Universitat Autònoma de Barcelona

Fuzzy Horn clauses in artificial intelligence: a study of free models, and applications in art painting style categorization

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UNIVERSITAT AUTÒNOMA DE BARCELONA

DOCTORAL THESIS

Fuzzy Horn clauses in artificial intelligence: a study of free models, and applications in art painting style categorization

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A **Giulio Aurelio Spina**.

Que els teoremes,

dels quals diuen que perduraran

més enllà de la humanitat,

siguen per sempre

els guardians del teu record.

(Mentrestant, jo t'estimaré fins al meu últim alé.)

"*E come tutte le più belle cose,*
vivesti solo un giorno come le rose"

Fabrizio de André – Elvio Monti.

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Abstract

PhD in Cognitive Science and Language
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by Vicent COSTA

This PhD thesis contributes to the systematic study of Horn clauses of predicate fuzzy logics and their use in knowledge representation for the design of an art painting style classification algorithm. We first focus the study on relevant notions in logic programming, such as free models and Herbrand structures in mathematical fuzzy logic. We show the existence of free models in fuzzy universal Horn classes, and we prove that every equality-free consistent universal Horn fuzzy theory has a Herbrand model. Two notions of minimality of free models are introduced, and we show that these notions are equivalent in the case of fully named structures. Then, we use Horn clauses combined with qualitative modeling as a fuzzy knowledge representation framework for art painting style categorization. Finally, we design a style painting classifier based on evaluated Horn clauses, qualitative color descriptors, and explanations. This algorithm, called ℓ-SHE, provides reasons for the obtained results and obtains percentages of accuracy in the experimentation that are competitive.

UNIVERSITAT AUTÒNOMA DE BARCELONA

Resum

PhD in Cognitive Science and Language
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Aquesta tesi doctoral contribueix a l'estudi de les clàusules de Horn en lògiques difuses, així com al seu ús en representació difusa del coneixement aplicada al disseny d'un algorisme de classificació de pintures segons el seu estil artístic. En la primera part del treball ens centrem en algunes nocions rellevants per a la programació lògica, com ho són per exemple els models lliures i les estructures de Herbrand en lògica matemàtica difusa. Així doncs, provem l'existència de models lliures en classes universals difuses de Horn, i demostrem que tota teoria difusa universal de Horn sense igualtat té un model de Herbrand. A més, introduïm dues nocions de *minimalitat* per a models lliures, i demostrem que aquestes nocions són equivalents en el cas de les *fully named structures*. En la segona part de la tesi doctoral, utilitzem les clàusules de Horn combinades amb el modelatge qualitatiu com a marc de representació difusa del coneixement per a la categorització d'estils de pintura artística. Finalment, dissenyem un classificador de pintures basat en clàusules de Horn avaluades, descriptors qualitatius de colors i explicacions. Aquest algorisme, anomenat ℓ-SHE, proporciona raons dels resultats obtinguts i mostra percentatges competitius de precisió a l'experimentació.

Publications arising from this PhD thesis[1]**:**

- **Vicent Costa**, and Pilar Dellunde (2015): On Free Models for Horn Clauses over Predicate Fuzzy Logics, Frontiers in Artificial Intelligence and Applications 277, IOS Press, 2015, pp. 49-58.

- **Vicent Costa**, and Pilar Dellunde (2017): On the existence of Free Models in Fuzzy Universal Horn Classes, Journal of Applied Logic, 23C, pp 3-15.

- **Vicent Costa**, and Pilar Dellunde (2017): Term Models of Horn Clauses over Rational Pavelka Predicate Logic, Proceedings of the 47th IEEE International Symposium on Multiple-Valued Logic, IEEE Computer Society 2017, pp 112-117.

- **Vicent Costa**, Pilar Dellunde, and Zoe Falomir (2018): Style painting classifier based on Horn clauses and Explanations (SHE), Frontiers in Artificial Intelligence and Applications. Artificial Intelligence Research and Development 308, IOS Press, 2018, pp. 37-46.

- **Vicent Costa** (2020): The art painting style Classifier based on Logic Aggregators and qualitative colour Descriptors (C-LAD). In: Rudolph, S., Marreiros, G.(eds.): Proceedings of STAIRS 2020, http:// ceur-ws.org.

- **Vicent Costa**, Pilar Dellunde, and Zoe Falomir (2021): The logical style painting classifier based on Horn clauses and explanations (ℓ-SHE), Logic Journal of the IGPL, Volume 29, Issue 1, February 2021, Pages 96–119.

[1] During this PhD thesis' writing process, another three journal articles have been published: Badia et al. (2019), Falomir et al. (2019), and Falomir et al. (2020).

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Contents

Chapter 1

Introduction and research objectives

Two interrelated parts, the theoretical one and the practical other, divide the research presented in this doctoral thesis. Both parts focus on Horn clauses, which are among the most widely used formulas in computer science, and they are significant in classical logic. This chapter indicates the principal objectives of this PhD thesis and introduces our main topics' background.

Part I: free models in predicate fuzzy logics Although the general semantics presented in the seminal book of Hájek (1998) prompted the development of a systematic study of mathematical fuzzy logic, Horn clauses had not been extensively investigated. Regarding the primary goal of Part I, we intend to show some important properties of the universal Horn fragment of predicate fuzzy logics, focusing on free models of universal Horn theories and Herbrand structures. Specifically, we aim to enhance our understanding of the universal Horn fragment by obtaining results on the existence and characterization of free models of universal Horn theories.

Part II: art painting style categorization Concerning Part II's principal objective, we aim to use fuzzy Horn clauses combined with qualitative modeling as a knowledge representation framework for dealing with the problem of art painting style categorization. The classification of paintings in art styles has been widely explored from different approaches in artificial intelligence; however, according to the best of our knowledge, there are no research works that integrate fuzzy logics and qualitative modeling for this aim, allowing for explanations in human-machine interactions.

The work related to this doctoral thesis was conducted at the Artificial Intelligence Research Institute (Spanish National Research Council) and the Philosophy Department of the Autonomous University of Barcelona. Besides, essential contributions related to Part II were obtained during three research stays at the Bremen Spatial Cognition Center (University of Bremen), for which a YERUN Research Mobility Award (2018) and a DAAD short-term scholarship were obtained. The coordinator of these stays was Dr.-Ing. Zoe Falomir. Related to Part I, we first published the book chapter Costa & Dellunde (2015) and presented it at the 18th International Conference of the Catalan Association for Artificial Intelligence (University of Valencia). We also published the book chapter Costa & Dellunde (2017*b*) and presented it at the 47th International Symposium on Multiple-Valued Logic (University of Novi Sad). Furthermore, we published the journal paper Costa & Dellunde (2017*a*). Part of the results shown in this article was presented at the international conference Syntax Meets Semantics

2016 (University of Barcelona). The results obtained in Part I also contributed to the writing of the journal paper Badia et al. (2019), also authored by Dr. Badia (University of Queensland), Dr. Noguera (Institute of Information Theory and Automation, Czech Academy of Sciences), apart from the supervisor of this doctoral thesis Prof. Dr. Dellunde. Part of the results presented in Badia et al. (2019) was presented at the 7th international workshop on Many-Valued Logic (Paul Sabatier University) and the international conference Syntax Meets Semantics 2019 (University of Amsterdam). Concerning Part II, we first published the book chapter Costa et al. (2018), also authored by Dr.-Ing Falomir (Bremen Spatial Cognition Center, University of Bremen), and presented it at the 21st International Conference of the Catalan Association for Artificial Intelligence. We extended this work and published the journal paper Costa et al. (2021). During the research stays, we also published the journal articles Falomir et al. (2019), also authored by Prof. Dr. Gonzalez-Abril (University of Seville) and Dr.-Ing Falomir, and Falomir et al. (2020), also authored by Mr. Pich and Dr.-Ing Falomir. Finally, as the first step to a further research direction related to this doctoral thesis, the book chapter Costa (2020) has been published.

Next, to clarify the above-stated objectives, we introduce the main background of the topics related to this doctoral thesis.

Fuzzy model theory Classical model theory is the branch of mathematical logic that studies the construction and classification of structures. It started at the beginning of the last century, and nowadays, the field is an active part of mathematical logic. Chang & Keisler (1973) described model theory as the result of logic plus universal algebra. The recent roadmap of model theory has been focused on the development of stability theory (see, for instance, Pillay (2013)), whereas the field of applied model theory has repeatedly merged with stability theory, rising to geometric model theory. For a general introduction to model theory, see Kirby (2019) or Jahnke et al. (2018). In parallel to the development of fuzzy model theory, the stream of research of continuous model theory, Łukasiewicz logic expanded with connectives for each continuous function, has also been quite relevant (see, for instance, Yaacov et al. (2008)).

An important sub-area that grew out of computer science applications is finite model theory, which focuses on structures. Finite model theory benefited from a continuous interaction of different sub-areas of computer science as database theory and computational complexity. An overview of the relation between finite model theory and database theory has been presented by Vianu (1996). Cozman & Mauá (2019) discussed various facets of finite model theory of Bayesian networks, including definability and complexity of inferences. For an overview of finite model theory, the reader Grädel et al. (2007).

Mathematical fuzzy logic is concerned with the logical systems which involve some notion of truth degree. Its birth is traditionally placed at the crossroads of three areas: philosophy of science (the need to model reasoning in contexts with vague predicates mainly), fuzzy set theory (Zadeh (1965) defined the notion of fuzzy set for working with imprecision in the context of engineering applications), and many-valued logics (logical systems whose intended algebraic semantics present three or more truth values - for instance, Łukasiewicz (1920)). Hájek et al. (1996), Novák (1990*a*,*b*), and Pavelka (1979*a*,*b*,*c*) are part of the seminal works concerning mathematical fuzzy logic, which was first systematized by Hájek (1998). The series of handbooks Cintula et al. (2011*a*,*b*, 2015) have collected in-depth knowledge of the discipline. We refer the reader to XX for a comprehensive account of the history of mathematical fuzzy logic.

Fuzzy model theory (also named *graded model theory* or *model theory of predicate fuzzy logics*) is the generalized study, in mathematical fuzzy logic, of the construction and classification of graded structures. The field was properly started by Cintula & Hájek (2006), and it has received quite an attention recently. Bagheri & Moniri (2013) proved preservation theorems for Łukasiewicz logic. Ultraproducts in predicate fuzzy logics have been deeply analyzed by Dellunde (2012) and Dellunde (2014). Dellunde et al. (2016) obtained downward and upward L¨wenheim–Skolem theorems in the context of mathematical fuzzy logic. Badia & Noguera (2018) presented a graded analog of the Fraïssé limit.

Fuzzy Horn clauses Classical Horn clauses were firstly studied by McKinsey (1943) in the context of a decision problem for classes of first-order sentences without quantifiers. Since then, the influence of Horn clauses has been very remarkable in different areas such as, for instance, universal algebra or computer science. In classical predicate logic, a basic Horn formula is an implication where the antecedent is a conjunction of atomic formulas, and the consequent is an atomic formula. A quantifier-free Horn formula is built from basic Horn formulas with the conjunction. A Horn formula is built up from basic Horn formulas with the conjunction, the universal quantifier, and the existential quantifier. Furthermore, a universal Horn formula is the universal quantification of a quantifier-free Horn formula.

In universal algebra, Burris & Sankappanavar (1981) proved that a universal Horn class of structures is characterized by the closure under isomorphisms, substructures, and reduced products (alternatively, by the closure under isomorphisms, substructures, products, and ultraproducts). They also characterized the set of sentences logically equivalent to Horn clauses in terms of preservation under substructures and reduced products. The literature shows that Horn clauses also play a very significant role in the development of model theory. For instance, Fujiwara (1971) and Mal'tsev et al. (1971) obtained the least universal Horn classes containing different types of classes of first-order structures, and Givant (1978) described the universal Horn classes which could be categorical in infinite powers. The categoricity in power for universal Horn logic was independently analyzed by Abakumov et al. (1972) and Baldwin & Lachlan (1973). Furthermore, Chang & Keisler (1973) found the required conditions to characterize a reduced product sentence in terms of Horn sentences. McNulty (1977) presented two characterizations of the notion of universal Horn equivalence, proved a version of Beth's definability theorem for universal Horn logic, presented a modification of the notion of the consistent property more useful for Horn logic with the corresponding model existence theorem, and proved that the set of universal Horn sentences preserved under the formation of homomorphic images is not recursive. Preservation theorems for the universal Horn fragments of equality-free classical logic were showed by Dellunde & Jansana (1996). For a general discussion on the importance of Horn clauses in classical logic, including model theory, we refer the reader to Hodges (1993)).

In computer science, Horn clauses play an outstanding role, especially in logic programming (see Makowsky (1987) for a general presentation). Based on a procedural interpretation of Horn clauses developed by Kowalsky, Alain Colmerauer designed the programming language Prolog in 1972 (we refer the reader to Colmerauer & Roussel (1996) for an exposition of the birth of Prolog). An important problem in artificial intelligence, specifically for its applications in rule-based systems, is the SAT problem. The works of Yamasaki & Doshita (1983) and Dowling & Gallier (1984) presented algorithms for deciding whether a propositional Horn formula is satisfiable and proved that this problem could be decided in polynomial time. Minoux (1992) presented an algorithm for testing whether a given irreducible Horn formula is uniquely satisfiable. Escalada-Imaz & Manyà (1994) presented an almost linear algorithm for testing the multiple-valued Horn formulas' satisfiability. In the context of efficient knowledge representation systems, Selman & Kautz (1996) showed how propositional logical theories could be compiled into Horn theories that approximate the original information. Furthermore, Kanovich & Vauzeilles (2001) presented Horn logic as a comprehensive logical system capable of handling the usual artificial intelligence problem of planning the actions performed by a robot.

In fuzzy logic programming, there is a rich battery of proposals of Horn clauses which differ depending on the programming approach selected. For instance, Vojtás (2001) presented a truth-functional logic approach where the rules of the programs were many-valued functions. Ebrahim (2001) defined a Horn clause as a definite program clause or a definite goal. Both authors proved the soundness and completeness of their systems. Fuzzy logics have been applied to image interpretation; for example, Hudelot et al. (2008) developed a fuzzy spatial relation ontology to deal with brain structures in 3D magnetic resonance images. They have also been used in landslide identification and classification (Aksoy & Ercanoglu (2012)). Furthermore, Almubarak et al. (2017) proposed a fuzzy logic-based color histogram analysis for discriminating benign skin lesions from malignant melanomas in dermoscopy images. González et al. (2017), Rubio et al. (2017) applied a general type-2 fuzzy logic method for edge detection to color format images. Moreover, Dasiopoulou et al. (2010) developed a fuzzy description logic-based reasoning framework for an extracted description of an outdoor image.

In recent years several definitions of Horn clause have been proposed in the literature of mathematical fuzzy logic. One of the firsts was introduced by (Manyà 1999, Def 2.20). Regarding the universal Horn fragment of fuzzy equational logic, in the works of Belohlávek & Vychodil (2006*a*,*b*, 2005), Belohlávek (2003, 2002) and Vychodil (2015) the authors presented fuzzy equalities. They worked with theories that consisted of formulas that are implications between identities with premises weighted by truth degrees. They adopted Pavelka style: theories were fuzzy sets of formulas, and they considered degrees of provability of formulas from theories. The authors derived a Pavelka-style completeness theorem (the degree of provability equals the degree of truth) from which they got some particular cases by imposing restrictions on the formulas under consideration.

Art painting style classification Artificial intelligence is the science and engineering devoted to the design of virtual and physical machines that intend to perform intelligent tasks and activities involving other human faculties (e.g., association, perception, or motor control). In the last years, virtual art encyclopedias and virtual museum tours have increased the number of online images of art paintings, which may explain the interest in the challenge of applying artificial intelligence for the classification of art paintings. In the literature, research works that deal with this challenge are quite diverse. Traditional Chinese paintings were classified by Jiang et al. (2006) using color and SVMs. The authors proposed a scheme to detect traditional Chinese paintings from general images and categorize them into Gongbi (traditional Chinese realistic painting) and Xieyi (freehand style) schools. The classifier algorithm was tested on a dataset of 9515 images. Karayev et al. (2014) trained deep neural networks on object recognition for style categorization of artworks of Baroque, Impressionism, and Post-Impressionism. The work of Condorovici et al. (2015) presents a fusion scheme based on combining Multi-Layer Perceptron classified data with SVMs. The authors considered eight art painting styles (Baroque, Cubism, Renaissance, Byzantine Icons, Impressionism, Greek Pottery Paintings, Rococo, and Romanticism) and tested the classification on a dataset more than 4000 paintings.

Chapter 2

Presentation of the research

In this chapter, we present the main research of the doctoral thesis, divided into Parts I and II. To do it, we use some research papers published throughout the work related to this PhD thesis. The limited length of journal articles sometimes hampers a good exposition of the preliminaries on the studied topic. For this reason, in this chapter, it has been considered appropriate to include extended preliminaries sections for both parts, before presenting the research works.

2.1 Part I: free models of fuzzy universal Horn theories

The study of free models of fuzzy universal Horn theories is presented in the journal paper Costa & Dellunde (2017*a*) and the book chapter Costa & Dellunde (2017*b*). The research work Costa & Dellunde (2017*a*) is an extended and revised version of the book chapter Costa & Dellunde (2015), which got the best paper award of the 18th International Conference of the Catalan Association for Artificial Intelligence. Next, we present the preliminaries of Part I.

2.1.1 Preliminaries: t-norms and predicate fuzzy logics

Following the *Handbook of Mathematical Fuzzy Logic* (Cintula et al. (2011*a*,*b*, 2015)), this chapter sets up notation and presents some preliminaries on the predicate fuzzy logics studied in this thesis.

Monoidal t-norm based logic Triangular norms (or t-norms for short) appeared for the first time in 1942, in the work of Menger (1942). The aim of K. Menger, i.e., to yield metric spaces where probability distributions describe the distance between elements, led the author through a generalization of the classical triangle inequality. Initially, the set axioms for t-norms included parts that later were discarded and included as axioms for t-conorms. The set of axioms used currently was proposed by Schweizer & Sklar (1958, 1960, 1961). For a fuller treatment of related historical remarks, see Klement et al. (2013). Nowadays, the use of t-norms spreads throughout different disciplines. Although the origin of fuzzy logics goes back to the work of Łukasiewicz (1920), the term *fuzzy logic* was introduced by Zadeh (1965). In the last two decades, the systematization presented by Hájek (1998) has significantly contributed to the study of fuzzy logics. In the context of fuzzy logics, t-norms have been thoroughly applied. From now on, let $[0,1] \subseteq \mathbb{R}$ denote the closed unit interval of real numbers.

Definition 1 (Continuous and left-continuous t-norms (Hájek 1998, Def 2.1.1)) *A* t-norm $*$ *is a binary operation on* $[0,1]$ *satisfying the following conditions:*

(i) $*$ *is commutative and associative;*

(ii) $*$ *is non-decreasing in both arguments, that is, for any* $x, y, x_1, y_1, x_2, y_2 \in [0,1] \subseteq \mathbb{R}$*,*

$$x_1 \leq x_2 \textit{ implies } x_1 * y \leq x_2 * y \textit{ and}$$
$$y_1 \leq y_2 \textit{ implies } x * y_1 \leq x * y_2.$$

(iii) For any $x \in [0,1]$*,* $x * 1 = x$*.*

A t-norm $*$ *is said to be* continuous *(*left-continuous*) if it is a continuous (left-continuous) mapping of* $[0,1]^2$ *into* $[0,1]$*.*

The logic of all continuous t-norms is called *basic logic* (BL for short), and it was introduced by Hájek (1998). Monoidal t-norm based logic (MTL for short) is a propositional fuzzy logic introduced by Esteva & Godo (2001). This logic is weaker than BL, and it is the logic of all left continuous t-norms and their residua (Jenei & Montagna (2002)). The algebras corresponding to MTL are called MTL-algebras, and are defined as bounded residuated lattices $(A, \sqcap, \sqcup, *, \Rightarrow_*, 0, 1)$, where $\sqcap$ and $\sqcup$ are respectively the lattice meet and join operations and $(*, \Rightarrow_*)$ is a residuated pair satisfying the pre-linearity equation $(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1$ (see Noguera et al. (2005) for more details). Completeness of MTL with respect to MTL-algebras is proven in (Esteva & Godo 2001, Thm 1). Next, we present a set of axioms for MTL.

Definition 2 (Axiomatic system for MTL)

MTL1 $(\varphi \to \psi) \to ((\psi \to \xi) \to (\varphi \to \xi))$.

MTL2 $(\varphi \& \psi) \to \varphi$.

MTL3 $(\varphi \& \psi) \to (\psi \& \varphi)$.

MTL4 $(\varphi \wedge \psi) \to \varphi$.

MTL5 $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$.

MTL6 $(\varphi \& (\varphi \to \psi)) \to (\varphi \wedge \psi)$.

MTL7a $(\varphi \to (\psi \to \xi)) \to ((\varphi \& \psi) \to \xi)$.

MTL7b $((\varphi \& \psi) \to \xi) \to (\varphi \to (\psi \to \xi))$.

MTL8 $((\varphi \to \psi) \to \xi) \to (((\psi \to \varphi) \to \xi) \to \xi)$.

MTL9 $\overline{0} \to \varphi$.

Gödel logic, Product logic, and Łukasiewicz logic Gödel logic, Product logic, and Łukasiewicz logic (denoted by G, $\sqcap$ and Ł, respectively) are some of the most significant and well-known t-norm based logics.

Let us start with Gödel logic. Finite valued Gödel logics were first introduced by Gödel (1932), and expanding this logic with truth-constants has been explored by Guller (2015).

The *Gödel t-norm* $*_{\mathrm{G}}$ is defined as

$$x *_{\mathrm{G}} y = min\{x, y\}, \text{ for any } x, y \in [0,1].$$

Thus for any $x, y \in [0,1]$,

$$x \Rightarrow_{\mathrm{G}} y = \begin{cases} 1 & \text{if } x \leq y \\ y, & \text{otherwise.} \end{cases}$$

For any $x, y \in [0,1]$, the negation associated to $*_{\mathrm{G}}$ is defined as

$$\neg_{*_{\mathrm{G}}} x = \begin{cases} 1 & \text{if } x = 0 \\ 0, & \text{otherwise.} \end{cases}$$

An axiomatic system for G is obtained by adding to the axiomatic system of MTL (Definition 2) the following axioms:

BL $(\varphi \& (\varphi \to \psi)) \leftrightarrow (\varphi \wedge \psi)$.

G $\varphi \to (\varphi \& \varphi)$.

For a fuller exposition of Gödel logic the reader is referred to (Hájek 1998, Ch. 4 Section 2, and Ch. 5 Section 3) and (Cintula et al. 2011*b*, Ch. VII).

Let us present the Product logic, $\sqcap$. This logic was first introduced by Hájek, Godo and Esteva Hájek et al. (1996).

The *product t-norm* (also named *minimum t-norm*) $*_{\sqcap}$ is the product of two real numbers, that is,

$$x *_{\sqcap} y = xy, \text{ for any } x, y \in [0,1].$$

For any $x, y \in [0,1]$, the residuum associated to $*_{\sqcap}$, which is the truth function of the Goguen implication, is defined as:

$$x \Rightarrow_{\sqcap} y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x}, & \text{otherwise.} \end{cases}$$

For any $x, y \in [0,1]$, the negation associated to $*_{\sqcap}$ is defined as $\neg_{*_{\mathrm{G}}}$, that is,

$$\neg_{*_\sqcap} x = \begin{cases} 1 & \text{if } x = 0 \\ 0, & \text{otherwise.} \end{cases}$$

An axiomatic system for $\sqcap$ is obtained by adding to the axiomatic system of MTL (Definition 2) the following axioms:

⊓1 $\neg\neg\varphi \to (((\varphi \& \psi) \to (\varphi \& \xi)) \to (\psi \to \xi))$.

⊓2 $(\varphi \& \neg\varphi) \to \overline{0}$.

Finally let us present some basics on the infinite-valued Łukasiewicz logic (in this thesis the infinite-valued Łukasiewicz logic is called *Łukasiewicz logic* for short), denoted by Ł.

The *Łukasiewicz t-norm* $*_Ł$ is defined as:

$x *_Ł y = max\{0, x+y-1\}$, for any $x, y \in [0,1]$.

The residuum associated to $*_Ł$, which is the truth function of the Łukasiewicz implication, is defined as:

$x \Rightarrow_Ł y = min\{1-x+y, 1\}$, for any $x, y \in [0,1]$.

Finally, for any $x \in [0,1]$, the negation associated to $*_Ł$ is defined as $\neg_{*_Ł} x = 1-x$.

An axiomatic system for Ł is obtained by adding to the axiomatic system of MTL (Definition 2) the following axioms:

BL $(\varphi \& (\varphi \to \psi)) \leftrightarrow (\varphi \wedge \psi)$.

Ł $\neg\neg\varphi \to \varphi$.

Let us refer the reader to (Hájek 1998, Ch. 3, Section 3.3, and Ch. 5, Section 5.4), (Cintula et al. 2011*b*, Ch. 6), and (Cintula et al. 2015, Ch. XVII, Section 2.2) for further details on Łukasiewicz logic.

Pavelka (1979*a*,*b*,*c*) introduced the rational Pavelka logic. Later Novák (1987) extended rational Pavelka logic to first-order logic. For an extensive presentation of RPL the reader is referred to (Hájek 1998, Ch. 3, Section 3.3 and Ch. 5, Section 5.4) and (Cintula et al. 2011*b*, Ch. VIII).

Rational Pavelka Predicate Logic RPL is the expansion of Ł∀ by adding a truth constant $\overline{r}$ for each rational number r in $[0,1]$ and by adding the axioms RPL1 and RPL2:

(RPL1) $(\overline{r} \to \overline{s}) \leftrightarrow \overline{r \Rightarrow s}$.

(RPL2) $(\overline{r} \& \overline{s}) \leftrightarrow \overline{r \& s}$.

Definition 3 (Evaluated formula and evaluated atomic formula) *An* evaluated formula (φ, r) *in a language of* RPL *is a formula of the form* $\overline{r} \to \varphi$*, where* $r \in [0,1]$ *is a rational number and* φ *is a formula without truth constants apart from* $\overline{0}$ *and* $\overline{1}$*. We say that an evaluated formula* (φ, r) *is* atomic *whenever* φ *is atomic.*

Definition 4 (The standard RPL-algebra) *The* standard RPL-algebra*, denoted by* $[0,1]_{\text{RPL}}$*, is the expansion of the standard* MV*-algebra* $[0,1]_{\text{Ł}}$ *with each truth constant* $\overline{r}$ *interpreted as* r *and the 'book-keeping axioms' for each rational* $r,s \in [0,1]$*:*

- $\overline{r}\&\overline{s} \leftrightarrow \overline{r*s}$.
- $(\overline{r} \to \overline{s}) \leftrightarrow \overline{r \Rightarrow s}$.

Predicate fuzzy logics

Cintula & Hájek (2006) introduced the class of core fuzzy logics, a class of extensions of MTL, in order to provide a common framework to the study of first-order fuzzy logics. In this chapter we recall the syntax and semantics for predicate fuzzy logics based in core fuzzy logics, and later we will present our results only for some of these logics.

Definition 5 (Core fuzzy logic) *A logic* L *is said to be a* core fuzzy logic *if it expands* MTL*, has local deduction theorem and the substitution rule holds.*

For some important logical and algebraic properties of core fuzzy logics the reader is referred to (Cintula et al. 2011*a*, Ch. III, Thm 3.2.11). Throughout this section let L be a core fuzzy logic in a propositional language $\mathscr{L}$. As next definition shows, the language of first-order fuzzy logic is defined as in classical first-order logic.

Definition 6 (Language of first-order logic) *A* predicate language $\mathscr{P}$ *consists in a triple* $\langle Pred_{\mathscr{P}}, Func_{\mathscr{P}}, Ar_{\mathscr{P}}\rangle$*, where* $Pred_{\mathscr{P}}$ *is a nonempty set of* predicate symbols*,* $Func_{\mathscr{P}}$ *is a set of* function symbols *(disjoint with* $Pred_{\mathscr{P}}$*), and* $Ar_{\mathscr{P}}$ *represents the* arity function*, which assigns a natural number to each predicate symbol or function symbol. This natural number is called the* arity of the symbol*. The predicate symbols with arity zero are called* truth constants*, whereas the function symbols whose arity is zero are named* individual constants *(*constants *for short).*

The set of $\mathscr{P}$-terms, $\mathscr{P}$-formulas and the notions of free occurrence of a variable, open formula, substitutability and sentence (or closed formula) are defined as in classical predicate logic. From now on, when it is clear from the context, $\mathscr{P}$-terms and $\mathscr{P}$-formulas are simply called *terms* and *formulas*, respectively. A term t is *ground* if it has no variables. Throughout this thesis, a *theory* is a set of sentences. Observe that a theory is not necessarily closed under the consequence relation. We work with the first-order extension of L of models over linear algebras, L$\forall$. An axiomatic system of the logic L$\forall$ is defined as follows.

(P) Instances of the axioms of L (the propositional variables are substituted for first-order formulas).

($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$, where the term t is substitutable for x in φ.

($\exists$1) $\varphi(t) \to (\exists x)\varphi(x)$, where the term t is substitutable for x in φ.

($\forall$2) $(\forall x)(\xi \to \varphi) \to (\xi \to (\forall x)\varphi)$, where x is not free in ξ.

($\exists$2) $(\forall x)(\varphi \to \xi) \to ((\exists x)\varphi \to \xi)$, where x is not free in ξ.

($\forall$3) $(\forall x)(\xi \vee \varphi) \to \xi \vee (\forall x)\varphi$, where x is not free in ξ.

The deduction rules of L$\forall$ are those of L and the rule of generalization: from φ infer $(\forall x)\varphi$. The definitions of proof and provability are analogous to the classical ones. The expression $\Phi \vdash_{L\forall} \varphi$ denotes the fact that φ is provable in L$\forall$ from the set of formulas Φ. For the sake of clarity, when it is clear from the context we write $\vdash$ to refer to $\vdash_{L\forall}$. A theory Φ is *consistent* if $\Phi \nvdash \overline{0}$. Next we present some theorems of the logic L$\forall$.

Proposition 1 ((Cintula et al. 2011*a*, Ch. I, Thm 5.1.4)) *Let φ, ψ, ξ be formulas, x a variable not free in ξ, and x' a variable not occurring in φ. The following formulas are theorems of* L$\forall$*:*

(T$\forall$1) $\psi \leftrightarrow (\forall x)\psi$.

(T$\forall$2) $\psi \leftrightarrow (\exists x)\psi$.

(T$\forall$3) $(\forall x)\varphi(x) \leftrightarrow (\forall x')\varphi(x')$.

(T$\forall$4) $(\exists x)\varphi(x) \leftrightarrow (\exists x')\varphi(x')$.

(T$\forall$5) $(\forall x)(\forall y)\varphi \leftrightarrow (\forall y)(\forall x)\varphi$.

(T$\forall$6) $(\exists x)(\exists y)\varphi \leftrightarrow (\exists y)(\exists x)\varphi$.

(T$\forall$7) $((\forall x)(\varphi \to \psi)) \to ((\forall x)\varphi \to (\forall x)\psi)$.

(T$\forall$8) $((\forall x)(\varphi \to \psi)) \to ((\exists x)\varphi \to (\exists x)\psi)$.

(T$\forall$9) $(\xi \to (\forall x)\varphi) \leftrightarrow ((\forall x)(\xi \to \varphi))$.

(T$\forall$10) $((\exists x)\varphi \to \xi) \leftrightarrow ((\forall x)(\varphi \to \xi))$.

(T$\forall$11) $((\exists x)(\xi \to \varphi)) \to (\xi \to (\exists x)\varphi)$.

(T$\forall$12) $((\exists x)(\varphi \to \xi)) \to ((\forall x)\varphi \to \xi)$.

(T$\forall$13) $((\forall x)\varphi \wedge (\forall x)\psi) \leftrightarrow ((\forall x)(\varphi \wedge \psi))$.

(T$\forall$14) $((\exists x)(\varphi \vee \psi)) \leftrightarrow ((\exists x)\varphi \vee (\exists x)\psi)$.

(T$\forall$15) $((\forall x)\varphi \vee \psi) \leftrightarrow ((\forall x)(\varphi \vee \psi))$.

(T$\forall$16) $((\exists x)(\varphi \wedge \psi)) \leftrightarrow ((\exists x)\varphi \wedge \psi)$.

(T$\forall$17) $((\exists x)(\varphi \& \psi)) \leftrightarrow ((\exists x)\varphi \& \psi)$.

(T$\forall$18) $(\exists x)\varphi^n \leftrightarrow ((\exists x)\varphi)^n$.

(T$\forall$19) $(\exists x)\varphi \to (\neg(\forall x)\neg\varphi)$.

(T∀20) $(\neg(\exists x)\varphi) \to ((\forall x)\neg\varphi)$.

(T∀21) $((\exists x)\varphi \wedge \xi) \to ((\exists x)(\varphi \wedge \xi))$.

Let us now introduce the semantics of predicate core fuzzy logics. Throughout this section, let $\boldsymbol{A}$ be an L-algebra.

Definition 7 (Semantics of Predicate Fuzzy Logics)
Consider a predicate language $\mathscr{P} = \langle Pred_{\mathscr{P}}, Func_{\mathscr{P}}, Ar_{\mathscr{P}}\rangle$ and let $\mathbf{A}$ *be an* L*-algebra. An* $\boldsymbol{A}$-structure $\mathbf{M}$ *for* $\mathscr{P}$ *is a triple* $\langle M, (P_{\mathrm{M}})_{P\in Pred}, (F_{\mathrm{M}})_{F\in Func}\rangle$*, where* M *is a nonempty domain,* P_{M} *is a fuzzy relation on* M *for each n-ary predicate symbol, i.e., a function from* M^n *to* $\mathbf{A}$*, identified with an element of* $\mathbf{A}$ *if* $n = 0$*; and* F_{M} *is a function from* M^n *to* M*, identified with an element of* M *if* $n = 0$*. If* $\mathbf{M}$ *is an* $\mathbf{A}$*-structure for* $\mathscr{P}$*, an* $\mathbf{M}$*-evaluation of the object variables is a mapping* v *assigning to each object variable an element of* M*. The set of all object variables is denoted by* Var*. If* v *is an* $\mathbf{M}$*-evaluation,* x *is an object variable and* $a \in M$*, it is denoted by* $v[x \mapsto a]$ *the* $\mathbf{M}$*-evaluation so that* $v[x \mapsto a](x) = a$ *and* $v[x \mapsto a](y) = v(y)$ *for each object variable* y *such that* $y \neq x$*. If* $\mathbf{M}$ *is an* $\mathbf{A}$*-structure and* v *is an* $\mathbf{M}$*-evaluation, the* values *of terms and the* truth values *of formulas in* M *are defined recursively as follows:*

$||x||^{\boldsymbol{A}}_{\mathrm{M},v} = v(x)$;

$||F(t_1,\dots,t_n)||^{\boldsymbol{A}}_{\mathrm{M},v} = F_{\mathrm{M}}(||t_1||^{\boldsymbol{A}}_{\mathrm{M},v},\dots,||t_n||^{\boldsymbol{A}}_{\mathrm{M},v})$, *for* $F \in Func$;

$||P(t_1,\dots,t_n)||^{\boldsymbol{A}}_{\mathrm{M},v} = P_{\mathrm{M}}(||t_1||^{\boldsymbol{A}}_{\mathrm{M},v},\dots,||t_n||^{\boldsymbol{A}}_{\mathrm{M},v})$, *for* $P \in Pred$;

$||c(\varphi_1,\dots,\varphi_n)||^{\boldsymbol{A}}_{\mathrm{M},v} = c_{\boldsymbol{A}}(||\varphi_1||^{\boldsymbol{A}}_{\mathrm{M},v},\dots,||\varphi_n||^{\boldsymbol{A}}_{\mathrm{M},v})$, *for* $c \in \mathscr{L}$;

$||(\forall x)\varphi||^{\boldsymbol{A}}_{\mathrm{M},v} = inf\{||\varphi||^{\boldsymbol{A}}_{\mathrm{M},v[x\to a]} \mid a \in M\}$;

$||(\exists x)\varphi||^{\boldsymbol{A}}_{\mathrm{M},v} = sup\{||\varphi||^{\boldsymbol{A}}_{\mathrm{M},v[x\to a]} \mid a \in M\}$.

If the infimum or the supremum do not exist, the truth value of the quantified formula is undefined*. It is said that an* $\boldsymbol{A}$*-structure is* safe *if* $||\varphi||^{\boldsymbol{A}}_{\mathrm{M},v}$ *is defined for each formula* φ *and each* $\mathbf{M}$*-evaluation* v*.*

For a set of formulas Φ, $||\Phi||^{\boldsymbol{A}}_{\mathrm{M},v} = 1$ stands for $||\varphi||^{\boldsymbol{A}}_{\mathrm{M},v} = 1$ for every $\varphi \in \Phi$. It is said that $\langle \boldsymbol{A}, \mathrm{M}\rangle$ is a *model of a set of formulas* Φ if $||\varphi||^{\boldsymbol{A}}_{\mathrm{M},v} = 1$ for every $\varphi \in \Phi$ and every $\mathbf{M}$-evaluation v. The notation $||\varphi(a_1,\dots,a_n)||^{\boldsymbol{A}}_{\mathrm{M}}$ means that $||\varphi(x_1,\dots,x_n)||^{\boldsymbol{A}}_{\mathrm{M},v}$ for every $v(x_i) = a_i$ for $i \in \{1,\dots,n\}$. Besides, $||\varphi||^{\boldsymbol{A}}_{\mathbf{M}} = 1$ denotes that $||\varphi||^{\boldsymbol{A}}_{\mathbf{M},v} = 1$ for all $\mathbf{M}$-evaluation v. From now on when it is clear from the context $\boldsymbol{A}$-structures are simply referred as *structures*. A formula φ is called *satisfiable* if there exists a structure $\langle \boldsymbol{A}, \mathbf{M}\rangle$ such that $||\varphi||^{\boldsymbol{A}}_{\mathbf{M}} = 1$. In such case, it is also said that φ is *satisfied by* $\langle \boldsymbol{A}, \mathbf{M}\rangle$ or that $\langle \boldsymbol{A}, \mathbf{M}\rangle$ *satisfies* φ. In addition, for an $\mathbf{M}$-evaluation v, $\langle \boldsymbol{A}, \mathbf{M}\rangle \models \varphi[v]$ means that $||\varphi||^{\boldsymbol{A}}_{\mathrm{M},v} = 1$.

Theorem 1 (Linear strong completeness for L∀ (Cintula et al. 2011*a*, Ch. III, Thm 5.2.3)) *Let* T *be a* $\mathscr{P}$*-theory, and* φ *a* $\mathscr{P}$*-formula. The following are equivalent:*

- $T \vdash_{\mathrm{L}\forall} \varphi$.
- $\langle \boldsymbol{A}, \mathbf{M} \rangle \models \varphi$ *for each linear model* $\langle \boldsymbol{A}, \mathbf{M} \rangle$ *of the theory* T.

The notion of homomorphism is specially relevant for our study of predicate fuzzy logics, since it allows us to relate a pair of structures Several notions of homomorphism have been proposed in the literature. Thus clarifying which proposal is selected is mandatory whenever these functions are used. Indeed, the definition of homomorphism selected determines how the relation between two first-order structures is understood. An in-depth investigation of different definitions of homomorphism in the context of fuzzy logics was carried out by Dellunde et al. (2016), and here a generalization of the classical homomorphism is taken from this reference.

