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Chern degree functions and Prym semicanonical pencils

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CERTIFIQUEN

que la present memòria ha estat realitzada sota la seva direcció per Andrés Rojas González, i que constitueix la seva tesi per optar al grau de Doctor en Matemàtiques.

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Abstract

Abelian varieties are projective algebraic varieties endowed with a group structure. Over the field of the complex numbers, they are obtained as complex tori $\mathbb{C}^g/\Lambda$ admitting an embedding into some projective space.

Throughout the last decades, abelian varieties have been the object of an intensive research, playing a fundamental role in some of the most notable advances in Algebraic Geometry. This study has been performed from a twofold perspective. On the one hand, abelian varieties are interesting on their own right, as they are varieties possessing a rich geometry; on the other hand, their study is useful to understand other algebraic varieties.

In this thesis we investigate two problems motivated by the study of abelian varieties, which somehow reflect this dualism. Whereas in the first problem the picture is dominated by abelian surfaces (even if natural generalizations to arbitrary surfaces arise and may deserve attention), the second problem lies in the interplay between algebraic curves and abelian varieties, and has consequences on the geometry of cubic threefolds as well.

The first problem under consideration is that of understanding *cohomological rank functions* on abelian surfaces. The cohomological rank functions $h^i_{F,L}$ associated to a coherent sheaf (or more generally, a bounded complex of coherent sheaves) F on a polarized abelian variety (A, L) were introduced by Jiang and Pareschi in [JP20], as a generalization of the *continuous rank functions* defined by Barja, Pardini and Stoppino in [BPS20b]. For every $x \in \mathbb{Q}$, using the multiplication maps on A the number $h^i_{F,L}(x)$ makes sense of the i-th (hyper)cohomological rank of F twisted with (the general representative of) the fractional polarization xL.

The main results of Jiang and Pareschi about these functions are proved via an extensive use of the Fourier-Mukai transform on the abelian variety A. The Fourier-Mukai transform is an explicit equivalence between the derived categories of A and its dual abelian variety; it is induced by the Poincaré line bundle (namely the universal family of topologically trivial line bundles on A).

Among other applications, cohomological rank functions provide a more general context in which positivity notions like GV or *M-regularity* for coherent sheaves (resembling the usual Castelnuovo-Mumford regularity) can be considered. These notions were already introduced by Pareschi and Popa around twenty years ago ([PP03, PP11]), also by means of Fourier-Mukai techniques, and

received several important applications to the geometry of abelian and irregular varieties (see for instance [PP04, PP08, BLNP12, JLT13, PS14, CJ18]).

In the case of elliptic curves, it is well known that the cohomological rank functions of a coherent sheaf F can be described through its Harder-Narasimhan filtration (with respect to Mumford's slope stability). This filtration decomposes F into semistable factors of decreasing slope. Nevertheless, for higher-dimensional abelian varieties only a few concrete examples of functions are known, and a general structure is far from being understood.

On the other hand, Bridgeland ([Bri07]) defined new notions of stability on the derived category $\mathrm{D^b}(X)$ of a smooth projective variety X, which generalize and make more flexible the preceding notions for sheaves. Their definition includes a technical condition (*support property*), which allows to give the structure of a complex manifold to the space $\mathrm{Stab}(X)$ of stability conditions. In other words, Bridgeland stability conditions can be deformed.

The region of $\mathrm{Stab}(X)$ where an object $F \in \mathrm{D^b}(X)$ (or several of them) is semistable turns out to be an important invariant, providing interesting information about F. These regions behave following a wall and chamber structure, and their study (via *wall-crossing* techniques) has become in the recent years a powerful tool to attack many concrete geometric problems.

In the particular case of a smooth polarized projective surface (X, L), one can consider a stability condition $\sigma_{\alpha,\beta}$ for every $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$, as first observed by Bridgeland for K3 and abelian surfaces ([Bri08]) and generalized to arbitrary surfaces by Arcara and Bertram ([AB13]). In the corresponding region of stability conditions, often called the (α, β)*-plane*, the wall-crossing behaviour is very well understood.

When $\alpha = 0$, this construction may no longer produce stability conditions; even so, for $\beta \in \mathbb{Q}$ there exist Harder-Narasimhan filtrations with respect to the slope induced by $\sigma_{0,\beta}$. This defines a so-called *weak stability condition*, which (after several verifications) may be thought of as a limit of nearby stability conditions with $\alpha > 0$.

The information provided by these weak stability conditions is indeed weaker than the one provided by all the stability conditions in the (α, β)-plane, but still useful. In the first part of the thesis we aim to classify this information by defining, for every object $F \in \mathrm{D^b}(X)$, *Chern degree functions*

$$\mathrm{chd}^k_{F,L} : \mathbb{Q} \to \mathbb{Q}_{\geq 0}$$

determined by the Harder-Narasimhan filtrations of F for the weak stability conditions along the line $\alpha = 0$.

These functions admit local expressions and satisfy properties of continuity similar to those proved by Jiang and Pareschi. We also describe their differentiability in terms of stability. Our arguments make a systematic use of wall-crossing and may be of independent interest, especially for those surfaces where $\alpha = 0$ lies in the boundary of the stability manifold.

In the particular case of abelian surfaces, we prove that the Chern degree functions recover the cohomological rank functions of Jiang and Pareschi. This establishes a clear analogy with the case of elliptic curves, since we obtain that cohomological rank functions on abelian surfaces are determined by stability. This Fourier-Mukai-free approach gives, in many situations, a clear picture for properties like differentiability or rationality of critical points.

In addition, this new presentation is also useful for the computation of particular examples. The most relevant one is the ideal sheaf of one point, from which we will obtain new results on the syzygies of polarized abelian surfaces.

Syzygies are a notion of great interest to understand embedded projective varieties, since they essentially contain all the information that can be extracted from the equations in the ambient space. As such, they are in general very difficult to control; for integers $p \geq 0$, the properties (N_p) are requirements of simplicity for these syzygies. Roughly speaking, (N_p) means that the first p steps of the minimal graded free resolution of the homogeneous ideal of the variety are linear.

Green ([Gre84]) proved the fulfilment of the property (N_p) for curves embedded by line bundles of sufficiently high degree, generalizing in a unified way previous results of Castelnuovo, Mattuck, Fujita and Saint-Donat. Furthermore, he stated a famous conjecture relating the Clifford index with the failure of (N_p) for canonical curves, solved by Voisin for general curves ([Voi02, Voi05]).

In the case of abelian varieties, Lazarsfeld conjectured the fulfilment of the property (N_p) for powers L^m $(m \geq p+3)$ of any ample line bundle L, as a generalization of previous results of Koizumi, Mumford and Kempf. Pareschi ([Par00]) gave a proof in characteristic zero. In arbitrary characteristic, this was recently proved by Caucci ([Cau20]), as an application of a criterion relating the cohomological rank functions $h^i_{\mathcal{I}_0,L}$ and the properties (N_p). In virtue of Caucci's criterion (and some refinements by Ito [Ito21]), our explicit computations of $h^i_{\mathcal{I}_0,L}$ for abelian surfaces lead to new effective results on their syzygies.

The second problem treated in this thesis deals with double étale covers of curves with a semicanonical pencil, and their Prym varieties. Prym varieties are principally polarized abelian varieties (*ppav*'s in the sequel) associated to double étale covers of curves. They were already considered from an analytic viewpoint by Wirtinger in the late XIX century, but it was Mumford in his seminal work [Mum74] who established their current denomination (in honour of the German mathematician Friedrich Prym) and presented them in a modern algebraic language for the first time.

Since then, they have become an important tool in Algebraic Geometry, as they form a broader class of ppav's than Jacobians. For instance the *Prym map* $\mathcal{P}_g : \mathcal{R}_g \to \mathcal{A}_{g-1}$, which assigns to every double étale cover of a smooth curve of genus g its $(g-1)$-dimensional (principally polarized) Prym variety, is dominant for $g \leq 6$. This is the reason why ppav's are reasonably well understood up to dimension 5.

In fact, as in the case of Jacobians much information about the geometry of Pryms can be obtained from the geometry of curves. For instance, Mumford's work classified the singularities of the theta divisor of a Prym variety into *stable* and *exceptional* ones. The most elementary example of exceptional singularity arises from double étale covers $\tilde{C} \xrightarrow{f} C$ satisfying the following conditions:

- The curve C has a *semicanonical pencil*, namely a line bundle L of degree $g-1$ on C such that $L^2 \cong \omega_C$ and $h^0(C,L)$ is even and positive.
- The number $h^0(\tilde{C}, f^*L)$ is even.

We will call such a line bundle L an *even semicanonical pencil* for the cover f. In case that it exists, the corresponding Prym variety belongs to the divisor $\theta_{null} \subset \mathcal{A}_{g-1}$ of ppav's whose theta divisor contains a singular 2-torsion point.

In his paper [Bea77a], Beauville proved that the Andreotti-Mayer locus $\mathcal{N}_0 \subset \mathcal{A}_4$ (of principally polarized abelian fourfolds with a singular theta divisor) is the union of two irreducible divisors: the (closure of the) Jacobian locus $\mathcal{J}_4$ and θ_{null}. An essential tool was the extension of the Prym map to a proper map $\widetilde{\mathcal{P}}_g : \widetilde{\mathcal{R}}_g \to \mathcal{A}_{g-1}$, by considering admissible covers of (possibly nodal) curves. In the case $g = 5$, this guarantees that every 4-dimensional ppav is a Prym variety (i.e. the dominant map $\mathcal{P}_5$ is replaced by the surjective map $\widetilde{\mathcal{P}}_5$).

One of the key points in Beauville's work was the identification of the covers having Prym variety in θ_{null}; he showed that

$$\mathcal{T}^e = \text{(closure of) } \{[f : \widetilde{C} \longrightarrow C] \in \mathcal{R}_5 \mid \text{The cover } f \text{ has an even semicanonical pencil}\}$$

is irreducible and equals $\widetilde{\mathcal{P}}_5^{-1}(\theta_{null})$. Indeed, the proof of the irreducibility of θ_{null} relied on the irreducibility of $\mathcal{T}^e$, and started by noticing that the locus $\mathcal{T} \subset \mathcal{M}_5$ of genus 5 curves with a semicanonical pencil is an irreducible divisor of the moduli space $\mathcal{M}_5$.

Now, we consider the following situation: for a fixed genus $g \geq 3$, let $\mathcal{T}_g \subset \mathcal{M}_g$ be the locus of smooth genus g curves admitting a semicanonical pencil, which is well known to be a divisor of the moduli space $\mathcal{M}_g$. The general element of $\mathcal{T}_g$ has a unique such semicanonical pencil L, and hence the pullback of $\mathcal{T}_g$ to $\mathcal{R}_g$ decomposes as a union $\mathcal{T}_g^e \cup \mathcal{T}_g^o$ according to the parity of $h^0(\widetilde{C}, f^*L)$.

In view of Beauville's work, it seems natural to ask whether $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$ are irreducible divisors, and to ask about the behaviour of the restricted Prym maps $\mathcal{P}_g|_{\mathcal{T}_g^e}$ and $\mathcal{P}_g|_{\mathcal{T}_g^o}$ (or $\widetilde{\mathcal{P}}_g|_{\mathcal{T}_g^e}$ and $\widetilde{\mathcal{P}}_g|_{\mathcal{T}_g^o}$, if one considers the closures of $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$ in $\widetilde{\mathcal{R}}_g$). These are the two questions treated in the second part of the thesis.

With respect to the first question, the divisor $\mathcal{T}_g \subset \mathcal{M}_g$ was studied by Teixidor in [TiB88]. Using the theory of limit linear series developed by Eisenbud and Harris, she proved the irreducibility of $\mathcal{T}_g$ and computed the class of its closure in the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$. Following this approach, we have obtained natural analogues of these results, namely: $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$ are irreducible divisors of $\mathcal{R}_g$, and we have computed the classes in $\mathrm{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}$ of their closures, where $\overline{\mathcal{R}}_g$ stands for the Deligne-Mumford compactification.

Whereas we will only use these classes as a tool to understand the second question, it is worth mentioning that the computation of classes of effective divisors has been largely applied to study the birational geometry of certain moduli spaces. This point of view started with the pioneering work [HM82] of Harris and Mumford for $\overline{\mathcal{M}}_g$; the reader is referred to [Far09, Section 1] for an historical account of this problem, and to [FL10] for results in the case of $\overline{\mathcal{R}}_g$.

