obviate other possible cases. To see this, we prove previously a Lemma.

(13.5).- **Lemma.** The subspace

$$\{(\tilde{D}_0,D_0)\in\mathcal{H}'_{g,0}\mid D_0\text{ is reducible}\}$$

has codimension ≥ 3 in $\mathcal{H}'_{g,0}$.

PROOF: Let $(\tilde{D}_0,D_0)$ be a generic element of an irreducible component of the set of the statement. Let H_0 be a hyperelliptic curve such that D_0 is constructed from H_0 by identifying two pairs of points. By hypothesis H_0 is reducible, hence it is obtained by identifying two copies of $\mathbf{P}^1$ along g-1 points. The points in the second copy are not arbitrary. Therefore the component has big codimension and we are done. ∎

(13.6).- Now we end the proof of Proposition (13.4). Assume that the curve H is reducible. According to (13.5)

$$\dim P(\{(\tilde{D},D)\in\mathcal{H}'_{g,0}\mid D\text{ is reducible}\})\leq 2g-4.$$

Since $\dim P(\mathcal{R}_{B,g,t})=2g-3$ when $t\geq 1$, and $(\tilde{C},C)$ is general we get $(\tilde{C},C)\notin\mathcal{R}_{B,g,t}$ for $t\geq 1$. Analogously one has $\dim P(\mathcal{R}_{B,g,0})=\dim P(\mathcal{R}'_{B,g})=2g-2$. Hence $(\tilde{C},C)\notin\mathcal{R}_{B,g,0}\cup\mathcal{R}'_{B,g}$ and we get a contradiction. ∎

The following two facts will be very useful in the rest of the paper.

(13.7).- **Lemma.** Let $(\tilde{C},C)$ be a general element of $\mathcal{R}_{B,g,t}$ with $t\geq 1$. Then $P(\tilde{C},C)$ is isogenous to a product of two simple abelian varieties of dimensions t and g-t-1. If $(\tilde{C},C)$ is a generic element of $\mathcal{R}_{B,g,0}\cup\mathcal{R}'_{B,g}$, then $P(\tilde{C},C)$ is simple.

Proof: By (2.8) and (2.11) all we have to prove is simplicity. This is a consequence of Proposition (4.7) in [C-G-T] where the following result is proved: let Γ be a generic bi-elliptic curve, then $J\Gamma$ is isogenous to a product of a elliptic curve by a simple abelian variety A verifying End$(A)\cong \mathbb{Z}$. ∎

(13.8).- **Corollary.** Let $(\tilde{C}, C)$ be a generic element of $\mathcal{R}_{B,g}$ and let $(\tilde{D}, D) \in \mathcal{H}'_{g,t}$ with $t \geq 1$ such that $P(\tilde{C}, C) \cong P(\tilde{D}, D)$. We write $D = D_1 \cup_4 D_2$ where $g(D_1) = t-1$ and $g(D_2) = g - t - 2$. Then:

a) the curves D_1 and D_2 are irreducible,

b) $(\tilde{C}, C) \in \mathcal{R}_{B,g,t}$.

Proof: Recall that partial normalization at $\tilde{D}_1 \cap \tilde{D}_2$ gives an isogeny

$$P(\tilde{D}, D) \longrightarrow P(\tilde{D}_1, D_1) \times P(\tilde{D}_2, D_2).$$

Suppose, for instance, that D_1 is reducible. Then, normalization at the intersection of its components, gives an isogeny between $P(\tilde{D}_1, D_1)$ and a product of at least two non-trivial abelian varieties. This contradicts (11.8) and hence a) is proved. Moreover the dimensions of the abelian varieties that appear in the product above is an invariant of $P(\tilde{D}, D) \cong P(\tilde{C}, C)$. Thus the dimensions of $P(\tilde{D}_1, D_1)$ and $P(\tilde{D}_2, D_2)$ coincide with the dimensions of $P(C_1, E)$ and $P(C_2, E)$. This implies b). ∎

Next we consider the case (13.2.c).

(13.9).- **Proposition.** Let $(\tilde{C}, C)$ be a generic element of $\mathcal{R}_{B,g}$ and let $(\tilde{D}, D) \in \mathcal{H}'_{g,1}$ be such that $P(\tilde{D}, D) \cong P(\tilde{C}, C)$. Then $(\tilde{C}, C) \in \mathcal{R}_{B,g,1}$. Moreover either $(\tilde{D}, D)$ is tetragonally related with $(\tilde{C}, C)$ or is tetragonally related with an element of $\mathcal{R}'_{B,g,1}$ (i.e.: verifying (13.2.a)).

Proof: The first statement has been proved in (13.8.b). To see the second claim we write $D = \mathbf{P}^1 \cup_4 D_2$ where D_2 is a hyperelliptic curve (cf. (2.10)). By (13.8.a) D_2 is irreducible. Then, (12.2) says that there exists an element $(\tilde{C}', C') \in \mathcal{R}'_{B,g,1}$ tetragonally related with $(\tilde{D}, D)$. If this element belongs to $\mathcal{R}_{B,g,1}$ then Theorem (6.24) shows that $(\tilde{C}, C)$ and $(\tilde{D}, D)$ are tetragonally related. ∎

(13.10).- Summarizing: given $(\tilde{C}, C)$ and $(\tilde{D}, D)$ as in the statement of the Theorem (13.1), the second pair verifies at least one of the four conditions in (13.2). When condition b) holds then the theorem is true. On the other hand, if c) holds either the theorem is verified or we are lead to the case (13.2.a). We will consider the case a) in §14. Now we want to start the study of case d). This study will be completed in §§15 and 16.

(13.11).- Remark. Let $(\check{C},C)$ be a general element of $\mathcal{R}_{B,g}$ and

$$(13.12) \qquad (\tilde{D},D) \in \mathcal{H}'_{g,t} - (\mathcal{H}'_{g,0} \cup \mathcal{H}'_{g,1} \cup (\bigcup_{s=0}^{[\frac{t-1}{2}]} \mathcal{R}'_{B,g,s}) \cup \mathcal{R}''_{B,g}) \text{ with } t \geq 2,$$

(cf. (13.3) for definitions) such that $P(\check{C},C) \cong P(\tilde{D},D)$. By (13.8.b) $(\check{C},C) \in \mathcal{R}_{B,g,t}$. We shall write $D = D_1 \cup_4 D_2$. By (13.8.a) D_1 and D_2 are both irreducible. Recall that $P(\check{C},C)$ is not a Jacobian and that $g \geq 10$. All these properties make it possible to use the next Proposition, which is a particular case of (5.12) in [Sh2]:

(13.13).- Proposition. Let $q : \tilde{\Gamma} \longrightarrow \Gamma$ be an element of $\tilde{\mathcal{R}}_g$ such that $\dim \mathrm{Sing}\Xi = g-5$, $g \geq 10$, $P(\tilde{\Gamma},\Gamma)$ is not a Jacobian and Γ is either irreducible or has two irreducible components intersecting in ,at least, four points. Let X be an irreducible component of $\mathrm{Sing}\Xi$ such that $\dim X = g-5$. Then we are in one of the cases a), b), c), d), e) below and X, thought in the natural model Ξ^*, is contained in the respective varieties Z_a, Z_b, Z_c Z_d, or Z_e (cf §1 for definitions):

a) Γ is obtained by identifying two pairs of points on a curve H. There exists a morphism $\gamma : H \longrightarrow \mathbf{P}^1$ of degree 2 over the generic point of $\mathbf{P}^1$. Let

$$\begin{array}{ccc} \tilde{H} & \xrightarrow{\tilde{h}} & \tilde{\Gamma} \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q} \\ H & \longrightarrow & \Gamma \end{array}$$

be the partial desingularizations. Then

$$Z_a = \text{closure of } \{\tilde{L} \in P(\tilde{\Gamma},\Gamma)^* \mid \tilde{h}^0(\tilde{L}) = q_1^*(\gamma^*(\mathcal{O}_{\mathbf{P}^1}(1)))(\tilde{A})$$
$$\text{where } \tilde{A} \text{ is an effective divisor with non singular support}\}$$

b) $\tilde{\Gamma} = \tilde{\Gamma}_1 \cup_4 \tilde{\Gamma}_2$ and $\Gamma = \Gamma_1 \cup_4 \Gamma_2$. If $\tilde{f}$ is the partial desingularization of $\tilde{\Gamma}$ at $\tilde{\Gamma}_1 \cap \tilde{\Gamma}_2$, then

$$Z_b = (\tilde{f}^0)^{-1}(\Xi_1^* \times \Xi_2^*).$$

In this case the codimensions of Ξ_i^* in $P(\tilde{\Gamma}_i,\Gamma_i)^*$, $i = 1,2$ are exactly 2, that is to say $\dim Z_b = g-5$.

