He encontrado una prueba realmente maravillosa, que este margen, demasiado angosto, no puede contener.

PIERRE DE FERMAT (1601-1665)

ANEJO A1

FORMULACION TRADICIONAL PARA DERIVACION DE INTEGRALES

A1.1 INTRODUCCIÓN

Sea el dominio de referencia $E_\zeta \subset R^3$, y sea $\bar{x}$ el vector de variables de diseño. Para cada $\bar{x}$, supongamos que se puede definir la transformación suficientemente regular

$$T(\bar{x}): \bar{\zeta} \rightarrow \bar{r}$$

$$\bar{r} = \bar{\rho}(\bar{\zeta}, \bar{x}) \tag{a1.1}$$

$$E_r(\bar{x}) \equiv \bar{\rho}(E_\zeta, \bar{x}), \qquad E_r(\bar{x}) \subset R^3$$

donde $\bar{r}$ es el vector de coordenadas materiales asociadas por la variables de diseño $\bar{x}$ al punto de coordenadas de referencia $\bar{\zeta}$. Sean las superficies Γ_ζ frontera del dominio E_ζ, y $\Gamma_r(\bar{x})$ frontera del dominio transformado suficientemente regulares; las coordenadas de referencia de un punto de la frontera Γ_ζ las expresaremos en la forma paramétrica

$$\bar{\zeta} = \bar{\zeta}(u, v)$$

Sean $\bar{m}$ y $\bar{n}$ los campos de normales exteriores unitarias de las superficies Γ_ζ y $\Gamma_r(\bar{x})$ respectivamente.

Por hipótesis el dominio E_ζ y su frontera Γ_ζ no dependen de las variables de diseño $\bar{x}$.

(Fig. A1.1)

Definimos las funciones reales

$$P(\bar{x}) = \iiint_{E_r(\bar{x})} p(\bar{r}, \bar{x})\, dV \tag{a1.2}$$

$$Q(\bar{x}) = \iint_{\Gamma_r(\bar{x})} q(\bar{r}, \bar{x})\, d\Omega \tag{a1.3}$$

$$F(\bar{x}) = \int_{C_r(\bar{x})} f(\bar{r}, \bar{x})\, ds \tag{a1.4}$$

donde p, q y f son funciones suficientemente regulares de sus argumentos y definidas respectivamente en el dominio $E_r(\bar{x})$, su frontera $\Gamma_r(\bar{x})$, y la curva material $C_r(\bar{x})$ incluida en $E_r(\bar{x})$.

Supondremos que un punto genérico de la curva C_ζ, suficientemente regular, de E_ζ, cuya transformada es $C_r(\bar{x})$, se expresará en la forma paramétrica.