Regarding the second question we point out that, apart from the aforementioned analysis of $\mathcal{T}_5^e$ by Beauville, Izadi [Iza95] showed the surjectivity of the restricted Prym map $\widetilde{\mathcal{P}}_5|_{\mathcal{T}_5^o}$. As we will prove, this is a general phenomenon in the range $3 \leq g \leq 5$: whereas $\mathcal{T}_g^e$ equals the preimage of $\theta_{null} \subset \mathcal{A}_{g-1}$, the map $\widetilde{\mathcal{P}}_g|_{\mathcal{T}_g^o}$ is surjective. Furthermore, in those cases we give an explicit description of the general fibers of $\widetilde{\mathcal{P}}_g|_{\mathcal{T}_g^o}$, which are geometrically significant; for instance, our analysis of the general fiber of $\widetilde{\mathcal{P}}_5|_{\mathcal{T}_5^o}$ has a surprising application to the geometry of cubic threefolds.

Cubic threefolds have been largely investigated since the beginnings of Algebraic Geometry; for instance, their unirationality was already known by Max Noether. At the beginning of the XX century they were studied by Fano, in an attempt to prove their irrationality and hence to give a negative answer to the Lüroth problem. Their irrationality was finally proved by Clemens and Griffiths in their celebrated paper [CG72], by studying a ppav (the *intermediate Jacobian*) associated to every cubic threefold. Later work of Mumford, Tjurin, Beauville, Donagi, Smith, Casalaina-Martin and Friedman (among many others) provided more precise information about the intermediate Jacobian and its theta divisor, thanks to its presentation as a Prym variety.

Returning to our problem, the general fiber of the Prym map $\widetilde{\mathcal{P}}_5 : \widetilde{\mathcal{R}}_5 \to \mathcal{A}_4$ was described by Donagi ([Don92]) as a double étale cover of the Fano surface of lines $F(V)$ of a general cubic threefold V. In this context, the general fiber of $\widetilde{\mathcal{P}}_5|_{\mathcal{T}_5^o}$ arises as an irreducible curve, which is a partial desingularization of

$$\Gamma = \{l \in F(V) \mid \text{There exist a 2-plane } \pi \text{ and a line } r \in F(V) \text{ with } V \cdot \pi = l + 2r\} \subset F(V)$$

This curve Γ remains largely unexplored, in contrast to its natural counterpart (the curve Γ' formed by *lines of the second type*) that was studied in [CG72] among many other works.

Our main contribution with respect to Γ is the computation of its numerical class in $F(V)$, for which the expression of the class of $\mathcal{T}_5^o$ in $\mathrm{Pic}(\overline{\mathcal{R}}_5)_{\mathbb{Q}}$ plays a crucial role. The numerical class of Γ is surprisingly high in relation to that of Γ'; combining with an explicit analysis of the singular locus of Γ, we obtain enumerative results for lines contained in cubic threefolds.

Resum en català

Les varietats abelianes són varietats algebraiques projectives dotades d'una estructura de grup. Sobre el cos dels nombres complexos, s'obtenen com a tors complexos $\mathbb{C}^g/\Lambda$ que admeten una immersió en un espai projectiu.

Al llarg de les últimes dècades, les varietats abelianes han estat objecte d'una recerca intensiva, jugant un paper fonamental en alguns dels avenços més notables en la Geometria Algebraica. Aquest estudi ha tingut lloc des d'una perspectiva doble. D'una banda, les varietats abelianes són objectes interessants per ells mateixos, com a varietats que tenen una geometria rica; per altra banda, el seu estudi resulta útil per entendre altres varietats algebraiques.

En aquesta tesi investiguem dos problemes motivats per l'estudi de les varietats abelianes, que en certa manera reflecteixen aquesta dualitat. Mentre que en el primer problema la situació ve dominada per les superfícies abelianes (tot i que apareixen generalitzacions naturals a superfícies arbitràries que poden ser mereixedores d'atenció), el segon problema se situa al nexe entre les corbes algebraiques i les varietats abelianes, a més de tenir conseqüències en la geometria dels sòlids cúbics.

El primer problema considerat és el de comprendre les *cohomological rank functions* en superfícies abelianes. Les *cohomological rank functions* $h^i_{F,L}$ associades a un feix coherent (o més en general, a un complex acotat de feixos coherents) F en una varietat abeliana polaritzada (A, L) van ser introduïdes per Jiang i Pareschi a [JP20], com a generalització de les *continuous rank functions* definides per Barja, Pardini i Stoppino a [BPS20b]. Per cada $x \in \mathbb{Q}$, mitjançant els morfismes de multiplicació en A el nombre $h^i_{F,L}(x)$ dóna sentit al i-èsim rang de (hiper)cohomologia de F torçat amb (el representant general de) la polarització fraccionària xL.

Els resultats principals de Jiang i Pareschi sobre aquestes funcions van ser demostrats amb un ús exhaustiu de la transformada de Fourier-Mukai a la varietat abeliana A. La transformada de Fourier-Mukai és una equivalència explícita entre la categoria derivada d'A i la de la seva varietat abeliana dual, que ve induïda pel fibrat de Poincaré (i.e. la família universal de fibrats de línia topològicament trivials en A).

Entre altres aplicacions, les *cohomological rank functions* proporcionen un context més general on considerar nocions de positivitat com GV o la *M-regularitat* per feixos coherents (anàloga a la regularitat de Castelnuovo-Mumford habitual). Aquestes nocions ja havien estat introduïdes per

Pareschi i Popa fa uns vint anys ([PP03, PP11]), també mitjançant la transformada de Fourier-Mukai, i van donar lloc a importants aplicacions a la geometria de les varietats abelianes i irregulars (per exemple [PP04, PP08, BLNP12, JLT13, PS14, CJ18]).

En el cas de les corbes el·líptiques, és ben conegut que les *cohomological rank functions* d'un feix coherent F es poden descriure a través de la seva filtració de Harder-Narasimhan (respecte de l'estabilitat de Mumford). Aquesta filtració descomposa F en factors semiestables de pendent decreixent. No obstant això, per varietats abelianes de dimensió superior només es coneixen uns pocs exemples de funcions, i la seva estructura general està lluny de ser ben entesa.

D'altra banda, Bridgeland ([Bri07]) va definir noves nocions d'estabilitat en la categoria derivada $\mathrm{D}^{\mathrm{b}}(X)$ d'una varietat projectiva llisa X, que generalitzen i flexibilitzen les nocions precedents per feixos. La seva definició inclou una condició tècnica (la *support property*), que permet dotar l'espai $\mathrm{Stab}(X)$ de condicions d'estabilitat amb una estructura de varietat complexa. En altres paraules, les condicions d'estabilitat de Bridgeland es poden deformar.

La regió de $\mathrm{Stab}(X)$ on un objecte $F \in \mathrm{D}^{\mathrm{b}}(X)$ (o diversos) és semiestable resulta ser un invariant que proporciona informació interessant sobre F. Aquestes regions es comporten seguint una estructura de murs i cambres, i el seu estudi (via tècniques de *wall-crossing*) s'ha convertit els darrers anys en una eina potent per atacar molts problemes geomètrics concrets.

En el cas particular d'una superfície projectiva llisa polaritzada (X, L), es pot considerar una condició d'estabilitat $\sigma_{\alpha,\beta}$ per cada $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$, com primer va observar Bridgeland per superfícies K3 i abelianes ([Bri08]) i van generalitzar Arcara i Bertram a superfícies arbitràries ([AB13]). En la regió corresponent de condicions d'estabilitat, que rep el nom de (α, β)*-plane*, el comportament del *wall-crossing* està molt ben entès.

Per $\alpha = 0$, aquesta construcció pot no produir més condicions d'estabilitat; tanmateix, per $\beta \in \mathbb{Q}$ existeixen filtracions de Harder-Narasimhan respecte del pendent induït per $\sigma_{0,\beta}$. Això defineix una anomenada *condició d'estabilitat feble*, que pot ser pensada (després de diverses comprovacions) com un límit de condicions d'estabilitat properes amb $\alpha > 0$.

La informació que proporcionen aquestes condicions d'estabilitat febles és en efecte més feble que la que proporcionen totes les condicions d'estabilitat en el (α, β)*-plane*, però encara resulta útil. En la primera part d'aquesta tesi pretenem classificar aquesta informació definint, per cada objecte $F \in \mathrm{D}^{\mathrm{b}}(X)$, *Chern degree functions*

$$\mathrm{chd}^k_{F,L} : \mathbb{Q} \to \mathbb{Q}_{\geq 0}$$

determinades per les filtracions de Harder-Narasimhan de F respecte de les condicions d'estabilitat febles al llarg de la recta $\alpha = 0$.

Aquestes funcions admeten expressions locals i satisfan propietats de continuïtat similars a les obtingudes per Jiang i Pareschi. També descrivim la seva derivabilitat en termes d'estabilitat. Els nostres arguments fan servir sistemàticament el *wall-crossing* i poden ser interessants de manera independent, especialment en aquelles superfícies X per les quals la recta $\alpha = 0$ se situa en la frontera de la varietat Stab(X) de condicions d'estabilitat.

En el cas particular de les superfícies abelianes, demostrem que les *Chern degree functions* recuperen les *cohomological rank functions* de Jiang i Pareschi. Això estableix un paral·lelisme amb el cas de les corbes el·líptiques, ja que obtenim que les *cohomological rank functions* en superfícies abelianes venen determinades per l'estabilitat. Aquest acostament al problema, sense transformades de Fourier-Mukai, dóna en moltes situacions una descripció clara de propietats com la derivabilitat o la racionalitat de punts crítics.

A més, aquesta nova presentació també resulta útil per calcular exemples particulars. El més rellevant és el feix d'ideals d'un punt, a partir del qual obtenim nous resultats sobre syzygies de superfícies abelianes polaritzades.

Les syzygies són una noció de gran interès per entendre una varietat projectiva immersa, ja que bàsicament contenen tota la informació que es pot extreure de les equacions a l'espai ambient. Com a tals, en general són molt difícils de controlar; per enters $p \geq 0$, les propietats (N_p) són condicions de senzillesa per aquestes syzygies. A grans trets, la propietat (N_p) significa que els primers p passos de la resolució lliure graduada minimal de l'ideal homogeni de la varietat són lineals.

Green ([Gre84]) va demostrar que la propietat (N_p) se satisfà en corbes immerses per fibrats de línia de grau suficientment alt, tot generalitzant de manera unificada resultats anteriors de Castelnuovo, Mattuck, Fujita i Saint-Donat. A més, va enunciar una famosa conjectura relacionant l'índex de Clifford amb el no-compliment de (N_p) per corbes canòniques, que va ser resolta per Voisin per corbes generals ([Voi02, Voi05]).

En el cas de les varietats abelianes, Lazarsfeld va conjecturar el compliment de la propietat (N_p) per potències L^m $(m \geq p+3)$ de qualsevol fibrat de línia ample L, com a generalització de resultats anteriors de Koizumi, Mumford i Kempf. Pareschi ([Par00]) va donar una demostració en característica zero. En característica arbitrària, Caucci va donar recentment una prova ([Cau20]), com a aplicació d'un criteri relacionant les *cohomological rank functions* $h^i_{\mathcal{I}_0,L}$ amb les propietats (N_p). D'acord amb el criteri de Caucci (i alguns refinaments d'Ito [Ito21]), els nostres càlculs explícits per $h^i_{\mathcal{I}_0,L}$ en superfícies abelianes condueixen a nous resultats efectius sobre les seves syzygies.

El segon problema tractat en aquesta tesi aborda els recobriments dobles no ramificats de corbes amb un *semicanonical pencil*, i les seves varietats de Prym. Les varietats de Prym són varietats abelianes principalment polaritzades (*vapp*'s a partir d'ara) associades a recobriments dobles no ramificats de corbes. Des d'un punt de vista analític ja havien estat considerades per Wirtinger a finals del segle XIX, però va ser Mumford al seu treball fonamental [Mum74] qui va establir el nom actual (en honor al matemàtic alemany Friedrich Prym) i les va presentar en un llenguatge algebraic modern per primera vegada.

Des d'aleshores, han esdevingut una eina important en la Geometria Algebraica, ja que formen una classe de vapp's més general que no pas les Jacobianes. Per l'exemple l'*aplicació de Prym*

$\mathcal{P}_g : \mathcal{R}_g \to \mathcal{A}_{g-1}$ que associa a cada recobriment doble no ramificat d'una corba llisa de gènere g la seva varietat de Prym (principalment polaritzada) de dimensió $g-1$, és dominant per $g \leq 6$. Aquesta és la raó per la qual les vapp's de fins a dimensió 5 són ben enteses.

