Universitat Autònoma de Barcelona

On the relation between homology and K-theory of etale groupoids

Alvaro Sánchez Madrigal

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On the relation between homology and K-theory of étale groupoids

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Prof. Pere Ara Bertran

Prof. Joan Bosa Puigredon

by

Álvaro Sánchez Madrigal

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I certify that the present Memoir has been
written by Álvaro Sánchez Madrigal
under the supervision of Prof. Pere Ara Bertran and
Prof. Joan Bosa Puigredon

Signed: Prof. Pere Ara Bertran Signed: Prof. Joan Bosa Puigredon

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Contents

Chapter 1

Introduction

Led by the increasing demand of a strong mathematical theory to support the non-commutative nature of quantum physics phenomena, Murray and von Neumann defined *von Neumann algebras* in the 1920s, laying the first stone of modern non-commutative algebra, a field that has become a major undertaking worldwide. Later on, Gelfand and Naimark developed in the 1940s the notion of *C^*-algebra*, which turns out to generalize the concept of von Neumann algebras. Since those early days, the study of C^*-algebras has provided an elegant setting for many problems in mathematics and physics.

One of the major concerns across all mathematical areas (and for C^*-Algebra Theory in particular) is the problem of classification. Motivated by the advances of Connes in the classification of von Neumann algebras, G. Elliot initiated in [17] the classification of C^*-algebras. He proved that the class of approximately finite dimensional C^*-algebras (or simply AF) is completely determined by their ordered K_0-groups. Throughout the years, it has been proven that all simple separable unital nuclear C^*-algebras with finite nuclear dimension satisfying the UCT can be distinguished by their K-theoretic invariants, now known as Elliott invariants (see, for example, [53]). As a consequence, the relevance of the study of K-theory has grown larger over the years.

The search of concrete models for certain classes of C^*-algebras has been a topic of interest ever since. In this sense, the notion of *group C^*-algebra* was naturally generalized to the one of *groupoid C^*-algebra* by Renault in his PhD. thesis [52]. Groupoids are mathematical objects that generalize the concept of groups, where the unit space does not need to consist of a single element. In consequence, groupoids are provided with a partially defined multiplication. This *small* difference implies major changes in their structure. Indeed, groupoids provide a unifying model for groups and group actions, or even higher-rank graphs (and hence, for their associated C^*-algebras), among other structures.

In group C^*-algebras, it is common to restrict our study to a more approachable class of groups, usually the class of discrete groups. The groupoid analogue is what Renault himself called *étale groupoids*. This family of groupoids is sufficient to model a large number of examples (AF algebras, graph algebras, all Kirchberg algebras in the UCT class, etc.).

Therefore, it is relevant the study of the K-theory of étale groupoids C^*-algebras. In this line, Matui posed a conjecture in [38] where he claimed that the K-groups of the C^*-algebra associated to an étale groupoid could be computed as a direct sum of all the even/odd homology groups of the given groupoid, under certain topological conditions:

Conjecture 2.4.1. [38, Conjecture 2.6] *Let $\mathcal{G}$ be an effective minimal étale groupoid, such that $\mathcal{G}^{(0)}$ is a Cantor space. Then:*

$$\bigoplus_{i=0}^{\infty} H_{2i}(\mathcal{G}) \cong K_0(C_r^*(\mathcal{G}))$$

and

$$\bigoplus_{i=0}^{\infty} H_{2i+1}(\mathcal{G}) \cong K_1(C_r^*(\mathcal{G})).$$

□

This conjecture has been proven true for several families of groupoids, that include AF groupoids, transformation groupoids of Cantor minimal systems [38], or Cuntz-Krieger groupoids [21], [36]. In [46], it was proven that Katsura-Exel-Pardo groupoids $\mathcal{G}_{A,B}$ associated with square integer matrices, with $A \geq 0$, satisfy the conjecture. In a more recent work [6], the authors proved that the conjecture holds for principal groupoids with dynamic asymptotic dimension at most 2.
Matui also posed a weakened version of this conjecture, obtained by applying a tensor product $(\cdot \otimes \mathbb{Q})$ on both sides, in order to avoid extension and torsion problems.
A first counterexample to the strong conjecture was found by Scarparo in [56], and we found two more shortly after in [47], in a joint work with Eduard Ortega. However, the study of necessary and/or sufficient conditions for the conjecture to hold remains relevant. The main goal of this work is to further deepen the knowledge of this conjecture, providing some examples and counterexamples for it and, more importantly, developing new techniques for the study of the invariants of certain families of groupoids.
The contents of this thesis are outlined below:

Contents and structure:

Chapter 2 will be devoted to setting all the notions required for the study of Matui's conjecture. We begin the chapter by providing most of the background that we will need in the forthcoming pages. We state the basic definitions regarding groupoids and their C^*-algebras, following [1], [3], [52], [58] and [59], among others. We also recall some of the most important results concerning K-theory, appearing in [5], [53] and [51]. Later on, we provide the definition of homology groups of a groupoid, as appearing in [14], and show the computation of the homology for some basic groupoids. We show special interest in the canonical map Φ between $H_0(\mathcal{G})$ into $K_0(C_r^*(\mathcal{G}))$ (introduced in [36]), given by $[1_U]_{H_0} \to [1_U]_{K_0}$, where U is a compact open set of the unit space of $\mathcal{G}$. This map, always well defined, will play a key role in the discussion of Deaconu-Renault groupoids, later in Chapter 4.

With all the basic notions set out, we finally introduce the HK conjecture posed by H. Matui in [38], after a series of articles [36], [37]. We also introduce a second conjecture appearing in [38], named AH, which we will discuss briefly in Chapter 5. This conjecture predicts, for essentially principal, minimal étale groupoids whose unit space is a Cantor set, the existence of an exact sequence relating the homology groups with the topological full group $[[\mathcal{G}]]$:

$$H_0(\mathcal{G}) \otimes \mathbb{Z} \xrightarrow{j} [[\mathcal{G}]]_{ab} \xrightarrow{I_{ab}} H_1(\mathcal{G}) \to 0$$

In Chapter 3, we aim to collect some of the most important techniques used to compute groupoid invariants. First, we discuss the notion of *Kakutani equivalence*, also known as *Morita equivalence* (see [36]), appearing in [21], [52] or [58], among others. Kakutani equivalence has proven to be a powerful tool when studying the HK conjecture, since it preserves both the homology and the K-theory groups associated to a groupoid (see, for example, [21]).

After that, we study the concept of *groupoid cocycles*, their associated *skew-product groupoids*, and their relation with the original groupoid. In this line, we show how the study of groupoid invariants can be approached by studying the invariants of the associated skew-product groupoid, using the results appearing in [46], [36], and [21].

Crossed product C^*-algebras arise naturally from skew product groupoids due to Takai-Takesaki duality (see [62]-[63]). In order to study those algebras, we follow the strategies of [4], [29], [48], [51] or [67], among others. Since some of those results require certain level of understanding about spectral sequences, we facilitate an introduction to that subject, using [34], [35], [39] and [64] as guidelines.

We then state both Matui's ([36], [14]) and Kasparov's ([29]) spectral sequences in particular, and provide a certain explicit approach to the second one, appearing in [4], [54] and [57]. We discuss the *mapping torus* C^*-algebra $\mathcal{M}_\alpha(A)$ associated to a dynamical system $(A, \alpha, \mathbb{Z}^n)$, which is known to satisfy

$$K_{q+n}(\mathcal{M}_\alpha(A)) \cong K_q(A \rtimes_\alpha \mathbb{Z}^n).$$

Moreover, using similar techniques to the ones appearing in [48], we show a new approach to this isomorphism in **Subsection 3.3.4.2**, and we prove in **Theorem 3.50** that the

isomorphism satisfies certain naturality conditions which will be of major relevance in Chapter 4.
We finish this chapter by displaying, in **Corollary 3.59**, the first counterexample of the strong version of Matui's HK conjecture, developed by Scarparo in [56], and consisting of the transformation groupoid associated to a certain odometer.

The purpose of Chapter 4 is to investigate Matui's HK-conjecture for Deaconu-Renault groupoids, following the work of [21]. This family of groupoids is an important source of C^*-algebras; they provide models for crossed products associated to Cantor minimal dynamical systems or higher-rank graph C^*-algebras, among others. Matui's HK conjecture, if proven true for this family, could become a very useful tool for computing the K-theory of those C^*-algebras, since the homology groups of Deaconu-Renault groupoids were completely determined in [21], following the work of [18]. In the same work [21], the authors prove that Deaconu-Renault groupoids of rank 1 and 2 satisfy the HK conjecture, and raise the question of whether the HK isomorphism could be chosen to be natural, in some sense. We shed some light on this by using the picture of Kasparov's spectral sequence for K-theory appearing in [4] and [54]. By doing so, we prove the following:

Propositions 4.14 + 4.15. *Rank 2 Deaconu-Renault groupoids satisfy Matui's HK conjecture. Moreover, the isomorphism for K_0 can be chosen such that $H_0(\mathcal{G}(X,\sigma))$ embeds into $K_0(C^*_r(\mathcal{G}(X,\sigma)))$ via the canonical map Φ.*

□

Moreover, in [21], the authors conjectured that, for a rank 3 Deaconu-Renault groupoid, the injectivity of the canonical map Φ from $H_0(\mathcal{G})$ into $K_0(C^*_r(\mathcal{G}))$ may lead to the verification of HK conjecture for this family. In this chapter, we prove that their conjecture is true. To do so, we use a well-known result relating the reduced C^*-algebra of a Deaconu-Renault groupoid $C^*_r(\mathcal{G})$ with the crossed product by $\mathbb{Z}^k$ of the C^*-algebra associated to certain skew product groupoid $\mathcal{G} \times_c \mathbb{Z}^k$, via Takai-Takesaki duality [62], [63], and then we study Kasparov's spectral sequence following [4]. By doing so, we are able to identify each one of the components of the sequence, as well as the maps involved. We use this to construct an exact sequence containing, in a consecutive way, the only non-trivial map of the third page of the spectral sequence, and Φ. Those techniques are enough to prove that the spectral sequence meets its limit prematurely, which, in this case, is enough to verify HK conjecture.

Theorem 4.18. *Let $\mathcal{G}(X,\sigma)$ be a Deaconu-Renault groupoid of rank 3. Then, whenever the canonical map between $H_0(\mathcal{G}(X,\sigma))$ and $K_0(C_r^*(\mathcal{G}(X,\sigma)))$ is injective, $\mathcal{G}(X,\sigma)$ verifies Matui's weak HK conjecture for K_0, and the strong version of HK for K_1.*

□

We then use this in order to obtain:

Theorem 4.26. *Let σ be an action of $\mathbb{Z}^3$ **by homeomorphisms** on the Cantor set X. Then the associated Deaconu-Renault groupoid $\mathcal{G}(X,\sigma)$ satisfies the HK conjecture for K_1, and the weak HK conjecture for K_0.*

□

We conclude this chapter by providing a complete description of the group $K_1(C_r^*(\mathcal{G}(X,\sigma)\times_c \mathbb{Z}^k)\rtimes\mathbb{Z})$, using the techniques appearing in [3].

Chapter 5 revolves around the groupoid arising from the infinite dihedral group acting as a self-similar group over certain set X, as studied in [46]. Self-similar objects are well-known to provide relatively simple examples of exotic structures. In this line, we expected to find new counterexamples to Matui's HK conjecture encoded as self-similar objects.

We begin the chapter with the basic notions of self-similarity and the associated C^*-algebras, following [20], [44], [46] and [43]. We also show some results from Nekrashevych [41], [42] relating those algebras with ones constructed via crossed products. In particular, it was proven in [42] that the C^*-algebra associated to a self-similar groupoid $\mathcal{G}_{(\Gamma,X)}$ is isomorphic to the Cuntz-Pimsner algebra $C^*(\Gamma,X)$. In the same work, the author shows how this C^*-algebra $C^*(\Gamma,X)$ can be expressed as a certain crossed product C^*-algebra of the gauge-invariant subalgebra $\mathcal{M}_\Gamma$.

We use those results to compute the K-theory of the self-similar groupoid associated to the infinite dihedral group.

We then proceed to study the homology groups of said groupoid, obtaining, up to isomorphisms, the low homology groups, and some useful results on the torsion properties of the higher ones. We combine all those results to present the dihedrical self-similar groupoid as a complete counterexample for both strong and weak versions of the HK conjecture.

Theorem 5.21. *Let $\mathcal{D}_\infty$ be the infinite dihedral group, and $X = \{0,1\}$. Let $(\mathcal{D}_\infty, X)$ be the induced self-similar group, and let $\mathcal{G}_{(\mathcal{D}_\infty,X)}$ be the associated groupoid of germs. Then*

$$\mathbb{Q} \cong K_i(C_r^*(\mathcal{G}_{(\mathcal{D}_\infty,X)})) \otimes \mathbb{Q} \ncong \bigoplus_{k=0}^{\infty} H_{i+2k}(\mathcal{G}_{(\mathcal{D}_\infty,X)}) \otimes \mathbb{Q} = 0, \;\; i = 1, 0.$$

□

We finish this chapter with the verification of the AH conjecture for the groupoid $\mathcal{G}_{(\mathcal{D}_\infty,X)}$ appearing in Theorem 5.21, using techniques developed in [38] and [44].

Chapter 2

Background

The main purpose of this chapter is to introduce the HK-conjecture, posed by H. Matui in a series of articles [36], [37], and [38]. To do so, and since this work aims to be self-contained, we need first to provide all the definitions regarding this conjecture: groupoids, étaleness, homology, K-theory, etc. The chapter is structured as follows:

In Section 2.1 we provide the basic notions about groupoids, and at the same time we set most of the notation that we will use in this work. Here we show some of the most common families of groupoids, as well as the ones we will focus our work later on. A more detailed account on the groupoid subject can be found in [1], [52] or [59], among others.

In Section 2.2, we give the definition of homology groups associated to a groupoid introduced in [14]. These are obtained by computing the homology groups of a certain chain complex associated to the groupoid. We also provide some basic results regarding groupoid homology (see, for example, [14], [21] or [38]).

In Section 2.3 we describe how to naturally associate a (reduced) C^*-algebra to an étale groupoid, following [19], [52] and [59]. We also use this section to recall some basic results about K-theory.

Finally, in Section 2.4, we discuss the main motivation of this work, that is, Matui's HK conjecture introduced in [38]. This conjecture anticipated a relation between the homology and the associated K-theory of an étale groupoid, under certain structure conditions. As we will show throughout this document, a counterexample for the strong version of this conjecture was found by Scarparo in a recent article [56]. Shortly after, we found a counterexample for both the strong and weak version of this conjecture, in a work made with Eduard Ortega [47]. We finish the chapter by showing a second conjecture posed by Matui in [38], named AH, which we will discuss for self-similar groupoids in Chapter 5. The question of whether this conjecture is true or false remain unanswered yet.

2.1 Basic notions

2.1.1 Groupoids

As we exposed in the introduction, the main theme of this work is the study of groupoids, as well as the relation between their invariants. It is therefore mandatory to begin with a proper definition of *groupoid*.

In a few words, groupoids are a generalization of groups. More precisely, a groupoid is a group whose unit may fail to be unique. This change allows a richer and more versatile structure, making their study worthwile.

If the reader is familiar with category theory, the simpler way to define a groupoid is the following:

Definition 2.1. *(Groupoid 1). A* ***groupoid*** *is a small category with inverses.*

As mentioned, this is a quick and clean way to define a groupoid. However, it may means very little for a reader unfamiliar with category notions. This work aims to be self-contained, making it imperative to provide a less abstract, algebraic definition:

Definition 2.2. *(Groupoid 2). A* ***groupoid*** *is a set $\mathcal{G}$ with a distinguished subset $\mathcal{G}^{(0)}$, together with a collection of maps:*

Range and source maps $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$.

Inversion map $g \mapsto g^{-1}$, for all $g \in \mathcal{G}$.

Multiplication map $(g, h) \mapsto gh \in \mathcal{G}$, for all $(g, h) \in \mathcal{G}^{(2)} := \{(\alpha, \beta) \in \mathcal{G} : s(\alpha) = r(\beta)\}$.

Moreover, the following properties must be met:

- $r(x) = x = s(x)$, *for all* $x \in \mathcal{G}^{(0)}$.
- $r(g)g = g = gs(g)$ *for all* $g \in \mathcal{G}$.
- $r(g^{-1}) = s(g)$, *and* $s(g^{-1}) = r(g)$, *for all* $g \in \mathcal{G}$.
- $g^{-1}g = s(g)$, *and* $gg^{-1} = r(g)$, *for all* $g \in \mathcal{G}$.
- $r(gh) = r(g)$, *and* $s(gh) = s(h)$, *for all* $(g, h) \in \mathcal{G}^{(2)}$.
- $(gh)f = g(hf)$, *for all* $(g, h, f) \in \mathcal{G}^{(3)}$ *(defined similarly to* $\mathcal{G}^{(2)}$*).*

$\square$

An elemental exercise for a category course would be to prove that, indeed, those two definitions are equivalent.

Groupoid homomorphisms are defined similarly to group homomorphisms, that is, a map $\varphi : \mathcal{G} \to \mathcal{H}$ between two groupoids satisfying that, whenever $(g_1, g_2) \in \mathcal{G}^{(2)}$, then $(\varphi(g_1), \varphi(g_2)) \in \mathcal{H}^{(2)}$, and

$$\varphi(g_1)\varphi(g_2) = \varphi(g_1g_2).$$

Immediate consequences arise from this:

- $\varphi(x) \in \mathcal{H}^{(0)}$, for all $x \in \mathcal{G}^{(0)}$.
- $r(\varphi(g)) = \varphi(r(g))$, and $s(\varphi(g)) = \varphi(s(g))$, for all $g \in \mathcal{G}$.
- $\varphi(g^{-1}) = \varphi(g)^{-1}$.

Before advancing further, let us show some basic examples:

Example 2.3. *(Groups).*
Let G be a group, and let $e \in G$ be its unit. Set $G^{(0)} = \{e\}$. Then G, together with the group operations, is a groupoid. □

This goes in both directions. If the unit space of a groupoid has one single element, then the groupoid is a group.

Example 2.4. *(Product groupoid).*
Let $\mathcal{G}$, $\mathcal{H}$ be groupoids. Then the set $\mathcal{G} \times \mathcal{H}$, together with pointwise operations and $\mathcal{G}^{(0)} \times \mathcal{H}^{(0)}$ as unit space, is a groupoid. □

Example 2.5. *(Equivalence relations).*
Let R be an equivalence relation on a set X. Set $R^{(0)} := \{(x, x) : x \in X\} \subseteq R$. Define $r(x, y) = (x, x)$, and $s(x, y) = (y, y)$, multiplication given by $(x, y)(y, z) = (x, z)$, and inverse given by $(x, y)^{-1} = (y, x)$. Then R, together with those operations, is a groupoid. Usually, the elements (x, x) of $R^{(0)}$ are simply identified as elements x of X. □

Given a groupoid, one can always construct an equivalence relation subgroupoid as it follows:
Define $R(\mathcal{G}) := \{(r(g), s(g)) : g \in \mathcal{G}\} \subseteq \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$. It is straightforward to check that $R(\mathcal{G})$ is indeed a groupoid. Moreover, the map $g \mapsto (r(g), s(g))$ from $\mathcal{G}$ to $R(\mathcal{G})$ is always a surjective groupoid homomorphism.

Example 2.6. *(Group bundles).*
Let X be a set, and let G_x be a group for every $x \in X$. Define the sets $\mathcal{G} := \bigcup_{x \in X} \{x\} \times G_x$, and $\mathcal{G}^{(0)} := \{(x, e_{G_x}) : x \in X\}$. Identify $\mathcal{G}^{(0)}$ with X in the natural way. Then the set $\mathcal{G}$ is a groupoid, together with operations $r(x, g) = s(x, g) = x$, $(x, g)(x, h) = (x, gh)$, and inverse given by $(x, g)^{-1} = (x, g^{-1})$. □

Example 2.7. *(Transformation groupoids).*
*Let Γ be a group acting on a set X by bijections. Let $\mathcal{G} := \Gamma \times X$, and set $\mathcal{G}^{(0)} = \{e\} \times X$, identified with X via $(e, x) \mapsto x$. Define operations $r(g, x) = g \cdot x$, $s(g, x) = x$, multiplication given by $(g, h \cdot x)(h, x) = (gh, x)$, and inverse $(g, x)^{-1} = (g^{-1}, g \cdot x)$. Then $\mathcal{G}$ is a groupoid called **transformation groupoid**. The usual notation is $\mathcal{G} = X \rtimes \Gamma$.* □

Example 2.8. *Let $\varphi : \Gamma \curvearrowright \mathcal{G}$ be an action of a countable group on a groupoid $\mathcal{G}$. Define the **semi-direct product** groupoid $\mathcal{G} \rtimes_{\varphi} \Gamma$ as $\mathcal{G} \times \Gamma$, together with the following structure:*

- $(g, \gamma)(g', \gamma') = (g\varphi^{\gamma}(g'), \gamma\gamma')$, *whenever* $(g, \varphi^{\gamma}(g')) \in \mathcal{G}^{(2)}$.
- $(g, \gamma)^{-1} = (\varphi^{\gamma^{-1}}(g^{-1}), \gamma^{-1})$.

There exists a natural homomorphism $\hat{\varphi} : \mathcal{G} \rtimes_{\varphi} \Gamma \to \Gamma$ given by $(g, \gamma) \mapsto \gamma$.
It is straightforward to check that, whenever $\mathcal{G}$ is trivial, in the sense that $\mathcal{G} = \mathcal{G}^{(0)} = X$, then the semi-direct product groupoid is just the transformation groupoid. □

Some immediate unicity properties arise from the definition of a groupoid:
Lemma 2.9. *Let $\mathcal{G}$ be a groupoid, and let $g \in \mathcal{G}$. Then g^{-1} is the unique element such that $gg^{-1} = r(g)$. It is also the unique element such that $g^{-1}g = s(g)$.*

Proof. Let $gh = r(g)$. Then $h = r(h)h = s(g)h = (g^{-1}g)h = g^{-1}(gh) = g^{-1}r(g) = g^{-1}$. The second statement is analogous. □

Lemma 2.10. *Let $\mathcal{G}$ be a groupoid, and let $g, h \in \mathcal{G}$. Suppose that there exists some $\gamma \in \mathcal{G}$ such that $g\gamma = h\gamma$. Then $g = h$. The same happens if $\gamma g = \gamma h$.*

Proof. If $g\gamma = h\gamma$, then $g = g\gamma\gamma^{-1} = h\gamma\gamma^{-1} = h$. □

Now we want to introduce the concept of *isotropy*, in order to define some key properties for the groupoids. Before that, we need to set some notation:
Let $x, y \in \mathcal{G}^{(0)}$. We write

$$\begin{aligned} \mathcal{G}_x &:= \{g \in \mathcal{G} : s(g) = x\} = s^{-1}(x), \\ \mathcal{G}^y &:= \{g \in \mathcal{G} : r(g) = y\} = r^{-1}(x), \\ &\text{and } \mathcal{G}_x^y := \mathcal{G}_x \cap \mathcal{G}^y. \end{aligned}$$

Some texts refer to $\mathcal{G}_x$, $\mathcal{G}^x$ and $\mathcal{G}_x^y$ as $\mathcal{G}x$, $x\mathcal{G}$ and $y\mathcal{G}x$ respectively (see [36], for example).
Definition 2.11. *Let $\mathcal{G}$ be a groupoid. For every $x \in \mathcal{G}^{(0)}$, we define the **isotropy** of x as $\mathcal{G}_x^x$. We define the **isotropy** of $\mathcal{G}$ to be the set $Iso(\mathcal{G}) := \bigcup_{x \in \mathcal{G}^{(0)}} \mathcal{G}_x^x$.* □

The isotropy of a groupoid can be seen as the set of all elements $g \in \mathcal{G}$ such that $r(g) = s(g)$ or, in category language, the set of all endomorphisms of $\mathcal{G}$. It is clear that $\mathcal{G}^{(0)} \subseteq Iso(\mathcal{G})$.
Proposition 2.12. *For every $x \in \mathcal{G}^{(0)}$, the isotropy of x is a group with unit x. Moreover, $Iso(\mathcal{G})$ is a subgroupoid of $\mathcal{G}$.*

Proof. The first statement is straightforward to check: $\mathcal{G}_x^x$ is clearly a subgroupoid of $\mathcal{G}$, and its unit space contains a single element. Therefore, it is a group.
Then the second statement is immediate. Indeed, $Iso(\mathcal{G})$ is a group bundle as in Example 2.6. Moreover, the unit space of $Iso(\mathcal{G})$ is $\mathcal{G}^{(0)}$. □

Definition 2.13. *For every $x \in \mathcal{G}^{(0)}$, we define the* ***orbit*** *of x as $\mathcal{G}(x) := r(\mathcal{G}_x) = r(s^{-1}(x))$.* □

Notice that, since every element has a unique inverse, the definition is equivalent to $\mathcal{G}(x) = s(\mathcal{G}^x) = s(r^{-1}(x))$.
Definition 2.14. *Let $\mathcal{G}$ be a groupoid. A subset $F \subseteq \mathcal{G}^{(0)}$ is said to be $\mathcal{G}$-full if, for every $x \in \mathcal{G}^{(0)}$, $r^{-1}(x) \cap s^{-1}(F) \neq \emptyset$.* □

Saying that F is $\mathcal{G}$-full means, essentially, that F is connected with every unit of $\mathcal{G}^{(0)}$. Again, the reader can check that the definition does not change if we switch $r^{-1}(x) \cap s^{-1}(F)$ for $s^{-1}(x) \cap r^{-1}(F)$.
From now on, we will refer to $\mathcal{G}$-full subsets simply as *full*. At least, we will do that whenever the groupoid they belong to is clear.
The following definition is often denoted as *restriction subgroupoid*.
Definition 2.15. *Let F be a subset of $\mathcal{G}^{(0)}$. The* ***reduction*** *(or restriction) of $\mathcal{G}$ to F, denoted as $\mathcal{G}|_F$, is given by:*

$$\mathcal{G}|_F := r^{-1}(F) \cap s^{-1}(F).$$

□

It is clear that $\mathcal{G}|_F$ is a subgroupoid of $\mathcal{G}$. Indeed, it is the biggest subgroupoid of $\mathcal{G}$ with F as its unit space.
We now introduce a well known family of groupoids. In some texts, this definition is replaced with another (equivalent) one. We use here the one we think is more intuitive.
Definition 2.16. *We say that a groupoid $\mathcal{G}$ is* ***principal*** *if and only if $Iso(\mathcal{G}) = \mathcal{G}^{(0)}$.* □

As we noted before, there are few other equivalent definitions for principality. The most relevant one is the following:
Lemma 2.17. *A groupoid $\mathcal{G}$ is principal if and only if it is algebraically isomorphic to the equivalence relation $R(\mathcal{G})$ under the map $g \mapsto (r(g), s(g))$.*

Proof. Suppose that $\mathcal{G}$ is principal. The map $g \mapsto (r(g), s(g))$ is always a surjective homomorphism, so we just need to check its injectivity. Suppose that $(r(g), s(g)) = (r(h), s(h))$ for some $g, h \in \mathcal{G}$. Then $hg^{-1} \in Iso(\mathcal{G}) = \mathcal{G}^{(0)}$. Since the inverse is unique, we deduce that $g = h$.

For the other implication, suppose that $g \mapsto (r(g), s(g))$ is an isomorphism. Let $g \in Iso(\mathcal{G})$, and denote $x = r(g) = s(g) \in G^{(0)}$. Then $(r(g), s(g)) = (x, x) = (r(x), s(x))$. Since the map is an isomorphism, we deduce $g = x$, and thus $Iso(\mathcal{G}) = \mathcal{G}^{(0)}$, as desired. $\square$

We can easily study the principality for the examples we introduced in the previous lines.

Lemma 2.18. *The following statements hold:*

- *A group is principal if and only if it is trivial.*
- *$\mathcal{G} \times \mathcal{H}$ is principal if an only if both $\mathcal{G}$, $\mathcal{H}$ are.*
- *A group bundle groupoid is principal if and only if G_x is trivial, for every $x \in X$. Hence, a principal group bundle is just a set.*
- *A transformation groupoid is principal if and only if $g \cdot x \neq x$, for all $x \in X$ and all $g \in G$, that is, if and only if the action is free.*

$\square$

2.1.2 Topological groupoids.

We aim to build some C^*-algebra structures associated to groupoids. In this line, the first step is to associate some kind of topology to a groupoid, meeting certain standards.

Definition 2.19. *We say that a groupoid $\mathcal{G}$ is a* ***topological groupoid*** *whenever $\mathcal{G}$ is endowed with a topology, with the following conditions:*

- *The topology is locally compact.*
- *$\mathcal{G}^{(0)}$ is Hausdorff in the relative topology.*
- *The maps r, s, and $g \mapsto g^{-1}$ are all continuous.*
- *The map $(g, h) \mapsto gh$ from $\mathcal{G}^{(2)}$ onto $\mathcal{G}$ is continuous in the relative topology on $\mathcal{G}^{(2)}$.*

$\square$

In particular, when considering Hausdorff groupoids, the following property appears.

Lemma 2.20. *Let $\mathcal{G}$ be a topological groupoid. Then $\mathcal{G}^{(0)}$ is closed in $\mathcal{G}$ if and only if $\mathcal{G}$ is Hausdorff.*

Proof. Suppose $\mathcal{G}^{(0)}$ is not closed. Then there exists some $\{x_i\}_{i\in I} \subseteq \mathcal{G}^{(0)}$ converging to some $g \in \mathcal{G} \setminus \mathcal{G}^{(0)}$. However, r is continuous, and hence $\{r(x_i)\}_{i\in I} = \{x_i\}_{i\in I}$ converges to $r(g) \neq g$. Therefore, $\mathcal{G}$ is not Hausdorff.

Now suppose that $\mathcal{G}^{(0)}$ is closed. Let $\{g_i\}_{i\in I}$ converging to both α and β. By continuity, we have that $\{s(g_i)\} = \{g_i^{-1}g_i\}$ converges to $\alpha^{-1}\beta$. Since $\mathcal{G}^{(0)}$ is closed, and $s(g_i) \in \mathcal{G}^{(0)}$, we deduce that $\alpha^{-1}\beta \in \mathcal{G}^{(0)}$, and therefore $\alpha = \beta$. Hence, $\mathcal{G}$ is Hausdorff. □

We can easily add a topology to the groupoids listed previously. For example:

Example 2.21. *Every group is a topological groupoid with the discrete topology.*

Example 2.22. *If $\mathcal{G}$ and $\mathcal{H}$ are topological groupoids, then $\mathcal{G} \times \mathcal{H}$ is a topological groupoid with the product topology.* □

Example 2.23. *Let X be a locally compact Hausdorff space, and R an equivalence relation on X. Then R is a topological groupoid in the relative topology of $X \times X$.* □

Example 2.24. *Let X be a locally compact Hausdorff space, and G a locally compact group acting on X by homeomorphisms. Then the associated transformation groupoid is a topological groupoid in the product topology.* □

In most cases, when referring to topological groupoids, we will just name them as groupoids, since it will be clear that they are endowed with a topology.

We now show a couple of more sophisticated groupoids. More precisely, we introduce two of the families of groupoids that will be studied in the forthcoming chapters.

Example 2.25. *(Deaconu-Renault groupoids). Let X be a locally compact Hausdorff space, and let $\sigma := (\sigma_1, ..., \sigma_k)$ be an action of $\mathbb{N}^k$ on X by surjective local homeomorphisms. For $p = (p_1, ..., p_k) \in \mathbb{N}^k$, denote $\sigma^p := \sigma_1^{p_1} ... \sigma_k^{p_k}$.*

*The associated **Deaconu-Renault groupoid** $\mathcal{G}(X, \sigma)$ was defined in [16] as*

$$\mathcal{G}(X, \sigma) := \{(x, p - q, y) \in X \times \mathbb{Z}^k \times X : \sigma^p(x) = \sigma^q(y)\},$$

together with operations $r(x, n, y) = (x, 0, x)$, $s(x, n, y) = (y, 0, y)$, and $(x, n, y)(y, m, z) = (x, m + n, z)$. The set $\mathcal{G}(X, \sigma)^{(0)}$ is usually identified with X via $(x, 0, x) \mapsto x$.

The topology is given by the basis of sets $Z(U, p, q, V) := (U \times \{p - q\} \times V) \cap \mathcal{G}(X, \sigma)$, where U, V are open sets of X such that $\sigma^p(U) = \sigma^q(V)$. □

Remark 2.26. *For a Deaconu-Renault groupoid to be principal, one needs to ensure that, for every $x \in X$, there are no integers $p, q \in \mathbb{N}^k$, with $p \neq q$, such that $\sigma^p(x) = \sigma^q(x)$. In that (unusual) case, $\mathcal{G}(X, \sigma)$ is principal.* □

Remark 2.27. *Deaconu-Renault groupoids, as defined initially in [16], are a more general family of groupoids, since they do not impose that many topological conditions. However, whenever they are studied along with Matui's HK conjecture (which we will introduce later), they appear as defined above, in order to meet the conjecture's hypothesis.* □

Remark 2.28. *Let X be a locally compact Hausdorff space, and suppose that $\sigma := (\sigma_1, ..., \sigma_k)$ is an action of $\mathbb{Z}^k$ on X* ***by homeomorphisms****. Then the elements of the associated Deaconu-Renault groupoid are of the form*

$$(x, p-q, y) \in X \times \mathbb{Z}^k \times X,$$

where $\sigma^p(x) = \sigma^q(y)$. Since σ is an action by homeomorphisms, we can write this equality as $x = \sigma^{q-p}(y)$, and therefore $(x, p-q, y) = (\sigma^{q-p}(y), p-q, y)$. Then, straightforward computation shows that there is an isomorphism between the Deaconu-Renault groupoid $\mathcal{G}(X, \sigma)$ associated to the action ***by homeomorphisms*** *and the associated transformation groupoid (Definition 2.7), given by*

$$(\sigma^{q-p}(y), p-q, y) \mapsto (q-p, y) \in X \rtimes \mathbb{Z}^k.$$

□

We now show another example, used to encode the dynamics of inverse semigroup actions.

Example 2.29. *(Groupoids of germs).*

Let X be a locally compact Hausdorff space, and let G be an inverse semigroup of homeomorphisms between open subsets of X. Given $g \in G$, and whenever x is in the domain of g, we define the **germ** *of (g, x) as the equivalence class $[g, x]$, where :*
$[g, x] = [g', x'] \Leftrightarrow x = x'$ and g coincides with g' in a neighborhood of x. We define the ***groupoid of germs*** *$\mathcal{G}$ as the set of all germs, with structure:*

$$\begin{gathered}\mathcal{G}^{(0)} = \{[e, x] : x \in X\} \\ s([g, x]) = [e, x] \\ r([g, x]) = [e, g(x)] \\ [g_1, g_2(x)][g_2, x] = [g_1 g_2, x], \text{ and} \\ [g, x]^{-1} = [g^{-1}, g(x)].\end{gathered}$$

The topology of $\mathcal{G}$ is given by the basis

$$\mathcal{U}_{U,g} := \{[g, x] : x \in U\},$$

where $g \in G$ and U is an open subset of the domain of g. □

Once a groupoid is endowed with a certain topology, we can look for a certain set of *desirable* properties.

Definition 2.30. *A topological groupoid $\mathcal{G}$ is* ***minimal*** *if, for every $x \in \mathcal{G}^{(0)}$, the orbit of x is dense in $\mathcal{G}^{(0)}$.* □

Given a groupoid $\mathcal{G}$, one of the most natural conditions to expect is for $\mathcal{G}^{(0)}$ to be open. This, however, does not always happens. To meet that condition, we now introduce the family of groupoids we focus our work on. In some sense, this family is the groupoid equivalent to discrete groups.

Definition 2.31. *(Étale groupoids) A topological groupoid is said to be* ***étale*** *if the source map $s : \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism. An étale groupoid is said to be* ***ample*** *if its unit space is zero dimensional.* □

Before advancing forward, we give two remarks. More precisely, one remark and one warning:

- The definition is equivalent if instead of the source map, we use the range map. This one is clear.
- Early documents asked for the source (or range) map to be a local homeomorphism as a map from $\mathcal{G}$ to $\mathcal{G}$, not from $\mathcal{G}$ to $\mathcal{G}^{(0)}$, as a way to force $\mathcal{G}^{(0)}$ to be open in $\mathcal{G}$. Indeed, $\mathcal{G}^{(0)}$ can then be put as $\mathcal{G}^{(0)} = \bigcup_{g\in\mathcal{G}} s(U_g)$, where, for every g, U_g is an open neighbourhood of g such that $s : U_g \to r(U_g)$ is a homeomorphism. This condition is not needed, as we prove in the following Lemma.

Lemma 2.32. *[59, Lemma 8.4.2]. If $\mathcal{G}$ is étale (as in Definition 2.31), then $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$.*

Proof. Suppose that $\mathcal{G}^{(0)}$ is not open. Then there is a sequence $\{\gamma_n\}_n$ in $\mathcal{G} \setminus \mathcal{G}^{(0)}$ such that $\gamma_n \to x \in \mathcal{G}^{(0)}$. Continuity of s implies that $s(\gamma_n) \to s(x) = x$. Now take an open neighbourhood U of x in $\mathcal{G}$. There exists some n_0 such that $\gamma_{n_0}, s(\gamma_{n_0}) \in U$. Since $\gamma_{n_0} \in \mathcal{G} \setminus \mathcal{G}^{(0)}$, it is clear that $s(\gamma_{n_0}) \neq \gamma_{n_0}$. But $s(s(\gamma_{n_0})) = s(\gamma_{n_0})$, and therefore s is not locally injective. Thus, s is not a local homeomorphism, concluding the proof. □

From now on, we will ask any groupoid homomorphism between étale groupoids to be continuous. Some texts emphasize this by denoting continuous groupoid homomorphisms as **étale homomorphisms**.

Under the proper conditions, all the groupoids listed before are étale.

Lemma 2.33. *The following statements hold:*

- *Every discrete group is an étale groupoid.*
- *The product of two étale groupoids is étale.*
- *If $\mathcal{G}$ is an étale groupoid, and F is a subset of $\mathcal{G}^{(0)}$, then $\mathcal{G}|_F$, endowed with the induced topology, is an étale groupoid.*
- *A transformation groupoid is étale if and only if the acting group G is discrete.*
- *Deaconu-Renault groupoids, as defined in Example 2.25, are étale groupoids [59, Example 8.4.6].*

□

Example 2.34. *Let $\mathcal{G}$ be a Hausdorff étale groupoid, and let Y be a locally compact Hausdorff space. Supppose that there exists a local homeomorphism $\psi : Y \to \mathcal{G}^{(0)}$. Then the associated* ***ampliation*** *groupoid ([66, §3.3]) $\mathcal{G}^{\psi}$ is defined as*

$$\mathcal{G}^{\psi} := \{(x, \gamma, y) \in Y \times \mathcal{G} \times Y : \psi(x) = r(\gamma) \text{ and } \psi(y) = s(\gamma)\},$$

together with structure:

$$r(x, \gamma, y) = (x, \psi(x), x),\ s(x, \gamma, y) = (y, \psi(y), y),$$
$$(x, \gamma, y)(y, \mu, z) = (x, \gamma\mu, z), \text{ and}$$
$$(x, \gamma, y)^{-1} = (y, \gamma^{-1}, x).$$

This is a Hausdorff étale groupoid under the topology inherited from $Y \times \mathcal{G} \times Y$ (see [66]). The unit space $(\mathcal{G}^{\psi})^{(0)} = \{(y, \psi(y), y) \in Y \times \mathcal{G}^{(0)} \times Y\}$ is usually identified with Y via $(y, \psi(y), y) \mapsto y$. □

One of the most important properties of étale groupoids is that they always have a base consisting of *open bisections*. For that sentence to make any sense, we need, of course, to define the notion of *bisection*.

Definition 2.35. *([59, Definition 8.4.8]) A subset $B \subseteq \mathcal{G}$ of a topological groupoid is called a* ***bisection*** *(of $\mathcal{G}$-set) if there is an open set $U \supseteq B$ such that both $r : U \to r(U)$ and $s : U \to s(U)$ are homeomophisms in $\mathcal{G}$. A bisection U is called* ***full*** *if $r(U) = s(U) = \mathcal{G}^{(0)}$.* □

Then:

Lemma 2.36. *([59, Lemma 8.4.9]) Let $\mathcal{G}$ be an étale groupoid. Then $\mathcal{G}$ has a countable base of open bisections.*

Proof. Take a countable dense subset $\{g_n\}$ of $\mathcal{G}$. For each g_n, find countable neighbourhood bases $\{U_{n,i}\}_i$, $\{V_{n,i}\}_i$ such that $r : U_{n,i} \to r(U_{n,i})$ and $s : V_{n,i} \to s(V_{n,i})$ are all homeomorphisms. Then the family $\{U_{n,i} \cap V_{n,i} : n, i \in \mathbb{N}\}$ is a countable base of open bisections. □

Corollary 2.37. *Let $\mathcal{G}$ be an étale groupoid. Then, for each $x \in \mathcal{G}^{(0)}$, $\mathcal{G}_x$, $\mathcal{G}^x$ and $\mathcal{G}_x^x$ are all discrete in the relative topology.*

Proof. Take $g \in G_x$, and choose U_g open bisection containing it. Then it is clear that $U_g \cap \mathcal{G}_x = \{g\}$. Hence, $\{g\}$ is open in the relative topology of $\mathcal{G}_x$. The same argument applies to $\mathcal{G}^x$. For $\mathcal{G}_x^x$, it is then immediate by definition. □

As noted before, étale groupoids are a family of *well behaved* groupoids, granting some desirable properties. The following one appears, for example, in [59, Lemma 8.4.11].

Lemma 2.38. *If $\mathcal{G}$ is a topological groupoid with r an open map, then the multiplication map is open. In particular, every étale groupoid has open multiplication.*

Proof. Let $U, V \subseteq \mathcal{G}$ open sets, and let $(\alpha, \beta) \in (U \times V) \cap \mathcal{G}^{(2)}$. Let $\{g_i\}$ be a sequence converging to $g = \alpha\beta$. We need to prove that g_i eventually belong to UV. Let us show it:

Let $\{U_j\}_{j \in \mathbb{N}}$ be a decreasing neighbourhood base for α contained in U. By hypothesis, r is an open map, and thus $r(U_j)$ is an open neighbourhood of $r(\alpha)$, for every j. Using the continuity of the maps, $\{g_i\}$ converging to $\alpha\beta$ implies that $\{r(g_i)\}$ converges to $r(\alpha\beta) = r(\alpha)$, so for every j we eventually obtain $r(g_i) \in r(U_j)$. Now choose $\{\alpha_i\} \subset U$ with $r(\alpha_i) = r(g_i)$, and $\alpha_i \in U_j$ whenever $r(g_i) \in r(U_j)$. Then $\{\alpha_i\}$ converges to α. Therefore, $\{\alpha_i^{-1} g_i\}$ converges to β. In particular, $\alpha_i^{-1} g_i$ eventually belongs to V, and then $g_i = \alpha_i(\alpha_i^{-1} g_i) \in UV$ for large i. □

From now on, **all the groupoids appearing in this work will be second countable Hausdorff groupoids**. Many interesting families of groupoids exist outside these bounds but, at some point, the need to delimit the object of study appears.

Remark 2.39. *In the examples of Lemma 2.33, to meet the conditions of being second countable and Hausdorff we need the acting group to be countable and X to be second countable (in the case of transformation groupoids), and X to be second countable (in the case of Deaconu-Renault groupoids).* □

We define now two families of étale groupoid that play a key role in our study: *elementary* and AF groupoids. We will see later that, indeed, those are the groupoid versions of finite dimensional and AF C^*-algebras. The original definition forces their unit spaces to be compact. However, there is a more recent version developed in [21] that simply requires them to be locally compact. We provide both of them here.

Definition 2.40. *Let $\mathcal{G}$ be a second countable étale groupoid, and let $\mathcal{G}^{(0)}$ be compact and totally disconnected. We say that $\mathcal{K} \subseteq \mathcal{G}$ is a* ***compact elementary*** *groupoid if $\mathcal{K}$ is a compact, open and principal subgroupoid, such that $\mathcal{K}^{(0)} = \mathcal{G}^{(0)}$.* □

Definition 2.41. *([21, Definition 4.9]) Let X, Y be locally compact Hausdorff spaces, and let $\psi : Y \to X$ be a local homeomorphism. Then, considering X as a trivial groupoid, we can build the associated ampliation groupoid as in Example 2.34:*

$$R(\psi) := X^{\psi} = \{(y_1, y_2) \in Y \times Y : \psi(y_1) = \psi(y_2)\}.$$

*An ample groupoid is said to be **elementary** if it is isomorphic to the groupoid $R(\psi)$ for some local homeomorphism between two 0-dimensional spaces.* □

Straightforward computation shows that both definitions agree whenever the unit space is compact.

Definition 2.42. *Let $\mathcal{G}$ be a second countable étale groupoid. We say that $\mathcal{G}$ is an **AF** groupoid (with compact unit space) if it can be written as an increasing union of (compact) elementary subgroupoids.* □

Any AF groupoid is, by definition, principal, and therefore algebraically isomorphic to an equivalence relation. The topology, however, may not coincide.

We give now a few technical definitions, that we will require later.

Definition 2.43. *We say that a groupoid $\mathcal{G}$ is **effective** whenever the interior of $Iso(\mathcal{G})$ equals to $\mathcal{G}^{(0)}$.* □

Some disparities appear over the nomenclature of this notion. For example, some texts (see [36]), name this family of groupoids as *essentially principal* (or topologically principal) groupoids. Moreover, some other texts define essentially principal groupoids as the family of groupoids such that the set $\{x \in \mathcal{G}^{(0)} : \mathcal{G}_x^x = \{x\}\}$ is dense in $\mathcal{G}^{(0)}$ (see [8, Definition 2.1]). This condition is equivalent to the one stated above whenever $\mathcal{G}$ is a second-countable étale groupoid ([59, Lemma 10.2.3]), which is probably where the disparities arose from in the first place.

Definition 2.44. *([38, Definition 4.1]) Let $\mathcal{G}$ be an effective étale groupoid whose unit space is a Cantor set.*

1. *A clopen set $F \subset \mathcal{G}^{(0)}$ is said to be **properly infinite** if there exist compact open bisections $U, V \subseteq \mathcal{G}$ such that $s(U) = s(V) = F$, $r(U) \cup r(V) \subseteq F$, and $r(U) \cap r(V) = \emptyset$.*
2. *We say that $\mathcal{G}$ is **purely infinite** if every clopen set $F \subset \mathcal{G}^{(0)}$ is properly infinite.*

□

Purely infinite groupoids has been recently studied in-depth in [33].

Finally, we introduce the notion of *topological amenability*. There are a few equivalent definitions for this. The most standard one states that a groupoid is topologically amenable if it admits an approximate invariant continuous mean. We will not be treating with means in this document. Hence, we believe the following definition, given by [1, Proposition 2.2.13], is more adequate for this work:

Definition 2.45. *Let $\mathcal{G}$ be an étale groupoid. We say that $\mathcal{G}$ is amenable whenever there is a sequence $(h_i)_i \subset C_c(\mathcal{G})$ such that:*

- *The maps* $m_i : \mathcal{G}^{(0)} \to \mathbb{R}$ *given by* $m_i(x) = \sum_{g \in \mathcal{G}^x} |h_i(g)|^2$ *converge uniformly to 1 on every compact subset of* $\mathcal{G}^{(0)}$.
- *The maps* $n_i : \mathcal{G} \to \mathbb{R}$ *given by* $n_i(\gamma) = \sum_{g \in \mathcal{G}^{r(\gamma)}} |h_i(\gamma^{-1}g) - h_i(g)|$ *converge uniformly to 0 on every compact subset of* $\mathcal{G}$.

□

It is now time to introduce some of the most relevant invariants of groupoids. More precisely, we will focus our research in two objects:
On one hand, we will study groupoid homology, as defined by Crainic and Moerdijk in [14].
On the other hand, we will build two C^*-algebras associated to an étale groupoid, and then study its K-theory. This one requires a little bit more setup, so we will leave it for later.

2.2 The homology groups of a groupoid

Here we define the homology groups associated to a groupoid $\mathcal{G}$. This definition was first introduced by Crainic and Moerdijk in [14]. The main idea is to build a chain complex associated to $\mathcal{G}$, capturing its structure, and then defining the homology of $\mathcal{G}$ as the homology of the chain complex. Let us show it.

Definition 2.46. *Let A be a topological abelian group, X, Y be locally compact Hausdorff spaces, and let $C_c(X, A)$ be the set of continuous functions with compact support taking values in A. $C_c(X, A)$, together with pointwise addition, is trivially an abelian group. Let $\pi : X \to Y$ be a local homeomorphism. Then, for every $f \in C_c(X, A)$, we define the map $\pi_*(f) : Y \to A$ as:*

$$\pi_*(f)(y) := \sum_{\pi(x)=y} f(x)$$

□

Remark 2.47. *The reader may easily check that, indeed, π_* is a homomorphism between $C_c(X, A)$ and $C_c(Y, A)$.* □

We can now use this to build a certain chain complex associated to an étale groupoid, in the following way.

Let $\mathcal{G}$ be an étale groupoid, and for every $n \in \mathbb{N}$, let $\mathcal{G}^{(n)} \subseteq \mathcal{G}^n$ be the set of composable n-strings in $\mathcal{G}$, i.e.

$$\mathcal{G}^{(n)} := \{(g_1, g_2, ..., g_n) \in \mathcal{G}^n : s(g_i) = r(g_{i+1}), i = 1, ..., n-1\}.$$

For every $i = 0, 1, ..., n$, define the map $d_i : \mathcal{G}^{(n)} \to \mathcal{G}^{(n-1)}$ as:

$$d_i(g_1, ..., g_n) = \begin{cases} (g_2, g_3, ..., g_n) & i = 0 \\ (g_1, ..., g_i g_{i+1}, ..., g_n) & 1 \leq i \leq n-1 \\ (g_1, g_2, ..., g_{n-1}) & i = n \end{cases}$$

In the extreme case $n = 1$, we let $d_0, d_1 : \mathcal{G}^{(1)} \to \mathcal{G}^{(0)}$ be the source and the range maps, respectively.

Using the previous result, we can now define the maps $\delta_n : C_c(\mathcal{G}^{(n)}, A) \to C_c(\mathcal{G}^{(n-1)}, A)$ by:

$$\delta_n := \sum_{i=0}^{n} (-1)^i d_{i*}.$$

The reader can check that, indeed, $(C_c(\mathcal{G}^{(n)}, A), \delta_n)$ is a chain complex.

Definition 2.48. *Let $\mathcal{G}$ be an étale groupoid. We define the* ***homology groups*** *of $\mathcal{G}$ with coefficients in A to be the homology of the chain complex defined above, that is*

$$H_n(\mathcal{G}, A) := ker\delta_n / Im\delta_{n+1}.$$

Whenever $A = \mathbb{Z}$, we just write $H_n(\mathcal{G})$, and simply denote them as the homology groups of $\mathcal{G}$. □

There is an equivalent of the Kunneth formula for groupoids [38, Theorem 2.4]:

Lemma 2.49. *(Kunneth theorem for groupoid homology). Let $\mathcal{G}, \mathcal{H}$ be étale groupoids. For every $n \geq 0$, there exists a natural short exact sequence*

$$0 \to \bigoplus_{i+j=n} H_i(\mathcal{G}) \otimes H_j(\mathcal{H}) \to H_n(\mathcal{G} \times \mathcal{H}) \to \bigoplus_{i+j=n-1} Tor(H_i(\mathcal{G}), H_j(\mathcal{H})) \to 0$$

Moreover, the sequence split, but not canonically. □

There is also a long exact sequence (introduced in [14, Section 3.6]) induced by any short exact sequence of abelian groups

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

given by:

$$... \to H_{n+1}(\mathcal{G}, A_3) \to H_n(\mathcal{G}, A_1) \to H_n(\mathcal{G}, A_2) \to H_n(\mathcal{G}, A_3) \to ...$$

Finally, groupoid homology posess a really useful *functoriality* property. Let $\mathcal{G}, \mathcal{H}$ be ample Hausdorff groupoids, and let $\phi : \mathcal{G} \to \mathcal{H}$ be a groupoid homomorphism. Then the maps $\phi_*^{(n)} : C_c(\mathcal{G}^{(n)}, \mathbb{Z}) \to C_c(\mathcal{H}^{(n)}, \mathbb{Z})$ induce homomorphisms on homology $\phi_* : H_n(\mathcal{G}) \to H_n(\mathcal{H})$ (see [14, 3.7.2]). The induced maps are natural with respect to the composition, that is, if we have $\mathcal{G}_1 \xrightarrow{\phi_1} \mathcal{G}_2 \xrightarrow{\phi_2} \mathcal{G}_3$ homomorphisms between ample Hausdorff groupoids, then $(\phi_2 \circ \phi_1)_* = (\phi_2)_* \circ (\phi_1)_*$. This allows us to introduce the following result, appearing in [21, Proposition 4.7].

Proposition 2.50. *Let $\mathcal{G}$ be an ample Hausdorff groupoid, and let $\{\mathcal{G}_i\}$ be an increasing sequence of open subgroupoids of $\mathcal{G}$, such that $\bigcup_{i=1}^{\infty} \mathcal{G}_i = \mathcal{G}$. Then $H_*(\mathcal{G}) \cong \varinjlim(H_*(\mathcal{G}_i), \iota_*)$.*

Proof. Since $\mathcal{G}_i$ are open subgroupoids of $\mathcal{G}$, all of them are ample and Hausdorff. Then, each inclusion map $\iota_{i,\infty} : \mathcal{G}_i \to \mathcal{G}$ induces homomorphisms

$$(\iota_{i,\infty})_* : H_*(\mathcal{G}_i) \to H_*(\mathcal{G}).$$

Hence, the universal property of the direct limit provides homomorphisms

$$\theta_* : \varinjlim(H_*(\mathcal{G}_i), \iota_*) \to H_*(\mathcal{G}).$$

The maps θ_* are all surjective. Indeed, take an open compact subset $U \subseteq \mathcal{G}^{(n)}$. Then there exists some i such that $U \subseteq \mathcal{G}_i$, and hence $[1_U]_{H_n(\mathcal{G})} = (\iota_{i,\infty})_n [1_U]_{H_n(\mathcal{G}_i)}$.

On the other hand, the maps θ_* are all injective. Take some $a \in \varinjlim(H_*(\mathcal{G}_i), \iota_*)$, and suppose $\theta_n(a) = 0$. There exist some $i \in \mathbb{N}$ and some $f \in C_c(\mathcal{G}_i^{(n)}, \mathbb{Z})$ such that $\theta_n(a) = [\iota_{i,\infty}(f)]$. But, since $[f] = 0$ in $H_n(\mathcal{G}) = ker(\delta_n)/Im(\delta_{n+1})$, there exist some $j \in \mathbb{N}$ and some $g \in C_c(\mathcal{G}_j^{(n+1)}, \mathbb{Z})$ such that $\delta_{n+1}(g) = f$, and hence $[f] = 0$ in $H_n(\mathcal{G}_{max\{i,j\}})$. Hence, $a = 0$, and θ_n is an isomorphism, for every $n \in \mathbb{N}$.

$\square$

2.3 The C^*-algebras associated to a groupoid

As with groups, we can associate two C^*-algebras to any étale groupoid: the full C^*-algebra and the reduced one. Let us build both.

Lemma 2.51. *Let $\mathcal{G}$ be an étale groupoid, and let $\gamma \in \mathcal{G}$. Then the set $\{(\alpha, \beta) \in \mathcal{G}^{(2)} : \alpha\beta = \gamma\}$ is discrete in the topology of $\mathcal{G}$.*

Proof. The proof is standard. If $\alpha\beta = \gamma$, then $\alpha \in \mathcal{G}^{r(\gamma)}$ and $\beta \in \mathcal{G}_{s(\gamma)}$. Since $\mathcal{G}$ is étale, both $\mathcal{G}^{r(\gamma)}$ and $\mathcal{G}_{s(\gamma)}$ are discrete (see Lemma 2.37). Hence $\{(\alpha, \beta) \in \mathcal{G}^{(2)} : \alpha\beta = \gamma\} \subseteq \mathcal{G}^{r(\gamma)} \times \mathcal{G}_{s(\gamma)}$ is also discrete. □

Corollary 2.52. *Let $\mathcal{G}$ be an étale groupoid, and let $C_c(\mathcal{G})$ be the set of continuous functions over $\mathcal{G}$ with compact support (and taking values in $\mathbb{C}$). Then, for any given $\gamma \in \mathcal{G}$, and $f, g \in C_c(\mathcal{G})$, the set $\{(\alpha, \beta) \in \mathcal{G}^{(2)} : \alpha\beta = \gamma,\ and\ f(\alpha)g(\beta) \neq 0\}$ is finite.*

Proof. Using the previous lemma, $\{(\alpha, \beta) \in \mathcal{G}^{(2)} : \alpha\beta = \gamma,\ and\ f(\alpha)g(\beta) \neq 0\}$ is the intersection of a discrete set, with $supp(f)$ and $supp(g)$, both compact. Thus, it is finite. □

This result allows us to define the *convolution C^*-algebra* of an étale groupoid $\mathcal{G}$.

Definition 2.53. *Let $\mathcal{G}$ be an étale groupoid. We define the* ***convolution algebra*** *of $\mathcal{G}$ as the space $C_c(\mathcal{G})$, together with the usual vector structure, and operations given by:*

- $(f * g)(\gamma) := \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$.
- $f^*(\gamma) := \overline{f(\gamma^{-1})}$.

□

See that the *étaleness* provides a key property that allows to define the convolution product as a finite sum. There is a more general definition, covering non-étale groupoids, in which the finite sum is replaced with integrals (see [52]). All the groupoids appearing in this document will be étale, so we can stick to this definition.

As we noted before, the topology of an étale groupoid has a base of open bisections. In this line, we can also build the convolution algebra using this type of sets [59, Lemma 9.1.3]:

Lemma 2.54. *Let $\mathcal{G}$ be an étale groupoid. Then we have*

$$C_c(\mathcal{G}) = span\{f \in C_c(\mathcal{G}) : supp(f) \text{ is a bisection}\}.$$

Proof. Let $f \in C_c(\mathcal{G})$. Recall that $\mathcal{G}$ has a base of open bisections. Hence, since $supp(f)$ is compact, we can find a finite subcover $U_1, ..., U_n$ of $supp(f)$ consisting in open bisections.

Take a partition of unity $\{h_i\}$ on $\bigcup U_i$ subordinate to the U_i. Then $f_i := f \cdot h_i \in C_c(\mathcal{G})$ verifies $supp(f_i) \subseteq U_i$, for every $i = 1, ..., n$, and $f = \sum_{i=1}^{n} f_i$, as desired. □

Immediate consequences of this lemma arise:

Corollary 2.55. *Let $\mathcal{G}$ be and étale groupoid, $f, g \in C_c(\mathcal{G})$, and U, V open bisections such that $supp(f) \subseteq U$ and $supp(g) \subseteq V$. The the following statements are true:*

- $supp(f * g) \subseteq UV$.
- $(f * g)(\gamma) = f(\alpha)g(\beta)$, *for* $\gamma = \alpha\beta \in UV$.
- $supp(f^* * f) = s(supp(f))$.

Proof. Let $\gamma \in supp(f * g)$. Then $0 \neq (f * g)(\gamma) := \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$. We deduce that $s(\beta) = s(\gamma)$ and $r(\alpha) = r(\gamma)$. Recall that both f, g are supported on bisections, and thus the only non zero element of the sum must correspond to $\alpha\beta \in UV$. This proves the first two statements.

The third statement is then immediate. □

2.3.1 The full and reduced C^*-algebras

There are a few equivalent definitions for the full C^*-algebra of an étale groupoid. Once again, we show here the one we think is more intuitive. In that line, we will provide a certain C^*-norm, that will allow us to define the universal C^*-algebra of an étale groupoid. From now on, $C_c(\mathcal{G})$ will denote the convolution algebra of $\mathcal{G}$. The following description appears in [19, Definition 3.17]:

Proposition 2.56. *Let $\mathcal{G}$ be an étale groupoid. For each $f \in C_c(\mathcal{G})$, there is a constant $K_f \geq 0$ such that $\|\pi(f)\| \leq K_f$ for every $*$-representation $\pi : C_c(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ of $C_c(\mathcal{G})$ on Hilbert space. Moreover, if f is supported on a bisection, we can take $K_f = \|f\|_\infty$.*

Proof. Fix $f \in C_c(\mathcal{G})$. We can write $f = \sum_{i=1}^{n} f_i$, with f_i supported on a bisection, for all i, and then define $K_f := \sum_{i=1}^{n} \|f_i\|_\infty$.

Let π be a $*$-representation. Then $\pi|_{C_c(\mathcal{G}^{(0)})}$ is a $*$-representation of the commutative $*$-algebra $C_c(\mathcal{G}^{(0)})$. Therefore, $\|\pi(h)\| \leq \|h\|_\infty$, for every $h \in C_c(\mathcal{G}^{(0)})$. Since every $f_i^* * f_i$ is supported on $\mathcal{G}^{(0)}$, and $\|f_i^* * f_i\|_\infty = \|f_i\|_\infty^2$, we deduce:

$$\|\pi(f_i)\|^2 = \|\pi(f_i^* * f_i)\| \leq \|f_i^* * f_i\|_\infty = \|f_i\|_\infty^2,$$

and so each $\|\pi(f_i)\| \leq \|f_i\|_\infty^2$. The triangle inequality gives $\|\pi(f_i)\| \leq K_f$.

As we noted, if f is already supported on a bisection, then we can just take $K_f = \|f\|_\infty$. □

We can then define the universal C^*-algebra of an étale groupoid:

Theorem 2.57. *There exist a C^*-algebra $C^*(\mathcal{G})$ and a $*$-homomorphism $\pi_{max} : C_c(\mathcal{G}) \to C^*(\mathcal{G})$ such that $\pi_{max}(C_c(\mathcal{G}))$ is dense in $C^*(\mathcal{G})$, and such that for every representation $\pi : C_c(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ there is a representation ψ of $C^*(\mathcal{G})$ such that $\psi \circ \pi_{max} = \pi$. The norm on $C^*(\mathcal{G})$ satisfies*

$$\|\pi_{max}(f)\| = sup\{\|\pi(f)\| : \pi \text{ is a } *-representation \text{ of } C_c(\mathcal{G})\},$$

*for all $f \in C_c(\mathcal{G})$. The algebra $C^*(\mathcal{G})$ is denoted the **full** C^*-algebra of $\mathcal{G}$.*

Proof. See [59, Theorem 9.2.3]. □

There is an alternative, previous definition of the full C^*-algebra of a groupoid, given by Renault in [52]. This definition generalizes the one we just introduced, in the sense that it does not ask for the groupoid to be étale. It was proven in [59, Theorem 9.2.4] that, indeed, the two definitions agree whenever the groupoid is étale. Since all the groupoids involved in our work are étale, we will just stick to this simpler definition.

Example 2.58. *The full groupoid C^*-algebra of a group $\mathcal{G}$ coincides with the usual full group C^*-algebra.* □

Example 2.59. *The full C^*-algebra of an AF groupoid is an AF C^*-algebra.* □

We now introduce a second C^*-algebra associated to an étale groupoid, known as the *reduced* algebra.

Lemma 2.60. *Let $\mathcal{G}$ be an étale groupoid. For each $x \in \mathcal{G}^{(0)}$, there exists a $*$-representation $\pi_x : C_c(\mathcal{G}) \to \mathcal{B}(\ell^2(\mathcal{G}_x))$ such that*

$$\pi_x(f)\delta_\gamma = \sum_{\alpha \in \mathcal{G}_{r(\gamma)}} f(\alpha)\delta_{\alpha\gamma},$$

for each $f \in C_c(\mathcal{G})$, and each $\gamma \in \mathcal{G}_x$. Then π_x is called the regular representation of $C_c(\mathcal{G})$ associated to x.