Definition 8 (Homomorphism) *Let* $\langle \boldsymbol{A}, \mathrm{M} \rangle$ *and* $\langle \boldsymbol{B}, \mathrm{N} \rangle$ *be structures,* f *a mapping from* $\boldsymbol{A}$ *to* $\boldsymbol{B}$ *and* g *a mapping from* M *to* N. *The pair* (f, g) *is said to be a* homomorphism *from* $\langle \boldsymbol{A}, \mathrm{M} \rangle$ *to* $\langle \boldsymbol{B}, \mathrm{N} \rangle$ *if* f *is a homomorphism of* L*-algebras and for every* n*-ary function symbol* F *and* $d_1, \ldots, d_n \in M$,

$$g(F_{\mathrm{M}}(d_1, \ldots, d_n)) = F_{\mathrm{N}}(g(d_1), \ldots, g(d_n))$$

and for every n*-ary predicate symbol* P *and* $d_1, \ldots, d_n \in M$,

$$(*) \text{ If } P_{\mathrm{M}}(d_1, \ldots, d_n) = 1, \text{ then } P_{\mathrm{N}}(g(d_1), \ldots, g(d_n)) = 1.$$

It is said that (f, g) *is* strict *if instead of* (*) *it satisfies the stronger condition: for every* n*-ary predicate symbol* P *and* $d_1, \ldots, d_n \in M$,

$$P_{\mathrm{M}}(d_1, \ldots, d_n) = 1 \text{ if and only if } P_{\mathrm{N}}(g(d_1), \ldots, g(d_n)) = 1.$$

Moreover it is said that (f, g) *is an* embedding *if it is a strict homomorphism and both functions* f *and* g *are injective. And it is said that an embedding* (f, g) *is an* isomorphism *if both functions* f *and* g *are surjective.*

Definition 9 (Witnessed models) *A formula of the form* $(\exists x)\varphi(y_1, \ldots, y_n)$ *is* witnessed *in a structure* $\langle \boldsymbol{A}, \mathbf{M} \rangle$ *if for each* $\mathbf{M}$*-evaluation* v *with* $v(y_i) = a_i$ *for* $i \in \{1, \ldots, n\}$, *there exists* $b \in M$ *such that*

$$||(\exists x)\varphi(x, a_1, \ldots, a_n)||^{\boldsymbol{A}}_{\mathbf{M}} = ||\varphi(b, a_1, \ldots, a_n)||^{\boldsymbol{A}}_{\mathbf{M}};$$

and analogously for $(\forall x)$. *It is said that* $\langle \boldsymbol{A}, \mathbf{M} \rangle$ *is a* witnessed model *if each formula beginning with a quantifier is witnessed in* $\langle \boldsymbol{A}, \mathbf{M} \rangle$.

We assume that all the predicate languages contain a similarity symbol $\approx$, and the axiomatization of the logic is extended with the following axioms: Throughout this thesis the $\approx$ symbol is considered as a binary predicate symbol, not as a logical symbol, and thereby its interpretation is not fixed.

Definition 10 (Axioms of similarity and congruence) *Let* $\approx$ *be a binary predicate symbol, the following are the axioms of similarity and congruence:*

S1. $(\forall x) x \approx x$.

S2. $(\forall x)(\forall y)(x \approx y \to y \approx x)$.

S3. $(\forall x)(\forall y)(\forall z)(x \approx y \& y \approx z \to x \approx z)$.

C1. For each n-ary function symbol F,

$$(\forall x_1)\cdots(\forall x_n)(\forall y_1)\cdots(\forall y_n)(x_1 \approx y_1 \& \cdots \& x_n \approx y_n \to F(x_1,\ldots,x_n) \approx F(y_1,\ldots,y_n)).$$

C2. For each n-ary predicate symbol P,

$$(\forall x_1)\cdots(\forall x_n)(\forall y_1)\cdots(\forall y_n)(x_1 \approx y_1 \& \cdots \& x_n \approx y_n \to P(x_1,\ldots,x_n) \leftrightarrow P(y_1,\ldots,y_n)).$$

The extension of L$\forall$ with the similarity and congruence axioms is denoted by L$\forall_{\approx}$.

Definition 11 (Truth degree and provability degree) *Let Φ be a theory, and $r \in [0,1]$ a rational number.*

(i) *The* truth degree *of φ over Φ is $||\varphi||_\Phi =$*

$$inf\{||\varphi||_{\mathbf{M}}^{[0,1]_{\mathrm{RPL}}} \mid \langle [0,1]_{\mathrm{RPL}}, \mathbf{M}\rangle \textit{ is a model of } \Phi\}.$$

(ii) *The* provability degree *of φ over Φ is*

$$|\varphi|_\Phi = sup\{r \mid \Phi \vdash \overline{r} \to \varphi\}.$$

The logic RPL$\forall_{\approx}$ is not strongly complete with respect to its standard semantics, but it is complete in a weak sense. Indeed, it satisfies what is called Pavelka-style completeness, that is, truth degree and provability degree coincide.

Theorem 2 (Pavelka-style completeness (Hájek 1998, Thm 5.4.10)) *Let Φ be a theory and φ a formula. Then,*

$$|\varphi|_\Phi = ||\varphi||_\Phi.$$

2.1.2 A summary of the main original contributions of Part I

In this section, we first introduce the key notions related to Part I and the original contributions of the published papers included in this PhD thesis.

Free models The notion of free structure is especially important in our study of the universal Horn fragment of fuzzy logics. Free structures were introduced by Goguen et al. (1975) in category theory (see (Barr & Wells 1990, Def. 4.7.17) for a definition of free structure in category theory). They are very relevant in different areas. In classical logic, free structures have been used to prove the satisfiability of a set of consistent sentences ((Ebbinghaus et al. 1994, Ch.V)). In the context of logic programming, they allow a procedural interpretation of the programs, and admitting free structures makes reasonable the *negation as failure* (Makowsky (1987)). And in the context of abstract data types, Tarlecki (1985) characterized abstract algebraic institutions that admit free constructions.

Herbrand structures A Herbrand structure is defined only by the syntactical properties of its vocabulary. In classical logic, Herbrand structures have been widely used (see, for instance, (Ebbinghaus et al. 1994, Ch.11)). Regarding fuzzy logics, Baaz & Metcalfe (2008) presented approximate Herbrand theorems for first-order t-norm based logics, provided proof-theoretic proofs of Skolemization for its prenex fragments. Concerning Łukasiewicz logic, an approximate version of Herbrand's theorem was first presented by Novák (1996). Furthermore, Herbrand structures also have an important role in the foundation of logic programming (see, for example, the work of Dovier & Pontelli (2010)). And the same occurs in the case of fuzzy logic programming. In a series of papers Gerla (2005, 2001*a*,*b*) proposed to base fuzzy control on fuzzy logic programming and observed that the class of fuzzy Herbrand interpretations gives semantics for fuzzy programs. Gerla worked with a complete, completely distributive, lattice of truth-values. Other approaches are, for instance, Vojtás (2001) and Ebrahim (2001).

Original Contributions

- Definition of term structure associated to a consistent theory in MTL$\forall_{\approx}$. If the theory consists of universal Horn formulas, we show that the associated term structure is a model of the theory. [Definition 10 and Theorem 2 of Costa & Dellunde (2017*a*)]

- Proof that term structures that are models of its associated theory, are free in the class of reduced models of the theory (i.e., an analog of Mal'tsev theorem – see Mal'tsev (1971) – is proved for MTL$\forall_{\approx}$). Besides, the proof of this result provides an explicit criterion for defining such models. The possibility given by fuzzy logics of defining the term structure associated to a theory using the similarity symbol $\approx$ leads us to a notion of free structure restricted to the class of reduced models of that theory. In case that the interpretation of the similarity symbol $\approx$ is crisp, the above mentioned term model is free over the class of all models of the theory (Theorem 1 of Costa & Dellunde (2017*a*))

- Proof that consistent universal Horn theories over MTL$\forall_{\approx}$ (containing only the truth-constants $\overline{1}$ and $\overline{0}$) have classical models. [Corollary 1 of Costa & Dellunde (2017*a*)]

- Definition of Herbrand structures in MTL$\forall_{\approx}$. (Definition 12 of Costa & Dellunde (2017*a*))

- Proof that every equality-free universal Horn theory over MTL$\forall_{\approx}$ has a Herbrand model. [Corollary 2 of Costa & Dellunde (2017*a*)]

- Introduction of the notion of term structure associated to a theory in RPL$\forall_{\approx}$. [Definition 7 of Costa & Dellunde (2017*b*)]

- Proof that term structures that are models of its associated theory, are free in the class of RPL$\forall_{\approx}$ reduced models of the theory (in all models, if the similarity symbol has a crisp interpretation). [Theorem 2of Costa & Dellunde (2017*b*)]

- Proof of the existence of free models in fuzzy universal Horn classes of structures in RPL$\forall_{\approx}$. [Theorem 3 of Costa & Dellunde (2017*b*)]

2.1.3 On the existence of free models in fuzzy universal Horn classes

Next, we include in this chapter the journal article Costa & Dellunde (2017*a*), titled *On the existence of free models in fuzzy universal Horn classes*[1]

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On the existence of free models in fuzzy universal Horn classes

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ABSTRACT

This paper is a contribution to the study of the universal Horn fragment of predicate fuzzy logics, focusing on some relevant notions in logic programming. We introduce the notion of *term structure associated to a set of formulas* in the fuzzy context and we show the existence of free models in fuzzy universal Horn classes. We prove that every equality-free consistent universal Horn fuzzy theory has a Herbrand model.

1. Introduction

Since their introduction in [28], Horn clauses have shown to have good logic properties and have proven to be of importance for many disciplines, ranging from logic programming, abstract specification of data structures and relational data bases, to abstract algebra and model theory. However, the analysis of Horn clauses has been mainly restricted to the sphere of classical logic. For a good exposition of the most relevant results concerning Horn clauses in classical logic we refer to [24], and to [26] for a good study of their importance in computer science.

The interest in continuous t-norm based logics since its systematization by Hájek [23] and the subsequent study of core fuzzy logics [9] invite to a systematic development of a model theory of these logics (and of algebraizable non-classical logics in general). Cintula and Hájek raised the open question of characterizing theories of Horn clauses in predicate fuzzy logics [9]. Our first motivation to study the Horn fragment of predicate fuzzy logics was to solve this open problem, the present article is a first contribution towards its solution.

Some authors have contributed to the study of Horn clauses over fuzzy logic. In [6,5,4,2,3,31] Bělohlávek and Vychodil study fuzzy equalities, they work with theories that consist of formulas that are implications between identities with premises weighted by truth degrees. They adopt Pavelka style: theories are fuzzy sets

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of formulas and they consider degrees of provability of formulas from theories. Their basic structure of truth degrees is a complete residuated lattice. The authors derive a Pavelka-style completeness theorem (degree of provability equals degree of truth) from which they get some particular cases by imposing restrictions on the formulas under consideration. As a particular case, they obtain completeness of fuzzy equational logic. In different articles they study the main logical properties of varieties of algebras with fuzzy equalities. Taking a different approach, in a series of papers [21,20,19], Gerla proposes to base fuzzy control on fuzzy logic programming, and observes that the class of fuzzy Herbrand interpretations gives a semantics for fuzzy programs. Gerla works with a complete, completely distributive, lattice of truth-values. For a reference on fuzzy logic programming see [30,17].

Several definitions of Horn clause have been proposed in the literature of fuzzy logics, but there is not a canonical one yet. Cintula and Hájek affirm that the elegant approach of [2] is not the only possible one. In [15], Dubois and Prade discuss different possibilities of defining *fuzzy rules* and they show how these different semantics can be captured in the framework of fuzzy set theory and possibility theory. Following all these works, our contribution is a first step towards a systematic model-theoretic account of Horn clauses in the framework introduced by Hájek in [23]. We introduce a basic definition of Horn clause over the predicate fuzzy logic MTL$\forall^m$ that extends the classical one in a natural way. In future work we will explore different generalizations of our definitions for expanded languages. Our approach differs from the one of Bělohlávek and Vychodil because we do not restrict to fuzzy equalities. Another difference is that, unlike these authors and Gerla, our structures are not necessarily over the same complete algebra, because we work in the general semantics of [23].

In the present work we have focused on the study of *free models of Horn clauses*. Free structures have a relevant role in classical model theory and logic programming. Admitting free structures makes reasonable the concepts of *closed-word assumption* for databases and *negation as failure* for logic programming. These structures allow also a procedural interpretation for logic programs (for a reference see [26]). Free structures of a given class are minimal from an algebraic point of view, in the sense that there is a unique homomorphism from these structures to any other structure in the class. The free structures introduced here are *term structures*, structures whose domains consist of terms or equivalence classes of terms of the language. In classical logic, term structures have been used to prove the satisfiability of a set of consistent sentences, see for instance [16, Ch.5]. Notorious examples of term structures are Herbrand models, they play an important function in the foundations of logic programming. Several authors have been studied Herbrand models in the fuzzy context (for a reference see [19,30,17]), providing theoretical background for different classes of fuzzy expert systems. For a general reference on Herbrand Theorems for substructural logics we refer to [7].

The present paper is an extension of the work presented in the 18th International Conference of the Catalan Association for Artificial Intelligence (CCIA 2015) [11]. Our main original contributions are the following:

- Introduction of the notion of term structure associated to a theory over predicate fuzzy logics. If the theory consists of universal Horn formulas, we show that the associated term structure is a model of the theory (Theorem 2).
- Existence of free models in fuzzy universal Horn classes of structures. In the case that the language has an equality symbol $\approx$ interpreted as a similarity, we prove the existence of models which are free in the class of reduced models of the theory (Theorem 1). In the case that the language has the crisp identity, the class has free models in the usual sense.
- Consistent universal Horn theories over predicate fuzzy logics (that contains only the truth-constants $\overline{1}$ and $\overline{0}$) have classical models (Corollary 1).
- Introduction of Herbrand structures. We prove that every equality-free consistent universal Horn theory over predicate fuzzy logics have a Herbrand model (Corollary 2).

The paper is organized as follows. Section 2 contains the preliminaries on predicate fuzzy logics. In Section 3 we introduce the definition of Horn clause over predicate fuzzy logics. In Section 4 we study the term structures associated to universal Horn theories. In Section 5 we introduce Herbrand structures for equality-free theories. Finally, there is a section devoted to conclusions and future work.

2. Preliminaries

Our study of the model theory of Horn clauses is focused on the basic predicate fuzzy logic MTL$\forall^m$ and some of its extensions based on propositional core fuzzy logics in the sense of [9]. The logic MTL$\forall^m$ is the predicate extension of the left-continuous t-norm based logic MTL introduced in [18], where MTL-algebras are defined as bounded integral commutative residuated lattices $(A, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$, where $\sqcap$ and $\sqcup$ are respectively the lattice meet and join operations and $(\Rightarrow, *)$ is a residuated pair, satisfying the pre-linearity equation $(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1$ (for an exhaustive exposition of MTL-algebras, see [29]). In addition, completeness of this logic with respect to MTL-algebras is proven in [18, Th.1], and Jenei and Montagna shown that MTL is the logic of all left continuous t-norms and their residua [25]. Now we present the syntax and semantics of predicate fuzzy logics and we refer to [8, Ch.1] for a complete and extensive presentation.

Definition 1 *(Syntax of predicate languages).* A *predicate language* $\mathcal{P}$ is a triple $\langle Pred_{\mathcal{P}}, Func_{\mathcal{P}}, Ar_{\mathcal{P}}\rangle$, where $Pred_{\mathcal{P}}$ is a nonempty set of *predicate symbols*, $Func_{\mathcal{P}}$ is a set of *function symbols* (disjoint from $Pred_{\mathcal{P}}$), and $Ar_{\mathcal{P}}$ represents the *arity function*, which assigns a natural number to each predicate symbol or function symbol. We call this natural number the *arity of the symbol*. The predicate symbols with arity zero are called *truth constants*, while the function symbols whose arity is zero are named *individual constants* (*constants* for short) or *objects*.

The set of $\mathcal{P}$-terms, $\mathcal{P}$-formulas and the notions of free occurrence of a variable, open formula, substitutability and sentence are defined as in classical predicate logic. From now on, when it is clear from the context, we will refer to $\mathcal{P}$-terms and $\mathcal{P}$-formulas simply as *terms* and *formulas*. A term t is *ground* if it has no variables. Throughout the paper we consider the equality symbol as a binary predicate symbol, not as a logical symbol, that is, the equality symbol is not necessarily present in all the languages and its interpretation is not fixed. From now on, let L be a core fuzzy logic in a propositional language $\mathcal{L}$ that contains only the truth-constants $\overline{1}$ and $\overline{0}$ (for an extended study of core fuzzy logics, see [9]).

Definition 2. We introduce an axiomatic system for the predicate logic $L\forall^m$:

(P) Instances of the axioms of L (the propositional variables are substituted for first-order formulas).
($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$, where the term t is substitutable for x in φ.
($\exists$1) $\varphi(t) \to (\exists x)\varphi(x)$, where the term t is substitutable for x in φ.
($\forall$2) $(\forall x)(\xi \to \varphi) \to (\xi \to (\forall x)\varphi(x))$, where x is not free in ξ.
($\exists$2) $(\forall x)(\varphi \to \xi) \to ((\exists x)\varphi \to \xi)$, where x is not free in ξ.

The deduction rules of $L\forall^m$ are those of L and the rule of generalization: from φ infer $(\forall x)\varphi$. The definitions of proof and provability are analogous to the classical ones. We denote by $\Phi \vdash_{L\forall^m} \varphi$ the fact that φ is provable in $L\forall^m$ from the set of formulas Φ. For the sake of clarity, when it is clear from the context we will write $\vdash$ to refer to $\vdash_{L\forall^m}$. A set of formulas Φ is *consistent* if $\Phi \nvdash \overline{0}$.

Definition 3 *(Semantics of predicate fuzzy logics).* Consider a predicate language $\mathcal{P} = \langle Pred_{\mathcal{P}}, Func_{\mathcal{P}}, Ar_{\mathcal{P}}\rangle$ and let $\mathbf{A}$ be an L-algebra. We define an $\boldsymbol{A}$-*structure* $\mathbf{M}$ for $\mathcal{P}$ as the triple $\langle M, (P_M)_{P\in Pred}, (F_M)_{F\in Func}\rangle$, where M is a nonempty domain, $P_{\mathbf{M}}$ is an n-ary fuzzy relation for each n-ary predicate symbol, i.e., a function from M^n to $\boldsymbol{A}$, identified with an element of $\boldsymbol{A}$ if $n = 0$; and $F_{\mathbf{M}}$ is a function from M^n to M, identified with an element of M if $n = 0$. As usual, if $\mathbf{M}$ is an $\boldsymbol{A}$-structure for $\mathcal{P}$, an $\mathbf{M}$-evaluation of the object variables is a mapping v assigning to each object variable an element of M. The set of all object variables is denoted by Var. If v is an $\mathbf{M}$-evaluation, x is an object variable and $a \in M$, we denote by $v[x \mapsto a]$ the $\mathbf{M}$-evaluation so that $v[x \mapsto a](x) = a$ and $v[x \mapsto a](y) = v(y)$ for y an object variable such that $y \neq x$. If $\mathbf{M}$ is an $\boldsymbol{A}$-structure and v is an $\mathbf{M}$-evaluation, we define the *values* of terms and the *truth values* of formulas in M for an evaluation v recursively as follows:

$\|x\|^{\boldsymbol{A}}_{\mathbf{M},v} = v(x)$;
$\|F(t_1, \ldots, t_n)\|^{\boldsymbol{A}}_{\mathbf{M},v} = F_{\mathbf{M}}(\|t_1\|^{\boldsymbol{A}}_{\mathbf{M},v}, \ldots, \|t_n\|^{\boldsymbol{A}}_{\mathbf{M},v})$, for $F \in Func$;
$\|P(t_1, \ldots, t_n)\|^{\boldsymbol{A}}_{\mathbf{M},v} = P_{\mathbf{M}}(\|t_1\|^{\boldsymbol{A}}_{\mathbf{M},v}, \ldots, \|t_n\|^{\boldsymbol{A}}_{\mathbf{M},v})$, for $P \in Pred$;
$\|c(\varphi_1, \ldots, \varphi_n)\|^{\boldsymbol{A}}_{\mathbf{M},v} = c_{\boldsymbol{A}}(\|\varphi_1\|^{\boldsymbol{A}}_{\mathbf{M},v}, \ldots, \|\varphi_n\|^{\boldsymbol{A}}_{\mathbf{M},v})$, for $c \in \mathcal{L}$;
$\|(\forall x)\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} = inf\{\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v[x\to a]} \mid a \in M\}$;
$\|(\exists x)\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} = sup\{\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v[x\to a]} \mid a \in M\}$.

If the infimum or the supremum do not exist, we take the truth value of the formula as undefined. We say that an $\boldsymbol{A}$-structure is *safe* if $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v}$ is defined for each formula φ and each $\mathbf{M}$-evaluation v.

For a set of formulas Φ, we write $\|\Phi\|^{\boldsymbol{A}}_{\mathbf{M},v} = 1$ if $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} = 1$ for every $\varphi \in \Phi$. We say that $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is a *model of a set of formulas* Φ if $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} = 1$ for any $\varphi \in \Phi$ and any $\mathbf{M}$-evaluation v. We denote by $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$ that $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} = 1$ for all $\mathbf{M}$-evaluation v. We say that a formula φ is *satisfiable* if there exists a structure $\langle \boldsymbol{A}, \mathbf{M}\rangle$ such that $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$. In such case, we also say that φ is *satisfied by* $\langle \boldsymbol{A}, \mathbf{M}\rangle$ or that $\langle \boldsymbol{A}, \mathbf{M}\rangle$ *satisfies* φ. Unless otherwise stated, from now on $\boldsymbol{A}$ denotes an MTL-algebra and we refer to $\boldsymbol{A}$-structures simply as *structures*.

Now we recall the notion of homomorphism between fuzzy structures.

Definition 4 *([12, Definition 6])*. Let $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and $\langle \boldsymbol{B}, \mathbf{N}\rangle$ be structures, f be a mapping from $\boldsymbol{A}$ to $\boldsymbol{B}$ and g be a mapping from M to N. The pair $\langle f, g\rangle$ is said to be a *homomorphism* from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ to $\langle \boldsymbol{B}, \mathbf{N}\rangle$ if f is a homomorphism of L-algebras and for every n-ary function symbol F and $d_1, \ldots, d_n \in M$,

$$g(F_{\mathbf{M}}(d_1, \ldots, d_n)) = F_{\mathbf{N}}(g(d_1), \ldots, g(d_n))$$

and for every n-ary predicate symbol P and $d_1, \ldots, d_n \in M$,

$$(*) \text{ If } P_{\mathbf{M}}(d_1, \ldots, d_n) = 1, \text{ then } P_{\mathbf{N}}(g(d_1), \ldots, g(d_n)) = 1.$$

We say that a homomorphism $\langle f, g\rangle$ is *strict* if instead of (*) it satisfies the stronger condition: for every n-ary predicate symbol P and $d_1, \ldots, d_n \in M$,

$$P_{\mathbf{M}}(d_1, \ldots, d_n) = 1 \text{ if and only if } P_{\mathbf{N}}(g(d_1), \ldots, g(d_n)) = 1.$$

Moreover we say that $\langle f, g\rangle$ is an *embedding* if it is a strict homomorphism and both functions f and g are injective. And we say that an embedding $\langle f, g\rangle$ is an *isomorphism* if both functions f and g are surjective.

3. Horn clauses

In this section we present a definition of Horn clause over predicate fuzzy logics that extends the classical definition in a natural way. In classical predicate logic, a *basic Horn formula* is a formula of the form $\alpha_1 \wedge \cdots \wedge \alpha_n \to \beta$, where $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n, \beta$ are atomic formulas. Now we extend these definitions to work with predicate fuzzy logics. Observe that there is not a unique way to extend them due to the fact that, in predicate fuzzy logic, we have different conjunctions and implications.

Definition 5 *(Basic Horn formula).* A *basic Horn formula* is a formula of the form

$$\alpha_1 \& \cdots \& \alpha_n \to \beta \tag{1}$$

where $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n, \beta$ are atomic formulas.

The formula obtained by substitution in expression (1) of the strong conjunction & by the weak conjunction $\wedge$ will be called *basic weak Horn formula*. From now on, for the sake of clarity, we will refer to the basic weak Horn formulas as *basic w-Horn formulas*.

Analogously to classical logic, disjunctive definitions of basic Horn formulas can be defined. Nevertheless, it is an easy exercise to check that, for predicate fuzzy logics, these disjunctive forms are not in general equivalent to the implicational ones that we have introduced here. Here we focus our analysis on the implicational Horn clauses and we leave for future work the study of the properties of disjunctive Horn clauses.

Definition 6. A *quantifier-free Horn formula* is a formula of the form $\phi_1 \& \cdots \& \phi_m$ where $m \in \mathbb{N}$ and ϕ_i is a basic Horn formula for every $1 \leq i \leq m$. If ϕ_i is a basic w-Horn formula for every $1 \leq i \leq m$, we say that $\phi_1 \wedge \cdots \wedge \phi_m$ is a *quantifier-free w-Horn formula.*

From now on, whenever it is possible, we present a unique definition for both the strong and the weak version, we use the *w-* symbol into parenthesis.

Definition 7. A *(w-)Horn formula* is a formula of the form $Q\gamma$, where Q is a (possibly empty) string of quantifiers $(\forall x), (\exists x)$... and γ is a quantifier-free (w-)Horn formula. A *(w-)Horn clause* (or *universal (w-)Horn formula*) is a (w-)Horn formula in which the quantifier prefix (if any) has only universal quantifiers. A *(w-)universal Horn theory* is a set of (w-)Horn clauses.

Observe that, in classical logic, the formula $(\forall x)\varphi \wedge (\forall x)\psi$ is logically equivalent to $(\forall x)(\varphi \wedge \psi)$. This result can be used to prove that every Horn clause is equivalent in classical logic to a conjunction of formulas of the form $(\forall x_1) \ldots (\forall x_k)\varphi$, where φ is a basic Horn formula. Having in mind these equivalences, it is easy to see that the set of all Horn clauses is recursively defined in classical logic by the following rules:

1. If φ is a basic Horn formula, then φ is a Horn clause;
2. If φ and ψ are Horn clauses, then $\varphi \wedge \psi$ is a Horn clause;
3. If φ is a Horn clause, then $(\forall x)\varphi$ is a Horn clause.

In MTL$\forall^m$ we can deduce $(\forall x)\varphi \wedge (\forall x)\psi \leftrightarrow (\forall x)(\varphi \wedge \psi)$. This fact allows us to show that in MTL$\forall^m$, any w-Horn clause is equivalent to a weak conjunction of formulas of the form $(\forall x_1) \cdots (\forall x_k)(\varphi)$ where φ is a basic w-Horn formula. Thus, w-Horn clauses can be recursively defined in MTL$\forall^m$ as above. But it is not the case for the strong conjunction since $(\forall x)\varphi \& (\forall x)\psi \leftrightarrow (\forall x)(\varphi \& \psi)$ can not be deduced from MTL$\forall^m$ (we refer to [18, Remark p.281]). So the set of Horn clauses is not recursively defined in MTL$\forall^m$.

4. Term structures associated to a set of formulas

In this section we introduce the notion of term structure associated to a set of formulas over predicate fuzzy logics. We study the particular case of sets of universal Horn formulas and prove that the term structure associated to these sets of formulas is free. Term structures have been used in classical logic to prove the satisfiability of a set of consistent sentences, see for instance [16, Ch.5]. From now on we assume that we work in a language with a binary predicate symbol $\approx$ interpreted as a similarity. We assume also that the axiomatization of the logic $L\forall^m$ contains the following axioms for $\approx$.

Definition 8 *([23, Definitions 5.6.1, 5.6.5]).* Let $\approx$ be a binary predicate symbol, the following are the axioms of similarity and congruence:

S1. $(\forall x)x \approx x$
S2. $(\forall x)(\forall y)(x \approx y \to y \approx x)$
S3. $(\forall x)(\forall y)(\forall z)(x \approx y \& y \approx z \to x \approx z)$

C1. For each n-ary function symbol F,

$$(\forall x_1)\cdots(\forall x_n)(\forall y_1)\cdots(\forall y_n)(x_1 \approx y_1 \& \cdots \& x_n \approx y_n \to F(x_1,\dots,x_n) \approx F(y_1,\dots,y_n))$$

C2. For each n-ary predicate symbol P,

$$(\forall x_1)\cdots(\forall x_n)(\forall y_1)\cdots(\forall y_n)(x_1 \approx y_1 \& \cdots \& x_n \approx y_n \to (P(x_1,\dots,x_n) \leftrightarrow P(y_1,\dots,y_n)))$$

Definition 9. Let Φ be a set of formulas, we define a binary relation on the set of terms, denoted by $\sim$, in the following way: for every terms t_1, t_2,

$$t_1 \sim t_2 \text{ if and only if } \Phi \vdash t_1 \approx t_2.$$

By using [18, Prop.1(5)], it is easy to check that for every set of formulas Φ, $\sim$ is an equivalence relation. From now on we denote by $\bar{t}$ the $\sim$-class of the term t. The next result, which states that $\sim$ is compatible with the symbols of the language, can be easily proven using the Axioms of Congruence of Definition 8.

Lemma 1. *Let Φ be a set of formulas. The relation $\sim$ has the following property: if for every $1 \le i \le n$, $t_i \sim t'_i$, then*

(i) *For any n-ary function symbol F, $F(t_1,\dots,t_n) \sim F(t'_1,\dots,t'_n)$,*
(ii) *For any n-ary predicate symbol P, $\Phi \vdash P(t_1,\dots,t_n)$ iff $\Phi \vdash P(t'_1,\dots,t'_n)$*

Definition 10 *(Term structure).* Let Φ be a consistent set of formulas. We define the following structure $\langle \boldsymbol{B}, \mathbf{T}^\Phi \rangle$, where $\boldsymbol{B}$ is the two-valued Boolean algebra, $\mathbf{T}^\Phi$ is the set of all equivalence classes of the relation $\sim$ and

- For any n-ary function symbol F,

$$F_{\mathbf{T}^\Phi}(\bar{t}_1,\dots,\bar{t}_n) = \overline{F(t_1,\dots,t_n)}$$

- For any n-ary predicate symbol P,

$$P_{\mathbf{T}^\Phi}(\bar{t}_1,\dots,\bar{t}_n) = \begin{cases} 1, & \text{if } \Phi \vdash P(t_1,\dots,t_n) \\ 0, & \text{otherwise} \end{cases}$$

We call $\langle \boldsymbol{B}, \mathbf{T}^\Phi \rangle$ the *term structure associated to* Φ.

Notice that for every 0-ary function symbol c, $c_{\mathbf{T}^\Phi} = \overline{c}$. By using Lemma 1, it is easy to prove that the structure $\langle \boldsymbol{B}, \mathbf{T}^\Phi \rangle$ is well-defined, because the conditions are independent from the choice of the representatives. Observe that, so defined, $\langle \boldsymbol{B}, \mathbf{T}^\Phi \rangle$ is a classical structure. The following lemma agrees with this classical character.

Lemma 2. *Let Φ be a consistent set of formulas. The interpretation of the $\approx$ symbol in the structure $\langle \boldsymbol{B}, \mathbf{T}^\Phi \rangle$ is the crisp equality.*

Proof. Let t_1, t_2 be terms. We have $\overline{t}_1 = \overline{t}_2$ iff $t_1 \sim t_2$ iff $\Phi \vdash t_1 \approx t_2$ iff $\overline{t_1} \approx_{\mathbf{T}^\Phi} \overline{t_2}$ (this last step by Definition 10). □

Now we prove some technical lemmas that will allow us to show that the term structure $\langle \boldsymbol{B}, \mathbf{T}^\Phi \rangle$ is free.

Definition 11. Given a consistent set of formulas Φ, let e^Φ be the following $\mathbf{T}^\Phi$-evaluation: $e^\Phi(x) = \overline{x}$. We call e^Φ the *canonical evaluation of* $\langle \boldsymbol{B}, \mathbf{T}^\Phi \rangle$.

Lemma 3. *Let Φ be a consistent set of formulas, the following holds:*

(i) *For any term t, $\|t\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi} = \overline{t}$.*
(ii) *For any atomic formula φ, $\|\varphi\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi} = 1$ if and only if $\Phi \vdash \varphi$.*
(iii) *For any atomic formula φ, $\|\varphi\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi} = 0$ if and only if $\Phi \nvdash \varphi$.*

Proof. (i) By induction on the complexity of t and Definitions 10 and 11.

(ii) Let P be an n-ary predicate symbol and $t_1, \dots, t_n$ be terms, we have:

$$\begin{aligned} \|P(t_1, \dots, t_n)\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi} = 1 \quad & \text{iff} \\ P_{\mathbf{T}^\Phi}(\|t_1\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi}, \dots, \|t_n\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi}) = 1 \quad & \text{iff} \\ P_{\mathbf{T}^\Phi}(\overline{t}_1, \dots, \overline{t}_n) = 1 \quad & \text{iff} \\ \Phi \vdash P(t_1, \dots, t_n) \quad & \end{aligned}$$

The second equivalence is by (i) of the present Lemma, and the third one by Definition 10. (iii) holds because $\langle \boldsymbol{B}, \mathbf{T}^\Phi \rangle$ is a classical structure. □

Observe that, since terms are the smallest significance components of a first-order language, Lemma 3 (ii) and (iii) can be read as saying that term structures are minimal with respect to atomic formulas. Intuitively speaking, the term structure picks up the positive atomic information associated to Φ.

Lemma 4. *Let Φ be a consistent set of formulas. The set $\{\overline{x} \mid x \in Var\}$ generates the universe T^Φ of the term structure associated to Φ.*

Proof. Let $\overline{t(x_1, \dots, x_n)} \in T^\Phi$. By Lemma 3,

$$\overline{t(x_1, \dots, x_n)} = \|t(x_1, \dots, x_n)\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi}$$

and by the semantics of predicate fuzzy logics (Definition 3),

$$\|t(x_1, \dots, x_n)\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi} = t_{\mathbf{T}^\Phi}(\|x_1\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi}, \dots, \|x_n\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi}) = t_{\mathbf{T}^\Phi}(\overline{x}_1, \dots, \overline{x}_n). \qquad \square$$

Term structures do not necessarily satisfy the theory to which they are associated. In classical logic, if it is the case, from an algebraic point of view, the minimality of the term structure is revealed by the fact that the structure is *free*. A model of a theory is free if there is a unique homomorphism from this model to any other model of the theory. Free structures have their origin in category theory, as a generalization of free groups (for a definition of free structure in category theory, see [1, Def. 4.7.17]). Free structures are also named *initial* in [26, Def. 2.1 (i)]. In the context of computer science, they appeared for the first time in [22].

The possibility given by fuzzy logic of defining the term structure associated to a theory using the similarity symbol $\approx$ leads us to a notion of free structure restricted to the class of reduced models of that theory, as we will prove in next theorem. Remember that *reduced structures* are those whose Leibniz congruence is the identity. By [13, Lemma 20], a structure $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is reduced iff it has the *equality property* (EQP) (that is, for any $d, e \in M$, $d \approx_{\mathbf{M}} e$ iff $d = e$).

Theorem 1. *Let Φ be a consistent set of formulas with $\|\Phi\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi} = 1$. Then, $\langle \boldsymbol{B}, \mathbf{T}^\Phi\rangle$ is a free structure in the class of the reduced models of Φ, i.e., for every reduced structure $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and every evaluation v such that $\|\Phi\|^{\boldsymbol{A}}_{\mathbf{M},v} = 1$, there is a unique homomorphism $\langle f, g\rangle$ from $\langle \boldsymbol{B}, \mathbf{T}^\Phi\rangle$ to $\langle \boldsymbol{A}, \mathbf{M}\rangle$ such that for every $x \in Var$, $g(\overline{x}) = v(x)$.*

Proof. Let $\langle \boldsymbol{A}, \mathbf{M}\rangle$ be a reduced structure and v an $\mathbf{M}$-evaluation such that $\|\Phi\|^{\boldsymbol{A}}_{\mathbf{M},v} = 1$. Now let $f : \boldsymbol{B} \to \boldsymbol{A}$ be the identity and define g by: $g(\overline{t}) = \|t\|^{\boldsymbol{A}}_{\mathbf{M},v}$ for every term t. We show that $\langle f, g\rangle$ is the desired homomorphism (for the definition of homomorphism see the Preliminaries section, Definition 4).

First let us check that g is well-defined. Given terms t_1, t_2 with $\overline{t}_1 = \overline{t}_2$, that is, $t_1 \sim t_2$, by Definition 9, $\Phi \vdash t_1 \approx t_2$. Then, since $\|\Phi\|^{\boldsymbol{A}}_{\mathbf{M},v} = 1$, we have $\|t_1 \approx t_2\|^{\boldsymbol{A}}_{\mathbf{M},v} = 1$. But $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is reduced, which by [13, Lemma 20] is equivalent to have the EQP; therefore $\|t_1\|^{\boldsymbol{A}}_{\mathbf{M},v} = \|t_2\|^{\boldsymbol{A}}_{\mathbf{M},v}$, that is, $g(\overline{t_1}) = g(\overline{t_2})$.

Now, let us see that g is a homomorphism. Let $\overline{t}_1, \ldots, \overline{t}_n \in T^\Phi$ be terms and F be an n-ary function symbol. By Definition 10, we have that

$$F_{\mathbf{T}^\Phi}(\overline{t}_1, \ldots, \overline{t}_n) = \overline{F(t_1, \ldots, t_n)}$$

and then $g(F_{\mathbf{T}^\Phi}(\overline{t}_1, \ldots, \overline{t}_n)) = g(\overline{F(t_1, \ldots, t_n)}) = \|F(t_1, \ldots, t_n)\|^{\boldsymbol{A}}_{\mathbf{M},v} = F_{\mathbf{M}}(\|t_1\|^{\boldsymbol{A}}_{\mathbf{M},v}, \ldots, \|t_n\|^{\boldsymbol{A}}_{\mathbf{M},v}) = F_{\mathbf{M}}(g(\overline{t}_1), \ldots, g(\overline{t}_n))$.

Let P be an n-ary predicate symbol such that $P_{\mathbf{T}^\Phi}(\overline{t}_1, \ldots, \overline{t}_n) = 1$. By Definition 10, $\Phi \vdash P(t_1, \ldots, t_n)$. Since $\|\Phi\|^{\boldsymbol{A}}_{\mathbf{M},v} = 1$, we have

$$\|P(t_1, \ldots, t_n)\|^{\boldsymbol{A}}_{\mathbf{M},v} = 1$$

and then $P_{\mathbf{M}}(\|t_1\|^{\boldsymbol{A}}_{\mathbf{M},v}, \ldots, \|t_n\|^{\boldsymbol{A}}_{\mathbf{M},v}) = 1$, that is, $P_{\mathbf{M}}(g(\overline{t}_1), \ldots, g(\overline{t}_n)) = 1$.

Finally, since by Lemma 4 the set $\{\overline{x} \mid x \in Var\}$ generates the universe T^Φ of the term structure associated to Φ, $\langle f, g\rangle$ is the unique homomorphism such that for every $x \in Var$, $g(\overline{x}) = v(x)$. $\square$

Observe that in languages in which the similarity symbol is interpreted by the crisp identity, by using an analogous argument to the one in Theorem 1, we obtain that the term structure is free in all the models of the theory and not only in the class of reduced models.

To end this section we prove that the term structure associated to a universal Horn theory is a model of this theory. We have shown above in Section 3 that the set of Horn clauses is not recursively defined in MTL$\forall^m$. For that reason we will present here proofs that differ from the proofs of the corresponding results in classical logic, using induction on the rank of a formula instead of induction on the set of the (w-)Horn clauses. We introduce first the notion of *rank of a formula* φ. Our definition is a variant of the notion of *syntactic degree of a formula* in [23, Definition 5.6.7]).