c) $\tilde{\Gamma} = \tilde{\Gamma}_1 \cup_4 \tilde{\Gamma}_2$, $\Gamma = \Gamma_1 \cup_4 \Gamma_2$ and, say, Γ_1 is hyperelliptic with γ the attached (2:1) map. If $\tilde{f}$ is the partial desingularization of $\tilde{\Gamma}$ at $\tilde{\Gamma}_1 \cap \tilde{\Gamma}_2$, then

$$Z_c = (\tilde{f}^0)^{-1}(ex_1^* \times P(\tilde{\Gamma}_2,\Gamma_2)^*),$$

where

$$ex_1^* = \text{ closure of } \{q^*(\gamma^*(\mathcal{O}_{\mathbb{P}^1}(1)))^*(\tilde{A}) \in P(\tilde{\Gamma}_1,\Gamma_1) \mid \text{ where } \tilde{A} \text{ is an effective divisor with non singular support}\}.$$

d) $\tilde{\Gamma} = \tilde{\Gamma}_1 \cup_4 \tilde{\Gamma}_2$, $\Gamma = \Gamma_1 \cup_4 \Gamma_2$ and Γ_1 is a plane quartic. Writing $\Gamma_1 \cap \Gamma_2 = \{x_1 + \cdots + x_4\}$, it is $\mathcal{O}_{\Gamma_1}(x_1 + \cdots + x_4) = \omega_{\Gamma_1}$. One has

$$Z_d = \text{ closure of } \{\tilde{L} = q^*(M)(\tilde{A}) \in P(\tilde{\Gamma},\Gamma)^* \mid \tilde{A} \text{ is an effective divisor with non singular support and } M \in \mathrm{Pic}^4(\Gamma) \text{ with } h^0(M) \geq 2 \text{ and } M_{|\Gamma_1} = \omega_{\Gamma_1}\}.$$

e) There exists a morphism $\varepsilon : \Gamma \longrightarrow E_0$ onto a curve E_0 consisting of at most two irreducible components; the genus of E_0 is equal to 1 and the morphism ε has degree 2 over the generic points of E_0. We will not need the description of Z_e.

We shall call in each case Z_a^m, Z_b^m, Z_c^m, Z_d^m, $(\Xi^*)^m$ and $(ex^*)^m$ the union of the components of maximal dimension.

(13.14).- **Proposition.** Let $(\tilde{C},C)$ be a generic element of $\mathcal{R}_{B,g}$ and let $(\tilde{D},D) \in \tilde{\mathcal{R}}_g$ such that $P(\tilde{C},C) \cong P(\tilde{D},D)$. Then $(\tilde{C},C)$ and $(\tilde{D},D)$ are tetragonally related or at least one of the following facts occurs:

a)

$$(\tilde{D},D) \in \bigcup_{t=0}^{[\frac{g-1}{2}]} (\mathcal{R}'_{B,g,t}) \cup \mathcal{R}''_{B,g}$$

(i.e. $(\tilde{D},D)$ verifies (13.2.a); cf. (13.3)).

b) $\tilde{D} = \tilde{D}_1 \cup_4 \tilde{D}_2$, $D = D_1 \cup_4 D_2$ and D_1 is an irreducible plane quartic. Writing $D_1 \cap D_2 = \{x_1 + \cdots + x_4\}$, it is $\mathcal{O}_{D_1}(x_1 + \cdots + x_4) = \omega_{D_1}$. The curve D_2 is irreducible and hyperelliptic of genus $g-5$. In this case $(\tilde{C},C) \in \mathcal{R}_{B,g,4}$ and the isomorphism $P(\tilde{D},D) \cong P(\tilde{C},C)$ identifies Z_b^m with W_0, Z_c^m with W_2 and $Z_d^m = Z_d$ with W_{-2} (see (13.13) above for notations).

c) $\tilde{D} = \tilde{D}_1 \cup_4 \tilde{D}_2$ and $D = D_1 \cup_4 D_2$ with D_1, D_2 irreducible hyperelliptic curves of genus $t-1$ and $g-t-2$ respectively, with $t \geq 2$. In particular $(\tilde{D},D) \in \mathcal{H}'_{g,t}$. In this case $(\tilde{C},C) \in \mathcal{R}_{B,g,t}$ and with the notations of (13.13), the isomorphism $P(\tilde{D},D) \cong P(\tilde{C},C)$ identifies Z_b^m with W_0 and the two varieties of type Z_c^m corresponding to the two hyperelliptic components with W_2 and W_{-2} (one of them is empty exactly when $W_{-2} = \emptyset$).

PROOF: We already mentioned that, with this hypothesis, $(\tilde{D}, D)$ verifies at least one of the four cases in (13.2). On the other hand two of these cases have been treated in (13.4) and (13.9). Since (13.2.a) coincides with a) above it only remains to prove:

(13.15).- If in addition $(\tilde{D}, D)$ belongs to

$$\mathcal{H}'_{g,t} - (\mathcal{H}'_{g,0} \cup \mathcal{H}'_{g,1} \cup (\bigcup_{s=0}^{[\frac{t-1}{2}]} \mathcal{R}'_{B,g,s}) \cup \mathcal{R}''_{B,g}) \text{ with } t \geq 2,$$

then either (13.14.b) or (13.14.c) holds. We keep this hypothesis on $(\tilde{D}, D)$ in the rest of the proof. In particular we can write $D = D_1 \cup_4 D_2$ with D_1 and D_2 irreducible (cf. (13.8.b)). Moreover $(\check{C}, C)$ belongs to $\mathcal{R}_{B,g,t}$ (cf. (13.8.a)). By applying (13.13) we find the possible descriptions of the components of maximal dimension of $\mathrm{Sing}\Xi^*$. Recall that $\dim Z_b^m = g - 5$.

(13.16).- **Lemma.** Let $(\check{C}, C)$ and $(\tilde{D}, D)$ be as above. Then Z_b^m is irreducible and via the isomorphism $P(\tilde{D}, D) \cong P(\check{C}, C)$ it corresponds to the component W_0 of $\mathrm{Sing}\Xi^*$ (cf. (2.7) and (13.13) for definitions and notations).

PROOF: Indeed, let X_1 and X_2 be components of $(\Xi_1^*)^m$ and $(\Xi_2^*)^m$ respectively. Then $(\tilde{f}^0)^{-1}(X_1 \times X_2)$ is irreducible: suppose not, then different components of $\mathrm{Sing}\Xi^*$ of dimension $g - 5$ are exchanged by translations. From the definitions of W_i, i=-2,0,2 (cf. (2.6), (3.7)) it is easy to check this is not possible in $P(\check{C}, C)$ and we get a contradiction.