Proof. See [59, Theorem 9.3.1]. □

Definition 2.61. *Let $\mathcal{G}$ be an étale groupoid. The **reduced** C^*-algebra $C^*_r(\mathcal{G})$ of an étale groupoid $\mathcal{G}$ is the completion of*

$$\Big(\bigoplus_{x \in \mathcal{G}^{(0)}} \pi_x\Big)(C_c(\mathcal{G})) \subseteq \bigoplus_{x \in \mathcal{G}^{(0)}} (\mathcal{B}(\ell^2(\mathcal{G}_x))).$$

□

The following one is a well known result when dealing with the two C^*-algebras of a groupoid. The first statement can be found, for example, in [59, Theorems 10.1.4-10.1.5]. A counterexample for the converse can be found in [65].

Lemma 2.62. *Let $\mathcal{G}$ be a locally compact, Hausdorff étale groupoid. Then $C_r^*(\mathcal{G})$ is nuclear if and only if $\mathcal{G}$ is amenable.*
Moreover, if $\mathcal{G}$ is amenable, then $C^*(\mathcal{G}) = C_r^*(\mathcal{G})$. However, the converse is false. □

Most of the groupoids we will be using throughout the document are amenable. Thus, the C^*-algebras will coincide.

Example 2.63. *[59, Example 9.3.8] Let $X \rtimes \Gamma$ be the transformation groupoid associated to an action of a discrete group Γ on a compact Hausdorff space X. Recall that the unit space $(X \rtimes \Gamma)^{(0)}$ can be identified with X. Let α be the induced action of Γ on $C(X)$, and let $C_c(\Gamma, C(X))$ be the convolution algebra associated to the dynamical system, as in [67, Section 1.3.2]. Then the map $\varphi : C_c(\mathcal{G}) \to C_c(\Gamma, C(X))$ given by $\varphi(f)(g)(x) := f(g, x)$ is an isomorphism. For any fixed $x \in X$, this isomorphism intertwines the regular representation π_x of $C_c(\mathcal{G})$ with the induced representation of $C_c(\Gamma, C(X))$ associated to the character of $C(X)$ given by the evaluation at x. Then, using [67, Example 2.4.2], we deduce that $C_r^*(\mathcal{G})$ is isomorphic to the reduced crossed product $C(X) \rtimes_r \Gamma$.*

□

2.3.2 K-theory

The study of K-theory of C^*-algebras has brought the interest of researchers for a few decades, making the literature on this subject quite extensive. If the reader wishes to expand its study in this area, a broad study of it can be found in [53] or [5], among others.

In this brief section, we remind the basic notions of K-theory.

Definition 2.64. *Let A be a C^*-algebra, let $\mathcal{P}_n(A)$ be the set of all projections over the matrix C^*-algebra $M_n(A)$. Denote $\mathcal{P}_\infty(A) := \bigcup_{n=1}^{\infty} \mathcal{P}_n(A)$. Given two projections $p \in \mathcal{P}_m(A)$, $q \in \mathcal{P}_n(A)$, we say that $p \sim_0 q$ if there is an element $v \in M_{m,n}(A)$ such that $p = v^*v$, and $q = vv^*$.*
One can check that, indeed, $\sim_0$ is an equivalence relation on $\mathcal{P}_\infty(A)$. We denote the equivalence class of p as $[p]_0$.
Then, the set $(\mathcal{P}_\infty(A)/\sim_0, +)$ is an abelian semigroup, with operation given by:

$$[p]_0 + [q]_0 = [p \oplus q]_0$$

□

Definition 2.65. *([53, Definition 3.1.4]) (K_0 **group of a unital C^*-algebra**).*
Let A be a unital C^*-algebra, and let $(\mathcal{D}(A), +)$ be the abelian semigroup defined above. Then we define the K_0 group of A as the Grothendiek group of $\mathcal{D}(A)$. We denote it as $K_0(A)$. □

It is a well known result that any homomorphism between C^*-algebras $\varphi : A \to B$ induces a group homomorphism between K_0 groups, given by $K_0(\varphi)[p]_0 := [\varphi(p)]_0$.

Definition 2.66. *(**K_0 group of a non-unital C^*-algebra**).*
Let A be a non-unital C^*-algebra, and denote by $\tilde{A}$ its unitization. Consider the split exact sequence

$$0 \to A \to \tilde{A} \xrightarrow{\pi} \mathbb{C} \to 0$$

*We define $K_0(A)$ to be the kernel of the homomorphism $K_0(\pi) : K_0(\tilde{A}) \to K_0(\mathbb{C})$. We denote by $\lambda : \mathbb{C} \to \tilde{A}$ the lifting of π. The **scalar mapping** is defined as*

$$s = \lambda \circ \pi : \tilde{A} \to \tilde{A}$$

□

By definition, the above short exact sequence induces a short exact sequence in K-theory. This, however, must not be expected to happen in general. Instead, short exact sequences of C^*-algebras induce six-term exact sequences in K-theory. We will show that later on.

Proposition 2.67. *([53, Proposition 4.2.2]) **(The standard picture of $K_0(A)$).***
For each C^*-algebra A, the K_0 group is given by:

$$K_0(A) = \{[p]_0 - [s(p)]_0 : p \in \mathcal{P}_\infty(\tilde{A})\}.$$

Moreover, the following conditions hold:

- *For each $p, q \in \mathcal{P}_\infty(\tilde{A})$, the following are equivalent.*
  1. *$[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$.*
  2. *There exist natural numbers k, l such that $p \oplus 1_k \sim_0 q \oplus 1_l \in \mathcal{P}_\infty(\tilde{A})$.*
  3. *There exist scalar projections r_1, r_2 such that $p \oplus r_1 \sim_0 q \oplus r_2$.*
- *Whenever $[p]_0 - [s(p)]_0 = 0$, there exists a natural number m such that*
$$p \oplus 1_m \sim_0 s(p) \oplus 1_m.$$
- *If $\varphi : A \to B$ is a $*$-homomorphism, then:*
$$K_0(\varphi)([p]_0 - [s(p)]_0) = [\tilde{\varphi}(p)]_0 - [s(\tilde{\varphi}(p))]_0,$$
  where $\tilde{\varphi} : \tilde{A} \to \tilde{B}$ is the map induced by φ.

□

On the other hand, the K_1 group is defined using the unitaries of the matrix algebra:

Definition 2.68. *Let A be a unital C^*-algebra, let $\mathcal{U}_n(A)$ be the set of unitary elements on $M_n(A)$, and define $\mathcal{U}_\infty := \bigcup_{n=1}^{\infty} \mathcal{U}_n(A)$. We define a relation $\sim_1$ on $\mathcal{U}_\infty$ in the following way:*
For $u \in \mathcal{U}_n(A), v \in \mathcal{U}_m(A)$, $u \sim_1 v$ if there exists a natural number $k \geq max\{m, n\}$ such

that there exists a continuous path in $\mathcal{U}_k(A)$ between $u \oplus 1_{k-n}$ and $v \oplus 1_{k-m}$, that is, some $f : [0,1] \to \mathcal{U}_k(A)$, such that $f(0) = u \oplus 1_{k-n}$, and $f(1) = v \oplus 1_{k-m}$. □

One can check that this is indeed an equivalence relation (see [53, Proposition 8.1.2]). Then, we can define the K_1 group:

Definition 2.69. *For each C^*-algebra A, the group $K_1(A)$ is given by:*

$$K_1(A) := \mathcal{U}_\infty(\tilde{A})/\sim_1,$$

with operation given by $[u]_1 + [v]_1 = [u \oplus v]_1$. □

As with the K_0 group, we can give an standard picture of K_1.

Proposition 2.70. *([53, Proposition 8.1.4])**(The standard picture of K_1)**. Let A be a C^*-algebra. Then:*

$$K_1(A) = \{[u]_1 : u \in \mathcal{U}_\infty(\tilde{A})\}.$$

Moreover, the map $[\cdot]_1 : \mathcal{U}_\infty(\tilde{A}) \to K_1(A)$ verifies:

- $[u \oplus v]_1 = [u]_1 + [v]_1$.
- $[1]_1 = 0$.
- *If $u, v \in \mathcal{U}_n(\tilde{A})$, then $[uv]_1 = [vu]_1 = [u]_1 + [v]_1$.*
- *For $u, v \in \mathcal{U}_\infty(\tilde{A})$, $[u]_1 = [v]_1$ if and only if $u \sim_1 v$.*

□

As we said before, there is more literature on K-theory that we could possibly put in this document. However, there are a few statements that we believe we must write down here, to serve, at least, as a reminder of their existence. The following results will probably sound familiar for the reader.

Theorem 2.71. *([53, Theorem 10.1.3]) For every C^*-algebra A, denote by SA the suspension of A, that is, $SA := \{f \in C(\mathbb{T}, A) : f(1) = 0\}$. The suspension is an endofunctor on the category of C^*-algebras. Then the groups $K_1(A)$ and $K_0(SA)$ are always isomorphic. Moreover, the isomorphism is natural in the following sense:*
Let A, B be C^*-algebras, and denote by θ_A, θ_B the respective isomorphisms. Suppose that $\varphi : A \to B$ is a *-homomorphism. Then the following diagram commutes:

$$\begin{tikzcd}
K_1(A) \arrow[r, "K_1(\varphi)"] \arrow[d, "\theta_A"'] & K_1(B) \arrow[d, "\theta_B"] \\
K_0(SA) \arrow[r, "K_0(S_\varphi)"'] & K_0(SB)
\end{tikzcd}$$

The isomorphism θ_A has the following explicit description. Let $u \in \mathcal{U}_n(\tilde{A})$, such that $s(u) = 1_n$. Let $g \in C([0,1], \mathcal{U}_{2n}(\tilde{A}))$ such that $g(0) = 1_{2n}$, $g(1) = diag(u, u^*)$, and $s(g(t)) = 1_{2n}$ for every $t \in [0,1]$. Put

$$p = g \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} g^*$$

Then p is a projection in $\mathcal{P}_{2n}(\widetilde{SA})$, $s(p) = diag(1_n, 0_n)$, and

$$\theta_A([u]_1) = [p]_0 - [s(p)]_0.$$

□

Theorem 2.72. *([53, Theorem 11.1.2]) (**Bott's periodicity**) For every C^*-algebra A, there exists an isomorphism $\beta_A : K_0(A) \to K_1(SA)$. Moreover, β_A has the following explicit description for unital C^*-algebras:*
For every $n \in \mathbb{N}$, and every $p \in \mathcal{P}_n(A)$, define $f_p : \mathbb{T} \to \mathcal{U}_n(A)$ by

$$f_p(z) = zp + (1_n - p),\ z \in \mathbb{T}.$$

*By identifying $M_n(\tilde{SA})$ with $\{f \in C(\mathbb{T}, M_n(A)) : f(1) \in M_n(\mathbb{C}1_A)\}$, we obtain that $f_p \in \mathcal{U}_n(\tilde{SA})$. Then the isomorphism is given by $\beta_A([p]_0) = [f_p]_1$, and it is called **Bott map**.*
For non-unital algebras, the Bott map is defined using the following universal property: Let A be a non-unital C^*-algebra. Then there is a unique group homomorphism $\beta_A : K_0(A) \to K_1(SA)$, making the next diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & K_0(A) \arrow[r] \arrow[d, "\beta_A"'] & K_0(\tilde{A}) \arrow[r] \arrow[d, "\beta_{\tilde{A}}"'] & K_0(\mathbb{C}) \arrow[r] \arrow[d, "\beta_{\mathbb{C}}"'] & 0 \\
0 \arrow[r] & K_1(SA) \arrow[r] & K_1(\tilde{SA}) \arrow[r] & K_1(S\mathbb{C}) \arrow[r] & 0
\end{tikzcd}$$

□

Theorem 2.73. *([53, Theorem 12.1.2])(**The six-term exact sequence**). Every short exact sequence of C^*-algebras*

$$0 \to I \to A \to B \to 0$$

induces a six-term exact sequence

$$\begin{tikzcd}
K_0(I) \arrow[r] & K_0(A) \arrow[r] & K_0(B) \arrow[d] \\
K_1(B) \arrow[u] & K_1(A) \arrow[l] & K_1(I) \arrow[l]
\end{tikzcd}$$

where the horizontal maps are induced by the ones of the short exact sequence. Moreover, the vertical homomorphisms are explicitly described in [53]. □

We conclude this section by providing a picture of the canonical map between the groups H_0 and K_0 associated to an étale groupoid, appearing in [36].

Definition 2.74. *(**Canonical map** $\Phi : H_0(\mathcal{G}) \to K_0(C_r^*(\mathcal{G}))$).*
Given an étale groupoid $\mathcal{G}$ such that $\mathcal{G}^{(0)}$ is locally compact, metrizable and totally disconnected, one can always build a canonical map between $H_0(\mathcal{G})$ and $K_0(C_r^*(\mathcal{G})$. It is deduced from the following reasoning:
By definition of $C_r^*(\mathcal{G})$, we can consider the canonical inclusion $\iota : C_c(\mathcal{G}^{(0)}) \to C_r^*(\mathcal{G})$, which induces an homomorphism in K-theory $K_0(\iota) : K_0(C_c(\mathcal{G}^{(0)})) \to K_0(C_r^*(\mathcal{G}))$. Now, since there are no non-unit elements in $\mathcal{G}^{(0)}$, a straightforward computation shows that $K_0(C_c(\mathcal{G}^{(0)})) = C_c(\mathcal{G}^{(0)}, \mathbb{Z})$. Let us take U some compact open bisection of $\mathcal{G}$, and define $u = \chi_U$. Then u is a partial isometry of $C_r^*(\mathcal{G})$ such that $uu^* = \chi_{s(U)}$, and $u^*u = \chi_{r(U)}$, which means $\chi_{s(U)}$ and $\chi_{r(U)}$ belong to the same equivalence class in $K_0(C_r^*(\mathcal{G}))$.
Then the differential map of Definition 2.48, $\delta_1 : C_c(\mathcal{G}, \mathbb{Z}) \to C_c(\mathcal{G}^{(0)}, \mathbb{Z})$ defining the homology groups verifies $(K_0(\iota) \circ \delta_1)(u) = 0$, since $\delta_1(u) = \chi_{s(U)} - \chi_{r(U)}$, by definition, from where we deduce that $Im(\delta_1) \subseteq ker(K_0(\iota))$. Since every étale groupoid has a countable basis consisting in open bisections, it follows that there exists a canonical homomorphism $\Phi : H_0(\mathcal{G}) \to K_0(C_r^*(\mathcal{G}))$ such that $\Phi([f]) := K_0(\iota)(f)$. The question about when this map is injective is open, even for some simple groupoids. In chapter 3 we study this problem for Deaconu-Renault groupoids. □

It is worth to mention that there exists a counterpart of this result providing a canonical map between $H_1(\mathcal{G})$ and $K_1(C_r^*(\mathcal{G})$, for a certain class of groupoids (see [36, Corollary 7.15]).

2.4 Matui's conjectures

In 2016, two conjectures were presented by H. Matui in [38], both involving some groupoids invariants. The first one predicts some relation between the homology groups and the K-theory of the associated C^*-algebra of a certain family of étale groupoids. The strong version of this conjecture was disproven by Scarparo in [56], using a counterexample we will show later. In later chapters, we will also provide the first complete counterexample for both strong and weak versions of the conjecture. This counterexample was obtained with the help of Eduard Ortega, appearing in [47].
Even though the main object of study in this work is this first conjecture, there is a second one relating the lower homology groups of a certain family of groupoids, with its topological full group. So far, the conjecture still holds, meaning that no counterexample has been found. We will show that conjecture in this chapter, as well as some results involving it. In chapter 5, we will show how the counterexample to the first conjecture verifies the second one.

2.4.1 Matui's HK conjecture

[38, Conjecture 2.6] Let $\mathcal{G}$ be an effective minimal étale groupoid, such that $\mathcal{G}^{(0)}$ is a Cantor space. The (strong) *HK conjecture* states that:

$$\bigoplus_{i=0}^{\infty} H_{2i}(\mathcal{G}) \cong K_0(C_r^*(\mathcal{G}))$$

and

$$\bigoplus_{i=0}^{\infty} H_{2i+1}(\mathcal{G}) \cong K_1(C_r^*(\mathcal{G})).$$

There is, however, a weakened version of HK conjecture that ignores any torsion problems. Under the same hypothesis, the *weak HK conjecture* states that:

$$\bigoplus_{i=0}^{\infty} H_{2i}(\mathcal{G}) \otimes \mathbb{Q} \cong K_0(C_r^*(\mathcal{G})) \otimes \mathbb{Q}$$

and

$$\bigoplus_{i=0}^{\infty} H_{2i+1}(\mathcal{G}) \otimes \mathbb{Q} \cong K_1(C_r^*(\mathcal{G})) \otimes \mathbb{Q}.$$

In the next chapter we will provide some of the most common tools used to compute the groupoid invariants, and then use them to verify Matui's HK conjecture for some basic groupoids. Here we just present a couple of immediate results.

Lemma 2.75. *Let $\mathcal{G} = \mathcal{G}^{(0)}$ be a trivial groupoid, with $\mathcal{G}^{(0)}$ the Cantor set. Then $\mathcal{G}$ satisfies HK-conjecture.*

Proof. The proof is standard:
Since $\mathcal{G} = \mathcal{G}^{(0)}$, we deduce that $\mathcal{G}^{(n)} = \mathcal{G}^{(0)}$, for all $n \in \mathbb{N}$. Hence, definition of homology implies $\delta_1 = 0$, and, in general, $\delta_n(f)(x) = \sum_{i=0}^{n}(-1)^i f(x)$, for all $n > 1$. Thus, $\delta_n = 0$ if n is odd, and $\delta_n = id$ whenever n is even. We deduce that $H_0(\mathcal{G}) = C_c(\mathcal{G}^{(0)}, \mathbb{Z})$, and $H_n(\mathcal{G}) = 0$ for all $n > 1$.
On the other hand, since $\mathcal{G}$ has no non-trivial composable elements, it is clear that its associated algebra is $C_c(\mathcal{G}^{(0)}, \mathbb{C})$. Hence, $K_0(C_r^*(\mathcal{G})) \cong C_c(\mathcal{G}^{(0)}, \mathbb{Z})$, and $K_1(C_r^*(\mathcal{G})) = 0$ (see,for example, [53]). Therefore, Matui's HK conjecture holds, and the isomorphism is given by the natural map $[1_U]_{H_0} \mapsto [1_U]_{K_0}$, for any compact open $U \subseteq \mathcal{G}^{(0)}$. □

Proposition 2.76. *Let $\mathcal{G}$, $\mathcal{H}$ be effective, minimal étale groupoids whose unit space is a Cantor set, and suppose $C_r^*(\mathcal{G})$ is nuclear and verifies the UCT. Then, if HK is true for both $\mathcal{G}$ and $\mathcal{H}$, it is also true for $\mathcal{G} \times \mathcal{H}$.*

Proof. The result is a consequence of combining the Kunneth theorem for groupoids (Lemma 2.49) and the Kunneth theorem for C^*-algebras (see [53]). □

2.4.2 Matui's AH conjecture

Here we introduce all the notions involved in Matui's second conjecture, named *AH*. Even though it is not the main object of study of this work, it will be discussed in chapter 5 for a certain groupoid. If the reader is only interested in Matui's HK conjecture, this section can be skipped. Most of the early literature about this conjecture can be found in [36] and [38].

Definition 2.77. *Let $\mathcal{G}$ be an étale groupoid, and let U be a full compact open bisection of $\mathcal{G}$. We define the map $\pi_U : \mathcal{G}^{(0)} \to \mathcal{G}^{(0)}$ as:*

$$s(\gamma) \mapsto r(\gamma), \text{ for } \gamma \in U.$$

If U verifies that $r(U) \cap s(U) = \emptyset$, we can define $\hat{U}$ as $U \sqcup U^{-1} \sqcup \mathcal{G}^{(0)} \setminus r(U) \sqcup s(U)$. Then $\hat{U}$ is clearly full, and the map $\pi_{\hat{U}}$ is said to be a ***transposition****.* □

The maps π_U capture the dynamical properties of a groupoid. In this line, one can note the reason behind the name *transposition*: by definition, $\pi_{\hat{U}}^2$ is always the identity map in $\mathcal{G}^{(0)}$. It is natural to study the structure of the set of all π_U.

Definition 2.78. *The topological full group is defined as the set*

$$[[\mathcal{G}]] := \{\pi_U : U \text{ full bisection}\},$$

together with the product given by composition, i.e. , $\pi_U \pi_V = \pi_{UV}$. The unit of $[[\mathcal{G}]]$ is $\pi_{\mathcal{G}^{(0)}} = id_{\mathcal{G}^{(0)}}$. □

Notice that this product always makes sense, since both U and V are full, and hence $U \times V \in \mathcal{G}^{(2)}$, and UV is full.

There is always a natural map between the topological full group and the homology group $H_1(\mathcal{G})$.

Definition 2.79. *([36, Definition 7.1]) We define the* ***index map*** $I : [[\mathcal{G}]] \to H_1(\mathcal{G})$ *as the map given by* $\pi_U \mapsto [1_U]$. □

Immediate properties arise for this map.

Lemma 2.80. *([36, Lemma 7.3]) The following statements are true:*

1. *Let* $U, V \subseteq \mathcal{G}$ *full compact open bisections. Then the set*
$$O = \{(g_1, g_2) \in \mathcal{G}^{(2)} : g_1 \in U, g_2 \in V\}$$
is a compact open subset of $\mathcal{G}^{(2)}$, *and* $\delta_2(1_O) = 1_U - 1_{UV} + 1_V$.
2. $[1_U] = 0 \in H_1(\mathcal{G})$, *for any clopen subset* $U \subseteq \mathcal{G}^{(0)}$. *In particular,* $I(\pi_U) = 0$.
3. $I(\pi_U \pi_V) = [1_U] + [1_V] = I(\pi_U) + I(\pi_V) \in H_1(\mathcal{G})$, *for* U, V *full compact open bisections of* $\mathcal{G}$.
4. $[1_U] + [1_{U^{-1}}] = 0 \in H_1(\mathcal{G})$, *for any full compact open of* $\mathcal{G}$. *In particular,*
$$I(\pi_U) + I(\pi_{U^{-1}}) = 0.$$
5. *The index map is a group homomorphism.*

Proof. The statements are straightforward:

1. This follows directly from definition of the homology maps.
2. The claim is immmediate after noticing that, whenever $U \subseteq \mathcal{G}^{(0)}$, then $U = UU$.
3. Using the first result, we obtain that $[1_U] - [1_{UV}] + [1_V] = 0 \in H_1(\mathcal{G})$. Therefore, $I(\pi_U \pi_V) = I(\pi_{UV}) = [1_{UV}] = [1_U] + [1_V]$.
4. Note that $UU^{-1} = \mathcal{G}^{(0)}$. Thus, $[1_U] + [1_{U^{-1}}] = [1_{UU^{-1}}] = 0 \in H_1(\mathcal{G})$.
5. Follows from $1 - 4$.

□

Remark 2.81. *Recall that* $H_1(\mathcal{G})$ *is an abelian group. Thus, we can induce a homomorphism* $I_{ab} : [[\mathcal{G}]]_{ab} \to H_1(\mathcal{G})$, *where* $[[\mathcal{G}]]_{ab}$ *denotes the abelianization of* $[[\mathcal{G}]]$. □

With all the previous notions, we can already enunciate the conjecture. It appeared firstly in [38], and predicts the existence of a certain exact sequence

$$H_0(\mathcal{G}) \otimes \mathbb{Z} \xrightarrow{j} [[\mathcal{G}]]_{ab} \xrightarrow{I_{ab}} H_1(\mathcal{G}) \to 0$$

for essentially principal minimal étale groupoids whose unit space is a Cantor set [38, Conjecture 2.9]. This conjecture is known as *Matui's AH conjecture.*

The map $j : H_0(\mathcal{G}) \otimes \mathbb{Z} \to [[\mathcal{G}]]_{ab}$ is explicitly described in [38].

The first (strong) version of the conjecture predicted the existence of a *short* exact sequence. This was, however, early discarded by Nekrashevych (see [38, Remark 2.10]). This *weakened* conjecture has been confirmed for several cases, and so far no counter-examples have been found. It has been proven, for example, for principal almost finite groupoids [36], or transformation groupoids associated to odometers [56], which happen to provide the first counterexample for Matui's HK conjecture. Among other cases the conjecture has been confirmed there are the Katsura-Exel-Pardo groupoids (see [44]), or what is the same, the groupoids of a special self-similar action of $\mathbb{Z}$ over a finite graph. We remark those because we will use some of the techniques developed there in later chapters.

There is quite a extensive literature around this conjecture. For example, an in-depth study of the index map can be found in [36]. However, since this is not the main theme of this work, we will just state the results we will use later in chapter 5. The proof of those results can be found in [36] and [38].

Denote by $\mathcal{T}(\mathcal{G})$ to be the subgroup of $[[\mathcal{G}]]$ generated by all transpositions.

Lemma 2.82. *Let $I : [[\mathcal{G}]] \to H_1(\mathcal{G})$ be the index map. Then $\mathcal{T}(\mathcal{G}) \subseteq ker(I)$ always holds.*

Proof. See [36, Lemma 7.7 (3)]. □

The relation between $\mathcal{T}(\mathcal{G})$ and $ker(I)$ is closely related to the verification of the AH-conjecture, in the following way.

Definition 2.83. *Let $\mathcal{G}$ effective, Hausdorff, ample groupoid, and let $\mathcal{T}(\mathcal{G})$ be the subgroup of $[[\mathcal{G}]]$ generated by all transpositions. We say that $\mathcal{G}$ has Property TR whenever $\mathcal{T}(\mathcal{G}) = ker(I)$.* □

Remark 2.84. *It was proven in [38, Theorem 4.4] that, whenever $\mathcal{G}$ is minimal and purely infinite, then Property TR is equivalent to the verification of Matui's AH conjecture. However, it is not known if all minimal, purely infinite groupoids have Property TR.* □

Chapter 3

Useful strategies for the computation of the groupoids invariants

Throughout the last decades, the study of groupoids and their invariants has been approached using different strategies. This chapter provides to the reader with some of the most common tools about groupoids used when studying the invariants involved in Matui's HK-conjecture. More techniques can be found, for example, in [38], [14], [18] or [20], among others. The chapter is structured as follows:

In Section 3.1, we study one of the many notions of *groupoid equivalence* existing in the literature. More precisely, we study the concept of *Kakutani equivalence*, sometimes named as *Morita equivalence* (see [36]). Other types of equivalence can be found, for example, in [21], as well as the relation among them. Kakutani equivalence is known to preserve all the invariants involved in Matui's HK conjecture. Hence, it is a common tool in its study.

In Section 3.2 we introduce the notion of cocycles, that is, groupoid homomorphisms from $\mathcal{G}$ to an abelian group Γ (considered as a groupoid). We then relate the invariants of a groupoid, under certain conditions, with the invariants of the associated *skew-product* groupoid arising from the cocycle. We show special interest in the case $\Gamma = \mathbb{Z}^k$. This technique has been broadly used in the study of Matui's HK conjecture (see, for example, [46]), usually in combination with Kakutani equivalence (see [21]).

We devote Section 3.3 to give the basic notions concerning spectral sequences. For a more detailed account, we direct the reader to [39] or [64]. Following that, we use the techniques appearing in [4] in order to explicitly build Kasparov's spectral sequence for K-theory (see [29]).

We finish the chapter recalling in Section 3.4 the work of Scarparo, published in [56], where the author builds the first counterexample to the strong version of Matui's HK-conjecture.

3.1 Kakutani equivalence

In [21], the authors introduce a serie of different types of groupoids equivalence, and the relations among them. Here we describe one of them: *Kakutani equivalence.* This notion will play an important role throughout this document. Indeed, it has been proven that Kakutani equivalence preserves all the invariant objects involved in our work, that is, the homology groups and the associated K-theory (see, for example, [21]). We recall those results here.

We should note that some texts refer to this notion as *Morita equivalence* (see [38]), since whenever two étale groupoids are Kakutani equivalent, their reduced C^*-algebras are strongly Morita equivalent.

Definition 3.1. *Let $\mathcal{G}, \mathcal{H}$ be étale groupoids, with both $\mathcal{G}^{(0)}$ and $\mathcal{H}^{(0)}$ compact and totally disconnected spaces. We say that $\mathcal{G}$ is* ***Kakutani equivalent*** *to $\mathcal{H}$ if there exist full clopen subsets $Y \subseteq \mathcal{G}^{(0)}$, $Z \subseteq \mathcal{H}^{(0)}$ such that $\mathcal{G}|_Y \cong \mathcal{H}|_Z$.* □

Kakutani equivalence is, indeed, an equivalence relation. Symmetry and reflexivity are trivial, so we must just prove transitivity. Before proving it in Lemma 3.5, we need to show a couple of results. Throughout this subsection, all groupoids will be étale, with compact and totally disconnected unit space. We also keep our standing assumptions stated in section 2.1.2.

Definition 3.2. *Let $\mathcal{G}$ be a groupoid, and let $f \in C(\mathcal{G}^{(0)}, \mathbb{Z})$, with $f \geq 0$. Define*

$$\mathcal{G}_f := \{(g, i, j) \in \mathcal{G} \times \mathbb{Z} \times \mathbb{Z} : 0 \leq i \leq f(r(g)), \text{ and } 0 \leq j \leq f(s(g))\}.$$

Then $\mathcal{G}_f$ is a groupoid, under the following structure:

$$\begin{gathered}\mathcal{G}_f^{(0)} := \{(x, i, i) \in \mathcal{G}^{(0)} \times \mathbb{Z} \times \mathbb{Z} : 0 \leq i \leq f(x)\},\\ (g, i, j)^{-1} = (g^{-1}, j, i), \text{ and}\\ (g, i, j)(h, j, l) = (gh, i, l),\end{gathered}$$

whenever $(g, h) \in \mathcal{G}^{(2)}$. □

Straightforward computation shows that the groupoid $\mathcal{G}_f$, together with the topology induced from $\mathcal{G} \times \mathbb{Z} \times \mathbb{Z}$, is an étale groupoid. Moreover, the subset

$$\{(x, 0, 0) : x \in \mathcal{G}^{(0)}\} \subseteq \mathcal{G}_f^{(0)}$$

is full in $\mathcal{G}_f$.

Lemma 3.3. *([36, Lemma 4.3]) Let $\mathcal{G}$ be a groupoid, and let $Y \subseteq \mathcal{G}^{(0)}$ be a full clopen subset of $\mathcal{G}$. Then there exists $f \in C(Y, \mathbb{Z})$, and an isomorphism π between $(\mathcal{G}|_Y)_f$ and $\mathcal{G}$, satisfying $\pi(g, 0, 0) \mapsto g$, for all $g \in \mathcal{G}|_Y$.*

Proof. Take $x \in \mathcal{G}^{(0)} \setminus Y$. Since Y is full, we can always find $g \in r^{-1}(x) \cap s^{-1}(Y)$. Take a compact open bisection U_x such that $g \in U_x$, $r(U_x) \subset \mathcal{G}^{(0)} \setminus Y$, and $s(U_x) \subset Y$. It is clear that $\{r(U_x) : x \in \mathcal{G}^{(0)} \setminus Y\}$ is an open covering of $\mathcal{G}^{(0)} \setminus Y$. Hence, we can find $x_1, ..., x_n \in \mathcal{G}^{(0)} \setminus Y$ such that $r(U_{x_1}), ..., r(U_{x_n})$ form a finite subcover of $\mathcal{G}^{(0)} \setminus Y$. We can refine this to make it a mutually disjoint cover, defining the following open compact bisections inductively:

$$V_1 = U_{x_1},$$
$$V_k = U_{x_k} \setminus r^{-1}(r(V_1 \cup ... \cup V_{k-1})).$$

Then $r(V_1), ..., r(V_n)$ are mutually disjoint, and their union equals to $\mathcal{G}^{(0)} \setminus Y$.
Now, for each subset $\Lambda \subset \{1, 2, ..., n\}$, fix a bijection $\alpha_\Lambda : \{1, ..., |\Lambda|\} \to \Lambda$. For $y \in Y$, define $\Lambda(y) := \{l \in \{1, ..., n\} : y \in s(V_k)\}$, and $f : Y \to \mathbb{Z}$ as $f(y) = |\Lambda(y)|$. Each $s(V_k)$ is clopen, hence $f \in C(Y, \mathbb{Z})$. Fix $y \in Y$. Then, for any $i \leq f(y)$, define $l_i := \alpha_{\Lambda(y)}(i)$. Note that this is just a way to enumerate the $V_j's$ whose source contains y, and then choosing the $i - th$ one. Then we can define $\theta : (\mathcal{G}|_Y)_f^{(0)} \to \mathcal{G}$ as

$$\theta(y, i, i) = \begin{cases} y & i = 0 \\ (s|_{V_{l_i}})^{-1}(y) & otherwise \end{cases}$$

The reader may check that

$$\pi(g, i, j) = \theta(r(g), i, i) \cdot g \cdot \theta(s(g), j, j)^{-1}$$

defines an isomorphism between $(\mathcal{G}|_Y)_f$ and $\mathcal{G}$. □

The following result, appearing in [36, Lemma 4.4], shows that two reduction subgroupoids of the same groupoid are Kakutani equivalent whenever the subsets are full.

Lemma 3.4. *Let $\mathcal{G}$ be as above, and let $Y, Y' \subseteq \mathcal{G}^{(0)}$ be full subsets of $\mathcal{G}$. Then $\mathcal{G}|_Y$ is Kakutani equivalent to $\mathcal{G}|_{Y'}$.*

Proof. Take $f \in C(Y, \mathbb{Z})$, and let $\pi : (\mathcal{G}|_Y)_f \to \mathcal{G}$ be as in Lemma 3.3. Define the clopen subset $Z \subset Y$ given by

$$Z = \{y \in Y : \pi(y, k, k) \in Y', \ for \ some \ 0 \leq k \leq f(y)\}.$$

Since Y' is full, we can see that Z is also full. Let

$$g(z) := min\{k \in \{0, ..., f(z)\} : \pi(z, k, k) \in Y'\},$$

for each $z \in Z$, and define $U = \{\pi(z, g(z), 0) : z \in Z\}$. Then U is a compact open bisection of $\mathcal{G}$, satisfying $s(U) = Z$, and $r(U) \subset Y'$. Thus, $r(U)$ is clearly a full subset of $\mathcal{G}^{(0)}$ contained in Y', such that $\mathcal{G}|_Z$ is isomorphic to $\mathcal{G}|_{r(U)}$. Hence, $\mathcal{G}|_Y$ and $\mathcal{G}|_{Y'}$ are Kakutani equivalent. □

Lemma 3.5. *([36, Lemma 4.5]) Let $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ be étale groupoids with compact, totally disconnected unit spaces, such that $\mathcal{G}_1$ is Kakutani equivalent to $\mathcal{G}_2$, and $\mathcal{G}_2$ is Kakutani equivalent to $\mathcal{G}_3$. Then $\mathcal{G}_1$ is Kakutani equivalent to $\mathcal{G}_3$.*

Proof. The hypothesis implies the existence of Y_1, Y_2, Y_2', Y_3 full clopen subsets of $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ respectively, together with maps π, π', such that $\pi : \mathcal{G}_2|_{Y_2} \to \mathcal{G}_1|_{Y_1}$, and $\pi' : \mathcal{G}_2|_{Y_2'} \to \mathcal{G}_3|_{Y_3}$ are both isomorphisms. Using the previous lemma, there exist $Z \subseteq Y_2$, $Z' \subseteq Y_2'$ full clopen subsets of $\mathcal{G}_2$ such that $\mathcal{G}_2|_Z \cong \mathcal{G}_2|_{Z'}$. Hence, $\mathcal{G}_1|_{\pi(Z)} \cong \mathcal{G}_3|_{\pi'(Z')}$, concluding the proof. □

Corollary 3.6. *Kakutani equivalence is an equivalence relation.* □

A different approach to the proof of the last statement can be found in [11, Theorem 3.2], where the authors prove that, for ample groupoids $\mathcal{G}, \mathcal{H}$ with σ-compact unit spaces, Kakutani equivalence is equivalent to the condition $\mathcal{G} \times \mathcal{R} \cong \mathcal{H} \times \mathcal{R}$, where $\mathcal{R}$ is the full countable equivalence relation, that is, $\mathcal{R} = \mathbb{N} \times \mathbb{N}$. This condition is referred to as *groupoid equivalence*.

Among other applications, we can use Kakutani equivalence to study the HK-conjecture:

Lemma 3.7. *Let $\mathcal{G}, \mathcal{H}$ be Kakutani equivalent groupoids. Then $C_r^*(\mathcal{G})$ is strongly Morita equivalent to $C_r^*(\mathcal{H})$, that is, $C_r^*(\mathcal{G}) \otimes \mathbb{K} \cong C_r^*(\mathcal{H}) \otimes \mathbb{K}$. Hence, $K_i(C_r^*(\mathcal{G})) \cong K_i(C_r^*(\mathcal{H}))$, for $i = 0, 1$.*

Proof. Let $Y \subseteq \mathcal{G}^{(0)}$ be a full clopen subset. Then $C_r^*(\mathcal{G}|_Y)$ is canonically isomorphic to the hereditary subalgebra $1_Y C_r^*(\mathcal{G}) 1_Y$ of $C_r^*(\mathcal{G})$. Therefore, if $\mathcal{G}$ and $\mathcal{H}$ are Kakutani equivalent, their reduced C^*-algebras are strongly Morita equivalent. Thus, their K-theory groups coincide. □

The first proof of the following theorem appeared in [14, Corollary 4.6]. However, it is quite technical, and so we will go with the one appearing in [36, Propositions 3.5-3.6]. Before that, we need to define the notion of *similarity*.

Definition 3.8. *Let $\mathcal{G}, \mathcal{H}$ be étale groupoids. Two étale homomorphisms $\rho, \sigma : \mathcal{G} \to \mathcal{H}$ are said to be **similar** if there exists a continuous map $\theta : \mathcal{G}^{(0)} \to \mathcal{H}$ such that*

$$\theta(r(g))\rho(g) = \sigma(g)\theta(s(g)),$$

for all $g \in \mathcal{G}$. If such θ exists, it is automatically étale.

*Two groupoids $\mathcal{G}$ and $\mathcal{H}$ are said to be **homologically similar** if there exist two étale homomorphisms $\rho : \mathcal{G} \to \mathcal{H}$ and $\sigma : \mathcal{H} \to \mathcal{G}$ such that $\sigma \circ \rho$ is similar to $id_{\mathcal{G}}$, and $\rho \circ \sigma$ is similar to $id_{\mathcal{H}}$.*

□

With this in mind, we can use the results of [36] to prove the theorem below.

Theorem 3.9. *Let $\mathcal{G}, \mathcal{H}$ be Kakutani equivalent ample groupoids, and let A be an abelian group. Then*

$$H_n(\mathcal{G}, A) \cong H_n(\mathcal{H}, A),$$

for all $n \in \mathbb{N}$.

Proof. The proof has two steps. First, we prove that groupoid homology is invariant under similarities. Second, we will prove that $\mathcal{G}$ is homologically similar to $\mathcal{G}|_F$. Let us show it:

Suppose that $\rho, \sigma : \mathcal{G} \to \mathcal{H}$ are similar étale homomorphisms. Then there exists a continuous map $\theta : \mathcal{G}^{(0)} \to \mathcal{H}$ such that $\theta(r(g))\rho(g) = \sigma(g)\theta(s(g))$, for all $g \in \mathcal{G}$. For each $n \in \mathbb{N} \cup \{0\}$, define $h_n : C_c(\mathcal{G}^{(n)}, A) \to C_c(\mathcal{H}^{(n+1)}, A)$ as follows. Let $h_0 := \theta_*$, and $h_n := \sum_{j=0}^{n} (-1)^j k_{j*}$, with $k_j : \mathcal{G}^{(n)} \to \mathcal{H}^{(n+1)}$ given by:

$$k_j(g_1, ..., g_n) = \begin{cases} (\theta(r(g_1)), \rho(g_1), \rho(g_2), ..., \rho(g_n)) & j = 0 \\ (\sigma(g_1), ..., \sigma(g_j), \theta(s(g_j)), \rho(g_{j+1}), ..., \rho(g_n)) & 1 \leq j \leq n-1 \\ (\sigma(g_1), \sigma(g_2), ..., \sigma(g_n), \theta(s(g_n))) & j = n. \end{cases}$$

The reader may check that $\delta_1 \circ h_0 = \rho_*^{(0)} - \sigma_*^{(0)}$, and

$$\delta_{n+1} \circ h_n + h_{n-1} \circ \delta_n = \rho_*^{(n)} - \sigma_*^{(n)},$$

where $\rho^{(n)}$ denotes the induced map $\rho^{(n)} : \mathcal{G}^{(n)} \to \mathcal{H}^{(n)}$. Note that this means that $\rho_*^{(n)}$ and $\sigma_*^{(n)}$ are homotopic. Hence, we get $H_n(\rho) = H_n(\sigma)$.

Now, if $\mathcal{G}$ is homologically similar to $\mathcal{H}$, we have two homomorphisms ρ, σ such that $\rho \circ \sigma$ is similar to $id_{\mathcal{H}}$, and $\sigma \circ \rho$ is similar to $id_{\mathcal{G}}$. But then:

$$H_n(id_{\mathcal{G}}) = id_{H_n(\mathcal{G})} = H_n(\sigma \circ \rho) = H_n(\sigma) \circ H_n(\rho), \text{ and}$$
$$H_n(id_{\mathcal{H}}) = id_{H_n(\mathcal{H})} = H_n(\rho \circ \sigma) = H_n(\rho) \circ H_n(\sigma).$$

Hence, $H_n(\rho) = H_n(\sigma)^{-1}$, and $H_n(\mathcal{G}, A) \cong H_n(\mathcal{H}, A)$. This completes the first step.

For the second step, we use similar strategies to the ones used in Lemma 3.3.

First, take an open full subset $F \subseteq \mathcal{G}^{(0)}$, and suppose that there exist some continuous map $\theta : \mathcal{G}^{(0)} \to \mathcal{G}$, such that $r(\theta(x)) = x$, and $s(\theta(x)) \in F$, for all $x \in \mathcal{G}^{(0)}$. Define $\rho : \mathcal{G} \to \mathcal{G}|_F$ as $\rho(g) = \theta(r(g))^{-1} g \theta(s(g))$, for all $g \in \mathcal{G}$, and $\sigma : \mathcal{G}|_F \to \mathcal{G}$ as $\sigma(g) = g$, for all $g \in \mathcal{G}|_F$. The reader may check that, under those maps, $\mathcal{G}$ is homologically similar to $\mathcal{G}|_F$. Hence, all is left to do is to prove that the map θ always exists.

Take a countable family of compact open bisections $\{U_n\}$ such that $\{r(U_n)\}$ covers $\mathcal{G}^{(0)}$, and $s(U_n) \subset F$, for all n. Define compact open bisections $V_1, ..., V_n$ inductively by $V_1 = U_1$, and

$$V_n = U_n \setminus r^{-1}(r(V_1 \cup ... \cup V_{n-1})).$$

Then the map $\theta : \mathcal{G}^{(0)} \to \mathcal{G}$ given by $\theta(x) = (r|_{V_n})^{-1}(x)$, for $x \in V_n$, satisfies the assumptions stated above, concluding the proof. □

Corollary 3.10. *Let $\mathcal{G}, \mathcal{H}$ be Kakutani equivalent groupoids. Then, if $\mathcal{G}$ satisfies HK conjecture, so does $\mathcal{H}$.*

Proof. Immediate combining the previous results. □

As shown above, Kakutani equivalence is a powerful tool in the study of the HK conjecture for certain families of groupoids. Here we show some of the most direct results.

Lemma 3.11. *Let $\mathcal{G}$ be a compact elementary groupoid, that is, a compact and principal étale groupoid. Then $\mathcal{G}$ is Kakutani equivalent to a trivial groupoid, i.e. a groupoid $\mathcal{H}$ such that $\mathcal{H} = \mathcal{H}^{(0)}$.*

Proof. The proof is deduced from [22, Lemma 3.4]. Since $\mathcal{G}^{(0)}$ is totally disconnected, we can find a clopen partition $\mathcal{G}^{(0)} = \bigsqcup_{i=1}^{\infty} X_i$ where each X_i satisfies that $|\mathcal{G}(x)| = i$ for all $x \in X_i$, where $\mathcal{G}(x)$ denotes the orbit of x. Compactness of $\mathcal{G}^{(0)}$ implies that this partition is finite, and that $X_\infty = \emptyset$. Indeed, suppose $x \in \mathcal{G}^{(0)}$ such that $|\mathcal{G}(x)| = \infty$, and let $\{\gamma_n\}_n := \{x\mathcal{G}\}$, where $x\mathcal{G} := r^{-1}(x)$, as in Corollary 2.37. Since $\mathcal{G}$ is compact, $\{\gamma_n\}_n$ must have an accumulation point $\gamma \in \mathcal{G}$. Then $\{x\}_n = \{r(\gamma_n)\}_n \to r(\gamma) = x$, by continuity of the range map. Finally, since $x\mathcal{G}$ is discrete in $\mathcal{G}$ (see Corollary 2.37), we deduce that there exists some $m \in \mathbb{N}$ such that $\gamma_n = \gamma$ for all $n \geq m$. Therefore, the orbit $\mathcal{G}(x)$ is not infinite, and $X_\infty = \emptyset$.

With this in mind, we can write $\mathcal{G}$ as the disjoint finite union $\mathcal{G} = \bigsqcup_{i=1}^{n} \mathcal{G}|_{X_i}$, for some $n \in \mathbb{N}$. Since $\mathcal{G}$ is principal, all $\mathcal{G}|_{X_i}$ are also principal. Each $\mathcal{G}|_{X_i}$ is isomorphic to an equivalence relation where each class contains exactly i elements, and hence, for each i, there exist some $\mathcal{G}|_{X_i}$-full clopen subsets $Y_i^j \subseteq X_i$ such that $X_i = \bigsqcup_{j=1}^{i} Y_i^j$, and $\mathcal{G}|_{Y_i^j} = Y_i^j$, for all $1 \leq j \leq i$. Therefore, denoting $Z := \bigsqcup\limits_{i=1}^{n} Y_i^1$, we have that $\mathcal{G}|_Z = Z$, as desired. □

Corollary 3.12. *Let $\mathcal{G}$ be a compact elementary groupoid. Then $\mathcal{G}$ verifies the HK-conjecture, with $K_1(C_r^*(\mathcal{G})) = 0$, $H_n(\mathcal{G}) = 0$ for all $n > 0$, and $H_0(\mathcal{G}) \cong K_0(C_r^*(\mathcal{G}))$. Moreover, the isomorphism is the natural one, given by $[1_U]_{H_0} \mapsto [1_U]_{K_0}$, for any compact open $U \subseteq \mathcal{G}^{(0)}$.*

Proof. Immediate after using Corollaries 2.75 and 3.10, combined with the last lemma. □

We can extend this result to AF groupoids.

Lemma 3.13. *Let $\mathcal{G}$ be an AF groupoid with compact unit space. Then $\mathcal{G}$ satisfies the HK-conjecture. Moreover, the isomorphism is the natural one, given by $[1_U]_{H_0} \mapsto [1_U]_{K_0}$, for any compact open $U \subseteq \mathcal{G}^{(0)}$.*

Proof. Let $\mathcal{G} = \bigcup_{i=0}^{\infty} \mathcal{K}_i$, with $\mathcal{K}_i$ elementary groupoids. Then $C_r^*(\mathcal{G})$ is given by the closure of the increasing union of $C_r^*(\mathcal{K}_i)$ (i.e., it is an AF algebra). Since K_* is a continuous functor, we can compute $K_1(C_r^*(\mathcal{G})) = 0$, and $K_0(C_r^*(\mathcal{G})) = \varinjlim(K_0(C_r^*(\mathcal{K}_i)))$.
On the other hand, combining Proposition 2.50, Lemma 2.75 and the previous corollary, we deduce that $H_n(\mathcal{G}) = 0$ for any $n > 0$, and

$$H_0(\mathcal{G}) \cong \varinjlim(H_0(\mathcal{K}_i)) \cong \varinjlim(K_0(C_r^*(\mathcal{K}_i))) \cong K_0(C_r^*(\mathcal{G})),$$

concluding the proof. □

Moreover, it was shown in [21] that this result extends to the case of non-compact elementary and AF groupoids. The techniques used in the proof are similar to the ones used for the compact version. Hence, we just show here the result, and direct the reader to the original source for further details.

Theorem 3.14. *([21, Theorem 4.10]) Let X, Y be locally compact Hausdorff spaces such that Y is σ-compact and totally disconnected, and let $\psi : Y \to X$ be a local homeomorphism. Then $H_n(R(\psi)) = H_n(X) = 0$ for $n \geq 1$, and $H_0(R(\psi)) = H_0(X) \cong C_c(X, \mathbb{Z})$ under the map given by $[\psi_*(1_U)] \mapsto [1_U]$.*
Moreover, the groupoid C^*-algebra $C^*(R(\psi))$ is an AF algebra, the map ψ_* induces an isomorphism $K_0(C^*(R(\psi))) \cong C_c(X, \mathbb{Z})$ such that the diagram

$$\begin{tikzcd}
C_c(Y,\mathbb{Z}) \arrow[d, "i_*"'] \arrow[dr, "\psi_*"] & \\
K_0(C^*(R(\psi))) \arrow[r, "\cong"] & C_c(X,\mathbb{Z})
\end{tikzcd}$$

commutes, and $K_1(C^*(R(\psi))) = 0$. In particular, $R(\psi)$ verifies the HK-conjecture. □

Corollary 3.15. *AF groupoids satisfy Matui's HK-conjecture.* □

3.2 Cocycles

Cocycles usually provide a way to simplify the study of both homology and K-theory associated to an étale groupoid. They transform the direct computation to the study of either a certain exact sequence of homology or a crossed product algebra. In this line, we will show special interest in cocycles over $\mathbb{Z}^k$.

A cocycle is a continuous groupoid homomorphism $c : \mathcal{G} \to \Gamma$, with Γ an abelian group (considered as a groupoid).

Definition 3.16. *Let $c : \mathcal{G} \to \Gamma$ be a cocycle. We define the associated* ***skew product groupoid*** *$\mathcal{G} \times_c \Gamma$ as the set $\mathcal{G} \times \Gamma$, together with structure:*

$$r(g,\gamma) = (r(g),\gamma),$$
$$s(g,\gamma) = (s(g),\gamma + c(g)),$$
$$(g,\gamma)(h,\gamma + c(g)) = (gh,\gamma), \text{ and}$$
$$(g,\gamma)^{-1} = (g^{-1},\gamma + c(g)), \text{ whenever } (g,h) \in \mathcal{G}^{(2)}.$$

Moreover, there is a natural action $\hat{c} : \Gamma \curvearrowright \mathcal{G} \times_c \Gamma$ given by $\hat{c}_\mu(g,\gamma) = (g,\gamma+\mu)$. □

One can usually study $\mathcal{G} \times_c \Gamma$ indirectly, through the consideration of $ker(c)$:

Lemma 3.17. *Let $\mathcal{G}$ be a groupoid, and let $c : \mathcal{G} \to \Gamma$ be a cocycle. Suppose that $\mathcal{G}^{(0)} \times \{0\}$ is full in $\mathcal{G} \times_c \Gamma$. Then $\mathcal{G} \times_c \Gamma$ is Kakutani equivalent to $ker(c)$.*

Proof. It is clear that $\mathcal{G}^{(0)} \times \{0\}$ is a clopen subset of $(\mathcal{G} \times_c \Gamma)^{(0)} = \mathcal{G}^{(0)} \times \Gamma$. Then $(\mathcal{G} \times_c \Gamma)|_{\mathcal{G}^{(0)} \times \{0\}} = \{(g,0) \in \mathcal{G} \times_c \Gamma : c(g) = 0\} \cong ker(c)$, as desired. □

Some of the most useful applications of cocycles arise when considering $\Gamma = \mathbb{Z}^k$. The following results will provide a powerful tool when computing either the homology or the K-theory associated to a groupoid.

3.2.1 Long exact sequence in homology

Given a cocycle $c : \mathcal{G} \to \mathbb{Z}$, we can compute the homology groups of $\mathcal{G}$ by studying a certain long exact sequence in homology. Note that, for clarity matters, we make a slight abuse of notation, using the same symbols to denote the maps induced in homology and the maps induced in the algebras.

Theorem 3.18. *([46, Lemma 1.3]) Let $\mathcal{G}$ be an étale groupoid with $\mathcal{G}^{(0)}$ a locally compact, Hausdorff and totally disconnected space, and let $c : \mathcal{G} \to \mathbb{Z}$ be a cocycle. Then there exists a long exact sequence*

$$\begin{tikzcd}
0 & H_0(\mathcal{G}) \arrow[l] & H_0(\mathcal{G}\times_c\mathbb{Z}) \arrow[l] & H_0(\mathcal{G}\times_c\mathbb{Z}) \arrow[l, "Id-\hat{c}^{(0)}_{1*}"'] & H_1(\mathcal{G}) \arrow[l] & H_1(\mathcal{G}\times_c\mathbb{Z}) \arrow[l] \\
\cdots \arrow[r] & H_3(\mathcal{G}) \arrow[r] & H_2(\mathcal{G}\times_c\mathbb{Z}) \arrow[r, "Id-\hat{c}^{(2)}_{1*}"] & H_2(\mathcal{G}\times_c\mathbb{Z}) \arrow[r] & H_2(\mathcal{G}) \arrow[r] & H_1(\mathcal{G}\times_c\mathbb{Z}) \arrow[u, "Id-\hat{c}^{(1)}_{1*}"']
\end{tikzcd} \tag{3.1}$$

where the maps $\hat{c}_{1*}^{(n)}$ *are induced by the action* $\hat{c} : \mathbb{Z} \curvearrowright \mathcal{G} \times_c \mathbb{Z}$, *with* $\hat{c}_1(g, i) = (g, i + 1)$.

Proof. Let $\hat{c}_1$ be as above, and consider

$$0 \to C_c((\mathcal{G} \times_c \mathbb{Z})^{(n)}, \mathbb{Z}) \overset{Id-\hat{c}_{1*}^{(n)}}{\to} C_c((\mathcal{G} \times_c \mathbb{Z})^{(n)}, \mathbb{Z}) \overset{\pi_*^{(n)}}{\to} C_c(\mathcal{G}^{(n)}, \mathbb{Z}) \to 0, \tag{3.2}$$

where $\pi_*^{(n)}$ is induced by the map $\pi^{(n)} : (\mathcal{G} \times_c \mathbb{Z})^{(n)} \to \mathcal{G}^{(n)}$ given by $((g_1, i_1), ...(g_n, i_n)) \mapsto (g_1, ..., g_n)$. Those are short exact sequences. Clearly, $\pi_*^{(n)}$ is a surjective map, and $Id - \hat{c}_{1*}^{(n)}$ is injective. Moreover, $\pi_*^{(n)} \circ (Id - \hat{c}_{1*}^{(n)})(f) = 0$, for all $f \in C_c((\mathcal{G} \times_c \mathbb{Z})^{(n)}, \mathbb{Z})$. Thus, all is left to do is to prove that $ker(\pi_*^{(n)}) \subseteq Im(Id - \hat{c}_{1*}^{(n)})$. Let us show it.

Denote by $(\mathcal{G} \times_c \mathbb{Z})_i^{(n)}$ the set of elements

$$\{((g_1, i), (g_2, i + c(g_1)), ..., (g_n, i + c(g_1) + ... + c(g_{n-1}))) : (g_1, ..., g_n) \in \mathcal{G}^{(n)}\} \subseteq (\mathcal{G} \times_c \mathbb{Z})^{(n)}.$$

For clarity matters, we will write the elements of $(\mathcal{G} \times_c \mathbb{Z})_i^{(n)}$ as $(g_1, ..., g_n)_i$.

Now take $f \in ker(\pi_*^{(n)})$. Since it has compact support, we can always find some $m \in \mathbb{N}$, and write

$$f = \sum_{i=-m}^{m} f_i,$$

for some $f_i \in C_c((\mathcal{G} \times_c \mathbb{Z})^{(n)}, \mathbb{Z})$ verifying $supp(f_i) \subseteq (\mathcal{G} \times_c \mathbb{Z})_i^{(n)}$, for each i.

We claim that

$$f_m + \hat{c}_{1*}^{(n)}(f_{m-1}) + (\hat{c}_{1*}^{(n)})^2(f_{m-2}) + ... + (\hat{c}_{1*}^{(n)})^{2m}(f_{-m}) = 0.$$

Indeed, if $f \in ker(\pi_*^{(n)})$, then

$$\sum_{i=-m}^{m} f((g_1, ..., g_n)_i) = 0,$$

for all $(g_1, ..., g_n) \in \mathcal{G}^{(n)}$. Then, for an element of $(\mathcal{G} \times_c \mathbb{Z})_m^{(n)}$, we have:

$$\begin{aligned}
(f_m + &\hat{c}_{1*}^{(n)}(f_{m-1}) + (\hat{c}_{1*}^{(n)})^2(f_{m-2}) + ... + (\hat{c}_{1*}^{(n)})^{2m}(f_{-m}))((g_1, ..., g_n)_m) = \\
&= f_m((g_1, ..., g_n)_m) + f_{m-1}((g_1, ..., g_n)_{m-1}) + ... + f_{-m}((g_1, ..., g_n)_{-m}) = \\
&= \sum_{i=-m}^{m} f((g_1, ..., g_n)_i) = 0\,.
\end{aligned}$$

Moreover, for an element of $(\mathcal{G} \times_c \mathbb{Z})_j^{(n)}$, with $j \neq m$, we obtain:

$$\begin{aligned}
(f_m + &\hat{c}_{1*}^{(n)}(f_{m-1}) + (\hat{c}_{1*}^{(n)})^2(f_{m-2}) + ... + (\hat{c}_{1*}^{(n)})^{2m}(f_{-m}))((g_1, ..., g_n)_j) = \\
&= f_m((g_1, ..., g_n)_j) + f_{m-1}((g_1, ..., g_n)_{j-1}) + ... + f_{-m}((g_1, ..., g_n)_{-j}) = \\
&= 0 + ... + 0 = 0\,,
\end{aligned}$$

hence concluding that the equality

$$f_m + \hat{c}_{1*}^{(n)}(f_{m-1}) + (\hat{c}_{1*}^{(n)})^2(f_{m-2}) + ... + (\hat{c}_{1*}^{(n)})^{2m}(f_{-m}) = 0$$

holds. Therefore, using this equality, we deduce

$$\begin{aligned} f = \sum_{i=-m}^{m} f_i = f_{m-1} + ... + f_{-m} - \hat{c}_{1*}^{(n)}(f_{m-1}) - (\hat{c}_{1*}^{(n)})^2(f_{m-2}) - ... - (\hat{c}_{1*}^{(n)})^{2m}(f_{-m}) = \\ = (f_{m-1} - \hat{c}_{1*}^{(n)}(f_{m-1})) + (f_{m-2} - \hat{c}_{1*}^{(n)}(f_{m-2})) + ((\hat{c}_{1*}^{(n)}(f_{m-2}) - \hat{c}_{1*}^{(n)}(\hat{c}_{1*}^{(n)}(f_{m-2}))) \ + \\ + ... + (f_{-m} - \hat{c}_{1*}^{(n)}(f_{-m})) + ... + ((\hat{c}_{1*}^{(n)})^{2m-1}(f_{-m}) - \hat{c}_{1*}^{(n)}(\hat{c}_{1*}^{(n)})^{2m-1}(f_{-m}))) \in Im(Id - \hat{c}_{1*}^{(n)}) \,, \end{aligned}$$

making (3.2) a short exact sequence, as desired.
Then, the long exact sequence of homology associated to (3.2) gives us the exact sequence (3.1) as claimed. □

There is a more general result relating the homology of a groupoid with the homology of the associated skew product groupoid $\mathcal{G} \times_c \Gamma$, involving a certain spectral sequence. If the reader is unfamiliar with the basics of spectral sequences, we will devote a short introductory section about them later in this chapter.

Theorem 3.19. *([36, Theorem 3.8]). Let $\mathcal{G}$ be an étale groupoid, A an abelian group, and let Γ be a countable discrete group, such that there exists some cocycle $c : \mathcal{G} \to \Gamma$. Then there exists a spectral sequence*

$$E^2_{p,q} = H_p(\Gamma, H_q(\mathcal{G} \times_c \Gamma, A)) \Rightarrow H_{p+q}(\mathcal{G}, A),$$

where $H_q(\mathcal{G} \times_c \Gamma, A)$ is regarded as a Γ-module via the action $\hat{c} : \Gamma \curvearrowright \mathcal{G} \times_c \Gamma$.
Moreover, whenever there exists an action $\varphi : \Gamma \curvearrowright \mathcal{G}$, then there exists a spectral sequence

$$E^2_{p,q} = H_p(\Gamma, H_q(\mathcal{G}, A)) \Rightarrow H_{p+q}(\mathcal{G} \rtimes_\varphi \Gamma, A).$$

□

The proof of this theorem is out of the scope of this work.
We have shown how groupoid homology can be studied when there is a cocycle involved. Let us now introduce some results concerning its associated K-theory.

3.2.2 Crossed product C^*-algebra associated to a cocycle

Recall that, given a cocycle $c : \mathcal{G} \to \Gamma$, there is an induced action $\hat{c} : \Gamma \curvearrowright \mathcal{G} \times_c \Gamma$ given by $\hat{c}_\mu(g, \gamma) = (g, \gamma + \mu)$, for $\mu \in \Gamma$. Since every automorphism of a groupoid induces an automorphism on its associated C^*-algebra, we obtain an induced action α of Γ on $C^*(\mathcal{G} \times_c \Gamma)$. We can use this in order to study the C^*-algebra of $\mathcal{G}$.

Lemma 3.20. *([21, Lemma 6.6]) Let $\mathcal{G}$ be an amenable, Hausdorff, second countable étale groupoid, and let $c : \mathcal{G} \to \mathbb{Z}^k$ be a cocycle. Denote by $\alpha : \mathbb{Z}^k \curvearrowright C^*(\mathcal{G} \times_c \mathbb{Z}^k)$ the associated action. Then the crossed product $C^*(\mathcal{G} \times_c \mathbb{Z}^k) \rtimes_\alpha \mathbb{Z}^k$ is stably isomorphic to $C^*(\mathcal{G})$. This isomorphism extends to an isomorphism $C_r^*(\mathcal{G} \times_c \mathbb{Z}^k) \rtimes_\alpha \mathbb{Z}^k \cong C_r^*(\mathcal{G}) \otimes \mathbb{K}$.*

Proof. A cocycle $c : \mathcal{G} \to \mathbb{Z}^k$ determines an action of $\mathbb{T}^k$ by automorphisms of $C_c(\mathcal{G})$ given by $(z \cdot f)(\gamma) = z^{c(\gamma)} f(\gamma)$, for all $\gamma \in \mathcal{G}$. This extends to an action φ of $\mathbb{T}^k$ by automorphisms on $C^*(\mathcal{G})$. Straightforward computation shows that the map $\theta : C^*(\mathcal{G} \times_c \mathbb{Z}^k) \to C^*(\mathcal{G}) \rtimes_\varphi \mathbb{T}^k$ that carries $f \in C_c(\mathcal{G} \times \{n\}) \subseteq C_c(\mathcal{G} \times_c \mathbb{Z}^k)$ to the function $z \mapsto (\gamma \mapsto z^n f(\gamma, n)) \in C(\mathbb{T}^k, C^*(\mathcal{G})) \subseteq C^*(\mathcal{G}) \rtimes_\varphi \mathbb{T}^k$ is an isomorphism.
By construction, it follows that the map θ intertwines with both α and the dual action $\hat{\varphi}$ on $C^*(\mathcal{G}) \rtimes_\varphi \mathbb{T}^k$. Hence, Takesaki-Takai duality (see [63, Theorem 4.5]-[62, Theorem 3.4]) implies that $C^*(\mathcal{G} \times_c \mathbb{Z}^k) \rtimes_\alpha \mathbb{Z}^k$ is stably isomorphic to $C^*(\mathcal{G})$. Indeed, we have that

$$C^*(\mathcal{G} \times_c \mathbb{Z}^k) \rtimes_\alpha \mathbb{Z}^k \cong (C^*(\mathcal{G}) \rtimes_\varphi \mathbb{T}^k) \rtimes_{\hat{\varphi}} \mathbb{Z}^k \cong C^*(\mathcal{G}) \otimes \mathbb{K}.$$

For further details about this isomorphism, see [49, Theorem 7.9.3].
Finally, the amenability of the skew product $\mathcal{G} \times_c \mathbb{Z}^k$ follows from that of $\mathcal{G}$ (see [52, Proposition II.3.8]), extending the result to the reduced C^*-algebras. □

The above result is a particular case of [27, Theorem 5.9], where the group $\mathbb{Z}^k$ can be replaced for any discrete group, and dual actions are replaced by coactions in the non-abelian case. Both of the families of groupoid involved in our work (Deaconu-Renault groupoids and self-similar groups) can be studied using a cocycle $c : \mathcal{G} \to \mathbb{Z}^k$. Thus, we stick to the simpler statement, and direct the reader to the work of Kaliszewski, Quigg and Raeburn for further details.