$rk(\varphi) = 0$, if φ is atomic;
$rk(\neg\varphi) = rk((\exists x)\varphi) = rk((\forall x)\varphi) = rk(\varphi) + 1$;
$rk(\varphi \circ \psi) = rk(\varphi) + rk(\psi)$, for every binary propositional connective $\circ$.

Lemma 5. *Let φ be a (w-)Horn clause where $x_1, \ldots, x_m$ are pairwise distinct free variables. Then, for every terms $t_1, \ldots, t_m$,*

$$\varphi(t_1, \ldots, t_m/x_1, \ldots, x_m)$$

is a (w-)Horn clause.

Proof. We prove it for the strong conjunction but the proof is analogous for the weak conjunction. By induction on $rk(\varphi)$.

Case $rk(\varphi) = 0$. If φ is a basic Horn formula of the form $\psi_1 \& \cdots \& \psi_n \to \psi$, it is clear that $\varphi(t_1, \ldots, t_m/x_1, \ldots, x_m)$ is still a basic Horn formula. In case that $\varphi = \phi_1 \& \cdots \& \phi_l$ is a conjunction of basic Horn formulas, note that $\varphi(t_1, \ldots, t_m/x_1, \ldots, x_m)$ has the same form as φ.

Case $rk(\varphi) = n + 1$. Assume inductively that for any Horn clause ψ where $x_1, \ldots, x_m$ are pairwise distinct free variables in ψ and whose rank is n, the formula $\psi(t_1, \ldots, t_m/x_1, \ldots, x_m)$ is a Horn clause. Let φ be a Horn clause of rank $n + 1$, then φ is of the form $(\forall y)\psi$, where ψ has rank n. Assume without loss of generality that and $y \notin \{x_1, \ldots, x_m\}$, then

$$[(\forall y)\psi](t_1, \ldots, t_m/x_1, \ldots, x_m) = (\forall y)[\psi(t_1, \ldots, t_m/x_1, \ldots, x_m)]$$

thus we can apply the inductive hypothesis to obtain the desired result. □

Theorem 2. *Let Φ be a consistent set of formulas. For every (w-)Horn clause φ, if $\Phi \vdash \varphi$, then $\|\varphi\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi} = 1$.*

Proof. We prove it for the strong conjunction but the proof is analogous for the weak conjunction. By induction on $rk(\varphi)$.

Case $rk(\varphi) = 0$. We can distinguish two subcases:

1) If $\varphi = \psi_1 \& \cdots \& \psi_n \to \psi$ is a basic Horn formula, we have to show that $\|\psi_1 \& \cdots \& \psi_n\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi} \leq \|\psi\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi}$. If $\|\psi\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi} = 1$, we are done. Otherwise, by Definition 10, $\Phi \nvdash \psi$. Consequently, since $\Phi \vdash \psi_1 \& \cdots \& \psi_n \to \psi$, $\Phi \nvdash \psi_1 \& \cdots \& \psi_n$ and thus for some $1 \leq i \leq n$, $\Phi \nvdash \psi_i$. By Lemma 3 (ii), we have $\|\psi_i\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi} = 0$ and then $\|\psi_1 \& \cdots \& \psi_n\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi} = 0$. Therefore, we can conclude $\|\psi_1 \& \cdots \& \psi_n\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi} \leq \|\psi\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi}$. Note that if $n = 0$, φ is an atomic formula and the property holds by Lemma 3 (ii).

2) If $\varphi = \psi_1 \& \cdots \& \psi_n$ is a conjunction of basic Horn formulas and $\Phi \vdash \varphi$, then for every $1 \leq i \leq n$, $\Phi \vdash \psi_i$. Thus, by 1), for every $1 \leq i \leq n$, $\|\psi_i\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi} = 1$ and then $\|\varphi\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi} = 1$.

Case $rk(\varphi) = n + 1$.

If $\varphi = (\forall x)\phi(x)$ is a Horn clause, where $rk(\phi(x)) = n$ and $\Phi \vdash \varphi$, by Axiom $\forall 1$ of $L\forall^m$, for every term t, $\Phi \vdash \phi(t/x)$. Since by Lemma 5, $\phi(t/x)$ is also a Horn clause and $rk(\phi(t/x)) = n$, we can apply the inductive hypothesis and hence for every term t, $\|\phi(t/x)\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi} = 1$, that is, by Lemma 3 (i), for every element $\bar{t}$ of the domain, $\|\phi(x)\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi(x \to \bar{t})} = 1$. Therefore, we can conclude that $\|(\forall x)\phi(x)\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi, e^\Phi} = 1$. □

Observe that the inverse direction of Theorem 2 is not true. Assume that we work in Gödel predicate fuzzy logic G∀. Let P be a 1-ary predicate symbol, $\bar{c}$ be an individual constant, $\Phi = \{\neg(P(\bar{c}) \to \bar{0})\}$ and

$\varphi = P(\overline{c}) \to \overline{0}$. Now we show that $\|\varphi\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi} = 1$, but $\Phi \not\vdash \varphi$. First, in order to show that $\Phi \not\vdash \varphi$, consider a G-algebra $\boldsymbol{A}$ with domain the real interval $[0,1]$ and a structure $\langle \boldsymbol{A}, \mathbf{M}\rangle$ such that $\|P(\overline{c})\|^{\boldsymbol{A}}_{\mathbf{M}} = 0.8$, then we have that $\|\Phi\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$ and $\|P(\overline{c}) \to \overline{0}\|^{\boldsymbol{A}}_{\mathbf{M}} \neq 1$ consequently $\Phi \not\vdash_G P(\overline{c}) \to \overline{0}$. Using the same structure we obtain also that $\Phi \not\vdash_G P(\overline{c})$. Finally, since $\Phi \not\vdash_G P(\overline{c})$, by Lemma 3, $\|P(\overline{c})\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi} = 0$ and then $\|\varphi\|^{\boldsymbol{B}}_{\mathbf{T}^\Phi} = 1$.

Remark that, as a corollary of Theorem 2, we have that the substructure of $\langle \boldsymbol{B}, \mathbf{T}^\Phi\rangle$ generated by the set of ground terms is also a model for all universal Horn sentences that are consequences of the theory. Another important corollary of Theorem 2 is the following:

Corollary 1. *Every consistent set of (w-)Horn clauses without free variables has a classical model.*

Observe that Corollary 1 is not true in general. The consistent sentence $\neg(\overline{1} \to Pa)\&\neg(Pa \to \overline{0})$ has no classical model.

5. Herbrand structures

In this section we introduce Herbrand structures for fuzzy universal Horn theories. They are a prominent form of term structures, specially helpful when dealing with sets of equality-free formulas (that is, formulas in which the symbol $\approx$ does not occur), the reason is that, as it is shown below in Lemma 6, no non-trivial equations are derivable from a set of equality-free formulas. In classical logic, Herbrand structures have been used to present a simplified version of a term structure associated to a consistent theory [16, Ch.11], and they have also a relevant role in the foundation of logic programming (see for instance [14]). Regarding Herbrand structures in fuzzy logic programming, we refer to the works [19,30,17]. Throughout this section we assume that the symbol $\approx$ is interpreted always as the crisp identity and that there is at least an individual constant in the language.

Lemma 6. *Let Φ be a consistent set of equality-free formulas, then for every terms t_1, t_2,*

$$\textit{If } \Phi \vdash t_1 \approx t_2, \ \textit{then } t_1 = t_2.$$

Proof. Assume that Φ is a consistent set of equality-free formulas and $\Phi \vdash t_1 \approx t_2$ for terms t_1, t_2 of the language. Since CL∀ is an extension of MTL$\forall^m$, $\Phi \vdash t_1 \approx t_2$ in CL∀. Then, by the analogous classical result [16, Ch. 11, Th. 3.1], we have $t_1 = t_2$. □

Definition 12 *(Herbrand structure).* The *Herbrand universe of a predicate language* is the set of all ground terms of the language. A *Herbrand structure* is a structure $\langle \boldsymbol{A}, \mathbf{H}\rangle$, where $\mathbf{H}$ is the Herbrand universe, and:

For any individual constant symbol c, $c_{\mathbf{H}} = c$.
For any n-ary function symbol F and any $t_1, \dots, t_n \in H$,

$$F_{\mathbf{H}}(t_1, \dots, t_n) = F(t_1, \dots, t_n)$$

Observe that in Definition 12 no restrictions are placed on the interpretations of the predicate symbols and on the algebra we work over. The canonical models $\langle \boldsymbol{Lind}_T, \mathbf{CM}(T)\rangle$ introduced in [10, Def.9] are examples of Herbrand structures. In these structures $\boldsymbol{Lind}_T$ is the Lindenbaum algebra of a theory T and the domain of $\mathbf{CM}(T)$ is the set of individual constants (the language in [10] does not contain function symbols). Now we introduce a particular case of Herbrand structure and we show that every consistent Horn clause without free variables has a model of this kind.

Definition 13 *(H-structure and H-model).* Let $\overline{H}$ be the set of all equality-free sentences of the form $P(t_1, \dots, t_n)$, where $t_1, \dots, t_n$ are ground terms, $n \geq 1$ and P is an n-ary predicate symbol. For every subset H of $\overline{H}$, we define the Herbrand structure $\langle \boldsymbol{B}, \mathbf{N}^{\mathrm{H}}\rangle$, where $\boldsymbol{B}$ is the two-valued Boolean algebra, the domain $\mathbf{N}^{\mathrm{H}}$ is the set of all ground terms of the language, the interpretation of the function symbols is as in every Herbrand structure and the interpretation of the predicate symbols is as follows: for every $n \geq 1$ and every n-ary predicate symbol P,

$$P_{\mathbf{N}^{\mathrm{H}}}(t_1, \dots, t_n) = \begin{cases} 1, & \text{if } P(t_1, \dots, t_n) \in H \\ 0, & \text{otherwise.} \end{cases}$$

We call this type of Herbrand structures *H-structures*. If Φ is a set of sentences, we say that an H-structure is an *H-model* of Φ if it is a model of Φ.

Proposition 1. *Let $\langle \boldsymbol{A}, \mathbf{M}\rangle$ be a structure and H be the set of all atomic equality-free sentences σ such that $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$. Then, for every equality-free sentence φ which is a (w-)Horn clause, if $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$, then $\|\varphi\|^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}} = 1$, where $\langle \boldsymbol{B}, \mathbf{N}^{\mathrm{H}}\rangle$ is an* H*-structure as in Definition 13.*

Proof. We prove it for the strong conjunction but the proof is analogous for the weak conjunction. Assume that φ is an equality-free sentence which is a Horn clause and $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$. We proceed by induction on the rank of φ

Case $rk(\varphi) = 0$. We distinguish two cases:

1) If $\varphi = \psi_1 \& \cdots \& \psi_n \to \psi$ is a basic Horn formula, we have to show that $\|\psi_1 \& \cdots \& \psi_n\|^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}} \leq \|\psi\|^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}}$. If $\|\psi\|^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}} = 1$, we are done. Otherwise, by Definition 13, $\psi \notin \mathrm{H}$, and thus $\|\psi\|^{\boldsymbol{A}}_{\mathbf{M}} \neq 1$. Therefore, since $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$, we have that $\|\psi_1 \& \cdots \& \psi_n\|^{\boldsymbol{A}}_{\mathbf{M}} \neq 1$. Consequently for some $1 \leq i \leq n$, $\|\psi_i\|^{\boldsymbol{A}}_{\mathbf{M}} \neq 1$, therefore $\psi_i \notin \mathrm{H}$, i.e., $\|\psi_i\|^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}} = 0$, and then $\|\psi_1 \& \cdots \& \psi_n\|^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}} = 0$. Hence, $\|\psi_1 \& \cdots \& \psi_n\|^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}} \leq \|\psi\|^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}}$.

2) If $\varphi = \psi_1 \& \cdots \& \psi_n$ is a strong conjunction of basic Horn formulas, then by 1) we have that $\|\psi_i\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$ implies $\|\psi_i\|^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}} = 1$, for each $i \in \{1, \dots, n\}$. Thus, if $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$, then $\|\varphi\|^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}} = 1$.

Case $rk(\varphi) = n + 1$.

Let $\varphi = (\forall x)\phi(x)$ be a Horn clause with $rk(\phi(x)) = n$. Since $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$, by Axiom $\forall 1$ of $L\forall^m$, for every ground term t, $\|\phi(t/x)\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$. By Lemma 5, $\phi(t/x)$ is also a Horn clause, and since $rk(\phi(t/x)) = n$, we can apply the inductive hypothesis and hence for every ground term t, $\|\phi(t/x)|^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}} = 1$. Finally, since $\langle \boldsymbol{B}, \mathbf{N}^{\mathrm{H}}\rangle$ is a Herbrand structure, we have that for every element t of its domain $\|\phi(t/x)\|^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}} = 1$, and consequently $\|(\forall x)\phi(x)\|^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}} = 1$. □

Notice that Proposition 1 does not assert that given a structure $\langle \boldsymbol{A}, \mathbf{M}\rangle$, $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and $\langle \boldsymbol{B}, \mathbf{N}^{\mathrm{H}}\rangle$ satisfies exactly the same equality-free sentences which are Horn clauses. Actually, this is not true. Let $\mathcal{P}$ be a predicate language with three monadic predicate symbols P_1, P_2, P_3 and one individual constant c. Suppose that $\boldsymbol{A}$ is the Łukasiewicz algebra $[0,1]_{Ł}$ and let $\langle \boldsymbol{A}, \mathbf{M}\rangle$ be a structure over $\mathcal{P}$ such that $\|P_1(c)\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$, $\|P_2(c)\|^{\boldsymbol{A}}_{\mathbf{M}} = 0.9$ and $\|P_3(c)\|^{\boldsymbol{A}}_{\mathbf{M}} = 0.5$. Let φ be $P_1(c)\&P_2(c) \to P_3(c)$, φ is an equality-free sentence which is a Horn clause with $\|P_1(c)\&P_2(c) \to P_3(c)\|^{\boldsymbol{A}}_{\mathbf{M}} = 0.6$, but if we consider its associated H-structure, $\langle \boldsymbol{B}, \mathbf{N}^{\mathrm{H}}\rangle$, we have H$= \{P_1(c)\}$ and thus $\|P_1(c)\&P_2(c) \to P_3(c)\|^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}} = 1$.

Corollary 2. *An equality-free sentence which is a (w-)Horn clause has a model if and only if it has an* H*-model.*

We can conclude here, in the same sense as in Corollary 1, that every consistent equality-free sentence which is a (w-)Horn clause has a classical Herbrand model.

6. Discussion, conclusions and future work

The present paper is a first step towards a systematic study of universal Horn theories over predicate fuzzy logics from a model-theoretic perspective. We have proved the existence of free models in universal Horn classes of structures. In the future we will pay special attention to the study of possible characterizations of universal Horn theories in terms of the existence of these free models and its relevance for fuzzy logic programming.

Future work will be devoted also to the analysis of the logical properties of the different definitions of Horn clauses introduced so far in the literature of fuzzy logics, for instance see [2,3,27]. It is important to underline here some differences between our work and some important related references. Our paper differs from the approaches of Bělohlávek and Vychodil and also the one of Gerla, due to mainly three reasons: it is not restricted to fuzzy equalities, it does not adopt the Pavelka-style definition of the Horn clauses and it does not assume the completeness of the algebra. Our choice is taken because it gives more generality to the results we wanted to obtain, even if in this first work our Horn clauses are defined very basically.

We take as a future task to explore how a Pavelka-style definition of Horn clauses in the framework developed by Hájek [23] could change or even improve the results we have obtained on free models. We will follow the broad approach taken in [8, Ch.8] about fuzzy logics with enriched languages. Finally we will study also quasivarieties over fuzzy logic, and closure properties of fuzzy universal Horn classes by using recent results on direct and reduced products over fuzzy logic like [13]. Our next objective is to solve the open problem of characterizing theories of Horn clauses in predicate fuzzy logics, formulated by Cintula and Hájek in [9].

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2.1.4 Term models of Horn clauses over rational Pavelka predicate logic

Next, we include in this doctoral thesis the book chapter Costa & Dellunde (2017*b*), titled *Term Models of Horn Clauses over Rational Pavelka Predicate Logic*[2].

Term Models of Horn Clauses over Rational Pavelka Predicate Logic

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***Abstract*—This paper is a contribution to the study of the universal Horn fragment of predicate fuzzy logics, focusing on the proof of the existence of free models of theories of Horn clauses over Rational Pavelka predicate logic. We define the notion of a term structure associated to every consistent theory T over Rational Pavelka predicate logic and we prove that the term models of T are free on the class of all models of T. Finally, it is shown that if T is a set of Horn clauses, the term structure associated to T is a model of T.**

Keywords–Horn clause; term model; free model; Rational Pavelka predicate logic.

1. Introduction

Free models and Horn clauses have a relevant role in classical logic and logic programming. On the one hand, free models, which appeared first in category theory (see for instance [1, Def. 4.7.17]), are crucial in universal algebra and, thereby, in model theory. In the context of logic programming, free structures, introduced in [16] and also named *initial* (as for instance in [19, Def. 2.1 (i)]), are important in logic programming, since they allow a procedural interpretation of the programs and admitting free structures makes reasonable the *negation as failure* (see for instance [19]). In the context of abstract data types, Tarlecki [20] characterizes abstract algebraic institutions which admit free constructions. On the other hand, the significant importance of Horn clauses in classical logic was detailed in [18], while it is well-known that Horn clauses are used both as a specification and as a programming language in Prolog, the most common language in logic programming.

In the context of fuzzy logics, several definitions of Horn clause have been proposed in the literature, but there is not a canonical one yet. An extensive and important work in predicate fuzzy logics has been done by Bělohlávek and Vychodil (see [2], [3], [4], [5], [6], [22]). Even if the work of these authors also adopts Pavelka-style, it differs from our approach: we do not restrict Horn clauses to fuzzy equalities and we work in the general semantics of [17]. Another approach is shown in [13], where Dubois and Prade discuss different possibilities of defining *fuzzy rules* and they show how these different semantics can be captured in the framework of fuzzy set theory and possibility theory. We find also that, in the context of fuzzy logic programming, there is a rich battery of proposals of Horn clauses which differ depending on the programming approach chosen. Some reference here are [15], [21].

With the goal of developing a systematic study of the universal Horn fragment of predicate fuzzy logics from a model-theoretic point of view, we took in [10] the syntactical definition of Horn clause of classical logic. Starting by this general and basic definition we studied the existence of free models of theories of Horn clauses in MTL∀. As a generalisation of a group-theoretic construction, Mal'tsev showed in classical logic that any theory of Horn clauses has a free model. In the present paper, a definition of Horn clause in RPL∀ using evaluated formulas is introduced. Consequently, we prove the existence of free models of theories of RPL∀- Horn clauses showing in RPL∀ an analogous result to Mal'tsev's one. The advantage of using these RPL∀-Horn clauses instead of the ones of [10] lies in the fact that the former can be better settled in the context of fuzzy logic programming. For instance, from a syntactical point of view, basic RPL∀-Horn clauses are a particular case of the clauses used in [7]

The paper is organized as follows. Section 2 contains the preliminaries on RPL∀. In Section 3 we introduce the definition of a term structure associated to a consistent theory and prove that when this structure is a model of the associated theory, the term structure is free on the class of all models of the theory. In Section 4 we define the notion of RPL∀-Horn clause and it is shown that whenever the associated theory is a set of RPL∀-Horn clauses, the term structure is a model of this theory.

2. Preliminaries

In this section we introduce the basic notions and results of RPL∀, the first-order extension of Rational Pavelka Logic.

For an extensive presentation of RPL∀ see [17, Ch.3.3 and Ch.5.4] and [8, Ch.VIII].

Definition 1. **Rational Pavelka Predicate Logic [8, Ch.VIII]** Rational Pavelka Predicate Logic RPL∀ is the expansion of Ł∀ by adding a truth constant for each rational number r in $[0,1]$ and by adding the axioms RPL1 and RPL2. The following is an axiomatic sytem for RPL∀:

(Ł1) $\varphi \to (\psi \to \varphi)$
(Ł2) $(\varphi \to \psi) \to ((\psi \to \xi) \to (\varphi \to \xi))$
(Ł3) $(\neg\psi \to \neg\varphi) \to (\psi \to \varphi)$
(Ł4) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$
(RPL1) $(\overline{r} \to \overline{s}) \leftrightarrow \overline{r \to s}$
(RPL2) $(\overline{r}\&\overline{s}) \leftrightarrow \overline{r\&s}$
(∀1) $(\forall x)\varphi(x) \to \varphi(t)$, where the term t is substitutable for x in φ.
(∀2) $(\forall x)(\xi \to \varphi) \to (\xi \to (\forall x)\varphi(x))$, where x is not free in ξ.

The rules are Modus Ponens and Generalization, that is, from φ infer $(\forall x)\varphi$.

A *theory* Φ is a set of sentences. We denote by $\Phi \vdash_{\text{RPL}\forall} \varphi$ the fact that φ is provable in RPL∀ from the set of formulas Φ. From now on, when it is clear from the context, we will write $\vdash$ to refer to $\vdash_{\text{RPL}\forall}$. We say that a theory Φ is *consistent* if $\Phi \nvdash \overline{0}$.

Definition 2. An *evaluated formula* (φ, r) in a language of RPL∀ is a formula of the form $\overline{r} \to \varphi$, where $r \in [0,1]$ is a rational number and φ is a formula without truth constants apart from $\overline{0}$ and $\overline{1}$. We say that an evaluated formula (φ, r) is *atomic* whenever φ is atomic.

Now we introduce the semantics of the predicate languages. Let $[0,1]_{\text{RPL}}$ be the standard RPL-algebra [8, Def.2.2.5, Ch.II], a *structure* for a predicate language $\mathcal{P}$ of the logic RPL∀ has the form $\langle [0,1]_{\text{RPL}}, \mathbf{M}\rangle$, where $\mathbf{M} = \langle M, (P_M)_{P\in Pred}, (F_M)_{F\in Func}\rangle$, M is a non-empty domain; for each n-ary predicate symbol $P \in Pred$, $P_{\mathbf{M}}$ is an n-ary fuzzy relation M, i.e., a function $M^n \to [0,1]_{\text{RPL}}$ (identified with an element of $[0,1]_{\text{RPL}}$ if $n=0$); for each n-ary function symbol $F \in Func$, $F_{\mathbf{M}}$ is a function $M^n \to M$ (identified with an element of M if $n = 0$).

An $\mathbf{M}$-*evaluation* of the object variables is a mapping v which assigns an element from M to each object variable. Let v be an $\mathbf{M}$-evaluation, x a variable, and $a \in M$. Then by $v[x \mapsto a]$ we denote the $\mathbf{M}$-evaluation such that $v[x \mapsto a](x) = a$ and $v[x \mapsto a](y) = v(y)$ for each object variable y different from x. We define the *values* of terms and the *truth values* of formulas in the structure $\langle [0,1]_{\text{RPL}}, \mathbf{M}\rangle$ for an evaluation v recursively as follows: given $F \in Func, P \in Pred$ and c a connective of RPL:

$$||x||^{[0,1]_{\text{RPL}}}_{\mathbf{M},v} = v(x)$$

$$||F(t_1,\ldots,t_n)||^{[0,1]_{\text{RPL}}}_{\mathbf{M},v} = F_{\mathbf{M}}(||t_1||^{[0,1]_{\text{RPL}}}_{\mathbf{M},v},\ldots,||t_n||^{[0,1]_{\text{RPL}}}_{\mathbf{M},v})$$

$$||P(t_1,\ldots,t_n)||^{[0,1]_{\text{RPL}}}_{\mathbf{M},v} = P_{\mathbf{M}}(||t_1||^{[0,1]_{\text{RPL}}}_{\mathbf{M},v},\ldots,||t_n||^{[0,1]_{\text{RPL}}}_{\mathbf{M},v})$$

$$||c(\varphi_1,\ldots,\varphi_n)||^{[0,1]_{\text{RPL}}}_{\mathbf{M},v} =$$
$$c_{[0,1]_{\text{RPL}}}(||\varphi_1||^{[0,1]_{\text{RPL}}}_{\mathbf{M},v},\ldots,||\varphi_n||^{[0,1]_{\text{RPL}}}_{\mathbf{M},v})$$
$$||(\forall x)\varphi||^{[0,1]_{\text{RPL}}}_{\mathbf{M},v} = inf\{||\varphi||^{[0,1]_{\text{RPL}}}_{\mathbf{M},v[x\to a]} \mid a \in M\}$$
$$||(\exists x)\varphi||^{[0,1]_{\text{RPL}}}_{\mathbf{M},v} = sup\{||\varphi||^{[0,1]_{\text{RPL}}}_{\mathbf{M},v[x\to a]} \mid a \in M\}.$$

Observe that, since the universe of the standard RPL-algebra is the interval of real numbers $[0,1]$, which is complete, all the infima and suprema in the definition of the semantics of the quantifiers exist.

For every formula φ, possibly with variables, we write $||\varphi||^{[0,1]_{\text{RPL}}}_{\mathbf{M}} =$

$$inf\{||\varphi||^{[0,1]_{\text{RPL}}}_{\mathbf{M},v} \mid \text{ for every } \mathbf{M}\text{ -evaluation } v\},$$

we say that $\langle [0,1]_{\text{RPL}}, \mathbf{M}\rangle$ is a *model of a sentence* φ if $||\varphi||^{[0,1]_{\text{RPL}}}_{\mathbf{M}} = 1$; and that $\langle [0,1]_{\text{RPL}}, \mathbf{M}\rangle$ is a *model of a theory* Φ if $||\varphi||^{[0,1]_{\text{RPL}}}_{\mathbf{M}} = 1$ for every $\varphi \in \Phi$.

In particular, given a structure $\langle [0,1]_{\text{RPL}}, \mathbf{M}\rangle$ and two formulas φ and ψ:

$$||\varphi\&\psi||^{[0,1]_{\text{RPL}}}_{\mathbf{M}} = max\{||\varphi||^{[0,1]_{\text{RPL}}}_{\mathbf{M}} + ||\psi||^{[0,1]_{\text{RPL}}}_{\mathbf{M}} - 1, 0\}$$
$$||\varphi \to \psi||^{[0,1]_{\text{RPL}}}_{\mathbf{M}} = min\{1 - ||\varphi||^{[0,1]_{\text{RPL}}}_{\mathbf{M}} + ||\psi||^{[0,1]_{\text{RPL}}}_{\mathbf{M}}, 1\}.$$

Definition 3. Let $\langle [0,1]_{\text{RPL}}, \mathbf{M}\rangle$ and $\langle [0,1]_{\text{RPL}}, \mathbf{N}\rangle$ be structures, and g be a mapping from M to N. We say that g is a *homomorphism from* $\langle [0,1]_{\text{RPL}}, \mathbf{M}\rangle$ *to* $\langle [0,1]_{\text{RPL}}, \mathbf{N}\rangle$ if for every n-ary function symbol F, any n-ary predicate symbol P and $d_1,\ldots,d_n \in M$,

(1) $g(F_{\mathbf{M}}(d_1,\ldots,d_n)) = F_{\mathbf{N}}(g(d_1),\ldots,g(d_n))$, and

(2) $P_{\mathbf{M}}(d_1,\ldots,d_n) = 1 \Rightarrow P_{\mathbf{N}}(g(d_1),\ldots,g(d_n)) = 1$.

Throughout the paper we assume that all our languages have a binary predicate symbol $\approx$ and we extend the axiomatic system of RPL∀ in [8, Ch.VIII] with the following axioms of similarity and congruence.

Definition 4. **[17, Definitions 5.6.1, 5.6.5]**

S1. $(\forall x)x \approx x$
S2. $(\forall x)(\forall y)(x \approx y \to y \approx x)$
S3. $(\forall x)(\forall y)(\forall z)(x \approx y\&y \approx z \to x \approx z)$

C1. For each n-ary function symbol F,

$(\forall x_1)\cdots(\forall x_n)(\forall y_1)\cdots(\forall y_n)(x_1 \approx y_1\&\cdots\&x_n \approx y_n \to F(x_1,\ldots,x_n) \approx F(y_1,\ldots,y_n))$

C2. For each n-ary predicate symbol P,

$(\forall x_1)\cdots(\forall x_n)(\forall y_1)\cdots(\forall y_n)(x_1 \approx y_1\&\cdots\&x_n \approx y_n \to (P(x_1,\ldots,x_n) \leftrightarrow P(y_1,\ldots,y_n))$

Definition 5. Let Φ be a theory over RPL∀, φ a formula in a language of RPL∀ and $r \in [0,1]$ a rational number.

(i) The *truth degree* of φ over Φ is $||\varphi||_\Phi =$

$$inf\{||\varphi||^{[0,1]_{\text{RPL}}}_{\mathbf{M}} \mid \langle[0,1]_{\text{RPL}}, \mathbf{M}\rangle \text{ is a model of } \Phi\}.$$

(ii) The *provability degree* of φ over Φ is

$$|\varphi|_\Phi = sup\{r \mid \Phi \vdash \overline{r} \to \varphi\}.$$

Theorem 1. **Pavelka-style completeness [17, Th.5.4.10]** Let Φ be a theory over RPL∀ and φ a formula in a language of RPL∀. Then, $|\varphi|_\Phi = ||\varphi||_\Phi$.

3. Term structures

In this section we introduce the notion of term structure associated to a consistent theory Φ over RPL∀, and prove that whenever the term structure is a model of Φ, the structure is free on the class of models of Φ. Term structures have been used extensively in classical logic, for instance, to prove the satisfiability of a set of consistent sentences (see for example [14, Ch.V]).

Definition 6. Let Φ be a consistent theory, we define a binary relation on the set of terms, denoted by $\sim$, in the following way: For every terms t_1, t_2,

$$t_1 \sim t_2 \text{ if and only if } |t_1 \approx t_2|_\Phi = 1.$$

Using Axioms ∀1, S1, S2 and S3, it can be proven that $\sim$ is an equivalence relation. Next lemma, which states that the equivalence relation $\sim$ is compatible with the symbols of the language, is proved using Axioms ∀1, C1, C2 and [17, Remark 3.18].

Lemma 1. For any consistent theory Φ, the following holds: If $t_i \sim t'_i$ for every $1 \leq i \leq n$, then

(i) For any n-ary function symbol F, $F(t_1, \ldots, t_n) \sim F(t'_1, \ldots, t'_n)$.

(ii) For any n-ary predicate symbol P and rational number $r \in [0,1]$,

$$|(\overline{r} \to P(t_1, ..., t_n)) \leftrightarrow (\overline{r} \to P(t'_1, ..., t'_n))|_\Phi = 1$$

From now on, for any term t we denote by $\overline{t}$ the $\sim$-class of t.

Definition 7. **Term Structure** Let Φ be a consistent theory. We define the following structure $\langle[0,1]_{\text{RPL}}, \mathbf{T}^\Phi\rangle$, where T^Φ is the set of all equivalence classes of the relation $\sim$ and

- For any n-ary function symbol F and terms $t_1, \ldots, t_n$,

$$F_{\mathbf{T}^\Phi}(\overline{t_1}, \ldots, \overline{t_n}) = \overline{F(t_1, \ldots, t_n)}$$

- For any n-ary predicate symbol P and terms $t_1, \ldots, t_n$,

$$P_{\mathbf{T}^\Phi}(\overline{t_1}, \ldots, \overline{t_n}) = |P(t_1, \ldots, t_n)|_\Phi$$

We call $\langle[0,1]_{\text{RPL}}, \mathbf{T}^\Phi\rangle$ the *term structure associated to* Φ.

Notice that for 0-ary functions, that is, for individual constants, $c_{\mathbf{T}^\Phi} = \overline{c}$. Given a consistent theory Φ, let e^Φ be the following $\mathbf{T}^\Phi$-evaluation: $e^\Phi(x) = \overline{x}$ for every variable x. We call e^Φ the *canonical evaluation of* $\langle[0,1]_{\text{RPL}}, \mathbf{T}^\Phi\rangle$.

Lemma 2. Let Φ be a consistent theory, the following holds:

(i) For any term t, $||t||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi, e^\Phi} = \overline{t}$.

(ii) For any atomic formula φ, $||\varphi||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi, e^\Phi} = 1$ if and only if $|\varphi|_\Phi = 1$.

(iii) For any evaluated atomic formula (φ, s), $||(\varphi, s)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi, e^\Phi} = 1$ if and only if $|(\varphi, s)|_\Phi = 1$.

Proof: The proofs of (i) and (ii) are straightforward. Regarding (iii), let $(\varphi, s) = (P(t_1 \ldots, t_n), s)$, we have:

$$||(P(t_1 \ldots, t_n), s)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi, e^\Phi} = 1 \quad \text{iff}$$

$$s \leq ||P(t_1 \ldots, t_n)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi, e^\Phi} \quad \text{iff}$$

$$s \leq P_{\mathbf{T}^\Phi}(\overline{t_1} \ldots, \overline{t_n}) \quad \text{iff}$$

$$s \leq |P(t_1 \ldots, t_n)|_\Phi \text{ iff } |\overline{s} \to P(t_1, \ldots, t_n)|_\Phi = 1.$$

The last equivalence is proved from [17, Remark 3.18]. □

Since the simplest well-formed formulas are atomic formulas, Lemma 2 (ii) can be read as saying that term structures are minimal with respect to atomic formulas. By Theorem 1, $|\varphi|_\Phi = ||\varphi||_\Phi$ and, by Lemma 2 (ii), the term structure $\langle[0,1]_{\text{RPL}}, \mathbf{T}^\Phi\rangle$ only assigns the truth value 1 to those atomic formulas that have 1 as their truth value in every model $\langle[0,1]_{\text{RPL}}, \mathbf{M}\rangle$ of Φ. By a similar argument, Lemma 2 (iii) states that the term structure $\langle[0,1]_{\text{RPL}}, \mathbf{T}^\Phi\rangle$ is minimal with respect to evaluated atomic formulas.

From an algebraic point of view, the minimality of the term structure is revealed by the fact that the structure is *free*. The following theorem proves that in case that the term structure associated to a theory is a model of that theory, the term structure is free.

Working in predicate fuzzy logics (and, in particular, in RPL∀) allows to define the term structure associated to a theory using similarities instead of crisp identities. This leads us to a notion of free structure restricted to the class of reduced models of that theory. Remember that *reduced structures* are those whose Leibniz congruence is the identity. By [12, Lemma 20], a structure $\langle[0,1]_{\text{RPL}}, \mathbf{M}\rangle$ is reduced iff it has the *equality property* (EQP) (that is, for any $d, e \in M$, $||d \approx e||^{[0,1]_{\text{RPL}}}_{\mathbf{M}} = 1$ iff $d = e$). Observe that, by using Definitions 6 and 7 and the fact that $\sim$ is an equivalence relation, it can be proven that $\langle[0,1]_{\text{RPL}}, \mathbf{T}^\Phi\rangle$ is a reduced structure.

Theorem 2. Let Φ be a consistent theory such that $\langle[0,1]_{\text{RPL}}, \mathbf{T}^\Phi\rangle$ is a model of Φ. Then $\langle[0,1]_{\text{RPL}}, \mathbf{T}^\Phi\rangle$ is free on the class of all the reduced models $\langle[0,1]_{\text{RPL}}, \mathbf{N}\rangle$ of Φ. That is, for every reduced model of Φ $\langle[0,1]_{\text{RPL}}, \mathbf{N}\rangle$ and every $\mathbf{N}$-evaluation v, there is a unique homomorphism g from $\langle[0,1]_{\text{RPL}}, \mathbf{T}^\Phi\rangle$ to $\langle[0,1]_{\text{RPL}}, \mathbf{N}\rangle$ such that for every variable x, $g(\overline{x}) = v(x)$.

Proof: Let $\langle[0,1]_{\text{RPL}}, \mathbf{N}\rangle$ be a reduced model of Φ and v an $\mathbf{N}$-evaluation. We define g by: $g(\overline{t}) = ||t||^{[0,1]_{\text{RPL}}}_{\mathbf{N}, v}$ for every term t. We show that g is the claimed homomorphism.

Let us first check that g is well-defined. Let t_1, t_2 be terms with $\overline{t_1} = \overline{t_2}$, i.e., $t_1 \sim t_2$, that is, $|t_1 \approx t_2|_\Phi = 1$. From Theorem 1 we have $||t_1 \approx t_2||_\Phi = 1$. Since $||\Phi||_{\mathbf{N}}^{[0,1]_{\text{RPL}}} = 1$, it follows that $||t_1 \approx t_2||_{\mathbf{N}}^{[0,1]_{\text{RPL}}} = 1$ and, in particular, $||t_1 \approx t_2||_{\mathbf{N},v}^{[0,1]_{\text{RPL}}} = 1$. From this and the fact that $\langle [0,1]_{\text{RPL}}, \mathbf{N}\rangle$ is reduced, we deduce, by [12, Lemma 20], that $||t_1||_{\mathbf{N},v}^{[0,1]_{\text{RPL}}} = ||t_2||_{\mathbf{N},v}^{[0,1]_{\text{RPL}}}$, i.e., $g(\overline{t_1}) = g(\overline{t_2})$.

The task is now to see that g satisfies the conditions (1) and (2) of Definiton 3. For any 0-function symbol c, $c_{\mathbf{T}^\Phi} = \overline{c} = c_{\mathbf{N}}$ by Definition 7. Let $\overline{t_1}, \ldots, \overline{t_n} \in T^\Phi$ and F be an n-ary function symbol, $F_{\mathbf{T}^\Phi}(\overline{t_1}, \ldots, \overline{t_n}) = \overline{F(t_1, \ldots, t_n)}$ by Definition 7. Then, by the definition of g,

$$g(F_{\mathbf{T}^\Phi}(\overline{t_1}, \ldots, \overline{t_n})) = g(\overline{F(t_1, \ldots, t_n)}) =$$

$$F_{\mathbf{N}}(||t_1||_{\mathbf{N},v}^{[0,1]_{\text{RPL}}}, \ldots, ||t_n||_{\mathbf{N},v}^{[0,1]_{\text{RPL}}}) = F_{\mathbf{N}}(g(\overline{t_1}), \ldots, g(\overline{t_n})).$$

Let P be an n-ary predicate symbol such that $P_{\mathbf{T}^\Phi}(\overline{t_1}, \ldots, \overline{t_n}) = 1$. By Definition 7 and Theorem 1, $1 =$

$$P_{\mathbf{T}^\Phi}(\overline{t_1}, \ldots, \overline{t_n}) = |P(t_1, \ldots, t_n)|_\Phi = ||P(t_1, \ldots, t_n)||_\Phi.$$

Consequently, $||P(t_1, \ldots, t_n)||_{\mathbf{N}}^{[0,1]_{\text{RPL}}} = 1$, because $||\Phi||_{\mathbf{N}}^{[0,1]_{\text{RPL}}} = 1$. Thus $||P(t_1, \ldots, t_n)||_{\mathbf{N},v}^{[0,1]_{\text{RPL}}} = 1$. Therefore $P_{\mathbf{N}}(||t_1||_{\mathbf{N},v}^{[0,1]_{\text{RPL}}}, \ldots, ||t_n||_{\mathbf{N},v}^{[0,1]_{\text{RPL}}}) = 1$, that is, $P_{\mathbf{N}}(g(\overline{t_1}), \ldots, g(\overline{t_n})) = 1$.

Finally, since the set $\{\overline{x} \mid x \text{ is a variable}\}$ generates the universe T^Φ of the term structure associated to Φ, g is the unique homomorphism such that for every variable x, $g(\overline{x}) = v(x)$. □

Observe that in languages in which the similarity symbol is interpreted by the crisp identity, by using an analogous argument to the one in Theorem 2, we obtain that the term structure is free in the class of all the models $\langle [0,1]_{\text{RPL}}, \mathbf{M}\rangle$ of the theory and not only in the class of the reduced ones.