On the other hand

$$\tilde{f}^*(I((\tilde{f}^0)^{-1}(X_1 \times X_2))) = I(X_1) \times I(X_2).$$

By (13.7) $P(\tilde{D}_1, D_1)$ and $P(\tilde{D}_2, D_2)$ are simple. Thus, for i=1,2 either $I(X_i)$ is finite or $I(X_i) = P(\tilde{D}_i, D_i)$. Let $\tilde{L}_i$ be a generic element of X_i, i=1,2. Then $h^0(\tilde{L}_i) = 1$ (recall that $\mathrm{codim}_{P(\tilde{D}_i, D_i)} X_i = 2$). Now (cf. e.g. (3.14) of [Sh2]) $h^0(\tilde{L}_i(\tilde{x}_i - \iota'(\tilde{x}_i))) = 0$, where $\tilde{x}_i$ is a generic point in $\tilde{D}_i$ and ι' is the natural involution. Therefore $\tilde{x}_i - \iota'(\tilde{x}_i) \notin I(X_i)$. We conclude that $I(X_1)$, $I(X_2)$ and $I((\tilde{f}^0)^{-1}(X_1 \times X_2))$ are finite. Hence $(\tilde{f}^0)^{-1}(X_1 \times X_2)$ is an irreducible component of $\mathrm{Sing}\Xi^*$ invariant only by a finite group. Only the component W_0 verifies this property (cf. (5.12)), therefore $X_i = (\Xi_i^*)^m$, for i=1,2 and Z_b^m is an irreducible component of $\mathrm{Sing}\Xi^*$ corresponding to W_0. ∎

In the situation of (13.16), $\deg(\tilde{f}^*) = 4$ (cf. [Be1], (3.6)), thus from the proof of (13.16) one also obtains that $I((\Xi_i^*)^m) = 0$, i=1,2 and $I(Z_b^m) = \ker \tilde{f}^*$.

(13.17).- **Lemma.** Assume that one of the components of D, say D_1, is hyperelliptic and that $\dim Z_c = g - 5$ (cf. (13.13)). Then the corresponding variety Z_c^m is irreducible.

Proof: Arguing as in Lemma (13.16), if X is a component of $(ex_1^*)^m$, then $(\tilde{f}^0)^{-1}(X \times P(\tilde{D}_2, D_2)^*)$ is irreducible. Suppose that Y is another component of $(ex_1^*)^m$. Since Z_b^m is non empty and corresponds to W_0, then the isomorphism $P(\tilde{D}, D) \cong P(\tilde{C}, C)$ sends $(\tilde{f}^0)^{-1}(X \times P(\tilde{D}_2, D_2)^*) \cup (\tilde{f}^0)^{-1}(Y \times P(\tilde{D}_2, D_2))$ to $W_{-2} \cup W_2$. On the other hand

$$\tilde{f}^*(I((\tilde{f}^0)^{-1}(X \times P(\tilde{D}_2, D_2)^*))) \cap I((\tilde{f}^0)^{-1}(Y \times P(\tilde{D}_2, D_2)^*))) \supset \{0\} \times P(\tilde{D}_2, D_2).$$

Hence we get a contradiction because

$$I(W_2) \cap I(W_{-2}) \text{ is finite.}$$

Therefore $(ex_1^*)^m$ and Z_c^m are irreducibles. ∎

(13.18).- **Lemma.** With our hypothesis (cf. (13.15)), if $(\tilde{D}, D)$ verifies also the assumptions of (13.13.a), then $\dim Z_a^m < g - 5$.

Proof: The unique configuration of the type of (13.13.a) compatible with $D = D_1 \cup_4 D_2$, D_1 and D_2 irreducible, and $(\tilde{D}, D) \notin \mathcal{H}'_{g,0}$ is the following one:

Normalizing D at two points of $D_1 \cap D_2$ we obtain a curve H admiting a (2:1) map $\gamma : H \longrightarrow \mathbf{P}^1$ which is constant on one of the curves, say D_2.

Assume that $\dim Z_a^m = g - 5$. We call $\tilde{H}$ the curve obtained by normalizing $\tilde{D}$ at the two points corresponding to the above ones, and we write q_1 for the double cover $\tilde{H} \longrightarrow H$. Let $\tilde{d}_1, \tilde{d}_2 \in \tilde{H}$ be the preimages of the remaining points in $\tilde{D}_1 \cap \tilde{D}_2$. Let $\tilde{g}$ the partial desingularization of $\tilde{H}$ in $\tilde{d}_1, \tilde{d}_2$. One has the isogenies (cf. §1)

$$P(\tilde{D}, D)^* \xrightarrow{\tilde{h}^0} P(\tilde{H}, H)^* \xrightarrow{\tilde{g}^0} P(\tilde{D}_1, D_1)^* \times P(\tilde{D}_2, D_2)^*$$

where $\tilde{h}$ is the desingularization of $\tilde{D}$ at $\tilde{D}_1 \cap \tilde{D}_2$. Let $\tilde{L}$ be a general element of Z_a, then $\tilde{h}^0(\tilde{L}) = q_1^*(\gamma^*(\mathcal{O}_{\mathbf{P}^1}(1)))(\tilde{A})$, with $\tilde{A}$ an effective divisor with non singular support. Thus

$$\begin{aligned}
\tilde{g}^0(\tilde{h}^0(\tilde{L})) &= \tilde{g}^0(q_1^*(\gamma^*(\mathcal{O}_{\mathbf{P}^1}(1)))(\tilde{A})) = \\
&= (q_1^*(\gamma^*(\mathcal{O}_{\mathbf{P}^1}(1)))(\tilde{A})_{|\tilde{D}_1}(-\tilde{d}_1 - \tilde{d}_2), q_1^*(\gamma^*(\mathcal{O}_{\mathbf{P}^1}(1)))(\tilde{A})_{|\tilde{D}_2}(-\tilde{d}_1 - \tilde{d}_2)) = \\
&= (\mathcal{O}_{\tilde{D}_1}(2\tilde{d}_1 + 2\tilde{d}_2)(\tilde{A}_1)(-\tilde{d}_1 - \tilde{d}_2), \mathcal{O}_{\tilde{D}_2}(-\tilde{d}_1 - \tilde{d}_2)(\tilde{A}_2)) = \\
&= (\mathcal{O}_{\tilde{D}_1}(\tilde{d}_1 + \tilde{d}_2)(\tilde{A}_1), \mathcal{O}_{\tilde{D}_2}(-\tilde{d}_1 - \tilde{d}_2)(\tilde{A}_2)),
\end{aligned}$$

where $\mathcal{O}_{\tilde{D}}(\tilde{A})_{|\tilde{D}_i} = \mathcal{O}_{\tilde{D}_i}(\tilde{A}_i)$, i=1,2. Hence:

$$\tilde{g}^0\tilde{h}^0(Z_a) \subset \{\tilde{L}_1 \in \Xi_1^* \mid h^0(\tilde{L}_1(-\tilde{d}_1 - \tilde{d}_2)) > 0\} \times \{\tilde{L}_2 \in P(\tilde{D}_2, D_2)^* \mid h^0(\tilde{L}_2(\tilde{d}_1 + \tilde{d}_2)) > 0\}.$$

It is easy to check that the dimensions of the sets on the right hand side are less than or equal to (a posteriori equal to) $\dim P(\tilde{D}_1, D_1)$-3 and $\dim P(\tilde{D}_2, D_2)$-1 respectively. Therefore, if X is a component of Z_a^m, there exist irreducible components X_1 and X_2 of the sets on the right hand side such that $\tilde{g}^0(\tilde{h}^0(X)) = X_1 \times X_2$. Arguing as in Lemma (13.16), we find that the elements of the form $\tilde{x} - \iota'(\tilde{x})$ do not belong to $I(X_i)$ if $\tilde{x}$ is general in $\tilde{D}$ and ι' is the involution. Therefore the simplicity of $P(\tilde{D}_i, D_i)$ (cf. (11.8)) implies that $I(X_i)$ are finite for i=1,2. In particular $I(X)$ is finite. Hence X corresponds to W_0 by the isomorphism $P(\tilde{D}, D) \cong P(\check{C}, C)$. Since the components Z_a^m and Z_b^m are different (take $f = g \circ h$ and compare $\tilde{f}^0(Z_a)$ computed above with $\tilde{f}^0(Z_b) = \Xi_1^* \times \Xi_2^*$) one gets a contradiction with (13.16). ∎

(13.19).- **Lemma.** Keeping our assumptions (cf.(13.15)), suppose that $(\tilde{D}, D)$ verifies (13.13.d) and that $\dim Z_d = g - 5$. Then Z_d is irreducible (in particular $Z_d = Z_d^m$).