Remark 3.21. *Throught this work, we will usually write $A \rtimes_\alpha \mathbb{Z}$ simply as $A \rtimes \mathbb{Z}$, with α an action of $\mathbb{Z}$ on a C^*-algebra A. We will, however, keep the subindex whenever we want to outline the action involved.* □

Lemma 3.20 stablishes a stable isomorphism between the C^*-algebra associated to a groupoid and a certain crossed product C^*-algebra associated to its skew product groupoid. It is a well known fact that stable isomorphisms of C^*-algebras preserve the K-groups (see, for example, [53]). Hence, all is left to do is to find a way to describe the K-theory of a crossed product algebra $A \rtimes \mathbb{Z}^k$, given that we know $K_*(A)$. There are two major results in this matter. The first one is the *Pimsner-Voiculescu* exact sequence, a six-term exact sequence that relates $K_*(A)$ with $K_*(A \rtimes \mathbb{Z})$. The second one is *Kasparov's* spectral sequence, a more general result that allows us to study $K_*(A \rtimes \mathbb{Z}^k)$. Both approaches are strongly related. In fact, it is shown in [5, Proposition 10.4.1] that

the Pimsner-Voiculescu exact sequence can be deduced using the same techniques that the ones used to study Kasparov's spectral sequence.
We begin studying the Pimsner-Voiculescu exact sequence. We will provide a sketch of the proof, and direct the reader to the original source for further details: [51, Theorem 2.4] for the unital case, and [51, Remark 2.7] for the non-unital one.

Theorem 3.22. ***(Pimsner-Voiculescu exact sequence for K-theory)****.*
Let A be a C^*-algebra, and let α be an action of $\mathbb{Z}$ on A by automorphisms. Denote by $A \rtimes \mathbb{Z}$ the associated crossed product C^*-algebra (see, for example, [53]). Then the diagram

$$\begin{tikzcd}
K_0(A) \arrow[r, "id - K_0(\alpha)"] & K_0(A) \arrow[r, "K_0(j)"] & K_0(A \rtimes \mathbb{Z}) \arrow[d] \\
K_1(A \rtimes \mathbb{Z}) \arrow[u] & K_1(A) \arrow[l, "K_1(j)"] & K_1(A) \arrow[l, "id - K_1(\alpha)"]
\end{tikzcd}$$

is exact, where A is considered as a subalgebra of $A \rtimes \mathbb{Z}$ under the embedding j.

Proof. Let $C^*(S)$ be the C^*-algebra generated by a non-unitary isometry S. The *Toeplitz* algebra $\mathcal{T}_{(A,\alpha)}$ associated to the action $\alpha : \mathbb{Z} \curvearrowright A$ is defined as the subalgebra of $(A \rtimes \mathbb{Z}) \otimes C^*(S)$, generated by $A \otimes I$, and $u \otimes S^*$, where u denotes the unitary associated to the crossed product. We will use that there exists an isomorphism $K_*(\mathcal{T}_{(A,\alpha)}) \cong K_*(A)$ (see, for example, [51]).
By construction, one has the short exact sequence

$$0 \to \mathbb{K} \to C^*(S) \to C^*(\mathbb{T}) \to 0,$$

which induces a short exact sequence

$$0 \to A \otimes \mathbb{K} \xrightarrow{\iota} \mathcal{T}_{(A,\alpha)} \to A \rtimes \mathbb{Z} \to 0.$$

It can be proven that the isomorphism $K_*(\mathcal{T}_{(A,\alpha)}) \cong K_*(A)$ can be described by the following commutative diagram

$$\begin{tikzcd}
K_*(A \otimes \mathbb{K}) \arrow[r, "K_*(\iota)"] & K_*(\mathcal{T}_{(A,\alpha)}) \\
K_*(A) \arrow[u, "\cong"] \arrow[r, "id - K_*(\alpha)"'] & K_*(A) \arrow[u, "\cong"']
\end{tikzcd}$$

The Pimsner-Voiculescu exact sequence is then obtained as the six-term exact sequence of the extension above. □

In a more general setting, the following result studies the K-theory of a crossed product by $\mathbb{Z}^k$, via spectral sequences. Here we just state the theorem, appearing in [29, Theorem 2]. In the following section, we fully build the cohomological spectral sequence using the strategies of [4] and [54].

Theorem 3.23. *(**Kasparov's spectral sequence**).*
Let A be a C^*-algebra, and let α be an action of $\mathbb{Z}^k$ on A by automorphisms. Then, there is a homological spectral sequence:

$$E^2_{p,q} = H_p(\mathbb{Z}^k, K_q(A)) \Rightarrow K_{p+q}(A \rtimes \mathbb{Z}^k),$$

and a cohomological spectral sequence

$$E_2^{p,q} = H^p(\mathbb{Z}^k, K_q(A)) \Rightarrow K_{p+q+k}(A \rtimes \mathbb{Z}^k),$$

both converging to the K-theory of the associated crossed product. □

3.3 Spectral sequences

So far, we have introduced two major results involving spectral sequences, without talking the least about *what is* a spectral sequence. For this reason, in this section we provide a (very brief) guide to spectral sequences, for those who just want to understand the statements above. An in-depth guide on spectral sequences can be found in [39].

3.3.1 Terminology

Definition 3.24. *A (cohomological)* ***spectral sequence*** *in an abelian category $\mathcal{A}$ is a family $\{E_r^{p,q}\}$ of objects in $\mathcal{A}$, for all $p,q \in \mathbb{Z}$, $r \in \mathbb{N}$, together with maps*

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

that are differentials, in the sense that $d_r^{p,q} \circ d_r^{p-r,q+r-1} = 0$, satisfying that

$$E_{r+1}^{p,q} \cong ker(d_r^{p,q})/Im(d_r^{p-r,q+r-1}).$$

The bigraded object $\{E_r^{p,q}\}$ will be called the r-th page of the spectral sequence. One can picture the first page of a spectral sequence as follows

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\cdots & \longrightarrow & E_1^{0,3} & \longrightarrow & E_1^{1,3} & \longrightarrow & E_1^{2,3} & \longrightarrow & E_1^{3,3} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & E_1^{0,2} & \longrightarrow & E_1^{1,2} & \longrightarrow & E_1^{2,2} & \longrightarrow & E_1^{3,2} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & E_1^{0,1} & \longrightarrow & E_1^{1,1} & \longrightarrow & E_1^{2,1} & \longrightarrow & E_1^{3,1} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & E_1^{0,0} & \longrightarrow & E_1^{1,0} & \longrightarrow & E_1^{2,0} & \longrightarrow & E_1^{3,0} & \longrightarrow & \cdots
\end{array}$$

We say that a spectral sequence ***reaches its limit*** *at the r_0-th page if $E_r^{p,q} = E_{r_0}^{p,q}$, for every $p,q \in \mathbb{Z}$, and for every $r \geq r_0$.* □

Remark 3.25. *Analogously, one can define a (homological) spectral sequence by inverting the bidegree of the differential maps, that is, $d_{p,q}^r : E_{p,q}^r \to E_{p-r,q+r-1}^r$. All of the results shown here can be rephrased to the homology counterpart.* □

One can see that $E_{r+1}^{p,q}$ is a subquotient of $E_r^{p,q}$, for all r,p,q. More precisely, denote $Z_1 = ker(d_1)$, and $B_1 = Im(d_1)$. Although Z_1, B_1 are bigraded objects, we drop the superindexes p,q for a matter of clarity. The differential condition over the maps d_r implies $B_1 \subseteq Z_1 \subseteq E_1$. Write $\overline{Z_2} = ker(d_2 : E_2 \to E_2)$. Clearly $\overline{Z_2} \subseteq E_2 = Z_1/B_1$, and hence it can be written as Z_2/B_1, for some $Z_2 \subseteq Z_1$. Similarly, we can find some B_2 such that $Im(d_2) := \overline{B_2} = B_2/B_1$, and so

$$B_1 \subseteq B_2 \subseteq Z_2 \subseteq Z_1.$$

Moreover, $E_3 \cong \overline{Z_2}/\overline{B_2} \cong (Z_2/B_1)/(B_2/B_1) \cong Z_2/B_2$.
Iterating this process, we obtain an infinite tower

$$B_1 \subseteq B_2 \subseteq B_3 \subseteq ... \subseteq Z_3 \subseteq Z_2 \subseteq Z_1,$$

satisfying $E_{r+1} \cong Z_r/B_r$, for each $r \in \mathbb{N}$.

Definition 3.26. *Let $(E_r^{p,q}, d_r^{p,q})$ be a spectral sequence in $\mathcal{A}$. Define the bigraded objects:*

$$B_\infty = \bigcup_{r=1}^{\infty} B_r, \text{ and}$$
$$Z_\infty = \bigcap_{r=1}^{\infty} Z_r.$$

*Then we define the **limit term** as $E_\infty^{p,q} := Z_\infty^{p,q}/B_\infty^{p,q}$.*
If the spectral sequence reaches its limit at some r_0-th page, then $E_\infty^{p,q} = E_{r_0}^{p,q}$. □

We now provide the definition of convergence. We show two definitions: one for the homology and another for the cohomology case, since the extrapolation from one to another may not be trivial.

Definition 3.27. *([64, Definition 5.2.11]) Let $H_* := \{H_n\}_n$ be a family of objects in $\mathcal{A}$.*

- *We say that a homological spectral sequence $(E_{p,q}^r, d_{p,q}^r)$ **weakly converges** to H_* if, for every n, H_n has a filtration*

$$... \subseteq F_{p-1}H_n \subseteq F_pH_n \subseteq F_{p+1}H_n \subseteq ... \subseteq H_n,$$

  such that, for every p, q, there exists an isomorphism $E_{p,q}^\infty \cong F_pH_{p+q}/F_{p-1}H_{p+q}$.

- *We say that $(E_{p,q}^r, d_{p,q}^r)$ **approaches** H_* if it weakly converges to H_*, and it satisfies $H_n = \cup F_pH_n$, $0 = \cap F_pH_n$.*

- *Then $(E_{p,q}^r, d_{p,q}^r)$ **converges** to H_* if all of the following conditions hold:*
  1. *$(E_{p,q}^r, d_{p,q}^r)$ approaches H_*.*
  2. *$(E_{p,q}^r, d_{p,q}^r)$ is regular, meaning that, for each p, q, the differentials $d_{p,q}^r$ are all eventually zero for large enough r.*
  3. *For each n, $H_n = \varprojlim(H_n/F_pH_n)$.*

The usual notation for a convergent homological spectral sequence is:

$$E_{p,q}^r \Rightarrow H_{p+q}$$

□

Definition 3.28. *Let $H^* := \{H^n\}_n$ be a family of objects in $\mathcal{A}$.*

- *We say that a cohomological spectral sequence $(E_r^{p,q}, d_r^{p,q})$* ***weakly converges*** *to H^* if, for every n, H^n has a decreasing filtration*
$$... \subseteq F^{p+1}H^n \subseteq F^pH^n \subseteq F^{p-1}H^n \subseteq ... \subseteq H^n,$$
such that, for every p, q, there exists an isomorphism $E_\infty^{p,q} \cong F^pH^{p+q}/F^{p+1}H^{p+q}$.

- *We say that $(E_r^{p,q}, d_r^{p,q})$* ***approaches*** *H^* if it weakly converges to H^*, and it satisfies $H^n = \cup F^pH^n$, $0 = \cap F^pH^n$.*

- *Then $(E_r^{p,q}, d_r^{p,q})$* ***converges*** *to H^* if all of the following conditions hold:*
  1. *$(E_r^{p,q}, d_r^{p,q})$ approaches H^*.*
  2. *$(E_r^{p,q}, d_r^{p,q})$ is regular, meaning that, for each p, q, the differentials $d_r^{p,q}$ are all eventually zero for large enough r.*
  3. *For each n, $H^n = \varprojlim_{p\to-\infty}(H^n/F^pH^n)$.*

The usual notation for a convergent cohomological spectral sequence is:
$$E_r^{p,q} \Rightarrow H^{p+q}$$

□

Note that we can ignore the last condition whenever the filtration is finite, in the sense that, for all n, $F^tH^n = F^{t+1}H^n$ for large enough t, and $F^sH^n = F^{s-1}H^n$, for small enough s. Finite filtrations such that $F^tH^n = 0$ and $F^sH^n = H^n$ are usually called *bounded*. A similar argument can be made in the case of homological spectral sequences. We provide here a few examples of convergent spectral sequences, for both the homology and cohomology cases:

Example 3.29. *We say that a spectral sequence* ***collapses*** *at the r-th page ($r \geq 1$) if there is only one non-zero row (or column) in $\{E_{p,q}^r\}$. A collapsing spectral sequence reaches its limit at the r-th page, since the differential maps are forced to be zero. Straightforward computation shows that a collapsing spectral sequence converges to the limit term $H_n := E_{n-q,q}^r = E_{n-q,q}^\infty$, where q is the only non-zero row. Indeed, consider the trivial filtration F_pH_n of H_n given by*
$$... \subseteq 0 \subseteq ... \subseteq 0 \subseteq H_n \subseteq H_n \subseteq ...$$
where $F_jH_n = H_n$, whenever $j \geq n-q$, and 0 otherwise.
Then, for any fixed n, $E_{n-q,q}^\infty = H_n = H_n/\{0\} = F_{n-q}H_n/F_{n-q-1}H_n$, as desired.

□

It is important to note that a great number of applications of spectral sequences rely on spectral sequences that collapse in the early pages.

Example 3.30. *Suppose that a homological spectral sequence converging to H_* has $E^2_{p,q} = 0$ unless $p = 0, 1$. Then there are short exact sequences of the form*

$$0 \to E^2_{0,n} \to H_n \to E^2_{1,n-1} \to 0$$

Indeed, the differential maps $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ are trivial for all $r \geq 2$, since either domain or codomain lies on a row different than $0, 1$, forcing them to be zero. Hence the spectral sequence reaches its limit at the page 2. Then, convergence of the spectral sequence means that there exists a filtration F_pH_n of H_n such that, for each p, q, we have

$$F_pH_{p+q}/F_{p-1}H_{p+q} \cong E^2_{p,q}.$$

Then, for all $p < 0$, we have $F_pH_{p+q}/F_{p-1}H_{p+q} = 0$, and therefore $F_pH_{p+q} = F_{p-1}H_{p+q}$. Since the spectral sequence approaches H_*, we deduce that $F_pH_{p+q} = 0$ for all $p < 0$. A similar argument can be made to show that $F_pH_{p+q} = H_{p+q}$ for all $p \geq 1$.
Then, for $p = 1, q = n - 1$, we have that

$$F_1H_n/F_0H_n = H_n/F_0H_n \cong E^2_{1,n-1},$$

and for $p = 0, q = n$

$$F_0H_n/F_{-1}H_n = F_0H_n \cong E^2_{0,n},$$

Therefore, we deduce that there exists a short exact sequence

$$0 \to E^2_{0,n} \to H_n \to E^2_{1,n-1} \to 0$$

as claimed.

□

The following two examples will appear when dealing with Deaconu-Renault groupoids later in this document. They are, mostly, the same one, but one refers to a homological spectral sequence, and the other to a cohomological one. We believe it is worthwhile to put both here, so the reader may notice the subtle differences.

Example 3.31. *Suppose that a homological spectral sequence $(E^r_{p,q}, d^r)$ converging to H_* has $E^2_{p,q} = 0$ whenever q is odd, and suppose that $E^2_{p,q} = 0$ unless $p = 0, 1, 2$. Then we have*

$$E^2_{1,n-1} \cong H_n$$

whenever n is odd, and

$$0 \to E^2_{0,n} \to H_n \to E^2_{2,n-2} \to 0$$

whenever n is even. Let us show it:
First, note that, since all the odd rows are zero, the differentials of the second page $d^2_{p,q} : E^2_{p,q} \to E^2_{p-2,q+2-1}$ are all zero (since either domain or codomain lies in an odd row), and hence $E^2_{p,q} = E^3_{p,q}$. Moreover, since $E^2_{p,q}$ is zero unless $p = 0, 1, 2$, the differentials $d^r_{p,q}$ are also zero for every $r > 2$ (since either domain or codomain lies in a column different than $0, 1, 2$). Hence, we conclude that the spectral sequence reaches its limit at the page 2.
For each $n \in \mathbb{Z}$, let F_pH_n be the filtrations for H_n associated to the convergence of the spectral sequence. Using similar arguments as the ones in the previous example, it is straightforward to check that $F_pH_n = 0$ whenever $p < 0$, and $F_pH_n = H_n$ whenever $p \geq 2$. Hence, the filtrations are of the form

$$\{0\} = F_{-1}H_n \subseteq F_0H_n \subseteq F_1H_n \subseteq F_2H_n = H_n.$$

We study the filtrations of H_n considering separately the two cases, n odd and n even. First, suppose that n is odd. Then $F_2H_n/F_1H_n \cong E^2_{2,n-2} = 0$, and hence $H_n = F_2H_n = F_1H_n$. Moreover, $F_0H_n/F_{-1}H_n \cong E^2_{0,n} = 0$, and thus $F_0H_n = F_{-1}H_n = 0$. Therefore

$$F_1H_n/F_0H_n = H_n \cong E^2_{1,n-1},$$

as desired.
Now, suppose that n is even. Then $F_2H_n/F_1H_n \cong E^2_{2,n-2}$. Moreover, $F_1H_n/F_0H_n \cong E^2_{1,n-1} = 0$, meaning that $F_1H_n = F_0H_n$, and $F_0H_n/F_{-1}H_n = F_0H_n \cong E^2_{0,n}$. Combining those three results, together with the fact that $F_2H_n = H_n$, we obtain a short exact sequence

$$0 \to E^2_{0,n} \to H_n \to E^2_{2,n-2} \to 0$$

as claimed.
A similar study can be made for any finite number of columns, although the extension problems grow.
The reader may note that, if the rows repeat periodically (in the sense that $E^2_{p,i} = E^2_{p,2q+i}$, for $i = 0, 1$, and all p, q), then H_{2n+i} and H_i agree up to extensions problems, with $n \in \mathbb{Z}$ and $i = 0, 1$. This is usually the case when dealing with K-theory.

$\square$

Example 3.32. *Suppose that a cohomological spectral sequence $(E^{p,q}_r, d_r)$ converging to H^* has $E^{p,q}_2 = 0$ whenever q is odd, and suppose that $E^{p,q}_2 = 0$ unless $p = 0, 1, 2$. Then we have*

$$E_2^{1,n-1} \cong H^n$$

whenever n is odd, and

$$0 \to E_2^{2,n-2} \to H^n \to E_2^{0,n} \to 0$$

whenever n is even. Let us show it:
Using the same argument as before, we can conclude that the spectral sequence reaches its limit at the page 2.
For each $n \in \mathbb{Z}$, let F^pH^n be the decreasing filtrations for H^n associated to the convergence of the spectral sequence. As before, straightforward computation shows that $F^pH^n = 0$ whenever $p > 2$, and $F^pH^n = H^n$ whenever $p \leq 0$. Hence, the filtrations are of the form

$$\{0\} = F^3H^n \subseteq F^2H^n \subseteq F^1H^n \subseteq F^0H^n = H^n.$$

We study the filtrations of H^n considering separately the two cases, n odd and n even. First, suppose that n is odd. Then $F^2H^n/F^3H^n \cong E_2^{2,n-2} = 0$, and hence $F^2H^n = 0$. Moreover, $F^0H^n/F^1H^n \cong E_2^{0,n} = 0$, and thus $F^0H^n = F^1H^n = 0$. Therefore

$$H^n = F^0H^n = F^1H^n = F^1H^n/F^2H^n \cong E_2^{1,n-1},$$

as desired.
Now, suppose that n is even. Then $F^1H^n/F^2H^n \cong E_2^{1,n-1} = 0$, meaning that $F^1H^n = F^2H^n$. Therefore, since $F^3H^n = 0$, we have $E_2^{2,n-2} \cong F^2H^n/F^3H^n = F^2H^n = F^1H^n$. Then, using the isomorphism $E_2^{0,n} \cong F^0H^n/F^1H^n$, together with the fact $F^0H^n = H^n$, we obtain a short exact sequence

$$0 \to E_2^{2,n-2} \to H^n \to E_2^{0,n} \to 0$$

as claimed.
Both remarks at the end of Example 3.31 can also be used here.

□

We show one last example, similar to the previous one. We will use this in Subsection 4.4.1.

Example 3.33. *Suppose that a cohomological spectral sequence $(E_r^{p,q}, d_r)$ converging to H^* has $E_2^{p,q} = 0$ whenever q is odd, and suppose that $E_2^{p,q} = 0$ unless $p = 0, 1, 2, 3$. Then we have exact sequences*

$$0 \to coker(d_3^{0,n-1}) \to H^n \to E_2^{1,n-1} \to 0$$

whenever n is odd, and

$$0 \to E_2^{2,n-2} \to H^n \to ker(d_3^{0,n}) \to 0$$

whenever n is even. Let us show it:
Reasoning as in Example 3.32, we have that all the differential maps of the second page are zero, and hence $E_2 = E_3$. Moreover, the third page of the spectral sequence is of the form

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & E_2^{0,0} \arrow[ddrrr, "d_3^{0,0}"] & E_2^{1,0} & E_2^{2,0} & E_2^{3,0} & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & E_2^{0,-2} & E_2^{1,-2} & E_2^{2,-2} & E_2^{3,-2} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

where the only maps that may not be zero are $d_3^{0,n}$, with n even. Therefore, for all q even, we have:

$$E_\infty^{p,q} = E_2^{p,q}, \text{ whenever } p = 1, 2,$$
$$E_\infty^{0,q} = E_4^{0,q} = ker(d_3^{0,q}), \text{ and}$$
$$E_\infty^{3,q} = E_4^{3,q} = coker(d_3^{0,q+2}).$$

Trivially, we have $E_\infty^{p,q} = 0$ whenever q is odd.
We keep the convention $n = p + q$. Then, straightforward computation shows that $F^pH^n = 0$ whenever $p > 3$, and $F^pH^n = H^n$ whenever $p \leq 0$. Hence, the filtrations are of the form

$$\{0\} = F^4H^n \subseteq F^3H^n \subseteq F^2H^n \subseteq F^1H^n \subseteq F^0H^n = H^n.$$

Suppose that n is odd. Then we have:

$$F^0H^n/F^1H^n \cong E_\infty^{0,n} = 0, \text{ and hence } F^0H^n = F^1H^n = H^n.$$
$$F^1H^n/F^2H^n \cong E_\infty^{1,n-1} = E_2^{1,n-1}.$$
$$F^2H^n/F^3H^n \cong E_\infty^{2,n-2} = 0, \text{ and hence } F^2H^n = F^3H^n.$$
$$F^3H^n/F^4H^n \cong E_\infty^{3,n-3} = coker(d_3^{0,n-1}).$$

Since $F^4H^n = 0$, the last equality translates into $F^3H^n = coker(d_3^{0,n-1})$. Then, combining all four equalities, we have a short exact sequence

$$0 \to coker(d_3^{0,n-1}) \to H^n \to E_2^{1,n-1} \to 0,$$

as desired.
Now suppose n even. Then we have:

$$F^0H^n/F^1H^n \cong E_\infty^{0,n} = ker(d_3^{0,n}).$$
$$F^1H^n/F^2H^n \cong E_\infty^{1,n-1} = 0, \text{ and hence } F^1H^n = F^2H^n.$$
$$F^2H^n/F^3H^n \cong E_\infty^{2,n-2} = E_2^{2,n-2}.$$
$$F^3H^n/F^4H^n \cong E_\infty^{3,n-3} = 0, \text{ and hence } F^3H^n = F^4H^n = 0.$$

Therefore, combining all four equalities (and since $F^0H^n = H^n$), we obtain a short exact sequence

$$0 \to E_2^{2,n-2} \to H^n \to ker(d_3^{0,n}) \to 0,$$

concluding the proof. □

3.3.2 Exact couples

We have already shown *what* is a spectral sequence, but it remains to be seen *how* they are built. In this section, we introduce one of the more common ways of creating a spectral sequence. More methods can be found in [64, Chapter 5].

Definition 3.34. *An **exact couple** is a pair of objects (D, E) in an abelian category $\mathcal{A}$, together with three morphisms i, j, k, such that the following diagram*

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

is exact at every vertex, that is, $ker(j) = Im(i)$, $ker(k) = Im(j)$ and $ker(i) = Im(k)$. □

Given an exact couple, one can always build the associated *derived couple*.
Let (D, E, i, j, k) be an exact couple. The map $d_1 := j \circ k$ verifies $d_1 \circ d_1 = 0$, since $(j \circ k) \circ (j \circ k) = j \circ (k \circ j) \circ k = 0$. Then, it makes sense to consider the associated homology $E_2 := ker(d_1)/Im(d_1)$. Consider then the diagram

$$\begin{tikzcd} i(D) \arrow[rr, "i'"] & & i(D) \arrow[dl, "j'"] \\ & E_2 \arrow[ul, "k'"] & \end{tikzcd}$$

where the morphisms are defined as:

$$i' := i|_{i(D)},$$
$$j'(i(a)) := [j(a)],\ a \in D, \text{ and}$$
$$k'([e]) = k(e), \text{ for } [e] \in E_2.$$

Let us check that those maps are well defined. If $i(a) = 0$, then $a \in Im(k)$, and hence $a = k(e)$ for some $e \in E$. Therefore, $j(a) = j(k(e)) \in Im(j \circ k)$. In particular, $i(a) = 0$ implies $[j(a)] = 0$ in E_2.

On the other hand, $k'([e_1 + j(k(e_2))]) = k(e_1 + j(k(e_2))) = k(e_1) + (k \circ j \circ k)(e_2) = k(e_1) = k'([e_1])$. Also, $e \in ker(j \circ k)$ implies that $(j \circ k)(e) = 0$, and thus $k(e) \in ker(j) = Im(i)$. Hence $k'([e]) = k(e) \in i(D)$, and k' is well defined.

Proposition 3.35. *The triangle $(i(D), E_2, i', j', k')$ is an exact couple. We called this exact couple the* ***derived couple*** *of (D, E, i, j, k).*

Proof. To check the exactness, let's start with the upper left vertex:

Consider $[e] \in ker(j \circ k)/Im(j \circ k)$. Then $i'(k'[e]) = i(k(e)) = 0$, so $Im(k') \subseteq ker(i')$. For the other inclusion, take $a \in Im(i)$ such that $i'(a) = i(a) = 0$. Then $a = k(e)$ for some $e \in E$. Since $a \in Im(i)$, write $a = i(a')$ for some $a' \in D$. Then $j(k(e)) = j(a) = j(i(a')) = 0$, therefore $e \in ker(j \circ k)$. Taking $[e] \in ker(j \circ k)/Im(j \circ k)$, we have $k'[e] = k(e) = a$, and so $ker(i') = Im(k')$.

Let us now study the upper right vertex:

Let $a \in Im(i)$, and $a' \in D$ such that $i(a') = a$. Then $j'(i'(a)) = j'(i(a)) = [j(a)] = [j(i(a'))] = 0$, hence $Im(i') \subseteq ker(j')$. On the other hand, let $a = i(a')$ for some $a' \in D$, such that $j'(a) = [j(a')] = 0$. Then $j(a') \in Im(j \circ k)$, so there exists some $e \in E$ such that $j(a') = (j \circ k)(e)$. Then $j(a' - k(e)) = 0$, so $a' - k(e) \in ker(j) = Im(i)$. Hence we can find some $b \in D$ such that $a' = k(e) + i(b)$. In particular, a can be written as $a = i(a') = i(k(e)) + (i \circ i)(b) = (i \circ i)(b) \in Im(i')$, and so $ker(j') = Im(i')$.

Finally, for the lower vertex:

Let $a \in Im(i)$, with $a = i(a')$ for some $a' \in D$. Then we have $k'(j'(a)) = k'[j(a')] = k(j(a)) = 0$, and so $Im(j') \subseteq ker(k')$. On the other direction, consider $[e] \in ker(j \circ k)/Im(j \circ k)$ such that $k'([e]) = k(e) = 0$. Then $e \in ker(k) = Im(j)$. Hence $e = j(a)$ for some $a \in D$, and we have $[e] = [j(a)] = j'(i(a))$, concluding the proof. □

As the reader may have noticed, this process can be iterated indefinitely. We can use this to build a spectral sequence.

Proposition 3.36. *Let $D := D^{p,q}$ and $E := E^{p,q}$ be bigraded objects in $\mathcal{A}$, together with morphisms*

$$i : D \to D$$
$$j : D \to E$$
$$k : E \to D$$

of bidegree $(-1,1)$, $(1,0)$ *and* $(0,0)$ *respectively, and such that the triangle is exact, in the sense that*

$$Im(i) = ker(j),$$
$$Im(j) = ker(k), \text{ and}$$
$$Im(k) = ker(i).$$

Fix $E_1 := E$, $D_1 := D$, $i_1 := i$, $j_1 := j$, *and* $k_1 := k$. *Then, for each* $r \geq 1$, *define* $(E_{r+1}, D_{r+1}, i_{r+1}, j_{r+1}, k_{r+1})$ *to be the derived couple of* $(E_r, D_r, i_r, j_r, k_r)$, *and denote* $d_r := j_r \circ k_r$. *Then the pair* (E_r, d_r) *is a (cohomological) spectral sequence.*

Proof. By definition, the maps d_r are all differentials. Straightforward computation shows that their bidegree is $(r, -r+1)$, concluding the proof. □

Strictly speaking, one can allow k to have bidegree $(m, -m)$, with $m \in \mathbb{N}$, and still obtain a spectral sequence (starting at the $(m+1)$-page, instead to the first one).
Moreover, a homology exact couple can be defined in a similar way, replacing the bidegrees of i, j, k for $(1,-1)$, $(0,0)$ and $(-1,0)$, respectively. This allows the differential maps d^r of the homology spectral sequence to have bidegree $(-r, r-1)$.

3.3.3 Spectral sequences associated to cofiltrations of C^*-algebras

One of the most common ways to build convergent spectral sequences is associating an exact couple to a finite *filtration* or *cofiltration*. Here we show how.

Definition 3.37. *A finite cofiltration of a* C^**-algebra* B *is a family of* C^**-algebras* $F_n, F_{n-1}, ..., F_{-1}$, *together with surjective maps* π_i:

$$B = F_n \overset{\pi_n}{\to} F_{n-1} \overset{\pi_{n-1}}{\to} ... \overset{\pi_1}{\to} F_0 \overset{\pi_0}{\to} F_{-1} = 0.$$

□

Any finite cofiltration of a given C^*-algebra B induces a series of exact sequences:

$$0 \to I_k \overset{i_k}{\to} F_k \overset{\pi_k}{\to} F_{k-1} \to 0, \tag{3.3}$$

where $I_k := ker(\pi_k)$. We can use this to build an exact couple.

Proposition 3.38. *A finite cofiltration of a* C^**-algebra* B *induces a bigraded exact couple (i.e. an exact couple with bigraded objects) and, hence, a (cohomological) spectral sequence.*

Proof. Consider the bigraded objects D_1, E_1 given by $D_1 := D_1^{p,q}$, $E_1 := E_1^{p,q}$, where $D_1^{p,q} = K_{p+q}(F_p)$ and $E_1^{p,q} = K_{p+q}(I_p)$ Denote by $\delta_*^{(k)}$ the boundary maps from $K_*(F_{k-1})$ to $K_{*+1}(I_k)$ appearing in the respective six-term exact sequences. This, together with the induced maps in K-theory associated to i_k, π_k, defines the morphisms

$$\begin{aligned} K_*(\pi_*) &: D_1 \to D_1, \\ \delta_*^* &: D_1 \to E_1, \text{ and} \\ K_*(i_*) &: E_1 \to D_1, \end{aligned}$$

of bidegree $(-1, 1)$, $(1, 0)$ and $(0, 0)$, respectively. The six-term exact sequences of (3.3) ensures that the following triangle is exact

$$\begin{tikzcd} D_1 \arrow[rr, "K_*(\pi_*)"] & & D_1 \arrow[dl, "\delta_*^*"] \\ & E_1 \arrow[ul, "K_*(i_*)"] & \end{tikzcd}$$

and hence $(D_1, E_1, K_*(\pi_*), \delta_*^*, K_*(i_*))$ is a cohomology exact couple. ☐

This spectral sequence converges to the K-theory of the C^*-algebra B. This is consequence of the following two theorems, which we just state here, and direct the reader to their original sources for further details:

Theorem 3.39. *([57, Theorem 2.1]). Suppose given a filtered C^*-algebra*

$$A_0 \subset A_1 \subset \ldots \subset A_n \subset \ldots \subset B$$

with $\overline{\bigcup A_n} = B$. Define the bigraded objects $L^1 := L^1_{p,q} = K_{p+q}(A_p)$, and $E^1 := E^1_{p,q} = K_{p+q}(A_p/A_{p-1})$, and denote by $\theta_*^* : E^1_{*,*} \to L^1_{*,*}$ the respective index maps. Then the homology spectral sequence associated to the exact couple

$$\begin{tikzcd} L^1 \arrow[rr, "K_*(i_*)"] & & L^1 \arrow[dl, "K_*(\omega_*)"] \\ & E^1 \arrow[ul, "\theta_*^*"] & \end{tikzcd}$$

converges to $K_*(B)$, where i_*, ω_* are the respective inclusion and projection maps.
If the filtration is finite, with $A_n = B$ for $n \geq N$, then $E^1_{p,q} = 0$ for $p \geq N+1$, and $E^N = E^\infty$. ☐

Theorem 3.40. *([54, Theorem 9]). The spectral sequence for K-theory of a C^*-algebra B associated with a finite filtration of B by ideals A_p, and the spectral sequence associated with its corresponding finite cofiltration by quotients $F_p = B/A_{n-p-1}$, are isomorphic, in the sense that their associated exact couples are equivalent (see [54, Definition 17]).* □

Combining both statements, we conclude:

Corollary 3.41. *Let*

$$B = F_n \xrightarrow{\pi_n} F_{n-1} \xrightarrow{\pi_{n-1}} \dots \xrightarrow{\pi_1} F_0 \xrightarrow{\pi_0} F_{-1} = 0$$

be a finite cofiltration of a C^*-algebra B. The spectral sequence arising from the exact couple $(D_1, E_1, K_*(\pi_*), \delta_*^*, K_*(i_*))$ associated to Equation (3.3), converges to $K_*(B)$. Moreover, $E_1^{p,q} = 0$ for $p \geq n+1$, and $E_n = E_\infty$.

Proof. For each p, we have a surjection $\Delta_p : B \to F_p$, given as the composition of the consecutive maps π_*. Then the kernels of Δ_p provide a filtration of B, which satisfies the relation of Theorem 3.40. Using Theorem 3.39, this spectral sequence converges to $K_*(B)$, concluding the proof. □

Remark 3.42. *There is a disparity of notation between both sources [57] and [54]. In the first one, the author uses a filtration with* increasing *subindexes. In the second one, the authors use* decreasing *subindexes. We have adapted the notation appearing in the second one ([54, Theorem 9]), in order to match the convention of [57] (which is the one we use throughout this work).* □

Finally, we provide a picture of the consecutive differential maps of the spectral sequence. In [4], the author gives a general description of the differential maps of the r-page, following the work appearing in [54].

Lemma 3.43. *Let $[x] \in E_r^{p,q}$ be represented by $x \in K_{p+q}(I_p)$, and consider its image under the map induced by the natural inclusion, given by $K_{p+q}(i_p)(x) \in K_{p+q}(F_p)$. Then there exists a lift $y \in K_{p+q}(F_{p+r-1})$ for $K_{p+q}(i_p)(x)$ under the map $K_{p+q}(F_{p+r-1}) \to K_{p+q}(F_p)$, such that*

$$d_r^{p,q}([x]) = [\delta_{p+q}^{(p+r)}(y)] \in E_r^{p+r,q-r+1}.$$

□

3.3.4 A spectral sequence for the crossed product by an action of $\mathbb{Z}^k$

Here we show a method appearing in [4] and [54], in which Kasparov's cohomological spectral sequence (see Theorem 3.23) is built explicitly from a cofiltration associated to the crossed product algebra. Recall that, for a C^*-dynamical system $(A, \alpha, \mathbb{Z}^k)$, Kasparov's cohomological spectral sequence is of the form

$$E_2^{p,q} \cong H^p(\mathbb{Z}^k, K_q(A)) \Rightarrow K_{p+q+k}(A \rtimes \mathbb{Z}^k).$$

The construction goes as follows.
First, we define $\mathcal{M}_\alpha(A)$, the *mapping torus* C^*-algebra associated to a dynamical sistem $(A, \alpha, \mathbb{Z}^k)$. This C^*-algebra satisfies $K_{*+k}(\mathcal{M}_\alpha(A)) \cong K_*(A \rtimes \mathbb{Z}^k)$. After that, we find certain cofiltration of the mapping torus, and apply the previous section. This technique appeared originally in [54]. Let us show it.

3.3.4.1 The mapping torus

We give here a definition for the mapping torus associated to a $\mathbb{Z}^k$-action. It can be generalized to any abelian group (see [45]).

Definition 3.44. *Let $(A, \alpha, \mathbb{Z}^k)$ be a C^*-dynamical system. The associated **mapping torus** C^*-algebra is defined as*

$$\begin{gathered}\mathcal{M}_\alpha(A) := \{f \in C([0,1]^k, A) : f(t_1, ..., t_{i-1}, 1, t_{i+1}, ..., t_k) = \\ \alpha_i(f(t_1, ..., t_{i-1}, 0, t_{i+1}, ..., t_k)),\ for\ all\ 1 \leq i \leq k\},\end{gathered}$$

where α_i denotes the action of the generator e_i of $\mathbb{Z}^k$. □

Remark 3.45. *We give two remarks on this definition:*

- *There is an equivalent definition of $\mathcal{M}_\alpha$ given by*
$$\mathcal{M}_\alpha(A) := \{f \in C(\mathbb{R}^k, A) : f(x+z) = \alpha_z(f(x)),\ z \in \mathbb{Z}^k\}.$$
- *Moreover, the mapping torus of a $\mathbb{Z}^k$-dynamical system can be defined inductively. Indeed, let $\alpha = (\alpha_1, ..., \alpha_k)$. Then the reader may check that*
$$\mathcal{M}_\alpha(A) \cong \mathcal{M}_{\alpha_k}(\mathcal{M}_{(\alpha_1,...,\alpha_{k-1})}(A)).$$

□

The K-theory of the mapping torus has been broadly studied (see [5], [4], [48] or [54], for example). The following theorem, a special case of [45, Corollary 2.5], relates the K-theory of the mapping torus with the one of the crossed product. A proof of this particular case can be found in [4, Theorem 1.2.6], or [48]. In the following section, we show an explicit description of this result, appearing in [48]. Moreover, we prove some key properties for the isomorphism. We will use this to prove Theorem 3.50, one of the main technical results of this document, which will allow us to study the associated K-theory of Deaconu-Renault groupoids in Chapter 4.

Theorem 3.46. *([45, Corollary 2.5]) Given a C^*-dynamical system $(A, \alpha, \mathbb{Z}^k)$, there exists an isomorphism*

$$\Psi_A^{(k)} : K_{*+k}(\mathcal{M}_\alpha(A)) \cong K_*(A \rtimes_\alpha \mathbb{Z}^k).$$

□

3.3.4.2 Naturality of the isomorphism

Here we prove that the isomorphism Ψ_A is natural in various senses, similar to the axiomatics presented by Alain Connes [13]. We will use the following explicit description appearing in [48].

We first recall various basic definitions and facts.
Let A be a C^*-algebra and α an automorphism of A. There are two natural sequences associated to (A, α). The first one involves the mapping torus $\mathcal{M}_\alpha(A)$:

$$0 \longrightarrow SA \longrightarrow \mathcal{M}_\alpha(A) \longrightarrow A \longrightarrow 0 \tag{3.4}$$

where $\mathcal{M}_\alpha(A) \to A$ is given by evaluation at 0. The second one is the sequence

$$0 \longrightarrow A \otimes \mathbb{K} \longrightarrow \mathcal{T}_{A,\alpha} \longrightarrow A \rtimes_\alpha \mathbb{Z} \longrightarrow 0 \tag{3.5}$$

associated to the Toeplitz extension of the crossed product $A \rtimes_\alpha \mathbb{Z}$.
The first sequence (3.4) gives rise to a 6-term exact sequence

$$\begin{tikzcd} K_1(A) \arrow[r, "\zeta_A^1"] & K_0(\mathcal{M}_\alpha(A)) \arrow[r] & K_0(A) \arrow[d] \\ K_1(A) \arrow[u] & K_1(\mathcal{M}_\alpha(A)) \arrow[l] & K_0(A) \arrow[l, "\zeta_A^0"'] \end{tikzcd} \tag{3.6}$$

where $\zeta_A^1 \colon K_1(A) \to K_0(\mathcal{M}_\alpha(A))$ is the composition of the Bott isomorphism $\theta_A \colon K_1(A) \to K_0(SA)$ and the map $K_0(SA) \to K_0(\mathcal{M}_\alpha(A))$ induced by the inclusion $SA \to \mathcal{M}_\alpha(A)$, and similarly for ζ_A^0.
On the other hand, the Toeplitz extension (3.5) gives rise to the Pimsner-Voiculescu 6-term exact sequence (see Theorem 3.22 and its proof)

$$\begin{tikzcd} K_0(A) \arrow[r, "id-K_0(\alpha)"] & K_0(A) \arrow[r] & K_0(A \rtimes_\alpha \mathbb{Z}) \arrow[d] \\ K_1(A \rtimes_\alpha \mathbb{Z}) \arrow[u] & K_1(A) \arrow[l] & K_1(A) \arrow[l, "id-K_1(\alpha)"'] \end{tikzcd} \tag{3.7}$$

Paschke showed in [48] that the vertical maps in the sequence (3.6) are equal to $id - K_*(\alpha)$, and that there are isomorphisms $K_i(\mathcal{M}_\alpha(A)) \to K_{i+1}(A \rtimes_\alpha \mathbb{Z})$ which provide an isomorphism from the exact sequence (3.6) to the exact sequence (3.7).

We now recall the definition of Ψ_A, following [48].
Suppose first that A is unital and α is an automorphism of A. Then we define

$$\Psi_A\colon K_0(\mathcal{M}_\alpha(A)) \to K_1(A \rtimes_\alpha \mathbb{Z})$$

by

$$\Psi_A([p]) = [L^* w_1 p(0) + 1_n - p(0)].$$

Here p is a path of projections in $M_n(A)$ such that $p(1) = \alpha(p(0))$ and $\{w_t\}$ is an implementing path of unitaries in $M_n(A)$ for $p(t)$, and L is the canonical unitary associated to the action.
We give also the definition for non-unital A. We will denote by $\tilde{A}$ the unitization of A, and by $s\colon \tilde{A} \to \tilde{A}$ the scalar map of Definition 2.66. In this case the map $\Psi_A\colon K_0(\mathcal{M}_\alpha(A)) \to K_1(A \rtimes_\alpha \mathbb{Z})$ is the restriction of $\Psi_{\tilde{A}}\colon K_0(\mathcal{M}_{\tilde{\alpha}}(\tilde{A})) \to \tilde{A} \rtimes_{\tilde{\alpha}} \mathbb{Z}$ to $K_0(\mathcal{M}_\alpha(A))$, where $\tilde{\alpha}$ denotes the automorphism in $\tilde{A}$ induced by α. We shall always use a concrete picture of this map, compatible with the usual conventions for suspension maps.
An element of $K_0(\mathcal{M}_\alpha(A))$ is represented by an element of the form $[p] - [1_n]$, where p is a projection in $M_n(\mathcal{M}_{\tilde{\alpha}}(\tilde{A}))$, such that $s(p(t)) = 1_n$ for all $t \in [0,1]$. Now by inspecting the proof of [48, Lemma 2], one realizes that the implementing path $\{w_t\}$ can be chosen such that $w_0 = 1_n$ and $s(w_t) = 1_n$ for all $t \in [0,1]$. We will always use this representation.
Note that we obtain

$$\Psi_A([p] - [1_n]) = [L^* w_1 p(0) + 1_n - p(0)] - [L^* 1_n] = [w_1 p(0) + L(1_n - p(0))]$$

with $\tilde{s}(L^* w_1 p(0) + 1_n - p(0)) = L^*$, where $\tilde{s}\colon \tilde{A} \rtimes_{\tilde{\alpha}} \mathbb{Z} \to C(\mathbb{T})$ is the $C(\mathbb{T})$-scalar map, so that $s(w_1 p(0) + L(1_n - p(0))) = 1_n$.
Paschke showed in [48] that the diagram

$$\begin{tikzcd} K_1(A) \arrow[r, "\zeta_A^1"] \arrow[d, "="] & K_0(\mathcal{M}_\alpha(A)) \arrow[r] \arrow[d, "\Psi_A"] & K_0(A) \arrow[d, "="] \\ K_1(A) \arrow[r] & K_1(A \rtimes_\alpha \mathbb{Z}) \arrow[r, "\delta"] & K_0(A) \end{tikzcd} \tag{3.8}$$

is commutative.
We are now ready for our axiomatic approach.
The map Ψ_A satisfies the following axioms:

1. Normalization. If $A = \mathbb{C}$, then the map $\Psi_{\mathbb{C}}$ is the canonical map

$$K_0(C(\mathbb{T})) \to K_1(C(\mathbb{T}))$$

sending the generator $[1]$ of $K_0(C(\mathbb{T}))$ to the generator $[L^*]$ of $K_1(C(\mathbb{T}))$. (Note that $C^*(\mathbb{Z}) \cong C(\mathbb{T})$ by Fourier transform.)
2. Suspension. If α is an automorphism of A, then it induces an automorphism $S\alpha$ of SA and we have natural isomorphisms

$$(SA) \rtimes_{S\alpha} \mathbb{Z} \cong S(A \rtimes_\alpha \mathbb{Z}), \qquad \mathcal{M}_{S\alpha}(SA) \cong S(\mathcal{M}_\alpha(A)).$$

We have a commutative diagram

$$\begin{tikzcd}
K_0(\mathcal{M}_{S\alpha}(SA)) \arrow[r, "\Psi_{SA}"] & K_1(SA \rtimes_{S\alpha} \mathbb{Z}) \\
K_0(S\mathcal{M}_\alpha(A)) \arrow[u, "\cong"] & K_1(S(A \rtimes_\alpha \mathbb{Z})) \arrow[u, "\cong"] \\
K_1(\mathcal{M}_\alpha(A)) \arrow[u, "\cong"] \arrow[r, "\Psi_A^1"] & K_0(A \rtimes_\alpha \mathbb{Z}), \arrow[u, "\cong"]
\end{tikzcd}$$

where Ψ_A^1 is the unique map making the diagram commutative.
3. Naturality. If we have dynamical systems $(A, \alpha, \mathbb{Z})$, $(B, \beta, \mathbb{Z})$ and $f\colon A \to B$ is an equivariant homomorphism, then the following diagram is commutative:

$$\begin{tikzcd}
K_0(\mathcal{M}_\alpha(A)) \arrow[r, "\Psi_A"] \arrow[d] & K_1(A \rtimes_\alpha \mathbb{Z}) \arrow[d] \\
K_0(\mathcal{M}_\beta(B)) \arrow[r, "\Psi_B"] & K_1(B \rtimes_\beta \mathbb{Z}),
\end{tikzcd}$$

where the vertical maps are induced by f.
This is readily checked.
Lemma 3.47. *With the map Ψ_A^1 defined in 2. above, we have the following commutative diagram:*

$$\begin{tikzcd}
K_0(A) \arrow[r, "\zeta_A^0"] \arrow[d, "="] & K_1(\mathcal{M}_\alpha(A)) \arrow[r] \arrow[d, "\Psi_A^1"] & K_1(A) \arrow[d, "="] \\
K_0(A) \arrow[r] & K_0(A \rtimes_\alpha \mathbb{Z}) \arrow[r, "\delta"] & K_1(A).
\end{tikzcd}$$

In particular, the maps Ψ_A and Ψ_A^1 define an isomorphism from the 6*-term exact sequence* (3.6) *to the* 6*-term exact sequence* (3.7).

Proof. We will only check the commutativity of the left hand square, leaving to the reader to check the commutativity of the right hand square.

Let $j_A\colon A \to A \rtimes_\alpha \mathbb{Z}$ and $i_A\colon SA \to \mathcal{M}_\alpha(A)$ denote the natural inclusions.
By (3.8) applied to the pair $(SA, S\alpha)$, we have a commutative diagram:

$$\begin{tikzcd}
K_0(S^2A) \arrow[r, "K_0(i_{SA})"] & K_0(\mathcal{M}_{S\alpha}(SA)) \arrow[d, "\Psi_{SA}"] \\
K_1(SA) \arrow[u, "\theta_{SA}"] \arrow[r, "K_1(j_{SA})"] & K_1(SA \rtimes_{S\alpha} \mathbb{Z})
\end{tikzcd}$$

Applying the natural isomorphisms $\mathcal{M}_{S\alpha}(SA) \cong S\mathcal{M}_\alpha(A)$ and $SA \rtimes_{S\alpha} \mathbb{Z} \cong S(A \rtimes_\alpha \mathbb{Z})$, we get the diagram:

$$\begin{tikzcd}
 & K_0(S^2A) \arrow[rr, "K_0(i_{SA})"] \arrow[dl, "="'] & & K_0(\mathcal{M}_{S\alpha}(SA)) \arrow[dl, "\cong"] \arrow[dd, "\Psi_{SA}"] \\
K_0(S^2A) \arrow[rr, "K_0(Si_A)"] & & K_0(S(\mathcal{M}_\alpha(A))) \arrow[dd, "\tilde{\Psi}"] & \\
 & K_1(SA) \arrow[uu, "\theta_{SA}"] \arrow[rr, "K_1(j_{SA})"] & & K_1(SA \rtimes_{S\alpha} \mathbb{Z}) \arrow[dl, "\cong"] \\
K_1(SA) \arrow[uu, "\theta_{SA}"] \arrow[ur, "="] \arrow[rr, "K_1(Sj_A)"'] & & K_1(S(A \rtimes_\alpha \mathbb{Z})) &
\end{tikzcd} \tag{3.9}$$

Here the map $K_0(S^2A) \to K_0(S^2A)$ at the top face of the diagram is the map induced by the twist function $f \mapsto \tilde{f}$ from S^2A to itself, where $\tilde{f}(t,s) = f(s,t)$ for $s,t \in I$. It can be easily seen that the induced map on K-theory is the identity. For instance, if f is a path of projections, then $F_r(t,s) = f(rs + (1-r)t, (1-r)s + rt)$, for $r \in I$, is a homotopy of projections connecting f with $\tilde{f}$. It follows that the top face of the diagram is a commutative square.
The left hand and bottom faces are clearly commutative,and the back face is commutative by [48]. The map $\tilde{\Psi}\colon K_0(S(\mathcal{M}_\alpha(A))) \to K_1(S(A \rtimes_\alpha \mathbb{Z}))$ is defined as the unique homomorphism making the right hand square of the diagram commutative.
Hence all faces of the diagram but possibly the front face are commutative. Now a diagram chasing shows that the front face must also be commutative.
We now consider the following diagram:

$$\begin{tikzcd}
 & K_0(S^2A) \arrow[rr, "K_0(Si_A)"] & & K_0(S(\mathcal{M}_\alpha(A))) \arrow[dd, "\tilde{\Psi}"] \\
K_1(SA) \arrow[ur, "\theta_{SA}"] \arrow[rr, "K_1(i_A)"] & & K_1(\mathcal{M}_\alpha(A)) \arrow[ur, "\theta_{\mathcal{M}_\alpha(A)}"] \arrow[dd, "\Psi^1_A"'] & \\
 & K_1(SA) \arrow[uu, "\theta_{SA}"] \arrow[rr, "K_1(Sj_A)"] & & K_1(S(A \rtimes_\alpha \mathbb{Z})) \\
K_0(A) \arrow[uu, "\beta_A"] \arrow[ur, "\beta_A"] \arrow[rr, "K_0(j_A)"'] & & K_0(A \rtimes_\alpha \mathbb{Z}) \arrow[ur, "\beta_{A \rtimes_\alpha \mathbb{Z}}"'] &
\end{tikzcd} \tag{3.10}$$

Here the back square is the front square of the previous diagram (3.9), so it is commutative. The map Ψ^1_A has been defined so that the right hand square is commutative. The commutativity of the upper and bottom squares follows from the naturality of the Bott maps θ and β respectively. The left hand square is obviously commutative. Therefore diagram chasing shows that the front face is also commutative, which gives the desired result. $\square$

If α, β are two commuting automorphisms of a C^*-algebra A, then one can easily see that β defines an automorphism on $\mathcal{M}_\alpha(A)$ by the rule $\beta(f)(t) = \beta(f(t))$, for $f \in \mathcal{M}_\alpha(A)$ and $t \in [0,1]$. Similarly, α induces an automorphism on $A \rtimes_\beta \mathbb{Z}$ satisfying that $\alpha(av^i) = \alpha(a)v^i$, where $a \in A$ and v is the canonical unitary in $A \rtimes_\beta \mathbb{Z}$.

Lemma 3.48. *Let α and β two commuting automorphisms of a C^*-algebra A. Then there is a natural isomorphism*

$$\psi \colon \mathcal{M}_\alpha(A) \rtimes_\beta \mathbb{Z} \to \mathcal{M}_\alpha(A \rtimes_\beta \mathbb{Z}).$$

Naturality means that if we have another C^*-algebra B with two commuting automorphisms α', β' and $f \colon A \to B$ is an equivariant $*$-homomorphism then the following diagram

$$\begin{tikzcd}
\mathcal{M}_\alpha(A) \rtimes_\beta \mathbb{Z} \arrow[r, "\psi_A"] \arrow[d, "\mathcal{M}(f) \rtimes \mathbb{Z}"] & \mathcal{M}_\alpha(A \rtimes_\beta \mathbb{Z}) \arrow[d, "\mathcal{M}(f \rtimes \mathbb{Z})"] \\
\mathcal{M}_{\alpha'}(B) \rtimes_{\beta'} \mathbb{Z} \arrow[r, "\psi_B"] & \mathcal{M}_{\alpha'}(B \rtimes_{\beta'} \mathbb{Z})
\end{tikzcd}$$

is commutative.

Proof. We first assume A and B unital.

Let $j \colon A \to A \rtimes_\beta \mathbb{Z}$ be the canonical injective $*$-homomorphism. Since j is α-equivariant, there is a $*$-homomorphism

$$\mathcal{M}_\alpha(j) \colon \mathcal{M}_\alpha(A) \to \mathcal{M}_\alpha(A \rtimes_\beta \mathbb{Z})$$

defined by $\mathcal{M}_\alpha(j)(f) = j \circ f$ for $f \in \mathcal{M}_\alpha(A)$. Note that, since j is injective, $\mathcal{M}_\alpha(j)$ is also injective.
Let u and v be the canonical unitaries associated to the actions α and β in the respective crossed product algebras $A \rtimes_\alpha \mathbb{Z}$ and $A \rtimes_\beta \mathbb{Z}$. Define $c_v \in \mathcal{M}_\alpha(A \rtimes_\beta \mathbb{Z})$ as the constant function $c_v(t) = v$. Observe that $c_v \in \mathcal{M}_\alpha(A \rtimes_\beta \mathbb{Z})$ because $\alpha(v) = v$. We show the covariance property for the homomorphism $\mathcal{M} := \mathcal{M}_\alpha(j)$ and c_v, that is

$$\mathcal{M}(\beta(f)) = c_v \mathcal{M}(f) c_v^*$$

for $f \in \mathcal{M}_\alpha(A)$. For $t \in [0,1]$ we have

$$\mathcal{M}(\beta(f))(t) = (j \circ \beta(f))(t) = j(\beta(f(t))) = vj(f(t))v^* = (c_v \mathcal{M}(f) c_v^*)(t),$$

which proves the covariance. Hence we get an induced $*$-homomorphism

$$\psi\colon \mathcal{M}_\alpha(A) \rtimes_\beta \mathbb{Z} \to \mathcal{M}_\alpha(A \rtimes_\beta \mathbb{Z})$$

such that $\psi(fv^i) = \mathcal{M}(f)c_v^i$ for all $i \in \mathbb{Z}$.
We now show that ψ is injective. Let E and E' be the canonical conditional expectations from $A \rtimes_\beta \mathbb{Z}$ onto A and from $\mathcal{M}_\alpha(A) \rtimes_\beta \mathbb{Z}$ onto $\mathcal{M}_\alpha(A)$, respectively.
We claim that for each $x \in \mathcal{M}_\alpha(A) \rtimes_\beta \mathbb{Z}$ and $t \in [0,1]$ we have the equality

$$\mathcal{M}(E'(x))(t) = E(\psi(x)(t)). \tag{3.11}$$

By continuity and linearity it is enough to show this formula for an element of the form fv^i, where $f \in \mathcal{M}_\alpha(A)$ and $i \in \mathbb{Z}$. But this is trivially verified, so the formula holds.
Now suppose that x is a nonzero positive element in the kernel of ψ. Note that E' is faithful (see [9, Prop. 4.1.9 and Thm. 4.2.4]), and hence we have that $E'(x) \neq 0$. Now, since $\mathcal{M} = \mathcal{M}_\alpha(j)$ is injective, we have $\mathcal{M}(E'(x)) \neq 0$ and so there is some $t \in [0,1]$ such that $\mathcal{M}'(E'(x))(t) \neq 0$. Equation (3.11) tells us that $E(\psi(x)(t)) \neq 0$; in particular, $\psi(x) \neq 0$. This shows that ψ is injective.
To show that ψ is surjective, it suffices to show that its range is dense. This is done by a partition of unity argument, as follows. Let $a \in \mathcal{M}_\alpha(A \rtimes_\beta \mathbb{Z})$ and $\varepsilon > 0$ be given. Let $\mathcal{U}$ be a finite open cover of $[0,1]$, and elements $t_U \in U$ for each $U \in \mathcal{U}$ with the following properties:

1. $\|a(t) - a(t_U)\| < \varepsilon$ when $t \in U$.
2. There is a unique $U \in \mathcal{U}$ such that $0 \in U$, This unique set is denoted by U_0. Moreover $t_{U_0} = 0$.
3. There is a unique $U \in \mathcal{U}$ such that $1 \in U$, This unique set is denoted by U_1. Moreover $t_{U_1} = 1$.

Now let $\{f_U : U \in \mathcal{U}\}$ be a partition of unity subordinated to the cover $\mathcal{U}$. Define

$$b = \sum_{U \in \mathcal{U}} a(t_U) f_U.$$

Then $b \in \mathcal{M}_\alpha(A \rtimes_\beta \mathbb{Z})$, because using conditions (2) and (3) above we get

$$b(1) = \sum_{U \in \mathcal{U}} a(t_U) f_U(1) = a(t_{U_1}) f_{U_1}(1) = a(1) = \alpha(a(0)) = \alpha(b(0)).$$

Moreover we have that for all $t \in [0,1]$

$$\|a(t) - \sum_{U \in \mathcal{U}} a(t_U) f_U(t)\| \leq \sum_{U \in \mathcal{U}} f_U(t) \|a(t) - a(t_U)\| < \varepsilon.$$

Hence $\|a - b\| < \varepsilon$. Now $a(t_U) \in A \rtimes_\beta \mathbb{Z}$ for each $U \in \mathcal{U}$, and since $\mathcal{U}$ is finite we can find a positive integer N and elements $a_{U,i} \in A$, for $U \in \mathcal{U}$, $-N \leq i \leq N$, such that

$$\|a(t_U) - \sum_{i=-N}^{N} a_{U,i} v^i\| < \varepsilon.$$

Moreover we can clearly take $a_{U_1,i}$ such that $a_{U_1,i} = \alpha(a_{U_0,i})$ for all $-N \leq i \leq N$. We now can build elements $a_i \in \mathcal{M}_\alpha(A)$ as follows:

$$a_i = \sum_{U \in \mathcal{U}} a_{U,i} f_U.$$

By the same argument as before and since we have that $a_{U_1,i} = \alpha(a_{U_0,i})$ for all i, we get that $a_i \in \mathcal{M}_\alpha(A)$. Hence the element $\sum_{i=-N}^{N} a_i v^i$ belongs to $\mathcal{M}_\alpha(A) \rtimes_\beta \mathbb{Z}$ and for $t \in [0,1]$ we have

$$\begin{aligned} \|b(t) - (\psi(\sum_{i=-N}^{N} a_i v^i))(t)\| &= \|\sum_{U \in \mathcal{U}} a(t_U) f_U(t) - \sum_{i=-N}^{N} \Big(\sum_{U \in \mathcal{U}} a_{U,i} v^i f_U(t)\Big)\| \leq \\ &\leq \sum_{U \in \mathcal{U}} f_U(t) \|a(t_U) - \sum_{i=-N}^{N} a_{U,i} v^i\| < \varepsilon. \end{aligned}$$

Hence $\|b - \psi(\sum_{i=-N}^{N} a_i v^i)\| < \varepsilon$ and

$$\|a - \psi(\sum_{i=-N}^{N} a_i v^i)\| \leq \|a - b\| + \|b - \psi(\sum_{i=-N}^{N} a_i v^i)\| < 2\varepsilon,$$

proving the surjectivity.

We now show the non-unital case. Let α and β be two commuting actions on a C^*-algebra A and consider the unitization $\tilde{A}$ of A, and the unital extensions $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β to $\tilde{A}$. We then obtain a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M}_\alpha(A) \rtimes_\beta \mathbb{Z} \arrow[r] \arrow[d, "\psi_A"] & \mathcal{M}_{\tilde{\alpha}}(\tilde{A}) \rtimes_{\tilde{\beta}} \mathbb{Z} \arrow[r] \arrow[d, "\psi_{\tilde{A}}"] & C(\mathbb{T}) \otimes C(\mathbb{T}) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathcal{M}_\alpha(A \rtimes_\beta \mathbb{Z}) \arrow[r] & \mathcal{M}_{\tilde{\alpha}}(\tilde{A} \rtimes_{\tilde{\beta}} \mathbb{Z}) \arrow[r] & C(\mathbb{T}) \otimes C(\mathbb{T}) \arrow[r] & 0
\end{tikzcd}$$

Here the map $\psi_{\tilde{A}}$ is an isomorphism by what we have proved before, and the map ψ_A is the induced map, which must be also an isomorphism.
The naturality is easily proved. $\square$

Later on (see Theorem 3.50), we will use the above strategies to extend this naturality to the isomorphism associated to the n-mapping torus.