4. RPL∀-Horn Clauses

In the previous section we have seen that if the term structure associated to a theory Φ is a model of Φ, then the structure is free in the class of all models of Φ. In this section, we show in Theorem 3 that whenever Φ is a theory of RPL∀-Horn clauses, $\langle [0,1]_{\text{RPL}}, \mathbf{T}^\Phi\rangle$ is a model of Φ. Theorem 3 gains in interest if we realize that it proves (using Theorem 2) the existence of free models of theories of RPL∀-Horn clauses. Let us first define the notion of RPL∀-Horn clauses.

In predicate classical logic, a *basic Horn formula* is a formula of the form $\alpha_1 \wedge \cdots \wedge \alpha_n \rightarrow \beta$, where n is a natural number and $\alpha_1, \ldots, \alpha_n, \beta$ are atomic formulas. Notice that there is not a unique way to extend this definition in fuzzy logics, where we have different conjunctions and implications. In this section we present one way to define Horn clauses over RPL∀ extending the classical definition.

Definition 8. **Basic RPL∀-Horn Formula** A *basic* RPL∀-*Horn formula* is a formula of the form

$$(\alpha_1, r_1)\& \cdots \&(\alpha_n, r_n) \rightarrow (\beta, s)$$

where $(\alpha_1, r_1) \ldots, (\alpha_n, r_n), (\beta, s)$ are evaluated atomic formulas and n is a natural number. Observe that n can be 0. In that case the basic RPL∀-Horn formula is an evaluated atomic formula.

Definition 9. **Quantifier-free RPL∀-Horn Formula** A *quantifier-free* RPL∀-*Horn formula* is a formula of the form $\phi_1 \& \cdots \& \phi_m$, where m is a natural number and ϕ_i is a basic RPL∀-Horn formula for every $1 \leq i \leq m$.

Definition 10. **RPL∀-Horn Clause** A RPL∀-*Horn clause* is a formula of the form $Q\gamma$, where Q is a (possibly empty) string of universal quantifiers $(\forall x)$ and γ is a quantifier-free RPL∀-Horn formula.

Example 1. Let $\mathcal{P}$ be a predicate language with a unary predicate symbol P, a binary predicate symbol R and a an individual constant. The following formulas are examples of RPL∀-Horn clauses:

(1) $(P(a), 0.5)$,
(2) $(P(a), 0.6)\&(R(a, x), 0.3)$,
(3) $(P(a), 0.5) \rightarrow (R(a, a), 0.1)$,
(4) $(P(a), 0.6)\&(R(a, x), 0.3) \rightarrow (P(x), 0.8)$,
(5) $(\forall x)((P(x), 0.6)\&(R(a, x), 0.3))$,
(6) $(\forall x)((P(x), 0.6)\&(R(a, x), 0.3) \rightarrow (P(a), 0.9))$.

Observe that, in general, RPL∀-Horn clauses are not evaluated, only the atomic RPL∀-Horn clauses are evaluated formulas.

A weak version of RPL∀-Horn clauses can be introduced by substituting each strong conjunction & appearing in the formula by the weak conjunction $\wedge$. Although in this paper we do not present this weak version, all the results we prove are also true for weak RPL∀-Horn clauses. In classical logic, the set of all Horn clauses is recursively defined, because the formula $(\forall x)(\varphi \wedge \psi)$ is logically equivalent to $(\forall x)\varphi \wedge (\forall x)\psi$. In RPL∀ these two formulas are also logically equivalent, so the set of the weak version of fuzzy RPL∀-Horn clauses is recursively definable. However, this is not the case for fuzzy RPL∀-Horn clauses. Indeed, let P and R be unary predicate symbols, consider the structure $\langle [0,1]_{\text{RPL}}, \mathbf{M}\rangle$ such that $M = \{a, b\}$, $P_{\mathbf{M}}(a) = R_{\mathbf{M}}(b) = 0.4$ and $P_{\mathbf{M}}(b) = R_{\mathbf{M}}(a) = 0.7$. Then, $||(\forall x)((P(x), 1)\&(R(x), 1))||_{\mathbf{M}}^{[0,1]_{\text{RPL}}} = 0.1$, but $||(\forall x)((P(x), 1))\&(\forall x)((R(x), 1))||_{\mathbf{M}}^{[0,1]_{\text{RPL}}} = 0$.

We now see that for any consistent theory of RPL∀-Horn clauses Φ, the term structure associated to Φ is a model of Φ. To show that, we need the following lemmas and the notion of rank of a formula. Our definition of rank is a variant of the notion of *syntactic degree of a formula* of [17, Def. 5.6.7]). Let φ be a formula, the *rank* of φ, denoted by $rk(\varphi)$ is defined by:

- $rk(\varphi) = 0$ if φ is atomic;
- $rk(\neg\varphi) = rk((\exists x)\varphi) = rk((\forall x)\varphi) = rk(\varphi) + 1$;

- $rk(\varphi \circ \psi) = rk(\varphi) + rk(\psi)$ for every binary propositional connective $\circ$.

Note that since the set of RPL∀-Horn clauses is not recursively definable, induction on the complexity of the clause cannot be applied. Hence it is applied on the rank of the clauses. Such induction can be used to prove next lemma.

Lemma 3. Let φ be an RPL∀-Horn clause where $x_1, \ldots, x_m$ are pairwise distinct free variables. Then, for every terms $t_1, \ldots, t_m$, the substitution

$$\varphi(t_1, \ldots, t_m/x_1, \ldots, x_m)$$

is an RPL∀-Horn clause.

Lemma 4. For any consistent theory Φ and any evaluated atomic formula (φ, s),

$$||(\varphi, s)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi} = ||(\varphi, s)||_\Phi.$$

Proof: It is enough to show that for any rational number $t \in [0, 1]$, $||(\varphi, s)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi} \geq t$ iff $||(\varphi, s)||_\Phi \geq t$.

Let $t \in [0, 1]$ be a rational number, we have:

$$||(\varphi, s)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi} \geq t \text{ iff } ||\bar{t} \to (\bar{s} \to \varphi)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi} = 1 \text{ iff}$$

$$||\bar{t}\&\bar{s} \to \varphi||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi} = 1 \text{ iff } ||\varphi||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi} \geq t *_{\text{Ł}} s \text{ iff}$$

$$||\varphi||^{[0,1]_{\text{RPL}}}_{\mathbf{M}} \geq t *_{\text{Ł}} s \text{ for every model } \langle [0, 1]_{\text{RPL}}, \mathbf{M}\rangle \text{ of } \Phi \text{ iff}$$

$$\text{for any model } \langle [0, 1]_{\text{RPL}}, \mathbf{M}\rangle \text{ of } \Phi,$$

$$||\bar{t} \to (\bar{s} \to \varphi)||^{[0,1]_{\text{RPL}}}_{\mathbf{M}} = 1.$$

The second and latter equivalence are proved by using [17, Def.2.2.4 (Axioms 5a and 5b)]. The latter expression is equivalent to $||(\varphi, s)||^{[0,1]_{\text{RPL}}}_{\mathbf{M}} \geq t$ for every model $\langle [0, 1]_{\text{RPL}}, \mathbf{M}\rangle$ of Φ, i.e., $||(\varphi, s)||_\Phi \geq t$. □

Lemma 5. For any consistent theory Φ and any evaluated atomic sentences $(\varphi_1, s_1), \ldots, (\varphi_n, s_n)$,

$$||(\varphi_1, s_1)\& \cdots \&(\varphi_n, s_n)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi} \leq ||(\varphi_1, s_1)\& \cdots \&(\varphi_n, s_n)||_\Phi.$$

Proof: By Lemma 4, it is clear for $n = 1$. For the sake of clarity, we present the proof for the case $n = 2$, but the argument is analogous for the cases with $n > 2$. First, by Lemma 4 we have:

$$||(\varphi_1, s_1)\&(\varphi_2, s_2)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi} = ||(\varphi_1, s_1)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi} *_{\text{Ł}} ||(\varphi_2, s_2)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi} = ||(\varphi_1, s_1)||_\Phi *_{\text{Ł}} ||(\varphi_2, s_2)||_\Phi.$$

Since for any model $\langle [0, 1]_{\text{RPL}}, \mathbf{M}\rangle$ of Φ, $||(\varphi_1, s_1)||_\Phi \leq ||(\varphi_1, s_1)||^{[0,1]_{\text{RPL}}}_{\mathbf{M}}$ and $||(\varphi_2, s_2)||_\Phi \leq ||(\varphi_2, s_2)||^{[0,1]_{\text{RPL}}}_{\mathbf{M}}$, we have that for any model $\langle [0, 1]_{\text{RPL}}, \mathbf{M}\rangle$ of Φ,

$$||(\varphi_1, s_1)||_\Phi *_{\text{Ł}} ||(\varphi_2, s_2)||_\Phi \leq ||(\varphi_1, s_1)||^{[0,1]_{\text{RPL}}}_{\mathbf{M}} *_{\text{Ł}} ||(\varphi_2, s_2)||^{[0,1]_{\text{RPL}}}_{\mathbf{M}} = ||(\varphi_1, s_1)\&(\varphi_2, s_2)||^{[0,1]_{\text{RPL}}}_{\mathbf{M}}.$$

Therefore, since $||(\varphi_1, s_1)\&(\varphi_2, s_2)||_\Phi$ is the infimum, we have

$$||(\varphi_1, s_1)||_\Phi * ||(\varphi_2, s_2)||_\Phi \leq ||(\varphi_1, s_1)\&(\varphi_2, s_2)||_\Phi.$$

Consequently,

$$||(\varphi_1, s_1)\&(\varphi_2, s_2)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi} \leq ||(\varphi_1, s_1)\&(\varphi_2, s_2)||_\Phi.$$ □

Theorem 3. Let Φ be a consistent theory. For every RPL∀-Horn clause φ without free variables,

$$\text{If } |\varphi|_\Phi = 1, \text{ then } ||\varphi||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi} = 1.$$

Proof: Let φ be an RPL∀-Horn clause without free variables. We proceed by induction on $rk(\varphi)$.

$rk(\varphi) = 0$. We can distinguish three subcases:

1) If $\varphi = (\psi, s)$ is an evaluated atomic formula, the statement holds by Lemma 4 (iii).

2) Let $\varphi = (\psi_1, s_1)\& \cdots \&(\psi_n, s_n) \to (\psi, s)$ be a basic RPL∀-Horn formula, where $(\psi_1, s_1), \ldots, (\psi_n, s_n), (\psi, s)$ are evaluated atomic formulas. By hypothesis and Theorem 1 we have:

$$1 = |(\psi_1, s_1)\& \cdots \&(\psi_n, s_n) \to (\psi, s)|_\Phi =$$

$$||(\psi_1, s_1)\& \cdots \&(\psi_n, s_n) \to (\psi, s)||_\Phi.$$

Therefore, $||(\psi_1, s_1)\& \cdots \&(\psi_n, s_n)||_\Phi \leq ||(\psi, s)||_\Phi$.

By Lemmas 4 and 5,

$$||(\psi, s)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi} = ||(\psi, s)||_\Phi$$

and

$$||(\psi_1, s_1)\& \cdots \&(\psi_n, s_n)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi} \leq ||(\psi_1, s_1)\& \cdots \&(\psi_n, s_n)||_\Phi.$$

Therefore

$$||(\psi_1, s_1)\& \cdots \&(\psi_n, s_n)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi} \leq ||(\psi, s)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi}.$$

That is,

$$||(\psi_1, s_1)\& \cdots \&(\psi_n, s_n) \to (\psi, s)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi} = 1.$$

3) If $\varphi = \phi_1\& \cdots \&\phi_m$ is a conjunction of basic RPL∀-Horn formulas,

$$||\phi_1\& \cdots \&\phi_m||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi} = 1 \text{ iff}$$

$$||\phi_i||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi} = 1 \text{ for every } 1 \leq i \leq m.$$

From 1) and 2), $|\phi_i|_\Phi = 1$ for every $1 \leq i \leq m$ and thus $|\phi_1\& \cdots \&\phi_m|_\Phi = 1$.

$rk(\varphi) = n + 1$. Let $\varphi = (\forall x)\psi$ be such that ψ is an RPL∀-Horn clause of rank n. Assume inductively that for any RPL∀-Horn clause without free variables ξ of rank less or equal than n and such that $|\xi|_\Phi = 1$, $||\xi||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^\Phi} = 1$. By assumption and Axiom ∀1,

$$\Phi \vdash (\forall x)\psi \to \psi(t/x) \text{ for every term } t.$$

From Axiom Ł2, $sup\{r \mid \Phi \vdash \overline{r} \to \varphi\} = 1$ implies that $sup\{r \mid \Phi \vdash \overline{r} \to \psi(t/x)\} = 1$ for any term t. That is,

$$|\psi(t/x)|_\Phi = 1 \text{ for every term } t.$$

Since ψ has rank n and is an RPL∀-Horn clause by Lemma 3, we can apply the inductive hypothesis and conclude that $||\psi(t/x)||^{[0,1]_{\mathrm{RPL}}}_{\mathbf{T}^\Phi} = 1$ for any term t. So, by Lemma 2 (i),

$$||\psi(x)||^{[0,1]_{\mathrm{RPL}}}_{\mathbf{T}^\Phi, v[x \mapsto \bar{t}]} = 1 \text{ for every element } \bar{t} \text{ of the domain,}$$

and thus we get $||(\forall x)\psi||^{[0,1]_{\mathrm{RPL}}}_{\mathbf{T}^\Phi} = 1$. □

5. Conclusions and Future Work

The present paper is another step towards a systematic study of theories of Horn clauses over predicate fuzzy logics from a model-theoretic point of view, a study that we started in [10] and which is still in progress. In particular, here we have proved the existence of free models of theories of Horn clauses in RPL∀.

Future work will be devoted to study the broad approach taken in [8, Ch.8] to fuzzy logics with enriched languages. We shall see if RPL∀-Horn clauses introduced here can be generalized to that logics with enriched languages. Later, since one of our next goals is to solve the open problem (formulated by Cintula and Hájek in [9]) about the characterization of theories of fuzzy Horn clauses in terms of quasivarieties, we will analyze quasivarieties and try to define them in the context of fuzzy logics using recent results on products over fuzzy logics like [12].

Herbrand structures have been important in model theory and in the foundations of logic programming. Therefore, as a continuation of the present work, we would like to characterize the free Herbrand model in the class of the Herbrand models of theories of RPL∀-Horn clauses without equality. Finally, we will focus on a generalization of Herbrand structure, *fully named models*, in order to show that two types of minimality for these models (specifically free models and A-generic models) are equivalent.

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2.2 Part II: art painting style categorization

The work-related to Part II is presented in the journal paper Costa et al. (2021). The research work Costa et al. (2021) is an extended and revised version of the book chapter Costa et al. (2018). We presented the results included in this book chapter at the 21st International Conference of the Catalan Association for Artificial Intelligence, where we obtained the best presentation award.

2.2.1 Preliminaries: color spaces, the QCD model and the datasets QArt-Dataset and Painting-91-BIP

In this chapter we first introduce the concept of color space and give an example of its use. Then, we present the QCD model defined by Falomir, Museros & Gonzalez-Abril (2015), and describe the datasets used in this PhD thesis, the QArt-Dataset and the Painting-91-BIP. Finally, we explain the procedure we follow in order to obtain a color description of each image in the datasets.

Color is defined as the human perception of the visible electromagnetic spectrum, and a color space is a way by which we can specify colors (Tkalcic & Tasic (2003)). The latter concept was defined by Hermann Grassmann in the middle of the nineteenth century. Nowadays there are different classes of color spaces (for more details see Tkalcic & Tasic (2003)). The human visual system is composed of three components which produce color sensation (Joblove & Greenberg (1978)). The color spaces we use belong to the class of those spaces inspired by the properties of the human visual system. In this way, as we shown next a color space is a coordinate system that makes it possible to describe any displayable color by the corresponding values of its coordinates.

Let us now introduce one of the most well-known color spaces, the RGB space (RGB stands for Red, Green, Blue), which creates the entire range of displayable colors as a mixture of the three primary colors. The RGB color space intends to describe the visible electromagnetic spectrum by simulating the very first detection of light. According to the trichromatic theory (based on the work of Thomas Young and Hermann von Helmholtz in the 19th century), the components of the human visual system are three types of photoreceptors, sensitive to particular wavelengths of red, green and blue. They are the responsable of the perception of color, and nowadays they are known as *cell cones* (or simply *cones*. The human eye has three types of cones: S-cones, M-cones and L-cones (Long, Middle and Short wavelength sensitivity, respectively), depending on the visible wavelengths of light for which the cones are sensitive to. In this way, the color is can be described by the three numerical components. Thus any displayable color can be regarded as Cartesian coordinates in a Euclidean space (in this case, as a point in the RGB system). In the RGB model, the coordinates of any color must be contained in the cube shown in Figure 2.1.

The range of the three coordinates is $[0,1]$; and the origin at the vertex (i.e., $(0,0,0)$) corresponds to the black color. The intensity of the color increases along the three axes, up to the vertex (i.e., $(1,1,1)$), which corresponds to the white color.

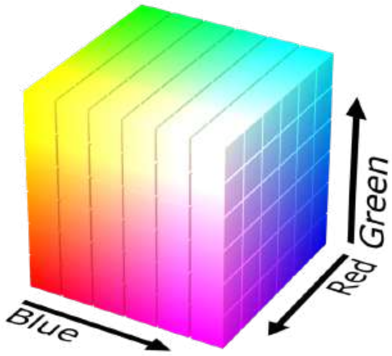

FIGURE 2.1: RGB color space mapped to a cube. Author of the image: SharkD (own work Source-code available at the POV-Ray Object Collection., CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=3375025).

For instance, the coordinates for a color with $R = 0.2, G = 0.6$ and $B = 0.8$ in the RGB cube are shown in Figure 2.2.

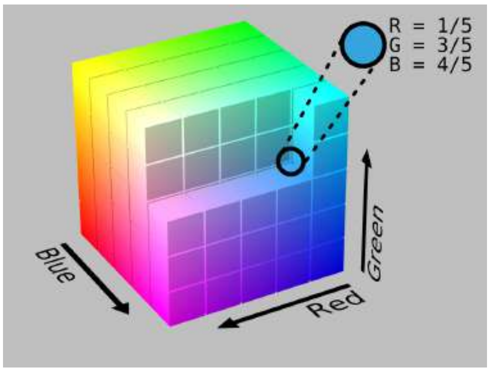

FIGURE 2.2: Example of a color in the RGB color space. Author of the image: SharkD (own work Source-code available at the POV-Ray Object Collection., CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=3375025).

Other color spaces inspired by the properties of the human visual system have been introduced in the literature, as for example the HSI color space (Hue,Saturation and Intensity) or the HSV/HSB color space (Hue, Saturation and Value or Brightness). In this thesis we work with the HSL color space (Hue, Saturation and Lightness), described below, which is the space used to define the QCD model. The coordinates of the HSL are obtained by linear transformations from the RGB space, but these transformations are not unique. The QCD model translates the RGB color channels into coordinates of the HSL (for the details on these linear transformations the reader is referred to Falomir, Museros & Gonzalez-Abril (2015)).

The QCD model was defined by Falomir, Museros & Gonzalez-Abril (2015). With this model, one can extract the color coordinates of each pixel of any digital image, describe it using the HSL color space, and obtain a color label for the pixel. The HSL color space is described by 3 coordinates $(uH, uL, uS) \in [0,360] \times [0,100] \times [0,100] \subseteq \mathbb{N}^3$, where $\mathbb{N}$ stands for the set of natural numbers, and:

$\underline{uH}$: The hue refers to the pure spectrum colors and corresponds to the dominant color as perceived by a human. The uH takes any value from the interval $[0,360]$, that is, $0 \leq uH \leq 360$.

$\underline{uS}$: The saturation refers to the relative purity or the amount of white light mixed with hue. The uS takes any value from the interval $[0,100]$, i.e., $0 \leq uS \leq 100$.

$\underline{uL}$: The luminance corresponds to the amount of light in a color. The uL takes any value from the interval $[0,100]$, that is, $0 \leq uL \leq 100$.

For each pixel in a digital image, using the QCD model one can extract its HSL color coordinates and obtain its corresponding color name according to the following Qualitative Color Reference System (QCRS) (see Figure 2.3), which discretizes the HSL color space as follows:

$$QCRS = \{uH, uS, uL, QC_{NAME1,\ldots,5}, QC_{INT1,\ldots,5}\},$$

where uH, uS and uL stand for the previous definitions, and the color names $QC_{NAME1,\ldots,5}$ and its corresponding HSL interval values $QC_{INT1,\ldots,5}$ are shown by Table 2.1. The QCRS was calibrated by Falomir, Museros & Gonzalez-Abril (2015) using machine learning on data obtained from surveys to people.

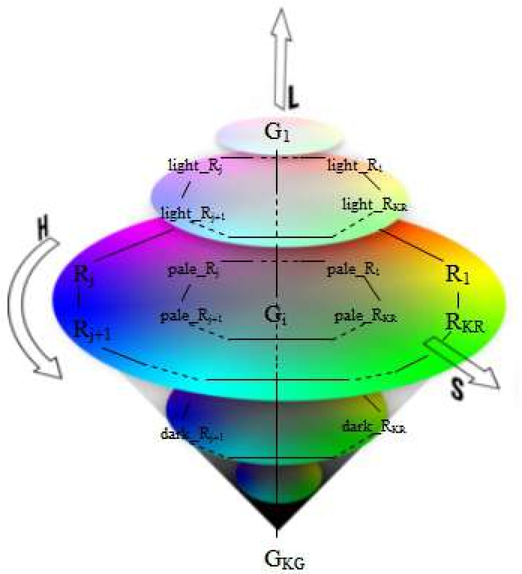

FIGURE 2.3: HSL colour space and QCD discretisation according to the QCRS.

The QCD model considers 37 labels for color names, which are grouped into 5 sets according to their spatial properties in the color space:

$QC_{NAME_1} = \{black,\ light\_grey,\ grey,\ dark\_grey,\ white\}$,

$QC_{NAME_2} = \{red,\ orange,\ yellow,\ green,\ turquoise,\ blue,\ purple,\ pink\}$,

$QC_{NAME_3} = \{pale\_red,\ pale\_orange,\ pale\_yellow,\ pale\_green,\ \ldots,\ pale\_pink\}$,

$QC_{NAME_4} = \{light\_red,\ light\_orange,\ light\_yellow,\ light\_green,\ \ldots,\ light\_pink\}$,

and $QC_{NAME_5} = \{dark\_red,\ dark\_orange,\ dark\_yellow,\ \ldots,\ dark\_pink\}$.

Let $QC_{NAME1,\ldots,5}$ be the union of the five sets above-defined, and let us note the 37 elements of $QC_{NAME1,\ldots,5}$ by QC_i with $1 \leq i \leq 37$ (i.e., QC_i denotes a color name for $1 \leq i \leq 37$). In order to determine a color name, QC_i, for the color displayed by a pixel, the QCD model considers the QCRS, which can also be expressed as the following function:

$$f_{QCRS}(uH, uL, uS):\ [0,360] \times [0,100] \times [0,100] \to QC_{NAME1,\ldots,5}$$
$$(uH, uL, uS) \in [0,360] \times [0,100] \times [0,100] \mapsto QC_i \in QC_{NAME1,\ldots,5},$$

defined in Table 2.1.

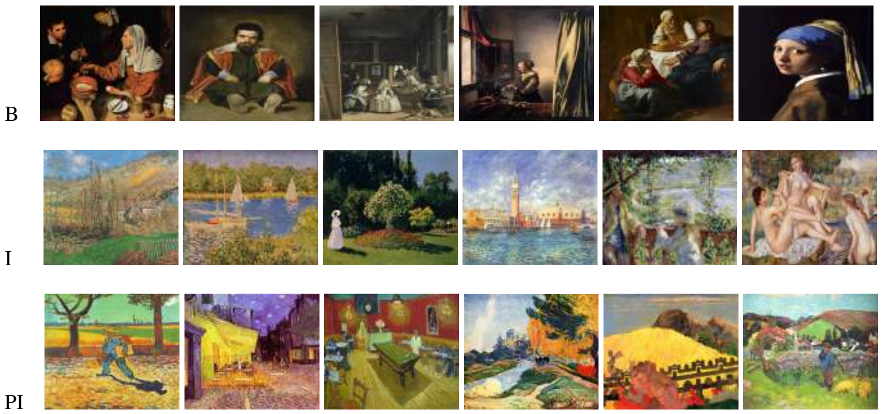

FIGURE 2.4: Extract from the QArt-Dataset: paintings corresponding to the Baroque style (B), the Impressionist style (I) and the Post-Impressionist style (PI). All rights by Wikimedia commons, public domain.

In this thesis we consider two datasets, the QArt-Dataset and the Painting-91-BIP dataset. The QArt-Dataset contains 90 images (30 Baroque paintings, 30 Impressionist paintings and 30 Post-Impressionist paintings). For each art style, the QArt-Dataset considers two representative authors: Velázquez and Vermeer for the Baroque style, Monet and Renoir for the Impressionism style, and Gauguin and van Gogh for the Post-Impressionism style (see Figure 2.4 for some examples). From the Painting-91 dataset introduced by Khan et al. (2014), Falomir et al. (2018) extracted the paintings of the six authors considered in the QArt-Dataset, 252 in total. We consider a total of 247 of images from the Painting-91 dataset: 74 for the Baroque style (39 by Velázquez and 35 by Vermeer), 82 for the Impressionism style (46 by Renoir and 36 by Monet),

TABLE 2.1: The definition of the function f_{QCRS}.

uH	**uL**	**uS**	$f_{QCRS}(uH, uL, uS)$ $QC_{NAME1,...,5}$
	(0, 20]		*black*
	(20, 40]		*dark_grey*
[0, 360]	(40, 60]	[0,20]	*grey*
	(60, 80]		*light_grey*
	(80, 100]		*white*
[0, 20] ∪ (335, 360]			*red*
(20, 50]			*orange*
(50, 80]			*yellow*
(80, 160]	(40, 55]	(50,100]	*green*
(160, 200]			*turquoise*
(200, 239]			*blue*
(239, 297]			*purple*
(297, 335]			*pink*
[0, 20] ∪ (335, 360]			*pale_red*
(20, 50]			*pale_orange*
(50, 80]			*pale_yellow*
(80, 160]	(40, 55]	(20, 50)	*pale_green*
(160, 200]			*pale_turquoise*
(200, 239]			*pale_blue*
(239, 297]			*pale_purple*
(297, 335]			*pale_pink*
[0, 20] ∪ (335, 360]			*ligth_red*
(20, 50]			*ligth_orange*
(50, 80]			*ligth_yellow*
(80, 160]	(55, 100]	(50,100]	*ligth_green*
(160, 200]			*ligth_turquoise*
(200, 239]			*ligth_blue*
(239, 297]			*ligth_purple*
(297, 335]			*ligth_pink*
[0, 20] ∪ (335, 360]			*dark_red*
(20, 50]			*dark_orange*
(50, 80]			*dark_yellow*
(80, 160]	(20, 40]	(50, 100)	*dark_green*
(160, 200]			*dark_turquoise*
(200, 239]			*dark_blue*
(239, 297]			*dark_purple*
(297, 335]			*dark_pink*

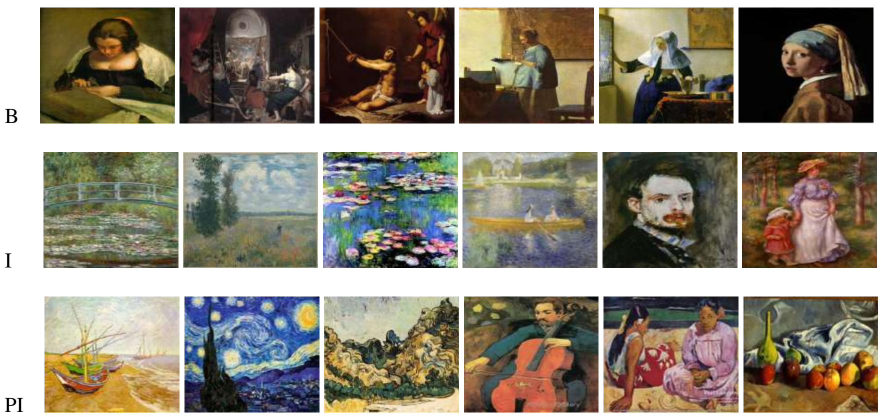

FIGURE 2.5: Extract from the Painting-91-BIP dataset. All rights under © creative commons, public license.

and 91 for the Post-Impressionism style (40 by van Gogh and 51 by Gauguin). The dataset containing these 247 images is called Painting-91-BIP (Figure 2.5). Details of both datasets, the QArt-Dataset and the Painting-91-BIP, can be found in Appendix A.

In order to obtain a color description of the images in the datasets, each fine-art painting image (*Img*) is described by applying computer vision techniques (see Falomir et al. (2018) for more details), from which we can extract a color vector histogram:

$$(f_1(Img), f_2(Img), \dots, f_{37}(Img)) \in \mathbb{N}^{37},$$

where $f_i(Img)$ corresponds to the number of pixels labeled as QC_i in Img.

Let $T(Img)$ be the number of pixels in Img, we define the frequency of the color QC_i, $F_i(Img)$, as $f_i(Img)/T(Img)$ for $1 \leq i \leq 37$. Note that for any image Img, $f_i(Img), F_i(Img) \geq 0$ for $1 \leq i \leq 37$. Then we transform the color traits in each painting to expressions with the following sintaxis:

$$color\_painting(P, QC_i, F_i),$$

where P corresponds to the digital image identifier (provided by the chosen dataset), $QC_i \in QC_{NAME1,\dots,5}$, F_i is defined as indicated above and $1 \leq i \leq 37$.

We have described all the images of the datasets considered using these formulas. Examples of paintings described by these expressions are shown in Figures 2.6 – 2.11.

```
colour_painting(v10, black, 0.362).
colour_painting(v10, dark_turquoise, 0.056).
colour_painting(v10, dark_green, 0.025).
colour_painting(v10, dark_grey, 0.117).
colour_painting(v10, dark_orange, 0.022).
...
colour_painting(v10, light_green, 0.014).
colour_painting(v10, light_grey, 0.054).
colour_painting(v10, light_orange, 0.010).
...
colour_painting(v10, pale_yellow, 0.0128).
colour_painting(v10, pale_green, 0.046).
colour_painting(v10, turquoise, 0.0004).
colour_painting(v10, white, 0.021).
```

FIGURE 2.6: *Equestrian Portrait of Prince Balthasar Charles* ($v10$ in the QArt-Dataset) by Velázquez. All rights under © creative commons, public license.

2.2.2 A summary of the main original contributions of Part II

First, we motivate and contextualize the work related to Part II. Then, before including the journal paper related to this part, we outline the original contributions of this part.

Qualitative descriptors and fuzzy knowledge representation for art painting style categorization As we have shown in the previous section, art painting classification tasks in artificial intelligence have been recently fostered by machine learning algorithms (i.e., neural networks, support vector machines, deep learning, etc.).

```
colour_painting(jan_vermeer_6, black, 0.329).
colour_painting(jan_vermeer_6, dark_grey, 0.043).
colour_painting(jan_vermeer_6, grey, 0.063).
colour_painting(jan_vermeer_6, light_grey, 0.165).
...
colour_painting(jan_vermeer_6, orange, 0.016).
colour_painting(jan_vermeer_6, light_orange, 0.070).
colour_painting(jan_vermeer_6, light_yellow, 0.030).
...
colour_painting(jan_vermeer_6, pale_orange, 0.050).
colour_painting(jan_vermeer_6, dark_orange, 0.101).
colour_painting(jan_vermeer_6, dark_blue, 0.004).
colour_painting(jan_vermeer_6, white, 0.095).
```

FIGURE 2.7: *The Lacemaker* (*jan_vermeer_*6 in the Painting-91-BIP dataset) by Vermeer. All rights under © creative commons, public license.

```
colour_painting(rn3, black, 0.206).
colour_painting(rn3, dark_grey, 0.213).
colour_painting(rn3, dark_orange, 0.034).
...
colour_painting(rn3, green, 0.151).
colour_painting(rn3, light_grey, 0.1245).
colour_painting(rn3, light_orange, 0.033).
...
colour_painting(rn3, pale_orange, 0.068).
colour_painting(rn3, pale_green, 0.046).
colour_painting(rn3, white, 0.034).
```

FIGURE 2.8: *Luncheon of the Boating Party* (*rn*3 in the QArt-Dataset) by Renoir. All rights under © creative commons, public license.

```
colour_painting(claude_monet13, black, 0.096).
colour_painting(claude_monet13, dark_turquoise, 0.019).
colour_painting(claude_monet13, grey, 0.068).
colour_painting(claude_monet13, light_yellow,  0.00).
...
colour_painting(claude_monet13, orange, 0.002).
colour_painting(claude_monet13, pale_orange, 0.006).
colour_painting(claude_monet13, light_blue, 0.178).
...
colour_painting(claude_monet13, pale_turquoise, 0.007).
colour_painting(claude_monet13, blue, 0.022).
colour_painting(claude_monet13, dark_blue, 0.090).
colour_painting(claude_monet13, white, 0.040).
```

FIGURE 2.9: *Argenteuil Red Boats* (*claude_monet*13 in the Painting-91-BIP dataset) by Monet. All rights under © creative commons, public license.

Machine learning methods provide high categorization accuracies, but they usually cannot provide reasons to users regarding why an item is classified in a category.

Artificial intelligence is today present in almost all aspects of daily life, drawing much interest from governments, universities, and societies in general. For this reason, human-machine interaction is one of the ongoing reality to be expected in the foreseeable future, or one could even say that these interactions are the current realities, so intuitive and easy communications are crucial for making possible satisfactory interactions. In order to improve these interactions, in this doctoral thesis we make

```
colour_painting(vg12, black, 0.003).
colour_painting(vg12, dark_green,0.001).
colour_painting(vg12, dark_grey, 0.015).
...
colour_painting(vg12, grey, 0.018).
colour_painting(vg12, light_orange, 0.057).
colour_painting(vg12, light_yellow, 0.009).
...
colour_painting(vg12, pale_orange, 0.089).
colour_painting(vg12, pale_green, 0.002).
colour_painting(vg12, white, 0.017).
```

FIGURE 2.10: *Sunflowers* (*vg*12 in the QArt-Dataset) by van Gogh. All rights under © creative commons, public license.

```
colour_painting(paul_gauguin_41, black, 0.107).
colour_painting(paul_gauguin_41, dark_grey, 0.225).
colour_painting(paul_gauguin_41, grey, 0.143).
colour_painting(paul_gauguin_41, light_grey, 0.060).
...
colour_painting(paul_gauguin_41, dark_red, 0.010).
colour_painting(paul_gauguin_41, pale_red, 0.007).
colour_painting(paul_gauguin_41, light_blue, ).
...
colour_painting(paul_gauguin_41, dark_turquoise, 0.03).
colour_painting(paul_gauguin_41, orange, orange, 0.010).
colour_painting(paul_gauguin_41, white,  0.060).
```

FIGURE 2.11: *The Ancestors of Tehamana* (*paul_gauguin*_41 in the Painting-91-BIP dataset) by Gauguin. All rights under © creative commons, public license.

use of qualitative descriptors in the design of the classification algorithms presented. Qualitative descriptors are theoretical tools which describe within the same formalism different approaches to transform quantitative data into qualitative data Baillie & Ganascia (2000). In this way, they extract qualitative phases from a data flow and can serve to provide evidences to users regarding why an item is classified in a category. Indeed, qualitative descriptors (see Falomir & Kluth (2018)) and conceptual spaces (see for instance Mast et al. (2016) and Banaee et al. (2018)) have proven to be favorable in providing human understandable narratives of scenes. Qualitative color descriptors have been used by Falomir, Cabedo, Sanz & Abril (2015) to categorize painting styles using some machine learning techniques (e.g., k-nearest neighbors algorithms and SVMs). Later, Falomir et al. (2018) extended this approach by the *QArt-Learn*, adding quantitative global features to the qualitative color descriptors. In concordance to these works (Falomir, Cabedo, Sanz & Abril (2015) and Falomir et al. (2018)), in order to classify paintings into art styles, we focus on the color features of the images. In addition, in this doctoral thesis we integrate fuzzy logics to express the knowledge we have about distinctive color traits of each style under study (i.e., the Baroque, the Impressionism, and the Post-Impressionism styles). The use of fuzzy notions helps us to interpret the algorithm designed, ℓ-SHE, so that explaining its classifications becomes possible.

Machine learning methods in general provide high categorization accuracies, but they need great amounts of training data. In this PhD thesis we consider two datasets, the QArt-Dataset and the Painting-91-BIP dataset. The QArt-Dataset only contains 90 images (30 Baroque paintings, 30 Impressionist paintings and 30 Post-Impressionist paintings), and this is our training dataset. The Painting-91-BIP contains 247 images (74 Baroque paintings, 82 Impressionist paintings and 91 Post-Impressionist painting), and we use it to evaluate the classifier algorithm, ℓ-SHE.

Original Contributions

- Introduction of distinctive color traits of each style based on qualitative color descriptors and fuzzy notions. Using the Qualitative Colour Descriptor model (QCD model) introduced by Falomir, Museros & Gonzalez-Abril (2015), we propose a method to obtain a color description of each image in the datasets (Section 2 of Costa et al. (2021)).
- On the basis of the characteristic color features outlined by the art experts for each art style under study, we present distinctive color traits of the three styles as fuzzy notions (Section 3 of Costa et al. (2021)).
- Categorization of art styles. We introduce evaluated Horn clauses that express characteristic color traits of the Baroque, the Impressionism, and the Post-Impressionism styles (Section 4 of Costa et al. (2021).
- Design of the ℓogical painting Style classifier based on Horn clauses and Explanations, called ℓ-SHE (Section 5 of Costa et al. (2021)).

2.2.3 The logical style painting classifier based on Horn clauses and explanations (ℓ-SHE)

Next, we include in this doctoral thesis the journal article Costa et al. (2021), titled *The logical style painting classifier based on Horn clauses and explanations (ℓ-SHE)*[3].

[3]We acknowledge to the *Logic Journal of the IGPL*, and to the Oxford University Press. https://academic.oup.com/jigpal/advance-article-abstract/doi/10.1093/jigpal/jzz029/5618973

The logical style painting classifier based on Horn clauses and explanations (ℓ-SHE)

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Abstract

This paper presents a logical Style painting classifier based on evaluated Horn clauses, qualitative colour descriptors and Explanations (ℓ-SHE). Three versions of ℓ-SHE are defined, using rational Pavelka logic (RPL), and expansions of Gödel logic and product logic with rational constants: RPL, $G(\mathbb{Q})$ and $\sqcap(\mathbb{Q})$, respectively. We introduce a fuzzy representation of the more representative colour traits for the Baroque, the Impressionism and the Post-Impressionism art styles. The ℓ-SHE algorithm has been implemented in Swi-Prolog and tested on 90 paintings of the QArt-Dataset and on 247 paintings of the Paintings-91-PIB dataset. The percentages of accuracy obtained in the QArt-Dataset for each ℓ-SHE version are 73.3% (RPL), 65.6% ($G(\mathbb{Q})$) and 68.9% ($\sqcap(\mathbb{Q})$). Regarding the Paintings-91-PIB dataset, the percentages of accuracy obtained for each ℓ-SHE version are 60.2% (RPL), 48.2% ($G(\mathbb{Q})$) and 57.0% ($\sqcap(\mathbb{Q})$). Our logic definition for the Baroque style has obtained the highest accuracy in both datasets, for all the ℓ-SHE versions (the lowest Baroque case gets 85.6% of accuracy). An important feature of the classifier is that it provides reasons regarding why a painting belongs to a certain style. The classifier also provides reasons about why outliers of one art style may belong to another art style, giving a second classification option depending on its membership degrees to these styles.