En efecte, com en el cas de les Jacobianes la geometria de corbes permet extreure molta informació sobre la geometria de les Pryms. Per exemple, el treball de Mumford va classificar les singularitats del divisor theta d'una varietat de Prym en *singularitats estables* i *singularitats excepcionals*. L'exemple més elemental de singularitat excepcional té lloc per recobriments dobles no ramificats $\tilde{C} \xrightarrow{f} C$ que satisfan les següents condicions:

- La corba C té un *semicanonical pencil*, és a dir, un fibrat de línia L de grau $g-1$ en C tal que $L^2 \cong \omega_C$ i $h^0(C, L)$ és parell i positiu.
- El nombre $h^0(\tilde{C}, f^*L)$ és parell.

Un tal fibrat de línia L l'anomenarem un *semicanonical pencil parell* pel recobriment f. En cas que existeixi, la corresponent varietat de Prym pertany al divisor $\theta_{null} \subset \mathcal{A}_{g-1}$ de vapp's amb divisor theta contenint un punt singular de 2-torsió.

Al seu article [Bea77a], Beauville va demostrar que el lloc d'Andreotti-Mayer $\mathcal{N}_0 \subset \mathcal{A}_4$ (de vapp's de dimensió 4 amb divisor theta singular) és la unió de dos divisors irreductibles: (la clausura de) el lloc Jacobià $\mathcal{J}_4$ i θ_{null}. Una eina essencial va ser l'extensió de l'aplicació de Prym a una aplicació pròpia $\widetilde{\mathcal{P}}_g : \widetilde{\mathcal{R}}_g \to \mathcal{A}_{g-1}$, considerant recobriments admissibles de corbes (possiblement nodals). En el cas $g = 5$, això garanteix que tota vapp de dimensió 4 és una varietat de Prym (és a dir, es reemplaça el morfisme dominant $\mathcal{P}_5$ per l'epimorfisme $\widetilde{\mathcal{P}}_5$).

Un dels punts clau del treball de Beauville és la identificació dels recobriments que tenen varietat de Prym a θ_{null}; Beauville demostra que

$$\mathcal{T}^e = (\text{clausura de}) \; \{[f : \widetilde{C} \longrightarrow C] \in \mathcal{R}_5 \mid \text{El recobriment } f \text{ té un } \textit{semicanonical pencil} \text{ parell}\}$$

és irreductible i igual a $\widetilde{\mathcal{P}}_5^{-1}(\theta_{null})$. En efecte, la demostració de la irreductibilitat de θ_{null} es basa en la irreductibilitat de $\mathcal{T}^e$, i comença amb l'observació que el lloc $\mathcal{T} \subset \mathcal{M}_5$ de corbes de gènere 5 amb un *semicanonical pencil* és un divisor irreductible en l'espai de moduli $\mathcal{M}_5$.

Ara considerem la següent situació: per un gènere fixat $g \geq 3$, sigui $\mathcal{T}_g \subset \mathcal{M}_g$ el lloc de corbes llises de gènere g que admeten un *semicanonical pencil*, que correspon a un divisor en l'espai de moduli $\mathcal{M}_g$ com és ben conegut. L'element general de $\mathcal{T}_g$ té un únic *semicanonical pencil* L, i per tant la preimatge de $\mathcal{T}_g$ a $\mathcal{R}_g$ descomposa com una unió $\mathcal{T}_g^e \cup \mathcal{T}_g^o$ d'acord amb la paritat de $h^0(\widetilde{C}, f^*L)$.

D'acord amb el treball de Beauville, sembla natural preguntar-se si $\mathcal{T}_g^e$ i $\mathcal{T}_g^o$ són divisors irreductibles, i també preguntar-se sobre el comportament de les aplicacions de Prym restringides $\mathcal{P}_g|_{\mathcal{T}_g^e}$ i $\mathcal{P}_g|_{\mathcal{T}_g^o}$ (o $\widetilde{\mathcal{P}}_g|_{\mathcal{T}_g^e}$ i $\widetilde{\mathcal{P}}_g|_{\mathcal{T}_g^o}$, si hom considera les clausures de $\mathcal{T}_g^e$ i $\mathcal{T}_g^o$ en $\widetilde{\mathcal{R}}_g$). Aquestes dues preguntes són l'objecte de la segona part de la tesi.

Respecte de la primera pregunta, el divisor $\mathcal{T}_g \subset \mathcal{M}_g$ va ser estudiat per Teixidor a [TiB88]. Fent servir la teoria de *limit linear series* desenvolupada per Eisenbud i Harris, Teixidor va demostrar la irreductibilitat de $\mathcal{T}_g$ i va calcular la classe de la seva clausura a la compactificació de Deligne-Mumford $\overline{\mathcal{M}}_g$. Seguint aquest enfocament, hem obtingut anàlegs naturals d'aquests resultats, és a dir: $\mathcal{T}_g^e$ i $\mathcal{T}_g^o$ són divisors irreductibles de $\mathcal{R}_g$, i hem calculat les classes de les seves clausures en $\mathrm{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}$, on $\overline{\mathcal{R}}_g$ denota la compactificació de Deligne-Mumford.

Mentre que nosaltres només farem servir aquestes classes com a eina per atacar la segona pregunta, val a dir que el càlcul de classes de divisors efectius ha estat molt aplicat a l'estudi de la geometria birracional de certs espais de moduli. Aquest punt de vista va començar amb el treball pioner [HM82] de Harris i Mumford per $\overline{\mathcal{M}}_g$; el lector pot trobar a [Far09, Section 1] una síntesi de caire històric sobre aquest problema, i resultats pel cas de $\overline{\mathcal{R}}_g$ a [FL10].

Respecte de la segona pregunta cal assenyalar que, a part de la ja comentada anàlisi de $\mathcal{T}_5^e$ per part de Beauville, Izadi [Iza95] va demostrar l'exhaustivitat de l'aplicació de Prym restringida $\widetilde{\mathcal{P}}_5|_{\mathcal{T}_5^o}$. Com veurem, aquest és un fenòmen general quan $3 \leq g \leq 5$: mentre que $\mathcal{T}_g^e$ és la preimatge de $\theta_{null} \subset \mathcal{A}_{g-1}$, l'aplicació $\widetilde{\mathcal{P}}_g|_{\mathcal{T}_g^o}$ és exhaustiva. A més, en aquest casos descrivim explícitament la fibra general de $\widetilde{\mathcal{P}}_g|_{\mathcal{T}_g^o}$, que tenen una geometria rellevant; per exemple, la nostra anàlisi de la fibra general de $\widetilde{\mathcal{P}}_5|_{\mathcal{T}_5^o}$ presenta una aplicació sorprenent a la geometria dels sòlids cúbics.

Els sòlids cúbics han estat objectes molt investigats des dels inicis de la Geometria Algebraica; per exemple, la seva uniracionalitat ja era coneguda per Max Noether. A principis del segle XX van ser estudiats per Fano, en un intent de demostrar la seva irracionalitat i per tant de donar una resposta negativa al problema de Lüroth. La irracionalitat va ser demostrada finalment per Clemens i Griffiths al seu cèlebre article [CG72], estudiant una ppav (la *Jacobiana intermèdia*) associada a cada sòlid cúbic. Resultats posteriors de Mumford, Tjurin, Beauville, Donagi, Smith, Casalaina-Martin i Friedman (entre molts altres) van donar informació més precisa sobre la Jacobiana intermèdia i el seu divisor theta, a través de la seva presentació com a varietat de Prym.

Tornant al nostre problema, Donagi ([Don92]) va descriure la fibra general de l'aplicació de Prym $\widetilde{\mathcal{P}}_5 : \widetilde{\mathcal{R}}_5 \to \mathcal{A}_4$ com un recobriment doble no ramificat de la superfície de Fano de rectes $F(V)$ d'un sòlid cúbic general V. En aquest context, la fibra general de $\widetilde{\mathcal{P}}_5|_{\mathcal{T}_5^o}$ és una corba irreductible, que és una desingularització parcial de

$$\Gamma = \{l \in F(V) \mid \text{Existeixen un 2-pla } \pi \text{ i una recta } r \in F(V) \text{ tals que } V \cdot \pi = l + 2r\} \subset F(V)$$

Aquesta corba Γ roman molt poc explorada, a diferència de la seva contrapartida natural (la corba Γ' formada per les *rectes de segon tipus*), que va ser estudiada a [CG72] entre molts altres treballs.

La nostra contribució principal respecte de Γ és el càlcul de la seva classe numèrica a $F(V)$, pel qual l'expressió de la classe de $\mathcal{T}_5^o$ a $\mathrm{Pic}(\overline{\mathcal{R}}_5)_{\mathbb{Q}}$ juga un paper crucial. La classe numèrica de Γ és sorprenentment alta en relació a la de Γ'; juntament amb una anàlisi explicita del lloc singular de Γ, obtenim resultats enumeratius per rectes en sòlids cúbics.

Contents

Introduction

This thesis splits into two independent parts; each of them corresponds to the study of a problem concerning abelian varieties. The subsequent pages introduce the two problems separately and report on the main results obtained.

Chern degree functions

The results obtained in this first part can be found in the preprints [LR21] and [Roj21].

In the context of irregular varieties, Barja, Pardini and Stoppino [BPS20b] introduced the *continuous rank function* associated to a line bundle, a continuous function defined on a line in the space of $\mathbb{R}$-divisor classes with similar properties to those of the volume function. Shortly after, in [JP20] Jiang and Pareschi generalized this notion to the *cohomological rank functions* $h^i_{F,L}$ associated to a coherent sheaf (or more generally, a bounded complex of coherent sheaves) F on a polarized abelian variety (A, L).

These functions have received several applications. For instance, they have been used to prove new Clifford–Severi inequalities, i.e. geographical lower bounds of the volume of a line bundle and characterize the polarized varieties where the bound is attained (see [BPS20a, Jia21]). They have been also applied to the study of syzygies of abelian varieties, as we will specify later.

Given $x \in \mathbb{Q}$, the number $h^i_{F,L}(x)$ makes sense of the (generic) i-th cohomological rank of F twisted with the fractional polarization xL, thanks to the multiplication maps on A. The two main results of Jiang and Pareschi about the general structure of these functions, a priori only defined over the rational numbers, can be summarized as follows:

(1) [JP20, Corollary 2.6] Every $x_0 \in \mathbb{Q}$ admits a left (resp. right) neighborhood where the function $h^i_{F,L}$ is given by an explicit polynomial P^- (resp. P^+) depending on x_0.
(2) [JP20, Theorem 3.2] The functions extend to continuous real functions of real variable.

These results were proved via an extensive use of the Fourier-Mukai transform on the abelian variety, which also justifies the need to extend the definition of the cohomological rank functions to arbitrary objects in the derived category of coherent sheaves.

In the case of elliptic curves, it is well known that the Harder-Narasimhan filtration of a coherent sheaf F provides a precise description of its cohomological rank functions (see Proposition I.2.3 for details). Nevertheless for higher-dimensional abelian varieties only a few concrete examples of functions are known, and their general structure is far from being understood. For instance, the question of whether cohomological rank functions are always piecewise polynomial remains open (see [JP20, Remark 2.8]).

The first part of the present thesis investigates the relation between the functions and stability in the case of surfaces, from a twofold perspective: not only that of obtaining a better understanding of cohomological rank functions on abelian surfaces, but also that of proposing similar invariants on arbitrary (i.e. not necessarily abelian) polarized surfaces. This can be done in a unified way, by means of the *Chern degree functions* that we attach to objects in the derived category $\mathrm{D^b}(X)$ of any smooth polarized surface (X, L). These functions are our main object of study in this first part, and their definition relies on a certain set of (weak) stability conditions on $\mathrm{D^b}(X)$.

More precisely, let (X, L) be a smooth polarized surface. Since the seminal work of Bridgeland [Bri08] (for K3 surfaces), then extended by Arcara and Bertram ([AB13]) to arbitrary smooth surfaces, one can consider a Bridgeland stability condition $\sigma_{\alpha,\beta} = (Z_{\alpha,\beta}, \mathrm{Coh}^\beta(X))$ for every $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$, where $\mathrm{Coh}^\beta(X)$ is the heart of a bounded t-structure on $\mathrm{D^b}(X)$. In the corresponding (α, β)*-plane* of stability conditions, the wall-crossing behaviour is very well understood. A similar construction holds in positive characteristic as well, thanks to work of Koseki [Kos20] (even though the cases we will consider were already covered by the work of Langer [Lan16]).

When $\alpha = 0$, one cannot ensure in general that $\sigma_{0,\beta}$ is a Bridgeland stability condition. Roughly speaking, this depends on whether the classical Bogomolov inequality for slope-semistable sheaves of slope β is sharp or not. In case of non-sharpness, the construction of Bridgeland stability conditions can be extended to a larger region containing $\sigma_{0,\beta}$ in its interior, as explained in the recent work [FLZ21, section 3].