Proof: Writing $\tilde{f}$ for the partial normalization of $\tilde{D}$ at $\tilde{D}_1 \cap \tilde{D}_2$ one easily checks that

$$\tilde{f}^0(Z_d) \subset \{\tilde{l}\} \times P(\tilde{D}_2, D_2)^*$$

where $\tilde{l}$ is the ramification divisor of $\tilde{D}_1 \longrightarrow D_1$. Since $(\tilde{f}^0)^{-1}(\{\tilde{l}\} \times P(\tilde{D}_2, D_2)^*)$ is irreducible and has dimension $g - 5$ the result follows. ∎

Now we end the proof of Proposition (13.14). Since the element $(\tilde{D}, D)$ verifies the hypothesis given in (13.15) we can apply (13.13) in order to recognize the components of maximal dimension in $\mathrm{Sing}\Xi^*$. By (13.16) the component W_0 corresponds to Z_b^m. Since $t \geq 2$ other components of maximal dimension exist (cf.(2.7)). According to (16.18) case (13.13.a) does not provide any component. Let us consider case e). One obtains that the only configuration of type (13.13.e) compatible with (13.15) is the following one:

D_1, D_2 are two hyperelliptic curves and $D_1 \cap D_2$ consists of two pairs of hyperelliptic points for both curves.

This kind of elements parametrize a subspace of $\mathcal{R}_g$ of dimension 2g-4. Therefore $P(\tilde{D}, D) \cong P(\check{C}, C)$ contradicts the genericity of $(\check{C}, C)$ (see (13.6) for a similar argument).

We conclude that the components W_{-2} (if non empty) and W_2 come from the cases (13.13.c) and (13.13.d). By (13.17) and (13.19), the components of type Z_c^m appear twice when $t \neq 4$ and (13.14.c) is verified. Moreover when $t = 4$ we are lead to the possibilities b) and c) of the statement. ∎

(13.20).- In the rest of Part III we shall prove the following results:

- If $(\tilde{D}, D)$ verifies (13.14.a), then $(\tilde{D}, D)$ is tetragonally related with $(\check{C}, C)$ (§14).

• If $(\tilde{D}, D)$ verifies (13.14.b), then $(\tilde{D}, D)$ is constructed from $(\tilde{C}, C)$ as in Part II (§15).
• If $(\tilde{D}, D)$ verifies (13.14.c), then $(\tilde{D}, D)$ is tetragonally related with $(\tilde{C}, C)$ (§16).
Clearly (13.14) plus these three facts imply Theorem (13.1).

14. The case (13.14.a).

The aim of this section is to prove the following result:

(14.1).- **Proposition.** Let $(\tilde{C}, C)$ be a general element of $\mathcal{R}_{B,g}$ and let $(\tilde{D}, D) \in \bar{\mathcal{R}}_g$ be such that D is a double cover of a stable curve E_0 of genus 1 and $P(\tilde{D}, D) \cong P(\tilde{C}, C)$. Then $(\tilde{C}, C)$ and $(\tilde{D}, D)$ are tetragonally related.

(14.2).- **Remark.** Notice that (14.1) finishes the proof of Theorem (13.1) in case (13.14.a) (or alternatively (13.2.a)).

PROOF: If D is smooth, then the statement is a consequence of the results of Part I. Assume that D is singular. Observe that a stable curve of genus 1 is irreducible with, at most, one double point.

We first prove that D is irreducible: suppose not, then D consists in the union of two curves of genus ≤ 1 intersecting in, at most, g+1 points. This kind of elements parametrizes a subspace of high codimension (greater than 2) in $\bar{\mathcal{R}}_{B,g}$. On the other hand the dimension of the generic fibre of $P_{|\mathcal{R}_{B,g,t}}$ is 1 if t$\geq$ 1 and 0 if t=0 (cf. the summary in §17). Hence

$$\dim P(\mathcal{R}_{B,g,t}) = \begin{cases} 2g-2 & \text{for } t \geq 1 \\ 2g-3 & \text{for } t = 0 \end{cases}$$

and the genericity of $(\tilde{C}, C)$ allows to avoid this possibility (cf. (13.6) for a similar argument). In fact by the same reason D is supposed to have either one singularity or two singularities with image a singularity of E_0. In the second case the element $(\tilde{D}, D)$ belongs to $\mathcal{H}'_{g,0}$ and by (13.4) the statement follows. In the rest of the proof we assume that D has one singularity.

If E_0 is singular then D is obtained by identifying a pair of points in a hyperelliptice curve. By [Be1], (4.10) this implies that $P(\tilde{D}, D)$ is the Jacobian of a curve and we get a contradiction with [Sh1]. Hence E_0 is smooth.

We treat first the case $\mathrm{Gal}_{E_0}(\tilde{D}) \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. There exist two involutions ι'_1 and ι'_2 on $\tilde{D}$ lifting the involution on D. By construction, ι'_1 and ι'_2 exchange the branches of the singularity of $\tilde{D}$. Then one obtains the following commutative diagram:

$$\begin{tikzcd} & \tilde{D} \arrow[dl] \arrow[d] \arrow[dr] & \\ D \arrow[dr] & D_1 \arrow[d] & D_2 \arrow[dl] \\ & E_0 & \end{tikzcd}$$

where $D_i := \tilde{D}/\iota'_i$, i=1,2 are smooth curves and the discriminant divisors of $D_1 \longrightarrow E$ and $D_2 \longrightarrow E_0$ intersect in a point (in particular $t \geq 1$). By (2.10) this element is obtained by applying the tetragonal construction to an element of $\mathcal{R}_{B,g,t}$ for some t. By the results of Part I $(\tilde{D},D)$ and $(\tilde{C},C)$ are tetragonally related.

Finally assume that $\mathrm{Gal}_E(\tilde{D}) \cong \mathbb{Z}/2\mathbb{Z}$. Then $(\tilde{D},D) \in \mathcal{R}''_{B,g}$ (cf. (13.3)). The statement is a consequence of the following Lemma and the results of Part I.

(14.3).- **Lemma.** With these assumptions, there exists an element $(\tilde{C}',C') \in \mathcal{R}_{B,g,0}$ tetragonally related with $(\tilde{D},D)$.