Keywords: qualitative colour, art, fuzzy logics, Horn clause, logic programming, classifier, explainable AI

1 Introduction

Classification tasks in artificial intelligence (AI) have been recently fostered by machine learning algorithms (i.e. neural networks, support vector machines, deep learning, etc.). In the literature, research works that deal with the challenge of classifying paintings in art styles are the following: traditional Chinese paintings were classified using colour and support vector machines (SVMs) [22]; 2-way classification of paintings by Renoir/Monet, Pollock/Ernst, Dalí/Ernst, Renoir/ Rothko and Dalí/Kandinsky were categorized using signature styles (computer vision statistical features) and SVMs [34, 35]; deep neural networks achieved a separation of image content from style, which allowed to recast the content of one image in the style of another image [16]; deep neural networks were also trained on object recognition for style categorization of artworks [23] and obtained

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81.45% accuracy for Baroque paintings, 82.15% for the Impressionism style and 74.51% for the Post-Impressionism style. However, although machine learning methods provide high categorization accuracies, they need great amounts of training data and they usually cannot provide reasons to users regarding why an item is classified in a category. Providing reasons for a decision is very important in human–machine interactions, because users expect intelligent systems to explain themselves in a rational or human-like way when they take decisions. Moreover, although the expressiveness of deep neural networks is the reason they succeed, it also causes them to learn uninterpretable solutions that could have counter-intuitive properties [37]: (i) the individual units in the learning algorithm does not contain semantic information; and (ii) the stability of neural networks can be affected by small perturbations to their inputs (adversarial examples). Our research differs from all these approaches in that it does not use machine learning, but logic representation and moreover, it can generate explanations of the outliers (i.e. items classified in a wrong category) using qualitative concepts (as individual units with semantic meaning), which are used for the classification as features.

Logical reasoning has been also associated with image interpretation: non-monotonic reasoning has been applied to image description [30]; description logics have been used to interpret digital images by describing each object by its colour and qualitative shape and by its main spatial features (location, relative orientation and topology), which allows to infer new object categories (i.e. doors) by reasoning [10], etc. Fuzzy descriptions logics have been also applied to image interpretation: a fuzzy spatial relation ontology have been developed to deal with brain structures in 3D magnetic resonance images [21]; a fuzzy logic-based colour histogram analysis for discriminating benign skin lesions from malignant melanomas in dermoscopy images has been proposed [2]; fuzzy logics have been also used in landslide identification and classification [1]; a general type-2 fuzzy logic method for edge detection has also been applied to colour format images [17, 31]; a fuzzy description logics-based reasoning framework has been developed, which reasons over an extracted description of an outdoor image and it handles the underlying vagueness in a formal way providing well-defined reasoning services [8]. The work presented in this paper differs from all these logic approaches for image interpretation in that it uses qualitative features of colour and do classify images of paintings into art styles, also providing explanations when there are reasons to believe that a painting may fit in two styles. In the literature, some research works tend to follow the *explainable AI* principle too. Recently, research works have appeared, which provide reasons for a concept/object to be classified in a category: when categorizing leaves [3] or when categorizing places, movies and wines [9]. In addition, Sørmo *et al.* [36] consider different theories of explanation from the philosophy and cognitive science communities. Using these studies, the authors present a framework for explanation in case-based reasoning. Moreover, the SWALE project[1] studies creative explanation of anomalous events. The current paper follows also this *explainable AI* principle.

Qualitative descriptors have been shown to be successful in managing incomplete, imprecise and ambiguous information [5, 15] when reasoning. In addition, they use linguistic concepts that align with human perception and can be easily used to generate narratives that explain the reasoning process in order to give feedback to the users. Regarding human–machine interaction, qualitative descriptors [11] and conceptual spaces [3, 27] have proven to be successful in providing human understandable narratives of scenes. In the literature, few works have used qualitative colours for image interpretation. Semantic categories (e.g. *warm*, *cold*) and colour names have shown to be effective for painting retrieval in databases [26]. Qualitative colour descriptors (QCDs) have been used to categorize painting styles using machine learning techniques (i.e. k-Nearest Neighbors and

[1] http://www.cs.indiana.edu/\ignorespacesleake/projects/swale

SVMs) and the results obtained an accuracy of 75% for a dataset of 70 paintings [13]. Later, this approach was extended by *QArt-Learn* [14] adding quantitative global features to the QCDs and the accuracy obtained was 65% for 252 paintings. However, as far as the authors are concerned, there are no research works that integrate qualitative descriptors and logics for art style categorization that also can provide explanations of decisions and outliers. This is the main contribution of this paper, i.e. the definition of a ℓogical **S**tyle painting classifier based on **H**orn clauses and **E**xplanations (**ℓ-SHE**). This paper extends the pilot study [7] that formalizes distinctive colour traits for the Baroque, the Impressionism and the Post-Impressionism styles, and introduces an evaluated Horn clause based on these colour features as a categorization of each style. ℓ-SHE has been tested using the above-mentioned fuzzy propositional logics using 90 painting of the QArt-dataset and on a wider and different dataset, the Paintings-91-PIB dataset containing 247 paintings.

The rest of the paper is organized as follows. Section 2 introduces the QCD as preliminaries. Section 3 presents the colour traits that characterize the Baroque, the Impressionism and the Post-Impressionism styles in the literature and it explains how these traits are obtained. Section 4 presents three different logics that can be used to categorize the art styles, rational Pavelka logic (RPL) and expansions of Gödel logic and product logic with rational constants, and explains how the definitions for each art style are parameterized using the QArt-dataset. Section 5 describes in detail the ℓ-SHE categorization. Section 6 presents and discusses the results obtained when classifying the 90 images in the QArt-Dataset with the three art style painting classifiers defined: ℓ-SHE$^{\mathrm{RPL}}$, ℓ-SHE$^{G(\mathbb{Q})}$ and ℓ-SHE$^{\sqcap(\mathbb{Q})}$. Section 8 shows and analyses the results obtained when classifying the 247 images in the Paintings-91-PIB dataset with the three classifiers. Finally, in Section 9 conclusions and future work are presented.

2 Preliminaries

This section introduces the QCD model and shows how the colour frequencies of any digital image are extracted and expressed as facts for reasoning using Prolog Horn clauses. The datasets used in this paper are also introduced here: the QArt-Dataset and the Painting-91-BIP.

The QCD model was defined by Falomir *et al.* [12]. It extracts the colour coordinates of each pixel of any digital image and it describes it using the Hue, Saturation and Lightness (HSL) colour space.

The HSL colour space is described by 3 coordinates $(uH, uL, uS) \in [0, 360]\times[0, 100]\times[0, 100] \subseteq \mathbb{N}^3$, where $\mathbb{N}$ stands for the set of natural numbers.

1. <u>*uH*:</u> The hue refers to the pure spectrum colours and corresponds to the dominant colour as perceived by a human. The uH takes any value from the interval $[0, 360]$, i.e. $0 \leq uH \leq 360$.
2. <u>*uS*:</u> The saturation refers to the relative purity or the amount of white light mixed with hue. The uS takes any value from the interval $[0, 100]$, i.e. $0 \leq uS \leq 100$.
3. <u>*uL*:</u> The luminance corresponds to the amount of light in a colour. The uL takes any value from the interval $[0, 100]$, i.e. $0 \leq uL \leq 100$.

For each pixel in a digital image, the QCD model [12] extracts its HSL colour coordinates and it obtains its corresponding colour name according to the following Qualitative Colour Reference System (QCRS) (see Figure 1), which discretizes the HSL colour space as follows:

$QCRS = \{uH, uS, uL, QC_{NAME1...5}, QC_{INT1...5}\}$,

where uH, uS and uL stands for the previous definitions, and the colour names $QC_{NAME1...5}$ and its corresponding HSL interval values $QC_{INT1...5}$ are shown by Table 1. The QCRS was calibrated using machine learning on data obtained from surveys to people[12, 33].

The QCD model considers 37 labels for colour names [12], which are grouped into 5 sets according to their spatial properties in the colour space:

1. $QC_{NAME_1} = \{$*black, light_grey, grey, dark_grey, white*$\}$,

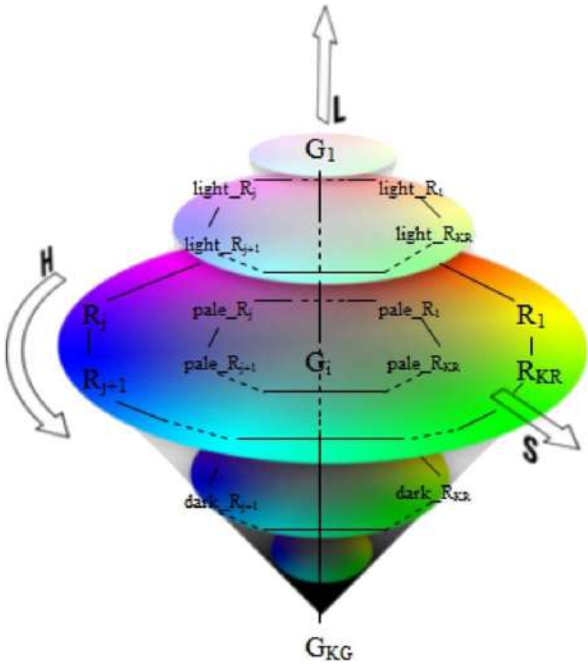

FIGURE 1. HSL colour space and QCD discretization according to the QCRS. The colour version of the figure is available on the online version of this paper.

2. QC_{NAME_2} = {*red, orange, yellow, green, turquoise, blue, purple, pink*},
3. QC_{NAME_3} = {*pale_red, pale_orange, pale_yellow, pale_green, ..., pale_pink*},
4. QC_{NAME_4} = {*light_red, light_orange, light_yellow, light_green, ..., light_pink*} and
5. QC_{NAME_5} = {*dark_red, dark_orange, dark_yellow, dark_green, ..., dark_pink*}.

Let $QC_{NAME1...5}$ = {*black, red, orange, ..., dark_purple, dark_pink*} $= \{QC_i \mid 1 \leq i \leq 37\}$ be the set of all the 37 colour names considered by the QCD model, where each QC_i denotes a colour name for $1 \leq i \leq 37$. In order to determine a colour name, QC_i, for the colour displayed by a pixel, the QCD considers the QCRS (see Figure 1), which can also be expressed as the following function:

$$f_{QCRS}(uH, uL, uS)\colon [0, 360] \times [0, 100] \times [0, 100] \to QC_{NAME1...5}$$
$$(uH, uL, uS) \in [0, 360] \times [0, 100] \times [0, 100] \mapsto QC_i \in QC_{NAME1...5},$$

defined by Table 1.

In order to obtain a colour description of the images in the datasets, each fine-art painting image (*Img*) is described by applying computer vision techniques [14] that extract a colour vector histogram: $(f_1(Img), f_2(Img), \ldots, f_{37}(Img)) \in \mathbb{N}^{37}$, where $f_i(Img)$ corresponds to the number of pixels labelled as QC_i in *Img*. Let $T(Img)$ be number of pixels in *Img*, we define the frequency of the colour QC_i, $F_i(Img)$, as $f_i(Img)/T(Img)$ for $1 \leq i \leq 37$. Note that for any image *Img*, $f_i(Img), F_i(Img) \geq 0$ for $1 \leq i \leq 37$.

We transform the colour traits in each painting to Prolog facts with the following sintaxis:

$$colour\_painting(P, QC_i, F_i),$$

TABLE 1 The definition of the function f_{QCRS}

uH	uL	uS	$f_{QCRS}(uH, uL, uS) QC_{NAME1...5}$
[0, 360]	(0, 20]	[0,20]	*black*
	(20, 40]		*dark_grey*
	(40, 60]		*grey*
	(60, 80]		*light_grey*
	(80, 100]		*white*
[0, 20] (335, 360]	(40, 55]	(50,100]	*red*
(20, 50]			*orange*
(50, 80]			*yellow*
(80, 160]			*green*
(160, 200]			*turquoise*
(200, 239]			*blue*
(239, 297]			*purple*
(297, 335]			*pink*
[0, 20] (335, 360]	(40, 55]	(20, 50)	*pale_red*
(20, 50]			*pale_orange*
(50, 80]			*pale_yellow*
(80, 160]			*pale_green*
(160, 200]			*pale_turquoise*
(200, 239]			*pale_blue*
(239, 297]			*pale_purple*
(297, 335]			*pale_pink*
[0, 20] (335, 360]	(55, 100]	(50,100]	*ligth_red*
(20, 50]			*ligth_orange*
(50, 80]			*ligth_yellow*
(80, 160]			*ligth_green*
(160, 200]			*ligth_turquoise*
(200, 239]			*ligth_blue*
(239, 297]			*ligth_purple*
(297, 335]			*ligth_pink*
[0, 20] (335, 360]	(20, 40]	(50, 100)	*dark_red*
(20, 50]			*dark_orange*
(50, 80]			*dark_yellow*
(80, 160]			*dark_green*
(160, 200]			*dark_turquoise*
(200, 239]			*dark_blue*
(239, 297]			*dark_purple*
(297, 335]			*dark_pink*

where P corresponds to the digital image identifier (provided by the chosen dataset), $QC_i \in QC_{NAME1...5}$, F_i is defined as indicated above and $1 \leq i \leq 37$.

The QArt-Dataset contains 90 images (30 Baroque paintings, 30 Impressionist paintings and 30 Post-Impressionist paintings) and we have used its colour histograms describing each image to obtain

FIGURE 2. Extract from the QArt-Dataset: paintings corresponding to the Baroque style (B), Impressionist style (I) and Post-Impressionist style (PI). All rights by Wikimedia commons, public domain. The colour version of this figure is available on the online version of this paper.

```
colour_painting(v10, black, 0.362).
colour_painting(v10, dark_turquoise, 0.056).
colour_painting(v10, dark_green, 0.025).
colour_painting(v10, dark_grey, 0.117).
colour_painting(v10, dark_orange, 0.022).
...
colour_painting(v10, light_green, 0.014).
colour_painting(v10, light_grey, 0.054).
colour_painting(v10, light_orange, 0.010).
...
colour_painting(v10, pale_yellow, 0.0128).
colour_painting(v10, pale_green, 0.046).
colour_painting(v10, turquoise, 0.0004).
colour_painting(v10, white, 0.021).
```

Extracted Prolog facts corresponding to this painting (*v*10, QArt-Dataset).

FIGURE 3. Extracted Prolog facts from Equestrian Portrait of Prince Balthasar Charles by Velázquez. All rights under © creative commons, public license. The colour version of this figure is available on the online version of this paper.

the parameters in ℓ-SHE. For each art style, the QArt-Dataset considers two representative authors: Velázquez and Vermeer for the Baroque style, Monet and Renoir for the Impressionism style and Gauguin and van Gogh for the Post-Impressionism style (see Figure 2 for some examples). From the Painting-91 dataset introduced in [24], Falomir *et al.* [14] extracted the paintings of the six authors considered in the QArt-Dataset, 252 in total. This paper considers a total of 247 of images from the Painting-91 dataset: 74 for the Baroque style (39 by Velázquez and 35 by Vermeer), 82 for the Impressionism style (46 by Renoir and 36 by Monet) and 91 for the Post-Impressionism style (40 by Van Gogh and 51 by Gauguin). This paper renames this new dataset as Painting-91-BIP (Figure 6). Section 8 uses the Painting-91-BIP dataset in order to test ℓ-SHE. In order to name the images in both datasets, the QArt-Dataset and the Painting-91-BIP dataset, we use the notation established in each dataset. Examples of paintings described by the Prolog facts are shown in Figures 3, 4 and 5.

```
colour_painting(rn3, black, 0.206).
colour_painting(rn3, dark_grey, 0.213).
colour_painting(rn3, dark_orange, 0.034).
...
colour_painting(rn3, green, 0.151).
colour_painting(rn3, light_grey, 0.1245).
colour_painting(rn3, light_orange, 0.033).
...
colour_painting(rn3, pale_orange, 0.068).
colour_painting(rn3, pale_green, 0.046).
colour_painting(rn3, white, 0.034).
```

Extracted Prolog facts corresponding to this painting (rn_3, QArt-Dataset).

FIGURE 4. Extracted Prolog facts from Luncheon of the Boating Party by Renoir. All rights under © creative commons, public license. The colour version of this figure is available on the online version of this paper.

```
colour_painting(vg12, black, 0.003).
colour_painting(vg12, dark_green,0.001).
colour_painting(vg12, dark_grey, 0.015).
...
colour_painting(vg12, grey, 0.018).
colour_painting(vg12, light_orange, 0.057).
colour_painting(vg12, light_yellow, 0.009).
...
colour_painting(vg12, pale_orange, 0.089).
colour_painting(vg12, pale_green, 0.002).
colour_painting(vg12, white, 0.017).
```

Extracted Prolog facts corresponding to the digital image ($vg12$, QArt-Dataset).

FIGURE 5. Extracted Prolog facts from Sunflowers by van Gogh. All rights under © creative commons, public license. The colour version of this figure is available on the online version of this paper.

3 Art style representation based on fuzzy QCDs

In this section we present a representation of the characteristic colour traits of the Baroque, the Impressionism and the Post-Impressionism styles using fuzzy sets. The fuzzy sets are defined using the frequencies of the colours.

The literature explains that **Baroque paintings** show mainly indoor scenes where lighting is exaggerated by contrasting *dark* colours to *light/pale* colours [32]. Regarding colour features in the **Impressionist style**, the literature [25, 28] explains that the development of synthetic pigments provided artists with vibrant shades of *blue* and *green*, among others. Moreover, the Impressionists captured the effects of sunlight by painting *en plein air* (outdoors), and thereby the *blue* of the sky, *light* colours and *grey* shadows are common colour traits in this style. In contrast to the Impressionism style, the **Post-Impressionist style** [19] breaks the tendency of representing colours as appearing in reality [20]. The Post-Impressionists looked for expressiveness using colours arbitrarily [38]. Thus, colours with pure hues (i.e. vivid colours) are present in the Post-Impressionist paintings.

Hence, with the goal of defining the different colour features for the art styles selected, we extend the QCD model.

DEFINITION 1
The following is an extension of the QCD model:

dark_colours = {*black, dark_red, dark_orange, dark_yellow, dark_green, dark_turquoise, dark_blue, dark_purple, dark_pink, dark_grey*},
pale_colours = {*pale_red, pale_orange, pale_yellow, pale_green, pale_turquoise, pale_blue, pale_purple, pale_pink, grey*},
light_colours = {*white,light_red, light_orange, light_yellow, light_green, light_turquoise, light_blue, light_purple, light_pink*},
grey_hue = {*grey*, *pale_grey*, *light_grey*, *dark_grey*},
red_hue = {*red*, *pale_red*, *light_red*, *dark_red*},
orange_hue = {*orange*, *pale_orange*, *light_orange*, *dark_orange*},
yellow_hue = {*yellow*, *pale_yellow*, *light_yellow*, *dark_yellow*},
green_hue = {*green*, *pale_green*, *light_green*, *dark_green*},
turquoise_hue = {*turquoise*, *pale_turquoise*, *light_turquoise*, *dark_turquoise*},
blue_hue = {*blue*, *pale_blue*, *light_blue*, *dark_blue*},
purple_hue = {*purple*, *pale_purple*, *light_purple*, *dark_purple*} and
pink_hue = {*pink*, *pale_pink*, *light_pink*, *dark_pink*}.

Vivid colours, warm hues and cold hues are also defined:

warm_hue = {*red_hue,orange_hue,yellow_hue*},
vivid_colours = {*red, orange, yellow, green, turquoise, blue, purple, pink*} and
cold_hue={*green_hue,turquoise_hue,blue_hue,purple_hue,pink_hue*}.

Considering these colour features outlined by the art experts and the extension of the QCD presented above, we propose to use the following distinctive colour traits for the Baroque style:

darkness_level: the accumulative sum of the frequencies of *dark_colours*.
no_paleness_level: the total frequency of colours that are not *pale_colours*.
contrast_level: the total frequency of *dark* and *pale* colours bounded to 1.

Regarding the Impressionism style, four characteristic colour features are proposed:

bluish_level: the total frequency of the QCs extracted as having blue hue (see Definition 1).
greyish_level: the total frequency of the QCs extracted as having grey hue (see Definition 1).
diversityofHues: all the QCs in a painting are grouped according to their hues (see Definition 1) and they are related to the total number of hues in QCD, which is 11 ($|vivid\_colours \cup \{black, white\}| = 11$).
diversityofQCDs: the relation between the amount of qualitative colours (including all their pale-, light- and dark- variants) in a painting, and the total number of QCs possible (i.e. 37).

Two distinctive colour traits for the Post-Impressionism are suggested:

vividness_level: the total frequency of the QCs extracted as having pure hue (see Definition 1).
warm_colours_level: the total frequency of the QCs extracted as having warm hue (see Definition 1).

4 Art style categorization based on evaluated Horn clauses

Since the distinctive colour traits presented in Section 3 can be regarded in a natural way as fuzzy notions, in this section we introduce one evaluated Horn clause for each painting style using the

FIGURE 6. Extract from the Painting-91-BIP dataset. All rights under © creative commons, public license. The colour version of this figure is available on the online version of this paper.

RPL, and other propositional fuzzy languages expanded with truth-constants. These evaluated Horn clauses give a categorization of the different painting styles. First we recall the syntax and semantics of these formal languages and then introduce a propositional variable for each main colour trait of the different painting styles. Finally, we show how the rational parameters of the evaluated Horn clauses are obtained using the data on qualitative colours and frequencies extracted from the QArt-Dataset.

DEFINITION 2
(Syntax and semantics of continuous t-norm based propositional fuzzy logics [4, Chapter I, Definition 1.1.13]) The language of continuous t-norm based propositional fuzzy logics contains a set of propositional variables *Var*, the binary connectives in the set $\{\rightarrow, \&, \wedge, \vee, \leftrightarrow\}$, the unary connective $\neg$ and the truth-constants $\overline{0}, \overline{1}$. Let $[0,1] \subseteq$, where denotes the set of real numbers, a $[0,1]$-evaluation e is a mapping $e : Var \rightarrow [0,1]$. Let $*$ be a continuous t-norm, an evaluation e extends uniquely to an evaluation e^* of the set of well-formed formulas as usual.

For the sake of simplicity, no distinction between e and e^* is made and the notation is simplified to e in both cases. For each rational number $r \in [0,1]$, we consider the truth-constant $\overline{r}$ so that $e^*(\overline{r}) = r$. In order to categorize the three art styles considered, we use RPL and expansions with rational constants of Gödel logic ($G(\mathbb{Q})$ for short) and product logic ($\sqcap(\mathbb{Q})$ for short). Let φ, ψ be two formulas, we recall the interpretation of $\&$ and $\rightarrow$ in RPL:

$$e(\varphi \& \psi) = max\{0, e(\varphi) + e(\psi) - 1\}, \text{ and}$$
$$e(\varphi \rightarrow \psi) = min\{1 - e^*(\varphi) + e^*(\psi), 1\}.$$

Regarding $G(\mathbb{Q})$, let us remember that

$$e(\varphi \& \psi) = min\{e(\varphi), e(\psi)\}, \text{ and}$$
$$e(\varphi \rightarrow \psi) = \begin{cases} 1 & \text{if} \varphi \leq \psi \\ \psi, & \text{otherwise.} \end{cases}$$

Finally, with respect to $\sqcap(\mathbb{Q})$, let us recall that

$$e(\varphi \& \psi) = e(\varphi)e(\psi), \text{ and}$$
$$e(\varphi \rightarrow \psi) = \begin{cases} 1 & \text{if } \varphi \leq \psi \\ \frac{\psi}{\varphi}, & \text{otherwise.} \end{cases}$$

Next definition is a generalization of the definition of RPL∀-Horn clause introduced in [6].

DEFINITION 3
(Evaluated Horn clause [6, Definition 10]) An atomic evaluated formula (φ, r) is defined as $\overline{r} \rightarrow \varphi$, where $r \in [0,1]$ is a rational number and φ is an atomic formula without truth constants apart from $\overline{0}$ and $\overline{1}$. An evaluated Horn clause has the form

$$(\varphi_1, r_1)\& \ldots \&(\varphi_n, r_n) \rightarrow (\varphi, s),$$

where $(\varphi_1, r_1), \ldots, (\varphi_n, r_n)$ and (φ, s) are atomic evaluated formulas.

For the sake of clarity, evaluated Horn clauses are simply named Horn clauses. Consider the following propositional variables referring to the colour features defined in Section 3: *darkness_level*, *no_paleness_level*, *contrast_level*, *bluish_level*, *greyish_level*,*diversityofHues*, *diversityofQCDs*, *vividness_level*, *warm_colours_level*; and consider also the following propositional variables referring to the styles Baroque, Impressionism and Post-Impressionism, respectively: *baroque*, *impressionism*,*post_impressionism*. We propose the following Horn clauses to categorize the different art styles selected. H_B represents the Baroque style:

$$(darkness\_level, 0.76)\&(no\_paleness\_level, 0.84)\& \\ (contrast\_level, 0.90) \rightarrow (baroque, 1).$$

H_I represents the Impressionist style:

$$(diversityofQCDs, 0.60)\&(diversityofHues, 0.75)\&(bluish\_level, 0.05) \\ \&(greyish\_level, 0.44) \rightarrow (impressionism, 1).$$

And H_{PI} represents the Post-Impressionist style:

$$(vividness\_level, 0.14)\&(warm\_colours\_level, 0.53) \rightarrow (post\_impressionism, 1).$$

Observe that the semantics of the three logics selected in this paper are different, and thus the interpretation of the Horn clauses depends on the logic used. Since their systematization by Hájek [18], these three logics have shown to be some of the most significant and well-known t-norm based logics.

For any digital painting p, it can be associated an evaluation e_p of the variables in the antecedent of the Horn clauses H_B, H_I, H_{PI}. For instance, for painting $v10$ (see Figure 3), in RPL we obtain that $e_{v10}(contrast\_level, 0.90) = min\{1 - 0.9 + e_{v10}(contrast\_level), 1\} = min\{1 - 0.9 + 0.87, 1\} = 0.97$. Given a painting p, the antecedent of clause H_B is evaluated using e_p in order to obtain a membership degree for the Baroque style. For the sake of clarity, let us introduce some notation: $B_1(p) = e_p(darkness\_level, 0.76)$ and

$$\begin{array}{rr} B_2(p) = e_p(no\_paleness\_level, 0.84) & I_1(p) = e_p(diversityofQCDs, 0.60) \\ B_3(p) = e_p(contrast\_level, 0.90) & I_2(p) = e_p(diversityofHues, 0.75) \\ PI_1(p) = e_p(vividness\_level, 0.14) & I_3(p) = e_p(bluish\_level, 0.05) \\ PI_2(p) = e_p(warm\_colours\_level, 0.53) & I_4(p) = e_p(greyish\_level, 0.44). \end{array}$$

According to the semantics of each logic selected, the membership degrees to the Baroque, the Impressionism and the Post-Impressionism styles are next calculated. For the sake of clarity, throughout this section we focus the presentation on the ℓ-SHE$^{\text{RPL}}$ version. For ℓ-SHE$^{\text{RPL}}$,

$$B(p) = max\{0, B_1(p) + B_2(p) + B_3(p) - 2\}$$

(i.e. $B(p) = e_p((darkness\_level, 0.76)\&(no\_paleness\_level, 0.84)\&(contrast\_level, 0.9)))$, $I(p) = max\{0, I_1(p) + I_2(p) + I_3(p) + I_4(p) - 3\}$, and $PI = max\{0, PI_1(p) + PI_2(p) - 1\}$.

Let us now consider the painting $v10$ in the QArt-Dataset, we take ℓ-SHE^{RPL} and we show how the membership degree of $v10$ to each art style is obtained. First, the levels of the characteristic colour traits of the Baroque style are obtained:

$B_1(v10) = e_{v10}(darkness\_level, 0.76) = min\{1 - 0.76 + e_{v10}(darkness\_level), 1\} = min\{0.24 + 0.67, 1\} = 0.91$,
$B_2(v10) = e_{v10}(no\_paleness\_level, 0.84) = min\{1 - 0.84 + e_{v10}(no\_paleness\_level), 1\} = min\{0.16 + 0.80, 1\} = 0.96$ and
$B_3(v10) = e_{v10}(contrast\_level, 0.90) = min\{1 - 0.9 + e_{v10}(contrast\_level), 1\} = min\{1 - 0.9 + 0.87, 1\} = 0.97$.

The membership degree to the Baroque style given to $v10$ is $B(v10) = max\{0, B_1(p) + B_2(p) + B_3(p) - 2\} = \{0.91 + 0.96 + 0.97 - 2, 0\} = 0.84$. The other membership degrees are obtained similarly: $I(v10) = 0.78$, and $PI(v10) = 0.53$.

We explain the procedure for the colour trait *darkness_level* of clause H_B. The median and the standard deviation for *darkness_level* are denoted by $\overline{x_{dly}}$ and σ_{dly}, respectively, where *dl* denotes *darkness_level* and y is substituted by B, I, PI, depending on the art style selected. For instance, $\overline{x_{dlB}}$ denotes the median of the *darkness_level* of the 30 Baroque paintings in the QArt-Dataset. We also define the interval $R_{dly} = [\overline{x_{dly}} - \sigma_{dly}, \overline{x_{dly}} + \sigma_{dly}]$. For each art style y, $\overline{x_{dly}}, \sigma_{dly}, R_{dly}$ are computed from the colour frequencies using the R platform [29] (observe that for any digital painting, each colour trait yields a degree in $[0, 1]$):

darkness_level(dl)	$\overline{x_{dly}}$	σ_{dly}	$R_{dly} = [\overline{x_{dly}} - \sigma_{dly}, \overline{x_{dly}} + \sigma_{dly}]$
$Baroque(B)$	0.76	0.15	$[0.61, 0.91]$
$Impressionism(I)$	0.42	0.20	$[0.22, 0.62]$
$Post-Impressionism(PI)$	0.33	0.19	$[0.14, 0.52]$.

Let $a, b, c, d \in [0, 1]$, where $b \geq a$ and $d \geq c$, and $R_1 = [a, b], R_2 = [c, d]$ be two intervals, the intersection of R_1 and R_2 is defined as

$$R_1 \cap R_2 = \begin{cases} \emptyset & \text{if } c > b \text{ or } a > d \\ [max\{a, c\}, min\{b, d\}] & \text{otherwise.} \end{cases}$$

Let $R = [a, b]$ be an interval, the length of R, denoted by $|R|$, is defined as $|R| = b - a$; and by convention the length of the empty set is 0.

Note that $|R_{dlB} \cap R_{dlI}| = 0.009$, i.e. the darkness level interval corresponding to the Baroque and the Impressionism styles presents an intersection that corresponds to 2.98% of the length of R_{dlB} and 2.22% of the length of R_{dlI}. This shows that these styles have very few in common regarding the *darkness_level* feature. Moreover, $|R_{dlB} \cap R_{dlPI}| = 0$, but $|R_{dlI} \cap R_{dlPI}| = 0.303$. Thus, the *darkness_level* feature is very similar in both styles, the Impressionism and the Post-Impressionism styles. These results suggest that using the level of darkness to categorize Baroque paintings is reasonable, considering the large difference between $\overline{x_{dlB}}$ and both $\overline{x_{dlI}}$ and $\overline{x_{dlPI}}$. They also show that darkness is not a useful colour feature for separating the Impressionism and the Post-Impressionism styles. Furthermore, $\overline{x_{dlB}}$ is much larger than $\overline{x_{dlI}}$ and $\overline{x_{dlPI}}$, and σ_{dlI} and σ_{dlPI} represent around the half part of $\overline{X_{dlI}}$ and $\overline{x_{dlPI}}$, respectively. Considering these, it has been deemed advisable to consider 0.76 as the parameter for *darkness_level*. This simple method avoids hard computation such as other training methods used in machine learning.

5 The ℓ-SHE categorization

The aim of this section is to describe the ℓ-SHE algorithm that is intended to generate human-understandable explanations based on colour traits according to the categorization obtained. We recall that ℓ-SHE has been defined for RPL, $G(\mathbb{Q}), \sqcap(\mathbb{Q})$: ℓ-SHE$^{\mathrm{RPL}}$, ℓ-SHE$^{G(\mathbb{Q})}$ and ℓ-SHE$^{\sqcap(\mathbb{Q})}$, respectively. Note that ℓ-SHE categorizations are not crisp, i.e. a membership degree for each art style—Baroque, Impressionism and Post-Impressionism—is provided, as detailed in Section 4.

From now on, let p denote any digital painting. We define the belief degree for p to belong to an art style as

$$dbAS(p) = \begin{cases} (B_{st}, B(p)) & \text{if } max\{B(p), I(p), PI(p)\} = B(p) \text{ and } B(p) \neq I(p) \\ (I_{st}, I(p)) & \text{if } max\{B(p), I(p), PI(p)\} = I(p) \\ (PI_{st}, PI(p)) & \text{if } max\{B(p), I(p), PI(p)\} = PI(p) \text{ and } B(p) \neq PI(p) \neq I(p). \end{cases}$$

Note that in the event of a tied membership degree, *dbAS* chooses the most restrictive art style. Since ℓ-SHE has to give a second option in difficult cases, a similarity between membership degrees, *Sim*, is defined: $Sim_{B,I}(p) = |B(p) - I(p)|$, $Sim_{B,PI}(p) = |B(p) - PI(p)|$ and $Sim_{I,PI}(p) = |I(p) - PI(p)|$, where $Sim_{B,I}(p)$ stands for the closeness between the Baroque and the Impressionism membership degrees of p, and $Sim_{B,PI}(p)$ and $Sim_{I,PI}(p)$ are described analogously. From data analysis obtained in the ℓ-SHE classification of the QArt-Dataset, we considered different values for determining doubt between art styles: $0.10, 0.15, 0.20$ and 0.25. Finally, it was found by experimentation that 0.15 is the best option for this parameter.

For the sake of clarity, throughout this section we introduce only the ℓ-SHE$^{\mathrm{RPL}}$ version. The rest of the ℓ-SHE versions are defined analogously. The ℓ-SHE$^{\mathrm{RPL}}$ algorithm categorizes paintings in the three following styles.

(1) If $dbAS = (B_{st}, B(p))$, then 'p is a Baroque painting.' & $explanation_{\mathrm{RPL}}(B, p)$.
- If $Sim_{B(p),I(p)} \leq 0.15$, then 'Although p is categorised in the Baroque style, there are reasons to believe that it may belong to the Impressionism.' & $explanation_{\mathrm{RPL}}(I, p)$.
- If $Sim_{B(p),PI(p)} \leq 0.15$, then 'Although p is categorised in the Baroque style, there are reasons to believe that it may belong to the Post-Impressionist.' & $explanation_{\mathrm{RPL}}(PI, p)$.

(2) If $dbAS = (I_{st}, I(p))$, then 'p is an Impressionist painting (I).' & $explanation_{\mathrm{RPL}}(I, p)$.
- If $Sim_{B(p),I(p)} \leq 0.15$, then 'Although p is categorised in the Impressionist style, there are reasons to believe that it may belong to the Baroque.' & $explanation_{\mathrm{RPL}}(B, p)$.
- If $Sim_{I(p),PI(p)} \leq 0.15$, then 'Although p is categorised in the Impressionist style, there are reasons to believe that it may belong to the Post-Impressionism.' & $explanation_{\mathrm{RPL}}(PI, p)$.

(3) If $dbAS = (PI_{st}, PI(p))$, then 'p is a Post-Impressionist painting (PI).' & $explanation_{\mathrm{RPL}}(PI, p)$.
- If $Sim_{B(p),PI(p)} \leq 0.15$, then 'Although p is categorised in the Post-Impressionist style, there are reasons to believe that it may belong to the Baroque.' & $explanation_{\mathrm{RPL}}(B, p)$.
- If $Sim_{I(p),PI(p)} \leq 0.15$, then 'Although p is categorised in the Post-Impressionist style, there are reasons to believe that it may belong to the Impressionism.' & $explanation_{\mathrm{RPL}}(I, p)$.

TABLE 2 Thresholds used by the explanations corresponding to each colour trait and logic

Logic/thresholds	RPL	$G(\mathbb{Q})$	$\sqcap(\mathbb{Q})$
T_{B1}	0.82	0.94	0.90
T_{B2}	0.90	0.98	0.97
T_{B3}	0.78	0.98	0.95
T_{I1}	0.87	0.89	0.90
T_{I2}	0.89	0.95	0.94
T_{I3}	0.97	1.00	0.87
T_{I4}	0.84	0.92	0.80
T_{PI1}	0.96	0.89	0.47
T_{PI2}	0.76	0.89	0.75

In addition, explanations for specific characteristics in each art style can also be provided, as explained below. Let us consider the feature *darkness_level* as significant for classifying a painting p into the Baroque style, whenever $B1(p)$ is higher than a threshold T_{B1}. If this is the case, the presence of this feature must appear as an explanation/evidence for p classified into this style. This threshold T_{B1} is calculated as $\overline{x_{B_1}} - \sigma_{B_1}$, where $\overline{x_{B_1}}$ is the mean of $\{B_1(p) \mid p \text{ is a Baroque painting}\}$ and σ_{B_1} is the corresponding standard deviation. Notice that the thresholds are obtained from the truth value of an implication, and this truth value depends on the t-norm used. Hence, each threshold depends on the selected logic. Thus, a superscript indicating the logic used is provided. For instance, the three cases regarding the threshold B_1 are noted as $T_{B1}^{\text{RPL}}, T_{B1}^{G(\mathbb{Q})}, T_{B1}^{\sqcap(\mathbb{Q})}$. Table 2 shows the value of each threshold for the three logics considered.

For the Baroque style, $explanation_{\text{RPL}}(B,p)$ provided by ℓ-SHE are the following:

If $B_1 \geq T_{B1}^{\text{RPL}}$, then 'The darkness evidences the Baroque style.'
If $B_2 \geq T_{B2}^{\text{RPL}}$, then 'Due to the contrast of *dark* and *pale* colours.'
If $B_3 \geq T_{B3}^{\text{RPL}}$, then 'The lack of *pale* colours evidences this style.'

The ℓ-SHE also provides the following explanations, $explanation_{\text{RPL}}(I,p)$, for the Impressionist style:

If $I_1 \geq T_{I1}^{\text{RPL}}$, then 'The diversity of qualitative colours evidences the Impressionism style.'
If $I_2 \geq T_{I2}^{\text{RPL}}$, then 'The variety of hues evidences the Impressionism style.'
If $I_3 \geq T_{I3}^{\text{RPL}}$, then 'The amount of bluish evidences this style.'
If $I_4 \geq T_{I4}^{\text{RPL}}$, then 'The amount of grey colour evidences this style.'

The explanations by ℓ-SHE for the Post-Impressionist style, $explanation_{\text{RPL}}(PI,p)$, are the following:

If $PI_1 \geq T_{PI1}^{\text{RPL}}$, then 'The presence of vivid colours evidences the Post-Impressionism style.'
If $PI_2 \geq T_{PI2}^{\text{RPL}}$, then 'The high level of warm colours evidences the Post-Impressionism style.'

6 Implementing ℓ-SHE

This section shows the implementation of the ℓ-SHE algorithm and provides some examples of responses produced by ℓ-SHE.

ℓ-SHE has been implemented in Prolog using Swi-Prolog [39] as the testing platform, whereas some thresholds have been obtained using the platform R [29], as indicated previously.

Some of the clauses implemented in Prolog for evaluating RPL formulae are shown next:

```
evaluate_formulaRPL(Formula,Rational,DoB):-
 Formula >= Rational,
 DoB is 1.

evaluate_formulaRPL(Formula,Rational,DoB):-
  Formula < Rational,
  DoB is 1-Rational+Formula.
```

The Prolog implementation clauses for obtaining the degree of believing of a painting P to be classified into the Baroque style is the following:

```
baroque_style(P,DoBa, DoBb, DoBc, DoB):-
  darkness_level(P,_,DLevel, _, _, _, _),
  darknessTh(DT), evaluate_formulaRPL(DLevel,DT,DoBa),
  contrast_level(P,ContrLevel), contrastTh(CT),
  evaluate_formulaRPL(ContrLevel,CT,DoBb),
  no_paleness_level(P,NoPaleness), noPalenessTh(NPT),
  evaluate_formulaRPL(NoPaleness,NPT,DoBc),
  DoBaux is DoBa+DoBb+DoBc-2, DoB is max(0,DoBaux).
```

For the sake of simplicity the exposition is focused on the ℓ-SHE$^{\mathrm{RPL}}$ version (the versions ℓ-SHE$^{G(\mathbb{Q})}$ and ℓ-SHE$^{\sqcap(\mathbb{Q})}$ are implemented analogously).