$$\bar{\zeta} = \bar{\zeta}(\tau)$$

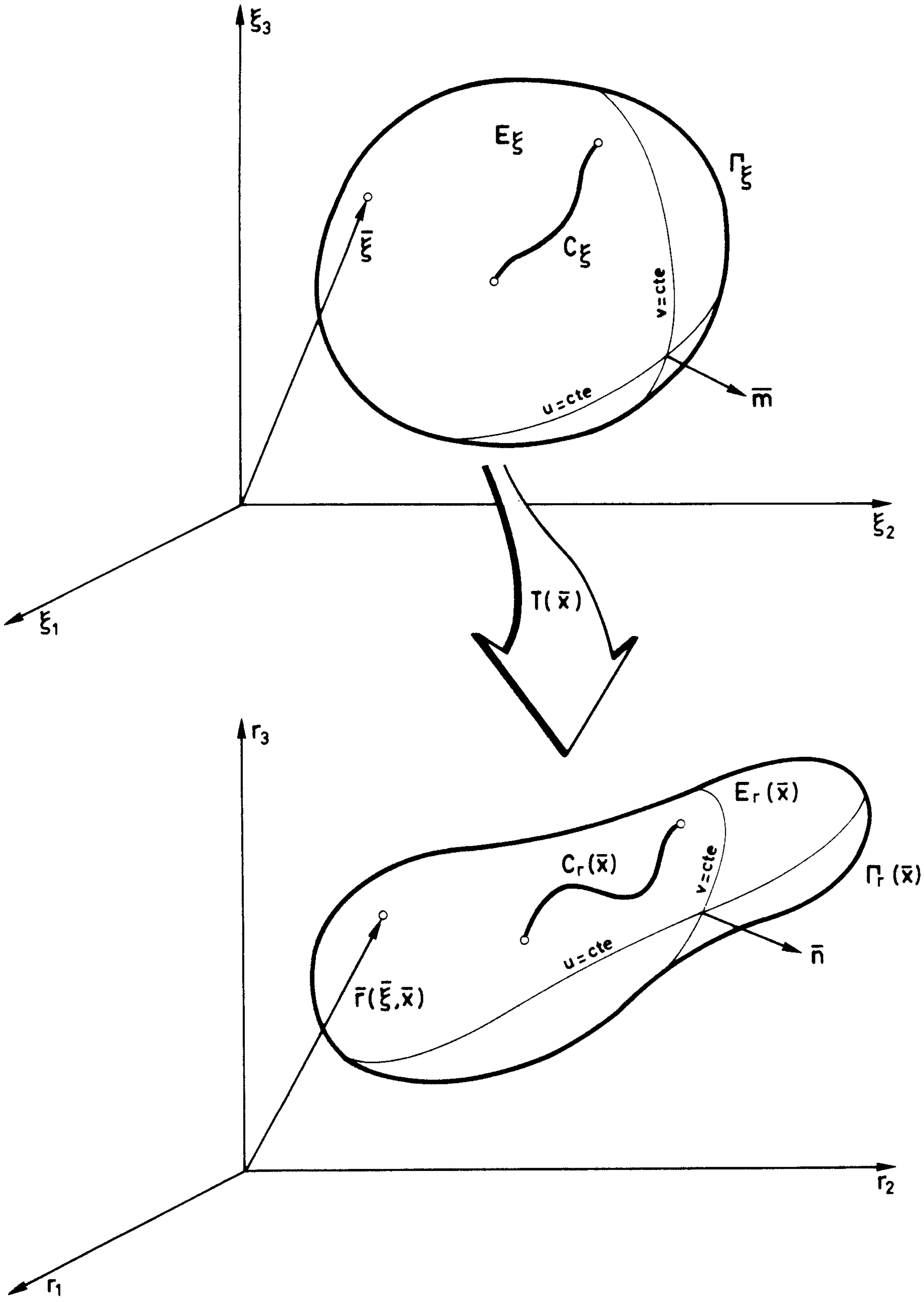

Figura A1.1.- Transformación del espacio de referencia en el espacio material.

A1.2 PUNTOS PREVIOS

Sea $\underset{\sim}{J}$ la matriz Jacobiana de la transformación $T(\bar{x})$

$$\underset{\sim}{J} = \frac{\partial \bar{\rho}}{\partial \bar{\zeta}} \tag{a1.5}$$

y sea J su determinante.

Sea $\underset{\sim}{W}$ la matriz

$$\underset{\sim}{W} = \frac{\partial \bar{\rho}}{\partial \bar{x}} \tag{a1.6}$$

y sea $\bar{W}_j$ su columna j-ésima, es decir

$$\bar{W}_j = \frac{\partial \bar{\rho}}{\partial x_j} \tag{a1.7}$$

A1.2.1 Derivación del determinante Jacobiano de la transformación

Siendo J el determinante Jacobiano, su derivada es:

$$\frac{\partial J}{\partial x_i} = \frac{\partial J}{\partial J_{\alpha\beta}} \frac{\partial J_{\alpha\beta}}{\partial x_i}$$

donde

$$\frac{\partial J}{\partial J_{\alpha\beta}} = \mathrm{cof}\,(J_{\alpha\beta})$$

$$J_{\alpha\beta} = \frac{\partial \rho_\alpha}{\partial \zeta_\beta}$$

y por definición de cofactor

$$\mathrm{cof}\,(J_{\alpha\beta}) = J \frac{\partial \zeta_\beta}{\partial r_\alpha}$$

luego

$$\frac{\partial J}{\partial x_i} = J \frac{\partial \zeta_\beta}{\partial r_\alpha} \frac{\partial}{\partial x_i}\left(\frac{\partial \rho_\alpha}{\partial \zeta_\beta}\right) = J \frac{\partial \zeta_\beta}{\partial r_\alpha} \frac{\partial}{\partial \zeta_\beta}\left(\frac{\partial \rho_\alpha}{\partial x_i}\right) = J \frac{\partial}{\partial r_\alpha}\left(\frac{\partial \rho_\alpha}{\partial x_i}\right) = J\,\mathrm{div}(\bar{W}_i)$$

por tanto

$$\boxed{\frac{\partial J}{\partial x_i} = J\,\mathrm{div}(\overline{W}_i)} \qquad \text{(a1.8)}$$