3.3.4.3 A cofiltration of the mapping torus

First, we set some notation. Let $i \leq k$, and consider $\mu = (\mu_1, \mu_2, ..., \mu_i)$ such that $1 \leq \mu_1 < \mu_2 < ... < \mu_i \leq k$. Let $T(i,k)$ be the set of all such elements μ. Consider $\{e_1, e_2, ..., e_k\}$ to be the usual basis of $\mathbb{Z}^k$, and let $\bigwedge^i(\mathbb{Z}^k)$ be the $\mathbb{Z}$-linear span of the elements $e_\mu := e_{\mu_1} \wedge e_{\mu_2} \wedge ... \wedge e_{\mu_i}$, with $\mu = (\mu_1, \mu_2, ..., \mu_i) \in T(i,k)$. By convention, we put $\bigwedge^0(\mathbb{Z}^k) := \mathbb{Z}$ and $\bigwedge^i(\mathbb{Z}^k) = 0$ for all $i > k$ or $i < 0$. We also consider $T(0,k)$ to be the set which contains only the empty tuple, and set $e_\mu := 1$ when $\mu = \emptyset \in T(0,k)$.
The group $\bigwedge^i(\mathbb{Z}^k)$ has the set $\{e_\mu : \mu \in T(i,k)\}$ as a basis. By equipping $T(i,k)$ with the lexicographical order, we obtain an order-preserving bijection between $T(i,k)$ and $\{1, 2, ..., \binom{k}{i}\}$, which induces a group isomorphism $\bigwedge^i(\mathbb{Z}^k) \cong \mathbb{Z}^{\binom{k}{i}}$.
With this in mind, we can build a cofiltration of the mapping torus associated to a C^*-dynamical system $(A, \alpha, \mathbb{Z}^k)$.
For $i = 0, ..., k$, define

$$X_i := \{t \in [0,1]^k : t_{\mu_1} = t_{\mu_2} = ... = t_{\mu_{k-i}} = 0,\ \textit{for some}\ (\mu_1, \mu_2, ..., \mu_{k-i}) \in T(k-i,k)\},$$

and set $X_{-1} := \emptyset$. The sequence

$$\emptyset = X_{-1} \subseteq X_0 = \{0\} \subseteq ... \subseteq X_k = [0,1]^k$$

is a filtration of the k-cube. From this, we can build a cofiltration of the mapping torus

$$\mathcal{M}_\alpha(A) = F_k \overset{\pi_k}{\to} F_{k-1} \overset{\pi_{k-1}}{\to} ... \to F_0 = A \overset{\pi_0}{\to} F_{-1} = 0,$$

where each F_i is defined by restricting the respective domain to X_i, and the π_i maps are obtained by restricting each domain to X_{i-1}.

We have enough data to build a spectral sequence using the techniques described in Section 3.3.3. We show here the most relevant results.

3.3.4.4 E_1-page of the spectral sequence

The first page of the spectral sequence is defined as $E_1^{p,q} := K_{p+q}(I_p)$, where $I_p := ker(\pi_p : F_p \to F_{p-1})$. Let us compute those K-theory groups, following the techniques of [4] and [54]:
For $p = 1$, we have that

$$X_1 = [0,1] \times \{0\}^{k-1} \cup \{0\} \times [0,1] \times \{0\}^{k-2} \cup ... \cup \{0\}^{k-1} \times [0,1],$$

that is, the union of all the edges of the n-cube converging at the origin. Moreover, $X_0 = \{0\}^k$, and then

$$I_1 := ker(\pi_1 : F_1 \to F_0)$$

is given by the set of continuous functions $f \in C(X_1, A)$ such that f is zero in every vertex of X_1. Hence, I_1 can be identified as:

$$I_1 = SA \oplus SA \oplus ... \oplus SA \cong (SA)^k.$$

The sets X_p can be studied similarly to X_1. In this line, X_2 can be seen as the union of all the faces converging at the origin, X_3 as the set of all 3-cubes converging at the origin, and so on. In general, for $1 < p \leq k$, and $\mu = (\mu_1, ..., \mu_p) \in T(p,k)$ we can write

$$X_p = \bigcup_{\mu \in T(p,k)} X(\mu),$$

with $X(\mu)$ defined as:

$$X(\mu) := \{t \in [0,1]^k : t_{\mu_1^\perp} = ... = t_{\mu_{k-p}^\perp} = 0\} \subseteq X_p,$$

where $\mu^\perp \in T(k-p,k)$ is the unique $(k-p)$-string disjoint to μ. We can consider the induced $\mathbb{Z}^p$-action generated by $\alpha_{\mu_1}, ..., \alpha_{\mu_p}$, which we denote by $\alpha(\mu)$. Then $(A, \alpha(\mu), \mathbb{Z}^p)$ is a dynamical system, and we can build the associated mapping torus $\mathcal{M}_{\alpha(\mu)}(A)$. Using the same strategy as before, we obtain a cofiltration of $\mathcal{M}_{\alpha(\mu)}(A)$, which we will denote by $F(\mu)_i$. Notice that we can identify $\mathcal{M}_{\alpha(\mu)}(A)$ as the quotient of F_p by restricting its domain to $X(\mu) \subseteq X_p$. By doing so, the following commutative diagram for each p and each $\mu \in T(p,k)$ arises:

$$\begin{tikzcd}
0 \arrow[r] & I_p \arrow[r] \arrow[d] & F_p \arrow[r] \arrow[d] & F_{p-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S^p A \arrow[r] & \mathcal{M}_{\alpha(\mu)(A)} \arrow[r] & F(\mu)_{p-1} \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are obtained by projecting into the μ-component. Therefore, we can obtain F_p as the iterative pullback of the $\binom{k}{p}$ mapping tori $\mathcal{M}_{\alpha(\mu)}(A)$ of the natural $\mathbb{Z}^p$-subactions, *glued* together over the $\binom{k}{p-1}$ many mapping tori $\mathcal{M}_{\alpha(\lambda)}(A)$ for $\lambda \in T(p-1,k)$ arising from the natural $\mathbb{Z}^{p-1}$ subactions. Denote by $\chi(\mu) : I_p \to S^p A$ the respective map of the diagram above. One can see that the $*$-homomorphism

$$\chi = (\chi(\mu))_{\mu \in T(p,k)} : I_p \to (S^p A)^{\binom{k}{p}}$$

is an isomorphism. Therefore, we obtain isomorphisms

$$E_1^{p,q} := K_{p+q}(I_p) \cong K_{p+q}((S^p A)^{\binom{k}{p}}) \cong K_q(A)^{\binom{k}{p}}$$

whenever $0 \leq p \leq k$, and 0 otherwise. We use the convention

$$K_q(A)^{\binom{k}{p}} \cong K_q(A) \otimes_{\mathbb{Z}} \bigwedge\nolimits^p(\mathbb{Z}^k).$$

Moreover, Savignen and Bellisard developed the following method to compute the E_2-page of the spectral sequence. We direct the reader to the original source [54, Theorem 2] for its proof.

Theorem 3.49. *Given a C^*-dynamical system $(A, \alpha, \mathbb{Z}^k)$, the Pimsner-Voiculescu complex $(C_{PV}, d_{PV}^{p,q})$ is defined as:*

$$\begin{gathered} C_{PV}^{p,q} := K_q(A) \otimes_{\mathbb{Z}} \bigwedge\nolimits^p(\mathbb{Z}^k) \\ d_{PV}^{p,q} : \; C_{PV}^{p,q} \longrightarrow C_{PV}^{p+1,q} \\ x \otimes e \mapsto \sum_{i=1}^{k} (K_q(\alpha_i) - id)(x) \otimes (e \wedge e_i) \end{gathered}$$

Then the isomorphism $E_1^{p,q} = K_{p+q}(I_p) \cong C_{PV}^{p,q}$ intertwines the differentials d_1 and d_{PV}. Therefore, the E_2-term of the spectral sequence is obtained as the cohomology of (C_{PV}, d_{PV}). □

We can now state and prove the following result, main technical Theorem of this Chapter. It shows that the isomorphism between $E_1^{p,q}$ and $C_{PV}^{p,q}$ given in Theorem 3.49 is natural with respect to the isomorphism Ψ described in Subsection 3.3.4.2. We will follow the notation of [4].

Recall that for $n \geq 2$, the functor K_n is defined inductively by $K_n = K_{n-1} \circ S$ ([53, Definition 10.2.1]). We thus have $K_n(A) = K_1(S^{n-1}A)$ for any C^*-algebra A and all $n \geq 2$.

Theorem 3.50. *Let α be an action of $\mathbb{Z}^n$ on A and $q \in \{0, 1\}$. Then there exists a commutative diagram:*

$$\begin{tikzcd}
K_{q+n-1}(I_{n-1}) \arrow[r, "d_1^{n-1,q}"] \arrow[d, "\cong"'] & K_{q+n}(I_n) \arrow[r, "K_{q+n}(i_n)"] \arrow[d, "\cong"'] & K_{q+n}(\mathcal{M}_\alpha(A)) \arrow[d, "\Psi_A^{(n)}"] \\
K_q(A) \otimes \bigwedge^{n-1}(\mathbb{Z}^n) \arrow[r, "d_{PV}^{n-1,q}"'] & K_q(A) \arrow[r, "K_q(j_n)"'] & K_q(A \rtimes_\alpha \mathbb{Z}^n)
\end{tikzcd}$$

where $\Psi_A^{(n)}$ is a natural isomorphism, and the vertical arrows are the natural isomorphisms between $E_1^{p,q}$ and $C_{PV}^{p,q}$, induced by the canonical Bott isomorphisms $\beta_n \colon K_q(A) \to K_{q+n}(S^n A) = K_{q+n}(I_n)$ and

$$\beta_{n-1} \otimes id\colon K_q(A) \otimes \bigwedge^{n-1}(\mathbb{Z}^n) \to K_{q+n-1}(S^{n-1}A) \otimes \bigwedge^{n-1}(\mathbb{Z}^n) \cong K_{q+n-1}(I_{n-1}).$$

The maps $i_n \colon I_n \to \mathcal{M}_\alpha(A)$ and $j_n \colon A \to A \rtimes_\alpha \mathbb{Z}^n$ are the natural inclusions.

Proof. The left hand diagram is commutative for all n by [4, Corollary 7.2.3].
We show the result for the right hand diagram by induction on n. If $n = 1$, the result follows from [48]. Indeed, for $q = 0$, the right hand square is exactly the left hand square of the diagram from Lemma 3.47. For $q = 1$, the right hand diagram is obtained as the composite

$$\begin{tikzcd}
K_1(S^2A) \arrow[r, "K_1(Si_A)"] & K_1(S(\mathcal{M}_\alpha A)) \\
K_0(SA) \arrow[r, "K_0(i_A)"] \arrow[u, "\beta_{SA}"] & K_0(\mathcal{M}_\alpha(A)) \arrow[u, "\beta_{\mathcal{M}_\alpha(A)}"'] \arrow[d, "\Psi_A"] \\
K_1(A) \arrow[r, "K_1(j_A)"] \arrow[u, "\theta_A"] & K_1(A \rtimes_\alpha \mathbb{Z}),
\end{tikzcd}$$

where the lower square is commutative by (3.8), and the upper square is commutative by naturality of the Bott map β.
Assume that the result holds for some $n \geq 1$. We will show that it also holds for $n+1$. So let α be an action by automorphisms of $\mathbb{Z}^{n+1}$ on A. We will denote by $\alpha^{(n)}$ the subaction of $\mathbb{Z}^n$ on A given by the first n automorphisms $\alpha_1, \ldots, \alpha_n$ associated to the action α.
Observe that we have a natural isomorphism $\mathcal{M}_\alpha(A) \cong \mathcal{M}_{\alpha_{n+1}}(\mathcal{M}_{\alpha^{(n)}}(A))$. We obtain an exact sequence

$$0 \longrightarrow S\mathcal{M}_{\alpha^{(n)}}(A) \xrightarrow{i_n'} \mathcal{M}_\alpha(A) \longrightarrow \mathcal{M}_{\alpha^{(n)}}(A) \longrightarrow 0$$

and applying the case $k = 1$ we obtain a commutative diagram

$$\begin{tikzcd}
K_{q+n+1}(S\mathcal{M}_{\alpha^{(n)}}(A)) \arrow[r, "K_{q+n+1}(i'_n)"] & K_{q+n+1}(\mathcal{M}_{\alpha_{n+1}}(\mathcal{M}_{\alpha^{(n)}}(A))) \arrow[d, "\Psi_{\mathcal{M}_{\alpha^{(n)}}(A)}"] \\
K_{q+n}(\mathcal{M}_{\alpha^{(n)}}(A)) \arrow[u, "\beta"] \arrow[r, "K_{q+n}(j'_n)"'] & K_{q+n}(\mathcal{M}_{\alpha^{(n)}}(A) \rtimes_{\alpha_{n+1}} \mathbb{Z})
\end{tikzcd}$$

where $j'_n \colon \mathcal{M}_{\alpha^{(n)}}(A) \to \mathcal{M}_{\alpha^{(n)}}(A) \rtimes_{\alpha_{n+1}} \mathbb{Z}$ is the canonical map, and β is the Bott isomorphism.

By using Lemma 3.48 and an induction argument, we get that $\mathcal{M}_{\alpha^{(n)}}(A) \rtimes_{\alpha_{n+1}} \mathbb{Z} \cong \mathcal{M}_{\alpha^{(n)}}(A \rtimes_{\alpha_{n+1}} \mathbb{Z})$ canonically through a natural $*$-isomorphism λ_n. . Therefore we have another commutative diagram

$$\begin{tikzcd}
K_{q+n}(\mathcal{M}_{\alpha^{(n)}}(A)) \arrow[r, "K_{q+n}(j'_n)"] \arrow[d, "id"'] & K_{q+n}(\mathcal{M}_{\alpha^{(n)}}(A) \rtimes_{\alpha_{n+1}} \mathbb{Z}) \arrow[d, "\lambda_n"] \\
K_{q+n}(\mathcal{M}_{\alpha^{(n)}}(A)) \arrow[r, "K_{q+n}(\mathcal{M}_{\alpha^{(n)}}(j'))"'] & K_{q+n}(\mathcal{M}_{\alpha^{(n)}}(A \rtimes_{\alpha_{n+1}} \mathbb{Z}))
\end{tikzcd}$$

where $j' \colon A \to A \rtimes_{\alpha_{n+1}} \mathbb{Z}$ is the canonical map, which is $\alpha^{(n)}$-equivariant.

Now, since j' is $\alpha^{(n)}$-equivariant, naturality of the $*$-homomorphism $\Psi^{(n)}$ gives another commutative diagram:

$$\begin{tikzcd}
K_{q+n}(\mathcal{M}_{\alpha^{(n)}}(A)) \arrow[r, "K_{q+n}(\mathcal{M}_{\alpha^{(n)}}(j'))"] \arrow[d, "\Psi^{(n)}_A"'] & K_{q+n}(\mathcal{M}_{\alpha^{(n)}}(A \rtimes_{\alpha_{n+1}} \mathbb{Z})) \arrow[d, "\Psi^{(n)}_{A \rtimes_{\alpha_{n+1}} \mathbb{Z}}"] \\
K_q(A \rtimes_{\alpha^{(n)}} \mathbb{Z}^n) \arrow[r, "K_q(j' \rtimes_{\alpha^{(n)}} \mathbb{Z}^n)"'] & K_q((A \rtimes_{\alpha_{n+1}} \mathbb{Z}) \rtimes_{\alpha^{(n)}} \mathbb{Z}^n)
\end{tikzcd}$$

The map $\Psi^{(n+1)}_A \colon K_{q+n+1}(\mathcal{M}_\alpha(A)) \to K_q((A \rtimes_{\alpha_{n+1}} \mathbb{Z}) \rtimes_{\alpha^{(n)}} \mathbb{Z}^n) = K_q(A \rtimes_\alpha \mathbb{Z}^{n+1})$ is defined as the composition

$$\Psi^{(n)}_{A \rtimes_{\alpha_{n+1}} \mathbb{Z}} \circ \lambda_n \circ \Psi_{\mathcal{M}_{\alpha^{(n)}(A)}}.$$

It is natural because it is the composition of three natural $*$-homomorphisms. By induction, we have another commutative diagram

$$\begin{tikzcd}
K_{q+n}(S^n A) \arrow[r, "K_{q+n}(i_n)"] & K_{q+n}(\mathcal{M}_{\alpha^{(n)}}(A)) \arrow[d, "\Psi_A^{(n)}"] \\
K_q(A) \arrow[u, "\beta_n"] \arrow[r, "K_q(j_n)"'] & K_q(A \rtimes_{\alpha^{(n)}} \mathbb{Z}^n)
\end{tikzcd}$$

where $i_n \colon S^n A \to \mathcal{M}_{\alpha^{(n)}}(A)$ and $j_n \colon A \to A \rtimes_{\alpha^{(n)}} \mathbb{Z}^n$ are the natural maps.
We obtain thus the following diagram, where all the squares are commutative:

$$\begin{tikzcd}
K_{q+n+1}(S^{n+1}A) \arrow[r, "K_{q+n+1}(Si_n)"] & K_{q+n+1}(S\mathcal{M}_{\alpha^{(n)}}(A)) \arrow[r, "K_{q+n+1}(i'_n)"] & K_{q+n+1}(\mathcal{M}_{\alpha}(A)) \arrow[d, "\Psi_{\mathcal{M}_{\alpha^{(n)}}(A)}"] \\
K_{q+n}(S^n A) \arrow[u, "\beta"] \arrow[r, "K_{q+n}(i_n)"] & K_{q+n}(\mathcal{M}_{\alpha^{(n)}}(A)) \arrow[u, "\beta"] \arrow[r, "K_{q+n}(j'_n)"] & K_{q+n}(\mathcal{M}_{\alpha^{(n)}}(A) \rtimes_{\alpha_{n+1}} \mathbb{Z}) \arrow[d, "\lambda_n"] \\
K_{q+n}(S^n A) \arrow[u, "id"] \arrow[r, "K_{q+n}(i_n)"] & K_{q+n}(\mathcal{M}_{\alpha^{(n)}}(A)) \arrow[u, "id"] \arrow[d, "\Psi_A^{(n)}"] \arrow[r, "K_{q+n}(\mathcal{M}_{\alpha^{(n)}}(j'))"] & K_{q+n}(\mathcal{M}_{\alpha^{(n)}}(A \rtimes_{\alpha_{n+1}} \mathbb{Z})) \arrow[d, "\Psi^{(n)}_{A \rtimes_{\alpha_{n+1}} \mathbb{Z}}"] \\
K_q(A) \arrow[u, "\beta_n"] \arrow[r, "K_q(j_n)"] & K_q(A \rtimes_{\alpha^{(n)}} \mathbb{Z}^n) \arrow[r, "K_q(j' \rtimes_{\alpha^{(n)}} \mathbb{Z}^n)"] & K_q((A \rtimes_{\alpha_{n+1}} \mathbb{Z}) \rtimes_{\alpha^{(n)}} \mathbb{Z}^n)
\end{tikzcd}$$

Using that $i_{n+1} = i'_n \circ Si_n$ and that $j_{n+1} = (j' \rtimes_{\alpha^{(n)}} \mathbb{Z}^n) \circ j_n$, we obtain the following commutative diagram:

$$\begin{tikzcd}
K_{q+n+1}(I_{n+1}) \arrow[r, "K_{q+n+1}(i_{n+1})"] & K_{q+n+1}(\mathcal{M}_\alpha(A)) \arrow[d, "\Psi_A^{(n+1)}"] \\
K_q(A) \arrow[u, "\beta_{n+1}"] \arrow[r, "K_q(j_{n+1})"'] & K_q(A \rtimes_\alpha \mathbb{Z}^{n+1})
\end{tikzcd}$$

This completes the proof. □

Remark 3.51. *Observe that, by considering the cokernels associated to the horizontal rows of Theorem 3.50, we can extend this naturality of* $\Psi^{(n)}$ *to the second page of the spectral sequence, obtaining the following commutative diagram:*

$$\begin{tikzcd}
K_{q+n}(I_n)/Im(d_1^{n-1,q}) \arrow[r, "K_{q+n}(i_n)"] \arrow[d, "\cong"] & K_{q+n}(\mathcal{M}_\alpha(A)) \arrow[d, "\Psi_A^{(n)}"] \\
K_q(A)/Im(d_{PV}^{n-1,q}) \arrow[r, "K_q(j_n)"'] & K_q(A \rtimes_\alpha \mathbb{Z}^n)
\end{tikzcd}$$

where the horizontal maps are both injective.

□

In chapter 4, we will use the previous strategies in order to compute the K-theory of the C^*-algebras associated to certain Deaconu-Renault groupoids, generalizing the arguments of [21].

3.4 First counterexample to Matui's HK conjecture

With the basic notions about spectral sequences already explained, we can now deduce an immediate result arising from Theorem 3.19. This result allows us to compute the homology groups of any given transformation groupoid, in a straightforward way.

Lemma 3.52. *Let $\varphi : \Gamma \curvearrowright X$ be an action of a discrete group Γ on a totally disconnected compact Hausdorff space X. Then we have*

$$H_n(X \rtimes \Gamma) \cong H_n(\Gamma, C(X, \mathbb{Z})).$$

Proof. Recall from Example 2.8 that a transformation groupoid $X \rtimes \Gamma$ is just a particular case of a semi-direct product groupoid in which Γ acts on the trivial groupoid X. Hence, Theorem 3.19 ensures the existence of a spectral sequence

$$E^2_{p,q} = H_p(\Gamma, H_q(X)) \Rightarrow H_{p+q}(X \rtimes \Gamma).$$

The homology groups of a trivial groupoid X (with X totally disconnected) were computed in Lemma 2.75 to be $H_0(X) = C(X, \mathbb{Z})$, and $H_n(X) = 0$, for all $n \geq 1$. Therefore, the spectral sequence collapses at the second page, meaning that this page only has one non-zero row. Hence we obtain

$$E^\infty_{p,0} = E^2_{p,0} = H_p(\Gamma, C(X, \mathbb{Z})) \cong H_p(X \rtimes \Gamma)$$

as desired. □

This lemma, together with Example 2.63, allows us to compute all of the pieces involved in Matui's HK conjecture for any given transformation groupoid. In [56], Scarparo uses the above techniques to compute all the needed data to check that the transformation groupoid arising from the action of a certain odometer on the Cantor set does not satisfy Matui's strong HK conjecture. This is, indeed, the first counterexample to the strong version of the conjecture. Let us show it.

Definition 3.53. *Let Γ be a group, and let $(\Gamma_i)_{i\in\mathbb{N}}$ be a strictly decreasing sequence of finite index subgroups of Γ. Denote by $p_i : \Gamma/\Gamma_{i+1} \to \Gamma/\Gamma_i$ the natural surjections given by*

$$p_i(\gamma\Gamma_{i+1}) = \gamma\Gamma_i.$$

*Define $X := \varprojlim(\Gamma/\Gamma_i, p_i) = \{(x_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} \Gamma/\Gamma_i : p_i(x_{i+1}) = x_i, \forall i \in \mathbb{N}\}$. It is known that X is homeomorphic to the Cantor set. Moreover, there is a minimal action of Γ on X given by $\gamma(x_i) = (\gamma x_i)$, for all $\gamma \in \Gamma$, $(x_i) \in X$. This action is called an **odometer**.* □

Now we recall the definition of the infinite dihedral group.

Definition 3.54. *The* ***infinite dihedral group*** $\mathcal{D}_\infty$ *is given by the semidirect product* $\mathbb{Z} \rtimes \mathbb{Z}_2$ *under the action given by multiplication by* -1*. Thus, the group* $\mathcal{D}_\infty$ *has the following presentation:*

$$\mathcal{D}_\infty = \langle r, s : s^2 = 1, srs = r^{-1} \rangle.$$

□

Note that there exist a few equivalent presentations of this group. For example, in later chapters we use the one given as the free product $\mathcal{D}_\infty = \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \rangle$. The reader may check that the map $\theta : \mathbb{Z} \rtimes \mathbb{Z}_2 \to \mathbb{Z}_2 * \mathbb{Z}_2$ defined by $\theta(r) = ab$, $\theta(s) = a$ is a group isomorphism. Unless we state the contrary, in this section we will see $\mathcal{D}_\infty$ as $\mathbb{Z} \rtimes \mathbb{Z}_2$.

Let us now build an odometer arising from $\mathcal{D}_\infty$.

Example 3.55. *Denote* $\Gamma = \mathcal{D}_\infty$ *, and let* (n_i) *be a strictly increasing sequence of natural numbers, satisfying* $n_i | n_{i+1}$*, for all* i*. Set* $\Gamma_i := n_i\mathbb{Z} \rtimes \mathbb{Z}_2$*. Then* $(\Gamma_i)_{i \in \mathbb{N}}$ *is a strictly decreasing sequence of finite index subgroups of* Γ *with, and hence induces an odometer.* □

For each $i \in \mathbb{N}$, the quotient $\mathcal{D}_\infty / \Gamma_i$ is isomorphic to $\mathbb{Z}_{n_i}$. Moreover, it was shown in [56, Proposition 2.1, Example 2.2] that this odometer $\mathcal{D}_\infty \curvearrowright \varprojlim \mathbb{Z}_{n_i}$ is topologically free.

In particular, the object of study will be the transformation groupoid associated to this odometer. All we need to do is to compute both its homology and K-theory, using Lemmas 3.52 and 2.63. We just provide here a sketch of the proofs, and direct the reader to [56] for full details.

3.4.1 K-theory of the infinite dihedral odometer.

It was shown in [56, Proposition 2.3] that, given an odometer $\Gamma \curvearrowright X = \varprojlim \Gamma/\Gamma_i$, we have that $C(X) \rtimes_r \Gamma \cong \varinjlim M_{\Gamma/\Gamma_i}(\mathbb{C}) \otimes C^*_r(\Gamma_i)$, and hence

$$K_*(C(X) \rtimes_r \Gamma) \cong \varinjlim K_*(C^*_r(\Gamma_i)).$$

Recall that, in the case of the infinite dihedral odometer, we set $\Gamma = \mathcal{D}_\infty = \mathbb{Z} \rtimes \mathbb{Z}_2$, and all Γ_i are of the form $n_i\mathbb{Z} \rtimes \mathbb{Z}_2$, for certain n_i. Note that $\mathcal{D}_\infty \cong \Gamma_i = n_i\mathbb{Z} \rtimes \mathbb{Z}_2$ for all $i \in \mathbb{N}$.

The K_1-group of $C^*(\mathbb{Z} \rtimes \mathbb{Z}_2)$ is known to be trivial (see [5, 10.11.5(a)]), and therefore $K_1(C(X) \rtimes \mathcal{D}_\infty) = 0$. In fact, this algebra is AF (see [30] or [7]), and its K_0-group was determined in [7].

In order to ease the computation, we make a slight abuse of notation: given the action of $\mathcal{D}_\infty$ on the Cantor set X, we denote by $(a, b) \in \mathbb{Z} \rtimes \mathbb{Z}_2$ the automorphism on X associated to the element (a, b).

Theorem 3.56. *([7, Theorem 4.1]) Given an action of $\mathcal{D}_\infty = \mathbb{Z} \rtimes \mathbb{Z}_2$ on the Cantor set X such that the restricted $\mathbb{Z}$-action is minimal and $(0,1),(1,1) \in \mathbb{Z} \rtimes \mathbb{Z}_2$ have, at most, a finite number $m_{(0,1)}$ and $m_{(1,1)}$ of fixed points, with $m_{(0,1)} + m_{(1,1)} > 0$, then $K_0(C(X) \rtimes \mathcal{D}_\infty)$ is isomorphic to*

$$(Id + (0,1)_*)\left(\frac{C(X,\mathbb{Z})}{(Id-(1,0)_*)(C(X,\mathbb{Z}))}\right) \oplus \mathbb{Z}^{m_{(0,1)}+m_{(1,1)}}.$$

□

Using this result, we need now to study the number of fixed points of $(0,1)$ and $(1,1)$. Here we provide a sketch of the proof.

Theorem 3.57. *([56, Lemma 3.2-3.3]) Let $\mathcal{D}_\infty \curvearrowright \varprojlim \mathbb{Z}_{n_i}$ be an odometer as in Example 3.55. Then:*

$$K_0(C(X) \rtimes \mathcal{D}_\infty) \cong \begin{cases} \{\frac{m}{n_i} : m \in \mathbb{Z}, i \geq 1\} \oplus \mathbb{Z} & if\ \frac{n_{i+1}}{n_i}\ is\ even\ for\ infinitely\ many\ i \\ \{\frac{m}{n_i} : m \in \mathbb{Z}, i \geq 1\} \oplus \mathbb{Z}^2 & otherwise. \end{cases}$$

Proof. First, we compute the number of fixed points. Recall that $\mathbb{Z} \rtimes \mathbb{Z}_2 / n_i\mathbb{Z} \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_{n_i}$, and that, for every $(\overline{x_i}) \in \varprojlim \mathbb{Z}_{n_i}$, $(0,1)(\overline{x_i}) = (\overline{-x_i})$. Hence $(\overline{x_i})$ is a fixed point of $(0,1)$ if and only if, for every i, $x_i = 0$ or x_i is even and $x_i = \frac{n_i}{2}$.
On the other hand, the action of $(1,1)$ is given by $(1,1)(\overline{x_i}) = (\overline{-x_i+1})$. Hence, any fixed point is a list of elements $x_i = \frac{n_i+1}{2}$, with n_i odd. Straightforward computation shows that

$$m_{(0,1)} = \begin{cases} 1 & if\ \frac{n_{i+1}}{n_i}\ is\ even\ for\ infinitely\ many\ i \\ 1 & if\ n_i\ is\ odd\ for\ every\ i \\ 2 & otherwise \end{cases}$$

$$m_{(1,1)} = \begin{cases} 1 & if\ n_i\ is\ odd\ for\ every\ i \\ 0 & otherwise \end{cases}$$

Now we apply Theorem 3.56. Then the result follows after noting that $(0,1)_*$ acts trivially on $\frac{C(X,\mathbb{Z})}{(Id-(1,0)_*)(C(X,\mathbb{Z}))} \cong \{\frac{m}{n_i} : m \in \mathbb{Z}, i \geq 1\}$. □

We now investigate the homology groups $H_*(\mathcal{D}_\infty, C(X,\mathbb{Z}))$.

3.4.2 Homology groups of the infinite dihedral odometer

In order to study $H_k(\mathcal{D}_\infty, C(X,\mathbb{Z}))$, we see $\mathcal{D}_\infty$ as the free product $\mathbb{Z}_2 * \mathbb{Z}_2$, and then use [64, Corollary 6.2.10] to compute $H_k(\mathcal{D}_\infty, C(X,\mathbb{Z}))$ as the direct sum $H_k(\mathbb{Z}_2, C(X,\mathbb{Z})) \oplus H_k(\mathbb{Z}_2, C(X,\mathbb{Z}))$, for $k \geq 2$. We investigate the lower homology groups in a separate way. Those particular computations are quite extensive, hence we choose to skip the proof, and redirect the reader to [56, Proposition 2.4-Theorem 3.5] for the full computation. Note that in Chapter 5 we use similar techniques when computing $H_k(\mathcal{D}_\infty, C(X,\mathbb{Z}))$ (for a different action). The reader may just check that.

Theorem 3.58. *Let $\mathcal{D}_\infty \curvearrowright \varprojlim \mathbb{Z}_{n_i} = X$ be the odometer of Example 3.55. Then we have $H_0(\mathcal{D}_\infty, C(X,\mathbb{Z})) = \{\frac{m}{n_i} : m \in \mathbb{Z}, i \geq 1\}$, and for $k \geq 1$, we have that $H_{2k}(\mathcal{D}_\infty, C(X,\mathbb{Z})) = 0$, and*

$$H_{2k-1}(\mathcal{D}_\infty, C(X,\mathbb{Z})) = \begin{cases} \mathbb{Z}_2 & if\ \frac{n_{i+1}}{n_i}\ is\ even\ for\ infinitely\ many\ i \\ (\mathbb{Z}_2)^2 & otherwise. \end{cases}$$

□

Combining Theorems 3.57 and 3.58, together with Lemmas 3.52 and 2.63, we conclude that:

Corollary 3.59. *Let $\mathcal{D}_\infty \curvearrowright \varprojlim \mathbb{Z}_{n_i}$ be the odometer of Example 3.55. Then, Matui's HK conjecture* **does not** *hold for the associated transformation groupoid.* □

Remark 3.60. *In relation with the weak HK conjecture (see paragraph 2.4.1), we can see that Scarparo's counter-example satisfies the conjecture for K_1, but not for K_0. In Chapter 5, we present a complete counter-example for the weak HK conjecture, arising from the self-similar action of the infinite dihedral group over a Cantor set. Note that, even though Scarparo's groupoid is also built upon the infinite dihedral groupoid, it is different to the one we show in Chapter 5. Most important, the C^*-algebra of this groupoid is an AF algebra, while the C^*-algebra studied in Chapter 5 provides, as we will see later, a richer structure.*

Chapter 4

A study of HK conjecture for Deaconu-Renault groupoids

The purpose of this chapter is to investigate the HK-conjecture for groupoids arising from an action of $\mathbb{N}^k$ by local homeomorphisms on a locally compact Hausdorff zero-dimensional space. The associated Deaconu-Renault groupoids were introduced by Deaconu in [16], and later studied by Farsi, Kumjian, Pask and Sims in [21]. In their recent work [21], the authors studied the HK conjecture for this family of groupoids, generalizing the techniques of Evans developed in [18] regarding k-graphs. In this work, the authors prove that rank 1 and rank 2 Deaconu-Renault groupoids satisfy the strong version of the conjecture. However, this approach cannot be easily extended to rank 3 and higher, requiring more explicit techniques. The chapter is structured as follows:

In Section 4.1, we show the computations of [18] and [21], in which the homology groups of Deaconu-Renault groupoids of arbitrary rank are completely determined.

In Section 4.2, we use both Pimsner-Voiculescu exact sequence and Kasparov's spectral sequence to compute the K-theory of Deaconu-Renault groupoids of rank 1 and 2. We show that Deaconu-Renault groupoids of rank 1 and 2 satisfy the HK-conjecture. Those results were obtained by the authors in [21]. Two major questions arise from their work. The first one asks for an explicit expression of the HK isomorphism for rank 2 Deaconu-Renault groupoids. The second, wonders whether this result can be extended to rank 3 and higher Deaconu-Renault groupoids. We will discuss both questions in the following sections.

In Section 4.3, we describe the isomorphism mentioned in their first question. To do so, we explicitly build the spectral sequence associated to our C^*-algebra as in Section 3.3.4, using the techniques of [4] and [54]. In particular, we find that the isomorphism can be chosen in a way that the embedding $H_0(\mathcal{G}(X,\sigma)) \hookrightarrow K_0(C_r^*(\mathcal{G}(X,\sigma)))$ is canonical (see Definition 2.74).

Finally, in Section 4.4, we attack the second question mentioned above. We provide a sufficient condition for Deaconu-Renault groupoids of rank 3 to satisfy the conjecture. Indeed, we show that Deaconu-Renault groupoids of rank 3 may satisfy the HK conjecture whenever the canonical map $\Phi : H_0(\mathcal{G}(X,\sigma)) \to K_0(C_r^*(\mathcal{G}(X,\sigma)))$ of Definition 2.74 is injective, as the authors suggested in [21]. To this end, we build the associated spectral sequence to our algebra, and determine the relationship between the respective differential map and Φ. We also prove that this condition is met whenever the group $\mathbb{N}^3$ acts on X by homeomorphisms.

4.1 Homology of Deaconu-Renault groupoids

The homology of Deaconu-Renault groupoids has been completely determined by Farsi, Kumjian, Pask and Sims in a recent work (see [21]), where they generalize Evans' study of the homology of k-graphs (see [18]). In this section, we show the techniques of [21] to give an explicit formula for all the homology groups of a Deaconu-Renault groupoid. Throughout this chapter, we use the convention:

$$\sigma^n := \sigma_1^{n_1}...\sigma_k^{n_k},$$

whenever $\sigma := (\sigma_1, ..., \sigma_k)$ is an action of $\mathbb{N}^k$ on a space X, and $n := (n_1, ..., n_k) \in \mathbb{N}^k$. We recall the definition of a Deacounu-Renault grupoid, given in Example 2.25:

Definition 4.1. *(Deaconu-Renault groupoids). Let X be a locally compact Hausdorff space, and let σ be an action of $\mathbb{N}^k$ on X by surjective local homeomorphisms. The associated* ***Deaconu-Renault groupoid*** *$\mathcal{G}(X, \sigma)$ is given by*

$$\mathcal{G}(X, \sigma) := \{(x, p - q, y) \in X \times \mathbb{Z}^k \times X : \sigma^p(x) = \sigma^q(y)\},$$

together with structure operations $r(x, n, y) = (x, 0, x)$, $s(x, n, y) = (y, 0, y)$, and $(x, n, y)(y, m, z) = (x, m + n, z)$. The set $\mathcal{G}(X, \sigma)^{(0)}$ is usually identified with X via $(x, 0, x) \mapsto x$.
The topology is given by the basis of sets $Z(U, p, q, V) := (U \times \{p - q\} \times V) \cap \mathcal{G}(X, \sigma)$, where U, V are open sets of X such that $\sigma^p(U) = \sigma^q(V)$.

□

Note that, if we denote $X' := X \times \mathbb{Z}^k$, there is an induced action τ of $\mathbb{N}^k$ by surjective local homeomorphisms on X' defined by $\tau^q(x, p) = (\sigma^q(x), p + q)$.

Lemma 4.2. *Let X be a locally compact Hausdorff space, and let σ be an action of $\mathbb{N}^k$ on X by surjective local homeomorphisms. Then the map $c : \mathcal{G}(X, \sigma) \to \mathbb{Z}^k$ given by $(x, p - q, y) \mapsto p - q$ is a cocycle. Moreover, there exists an isomorphism between $\mathcal{G}(X, \sigma) \times_c \mathbb{Z}^k$ and $\mathcal{G}(X', \tau)$ given by*

$$((x, n, y), p) \mapsto ((x, p), n, (y, p + n)).$$

Proof. For any $m, n \in \mathbb{Z}$, and $x, y, z \in X$, we have that $(x, m, y)(y, n, z) = (x, m+n, z) \in \mathcal{G}(X, \sigma)$, and then:

$$m + n = c(x, m + n, z) = c((x, m, y)(y, n, z)) = c(x, m, y) + c(y, n, z) = m + n,$$

hence c is a cocycle.
The second statement is straightforward to check.

□

In order to compute the homology groups of $\mathcal{G}(X,\sigma)$, we will use Matui's spectral sequence given in Theorem 3.19. To do so, we first need to study the homology groups of $\mathcal{G}(X,\sigma)\times_c\mathbb{Z}^k$.

Lemma 4.3. *Let X be a locally compact Hausdorff space, and let σ be an action of $\mathbb{N}^k$ on X by surjective local homeomorphisms. Then the set $X\times\{0\}\subseteq X\times\mathbb{Z}^k$ is a clopen $\mathcal{G}(X',\tau)$-full subset of $\mathcal{G}(X',\tau)^{(0)}$, and $\mathcal{G}(X,\sigma)\times_c\mathbb{Z}^k$ is Kakutani equivalent to $ker(c)$.*

Proof. ([21, Lemma 6.1]) It is clear that $X\times\{0\}$ is clopen. We need to see that it is full. Fix $(x,n)\in X\times\mathbb{Z}^k$, and let $n=p-q$, with $p,q\in\mathbb{N}^k$. Since σ acts via surjections, there exists some $y\in X$ such that $\sigma^p(y)=\sigma^q(x)$. Then we have

$$\tau^p(y,0)=(\sigma^p(y),p)=(\sigma^q(x),n+q)=\tau^q(x,n).$$

Therefore, if we define $\gamma:=((y,0),p-q,(x,n))\in\mathcal{G}(X',\tau)$, we have that $r(\gamma)=(y,0)\in X\times\{0\}$, and $s(\gamma)=(x,n)$. Hence, $X\times\{0\}$ is full.

The last statement is then immediate after using Lemma 3.17 and the isomorphism $\mathcal{G}(X,\sigma)\times_c\mathbb{Z}^k\cong\mathcal{G}(X',\tau)$. □

Corollary 4.4. *The skew product $\mathcal{G}(X,\sigma)\times_c\mathbb{Z}^k$ is an AF groupoid.*

Proof. For each $n\in\mathbb{N}^k$, define $c_n:=\{(x,0,y)\in\mathcal{G}(X,\sigma):\sigma^n(x)=\sigma^n(y)\}\subseteq ker(c)$. Each c_n is an elementary groupoid and, for each $n,m\in\mathbb{N}^k$, we have $c_n\subseteq c_{n+m}$. Moreover,

$$ker(c)=\bigcup_{n\in\mathbb{N}^k}c_n.$$

Combining this with the last lemma, we deduce that $\mathcal{G}(X,\sigma)\times_c\mathbb{Z}^k$ is an AF groupoid. □

The following Lemma appears in [21, Lemma 6.2]. We write here a sketch of its proof for a matter of completeness.

Lemma 4.5. *Let X be a totally disconnected locally compact Hausdorff space, and let σ be an action of $\mathbb{N}^k$ on X by surjective local homeomorphisms. There is an isomorphism $\varinjlim_{n\in\mathbb{N}^k}(C_c(X,\mathbb{Z}),\sigma_*^n)\cong H_0(ker(c))$ that takes $\sigma_*^{0,\infty}(1_U)$ to $[1_U]$ for every compact open $U\subseteq X$, where σ_*^n denotes the induced action on $C_c(X,\mathbb{Z})$. Then we have*

$$H_q(\mathcal{G}(X,\sigma)\times_c\mathbb{Z}^k)=\begin{cases}\varinjlim_{n\in\mathbb{N}^k}(C_c(X,\mathbb{Z}),\sigma_*^n) & if \quad q=0\\ \\ 0 & otherwise\end{cases}$$

Moreover, the isomorphism $H_0(\mathcal{G}(X,\sigma)\times_c\mathbb{Z}^k)\cong\varinjlim_{n\in\mathbb{N}^k}(C_c(X,\mathbb{Z}),\sigma_*^n)$ intertwines the action α of $\mathbb{Z}^k$ on $H_0(\mathcal{G}(X,\sigma)\times_c\mathbb{Z}^k)$ given by $\alpha_p((x,m,y),n)=((x,m,y),n+p)$ with the action of $\mathbb{Z}^k$ on $\varinjlim_{n\in\mathbb{N}^k}(C_c(X,\mathbb{Z}),\sigma_*^n)$ induced by σ_*^n.

Proof. The fact that $H_q(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k) = 0$ for all $q > 0$ is immediate consequence of $\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k$ being an AF groupoid. Let us study the case $q = 0$.
Let $U, V \subseteq X$ compact open sets on which σ^n is injective, and such that $\sigma^n(V) = \sigma^n(U)$. It is clear that $[\chi_U] = [\chi_V]$ in $H_0(c_n)$, given that $r(U,0,V) = U$, and $s(U,0,V) = V$. Then, since every $W \subseteq X$ can be expressed as a finite disjoint union of open compact sets $W = \bigsqcup \sigma^n(U_j)$, such that $\sigma^n|_{U_j}$ is injective, it follows that there exists a unique homomorphism $\varphi_n : C_c(X,\mathbb{Z}) \to H_0(c_n)$ such that $\varphi_n(\sigma^n_*(\chi_U)) = [\chi_U]$, for every compact open U. This map is an isomorphism (see [21]). Let $i_{m,n} : c_m \to c_{m+n}$ be the natural inclusion for each $m, n \in \mathbb{N}^k$. Then we have the following commutative diagram:

$$\begin{array}{ccc}
C_c(X,\mathbb{Z}) & \xrightarrow{\sigma^n_*} & C_c(X,\mathbb{Z}) \\
\Big\downarrow{\scriptstyle \varphi_m} & & \Big\downarrow{\scriptstyle \varphi_{m+n}} \\
H_0(c_m) & \xrightarrow[(i_{m,n})_*]{} & H_0(c_{n+m})
\end{array}$$

Then, using Corollary 4.4, we have that

$$H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k) \cong H_0(ker(c)) \cong \varinjlim_{n\in\mathbb{N}^k} (H_0(c_n), i_*) \cong \varinjlim_{n\in\mathbb{N}^k} (C_c(X,\mathbb{Z}), \sigma^n_*).$$

□

Since $H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k) = 0$ for all $q \geq 1$, Matui's spectral sequence (3.19) collapses at the second page, inducing an isomorphism

$$H_n(\mathcal{G}(X,\sigma)) \cong H_n(\mathbb{Z}^k, H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k)), \quad for\ 0 \leq n \leq k,$$

and zero for all $n > k$, where $H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k)$ is considered as a $\mathbb{Z}^k$-module under the action induced by α on $\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k$.
The homology groups $H_n(\mathbb{Z}^k, \mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k)$ were computed in [21], using techniques previously developed in [18]. We skip some of the technical proofs, directing the reader to [21].
The following lemmas allow us to study the homology groups when direct limits are involved.
Lemma 4.6. *([21, Lemma 6.3]) Let A be an abelian group, and let σ be an action of $\mathbb{N}^k$ on A. Denote by σ_i the subaction associated to the generator e_i and, for each $1 \leq p \leq k$, we write $\bigwedge^p \mathbb{Z}^k$ as in Chapter 3. Define*

$$\delta_p : \bigwedge^p \mathbb{Z}^k \otimes A \to \bigwedge^{p-1} \mathbb{Z}^k \otimes A$$

by

$$\delta_p(e_{i_1} \wedge ... \wedge e_{i_p} \otimes a) = \begin{cases} \sum_j (-1)^{j+1} e_{i_1} \wedge ... \wedge \hat{e}_{i_j} \wedge ... \wedge e_{i_p} \otimes (id - \sigma_{i_j})a & if \quad p > 1 \\ \\ 1 \otimes (1 - \sigma_{i_1})a & if \quad p = 1 \end{cases}$$

where the symbol $\hat{e}_{i_j}$ means that this element is removed.
Then the maps δ_p are differentials, in the sense that $\delta_p \circ \delta_{p+1} = 0$, and $(\bigwedge^* \mathbb{Z}^k \otimes A, \delta_*)$ is a complex. Moreover, the homomorphism $id \otimes \sigma_i : \bigwedge^* \mathbb{Z}^k \otimes A \to \bigwedge^* \mathbb{Z}^k \otimes A$ commutes with δ_*, and the induced map $(id \otimes \sigma_{i*})$ in homology is the identity map.

□

This extends to a more general result.
Lemma 4.7. *([21, Lemma 6.4]) Let A be an abelian group, and $\sigma : \mathbb{N}^k \curvearrowright A$ be an action. Let $\delta_p : \bigwedge^p \mathbb{Z}^k \otimes A \to \bigwedge^{p-1} \mathbb{Z}^k \otimes A$ be as in Lemma 4.6, and denote $\tilde{A} := \varinjlim_{n \in \mathbb{N}^k} (A, \sigma^n)$. For $i \leq k$, let $\tilde{\sigma}_i$ be the automorphism of $\tilde{A}$ induced by σ_i, and let $\tilde{\delta}_p : \bigwedge^p \mathbb{Z}^k \otimes \tilde{A} \to \bigwedge^{p-1} \mathbb{Z}^k \otimes \tilde{A}$ be the boundary maps of Lemma 4.6, applied to $\tilde{A}$ and $\tilde{\sigma}_i$. Then the homomorphism $\sigma^{(0,\infty)} : A \to \tilde{A}$ induces an isomorphism $H_*(\bigwedge^* \mathbb{Z}^k \otimes A) \cong H_*(\bigwedge^* \mathbb{Z}^k \otimes \tilde{A})$.*

Proof. The homology is a continuous functor (see [61, Theorem 4.1.7]), and hence $H_*(\bigwedge^* \mathbb{Z}^k \otimes \tilde{A}) \cong \varinjlim_{\mathbb{N}^k} (H_*(\bigwedge^* \mathbb{Z}^k \otimes A), (id \otimes \sigma^n)_*)$. Lemma 4.6 ensures that $(id \otimes \sigma^n)_*$ is the identity in homology, and therefore

$$H_*(\bigwedge^* \mathbb{Z}^k \otimes \tilde{A}) \cong H_*(\bigwedge^* \mathbb{Z}^k \otimes A).$$

□

Combining all the previous results, we can now compute the homology groups $H_n(\mathcal{G}(X, \sigma))$.
Theorem 4.8. *([21, Theorem 6.5]) Let X be a second countable totally disconnected locally compact space, and let σ be an action of $\mathbb{N}^k$ by surjective local homeomorphisms on X. For $1 \leq p \leq k$, let $A_p^\sigma := \bigwedge^p \mathbb{Z}^k \otimes C_c(X, \mathbb{Z})$, and define $A_p^\sigma = \{0\}$ for $p > k$. Define $\delta_p : A_p^\sigma \to A_{p-1}^\sigma$ by*

$$\delta_p(e_{i_1} \wedge ... \wedge e_{i_p} \otimes f) = \begin{cases} \sum_j (-1)^{j+1} e_{i_1} \wedge ... \wedge \hat{e}_{i_j} \wedge ... \wedge e_{i_p} \otimes (id - \sigma_{i_j *})f & if \quad 2 \leq p \leq k \\ \\ 1 \otimes (1 - \sigma_{i_1 *})f & if \quad p = 1 \\ \\ 0 & otherwise \end{cases}$$

Then (A_*^σ, δ_*) is a complex, and

$$H_*(\mathcal{G}(X, \sigma)) \cong H_*(\mathbb{Z}^k, H_0(\mathcal{G}(X, \sigma) \times_c \mathbb{Z}^k)) \cong H_*(A_*^\sigma, \delta_*).$$

In particular, $H_n(\mathcal{G}(X,\sigma)) = 0$ *for all* $n > k$.

Proof. For an abelian group A and an action $\sigma : \mathbb{N}^k \curvearrowright A$, the homology groups $H_*(\bigwedge^* \mathbb{Z}^k \otimes \tilde{A})$ were computed in [18, Lemma 3.12] and [21, Lemma 6.4] to be:

$$H_*(\bigwedge\nolimits^* \mathbb{Z}^k \otimes \tilde{A}) \cong H_*(\mathbb{Z}^k, \tilde{A}),$$

where $\tilde{A}$ is a $\mathbb{Z}^k$-module under the action induced by $\tilde{\sigma}$.
Now, define $A := C_c(X,\mathbb{Z})$, and then $\tilde{A} := \varinjlim_{n\in\mathbb{N}}(C_c(X,\mathbb{Z}), \sigma_*^n)$. Using Matui's spectral sequence, together with the previous results, we obtain an isomorphism:

$$\begin{aligned} H_*(\mathcal{G}(X,\sigma)) &\cong H_*(\mathbb{Z}^k, H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k)) \cong H_*(\mathbb{Z}^k, \varinjlim_{n\in\mathbb{N}}(C_c(X,\mathbb{Z}), \sigma_*^n)) \cong \\ &\cong H_*(\bigwedge\nolimits^* \mathbb{Z}^k \otimes \tilde{A}) \cong H_*(\bigwedge\nolimits^* \mathbb{Z}^k \otimes A) \cong H_*(A_*^\sigma, \delta_*). \end{aligned}$$

The last statement is immediate. □

4.2 Rank 1 and rank 2 Deaconu-Renault groupoids

In [21], the authors use the above results, together with Lemma 3.20 in order to verify Matui's HK conjecture. We show their strategies here, as well as the open questions arising from their reasoning.

Theorem 4.9. *([21, Theorem 6.7]) Let X be a second countable locally compact totally disconnected space, and let $\sigma : X \to X$ be a surjective local homeomorphism (that is, an action of $\mathbb{N}$ on X). Denote by $\sigma_* : C_c(X, \mathbb{Z}) \to C_c(X, \mathbb{Z})$ the induced map. Then the groupoid $\mathcal{G}(X, \sigma)$ satisfies Matui's HK conjecture, with*

$$K_0(C_r^*(\mathcal{G}(X,\sigma))) \cong H_0(\mathcal{G}(X,\sigma)) \cong coker(id - \sigma_*),$$
$$K_1(C_r^*(\mathcal{G}(X,\sigma))) \cong H_1(\mathcal{G}(X,\sigma)) \cong ker(id - \sigma_*), \text{ and}$$
$$H_n(\mathcal{G}(X,\sigma)) = 0, \text{ for all } n \geq 2.$$

Proof. Lemma 3.20 provides a stably isomorphism

$$C_r^*(\mathcal{G}(X,\sigma)) \otimes \mathbb{K} \cong C_r^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}) \rtimes_\alpha \mathbb{Z}.$$

As noted before, we will drop the subindex α of the crossed product whenever the involved action is clear.

The groupoid $\mathcal{G}(X,\sigma) \times_c \mathbb{Z}$ is an AF groupoid (see Corollary 4.4) and hence, using Lemma 3.13, we have that:

$$K_1(C_r^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z})) = 0, \text{ and}$$
$$K_0(C_r^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z})) \cong H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}),$$

under the map given by $[1_{U\times\{n\}}]_{K_0} \mapsto [1_{U\times\{n\}}]_{H_0}$ for any compact open $U \subseteq X$. Therefore if we denote $A := C_r^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z})$, the Pimsner-Voiculescu exact sequence (3.22) is of the form:

$$\begin{tikzcd}
K_0(A) \arrow[r, "id - K_0(\alpha)"] & K_0(A) \arrow[r] & K_0(A \rtimes \mathbb{Z}) \arrow[d] \\
K_1(A \rtimes \mathbb{Z}) \arrow[u] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

In particular, we have that

$$K_0(C_r^*(\mathcal{G}(X,\sigma))) \cong K_0(A \rtimes \mathbb{Z}) \cong coker(id - K_0(\alpha)), \text{ and}$$
$$K_1(C_r^*(\mathcal{G}(X,\sigma))) \cong K_1(A \rtimes \mathbb{Z}) \cong ker(id - K_0(\alpha)).$$

On the other hand, we can use Theorem 4.8 to compute the homology groups of $\mathcal{G}(X,\sigma)$ as the homology groups of the chain complex given by

$$0 \to C_c(X,\mathbb{Z}) \overset{id-\sigma_*}{\to} C_c(X,\mathbb{Z}) \to 0$$

and therefore

$$H_0(\mathcal{G}(X,\sigma)) \cong coker(id-\sigma_*),$$
$$H_1(\mathcal{G}(X,\sigma)) \cong ker(id-\sigma_*), \text{ and}$$
$$H_n(\mathcal{G}(X,\sigma)) = 0 \text{ for } n > 1.$$

Moreover, it was shown in Lemma 4.5 that the isomorphism

$$H_0(\mathcal{G}(X,\sigma)\times_c \mathbb{Z}) \cong \varinjlim_{n\in\mathbb{N}}(C_c(X,\mathbb{Z}),\sigma_*^n)$$

intertwines the action of $\mathbb{Z}$ on $\varinjlim_{n\in\mathbb{N}}(C_c(X,\mathbb{Z}),\sigma_*^n)$ induced by σ_* with the action α of $\mathbb{Z}$ on $H_0(\mathcal{G}(X,\sigma)\times_c \mathbb{Z})$. Then, using Lemma 4.7 and the canonical isomorphism $H_0(\mathcal{G}(X,\sigma)\times_c \mathbb{Z}) \cong K_0(C_r^*(\mathcal{G}(X,\sigma)\times_c \mathbb{Z}))$, we conclude that

$$K_0(C_r^*(\mathcal{G}(X,\sigma))) \cong coker(id-K_0(\alpha)) \cong coker(id-\sigma_*) \cong H_0(\mathcal{G}(X,\sigma)), \text{ and}$$
$$K_1(C_r^*(\mathcal{G}(X,\sigma))) \cong ker(id-K_0(\alpha)) \cong ker(id-\sigma_*) \cong H_1(\mathcal{G}(X,\sigma)),$$

as desired. □

Theorem 4.10. *([21, Theorem 6.10]) Let X be a second countable locally compact totally disconnected space, and let $\sigma^1,\sigma^2 : X \to X$ be a pair of commuting surjective local homeomorphisms (that is, an action of $\mathbb{N}^2$ on X). Define $d_2 : C_c(X,\mathbb{Z}) \to C_c(X,\mathbb{Z})\oplus C_c(X,\mathbb{Z})$ by $d_2(f) := ((\sigma_{2*}-id)f,(id-\sigma_{1*})f)$, and $d_1 : C_c(X,\mathbb{Z})\oplus C_c(X,\mathbb{Z}) \to C_c(X,\mathbb{Z})$ by $d_1(f,g) := (id-\sigma_{1*})f + (id-\sigma_{2*})g$. Then the groupoid $\mathcal{G}(X,\sigma)$ satisfies Matui's HK conjecture, with*

$$K_0(C_r^*(\mathcal{G}(X,\sigma))) \cong H_0(\mathcal{G}(X,\sigma)) \oplus H_2(\mathcal{G}(X,\sigma)) \cong coker(d_1)\oplus ker(d_2),$$
$$K_1(C_r^*(\mathcal{G}(X,\sigma))) \cong H_1(\mathcal{G}(X,\sigma)) \cong ker(d_1)/Im(d_2), \text{ and}$$
$$H_n(\mathcal{G}(X,\sigma)) = 0, \text{ for all } n \geq 3.$$

Proof. First, recall that $C_r^*(\mathcal{G}(X,\sigma))$ is stably isomorphic to $C_r^*(\mathcal{G}(X,\sigma)\times_c \mathbb{Z}^2)\rtimes_\alpha \mathbb{Z}^2$ (see Lemma 3.20), and hence we can consider Kasparov's homological spectral sequence from Theorem 3.23:

$$E^2_{p,q} = H_p(\mathbb{Z}^2, K_q(C_r^*(\mathcal{G}(X,\sigma)\times_c \mathbb{Z}^2))) \Rightarrow K_{p+q}(C_r^*(\mathcal{G}(X,\sigma))).$$

Before advancing further, we should note that the spectral sequence above, used by the authors in [21], is a *homological* spectral sequence, in contrast with the cohomological one introduced in Section 3.3.4. In this line, the reader may note that both spectral sequences are *symmetrical* with respect to each other. In this subsection, since we are just stating the results obtained by the authors in [21], as well as their reasoning, we have chosen to keep their convention. We will, however, use the cohomological approach in the next sections.

Recall that $\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2$ is an AF groupoid (see Corollary 4.4). Hence, we have that $K_1(C_r^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2)) = 0$. In particular, we deduce that $E^2_{p,q} = 0$ whenever q is an odd integer. Moreover, Lemma 3.13 provides an isomorphism

$$K_0(C_r^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2)) \cong H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2)$$

under the map given by $[1_{U\times\{n\}}]_{K_0} \mapsto [1_{U\times\{n\}}]_{H_0}$. This isomorphism intertwines the actions of $\mathbb{Z}^2$ on $K_0(C_r^*(\mathcal{G}(X,\sigma)\times_c\mathbb{Z}^2))$ and $H_0(\mathcal{G}(X,\sigma)\times_c\mathbb{Z}^2)$ induced by the translation on the second coordinate $\alpha_m((x,n,y),p) = ((x,n,y),p+m)$. Thus, it follows that

$$E^2_{n,q} \cong H_n(\mathbb{Z}^2, K_0(C_r^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2))) \cong H_n(\mathbb{Z}^2, H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2)),$$

whenever q is even. Therefore, using Theorem 4.8, the differential maps in the second page of the spectral sequence have bidegree $(-2,1)$, and the second page of the spectral sequence is of the form

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & H_0(\mathcal{G}(X,\sigma)) & H_1(\mathcal{G}(X,\sigma)) & H_2(\mathcal{G}(X,\sigma)) & 0 & \cdots \\
\cdots & 0 & 0 & 0 \arrow[llu] & 0 & \cdots \\
\cdots & H_0(\mathcal{G}(X,\sigma)) & H_1(\mathcal{G}(X,\sigma)) & H_2(\mathcal{G}(X,\sigma)) \arrow[llu] & 0 & \cdots
\end{tikzcd}$$

Since all the odd rows of the page are zero, we deduce that all the differentials $d^2_{p,q}$ are trivial, and hence $E^2_{p,q} = E^3_{p,q}$. Moreover, for any integer $r > 2$, the bidegree of $d^r_{*,*}$ is $(-r, r-1)$. Since there are only three non-trivial columns, we conclude that the differential maps are trivial for all $r \geq 2$, and then the spectral sequence reaches its limit at the second page, that is, $E^2_{p,q} = E^\infty_{p,q}$. Then, as in Example 3.31, convergence of the spectral sequence implies the existence of an isomorphism

$$K_1(C_r^*(\mathcal{G}(X,\sigma))) \cong E^2_{0,1} \cong H_1(\mathbb{Z}^2, H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2)) \cong H_1(\mathcal{G}(X,\sigma)),$$

and an extension

$$0 \to E^2_{0,2} \to K_0(C_r^*(\mathcal{G}(X,\sigma))) \to E^2_{2,0} \to 0$$

where

$$E^2_{0,2} \cong H_0(\mathbb{Z}^2, H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2)) \cong H_0(\mathcal{G}(X,\sigma)), \text{ and}$$
$$E^2_{2,0} \cong H_2(\mathbb{Z}^2, H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2)) \cong H_2(\mathcal{G}(X,\sigma)).$$

The groups $H_n(\mathbb{Z}^2, H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2))$ were computed in Theorem 4.8 to be isomorphic to the homology groups of the chain complex (A^σ_*, δ_*), which is defined as:

$$0 \to C_c(X,\mathbb{Z}) \overset{d_2}{\to} C_c(X,\mathbb{Z}) \oplus C_c(X,Z) \overset{d_1}{\to} C_c(X,\mathbb{Z}) \to 0$$

Therefore, we deduce that

$$K_1(C^*_r(\mathcal{G}(X,\sigma))) \cong ker(d_1)/Im(d_2) \cong H_1(\mathcal{G}(X,\sigma)).$$

On the other hand, it was shown in [21, Lemma 6.9] that $C_c(X,\mathbb{Z})$ is a free abelian group whenever X is a second countable locally compact Hausdorff space. Hence, since $E^2_{2,0} \cong ker(d_2) \subseteq C_c(X,\mathbb{Z})$, the extension of K_0 splits, and we obtain an isomorphism

$$K_0(C^*_r(\mathcal{G}(X,\sigma))) \cong coker(d_1) \oplus ker(d_2) \cong H_0(\mathcal{G}(X,\sigma)) \oplus H_2(\mathcal{G}(X,\sigma)),$$

concluding the proof. □

Two major questions arise in [21, Remarks 6.11-6.13]. The first one aims to determine the maps involved in the verification of Matui's HK conjecture for Deaconu-Renault groupoids of rank 2. As we just saw, there is an isomorphism

$$K_0(C^*_r(\mathcal{G}(X,\sigma))) \cong H_0(\mathcal{G}(X,\sigma)) \oplus H_2(\mathcal{G}(X,\sigma)) \cong coker(d_1) \oplus ker(d_2).$$

However, the explicit expression of the isomorphism is unknown.
The second question tries to determine if Matui's HK conjecture is still satisfied for Deaconu-Renault groupoids of higher ranks. In particular, the authors suggest that, for the rank 3, the conjecture could be related with the injectivity of the natural map

$$\Phi : H_0(\mathcal{G}(X,\sigma)) \to K_0(C^*_r(\mathcal{G}(X,\sigma)))$$

described in Definition 2.74.
Throughout the remaining of this chapter, we answer both of this questions. To do so, we need to study Kasparov's spectral sequence under last chapter's point of view (Section 3.3.4).

4.3 The HK isomorphisms for Deaconu-Renault groupoids of rank 2

In this section, we provide information about the HK isomorphisms previously obtained for Deaconu-Renault groupoids of rank 2:

$$K_0(C_r^*(\mathcal{G}(X,\sigma))) \cong H_0(\mathcal{G}(X,\sigma)) \oplus H_2(\mathcal{G}(X,\sigma)),$$
$$K_1(C_r^*(\mathcal{G}(X,\sigma))) \cong H_1(\mathcal{G}(X,\sigma)).$$

To this end, we will use two different approaches, leading both to the same conclusion: the HK isomorphism for K_0 can be chosen such that the map between $H_0(\mathcal{G}(X,\sigma))$ and $K_0(C_r^*(\mathcal{G}(X,\sigma)))$ is the natural one, given in Definition 2.74. In this line, we show the strengths and limitations of each strategy.