Regardless of whether $\sigma_{0,\beta}$ is a Bridgeland stability condition or not, for every $\beta \in \mathbb{Q}$ one can at least ensure the existence of Harder-Narasimhan filtrations with respect to the tilt slope $\nu_{0,\beta}$ induced by $\sigma_{0,\beta}$: this defines a *weak stability condition*. For instance, in the literature $\sigma_{0,0}$ has also received the name of *Brill-Noether stability condition* ([Li19, Definition 2.11]).

The Chern degree functions may be thought of as a way of classifying the information provided by these (possibly weak) stability conditions. Basically, to every object $E \in \mathrm{D^b}(X)$ we associate nonnegative functions

$$\mathrm{chd}^k_{E,L} : \mathbb{Q} \to \mathbb{Q}_{\geq 0}$$

for every $k \in \mathbb{Z}$, related by the identity $\sum (-1)^k \cdot \mathrm{chd}^k_{E,L}(x) = \mathrm{ch}_2^{-x}(E)$. For instance, if $F \in \mathrm{Coh}^\beta(X)$ then only $\mathrm{chd}^0_{F,L}$ and $\mathrm{chd}^1_{F,L}$ are nonzero at $-\beta$, and

$$\mathrm{chd}^0_{F,L}(-\beta) = \mathrm{ch}_2^\beta(\widetilde{F}),$$

where $\widetilde{F}$ is the maximal subobject $\widetilde{F} \subset F$ in $\mathrm{Coh}^\beta(X)$ with the property that any quotient of $\widetilde{F}$ has positive tilt slope $\nu_{0,\beta}$ (see section II.1 for the general definition).

Our first main result about these functions is:

Theorem A. *Let (X, L) be a smooth polarized projective surface over an algebraically closed field $\mathbb{K}$. If $\operatorname{char}\mathbb{K} > 0$, assume that X is neither of general type nor quasi-elliptic with $\kappa(X) = 1$.*

Then, for every object $E \in \mathrm{D}^{\mathrm{b}}(X)$ and $k \in \mathbb{Z}$ the following hold:

(1) (Theorem II.2.21) *Every rational number $x_0 \in \mathbb{Q}$ admits a left (resp. right) neighborhood where the function $\mathrm{chd}^k_{E,L}$ is given by an explicit polynomial P^- (resp. P^+) depending on x_0, satisfying $P^-(x_0) = \mathrm{chd}^k_{E,L}(x_0) = P^+(x_0)$.*

(2) (Corollary II.3.3) *The function $\mathrm{chd}^k_{E,L}$ extends to a continuous real function of real variable.*

Let us specify that, even in the case where $F \in \mathrm{Coh}^{\beta_0}(X)$, the local polynomial expressions for $\mathrm{chd}^0_{F,L}$ at $-\beta_0$ are not necessarily $\mathrm{ch}_2^{-x}(\widetilde{F})$. For instance, the polynomial expression in a left neighborhood of $-\beta_0$ is given by $\mathrm{ch}_2^{-x}(G)$, where $G \subset \widetilde{F}$ is a certain subobject in $\mathrm{Coh}^{\beta_0}(X)$ satisfying $Z_{0,\beta_0}(\widetilde{F}/G) = 0$ and appearing in the Harder-Narasimhan filtration of F with respect to $\sigma_{0,\beta}$ for all small enough $\beta > \beta_0$.

In consequence, in order to control the local polynomial expressions one is led to determine the Harder-Narasimhan filtrations of F with respect to $\sigma_{0,\beta}$ as β tends to β_0; we call such filtrations the *weak limit filtrations* of F at β_0 (see Definition II.2.9 for details). Their study requires a good understanding of *Bridgeland limit filtrations* as a first step; these are Harder-Narasimhan filtrations with respect to the (honest) Bridgeland stability conditions $\sigma_{\alpha,\beta}$ as α tends to 0 (see Definition II.2.1).

These auxiliary notions may be of independent interest to study the boundary of the stability manifold (see also Remark II.1.7); as a tool for Theorem A.(1), we prove their existence for objects without Harder-Narasimhan factors of vanishing tilt slope.

Theorem B. *Let $\beta_0 \in \mathbb{Q}$ and $F \in \mathrm{Coh}^{\beta_0}(X)$ be an object having no Harder-Narasimhan factor with respect to σ_{0,β_0} of vanishing tilt slope ν_{0,β_0}. Then,*

(1) (Theorem II.2.8) *F admits a Bridgeland limit Harder-Narasimhan filtration at β_0.*

(2) (Theorem II.2.14) *F admits a weak limit Harder-Narasimhan filtration at β_0.*

Theorem A.(1) almost follows from Theorem B.(2); the proof is completed after considering the objects with a Harder-Narasimhan factor of tilt slope 0. The treatment of these exceptional situations, as well as the proof of Theorem B, entails a systematic exploitation of wall-crossing.

On the other hand, the extension to continuous real functions (i.e. Theorem A.(2)) is obtained by integration after using the local polynomial expression to bound the derivative, following the approach of [JP20] for the cohomological rank functions.

Once the continuity is settled, it is natural to study the rational *critical points* (that is, rational points where the Chern degree functions are not of class $\mathcal{C}^\infty$). In particular we obtain the following result characterizing the rational points where the functions are not of class $\mathcal{C}^1$, as a corollary of our characterization of the critical points in terms of stability (see Proposition II.3.4).

Proposition C. *Let $\beta \in \mathbb{Q}$. If $F \in \mathrm{Coh}^\beta(X)$, then the Chern degree functions of F are not differentiable at $-\beta$ if and only if F has a Harder-Narasimhan factor (with respect to $\sigma_{0,\beta}$) of vanishing tilt slope.*

We must point out that the Chern degree functions seem to be especially nontrivial for polarized surfaces on which $\sigma_{0,\beta}$ fails to be a Bridgeland stability condition for every $\beta \in \mathbb{Q}$. Indeed, in those cases we do not know how to conclude from Theorem A that the Chern degree functions are piecewise polynomial, as one would expect.

Among these polarized surfaces, it is well known that one finds abelian surfaces. As we explain in Example I.1.15, this is also the case for surfaces with finite Albanese map (and polarization pulled back from the Albanese variety), which in particular covers polarized irregular surfaces with Picard rank 1[1]. In such cases of irregular (non-abelian) surfaces, the relation of the Chern degree functions to the original continuous rank functions, or more generally to the cohomological rank functions of the push-forwarded object via the Albanese map, remains mysterious.

On the opposite side, for regular surfaces such as K3 surfaces, the Chern degree functions are clearly piecewise polynomial, since their stability manifold is certainly larger than the (α, β)-plane. This indicates that these functions may not be the most challenging ones for regular surfaces; in Remark II.1.7 we propose a variant (explicitly for K3 surfaces of Picard rank 1).

While for arbitrary polarized surfaces these functions (and variants) may be the object of future work, in the case of abelian surfaces the Chern degree functions recover the cohomological rank functions of Jiang and Pareschi. This establishes a natural analogue with the structure of cohomological rank functions on elliptic curves:

Theorem D (Theorem II.4.3)**.** *Let (X, L) be a polarized abelian surface. Then, the Chern degree function $\mathrm{chd}^k_{E,L}$ equals the cohomological rank function $h^k_{E,L}$ for every object $E \in \mathrm{D^b}(X)$ and $k \in \mathbb{Z}$.*

Note that this shows that cohomological rank functions are determined by stability. Basically, the cohomological rank functions of an object at $-\beta$ split into simpler pieces, corresponding to its Harder-Narasimhan factors with respect to $\sigma_{0,\beta}$. Conversely, knowing the cohomological rank functions of an object one has certain constraints on its behaviour with respect to stability (the easiest case being an object that never destabilizes, see Proposition II.5.1).

This new presentation turns out to be significantly useful to understand cohomological rank functions on abelian surfaces. For instance, Proposition C (and more generally Proposition II.3.4) characterizes their differentiability at rational critical points, which in [JP20, Proposition 4.4] was

[1]This provides new examples (X, L) where the Le Potier function $\Phi_{X,L}$ is explicitly known, which may be of independent interest since allows to describe their Bridgeland stability manifold (cf. [FLZ21, Section 3]).

settled a sufficient condition to control the dimension of certain jump loci. Observe also that, thanks to Theorem A.(2), we obtain that cohomological rank functions extend to continuous real functions in positive characteristic as well (a case not covered by Jiang and Pareschi).

Furthermore, the stability viewpoint is fruitful for the computation of particular examples, when one exploits the wall-crossing behaviour in the (α, β)-plane. This is particularly affordable for Gieseker semistable sheaves, since they are semistable objects for large values of α (at the so-called *Gieseker chamber*).

For them, we propose a method based on successive destabilizations along the (α, β)-plane; in many concrete situations, it gives an explicit description of $\mathrm{chd}^0_{F,L}$ as a piecewise polynomial function. We illustrate this method with two geometric examples: the ideal sheaves of 0-dimensional subschemes of low length on principally polarized abelian surfaces, and the ideal sheaf of one point for abelian surfaces endowed with a polarization of arbitrary type.

The cohomological rank functions of the ideal sheaf of one point have recently received considerable attention as a tool to understand syzygies of abelian varieties. For a polarized abelian variety (A, L) (of arbitrary dimension) and a point $q \in A$, Jiang and Pareschi already observed in [JP20, Section 8] that the *basepoint-freeness threshold*

$$\epsilon_1(L) = \inf \left\{ x \in \mathbb{Q} \mid h^1_{\mathcal{I}_q,L}(x) = 0 \right\}$$

(independent of the choice of q) encodes interesting positivity properties of the polarization L:

(1) $\epsilon_1(L) \leq 1$, with equality if and only if any line bundle representing L has base points.
(2) [JP20, Corollary E] If $\epsilon_1(L) < \frac{1}{2}$, then any line bundle representing L is projectively normal.

Subsequent work of Caucci generalized (2) to higher syzygies, proving that every line bundle representing L satisfies the property (N_p) as long as $\epsilon_1(L) < \frac{1}{p+2}$ ([Cau20, Theorem 1.1]). Roughly, the property (N_p) means that we have a very ample line bundle, with certain conditions of simplicity on the syzygies of the homogeneous ideal of the embedded variety. For instance, (N_0) is equivalent to the embedding being projectively normal, and (N_1) also requires that the homogeneous ideal is generated by quadrics.

As a consequence, Caucci obtained a proof of Lazarsfeld's conjecture in arbitrary characteristic: any m-th power $(m \geq p + 3)$ of an ample line bundle on an abelian variety satisfies the property (N_p). This conjecture had originally been proved in $\mathrm{char}(\mathbb{K}) = 0$ by Pareschi ([Par00]).

Caucci's criterion has also been applied to understand the syzygies of abelian varieties endowed with a primitive polarization (i.e. a polarization which is not a multiple of another one), by means of upper bounds for the basepoint-freeness threshold (see [Jia20, Ito20a, Ito20b]). Furthermore, for $p \geq 1$ the hypothesis $\epsilon_1(L) < \frac{1}{p+2}$ ensuring (N_p) has recently been slightly weakened by Ito ([Ito21, Theorem 1.5]).

In the case of abelian surfaces, our stability approach allows to give explicit expressions for the cohomological rank function $h^0_{\mathcal{I}_q,L}$ (which is enough for determining $h^1_{\mathcal{I}_q,L}$ and hence $\epsilon_1(L)$). We do this for a certain class of polarized abelian surfaces which includes those with Picard rank 1. More precisely, we prove the following (see Theorem III.2.2 and Corollary III.3.2):

Theorem E. *Let (X, L) be a $(1, d)$-polarized abelian surface over an algebraically closed field $\mathbb{K}$, and let $q \in X$ be a (closed) point. Assume that $D \cdot L$ is a multiple of L^2 for every divisor class D.*

(1) *If d is a perfect square, then the cohomological rank function $h^0_{\mathcal{I}_q,L}$ reads*

$$h^0_{\mathcal{I}_q,L}(x) = \begin{cases} 0 & x \leq \frac{\sqrt{d}}{d} \\ dx^2 - 1 & x \geq \frac{\sqrt{d}}{d} \end{cases}$$

In particular, $\epsilon_1(L) = \frac{\sqrt{d}}{d}$.

(2) *If d is not a perfect square, then the function $h^0_{\mathcal{I}_q,L}$ is either that of (1) or*

$$h^0_{\mathcal{I}_q,L}(x) = \begin{cases} 0 & x \leq \frac{2\tilde{y}}{\tilde{x}+1} \\ \frac{d(\tilde{x}+1)}{2}x^2 - 2d\tilde{y} \cdot x + \frac{\tilde{x}-1}{2} & \frac{2\tilde{y}}{\tilde{x}+1} \leq x \leq \frac{2\tilde{y}}{\tilde{x}-1} \\ dx^2 - 1 & x \geq \frac{2\tilde{y}}{\tilde{x}-1} \end{cases}$$

where $(\tilde{x}, \tilde{y})$ is a nontrivial positive solution to Pell's equation $x^2 - 4d \cdot y^2 = 1$. In particular, if (x_0, y_0) is the minimal positive solution to this equation, then $\epsilon_1(L) \leq \frac{2y_0}{x_0-1}$.