A1.2.2 Derivación del vector normal a una superficie

Sea

$$\bar{r} = \bar{\rho}\,(\bar{\zeta}\,(u, v), \bar{x})$$

el vector de posición de un punto de la superficie $\Gamma_r(\bar{x})$. El versor $\bar{n}$, normal exterior a la superficie en un punto arbitrario, se obtiene en la forma:

$$\bar{n} = \frac{1}{|\overline{N}|}\,\overline{N}$$

siendo

$$\overline{N} = \bar{r}'_u \wedge \bar{r}'_u$$

y desarrollando el producto vectorial en notación tensorial resulta

$$N_i = e_{ijk}\,\frac{\partial r_j}{\partial u}\,\frac{\partial r_k}{\partial v} = e_{ijk}\,\frac{\partial r_j}{\partial \zeta_\beta}\,\frac{\partial r_k}{\partial \zeta_\gamma}\,\frac{\partial \zeta_\beta}{\partial u}\,\frac{\partial \zeta_\gamma}{\partial v}$$

multiplicando ambos términos por

$$\frac{\partial r_i}{\partial \zeta_\alpha}$$

y sumando en i

$$\frac{\partial r_i}{\partial \zeta_\alpha}\,N_i = e_{ijk}\,\frac{\partial r_i}{\partial \zeta_\alpha}\,\frac{\partial r_j}{\partial \zeta_\beta}\,\frac{\partial r_k}{\partial \zeta_\gamma}\,\frac{\partial \zeta_\beta}{\partial u}\,\frac{\partial \zeta_\gamma}{\partial v} =$$

$$= e_{\alpha\beta\gamma}\,J\,\frac{\partial \zeta_\beta}{\partial u}\,\frac{\partial \zeta_\gamma}{\partial v}$$

Escribiendo la expresión anterior en forma vectorial.

$$\underset{\sim}{J}^T\,\overline{N} = J\left(\bar{\zeta}'_u \wedge \bar{\zeta}'_v\right)$$

luego

$$\overline{N} = J\,\underset{\sim}{J}^{-T}\left(\overline{\zeta}'_u \wedge \overline{\zeta}'_v\right)$$

llamemos

$$\overline{M} = \overline{\zeta}'_u \wedge \overline{\zeta}'_v$$

y por tanto

$$\overline{m} = \frac{\overline{M}}{|\overline{M}|}$$

por lo que concluimos

$$\overline{N} = J\,\underset{\sim}{J}^{-T}\,\overline{M}$$

Derivando esta última expresión

$$\frac{\partial \overline{N}}{\partial x_i} = \frac{\partial}{\partial x_i}(J\,\underset{\sim}{J}^{-T})\overline{M} = \frac{\partial J}{\partial x_i}\underset{\sim}{J}^{-T}\;\overline{M} + J\,\frac{\partial \underset{\sim}{J}^{-T}}{\partial x_i}\;\overline{M} =$$

$$= \frac{\partial J}{\partial x_i}\underset{\sim}{J}^{-T}\;\overline{M} - J\left(\underset{\sim}{J}^{-T}\,\frac{\partial \underset{\sim}{J}^{T}}{\partial x_i}\underset{\sim}{J}^{-T}\right)\overline{M} =$$

$$= J\,\mathrm{div}(\overline{W}_i)\underset{\sim}{J}^{-T}\;\overline{M} - J\;\underset{\sim}{J}^{-T}\,\frac{\partial \underset{\sim}{J}^{T}}{\partial x_i}\underset{\sim}{J}^{-T}\;\overline{M} =$$

$$= \mathrm{div}(\overline{W}_i)\,\overline{N} - \underset{\sim}{J}^{-T}\,\frac{\partial \underset{\sim}{J}^{T}}{\partial x_i}\,\overline{N}$$

luego

$$\boxed{\frac{\partial \overline{N}}{\partial x_i} = \mathrm{div}\,\overline{W}_i\,\overline{N} - \underset{\sim}{J}^{-T}\,\frac{\partial \underset{\sim}{J}^{T}}{\partial x_i}\overline{N}} \qquad \text{(a1.9)}$$