PROOF: We extend the injection $j : \mathcal{R}'_{B,g} \hookrightarrow \mathcal{R}_{B,g,0}$ (commuting with the Prym map) given in §7 to elements $(\tilde{D},D)$ as above. To do this we replace in the definition of j the symmetric products $\tilde{D}^{(2)}$, $D^{(2)}$ by the varieties of effective Cartier divisors of degree 2 $\mathrm{Div}^2(\tilde{D})$, $\mathrm{Div}^2(D)$. In other words, take the curve C'_2 given by the pull-back diagram

$$\begin{array}{ccc} C'_2 & \longrightarrow & \mathrm{Div}^2(\tilde{D}) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Nm}} \\ E_0 & \longrightarrow & \mathrm{Div}^2(D) \end{array}$$

Then, local computations show that C'_2 is smooth. The involution on $\mathrm{Div}^2(\tilde{D})$ restricts to an involution on C'_2. Taking quotient we get an elliptic curve E'. The fibre product of $C'_1 \longrightarrow E'$ with the transposed morphism of $E' \longrightarrow E_0$ gives a curve $\tilde{C}'$. The curve $\tilde{C}'$ has two involutions attached to the projections; call ι' the composition of these involutions. Then $(\tilde{C}',\tilde{C}'/\iota') \in \mathcal{R}_{B,g,0}$. Since for all $(\tilde{D},D)$, the elements $(\tilde{D},D)$ and $j(\tilde{D},D)$ are tetragonally related (cf. §7) we are done. ∎

15. The case (13.14.b).

This section is devoted to prove the following

(15.1).- **Proposition.** Let $(\tilde{C}, C)$ be a generic element of $\mathcal{R}_{B,g}$ and let $(\tilde{D}, D) \in \mathcal{H}'_{g,4}$ be such that $P(\tilde{D}, D) \cong P(\tilde{C}, C)$, $\tilde{D} = \tilde{D}_1 \cup_4 \tilde{D}_2$, $D = D_1 \cup_4 D_2$ and D_1 is an irreducible plane quartic. Suppose also that writing $D_1 \cap D_2 = \{x_1 + \cdots + x_4\}$, it is $\mathcal{O}_{D_1}(x_1 + \cdots + x_4) = \omega_{D_1}$, and that the curve D_2 is irreducible and hyperelliptic of genus $g - 5$.

Then $(\tilde{D}, D)$ is constructed from $(\tilde{C}, C)$ as in §8.

PROOF: Recall that in this case $(\tilde{C}, C) \in \mathcal{R}_{B,g,4}$ and the isomorphism $P(\tilde{D}, D) \cong P(\tilde{C}, C)$ identifies Z_b^m with W_0, Z_c^m with W_2 and $Z_d^m = Z_d$ with W_{-2} (see (13.13) and (13.14.b)).

We shall use again the variety

$$\Lambda_2 = \{\tilde{a} \in P(\tilde{C}, C) \mid \tilde{a} + W_0 \cap W_2 \subset W_0\}$$

defined in (5.5).

One has

(15.2).- **Lemma.** With the hypothesis of (15.1) the following facts hold:

a) There exists a birational isomorphism between the curve $\Lambda_2 \cap 2\Lambda_2$ and the curve $\tilde{B}_2$ obtained by the following pull-back diagram

(15.3)

$$\begin{array}{ccc} \tilde{B}_2 & \longrightarrow & \tilde{N}_2^{(2)} \\ \downarrow & & \downarrow \\ \mathbf{P}^1 & \xrightarrow{g_2^1} & N_2^{(2)} \end{array}$$

where $\tilde{N}_2$ and N_2 are the normalizations of $\tilde{D}_2$ and D_2 respectively, and g_2^1 is the linear series induced by the hyperelliptic structure of D_2.

b) The curve C_2 (see (2.1)) is the normalization of $\tilde{B}_2$.

c) The involution τ_2 in C_2 corresponds to the involution of $\tilde{B}_2$ given by the restriction of the natural involution of $\tilde{N}_2^{(2)}$.

d) There exists a linear series g_2^1 on E such that one gets a pull-back diagram

$$\begin{array}{ccc} \tilde{D}_2 & \longrightarrow & C_2^{(2)} \\ \downarrow & & \downarrow \\ \mathbf{P}^1 & \xrightarrow{g_2^1} & E^{(2)}. \end{array}$$

Moreover the involution $(\tau_2^{(2)})_{|\tilde{D}_2}$ coincides with the exchange of sheets on $\tilde{D}_2$.

Proof: We first see a). By using the identifications $W_6 = Z_b^m$ and $W_2 = Z_c^m$, and the definitions of Z_b^m, Z_c^m (cf. (13.13)) it is easy to see that

$$W_6 \cap W_2 = (\tilde{f}^*)^{-1}((\Xi_1^*)^m \times ((ex_2^*)^m \cap (\Xi_2^*)^m)),$$

where $\tilde{f}$ is the normalization of $\tilde{D}$ at $\tilde{D}_1 \cap \tilde{D}_2$. On the other hand, by (5.3) the dimension of this set is g-7. This forces to have $(ex_2^*)^m \subset (\Xi_2^*)^m$. Hence

$$\begin{aligned}\Lambda_2 = (\tilde{f}^*)^{-1}(\{(\tilde{a}_1, \tilde{a}_2) \in P(\tilde{D}_1, D_1) \times P(\tilde{D}_2, D_2) \mid \tilde{a}_1 + (\Xi_1^*)^m \subset (\Xi_1^*)^m,\\ \tilde{a}_2 + (ex_2^*)^m \subset (\Xi_2^*)^m\}).\end{aligned}$$

In the proof of (13.16) we saw that $\mathrm{I}((\Xi_1^*)^m) = (0)$. Therefore

$$\Lambda_2 = (\tilde{f}^*)^{-1}(\{0\} \times \{\tilde{a}_2 \in P(\tilde{D}_2, D_2) \mid \tilde{a}_2 + (ex_2^*)^m \subset (\Xi_2^*)^m\}).$$

Since $(\Xi_2^*)^m$ is irreducible (cf. (13.16)) and $\mathrm{Sing}\Xi^*$ has not components of dimension g-6, it is not hard to see that $(\Xi_2^*)^m$ is the closure of the set of effective divisors with non-singular support $\tilde{A}$ such that $\mathrm{Nm}(\tilde{A}) = \omega_{D_2}$. By using this one checks the inclusion

$$\{\tilde{x} + \tilde{y} - \tilde{r} - \tilde{s} \in P(\tilde{D}_2, D_2) \mid \tilde{x}, \tilde{y}, \tilde{r}, \tilde{s} \in (\tilde{D}_2)_{reg} \quad \mathrm{Nm}(\tilde{x} + \tilde{y}) \in g_2^1\} + ex_2^* \subset (\Xi_2^*)^m.$$

Thus one has

$$\begin{aligned}(\tilde{f}^*)^{-1}(\{0\} \times \text{closure } \{\tilde{x} + \tilde{y} - \tilde{r} - \tilde{s} \in P(\tilde{D}_2, D_2) \mid \tilde{x}, \tilde{y}, \tilde{r}, \tilde{s} \in (\tilde{D}_2)_{reg},\\ \mathrm{Nm}(x + y) \in g_2^1\}) \subset \Lambda_2.\end{aligned}$$

From this inclusion a straightforward computation gives

$$\begin{aligned}\{0\} \times \text{closure } \{\tilde{x} + \tilde{y} - \iota'(\tilde{x}) - \iota'(\tilde{y}) \in P(\tilde{D}_2, D_2) \mid \tilde{x}, \tilde{y} \in (\tilde{D}_2)_{reg},\\ \mathrm{Nm}(\tilde{x} + \tilde{y}) \in g_2^1\} \subset \tilde{f}^*(\Lambda_2 \cap 2\Lambda_2),\end{aligned}$$

where ι' is the natural involution on $\tilde{D}_2$. Since the curve on the right hand side is irreducible (cf. (5.7)) one has an equality. By using the description of $\Lambda_2 \cap 2\Lambda_2$ in $P(\tilde{C}, C)$ one obtains that $\Lambda_2 \cap 2\Lambda_2$ is birationally isomorphic with $\Lambda_2 \cap 2\Lambda_2 / \pi^*(\varepsilon^*({}_2JE)) = \tilde{f}^*(\Lambda_2 \cap 2\Lambda_2)$ (recall that $\mathrm{Ker}(\tilde{f}^*) = \pi^*(\varepsilon^*({}_2JE))$). On the other hand there exists a natural map from the normalization of $\tilde{B}_2$ to the set of the left hand side in the inclusion above. Since C_2 is the normalization of $\Lambda_2 \cap 2\Lambda_2$ we get a morphism from the normalization of $\tilde{B}_2$ to C_2. An elementary count says that $g(C_2)$ equals the genus of the normalization of $\tilde{B}_2$ (use (11.3)). Therefore C_2 and $\tilde{B}_2$ are isomorphic and a) is proved.