The clauses used for obtaining the *darkness_level* of a painting are shown next. The QCDs (Sections 2 and 4) are highlighted in blue:

```
darkness_level(P,DarkQCD,Total,N1,N2,R,Q):-
  find_fuzzy_colours_in_painting(P,CL,QL,N1,Q),
  get_dark_colours(CL,QL,DarkQCD,Total),
  length(DarkQCD,N2),R is (N2/N1)*100.

find_fuzzy_colours_in_painting(P,CL,QL,N,Q):-
  findall(Colour,colour_painting(P,Colour,_),CL),
  findall(Q, colour_painting(P,_,Q),QL),
  length(CL,N), sum_list(QL,Q).
```

```
get_dark_colours([],[],[],0).
get_dark_colours([C|CL],[_|QL],DL,Total):-
  not(dark(Q)),
  get_dark_colours(CL,QL,DL,T), Total is T+0.

get_dark_colours([C|CL],[Q|QL],[[C,Q]|DL],Total):-
  dark(Q),
  get_dark_colours(CL,QL,DL,T), Total is T+Q.
```

The examples highlighted in Section 2 are $v10, rn3$ and $vg12$. Our purpose was to choose 3 possibilities of correct classifications: (i) clear case with membership degree 1 (Figure 5); (ii) a clear case with membership degree different from 1 (Figure 4); and (iii) not clear case, so a second opinion is needed (Figure 3). Thus, Figure 5 shows a painting ($vg12$) classified in the Post-Impressionism style with membership degree 1. Figure 4 ($rn3$ painting) shows that ℓ-SHE correctly classifies a painting when the membership degree to Impressionism is 0.891 (not 1). As we will see, here it does not give a second opinion, because it is a clear case since the other membership degrees are 0.475 to Baroque and 0.604 to Post-Impressionism. Figure 3 shows a case where $v10$ painting has a membership degree of 0.875 to Baroque Style. As we will see, here ℓ-SHE provides a second opinion because the membership degree to Impressionism Style is 0.78, close to 0.875.

So an example of the response produced by ℓ-SHE$^{\mathrm{RPL}}$ regarding painting $v10$ is the following:

```
?- baroque_style(v10, DoBa, DoBb, DoBc, DoB).
SHE obtains ...
Darkness level:0.667
Contrast level:0.868
noPaleness level:0.799
DoB = 0.875.
```

Moreover, the categorization reasons provided by ℓ-SHE$^{\mathrm{RPL}}$ for the painting $v10$ (shown in Figure 3) are the following:

```
v10 is a Baroque painting. The darkness evidences the Baroque style.
Due to the contrast of dark and pale colours. The lack of pale colours
evidences this style. Although v10 is classified in the Baroque style, there are reasons
to believe that it may belong to the Impressionism. The diversity of qualitative colours
evidences the Impressionism style. The variety of hues evidences the Impressionism style.
The amount of bluish evidences this style.
```

An example of the response produced by ℓ-SHE$^{\mathrm{RPL}}$ regarding painting $rn3$ (Figure 4) is the following:

```
?- impr_style(rn3,DoB).
SHE obtains ...
Diversity of Hues:0.727
Diversity of QCDs:0.514
Bluish Level:0.105
Greyish Level:0.488
DoB = 0.891.
```

Moreover, the explanations provided by ℓ-SHE$^{\mathrm{RPL}}$ for the painting $rn3$ (shown in Figure 4) are the following:

```
rn3 is an Impressionist painting.
The diversity of qualitative colours evidences the Impressionism style.
The variety of hues evidences the Impressionism style.
The amount of bluish evidences this style.
The amount of grey colour evidences this style.
```

Finally, an example of the response produced by ℓ-SHE$^{\mathrm{RPL}}$ regarding painting $vg12$ (Figure 5) is the following:

```
?- postimpr_style_g(vg12,DoB).
Vividness level:0.389
Warm Level:0.658
DoB = 1.
```

Moreover, the explanations provided for $vg12$ by ℓ-SHE$^{\mathrm{RPL}}$ are the following:

```
vg12 is a Post-Impressionist painting.
The presence of vivid colours evidences the Post-Impressionism style.
The high level of warm colours evidences the Post-Impressionism style.
```

7 Testing the ℓ-SHE on the QArt-Dataset

This section presents and discusses the results obtained when classifying the 90 images in the QArt-Dataset using the three art style painting classifiers defined: ℓ-SHE$^{\mathrm{RPL}}$, ℓ-SHE$^{G(\mathbb{Q})}$ and ℓ-SHE$^{\sqcap(\mathbb{Q})}$.

TABLE 3 Confusion matrix for ℓ-SHE$^{\text{RPL}}$ using the QArt-Dataset

	Baroque		Impressionism		Post-Impressionism	
	✓	**?**	✓	**?**	✓	**?**
Baroque	24	3	1	0	2	0
Impressionism	3	0	16	5	2	4
Post-Impressionism	1	1	4	6	15	3

rn14, outlier for ℓ-SHE$^{\text{RPL}}$ gg2, outlier for ℓ-SHE$^{\text{RPL}}$ rn2, outlier for ℓ-SHE$^{G(\mathbb{Q})}$ gg6, outlier for ℓ-SHE$^{G(\mathbb{Q})}$ m11, outlier for ℓ-SHE$^{\sqcap(\mathbb{Q})}$

FIGURE 7. Examples of outliers or paintings misclassified from the QArt-Dataset. All rights under © creative commons, public license. The colour version of this figure is available on the online version of this paper.

The ℓ-SHE$^{\text{RPL}}$ version has been tested using the paintings in the QArt-Dataset. Table 3 shows the confusion matrix obtained for the three art styles. The blue cells correspond to the correct classifications: on the left, correct classifications where ℓ-SHE$^{\text{RPL}}$ is sure (✓); on the right, correct classifications where ℓ-SHE$^{\text{RPL}}$ is not sure and provides an alternative style as a second opinion (**?**). The rest of the cells correspond to the outliers: in each column, the cell on the left indicates the outliers in that ℓ-SHE$^{\text{RPL}}$ does not give a second opinion (✓); and the cell on the right shows the outliers in that the second opinion given by ℓ-SHE$^{\text{RPL}}$ classifies correctly the painting (**?**). The rest of the confusion matrices of this paper use the same notation. In order to clarify, let us indicate that, when the algorithm is doubting, it provides two possible styles as a result. The first option (highest certainty) is the one considered as a correct classification (column **?**). If the second opinion (lowest certainty) is the correct one, it is not counted as a correct classification and it appears in a column corresponding to a different style.

From the analysis of the data in Table 3 regarding ℓ-SHE$^{\text{RPL}}$, we can conclude that

- the highest accuracy is obtained for the Baroque style (90%), and that almost 67% of the outliers (i.e. paintings classified outside its art style) are classified in the Post-Impressionist style. From the data analysis, we have obtained that the membership degree to the Baroque style of 55.6% of these correct classifications is 1.
- The Impressionism style obtains an accuracy of 70%. In 22.2% of the misclassifications obtained (that is, the outliers), ℓ-SHE$^{\text{RPL}}$ warns that there is evidence to belong to the Impressionist style. In addition, almost 67% of the outliers are classified in the Post-Impressionist style. An example of an outlier in the Impressionist style is *Garden Scene in Brittanny* ($rn14$ in the QArt-Dataset, Figure 7) by Renoir, where $PI(rn14) = 0.877$, while $I(rn14) = 0.875$. In fact ℓ-SHE$^{\text{RPL}}$ points out to the diversity of colours, the variety of hues, the high level of bluish and the use of greys as strong reasons to believe that $rn14$ is an Impressionist painting, although ℓ-SHE$^{\text{RPL}}$ categorizes the painting as Post-Impressionist.

diego_velazquez_27, outlier for ℓ-SHE$^{\sqcap\mathbb{Q}}$

jan_vermeer_12, outlier for ℓ-SHE$^{\mathrm{RPL}}$

pierrer_auguste_renoir_10, outlier for ℓ-SHE$^{\mathrm{RPL}}$

claude_monet_8, outlier for ℓ-SHE$^{G(\mathbb{Q})}$

claude_monet_3, outlier for ℓ-SHE$^{\sqcap\mathbb{Q}}$

paul_gauguin_4, outlier for ℓ-SHE$^{\mathrm{RPL}}$

vincent_van_gogh_10, outlier for ℓ-SHE$^{G(\mathbb{Q})}$

vincent_van_gogh_41, outlier for ℓ-SHE$^{\sqcap\mathbb{Q}}$

FIGURE 8. Examples of outliers or paintings misclassified from the Painting-91-BIP dataset. All rights under © creative commons, public license. The colour version of this figure is available on the online version of this paper.

TABLE 4 Confusion matrix for ℓ-SHE$^{G(\mathbb{Q})}$ using the QArt-Dataset

	Baroque		Impressionism		Post-Impressionism	
	✓	?	✓	?	✓	?
Baroque	29	1	0	0	0	0
Impressionism	9	3	10	7	1	0
Post-Impressionism	10	2	6	0	12	0

– The Post-Impressionism style gets 60% of accuracy rate. In 41.7% of the outliers, ℓ-SHE$^{\mathrm{RPL}}$ warns that there is evidence to believe that a painting belongs to the Post-Impressionist style. Again, separating the Impressionism and Post-Impressionism features becomes difficult, since 83.3% of outliers are categorized as Impressionist paintings. For instance, Les Alyscamps (*gg*2 in the QArt-Dataset, Figure 7) by Gauguin is classified as an Impressionist painting with a membership of $I(gg2) = 0.90$, whereas $PI(gg2) = 0.84$. This misclassification is due to ℓ-SHE$^{\mathrm{RPL}}$ recognizes in *gg*2 the totality of the colour features that have been considered as distinctive of the Impressionism style.

Table 4 shows the confusion matrix corresponding to the art style classification obtained by ℓ-SHE$^{G(\mathbb{Q})}$ regarding the 90 paintings in the QArt-Dataset.

From the analysis of the data in Table 4 regarding ℓ-SHE$^{G(\mathbb{Q})}$, we have obtained the following results.

– Again, the highest accuracy is obtained for the Baroque style (100%). Notice that ℓ-SHE$^{G(\mathbb{Q})}$ provided another possible style for 1 painting although with less certainty, this is why it was not classified into the other categories, but it is not a piece which had a clear diagnostic. Observe also that the membership degree to the Baroque style of 63.3% of these correct classifications is 1.

TABLE 5 Confusion matrix for ℓ-SHE$^{\sqcap(\mathbb{Q})}$ using the QArt-Dataset

	Baroque		Impressionism		Post-Impressionism	
	✓	?	✓	?	✓	?
Baroque	28	0	0	1	1	0
Impressionism	5	5	18	0	2	0
Post-Impressionism	3	1	9	1	15	1

– The Impressionism style gets 56.7% of accuracy rate. In addition, in 23.1% of the obtained misclassifications, ℓ-SHE$^{G(\mathbb{Q})}$ warns that there is evidence to belong to the Impressionist style. With respect to outliers, 92.3% are classified as Baroque paintings, whereas the rest, 7.7%, are classified as Post-Impressionist paintings. An example of an outlier in this style is Bal du moulin de la Galette ($rn2$ in the QArt-Dataset, Figure 7) by Renoir, where $B(rn2) = 0.68$ and $I(rn2) = 0.51$.
– The Post-Impressionism style gets an accuracy of 40%, and the membership degree to the Post-Impressionist style of 83.3% of these correct classifications is 1. In 11.1% of the misclassifications ℓ-SHE$^{G(\mathbb{Q})}$ warns that there is evidence to believe that a painting belongs to the Post-Impressionist style. Most of the outliers, 66.7%, are classified in Baroque style. An example of an outlier in this style is Madame Roulin ($gg6$ in the QArt-Dataset, Figure 7) by Gauguin, for which $B(gg6) = 0.40$ and $PI(gg6) = 0.13$.

Table 5 shows the confusion matrix obtained by ℓ-SHE$^{\sqcap(\mathbb{Q})}$ for all the art styles in the QArt-Dataset.

From the analysis of the data in Table 5 regarding ℓ-SHE$^{\sqcap(\mathbb{Q})}$, we have obtained the following results.

– The highest accuracy is obtained for the Baroque style (93.3%), and that the membership degree to the Baroque style of 64.3% of these correct classifications is 1. Moreover, outliers are classified equally in the Impressionist and Post-Impressionist styles.
– The Impressionism style gets 60% of accuracy rate. In addition, in 33.3% of the obtained misclassifications, ℓ-SHE$^{\sqcap(\mathbb{Q})}$ warns that there is evidence to belong to the Impressionist style. Regarding outliers, 86.3% are categorized as Baroque paintings. An example of an outlier in the Impressionist style is The Waterlily Pond, green harmony ($m11$ in the QArt-Dataset, Figure 7) by Monet, where $B(m11) = 1$ and $I(m11) = 0.87$.
– Regarding the Post-Impressionism style, the accuracy obtained is 53.3%. In none of the misclassifications ℓ-SHE$^{\sqcap(\mathbb{Q})}$ warns that there is evidence to believe that a painting belongs to the Post-Impressionist style. Most of the outliers, 71.4%, are classified in Impressionist style. An example of an outlier in this style is The three graces on the temple of Venus ($gg2$ in the QArt-Dataset and shown in Figure 7) by Gauguin, where $I(gg2) = 0.77$ and $PI(gg2) = 0.65$.

Note that the total accuracies obtained with each ℓ-SHE version for the QArt-Dataset (Table 6) do not show a large difference, but ℓ-SHE$^{\text{RPL}}$ is the proposal with highest general accuracy, 73.3%. In addition, although the lowest accuracy for the Baroque style is obtained by ℓ-SHE$^{\text{RPL}}$, let us remark that the highest accuracies for the Impressionist and the Post-Impressionist styles are also obtained by ℓ-SHE$^{\text{RPL}}$. In summary, ℓ-SHE$^{\text{RPL}}$ shows the highest general accuracy and the highest accuracies for two of the three art styles considered. Therefore, we conclude that the ℓ-SHE$^{\text{RPL}}$ is the best proposal for the QArt-Dataset.

TABLE 6 Percentages of accuracy obtained in the QArt-Dataset for each ℓ-SHE version

ℓ-SHE version / Art style	ℓ-SHE^{RPL}	ℓ-SHE$^{G(\mathbb{Q})}$	ℓ-SHE$^{\sqcap(\mathbb{Q})}$
Baroque	90	100.0	93.3
Impressionism	70	56.7	60
Post-Impressionism	60	40	53.3
General accuracy	73.3	65.6	68.9

TABLE 7 Confusion matrix for ℓ-SHE^{RPL} using the Painting-91-BIP dataset

	Baroque		Impressionism		Post-Impressionism	
	✓	?	✓	?	✓	?
Baroque	61	9	4	1	6	2
Impressionism	22	2	33	9	27	6
Post-Impressionism	20	10	22	6	49	10

8 Evaluating ℓ-SHE using a different dataset: Paintings-91-PIB

This section presents the performance of the three versions of ℓ-SHE in a larger dataset, Paintings-91-PIB, which contains 247 paintings 74 for the Baroque style (39 by Velázquez and 35 by Vermeer), 82 for the Impressionism style (46 by Renoir and 36 by Monet) and 91 for the Post-Impressionism style (40 by Van Gogh and 51 by Gauguin). See Section 2 for more details. Let us recall that the Painting-91-BIP dataset is slightly unbalanced by author and also with respect to the number of paintings belonging to each style. For this reason, the general accuracy obtained for each ℓ-SHE version has been calibrated: the general accuracy has been obtained as the median of the accuracies of each art style, and not as the quotient between the total of correct classifications and the total of outliers. In this way, the adequacy of ℓ-SHE using RPL, $G(\mathbb{Q})$ and $\sqcap(\mathbb{Q})$ is evaluated again. Notice that the same dataset, QArt-Dataset, was used both to parametrise and test the ℓ-SHE algorithm. Consequently, it was important to test ℓ-SHE with another dataset.

Let us start the analysis with ℓ-SHE^{RPL}. Table 7 shows the confusion matrix obtained with ℓ-SHE^{RPL} for all the art styles in the Painting-91-BIP dataset.

From the analysis of the data in Table 7 regarding ℓ-SHE^{RPL} in the Painting-91-BIP dataset, we have obtained the following results.

- The Baroque style gets 86.5% of accuracy rate, and the membership degree to this style of 76.6% of these correct classifications is 1. In 30% of the misclassifications ℓ-SHE^{RPL} warns that there is evidence to believe that a painting belongs to the Baroque. With respect to the outliers, 40% are classified as Impressionist paintings and 60% are classified as Post-Impressionist paintings. An example of an outlier in this style is The Music Lesson (*jan_vermeer*_12 in the Painting-91-BIP dataset) by Vermeer, for which $B(jan\_vermeer\_12) = 0.74$, $I(jan\_vermeer\_12) = 0.12$ and $PI(jan\_vermeer\_12) = 0.87$.
- The Impressionism obtains an accuracy of only 40.2%. With respect to the outliers, 44.9% are classified as Baroque paintings, whereas the rest, 55.1%, are classified as Post-Impressionist paintings. An example of an outlier in this style is Woman in a boat (*pierre_auguste_renoir*_10 in the Painting-91-BIP dataset) by Renoir, for which $PI(pierre\_auguste\_renoir\_10) = 0.87$, whereas $I(pierre\_auguste\_renoir\_10) = 0.72$ and $B(pierre\_auguste\_renoir\_10) = 0$.

TABLE 8 Confusion matrix for ℓ-SHE$^{G(\mathbb{Q})}$ using Painting-91-BIP dataset

	Baroque		Impressionism		Post-Impressionism	
	✓	?	✓	?	✓	?
Baroque	74	0	0	0	0	0
Impressionism	60	4	7	8	3	0
Post-Impressionism	54	3	10	0	24	0

TABLE 9 Confusion matrix for ℓ-SHE$^{\sqcap(\mathbb{Q})}$ using the Painting-91-BIP dataset

	Baroque		Impressionism		Post-Impressionism	
	✓	?	✓	?	✓	?
Baroque	69	2	2	1	0	0
Impressionism	42	3	25	3	9	0
Post-Impressionism	29	2	20	3	35	2

– The Post-Impressionism style gets 53.9% of accuracy rate, and the membership degree to this style of 44.9% of these correct classifications is 1. With respect to outliers, 47.6% are classified as Baroque paintings and 52.4% are classified as Impressionist paintings. In addition, in 38.1% of the obtained misclassifications ℓ-SHE$^{\mathrm{RPL}}$ warns that there is evidence to belong to the Post-Impressionist style. An example of an outlier in the Post-Impressionist style is Arearea (*paul_gauguin*_4 in the Painting-91-BIP dataset) by Gauguin, for which $PI(paul\_gauguin\_4) = 0.83$, $B(paul\_gauguin\_4) = 0.44$ and $I(paul\_gauguin\_4) = 0.88$.

Let us consider ℓ-SHE$^{G(\mathbb{Q})}$. Table 8 shows the confusion matrix obtained with ℓ-SHE$^{G(\mathbb{Q})}$ for all the art styles in the Painting-91-BIP dataset.

From the analysis of the data in Table 8 regarding ℓ-SHE$^{G(\mathbb{Q})}$ in the Painting-91-BIP dataset, we have obtained the following results.

– Regarding the Baroque style, observe that this style gets 100% of accuracy rate, and we obtain that 71.7% of the Baroque classifications have Baroque membership degree 1.
– The Impressionism style gets 18.3% of accuracy rate. Most of the outliers, 95.5%, are classified as Baroque paintings, and only 4.5% of the misclassifications ℓ-SHE$^{G(\mathbb{Q})}$ warns that there is evidence to believe that a painting belongs to the Impressionist style. An example of an outlier is Water lilies (*claude_monet*_8 in the Painting-91-BIP dataset) by Monet: $B(claude\_monet\_8) = 0.37$, $I(claude\_monet\_8) = 0.32$ and $PI(claude\_monet\_8) = 0.02$.
– The Post-Impressionism style gets 26.4% of accuracy, and we obtain that 83.3% of the Post-Impressionist classifications have Post-Impressionist membership degree 1. Again, most of the outliers, 85.1%, are classified in the Baroque style. Besides, none of the misclassifications ℓ-SHE$^{G(\mathbb{Q})}$ warns that there is evidence to believe that a painting belongs to the Post-Impressionist style. An example of an outlier in the Post-Impressionist style is Two Cut Sunflowers (*vincent_van_gogh*_10 in the Painting-91-BIP dataset) by van Gogh, for which $B(vincent\_van\_gogh\_10) = 0.41$, $I(vincent\_van\_gogh\_10) = 0.21$ and $PI(vincent\_van\_gogh\_10) = 0.06$.

Let us now analyse ℓ-SHE$^{\sqcap(\mathbb{Q})}$. Table 9 shows the confusion matrix obtained with ℓ-SHE$^{\sqcap(\mathbb{Q})}$ for all the art styles in the Painting-91-BIP dataset.

TABLE 10 Percentages of accuracy obtained in the QArt-Dataset and the Painting-91-BIP datasets for each ℓ-SHE version

Dataset	Art style painting	ℓ-SHE version		
		ℓ-SHE$^{\mathrm{RPL}}$	ℓ-SHE$^{G(\mathbb{Q})}$	ℓ-SHE$^{\sqcap(\mathbb{Q})}$
Painting-91-BIP	Baroque	86.5	100.0	96.0
	Impressionism	40.2	18.3	34.2
	Post-Impressionism	53.9	26.4	40.7
	General accuracy	60.2	48.2	57.0
QArt-Dataset	Baroque	90	100.0	93.3
	Impressionism	70	56.7	60
	Post-Impressionism	60	40	53.3
	General accuracy	73.3	65.6	68.9

From the analysis of the data in Table 9 regarding ℓ-SHE$^{\sqcap(\mathbb{Q})}$ in the Painting-91-BIP dataset, we have obtained the following results.

- The highest accuracy is obtained for the Baroque style (96.0%), and that the membership degree to this style of 66.2% of these correct classifications is 1. In 33.3% of the misclassifications, ℓ-SHE$^{\sqcap(\mathbb{Q})}$ warns that there is evidence to believe that a painting belongs to the Baroque style. An example of an outlier is Tercio (*diego_velazquez*_27 in the Painting-91-BIP dataset) by Velázquez, for which $B(diego\_velazquez\_27) = 0.53$, $I(diego\_velazquez\_27) = 0.61$ and $PI(diego\_velazquez\_27) = 0.00$.
- The Impressionism style gets 34.2% of accuracy rate. Regarding the outliers, 83.3% are classified as Baroque paintings and 16.7% as Post-Impressionist paintings. Only 9.3% of the misclassifications ℓ-SHE$^{\sqcap(\mathbb{Q})}$ warns that the painting might belong to the Impressionism style. An example of an outlier is Water lilies (*claude_monet*_3 in the Painting-91-BIP dataset) by Monet: $B(claude\_monet\_3) = 1$, $I(claude\_monet\_3) = 0.87$ and $PI(claude\_monet\_3) = 0.02$.
- The Post-Impressionism style obtains 40.7% of accuracy, and that the membership degree to this style of 59.5% of these correct classifications is 1. Regarding outliers, 57.4% are classified in the Baroque style and 42.6% in the Impressionist style. In 13.5% of the misclassifications ℓ-SHE$^{\sqcap(\mathbb{Q})}$ warns that there is evidence to believe that a painting belongs to the Post-Impressionist style. An example of an outlier in the Post-Impressionist style is Sorrowing Old Man (*vincent_van_gogh*_41 in the Painting-91-BIP dataset) by van Gogh, where $B(vincent\_van\_gogh\_41) = 0.17$, $I(vincent\_van\_gogh\_41) = 0.87$ and $PI(vincent\_van\_gogh\_41) = 0.00$.

Results obtained by each ℓ-SHE for both datasets are presented in Table 10. Note first that ℓ-SHE$^{G(\mathbb{Q})}$ gets the lowest general accuracy, 48.2%. Besides, the ℓ-SHE$^{\mathrm{RPL}}$ version is again the proposal with highest general accuracy, but ℓ-SHE$^{\sqcap(\mathbb{Q})}$ gets 57.0% of general accuracy. Hence, similar accuracies to other were obtained for both proposals, ℓ-SHE$^{\mathrm{RPL}}$ and ℓ-SHE$^{\sqcap(\mathbb{Q})}$. However, accuracies for the Impressionist and the Post-Impressionist styles are higher for ℓ-SHE$^{\mathrm{RPL}}$, and the ℓ-SHE$^{\sqcap(\mathbb{Q})}$ version shows an accuracy for the Impressionism style close to a random classifier. Therefore, from the data analysis it might be concluded that ℓ-SHE$^{\mathrm{RPL}}$ is the most accurate classifier.

9 Conclusions and future work

The art style classification algorithm ℓ-SHE has been presented and analysed considering the three different versions defined, which are determined by the three logics RPL, $G(\mathbb{Q})$ and $\sqcap(\mathbb{Q})$. The accuracy acquired in the QArt-Dataset with the three logics (Table 10) is similar to other works that use QCDs [12]. Regarding Painting-91-BIP dataset, the results obtained (Table 10) are similar, but a bit the two classifiers built in [14]. However, contrary to those classifiers based on machine learning methods, the ℓ-SHE classification provides explanations of right classifications, and also of some of the outliers by giving a second option. Hence, each classification method has both advantages and disadvantages. In this way, comparing in detail different approaches for art style classification is future work. On the other hand, all the ℓ-SHE versions show a low accuracy for Impressionist style in the Painting-91-BIP dataset. This flaw in the classification might be explained in terms of art genres: individual portraits are scant in Renoir's paintings from the QArt-dataset, whereas this is the main type of painting in Renoir's paintings from Painting-91-BIP dataset. This is an important aspect to consider for future work.

In both datasets the ℓ-SHE$^{\mathrm{RPL}}$ version gets the highest general accuracy among the ℓ-SHE approaches. Indeed, the general accuracy for the QArt-Dataset obtained by the ℓ-SHE$^{\mathrm{RPL}}$ version is 73.3%, whereas the general accuracies for the QArt-Dataset obtained by the ℓ-SHE$^{G(\mathbb{Q})}$ and the ℓ-SHE$^{\sqcap(\mathbb{Q})}$ are 65.6% and 68.9%, respectively (see Table 10). In addition, the general accuracy for the Painting-91-BIP obtained by the ℓ-SHE$^{\mathrm{RPL}}$ version is 60.2%, and the general accuracies for the Painting-91-BIP dataset obtained by the ℓ-SHE$^{G(\mathbb{Q})}$ and the ℓ-SHE$^{\sqcap(\mathbb{Q})}$ are 48.2% and 57%, respectively (see Table 10). Thus, the ℓ-SHE$^{\mathrm{RPL}}$ seems to be the best approach on both datasets.

Other future work includes introducing other logical formalisms and other aggregation methods to represent the different art styles and the use of reasoning mechanisms to draw conclusions about the relationship between new and classical styles. For this purpose it would be important to add new art styles to the dataset. Moreover, we expect to show the ℓ-SHE outcomes to art experts in order to get feedback from them and use it to improve the ℓ-SHE algorithm. Moreover, adding art-genre information, studying the complexity of ℓ-SHE algorithm and comparing it to machine learning methods are future work. Also, it would be relevant to explore a SMT-based (Statistical Machine Translation) approach in future extensions of the ℓ-SHE algorithm. Finally, we will study the possibility of enriching our algorithm with abduction procedures to improve the accuracy.

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Chapter 3

Discussion and conclusions

In this doctoral thesis, we presented original contributions in the areas of mathematical fuzzy logic and artificial intelligence. First, in Part I, we developed a systematic study on free models of fuzzy universal Horn theories. Then, in Part II, we used qualitative descriptors and fuzzy knowledge representation for designing an art painting style classification algorithm. In the present chapter, we discuss the results presented in Chapter 2 and propose further research directions.

On minimality Free structures are minimal from an algebraic point of view. We can, however, consider the minimality from a model-theoretic point of view (in the sense of Lemma 3(ii) of Costa & Dellunde (2017*a*), intuitively speaking, term structures pick up the positive atomic information that follows from their associated theory). It is, thus, natural to wonder whether these are equivalent terms. Next, we discuss this question and prove that they are equivalent for some class of fully named models.

The name of fully named model appears first in predicate fuzzy logics in the work (Cintula et al. 2011*a*, Ch.2), but these models were used before by Cintula & Hájek (2006) for the completeness proof of predicate core fuzzy logics (see Theorem 6 of Cintula & Hájek (2006)). In these models, every element of the domain is an interpretation of a ground term of the language (i.e., a term without variables). A prominent type of fully named models is the class of Herbrand structures, studied in this doctoral thesis. For a reference to the notion of fully named model in classical logic, see Definition 2.1(iii) of Makowsky (1987). We assume that the predicate languages have at least one individual constant symbol. Let us remark that the results presented next hold in MTL$\forall_{\approx}$.

Definition 12 (Fully Named Model) *A structure* $\langle \boldsymbol{B}, \mathrm{N} \rangle$ *is a* fully named model *if for any element n of the domain N, there exists a ground term t such that* $||t||^{\boldsymbol{B}}_{\mathrm{N}} = n$.

Observe that the substructure of $\langle \boldsymbol{B}, \mathrm{T}^{\Phi} \rangle$ (Costa & Dellunde 2017*a*, Definition 10) generated by the equivalence classes of the ground terms is also a fully named model. Given a class **K** of structures over the same predicate language, on the one hand one structure can be minimal in the sense of *free in* **K**. On the other hand, a structure can be minimal in a model-theoretic point of view (*at-generic in* **K**), which means that the structure is a model exactly of those atomic sentences that all the structures of **K** are models of. In the work Makowsky (1987) the at-genericity is defined as a special case of the notion of *genericity*, notion that was first introduced in specification of data structures, relational data bases and logic programming. Now we extend to the fuzzy context the definition of at-genericity of Makowsky (see Definition 2.4 of Makowsky (1987)).

Definition 13 *Let* $\mathbf{K}$ *be a class of structures. Given* $\langle \boldsymbol{B},\mathbf{N}\rangle \in \mathbf{K}$*, we say that* $\langle \boldsymbol{B},\mathbf{N}\rangle$ *is* at-generic in $\mathbf{K}$ *if for every atomic sentence* φ*, we have that*

$$||\varphi||^{\boldsymbol{B}}_{\mathbf{N}} = 1 \text{ if and only if for every structure } \langle \boldsymbol{A},\mathrm{M}\rangle \in \mathbf{K}, ||\varphi||^{\boldsymbol{A}}_{\mathbf{M}} = 1.$$

Next, we present the proof that the two notions of minimality (the algebraic and the model-theoretic) coincide for reduced fully named models $\langle \boldsymbol{B},\mathbf{N}\rangle$ where $\mathbf{B}$ is the two-valued Boolean algebra.

Theorem 3 *Let* $\mathbf{K}$ *be a class of fuzzy reduced structures and* $\langle \boldsymbol{B},\mathbf{N}\rangle \in \mathbf{K}$ *be a fully named model where* $\boldsymbol{B}$ *is the two-valued boolean algebra. Then,*

$$\langle \boldsymbol{B},\mathbf{N}\rangle \text{ is free in } \mathbf{K} \text{ if and only if } \langle \boldsymbol{B},\mathbf{N}\rangle \text{ is at-generic in } \mathbf{K}.$$

Proof: ($\Rightarrow$) Assume that $\langle \boldsymbol{B},\mathbf{N}\rangle$ is free in $\mathbf{K}$, and let φ be an atomic sentence such that $||\varphi||^{\boldsymbol{B}}_{\mathbf{N}} = 1$ and $\langle \boldsymbol{A},\mathbf{M}\rangle \in \mathbf{K}$. Since $\langle \boldsymbol{B},\mathbf{N}\rangle$ is free in $\mathbf{K}$, there is a unique homomorphism from $\langle \boldsymbol{B},\mathbf{N}\rangle$ to $\langle \boldsymbol{A},\mathbf{M}\rangle$, say (f,g). By Theorem 16 of Dellunde et al. (2016) we have that (f,g) preserves atomic formulas, then $||\varphi||^{\boldsymbol{A}}_{\mathbf{M}} = 1$ and we can conclude that $\langle \boldsymbol{B},\mathbf{N}\rangle$ is at-generic in $\mathbf{K}$.

($\Leftarrow$) Assume that $\langle \boldsymbol{B},\mathbf{N}\rangle$ is at-generic in $\mathbf{K}$ and let $\langle \boldsymbol{A},\mathbf{M}\rangle \in \mathbf{K}$. We define $g : M \to N$ by: $g(t_{\mathbf{N}}) = t_{\mathbf{M}}$ for any ground term t. Let us see that g is well-defined: if t,t' are ground terms and $t_{\mathbf{N}} = t'_{\mathbf{N}}$, then $||t \approx t'||^{\boldsymbol{B}}_{\mathbf{N}} = 1$. Since $\langle \boldsymbol{B},\mathbf{N}\rangle$ is at-generic, $||t \approx t'||^{\boldsymbol{A}}_{\mathbf{M}} = 1$, and since $\langle \boldsymbol{A},\mathbf{M}\rangle$ is reduced we have $t_{\mathbf{M}} = t'_{\mathbf{M}}$.

Now let F be an n-ary function symbol and $t_1,\dots,t_n$ ground terms. Then, since $\langle \boldsymbol{B},\mathbf{N}\rangle$ is a fully named model, we have that:

$$g(F_{\mathbf{N}}((t_1)_{\mathbf{N}},\dots,(t_n)_{\mathbf{N}})) = g([F(t_1,\dots,t_n)]_{\mathbf{N}}) =$$

$$= [F(t_1,\dots,t_n)]_{\mathbf{M}} = F_{\mathbf{M}}((t_1)_{\mathbf{M}},\dots,(t_n)_{\mathbf{M}}) =$$

$$= F_{\mathbf{M}}(g((t_1)_{\mathbf{N}}),\dots,g((t_n)_{\mathbf{N}})).$$

Let P be an n-ary predicate symbol and $t_1,\dots,t_n$ ground terms. Suppose that $P_{\mathbf{N}}((t_1)_{\mathbf{N}},\dots,(t_n)_{\mathbf{N}}) = 1$, that is, $||P(t_1,\dots,t_n)||^{\boldsymbol{B}}_{\mathbf{N}} = 1$. Then, since $\langle \boldsymbol{A},\mathbf{M}\rangle \in \mathbf{K}$ and $\langle \boldsymbol{B},\mathbf{N}\rangle$ is at-generic in $\mathbf{K}$, we conclude that $||P(t_1,\dots,t_n)||^{\boldsymbol{A}}_{\mathbf{M}} = 1$, i.e., $P_{\mathbf{M}}((t_1)_{\mathbf{M}},\dots,(t_n)_{\mathbf{M}}) = 1$, which is equivalent by the definition of g to $P_{\mathbf{M}}(g((t_1)_{\mathbf{N}}),\dots,g((t_n)_{\mathbf{N}})) = 1$.

Moreover, given another homomorphism (f,g') from $\langle \boldsymbol{B},\mathbf{N}\rangle$ to $\langle \boldsymbol{A},\mathbf{M}\rangle$, we will have that $g'(t_{\mathbf{N}}) = t_{\mathbf{M}}$, thus $g = g'$. This fact give us the unicity of (f,g). Consequently, $\langle \boldsymbol{B},\mathbf{N}\rangle$ is free in $\mathbf{K}$. □

Notice that if $\langle \boldsymbol{B},\mathbf{N}\rangle$ is free in $\mathbf{K}$, since $\langle \boldsymbol{B},\mathbf{N}\rangle \in\mathbf{K}$, then $\langle \boldsymbol{B},\mathbf{N}\rangle$ is rigid, that is, there are no non-trivial automorphisms of $\langle \boldsymbol{B},\mathbf{N}\rangle$. Observe that in languages in which the similarity symbol is interpreted by the crisp identity the result holds for arbitrary classes $\mathbf{K}$, and not only for classes of reduced structures. Finally, we show that a class of structures $\mathbf{K}$ has up to isomorphism, at most one fully named model which is free in $\mathbf{K}$.

Corollary 1 *Let* $\mathbf{K}$ *be a class of fuzzy reduced structures, then there is, up to isomorphism, at most one fully named model* $\langle \boldsymbol{B}, \mathbf{N} \rangle \in \mathbf{K}$*, which is free in* $\mathbf{K}$ *(where* $\boldsymbol{B}$ *is the two-valued Boolean algebra).*

Proof: Assume that $\langle \boldsymbol{B}, \mathbf{N} \rangle$ and $\langle \boldsymbol{B}, \mathbf{M} \rangle$ are fully named models free in $\mathbf{K}$. We have to prove that $\langle \boldsymbol{B}, \mathbf{N} \rangle \cong \langle \boldsymbol{A}, \mathbf{M} \rangle$. Let (f, g) be the homomorphism in the proof of Theorem 3. We show that (f, g) is an isomorphism. First notice that f is the identity on $\boldsymbol{B}$; and since both are fully named models, g is clearly surjective. Now we prove the injectivity. Let t, t' be ground terms such that $t_{\mathbf{N}} \neq t'_{\mathbf{N}}$. To find a contradiction, suppose that $g(t_{\mathbf{N}}) = g(t'_{\mathbf{N}})$. That is, $t_{\mathbf{M}} = t'_{\mathbf{M}}$. Since $\langle \boldsymbol{A}, \mathbf{M} \rangle$ is also a free model, there is another homomorphism (f, g') from $\langle \boldsymbol{A}, \mathbf{M} \rangle$ to $\langle \boldsymbol{B}, \mathbf{N} \rangle$, defined as in the proof of Theorem 3. Consequently, $g'(t_{\mathbf{M}}) = g'(t'_{\mathbf{M}})$, i.e., $t_{\mathbf{N}} = t'_{\mathbf{N}}$, which is a contradiction. Therefore, $g(t_{\mathbf{N}}) \neq g(t'_{\mathbf{N}})$. □

Characterization of free models of equality-free fuzzy universal Horn theory Here we discuss the possibility of characterizing the free model of a class of structures as the intersection of a class of Herbrand structures.

Definition 14 *Let* I *be a nonempty set and for every* $i \in I$*,* $H_i \subset \overline{H}$*. We call* $\langle \boldsymbol{B}, \boldsymbol{N}^{\mathrm{H}} \rangle$ *the* intersection *of the family of* H*-structures* $\{\langle \boldsymbol{B}, \boldsymbol{N}^{\mathrm{H}_i} \rangle \mid i \in I\}$*, where* $\mathrm{H} = \bigcap_{i \in I} \mathrm{H}_i$*.*

Using Definition 13 of Costa & Dellunde (2017*a*), it is easy to check that any intersection of a family of H-structures is also an H-structure.

Proposition 2 *Let* φ *be an equality-free consistent sentence which is a Horn clause. If* $\{\langle \boldsymbol{B}, \boldsymbol{N}^{\mathrm{H}_i} \rangle \mid i \in I\}$ *is the family of all* H*-models of* φ *and* $\mathrm{H} = \bigcap_{i \in I} \mathrm{H}_i$*, then* $\langle \boldsymbol{B}, \boldsymbol{N}^{\mathrm{H}} \rangle$ *is also an* H*-model of* φ*.*

Proof: Assume that φ is an equality-free consistent sentence which is a Horn clause and $\{\langle \boldsymbol{B}, N^{\mathrm{H}_i} \rangle \mid i \in I\}$ is the family of all H-models of φ. By Corollary 2 of Costa & Dellunde (2017*a*), the family is not empty. We proceed by induction on the quantifier rank of φ.