A1.2.3 Derivación del vector tangente a una curva

Sea:

$$\bar{r} = \bar{\rho}\,(\,\bar{\zeta}\,(\tau),\ \bar{x}\,)$$

el vector de posición de un punto de la curva $C_r(\bar{x})$

El versor tangente a la curva en un punto arbitrario τ, se obtiene en la forma

$$\bar{t} = \frac{\bar{r}'_\tau}{|\bar{r}'_\tau|}$$

derivando r'_τ

$$\frac{\partial \bar{r}'_\tau}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial \bar{r}}{\partial \tau}\right) = \frac{\partial}{\partial \tau}\left(\frac{\partial \bar{r}}{\partial x_i}\right) = \frac{\partial}{\partial \tau}\,\overline{W}_i =$$

$$= \frac{\partial}{\partial \bar{\bar{\zeta}}}\,(\overline{W}_i)\,\frac{\partial \bar{\zeta}}{\partial \tau} = \frac{\partial}{\partial \bar{\bar{r}}}\,(\overline{W}_i)\,\frac{\partial \bar{r}}{\partial \bar{\zeta}}\,\frac{\partial \bar{\zeta}}{\partial \tau} =$$

$$= \frac{\partial}{\partial \bar{\bar{r}}}\,(\overline{W}_i)\,\frac{\partial \bar{r}}{\partial \tau} = \frac{\partial}{\partial \bar{\bar{r}}}\,(\overline{W}_i)\,\bar{r}'_\tau$$

luego

$$\boxed{\frac{\partial \bar{r}'_\tau}{\partial x_i} = \frac{\partial}{\partial \bar{\bar{r}}}\,(\overline{W}_i)\,\bar{r}'_\tau} \qquad \text{(a1.10)}$$

A1.3 DERIVACIÓN DE INTEGRALES DE VOLUMEN

Sea:

$$P(\bar{x}) = \iiint_{E_r(\bar{x})} p(\bar{r}, \bar{x})\, dV$$

realizando el cambio de variable $\bar{r} = \bar{\rho}(\bar{\zeta}, \bar{x})$ se obtiene

$$P(\bar{x}) = \iiint_{E_\zeta} p\left(\bar{\rho}(\bar{\zeta}, \bar{x}), \bar{x}\right) J\, d\zeta_1\, d\zeta_2\, d\zeta_3$$

Derivando la expresión anterior,

$$\frac{\partial}{\partial x_i} P(\bar{x}) = \iiint_{E_\zeta} \frac{\partial}{\partial x_i}\left[p(\bar{\rho}(\bar{\zeta}, \bar{x}), \bar{x}) J\right] d\zeta_1\, d\zeta_2\, d\zeta_3$$

La función subintegral puede escribirse en la forma:

$$\frac{\partial}{\partial x_i}\left[p(\bar{\rho}(\bar{\zeta}, \bar{x}), \bar{x}) J\right] = \frac{\partial p}{\partial x_i} J + \frac{\partial p}{\partial \bar{r}} \frac{\partial \bar{\rho}}{\partial x_i} J + p \frac{\partial J}{\partial x_i}$$

$$= \left(\frac{\partial p}{\partial x_i} + \overline{\mathrm{grad}}_r(p) . \overline{W}_i\right) J + p \frac{\partial J}{\partial x_i}$$