Our first approach is based on the Pimsner-Voiculescu exact sequences shown in Theorem 3.22. Since Lemma 3.20 implies a stable isomorphism between $C_r^*(\mathcal{G}(X,\sigma))$ and $C_r^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2) \rtimes_\alpha \mathbb{Z}^2$, we can find the desired K_0 group, up to isomorphism, encoded in the second iteration of the Pimsner-Voiculescu exact sequences. However, we will see that this approach fails to provide any information about the isomorphism associated to K_1. This strategy will be used later, in Section 4.4.2.

In order to obtain information about the K_1 isomorphism, we use a second approach based on Kasparov's spectral sequence, using the techniques from Section 3.3.4, that is, building the spectral sequence associated to a certain cofiltration of the mapping torus. We will also use this strategy in Section 4.4.

Let us start with the approach via Pimsner-Voiculescu sequences:

Lemma 4.11. *Let X be a second countable locally compact totally disconnected space, and let $\sigma_1, \sigma_2 : X \to X$ be a pair of commuting surjective local homeomorphisms (that is, an action of $\mathbb{N}^2$ on X). Then the homology group $H_0(\mathcal{G}(X,\sigma))$ of the Deaconu-Renault groupoid embeds canonically into $K_0(C_r^*(\mathcal{G}(X,\sigma)))$.*

Proof. Recall that $C_r^*(\mathcal{G}(X,\sigma))$ is stably isomorphic to $C_r^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2) \rtimes_\alpha \mathbb{Z}^2$ (see Lemma 3.20). The proof of Lemma 3.20 shows that this result is deduced from the isomorphism

$$C_r^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2) \cong C_r^*(\mathcal{G}(X,\sigma)) \rtimes_\varphi \mathbb{T}^2,$$

where φ is the gauge action. By inspecting this map as defined in Lemma 3.20, one can see that this isomorphism sends $1_{U\times\{0\}}$ to 1_U, for each open $U \subseteq X$. Then naturality of Takai-Takesaki duality implies that the stable isomorphism is canonical, in the sense that $1_U \otimes e_{00} \in C_r^*(\mathcal{G}(X,\sigma)) \otimes \mathbb{K}$ is sent to the element $1_{U\times\{0\}} \in C_r^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2) \rtimes_\alpha \mathbb{Z}^2$. From now on, we denote $A := C_r^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2)$, and drop the subindex α. Recall that $\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2$ is an AF groupoid (see Corollary 4.4), and hence $K_1(A) = 0$. Therefore, the Pimsner-Voiculescu exact sequence associated to the action α_1 is as it follows:

$$\begin{tikzcd}
K_0(A) \arrow[r, "id - K_0(\alpha_1)"] & K_0(A) \arrow[r, "j_1"] & K_0(A \rtimes \mathbb{Z}) \arrow[d] \\
K_1(A \rtimes \mathbb{Z}) \arrow[u, "\delta"] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

where, in order to ease the notation, we simply write j_1 instead of $K_0(j_1)$.

From the diagram above, we deduce that $K_0(A \rtimes \mathbb{Z}) \overset{j_1}{\cong} K_0(A)/Im(id - K_0(\alpha_1))$, and $K_1(A \rtimes \mathbb{Z}) \overset{\delta}{\cong} ker(id - K_0(\alpha_1))$. Then the second Pimsner-Voiculescu exact sequence associated to the action α_2 is of the form

$$\begin{tikzcd}
K_0(A \rtimes \mathbb{Z}) \arrow[r, "id - K_0(\alpha_2)"] & K_0(A \rtimes \mathbb{Z}) \arrow[r, "j_2"] & K_0(A \rtimes \mathbb{Z}^2) \arrow[d, "\rho"] \\
K_1(A \rtimes \mathbb{Z}^2) \arrow[u] & K_1(A \rtimes \mathbb{Z}) \arrow[l] & K_1(A \rtimes \mathbb{Z}) \arrow[l, "id - K_1(\alpha_2)"']
\end{tikzcd}$$

Naturality of the Pimsner-Voiculescu exact sequences implies that the following diagram commutes

$$\begin{tikzcd}
ker(id - K_0(\alpha_1)) \arrow[r, "id - K_0(\alpha_2)"] & ker(id - K_0(\alpha_1)) \\
K_1(A \rtimes \mathbb{Z}) \arrow[u, "\delta", "\cong"'] \arrow[r, "id - K_1(\alpha_2)"'] & K_1(A \rtimes \mathbb{Z}) \arrow[u, "\cong", "\delta"']
\end{tikzcd}$$

Hence we obtain an extension for $K_0(A \rtimes \mathbb{Z}^2)$ given by

$$0 \to K_0(A)/(\sum_{i=1}^{2} Im(id - K_0(\alpha_i))) \overset{j_2 \circ j_1}{\to} K_0(A \rtimes \mathbb{Z}^2) \overset{\delta \circ \rho}{\to} \bigcap_{i=1}^{2} ker(id - K_0(\alpha_i)) \to 0,$$

where the map $K_0(A)/(\sum_{i=1}^{2} Im(id - K_0(\alpha_i))) \overset{j_2\circ j_1}{\to} K_0(A \rtimes \mathbb{Z}^2)$ is induced by the canonical inclusions j_1, j_2, and thus is given by

$$[1_{U\times\{n\}}] \mapsto [1_{U\times\{n\}}].$$

This extension induces Matui's HK isomorphism for K_0. Indeed, by Theorem 4.10 we have isomorphisms

$$H_2(\mathcal{G}(X,\sigma)) \cong \bigcap_{i=1}^{2} ker(id - \sigma_{i*}) \cong \bigcap_{i=1}^{2} ker(id - K_0(\alpha_i)),$$

and

$$H_0(\mathcal{G}(X,\sigma)) \cong C_c(X,\mathbb{Z})/(\sum_{i=1}^{2} Im(id - \sigma_{i*})) \cong K_0(A)/(\sum_{i=1}^{2} Im(id - K_0(\alpha_i))).$$

The isomorphism $H_0(\mathcal{G}(X,\sigma)) \cong K_0(A)/(\sum_{i=1}^{2} Im(id - K_0(\alpha_i)))$ is given by $[1_U] \mapsto [1_{U\times\{0\}}]$. Indeed, the isomorphism is consequence of the following chain of isomorphisms:

$$H_0(\mathcal{G}(X,\sigma)) \cong H_0(\mathbb{Z}^2, H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2)),$$

provided by Matui's spectral sequence, which sends $[1_U]$ to $[1_{U\times\{0\}}]$ (see [14, Theorem 4.4] for further details), where $H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2)$ is a $\mathbb{Z}^2$-module under the action given by α.

$$H_0(\mathbb{Z}^2, H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2)) \cong H_0(\mathbb{Z}^2, ker(c)),$$

where $ker(c)$ is a $\mathbb{Z}^2$-module under the action given by σ. This isomorphism is consequence of the Kakutani equivalence between $H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2)$ and $ker(c)$ described in Lemmas 4.3 and 3.17, given by $[1_{U\times\{0\}}] \mapsto [1_U]$, which intertwines the respective actions α and σ (see Lemma 4.5).

$$H_0(\mathbb{Z}^2, ker(c)) \cong H_0(\mathbb{Z}^2, \varinjlim_{n\in\mathbb{N}^2} (C_c(X,\mathbb{Z}), \sigma_*^n)),$$

since $ker(c)$ is an AF groupoid.

$$H_0(\mathbb{Z}^2, \varinjlim_{n\in\mathbb{N}^2} (C_c(X,\mathbb{Z}), \sigma_*^n)) \cong \varinjlim_{n\in\mathbb{N}^2} (C_c(X,\mathbb{Z}), \sigma_*^n)/\sum Im(id - \sigma_{i*}),$$

by definition of group homology. Then, using Lemma 4.5, Lemma 4.7 and Theorem 4.8, we have

$$\varinjlim_{n\in\mathbb{N}^2} (C_c(X,\mathbb{Z}), \sigma_*^n)/\sum Im(id - \sigma_{i*}) \cong K_0(A)/\sum Im(id - K_0(\alpha_i)).$$

Hence, following the involved maps one has that the isomorphism $H_0(\mathcal{G}(X,\sigma)) \cong K_0(A)/(\sum_{i=1}^{2} Im(id - K_0(\alpha_i)))$ is given by $[1_U] \mapsto [1_{U\times\{0\}}]$.
The isomorphism for $H_2(\mathcal{G}(X,\sigma))$ comes from the same reasoning. Finally, using the isomorphism $K_0(A \rtimes \mathbb{Z}^2) \cong K_0(C_r^*(\mathcal{G}))$ deduced from Lemma 3.20, which sends $[1_{U\times\{0\}}]$ to $[1_U \otimes e_{00}]$ (as we noted at the beginning of the proof), we conclude that the HK isomorphim for K_0 can be chosen to be natural, in the sense that the group H_0 embeds canonically into the group K_0. $\square$

This approach, however, does not provide a description of the isomorphism for the K_1-group. From the last Pimsner-Voiculescu exact sequence, we deduce an extension of $K_1(A \rtimes \mathbb{Z}^2)$ of the form:

$$0 \to ker(id - K_0(\alpha_1))/Im(id - K_1(\alpha_2)) \to K_1(A \rtimes \mathbb{Z}^2) \to ker(id - K_0(\alpha_2))/Im(id - K_0(\alpha_1)) \to 0$$

which does not give us enough information. To this end, we follow a second strategy, and investigate Kasparov's spectral sequence as in Section 3.3.4.

Denote by $\mathcal{M}_\alpha(A)$ the mapping torus associated to the action α, as in Definition 3.44. Then we have

$$K_*(A \rtimes \mathbb{Z}^2) \cong K_*(\mathcal{M}_\alpha(A)),$$

as we noted in Theorem 3.46. In Subsection 3.3.4.3, we associated a finite cofiltration of the mapping torus $\mathcal{M}_\alpha(A)$ arising from an action of $\mathbb{Z}^k$ on A, given by:

$$\mathcal{M}_\alpha(A) = F_k \overset{\pi_k}{\to} F_{k-1} \overset{\pi_{k-1}}{\to} ... \overset{\pi_3}{\to} F_2 \overset{\pi_2}{\to} F_1 \overset{\pi_1}{\to} F_0 \overset{\pi_0}{\to} F_{-1} = \{0\}$$

where each F_i is obtained by restricting its domain to

$$X_i := \{t \in [0,1]^k : t_{\mu_1} = t_{\mu_2} = ... = t_{\mu_{k-i}} = 0, (\mu_1, \mu_2, ..., \mu_{k-i}) \in T(k-i, k)\}.$$

In our case $k = 2$, we have:

$$\begin{gathered} F_i = \{0\}, \text{ for all } i < 0, \\ F_0 = C(\{0\}, A) \cong A, \\ F_1 = \{f \in C(X_1, A) : f(0,1) = \alpha_2(f(0,0)), f(1,0) = \alpha_1(f(0,0))\}, \\ F_2 = \mathcal{M}_\alpha(A), \text{ and} \\ F_i = F_2, \text{ for all } i > 2, \end{gathered}$$

where $X_1 = [0,1] \times \{0\} \cup \{0\} \times [0,1]$. For each i, we have a short exact sequence

$$0 \to I_i \overset{i_i}{\to} F_i \overset{\pi_i}{\to} F_{i-1} \to 0 \tag{4.1}$$

where $I_i := ker(\pi_i)$. Recall that we have isomorphisms:

$$
\begin{aligned}
I_0 &= F_0 \cong A, \\
I_1 &\cong (SA)^2, \\
I_2 &= S^2A, \text{ and} \\
I_i &= 0, \text{ for } i > 2.
\end{aligned}
$$

Proposition 4.12. *Under the above assumptions, we have that* $K_0(F_1) \cong H_2(\mathcal{G}(X,\sigma))$.

Proof. All the information of the first page of the associated spectral sequence is contained in the following periodic diagram, obtained by concatenating the six term exact sequences (see Theorem 2.73) of the above extensions (4.1):

$$
\begin{tikzcd}
 & & & & \cdots & & \\
 & & 0 & & K_1(F_0) \arrow[u] & & \\
\cdots \arrow[r] & K_0(I_0) \arrow[r, "K_0(i_0)"] & K_0(F_0) \arrow[u] \arrow[r, "\psi"] & K_1(I_1) \arrow[r, "K_1(i_1)"] & K_1(F_1) \arrow[u, "K_1(\pi_1)"'] \arrow[r, "\phi"] & K_0(I_2) \arrow[r] & \cdots \\
\cdots \arrow[u] \arrow[r] & K_0(I_1) \arrow[r, "K_0(i_1)"] & K_0(F_1) \arrow[u, "K_0(\pi_1)"'] \arrow[r] & K_1(I_2) \arrow[r] & K_1(F_2) \arrow[u, "K_1(\pi_2)"'] \arrow[r] & \cdots & \\
K_1(F_1) \arrow[u] \arrow[r, "\phi"] & K_0(I_2) \arrow[r, "K_0(i_2)"] & K_0(F_2) \arrow[u, "K_0(\pi_2)"'] \arrow[r] & 0 \arrow[r] & \cdots \arrow[u] & & \\
K_1(F_2) \arrow[u] & & \cdots \arrow[u] & & & &
\end{tikzcd}
$$

In order to outline the maps appearing in our reasoning, we denote by ϕ, ψ the respective index maps appearing in the diagram.

First, note that, since $\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2$ is AF (see Corollary 4.4), we have that $K_0(I_1) \cong K_0(SA)^2 \cong K_1(A)^2 = 0$. Therefore, we deduce that $K_0(\pi_1) : K_0(F_1) \to K_0(F_0)$ is injective, and hence $K_0(F_1) \cong ker(\psi)$. Moreover, $I_0 = F_0$, and hence $K_0(I_0) = K_0(F_0)$. Therefore, the differential map $K_0(I_0) \to K_1(I_1)$ defined as $\psi \circ K_0(i_0)$ equals to ψ. Thus, $K_0(\pi_1)$ induces an isomorphism:

$$K_0(F_1) \overset{K_0(\pi_1)}{\cong} ker(\psi) = ker(\psi \circ K_0(i_0)),$$

where $ker(\psi \circ K_0(i_0))$ equals to the term $E_2^{0,0}$, by definition (see Proposition 3.38). Moreover, by Theorem 3.49, we have an isomorphism

$$E_2^{0,0} \cong ker(id - K_0(\alpha_1)) \cap ker(id - K_0(\alpha_2)) \subseteq K_0(A),$$

which is isomorphic to $ker(id - \sigma_{1*}) \cap ker(id - \sigma_{2*}) \subseteq C_c(X, \mathbb{Z})$ (see Lemmas 4.5 and 4.7). Therefore:

$$K_0(F_1) \overset{K_0(\pi_1)}{\cong} E_2^{0,0} \cong \bigcap_{i=1}^{2} ker(id - K_0(\alpha_i)) \cong \bigcap_{i=1}^{2} ker(id - \sigma_{i*}) = H_2(\mathcal{G}(X, \sigma)),$$

as desired. $\square$

Remark 4.13. *This result can be generalized for any Deaconu-Renault groupoid. Indeed, building the suitable cofiltration, and using the above strategy, we can deduce that the term $K_0(F_1)$ is always isomorphic to the higher homology group of $\mathcal{G}(X, \sigma)$. In the case of a Deaconu-Renault groupoid of rank n, we have that $K_0(F_1) \cong H_n(\mathcal{G}(X, \sigma))$.*

$\square$

We can now provide an alternative picture of the HK isomorphism

$$H_0(\mathcal{G}(X, \sigma)) \oplus H_2(\mathcal{G}(X, \sigma)) \cong K_0(C_r^*(\mathcal{G}(X, \sigma))).$$

Proposition 4.14. *Under the above assumptions, we have that $H_0(\mathcal{G}(X, \sigma)) \cong K_0(I_2)/Im(\phi)$, and $\mathcal{G}(X, \sigma)$ satisfies the HK conjecture for K_0. Moreover, the HK isomorphism for K_0 can be chosen such that $H_0(\mathcal{G}(X, \sigma))$ embeds canonically into $K_0(C_r^*(\mathcal{G}(X, \sigma)))$.*

Proof. Note that, since we can identify $F_0 = A$, we have that $K_1(F_0) = 0$, and hence $K_1(i_1)$ is surjective. In particular, $Im(\phi \circ K_1(i_1)) = Im(\phi)$, and then $E_2^{2,0} = K_0(I_2)/Im(\phi \circ K_1(i_1)) = K_0(I_2)/Im(\phi)$. This, together with Theorem 3.49 and Lemmas 4.5, and 4.7, shows that

$$K_0(I_2)/Im(\phi) = E_2^{2,0} \cong K_0(A)/\sum Im(id - K_0(\alpha_i)) \cong C_c(X, \mathbb{Z})/\sum Im(id - \sigma_{i*}) \cong H_0(\mathcal{G}(X, \sigma)).$$

Moreover, since $K_1(I_2) = 0$ and $K_0(F_1) \cong H_2(\mathcal{G}(X, \mathbb{Z}))$ is free, we have a split exact sequence:

$$0 \to K_0(I_2)/Im(\phi) \overset{K_0(i_2)}{\to} K_0(F_2) \overset{K_0(\pi_2)}{\to} K_0(F_1) \to 0$$

By using the isomorphism $K_0(I_2)/Im(\phi) \cong H_0(\mathcal{G}(X,\sigma))$, and Proposition 4.12, the above extension of $K_0(F_2)$ induces a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_0(I_2)/Im(\phi) \arrow[r, "K_0(i_2)"] \arrow[d, "\cong"] & K_0(F_2) \arrow[r, "K_0(\pi_2)"] \arrow[d, "\cong"] & K_0(F_1) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & H_0(\mathcal{G}(X,\sigma)) \arrow[r] & K_0(C_r^*(\mathcal{G}(X,\sigma))) \arrow[r] & H_2(\mathcal{G}(X,\sigma)) \arrow[r] & 0
\end{tikzcd}$$

where the vertical isomorphism between $K_0(F_2)$ and $K_0(C_r^*(\mathcal{G}(X,\sigma)))$ is given by Lemma 3.20 and Theorem 3.46 . Observe that we can apply Remark 3.51 (which is a consequence of Theorem 3.50) to the left square, and conclude that the map between $H_0(\mathcal{G}(X,\sigma))$ and $K_0(C_r^*(\mathcal{G}(X,\sigma)))$ is the canonical one given by $[1_U]_{H_0} \mapsto [1_U]_{K_0}$. This provides an alternative description for the HK isomorphism for the K_0 component. □

Finally, we provide a picture of the isomorphism for the K_1 component.

Proposition 4.15. *Under the above assumptions, we have that $H_1(\mathcal{G}(X,\sigma)) \cong ker(\phi)$, and $\mathcal{G}(X,\sigma)$ satisfies the HK conjecture.*

Proof. Reasoning as above, we have that $K_1(i_1)$ is surjective, and so it induces an isomorphism

$$ker(\phi) \cong ker(\phi \circ K_1(i_1))/ker(K_1(i_1)) = ker(\phi \circ K_1(i_1))/Im(\psi).$$

Also, $K_1(I_2) = 0$, and therefore $K_1(F_2) \overset{K_1(\pi_2)}{\cong} Im(K_1(\pi_2)) = ker(\phi)$. By Theorem 3.46 and exactness of the diagram, we have

$$K_1(C_r^*(\mathcal{G}(X,\sigma))) \cong K_1(F_2) \cong ker(\phi) \cong ker(\phi \circ K_1(i_1))/Im(\psi), \tag{4.2}$$

which equals to $E_2^{1,0}$ by definition (see Proposition 3.38). Moreover, by using Theorem 3.49 and Lemma 4.5, we have that

$$E_2^{1,0} \cong H_1(\mathcal{G}(X,\sigma)). \tag{4.3}$$

Combining equations (4.2) and (4.3), we obtain a picture of the HK isomorphism for the K_1 component, as desired.

□

4.4 Rank 3 and higher Deaconu-Renault groupoids: an open problem

In order to study the K-theory of the C^*-algebra associated to a Deaconu-Renault groupoid of rank k, we use two approaches previously introduced. The first one involves the use of the spectral sequence described in Section 3.3.4. The second one consists in iterating the Pimsner-Voiculescu exact sequences (see Theorem 3.49) k-times, and study the extensions arising. This section will be structured as follows:

- First, we will give a sufficient condition for the verification of HK-conjecture for a Deaconu-Renault groupoid of rank 3, as it was suggested in [21].
- In the second part, we use this in order to prove that rank 3 Deaconu-Renault groupoids arising from actions by homeomorphisms satisfy the rational HK-conjecture.
- Finally, we will provide an explicit picture of the group $K_1(A \rtimes \mathbb{Z})$, in order to set the first step in the study of whether rank-3 Deaconu-Renault groupoids satisfy HK-conjecture in the general setting.

4.4.1 A sufficient condition for the verification of HK for Deaconu-Renault groupoids of rank 3

In this subsection, we answer positively in Theorem 4.18 the question appearing in [21, Remark 6.13], where the authors suggested that, for Deaconu-Renault groupoids of rank 3, the verification of the HK conjecture may follow from the injectivity of the natural map $\Phi : H_0(\mathcal{G}(X,\sigma)) \to K_0(C_r^*(\mathcal{G}(X,\sigma)))$ described in Definition 2.74.
To do so, we use Kasparov's cohomological spectral sequence (Theorem 3.23), as described in Subsection 3.3.4. In this case, the spectral sequence is of the form

$$E_2^{p,q} \cong H^p(\mathbb{Z}^3, K_q(A)) \Rightarrow K_{p+q}(\mathcal{M}_\alpha(A)) \cong K_{p+q-1}(A \rtimes \mathbb{Z}^3) \cong K_{p+q-1}(C_r^*(\mathcal{G}(X,\sigma))),$$

where $A := C_r^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^3)$. Notice that the last isomorphisms follow by Lemma 3.20 and Theorem 3.46.
Since $\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^3$ is an AF groupoid (see Corollary 4.4), and hence $K_1(A) = 0$, we can provide the specific picture of the above spectral sequence. Indeed, all the odd rows of the spectral sequence are trivial. Then, reasoning as in Example 3.33, we have $E_2 = E_3$, and the third page of the spectral sequence converging to the K-theory of $\mathcal{M}_\alpha(A)$ is of the form:

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
E_2^{0,0} \arrow[rrrdd, "d_3^{0,0}"] & E_2^{1,0} & E_2^{2,0} & E_2^{3,0} & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
E_2^{0,-2} & E_2^{1,-2} & E_2^{2,-2} & E_2^{3,-2} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & &
\end{tikzcd}
$$

The spectral sequence reaches its limit, at most, at the fourth page, since either domain or codomain of the following differential maps lie in a zero row. Its convergence in the fourth page induces two short exact sequences (see Example 3.33):

$$0 \to coker(d_3^{0,0}) \to K_1(\mathcal{M}_\alpha(A)) \to E_2^{1,0} \to 0$$

and

$$0 \to E_2^{2,-2} \to K_0(\mathcal{M}_\alpha(A)) \to ker(d_3^{0,0}) \to 0$$

If the map $d_3^{0,0}$ happens to be trivial, then $ker(d_3^{0,0}) = E_2^{0,0}$ and $coker(d_3^{0,0}) = E_2^{3,-2}$. Observe that we always have $E_2^{p,2q} = E_2^{p,0}$, by periodicity of K-theory.
As we noted in Theorem 3.49, the objects $E_2^{p,0}$ can be computed as the cohomology of the Pimsner-Voiculescu complex (C_{PV}, d_{PV}). More explicitly, we have the cochain complex

$$0 \to K_0(A) \overset{d_{PV}^{0,0}}{\to} K_0(A)^3 \overset{d_{PV}^{1,0}}{\to} K_0(A)^3 \overset{d_{PV}^{2,0}}{\to} K_0(A) \to 0,$$

where $d_{PV}^{p,0} : K_0(A) \otimes \bigwedge^p \mathbb{Z}^3 \to K_0(A) \otimes \bigwedge^{p+1} \mathbb{Z}^3$ is given by

$$x \otimes e \mapsto \sum_{i=1}^{3} (K_0(\alpha_i) - id)(x) \otimes (e \wedge e_i).$$

Lemma 4.16. *Let X be a second countable totally disconnected locally compact space, let σ be an action of $\mathbb{N}^k$ by surjective local homeomorphisms on X, and denote by $\mathcal{G}(X,\sigma)$ the associated Deaconu-Renault groupoid. Let $\tilde{B} := \varinjlim_{n \in \mathbb{Z}^k} (C_c(X,\mathbb{Z}), \sigma_*^n)$, and* *let $B = C_c(X, \mathbb{Z})$. Denote by $(\tilde{B}^\sigma_*, \tilde{\delta}_*)$ and (B^σ, δ_*) the respective chain complexes as in Lemma 4.6. Then we have isomorphisms*

$$\begin{tikzcd}
0 \arrow[r] & K_0(A)\otimes\bigwedge^0\mathbb{Z}^k \arrow[r, "d_{PV}^{0,0}"] \arrow[d, "\cong"', "\tau_0"] & K_0(A)\otimes\bigwedge^1\mathbb{Z}^k \arrow[r] \arrow[d, "\cong"', "\tau_1"] & \cdots \arrow[r] & K_0(A)\otimes\bigwedge^{k-1}\mathbb{Z}^k \arrow[r, "d_{PV}^{k-1,0}"] \arrow[d, "\cong"', "\tau_{k-1}"] & K_0(A)\otimes\bigwedge^k\mathbb{Z}^k \arrow[r] \arrow[d, "\cong"', "\tau_k"] & 0 \\
0 \arrow[r] & \tilde{B}\otimes\bigwedge^k\mathbb{Z}^k \arrow[r, "\tilde{\delta}_k"'] & \tilde{B}\otimes\bigwedge^{k-1}\mathbb{Z}^k \arrow[r] & \cdots \arrow[r] & \tilde{B}\otimes\bigwedge^1\mathbb{Z}^k \arrow[r, "\tilde{\delta}_1"'] & \tilde{B}\otimes\bigwedge^0\mathbb{Z}^k \arrow[r] & 0
\end{tikzcd}$$

Moreover, the isomorphisms intertwine the differential maps $d_{PV}^{p,0}$ and $\tilde{\delta}_{k-p}$, making the diagram commutative. Therefore:

$$H^p(C_{PV}, d_{PV}) \cong H_{k-p}(\tilde{B}^\sigma_*, \tilde{\delta}_*) \cong H_{k-p}(B^\sigma_*, \delta_*) \cong H_{k-p}(\mathcal{G}(X,\sigma)). \tag{4.4}$$

Proof. The skew product groupoid $\mathcal{G}(X,\sigma)\times_c\mathbb{Z}^k$ is AF (see corollary 4.4), and hence $K_0(A) \cong H_0(\mathcal{G}(X,\sigma)\times_c\mathbb{Z}^k) \cong \tilde{B}$, as shown in Lemma 4.5. Then, by symmetry of the exterior product, we have the isomorphisms $\tau_p : K_0(A)\otimes\bigwedge^p\mathbb{Z}^k \cong \tilde{B}\otimes\bigwedge^{k-p}\mathbb{Z}^k$. Denoting by θ the isomorphism between $K_0(A)$ and $\tilde{B}$, we can write τ_p as:

$$\tau_p(z\otimes e_\mu) = \theta(z)\otimes e_{\mu^\perp},$$

where $\mu \in T(p,k)$, and $\mu^\perp \in T(k-p,k)$ is the unique $(k-p)$-string disjoint to μ (see paragraph 3.3.4.4).
The intertwining of the differential maps $d_{PV}^{p,0}$ and $\tilde{\delta}_{k-p}$ follows from Lemma 4.5 and definition of those maps. Therefore, we obtain

$$H^p(C_{PV}, d_{PV}) \cong H_{k-p}(\tilde{B}^\sigma_*, \tilde{\delta}_*).$$

The isomorphisms

$$H_{k-p}(\tilde{B}^\sigma_*, \tilde{\delta}_*) \cong H_{k-p}(B^\sigma_*, \delta_*) \cong H_{k-p}(\mathcal{G}(X,\sigma))$$

were proven in Lemma 4.7 and Theorem 4.8, respectively.

□

With the second page of the spectral sequence already computed, our goal is to study the necessary conditions for the map $d_3^{0,0}$ to be trivial. More precisely, we will prove that it is trivial whenever the canonical map

$$\Phi : H_0(\mathcal{G}(X,\sigma)) \to K_0(C^*_r(\mathcal{G}(X,\sigma)))$$

of Definition 2.74 is injective. We will do so in Lemma 4.17 and Theorem 4.18.

To this end, we first build the cohomological spectral sequence as in Subsection 3.3.4.3: The associated finite cofiltration of the mapping torus $\mathcal{M}_\alpha(A)$ arising from the action of $\mathbb{Z}^3$ on A is given by:

$$\mathcal{M}_\alpha(A) = F_3 \xrightarrow{\pi_3} F_2 \xrightarrow{\pi_2} F_1 \xrightarrow{\pi_1} F_0 \xrightarrow{\pi_0} F_{-1} = \{0\}.$$

Each F_j is defined by restricting the domain to X_j, where:

$$X_j = \emptyset, \text{ for all } j < 0,$$
$$X_0 = \{(0,0,0)\},$$
$$X_1 = ([0,1] \times \{0\} \times \{0\}) \cup (\{0\} \times [0,1] \times \{0\}) \cup (\{0\} \times \{0\} \times [0,1]),$$
$$X_2 = ([0,1] \times [0,1] \times \{0\}) \cup ([0,1] \times \{0\} \times [0,1]) \cup (\{0\} \times [0,1] \times [0,1]), \text{ and}$$
$$X_j = [0,1]^3, \text{ for all } j \geq 3.$$

For each j, we have a short exact sequence

$$0 \to I_j \xrightarrow{i_j} F_j \xrightarrow{\pi_j} F_{j-1} \to 0 \tag{4.5}$$

where $I_j := ker(\pi_j)$. Recall that, as we noted in Subsection 3.3.4.4 we have isomorphisms:

$$I_0 = F_0 \cong A,$$
$$I_1 \cong (SA)^3,$$
$$I_2 = (S^2A)^3,$$
$$I_3 = S^3A, \text{ and}$$
$$I_j = 0, \text{ for } j > 3.$$

Reasoning as in Proposition 4.12, we can put together all the six term exact sequences associated to Equation 4.5, in order to obtain the following periodic diagram:

$$\begin{tikzcd}
 & & & & \cdots & & \\
 & & 0 & & K_1(F_0) \arrow[u] & & \\
\cdots \arrow[r] & K_0(I_0) \arrow[r, "K_0(i_0)"] & K_0(F_0) \arrow[u] \arrow[r, "\psi"] & K_1(I_1) \arrow[r, "K_1(i_1)"] & K_1(F_1) \arrow[u, "K_1(\pi_1)"'] \arrow[r, "\phi"] & K_0(I_2) \arrow[r] & \cdots \\
\cdots \arrow[u] \arrow[r] & K_0(I_1) \arrow[r, "K_0(i_1)"] & K_0(F_1) \arrow[u, "K_0(\pi_1)"'] \arrow[r] & K_1(I_2) \arrow[r, "K_1(i_2)"] & K_1(F_2) \arrow[u, "K_1(\pi_2)"'] \arrow[r] & K_0(I_3) & \\
K_1(F_1) \arrow[u] \arrow[r, "\phi"] & K_0(I_2) \arrow[r, "K_0(i_2)"] & K_0(F_2) \arrow[u, "K_0(\pi_2)"'] \arrow[r, "\xi"] & K_1(I_3) \arrow[r, "K_1(i_3)"] & K_1(F_3) \arrow[u, "K_1(\pi_3)"'] & & \\
K_1(F_2) \arrow[u] \arrow[r] & K_0(I_3) \arrow[r] & K_0(F_3) \arrow[u, "K_0(\pi_3)"'] & & & & \\
K_1(F_3) \arrow[u] & & \cdots \arrow[u] & & & &
\end{tikzcd}$$

Recall that, by definition, $E_1^{p,q} := K_{p+q}(I_p)$. Since $K_1(A) = 0$, straightforward computation shows that $K_{p+q}(I_p) = 0$ whenever q is odd. The differential maps $d_1^{*,*} : E_1^{*,*} \to E_1^{*,*}$ are defined as the composition

$$d_1^{p,q} : K_{p+q}(I_p) \to K_{p+q}(F_p) \to K_{p+q+1}(I_{p+1}).$$

The second page is defined as $E_2^{p,q} := ker(d_1^{p,q})/Im(d_1^{p-1,q})$, and its differentials $d_2^{p,q}$ are all zero, since either domain or codomain are trivial. Therefore we have that $E_2^{p,q} = E_3^{p,q}$, for all p, q. In the third page, there is only one differential map that may not be zero (up to periodicity), more precisely:

$$d_3^{0,0} : ker(\psi) \to K_1(I_3)/Im(\xi \circ K_0(i_2)),$$

where we have used that $I_0 = F_0$, making $K_0(i_0) : K_0(I_0) \to K_0(F_0)$ the identity map, and hence $ker(\psi \circ K_0(i_0)) = ker(\psi)$. If the map $d_3^{0,0}$ happens to also be zero, then $E_2^{p,q} = E_\infty^{p,q}$, and the result would follow.

Lemma 4.17. *Under the previous assumptions, $d_3^{0,0} = 0$ if and only if the canonical map $E_2^{3,0} = K_1(I_3)/Im(\xi \circ K_0(i_2)) \to K_1(F_3)$ induced by $K_1(i_3)$ is injective. If that happens, the spectral sequence reaches its limit at the second page, and therefore we have two short exact sequences*

$$0 \to E_2^{3,0} \to K_1(F_3) \to E_2^{1,0} \to 0$$
$$0 \to E_2^{2,0} \to K_0(F_3) \to E_2^{0,0} \to 0$$

Proof. It is clear that $d_3^{0,0} = 0$ implies the injection of $E_2^{3,0}$ into $K_1(F_3)$, by the convergence of the spectral sequence, so we only need to prove the other implication.

Looking at $K_1(F_3)$, we have the following exact sequence:

$$0 \to ker(K_1(\pi_3)) \to K_1(F_3) \to Im(K_1(\pi_3)) \to 0$$

Considering $ker(K_1(\pi_3)) = Im(K_1(i_3)) \subseteq K_1(F_3)$, we obtain a canonical isomorphism $Im(K_1(i_3)) \cong K_1(I_3)/Im(\xi)$ via the map $[v] \mapsto [v]$ for every v unitary of $M_n(I_3)$, and therefore $K_1(I_3)/Im(\xi)$ embeds canonically onto $K_1(F_3)$.

We claim that, if the canonical map $E_2^{3,0} = K_1(I_3)/Im(\xi \circ K_0(i_2)) \to K_1(F_3)$ is also injective, then $Im(\xi) = Im(\xi \circ K_0(i_2))$. This is straightforward, since the kernel of the map $K_1(I_3)/Im(\xi \circ K_0(i_2)) \to K_1(F_3)$ is precisely $Im(\xi)/Im(\xi \circ K_0(i_2))$. Therefore, if the kernel is trivial, we have that $Im(\xi) = Im(\xi \circ K_0(i_2))$ as intended.

In our sequence, the map $d_3^{0,0}$ is defined as follows (see Lemma 3.43). Take an element f_0 of $ker(\psi)$. Find any lift $f_2 \in K_0(F_2)$, and consider $\xi(f_2) \in K_1(I_3)$. Then $d_3^{0,0}(f_0) := [\xi(f_2)] \in K_1(I_3)/Im(\xi \circ K_0(i_2))$, which is independent of the choice of f_2. Since $Im(\xi) = Im(\xi \circ K_0(i_2))$, we have that $\xi(f_2) \in Im(\xi \circ K_0(i_2))$, and therefore $d_3^{0,0}(f_0) = 0 \in E_2^{3,0} = K_1(I_3)/Im(\xi \circ K_0(i_2))$, for any $f_0 \in ker(\psi)$.

Since $d_3^{0,0} = 0$, we deduce that the spectral sequence reaches its limit in the second page. Then, reasoning as in Example 3.33, and using the periodicity $E_2^{p,2q} = E_2^{p,0}$, we obtain the two exact sequences

$$0 \to E_2^{3,0} \to K_1(F_3) \to E_2^{1,0} \to 0$$
$$0 \to E_2^{2,0} \to K_0(F_3) \to E_2^{0,0} \to 0$$

□

Theorem 4.18. *Let X be a second countable locally compact totally disconnected space, and let $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ be an action of $\mathbb{N}^3$ on X by surjective local homeomorphisms. Let $\mathcal{G}(X, \sigma)$ be the associated Deaconu-Renault groupoid. Then, whenever the canonical map between $H_0(\mathcal{G}(X, \sigma))$ and $K_0(C_r^*(\mathcal{G}(X, \sigma)))$ is injective, $\mathcal{G}(X, \sigma)$ verifies Matui's weak HK conjecture for K_0, and the strong version of HK for K_1.*

Proof. As we noted in Equation 4.4, we have

$$E_2^{0,0} \cong H_3(\mathcal{G}(X, \sigma)),$$
$$E_2^{1,0} \cong H_2(\mathcal{G}(X, \sigma)),$$
$$E_2^{2,0} \cong H_1(\mathcal{G}(X, \sigma)),$$

$$E_2^{3,0} \cong H_0(\mathcal{G}(X,\sigma)).$$

Observe that, by periodicity of K-theory, we have $E_2^{p,q} = E_2^{p,0}$ for all even q.
Since $K_*(F_3) = K_*(\mathcal{M}_\alpha(A)) \cong K_{*-1}(A \rtimes \mathbb{Z}^3) \cong K_{*-1}(C_r^*(\mathcal{G}(X,\sigma)))$ (see Theorem 3.46 and Lemma 3.20), the short exact sequences of Lemma 4.17 are enough to prove that Matui's weak HK conjecture is satisfied for $\mathcal{G}(X,\sigma)$, whenever the map

$$K_1(I_3)/Im(\xi \circ K_0(i_2)) \to K_1(F_3)$$

is injective.
Finally observe that, by using Remark 3.51, we have a commutative diagram

$$\begin{tikzcd} K_1(I_3)/Im(\xi \circ K_0(i_2)) \arrow[r, "K_1(i_3)"] \arrow[d, "\cong"] & K_1(F_3) \arrow[d, "\Psi_A^{(3)}"] \\ K_0(A)/\sum(Im(id - K_0(\alpha_i))) \arrow[r, "K_0(j)"'] & K_0(A \rtimes \mathbb{Z}^3) \end{tikzcd}$$

As we observed in Lemma 4.11, the isomorphism $K_0(A \rtimes \mathbb{Z}^3) \cong K_0(C_r^*(\mathcal{G}(X,\sigma)))$ induced by Takai-Takesaki duality is natural, and hence the following diagram commutes

$$\begin{tikzcd} K_0(A)/\sum(Im(id - K_0(\alpha_i))) \arrow[r, "K_0(j)"] \arrow[d, "\cong"] & K_0(A \rtimes \mathbb{Z}^3) \arrow[d, "\cong"] \\ C_c(X,\mathbb{Z})/\sum(Im(id - \sigma_{i*})) \arrow[r, "\Phi"'] & K_0(C_r^*(\mathcal{G}(X,\sigma))) \end{tikzcd}$$

Therefore, we can translate the injectivity condition of Lemma 4.17, to the injectivity of the canonical map

$$\Phi : H_0(\mathcal{G}(X,\sigma)) \to K_0(C_r^*(\mathcal{G}(X,\sigma))).$$

Hence, the injectivity of Φ implies the existence of exact sequences

$$0 \to H_0(\mathcal{G}(X,\sigma)) \to K_0(C_r^*(\mathcal{G}(X,\sigma))) \to H_2(\mathcal{G}(X,\sigma)) \to 0$$
$$0 \to H_1(\mathcal{G}(X,\sigma)) \to K_1(C_r^*(\mathcal{G}(X,\sigma))) \to H_3(\mathcal{G}(X,\sigma)) \to 0$$

Therefore $\mathcal{G}(X,\sigma)$ satisfies the weak HK conjecture whenever Φ is injective.
Finally, since $C_c(X,\mathbb{Z})$ is free (see [21, Lemma 6.9]), we deduce that $H_3(\mathcal{G}(X,\sigma)) \subseteq$

$C_c(X, \mathbb{Z})$ is also free, and hence the short exact sequence for K_1 splits, obtaining the isomorphism

$$K_1(C_r^*(\mathcal{G}(X, \sigma))) \cong H_1(\mathcal{G}(X, \sigma)) \oplus H_3(\mathcal{G}(X, \sigma)),$$

that is, $\mathcal{G}(X, \sigma)$ satisfies the strong version of the HK conjecture for K_1.

$\square$

4.4.2 Actions by homeomorphisms $\mathbb{Z}^3 \curvearrowright X$ on a Cantor set

We now study the specific case of a Deaconu-Renault groupoid associated to an action by homeomorphisms, with compact unit space. So let $\sigma = (\sigma_1, \sigma_2, \sigma_3) : \mathbb{Z}^3 \curvearrowright X$ be an action of $\mathbb{Z}^3$ by homeomorphisms on a Cantor set X. We keep our standing convention $\sigma^n := \sigma_1^{n_1}\sigma_2^{n_2}\sigma_3^{n_3}$, for $n = (n_1, n_2, n_3) \in \mathbb{Z}^3$, and denote σ_* the induced action in $C(X)$. Observe that, by using Remark 2.1.2, we can identify the Deaconu-Renault groupoid associated to the action σ with the transformation groupoid $X \rtimes \mathbb{Z}^3$. The C^*-algebra of $X \rtimes \mathbb{Z}^3$ is isomorphic to $C(C) \rtimes_{\sigma_*} \mathbb{Z}^3$ (see, for example, [58, Example 3.3.8]).

Throughout this section, we write $A = C(X)$ and, as usual, we write the crossed product $A \rtimes_{\sigma_*} \mathbb{Z}^3$ simply as $A \rtimes \mathbb{Z}^3$.

We begin by describing the K-groups appearing in the first iteration of the Pimsner-Voiculescu sequence. To do so, we use a certain family of elements $u_f \in A \rtimes \mathbb{Z}$, under the following construction:

Consider $f \in C(X, \mathbb{Z})$ such that $f = \sigma_{1*}(f)$. Since σ_1 acts on X by homeomorphisms, we can write $f = \sum n_i \chi_{U_i}$, for $n_i \in \mathbb{Z}$ distinct, $X = \bigsqcup U_i$, and $\sigma_1(U_i) = U_i$, for all i. We define $u_f \in A \rtimes \mathbb{Z}$ as

$$u_f := \sum_n \chi_{f^{-1}(\{n\})} u_1^n,$$

where $u_1 \in A \rtimes \mathbb{Z}$ is the unitary of the crossed product associated to the action σ_{1*}.

Lemma 4.19. *Let X be the Cantor set. Let σ be an action of $\mathbb{Z}^3$ on X by homeomorphisms, and denote $A = C(X)$. Let $f \in C(X, \mathbb{Z})$ be of the form $f = \sum n_i \chi_{U_i}$, for $n_i \in \mathbb{Z}$ distinct, $X = \bigsqcup U_i$, and $\sigma_1(U_i) = U_i$, for all i. Then the element u_f defined as above is an unitary in $A \rtimes \mathbb{Z}$.*

Proof. The proof is standard:

$$\begin{aligned} u_f^* u_f &= (\sum_m u_1^{*m} \chi_{f^{-1}(\{m\})})(\sum_n \chi_{f^{-1}(\{n\})} u_1^n) = \sum_n u_1^{*n} \chi_{f^{-1}(\{n\})} u_1^n = \\ &= \sum_i u_1^{*n_i} \chi_{f^{-1}(\{n_i\})} u_1^{n_i} = \sum_i \sigma_{1*}(\chi_{U_i}) = \\ &= \sum_i \chi_{\sigma_1(U_i)} = \sum_i \chi_{U_i} = 1_X. \end{aligned}$$

The proof that $u_f u_f^* = 1_X$ is analogous, after checking that, for each $n \neq m$, one has that

$$\chi_{f^{-1}(\{n\})} u_1^n u_1^{*m} \chi_{f^{-1}(\{m\})} = 0.$$

Indeed, suppose that $n > m$, and then we have

$$\begin{aligned} \chi_{f^{-1}(\{n\})} u_1^n u_1^{*m} \chi_{f^{-1}(\{m\})} &= \chi_{f^{-1}(\{n\})} u_1^{n-m} \chi_{f^{-1}(\{m\})} = \\ &= u_1^{n-m} u_1^{*n-m} \chi_{f^{-1}(\{n\})} u_1^{n-m} \chi_{f^{-1}(\{m\})} = \\ &= u_1^{n-m} \sigma_{1*}^{n-m}(\chi_{f^{-1}(\{n\})}) \chi_{f^{-1}(\{m\})} = \\ &= u_1^{n-m} \chi_{f^{-1}(\{n\})} \chi_{f^{-1}(\{m\})} = 0. \end{aligned}$$

The same result is obtained when considering $n < m$. □

We can use this in order to describe $K_1(A \rtimes \mathbb{Z})$.

Lemma 4.20. *Let X be the Cantor set. Let σ be an action of $\mathbb{Z}^3$ on X by homeomorphisms, and denote $A = C(X)$.*
Then, $K_0(A \rtimes \mathbb{Z}) \cong coker(id - K_0(\sigma_1)) \cong coker(id - \sigma_{1*} : C(X,\mathbb{Z}) \to C(X,\mathbb{Z}))$, and the index map induces the isomorphism:

$$K_1(A \rtimes \mathbb{Z}) \cong \{[u_f]_1 := [\textstyle\sum_n \chi_{f^{-1}(\{n\})} u_1^n]_1 : f \in C(X,\mathbb{Z}) \text{ } is \text{ } \sigma_{1_*} - invariant\},$$

where u_1 denotes the unitary of the crossed product associated to the action σ_{1*}.

Proof. Recall that $K_0(A) \cong C(X,\mathbb{Z})$. The first Pimsner-Voiculescu exact sequence (see Theorem 3.22) reads as follows:

$$\begin{tikzcd} K_0(A) \arrow[r, "id - K_0(\sigma_1)"] & K_0(A) \arrow[r] & K_0(A \rtimes \mathbb{Z}) \arrow[d] \\ K_1(A \rtimes \mathbb{Z}) \arrow[u, "\varphi"] & K_1(A) \arrow[l] & K_1(A) \arrow[l, "id - K_1(\sigma_1)"'] \end{tikzcd}$$

where φ denotes the respective index map.
The terms $K_1(A)$ are zero and, therefore, exactness of the diagram gives us the first isomorphism, and also $K_1(A \rtimes \mathbb{Z}) \cong ker(id - K_0(\sigma_1) : K_0(A) \to K_0(A))$.
Note that the latter map can be identified with $id - \sigma_{1*} \colon C(X, \mathbb{Z}) \to C(X, \mathbb{Z})$, where

$$(id - \sigma_{1*})(f) = f - f \circ \sigma_1^{-1}.$$

Therefore we can identify $ker(id - K_0(\sigma_1))$ with the set of functions $f \in C(X, \mathbb{Z})$ such that $f = f \circ \sigma_1$. If we write $f = \sum n_i \chi_{U_i}$ for $n_i \in \mathbb{Z}$ distinct and $X = \bigsqcup U_i$ mutually disjoint clopen subsets of X, then we have that $f \in ker(id - \sigma_{1_*})$ if and only if $\sigma_1(U_i) = U_i$ for all i.
Let's consider $[\chi_U]_0$ for some U clopen σ_1-invariant, generator of $ker(id - K_0(\sigma_1)) \subseteq K_0(A)$. This element identifies with $[\chi_U \otimes e_{00}]_0$ under the isomorphism $K_0(A) \cong K_0(A \otimes \mathbb{K})$. We claim that its preimage under the index map is $[u_{\chi_U}]_1 := [\chi_U u_1 + \chi_{U^c}]_1$. Let's see it:
Let $f = \chi_U$, and $v_f = (\chi_U \otimes 1)(u \otimes S^*) + (\chi_{U^c} \otimes 1) \in \mathcal{T}_{A,\sigma_{1_*}}$ be a preimage of u_f under π, where $\mathcal{T}_{A,\sigma_{1_*}}$ is the Toeplitz algebra as defined in Theorem 3.22 (for further details, see [15, Proposition 5.8]), that is, the subalgebra of $(A \rtimes \mathbb{Z}) \otimes C^*(S)$ generated by $A \otimes I$ and $u \otimes S^*$.
Straightforward computation shows that $v_f v_f^* = 1 \otimes 1$, and $v_f^* v_f = 1 \otimes 1 - \chi_U \otimes e_{00}$, which means that v_f is a partial isometry in $\mathcal{T}_{A,\sigma_{1_*}}$, and so the index map is defined as ([53], 9.2.4):

$$\delta[u_f]_1 := [1 - v_f^* v_f]_0 - [1 - v_f v_f^*]_0 = [\chi_U \otimes e_{00}]_0 \cong [\chi_U]_0$$

To extend this result for every $f \in C(X, \mathbb{Z})$ σ_{1_*}-invariant, it is enough to check that $u_f u_g = u_{f+g}$. Indeed,

$$u_f u_g = \sum_n \sum_m \chi_{f^{-1}\{n\}} \chi_{g^{-1}\{m\}} u_1^{n+m} = \sum_k \sum_{n+m=k} \chi_{f^{-1}\{n\} \cap g^{-1}\{m\}} u_1^k = \sum_k \chi_{(f+g)^{-1}\{k\}} u_1^k = u_{f+g}.$$

Having this identity, and since φ is a homomorphism,

$$\varphi([u_{f+g}]_1) = \varphi([u_f u_g]_1) = \varphi([u_f]_1 + [u_g]_1) = [f]_0 + [g]_0,$$

thus concluding the proof. □

Remark 4.21. *Iterating for the second time we obtain the following Pimsner-Voiculescu six term exact sequence:*

$$\begin{tikzcd}
K_0(A \rtimes \mathbb{Z}) \arrow[r, "id - K_0(\sigma_2)"] & K_0(A \rtimes \mathbb{Z}) \arrow[r] & K_0(A \rtimes \mathbb{Z}^2) \arrow[d, "\delta"] \\
K_1(A \rtimes \mathbb{Z}^2) \arrow[u] & K_1(A \rtimes \mathbb{Z}) \arrow[l] & K_1(A \rtimes \mathbb{Z}) \arrow[l, "id - K_1(\sigma_2)"]
\end{tikzcd}$$

Now, we aim to prove that there exists some lifting γ of δ (and so $K_0(A \rtimes \mathbb{Z}^2) = ker(\delta) \oplus \gamma(Im(\delta))$), which verifies a key property. It is clear that δ has a splitting because

$$Im(\delta) \subseteq K_1(A \rtimes \mathbb{Z}) \cong ker(id - K_0(\sigma_1)) \cong ker(id - \sigma_{1*}) \subseteq C(X, \mathbb{Z})$$

is free, but we need an additional key property. We will find an explicit expression of the lifting γ. ☐

Lemma 4.22. *With the above notation, there is a homomorphism*

$$\gamma\colon Im(\delta) \to K_0(A \rtimes \mathbb{Z}^2)$$

such that $K_0(\sigma_3)\gamma\delta = \gamma\delta K_0(\sigma_3)$.

Proof. To show the result, we will study the trivial case of $\mathbb{Z}^3$ acting on the trivial algebra $\mathbb{C}$, and then we will expand the result for $A = C(X)$. First, we set some conventions:

- When working with the algebras arising from this action over the trivial algebra, we will add a subindex 0 to all the maps involved, for clarity matters.
- Recall that, if $A_0 = \mathbb{C}$, then $A_0 \rtimes \mathbb{Z}^3$ can be identified with $C(\mathbb{T}^3)$.
- Bott's periodicity gives, for every C^*-algebra B, two isomorphisms $K_0(B) \cong K_1(S(B))$ and $K_1(B) \cong K_0(S(B))$, where $S(B)$ denotes the suspension of B (see [53]). For convenience, both isomorphisms will be denoted by θ, since it will be clear which one we are applying in each case.

Lastly, recall that, by 4.20, $K_1(A\rtimes\mathbb{Z}) = \{[u_f]_1 := [\sum \chi_{f^{-1}(\{n\})} u_1^n]_1 \ : \ f \in C(X,\mathbb{Z}) \ is \ \sigma_{1_*}-invariant\}$, where u_1 is the unitary of the crossed product of $A \rtimes \mathbb{Z}$ associated to the action σ_{1*}. In particular, one may check that $Im(\delta) = ker(id - K_1(\sigma_2)) \subseteq K_1(A \rtimes \mathbb{Z})$ can be identified as $\{[u_f]_1 := [\sum \chi_{f^{-1}(\{n\})} u_1^n]_1 \ : \ f \in C(X,\mathbb{Z}) \ is \ \sigma_{1_*}, \sigma_{2_*} - invariant\}$.
Let u, v be the canonical unitaries generating $C(\mathbb{T}^2)$. There exists a unique $*$-homomorphism $\varphi\colon C(\mathbb{T}^2) \to A \rtimes \mathbb{Z}^2$ sending u to u_1 and v to u_2.
We will make use of the following map between short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & C(\mathbb{T}) \otimes \mathcal{K} \arrow[r] \arrow[d, "\varphi_1"] & \mathcal{T}_{C(\mathbb{T}),\mathrm{id}} \arrow[r] \arrow[d, "\psi"] & C(\mathbb{T}) \rtimes \mathbb{Z} \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & (A \rtimes \mathbb{Z}) \otimes \mathcal{K} \arrow[r] & \mathcal{T}_{A\rtimes\mathbb{Z},\sigma_2} \arrow[r] & A \rtimes \mathbb{Z}^2 \arrow[r] & 0,
\end{tikzcd} \tag{4.6}$$

where we identify $C(\mathbb{T}) \rtimes \mathbb{Z} = C(\mathbb{T}) \otimes C(\mathbb{T})$ with $C(\mathbb{T}^2)$.

Consider the diagram:

$$\begin{tikzcd}
K_0(C(\mathbb{T}^2)) \arrow[r, "\theta_0"] \arrow[d, two heads, "\delta_0"'] & K_1(S(C(\mathbb{T}^2))) \arrow[d, two heads, "\delta_0'"] \\
K_1(C(\mathbb{T})) \arrow[r, "\theta_0"'] & K_0(S(C(\mathbb{T})))
\end{tikzcd}$$

associated to the second iteration of the Pimsner-Voiculescu exact sequence of the trivial algebra $\mathbb{C}$. We will take a lifting for δ_0, and extend it in order to find liftings $[V_f]_1 \in K_1(S(A \rtimes \mathbb{Z}^2))$ for the elements $\theta([u_f]_1) \in K_0(S(A \rtimes \mathbb{Z}))$ in the general case

$$\delta' : K_1(S(A \rtimes \mathbb{Z}^2)) \to K_0(S(A \rtimes \mathbb{Z})).$$

Let us show it:

Let $[V]_1 \in K_1(S(C(\mathbb{T}^2)))$ be such that $\delta_0'([V]_1) = \theta_0([u]_1))$, where $V \in \mathcal{U}_r(S(\widetilde{C(\mathbb{T}^2)}))$, and we assume without loss of generality that $s(V) = I_r$, where s is the scalar map of Definition 2.66. Define

$$V_f := \sum_{n\in\mathbb{Z}} \chi_{f^{-1}(\{n\})} V_1^n \in \mathcal{U}_r(S(\widetilde{A \rtimes \mathbb{Z}^2})),$$

where $V_1 := \widetilde{S\varphi}(V)$, and

$$\widetilde{S\varphi} : M_r(S(\widetilde{C(\mathbb{T}^2)})) \to M_r(S(\widetilde{A \rtimes \mathbb{Z}^2}))$$

is the homomorphism on matrices induced by $\widetilde{S\varphi}$. Then the map $\gamma' : \theta([u_f]_1) \mapsto [V_f]_1$ is a lifting of $\delta' : K_1(S(A \rtimes \mathbb{Z}^2)) \to K_0(S(A \rtimes \mathbb{Z}))$.

First, we need to prove that, for each $f \in C(X, \mathbb{Z})$ such that f is $\sigma_{1_*}, \sigma_{2_*}$-invariant, the element $V_f := \sum_{n\in\mathbb{Z}} \chi_{f^{-1}(\{n\})} \widetilde{S\varphi}(V)$ is a unitary in $M_r(S(\widetilde{A \rtimes \mathbb{Z}^2}))$. Let us show it:

Recall that we can identify $M_r(\widetilde{S(A \rtimes \mathbb{Z}^2)})$ with the set of those elements $a \in C(\mathbb{T}, M_r(A \rtimes \mathbb{Z}^2))$ such that $a(1) \in M_r(\mathbb{C}1_A)$ (see [53, Chapter 11]). We think of V as an element in $C(\mathbb{T}, M_r(C(\mathbb{T}^2)))$ such that $V(1) = I_r \in M_r(\mathbb{C})$.
We have $\widetilde{S\varphi}(V)(z) \in M_r(C^*(u_1, u_2))$ for all $z \in \mathbb{T}$. Since f is σ_{1*}-invariant and σ_{2*}-invariant, it follows that $\widetilde{S\varphi}(V)(z)$ commutes with $\chi_{f^{-1}(\{n\})}1_r$ for all $z \in \mathbb{T}$ and consequently $V_f(z)$ is a unitary in $M_r(A \rtimes \mathbb{Z}^2)$ for all $z \in \mathbb{T}$. Moreover, since $\widetilde{S\varphi}(V)(1) = I_r$, it follows that $V_f(1) = I_r$, and we conclude that $V_f \in \mathcal{U}_r(\widetilde{S(A \rtimes \mathbb{Z}^2)})$.
We can now define the map $\lambda\colon Im(\delta) \to K_1(S(A \rtimes \mathbb{Z}^2))$ by

$$\lambda([u_f]_1) = [V_f]_1$$

for each $f \in C(X, \mathbb{Z})$ which is σ_{1*}- and σ_{2*}-invariant. Let us prove some properties for the map λ. First, we prove that it is a homomorphism. For this, it suffices to realize that $u_f u_g = u_{f+g}$, as we proved earlier. The same computation shows that $V_f V_g = V_{f+g}$ for f, g σ_{1*}- and σ_{2*}-invariant functions in $C(X, \mathbb{Z})$. This implies

$$\lambda([u_f]_1 + [u_g]_1) = \lambda([u_{f+g}]_1) = [V_{f+g}]_1 = [V_f V_g]_1 = [V_f]_1 + [V_g]_1 = \lambda([u_f]_1) + \lambda([u_g]_1),$$

showing that λ is a homomorphism.
Naturality of both the index map and Bott's isomorphism implies that the following square is commutative, that is, $\theta\delta = \delta'\theta$:

$$\begin{array}{ccc} K_0(A \rtimes \mathbb{Z}^2) & \xrightarrow{\theta} & K_1(S(A \rtimes \mathbb{Z}^2)) \\ \Big\downarrow \delta & & \Big\downarrow \delta' \\ Im(\delta) \subseteq K_1(A \rtimes \mathbb{Z}) & \xrightarrow[\theta]{} & K_0(S(A \rtimes \mathbb{Z})) \end{array}$$

We want to show that

$$\gamma := \theta^{-1} \circ \lambda : Im(\delta) \xrightarrow{\lambda} K_1(S(A \rtimes \mathbb{Z}^2)) \xrightarrow{\theta^{-1}} K_0(A \rtimes \mathbb{Z}^2)$$

is a section of δ, that is, $\delta = \delta \circ \gamma \circ \delta$. By the commutativity of the above square, it suffices to check that

$$\gamma' := \lambda \circ \theta^{-1} : \theta(Im(\delta)) \xrightarrow{\theta^{-1}} Im(\delta) \xrightarrow{\lambda} K_1(S(A \rtimes \mathbb{Z}^2))$$

is a section of δ', that is, $\delta' = \delta' \circ \gamma' \circ \delta'$, and for this it is enough to check that $\delta'\lambda = \theta$. We first check that we have $\delta'(\lambda([u_1]_1)) = \theta([u_1]_1)$. This follows from the naturality of the involved maps. Indeed, we have

$$\begin{aligned}\delta'(\lambda([u_1]_1)) &= \delta'([\widetilde{S\varphi}(V)]_1) = (\delta' \circ K_1(\widetilde{S\varphi}))([V]_1) = \\ &= (K_0(\widetilde{S\varphi_1}) \circ \delta'_0)([V]_1) = (K_0(\widetilde{S\varphi_1}) \circ \theta_0)([u]_1) = \\ &= (\theta \circ K_1(\varphi_1))([u]_1) = \theta([u_1]_1).\end{aligned}$$

To show that $\delta' \circ \lambda = \theta$, it suffices to check that $\delta'(\lambda([u_p]_1)) = \theta([u_p]_1)$, for any projection $p = \chi_U$, where U is a clopen subset of X such that $\sigma_1(U) = \sigma_2(U) = U$. The element $V_1 = S\varphi(V)$ will play an important role in our computations.