Part b) is a corollary of a). To see c) it suffices to recall that the multiplication by (-1) induces on C_2 the involution τ_2. Note that in this context this multiplication coincides on $\tilde{B}_2$ with the restriction of the involution on $\tilde{N}_2^{(2)}$.

Finally, we prove d). We first observe that c) implies that E is the normalization of $\tilde{B}_2$/(involution). Since this last curve has an obvious hyperelliptic structure given by diagram (15.2) we obtain on E a linear series g_2^1. The rest is left to the reader. ∎

As a consequence $(\tilde{D}_2, D_2)$ is obtained from $((C_2, E), g_2^1)$ as in the Step 2 of §8.

Next we concentrate in the relation between (C_1, E) and $(\tilde{D}_1, D_1)$. We shall consider as above the surface

$$\Lambda_{-2} = \{\tilde{a} \in P(\tilde{C}, C) \mid \tilde{a} + W_0 \cap W_{-2} \subset W_0\}$$

defined in (5.5). From the descriptions of Z_b^m and Z_d (cf. (13.13)) one gets

$$\Lambda_{-2} = (\tilde{f}^*)^{-1}(((\Xi_1^*)^m - \{\tilde{l}\}) \times \{0\})$$

where $\tilde{l}$ is the ramification divisor of $\tilde{D}_1 \longrightarrow D_1$. We call S the surface $((\Xi_1^*)^m - \{\tilde{l}\} \times \{0\}.)$ That is to say the group

$$\mathrm{Ker}\tilde{f}^* = I(W_0) = \pi^*(\varepsilon^*({}_2JE))$$

acts on Λ_{-2} and the quotient is S. We study first this surface in the more transparent context of $P(\tilde{C}, C)$.

(15.4).- **Proposition.** The surface S is exactly singular at the origin and the minimal resolution of the singularity is

$$C_1^{(2)} \longrightarrow S.$$

PROOF: We borrow from (5.6) the equality

$$\Lambda_{-2} = \{\pi_1^*(\varepsilon_1^*(\bar{x}) - r - s) \mid x \in E, \quad r, s \in C_1, \quad 2\bar{x} \equiv \varepsilon_1(r) + \varepsilon_1(s)\}.$$

Let $X \subset C_1^{(2)} \times E$ be the preimage of Λ_{-2} by the morphism

$$C_1^{(2)} \times E \longrightarrow J\tilde{C}$$
$$(r + s, x) \longrightarrow \pi_1^*(r + s - \varepsilon_1^*(x)).$$

Then X is an unramified covering of degree 4 of $C_1^{(2)}$. One obtains the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \Lambda_{-2} \\ \downarrow & & \downarrow \\ C_1^{(2)} & \longrightarrow & \Lambda_{-2}/\pi^*(\varepsilon^*({}_2JE)) = S. \end{array}$$

The morphism $C_1^{(2)} \longrightarrow S$ is an isomorphism away from the origin 0 and the preimage of 0 is the irreducible curve $\epsilon_1^*(E)$, of positive genus. Thus S is exactly singular in the origin and $C_1^{(2)}$ is the minimal resolution of the singularity. ∎

We shall consider the plane quintic given by the union of D_1 and the line r containing the discriminant points of $\tilde{D}_1 \longrightarrow D_1$. We call E' the elliptic curve which obtained as the double cover of r with discriminant divisor $r \cap D_1$. By identifying in the natural way the ramification points of $\tilde{D}_1 \longrightarrow D_1$ and $E' \longrightarrow r$ one constructs an allowable double cover of the plane quintic mentioned above. By [Be3] (Proposition (6.23)), there exists a smooth non hyperelliptic curve Γ of genus 5 such that

$$\begin{array}{ccc} W_4^1(\Gamma) & \xrightarrow{\cong} & \tilde{D}_1 \cup E' \\ \big\downarrow & & \big\downarrow \\ W_4^1(\Gamma)/\text{involution} & \xrightarrow{\cong} & D_1 \cup r. \end{array}$$

Now to prove that $(\tilde{D}_1, D_1)$ is constructed from C_1 as in Step 1 of §8 it suffices to show that $\Gamma \cong C_1$.

(15.5)- **Proposition.** The surfaces S and $\Gamma^{(2)}$ are birationally equivalent.

PROOF: The description of S as a subset of $P(\tilde{D}_1, D_1) \times P(\tilde{D}_2, D_2)$ (cf. (13.13)) gives the isomorphism $S \cong (\Xi_1^*)^m$. The general element of $(\Xi_1^*)^m$ is an effective divisor of degree 4 with non-singular support. Its norm is a divisor on D_1 consisting of 4 points on a line. By construction the general point of $\tilde{D}_1$ corresponds to a linear series g_4^1 on Γ that does not come from linear series on E'.

Let x, y be general points of Γ. To contain the line $\overline{xy}$ is a linear condition for a quadric containing the canonical image of Γ in $\mathbf{P}^4$. The intersection of the pencil of quadrics so obtained with D_1 provides four singular quadrics containing $\overline{xy}$. Consequently there exist exactly four linear series g_4^1 on Γ passing through the divisor $x + y$. These four linear series define an effective divisor of degree 4 on $\tilde{D}_1$ and the image in D_1 are four collinear points. We obtain a generically injective rational map from $\Gamma^{(2)}$ to $(\Xi_1^*)^m$ and we are done. ∎

(15.6).- **Corollary.** The curves C_1 and Γ are isomorphic.

PROOF: By (15.4) and (15.5) it follows that $C_1^{(2)}$ and $\Gamma^{(2)}$ are birationally equivalent. Now the result is a consequence of a Theorem of Martens ([M]). ∎

Having established that $(\tilde{D}_i, D_i)$ are obtained from (C_i, E), i=1,2 as in Part II we end the proof of (15.1) showing that $(\tilde{D}, D)$ comes from $(\tilde{D}_1, D_1)$ and $(\tilde{D}_2, D_2)$ as in the Step

3 of §8. Note first that the results just obtained make possible to use all the parts of (9.1) except the part iv). In fact the isogenies g_i, h_i and the fact $P(\tilde D, D) \cong P(\tilde C, C)$ provide the tools to prove the property (9.1.iv). (By (9.14) this property is equivalent to the property required in Step 3 of the construction of $(\tilde D, D)$). In conclusion all we have to do to end the proof of (15.1) is to show that (9.1.iv) holds. Keeping this strategy in mind one construct a commutative diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & {}_2JE \arrow[r] \arrow[d, equal] & \varepsilon_1^*({}_2JE) \times \varepsilon_2^*({}_2JE) \arrow[r] \arrow[d] & \pi^*(\varepsilon^*({}_2JE)) = \mathrm{Ker}\tilde f^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & {}_2JE \arrow[r] & P(C_1, E) \times P(C_2, E) \arrow[r, "\varphi"] \arrow[d, "h_1 \times h_2"'] & P(\tilde C, C) \cong P(\tilde D, D) \arrow[r] \arrow[d, "\tilde f^*"] & 0 \\
 & & P(\tilde D_1, D_1) \times P(\tilde D_2, D_2) \arrow[r, "\delta", "\cong"'] \arrow[d] & P(\tilde D_1, D_1) \times P(\tilde D_2, D_2) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
$$

where $\tilde f$ is the normalization of $\tilde D$ at $\tilde D_1 \cap \tilde D_2$ (cf. (2.8) for the definition of φ and cf. (13.16) and (5.12) for the upperrightcorner). Since $\mathrm{End}P(\tilde D_i, D_i) \cong \mathbf{Z}$ (cf.[C-G-T], (4.7) or proof of (13.8) above), $\delta = (\pm\mathrm{Id}) + (\pm\mathrm{Id})$. Hence

$$(15.7) \qquad \tilde f^*({}_2P(\tilde D, D)) = (h_1 \times h_2)(\varphi^{-1}({}_2P(\tilde C, C))).$$