(3) *Under the hypothesis of (2), assume also that $\mathrm{char}(\mathbb{K})$ divides neither x_0^2 nor $x_0^2 - 1$. Then the expression for $h^0_{\mathcal{I}_q,L}$ is the one corresponding to either the minimal solution (x_0, y_0) or to the second smallest positive solution (x_1, y_1). In particular, $\epsilon_1(L) \in \{\frac{2y_0}{x_0-1}, \frac{2y_1}{x_1-1}\}$.*

The key point for the proof of (1) and (2) is the fact that potential destabilizing walls for $\mathcal{I}_q$ are in correspondence with positive solutions to Pell's equation $x^2 - 4d \cdot y^2 = 1$ (see Lemma III.2.1). The absence of such solutions when d is a perfect square shows (1), whereas for d not a perfect square one obtains (2).

The corresponding upper bounds for the basepoint-freeness threshold refine those given by Ito for general complex abelian surfaces ([Ito20b]). In addition, the expressions of (1) and (2) reveal the differentiability of $h^0_{\mathcal{I}_q,L}$ at certain rational points; this is relevant with regard to syzygies, since it enables us to apply Ito's refined version of Caucci's criterion. As a result, we obtain new effective statements for syzygies on abelian surfaces:

Corollary F (Corollary III.2.5)**.** *Let (X, L) be a $(1, d)$-polarized abelian surface over an algebraically closed field $\mathbb{K}$, such that $D \cdot L$ is a multiple of L^2 for every divisor class D.*

(1) *If $d \geq 7$, then any ample line bundle representing L is projective normal.*
(2) *If $d > (p+2)^2$ for $p \geq 1$, then any ample line bundle representing L satisfies the property (N_p).*

For $\mathbb{K} = \mathbb{C}$, we point out that Corollary F.(1) recovers a well-known result of Iyer ([Iye99], see also [Laz90] for some cases previously covered), and the case $p = 1$ of Corollary F.(2) recovers a result of Gross and Popescu ([GP98]). For arbitrary p, Corollary F.(2) improves the bound ensuring the property (N_p) that was given recently by Ito in [Ito20b, Corollary 4.5].

On the other hand, for d not a perfect square Theorem E.(3) establishes that only two of the potential functions described in Theorem E.(2) may occur (modulo certain arithmetic restrictions on char($\mathbb{K}$)): those corresponding to the two smallest positive solutions of Pell's equation.

The proof of Theorem E.(3) relies on the explicit construction of curves containing all the torsion points of an unexpectedly high order (see Proposition III.3.1); for this construction, we require the classical theory of theta groups developed by Mumford in [Mum66].

It is worth noting that, for all the non-perfect squares d for which we know the exact value of $\epsilon_1(L)$, the equality $\epsilon_1(L) = \frac{2y_0}{x_0-1}$ holds. We expect this to be true in general; it would follow from a small refinement of Proposition III.3.1, that at present we do not know how to prove (see Remark III.3.3.(2) for details).

Structure of the first part. This first part of the thesis consists of three chapters. Chapter I introduces, mostly without proofs, preliminary theory concerning stability conditions, cohomological rank functions on abelian varieties and theta groups of ample line bundles on abelian varieties.

Chapter II constitutes the core of this first part, as it studies Chern degree functions in the general context of a smooth polarized surface. First we define the functions and prove some elementary properties in section II.1. Then an important part of the work is contained in section II.2, which is devoted to find the local polynomial expressions (Theorem A.(1)): we introduce the notions of Bridgeland and weak limit filtrations, and prove their existence (Theorem B) along the way.

The proof of Theorem A.(2) occupies the first part of section II.3. The rest of section II.3 addresses the characterization of rational critical points and their differentiability, which leads to Proposition II.3.4 (and in particular, to Proposition C). In section II.4 we prove the equivalence with cohomological rank functions on abelian surfaces (Theorem D). Finally, section II.5 discusses the structure of Chern degree functions of (twisted) Gieseker semistable sheaves, as well as methods for their computation.

In chapter III we compute new examples of cohomological rank functions, by means of our stability approach. In section III.1 we treat the ideal sheaves of finite subschemes on principally polarized abelian surfaces; then the rest of the chapter is devoted to the ideal sheaf of one point. In section III.2 we study upper bounds for the basepoint-freeness threshold and their consequences on the syzygies of abelian surfaces (in particular, we prove Corollary F and most of Theorem E). The proof of Theorem E is completed in section III.3, where we find lower bounds for the basepoint-freeness threshold using the theory of theta groups.

Prym semicanonical pencils

The results obtained in this second part can be found in the preprints [MPR21] and [LNR21].

Let $\mathcal{T}_g \subset \mathcal{M}_g$ be the divisor of (isomorphism classes of) complex, smooth, irreducible curves C of genus $g \geq 3$ with a semicanonical pencil, that is, with a theta-characteristic $L \in \mathrm{Pic}^{g-1}(C)$ such that $h^0(C, L)$ is even and positive. This divisor was studied in [TiB88], where Teixidor proved its irreducibility and computed the class of its closure in the rational Picard group of the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$.

The pullback of $\mathcal{T}_g$ to the moduli space of smooth Prym curves

$$\mathcal{R}_g = \{(C, \eta) \mid [C] \in \mathcal{M}_g, \eta \in JC_2 \setminus \{\mathcal{O}_C\}\}/\cong$$

(i.e. of double étale irreducible covers of smooth curves) via the forgetful map breaks up into two divisors. Indeed, since the general element of $\mathcal{T}_g$ admits a unique semicanonical pencil and the parity of theta-characteristics remains constant in families, the pullback of $\mathcal{T}_g$ is the union of the following divisors:

$$\mathcal{T}_g^e = \left\{(C, \eta) \in \mathcal{R}_g \mid C \text{ has a semicanonical pencil } L \text{ with } h^0(C, L \otimes \eta) \text{ even}\right\}$$
$$\mathcal{T}_g^o = \left\{(C, \eta) \in \mathcal{R}_g \mid C \text{ has a semicanonical pencil } L \text{ with } h^0(C, L \otimes \eta) \text{ odd}\right\}$$

We will call $\mathcal{T}_g^e$ (resp. $\mathcal{T}_g^o$) the divisor of *even* (resp. *odd*) *semicanonical pencils*. For simplicity, we use the same notation for the divisors in $\mathcal{R}_g$ and for their closures in the Deligne-Mumford compactification $\overline{\mathcal{R}}_g$.

This second part of the dissertation investigates the divisors $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$ from two different viewpoints. On the one hand, we are interested in the geometry of $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$, and their relation to other divisors of $\mathcal{R}_g$ (and $\overline{\mathcal{R}}_g$). On the other hand, we aim to understand their interplay with the geometry of abelian varieties via the Prym map $\mathcal{P}_g : \mathcal{R}_g \to \mathcal{A}_{g-1}$ (and Beauville's extension $\widetilde{\mathcal{P}}_g$ to a proper map).

The Prym map has well-known generic fibers for $g \leq 6$ and is generically injective for $g \geq 7$; nonetheless, its restriction to divisors is often far from being understood. For instance, the restriction of $\widetilde{\mathcal{P}}_g$ to the divisor of Beauville admissible covers of nodal curves has recently received attention, since its study is equivalent to that of the so-called *ramified Prym map* (see [MP12] and [NO19]).

The problem we are facing up can be found in the literature for genus 5 in two very remarkable works. The even case $\mathcal{T}_5^e$ was considered by Beauville in [Bea77a], where he proved that $\mathcal{T}_5^e$ is irreducible and equals the preimage of $\theta_{null} \subset \mathcal{A}_4$ via $\widetilde{\mathcal{P}}_5$; in this way, he obtained the irreducibility of θ_{null}. Furthermore, Izadi proved that $\mathcal{T}_5^o$ dominates $\mathcal{A}_4$ (see [Iza95, Proof of Theorem 6.14], where $\mathcal{T}_5^o$ is denoted by $\theta_{null\,2}$).

With regard to the first question, the analysis of $\mathcal{T}_g$ performed by Teixidor in [TiB88] relies on the theory of limit linear series developed by Eisenbud and Harris. Following her approach, we have described the intersection of $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$ with the boundary divisors of covers of reducible curves (Proposition V.1.2), and have obtained natural analogues of Teixidor's results for $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$:

Theorem G (Theorem V.1.1)**.** *Let $g \geq 5$ and let $[\mathcal{T}_g^e], [\mathcal{T}_g^o] \in \mathrm{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}$ denote the classes of (the closures in $\overline{\mathcal{R}}_g$ of) the divisors $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$. Then, the following equalities hold:*

$$[\mathcal{T}_g^e] = a\lambda - b_0'\delta_0' - b_0''\delta_0'' - b_0^{ram}\delta_0^{ram} - \sum_{i=1}^{\lfloor g/2 \rfloor} (b_i\delta_i + b_{g-i}\delta_{g-i} + b_{i:g-i}\delta_{i:g-i}),$$

$$[\mathcal{T}_g^o] = c\lambda - d_0'\delta_0' - d_0''\delta_0'' - d_0^{ram}\delta_0^{ram} - \sum_{i=1}^{\lfloor g/2 \rfloor} (d_i\delta_i + d_{g-i}\delta_{g-i} + d_{i:g-i}\delta_{i:g-i}),$$

where

$$\begin{aligned} a &= 2^{g-3}(2^{g-1}+1), & c &= 2^{2g-4}, \\ b_0' &= 2^{2g-7}, & d_0' &= 2^{2g-7}, \\ b_0'' &= 0, & d_0'' &= 2^{2g-6}, \\ b_0^{ram} &= 2^{g-5}(2^{g-1}+1), & d_0^{ram} &= 2^{g-5}(2^{g-1}-1), \\ b_i &= 2^{g-3}(2^{g-i}-1)(2^{i-1}-1), & d_i &= 2^{g+i-4}(2^{g-i}-1), \\ b_{g-i} &= 2^{g-3}(2^{g-i-1}-1)(2^i-1), & d_{g-i} &= 2^{2g-i-4}(2^i-1), \\ b_{i:g-i} &= 2^{g-3}(2^{g-1}-2^{i-1}-2^{g-i-1}+1), & d_{i:g-i} &= 2^{g-3}(2^{g-1}-2^{g-i-1}-2^{i-1}). \end{aligned}$$

Theorem H. *For every $g \geq 3$ the divisors $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$ are irreducible.*

We must point out that Theorem G has been independently proved by Maestro during his doctoral research ([MP21]). The techniques are similar in both cases, and consist in the intersection of $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$ with certain test curves. This has given rise to the joint publication [MPR21].

On the other hand, the proof of Theorem H for $g \geq 5$ combines monodromy arguments with the intersection of $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$ with the boundary divisor Δ_1 of $\overline{\mathcal{R}}_g$. The case $g = 3$ is rather elementary, whereas the case $g = 4$ follows from our analysis of the Prym map on $\mathcal{T}_4^e$ and $\mathcal{T}_4^o$.

Aside from their independent interest, these results (especially Theorem G) constitute an important tool for our study of the Prym map, in which we extend the analysis of Beauville and Izadi to other values of g. Additionally, in the case of $\mathcal{T}_5^o$, we provide a more detailed description with unexpected connections to the geometry of cubic threefolds.

Let us first consider the even cases $\mathcal{T}_g^e$. According to Mumford's description [Mum74] of the singularities of the theta divisor of a Prym variety, it is well known that $\mathcal{P}_g$ maps $\mathcal{T}_g^e$ to the divisor $\theta_{null} \subset \mathcal{A}_{g-1}$ of principally polarized abelian varieties whose theta divisor contains a singular 2-torsion point. Combining this with results of Teixidor on the loci of curves with unexpected theta-characteristics ([TiB87]), we prove item (1) and part of (3) in the following theorem, whereas (2) is essentially a consequence of Recillas' trigonal construction:

Theorem I. *The divisors* $\mathcal{T}_g^e$ *of even semicanonical pencils satisfy:*

(1) $\mathcal{T}_g^e = \mathcal{P}_g^{-1}(\theta_{null})$ *for* $3 \leq g \leq 5$.