We now study Bott's isomorphism θ:
In order to simplify notation, we put $B := (A \rtimes \mathbb{Z}) \otimes \mathbb{K}$ and $\mathcal{T} := \mathcal{T}_{A\rtimes\mathbb{Z},\sigma_{2*}}$ and $j : j_{A\rtimes\mathbb{Z}}$. Then B is a closed ideal of the unital C^*-algebra $\mathcal{T}$ and we have $S(B) \subseteq S(\mathcal{T})$. We will look at elements in $M_k(\widetilde{S\mathcal{T}})$ as continuous functions $a\colon \mathbb{T} \to M_k(\mathcal{T})$ such that $a(1) \in M_k(\mathbb{C}1_{\mathcal{T}})$. Analogously, we will look at elements in $M_k(\widetilde{SB})$ as those elements $a \in M_k(\widetilde{S\mathcal{T}})$ such that $a(t) = a(1) + b(t)$, where $b(t) \in M_k(B)$ for all $t \in \mathbb{T}$.
With these definitions, note that the element $[u_p]_1 \in K_1(A \rtimes \mathbb{Z})$ is represented in $K_1(B) \cong K_1(A \otimes \mathbb{Z})$ by $[u_p \otimes e_{00} + (1 - e_{00})]_1$, where $e_{00} = 1 - SS^*$.
By definition of the Bott's isomorphism,

$$\theta([u_p \otimes e_{00} + (1 - e_{00})]_1) = \Big[w_p \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w_p^*\Big]_0 - \Big[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\Big]_0,$$

where

$$w_p = \begin{pmatrix} u_p \otimes e_{00} + (1 - e_{00}) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\frac{\pi}{2}t) & -\sin(\frac{\pi}{2}t) \\ \sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix} \begin{pmatrix} u_p^* \otimes e_{00} + (1 - e_{00}) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\frac{\pi}{2}t) & \sin(\frac{\pi}{2}t) \\ -\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix}$$

In our case, this can be simplified, since an easy computation shows that, if we denote by w_1 the element as above associated to $u_1 \otimes e_{00} + (1 - e_{00})$, then:

$$w_p = \begin{pmatrix} \chi_U \otimes e_{00} & 0 \\ 0 & \chi_U \otimes e_{00} \end{pmatrix} w_1 + \begin{pmatrix} 1 - \chi_U \otimes e_{00} & 0 \\ 0 & 1 - \chi_U \otimes e_{00} \end{pmatrix}$$

and therefore, since χ_U and χ_{U^c} are orthogonal central projections:

$$[w_p \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w_p^*]_0 - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]_0 = [\begin{pmatrix} \chi_U \otimes e_{00} & 0 \\ 0 & \chi_U \otimes e_{00} \end{pmatrix} w_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w_1^*]_0 - [\begin{pmatrix} \chi_U \otimes e_{00} & 0 \\ 0 & 0 \end{pmatrix}]_0.$$

On the other hand, we will apply the definition of the index as given in [53, Definition 9.1]. We first compute $\delta'([V_1]_1)$. There exists $\mu_1 \in \mathcal{U}_{2r}(\widetilde{S\mathcal{T}})$ such that $Sj(\mu_1) = \begin{pmatrix} V_1 & 0 \\ 0 & V_1^* \end{pmatrix}$. We then have $\delta'([V_1]_1) = [\mu_1 \begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix} \mu_1^*]_0 - [\begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix}]_0$. Indeed, it will be important to make a special choice of μ_1. First we take $\mu \in \mathcal{U}_{2r}(\widetilde{S\mathcal{T}_{C(\mathbb{T}),\mathrm{id}}})$ such that $Sj_{C(\mathbb{T})}(\mu) = \begin{pmatrix} V & 0 \\ 0 & V^* \end{pmatrix}$. Note that necessarily we have $\mu(1) = 1_{2r}$. Now we take $\mu_1 := \widetilde{S\psi}(\mu)$. It is clear that μ_1 satisfies the desired conditions. Observe that $\mu_1(1) = 1_{2r}$.
Since we already know that $\delta'([V_1]) = \delta'(\lambda([u_1]_1)) = \theta([u_1]_1)$, and

$$\theta([u_1]_1) = \theta([u_1]_1) = \Big[w_1 \begin{pmatrix} 1 \otimes e_{00} & 0 \\ 0 & 0 \end{pmatrix} w_1^*\Big]_0 - \Big[\begin{pmatrix} 1 \otimes e_{00} & 0 \\ 0 & 0 \end{pmatrix}\Big]_0,$$

we conclude that

$$\Big[w_1 \begin{pmatrix} 1 \otimes e_{00} & 0 \\ 0 & 0 \end{pmatrix} w_1^*\Big]_0 - \Big[\begin{pmatrix} 1 \otimes e_{00} & 0 \\ 0 & 0 \end{pmatrix}\Big]_0 = \Big[\mu_1 \begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix} \mu_1^*\Big]_0 - \Big[\begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix}\Big]_0. \tag{4.7}$$

in $K_0(\widetilde{SB})$.
For any projection $p = \chi_U$, where $\sigma_1(U) = \sigma_2(U) = U$, we have $V_p = \chi_U V_1 + \chi_{U^c} 1_r$. We need to find an element in $\mathcal{U}_{2r}(\widetilde{S\mathcal{T}_{A\rtimes\mathbb{Z},\sigma_{2*}}})$ such that its image is equal to $\begin{pmatrix} V_p & 0 \\ 0 & V_p^* \end{pmatrix}$. Since U is σ_1- and σ_2-invariant, our choice of μ_1 enables us to check that $\mu_p = (\chi_U \otimes 1)\mu_1 + (\chi_{U^c} \otimes 1)\begin{pmatrix} 1_r & 0 \\ 0 & 1_r \end{pmatrix}$ is a unitary matrix over the unitization of $S\mathcal{T}_{A\rtimes\mathbb{Z},\sigma_{2*}}$ satisfying the required conditions. Indeed, $\mu_1 = \widetilde{S\psi}(\mu)$, and so, for $t \in \mathbb{T}$, all the entries of $\mu_1(t)$ belong to the C^*-subalgebra of $\mathcal{T}$ generated by $u_1 \otimes 1$ and $u_2 \otimes S^*$ (where here S is the unilateral shift), so all these entries commute with $\chi_U \otimes 1$. Indeed, we have that $\chi_U \otimes 1$ is a central projection in $\mathcal{T}$. Using this, it is straightforward to check that $\mu_p \in \mathcal{U}_{2r}(\widetilde{S\mathcal{T}})$ and that $\mu_p(1) = 1_{2r}$.
Therefore we have

$$\delta'([V_p]_1) = [\mu_p \begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix} \mu_p^*]_0 - [\begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix}]_0 =$$
$$[(\chi_U \otimes 1)\mu_1 \begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix} \mu_1^* + (\chi_{U^c} \otimes 1)\begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix}]_0 - [\begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix}]_0,$$

Note that the projections $(\chi_U \otimes 1)\mu_1 \begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix} \mu_1^*$ and $(\chi_{U^c} \otimes 1)\begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix}$ belong to $C([0,1], M_{2r}(\mathcal{T}))$, but they do not belong to $M_{2r}(\widetilde{S\mathcal{T}})$ (except when $U = X$ or $U = \emptyset$). However their sum belongs to $M_{2r}(\widetilde{SB})$.
Now by (4.7) there exists $p \in \mathbb{N}$ and a matrix $X \in M_k(\widetilde{SB})$, where $k = 2r + 2 + p$, such that

$$XX^* = \begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix} \oplus w_1 \begin{pmatrix} \chi_U \otimes e_{00} & 0 \\ 0 & 0 \end{pmatrix} w_1^* \oplus 1_p$$

and

$$X^*X = \mu_1 \begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix} \mu_1^* \oplus \begin{pmatrix} \chi_U \otimes e_{00} & 0 \\ 0 & 0 \end{pmatrix} \oplus 1_p.$$

Write $sX(t) = X(1) = C \in M_k(\mathbb{C}1_{\mathcal{T}})$. Then we have $C^*C = CC^* = 1_r \oplus 0 \oplus 1_p$. Hence we can complete C to a unitary matrix $\widetilde{C}$ so that $(1_r \oplus 0 \oplus 1_p)\widetilde{C} = C = \widetilde{C}(1_r \oplus 0 \oplus 1_p)$ and $(0_r \oplus 1_{r+2} \oplus 0_p)\widetilde{C} = 0_r \oplus 1_{r+2} \oplus 0_p = \widetilde{C}(0_r \oplus 1_{r+2} \oplus 0_p)$.
Set $Y = \widetilde{C}^*X$. Then $Y \in M_k(\widetilde{SB})$ and $s(Y(t)) = Y(1) = \widetilde{C}^*(1_r \oplus 0 \oplus 1_p)C = C^*C = 1_r \oplus 0 \oplus 1_p$. Moreover we have $YY^* = XX^*$ and $Y^*Y = X^*X$. Hence, by replacing X by Y, we may assume that $s(X(t)) = X(1) = 1_r \oplus 0 \oplus 1_p$.
Now let $p = \chi_U$ as before, and define

$$X_p = (\chi_U \otimes 1)X + (\chi_{U^c} \otimes 1)(1_r \oplus 0 \oplus 1_p).$$

Then we have $sX_p(t) = X_p(1) = (\chi_U \otimes 1)X(1) + (\chi_{U^c} \otimes 1)(1_r \oplus 0 \oplus 1_p) = 1_r \oplus 0 \oplus 1_p$. Using this, it is easy to verify that $X_p \in M_k(\widetilde{SB})$. Moreover, we have

$$X_pX_p^* = \begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix} \oplus w_1 \begin{pmatrix} \chi_U \otimes e_{00} & 0 \\ 0 & 0 \end{pmatrix} w_1^* \oplus 1_p$$

and

$$X_p^*X_p = \left((\chi_U \otimes 1)\mu_1 \begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix} \mu_1^* + (\chi_{U_c} \otimes 1) \begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix}\right) \oplus \begin{pmatrix} \chi_U \otimes e_{00} & 0 \\ 0 & 0 \end{pmatrix} \oplus 1_p.$$

This shows that

$$\begin{aligned} \theta([u_p]_1) &= [w_1 \begin{pmatrix} \chi_U \otimes e_{00} & 0 \\ 0 & 0 \end{pmatrix} w_1^*] - [\begin{pmatrix} \chi_U \otimes e_{00} & 0 \\ 0 & 0 \end{pmatrix}]_0 = \\ &= [(\chi_U \otimes 1)\mu_1 \begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix} \mu_1^* + (\chi_{U^c} \otimes 1) \begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix}]_0 - [\begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix}]_0 = \\ &= \delta'(\lambda([u_p]_1)), \end{aligned}$$

as desired.

Finally, we check that $K_0(\sigma_3)(\gamma\delta) = (\gamma\delta)K_0(\sigma_3)$. Observe that we have:

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, "\sigma_3"] & \mathcal{T} \arrow[r] \arrow[d, "\sigma_3"] & A \rtimes \mathbb{Z}^2 \arrow[r] \arrow[d, "\sigma_3"] & 0 \\ 0 \arrow[r] & B \arrow[r] & \mathcal{T} \arrow[r] & A \rtimes \mathbb{Z}^2 \arrow[r] & 0, \end{tikzcd} \tag{4.8}$$

where the squares are commutative, and we simply denote by σ_3 all the maps induced by σ_3. By naturallity of the connecting map we get that $\delta K_0(\sigma_3) = K_1(\sigma_3)\delta$.
We now show that $\lambda K_1(\sigma_3) = K_1(S\sigma_3)\lambda$. Let f be a σ_{1*} and σ_{2*}-invariant function in $C(X, \mathbb{Z})$. Then we have

$$\lambda(K_1(\sigma_3)([u_f]_1)) = \lambda([u_{f\circ\sigma_3^{-1}}]_1) = [V_{f\circ\sigma_3^{-1}}]_1 = K_1(S\sigma_3)([V_f]_1) = K_1(S\sigma_3)(\lambda([u_f]_1)).$$

This shows our claim. Now using the naturallity of the Bott map we get that $\theta K_0(\sigma_3) = K_1(S\sigma_3)\theta$. Hence

$$\gamma K_1(\sigma_3) = \theta^{-1}\lambda K_1(\sigma_3) = \theta^{-1}K_1(S\sigma_3)\lambda = K_0(\sigma_3)\theta^{-1}\lambda = K_0(\sigma_3)\gamma.$$

Combining this with the equality $\delta K_0(\sigma_3) = K_1(\sigma_3)\delta$ we get that $K_0(\sigma_3)(\gamma\delta) = (\gamma\delta)K_0(\sigma_3)$, as desired. $\square$

We are now going to use the fact that δ splits. The third iteration of the Pimsner-Voiculescu exact sequences is of the form:

$$\begin{tikzcd} K_0(A \rtimes \mathbb{Z}^2) \arrow[r, "id - K_0(\sigma_3)"] & K_0(A \rtimes \mathbb{Z}^2) \arrow[r] & K_0(A \rtimes \mathbb{Z}^3) \arrow[d] \\ K_1(A \rtimes \mathbb{Z}^3) \arrow[u] & K_1(A \rtimes \mathbb{Z}^2) \arrow[l] & K_1(A \rtimes \mathbb{Z}^2) \arrow[l, "id - K_1(\sigma_3)"] \end{tikzcd}$$

If we study the upper row, we obtain the following exact sequence:

$$K_0(A \rtimes \mathbb{Z}^2) \stackrel{id - K_0(\sigma_3)}{\longrightarrow} K_0(A \rtimes \mathbb{Z}^2) \to G_2 \to 0$$

where $G_2 \subseteq K_0(A \rtimes \mathbb{Z}^3)$ denotes the image of the respective map. Our goal now is to split this exact sequence according to δ. To do that, we first need to prove that $id - K_0(\sigma_3)$ is well defined in both parts of this splitting.

Lemma 4.23. *With the above notation , we have that for $\delta : K_0(A \rtimes \mathbb{Z}^2) \to K_1(A \rtimes \mathbb{Z})$ as in Remark 4.21, the map $(id - K_0(\sigma_3))\colon \gamma(Im\delta) \to \gamma(Im\delta)$ is well defined.*

Proof. Immediate from Lemma 4.22. $\square$

Now, we need to find an analogous statement for the kernel of δ, in order to prove that the map $id - K_0(\sigma_3)$ respects the splitting.

Lemma 4.24. *With the above notation, with $\delta : K_0(A \rtimes \mathbb{Z}^2) \to K_1(A \rtimes \mathbb{Z})$ as in Remark 4.21, the map $(id - K_0(\sigma_3))\colon \ker\delta \to \ker\delta$ is well defined.*

Proof. Recall that δ arise from the following diagram:

$$\begin{tikzcd} K_0(A \rtimes \mathbb{Z}) \arrow[r, "id - K_0(\sigma_2)"] & K_0(A \rtimes \mathbb{Z}) \arrow[r, "K_0(\iota)"] & K_0(A \rtimes \mathbb{Z}^2) \arrow[d, "\delta"] \\ K_1(A \rtimes \mathbb{Z}^2) \arrow[u] & K_1(A \rtimes \mathbb{Z}) \arrow[l] & K_1(A \rtimes \mathbb{Z}) \arrow[l, "id - K_1(\sigma_2)"] \end{tikzcd}$$

where $A = C(X)$. By exactness of the diagram, $ker(\delta) = Im(K_0(\iota))$. Note that $K_0(\iota)$ is induced by the natural inclusion $\iota\colon A \rtimes \mathbb{Z} \to A \rtimes \mathbb{Z}^2$. Since $\iota\sigma_{3*} = \sigma_{3*}\iota$, we get that $K_0(\iota)K_0(\sigma_3) = K_0(\sigma_3)K_0(\iota)$, and the result follows. $\square$

Corollary 4.25. *Given an action σ of $\mathbb{Z}^3$ by homeomorphisms on the Cantor set X, the following diagram is commutative:*

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
ker(\delta) \arrow[rr, "id - K_0(\sigma_3)"] \arrow[d] & & ker(\delta) \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d, "\iota"] & 0 \\
K_0(A \rtimes \mathbb{Z}^2) \arrow[rr, "id - K_0(\sigma_3)"] \arrow[d] & & K_0(A \rtimes \mathbb{Z}^2) \arrow[r] \arrow[d] & G_2 \arrow[r] \arrow[d] & 0 \\
\gamma(Im(\delta)) \arrow[rr, "id - K_0(\sigma_3)"] \arrow[d] & & \gamma(Im(\delta)) \arrow[r] \arrow[d] & G_1 \arrow[r] \arrow[d] & 0 \\
0 & & 0 & 0 &
\end{tikzcd} \tag{I}$$

where $A = C(X)$, G_0, G_1, G_2 are the groups that make the respective rows exact, and all the columns are split-exact.
Moreover there is a commutative square

$$\begin{tikzcd}
G_0 \arrow[r] \arrow[d, "\iota"] & H_0(\mathcal{G}(X,\sigma)) \arrow[d, "\Phi"] \\
G_2 \arrow[r] & K_0(A \rtimes \mathbb{Z}^3),
\end{tikzcd} \tag{4.9}$$

where $\Phi\colon H_0(\mathcal{G}(X,\sigma)) \to K_0(A \rtimes \mathbb{Z}^3)$ is the natural map from H_0 of the Deaconu-Renault groupoid $\mathcal{G}(X,\sigma)$ to $K_0(A \rtimes \mathbb{Z}^3) \cong K_0(C^*(\mathcal{G}(X,\sigma)))$, the top horizontal map is an isomorphism and the bottom horizontal map is injective. Therefore, the canonical map between $H_0(\mathcal{G}(X,\sigma))$ and $K_0(C^*(\mathcal{G}(X,\sigma)))$ is injective.

Proof. Split-exactness of the columns is immediate after the last two lemmas, and $G_2 \subseteq K_0(A \rtimes \mathbb{Z}^3)$ is clear, by construction.
To prove $G_0 \cong H_0(\mathcal{G}(X,\sigma))$, diagram chasing suffices to prove

$$G_0 \cong K_0(A) / \sum_{i=1}^{3} Im(id - K_0(\sigma_i)).$$

Indeed one may check that

$$ker(\delta) = Im(K_0(\iota)) = K_0(A)/\sum_{i=1}^{2} Im(id - K_0(\sigma_i)) \cong C(X,\mathbb{Z})/\sum_{i=1}^{2} Im(id - \sigma_{i_*}),$$

and so

$$G_0 \cong K_0(A)/\sum_{i=1}^{3} Im(id - K_0(\sigma_i)) \cong C(X,\mathbb{Z})/\sum_{i=1}^{3} Im(id - \sigma_{i_*})$$

naturally. By [21, Theorem 6.5], $H_0(\mathcal{G}(X,\sigma)) \cong C(X,\mathbb{Z})/\sum_{i=1}^{3} Im(id - \sigma_{i_*})$, and so we obtain a commutative diagram

$$\begin{tikzcd}
G_0 = C(X,\mathbb{Z})/\sum_{i=1}^{3} Im(id - \sigma_{i_*}) \arrow[r] \arrow[d, "\iota"] & H_0(\mathcal{G}(X,\sigma)) \arrow[d, "\Phi"] \\
G_2 \arrow[r] & K_0(A \rtimes \mathbb{Z}^3),
\end{tikzcd}$$

concluding the proof. □

Theorem 4.26. *Let σ be an action of $\mathbb{Z}^3$ by homeomorphisms on the Cantor set X. Then the associated Deaconu-Renault groupoid $\mathcal{G}(X,\sigma)$ satisfies the HK conjecture for K_1, and the weak HK conjecture for K_0.*

Proof. Immediate after Corollary 4.25 and Theorem 4.18.

□

4.4.3 A complete description of $K_1(C^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k) \rtimes \mathbb{Z})$

Finally, we extend our previous study to the general case. This section aims to provide the first step in the study of the HK conjecture for rank-3 Deaconu-Renault groupoids (and higher), in the general setting.

Again, we focus on our second approach, that is, by iterations of the Pimsner-Voiculescu exact sequences. In order to successfully investigate the iterations, it is important to have an explicit expression of the elements of some of the K-groups arising from those sequences. Contrary to the previous section, the results that follow are valid for a Deaconu-Renault groupoid of arbitrary rank.

In this document's main technical section, we provide a complete description of the elements of $K_1(C^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k) \rtimes \mathbb{Z})$, as well as their images under the index map.

Before starting, we recall some important notation. Given an action α of $\mathbb{Z}$ on a C^*-

algebra A, we can build the Pimsner-Voiculescu exact sequence ([51]):

$$\begin{tikzcd}
K_0(A) \arrow[r, "id - K_0(\alpha)"] & K_0(A) \arrow[r] & K_0(A \rtimes \mathbb{Z}) \arrow[d] \\
K_1(A \rtimes \mathbb{Z}) \arrow[u, "\delta"] & K_1(A) \arrow[l] & K_1(A) \arrow[l, "id - K_1(\alpha)"]
\end{tikzcd}$$

which is obtained by applying the six-term exact sequence of K-theory to the short exact sequence:

$$0 \to A \otimes \mathbb{K} \to \mathcal{T}_{A,\alpha} \xrightarrow{\pi} A \rtimes \mathbb{Z} \to 0$$

Here, $\mathcal{T}_{A,\alpha}$ denotes the generalized Toeplitz algebra ([15], 5.8), which can be identified as the subalgebra of $(A \rtimes \mathbb{Z}) \otimes C^*(S)$ generated by the elements of the form $A \otimes 1$, $u \otimes S^*$.
Now, let X be a locally compact Hausdorff space, and let $\sigma := (\sigma_1, ..., \sigma_k)$ be an action of $\mathbb{N}^k$ on X by surjective local homeomorphisms. Denote by $\mathcal{G}(X,\sigma)$ the associated Deaconu-Renault groupoid of rank k, and let $\alpha := (\alpha_1, ..., \alpha_k)$ the induced action of $\mathbb{Z}^k$ on $\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k$. We simply write α when referring to the action on $C_r^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k)$ induced by α.
By identifying $C_r^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k) \rtimes_\alpha \mathbb{Z}^k \cong C_r^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k) \rtimes_{\alpha_{e_1}} \mathbb{Z} \rtimes_{\alpha_{e_2}} \mathbb{Z} \rtimes ... \rtimes_{\alpha_{e_k}} \mathbb{Z}$, we obtain k consecutive Pimsner-Voiculescu exact sequences, with the desired information contained in the k-th one. We warn the reader that, for clarity matters, we will drop the subindexes in the crossed product, since it will be clear which action is being used in each case.
Before advancing further, we provide a technical lemma:

Lemma 4.27. *Let X be a zero-dimensional locally compact Hausdorff space and let $\sigma\colon X \to X$ be a surjective local homeomorphism. Let U and V be open compact subsets of X such that $\sigma(V) = U$. Then there exist $n \geq 1$ and decompositions $V = \bigsqcup_{i=1}^n V_i$ into mutually disjoint open compact subsets V_i such that each V_i can be decomposed as $V_i = \bigsqcup_{j=1}^i V_{ij}$ for mutually disjoint open compact subsets V_{ij}, such that the following properties hold:*

1. *$\sigma|_{V_{ij}}$ is a homeomorphism from V_{ij} onto $\sigma(V_{ij})$ for all i, j,*
2. *For each $i = 2, \dots, n$ we have $\sigma(V_{i1}) = \sigma(V_{i2}) = \cdots = \sigma(V_{ii})$.*
3. *$U = \bigsqcup_{i=1}^n \sigma(V_{i1})$.*

In particular, there exists an open compact subset $W \subseteq V$ such that $\sigma|_W$ is injective and $\sigma(W) = U$.

Proof. Since σ is a local homeomorphism, X is zero-dimensional, and V is compact, there is a finite partition $V = \bigsqcup_{i=1}^n A_i$ of V such that $\sigma|_{A_i}$ is injective for all i. Setting $B_i := \sigma(A_i)$, we observe that $U = \sigma(V) = \bigcup_{i=1}^n B_i$. Now we can refine this decomposition to a disjoint decomposition into open compact subsets $U = \bigsqcup_{r=1}^n C_r$, where

$$C_n = B_1 \cap B_2 \cap \cdots \cap B_n,$$

and, for $1 \leq r \leq n-1$,

$$C_r = \bigsqcup_{1 \leq i_1 < i_2 \cdots < i_r \leq n} \Big[(B_{i_1} \cap B_{i_2} \cap \cdots \cap B_{i_r}) \setminus \Big[\bigcup_{j \notin \{i_1, \dots, i_r\}} (B_{i_1} \cap B_{i_2} \cap \cdots \cap B_{i_r} \cap B_j) \Big] \Big].$$

Now for $r = 1, \dots, n$ and $1 \leq j \leq r$, set

$$V_{rj} = \bigsqcup_{1 \leq i_1 < \cdots < i_r \leq n} (\sigma|_{A_{i_j}})^{-1} \Big((B_{i_1} \cap B_{i_2} \cap \cdots \cap B_{i_r}) \setminus \Big[\bigcup_{j \notin \{i_1, \dots, i_r\}} (B_{i_1} \cap B_{i_2} \cap \cdots \cap B_{i_r} \cap B_j). \Big] \Big).$$

Then the sets $V_r = \bigsqcup_{j=1}^r V_{rj}$ satisfy the required conditions.
Finally observe that $W := \bigsqcup_{r=1}^n V_{r1}$ is an open compact subset of X satisfying that $\sigma|_W$ is injective and $\sigma(W) = U$. $\square$

We now develop some basic properties of the projections and unitaries.

Lemma 4.28. *Let $A = C^*(\mathcal{G}(X, \sigma) \times_c \mathbb{Z}^k)$. For $p \in \mathbb{Z}^k$ and $n \in \mathbb{N}^k$ we have*

1. *Let U be an open compact subset of X, and suppose that $\sigma^n|_U$ is injective. Then we have*
$$[\chi_{U \times \{p\}}]_0 = [\chi_{\sigma^n(U) \times \{n+p\}}]_0$$
in $K_0(A)$.
2. *Let U be an open compact subset of X. Then we have*
$$[\chi_{U \times \{p\}}]_0 = [(\sigma^n)_*(\chi_U)_{n+p}]_0$$
in $K_0(A)$, where for $f \in C_c(X, \mathbb{Z})^+$, we denote by f_p the corresponding projection in matrices over $C_c(X \times \{p\})$,

Proof. (1) Suppose that $\sigma^n|_U$ is injective. Then we may consider the compact open bisection

$$Z = \{((x, n, \sigma^n(x)), p) : x \in U\}.$$

Then we have $\chi_Z \chi_Z^* = \chi_{U \times \{p\}}$ and $\chi_Z^* \chi_Z = \chi_{\sigma^n(U) \times \{n+p\}}$, and we get the result.
(2) Write $U = \bigsqcup_{i=1}^N U_i$, where $\sigma^n|_{U_i}$ is injective for all i. Then we have

$$[\chi_{U \times \{p\}}]_0 = \sum_{i=1}^N [\chi_{U_i \times \{p\}}]_0 = \sum_{i=1}^N [\chi_{\sigma^n(U_i) \times \{n+p\}}]_0 = \sum_{i=1}^N [(\sigma^n)_*(\chi_{U_i})_{n+p}]_0 = [(\sigma^n)_*(\chi_U)_{n+p}]_0.$$

$\square$

In particular, we obtain that all the projections are represented in degree 0, because given $n \in \mathbb{N}^k$, every open compact subset of X can be represented as a disjoint union of sets of the form $\sigma^n(U)$, where U is open compact and $\sigma^n|_U$ is injective. In this way we obtain an explicit realization of the isomorphism

$$K_0(A) \cong \varinjlim_{n\in\mathbb{N}^k} (C_c(X,\mathbb{Z}), (\sigma^n)_*).$$

Now suppose that we have a compact open subset U of X such that $\sigma_1(U) = U$ and $\sigma_1|_U$ is injective. Then we have

$$K_0(\alpha_1)([\chi_{U\times\{0\}}]_0) = [\chi_{U\times\{e_1\}}]_0 = [\chi_{\sigma_1(U)\times\{e_1\}}]_0 = [\chi_{U\times\{0\}}]_0$$

where we have used Lemma 4.28 in the last step.
Therefore we have identified an element in $ker(id - K_0(\alpha_1))$. For this element we can produce a unitary in $A \rtimes \mathbb{Z}$ which is the pre-image under the isomorphism

$$K_1(A \rtimes \mathbb{Z}) \xrightarrow{\delta} ker(id - K_0(\alpha_1))$$

Let U be a compact open subset of X such that $\sigma_1(U) = U$ and $\sigma_1|_U$ is injective. Set $f = \chi_{U\times\{0\}}$, and let $Z = Z(U)$ be the compact open bisection defined by

$$Z = \{((x, e_1, \sigma_1(x)), 0) : x \in U\}.$$

Then the element

$$u_f = \chi_Z u + \chi_{X\setminus U\times\{0\}}$$

is a partial unitary in $A \rtimes \mathbb{Z}$ with $u_f u_f^* = u_f^* u_f = \chi_{X\times\{0\}}$.

Lemma 4.29. *With the hypothesis above we have that $\delta([u_f]_1) = [\chi_U]_0$.*

Proof. Let $f = \chi_{U\times\{0\}}$, and $v_f = (\chi_Z \otimes 1)v^* \in \mathcal{T}_{A,\alpha_1}$ be a preimage of u_f under π. Recall that $v = u \otimes S^*$ is an isometry in $\mathcal{T}_{A,\alpha_1}$ such that

$$v^*(a \otimes 1)v = \alpha_1(a) \otimes 1,$$

and $\pi(v^*) = u$. Straightforward computations, using the representation in [15, Proposition 5.8], show that $v_f v_f^* = \chi_{U\times\{0\}} \otimes 1$, and $v_f^* v_f = \chi_{U\times\{0\}} \otimes SS^*$, which means that v_f is a partial isometry in $\mathcal{T}_{A,\alpha_1}$, and so the index map can be computed as ([53], 9.2.4):

$$\delta([u_f]_1) := [1 - v_f^* v_f]_0 - [1 - v_f v_f^*]_0 = [\chi_{U\times\{0\}} \otimes (1 - SS^*)]_0$$

which is the class of $\chi_{U\times\{0\}}$ in $K_0(A \otimes \mathbb{K})$. $\square$

Consider now nonzero integers a_i, with $i = 1, \ldots, n$ and let $\{U_i : 1 \leq i \leq n\}$ be pairwise disjoint compact open subsets of X such that $\sigma_1|_{U_i}$ are injective and $\sigma_1(U_i) = U_i$ for all i. Note that the above hypothesis imply that $\sigma_1^n|_{U_i}$ are injective and $\sigma_1^n(U_i) = U_i$ for all i and all $n \in \mathbb{N}$. Then the element $f = \sum_{i=1}^n a_i[\chi_{U_i \times \{0\}}]_0$ belongs to the kernel of $id - K_0(\alpha_1)$. For each i we consider the compact open bisection

$$Z_i = \{((x, |a_i|, \sigma_1^{|a_i|}(x)), 0) : x \in U_i\}.$$

Write $u_i := \chi_{Z_i} u^{|a_i|}$ for all i, and let

$$u_f := \sum_{i=1}^n \frac{a_i}{|a_i|} u_i.$$

Then u_f is a partial unitary in $A \rtimes \mathbb{Z}$ such that $u_f u_f^* = u_f^* u_f = \chi_{U \times \{0\}}$, where $U = \bigsqcup_{i=1}^n U_i$. As before, straightforward computation proves the following lemma:

Lemma 4.30. *With the hypothesis above, we have that $\delta([u_f]_1) = f$.* □

We now present more elements in the kernel of $id - K_0(\alpha_1)$. This is inspired by the theory of graph C^*-algebras.

Let $V_1, \ldots, V_n$ be pairwise disjoint non-empty compact open subsets of X. Suppose that for some non-negative integers a_{ji}, with $1 \leq i,\, j \leq n$, we have decompositions

$$V_i = \bigsqcup_{j=1}^n \bigsqcup_{k=1}^{a_{ji}} V_{ji}^{(k)},$$

where $V_{ji}^{(k)}$ are compact open subsets of X, $\sigma_1|_{V_{ji}^{(k)}}$ is injective for all $i.j, k$ and

$$\sigma_1(V_{ji}^{(k)}) = V_j, \qquad (1 \leq i, j \leq n, 1 \leq k \leq a_{ji}).$$

Consider the matrix $A = (a_{ji}) \in M_n(\mathbb{Z}^+)$. Note that, since $V_i \neq \emptyset$ for all i, we get that all columns of A are nonzero. We will also assume that all rows are nonzero.

Lemma 4.31. *With the above notation, let $x = (x_1, \ldots, x_n)^t$ be a column vector in $\mathbb{Z}^n$ such that $(A - I)x = 0$. Let $f = \sum_{i=1}^n x_i[\chi_{V_i \times \{0\}}]_0 \in K_0(A)$. Then $f \in ker(id - K_0(\alpha_1))$.*

Proof. We have

$$\begin{aligned}(id - K_0(\alpha_1))(\sum_{i=1}^n x_i[\chi_{V_i\times\{0\}}]_0) &= \sum_{i=1}^n x_i[\chi_{V_i\times\{0\}}]_0 - \sum_{i=1}^n x_i[\chi_{V_i\times\{e_1\}}]_0\\ &= \sum_{i=1}^n x_i\Big(\sum_{j=1}^n\sum_{k=1}^{a_{ji}}[\chi_{V_{ji}^{(k)}\times\{0\}}]_0\Big) - \sum_{i=1}^n x_i[\chi_{V_i\times\{e_1\}}]_0\\ &= \sum_{i=1}^n x_i\Big(\sum_{j=1}^n\sum_{k=1}^{a_{ji}}[\chi_{\sigma_1(V_{ji}^{(k)})\times\{e_1\}}]_0\Big) - \sum_{i=1}^n x_i[\chi_{V_i\times\{e_1\}}]_0\\ &= \sum_{j=1}^n \Big(\sum_{i=1}^n a_{ji}x_i\Big)[\chi_{V_j\times\{e_1\}}]_0 - \sum_{i=1}^n x_i[\chi_{V_i\times\{e_1\}}]_0 = 0.\end{aligned}$$

□

Note that this covers the case considered before, in which $A = I$.
We are going to build a suitable unitary element u_f, lifting for an f defined as in the statement of the lemma. This follows the construction for graph C^*-algebras. Indeed we are going to use the same notation as in [10]. For this, we consider a finite graph $E = (E^0, E^1, r, s)$ with $E^0 = \{1, \ldots, n\}$ and $E^1 = \{e_{ji}^{(k)} : 1 \leq k \leq a_{ji}, 1 \leq i, j \leq n\}$, where $s(e_{ji}^{(k)}) = i$ and $r(e_{ji}^{(k)}) = j$ for all allowed indices i, j, k.
Let $Z_{ji}^{(k)} = \{((y, e_1, \sigma_1(y)), 0) : y \in V_{ji}^k\}$. Then, as before, $Z_{ji}^{(k)}$ is a compact open bisection and the element $u_{ji}^{(k)} := \chi_{Z_{ji}^{(k)}} u$ is a partial isometry in $A \rtimes \mathbb{Z}$ such that

$$u_{ji}^{(k)}(u_{ji}^{(k)})^* = \chi_{V_{ji}^{(k)}\times\{0\}} \quad \text{and} \quad (u_{ji}^{(k)})^* u_{ji}^{(k)} = \chi_{V_j\times\{0\}}.$$

It is easy to show that the assignments $v \mapsto \chi_{V_v\times\{0\}}$ and $e_{ji}^{(k)} \mapsto u_{ji}^{(k)}$ define a $*$-homomorphism $\rho\colon C^*(E) \to A \rtimes \mathbb{Z}$ such that $\rho(1) = \chi_{V\times\{0\}}$, where $V = \bigsqcup_{v=1}^n V_v$.
Now observe that $x = (x_1, \ldots, x_n)^t$ defines an element in $K_1(C^*(E))$ as described in [10], and concretely we can define a unitary $U_x \in M_h(C^*(E))$ (see below) representing this element. We then define

$$u_f := \rho(U_x) \in M_h(A \rtimes \mathbb{Z}),$$

which is a partial unitary with $u_f u_f^* = u_f^* u_f = \chi_V \otimes 1_h$.
We now recall the definition of U_x. We keep the above notation. Define

$$\begin{aligned} L_x^+ &= \Big\{(e, i) \mid e \in E^1, 1 \leq i \leq -x_{s(e)}\Big\} \bigcup \Big\{(v, i) \mid v \in E^0, 1 \leq i \leq x_v\Big\}, \text{ and}\\ L_x^- &= \Big\{(e, i) \mid e \in E^1, 1 \leq i \leq x_{s(e)}\Big\} \bigcup \Big\{(v, i) \mid v \in E^0, 1 \leq i \leq -x_v\Big\}. \end{aligned} \tag{4.10}$$

By [10, Lemma 3.1] and [10, Lemma 3.2], L_x^+ and L_x^- have the same number of elements and there are bijections

$$[\cdot] : L_x^+ \longrightarrow \{1, \cdots, h\} \qquad \text{and} \qquad \langle\cdot\rangle : L_x^- \longrightarrow \{1, \cdots, h\}$$

with the property that $[x,i]=\langle y,j\rangle$ implies $r(x)=r(y)$. Here h is the common number of elements in L_x^+ and L_x^-.

With notation as above, define two matrices V_x and P_x by

$$V_x=\sum_{\substack{1\le i\le x_w\\ s(e)=w}} eE_{[w,i],\langle e,i\rangle}+\sum_{\substack{1\le i\le -x_w\\ s(e)=w}} e^*E_{[e,i],\langle w,i\rangle}$$

and

$$P_x=\sum_{1\le i\le x_w} wE_{[w,i],[w,i]}+\sum_{\substack{1\le i\le -x_w\\ s(e)=w,r(e)=v}} vE_{[e,i],[e,i]},$$

where $E_{\bullet,\bullet}$ denote the standard matrix units in the $h\times h$-matrix algebra $M_h(C^*(E))$ (see [10, Definition 3.3]).

In addition, we define

$$U_x=V_x+(1-P_x). \tag{4.11}$$

The matrix U_x is invertible and $U_x^{-1}=U_x^*=V_x^*+(1-P_x)$ (compare [10, Lemma 3.4] and [10, Fact 3.6]).

Now set $u_f:=\pi(U_x)\in M_h(A\times\mathbb{Z})$.

Lemma 4.32. *With the above notation, we have that $\delta([u_f]_1)=f$.*

Proof. As before, we consider the element $v=u^*\otimes S$ in $\mathcal{T}_{A,\alpha_1}$ and we set

$$v_{ji}^{(k)}:=(\chi_{Z_{ji}^{(k)}}\otimes 1)v^*=u_{ji}^{(k)}\otimes S^*.$$

Then replacing each occurrence of $u_{ji}^{(k)}$ in u_f by $v_{ji}^{(k)}$, we obtain an element $v_f\in\mathcal{T}_{A,\alpha_1}$.

We now compute

$$\begin{aligned}\delta([u_f]_1)&=[1-v_f^*v_f]_0-[1-v_fv_f^*]_0\\&=\Big(\sum_{w=1}^n(\sum_{v:x_v>0}a_{wv}x_v)[\chi_{V_w\times\{0\}}\otimes(1-SS^*)]_0\Big)-\Big(\sum_{w=1}^n(\sum_{v:x_v<0}a_{wv}(-x_v))[\chi_{V_w\times\{0\}}\otimes(1-SS^*)]_0\Big)\\&=\sum_{w=1}^n x_w[\chi_{V_w\times\{0\}}\otimes(1-SS^*)]_0=f.\end{aligned}$$

□

We now consider another type of element, this time connected with a certain separated graph. Suppose we have a set of vertices $\Gamma=\bigsqcup_{i=1}^{n+1}\Gamma_i$. We force $\Gamma_i\neq\emptyset$ for $i=1,\dots,n$, but allow Γ_{n+1} to be empty. Now suppose that we have non-empty compact open mutually disjoint open sets Z_v, $v \in \Gamma$. Set $V_i = \bigsqcup_{v\in\Gamma_i} Z_v$. Moreover suppose that for each $i \in \{1, ..., n\}$ there are decompositions

$$V_i = \bigsqcup_{v\in\Gamma} \bigsqcup_{k=1}^{a_{vi}} V_{vi}^{(k)},$$

where $V_{vi}^{(k)}$ are compact open sets such that $\sigma_1|_{V_{vi}^{(k)}}$ is injective and $\sigma_1|_{V_{vi}^{(k)}}(V_{vi}^{(k)}) = Z_v$ for all allowable values of v, i, k. Let $A = (a_{vi}) \in M_{\Gamma\times[1,n]}(\mathbb{Z}^+)$, where we denote $[1, n] = \{1, \dots, n\}$. Let also $I \in M_{\Gamma\times[1,n]}(\mathbb{Z}^+)$ be the matrix with all entries in $\{0, 1\}$ such that in the i^{th} column has 1's exactly at the position that belongs to Γ_i, for $i = 1, \dots, n$.

Lemma 4.33. *With the above notation, let $x = (x_1, \dots, x_n)^t \in \mathbb{Z}^n$ be a column vector such that $(A - I)x = 0$ and let $p \in \mathbb{N}^k$. Let $f = \sum_{i=1}^n x_i[\chi_{V_i\times\{p\}}]_0 \in K_0(A)$. Then $f \in ker(id - K_0(\alpha_1))$.*

Proof. We have

$$\begin{aligned}
(id - K_0(\alpha_1))(\sum_{i=1}^n x_i[\chi_{V_i\times\{p\}}]_0) &= \sum_{i=1}^n x_i[\chi_{V_i\times\{p\}}]_0 - \sum_{i=1}^n x_i[\chi_{V_i\times\{p+e_1\}}]_0 \\
&= \sum_{i=1}^n x_i\Big(\sum_{v\in\Gamma}\sum_{k=1}^{a_{vi}}[\chi_{V_{vi}^{(k)}\times\{p\}}]_0\Big) - \sum_{i=1}^n x_i[\chi_{V_i\times\{p+e_1\}}]_0 \\
&= \sum_{i=1}^n x_i\Big(\sum_{v\in\Gamma}\sum_{k=1}^{a_{vi}}[\chi_{\sigma_1(V_{vi}^{(k)})\times\{p+e_1\}}]_0\Big) - \sum_{j=1}^n x_j[\sum_{v\in\Gamma_j}\chi_{Z_v\times\{p+e_1\}}]_0 \\
&= \sum_{v\in\Gamma}\Big(\sum_{i=1}^n a_{vi}x_i\Big)[\chi_{Z_v\times\{p+e_1\}}]_0 - \sum_{j=1}^n\sum_{v\in\Gamma_j} x_j[\chi_{Z_v\times\{p+e_1\}}]_0 \\
&= \sum_{j=1}^n\sum_{v\in\Gamma_j}\Big(\sum_{i=1}^n a_{vi}x_i - x_j\Big)[\chi_{Z_v\times\{p+e_1\}}]_0 + \sum_{v\in\Gamma_{n+1}}\Big(\sum_{i=1}^n a_{vi}x_i\Big)[\chi_{Z_v\times\{p+e_1\}}]_0 = 0.
\end{aligned}$$

□

Our objective now is to show that the above covers indeed all the elements in $ker(id - K_0(\alpha_1))$.

Theorem 4.34. *Let $f \in ker(id - K_0(\alpha_1))$. Then there are non-empty mutually disjoint compact open sets Z_v, $v \in \Gamma$, where $\Gamma = \bigsqcup_{i=1}^{n+1}\Gamma_i$ is a finite set, $p \in \mathbb{N}^k$, such that with $V_i = \bigsqcup_{v\in\Gamma_i} Z_v$, we have for each $i \in \{1, ..., n\}$ a decomposition*

$$V_i = \bigsqcup_{v\in\Gamma} \bigsqcup_{k=1}^{a_{vi}} V_{vi}^{(k)},$$

where $V_{vi}^{(k)}$ are compact open sets such that $\sigma_1|_{V_{vi}^{(k)}}$ is injective and $\sigma_1|_{V_{vi}^{(k)}}(V_{vi}^{(k)}) = Z_v$ for all allowable values of v, i, k. Let $A = (a_{vi}) \in M_{\Gamma\times[1,n]}(\mathbb{Z}^+)$, and let $I \in M_{\Gamma\times[1,n]}(\mathbb{Z}^+)$ be *the matrix with all entries in* $\{0,1\}$ *such that in the* i^{th} *column has* 1*'s exactly at the positions that belong to* Γ_i*, for* $i = 1, \dots, n$*. There exists* $x = (x_1, \dots, x_n)^t \in ker(A - I)$ *such that* $f = \sum_{i=1}^n x_i[\chi_{V_i \times \{p\}}]_0$.

Proof. We can write $f = \sum_{i=1}^{n'} x_i'[\chi_{V_i' \times \{0\}}]_0$, where $x_i' \in \mathbb{Z}$ and $V_1', \dots, V_n'$ are non-empty compact open subsets of X.
Since $f \in ker(id - K_0(\alpha_1))$, we have

$$\sum_{i=1}^{n'} x_i'[(\sigma_1)_*(\chi_{V_i'})_{e_1}]_0 = \sum_{i=1}^{n'} x_i'[\chi_{V_i' \times \{e_1\}}]_0.$$

Since $K_0(A) \cong \varinjlim_{n \in \mathbb{N}^k}(C_c(X, \mathbb{Z}), (\sigma^n)_*)$, we get that there is some $p \in \mathbb{N}^k$ such that

$$\sum_{i=1}^{n'} x_i'(\sigma^{p+e_1})_*(\chi_{V_i'}) = \sum_{i=1}^{n'} x_i'(\sigma^p)_*(\chi_{V_i'}),$$

where now this is an identity in $C_c(X, \mathbb{Z})$, and

$$f = \sum_{i=1}^{n'} x_i'[(\sigma^{p+e_1})_*(\chi_{V_i'})_{p+e_1}]_0 = \sum_{i=1}^{n'} x_i'[(\sigma^p)_*(\chi_{V_i'})_p]_0.$$

Writing each $(\sigma^p)_*(\chi_{V_i'})$ as a $\mathbb{Z}$-linear combination of characteristic functions and rearranging, we can therefore write

$$f = \sum_{i=1}^{n} x_i[\chi_{V_i \times \{p\}}]_0,$$

where $x_i \in \mathbb{Z}$, $\{V_i : 1 \le i \le n\}$ are mutually orthogonal compact open subsets of X, and

$$\sum_{i=1}^{n} x_i(\sigma_1)_*(\chi_{V_i})_{p+e_1} = \sum_{i=1}^{n} x_i\chi_{V_i \times \{p+e_1\}}. \tag{4.12}$$

We will assume that $x_i > 0$ for $i = 1, \dots, r$ and $x_i < 0$ for $i = r+1, \dots, n$. Then we have

$$\sum_{i=1}^{r} x_i(\sigma_1)_*(\chi_{V_i})_{p+e_1} + \sum_{i=r+1}^{n}(-x_i)\chi_{V_i \times \{p+e_1\}} = \sum_{i=1}^{r} x_i\chi_{V_i \times \{p+e_1\}} + \sum_{i=r+1}^{n}(-x_i)(\sigma_1)_*(\chi_{V_i})_{p+e_1}. \tag{4.13}$$

Using Lemma 4.27 we get decompositions

$$V_i = \bigsqcup_{j=1}^{t_i} W_{i,j},$$

where $W_{i,j}$ are compact open sets, $\sigma_1|_{W_{i,j}}$ are injective for all i, j, and there is $t_i' \le t_i$ such that

$$\sigma_1(V_i) = \bigsqcup_{j=1}^{t_i'} \sigma_1(W_{i,j}),$$

and for each $l = t'_i + 1, \ldots, t_i$ we have $\sigma_1(W_{i,l}) = \sigma_1(W_{i,j})$ for some $j \in \{1, \ldots, t'_i\}$.
Looking at the supports of the functions in the equality (4.13), we obtain a set equality:

$$Y := \bigcup_{i=1}^{r} \Big(\bigsqcup_{j=1}^{t'_i} \sigma_1(W_{i,j}) \Big) \cup \Big(\bigsqcup_{i=r+1}^{n} \bigsqcup_{j=1}^{t_i} W_{i,j} \Big) = \Big(\bigsqcup_{i=1}^{r} \bigsqcup_{j=1}^{t_i} W_{i,j} \Big) \cup \Big(\bigcup_{i=r+1}^{n} \Big(\bigsqcup_{j=1}^{t'_i} \sigma_1(W_{i,j}) \Big) \Big). \tag{4.14}$$

Here we need to introduce some further notation. Set

$$B := \bigcup_{i=1}^{r} \Big(\bigsqcup_{j=1}^{t'_i} \sigma_1(W_{i,j}) \Big), \qquad C := \bigcup_{i=r+1}^{n} \Big(\bigsqcup_{j=1}^{t'_i} \sigma_1(W_{i,j}) \Big).$$

For $1 \leq i_1 < i_2 < \cdots < i_l \leq r$, and $1 \leq j_s \leq t'_{i_s}$, $s = 1, \ldots, l$, set

$$B_{(i_1,j_1),\ldots,(i_l,j_l)} = (\sigma_1(W_{i_1,j_1}) \cap \cdots \cap \sigma_1(W_{i_l,j_l})) \setminus \Big[\bigcup_{(i_{l+1},j_{l+1})} \sigma_1(W_{i_1,j_1}) \cap \cdots \cap \sigma_1(W_{i_l,j_l}) \cap \sigma_1(W_{i_{l+1},j_{l+1}}) \Big],$$

where (i_{l+1}, j_{l+1}) ranges on all pairs with $1 \leq i_{l+1} \leq r$ and $1 \leq j_{l+1} \leq t'_{i_{l+1}}$, with $i_{l+1} \notin \{i_1, \ldots, i_l\}$. By Lemma 4.27, we have $B = \bigsqcup B_{(i_1,j_1),\ldots,(i_l,j_l)}$. Similarly we define sets $C_{(i_1,j_1),\ldots,(i_l,j_l)}$, where now $r+1 \leq i_1 < \cdots < i_l \leq n$ and $1 \leq j_s \leq t'_{i_s}$ for $s = 1, \ldots, l$, so that $C = \bigsqcup C_{(i_1,j_1),\ldots,(i_l,j_l)}$.
Now write

$$B^0_{(i_1,j_1),\ldots,(i_l,j_l)} := B_{(i_1,j_1),\ldots,(i_l,j_l)} \setminus \Big[\bigsqcup_{i=r+1}^{n} V_i \Big], \qquad C^0_{(i_1,j_1),\ldots(i_l,j_l)} = C_{(i_1,j_1),\ldots,(i_l,j_l)} \setminus \Big[\bigsqcup_{i=1}^{r} V_i \Big].$$

Moreover, set, for $r+1 \leq i \leq n$,

$$W^0_{i,d} = W_{i,d} \setminus B,$$

and, for $1 \leq i \leq r$,

$$W^0_{i,d} = W_{i,d} \setminus C.$$

Applying the formula appearing in the proof of Lemma 4.27 to the middle term of (4.14) we get

$$Y = \bigsqcup B^0_{(i_1,j_1),\ldots,(i_l,j_l)} \sqcup \bigsqcup (B_{(i_1,j_1),\ldots,(i_l,j_l)} \cap W_{i,d}) \sqcup \bigsqcup W^0_{i,d}. \tag{4.15}$$

Here, in all the unions on the right, the indices $(i_1, j_1), \ldots, (i_l, j_l)$ are extended over all possible indices with $1 \leq i_1 < \cdots < i_l \leq r$ and $1 \leq j_s \leq t'_{i_s}$, $s = 1, \ldots, l$. For the indices (i, d), we have $r+1 \leq i \leq n$ and $1 \leq d \leq t_i$.
Similarly, we obtain, applying the same formula to the term appearing at the right hand side of (4.14),

$$Y = \bigsqcup C^0_{(i_1,j_1),\ldots,(i_l,j_l)} \sqcup \bigsqcup (C_{(i_1,j_1),\ldots,(i_l,j_l)} \cap W_{i,d}) \sqcup \bigsqcup W^0_{(i,d)}. \tag{4.16}$$

Here, in all the unions on the right, the indices $(i_1, j_1), \dots, (i_l, j_l)$ are extended over all possible indices with $r+1 \leq i_1 < \cdots < i_l \leq n$ and $1 \leq j_s \leq t'_{i_s}$, $s = 1, \dots, l$. For the indices (i, d), we have $1 \leq i \leq r$ and $1 \leq d \leq t_i$.

Let $1 \leq i_1 < \cdots i_l \leq r$ and $1 \leq j_s \leq t'_{i_s}$ for $s = 1, \dots, l$. Notice that for each non-empty subset T of $B_{(i_1,j_1),\dots,(i_l,j_l)}$ and each s with $1 \leq s \leq l$, the preimage $(\sigma_1|_{V_{i_s}})^{-1}(T)$ of T in V_{i_s} is the union of $m(i_s, j_s)$ sets, where $m(i_s, j_s)$ is the cardinality of the set of indices $j \in \{1, \dots, t_{i_s}\}$ such that $\sigma_1(W_{i_s,j}) = \sigma_1(W_{i_s,j_s})$. Moreover, we have $\sigma_1(Z) = T$ for each of these $m(i_s, j_s)$ sets. In addition, we have that $(\sigma_1|_{V_i})^{-1}(T) = \emptyset$ for $i \in \{1, \dots, r\} \setminus \{i_1, \dots, i_l\}$.

A similar observation holds for the non-empty subsets of $C_{(i_1,j_1),\dots,(i_l,j_l)}$, where now $r+1 \leq i_1 < \cdots < i_l \leq n$.

We can use a lighter notation, and set, using (4.15) and (4.16),

$$Y = \bigsqcup_{\alpha} B^0_\alpha \sqcup \bigsqcup_{\beta} \bigsqcup_{i=r+1}^{n} \bigsqcup_{j=1}^{t_i} (B_\beta \cap W_{i,d}) \sqcup \bigsqcup_{i=r+1}^{n} \bigsqcup_{j=1}^{t_i} W^0_{i,d},$$

where B^0_α, B_β are subsets of B satisfying the above conditions for suitable subsets of $\{1, \dots, r\}$, and $B^0_\alpha \cap (\sqcup_{i=r+1}^n V_i) = \emptyset$. Similarly, we write

$$Y = \bigsqcup_{\gamma} C^0_\gamma \sqcup \bigsqcup_{\delta} \bigsqcup_{i=1}^{r} \bigsqcup_{j=1}^{t_i} (C_\delta \cap W_{i,d}) \sqcup \bigsqcup_{i=1}^{r} \bigsqcup_{j=1}^{t_i} W^0_{i,d},$$

where C^0_γ, C_δ are subsets of C satisfying the above conditions for suitable subsets of $\{r+1, \dots, n\}$, and $C^0_\gamma \cap (\sqcup_{i=1}^r V_i) = \emptyset$.

We now refine these two decompositions of Y. Observe that for each i, d with $r+1 \leq i \leq n$ we have

$$W_{i,d} \subseteq \bigsqcup_{\gamma} C^0_\gamma.$$

Similarly, for each i, d with $1 \leq i \leq r$ we have

$$W_{i,d} \subseteq \bigsqcup_{\alpha} B^0_\alpha.$$

We thus obtain

$$Y = \bigsqcup_{\alpha,\gamma} B^0_\alpha \cap C^0_\gamma \sqcup \bigsqcup_{i=1}^{r} \bigsqcup_{\alpha,d} B^0_\alpha \cap W^0_{i,d} \sqcup \bigsqcup_{i=1}^{r} \bigsqcup_{\alpha,\gamma,,d} B^0_\alpha \cap C_\delta \cap W_{i,d} \sqcup \bigsqcup_{i=r+1}^{n} \bigsqcup_{\gamma,\beta,d} C^0_\gamma \cap B_\beta \cap W_{i,d} \sqcup \bigsqcup_{i=r+1}^{n} \bigsqcup_{\gamma,d} C^0_\gamma \cap W^0_{i,d}.$$

We are now ready to define the set $\Gamma = \bigsqcup_{i=1}^{n+1} \Gamma_i$. Let Γ_{n+1} be the set of all the non-empty sets $B^0_\alpha \cap C^0_\gamma$, for all values of α and β. For $i = 1, \dots, r$ let Γ_i be the set of all the non-empty sets of the forms $B^0_\alpha \cap W^0_{i,d}$ and $B^0_\alpha \cap C_\delta \cap W_{i,d}$, for all possible values of

α, δ, d. Finally, for $i = r+1, \dots, n$, let Γ_i be the set of all the non-empty sets of the forms $C^0_\gamma \cap W^0_{i,d}$ and $C^0_\gamma \cap B_\beta \cap W_{i,d}$, for all possible values of β, γ, d. For each $v \in \Gamma$, we set $Z_v = v$, to keep in line with the notation previously introduced.
Finally we define the matrix $A = (a_{vi})$. For each $i = 1, \dots, n$ and $v \in \Gamma$, there is $a_{vi} \in \mathbb{Z}^+$ such that V_i contains exactly a_{vi} mutually disjoint subsets, all of them having the property that the restriction of σ_1 to them is injective and that its image under σ_1 is exactly Z_v. Note that for v of the form $B^0_\alpha \cap W^0_{i,d}$, one has $a_{vi} = 0$ for $i = r+1, \dots, n$, and that, for v of the form $C^0_\gamma \cap W^0_{i,d}$, one has $a_{vi} = 0$ for $i = 1, \dots, r$.
We therefore have subsets $V^{(k)}_{vi}$ for each $v \in \Gamma$, $1 \le i \le n$ and $1 \le k \le a_{vi}$ such that

$$V_i = \bigsqcup_{v \in \Gamma} \bigsqcup_{k=1}^{a_{vi}} V^{(k)}_{vi}.$$

Observe that we have, for $i = 1, \dots, r$ and $1 \le d \le t_i$, that

$$W_{i,d} = \bigsqcup_{\gamma,\beta} C^0_\gamma \cap B_\beta \cap W_{i,d} \sqcup \bigsqcup_{\gamma} C^0_\gamma \cap W^0_{i,d},$$

so that we obtain $V_i = \bigsqcup_{v \in \Gamma_i} Z_v$. Similarly we get that $V_i = \bigsqcup_{v \in \Gamma_i} Z_v$ for $i = r+1, \dots, n$. There is an extra term (possibly empty), disjoint to all the V_i's with $i = 1, \dots, n$, which is $V_{n+1} := \bigsqcup_{\alpha,\gamma} B^0_\alpha \cap C^0_\gamma = \bigsqcup_{v \in \Gamma_{n+1}} Z_v$.
Now observe that equation (4.12) gives us

$$\sum_{v \in \Gamma} (\sum_{i=1}^{n} a_{vi} x_i) \chi_{Z_v} = \sum_{i=1}^{n} \sum_{v \in \Gamma_i} x_i \chi_{Z_v}.$$

Since Z_v are mutually disjoint and non-empty, we get that $(A - I)x = 0$ and so $\sum_{i=1}^{n} x_i \chi_{V_i \times \{p\}}$ is of the desired form. $\square$

We now observe that the formula for the unitaries in $K_1(C^*(E))$ can be generalized to separated graphs.
Let (E, C) be a finitely separated graph. Following [3], we define the map:

$$A \colon \mathbb{Z}^{(C)} \to \mathbb{Z}^{(E^0)}$$

by $A(\delta_X) = \sum_{w \in E^0} a_X(v, w) \delta_w$, where $X \in C_v$. We also define the map

$$I \colon \mathbb{Z}^{(C)} \to \mathbb{Z}^{(E^0)}$$

by $I(\delta_X) = \delta_v$ for $X \in C_v$.
It will be useful in the sequel to use the notation $s(X) = v$ for $X \in C_v$. Now recall from [3, Theorem 5.2] that

$$K_1(C^*(E, C)) \cong ker\Big((I - A) \colon \mathbb{Z}^{(C)} \to \mathbb{Z}^{(E^0)}\Big).$$

We are going to find an explicit formula for the isomorphism

$$\chi\colon ker\Big((I-A)\colon \mathbb{Z}^{(C)} \to \mathbb{Z}^{(E^0)}\Big) \to K_1(C^*(E,C)).$$

Let $x=(x_X)_{X\in C}\in ker(I-A)\subseteq \mathbb{Z}^{(C)}$. Then we set:

$$L_x^+ = \{(X,i)\mid 1\le i\le x_X\}\cap\{(e,i)\mid e\in X, 1\le i\le -x_X\},$$

$$L_x^- = \{(X,i)\mid 1\le i\le -x_X\}\cup\{(e,i)\mid e\in X, 1\le i\le x_X\}.$$

Note that the condition that $x\in ker(I-A)$ translates into the equations:

$$\sum_{X\in C_w} x_X = \sum_{Y\in C} a_Y(v,w)x_Y, \qquad (w\in E^0).$$

Therefore we get $|L_x^+(w)|=|L_x^-(w)|$ for all $w\in E^0$, where

$$L_x^+(w) = \{(X,i): X\in C_w, 1\le i\le x_X\}\cup\{(e,i): e\in Y, r(e)=w, 1\le i\le -x_Y\},$$

and similarly

$$L_x^-(w) = \{(X,i): X\in C_w, 1\le i\le -x_X\}\cup\{(e,i): e\in Y, r(e)=w, 1\le i\le x_Y\}.$$

In particular we get $|L_x^+|=|L_x^-|$ and we fix bijections

$$[\ ]\colon L_x^+\to\{1,\dots,h\} \quad \text{and} \quad \langle\ \rangle : L_x^-\to\{1,\dots,h\}$$

such that $[L_x^+(w)]=\langle L_x^-(w)\rangle$ for all $w\in E^0$.
We now consider:

$$V_x = \sum_{1\le i\le x_X, e\in X} eE_{[X,i],\langle e,i\rangle} + \sum_{1\le i\le -x_X, e\in X} e^*E_{[e,i],\langle X,i\rangle}.$$

Then we have $V_xV_x^*=P_x=V_x^*V_x$, where

$$P_x = \sum_{X:x_X>0}\sum_{1\le i\le x_X} s(X)E_{[X,i],[X,i]} + \sum_{X:x_X<0}\sum_{1\le i\le -x_X} r(e)E_{[e,i],[e,i]}.$$

In this way, we obtain that the map χ is defined by $\chi(x)=U_x:=V_x+(1-P_x)$ for all $x\in ker(I-A)$.
We are going to relate this with our previous situation. Let $f\in ker(id-K_0(\alpha_1))$. By Theorem 4.34 there are non-empty mutually disjoint compact open sets Z_v, $v\in\Gamma$, where $\Gamma=\bigsqcup_{i=1}^{n+1}\Gamma_i$ is a finite set, $p\in\mathbb{N}^k$, such that with $V_i=\bigsqcup_{v\in\Gamma_i} Z_v$, and we have for each i a decomposition

$$V_i = \bigsqcup_{v\in\Gamma}\bigsqcup_{k=1}^{a_{vi}} V_{vi}^{(k)},$$

where $V_{vi}^{(k)}$ are compact open sets such that $\sigma_1|_{V_{vi}^{(k)}}$ is injective and $\sigma_1|_{V_{vi}^{(k)}}(V_{vi}^{(k)}) = Z_v$ for all allowable values of v, i, k.

Let $A = (a_{vi}) \in M_{\Gamma\times[1,n]}(\mathbb{Z}^+)$, and let $I \in M_{\Gamma\times[1,n]}(\mathbb{Z}^+)$ be the matrix with all entries in $\{0,1\}$ such that in the i^{th} column has 1's exactly at the positions that belong to Γ_i, for $i = 1,\dots,n$. There exists $x = (x_1,\dots,x_n)^t \in ker(A-I)$ such that $f = \sum_{i=1}^n x_i[\chi_{V_i\times\{p\}}]$.

We define a separated graph (E,C) which locally reflects the above situation. This will be a bipartite separated graph, with a partition $E^0 = E^{0,0} \sqcup E^{0,1}$ such that all arrows from E go from $E^{0,0}$ to $E^{0,1}$. Set $E^{0,0} = \{v^1,\dots,v^n\}$ and $E^{0,1} = \Gamma$. For each $v^i \in E^{0,0}$ we have $C_{v^i} = \{X_i, Y_i\}$, where $|X_i| = |\Gamma_i|$ and there is an arrow e_{vi} in X_i from v^i to $v \in \Gamma_i$ for each $v \in \Gamma_i$. On the other hand, there are exactly a_{vi} arrows, denoted $f_{vi}^{(k)}$, $1 \le k \le a_{vi}$, in Y_i from v^i to v, for each $v \in \Gamma$. In this way, we obtain a bipartite separated graph (E,C).

The proof of the following lemma is straightforward.

Lemma 4.35. *With the above notation, we have a group isomorphism*

$$ker(I-A) \cong ker(I_{(E,C)} - A_{(E,C)}).$$

This isomorphism sends an element $x = (x_1,\dots,x_n) \in ker(I-A)$ to the element $\sum_{i=1}^n x_i(\delta_{Y_i} - \delta_{X_i}) \in ker(I_{(E,C)} - A_{(E,C)})$.

□

We now define a homomorphism $\rho\colon pC^*(E,C)p \to A \rtimes \mathbb{Z}$, where $p = \sum_{v\in\Gamma} v$ is a full projection in $C^*(E,C)$. We first obtain a presentation of the algebra $pC^*(E,C)p$. We have $X_i = \{e_{vi} : v \in \Gamma_i\}$ and $Y_i = \{f_{wi}^{(k)} : w \in \Gamma, 1 \le i \le a_{wi}\}$. Now for $v \in \Gamma_i$, $w \in \Gamma$ and $1 \le k \le a_{wi}$, set

$$g_{vw}^{(k)} := e_{vi}^* f_{wi}^{(k)} \in pC^*(E,C)p.$$

Then $pC^*(E,C)p$ is generated by the elements $g_{vw}^{(k)}$ with the relations:

$$\sum_{w\in\Gamma}\sum_{k=1}^{a_{wi}} g_{vw}^{(k)}(g_{vw}^{(k)})^* = v \qquad (v \in \Gamma_i). \tag{4.17}$$

$$\sum_{v\in\Gamma_i} (g_{vw}^{(k)})^* g_{vw}^{(k)} = w, \qquad (w \in \Gamma, 1 \le k \le a_{wi}). \tag{4.18}$$

Using this we can define a suitable representation. For $v \in \Gamma_i$, $w \in \Gamma$ and $1 \le k \le a_{wi}$, we set

$$Z_{vw}^{(k)} = \{((y, e_1, \sigma_1(y)), p) : y \in V_{wi}^{(k)} \cap Z_v\}.$$

and $u_{vw}^{(k)} = \chi_{Z_{vw}^{(k)}} u \in A \rtimes \mathbb{Z}$. Then we have

$$u_{vw}^{(k)}(u_{vw}^{(k)})^* = \chi_{Z_v \cap V_{wi}^{(k)} \times\{p\}}$$

and

$$(u_{vw}^{(k)})^* u_{vw}^{(k)} = \chi_{\sigma^1(Z_v \cap V_{wi}^{(k)}) \times \{p\}}.$$

Consequently we get

$$\sum_{w \in \Gamma} \sum_{k=1}^{a_{wi}} u_{vw}^{(k)} (u_{vw}^{(k)})^* = \chi_{Z_v \times \{p\}}, \qquad (v \in \Gamma_i)$$

and

$$\sum_{v \in \Gamma_i} (u_{vw}^{(k)})^* u_{vw}^{(k)} = \chi_{Z_w \times \{p\}}, \qquad (w \in \Gamma, 1 \leq k \leq a_{wi}).$$

By (4.17) and (4.18), we get a unique $*$-homomorphism $\rho\colon pC^*(E, C)p \to A \rtimes \mathbb{Z}$ defined by

$$\rho(w) = \chi_{Z_w \times \{p\}}, \qquad \rho(g_{vw}^{(k)}) = u_{vw}^{(k)}$$

for each $v \in \Gamma_i$, $w \in \Gamma$ and $1 \leq k \leq a_{wi}$.