Case $qr(\varphi) = 0$. We distinguish two cases:

1) In case that φ is atomic is clear by definition of the intersection. And, in general, if $\varphi = \psi_1 \& \cdots \& \psi_n \to \psi$ is a basic Horn formula, we have to show that $||\psi_1 \& \cdots \& \psi_n||^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}} \leq ||\psi||^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}}$. If $||\psi||^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}} = 1$, we are done. Otherwise, by by Definition 13 of Costa & Dellunde (2017*a*), $\psi \notin \mathrm{H}$ and thus there is some $i \in I$ such that $\langle \boldsymbol{B}, \mathbf{N}^{\mathrm{H}_i} \rangle$ is an H-model of φ and $||\psi||^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}_i}} = 0$. Hence, as $||\varphi||^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}_i}} = 1$, $||\psi_1 \& \cdots \& \psi_n||^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}_i}} = 0$. So for some $j \in \{1, \ldots, n\}$, $||\psi_j||^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}_i}} = 0$ and then $||\psi_j||^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}} = 0$, which implies that $||\psi_1 \& \cdots \& \psi_n||^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}} = 0$. Therefore $||\psi_1 \& \cdots \& \psi_n||^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}} \leq ||\psi||^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}}$.

2) If $\varphi = \psi_1 \& \cdots \& \psi_n$ is a strong conjunction of basic Horn formulas, by 1), for every $j \in \{1, \ldots, n\}$ the following holds: if for every $i \in I, ||\psi_j||^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}_i}} = 1$, then $||\psi_j||^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}} = 1$. Thus, $||\varphi||^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}} = 1$.

Case $qr(\varphi) = n+1$.

Let $\varphi = (\forall x)\phi(x)$ be a Horn clause with $qr(\phi(x)) = n$. For every $i \in I$, $||\varphi||^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}_i}} = 1$. By Axiom $\forall$1, for every ground term t and every $i \in I$, $||\phi(t/x)||^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}_i}} = 1$. Since $\phi(t/x)$ is also a Horn clause, and $qr(\phi(t/x)) = n$, we can apply the inductive hypothesis and then for every ground term t, $||\phi(t/x)||^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}} = 1$. Consequently, $||(\forall x)\phi(x)||^{\boldsymbol{B}}_{\mathbf{N}^{\mathrm{H}}} = 1$ because $\langle \boldsymbol{B}, N^{\mathrm{H}} \rangle$ is a Herbrand structure. □

Theorem 4 (Characterization Theorem) *Let* **K** *be the class of all models of a consistent set of equality-free sentences which are Horn clauses. The intersection of the family of all* H*-structures in* **K** *is the free model in* **K**.

Proof: By Proposition 2 of this chapter, the intersection of the family of all H-structures in **K** is also a member of **K**. Then by Proposition 1 of Costa & Dellunde (2017*a*), the intersection is an at-generic structure in **K**. Consequently, by Theorem 3 of this chapter, the intersection is the free model in **K**. □

Some counterexamples Another issue to be discussed is the improvement of some of the results presented in the papers. Next, we discuss a possible generalization of some of the article's results included in this PhD thesis. Specifically, we focus on two main theorems and present some counterexamples.

In Theorem 2 of Costa & Dellunde (2017*b*), we proved that for any consistent theory Φ and any RPL$\forall_\approx$-Horn clause φ,

$$\text{If } |\varphi|_\Phi = 1, \text{ then } ||\varphi||^{[0,1]_{\mathrm{RPL}}}_{\mathbf{T}^{\Phi}_{\mathrm{RPL}}, e^{\Phi}_{\mathrm{RPL}}} = 1.$$

Now, using a counterexample, we prove that the inverse of the theorem is not true. Indeed, consider a predicate language $\mathscr{P}$ with just a monadic predicate symbol P, and two individual constants a, c, and let Φ be the empty theory. Observe that, since Φ is equivalent to the set of all logically valid sentences, Φ is consistent. Moreover, it is easy to see that: (i) every P-structure $\langle [0,1]_{\mathrm{RPL}}, \mathbf{M} \rangle$ is a model of Φ; (ii) $||P(t)||_\Phi = 0$ for any term t; (iii) the term structure associated to the empty theory Φ is well-defined. Let $\varphi = (P(a), 1) \to (P(c), 1)$. Note that $||(P(a), 1)||^{[0,1]_{\mathrm{RPL}}}_{\mathbf{T}^{\Phi}_{\mathrm{RPL}\forall_\approx}, e^{\Phi}_{\mathrm{RPL}\forall_\approx}} = ||(P(c), 1)||^{[0,1]_{\mathrm{RPL}}}_{\mathbf{T}^{\Phi}_{\mathrm{RPL}\forall_\approx}, e^{\Phi}_{\mathrm{RPL}\forall_\approx}} = 0$, thereby $||\varphi||^{[0,1]_{\mathrm{RPL}}}_{\mathbf{T}^{\Phi}_{\mathrm{RPL}\forall_\approx}, e^{\Phi}_{\mathrm{RPL}\forall_\approx}} = 1$. However, $|\varphi|_\Phi \neq 1$. Indeed, consider the $\mathscr{P}$-structure $\langle [0,1]_{\mathrm{RPL}}, \mathbf{M} \rangle$ so that $P_{\mathbf{M}}(a) = 0.1$ and $P_{\mathbf{M}}(c) = 0$. Note that $||\varphi||^{[0,1]_{\mathrm{RPL}}}_{\mathbf{M}} = 0.9$, and then $||\varphi||_\Phi \neq 1$. By theorem 2, $|\varphi|_\Phi \neq 1$.

In Lemma 5 of Costa & Dellunde (2017*b*) we proved that for any consistent theory Φ and any evaluated atomic formulas $(\varphi_1, s_1), \ldots, (\varphi_n, s_n)$,

$$||(\varphi_1, s_1) \& \cdots \& (\varphi_n, s_n)||^{[0,1]_{\mathrm{RPL}}}_{\mathbf{T}^{\Phi}_{\mathrm{RPL}_\approx}, e^{\Phi}_{\mathrm{RPL}\forall_\approx}} \leq ||(\varphi_1, s_1) \& \cdots \& (\varphi_n, s_n)||_\Phi.$$

Using a counterexample, we show that the symbol $\leq$ of the lemma cannot be substituted by an identity. Indeed, consider a predicate language $\mathscr{P}$ with two monadic predicate symbols P, R, and one individual constant symbol c, and let Φ be the theory

$$\{\overline{0.1} \rightarrow (\forall x)((P(x),1)\&(R(x),1))\}.$$

We show that

$$||(P(c),1)\&(R(c),1)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^{\Phi}_{\text{RPL}\forall\approx}, e^{\Phi}_{\text{RPL}\forall\approx}} < ||(P(c),1)\&(R(c),1)||_{\Phi}.$$

On the one hand, for any model $\langle [0,1]_{\text{RPL}}, \mathbf{M}\rangle$ of Φ, $||(P(c),1)\&(R(c),1)||^{[0,1]_{\text{RPL}}}_{\mathbf{M}} \geq 0.1$. Then, $||(P(c),1)\&(R(c),1)||_{\Phi} \geq 0.1$.

On the other hand, we show that $||(P(c),1)\&(R(c),1)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^{\Phi}_{\text{RPL}\forall\approx}, e^{\Phi}_{\text{RPL}\forall\approx}} = 0$. Indeed, consider two $\mathscr{P}$-structures $\langle [0,1]_{\text{RPL}}, \mathbf{M}_1\rangle$, $\langle [0,1]_{\text{RPL}}, \mathbf{M}_2\rangle$ with $M_1 = M_2 = \{c\}$ and such that:

$$||P(c)||^{[0,1]_{\text{RPL}}}_{\mathbf{M}_1} = 0, \text{ and } ||R(c)||^{[0,1]_{\text{RPL}}}_{\mathbf{M}_1} = 0.6;$$

$$||P(c)||^{[0,1]_{\text{RPL}}}_{\mathbf{M}_2} = 0.6, \text{ and } ||R(c)||^{[0,1]_{\text{RPL}}}_{\mathbf{M}_2} = 0.$$

Notice that $\langle [0,1]_{\text{RPL}}, \mathbf{M}_1\rangle$ and $\langle [0,1]_{\text{RPL}}, \mathbf{M}_2\rangle$ are models of Φ. Then,

$$||P(c)||_{\Phi} = ||R(c)||_{\Phi} = 0.$$

By Theorem 2, $|P(c)|_{\Phi} = |R(c)|_{\Phi} = 0$, and then, by Lemma 5 of Costa & Dellunde (2017*b*),

$$||P(c)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^{\Phi}_{\text{RPL}\forall\approx}, e^{\Phi}_{\text{RPL}\forall\approx}} = ||R(c)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^{\Phi}_{\text{RPL}\forall\approx}, e^{\Phi}_{\text{RPL}\forall\approx}} = 0.$$

Moreover, observe that

$$\begin{gathered} ||(P(c),1)\&(R(c),1)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^{\Phi}_{\text{RPL}\forall\approx}, e^{\Phi}_{\text{RPL}\forall\approx}} = \\ max\{0, ||(P(c),1)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^{\Phi}_{\text{RPL}\forall\approx}, e^{\Phi}_{\text{RPL}\forall\approx}} + ||(R(c),1)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^{\Phi}_{\text{RPL}\forall\approx}, e^{\Phi}_{\text{RPL}\forall\approx}} - 1\} = \\ max\{0, ||P(c))||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^{\Phi}_{\text{RPL}\forall\approx}, e^{\Phi}_{\text{RPL}\forall\approx}} + ||R(c)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^{\Phi}_{\text{RPL}\forall\approx}, e^{\Phi}_{\text{RPL}\forall\approx}} - 1\} = 0. \end{gathered}$$

Therefore, we have proven the following inequality:

$$||(P(c),1)\&(R(c),1)||^{[0,1]_{\text{RPL}}}_{\mathbf{T}^{\Phi}_{\text{RPL}\forall\approx}, e^{\Phi}_{\text{RPL}\forall\approx}} < ||(P(c),1)\&(R(c),1)||_{\Phi},$$

and we have finished.

The performance of the ℓ-SHE algorithm The results obtained when classifying all the images in the datasets considered are presented and discussed in Costa et al. (2021). Details on the results are indicated in the Appendices B and C. In Appendix B, we show thoroughly the results obtained for the three versions of the ℓ-SHE algorithm when classifying the 90 images in the QArt-Dataset. In Appendix C, we show the results obtained for the three versions of the ℓ-SHE algorithm in detail when classifying the 247 images in the Painting-91-BIP dataset. Furthermore, all the paintings in the QArt-Dataset and the Painting-91-BIP datasets are shown in Appendix A.

In conclusion, the ℓ-SHE algorithm, using fuzzy Horn clauses as knowledge representation, obtains percentages of accuracy in the experimentation that are competitive. Indeed, the performance of the ℓ-SHE is similar to other classification algorithms like those presented by Falomir, Cabedo, Sanz & Abril (2015), Falomir et al. (2018). Contrary to these methods based on machine learning, the ℓ-SHE algorithm does not require hard computation, and its outcomes provide explanations of the correct classifications and some of the outliers by giving a second option. However, the ℓ-SHE algorithm does not automatically obtain the required thresholds like machine learning methods do. We conclude that both methods are complementary. The ideal case would be that the evaluated Horn clauses' thresholds and the explanations in the algorithm could be automatically learned from the dataset using machine learning techniques. This ideal method could learn semantic traits of an art style that would be later explainable to the users. In the future, we intend to tackle this challenge.

Since the explainability is an essential issue in the ongoing artificial intelligence, a promising future line could be introducing other logical formalisms and other aggregation methods to represent the different art styles, and the use of reasoning mechanisms to draw conclusions about the relationship between new and classical styles. According to the theory of Dujmović, logic aggregators can be used in mathematical models of human evaluation reasoning, since in general, they yield a closer approximation to related human evaluation processes than t-norm based logics do. For a presentation of the Dujmović's theory, we refer the reader to, for instance, Dujmovic (2018*b*,*a*, 2017, 2013). The first step in research into solving this problem is already in progress. In this way, we have recently published the work Costa (2020), which has received the Best STAIRS 2020 Paper Award. Furthermore, to further our research it would be important also to add new art styles to the datasets, as well as to show the ℓ-SHE outcomes to art experts to get feedback from them.

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Appendix A

Datasets: the QArt-Dataset and the Painting-91-BIP

A.1 The QArt-Dataset

All the paintings in the QArt-Dataset corresponding to the Baroque style (Figure A.1), the Impressionism style (Figure A.2), and the Post-Impressionism style (Figure A.3) are shown in this section.

FIGURE A.1: Baroque paintings by Velázquez and Vermeer in the QArt-Dataset. All rights by Wikimedia commons, public domain.

A.2 The Painting-91-BIP

All the paintings from the Painting-91-BIP dataset corresponding to the Baroque style are indicated in Figures A.4, A.5 and A.6. All the paintings from the Painting-91-BIP dataset corresponding to the Impressionism style are shown in Figures A.7, A.8, and

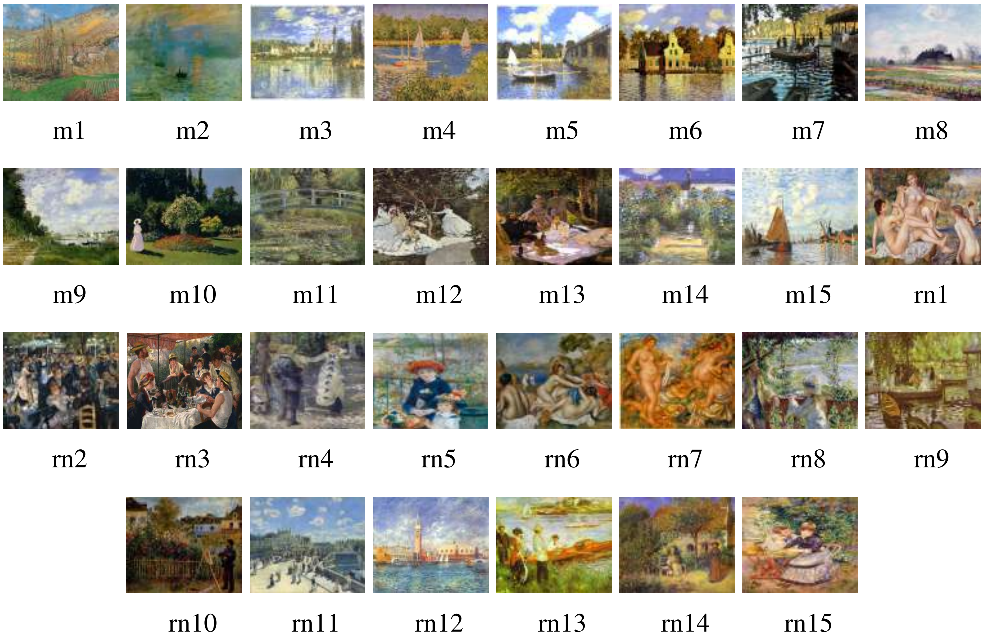

FIGURE A.2: Impressionist paintings by Monet and Renoir in the QArt-Datset. All rights by Wikimedia commons, public domain.

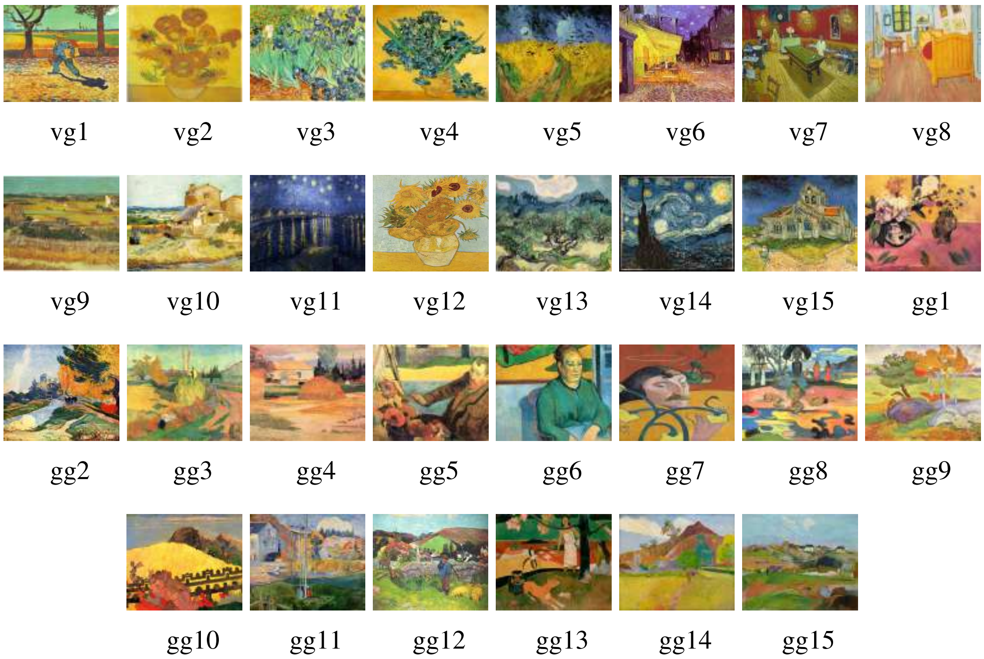

FIGURE A.3: Post-Impressionist paintings by van Gogh and Gauguin in the QArt-Dataset. All rights by Wikimedia commons, public domain.

A.9. Finally, all the paintings from the Painting-91-BIP dataset corresponding to the Post-Impressionism style are indicated in Figures A.10, A.11 and A.12.


diego_velazquez_1


diego_velazquez_2


diego_velazquez_3


diego_velazquez_4


diego_velazquez_5


diego_velazquez_6


diego_velazquez_7


diego_velazquez_8


diego_velazquez_9


diego_velazquez_10


diego_velazquez_11


diego_velazquez_12


diego_velazquez_13


diego_velazquez_14


diego_velazquez_15


diego_velazquez_16


diego_velazquez_17


diego_velazquez_18


diego_velazquez_19


diego_velazquez_20


diego_velazquez_21


diego_velazquez_22


diego_velazquez_23


diego_velazquez_24


diego_velazquez_25


diego_velazquez_26


diego_velazquez_27


diego_velazquez_28


diego_velazquez_29


diego_velazquez_30


diego_velazquez_31

FIGURE A.4: Some Baroque paintings by Velázquez in the Painting-91-BIP dataset. All rights by Wikimedia commons, public domain.

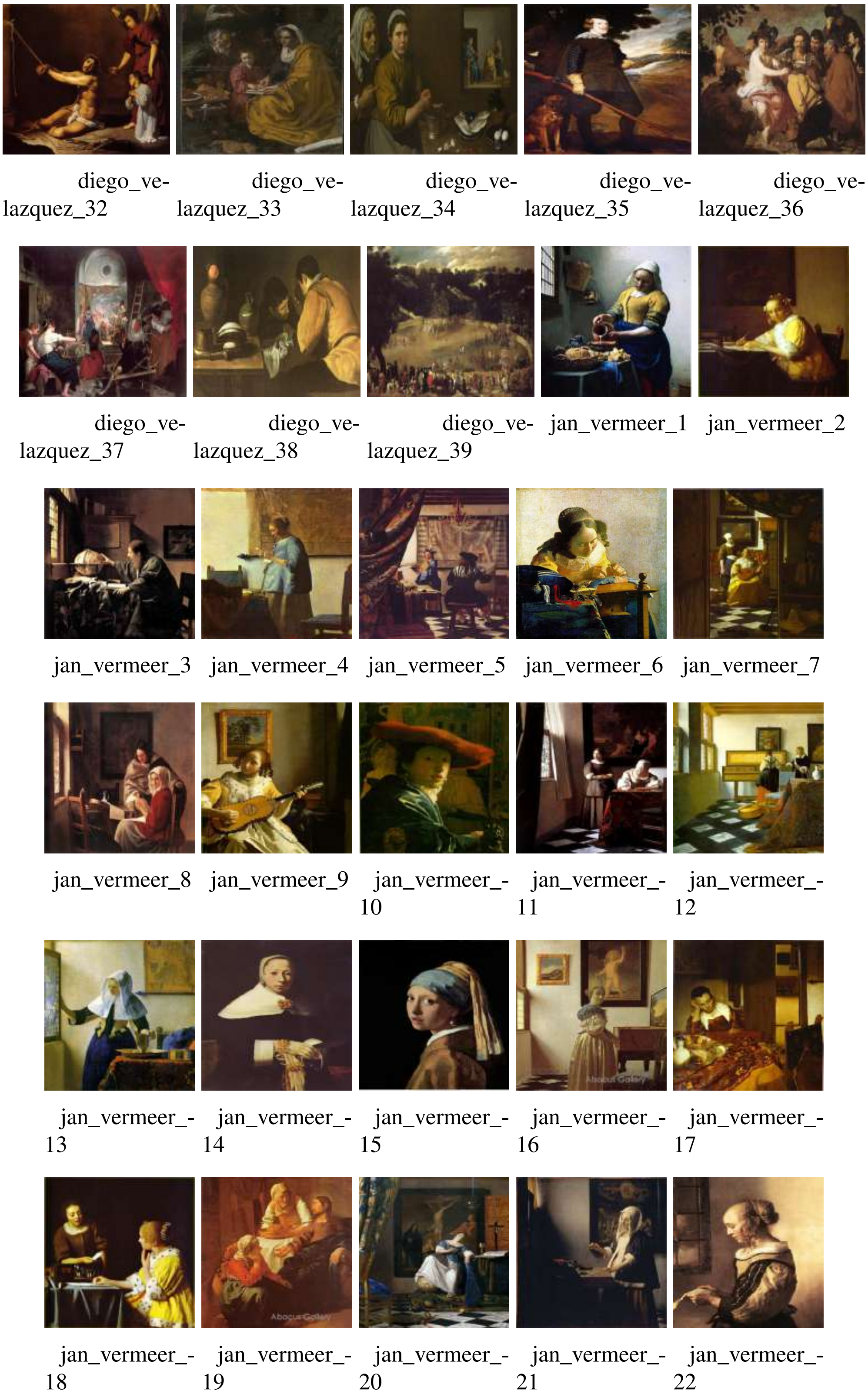

FIGURE A.5: Baroque paintings by Velázquez and Vermeer in the Painting-91-BIP dataset. All rights by Wikimedia commons, public domain.

FIGURE A.6: Baroque paintings by Vermeer in the Painting-91-BIP dataset. All rights by Wikimedia commons, public domain.

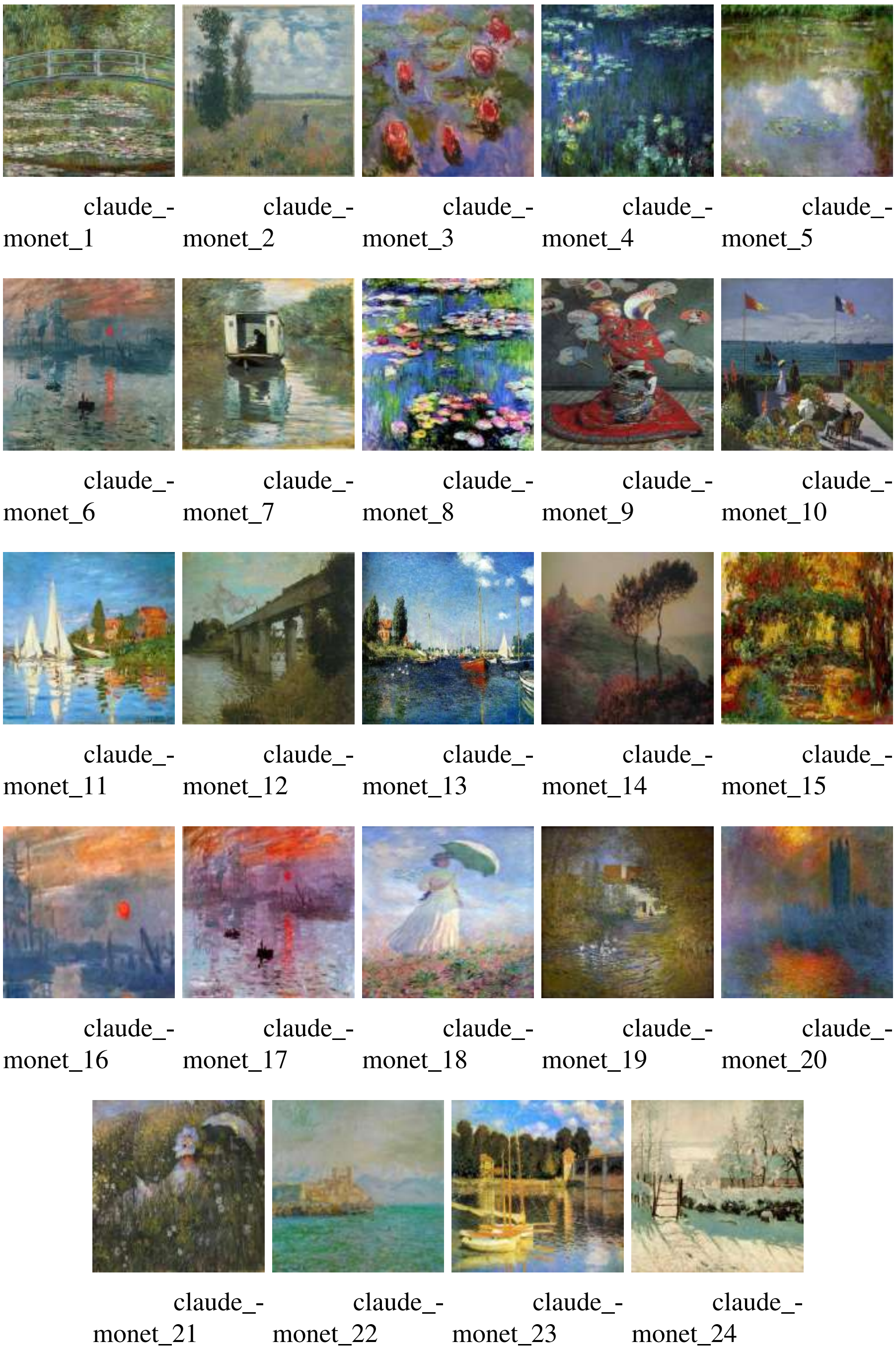

FIGURE A.7: Some Impressionist paintings by Monet in the Painting-91-BIP dataset. All rights by Wikimedia commons, public domain.

FIGURE A.8: Impressionist paintings by Monet and Renoir in the Painting-91-BIP dataset. All rights by Wikimedia commons, public domain.


pierre-auguste_-renoir_19


pierre-auguste_-renoir_21


pierre-auguste_-renoir_22


pierre-auguste_-renoir_24


pierre-auguste_-renoir_25


pierre-auguste_-renoir_27


pierre-auguste_-renoir_28


pierre-auguste_-renoir_29


pierre-auguste_-renoir_30


pierre-auguste_-renoir_31


pierre-auguste_-renoir_32


pierre-auguste_-renoir_33


pierre-auguste_-renoir_34


pierre-auguste_-renoir_35


pierre-auguste_-renoir_36


pierre-auguste_-renoir_37


pierre-auguste_-renoir_39


pierre-auguste_-renoir_40


pierre-auguste_-renoir_41


pierre-auguste_-renoir_42


pierre-auguste_-renoir_43


pierre-auguste_-renoir_44


pierre-auguste_-renoir_45


pierre-auguste_-renoir_46


pierre-auguste_-renoir_47


pierre-auguste_-renoir_48


pierre-auguste_-renoir_49


pierre-auguste_-renoir_50

FIGURE A.9: Impressionist paintings by Renoir in the Painting-91-BIP dataset. All rights by Wikimedia commons, public domain.

vincent_van_gogh_1 vincent_van_gogh_2 vincent_van_gogh_3 vincent_van_gogh_4 vincent_van_gogh_5 vincent_van_gogh_6

vincent_van_gogh_7 vincent_van_gogh_8 vincent_van_gogh_9 vincent_van_gogh_10 vincent_van_gogh_11

vincent_van_gogh_12 vincent_van_gogh_13 vincent_van_gogh_14 vincent_van_gogh_15 vincent_van_gogh_16

vincent_van_gogh_17 vincent_van_gogh_18 vincent_van_gogh_19 vincent_van_gogh_20 vincent_van_gogh_21

vincent_van_gogh_22 vincent_van_gogh_23 vincent_van_gogh_24 vincent_van_gogh_25 vincent_van_gogh_26

vincent_van_gogh_27 vincent_van_gogh_28 vincent_van_gogh_29 vincent_van_gogh_30 vincent_van_gogh_31

FIGURE A.10: Some Post-Impressionist paintings by van Gogh in the Painting-91-BIP dataset. All rights by Wikimedia commons, public domain.

FIGURE A.11: Post-Impressionist paintings by van Gogh and Gauguin in the Painting-91-BIP dataset. All rights by Wikimedia commons, public domain.

paul_gauguin_21 paul_gauguin_22 paul_gauguin_23 paul_gauguin_24 paul_gauguin_25 paul_gauguin_26

paul_gauguin_27 paul_gauguin_28 paul_gauguin_29 paul_gauguin_30 paul_gauguin_31

paul_gauguin_32 paul_gauguin_33 paul_gauguin_34 paul_gauguin_35 paul_gauguin_36

paul_gauguin_37 paul_gauguin_38 paul_gauguin_39 paul_gauguin_40 paul_gauguin_41

paul_gauguin_42 paul_gauguin_43 paul_gauguin_44 paul_gauguin_45 paul_gauguin_46

paul_gauguin_47 paul_gauguin_48 paul_gauguin_49 paul_gauguin_50 paul_gauguin_51

FIGURE A.12: Post-Impressionist paintings by Gauguin in the Painting-91-BIP dataset. All rights by Wikimedia commons, public domain.

Appendix B

Detailed results obtained when classifying the 90 images in the QArt-Dataset

This appendix provides a detailed exposition of the results obtained when classifying the 90 images in the QArt-Dataset using all the classification methods introduced in this PhD thesis: the three versions of the ℓ-SHE algorithm (i.e., the ℓ-SHE$^{\mathrm{RPL}}$, the ℓ-SHE$^{G(\mathbb{Q})}$, and the ℓ-SHE$^{\sqcap(\mathbb{Q})}$ versions).

B.1 The ℓ-SHE$^{\mathrm{RPL}}$ algorithm tested on the QArt-Dataset

This section presents in detail the quantitative results obtained by the ℓ-SHE$^{\mathrm{RPL}}$ algorithm when classifying all the paintings in the QArt-Dataset, indicating the membership degree to the three art styles (the Baroque, the Impressionism and the Post-Impressionism styles – B, I, PI, respectively) got for each image in the dataset. In addition, the tables of this section indicate whether the ℓ-SHE classification provides a second opinion (column *Second opinion?*). Table B.1 provides the results obtained from the 30 Baroque paintings and the 30 Impressionist paintings in the QArt-Dataset, and Table B.2 shows the results when classifying the 30 Post-Impressionist paintings in the dataset. Note that in all the tables of this appendix the last column indicates whether a second evaluation is given to the user.

B.2 The ℓ-SHE$^{G(\mathbb{Q})}$ algorithm tested on the QArt-Dataset

This section presents in detail the quantitative results obtained by the ℓ-SHE$^{G(\mathbb{Q})}$ algorithm when classifying all the paintings in the QArt-Dataset, indicating the membership degree to the three art styles (the Baroque, the Impressionism and the Post-Impressionism styles) got for each image in the dataset. Table B.4 provides the results obtained from the 30 Baroque paintings in the QArt-Dataset, and Table B.5

shows the results when classifying the 30 Impressionist 30 paintings and the 30 Post-Impressionist paintings in the dataset.

B.3 The ℓ-SHE$^{\sqcap(\mathbb{Q})}$ algorithm tested on the QArt-Dataset

This section presents in detail the quantitative results obtained by the ℓ-SHE$^{\sqcap(\mathbb{Q})}$ algorithm when classifying all the paintings in the QArt-Dataset, indicating the membership degree to the three art styles (the Baroque, the Impressionism and the Post-Impressionism styles) got for each image in the dataset. Table B.6 provides the results obtained from the 30 Baroque paintings in the QArt-Dataset, and Table B.7 shows the results when classifying the 30 Impressionist 30 Post-Impressionist paintings in the dataset.

TABLE B.1: ℓ-SHE$^{\text{RPL}}$: detailed results obtained when classifying the 30 Baroque images in the QArt-Dataset.

Art style	Painter	Name of the painting p in the QArt-Dataset	B(p)	I(p)	PI(p)	Second opinion?
Baroque	Vermeer	j1	0.974	0.414	0.533	No
		j2	0.631	0.334	0.415	No
		j3	0.820	0.564	0.352	No
		j4	1	0.153	0.580	No
		j5	1	0.645	0.426	No
		j6	1	0.295	0.363	No
		j7	1	0.616	0.508	No
		j8	0.975	0.601	0.409	No
		j9	1	0.413	0.445	No
		j10	1	0.440	0.458	No
		j11	1	0.588	0.667	No
		j12	0.661	0.384	0.577	Yes
		j13	0.841	0.431	0.538	No
		j14	1	0.401	0.520	No
	Velázquez	j15	0.405	0.406	0.666	No
		v1	0.995	0	0.760	No
		v2	1	0.376	0.456	No
		v3	0.681	0.837	0.432	No
		v4	1	0.615	0.457	No
		v5	1	0.470	0.430	No
		v6	0.986	0.355	0.706	No
		v7	0.867	0.284	0.750	Yes
		v8	0.975	0.574	0.538	No
		v9	1	0.119	0.509	No
		v10	0.835	0.781	0.526	Yes
		v11	1	0.437	0.455	No
		v12	0.117	0.161	0.942	No
		v13	1	0.261	0.511	No
		v14	0.630	0.611	0.382	Yes
		v15	1	0.305	0.571	No
Impressionism	Renoir	rn1	0.689	0.864	0.770	Yes
		rn2	0.855	0.891	0.463	Yes
		rn3	0.475	0.891	0.604	No
		rn4	0.430	0.665	0.355	No
		rn5	0.145	1	0.480	No
		rn6	0.532	0.918	0.661	No
		rn7	0.512	0.550	1	No
		rn8	0.332	0.557	0.549	Yes
		rn9	0.541	0.499	0.868	No
		rn10	1	0.795	0.659	No
		rn11	0	0.918	0.376	No
		rn12	0	0.948	0.464	No
		rn13	0.307	0.556	0.947	No
		rn14	0.637	0.876	0.877	Yes
		rn15	0.182	0.921	0.894	Yes
	Monet	m1	0.102	1	0.656	No
		m2	0.205	0.945	0.542	No
		m3	0	0.746	0.473	No
		m4	0.144	0.685	0.899	No
		m5	0	0.847	0.502	No
		m6	0.254	0.839	0.881	Yes
		m7	0.786	0.946	0.525	No
		m8	0	0.961	0.482	No
		m9	1	0.733	0.576	No
		m10	0.168	0.692	0.484	No
		m11	0.592	0.918	0.479	No
		m12	0.624	0.642	0.379	Yes
		m13	0.818	0.591	0.701	Yes
		m14	0.119	0.100	0.568	No
		m15	0	0.912	0.468	No

TABLE B.2: ℓ-SHE^{RPL}: detailed results obtained when classifying the 30 Impressionist images and 30 the Post-Impressionist paintings in the QArt-Dataset.

Art style	Painter	Name of the painting p in the QArt-Dataset	B(p)	I(p)	PI(p)	Doubt?
Post-Impressionism	van Gogh	vg1	0	0.810	0.989	No
		vg2	0	0.269	1	No
		vg3	0.049	0.785	0.719	Yes
		vg4	0.093	0.686	1	No
		vg5	0.665	0.679	1	No
		vg6	0.188	0.778	0.965	No
		vg7	0.891	0.723	1	Yes
		vg8	0	0.766	1	No
		vg9	0.023	0.750	0.100	No
		vg10	0	0.672	0.980	No
		vg11	0.943	0.559	0.369	No
		vg12	0	0.629	1	No
		vg13	0.259	1	0.481	No
		vg14	0.774	0.880	0.425	Yes
		vg15	0.182	0.921	0.894	Yes
	Gauguin	gg1	0	0.402	0.905	No
		gg2	0.160	0.899	0.844	Yes
		gg3	0	0.823	0.956	Yes
		gg4	0	0.521	0.919	No
		gg5	0.255	0.862	0.895	Yes
		gg6	0.298	0.809	0.695	Yes
		gg7	0.083	0.805	1	No
		gg8	0	0.938	0.772	No
		gg9	0	0.611	1	No
		gg10	0.067	0.816	1	No
		gg11	0.254	1	0.651	No
		gg12	0.001	0.958	0.726	No
		gg13	0.876	0.374	0.972	Yes
		gg14	0.007	0.818	1	No
		gg15	0.385	1	0.674	No

TABLE B.3: ℓ-SHE$^{G(\mathbb{Q})}$: detailed results obtained when classifying the 15 Baroque images by Vermeer in the QArt-Dataset.

Art style	Painter	Name of the painting p in the QArt-Dataset	B(p)	I(p)	PI(p)	Doubt?
	Vermeer	j1	0.874	0	0.003	Yes
		j2	0.584	0	0.004	No
		j3	0.660	0	0.000	No
		j4	1	0	0.001	No
		j5	1	0.378	0	No
		j6	1	0	0.000	No
		j7	1	0.056	0.003	No
		j8	0.758	0.406	0.000	No
		j9	1	0.080	0.001	No
		j10	1	0	0.003	No
		j11	1	0.083	0.006	No
		j12	0.597	0.304	0	No
		j13	0.688	0	0.002	No
		j14	1	0.149	0.012	No
		j15	0.453	0	0.000	No

TABLE B.4: ℓ-SHE$^{G(\mathbb{Q})}$: detailed results obtained when classifying the 15 Baroque images by Velázquez in the QArt-Dataset.

Art style	**Painter**	**Name of the painting p in the QArt-Dataset**	**B(p)**	**I(p)**	**PI(p)**	**Doubt?**
		v1	1	0	0	No
		v2	1	0	0	No
		v3	0.591	0.460	0	Yes
		v4	1	0	0	No
		v5	1	0	0	No
		v6	0.886	0	0.064	No
		v7	0.693	0	0	No
	Velázquez	v8	1	0.378	0.001	No
		v9	1	0	0.014	No
		v10	1	0.040	0.016	No
		v11	1	0.249	0.000	No
		v12	0.308	0	0.82	No
		v13	1	0	0.001	No
		v14	1	0.324	0	No
		v15	1	0	0	No

TABLE B.5: ℓ-SHE$^{G(\mathbb{Q})}$: detailed results obtained when classifying the 30 Impressionist images and 30 the Post-Impressionist paintings in the QArt-Dataset.