(2) *The fiber of* $\mathcal{P}_4$ *on a general hyperelliptic Jacobian* $JX \in \mathcal{A}_3$ *is birationally equivalent to its Kummer variety.*

(3) *For* $g \geq 6$, $\mathcal{T}_g^e$ *is the divisorial component of* $\mathcal{P}_g^{-1}(\theta_{null})$ *and the restricted Prym map* $\mathcal{P}_g|_{\mathcal{T}_g^e}$ *is generically finite onto its image. In particular,* $\deg(\mathcal{P}_6|_{\mathcal{T}_6^e}) = 27$.

By contrast, the behaviour of the Prym map on the divisors $\mathcal{T}_g^o$ of odd semicanonical pencils is considerably different to that of the even cases. Indeed, for low values of g (as long as $\dim \mathcal{T}_g^o \geq \dim \mathcal{A}_{g-1}$), $\mathcal{T}_g^o$ dominates $\mathcal{A}_{g-1}$.

The following theorem summarizes our results on the divisors $\mathcal{T}_g^o$. For the case $g = 5$, let us recall that Donagi [Don92] established a birational map between $\mathcal{A}_4$ and the set $\mathcal{RC}^+$ of pairs (V, δ), where $V \subset \mathbb{P}^4$ is a smooth cubic threefold and $\delta \in JV_2$ is a 2-torsion point of its intermediate Jacobian with a certain parity condition.

Theorem J. *The divisors* $\mathcal{T}_g^o$ *of odd semicanonical pencils satisfy:*

(1) *The map* $\mathcal{P}_3|_{\mathcal{T}_3^o} : \mathcal{T}_3^o \longrightarrow \mathcal{A}_2$ *is dominant, and its general fiber is isomorphic to the complement in the projective plane of six lines and a smooth conic. In particular,* $\mathcal{T}_3^o$ *is rationally connected.*

(2) *The map* $\mathcal{P}_4|_{\mathcal{T}_4^o} : \mathcal{T}_4^o \longrightarrow \mathcal{A}_3$ *is surjective, and the fiber of a general Jacobian* JX *is the complement in the projective plane of the union of the canonical model of* X *and its* 28 *bitangent lines. In particular,* $\mathcal{T}_4^o$ *is rationally connected.*

(3) (Izadi) *The restricted Prym map* $\widetilde{\mathcal{P}}_5|_{\mathcal{T}_5^o}$ *is dominant, and the fiber at a general* $(V, \delta) \in \mathcal{RC}^+$ *is a partial desingularization of the curve* $\Gamma \subset F(V)$ *defined by*

$$\Gamma = \{l \in F(V) \mid \textit{There exist a 2-plane } \pi \textit{ and a line } r \in F(V) \textit{ with } V \cdot \pi = l + 2r\}.$$

(4) *For every* $g \geq 6$ *the restricted Prym map* $\mathcal{P}_g|_{\mathcal{T}_g^o}$ *is generically finite onto its image.*

Observe that after (4) two natural questions come out: the computation of the degree of $\mathcal{P}_g|_{\mathcal{T}_g^o}$ (for $g \geq 6$, but especially for $g = 6$) and an intrinsic description of the divisor $\mathcal{P}_6(\mathcal{T}_6^o)$ in $\mathcal{A}_5$. Our analysis includes a partial answer to the first question: we prove that the degree of $\mathcal{P}_6|_{\mathcal{T}_6^o}$ is strictly smaller than 27 (which shows once again differences between the even and the odd cases). In addition, we propose a natural geometric description for $\mathcal{P}_6(\mathcal{T}_6^o)$, based on a close relation between $\mathcal{T}_g^o$ and the locus of Prym curves (C, η) for which the Brill-Noether locus $V^2(C, \eta)$ is singular.

The main tool for our reproof of (3) is the class of $\mathcal{T}_5^o$ in $\mathrm{Pic}(\overline{\mathcal{R}}_5)_{\mathbb{Q}}$, provided by Theorem G. Simultaneously, this cohomological approach reveals interesting properties of a general cubic threefold $V \subset \mathbb{P}^4$, which a priori seem difficult to detect via more direct techniques. Indeed, Donagi's description of the general fiber of $\widetilde{\mathcal{P}}_5$ ([Don92]) realizes (a double cover of) the Fano surface of lines $F(V)$ as a subvariety of $\overline{\mathcal{R}}_5$, where the rational Picard group and the canonical class are well understood. This enables us to prove:

Theorem K. *For every smooth cubic threefold $V \subset \mathbb{P}^4$, the curve $\Gamma \subset F(V)$ is numerically equivalent to $8K_{F(V)}$. Furthermore, for V general, Γ is irreducible and its singular locus consists of 1485 nodes.*

As far as we know, the numerical class of Γ had never been computed before. On the contrary, its natural counterpart

$$\Gamma' = \{r \in F(V) \mid \text{There exist a 2-plane } \pi \text{ and a line } l \in F(V) \text{ with } V \cdot \pi = l + 2r\}$$

(namely the curve formed by *lines of the second type*) had been largely explored in the literature (see e.g. [CG72, Section 10]). Theorem K has immediate consequences on the enumerative geometry of lines on a cubic threefold. For instance, the geometric interpretation of the nodes of Γ and the intersection points of Γ with Γ' establishes the following result:

Corollary L. *For a general smooth cubic threefold $V \subset \mathbb{P}^4$, the following statements hold:*

(1) (Corollary VI.4.7) *There are exactly 1485 lines $l \subset V$ for which there exist 2-planes $\pi_1, \pi_2 \subset \mathbb{P}^4$ and lines $r_1, r_2 \subset V$ satisfying $V \cdot \pi_i = l + 2r_i$ $(i = 1, 2)$.*
(2) (Corollary VI.4.8) *There are exactly 720 lines $l \subset V$ for which there exist 2-planes $\pi_1, \pi_2 \subset \mathbb{P}^4$ and lines $r_1, r_2 \subset V$ satisfying $V \cdot \pi_1 = l + 2r_1$ and $V \cdot \pi_2 = 2l + r_2$.*

Structure of the second part. This second part is also divided into three chapters. Chapter IV gathers all the preliminar results on Prym varieties and the moduli spaces $\overline{\mathcal{R}}_g$ and $\widetilde{\mathcal{R}}_g$, and introduces the two divisors of Prym semicanonical pencils.

Chapter V consists of two parts. In section V.1 we prove Theorem G, whereas in section V.2 we determine the irreducibility of $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$ for $g \neq 4$, which almost gives Theorem H.

Chapter VI is the core of this second part and examines the Prym map on $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$. In section VI.1 we determine $\mathcal{P}_g^{-1}(\theta_{null})$, which proves Theorem I.(1) and part of Theorem I.(3). The rest of the chapter essentially deals with the odd cases, and the study of each genus occupies a section.

Section VI.2 is devoted to Theorem J.(1), whose proof is based on Mumford's results on Prym varieties of covers of hyperelliptic curves. In section VI.3 we study the case of genus 4. Using involutions on certain moduli spaces, we prove that $\mathcal{T}_4^o$ corresponds under Recillas' trigonal construction to smooth genus 3 curves endowed with a non-complete g_4^1 linear series, which gives the arguments for proving Theorem J.(2). In this section, we also prove Theorem I.(2) and the irreducibility of $\mathcal{T}_4^e$ and $\mathcal{T}_4^o$ (which completes the proof of Theorem H).

Section VI.4 addresses the case of genus 5. As explained above, we prove Theorem J.(3) using a cohomological approach. A more detailed analysis also gives Theorem K and several enumerative consequences, including Corollary L as well as a more precise description of the desingularization appearing in Theorem J.(3) (see Corollary VI.4.9). Finally, in section VI.5 we study the cases of genus $g \geq 6$; after proving Theorem J.(4) and the rest of Theorem I.(3), we propose a natural geometric description for the divisor $\mathcal{P}_6(\mathcal{T}_6^o) \subset \mathcal{A}_5$.

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Chern degree functions

Chapter I

Preliminaries

This first chapter gives a brief account of the preliminary material needed for the first part of the thesis. Unless otherwise stated, throughout all this first part we will work over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic. As usual, a point of a variety will be called *general* if it lies outside a nontrivial Zariski-closed subset.

I.1 Preliminaries on stability conditions

In this section, we review the definitions and basic properties of (possibly weak) stability conditions, with a special view towards the (α, β)-plane defined by a polarization on a smooth projective surface as developed in [Bri07, Bri08, AB13]. We follow the notations of the excellent survey [MS17].

Weak and Bridgeland stability conditions

Let $\mathcal{A}$ be an abelian category with Grothendieck group $K_0(\mathcal{A})$.

Definition I.1.1. A *stability function* (resp. *weak stability function*) on $\mathcal{A}$ is a group homomorphism $Z : K_0(\mathcal{A}) \to \mathbb{C}$ such that every $F \in \mathcal{A} \setminus \{0\}$ satisfies

$$\Im Z(F) \geq 0, \text{ and } \Im Z(F) = 0 \Longrightarrow \Re Z(F) < 0 \quad (\text{resp. } \Im Z(F) = 0 \Longrightarrow \Re Z(F) \leq 0)$$

Given a (possibly weak) stability function Z on $\mathcal{A}$, one can define a slope for objects $F \in \mathcal{A}$ as $\mu(F) = \frac{-\Re Z(F)}{\Im Z(F)}$, with the convention $\mu(F) = +\infty$ whenever $\Im Z(F) = 0$.

Definition I.1.2. An object $F \in \mathcal{A} \setminus \{0\}$ is called *(semi)stable* if, for every nonzero subobject $E \subsetneq F$, the inequality $\mu(E) < (\leq)\ \mu(F/E)$ holds.

Remark I.1.3. Equivalently, F is semistable if and only if $\mu(E) \leq \mu(F)$ for every nonzero $E \subsetneq F$. The same holds (with strict inequality) for stability of F, whenever Z is a stability function. But if Z is strictly weak, then F may be stable and admit subobjects with $\mu(E) = \mu(F)$; indeed, if $Z(F/E) = 0$ then $\mu(E) = \mu(F) < +\infty = \mu(F/E)$ does not contradict stability for F.

The following lemma is certainly well known, and will be used throughout the first part of the thesis:

Lemma I.1.4. *Let $F, G \in \mathcal{A} \setminus \{0\}$ be semistable objects with respect to a weak stability function Z. If $\mu(F) > \mu(G)$, then* $\mathrm{Hom}(F, G) = 0$.

Now we give the definition of stability conditions on the bounded derived category $\mathrm{D^b}(X)$ of a smooth projective variety X (more generally, the same definition applies in the context of triangulated categories). To this end, we fix a finite rank lattice Λ together with a group epimorphism $v : K_0(X) \twoheadrightarrow \Lambda$, where $K_0(X)$ denotes the Grothendieck group of $\mathrm{D^b}(X)$.

Definition I.1.5. A *Bridgeland stability condition* (resp. *weak stability condition*) on $\mathrm{D^b}(X)$ is a pair $\sigma = (Z, \mathcal{A})$, where:

(1) $\mathcal{A}$ is the heart of a bounded t-structure on $\mathrm{D^b}(X)$, and $Z : K_0(X) \to \mathbb{C}$ (*central charge*) is a stability function (resp. weak stability function) on $\mathcal{A}$ factoring through v.

(2) Every object $F \in \mathcal{A} \setminus \{0\}$ has a *Harder-Narasimhan* (*HN* for short) *filtration*: there exists a (necessarily unique) chain of subobjects

$$0 = F_0 \subset F_1 \subset \ldots \subset F_{r-1} \subset F_r = F$$

with the *HN factors* F_i/F_{i-1} being semistable and $\mu(F_1) > \mu(F_2/F_1) > \ldots > \mu(F/F_{r-1})$.

(3) The *support property* is satisfied: Let $\Lambda_0 \subset \Lambda$ be the saturation of the subgroup generated by classes $v(F)$ of objects $F \in \mathcal{A}$ such that $Z(v(F)) = 0$. Then there exists a quadratic form Q on $(\Lambda/\Lambda_0) \otimes \mathbb{R}$, such that $Q(v, v) < 0$ for every nonzero $v \in (\Lambda/\Lambda_0) \otimes \mathbb{R}$ with $Z(v) = 0$, and $Q(v(F), v(F)) \geq 0$ for every semistable object $F \in \mathcal{A}$.

Example I.1.6. For (X, L) a polarized smooth projective variety of dimension n, the pair $\sigma = (Z, \mathrm{Coh}(X))$ (where $Z = -L^{n-1} \cdot \mathrm{ch}_1 + iL^n \cdot \mathrm{ch}_0$) defines a weak stability condition on $\mathrm{D^b}(X)$ (the usual *μ_L-stability*) with respect to the epimorphism $v : K_0(X) \twoheadrightarrow \mathbb{Z}^2$ given by $v(E) = (L^n \cdot \mathrm{ch}_0(E), L^{n-1} \cdot \mathrm{ch}_1(E))$. The support property is guaranteed by the quadratic form $Q = 0$.