Let $q = \sum_{v \in \cup_{i=1}^n \Gamma_i} v$. Then $q \leq p$ and we have an isomorphism

$$\psi\colon C^*(E, C) \to \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix} M_2(pC^*(E, C)p) \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix}$$

given by

$$\psi(a) = \begin{pmatrix} e^*ae & e^*ap \\ pae & pap \end{pmatrix},$$

where $e = \sum_{i=1}^n \sum_{v \in \Gamma_i} e_{vi}$. It follows that we have a homomorphism

$$\xi\colon C^*(E, C) \to \begin{pmatrix} \rho(q) & 0 \\ 0 & \rho(p) \end{pmatrix} M_2(A \rtimes \mathbb{Z}) \begin{pmatrix} \rho(q) & 0 \\ 0 & \rho(p) \end{pmatrix}$$

obtained by composing ψ with the extension to matrices of ρ. Using this morphism, we will define the unitary u_f as the image of the canonical unitary constructed before in matrices over $C^*(E, C)$. So let $f = \sum_{i=1}^n x_i \chi_{V_i \times \{p\}}$ as in Proposition 4.34, and keep all the notation above.

Since $x \in ker(I - A)$, by Lemma 4.35 we get the element $a = \sum_{i=1}^n x_i(\delta_{Y_i} - \delta_{X_i}) \in ker(I_{(E,C)} - A_{(E,C)})$. Let $[\ ]\colon L_a^+ \to \{1, \ldots, h\}$ and $\langle\ \rangle\colon L_a^- \to \{1, \ldots, h\}$ be bijections satisfying the conditions above.

We get

$$\begin{aligned} V_a = &\sum_{1 \leq i \leq -x_l} \sum_{v \in \Gamma_l} e_{vl} E_{[X_l, i], \langle e_{vl}, i \rangle} + \sum_{1 \leq i \leq x_l} \sum_{v \in \Gamma_l} e_{vl}^* E_{[e_{vl}, i], \langle X_l, i \rangle} \\ &+ \sum_{1 \leq i \leq x_l} \sum_{w \in \Gamma, 1 \leq k \leq a_{wl}} f_{wl}^{(k)} E_{[Y_l, i], \langle f_{wl}^{(k)}, i \rangle} + \sum_{1 \leq i \leq -x_l} \sum_{w \in \Gamma, 1 \leq k \leq a_{wl}} (f_{wl}^{(k)})^* E_{[f_{wl}^{(k)}, i], \langle Y_l, i \rangle} \end{aligned}$$

Then $U_a = V_a + (1 - P_a)$ is a unitary element in $M_h(C^*(E,C))$. Applying

$$\xi \colon M_h(C^*(E,C)) \to M_h(M_2(A \rtimes \mathbb{Z})) \cong M_2(M_h(A \rtimes \mathbb{Z})),$$

we obtain

$$\begin{aligned} u_f = &\begin{pmatrix} 0 & \sum_{1\le i\le -x_l} \sum_{v\in\Gamma_l} \chi_{Z_v\times\{p\}} E_{[X_l,i],\langle e_{vl},i\rangle} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \sum_{1\le i\le x_l} \sum_{v\in\Gamma_l} \chi_{Z_v\times\{p\}} E_{[e_{vl},i],\langle X_l,i\rangle} & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & \sum_{1\le i\le x_l} \sum_{w\in\Gamma, 1\le k\le a_{wl}} (\sum_{v\in\Gamma_l} u_{vw}^{(k)}) E_{[Y_l,i],\langle f_{wl}^{(k)},i\rangle} \\ 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ \sum_{1\le i\le -x_l} \sum_{w\in\Gamma, 1\le k\le a_{wl}} (\sum_{v\in\Gamma_l} (u_{vw}^{(k)})^*) E_{[f_{wl}^{(k)},i],\langle Y_l,i\rangle} & 0 \end{pmatrix} + \xi(1 - P_a) \end{aligned}$$

This is a partial unitary in $M_{2h}(A \rtimes \mathbb{Z})$ and thus it defines an element $[u_f] \in K_1(A \rtimes \mathbb{Z})$. Write

$$B_{wi}^{(k)} = \{((y, e_1, \sigma_1(y)), p) : y \in V_{wi}^{(k)}\}.$$

Then $B_{wi}^{(k)}$ are bisections and we may consider $u_{wi}^{(k)} = \chi_{B_{wi}^{(k)}} u \in A \rtimes \mathbb{Z}$. Note that $u_{wi}^{(k)} (u_{wi}^{(k)})^* = \chi_{V_{wi}^{(k)}\times\{p\}}$ and $(u_{wi}^{(k)})^* u_{wi}^{(k)} = \chi_{Z_w\times\{p\}}$. Moreover $\sum_{v\in\Gamma_l} u_{vw}^{(k)} = u_{wl}^{(k)}$, so that we can simplify the above formula for u_f as follows:

$$\begin{aligned} u_f = &\begin{pmatrix} 0 & \sum_{1\le i\le -x_l} \sum_{v\in\Gamma_l} \chi_{Z_v\times\{p\}} E_{[X_l,i],\langle e_{vl},i\rangle} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \sum_{1\le i\le x_l} \sum_{v\in\Gamma_l} \chi_{Z_v\times\{p\}} E_{[e_{vl},i],\langle X_l,i\rangle} & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & \sum_{1\le i\le x_l} \sum_{w\in\Gamma, 1\le k\le a_{wl}} u_{wl}^{(k)} E_{[Y_l,i],\langle f_{wl}^{(k)},i\rangle} \\ 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ \sum_{1\le i\le -x_l} \sum_{w\in\Gamma, 1\le k\le a_{wl}} (u_{wl}^{(k)})^* E_{[f_{wl}^{(k)},i],\langle Y_l,i\rangle} & 0 \end{pmatrix} + \xi(1 - P_a) \end{aligned}$$

We can now show the desired property of u_f.

Lemma 4.36. *With the above notation, we have that* $\delta([u_f]_1) = f$.

Proof. As before, we consider the element $v = u^* \otimes S$ in $\mathcal{T}_{A,\alpha_1}$ and we set

$$v_{wi}^{(k)} := (\chi_{B_{wi}^{(k)}} \otimes 1) v^* = u_{wi}^{(k)} \otimes S^*.$$

Then replacing each occurrence of $u_{wl}^{(k)}$ in u_f by $v_{wl}^{(k)}$, we obtain an element $v_f \in \mathcal{T}_{A,\alpha_1}$.

We now compute

$$\begin{aligned}
\delta([u_f]_1) &= [1 - v_f^* v_f]_0 - [1 - v_f v_f^*]_0 \\
&= \Big(\sum_{w \in \Gamma} \big(\sum_{l : x_l > 0} a_{wl} x_l \big) [\chi_{Z_w \times \{p\}} \otimes (1 - SS^*)]_0 \Big) - \Big(\sum_{w \in \Gamma} \big(\sum_{l : x_l < 0} a_{wl} (-x_l) \big) [\chi_{Z_w \times \{p\}} \otimes (1 - SS^*)]_0 \Big) \\
&= \sum_{w \in \Gamma} \big(\sum_{l=1}^{n} a_{wl} x_l \big) [\chi_{Z_w \times \{p\}} \otimes (1 - SS^*)]_0 \\
&= \sum_{i=1}^{n} x_i \big(\sum_{w \in \Gamma_i} [\chi_{Z_w \times \{p\}} \otimes (1 - SS^*)]_0 \big) \\
&= \sum_{i=1}^{n} x_i [\chi_{V_i \times \{p\}} \otimes (1 - SS^*)]_0 = f.
\end{aligned}$$

$\square$

Chapter 5

Self-similar groups and their groupoids

Self-similarity is a type of symmetry appearing in fractal geometry and dynamics. Self-similar structures first appeared in early eighties, when they were used to build a group which is neither of polynomial nor of exponential growth (see [25]). To some extent, an object is said to be self-similar if its structure repeats at all scales. A natural way to study a self-similar space is to encode it as the algebra associated to a certain groupoid of germs, as in [20]. In the last decades, their exotic nature has drawn the attention of researchers from many different areas. Indeed, very simple self-similar constructions could generate complicated structures with uncommon properties, hard to find with more *standard* approaches. In this line, new counterexamples to Matui's HK conjecture were expected to appear encoded as self-similar objects. This chapter is structured as follows:

Section 5.1 provides the basic facts and notions regarding self-similarity, as well as the groupoids associated to self-similar groups. The self-similar infinite dihedral group, as well as its associated groupoid of germs, is displayed as the main example of this chapter.

In Section 5.2 we analyze the C^*-algebra associated to the groupoid arising from the self-similar infinite dihedral group, as well as its K-theory. To do so, we build a suitable crossed product stably isomorphic to the initial C^*-algebra. This approach was developed by Nekrashevych in [42], where a more detailed account can be found. The section ends with the computation of the K-theory invariants for the mentioned C^*-algebra.

Section 5.3 displays a standard strategy for the computation of the homology of a groupoid associated to a self-similar group. The results are then applied to the particular case of the self-similar infinite dihedral groupoid.

Immediate consequences of the first two sections regarding the HK conjecture are shown in Section 5.4. There, we display the self-similar infinite dihedral groupoid as the first complete counterexample to both HK and weak HK conjectures, in contrast with the one found by Scarparo in [56]. Another minor counterexample will also be shown here. The chapter ends with a discussion on Matui's AH conjecture.

5.1 Self-similar groups (Γ, X)

The section begins stating the basic notions and notation that will be used throughout the chapter. The reader can find further details about self-similarity in [41], and about its associated C^*-algebras in [42].

Definition 5.1. *Let X be a finite alphabet, and let X^* be the set of finite words over X. A **self-similar group action** (Γ, X) is a faithful action of a group Γ over X^* such that for every $g \in \Gamma$ and every $x \in X$, there exist $h \in \Gamma$ and $y \in X$ such that*

$$g(xw) = yh(w),$$

for every $w \in X^*$. $\square$

It is clear that, given $g \in \Gamma$, $x \in X$, the element y is unique. Moreover, the faithful condition implies that h is also unique.
From now on, we will denote $h := g|_x$, and then we will just write

$$g \cdot x = y \cdot g|_x,$$

In general, for every $g \in \Gamma$, and every finite word $v \in X^*$, there exists a unique element $g|_v \in \Gamma$, and a unique $z \in X^{|v|}$ such that

$$g(vw) = zg|_v(w).$$

The word z will be denoted as $g(v)$, and the element $g|_v$ will be called the **restriction** of g at v. In particular, for every $g, g_1, g_2 \in \Gamma$ and every $v, v_1, v_2 \in X^*$ we have that:

1. $g|_{v_1 v_2} = (g|_{v_1})|_{v_2}$
2. $(g_1 g_2)|_v = g_1|_{g_2(v)} g_2|_v$

In the literature, self-similar group actions usually appear simply as self-similar groups. We will also stick to that nomenclature.
The following properties will play a notorious role throughout this chapter.
Definition 5.2. *Let (Γ, X) be a self-similar group.*

1. *(Γ, X) is **pseudo-free** if, whenever $g \cdot x = x \cdot e$ for some $g \in \Gamma$ and $x \in X$, then $g = e$.*
2. *(Γ, X) is **contractive** if there is a finite set $\mathcal{N} \subset \Gamma$ such that for every $g \in \Gamma$, there exists $n \in \mathbb{N}$ such that $g|_v \in \mathcal{N}$, for all $v \in X^*$ of length greater or equal that n.*
3. *(Γ, X) is **self-replicant** or **recurrent** if, for any $x, y \in X$, there exists $g \in \Gamma$ such that $g \cdot x = y \cdot e$.*

□

We now introduce the self-similar infinite dihedral group, which will be our main example throughout this chapter.

Example 5.3. *Let $\mathcal{D}_\infty$ be the infinite dihedral group, that is*

$$\mathcal{D}_\infty := \langle a, b : a^2 = b^2 = e \rangle,$$

and let $X = \{0, 1\}$. It was shown in [24, Example 4] that the group $(\mathcal{D}_\infty, X)$ is a self-similar group with relations

$$a \cdot 0 = 1 \cdot e, \qquad a \cdot 1 = 0 \cdot e, \qquad b \cdot 0 = 0 \cdot a \qquad b \cdot 1 = 1 \cdot b\,.$$

□

As it will be shown later, this simple example provides a groupoid with structure rich enough to contradict Matui's HK conjecture.

Proposition 5.4. *The self-similar infinite dihedral group $(\mathcal{D}_\infty, X)$ is pseudo-free. Moreover, $(\mathcal{D}_\infty, X)$ is contractive with $\mathcal{N} = \{e, a, b\}$.*

Proof. We begin with the first statement.
By definition, we need to prove that whenever $g \cdot x = x \cdot e$, with $x \in X$, then $g = e$.
Take $g \in \mathcal{D}_\infty$. Then g is of one of the following forms: $(ab)^n a, (ab)^n, (ba)^n b$, or $(ba)^n$, with $n \geq 0$. The reader may check that:

$$(ab) \cdot 0 = 1 \cdot a, \quad \text{and} \quad (ab) \cdot 1 = 0 \cdot b.$$

Therefore

$$(ab)^{2n} \cdot 0 = 0 \cdot (ba)^n, \qquad (ab)^{2n} \cdot 1 = 1 \cdot (ab)^n,$$

for the even exponents, and

$$(ab)^{2n+1} \cdot 0 = 1 \cdot a(ba)^n, \qquad (ab)^{2n+1} \cdot 1 = 0 \cdot b(ab)^n,$$

for the odd ones.
This shows that whenever g is of the form $(ab)^n$, $g \cdot x = x \cdot e$ implies $g = e$.
If $g = (ab)^n a$, we have:

$$(ab)^{2n} a \cdot 0 = 1 \cdot (ab)^n, \qquad (ab)^{2n} a \cdot 1 = 0 \cdot (ba)^n,$$
$$(ab)^{2n+1} a \cdot 0 = 0 \cdot b(ab)^n, \qquad (ab)^{2n+1} a \cdot 1 = 1 \cdot a(ba)^n$$

so then again, we have that $g \cdot x = x \cdot e$ implies $g = e$, whenever $g = (ab)^n a$.
The two remaining choices, $g = (ba)^n$ and $g = (ba)^n b$, are left to the reader.
We now prove that $(\mathcal{D}_\infty, X)$ is contractive with $\mathcal{N} = \{e, a, b\}$.

As noted before, the elements of $\mathcal{D}_\infty$ are of the form $(ab)^n, (ab)^n a, (ba)^n b$, or $(ba)^n$, with $n \geq 0$. The proof is analogous for every case; we check here the statement for the first one.

Let $g \in \mathcal{D}_\infty$ be of the form $(ab)^n$. Then we claim that, for every word v of length greater or equal than n, $(ab)^n|_v \in \mathcal{N}$. We use an induction argument:

If $n = 1$, then $(ab) \cdot 1 = 0 \cdot b$, and $(ab) \cdot 0 = 1 \cdot a$. Since for any word z of arbitrary length we have that $a|_z, b|_z \in \mathcal{N}$, the condition holds. Suppose now that the statement is true for n, and let us prove it for $n+1$. First, note that for every word w of length $n+1$, $(ab)|_w = e$ whenever $w \neq 1^{n+1}, 1^n 0$. Therefore, for every $w \neq 1^{n+1}, 1^n 0$:

$$(ab)^{n+1} w = (ab)^n (ab) w = (ab)^n w' e,$$

where $w' = (ab)(w)$. By induction, $(ab)^{n+1}|_w \in \mathcal{N}$.

Now suppose that $w = 1^{n+1}$, and we have:

$$(ab)^{n+1} \cdot 1^{n+1} = (ab)^n (ab) \cdot 1^{n+1} = (ab)^n \cdot 01^n \cdot b$$

By the induction hypothesis, $(ab)^n|_{01^{n-1}} \in \mathcal{N}$, and so we have that $(ab)^{n+1} \cdot 1^{n+1}$ is of one of the following forms:

$$\begin{gathered} w'e \cdot 1 \cdot b = w'1 \cdot b, \\ w'a \cdot 1 \cdot b = w'0 \cdot b, \text{ or} \\ w'b \cdot 1 \cdot b = w'1 \cdot e. \end{gathered}$$

Thus the statement holds.

On the other hand, if $w = 1^n 0$, we have:

$$(ab)^{n+1} \cdot 1^n 0 = (ab)^n (ab) \cdot 1^n 0 = (ab)^n \cdot 01^{n-1} 0a.$$

As before, by the induction hypothesis we have that $(ab)^n|_{01^{n-1}} \in \mathcal{N}$, and then $(ab)^{n+1} \cdot 1^n 0$ is of the form:

$$\begin{gathered} w'e \cdot 0 \cdot a = w'0 \cdot a, \\ w'a \cdot 0 \cdot a = w'1 \cdot a, \text{ or} \\ w'b \cdot 0 \cdot a = w'0 \cdot e. \end{gathered}$$

Since $a|_z, b|_z \in \mathcal{N}$ for any given word z, we conclude that for any word wz of length greater or equal than $n+1$, $(ab)^n|_{wz} \in \mathcal{N}$, as desired.

The three remaining choices, $g = (ab)^n a$, $g = (ba)^n$ and $g = (ba)^n b$, are left to the reader. □

Finally, it is shown that the self-similar infinite dihedral group verifies a strong version of recurrence.

Proposition 5.5. *Let $(\mathcal{D}_\infty, X)$ be the self-similar infinite dihedral group. Then, for any $n \in \mathbb{N}$ and any $\alpha, \beta \in X^n$, there exists $g \in \mathcal{D}_\infty$ such that $g \cdot \alpha = \beta \cdot e$. In particular, $(\mathcal{D}_\infty, X)$ is recurrent.*

Proof. Let us first show that, given $x, y \in \{0, 1\}$, there exist $g_{x,y}, h_{x,y} \in \mathcal{D}_\infty$ such that:

$$g_{x,y} \cdot x = y \cdot a,$$
$$h_{x,y} \cdot x = y \cdot b$$

The reader may check that:

$$g_{0,0} = b, \quad g_{0,1} = ab, \quad g_{1,0} = ba, \quad g_{1,1} = aba$$
$$h_{0,0} = aba, \quad h_{0,1} = ba, \quad h_{1,0} = ab, \quad h_{1,1} = b$$

Since a, b generate $\mathcal{D}_\infty$, we deduce that, for any $x, y \in X$ and any $h \in \mathcal{D}_\infty$, there exists $g \in \mathcal{D}_\infty$ such that $g \cdot x = y \cdot h$.
We now proceed by induction over $n = |\alpha| = |\beta|$.
If $n = 1$, then $\alpha, \beta \in X$, and then:

$$g_{\beta,\beta} g_{\alpha,\beta} \cdot \alpha = g_{\beta,\beta} \cdot \beta \cdot a = \beta \cdot a^2 = \beta \cdot e,$$

as desired. Note that this result already implies that $(\mathcal{D}_\infty, X)$ is recurrent. Suppose now that the property is true for all words of length n, and consider $\alpha = x_1 x_2 ... x_{n+1}$, and $\beta = y_1 y_2 ... y_{n+1}$. Let $\delta \in \mathcal{D}_\infty$ be the element such that $\delta \cdot x_2 x_3 ... x_{n+1} = y_2 y_3 ... y_{n+1} \cdot e$, and let $\gamma \in \mathcal{D}_\infty$ such that $\gamma \cdot x_1 = y_1 \cdot \delta$ (which exists by the first part of the proof). Then $\gamma \cdot x_1 x_2 ... x_{n+1} = y_1 \cdot \delta \cdot x_2 x_3 ... x_{n+1} = y_1 y_2 ... y_{n+1} \cdot e$, concluding the proof. □

In a similar fashion that it is done for groups and groupoids, we can associate a C^*-algebra to any self-similar group. The following definition was introduced in [42]; we recall it here for a matter of completeness:

Definition 5.6. *The (universal)* ***Cuntz-Pimsner algebra*** *$C^*(\Gamma, X)$ associated to a self-similar group (Γ, X) is the universal C^*-algebra generated by a set of unitaries $\{U_g\}_{g \in \Gamma}$ and isometries $\{\mathcal{S}_x\}_{x \in X}$ satisfying the following relations:*

1. *The map $g \mapsto U_g$ is a unitary representation (that is, U_g is a unitary element for every $g \in \Gamma$, $U_g U_h = U_{gh}$, and $U_e = 1$.)*
2. *$\sum_{x \in X} \mathcal{S}_x \mathcal{S}_x^* = 1$, and $\mathcal{S}_x^* \mathcal{S}_x = 1$ for every $x \in X$.*
3. *For all $g \in \Gamma$ and $x \in X$, whenever $g \cdot x = y \cdot g|_x$ we have that*

$$U_g \mathcal{S}_x = \mathcal{S}_y U_{g|_x}$$

□

With the basic notions about self-similarity already set up, we can now introduce the groupoid associated to a self-similar group.

5.1.1 Groupoid associated to a self-similar group.

In this part, we associate a groupoid of germs (see definition 2.29) to any self-similar group (Γ, X). This technique can be found, for example, in [20].
Define X^∞ to be the set of infinite words over the alphabet X, and associate to it the topology given by the cylinders:

$$Z(\alpha) := \{\alpha x : x \in X^\infty\},$$

where α is any finite word in X^*. Notice that, with this topology, X^∞ is homeomorphic to the Cantor set, whenever $|X| \geq 2$.
We define the natural action of Γ on X^∞ given by:

$$g \cdot x_1 x_2 x_3 ... = y_1 y_2 y_3 ...$$

with $g_n \cdot x_n = y_n \cdot g_{n+1}$, where we define $g_1 := g$, and $g_{n+1} := (g_n)|_{x_n}$.
Then, for each $(\alpha, g, \beta) \in X^n \times \Gamma \times X^m$ with $n, m \in \mathbb{N}$, there exists a homeomorphism

$$\begin{aligned} S_{(\alpha,g,\beta)} : Z(\beta) &\to Z(\alpha) \\ \beta x &\mapsto \alpha g \cdot x \end{aligned}$$

The set $\langle \Gamma, X \rangle = \{S_{(\alpha,g,\beta)} : \alpha, \beta \in X^*, g \in \Gamma\} \cup \{0\}$ defines an inverse semigroup of partial homeomorphisms on X^∞ (i.e., homeomorphisms between cylindrical subsets), with the product given by the composition, that is:

$$S_{(\alpha_1,g_1,\beta_1)} S_{(\alpha_2,g_2,\beta_2)} = \begin{cases} S_{(\alpha_1, g_1 g_2, \beta_2)} & if \ \beta_1 = \alpha_2 \\ S_{(\alpha_1 g_1(\gamma), g_1|_\gamma g_2, \beta_2)} & if \ \alpha_2 = \beta_1 \gamma \\ S_{(\alpha_1, g_1 g_2|_{g_2^{-1}(\mu)}, \beta_2 g_2^{-1}(\mu))} & if \ \beta_1 = \alpha_2 \mu \end{cases}$$

Indeed, if $\alpha_2 = \beta_1 \gamma$ for some $\gamma \in X^*$, then $S_{(\alpha_1,g_2,\beta_1)} S_{(\alpha_2,g_2,\beta_2)}(\beta_2 x) = S_{(\alpha_1,g_2,\beta_1)}(\beta_1 \gamma g_2 x) = \alpha_1 g_1 \gamma g_2 x = \alpha_1 g_1(\gamma) g_1|_\gamma g_2 x$.
The last row can be checked in a similar way.

Definition 5.7. *Given a self-similar group (Γ, X), we define the associated self-similar groupoid $\mathcal{G}_{(\Gamma,X)}$ as the groupoid of germs of $\langle \Gamma, X \rangle$.*

$$\mathcal{G}_{(\Gamma,X)} = \{[S_{(\alpha,g,\beta)}; \beta x] : \alpha, \beta \in X^*, g \in \Gamma, x \in X^\infty\},$$

where $[S_{(\alpha,g,\beta)}; \beta x] = [S_{(\alpha',g',\beta')}; \beta' x']$ if and only if $\beta x = \beta' x'$ and there exists a neighborhood U of βx such that $S_{(\alpha,g,\beta)}(y) = S_{(\alpha',g',\beta')}(y)$ for all $y \in U$.
The unit space is given by:

$$\mathcal{G}^{(0)}_{(\Gamma,X)} = \{[S_{(\emptyset,e,\emptyset)}; x] : x \in X^\infty\}.$$

The groupoid structure maps are of the form:

$$\begin{gathered} r([S_{(\alpha,g,\beta)}; \beta x]) = [S_{(\beta,e,\beta)}; \beta x] = [S_{(\emptyset,e,\emptyset)}; \beta x], \\ s([S_{(\alpha,g,\beta)}; \beta x]) = [S_{(\alpha,e,\alpha)}; \alpha g \cdot x] = [S_{(\emptyset,e,\emptyset)}; \alpha g \cdot x], \\ [S_{(\alpha',g',\beta')}; \beta' x'] \cdot [S_{(\alpha,g,\beta)}; \beta x] = [S_{(\alpha',g'g,\beta)}; \beta x] \end{gathered}$$

whenever $\beta' x' = \alpha g \cdot x$*. The inverse is given by*

$$[S_{(\alpha,g,\beta)}; \beta x]^{-1} = [S_{(\beta,g^{-1},\alpha)}; \alpha g \cdot x]$$

The topology of $\mathcal{G}_{(\Gamma,X)}$ *consists of open sets of the form:*

$$Z(\alpha, g, \beta; U) := \{[S_{(\alpha,g,\beta)}; x] : x \in U\},$$

with U *an open subset of* $Z(\beta)$*,* $\alpha, \beta \in X^*$*, and* $g \in \Gamma$*.* $\square$

With this topology, we have that $\mathcal{G}_{(\Gamma,X)}$ is a locally compact, effective, ample and locally contractive (in the sense of [20, Section 17]). Moreover, if (Γ, X) is recurrent, then $\mathcal{G}_{(\Gamma,X)}$ is minimal [20, Section 17].

It is shown in ([20, Propositions 8.6, 12.1]) that, whenever (Γ, X) is pseudo-free, the groupoid $\mathcal{G}_{(\Gamma,X)}$ is Hausdorff, and we have

$$[S_{(\alpha,g,\beta)}; \beta x] = [S_{(\alpha,g',\beta)}; \beta x]$$

if and only if $g = g'$.

From now on, we will identify $\mathcal{G}^{(0)}_{(\Gamma,X)}$ with X^∞ through the map $[S_{(\emptyset,e,\emptyset)}; x] \mapsto x$. We will also denote the elements $[S_{(\alpha,g,\beta)}; \beta x]$ simply as $[\alpha, g, \beta; \beta x]$.

Remark 5.8. *Notice that, following the definition of* $\mathcal{G}_{(\Gamma,X)}$*, we have that:*

$$[\alpha, \gamma, \beta_1; \beta_1\beta_2 x] = [\alpha\gamma \cdot \beta_2, \gamma|_{\beta_2}, \beta_1\beta_2; \beta_1\beta_2 x].$$

This gives us an easy way to enlarge the words defining a representative of an equivalence class. This fact will be used in later sections. $\square$

Let us finish this section relating the Cuntz-Pimsner C^*-algebra associated to a self-similar group (Γ, X), with the full C^*-algebra associated to its groupoid of germs. This relation was described in [42]; we provide here a sketch of the proof, for a matter of completeness.

Theorem 5.9. *Let* (Γ, X) *be a self-similar group. Then the* C^**-algebra* $C^*(\Gamma, X)$ *is isomorphic to the full convolution* C^**-algebra of* $\mathcal{G}_{(\Gamma,X)}$*.*

Proof. For every $\alpha, \beta \in X^*$, $g \in \Gamma$, consider the set of all germs of $S_{(\alpha,g,\beta)}$, that is, the cylinder $Z(\alpha, g, \beta; Z(\beta)) := [\alpha, g, \beta; Z(\beta)]$. This is a compact, open Hausdorff subset of $\mathcal{G}_{(\Gamma,X)}$, and such sets cover the groupoid. We deduce that every element of $C_c(\mathcal{G}_{(\Gamma,X)})$ is a finite sum of continuous complex valued functions f_i with $supp(f_i) \subseteq Z(\alpha, g, \beta; Z(\beta))$, that is:

$$C_c(\mathcal{G}_{(\Gamma,X)}) = \sum_{S_{(\alpha,g,\beta)} \in \langle \Gamma, X \rangle} C(Z(\alpha, g, \beta; Z(\beta))),$$

where $C(Z(\alpha, g, \beta; Z(\beta)))$ denotes the algebra of continuous complex valued functions over the cylinder $Z(\alpha, g, \beta; Z(\beta))$.
Let $\alpha = x_1 x_2 ... x_n \in X^n$, and define the elements $\mathcal{S}_\alpha := \mathcal{S}_{x_1}\mathcal{S}_{x_2}...\mathcal{S}_{x_n} \in C^*(\Gamma, X)$, and $\mathcal{S}^*_\alpha := (\mathcal{S}_\alpha)^*$. Then the map

$$\begin{aligned} F : C^*(\Gamma, X) &\to C^*(\mathcal{G}_{(\Gamma,X)}) \\ \mathcal{S}_\alpha U_g \mathcal{S}^*_\beta &\mapsto 1_{Z(\alpha,g,\beta;Z(\beta))} \end{aligned}$$

is an isomorphism [42, Theorem 5.1]. $\square$

It was proven in [42, Theorem 5.6] that, if (Γ, X) is a contractive recurrent group such that $\mathcal{G}_{(\Gamma,X)}$ is Hausdorff, then $\mathcal{G}_{(\Gamma,X)}$ is an amenable group. In particular,

$$C^*_r(\mathcal{G}_{(\Gamma,X)}) \cong C^*(\Gamma, X).$$

5.2 K-theory of a groupoid associated to a self-similar group

We are interested in Matui's HK conjecture for the self-similar group $(\mathcal{D}_\infty, X)$ defined before. To this end, we will use the techniques developed by Nekrashevych in [42], where he computed the K-theory of the C^*-algebra associated to a pseudo-free, recurrent and contractive self-similar groupoid as the K-theory of a certain crossed product C^*-algebra.

5.2.1 Nekrashevych's approach via crossed products

We begin this section defining a gauge action on the Cuntz-Pimsner C^*-algebra $C^*(\Gamma, X)$ associated to a self-similar group. This action will be used later in order to compute the K-theory of $C^*(\Gamma, X)$.

Definition 5.10. *Let (Γ, X) be a self-similar group, and denote $C^*(\Gamma, X)$ its associated Cuntz-Pimsner algebra. Then, for each $z \in \mathbb{T}$, we define*

$$z \cdot (U_g) = U_g, \text{ and}$$
$$z \cdot (\mathcal{S}_x) = z\mathcal{S}_x,$$

for all $g \in \Gamma$, and all $x \in X$. One can notice that this extends to an automorphism of $C^*(\Gamma, X)$, and defines an action of $\mathbb{T}$ on $C^*(\Gamma, X)$ [42]. This is called the ***gauge action****.* □

Now, given a self-similar group (Γ, X), consider the universal C^*-algebra $\mathcal{M}_\Gamma$ generated by elements of the form $\mathcal{S}_v U_g \mathcal{S}_u^*$, with $g \in \Gamma$ and $v, u \in X^*$ such that $|v| = |u|$, together with the relations given by:

1. $g \mapsto U_g$ is a unitary representation.
2. $\mathcal{S}_v U_g \mathcal{S}_u^* = \sum\limits_{x \in X} \mathcal{S}_{vg(x)} U_{g|_x} \mathcal{S}_{ux}^*$, and
3. For $|u_1| = |v_2|$, we have $\mathcal{S}_{v_1} U_{g_1} \mathcal{S}_{u_1}^* \mathcal{S}_{v_2} U_{g_2} \mathcal{S}_{u_2}^* = \mathcal{S}_{v_1} U_{g_1 g_2} \mathcal{S}_{u_2}^*$, whenever $u_1 = v_2$, and 0 otherwise.

Since the above relations are satisfied in $C^*(\Gamma, X)$, we deduce that the gauge-invariant sub-algebra of $C^*(\Gamma, X)$ is a quotient of $\mathcal{M}_\Gamma$. In fact, it was shown in [42, Theorem 3.7] that the gauge-invariant sub-algebra of $C^*(\Gamma, X)$ is isomorphic to $\mathcal{M}_\Gamma$.
From definition of $\mathcal{M}_\Gamma$, it follows that

$$\mathcal{M}_\Gamma \cong \mathcal{M}_\Gamma \otimes \mathbb{M}_d,$$

where $\mathbb{M}_d$ is the algebra of square matrices of size $d = |X|$ over $\mathbb{C}$. The isomorphism above is given by the map $\mathcal{S}_{xv} U_g \mathcal{S}_{yu}^* \mapsto \mathcal{S}_v U_g \mathcal{S}_u^* \otimes e_{x,y}^{(d)}$, for $x, y \in X$. It is then clear that

$$\mathcal{M}_\Gamma \cong \mathcal{M}_\Gamma \otimes \mathbb{M}_{d^n},$$

iterating the argument.

Any fixed $z \in X$ induces a non-unital homomorphism $E_z : \mathcal{M}_\Gamma \to \mathcal{M}_\Gamma$ given by

$$\mathcal{S}_v U_g \mathcal{S}_u^* \mapsto \mathcal{S}_{zv} U_g \mathcal{S}_{zu}^*.$$

If we use the identification $\mathcal{M}_\Gamma \cong \mathcal{M}_\Gamma \otimes \mathbb{M}_d$, then $E_z : \mathcal{M}_\Gamma \to \mathcal{M}_\Gamma \otimes \mathbb{M}_d$ is given by:

$$\mathcal{S}_v U_g \mathcal{S}_u^* \mapsto \mathcal{S}_v U_g \mathcal{S}_u^* \otimes e_{z,z}^{(d)}.$$

Let us denote by $\mathcal{B}_\Gamma$ the direct limit of

$$\mathcal{M}_\Gamma \xrightarrow{\quad E_z \quad} \mathcal{M}_\Gamma \xrightarrow{\quad E_z \quad} \mathcal{M}_\Gamma \xrightarrow{\quad E_z \quad} \mathcal{M}_\Gamma \xrightarrow{\quad E_z \quad} \cdots \longrightarrow \mathcal{B}_\Gamma$$

We have that $\mathcal{B}_\Gamma \cong \mathcal{M}_\Gamma \otimes \mathbb{K}$. Moreover, the map E_z induces an automorphism of $\mathcal{M}_\Gamma \otimes \mathbb{K}$, which will be denoted by $\hat{E}_z$.

This automorphism plays a key role in the study of the K-theory of $C^*(\Gamma, X)$. The following result appears in [42, Theorem 3.7]. We show here a sketch of the proof.

Theorem 5.11. *Let $C^*(\Gamma, X)$ be the Cuntz-Pimsner C^*-algebra associated to the self-similar group (Γ, X). Then we have an isomorphism*

$$C^*(\Gamma, X) \otimes \mathbb{K} \cong (\mathcal{M}_\Gamma \otimes \mathbb{K}) \rtimes_{\hat{E}_z} \mathbb{Z}$$

Proof. Fix $z \in X$, and denote by $\mathcal{M}_\Gamma^{(n)}$ the n-th component of the direct limit $\varinjlim(\mathcal{M}_\Gamma \otimes \mathbb{M}_{d^n}, E_z)$. Denote by Λ_n the subalgebra of $(\mathcal{M}_\Gamma \otimes \mathbb{K}) \rtimes_{\hat{E}_z} \mathbb{Z}$ generated by $\mathcal{M}_\Gamma^{(n)}$ and the elements:

$$T_x^{(n)} = \mathcal{S}_x \mathcal{S}_z^* u,$$

where u is the generator of $\mathbb{Z}$, and $\mathcal{S}_x \mathcal{S}_{z^*}$ is seen as an element of $\mathcal{M}_\Gamma^{(n)}$.

The subalgebras Λ_n are naturally isomorphic to $C^*(\Gamma, X)$ under the identification $T_x^{(n)} \mapsto \mathcal{S}_x$, and $S_v U_g S_u^* \mapsto S_v U_g S_u^*$, for $S_v U_g S_u^* \in \mathcal{M}_\Gamma^{(n)} \subseteq \Lambda_n$ on the left-hand side, and $S_v U_g S_u^* \in C^*(\Gamma, X)$ on the right-hand side of the equality. Moreover, the maps from Λ_n to Λ_{n+1}, after identifications of $C^*(\Gamma, X)$ with Λ_n and Λ_{n+1}, given by

$$\mathcal{S}_v U_g \mathcal{S}_u^* \mapsto \mathcal{S}_z \mathcal{S}_v U_g \mathcal{S}_u^* \mathcal{S}_z^*$$
$$\mathcal{S}_x \mapsto \mathcal{S}_z \mathcal{S}_x \mathcal{S}_z^*$$

are all injective.

It can be proved that the direct limit of these embeddings is isomorphic to $C^*(\Gamma, X) \otimes \mathbb{K}$, and $\bigcup \Lambda_n$ is dense in $(\mathcal{M}_\Gamma \otimes \mathbb{K}) \rtimes_{\hat{E}_z} \mathbb{Z}$, as desired. $\square$

Alternatively, we can build $\mathcal{M}_\Gamma$ as the following inductive limit. Let $\mathbb{C}(\Gamma)$ be the usual group algebra over $\mathbb{C}$. Let $d = |X|$, and note that we can embed both the group $*$-algebra $C^*(\Gamma)$ and $\mathcal{O}_d$ into $C^*(\Gamma, X)$, where $\mathcal{O}_d$ denotes the usual Cuntz algebra generated by d isometries. Now consider the map:

$$\begin{aligned}\phi_0 : \mathbb{C}(\Gamma) &\to \mathbb{C}(\Gamma) \otimes \mathbb{M}_d \\ U_g &\mapsto \sum_{x \in X} U_{g|_x} \otimes e^{(d)}_{y,x},\end{aligned}$$

when $g \cdot x = y \cdot g|_x$. We can then define ϕ_n by:

$$\begin{aligned}\phi_n : \mathbb{C}(\Gamma) \otimes \mathbb{M}_{d^n} &\to \mathbb{C}(\Gamma) \otimes \mathbb{M}_{d^{n+1}} \\ U_g \otimes e^{(d^n)}_{v,w} &\mapsto \sum_{x \in X} U_{g|_x} \otimes e^{(d^{n+1})}_{vy,wx},\end{aligned}$$

for $g \in \Gamma$, $v, w \in X^n$ and $y \in X$ such that $g \cdot x = y \cdot g|_x$.
One can easily check that every ϕ_n is an $*$-homomorphism of algebras. Thus, the maps ϕ_n extend naturally to unital $*$-homomorphisms

$$\Phi_n : C^*(\Gamma) \otimes \mathbb{M}_{d^n} \to C^*(\Gamma) \otimes \mathbb{M}_{d^{n+1}}.$$

The maps Φ_n will be called the **matrix recursion maps**.
Then

$$\mathcal{M}_\Gamma \cong \varinjlim (C^*(\Gamma) \otimes \mathbb{M}_{d^n}, \Phi_n)$$

under the identification $\mathcal{S}_v U_g \mathcal{S}_u^* \mapsto U_g \otimes e^{(d^n)}_{v,u}$, where $|v| = |u| = n$ ([42, Proposition 3.8]).

5.2.2 K-theory of the groupoid associated to the self-similar infinite dihedral group

In this section, we use Nekrashevych's approach in order to study the infinite dihedral self-similar group $(\mathcal{D}_\infty, X)$. Let us start with the basics:
The K-theory of $C^*(\mathcal{D}_\infty)$ is well-known. Indeed, the group $\mathcal{D}_\infty$ can be represented as the free product $\mathbb{Z}_2 * \mathbb{Z}_2$. Then, using for example [32, Theorem 5.4] , we have the short exact sequences:

$$0 \longrightarrow K_0(\mathbb{C}) \xrightarrow{(K_0(\chi_1), -K_0(\chi_2))} K_0(C^*(\mathbb{Z}_2)) \oplus K_0(C^*(\mathbb{Z}_2)) \xrightarrow{K_0(\epsilon_1) + K_0(\epsilon_2)} K_0(C^*(\mathcal{D}_\infty)) \to 0$$

$$0 \longrightarrow K_1(\mathbb{C}) \xrightarrow{(K_1(\chi_1), -K_1(\chi_2))} K_1(C^*(\mathbb{Z}_2)) \oplus K_1(C^*(\mathbb{Z}_2)) \xrightarrow{K_1(\epsilon_1) + K_1(\epsilon_2)} K_1(C^*(\mathcal{D}_\infty)) \to 0$$

where χ_i denote the embeddings of $\mathbb{C}$ into the respective algebras $C^*(\mathbb{Z}_2)$, and ϵ_i are the embeddings of each $C^*(\mathbb{Z}_2)$ into $C^*(\mathcal{D}_\infty)$. Then the following lemma is straightforward to check:

Lemma 5.12. *Let $\mathcal{D}_\infty$ be the infinite dihedral group. Then*

$$K_0(C^*(\mathcal{D}_\infty)) = \langle [1], [\tfrac{1+U_a}{2}], [\tfrac{1+U_b}{2}] \rangle \cong \mathbb{Z}^3$$
$$K_1(C^*(\mathcal{D}_\infty)) = 0$$

where U_a, U_b are the unitaries associated to a, b respectively. $\square$

For the self-similar infinite dihedral group $(\mathcal{D}_\infty, X)$, the matrix recursion maps are of the form:

$$\Phi_0 : C^*(\mathcal{D}_\infty) \to C^*(\mathcal{D}_\infty) \otimes \mathbb{M}_2$$
$$U_a \mapsto 1 \otimes e^{(2)}_{0,1} + 1 \otimes e^{(2)}_{1,0}$$
$$U_b \mapsto U_a \otimes e^{(2)}_{0,0} + U_b \otimes e^{(2)}_{1,1}\,,$$

and in general

$$\Phi_n : C^*(\mathcal{D}_\infty) \otimes \mathbb{M}_{2^n} \to C^*(\mathcal{D}_\infty) \otimes \mathbb{M}_{2^{n+1}}$$
$$U_a \otimes e^{(2^n)}_{v,w} \mapsto 1 \otimes e^{(2^{n+1})}_{v0,w1} + 1 \otimes e^{(2^{n+1})}_{v1,w0}$$
$$U_b \otimes e^{(2^n)}_{v,w} \mapsto U_a \otimes e^{(2^{n+1})}_{v0,w0} + U_b \otimes e^{(2^{n+1})}_{v1,w1}\,.$$

In particular, we will use the characterization

$$\mathcal{M}_{\mathcal{D}_\infty} \cong \varinjlim (C^*(\mathcal{D}_\infty) \otimes \mathbb{M}_{2^n}, \Phi_n)\,.$$

given in paragraph 5.2.1.

Proposition 5.13. *Let $(\mathcal{D}_\infty, X)$ be the self-similar infinite dihedral group. Then we have that:*

$$K_0(\mathcal{M}_{\mathcal{D}_\infty}) \cong \mathbb{Z}[\tfrac{1}{2}] \oplus \mathbb{Z}, \textit{ and}$$
$$K_1(\mathcal{M}_{\mathcal{D}_\infty}) = 0$$

Moreover, the isomorphism Ψ_0 between $K_0(\mathcal{M}_{\mathcal{D}_\infty})$ and $\mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}$ is given by:

$$\Psi_0([\Phi_{n,\infty}(1 \otimes e^{(2^n)}_{v,v})]) = (\tfrac{1}{2^n}, 0),$$
$$\Psi_0([\Phi_{n,\infty}(\tfrac{1+U_a}{2} \otimes e^{(2^n)}_{v,v})]) = (\tfrac{1}{2^{n+1}}, 0),$$
$$\Psi_0([\Phi_{n,\infty}(\tfrac{1+U_b}{2} \otimes e^{(2^n)}_{v,v})]) = (-\sum_{k=1}^{n} \tfrac{1}{2^{k+1}}, 1)$$

for every $v \in X^n$.

Proof. Since K_* is a continuous functor and $K_1(C^*(\mathcal{D}_\infty)) = 0$, one has the equality $K_1(\mathcal{M}_{\mathcal{D}_\infty}) = 0$ trivially. Let us now focus our effort into the computation of $K_0(\mathcal{M}_{\mathcal{D}_\infty})$. This will be computed using again the continuity of K_0, meaning that $K_0(\mathcal{M}_{\mathcal{D}_\infty}) \cong \varinjlim(K_0(C^*(\mathcal{D}_\infty) \otimes \mathbb{M}_{2^n}), \Phi_n^*)$, where Φ_n^* is the homomorphism in the K_0 groups induced by Φ_n. In particular, if we denote $P := \frac{1+U_a}{2}$, $Q := \frac{1+U_b}{2}$, then Φ_0^* is given by:

$$\Phi_0^*([1]) = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = 2[1] \in K_0(C^*(\mathcal{D}_\infty)),$$

$$\Phi_0^*([P]) = \left[\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right] = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = [1] \in K_0(C^*(\mathcal{D}_\infty)),$$

and

$$\Phi_0^*([Q]) = \left[\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}\right] = [P] + [Q] \in K_0(C^*(\mathcal{D}_\infty)).$$

For a general $n \geq 0$, if we consider $1_n := \Phi_{n,\infty}(1 \otimes e^{(2^n)}_{0^n,0^n})$, $P_n := \Phi_{n,\infty}(P \otimes e^{(2^n)}_{0^n,0^n})$ and $Q_n := \Phi_{n,\infty}(Q \otimes e^{(2^n)}_{0^n,0^n})$, we obtain the following relations in $K_0(\mathcal{M}_{\mathcal{D}_\infty})$:

$$\begin{aligned} [1_n] &= 2[1_{n+1}], \\ [P_n] &= [1_{n+1}], \\ [Q_n] &= [Q_{n+1}] + [P_{n+1}] = [Q_{n+1}] + [1_{n+2}]. \end{aligned}$$

We notice that $K_0(\mathcal{M}_{\mathcal{D}_\infty})$ is generated by the classes $[1_n]$ for $n \in \mathbb{N}$, and $[Q_0]$, since all $[Q_n]$ can be generated in an inductive way, using $[Q_n] = [Q_{n-1}] - [1_{n+1}]$, for $n > 0$. The identifications $[1_n] \mapsto (\frac{1}{2^n}, 0)$ and $[Q_0] \mapsto (0,1)$ determine the desired isomorphism between $K_0(\mathcal{M}_{\mathcal{D}_\infty})$ and $\mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}$. $\square$

The map $E_0 : \mathcal{M}_{\mathcal{D}_\infty} \to \mathcal{M}_{\mathcal{D}_\infty}$ defined in section 5.2.1 induces a map

$$\mathcal{E} : \varinjlim(C^*(\mathcal{D}_\infty) \otimes \mathbb{M}_{2^n}, \Phi_n) \to \varinjlim(C^*(\mathcal{D}_\infty) \otimes \mathbb{M}_{2^n}, \Phi_n)$$

given by

$$\Phi_{n,\infty}(x \otimes e^{(2^n)}_{v,w}) \mapsto \Phi_{n+1,\infty}(x \otimes e^{(2^{n+1})}_{0v,0w}).$$

With a slight abuse of notation, we define

$$\mathcal{B}_{\mathcal{D}_\infty} := \varinjlim(\mathcal{M}_{\mathcal{D}_\infty}, \mathcal{E}) \cong \mathcal{M}_{\mathcal{D}_\infty} \otimes \mathbb{K}.$$

Then clearly $K_0(\mathcal{M}_{\mathcal{D}_\infty})$ is isomorphic to $K_0(\mathcal{B}_{\mathcal{D}_\infty})$ under the map induced by the inclusion $m \mapsto \mathcal{E}_{0,\infty}(m)$, for $m \in \mathcal{M}_{\mathcal{D}_\infty}$.
Finally, we define the automorphism $\hat{\mathcal{E}} : \mathcal{B}_{\mathcal{D}_\infty} \to \mathcal{B}_{\mathcal{D}_\infty}$ given by:

$$\hat{\mathcal{E}}(\mathcal{E}_{k,\infty}(\Phi_{n,\infty}(x \otimes e_{v,w}^{(2^n)}))) = \mathcal{E}_{k-1,\infty}(\Phi_{n,\infty}(x \otimes e_{v,w}^{(2^n)})) = \mathcal{E}_{k,\infty}(\Phi_{n+1,\infty}(x \otimes e_{0v,0w}^{(2^{n+1})}))$$

Then, by Theorem 5.11, $\mathcal{B}_{\mathcal{D}_\infty} \rtimes_{\hat{\mathcal{E}}} \mathbb{Z}$ is isomorphic to $C^*(\mathcal{D}_\infty, X) \otimes \mathbb{K}$. Hence, using Proposition 5.9, and since $\mathcal{G}_{(\mathcal{D}_\infty,X)}$ is amenable, we deduce that

$$C_r^*(\mathcal{G}_{(\mathcal{D}_\infty,X)}) \otimes \mathbb{K} \cong \mathcal{B}_{\mathcal{D}_\infty} \rtimes_{\hat{\mathcal{E}}} \mathbb{Z}.$$

All is left to do is computing the K-theory of $\mathcal{B}_{\mathcal{D}_\infty} \rtimes_{\hat{\mathcal{E}}} \mathbb{Z}$.

Lemma 5.14. *Let $(\mathcal{D}_\infty, X)$ be the infinite dihedral self-similar group. Then the K-theory of $\mathcal{B}_{\mathcal{D}_\infty} \rtimes_{\hat{\mathcal{E}}} \mathbb{Z}$ is given by:*

$$K_0(\mathcal{B}_{\mathcal{D}_\infty} \rtimes_{\hat{\mathcal{E}}} \mathbb{Z}) \cong coker(1 - K_0(\hat{\mathcal{E}})) \cong \mathbb{Z}, \text{ and}$$
$$K_1(\mathcal{B}_{\mathcal{D}_\infty} \rtimes_{\hat{\mathcal{E}}} \mathbb{Z}) \cong ker(1 - K_0(\hat{\mathcal{E}})) \cong \mathbb{Z},$$

where $K_0(\hat{\mathcal{E}}) : K_0(\mathcal{B}_{\mathcal{D}_\infty}) \to K_0(\mathcal{B}_{\mathcal{D}_\infty})$ is the map induced in K_0 by $\hat{\mathcal{E}}$.

Proof. Using the Pimsner-Voiculescu exact sequence given in Theorem 3.22, we have

$$\begin{tikzcd}
K_0(\mathcal{B}_{\mathcal{D}_\infty}) \arrow[r, "id - K_0(\hat{\mathcal{E}})"] & K_0(\mathcal{B}_{\mathcal{D}_\infty}) \arrow[r] & K_0(\mathcal{B}_{\mathcal{D}_\infty} \rtimes_{\hat{\mathcal{E}}} \mathbb{Z}) \arrow[d] \\
K_1(\mathcal{B}_{\mathcal{D}_\infty} \rtimes_{\hat{\mathcal{E}}} \mathbb{Z}) \arrow[u] & K_1(\mathcal{B}_{\mathcal{D}_\infty}) \arrow[l] & K_1(\mathcal{B}_{\mathcal{D}_\infty}) \arrow[l, "id - K_1(\hat{\mathcal{E}})"']
\end{tikzcd}$$

Now, since $K_1(\mathcal{B}_{\mathcal{D}_\infty}) = K_1(\mathcal{M}_{\mathcal{D}_\infty}) = 0$, we have that $K_0(\mathcal{B}_{\mathcal{D}_\infty} \rtimes_{\hat{\mathcal{E}}} \mathbb{Z}) \cong coker(1 - K_0(\hat{\mathcal{E}}))$ and $K_1(\mathcal{B}_{\mathcal{D}_\infty} \rtimes_{\hat{\mathcal{E}}} \mathbb{Z}) \cong ker(1 - K_0(\hat{\mathcal{E}}))$. In order to prove the second set of isomorphisms, and using Lemma 4.7, we then need to study the map $K_0(\mathcal{E})$.

With the notation used in Proposition 5.13, the set $\{[1_n], [Q_0]\}_{n\in\mathbb{N}}$ generates $K_0(\mathcal{B}_{\mathcal{D}_\infty})$. Then we have:

$$K_0(\mathcal{E})([1_n]) = [\mathcal{E}(1_n)] = [1_{n+1}], \text{ and}$$
$$K_0(\mathcal{E})([Q_n]) = [\mathcal{E}(Q_n)] = [Q_{n+1}] = [Q_n] - [1_{n+2}],$$

for every $n \in \mathbb{N}$. Using the identification $K_0(\mathcal{M}_{\mathcal{D}_\infty}) \cong \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}$, with $[1_n] \mapsto (\frac{1}{2^n}, 0)$, and $[Q_n] \mapsto (0, 1)$, we have that $K_0(\mathcal{E}) : \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z} \to \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}$ is given by

$$K_0(\mathcal{E})(\tfrac{1}{2^n}, 0) = (\tfrac{1}{2^{n+1}}, 0),$$
$$K_0(\mathcal{E})(0, 1) = (-\tfrac{1}{2^2}, 1).$$

Therefore:

$$(1 - K_0(\mathcal{E}))(\tfrac{1}{2^n}, 0) = (\tfrac{1}{2^{n+1}}, 0),$$
$$(1 - K_0(\mathcal{E}))(0, 1) = (\tfrac{1}{2^2}, 0).$$

In particular, the kernel and image are given by:

$$ker(1 - K_0(\hat{\mathcal{E}})) \cong ker(1 - K_0(\mathcal{E})) = \langle (\tfrac{1}{2}, -1) \rangle \cong \mathbb{Z}, \text{ and}$$
$$Im(1 - K_0(\mathcal{E})) = (\mathbb{Z}[\tfrac{1}{2}], 0).$$

Hence $coker(1 - K_0(\hat{\mathcal{E}})) \cong coker(1 - K_0(\mathcal{E})) \cong \mathbb{Z}$, concluding the proof.

□

The following corollary is then immediate, after using the results obtained at the beginning of this chapter.

Corollary 5.15. *Let $(\mathcal{D}_\infty, X)$ be the self-similar infinite dihedral group, and let $\mathcal{G}_{(\mathcal{D}_\infty, X)}$ be its associated groupoid of germs. Then*

$$K_0(C_r^*(\mathcal{G}_{(\mathcal{D}_\infty, X)})) \cong K_1(C_r^*(\mathcal{G}_{(\mathcal{D}_\infty, X)})) \cong \mathbb{Z}.$$

□

5.3 Homology groups of a groupoid associated to a self-similar group

In this section, we provide a strategy for the computation of the homology groups defined in paragraph 2.2 of the groupoid associated to a self-similar group. To do so, we will make some use of the *skew-product* construction defined in chapter 3.

Lemma 5.16. *Let (Γ, X) be a self-similar group. Then*

$$\begin{gathered} c : \mathcal{G}_{(\Gamma,X)} \longrightarrow \mathbb{Z} \\ [\alpha, g, \beta; \beta x] \mapsto |\alpha| - |\beta| \end{gathered}$$

is a cocycle. Moreover, the subgroupoid $\mathcal{H}_{(\Gamma,X)} := c^{-1}(0) = \{[\alpha, g, \beta; \beta x] : |\alpha| = |\beta|\}$ is Kakutani equivalent to the skew-product groupoid $\mathcal{G}_{(\Gamma,X)} \times_c \mathbb{Z}$.

Proof. We begin by proving that c is a cocycle. First, we need to see that it is well defined:
Take $[\alpha, g, \beta; \beta x] = [\alpha', g', \beta'; \beta' x']$. By definition, we get that $\beta x = \beta' x'$. In particular, we can find a finite word $t \in X^*$ such that either $\beta = \beta' t$ or $\beta' = \beta t$ is true. Without any loss of generality, suppose that the latter equality, $\beta' = \beta t$, holds. Since $\beta x = \beta' x' = \beta t x'$, we deduce that $x = tx'$. Then we can use Remark 5.8 to write $[\alpha, g, \beta; \beta x] = [\alpha, g, \beta; \beta t x'] = [\alpha g \cdot t, g|_t, \beta t; \beta t x']$. Thus, we can assume that $\beta = \beta'$, and just study the case $[\alpha, g, \beta; \beta x] = [\alpha', g', \beta; \beta x]$. By definition, those two classes are equal if and only if $S_{(\alpha,g,\beta)}$ and $S_{(\alpha',g',\beta)}$ coincide in a neighbourhood U of βx. Without loss of generality, assume that $U = Z(\beta x_0)$, where $x = x_0 x'$ for some $x' \in X^\infty$. Then, for any $y \in X^\infty$, the elements $S_{(\alpha,g,\beta)}(\beta x_0 y) = \alpha g \cdot x_0 y$, and $S_{(\alpha',g',\beta)}(\beta x_0 y) = \alpha' g' \cdot x_0 y$ must be equal. Denote $A = \alpha g(x_0)$, $A' = \alpha' g'(x_0)$. Then the previous equality can be written as $Ag|_{x_0} \cdot y = A' g'|_{x_0} \cdot y$. Hence, we deduce that either $A' = A\alpha_0$, $A = A'\alpha_0$, or $A = A'$, for some α_0 such that $|\alpha_0| > 0$.
Suppose that the first statement is true. Then, for every $y \in X^\infty$ we have the equality $Ag|_{x_0} \cdot y = A\alpha_0 g'|_{x_0} \cdot y$, and therefore $g|_{x_0} \cdot y = \alpha_0 g'|_{x_0} \cdot y$, for every $y \in X^\infty$. But we can always choose $y = y_0 y'$ such that $g|_{x_0}(y_0)$ is different than the first letter of α_0, arriving at a contradiction. Thus, $A' \neq A\alpha_0$. In a similar fashion, we can also discard the equality $A = A'\alpha_0$, concluding that $A = A'$, that is, $\alpha g(x_0) = \alpha' g'(x_0)$, and therefore $|\alpha| = |\alpha'|$. Hence, the difference $|\alpha| - |\beta|$ remains constant, and the map c is well-defined.
Let us see that c is a groupoid homomorphism:
It is clear that c maps any $[\emptyset, e, \emptyset; x] \in \mathcal{G}^{(0)}_{(\Gamma,X)}$ to 0. Also, recall that

$$[\alpha', g', \alpha; \alpha(g \cdot x)][\alpha, g, \beta; \beta x] = [\alpha', g'g, \beta; \beta x],$$

and so we have:

$$c([\alpha', g', \alpha; \alpha(g\cdot x)][\alpha, g, \beta; \beta x]) = c([\alpha', gg', \beta; \beta x]) = |\alpha'| - |\beta|, \text{ and}$$
$$c([\alpha', g', \alpha); \alpha(g\cdot x)]) + c([\alpha, g, \beta; \beta x]) = |\alpha'| - |\alpha| + |\alpha| - |\beta| = |\alpha'| - |\beta|.$$

Indeed, c is a groupoid homomorphism.

To check Kakutani equivalence, notice that $\mathcal{G}^{(0)}_{(\Gamma,X)} \times \{0\} \subseteq (\mathcal{G}_{(\Gamma,X)} \times_c \mathbb{Z})^{(0)}$ is a full clopen subset of $\mathcal{G}_{(\Gamma,X)} \times_c \mathbb{Z}$, and then $(\mathcal{G}_{(\Gamma,X)} \times_c \mathbb{Z})|_{\mathcal{G}^{(0)}_{(\Gamma,X)}\times\{0\}} \cong \mathcal{H}_{(\Gamma,X)}$, under the map given by:

$$[\alpha, g, \beta; \beta x] \times \{0\} \mapsto [\alpha, g, \beta; \beta x].$$

Notice that $|\alpha| \neq |\beta|$ implies that $c([\alpha, g, \beta; \beta x]) = |\alpha| - |\beta| \neq 0$, and therefore $s([\alpha, g, \beta; \beta x] \times \{0\}) = (s([\alpha, g, \beta; \beta x]), |\alpha| - |\beta|) \notin (\mathcal{G} \times_c \mathbb{Z})|_{\mathcal{G}^{(0)}_{(\Gamma,X)}\times\{0\}}$. Hence, $[\alpha, g, \beta; \beta x] \times \{0\} \in (\mathcal{G} \times_c \mathbb{Z})|_{\mathcal{G}^{(0)}_{(\Gamma,X)}\times\{0\}}$ if and only if $|\alpha| = |\beta|$, and so the map above is a well defined groupoid homomorphism. $\square$

We can then study the homology of $\mathcal{G}_{(\Gamma,X)}$ combining this result with Theorem 3.18. To do so, we decompose $\mathcal{H}_{(\Gamma,X)}$ as the union $\bigcup\limits_{n\in\mathbb{N}} (\mathcal{H}_{(\Gamma,X)})_n$, where $(\mathcal{H}_{(\Gamma,X)})_n$ is defined as the subgroupoid of all equivalence classes $[\alpha, g, \beta; \beta x]$ that have a representative such that $|\alpha| = |\beta| = n$.

Remark 5.8 shows that $(\mathcal{H}_{(\Gamma,X)})_n \subseteq (\mathcal{H}_{(\Gamma,X)})_{n+1}$. Denoting by $\iota_{n,m}$ the inclusion map from $(\mathcal{H}_{(\Gamma,X)})_n$ into $(\mathcal{H}_{(\Gamma,X)})_m$, and $\iota_n := \iota_{n,n+1}$, we have that

$$\iota_{n,m} : (\mathcal{H}_{(\Gamma,X)})_n \to (\mathcal{H}_{(\Gamma,X)})_m$$

is given by $[\alpha, g, \beta; \beta\gamma x] \mapsto [\alpha(g\cdot\gamma), g|_\gamma, \beta\gamma; \beta\gamma x]$, for $\alpha, \beta \in X^n$, $\gamma \in X^{m-n}$, $x \in X^\infty$ and $g \in \Gamma$.

Since the homology is a continuous functor [36], we can compute it as the direct limit $H_*(\mathcal{H}_{(\Gamma,X)}) \cong \varinjlim(H_*((\mathcal{H}_{(\Gamma,X)})_n), (\iota_n)_*)$, where $(\iota_n)_*$ is the induced map in homology.

The topology of $(\mathcal{H}_{(\Gamma,X)})_n$ has a basis of open compact sets of the form $[\alpha, g, \beta; Z(\beta\beta')]$, with $\alpha, \beta \in X^n$, $\beta' \in X^*$.

5.3.1 Homology of the groupoid associated to the self-similar infinite dihedral group.

In this subsection, we study the homology groups of $\mathcal{G}_{(\mathcal{D}_\infty,X)}$. We first compute the lower homology groups in a separate way, and then we prove that the higher ones are all torsion groups.