Art style	**Painter**	**Name of the painting *p* in the QArt-Dataset**	**B(p)**	**I(p)**	**PI(p)**	**Doubt?**
Impressionism	Renoir	rn1	0.284	0.487	0.001	No
		rn2	0.677	0.514	0.001	No
		rn3	0.487	0.514	0.003	Yes
		rn4	0.465	0.378	0	Yes
		rn5	0.322	1	0.010	No
		rn6	0.516	0.541	0.003	Yes
		rn7	0.507	0.126	1	No
		rn8	0.745	1	0.002	No
		rn9	0.523	0.193	0.008	No
		rn10	1	0.294	0.006	No
		rn11	0.201	0.541	0	No
		rn12	0.054	0.388	0.015	No
		rn13	0.401	0	0.087	No
		rn14	0.575	0.316	0.017	No
		rn15	0.329	0.361	0.034	Yes
	Monet	m1	0.301	1	0.007	No
		m2	0.384	0.568	0.030	No
		m3	0.107	0.460	0.000	No
		m4	0.322	0.416	0.039	Yes
		m5	0.207	0.346	0.005	Yes
		m6	0.377	0.279	0.053	Yes
		m7	0.643	0.386	0.012	No
		m8	0.160	0.401	0.009	No
		m9	0.378	0.406	0	Yes
		m10	1	0.174	0.008	No
		m11	0.562	0.541	0	Yes
		m12	0.565	0	0	No
		m13	0.721	0.268	0	No
		m14	0.310	1	0.007	No
		m15	0.119	0.352	0.017	No
Post-Impressionism	van Gogh	vg1	0.227	0.250	0.519	No
		vg2	0.082	0	1	No
		vg3	0.175	0.225	0.068	Yes
		vg4	0.307	0.145	1	No
		vg5	0.583	0.119	1	No
		vg6	0.344	0.218	0.495	No
		vg7	0.723	0.163	0.007	No
		vg8	0.037	0.206	1	No
		vg9	0.261	0	1	No
		vg10	0.170	0.112	0.120	Yes
		vg11	0.732	0.226	0.005	No
		vg12	0.121	0	1	No
		vg13	0.379	1	0.001	No
		vg14	0.637	0.320	0.005	No
		vg15	0.431	0.273	0.057	No
	Gauguin	gg1	0.220	0	0.045	No
		gg2	0.343	0.330	0.125	Yes
		gg3	0.158	0.265	0.010	Yes
		gg4	0.118	0	0.059	Yes
		gg5	0.377	0.044	0.035	No
		gg6	0.399	0.250	0.128	Yes
		gg7	0.292	0.044	1	No
		gg8	0.240	0.378	0.017	Yes
		gg9	0.073	0	1	No
		gg10	0.284	0.311	1	No
		gg11	0.377	1	0.036	No
		gg12	0.251	0.398	0.033	Yes
		gg13	0.176	0	0.112	Yes
		gg14	0.253	0.258	1	No
		gg15	0.442	1	0.005	No

TABLE B.6: ℓ-SHE$^{\sqcap(\mathbb{Q})}$: detailed results obtained when classifying the 30 Baroque images in the QArt-Dataset.

Art style	**Painter**	**Name of the painting p in the QArt-Dataset**	**B(p)**	**I(p)**	**PI(p)**	**Doubt?**
Baroque	Vermeer	j1	0.971	0	0.009	No
		j2	0.508	0	0.004	No
		j3	0.811	0	0.000	No
		j4	1	0	0.004	No
		j5	1	0.512	0	No
		j6	1	0	0.004	No
		j7	1	0.127	0.008	No
		j8	1	0.491	0.000	No
		j9	1	0.125	0.001	No
		j10	1	0	0.004	No
		j11	1	0.172	0.028	No
		j12	0.545	0.271	0	No
		j13	0.791	0	0.005	No
		j14	1	0.194	0.028	No
		j15	0.337	0	0.002	No
	Velázquez	v1	1	0	0	No
		v2	1	0	0	No
		v3	0.668	0.743	0	Yes
		v4	1	0	0	No
		v5	1	0	0	No
		v6	0.984	0	0.269	No
		v7	0.839	0	0	No
		v8	1	0.459	0.003	No
		v9	1	0	0.032	No
		v10	0.847	0.420	0.040	No
		v11	1	0.297	0.001	No
		v12	0.321	0	0.588	No
		v13	1	0	0.003	No
		v14	0.717	0.459	0	No
		v15	1	0	0	No

TABLE B.7: ℓ-SHE$^{\sqcap(\mathbb{Q})}$: detailed results obtained when classifying the 30 Impressionist images and 30 the Post-Impressionist paintings in the QArt-Dataset.

Art style	**Painter**	**Name of the painting *p* in the QArt-Dataset**	**B(p)**	**I(p)**	**PI(p)**	**Doubt?**
Impressionism	Renoir	rn1	0.190	0.786	0.006	No
		rn2	0.891	0.830	0.001	Yes
		rn3	0.517	0.830	0.011	No
		rn4	0.578	0.535	0	Yes
		rn5	0.287	1	0.019	No
		rn6	0.676	0.874	0.012	No
		rn7	0.391	0.225	1	No
		rn8	0.402	1	0.004	No
		rn9	1	0.286	0.054	No
		rn10	1	0.602	0.028	No
		rn11	0.062	0.874	0	No
		rn12	0.003	0.882	0.024	No
		rn13	0.222	0	0.624	No
		rn14	1	0.718	0.124	No
		rn15	0.262	0.821	0.243	No
	Monet	m1	0.327	1	0.030	No
		m2	1	0.917	0.074	Yes
		m3	0.025	0.650	0.001	No
		m4	0.403	0.588	0.275	No
		m5	0.065	0.709	0.012	No
		m6	0.213	0.635	0.353	No
		m7	0.802	0.877	0.029	Yes
		m8	0.032	0.910	0.017	No
		m9	0.133	0.573	0	No
		m10	1	0.394	0.025	No
		m11	1	0.874	0	Yes
		m12	0.508	0	0	No
		m13	0.719	0.419	0	No
		m14	0.274	1	0.020	No
		m15	0.011	0.801	0.027	No
Post-Impressionism	van Gogh	vg1	0.078	0.567	0.979	No
		vg2	0.007	0	1	No
		vg3	0.147	0.512	0.295	No
		vg4	0.106	0.328	1	No
		vg5	0.640	0.270	1	No
		vg6	0.221	0.496	0.935	No
		vg7	1	0.370	0.049	No
		vg8	0.012	0.467	1	No
		vg9	0.261	0	1	No
		vg10	0.040	0.254	0.858	No
		vg11	1	0.354	0.002	No
		vg12	0.012	0	1	No
		vg13	0.323	1	0.002	No
		vg14	0.767	0.727	0.007	Yes
		vg15	0.406	0.621	0.298	No
	Gauguin	gg1	0.104	0	0.319	No
		gg2	0.136	0.769	0.654	Yes
		gg3	0.072	0.602	0.711	Yes
		gg4	0.024	0	0.424	No
		gg5	0.302	0.611	0.252	No
		gg6	0.308	0.567	0.409	No
		gg7	0.207	0.496	1	No
		gg8	0.126	0.860	0.095	No
		gg9	0.020	0	1	No
		gg10	0.170	0.649	1	No
		gg11	0.307	1	0.138	No
		gg12	0.200	0.904	0.161	No
		gg13	0.838	0	0.798	Yes
		gg14	0.128	0.586	1	No
		gg15	0.322	1	0.021	No

Appendix C

Detailed results obtained when classifying the 247 images in the Painting-91-BIP dataset

This appendix provides a detailed exposition of the results obtained when classifying the 247 images in the Painting-91-BIP dataset using the different methods presented in this thesis, ℓ-SHE$^{\text{RPL}}$, ℓ-SHE$^{G(\mathbb{Q})}$, and ℓ-SHE$^{\sqcap(\mathbb{Q})}$. Again, let us recall that the experimentation was run using Swi-Prolog (Wielemaker et al. (2012)).

C.1 The ℓ-SHE$^{\text{RPL}}$ algorithm tested on the Painting-91-BIP dataset

This section presents in detail the quantitative results obtained by ℓ-SHE$^{\text{RPL}}$ when classifying all the paintings in the Painting-91-BIP dataset, indicating the membership degree to the three art styles (Baroque, Impressionism and Post-Impressionism – B, I, PI, respectively) got for each image in the dataset. In this way, tables C.1–C.4 provide the results obtained for the 247 in the Painting-91-BIP dataset. Note again that in all the tables of this appendix the last column indicates whether a second evaluation is given to the user.

C.2 The ℓ-SHE$^{G(\mathbb{Q})}$ algorithm tested on the Painting-91-BIP dataset

This section presents in detail (Tables C.5–C.8) the quantitative results obtained by ℓ-SHE$^{G(\mathbb{Q})}$ when classifying all the paintings in the Painting-91-BIP dataset, indicating the membership degree to the three art styles (Baroque, Impressionism and Post-Impressionism) got for each image in the dataset.

C.3 The ℓ-SHE$^{\sqcap(\mathbb{Q})}$ algorithm tested on the Painting-91-BIP dataset

This section presents in detail (Tables C.9–C.12) the quantitative results obtained by ℓ-SHE$^{\sqcap(\mathbb{Q})}$ when classifying all the paintings in the Painting-91-BIP dataset, indicating the membership degree to the three art styles (Baroque, Impressionism and Post-Impressionism) got for each image in the dataset.

TABLE C.1: ℓ-SHE$^{\text{RPL}}$: detailed results obtained when classifying the 35 images from Vermeer and 25 paintings from Velázquez in the Painting-91-BIP dataset.

Art style	Painter	i	B(p)	I(p)	PI(p)	Doubt?
Baroque	jan_vermeer_i	1	0.610	0.667	0.406	Yes
		2	1	0.119	0.701	No
		3	1	1	0.270	Yes
		4	0.642	0.355	0.864	No
		5	1	0.250	0.595	No
		6	0.243	0.718	0.649	Yes
		7	1	0.245	0.598	No
		8	1	0.312	0.636	No
		9	0.980	0.080	0.885	Yes
		10	1	0.108	0.615	No
		11	1	0.141	0.401	No
		12	0.744	0.118	0.872	Yes
		13	0.577	0.600	0.734	Yes
		14	1	0.208	0.480	No
		15	1	0.093	0.482	No
		16	0.552	0.364	0.862	No
		17	1	0.127	0.907	Yes
		18	0.963	0.200	0.641	No
		19	1	0.015	0.950	Yes
		20	1	0.582	0.436	No
		21	1	0.126	0.391	No
		22	0.591	0.248	0.808	No
		23	0.780	0.234	0.861	Yes
		24	0.782	0.218	0.761	Yes
		25	1	0.408	0.601	No
		26	0.924	0.121	0.519	No
		27	1	0.386	0.609	No
		28	1	0	0.909	Yes
		29	1	0.199	0.696	No
		30	0.047	0.520	0.414	Yes
		31	1	0.237	0.569	No
		32	1	0.256	0.396	No
		33	1	0.152	0.537	No
		34	0.938	0.300	0.703	No
		35	1	0.510	0.373	No
	diego_-velazquez_i	1	1	0.073	0.853	Yes
		2	1	0.092	0.789	No
		3	0.913	0.110	0.865	Yes
		4	0.906	0.280	0.766	Yes
		5	1	0.297	0.715	No
		6	1	0	0.526	No
		7	1	0.024	0.696	No
		8	0.981	0.128	0.705	No
		9	1	0.378	0.433	No
		10	1	0.196	0.542	No
		11	1	0.285	0.423	No
		12	1	0.231	0.654	No
		13	1	0.192	0.758	No
		14	1	0.524	0.627	No
		15	1	0.050	0.887	Yes
		16	1	0.116	0.384	No
		17	1	0.076	0.554	No
		18	0.964	0.287	0.510	No
		19	1	0.470	0.438	No
		20	1	0	0.535	No
		21	0.973	0.294	0.807	No
		22	1	0	0.675	No
		23	0.990	0.105	0.879	Yes
		24	1	0.180	0.621	No
		25	0.955	0.126	0.708	No

TABLE C.2: ℓ-SHE^{RPL}: detailed results obtained when classifying the rest of the images from Velázquez and the 46 images from Renoir in the Painting-91-BIP dataset.

Art style	Painter	i	B(p)	I(p)	PI(p)	Doubt?
Baroque	diego_-velazquez_i	26	1	0	0.600	No
		27	0.561	0.719	0.506	No
		28	1	0.352	0.357	No
		29	1	0.374	0.509	No
		30	0.951	0.063	0.839	Yes
		31	0.854	0.416	0.493	No
		32	1	0	0.656	No
		33	1	0.352	0.357	No
		34	1	0.057	0.439	No
		35	1	0.160	0.567	No
		36	1	0.199	0.656	No
		37	1	0.341	0.529	No
		38	0.992	0.314	0.721	No
		39	1	0.362	0.646	No
Impressionism	pierre-auguste_-renoir_i	1	0.299	0.415	0.678	No
		2	0.691	0.406	0.752	Yes
		3	0	0.182	0.900	No
		4	0.373	0.669	0.738	Yes
		5	0.557	0.170	0.860	No
		6	0.736	0.005	1	No
		7	0.876	0.995	0.484	Yes
		8	0.369	1	1	Yes
		9	0	0.959	0.712	No
		10	0	0.721	0.866	Yes
		11	0.245	0.692	0.518	No
		12	1	0.574	0.375	No
		13	0	0.497	0.500	Yes
		14	0.414	0.722	0.799	Yes
		15	0.926	0.568	0.861	Yes
		16	0	0.496	0.837	No
		17	0.527	0.918	0.441	No
		18	0.850	0	0.902	Yes
		19	1	0.837	0.703	No
		20	0	0.592	0.925	No
		21	0	0.382	0.735	No
		22	0.989	0.496	0.767	No
		23	0.212	0.997	0.473	No
		24	1	0.437	0.864	No
		25	0.513	0.607	0.805	No
		27	0.373	0.571	0.911	No
		28	0.947	0.397	0.657	No
		29	0.207	0.173	0.624	No
		30	0.031	0	1	No
		31	0.069	0.860	0.639	No
		32	0.680	0.491	0.970	No
		33	0.872	0.024	0.860	Yes
		34	0.811	0.484	0.824	Yes
		35	1	0	0.875	Yes
		36	0.968	0	0.860	Yes
		37	0.703	0.159	0.860	No
		38	0.754	0.388	0.751	Yes
		39	0	0.249	0.962	No
		40	0.413	0.497	0.384	Yes
		41	0.679	0.432	0.555	Yes
		43	0.400	0.203	0.925	No
		44	0.619	0.497	0.742	Yes
		45	1	0.130	0.865	Yes
		46	0.399	0.723	0.662	Yes
		47	0	0.677	0.786	Yes
		48	0.077	0.897	0.727	No

TABLE C.3: ℓ-SHE$^{\text{RPL}}$: detailed results obtained when classifying the 36 images from Monet and 24 from van Gogh in the Painting-91-BIP dataset.

Art style	Painter	i	B(p)	I(p)	PI(p)	Doubt?
Impressionism	claude_monet_i	1	0.732	0.941	0.460	No
		2	0.132	0.783	0.367	No
		3	0.602	0.918	0.528	No
		4	1	1	0.340	Yes
		5	0.548	0.837	0.543	No
		6	1	0.574	0.450	No
		7	0.454	0.642	0.446	No
		8	0.304	0.884	0.444	No
		9	0.938	0.719	0.525	No
		10	0.756	0.941	0.415	No
		11	0	0.735	0.563	No
		12	0.821	0.443	0.369	No
		13	0.145	0.934	0.393	No
		14	0.925	0.180	0.425	No
		15	1	0.297	0.918	Yes
		16	0.205	0.574	0.627	Yes
		17	0	0.918	0.594	No
		18	0	1	0.410	No
		19	1	0.719	0.697	No
		20	1	0.887	0.602	Yes
		21	1	0.719	0.398	No
		22	0.001	0.665	0.384	No
		23	0.132	0.837	0.700	Yes
		24	0	0.615	0.498	Yes
		25	0	0.923	0.394	No
		26	0.606	0.696	0.419	Yes
		27	0.084	0.180	0.440	No
		28	1	0.207	0.767	No
		29	0.427	0.574	0.411	Yes
		30	0.917	0.834	0.685	Yes
		31	0.623	0.891	0.376	No
		32	0.495	0.035	0.348	Yes
		33	0.196	0.827	0.655	No
		34	0.481	0.810	0.338	No
		35	0	0.972	0.366	No
		36	0.602	0.918	0.528	No
Post-Impressionism	vincent_van_-gogh_i	1	0.681	0.442	0.900	No
		2	0	0.746	1	No
		3	0.022	0.810	0.425	No
		4	0.167	0.414	0.926	No
		5	0	0.494	0.793	No
		6	0	0.864	0.696	No
		7	1	0.352	0.432	No
		8	0	0.690	0.970	No
		9	0	1	0.366	No
		10	0.408	0.767	0.785	Yes
		11	0.566	0.230	1	No
		12	1	0	0.701	No
		13	0.951	0.041	0.968	Yes
		14	0.339	0.945	0.471	No
		15	0	0.956	0.658	No
		16	0.417	0.864	0.398	No
		17	0.961	0.703	0.711	No
		18	1	0.574	0.458	No
		19	0.535	0.703	0.893	No
		20	0.708	0.750	0.682	Yes
		21	0.481	0.416	0.373	Yes
		22	0.725	0.918	0.513	No
		23	0	0.863	0.766	Yes
		24	0.116	0.723	0.795	Yes

TABLE C.4: ℓ-SHE^{RPL}: detailed results obtained when classifying the 40 images from van Gogh in the Painting-91-BIP dataset.

Art style	Painter	i	B(p)	I(p)	PI(p)	Doubt?
Post-Impressionism	vincent_van_-gogh_i	26	0	0.747	0.922	No
		27	0.318	0.495	0.976	No
		28	0.046	0.433	0.891	No
		29	0	0.705	0.887	No
		30	0	0.854	0.911	Yes
		31	0.205	0.485	1	No
		32	0.678	0.727	0.903	No
		33	1	0.374	0.402	No
		34	0	0.780	0.875	Yes
		35	0.231	0.539	1	No
		36	0	0.714	0.901	No
		37	0.544	0.750	0.397	No
		38	0	1	0.678	No
		39	0.376	0	1	No
		40	0.970	0.918	0.702	Yes
		41	0.073	0.918	0.702	No
	paul_gauguin_i	1	0	0.627	1	No
		2	0.287	0.813	0.951	Yes
		3	0.527	1	0.623	No
		4	0.443	0.884	0.828	Yes
		5	0.490	0.655	0.752	Yes
		6	0.262	0.642	0.798	No
		7	0.809	0.730	1	No
		8	0.982	0.253	0.856	Yes
		9	1	0.671	0.709	No
		10	1	0.827	0.928	Yes
		11	1	0.478	0.884	Yes
		12	0	0.689	1	No
		13	0.517	1	0.781	No
		14	0.605	0.460	1	No
		15	0.868	0.817	0.735	Yes
		16	0.894	0.759	0.887	Yes
		17	0.699	0.757	0.884	Yes
		18	0.362	0.995	0.468	No
		19	0	0.719	1	No
		20	0	0.503	1	No
		21	0.221	0.561	1	No
		22	0.040	0.748	1	No
		23	0.291	0.202	0.955	No
		24	1	0.624	0.690	No
		25	0.285	0.682	1	No
		26	0.102	0.692	0.987	No
		27	0.517	0.218	0.931	No
		28	0.342	0.955	0.929	Yes
		29	0.374	0.700	1	No
		30	0.991	0.594	0.968	Yes
		31	0.754	0.500	0.758	Yes
		32	0.849	0.722	0.857	Yes
		33	0.550	1	0.603	No
		34	0.523	0.913	0.773	No
		35	0	0.738	1	No
		36	0	0.548	0.904	No
		37	0.546	0.690	1	No
		38	0.411	0.120	1	No
		39	0.908	0.864	0.787	Yes
		40	0.149	0.392	1	No
		41	0.559	0.824	0.714	Yes
		42	0.401	0.977	0.585	No
		43	1	0.642	0.442	No
		44	0	0.420	1	No
		45	0.009	0.351	1	No
		46	1	0.381	0.625	No
		47	0.756	0.417	0.712	Yes
		48	0.472	0.684	0.901	No
		49	1	0.378	0.891	Yes
		50	0.691	0.874	0.997	Yes
		51	0.819	0.500	1	No

TABLE C.5: ℓ-SHE$^{G(\mathbb{Q})}$: detailed results obtained when classifying the 35 images from Vermeer and 25 paintings from Velázquez in the Painting-91-BIP dataset.

Art style	Painter	i	B(p)	I(p)	PI(p)	Doubt?
Baroque	jan_vermeer_i	1	0.738	0.307	0.003	No
		2	1	0	0.015	No
		3	1	0	0	No
		4	0.761	0	0.004	No
		5	1	0	0.007	No
		6	0.622	0.271	0.018	No
		7	1	0	0.007	No
		8	1	0	0.002	No
		9	0.880	0	0.025	No
		10	1	0	0.004	No
		11	1	0	0.000	No
		12	0.776	0	0.024	No
		13	0.752	0.294	0.005	No
		14	1	0	0.001	No
		15	1	0	0	No
		16	0.765	0	0.007	No
		17	1	0	0.047	No
		18	0.863	0	0.062	No
		19	0.871	0	0.090	No
		20	1	0.303	0	No
		21	1	0	0	No
		22	0.734	0	0	No
		23	1	0	0.001	No
		24	0.643	0	0	No
		25	1	0	0.000	No
		26	0.832	0	0	No
		27	1	0	0.001	No
		28	1	0	0.072	No
		29	1	0	0.022	No
		30	0.546	0.324	0	No
		31	1	0	0	No
		32	1	0	0	No
		33	1	0	0.003	No
		34	0.698	0	0.004	No
		35	1	0.143	0	No
	diego_-velazquez_i	1	1	0	0.004	No
		2	1	0	0.025	No
		3	0.834	0	0.042	No
		4	1	0	0.014	No
		5	1	0	0.016	No
		6	1	0	0.000	No
		7	1	0	0.012	No
		8	0.881	0	0.026	No
		9	1	0.217	0.001	No
		10	1	0	0.002	No
		11	1	0	0	No
		12	1	0	0.007	No
		13	1	0	0.015	No
		14	1	0	0	No
		15	1	0	0.027	No
		16	1	0	0	No
		17	1	0	0	No
		18	0.864	0	0	No
		19	1	0	0	No
		20	1	0	0.015	No
		21	1	0	0.013	No
		22	1	0	0.017	No
		23	0.890	0	0.099	No
		24	1	0	0.000	No
		25	0.855	0	0.058	No

TABLE C.6: ℓ-SHE$^{G(\mathbb{Q})}$: detailed results obtained when classifying the rest of the images from Velázquez and the 46 images from Renoir in the Painting-91-BIP dataset.

Art style	Painter	i	B(p)	I(p)	PI(p)	Doubt?
Baroque	diego_-velazquez_i	26	1	0	0.037	No
		27	0.775	0.432	0.000	No
		28	1	0	0	No
		29	1	0.213	0.001	No
		30	1	0	0.040	No
		31	1	0	0	No
		32	1	0	0.039	No
		33	1	0	0	No
		34	1	0	0	No
		35	1	0	0.002	No
		36	1	0	0.002	No
		37	1	0	0.002	No
		38	0.892	0	0.062	No
		39	1	0	0.001	No
Impressionism	pierre-auguste_-renoir_i	1	0.661	0	0	No
		2	1	0.299	0.000	No
		3	0.479	0	0.040	No
		4	0.745	0	0.000	No
		5	0.758	0	0	No
		6	0.797	0	1	No
		7	0.668	0.595	0.011	Yes
		8	0.778	0	1	No
		9	0.274	0.399	0.039	Yes
		10	0.241	0	0.006	No
		11	0.209	0.405	0.013	No
		12	1	0.378	0.000	No
		13	0.451	0	0	No
		14	0.776	0	0	No
		15	0.702	0	0.011	No
		16	0.634	0	0.023	No
		17	0.483	0.541	0	Yes
		18	0.825	0	0.010	No
		19	1	0.460	0.005	No
		20	0.569	0	0.065	No
		21	0.529	0	0.048	No
		22	1	0.217	0.011	No
		23	0.641	0.437	0.05	No
		24	1	0.131	0.004	No
		25	0.734	0.328	0.079	No
		27	0.429	0	0.051	No
		28	1	0	0.001	No
		29	0.604	0	0.010	No
		30	0.551	0	1	No
		31	0.337	0.460	0	Yes
		32	0.761	0	0.110	No
		33	1	0	0.004	No
		34	1	0	0.018	No
		35	1	0	0.015	No
		36	1	0	0.000	No
		37	1	0	0.000	No
		38	0.828	0	0.005	No
		39	0.238	0	0.102	Yes
		40	0.486	0	0	No
		41	0.684	0.324	0	No
		43	0.765	0	0.065	No
		44	0.831	0	0	No
		45	1	0	0.005	No
		46	0.461	0	0.013	No
		47	0.553	0.371	0.001	No
		48	0.420	0.337	0.029	Yes

TABLE C.7: ℓ-SHE$^{G(\mathbb{Q})}$: detailed results obtained when classifying the 36 images from Monet and 24 from van Gogh in the Painting-91-BIP dataset.

Art style	Painter	i	B(p)	I(p)	PI(p)	Doubt?
Impressionism	claude_monet_i	1	1	0.541	0.000	No
		2	0.311	0.405	0	Yes
		3	0.362	0.541	0.008	No
		4	1	1	0.000	Yes
		5	0.408	0.460	0	Yes
		6	1	0.378	0.001	No
		7	0.757	0	0.002	No
		8	0.372	0.324	0.021	Yes
		9	1	0.432	0.005	No
		10	0.531	0.541	0.001	Yes
		11	0.473	0.175	0.041	No
		12	1	0	0	No
		13	0.619	0.374	0.025	No
		14	1	0	0	No
		15	1	0	0.058	No
		16	0.211	0.378	0.025	No
		17	0.607	0.541	0.057	Yes
		18	0.069	1	0	No
		19	1	0.432	0	No
		20	1	0.487	0.037	No
		21	1	0.432	0	No
		22	0.124	0.378	0	No
		23	0.380	0.386	0.045	Yes
		24	0.097	0	0	Yes
		25	0.099	0.369	0.002	No
		26	0.742	0	0.000	No
		27	0.388	0	0	No
		28	0.470	0	0	No
		29	0.723	0.378	0	No
		30	0.689	0.357	0.035	No
		31	0.791	0.514	0.002	No
		32	1	0	0	No
		33	0.679	0.541	0.023	Yes
		34	0.727	0.432	0	No
		35	0.083	0.595	0.010	No
		36	1	0	0.008	No
Post-Impressionism	vincent_van_-gogh_i	1	0.508	0	0.040	No
		2	0.531	0.186	1	No
		3	0.172	0.432	0	No
		4	0.727	0	0.066	No
		5	0.126	0	0.323	No
		6	0.388	0.487	0.025	Yes
		7	1	0	0.025	No
		8	0.395	0.130	0	No
		9	0.227	1	0.001	No
		10	0.406	0.213	0.059	No
		11	0.627	0	1	No
		12	1	0	0.074	No
		13	0.733	0	0.075	No
		14	0.734	0.568	0.000	No
		15	0.607	0.044	0.008	No
		16	0.836	0.487	0	No
		17	1	0	0.035	No
		18	1	0.378	0.035	No
		19	0.489	0.425	0.033	Yes
		20	1	0	0.012	No
		21	0.790	0	0	No
		22	0.485	0.541	0.001	Yes
		23	0.588	0.302	0.096	No
		24	0.222	0	0.003	No

TABLE C.8: ℓ-SHE$^{G(\mathbb{Q})}$: detailed results obtained when classifying the 40 images from van Gogh in the Painting-91-BIP dataset.

Art style	Painter	i	B(p)	I(p)	PI(p)	Doubt?
Post-Impressionism	vincent_van_-gogh_i	26	0.093	0.046	0.452	No
		27	0.517	0.017	0.506	Yes
		28	0.356	0	0.031	No
		29	0.494	0.018	0.417	Yes
		30	0.478	0.322	0.085	No
		31	0.604	0	1	No
		32	0.777	0.168	0.433	No
		33	1	0	0	No
		34	0.415	0.252	0.015	No
		35	0.708	0	0	No
		36	0.486	0	0.041	No
		37	0.778	0.196	0.017	No
		38	0.435	1	0	No
		39	0.730	0	1	No
		40	1	0.287	0	No
		41	0.216	0.541	0.000	No
	paul_gauguin_i	1	0.400	0.067	1	No
		2	0.468	0.252	0.091	No
		3	0.522	1	0.036	No
		4	0.468	0.324	0.110	Yes
		5	0.550	0.460	0.282	Yes
		6	0.649	0	0.110	No
		7	0.686	0.170	1	No
		8	0.882	0	0.056	No
		9	1	0	0.016	No
		10	1	0.295	0.068	No
		11	1	0	0.086	No
		12	0.483	0.126	1	No
		13	0.741	1	0.120	No
		14	0.593	0	1	No
		15	1	0	0.027	No
		16	1	0.221	0.090	No
		17	0.768	0.046	0.024	No
		18	0.430	0.595	0.003	No
		19	0.268	0.165	1	No
		20	0.294	0.143	1	No
		21	0.599	0	1	No
		22	0.442	0.271	1	No
		23	0.420	0	0.095	No
		24	1	0	0.006	No
		25	0.615	0.123	1	No
		26	0.556	0.165	0.517	Yes
		27	0.687	0	0.461	No
		28	0.735	0.400	0.069	No
		29	0.636	0.145	1	No
		30	0.891	0.121	0.108	No
		31	0.783	0	0.078	No
		32	1	0.390	0.115	No
		33	0.796	1	0.021	No
		34	0.451	0.386	0.002	Yes
		35	0.440	0.314	1	No
		36	0.536	0	0.044	No
		37	0.731	0.130	1	No
		38	0.697	0	1	No
		39	0.708	0.487	0.044	No
		40	0.601	0	1	No
		41	0.801	0.427	0.011	No
		42	0.781	0.727	0.002	Yes
		43	1	0	0.001	No
		44	0.535	0	1	No
		45	0.551	0	1	No
		46	1	0	0.005	No
		47	1	0	0.004	No
		48	0.769	0.233	0.056	No
		49	1	0.099	0.048	No
		50	0.781	0.314	0.137	No
		51	0.814	0	1	No

TABLE C.9: ℓ-SHE$^{\sqcap(\mathbb{Q})}$: detailed results obtained when classifying the 35 images from Vermeer and 25 paintings from Velázquez in the Painting-91-BIP dataset.

Art style	Painter	i	B(p)	I(p)	PI(p)	Doubt?
Baroque	jan_vermeer_i	1	0.601	0.507	0.003	Yes
		2	1	0	0.073	No
		3	1	0	0	No
		4	0.625	0	0.029	No
		5	1	0	0.005	No
		6	0.336	0.499	0.074	No
		7	1	0	0.023	No
		8	1	0	0.009	No
		9	0.977	0	0.182	No
		10	1	0	0.014	No
		11	1	0	0.000	No
		12	0.725	0	0.167	No
		13	0.564	0.434	0.024	Yes
		14	1	0	0.002	No
		15	1	0	0	No
		16	0.532	0	0.048	No
		17	1	0	0.334	No
		18	1	0	0.207	No
		19	0.967	0	0.646	No
		20	1	0.421	0	No
		21	1	0	0	No
		22	0.584	0	0	No
		23	0.734	0	0.005	No
		24	0.751	0	0	No
		25	1	0	0.002	No
		26	0.915	0	0	No
		27	1	0	0.005	No
		28	1	0	0.490	No
		29	1	0	0.103	No
		30	0.236	0.393	0	No
		31	1	0	0	No
		32	1	0	0	No
		33	1	0	0.007	No
		34	1	0	0.022	No
		35	1	0.347	0	No
	diego_-velazquez_i	1	1	0	0.025	No
		2	1	0	0.145	No
		3	0.908	0	0.280	No
		4	0.888	0	0.082	No
		5	1	0	0.081	No
		6	1	0	0.001	No
		7	1	0	0.057	No
		8	0.979	0	0.121	No
		9	1	0.243	0.001	No
		10	1	0	0.003	No
		11	1	0	0	No
		12	1	0	0.031	No
		13	1	0	0.084	No
		14	0.806	0	0	No
		15	1	0	0.192	No
		16	1	0	0	No
		17	1	0	0	No
		18	0.906	0	0	No
		19	1	0	0	No
		20	1	0	0.038	No
		21	0.965	0	0.079	No
		22	1	0	0.074	No
		23	0.989	0	0.601	No
		24	1	0	0.011	No
		25	0.950	0	0.250	No

TABLE C.10: ℓ-SHE$^{\sqcap(\mathbb{Q})}$: detailed results obtained when classifying the rest of the images from Velázquez and the 46 images from Renoir in the Painting-91-BIP dataset.

Art style	Painter	i	B(p)	I(p)	PI(p)	Doubt?
Baroque	diego_-velazquez_i	26	1	0	0.117	No
		27	0.533	0.612	0.000	Yes
		28	1	0	0	No
		29	1	0.238	0.002	No
		30	0.935	0	0.253	No
		31	0.807	0	0	No
		32	1	0	0	No
		33	1	0	0	No
		34	1	0	0	No
		35	1	0	0.006	No
		36	1	0	0.007	No
		37	1	0	0.005	No
		38	1	0	0.273	No
		39	1	0	0.003	No
Impressionism	pierre-auguste_-renoir_i	1	0.363	0	0	No
		2	0.616	0.290	0.001	No
		3	0.068	0	0.283	No
		4	0.370	0	0.001	No
		5	0.541	0	0	No
		6	0.709	0	1	No
		7	0.848	0.991	0.020	Yes
		8	0.330	0	1	No
		9	0.136	0.907	0.179	No
		10	0.165	0	0.045	No
		11	0.215	0.573	0.031	No
		12	1	0.459	0.000	No
		13	0.067	0	0	Yes
		14	0.379	0	0	No
		15	0.908	0	0.075	No
		16	0.074	0	0.150	Yes
		17	0.501	0.874	0	No
		18	0.829	0	0.634	No
		19	1	0.743	0.025	No
		20	1	0	0.465	No
		21	0.188	0	0.229	Yes
		22	1	0.302	0.066	No
		23	0.308	0.992	0.009	No
		24	1	0.193	0.028	No
		25	0.511	0.456	0.423	Yes
		27	0.390	0	0.361	Yes
		28	1	0	0.003	No
		29	0.317	0	0.037	No
		30	0.227	0	1	No
		31	0.223	0.766	0	No
		32	0.663	0	0.786	Yes
		33	1	0	0.003	No
		34	0.766	0	0.117	No
		35	1	0	0.110	No
		36	1	0	0.001	No
		37	1	0	0.001	No
		38	0.714	0	0.026	No
		39	0.094	0	0.729	No
		40	0.430	0	0	No
		41	0.665	0.315	0	No
		43	0.377	0	0.461	Yes
		44	0.549	0	0	No
		45	1	0	0.033	No
		46	0.415	0	0.057	No
		47	0.130	0.548	0.005	No
		48	0.245	0.765	0.057	No

TABLE C.11: ℓ-SHE$^{\sqcap(\mathbb{Q})}$: detailed results obtained when classifying the 36 images from Monet and 24 from van Gogh in the Painting-91-BIP dataset.

Art style	Painter	i	B(p)	I(p)	PI(p)	Doubt?
Impressionism	claude_monet_i	1	0.648	0.901	0.000	No
		2	0.236	0.655	0	No
		3	1	0.874	0.021	No
		4	1	1	0.000	Yes
		5	0.478	0.743	0	No
		6	1	0.459	0.001	No
		7	0.439	0	0.003	No
		8	0.332	0.736	0.026	No
		9	1	0.612	0.014	No
		10	0.687	0.901	0.001	No
		11	0.072	0.397	0.107	No
		12	0.784	0	0	No
		13	0.272	0.849	0.013	No
		14	1	0	0	No
		15	1	0	0.416	No
		16	0.205	0.459	0.090	No
		17	0.094	0.874	0.160	No
		18	0.010	1	0	No
		19	1	0.612	0	No
		20	1	0.811	0.117	No
		21	1	0.612	0	No
		22	0.102	0.535	0	No
		23	0.259	0.728	0.198	No
		24	0.023	0	0	Yes
		25	0.009	0.830	0.001	No
		26	0.596	0	0.000	No
		27	0.241	0	0	No
		28	1	0	0	No
		29	0.438	0.459	0	Yes
		30	0.894	0.708	0.151	No
		31	0.589	0.830	0.001	No
		32	1	0	0	No
		33	0.272	0.764	0.093	No
		34	0.485	0.699	0	No
		35	0.027	0.961	0.004	No
		36	1	0.874	0.021	Yes
Post-Impressionism	vincent_van_-gogh_i	1	0.619	0	0.285	No
		2	0.115	0.423	1	No
		3	0.136	0.699	0	No
		4	0.195	0	0.472	No
		5	0.014	0	0.610	No
		6	0.204	0.786	0.117	No
		7	1	0	0	No
		8	0.040	0.296	0.782	No
		9	0.153	1	0.000	No
		10	0.398	0.479	0.315	Yes
		11	0.565	0	1	No
		12	1	0	0.296	No
		13	0.941	0	0.533	No
		14	0.349	0.917	0.000	No
		15	0.133	0.795	0.036	No
		16	0.298	0.786	0	No
		17	1	0	0.163	No
		18	1	0.459	0.013	No
		19	0.508	0.590	0.238	Yes
		20	1	0	0.053	No
		21	0.436	0	0	No
		22	1	0.874	0.003	Yes
		23	0.132	0.688	0.441	No
		24	0.188	0	0.017	No

TABLE C.12: ℓ-SHE$^{\sqcap(\mathbb{Q})}$: detailed results obtained when classifying the 40 images from van Gogh in the Painting-91-BIP dataset.

Art style	Painter	i	B(p)	I(p)	PI(p)	Doubt?
Post-Impressionism	vincent_van_-gogh_i	26	0.006	0.463	0.853	No
		27	0.381	0.034	0.955	No
		28	0.220	0	0.222	Yes
		29	0.122	0.369	0.786	No
		30	0.097	0.703	0.567	Yes
		31	0.316	0	1	No
		32	0.656	0.379	0.817	No
		33	1	0	0	No
		34	0.041	0.543	0.108	No
		35	0.274	0	1	No
		36	0.010	0	0.292	No
		37	0.514	0.441	0.012	Yes
		38	0.062	1	0	No
		39	0.385	0	1	No
		40	1	0.299	0	No
		41	0.173	0.874	0.002	No
	paul_gauguin_i	1	0.093	0.152	1	No
		2	0.349	0.576	0.650	No
		3	0.514	1	0.126	No
		4	0.449	0.737	0.573	No
		5	0.496	0.557	0.532	Yes
		6	0.340	0	0.529	No
		7	0.786	0.386	1	No
		8	0.980	1	0.353	Yes
		9	1	0	0.077	No
		10	1	0.645	0.485	No
		11	1	0	0.508	No
		12	0.165	0.285	1	No
		13	0.511	1	0.535	No
		14	0.590	0	1	No
		15	0.829	0	0.139	No
		16	0.872	0.488	0.566	No
		17	0.680	0.424	0.172	No
		18	0.384	0.991	0.006	No
		19	0.115	0.371	1	No
		20	0.105	0.236	1	No
		21	0.327	0	1	No
		22	0.232	0.537	1	No
		23	0.343	0	0.679	No
		24	1	0	0.029	No
		25	0.348	0.279	1	No
		26	0.264	0.353	0.976	No
		27	0.528	0	0.869	No
		28	0.350	0.902	0.493	No
		29	0.424	0.327	1	No
		30	0.990	0.234	0.769	No
		31	0.739	0	0.362	No
		32	0.819	0.609	0.639	No
		33	0.504	1	0.071	No
		34	0.485	0.829	0.009	No
		35	0.046	0.561	1	No
		36	0.106	0	0.032	Yes
		37	0.543	0.294	1	No
		38	0.437	0	1	No
		39	0.891	0.786	0.247	Yes
		40	0.281	0	1	No
		41	0.510	0.721	0.054	No
		42	0.360	0.970	0.005	No
		43	1	0	0.000	No
		44	0.197	0	1	No
		45	0.215	0	1	No
		46	1	0	0.019	No
		47	1	0	0.022	No
		48	0.447	0.440	0.388	No
		49	1	0.138	0.329	No
		50	1	0.713	0.977	Yes
		51	0.797	0	1	No