Let $\mathrm{Stab}_\Lambda(X)$ be the set of Bridgeland stability conditions on $\mathrm{D^b}(X)$ (with respect to Λ and v). The support property may be thought of as a technical condition allowing to give $\mathrm{Stab}_\Lambda(X)$, when endowed with a certain topology, the structure of a complex manifold. This result is usually known as *Bridgeland's deformation theorem* ([Bri07], see also [Bay19]). The main consequence is that the regions where objects of a fixed class are (semi)stable behave following a locally finite wall and chamber structure (see [BM11, Proposition 3.3] for further details).

The (α, β)-plane of a polarized surface

Let X be a smooth projective surface over an algebraically closed field $\mathbb{K}$; if $\operatorname{char} \mathbb{K} > 0$, we will assume that X is neither of general type nor quasi-elliptic with $\kappa(X) = 1$. For a fixed polarization $L \in \mathrm{NS}(X)$ on X, we review the construction of the stability conditions forming the so-called (α, β)-*plane of* L. This is a slice of the connected component of the stability manifold $\mathrm{Stab}(X)$ constructed by Bridgeland ([Bri08]) in the case of K3 surfaces, and generalized to arbitrary surfaces in [AB13].

For the rest of this section we fix the lattice $\Lambda = \mathrm{Im}(v)$, where $v : K_0(X) \to \mathbb{Z}^2 \oplus \frac{1}{2}\mathbb{Z}$ is the map defined by

$$v(E) = (L^2 \cdot \mathrm{ch}_0(E), L \cdot \mathrm{ch}_1(E), \mathrm{ch}_2(E)).$$

Given $\beta \in \mathbb{R}$, consider the full subcategories

$$\mathcal{F}_\beta := \{E \in \mathrm{Coh}(X) \mid \mu_L^+(E) \leq \beta\}, \;\; \mathcal{T}_\beta := \{E \in \mathrm{Coh}(X) \mid \mu_L^-(E) > \beta\}$$

of $\mathrm{Coh}(X)$, where μ_L^+ (resp. μ_L^-) denotes the maximum (resp. minimum) slope of a HN factor in μ_L-stability. They form a torsion pair, and thus according to [HRS96] their tilt

$$\mathrm{Coh}^\beta(X) := \left\{E \in \mathrm{D^b}(X) \mid \mathcal{H}^{-1}(E) \in \mathcal{F}_\beta,\ \mathcal{H}^0(E) \in \mathcal{T}_\beta,\ \mathcal{H}^i(E) = 0 \text{ for } i \neq 0, -1\right\}$$

is the heart of a bounded t-structure on $\mathrm{D^b}(X)$; in particular $\mathrm{Coh}^\beta(X)$ is abelian, with short exact sequences in $\mathrm{Coh}^\beta(X)$ corresponding to distinguished triangles in $\mathrm{D^b}(X)$.

Remark I.1.7. If an object $E \in \mathrm{D^b}(X)$ satisfies $\mathcal{H}^i(E) = 0$ for $i \neq 0, -1$, then $E \in \mathrm{Coh}^\beta(X)$ is equivalent to $\mu_L^+(\mathcal{H}^{-1}(E)) \leq \beta < \mu_L^-(\mathcal{H}^0(E))$. In particular, as a condition on $\beta \in \mathbb{R}$, $E \in \mathrm{Coh}^\beta(X)$ is open on the right.

For every $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$, we define a central charge $Z_{\alpha,\beta} : K_0(X) \to \mathbb{C}$ by

$$Z_{\alpha,\beta}(E) = -\left(\mathrm{ch}_2^\beta(E) - \frac{\alpha^2}{2} L^2 \cdot \mathrm{ch}_0^\beta(E)\right) + i\left(L \cdot \mathrm{ch}_1^\beta(E)\right)$$

where $\mathrm{ch}^\beta = e^{-\beta L} \cdot \mathrm{ch}$ is the Chern character twisted by βL, namely:

$$\mathrm{ch}_0^\beta = \mathrm{ch}_0, \;\; \mathrm{ch}_1^\beta = \mathrm{ch}_1 - \beta L \cdot \mathrm{ch}_0, \;\; \mathrm{ch}_2^\beta = \mathrm{ch}_2 - \beta L \cdot \mathrm{ch}_1 + \frac{\beta^2}{2} L^2 \cdot \mathrm{ch}_0$$

Note that $Z_{\alpha,\beta}$ factors through v. We will denote by $\nu_{\alpha,\beta}$ the *tilt slope* defined by the central charge $Z_{\alpha,\beta}$.

The main result of this part, for which we adopt the version in [MS17, Theorems 6.10 and 6.13], strongly relies on the classical Bogomolov inequality for μ_L-semistable sheaves (see [Lan16, Theorem 1.3] for positive characteristic):

Theorem I.1.8 ([Bri08, AB13])**.** *For every $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$, the pair $\sigma_{\alpha,\beta} = (\mathrm{Coh}^{\beta}(X), Z_{\alpha,\beta})$ is a Bridgeland stability condition on $\mathrm{D}^{\mathrm{b}}(X)$, satisfying the support property with respect to the quadratic form $\overline{\Delta} := (L \cdot \mathrm{ch}_1)^2 - 2(L^2 \cdot \mathrm{ch}_0)\,\mathrm{ch}_2$.*

Remark I.1.9. In the recent work [Kos20] Koseki proved a modified version of the Bogomolov inequality in positive characteristic, for surfaces of general type and quasi-elliptic surfaces with $\kappa = 1$. This enables him to construct (a smaller region of) Bridgeland stability conditions in such cases, that we will not consider.

Given a class $v \in \Lambda$, a *numerical wall* for v is the region of $\mathbb{R}_{>0} \times \mathbb{R}$ determined by an equation of the form $\nu_{\alpha,\beta}(v) = \nu_{\alpha,\beta}(w)$, where $w \in \Lambda$ is a class non-proportional to v. An *actual wall* for v is a subset of a numerical wall, along which the set of semistable objects of class v changes.

The structure of the walls in this (α, β)-plane is well understood. Items (1)–(6) of the following theorem are called *Bertram's Nested Wall Theorem*, and were proved in [Mac14]. The last item is a consequence of [BMS16, Lemma A.7], as part of a systematic study of the support property in terms of the quadratic form.

Theorem I.1.10. *Let $v \in \Lambda$ be a class with $\overline{\Delta}(v) \geq 0$.*

(1) *All numerical walls for v are either semicircles centered on the β-axis or lines parallel to the α-axis.*
(2) *The numerical walls defined by classes $u, w \in \Lambda$ intersect if and only if v, u, w are linearly dependent. In such a case, the two walls are identical.*
(3) *If $v_0 \neq 0$, there is a unique vertical wall with equation $\beta = \frac{v_1}{v_0}$. At each side of this vertical wall, all semicircular walls are strictly nested (see Figure I.1).*
(4) *If $v_0 = 0$, there is no vertical wall and all numerical walls are strictly nested semicircles (see Figure I.2).*
(5) *The curve $H_v : \nu_{\alpha,\beta}(v) = 0$ intersects every semicircular wall at its top point. This curve is an hyperbola (if $v_0 \neq 0$ and $\overline{\Delta}(v) > 0$), a pair of lines (if $v_0 \neq 0$ and $\overline{\Delta}(v) = 0$) or a single vertical line (if $v_0 = 0$).*
(6) *If a numerical wall is an actual wall at some of its points, then it is an actual wall at all of its points.*
(7) *If an actual wall is defined by a short exact sequence $0 \to E \to F \to Q \to 0$ of semistable objects (with $v(F) = v$), then $\overline{\Delta}(E) + \overline{\Delta}(Q) < \overline{\Delta}(F)$. In particular, if $\overline{\Delta}(v) = 0$ then semistable objects F with $v(F) = v$ can only be destabilized at the vertical wall.*

Remark I.1.11. Item (7) allows to prove that any vertical line of rational β-coordinate intersects finitely many actual walls (see [Sch20, Appendix]); in particular, this gives the existence of a largest actual wall, and reproves the local finiteness of actual walls in the region with $\alpha > 0$. However, actual walls may accumulate towards irrational points of $H_v \cap \{\alpha = 0\}$ (see [Mea12, Chapter 4]).

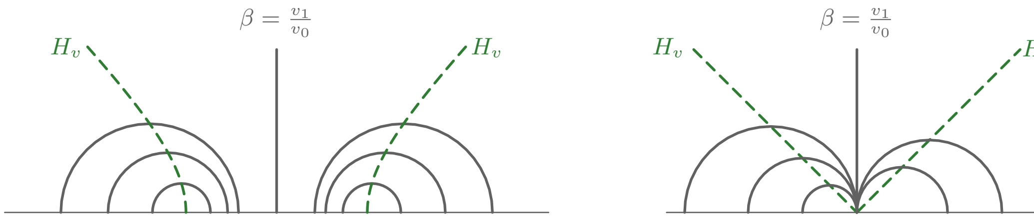


Figure I.1: Numerical walls for v when $v_0 \neq 0$

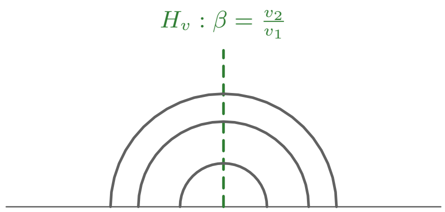


Figure I.2: Numerical walls for v when $v_0 = 0$

Remark I.1.12. If $F \in \mathrm{Coh}^\beta(X)$ for some $\beta \in \mathbb{R}$, then $L \cdot \mathrm{ch}_1^\beta(F) \geq 0$ gives $\beta \leq \mu_L(F)$ (resp. $\beta \geq \mu_L(F)$) if $\mathrm{ch}_0(F) > 0$ (resp. $\mathrm{ch}_0(F) < 0$). There is no condition on β, if $\mathrm{ch}_0(F) = 0$.

Thus in the study of a particular object F, one is led to consider just one of the regions separated by the vertical wall (if exists). Such a region is divided into two parts by a component of $H_{v(F)}$; abusing of notation, this component will be called *hyperbola of* F and denoted by H_F. Note that at the left-hand (resp. right-hand) side of H_F, F has positive (resp. negative) tilt slope. The β-coordinate of the intersection point of H_F with $\alpha = 0$ will be denoted by p_F.

The next result, originally due to Bridgeland, motivates the name of *Gieseker chamber* for the chamber above the largest wall, in the case of a class with positive rank:

Proposition I.1.13 ([Bri08], [MS17, Exercise 6.27])**.** *Let $v \in \Lambda$ be a class with $\overline{\Delta}(v) \geq 0$ and $v_0 > 0$, and let $\beta < \frac{v_1}{v_0}$. Then an object $F \in \mathrm{Coh}^\beta(X)$ of class $v(F) = v$ is $\sigma_{\alpha,\beta}$-semistable for every $\alpha \gg 0$ if, and only if, F is a twisted $(L, -\frac{1}{2}K_X)$-Gieseker semistable sheaf.*

When $\alpha = 0$ one cannot ensure in general that this construction gives Bridgeland stability conditions. This is due to the fact that for $\beta \in \mathbb{Q}$ the central charge $Z_{0,\beta}$ may send to 0 certain objects of $\mathrm{Coh}^\beta(X)$, as we describe in the following proposition:

Proposition I.1.14. *If $\beta \in \mathbb{Q}$, then an object $F \in \mathrm{Coh}^\beta(X)$ satisfies:*

(1) *$L \cdot \mathrm{ch}_1^\beta(F) = 0$ if and only if the following hold:*
- *$\mathcal{H}^{-1}(F)$ is either 0 or a μ_L-semistable torsion-free sheaf with $\mu_L = \beta$.*
- *$\mathcal{H}^0(F)$ is either 0 or a sheaf supported in dimension 0.*

(2) *$Z_{0,\beta}(F) = 0$ if and only if $F = S[1]$, where S is a twisted $(L, -\frac{1}{2}K_X)$-Gieseker semistable vector bundle with $\mu_L(S) = \beta$ and $\overline{\Delta}(S) = 0$.*

Proof. The first item is an easy consequence of $\mathcal{H}^{-1}(F) \in \mathcal{F}_\beta$ and $\mathcal{H}^0(F) \in \mathcal{T}_\beta$. We explain how to check the "only if" part of (2), the converse implication being immediate.