Lemma 5.17. *Let $(\mathcal{D}_\infty, X)$ be the self-similar infinite dihedral group, and let $\mathcal{G}_{(\mathcal{D}_\infty,X)}$ be its associated groupoid of germs. Consider $\mathcal{H}_{(\mathcal{D}_\infty,X)}$ as in Lemma 5.16. Then $H_0(\mathcal{H}_{(\mathcal{D}_\infty,X)}) \cong \mathbb{Z}[\frac{1}{2}]$.*

Proof. We identify $(\mathcal{H}_{(\mathcal{D}_\infty,X)})_n^{(0)}$ with X^∞ under $[\beta,e,\beta;\beta x]\mapsto \beta x$. Then the equivalence classes on $H_0((\mathcal{H}_{(\mathcal{D}_\infty,X)})_n)$ are generated by elements of the form $[1_{Z(\alpha)}]_0$, with $\alpha\in X^*$. We claim that $[1_{Z(\alpha)}]_0=[1_{Z(\beta)}]_0$ if and only if $|\alpha|=|\beta|$.
To show the forward implication, it is enough to observe that if $V=[\alpha_0,g,\beta_0;Z(\beta_0\beta')]$ is a basic compact open bisection of $(\mathcal{H}_{(\mathcal{D}_\infty,X)})_n$, with $|\alpha_0|=|\beta_0|=n$, then $s(V)=Z(\beta_0\beta')$, and $r(V)=Z(\alpha_0 g\beta')$, and so $|\beta_0\beta'|=|\alpha_0 g\beta'|$.
On the other hand, suppose that $|\alpha|=|\beta|$. We can assume $|\alpha|=|\beta|\geq n$. Set $\alpha=\alpha_0\alpha'$, $\beta=\beta_0\beta'$, with $|\alpha_0|=|\beta_0|=n$. Proposition 5.5 ensures the existence of some $g\in\mathcal{D}_\infty$ such that $g\cdot\beta'=\alpha'$. Take $V=[\alpha_0,g,\beta_0;Z(\beta_0\beta')]$. Then $s(V)=Z(\beta_0\beta')=Z(\beta)$, and $r(V)=Z(\alpha_0 g\beta')=Z(\alpha_0\alpha')=Z(\alpha)$, as desired.
Finally, since $1_{Z(\alpha)}=1_{Z(\alpha 0)}+1_{Z(\alpha 1)}$ for every $\alpha\in X^*$, it follows that $[1_{Z(0^n)}]_0=2[1_{Z(0^{n+1})}]_0$, for every $n\in\mathbb{N}$. Hence we deduce that the map $H_0((\mathcal{H}_{(\mathcal{D}_\infty,X)})_n)\to\mathbb{Z}[\frac{1}{2}]$ given by $[1_{Z(\alpha)}]_0\mapsto\frac{1}{2^{|\alpha|}}$ is an isomorphism for all $n\in\mathbb{N}$.
It remains to study the respective maps $(\iota_{n,m})_*$ in H_0. Those maps are given by

$$\begin{aligned}(\iota_{n,m})_*: H_0((\mathcal{H}_{(\mathcal{D}_\infty,X)})_n) &\to H_0((\mathcal{H}_{(\mathcal{D}_\infty,X)})_m)\\ [1_{Z(\alpha)}]_0 &\mapsto [1_{Z(\alpha)}]_0,\end{aligned}$$

and hence $(\iota_{n,m})_*$ are all equal to the identity map. Therefore

$$H_0(\mathcal{H}_{(\mathcal{D}_\infty,X)})\cong\varinjlim(H_*((\mathcal{H}_{(\mathcal{D}_\infty,X)})_n),(\iota_n)_*)\cong\varinjlim(\mathbb{Z}[\tfrac{1}{2}],id)\cong\mathbb{Z}[\tfrac{1}{2}],$$

concluding the proof. □

We now compute the H_1 group. Next lemma's original proof was developed with the help of professor Ortega in [47]. Here we show a slightly different one, developed afterwards.
Lemma 5.18. *Let $(\mathcal{D}_\infty,X)$ be the self-similar infinite dihedral group, and let $\mathcal{G}_{(\mathcal{D}_\infty,X)}$ be its associated groupoid of germs. Consider $\mathcal{H}_{(\mathcal{D}_\infty,X)}$ as in Lemma 5.16. Then $H_1(\mathcal{H}_{(\mathcal{D}_\infty,X)})\cong\mathbb{Z}_2$.*

Proof. By definition, we have

$$H_1((\mathcal{H}_{(\mathcal{D}_\infty,X)})_n):=ker\delta_1/Im\delta_2=\{[f]: f\in C_c((\mathcal{H}_{(\mathcal{D}_\infty,X)})_n,\mathbb{Z}), s.t.\ \delta_1(f)=0\}$$

with δ_1,δ_2 as defined in paragraph 2.2. Let $f\in C_c((\mathcal{H}_{(\mathcal{D}_\infty,X)})_n,\mathbb{Z})$ such that $\delta_1(f)=0$, and write it as

$$f=\sum_{i=0}^{k}\lambda_i 1_{Z(\alpha_i',g_i',\beta_i';U_i)},$$

with $\alpha_i',\beta_i'\in X^n$, $\lambda_i\in\mathbb{Z}$, $g_i'\in\mathcal{D}_\infty$ and U_i clopen subsets of $Z(\beta_i')$. Notice that, replacing f with $\iota_{n,m}(f)\in C_c((\mathcal{H}_{(\mathcal{D}_\infty,X)})_m,\mathbb{Z})$ for a large enough m, and using Remark 5.8, we can take

$$f = \sum_{i=0}^{k} \lambda_i 1_{Z(\alpha_i, g_i, \beta_i; Z(\beta_i))},$$

where $\alpha_i, \beta_i \in X^m$, $\lambda_i \in \mathbb{Z}$, and $g_i \in \mathcal{D}_\infty$.
Then, whenever $Z(\alpha, g, \beta; Z(\beta)) \times Z(\beta, h, \gamma; Z(\gamma)) \subset (\mathcal{H}_{(\mathcal{D}_\infty, X)})^{(2)}_m$, we have that

$$\delta_2(1_{(Z(\alpha,g,\beta;Z(\beta))\times Z(\beta,h,\gamma;Z(\gamma)))}) = 1_{Z(\alpha,g,\beta;Z(\beta))} + 1_{Z(\beta,h,\gamma;Z(\gamma))} - 1_{Z(\alpha,gh,\gamma;Z(\gamma))},$$

for every $\alpha, \beta, \gamma \in X^m$, $g, h \in \mathcal{D}_\infty$. Hence

$$[1_{Z(\alpha,gh,\gamma;Z(\gamma))}] = [1_{Z(\alpha,g,\beta;Z(\beta))}] + [1_{Z(\beta,h,\gamma;Z(\gamma))}] \in H_1((\mathcal{H}_{(\mathcal{D}_\infty,X)})_m). \tag{5.1}$$

In particular, if we choose $\beta = 0^m$, $h = e$ we obtain

$$[1_{Z(\alpha,g,\gamma;Z(\gamma))}] = [1_{Z(\alpha,g,0^m;Z(0^m))}] + [1_{Z(0^m,e,\gamma;Z(\gamma))}] \in H_1((\mathcal{H}_{(\mathcal{D}_\infty,X)})_m). \tag{5.2}$$

Using the same argument,

$$[1_{Z(\alpha,g,0^m;Z(0^m))}] = [1_{Z(\alpha,e,0^m;Z(0^m))}] + [1_{Z(0^m,g,0^m;Z(0^m))}] \in H_1((\mathcal{H}_{(\mathcal{D}_\infty,X)})_m),$$

and thus we get:

$$[1_{Z(\alpha,g,\gamma;Z(\gamma))}] = [1_{Z(\alpha,e,0^m;Z(0^m))}] + [1_{Z(0^m,g,0^m;Z(0^m))}] + [1_{Z(0^m,e,\gamma;Z(\gamma))}] \in H_1((\mathcal{H}_{(\mathcal{D}_\infty,X)})_m). \tag{5.3}$$

Moreover, if we consider $\alpha = \beta = \gamma$, and $g = h = e$ in equation (5.1), then we obtain:

$$[1_{Z(\alpha,e,\alpha;Z(\alpha))}] = [1_{Z(\alpha,e,\alpha;Z(\alpha))}] + [1_{Z(\alpha,e,\alpha;Z(\alpha))}] \in H_1((\mathcal{H}_{(\mathcal{D}_\infty,X)})_m).$$

We conclude that $[1_{Z(\alpha,e,\alpha;Z(\alpha))}] = 0$. Combining this result with equation (5.2), we also deduce that

$$[1_{Z(\alpha,e,0^m;Z(0^m))}] + [1_{Z(0^m,e,\alpha;Z(\alpha))}] = 0. \tag{5.4}$$

in $H_1((\mathcal{H}_{(\mathcal{D}_\infty,X)})_m)$. We then use all of the above results to find a representative h of $[f]$ of the form:

$$h = \sum_{i=0}^{k} \lambda_i 1_{Z(0^m,g_i,0^m;Z(0^m))} + \sum_{\substack{\alpha\in X^m \\ \alpha\neq 0^m}} \mu_\alpha 1_{Z(\alpha,e,0^m;Z(0^m))},$$

for $\lambda_i, \mu_\alpha \in \mathbb{Z}$. Using that $f, h \in ker\delta_1$, we have:

$$\delta_1(h) = \delta_1(\sum_{i=0}^{k} \lambda_i 1_{Z(0^m,g_i,0^m;Z(0^m))} + \sum_{\substack{\alpha\in X^m \\ \alpha\neq 0^m}} \mu_\alpha 1_{Z(\alpha,e,0^m;Z(0^m))}) = 0.$$

Notice that $r(Z(0^m, g_i, 0^m; Z(0^m))) = s(Z(0^m, g_i, 0^m; Z(0^m))) = Z(0^m)$, and therefore $\delta_1(1_{Z(0^m,g_i,0^m;Z(0^m))}) = 1_{s(Z(0^m,g_i,0^m;Z(0^m)))} - 1_{r(Z(0^m,g_i,0^m;Z(0^m)))} = 0$, for all $m \in \mathbb{N}$. Hence

$$\delta_1(h) = \delta_1(\sum_{\substack{\alpha\in X^m \\ \alpha\neq 0^m}} \mu_\alpha 1_{Z(\alpha,e,0^m;Z(0^m))})) = \sum_{\substack{\alpha\in X^m \\ \alpha\neq 0^m}} \mu_\alpha (1_{Z(\alpha)} - 1_{Z(0^m)}) = 0.$$

This forces $\mu_\alpha = 0$ for all $\alpha \in X^m \setminus \{0^m\}$.

We have proven that, for every $[f] \in H_1((\mathcal{H}_{(\mathcal{D}_\infty,X)})_n)$, there is a representative $\hat{f}$ of $[f]$ of the form:

$$\hat{f} = \sum_{i=0}^{k} \lambda_i 1_{Z(0^m,g_i,0^m;Z(0^m))}.$$

Applying equation (5.1) again, we obtain the relation

$$[1_{Z(0^m,g,0^m;Z(0^m))}] + [1_{Z(0^m,h,0^m;Z(0^m))}] = [1_{Z(0^m,gh,0^m;Z(0^m))}]$$

for all $g, h \in \mathcal{D}_\infty$. Since the homology groups are abelian groups, it follows that

$$[1_{Z(0^m,hg,0^m;Z(0^m))}] = [1_{Z(0^m,gh,0^m;Z(0^m))}].$$

Hence, since $\mathcal{D}_\infty/[\mathcal{D}_\infty, \mathcal{D}_\infty] = \langle \bar{a}, \bar{b} \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, we deduce:

$$[\hat{f}] = \lambda_a[1_{Z(0^m,a,0^m;Z(0^m))}] + \lambda_b[1_{Z(0^m,b,0^m;Z(0^m))}],$$

with $\lambda_a, \lambda_b \in \{0,1\}$. In particular, we deduce that $H_1(\mathcal{H}_{(\mathcal{D}_\infty,X)})$ is generated by $[1_{Z(0^m,a,0^m;Z(0^m))}]$ and $[1_{Z(0^m,b,0^m;Z(0^m))}]$, for $m \in \mathbb{N}$.

We now study their behaviour under the inductive limit

$$H_1(\mathcal{H}_{(\mathcal{D}_\infty,X)}) \cong \varinjlim(H_1((\mathcal{H}_{(\mathcal{D}_\infty,X)})_m), (\iota_m)_*),$$

that is, we need to study the maps

$$(\iota_{m,m+1})_* : H_1((\mathcal{H}_{(\mathcal{D}_\infty,X)})_m) \to H_1((\mathcal{H}_{(\mathcal{D}_\infty,X)})_{m+1})$$

Those are given by

$$\begin{aligned}(\iota_{m,m+1})_*([1_{Z(0^m,g,0^m;Z(0^m))}]) &= (\iota_{m,m+1})_*([1_{Z(0^m,g,0^m;Z(0^m0))} + 1_{Z(0^m,g,0^m;Z(0^m1))}]) = \\ &= [1_{Z(0^m(g\cdot 0),g|_0,0^{m+1};Z(0^{m+1}))} + 1_{Z(0^m(g\cdot 1),g|_1,0^m1;Z(0^m1))}].\end{aligned}$$

for every $g \in \mathcal{D}_\infty$. Evaluating this map for $g = a$, and using equation (5.4), we obtain:

$$(\iota_{m,m+1})_*([1_{Z(0^m,a,0^m;Z(0^m))}]) = [1_{Z(0^m1,e,0^{m+1};Z(0^{m+1}))}] + [1_{Z(0^{m+1},e,0^m1;Z(0^m1))}] = 0.$$

On the other hand, choosing $g = b$:

$$(\iota_{m,m+1})_*([1_{Z(0^m,b,0^m;Z(0^m))}]) = [1_{Z(0^{m+1},a,0^{m+1};Z(0^{m+1}))}] + [1_{Z(0^m1,b,0^m1;Z(0^m1))}].$$

Then, using equation (5.3), we have

$$[1_{Z(0^m1,b,0^m1;Z(0^m1))}] = [1_{Z(0^m1,e,0^{m+1};Z(0^{m+1}))}]+[1_{Z(0^{m+1},b,0^{m+1};Z(0^{m+1}))}]+[1_{Z(0^{m+1},e,0^m1;Z(0^m1))}],$$

which equals to $[1_{Z(0^{m+1},b,0^{m+1};Z(0^{m+1}))}]$ after applying equation (5.4). Therefore:

$$(\iota_{m,m+1})_*([1_{Z(0^m,b,0^m;Z(0^m))}]) = [1_{Z(0^{m+1},a,0^{m+1};Z(0^{m+1}))}] + [1_{Z(0^{m+1},b,0^{m+1};Z(0^{m+1}))}].$$

Gathering all of the above equalities, we conclude that, given $f \in C_c((\mathcal{H}_{\mathcal{D}_\infty,X})_n, \mathbb{Z})$ with $\delta_1(f) = 0$, there exists a large enough $m \in \mathbb{N}$ such that

$$[f] = (\iota_{n,m})_*([f]) = [\iota_{n,m}(f)] = [\lambda_b 1_{Z(0^m,b,0^m;Z(0^m))}] \in H_1(\mathcal{H}_{\mathcal{D}_\infty,X}),$$

with $\lambda_b \in \{0,1\}$. Suppose that, for every $m \in \mathbb{N}$, $[1_{Z(0^m,b,0^m;Z(0^m))}] \neq 0$ in $H_1((\mathcal{H}_{\mathcal{D}_\infty,X})_m)$. Then we could compute the homology as:

$$H_1(\mathcal{H}_{(\mathcal{D}_\infty,X)}) \cong \varinjlim(H_1((\mathcal{H}_{(\mathcal{D}_\infty,X)})_m), (\iota_m)_*) \cong \mathbb{Z}_2,$$

under the isomorphism given by

$$(\iota_{m,\infty})_*([1_{Z(0^m,b,0^m;Z(0^m))}]) \mapsto \bar{1},$$
$$(\iota_{m,\infty})_*([1_{Z(0^m,a,0^m;Z(0^m))}]) \mapsto \bar{0}.$$

It remains to be proven that $[1_{Z(0^m,b,0^m;Z(0^m))}]$ is indeed different than zero. Let us show it. For every $m \in \mathbb{N}$, $(\mathcal{H}_{(\mathcal{D}_\infty,X)})_m$ is Kakutani equivalent to $X^\infty \rtimes \mathcal{D}_\infty$, and so

$$H_*(X^\infty \rtimes \mathcal{D}_\infty) \cong H_*((\mathcal{H}_{(\mathcal{D}_\infty,X)})_m).$$

Indeed, for any $m \in \mathbb{N}$, the clopen $Z(0^m) \subseteq X^\infty$ is full in $(\mathcal{H}_{(\mathcal{D}_\infty,X)})_m$, and the elements of the restriction groupoid are of the form $[0^m, g, 0^m; 0^m x]$. Then the map $[0^m, g, 0^m; 0^m x] \mapsto (g, x)$ defines an isomorphism between $((\mathcal{H}_{(\mathcal{D}_\infty,X)})_m)|_{Z(0^m)}$ and $X^\infty \rtimes \mathcal{D}_\infty$.
Recall that, by Lemma 3.52, $H_n(X^\infty \rtimes \mathcal{D}_\infty) \cong H_n(\mathcal{D}_\infty, C(X^\infty))$, which is defined as $Tor_1^{\mathbb{Z}[\mathcal{D}_\infty]}(\mathbb{Z}, C(X^\infty))$ (see [64, Definition 6.1.2]). This isomorphism is natural, in the sense that the following diagram commutes (see [14]):

$$\begin{tikzcd} H_n(\mathcal{D}_\infty, C(X^\infty)) \arrow[r, "\cong"] & H_n(X^\infty \rtimes \mathcal{D}_\infty) \\ H_n(\langle b\rangle, C(X^\infty)) \arrow[u] \arrow[r, "\cong"'] & H_n(X^\infty \rtimes \langle b\rangle) \arrow[u] \end{tikzcd}$$

Let $\mathcal{I}_{\mathcal{D}_\infty}$ denote the augmentation ideal of $\mathbb{Z}[\mathcal{D}_\infty]$. It was shown in [64, Proposition 6.2.9] that

$$\mathcal{I}_{\mathcal{D}_\infty} \cong (\mathcal{I}_{\langle a\rangle} \otimes_{\mathbb{Z}[\langle a\rangle]} \mathbb{Z}[\mathcal{D}_\infty]) \oplus (\mathcal{I}_{\langle b\rangle} \otimes_{\mathbb{Z}[\langle b\rangle]} \mathbb{Z}[\mathcal{D}_\infty]).$$

Applying the functor $Tor_*^{\mathbb{Z}[\mathcal{D}_\infty]}(\cdot, C(X^\infty))$ to the exact sequence

$$0 \to \mathcal{I}_{\mathbb{Z}[\mathcal{D}_\infty]} \to \mathbb{Z}[\mathcal{D}_\infty] \to \mathbb{Z} \to 0$$

we obtain an exact sequence

$$\begin{tikzcd} 0 & \mathbb{Z} \otimes_{\mathbb{Z}[\mathcal{D}_\infty]} C(X^\infty) \arrow[l] & \mathbb{Z}[\mathcal{D}_\infty] \otimes_{\mathbb{Z}[\mathcal{D}_\infty]} C(X^\infty) \arrow[l] & \mathcal{I}_{\mathcal{D}_\infty} \otimes C(X^\infty) \arrow[l, "\omega"'] \\ \cdots \arrow[r] & Tor_1^{\mathbb{Z}[\mathcal{D}_\infty]}(\mathcal{I}_{\mathbb{Z}[\mathcal{D}_\infty]}, C(X^\infty)) \arrow[r] & Tor_1^{\mathbb{Z}[\mathcal{D}_\infty]}(\mathbb{Z}[\mathcal{D}_\infty], C(X^\infty)) \arrow[r] & H_1(\mathcal{D}_\infty, C(X^\infty)) \arrow[u, "\xi"'] \end{tikzcd} \tag{5.5}$$

Note that $\mathbb{Z}[\mathcal{D}_\infty]$ is flat, and therefore $Tor_1^{\mathbb{Z}[\mathcal{D}_\infty]}(\mathbb{Z}[\mathcal{D}_\infty], C(X^\infty)) = 0$. Hence, ξ is injective, and $H_1(\mathcal{D}_\infty, C(X^\infty)) \cong Im(\xi) = ker(\omega)$.
The kernel of ω is given by:

$$ker(\omega) = \{(1-a) \otimes_{\mathbb{Z}[\langle a\rangle]} f + (1-b) \otimes_{\mathbb{Z}[\langle b\rangle]} g : f + g = a \cdot f + b \cdot g\}.$$

Using the same argument,

$$H_1(\langle b\rangle, C(X^\infty)) = \{(1-b) \otimes_{\mathbb{Z}[\langle b\rangle]} g : g = b \cdot g\}.$$

Observe that given $N > 0$ we can write X^∞ as the disjoint union

$$X^\infty = (\bigsqcup_{n=0}^{N-1} (Z(1^n00) \sqcup Z(1^n01))) \sqcup Z(1^N)$$

Then, for any $g \in C(X^\infty, \mathbb{Z})$, we can write
$g = \sum_{n=0}^{N-1}(\sum_{\alpha\in X^*} (\lambda_{1^n00\alpha} 1_{Z(1^n00\alpha)} + \lambda_{1^n01\alpha} 1_{Z(1^n01\alpha)})) + \lambda_{1^N} 1_{Z(1^N)}$ for some $N > 0$ and with only finitely many λ's not equal to zero. If we consider that g is in the kernel of $(1-b)$, we have that

$$\lambda_{1^n00\alpha} = \lambda_{1^n01\alpha},$$

for each $n \in \mathbb{N}$, and each $\alpha \in X^*$. But $1_{Z(1^n00\alpha)} + 1_{Z(1^n01\alpha)} = (1+b)1_{Z(1^n00\alpha)}$, and therefore

$$(1-b) \otimes_{\mathbb{Z}[\langle b\rangle]} g = (1-b) \otimes_{\mathbb{Z}[\langle b\rangle]} \lambda_{1^N} 1_{Z(1^N)}.$$

Moreover,

$$(1-b) \otimes_{\mathbb{Z}[\langle b\rangle]} 2 \cdot 1_{Z(1^N)} = (1-b) \otimes_{\mathbb{Z}[\langle b\rangle]} (1+b) \cdot 1_{Z(1^N)} = 0,$$

hence $\lambda_{1^N} \in \{0, 1\}$. Finally, note that $1_{Z(1^N)} \setminus 1_{Z(1^{N+1})} = 1_{Z(1^N0)} = (1+b) \cdot 1_{Z(1^N00)}$. This implies that

$$(1-b) \otimes_{\mathbb{Z}[\langle b\rangle]} 1_{Z(1^N)} = (1-b) \otimes_{\mathbb{Z}[\langle b\rangle]} 1_{Z(1^M)}$$

for each N, M. Gathering all of the above information, we conclude that

$$H_1(\langle b\rangle, C(X^\infty)) \cong \mathbb{Z}_2,$$

generated by the element $(1-b)\otimes_{\mathbb{Z}[\langle b\rangle]} 1_{X^\infty}$. This element is not zero in $ker(\omega)$. Indeed, $(1-b)\otimes_{\mathbb{Z}[\langle b\rangle]} 1_{X^\infty}=0$ if and only if there exists some $g\in C(X^\infty)$ such that $1_{X^\infty}=(1+b)\cdot g$. Suppose that this is true. Then. evaluating in $1^\infty\in X^\infty$, we obtain

$$1=1_{X^\infty}(1^\infty)=(1+b)\cdot g(1^\infty)=g(1^\infty)+g(b\cdot 1^\infty)=2g(1^\infty),$$

which is impossible. Hence, $(1-b)\otimes_{\mathbb{Z}[\langle b\rangle]} 1_{X^\infty}\neq 0$.

It is not too difficult to check that, whenever ψ is a homeomorphism of a compact Hausdorff space X with ψ_* its induced map in $C(X,\mathbb{Z})$, such that $\psi^2=Id$, then the map

$$\begin{aligned}\chi: C(X,\mathbb{Z}) &\to C(X\rtimes\langle\psi\rangle,\mathbb{Z})\\ g&\mapsto \tilde{g}\end{aligned}$$

given by $\tilde{g}(\psi,x):=g(x)$, and $\tilde{g}(e,x)=0$ induces an isomorphism

$$H_1(\langle\psi\rangle,C(X,\mathbb{Z}))\overset{(*)}{\cong} ker(Id-\psi_*)/Im(Id+\psi_*)\overset{\chi}{\cong} H_1(X\rtimes\langle\psi\rangle),$$

where the isomorphism $(*)$, shown in [64, Theorem 6.2.2], is given by $(1-\psi)\otimes_{\mathbb{Z}[\langle\psi\rangle]} g\mapsto [g]$. In particular, the map

$$(1-b)\otimes_{\mathbb{Z}[\langle b\rangle]} 1_{X^\infty}\mapsto [1_{(b,X^\infty)}]$$

yields an isomorphism between $H_1(\langle b\rangle, C(X^\infty))$ and $H_1(X^\infty\rtimes\langle b\rangle)$, and hence

$$[1_{(b,X^\infty)}]\neq 0.$$

This, together with the commutative diagram shown above, gives us an isomorphism between $ker(\omega)\cong H_1(\mathcal{D}_\infty, C(X^\infty))$ and $H_1(X^\infty\rtimes\mathcal{D}_\infty)$ that sends $(1-b)\otimes_{\mathbb{Z}[\langle b\rangle]} 1_{X^\infty}$ to $[1_{(b,X^\infty)}]$. Finally, the Kakutani equivalence between $(\mathcal{H}_{(\mathcal{D}_\infty,X)})_n$ and $X^\infty\rtimes\mathcal{D}_\infty$ described above gives us an isomorphism that sends

$$0\neq[1_{(b,X^\infty)}]\mapsto[1_{Z(0^m,b,0^m;Z(0^m))}],$$

concluding the proof.

□

Once computed the low homology groups of $\mathcal{H}_{(\mathcal{D}_\infty,X)}$, we focus our study on the higher homology groups.

Lemma 5.19. *Let $(\mathcal{D}_\infty,X)$ be the self-similar infinite dihedral group, $\mathcal{G}_{(\mathcal{D}_\infty,X)}$ its associated groupoid of germs, and $\mathcal{H}_{(\mathcal{D}_\infty,X)}$ be as in Lemma 5.16. Then $H_{2k}(\mathcal{H}_{(\mathcal{D}_\infty,X)})=0$, and $H_{2k-1}(\mathcal{H}_{(\mathcal{D}_\infty,X)})\cong\mathbb{Z}_2$, for every $k\geq 1$.*

Proof. As noted before, for every $n \in \mathbb{N}$, $(\mathcal{H}_{(\mathcal{D}_\infty,X)})_n$ is Kakutani equivalent to $X^\infty \rtimes \mathcal{D}_\infty$, and so $H_*(X^\infty \rtimes \mathcal{D}_\infty) \cong H_*((\mathcal{H}_{(\mathcal{D}_\infty,X)})_n) \cong H_*(X^\infty \rtimes \mathcal{D}_\infty) \cong H_*(\mathcal{D}_\infty, C(X^\infty,\mathbb{Z}))$, where $C(X^\infty,\mathbb{Z})$ is a left $\mathcal{D}_\infty$-module by the induced action $\mathcal{D}_\infty \curvearrowright C(X^\infty,\mathbb{Z})$. Using [64, Corollary 6.2.10], we obtain an isomoprhism

$$H_k(\langle a\rangle * \langle b\rangle, C(X^\infty,\mathbb{Z})) \cong H_k(\langle a\rangle, C(X^\infty,\mathbb{Z})) \oplus H_k(\langle b\rangle, C(X^\infty,\mathbb{Z}))\,,$$

for every $k \geq 2$.

Observe that $H_*(\langle a\rangle, C(X^\infty,\mathbb{Z})) \cong H_*(X^\infty \rtimes \langle a\rangle)$. Note that $X^\infty \rtimes \langle a\rangle$ is an elementary groupoid (definition 2.40): it is clearly compact, and for any $(a,x) \in X^\infty \rtimes \langle a\rangle$, we have that $s(a,x) = (e,x)$, and $r(a,x) = (e, a\cdot x)$. For any $x \in X^\infty$, $x \neq a\cdot x$, and therefore there is no isotropy (i.e. $X^\infty \rtimes \langle a\rangle$ is principal). Hence, $X^\infty \rtimes \langle a\rangle$ is elementary. Therefore, $H_k(\langle a\rangle, C(X^\infty,\mathbb{Z})) = 0$ for $k \geq 1$.

Let's now show that $H_{2k}(\langle b\rangle, C(X^\infty,\mathbb{Z})) = 0$ and $H_{2k+1}(\langle b\rangle, C(X^\infty,\mathbb{Z})) \cong \mathbb{Z}_2$ for $k \geq 1$. It is shown in [64, Theorem 6.2.2] that

$$H_{2k}(\langle b\rangle, C(X^\infty,\mathbb{Z})) \cong ker(1+b)/(b-1)(C(X^\infty,\mathbb{Z}))$$

and

$$H_{2k-1}(\langle b\rangle, C(X^\infty,\mathbb{Z})) \cong ker(b-1)/(b+1)(C(X^\infty,\mathbb{Z}))\,,$$

for $k \geq 1$. We begin studying $H_{2k}(\langle b\rangle, C(X^\infty,\mathbb{Z}))$. As before, observe that given $N > 0$ we can write X^∞ as the disjoint union

$$X^\infty = (\bigsqcup_{n=0}^{N-1}(Z(1^n00) \sqcup Z(1^n01))) \sqcup Z(1^N)$$

Then, for any $f \in C(X^\infty,\mathbb{Z})$, we can write
$f = \sum_{n=0}^{N-1}(\sum_{\alpha\in X^*}(\lambda_{1^n00\alpha}1_{Z(1^n00\alpha)} + \lambda_{1^n01\alpha}1_{Z(1^n01\alpha)})) + \lambda_{1^N}1_{Z(1^N)}$ for some $N > 0$ and with only finitely many λ's not equal to zero. If we consider that f is in the kernel of $(1+b)$, we have that

$$0=(1+b)f=(1+b)(\sum_{n=0}^{N-1}(\sum_{\alpha\in X^*}(\lambda_{1^n00\alpha}1_{Z(1^n00\alpha)}+\lambda_{1^n01\alpha}1_{Z(1^n01\alpha)}))+\lambda_{1^N}1_{Z(1^N)})=$$

$$=\sum_{n=0}^{N-1}(\sum_{\alpha\in X^*}(\lambda_{1^n00\alpha}1_{Z(1^n00\alpha)}+\lambda_{1^n01\alpha}1_{Z(1^n01\alpha)}))+\lambda_{1^N}1_{Z(1^N)}+$$

$$+\sum_{n=0}^{N-1}(\sum_{\alpha\in X^*}(\lambda_{1^n00\alpha}1_{Z(b\cdot 1^n00\alpha)}+\lambda_{1^n01\alpha}1_{Z(b\cdot 1^n01\alpha)}))+\lambda_{1^N}1_{Z(b\cdot 1^N)}=$$

$$=\sum_{n=0}^{N-1}(\sum_{\alpha\in X^*}(\lambda_{1^n00\alpha}1_{Z(1^n00\alpha)}+\lambda_{1^n01\alpha}1_{Z(1^n01\alpha)}))+\lambda_{1^N}1_{Z(1^N)}+$$

$$+\sum_{n=0}^{N-1}(\sum_{\alpha\in X^*}(\lambda_{1^n00\alpha}1_{Z(1^n01\alpha)}+\lambda_{1^n01\alpha}1_{Z(1^n00\alpha)}))+\lambda_{1^N}1_{Z(1^N)}=$$

$$=\sum_{n=0}^{N-1}(\sum_{\alpha\in X^*}((\lambda_{1^n00\alpha}+\lambda_{1^n01\alpha})1_{Z(1^n00\alpha)}+(\lambda_{1^n01\alpha}+\lambda_{1^n00\alpha})1_{Z(1^n01\alpha)}))+2\lambda_{1^N}1_{Z(1^N)}\,.$$

But this forces $\lambda_{1^n01\alpha}+\lambda_{1^n00\alpha}=0$ and $\lambda_{1^N}=0$. Thus, we have that

$$f=\sum_{n=0}^{N-1}(\sum_{\alpha\in X^*}(\lambda_{1^n00\alpha}1_{Z(1^n00\alpha)}-\lambda_{1^n00\alpha}1_{Z(1^n01\alpha)}))=(1-b)(\sum_{n=0}^{N-1}\sum_{\alpha\in X^*}\lambda_{1^n00\alpha}1_{Z(1^n00\alpha)})$$

Hence, we have that $ker(1+b)=(1-b)(C(X^\infty,\mathbb{Z}))$, and so $H_{2k}(\langle b\rangle,C(X^\infty,\mathbb{Z})=0$. We now study $H_{2k-1}(\langle b\rangle,C(X^\infty,\mathbb{Z}))$. Let $f\in ker(b-1)$. Here, as before, we write any element of $C(X^\infty,\mathbb{Z})$ as $f=\sum_{n=0}^{N-1}(\sum\limits_{\alpha\in X^*}(\lambda_{1^n00\alpha}1_{Z(1^n00\alpha)}+\lambda_{1^n01\alpha}1_{Z(1^n01\alpha)}))+\lambda_{1^N}1_{Z(1^N)}$ for some $N>0$ and with only finitely many λ's not equal to zero. Hence, since $f\in ker(b-1)$, replicating the previous techniques we get that $\lambda_{1^n01\alpha}=\lambda_{1^n00\alpha}$, so

$$f=\sum_{n=0}^{N-1}(\sum_{\alpha\in X^*}(\lambda_{1^n00\alpha}1_{Z(1^n00\alpha)}+\lambda_{1^n00\alpha}1_{Z(1^n01\alpha)}))+\lambda_{1^N}1_{Z(1^N)}.$$

However, if we realize that

$$(1+b)(\sum_{n=0}^{N-1}\sum_{\alpha\in X^*}(\lambda_{1^n00\alpha}1_{Z(1^n00\alpha)})+\lambda_{1^N}1_{Z(1^N)})=$$

$$=\sum_{n=0}^{N-1}(\sum_{\alpha\in X^*}(\lambda_{1^n00\alpha}1_{Z(1^n00\alpha)}+\lambda_{1^n00\alpha}1_{Z(1^n01\alpha)}))+2\lambda_{1^N}1_{Z(1^N)}\,.$$

we deduce that $[f]=0$ in $ker(b-1)/(b+1)(C(X^\infty,\mathbb{Z}))$ whenever λ_{1^N} is even, and $[f]=[1_{Z(1^N)}]$, whenever λ_{1^N} is odd. All is left to see is that $[1_{Z(1^N)}]=[1_{Z(1^{N+1})}]$, for all N, but this is straightforward to check, since $1_{Z(1^N)}-1_{Z(1^{N+1})}=1_{Z(1^N0)}=(1+b)1_{Z(1^N00)}$, which is zero in the quotient. Hence, we conclude that $ker(b-1)/(b+1)(C(X^\infty,\mathbb{Z}))\cong\mathbb{Z}_2$, concluding the proof.

$\square$

Theorem 5.20. *Let $(\mathcal{D}_\infty, X)$ be the self-similar infinite dihedral group, and consider $\mathcal{G}_{(\mathcal{D}_\infty,X)}$ its associated groupoid of germs. Then $H_0(\mathcal{G}_{(\mathcal{D}_\infty,X)})=0$, and $H_1(\mathcal{G}_{(\mathcal{D}_\infty,X)})\cong H_2(\mathcal{G}_{(\mathcal{D}_\infty,X)})\cong H_1(\mathcal{H}_{(\mathcal{D}_\infty,X)})\cong\mathbb{Z}_2$. Moreover, $H_n(\mathcal{G}_{(\mathcal{D}_\infty,X)})$ is a torsion group for all $n\geq 3$.*

Proof. We will make use of the long exact sequence of homology given in Theorem 3.18. As we noted in Lemma 5.16, $\mathcal{H}_{(\mathcal{D}_\infty,X)}$ is Kakutani equivalent to $\mathcal{G}_{(\mathcal{D}_\infty,X)}\times_c\mathbb{Z}$, where c is the cocycle defined as $[\alpha,g,\beta;\beta x]\mapsto|\alpha|-|\beta|$. Moreover, the isomorphism $H_*(\mathcal{H}_{(\mathcal{D}_\infty,X)})\cong H_*(\mathcal{G}_{(\mathcal{D}_\infty,X)}\times_c\mathbb{Z})$ is induced by the natural inclusion $[\alpha,g,\beta;\beta x]\mapsto[\alpha,g,\beta;\beta x]\times\{0\}$.
Here, we study the behaviour of the map in $H_*(\mathcal{H}_{(\mathcal{D}_\infty,X)})$ induced by the shift $\sigma:\mathcal{G}_{(\mathcal{D}_\infty,X)}\times_c\mathbb{Z}\to\mathcal{G}_{(\mathcal{D}_\infty,X)}\times_c\mathbb{Z}$ given by $g\times\{n\}\mapsto g\times\{n+1\}$, that is, the map $\hat{\sigma}_*$ that makes the following diagram commute.

$$\begin{tikzcd}
H_*(\mathcal{H}_{(\mathcal{D}_\infty,X)}) \arrow[r, "\hat{\sigma}_*"] \arrow[d, "\cong"'] & H_*(\mathcal{H}_{(\mathcal{D}_\infty,X)}) \arrow[d, "\cong"] \\
H_*(\mathcal{G}_{(\mathcal{D}_\infty,X)}\times_c\mathbb{Z}) \arrow[r, "\sigma_*"'] & H_*(\mathcal{G}_{(\mathcal{D}_\infty,X)}\times_c\mathbb{Z})
\end{tikzcd}$$

We will focus our study in the cases $*=0,1$.
For $*=0$, the map $\sigma_0:H_0(\mathcal{G}_{(\mathcal{D}_\infty,X)}\times_c\mathbb{Z})\to H_0(\mathcal{G}_{(\mathcal{D}_\infty,X)}\times_c\mathbb{Z})$ is given by $[1_{Z(0^n,e,0^n;Z(0^n))\times\{0\}}]\mapsto[1_{Z(0^n,e,0^n;Z(0^n))\times\{1\}}]$.
Consider the bisection $U:=Z(0^n,e,0^{n+1};Z(0^{n+1}))\times\{1\}\subseteq\mathcal{G}_{(\mathcal{D}_\infty,X)}\times_c\mathbb{Z}$, which verifies

$$\begin{aligned}
r(U)&=Z(0^n,e,0^n;Z(0^n))\times\{1\}, \text{ and}\\
s(U)&=Z(0^{n+1},e,0^{n+1};Z(0^{n+1}))\times\{0\}.
\end{aligned}$$

Then $\delta_1(1_U):=s_*(1_U)-r_*(1_U)=1_{s(U)}-1_{r(U)}$ implies that $[1_{Z(0^n,e,0^n;Z(0^n))\times\{1\}}]=[1_{Z(0^{n+1},e,0^{n+1};Z(0^{n+1}))\times\{0\}}]$ in $H_0(\mathcal{G}_{(\mathcal{D}_\infty,X)}\times_c\mathbb{Z})$, and hence the map $\hat{\sigma}_0:H_0(\mathcal{H}_{(\mathcal{D}_\infty,X)})\to H_0(\mathcal{H}_{(\mathcal{D}_\infty,X)})$ is given by $[1_{Z(0^n,e,0^n;Z(0^n))}]\mapsto[1_{Z(0^{n+1},e,0^{n+1};Z(0^{n+1}))}]$. Using the isomorphism $H_0(\mathcal{H}_{(\mathcal{D}_\infty,X)})\cong\mathbb{Z}[\frac{1}{2}]$ given in Lemma 5.17, we deduce the following description of $\hat{\sigma}_0$:

$$\begin{aligned}
\hat{\sigma}_0:H_0(\mathcal{H}_{(\mathcal{D}_\infty,X)})\cong\mathbb{Z}[\tfrac{1}{2}]&\to\mathbb{Z}[\tfrac{1}{2}]\cong H_0(\mathcal{H}_{(\mathcal{D}_\infty,X)})\\
x&\mapsto\tfrac{x}{2}
\end{aligned}$$

We now the study the case $*=1$.

Let $[1_{Z(0^n,g,0^n;Z(0^n))\times\{0\}}] \in H_1(\mathcal{G}_{(\mathcal{D}_\infty,X)} \times_c \mathbb{Z})$, and let $[1_{Z(0^n,g,0^n;Z(0^n))\times\{1\}}]$ be its image under σ_1. Using the same bisection U as before, we obtain

$$Z(0^n,g,0^n;Z(0^n))\times\{1\}\cdot U = U\cdot Z(0^{n+1},g,0^{n+1};Z(0^{n+1}))\times\{0\}.$$

In particular,

$$(Z(0^n,g,0^n;Z(0^n))\times\{1\},U) \subset (\mathcal{G}_{(\mathcal{D}_\infty,X)} \times_c \mathbb{Z})^{(2)}, \text{ and}$$
$$(U, Z(0^{n+1},g,0^{n+1};Z(0^{n+1}))\times\{0\}) \subset (\mathcal{G}_{(\mathcal{D}_\infty,X)} \times_c \mathbb{Z})^{(2)}.$$

Then we have:

$$\delta_2(1_{((Z(0^n,g,0^n;Z(0^n))\times\{1\},U))}) = 1_{Z(0^n,g,0^n;Z(0^n))\times\{1\}} + 1_U - 1_{Z(0^n,g,0^n;Z(0^n))\times\{1\}\cdot U},$$

and

$$\delta_2(1_{(U,Z(0^{n+1},g,0^{n+1};Z(0^{n+1}))\times\{0\})}) = 1_U + 1_{Z(0^{n+1},g,0^{n+1};Z(0^{n+1}))\times\{0\}} - 1_{(U\cdot Z(0^{n+1},g,0^{n+1};Z(0^{n+1}))\times\{0\})}.$$

Gathering all the above facts, we deduce that

$$1_{Z(0^n,g,0^n;Z(0^n))\times\{1\}} - 1_{Z(0^{n+1},g,0^{n+1};Z(0^{n+1}))\times\{0\}} \in Im(\delta_2),$$

and thus it is 0 in $H_1(\mathcal{G}_{(\mathcal{D}_\infty,X)} \times_c \mathbb{Z})$.

Then we obtain that

$$\sigma_1([1_{Z(0^n,g,0^n;Z(0^n))\times\{0\}}]) = [1_{Z(0^{n+1},g,0^{n+1};Z(0^{n+1}))\times\{0\}}] \in H_1(\mathcal{G}_{(\mathcal{D}_\infty,X)} \times_c \mathbb{Z}).$$

Finally, using the isomorphism $H_1(\mathcal{H}_{(\mathcal{D}_\infty,X)}) \cong \mathbb{Z}_2$ given in Lemma 5.18, we conclude that

$$\hat{\sigma}_1 : H_1(\mathcal{H}_{(\mathcal{D}_\infty,X)}) \cong \mathbb{Z}_2 \to \mathbb{Z}_2 \cong H_1(\mathcal{H}_{(\mathcal{D}_\infty,X)})$$

is the identity map.

We can now use the long exact sequence introduced in Theorem 3.18, that is

$$\begin{tikzcd}
0 & H_0(\mathcal{G}_{(\mathcal{D}_\infty,X)}) \arrow[l] & H_0(\mathcal{H}_{(\mathcal{D}_\infty,X)}) \arrow[l] & H_0(\mathcal{H}_{(\mathcal{D}_\infty,X)}) \arrow[l, "Id-\hat{\sigma}_0"'] & H_1(\mathcal{G}_{(\mathcal{D}_\infty,X)}) \arrow[l] & H_1(\mathcal{H}_{(\mathcal{D}_\infty,X)}) \arrow[l, "\varphi"'] \\
\cdots \arrow[r] & H_3(\mathcal{G}_{(\mathcal{D}_\infty,X)}) \arrow[r] & H_2(\mathcal{H}_{(\mathcal{D}_\infty,X)}) \arrow[r, "Id-\hat{\sigma}_2"] & H_2(\mathcal{H}_{(\mathcal{D}_\infty,X)}) \arrow[r] & H_2(\mathcal{G}_{(\mathcal{D}_\infty,X)}) \arrow[r] & H_1(\mathcal{H}_{(\mathcal{D}_\infty,X)}) \arrow[u, "Id-\hat{\sigma}_1"']
\end{tikzcd} \tag{5.6}$$

which, combined with all the previous results, is of the form

$$\begin{tikzcd} 0 & H_0(\mathcal{G}_{(\mathcal{D}_\infty,X)}) \arrow[l] & \mathbb{Z}[\frac{1}{2}] \arrow[l] & \mathbb{Z}[\frac{1}{2}] \arrow[l, "\frac{x}{2}"'] & H_1(\mathcal{G}_{(\mathcal{D}_\infty,X)}) \arrow[l] & \mathbb{Z}_2 \arrow[l, "\varphi"'] \\ \cdots \arrow[r] & H_3(\mathcal{G}_{(\mathcal{D}_\infty,X)}) \arrow[r] & 0 \arrow[r, "Id-\hat{\sigma_*}"] & 0 \arrow[r] & H_2(\mathcal{G}_{(\mathcal{D}_\infty,X)}) \arrow[r] & \mathbb{Z}_2 \arrow[u, "0"'] \end{tikzcd} \tag{5.7}$$

It is immediate from this sequence that $H_0(\mathcal{G}_{(\mathcal{D}_\infty,X)}) = 0$, and $H_2(\mathcal{G}_{(\mathcal{D}_\infty,X)}) \cong \mathbb{Z}_2$. Moreover, since $\frac{x}{2} : \mathbb{Z}[\frac{1}{2}] \to \mathbb{Z}[\frac{1}{2}]$ is injective, the map φ is an isomorphism between $H_1(\mathcal{G}_{(\mathcal{D}_\infty,X)})$ and $\mathbb{Z}_2$.

Finally, by Lemma 5.19, we obtain exact sequences for $k \geq 1$:

$$0 \longleftarrow H_{2k+1}(\mathcal{G}_{(\mathcal{D}_\infty,X)}) \longleftarrow \mathbb{Z}_2 \overset{Id-\hat{\sigma_*}}{\longleftarrow} \mathbb{Z}_2 \longleftarrow H_{2k+2}(\mathcal{G}_{(\mathcal{D}_\infty,X)}) \longleftarrow 0$$

from which we deduce that for all $n \geq 3$, $H_n(\mathcal{G}_{(\mathcal{D}_\infty,X)})$ is either 0 or $\mathbb{Z}_2$ (and thus a torsion group).

$\square$

5.4 HK and AH conjectures

This section sums up all the results obtained and relate them with the HK-conjecture. In particular, we present the first complete counterexample for both HK and weak HK conjecture. Moreover, we also draw the attention a minor counterexample for HK conjecture obtained in the process. The chapter ends with a discussion of Matui's AH conjecture for the groupoids involved.

5.4.1 A complete counterexample for Matui's HK conjecture

The computation of the homology groups in Theorem 5.20, together with the computation of the K-groups in Lemma 5.14, leads us to the main result of this chapter. Indeed, below we show that the groupoid $\mathcal{G}_{(\mathcal{D}_\infty,X)}$ associated to the infinite dihedral self-similar group contradicts both Matui's HK and weak HK conjectures. Recall that, as we pointed out in Definition 5.7, the groupoid $\mathcal{G}_{(\mathcal{D}_\infty,X)}$ is minimal, effective, Hausdorff and ample, so it lies under the conditions of the mentioned conjecture. Moreover, $C_r^*(\mathcal{G}_{(\mathcal{D}_\infty,X)})$ is a unital, purely infinite simple C^*-algebra ([20, section 17]).

Theorem 5.21. *Let $\mathcal{D}_\infty$ be the infinite dihedral group, and $X = \{0,1\}$. Let $(\mathcal{D}_\infty, X)$ be the induced self-similar group, and let $\mathcal{G}_{(\mathcal{D}_\infty,X)}$ be the associated groupoid of germs. Then*

$$\mathbb{Q} \cong K_i(C_r^*(\mathcal{G}_{(\mathcal{D}_\infty,X)})) \otimes \mathbb{Q} \ncong \bigoplus_{k=0}^{\infty} H_{i+2k}(\mathcal{G}_{(\mathcal{D}_\infty,X)}) \otimes \mathbb{Q} = 0, \ \ i = 1, 0$$

This means that $\mathcal{G}_{(\mathcal{D}_\infty,X)}$ provides a complete counterexample for Matui's weak HK conjecture and, therefore, for Matui's HK conjecture.

Proof. Since $(\mathcal{D}_\infty, X)$ is recurrent, the groupoid $\mathcal{G}_{(\mathcal{D}_\infty,X)}$ is amenable, and thus $C^*(\mathcal{D}_\infty, X) \cong C_r^*(\mathcal{G}_{(\mathcal{D}_\infty,X)})$. The proof is then immediate after combining the computation of the K-groups in Lemma 5.14, where it is shown that $K_0(C_r^*(\mathcal{G}_{(\mathcal{D}_\infty,X)})) \cong K_1(C_r^*(\mathcal{G}_{(\mathcal{D}_\infty,X)})) \cong \mathbb{Z}$, and noticing that all the homology groups are torsion groups, by Theorem 5.20. $\square$

We close this subsection showing that, in fact, there is a second complete counterexample for Matui's HK conjecture (but not for the weak version) hidden among the previous pages. The homology of $\mathcal{H}_{(\mathcal{D}_\infty,X)}$ was given by:

$$\begin{gathered} H_0(\mathcal{H}_{(\mathcal{D}_\infty,X)}) \cong \mathbb{Z}[\tfrac{1}{2}], \\ H_1(\mathcal{H}_{(\mathcal{D}_\infty,X)}) \cong \mathbb{Z}_2, \\ H_{2k}(\mathcal{H}_{(\mathcal{D}_\infty,X)}) = 0, \text{ for all } k \geq 1, \text{ and} \\ H_{2k+1}(\mathcal{H}_{(\mathcal{D}_\infty,X)}) \cong \mathbb{Z}_2, \text{ for all } k \geq 1. \end{gathered}$$

On the other hand, it was shown in [42, Theorem 5.3] that, whenever the self-similar group is recurrent, then $\mathcal{M}_{\mathcal{D}_\infty}$ is isomorphic to the full convolution algebra of $\mathcal{H}_{(\mathcal{D}_\infty,X)}$. Moreover, $\mathcal{H}_{(\mathcal{D}_\infty,X)}$ is amenable [42, Theorem 5.6], and thus $C^*_r(\mathcal{H}_{(\mathcal{D}_\infty,X)}) \cong \mathcal{M}_{\mathcal{D}_\infty}$. The K-theory of $\mathcal{M}_{\mathcal{D}_\infty}$ was computed in Proposition 5.13 to be:

$$K_0(\mathcal{M}_{\mathcal{D}_\infty}) \cong \mathbb{Z}[\tfrac{1}{2}] \oplus \mathbb{Z}, \text{ and } K_1(\mathcal{M}_{\mathcal{D}_\infty}) = 0.$$

Finally, it is straightforward to check that $\mathcal{H}_{(\mathcal{D}_\infty,X)}$ lies under Matui's HK hypothesis. As we can see, this proves to be another complete counterexample for Matui's HK-conjecture. In the case of the weak version, the conjecture fails at the K_0-term, but it holds for the K_1.

5.4.2 AH conjecture

As noted in Section 2.4, Matui posed, in addition to the HK, a second conjecture regarding groupoids, which remains yet to be disproven. Even though it is not the main theme of this text, it is worthwhile to save a few lines to the study of this AH conjecture for the self-similar infinite dihedral groupoid, since most of the work is already done. As pointed out in Section 2.4, this conjecture claimed the existence of an exact sequence:

$$H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \xrightarrow{j} [[\mathcal{G}]]_{ab} \xrightarrow{I_{ab}} H_1(\mathcal{G}) \to 0,$$

where $[[\mathcal{G}]]_{ab}$ is the abelianization of the full group defined in Definition 2.78, and I_{ab} is the index map defined in 2.79.

It was shown in [38, Theorem 4.4] that, whenever a groupoid is minimal, purely infinite (Definition 2.44) and ample, then the verification of AH conjecture is equivalent to satisfying Property TR introduced in paragraph 2.83, that is, the condition that every element of the kernel of the index map can be put as a product of transpositions (definition 2.77). We will use the notation and results displayed in Section 2.4.

Proposition 5.22. *Let $(\mathcal{D}_\infty, X)$ be the self-similar infinite dihedral group. Then the groupoid $\mathcal{H}_{(\mathcal{D}_\infty,X)}$ defined in Lemma 5.16 satisfies Property TR.*

Proof. We use a similar strategy to the one used in [44, Lemma 5.3, Proposition 5.4].

First, recall that $\mathcal{H}_{(\mathcal{D}_\infty,X)} = \bigcup_{n=0}^{\infty} (\mathcal{H}_{(\mathcal{D}_\infty,X)})_n$, with $(\mathcal{H}_{(\mathcal{D}_\infty,X)})_n^{(0)} = \mathcal{G}^{(0)}_{(\mathcal{D}_\infty,X)}$, and $(\mathcal{H}_{(\mathcal{D}_\infty,X)})_n \subseteq (\mathcal{H}_{(\mathcal{D}_\infty,X)})_{n+1}$ for all n (see Remark 5.8). Then, the definition of topological full group implies that $[[(\mathcal{H}_{(\mathcal{D}_\infty,X)})_n]] \subseteq [[(\mathcal{H}_{(\mathcal{D}_\infty,X)})_{n+1}]]$, and $[[\mathcal{H}_{(\mathcal{D}_\infty,X)}]] = \bigcup_{n=0}^{\infty} [[(\mathcal{H}_{(\mathcal{D}_\infty,X)})_n]]$.

Let $\pi_U \in [[(\mathcal{H}_{(\mathcal{D}_\infty,X)})_n]]$ such that $I(\pi_U) = 0$, for some $n \in \mathbb{N}$. We can assume, without loss of generality, that U is of the form

$$U = \bigsqcup_{i=1}^{k} Z(\alpha_i, g_i, \beta_i; Z(\beta_i)),$$

where $\alpha_i, \beta_i \in X^n$, $g_i \in \mathcal{D}_\infty$, and $\bigsqcup_{i=1}^{k} Z(\alpha_i) = \bigsqcup_{i=1}^{k} Z(\beta_i) = X^\infty$. Then, using the results obtained in Lemma 5.18, we can write:

$$I(\pi_U) = \sum_{i=1}^{k} [1_{Z(0^n, g_i, 0^n; Z(0^n))}] = [1_{Z(0^n, \prod_{i=1}^{k} g_i, 0^n; Z(0^n))}],$$

where $[1_{Z(0^n,a,0^n;Z(0^n))}] \mapsto 0$, and $[1_{Z(0^n,b,0^n;Z(0^n))}] \mapsto 1$ under the isomorphism $H_1(\mathcal{H}_{(\mathcal{D}_\infty, X)}) \cong \mathbb{Z}_2$.
Therefore, we deduce that the condition $I(\pi_U) = 0$ in $H_1(\mathcal{H}_{(\mathcal{D}_\infty, X)})$ implies that $\prod_{i=1}^{k} g_i$ must be an arbitrary product of elements a, and bab.
Now, define U_1, U_2 as

$$U_1 := \bigsqcup_{i=1}^{k} Z(\alpha_i, g_i, \alpha_i; Z(\alpha_i)),$$
$$U_2 := \bigsqcup_{i=1}^{k} Z(\alpha_i, e_i, \beta_i; Z(\beta_i)).$$

It is clear that $U = U_1 \cdot U_2$, and hence $\pi_U = \pi_{U_1} \circ \pi_{U_2}$. It was shown in [44, Lemma 5.3] that π_{U_2} is product of transpositions. Let us show that π_{U_1} is also a product of transpositions.
For each $1 < i \leq k$, define the bisections

$$V_i := Z(\alpha_1, g_i, \alpha_i; Z(\alpha_i)), \quad W_i := Z(\alpha_1, e, \alpha_i; Z(\alpha_i)).$$

One can check that

$$U_1 \hat{V}_2 \hat{W}_2 ... \hat{V}_k \hat{W}_k = Z(\alpha_1, \prod_{i=1}^{k} g_i, \alpha_1; Z(\alpha_1)) \sqcup \bigsqcup_{i=2}^{k} Z(\alpha_i, e, \alpha_i; Z(\alpha_i)).$$

Moreover, if we define

$$W_a := Z(\alpha_1 1, e, \alpha_1 0; Z(\alpha_1 0)), \text{ and } W_{bab} := Z(\alpha_1 1, ba, \alpha_1 0; Z(\alpha_1 0)),$$

then we obtain

$$Z(\alpha_1, a, \alpha_1; Z(\alpha_1)) \sqcup \bigsqcup_{i=2}^{k} Z(\alpha_i, e, \alpha_i; Z(\alpha_i)) = \hat{W}_a,$$

and

$$Z(\alpha_1, bab, \alpha_1; Z(\alpha_1)) \sqcup \bigsqcup_{i=2}^{k} Z(\alpha_i, e, \alpha_i; Z(\alpha_i)) = \hat{W}_{bab}.$$

Indeed, since $a \cdot 0 = 1 \cdot e$ and $a \cdot 1 = 0 \cdot e$, we have
$\hat{W}_a = Z(\alpha_1 1, e, \alpha_1 0; Z(\alpha_1 0)) \sqcup Z(\alpha_1 0, e, \alpha_1 1; Z(\alpha_1 1)) \sqcup X^\infty \setminus Z(\alpha_1) =$
$= Z(\alpha_1, a, \alpha_1; Z(\alpha_1)) \sqcup \bigsqcup_{i=2}^{k} Z(\alpha_i, e, \alpha_i; Z(\alpha_i))$, where we identify $Z(\alpha_i, e, \alpha_i; Z(\alpha_i))$ with

$Z(\alpha_i)$, as usual. A similar argument applies to $\hat{W}_{bab}$. Now, since $\prod_{i=1}^{k} g_i$ is a product of elements a, and bab, we deduce that $U_1\hat{V}_2\hat{W}_2...\hat{V}_k\hat{W}_k$ is a product of elements $\hat{W}_a$, and $\hat{W}_{bab}$, and therefore

$$\pi_{U_1}(\pi_{\hat{V}_2}\pi_{\hat{W}_2}...\pi_{\hat{V}_k}\pi_{\hat{W}_k}) = ...\pi_{\hat{W}_a}\pi_{\hat{W}_{bab}}\pi_{\hat{W}_a}...,$$

thus π_{U_1} is a product of transpositions. We deduce that $\pi_{U_1} \in \mathcal{T}(\mathcal{H}_{(\mathcal{D}_\infty,X)})$, and hence $\pi_U \in \mathcal{T}(\mathcal{H}_{(\mathcal{D}_\infty,X)})$, concluding the proof. □

Finally, let us show that the $\mathcal{G}_{(\mathcal{D}_\infty,X)}$ also satisfies Property TR, concluding that the AH-conjecture holds for $\mathcal{G}_{(\mathcal{D}_\infty,X)}$.

Theorem 5.23. *Let $(\mathcal{D}_\infty, X)$ be the self-similar infinite dihedral group. Then the groupoid $\mathcal{G}_{(\mathcal{D}_\infty,X)}$ satisfies the AH-conjecture.*

Proof. As we noted before, since $\mathcal{G}_{(\mathcal{D}_\infty,X)}$ is a minimal purely infinite ample groupoid, all we need to do is to check that, indeed, $\mathcal{G}_{(\mathcal{D}_\infty,X)}$ satisfies Property TR.

Let $I_\mathcal{G}, I_\mathcal{H}$ be the respective index maps for $\mathcal{G}_{(\mathcal{D}_\infty,X)}$ and $\mathcal{H}_{(\mathcal{D}_\infty,X)}$, and let $\varphi : H_1(\mathcal{H}_{(\mathcal{D}_\infty,X)}) \to H_1(\mathcal{G}_{(\mathcal{D}_\infty,X)})$ be the map appearing in the long exact sequence shown in Lemma 3.18. Recall that this map was proven to be an isomorphism in Theorem 5.20, and by definition it is given by $\varphi([1_{Z(0^n,b,0^n;Z(0^n))}]) = [1_{Z(0^n,b,0^n;Z(0^n))}]$. It was shown in [44, Lemma 4.6] that this map is natural with respect to the index map, in the sense that $I_\mathcal{G}(\pi_U) = \varphi(I_\mathcal{H}(\pi_U))$, for every full bisection $U \subseteq \mathcal{H}_{(\mathcal{D}_\infty,X)}$. Since φ is an isomorphism, we deduce that, for every $U \subseteq \mathcal{H}_{(\mathcal{D}_\infty,X)}$, $I_\mathcal{G}(\pi_U) = 0$ if and only if $I_\mathcal{H}(\pi_U) = 0$.

Suppose $U \subseteq \mathcal{G}_{(\mathcal{D}_\infty,X)}$ is a full bisection such that $I_\mathcal{G}(\pi_U) = 0$, and recall that any full bisection can be written as $U = \bigsqcup_{i=1}^{k} Z(\alpha_i, g_i, \beta_i; Z(\beta_i))$, for $g_i \in \mathcal{D}_\infty$, and $\alpha_i, \beta_i \in X^*$ such that $\bigsqcup_{i=1}^{k} Z(\alpha_i) = \bigsqcup_{i=1}^{k} Z(\beta_i)$. Define

$$V = \bigsqcup_{i=1}^{k} Z(\alpha_i, g_i, \alpha_i; Z(\alpha_i)) \subseteq \mathcal{H}_{(\mathcal{D}_\infty,X)}, \text{ and}$$
$$W = \bigsqcup_{i=1}^{k} Z(\alpha_i, e, \beta_i; Z(\beta_i)) \subseteq \mathcal{G}_{(\mathcal{D}_\infty,X)}.$$

Then we have that $U = VW$, and hence

$$0 = I_\mathcal{G}(\pi_U) = I_\mathcal{G}(\pi_V) + I_\mathcal{G}(\pi_W).$$

Define now Ω as the subgroupoid of $\mathcal{G}_{(\mathcal{D}_\infty,X)}$ given by

$$\Omega := \{[\alpha, e, \beta; \beta x] \in \mathcal{G}_{(\mathcal{D}_\infty,X)} : \alpha, \beta \in X^*, x \in X^\infty\}$$

This subgroupoid is isomorphic to the groupoid of one-sided full shift with two letters (see [37]).

Then we have that $\pi_W \in [[\Omega]] \subseteq [[\mathcal{G}_{(\mathcal{D}_\infty,X)}]]$. It was shown in [37, Lemma 6.10] that $\pi_W \in \mathcal{T}(\Omega) \subseteq \mathcal{T}(\mathcal{G}_{(\mathcal{D}_\infty,X)})$, and hence $I_{\mathcal{G}}(\pi_W) = 0$.

Finally, $I_{\mathcal{G}}(\pi_W) = 0$ implies that $I_{\mathcal{G}}(\pi_U) = I_{\mathcal{G}}(\pi_V) = I_{\mathcal{H}}(\pi_V) = 0$, and therefore we obtain that $\pi_V \in \mathcal{T}(\mathcal{H}_{(\mathcal{D}_\infty,X)}) \subseteq \mathcal{T}(\mathcal{G}_{(\mathcal{D}_\infty,X)})$. We deduce that $\pi_U \in \mathcal{T}(\mathcal{G}_{\mathcal{D}_\infty,X})$, and so $\mathcal{G}_{(\mathcal{D}_\infty,X)}$ satisfies Property TR, concluding the proof.

$\square$

As we can see, the groupoid $\mathcal{G}_{(\mathcal{D}_\infty,X)}$ satisfies the AH conjecture, which remains yet to be disproven.

Matui's AH conjecture provides a useful tool for the computation of the topological full group of a groupoid. In this line, the last theorem, in addition to Theorem 5.20, implies an isomorphism $[[\mathcal{G}_{(\mathcal{D}_\infty,X)}]]_{ab} \cong \mathbb{Z}_2$ given by the index map. $[[\mathcal{G}_{(\mathcal{D}_\infty,X)}]]_{ab}$ is then generated by the class π_U, where $U = Z(\emptyset, b, \emptyset; X^\infty)$.

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