Empleando la fórmula

$$\mathrm{div}(p\, \overline{W}_i) = \overline{\mathrm{grad}}_r(p) . \overline{W}_i + p\, \mathrm{div}(\overline{W}_i) \qquad \text{(a1.11)}$$

y sustituyendo la derivada del determinante jacobiano J obtenida en (a1.8) resulta

$$\frac{\partial}{\partial x_i}\left[p(\bar{\rho}(\bar{\zeta}, \bar{x}), \bar{x}) J\right] = \left[\frac{\partial p}{\partial x_i} + \mathrm{div}(p\, \overline{W}_i)\right] J - p\, \mathrm{div}(\overline{W}_i) J +$$

$$+ p\, \mathrm{div}(\overline{W}_i) J = \left[\frac{\partial p}{\partial x_i} + \mathrm{div}(p\, \overline{W}_i)\right] J$$

Sustituyendo este desarrollo en la derivada de la integral,

$$\frac{\partial P(\bar{x})}{\partial x_i} = \iiint_{E_\zeta} \left[\frac{\partial p}{\partial x_i} + \mathrm{div}(p\, \overline{W}_i)\right] J\, d\zeta_1\, d\zeta_2\, d\zeta_3$$

y deshaciendo el cambio de variable,

$$\frac{\partial P(\bar{x})}{\partial x_i} = \iiint_{E_r(\bar{x})} \left[\frac{\partial p}{\partial x_i} + \mathrm{div}(p\,\overline{W}_i) \right] dV \qquad \text{(a1.12)}$$

Aplicando por último el teorema de la divergencia,

$$\boxed{\frac{\partial P(\bar{x})}{\partial x_i} = \iiint_{E_r(\bar{x})} \frac{\partial p}{\partial x_i} dV + \iint_{\Gamma_r(\bar{x})} p\,\overline{W}_i \cdot \bar{n}\, d\Omega} \qquad \text{(a1.13)}$$

A1.4 DERIVACIÓN DE INTEGRALES DE SUPERFICIE

Sea:

$$Q(\bar{x}) = \int\int_{\Gamma_r(\bar{x})} q(\bar{r}, \bar{x}) d\Omega$$

realizando los cambios de variable

$$\bar{r} = \bar{\rho}(\bar{\zeta}, \bar{x})$$

$$\bar{\zeta} = \bar{\zeta}(u, v)$$

se obtiene

$$Q(\bar{x}) = \int\int_{\Gamma_\zeta} q(\bar{\rho}(\bar{\zeta}(u, v), \bar{x}), \bar{x}) |\bar{N}| du\, dv$$

siendo

$$\bar{N} = \bar{r}'_u \wedge \bar{r}'_v$$

Derivando la función $Q(\bar{x})$

$$\frac{\partial Q(\bar{x})}{\partial x_i} = \int\int_{\Gamma_\zeta} \frac{\partial}{\partial x_i} \left[q(\bar{\rho}(\bar{\zeta}(u, v), \bar{x}), \bar{x}) |\bar{N}| \right] du\, dv$$

La función subintegral puede escribirse en la forma:

$$\frac{\partial}{\partial x_i} \left[q(\bar{\rho}(\bar{\zeta}(u, v), \bar{x}), \bar{x}) |\bar{N}| \right] = \frac{\partial q}{\partial x_i} |\bar{N}| +$$

$$+ \frac{\partial q}{\partial \bar{r}} \frac{\partial \bar{\rho}}{\partial x_i} |\bar{N}| + q \frac{\partial |\bar{N}|}{\partial x_i} =$$

$$= \left(\frac{\partial q}{\partial x_i} + \overline{\text{grad}}_r(q) . \bar{W}_i \right) |\bar{N}| + q \frac{\partial |\bar{N}|}{\partial x_i}$$

Empleando la fórmula (a1.11), resulta

$$\frac{\partial}{\partial x_i} \left[q(\bar{\rho}(\bar{\zeta}(u, v), \bar{x}) |\bar{N}| \right] = \left[\frac{\partial q}{\partial x_i} + \text{div}(q\, \bar{W}_i) \right] |\bar{N}| -$$

$$- q \,\mathrm{div}\, \overline{W}_i \,|\overline{N}| + q \frac{\partial |\overline{N}|}{\partial x_i}$$

además

$$\frac{\partial |\overline{N}|}{\partial x_i} = \frac{1}{|\overline{N}|} \overline{N} . \frac{\partial \overline{N}}{\partial x_i} = \overline{n} . \frac{\partial \overline{N}}{\partial x_i}$$