In (10.14) we saw that

$$\tilde f^*({}_2P(\tilde D, D)) = \{(\tilde\alpha_1, \tilde\alpha_2) \in {}_2P(\tilde D_1, D_1) \times {}_2P(\tilde D_2, D_2) \mid v_1(\tilde\alpha_1) = v_2(\tilde\alpha_2)\}$$

(cf. §§ 4 and 9 for definitions). On the other hand it is easy to check that

$$\varphi^{-1}({}_2P(\tilde C, C)) = \{(\tilde\alpha_1, \tilde\alpha_2) \in {}_2P(C_1, E) \times {}_2P(C_2, E) \mid \exists \tilde\rho \in {}_2JE \text{ such that } 2\tilde\alpha_1 = \varepsilon_1^*(\tilde\rho), 2\tilde\alpha_2 = \varepsilon_2^*(\tilde\rho)\}.$$

Thus by applying $g_1 \times g_2$ to (15.7) one has

$$(15.8) \qquad g_1 \times g_2(\{(\tilde\alpha_1, \tilde\alpha_2) \in {}_2P(\tilde D_1, D_1) \times P(\tilde D_2, D_2) \mid v_1(\tilde\alpha_1) = v_2(\tilde\alpha_2)\}) = \\ = \{(\varepsilon_1^*(\tilde\rho), \varepsilon_2^*(\tilde\rho)) \mid \tilde\rho \in {}_2JE\}.$$

Finally we show that (15.8) implies

$$v_1(\tilde\alpha_1) = v_2(\tilde\alpha_2) \text{ iff } \exists \tilde\rho \in {}_2JE \text{ such that } g_i(\tilde\alpha_i) = \varepsilon_i^*(\tilde\rho)$$

for all $\tilde{a}_1 \in_2 P(\tilde{D}_1)$ and $\tilde{a}_2 \in_2 P(\tilde{D}_2, D_2)$. The part $\Rightarrow$ is clear. Suppose that $g_1(\tilde{a}_1) = \epsilon_1^*(\bar{\rho})$ and $g_2(\tilde{a}_2) = \epsilon_2^*(\bar{\rho})$ for $\bar{\rho} \in_2 JE$. Then by (15.8) there exist $(\tilde{a}_1', \tilde{a}_2')$ such that $v_1(\tilde{a}_1') = v_2(\tilde{a}_2')$ and $g_1(\tilde{a}_1) = g_1(\tilde{a}_1')$, $g_2(\tilde{a}_2) = g_2(\tilde{a}_2')$. Since $\operatorname{Ker} g_i = p_i^*({}_2JD_i)$, i=1,2 (cf. (9.1.i)) and these elements do not change the value of v_i the part $\Leftarrow$ follows. This finishes the proof of (15.1). ∎

16. The case (13.14.c).

In this section we end the proof of Theorem (13.1). Recall that (13.14) reduced the proof to three cases. In (14.1) and (15.1) we have treated the first and the second respectively. So, to finish the proof of Theorem it suffices to prove the following

(16.1).- **Proposition.** Let $(\tilde{C}, C)$ be a general element of $\mathcal{R}_{B,g}$ and let $(\tilde{D}, D) \in \mathcal{H}'_{g,t}$, $t \geq 2$ such that $P(\tilde{C}, C) \cong P(\tilde{D}, D)$. We write $D = D_1 \cup_4 D_2$. Assume that D_1, D_2 are irreducible hyperelliptic curves of genus $t-1$ and $g-t-2$ respectively. Then $(\tilde{C}, C)$ and $(\tilde{D}, D)$ are tetragonally related.

(16.2).- **Remark.** Recall that in this case $(\tilde{C}, C) \in \mathcal{R}_{B,g,t}$ and with the notations of (13.13), the isomorphism $P(\tilde{D}, D) \cong P(\tilde{C}, C)$ identifies Z_b^m with W_0 and the two varieties of type Z_c^m corresponding to the two hyperelliptic components with W_2 and W_{-2} (one of them is empty exactly when $W_{-2} = \emptyset$).

PROOF: If we are able to prove that $(\tilde{D}, D)$ verifies he hypothesis of the construction given in (12.2), then there will exist elements of $\mathcal{R}'_{B,g,t}$ tetragonally related with $(\tilde{D}, D)$. Then, by (14.1), these elements will be tetragonally related with elements of $\mathcal{R}_{B,g,t}$ and $(\tilde{C}, C)$ and $(\tilde{D}, D)$ will be tetragonally related. Essentially we only have to prove that D is tetragonal. Therefore the Proposition is a consequence of the following fact.

(16.3).- **Proposition.** There exists a finite morphism of degree four, $\gamma : D \longrightarrow \mathbf{P}^1$, whose restrictions to D_1 and D_2 coincide with the respective hyperelliptic morphism and such that $\gamma(D_1 \cap D_2)$ consists of four different points.

PROOF: What we have to do is to glue the hyperelliptic morphisms $\gamma_i : D_i \longrightarrow \mathbf{P}^1$. Let $D_1 \cap D_2 = \{d_1, \ldots, d_4\}$. It suffices to prove the equality of cross ratios

$$(16.4) \qquad | \gamma_1(d_1) : \gamma_1(d_2) : \gamma_1(d_3) : \gamma_1(d_4) | = | \gamma_2(d_1) : \gamma_2(d_2) : \gamma_2(d_3) : \gamma_2(d_4) | .$$

Recall that we obtained in (15.2) that the irreducible curve $\Lambda_2 \cap 2\Lambda_2$ (cf. (5.5) and (5.7)) is birationally equivalent to the curve $\tilde{B}_2$ given by the pull-back diagram

(16.5)

$$\begin{array}{ccc} \tilde{B}_2 & \longrightarrow & \tilde{N}_2^{(2)} \\ \downarrow & & \downarrow \\ \mathbf{P}^1 & \longrightarrow & N_2^{(2)} \end{array}$$

where $\tilde{N}_2$ and N_2 are the normalizations of $\tilde{D}_2$ and D_2 respectively. Moreover the involution on $\Lambda_2 \cap 2\Lambda_2$ attached to the multiplication by -1 equals the involution on $\tilde{B}_2$ inhereted from the involution of $\tilde{N}_2^{(2)}$. According to (5.7) we have that C_2 is the normalization of $\tilde{B}_2$ and therefore E is the normalization of $\tilde{B}_2/(\text{involution})$. Then from the analysis of the diagram (16.5) we get that the cross ratio $|\, \gamma_1(d_1) : \gamma_1(d_2) : \gamma_1(d_3) : \gamma_1(d_4) \,|$ coincides with the cross ratio of the four discriminant points of the obvious two-to-one covering $E \longrightarrow \mathbf{P}^1$. In particular the points $\gamma(d_i), i = 1, \ldots, 4$ are all different.

On the other side when $t \geq 4$ the same argument works when replacing $\Lambda_2 \cap 2\Lambda_2$ by $\Lambda_{-2} \cap 2\Lambda_{-2}$ and $\tilde{B}_2$ by the curve $\tilde{B}_1$ given by the pull-back diagram analogous to (16.5). So the cross ratio at the right hand side in (16.4) also equals the cross ratio of the four discriminant points of certain two-to-one morphism from E to a projective line. This clearly implies the equality (16.4).

To conclude the proof we only need to consider the cases $t = 3, 2$. In the first case we imitate the procedure of Part I (cf. proof of (5.16)) in order to recover the set of data (C_1, E).