y sustituyendo la expresión (al.9) resulta

$$\frac{\partial |\overline{N}|}{\partial x_i} = \overline{n} . \left[\mathrm{div}(\overline{W}_i)\, \overline{N} - \underset{\sim}{J}^{-T} \frac{\partial \underset{\sim}{J}^T}{\partial x_i} \overline{N} \right] =$$

$$= \mathrm{div}(\overline{W}_i)\, (\overline{n} . \overline{N}) - \overline{n} . \left[\underset{\sim}{J}^{-T} \frac{\partial \underset{\sim}{J}^T}{\partial x_i} \overline{N} \right]$$

teniendo en cuenta que

$$\underset{\sim}{J}^{-T} \frac{\partial \underset{\sim}{J}^T}{\partial x_i} = \left(\frac{\partial \underset{\sim}{J}}{\partial x_i} \underset{\sim}{J}^{-1} \right)^T = \left(\frac{\partial}{\partial x_i} \left(\frac{\partial \overline{\rho}}{\partial \overline{\zeta}} \right) \frac{\partial \overline{\zeta}}{\partial \overline{r}} \right)^T =$$

$$= \left(\frac{\partial}{\partial \overline{\zeta}} \left(\frac{\partial \overline{\rho}}{\partial x_i} \right) \frac{\partial \overline{\zeta}}{\partial \overline{r}} \right)^T = \left(\frac{\partial}{\partial \overline{r}} \left(\frac{\partial \overline{\rho}}{\partial x_i} \right) \right)^T = \left(\frac{\partial}{\partial \overline{r}} \overline{W}_i \right)^T$$

resulta finalmente

$$\frac{\partial |\overline{N}|}{\partial x_i} = \mathrm{div}\, (\overline{W}_i) |\overline{N}| - < \overline{n} , \left(\frac{\partial \overline{W}_i}{\partial \overline{r}} \right)^T \overline{n} > |\overline{N}| \qquad \text{(al.14)}$$

y por lo tanto

$$\frac{\partial}{\partial x_i} \left[q(\overline{\rho}\, (\overline{\zeta}\,(u, v),\ \overline{x})\, |\overline{N}| \right] =$$

$$\left[\frac{\partial q}{\partial x_i} + \mathrm{div}\, (q\, \overline{W}_i) \right] |\overline{N}| - q < \overline{n} , \left(\frac{\partial \overline{W}_i}{\partial \overline{r}} \right)^T \overline{n} > |\overline{N}|$$

sustituyendo este desarrollo en la derivada de la integral,

$$\frac{\partial Q(\overline{x})}{\partial x_i} = \int \int_{\Gamma_\zeta} \left[\frac{\partial q}{\partial x_i} + \mathrm{div}(q\, \overline{W}_i) - q < \overline{n} , \left(\frac{\partial \overline{W}_i}{\partial \overline{r}} \right)^T \overline{n} > \right] |\overline{N}| \ du\, dv$$

y deshaciendo el cambio de variable,

$$\frac{\partial Q(\bar{x})}{\partial x_i} = \iint_{\Gamma_{r(\bar{x})}} \left[\frac{\partial q}{\partial x_i} + \operatorname{div}(q\, \overline{W}_i) - q < \bar{n}, \left(\frac{\partial \overline{W}_i}{\partial \bar{r}} \right)^T \bar{n} > \right] d\Omega$$

y finalmente

$$\boxed{\frac{\partial Q(\bar{x})}{\partial x_i} = \iint_{\Gamma_{r(\bar{x})}} \left[\frac{\partial q}{\partial x_i} + \operatorname{div}(q\, \overline{W}_i) - q < \frac{\partial \overline{W}_i}{\partial n}, \bar{n} > \right] d\Omega} \qquad \text{(a1.15)}$$

A1.5 DERIVACION DE INTEGRALES DE LINEA

Sea

$$F(\bar{x}) = \int_{C_{r(\bar{x})}} f(\bar{r}, \bar{x})ds$$

realizando los cambios de variable

$$\bar{r} = \bar{\rho}(\bar{\zeta}, \bar{x})$$

$$\bar{\zeta} = \bar{\zeta}(\tau)$$

se obtiene

$$F(\bar{x}) = \int_{C_\zeta} f(\bar{\rho}(\bar{\zeta}(\tau), \bar{x}), \bar{x})|\bar{r}'_\tau|d\tau$$