Assume first t=3. We denote by $\hat{f}$ the desingularization of $\tilde{D}$ at $\tilde{D}_1 \cap \tilde{D}_2$. We call π_1 and π_2 to the ramified double covers $\tilde{D}_i \longrightarrow D_i$, i=1,2 induced by the partial desingularization. One has (compare with (5.12.i) and (5.13)):

(16.6).- **Lemma.** The following equalities hold (cf. (13.3) for definitions):

a) $I(Z_c^m) = (\hat{f}^*)^{-1}(P(\tilde{D}_1, D_1) \times \{0\})$ (this is true for $t \geq 1$).

b)

$$\bigcup_{\tilde{L} \in Z_b^m} ((Z_b^m)_{-\tilde{L}} \cap I(Z_c^m)) = (\hat{f}^*)^{-1}(\{\tilde{L} - \tilde{M} \in P(\tilde{D}_1, D_1) \mid \tilde{L}, \tilde{M} \in (\Xi_1^*)^m\} \times \{0\}).$$

PROOF: We first see a). According to (5.12.i) and (16.2) one has that $I(Z_c^m)$ is an abelian variety of dimension t containing $I(W_0) = I(Z_b^m) = \mathrm{Ker}(\hat{f}^*)$ (see (13.16)). On the other hand the very definitions imply that $\hat{f}^*(I(Z_c^m)) \supset P(\tilde{D}_1, D_1) \times \{0\}$. Hence

$$I(Z_c^m) \supset (\hat{f}^*)^{-1}(P(\tilde{D}_1, D_1) \times \{0\}).$$

The equality of dimensions concludes the proof of a).

In part b) we only show the inclusion of the left hand side member in the right hand side member. The opposite inclusion is left to the reader. Fix $\tilde{L} \in Z_b^m$. By definition $f^0(\tilde{L}) = (\tilde{L}_1, \tilde{L}_2) \in (\Xi_1^*)^m \times (\Xi_2^*)^m$. Then

$$\begin{aligned}(Z_b^m)_{-\tilde{L}} \cap I(Z_c^m) &= \{\tilde{\alpha} \in P(\tilde{D}, D) \mid \hat{f}^*(\tilde{\alpha}) = (\tilde{\alpha}_1, 0) \text{ and } \tilde{\alpha} + \tilde{L} \in Z_b^m\} = \\ &= \{\tilde{\alpha} \in P(\tilde{D}, D) \mid \hat{f}^*(\tilde{\alpha}) = (\tilde{\alpha}_1, 0) \text{ and } \tilde{\alpha}_1 + \tilde{L}_1 \in (\Xi_1^*)^m\}\end{aligned}$$

and we are done. ∎

Let us denote by Λ_{-2} the 2-dimensional variety obtained in (16.6.b) (observe that $\dim(\Xi_1^*)^m = \dim P(\tilde{D}_1, D_1) - 2 = t - 2 = 1$).

(16.7).- Lemma. One has the equality:

$$\tilde{f}^*(\Lambda_{-2} \cap 2\Lambda_{-2}) = \{\tilde{L} - \iota_1^*(\tilde{L}) \in P(\tilde{D}_1, D_1)^* \mid \tilde{L} \in (\Xi_1^*)^m, \mathrm{Nm}_{\pi_1}(\tilde{L}) = \gamma_1^*(\mathcal{O}_{\mathbf{P}^1}(1))\} \times \{0\}.$$

PROOF: One has $\tilde{f}^*(\Lambda_{-2} \cap 2\Lambda_{-2}) = \tilde{f}^*(\Lambda_{-2}) \cap 2\tilde{f}^*(\Lambda_{-2})$. According to (5.16) this set is an irreducible curve. Since both sets in the equality of the statement have dimension 1, we only have to prove the inclusion of the right hand side member in the left hand side member and this is straightforward. ∎

Observe that the normalization of the curve $\tilde{B}$. given by the pull-back diagram

$$\begin{array}{ccc} \tilde{B}_1 & \longrightarrow & \tilde{N}_1^{(2)} \\ \downarrow & & \downarrow \\ \mathbf{P}^1 & \longrightarrow & N_1^{(2)} \end{array}$$

has a natural morphism onto $\{\tilde{L} - \iota_1^*(\tilde{L}) \mid \tilde{L} \in (\Xi_1^*)^m, \mathrm{Nm}_{\pi_1}(\tilde{L}) = \gamma_1^*(\mathcal{O}_{\mathbf{P}^1}(1))\}$. Since C_1 is the normalization of $\Lambda_{-2} \cap 2\Lambda_{-2}$ and $\Lambda_{-2} \cap 2\Lambda_{-2}$ is birationally equivalent to $\tilde{f}^*(\Lambda_{-2} \cap 2\Lambda_{-2})$ (use the explicit description of $\Lambda_{-2} \cap 2\Lambda_{-2}$ in $P(\tilde{C}, C)$ and that $\mathrm{Ker}\,\tilde{f}^* = \pi^*(\varepsilon^*({}_2JE))$) we obtain a morphism from the normalization of $\tilde{B}$. to C_1. By comparing genera one gets that C_1 is also the desingularization of $\tilde{B}_1$. The proof of (16.3) follows as in the case $t \geq 4$.

Finally we observe that in case $t = 2$ the curve D is always tetragonal. Indeed, in this case the genus of D_1 is 1. To simplify assume it is smooth. Then the cross ratio of the images of the four points $D_1 \cap D_2$ by the two-to-one morphisms $D_1 \longrightarrow \mathbf{P}^1$ induced by the linear series g_2^1 on D_1 is not constant. Hence with a suitable such morphism we construct a four-to-one morphism $D \longrightarrow \mathbf{P}^1$. This concludes the proof of (16.3) and therefore of Theorem (13.1). ∎

17. Description of the fibre.

As a consequence of the description (2.10), the construction of §8 and Theorems (5.11), (5.16), (6.11), (6.24), (7.23) and (13.1) we obtain the following facts (we keep the notations of §2):

a) Let $(\tilde{C}, C)$ be a generic element of $\mathcal{R}_{B,g,t}$ with $t \neq 0, 1, 4$. Then $\bar{P}^{-1}(P(\tilde{C}, C))$ consists of:

- two elliptic curves isomorphic to E contained in $\mathcal{R}'_{B,g,t}$ (note that $\mathrm{Aut}(E) \cong \mathbb{Z}/2\mathbb{Z} \times E$ acts on this part of the fibre),
- an irreducible surface contained in $\mathcal{H}'_{g,t}$. If $t \neq 2$ it is isomorphic to $E \times E$.

b) Let $(\tilde{C}, C)$ be a generic element of $\mathcal{R}_{B,g,4}$. Then $\bar{P}^{-1}(P(\tilde{C}, C))$ consists of:

- two elliptic curves isomorphic to E contained in $\mathcal{R}'_{B,g,4}$,
- a surface isomorphic to $E \times E$ contained in $\mathcal{H}'_{g,4}$.
- a subvariety of dimension one contained in $\mathcal{H}'_{g,4}$ (these are the unique elements of the fibre not obtained in a tetragonal way).

c) Let $(\tilde{C}, C)$ be a generic element of $\mathcal{R}_{B,g,1}$. Then $\bar{P}^{-1}(P(\tilde{C}, C))$ consists of:

- two elliptic curves isomorphic to E contained in $\mathcal{R}'_{B,g,1}$,
- an irreducible curve contained in $\mathcal{H}'_{g,1}$.

d) Let $(\tilde{C}, C)$ be a generic element of $\mathcal{R}_{B,g,0} \cup \mathcal{R}'_{B,g}$. Then $\bar{P}^{-1}(P(\tilde{C}, C))$ consists of:

- a single point in each component $\mathcal{R}_{B,g,0}$ and $\mathcal{R}'_{B,g}$,
- an elliptic curve isomorphic to E contained in $\mathcal{H}'_{g,0}$.

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