Derivando la expresión anterior

$$\frac{\partial F(\bar{x})}{\partial x_i} = \int_{C_\zeta} \frac{\partial}{\partial x_i}\left[f(\bar{\rho}(\bar{\zeta}(\tau), \bar{x}), \bar{x})|\bar{r}'_\tau|\right]d\tau$$

La función subintegral puede excribirse en la forma

$$\frac{\partial}{\partial x_i}\left[f(\bar{\rho}(\bar{\zeta}(\tau), \bar{x}), \bar{x})|\bar{r}'_\tau|\right] = \frac{\partial f}{\partial x_i}|\bar{r}'_\tau| +$$

$$+ \frac{\partial f}{\partial \bar{r}}\frac{\partial \bar{\rho}}{\partial x_i}|\bar{r}'_\tau| + f\frac{\partial|\bar{r}'_\tau|}{\partial x_i} =$$

$$= \left(\frac{\partial f}{\partial x_i} + \overline{\mathrm{grad}}_r(f).\overline{W}_i\right)|\bar{r}'_\tau| + f\frac{\partial|\bar{r}'_\tau|}{\partial x_i}$$

empleando la fórmula (a1.11) resulta

$$\frac{\partial}{\partial x_i}\left[f(\bar{\rho}(\bar{\zeta}(\tau), \bar{x}), \bar{x})|\bar{r}'_\tau|\right] = \left[\frac{\partial f}{\partial x_j} + \mathrm{div}(f\overline{W}_j)\right]|\bar{r}'_\tau| -$$

$$f\,\mathrm{div}(\overline{W}_i)|\bar{r}'_\tau| + f\frac{\partial|\bar{r}'_\tau|}{\partial x_i}$$

además

$$\frac{\partial|\bar{r}'_\tau|}{\partial x_i} = \frac{1}{|\bar{r}'_\tau|}\,\bar{r}'_\tau.\frac{\partial \bar{r}'_\tau}{\partial x_i} = \bar{t}.\frac{\partial \bar{r}'_\tau}{\partial x_i}$$

y sustituyendo la expresión (a1.10) resulta

$$\frac{\partial|\bar{r}'_\tau|}{\partial x_i} = \bar{t}.\left(\frac{\partial \overline{W}_i}{\partial \bar{r}}\,\bar{r}'_\tau\right) = <\bar{t}\,,\,\frac{\partial \overline{W}_i}{\partial \bar{r}}\,\bar{t}>|\bar{r}'_\tau|$$

y por tanto

$$\frac{\partial}{\partial x_i}\left[f(\bar{\rho}(\bar{\zeta}(\tau),\bar{x}),\,\bar{x})|\bar{r}'_\tau|\right] =$$

$$\left[\frac{\partial f}{\partial x_i} + \mathrm{div}\,(f\,\overline{W}_i) - f\,\mathrm{div}\,(\overline{W}_i) + f<\bar{t}\,,\,\frac{\partial \overline{W}_i}{\partial \bar{r}}\,\bar{t}>\right]|\bar{r}'_\tau|$$

y sustituyendo este desarrollo en la derivada de la integral y deshaciendo el cambio de variable, obtenemos

$$\boxed{\frac{\partial F(\bar{x})}{\partial x_i} = \int_{C_{r(\bar{x})}}\left[\frac{\partial f}{\partial x_i} + \mathrm{div}\,(f\,\overline{W}_i) - f\,\mathrm{div}\,(\overline{W}_i) + f<\bar{t}\,,\,\frac{\partial \overline{W}_i}{\partial t}>\right]ds} \qquad (a1.16)$$

A1.6 BREVES COMENTARIOS SOBRE LA EXTENSIÓN DE LOS RESULTADOS PRECEDENTES

- Son independientes de la dimensión del espacio las ecuaciones (a1.13) y (a1.16), es decir las fórmulas de derivación para las integrales de volumen y linea respectivamente.

- La ecuación (a1.15) es válida para derivación de integrales extendidas a superficies suficientemente regulares en $E_r(x)$.