Note that $Z_{0,\beta}(F) = 0$ implies $L \cdot \mathrm{ch}_1^\beta(F) = 0$, so $\mathcal{H}^{-1}(F)$ and $\mathcal{H}^0(F)$ must be as stated in (1). If it were $\mathcal{H}^0(F) \neq 0$, then $Z_{0,\beta}(\mathcal{H}^0(F)) \in \mathbb{R}_{<0}$; this would force $Z_{0,\beta}(\mathcal{H}^{-1}(F)[1]) \in \mathbb{R}_{>0}$, which contradicts that $Z_{0,\beta}$ is a weak stability function on $\mathrm{Coh}^\beta(X)$.

Therefore $F = \mathcal{H}^{-1}(F)[1]$. If $\mathcal{H}^{-1}(F)$ were not locally free, then $\mathcal{H}^{-1}(F)^{**}/\mathcal{H}^{-1}(F)$ would be a nontrivial subobject of F (in $\mathrm{Coh}^\beta(X)$) satisfying $Z_{0,\beta}(\mathcal{H}^{-1}(F)^{**}/\mathcal{H}^{-1}(F)) \in \mathbb{R}_{<0}$, a contradiction. The maximal destabilizing subobject of $\mathcal{H}^{-1}(F)$ with respect to twisted $(L, -\frac{1}{2}K_X)$-Gieseker stability, if different from $\mathcal{H}^{-1}(F)$, would give a similar contradiction.

Finally, $\overline{\Delta}(\mathcal{H}^{-1}(F)) = 0$ immediately follows from $\mathrm{ch}_2^\beta(\mathcal{H}^{-1}(F)) = 0 = L \cdot \mathrm{ch}_1^\beta(\mathcal{H}^{-1}(F))$. □

Example I.1.15. For $\beta \in \mathbb{Q}$, the fact that $\sigma_{0,\beta}$ is a Bridgeland stability condition depends on the sharpness of the Bogomolov inequality for μ_L-semistable sheaves of slope β, which is encoded by the Le Potier function

$$\Phi_{X,L}(\beta) := \sup \left\{ \frac{\mathrm{ch}_2(E)}{L^2 \cdot \mathrm{ch}_0(E)} \mid E \text{ is } \mu_L\text{-semistable with } \mu_L(E) = \beta \right\}$$

(see [FLZ21, Definition 3.1] for its extension as a real function). This function depends on the particular geometry of the surface:

(1) If X is a complex abelian surface, the vector bundles S of Proposition I.1.14 are semihomogeneous. This is a consequence of [Kob87, Theorem IV.4.7] and [Yan89, Theorem 5.12].

Conversely, for X an arbitrary abelian surface and for every $\beta \in \mathbb{Q}$ there exist semihomogeneous vector bundles S with $\frac{\mathrm{ch}_1(S)}{\mathrm{rk}(S)} = \beta L \in \mathrm{NS}(X)_\mathbb{Q}$ by [Muk78]; for such bundles S, one has $S[1] \in \mathrm{Coh}^\beta(X)$ and $Z_{0,\beta}(S[1]) = 0$. Therefore, $\sigma_{0,\beta}$ fails to be a Bridgeland stability condition for every $\beta \in \mathbb{Q}$.

(2) More generally, let X be a surface whose Albanese map $a : X \to A$ is finite onto its image, endowed with a polarization $L = a^*\widetilde{L}$ pulled back from a polarization $\widetilde{L}$ on A. Then, for every $\beta \in \mathbb{Q}$ one can find objects $F \in \mathrm{Coh}^\beta(X)$ with $Z_{0,\beta}(F) = 0$, for instance $F = a^*S[1]$ for simple semihomogeneous vector bundles S with $\frac{\mathrm{ch}_1(S)}{\mathrm{rk}(S)} = \beta\widetilde{L} \in \mathrm{NS}(A)_\mathbb{Q}$.

The equality $Z_{0,\beta}(a^*S[1]) = 0$ being straightforward, let us check that $a^*S[1] \in \mathrm{Coh}^\beta(X)$. Since $\mu_L(a^*S) = \beta$, we need to prove that a^*S is μ_L-semistable; by [HL10, Lemma 3.2.2], it suffices to check that $S|_{a(X)}$ is slope semistable with respect to $\widetilde{L}|_{a(X)}$.

And indeed, according to [Muk78, Proposition 7.3], there exists an isogeny $\pi : B \to A$ and a line bundle M on B with $\pi^*S = M^{\oplus \mathrm{rk}\, S}$. This description implies that $\pi^*S|_{\pi^{-1}(a(X))}$ is slope semistable with respect to $\pi^*\widetilde{L}|_{\pi^{-1}(a(X))}$, and therefore the semistability of $S|_{a(X)}$ again follows from [HL10, Lemma 3.2.2].

In particular, this shows that $\Phi_{X,L}(x) = \frac{x^2}{2}$ for every $x \in \mathbb{R}$ adding an instance to [FLZ21, Remark 3.3].

(3) If X is a simply connected complex surface and $Z_{0,\beta}(F) = 0$ for $F \in \mathrm{Coh}^{\beta}(X)$, then $\beta \in \mathbb{Z}$ and the vector bundle S of Proposition I.1.14 satisfies $S = (L^{\beta})^{\oplus \operatorname{rk} S}$; this follows from [Kob87, Theorem IV.4.7 and Corollary I.2.7].

In terms of the Le Potier function, this means that $\Phi_{X,L}(x) = \frac{x^2}{2}$ if and only if $x \in \mathbb{Z}$.

In any case, $Z_{0,\beta}$ is a (possibly weak) stability function on $\mathrm{Coh}^{\beta}(X)$ for every $\beta \in \mathbb{Q}$; since $\mathrm{Coh}^{\beta}(X)$ is Noetherian, the existence of HN filtrations with respect to the tilt slope $\nu_{0,\beta}$ is guaranteed (see [Bri07, Lemma 2.4] or [MS17, Proposition 4.10 and Remark 4.14]). Therefore, $\sigma_{0,\beta} = (\mathrm{Coh}^{\beta}(X), Z_{0,\beta})$ is a (possibly weak) stability condition on $\mathrm{D^b}(X)$[1].

We finish this section describing the behaviour of semistable objects under the derived dual. This is certainly well known to the experts but we include it for easy reference. For an object $E \in \mathrm{D^b}(X)$, we write $E^{\vee} = R\mathcal{H}om(E, \mathcal{O}_X)$.

Proposition I.1.16. *For $\beta \in \mathbb{R}$, let $F \in \mathrm{Coh}^{\beta}(X)$ be an object such that $\nu^{+}_{\alpha,\beta}(E) < +\infty$ for every $\alpha \geq 0$ (i.e. F contains no subobject with $L \cdot \mathrm{ch}_1^{\beta} = 0$). Then:*

(1) *$F^{\vee}[1] \in \mathrm{Coh}^{-\beta}(X)$.*

(2) *For every $\alpha \geq 0$, F is $\sigma_{\alpha,\beta}$-(semi)stable if and only if $F^{\vee}[1]$ is $\sigma_{\alpha,-\beta}$-(semi)stable.*

Proof. The first item is the particular case for surfaces of the more general result [BLMS17, Lemma 2.19.a]. As a consequence of it, the contravariant functor $\_^{\vee}[1]$ induces a bijection between subobjects of F (in $\mathrm{Coh}^{\beta}(X)$) and quotients of $F^{\vee}[1]$ (in $\mathrm{Coh}^{-\beta}(X)$). Taking into account the Chern character of the derived dual as well, item (2) is immediately checked. □

Remark I.1.17. A combination of Proposition I.1.13 and Proposition I.1.16 describes the semistable objects above the largest wall, for classes with negative rank.

I.2 Cohomological rank functions on abelian varieties

In this section we recall the definition of *cohomological rank functions* given by Jiang and Pareschi in [JP20], together with some of their most important properties. We also recall their relation to syzygies of abelian varieties.

Let (A, L) be a g-dimensional polarized abelian variety over $\mathbb{K}$, and let $M \in \mathrm{Pic}(A)$ be any ample line bundle representing the polarization L. We will denote by

$$\varphi_L : A \to \mathrm{Pic}^0(A), \ \ p \mapsto t_p^* M \otimes M^{-1}$$

the corresponding isogeny, where t_p stands for the translation by $p \in A$.

[1] If $Z_{0,\beta}$ is a stability function, the support property for the Bridgeland stability condition $\sigma_{0,\beta}$ holds as explained in [FLZ21, Remark 3.5]; otherwise, the quadratic form $Q = 0$ guarantees the support property for the weak stability condition $\sigma_{0,\beta}$.

Given an object $F \in \mathrm{D^b}(A)$ and $i \in \mathbb{Z}$, the cohomological rank function $h^i_{F,L} : \mathbb{Q} \to \mathbb{Q}_{\geq 0}$ is defined as follows: given a rational $x_0 = \frac{a}{b}$ with $b > 0$, then

$$h^i_{F,L}(x_0) := \frac{1}{b^{2g}} h^i(A, \mu_b^* F \otimes M^{ab} \otimes \alpha)$$

for general $\alpha \in \mathrm{Pic}^0(A)$, where $\mu_b : A \to A$ is the multiplication-by-b isogeny.

Since $\mu_b^* L = b^2 L$ (hence $\mu_b^*(x_0 L) = abL$) and $\deg \mu_b = b^{2g}$, the number $h^i_{F,L}(x_0)$ gives a meaning to the (hyper)cohomological rank $h^i(A, F \otimes L^{x_0})$ of F twisted with a general representative of the fractional polarization $x_0 L$. Note that, by semicontinuity and base change, these cohomological ranks are related by the equation

$$\sum_{i \in \mathbb{Z}} (-1)^i \cdot h^i_{F,L}(x_0) = \chi_{F,L}(x_0),$$

where $\chi_{F,L}$ is the Hilbert polynomial of F with respect to L.

Remark I.2.1. The definition in [JP20] is given under the assumption $\mathrm{char}\, \mathbb{K} = 0$, but the same definition works in arbitrary characteristic as observed in [Cau20, Section 2].

The main results of Jiang and Pareschi about these functions can be summarized as:

Theorem I.2.2. *Let $F \in \mathrm{D^b}(A)$ be an object and $i \in \mathbb{Z}$. Then:*

(1) [JP20, Corollary 2.6] *For every $x_0 \in \mathbb{Q}$, there exists a left (resp. right) neighborhood of x_0 where the function $h^i_{F,L}$ is given by a polynomial. Explicitly, there exists $\epsilon > 0$ such that*

$$h^i_{F,L}(x) = \frac{(x_0 - x)^g}{\chi(L)} \cdot \chi_{\varphi_L^* R^i \Phi_{\mathcal{P}}(\mu_b^* F \otimes M^{ab}), L}\left(\frac{1}{b^2(x - x_0)}\right) \text{ for } x \in (x_0 - \epsilon, x_0] \cap \mathbb{Q}$$
$$h^i_{F,L}(x) = \frac{(x - x_0)^g}{\chi(L)} \cdot \chi_{\varphi_L^* R^{g-i} \Phi_{\mathcal{P}^\vee}((\mu_b^* F \otimes M^{ab})^\vee), L}\left(\frac{1}{b^2(x - x_0)}\right) \text{ for } x \in [x_0, x_0 + \epsilon) \cap \mathbb{Q}$$

(2) [JP20, Theorem 3.2] *If $\mathrm{char}\, \mathbb{K} = 0$, the function $h^i_{F,L}$ extends to a continuous function of real variable $h^i_{F,L} : \mathbb{R} \to \mathbb{R}_{\geq 0}$.*

It is expected (see [JP20, Remark 2.8]) that these real functions are piecewise polynomial; in other words, that their critical points (i.e. the points where they are not of class $\mathcal{C}^\infty$) do not accumulate towards an irrational number.

In the case of elliptic curves, this is true thanks to a precise description admitted by the functions in terms of μ_L-stability. The key point of this description is that μ_L-semistable coherent sheaves have *trivial* functions, that is, the support of any of its functions is disjoint with the support of all the other functions. This is certainly well known to the experts, but we include a proof since we could not find a published reference.

Proposition I.2.3. *Let (E, L) be an elliptic curve endowed with a polarization of degree 1.*

(1) *If $F \in \mathrm{Coh}(E)$ is a μ_L-semistable coherent sheaf, then it has trivial functions*

$$h^0_{F,L}(x) = \begin{cases} 0 & \chi_{F,L}(x) \leq 0 \\ \chi_{F,L}(x) & \chi_{F,L}(x) \geq 0 \end{cases} \qquad h^1_{F,L}(x) = \begin{cases} -\chi_{F,L}(x) & \chi_{F,L}(x) \leq 0 \\ 0 & \chi_{F,L}(x) \geq 0 \end{cases}$$

where $\chi_{F,L}(x) = \mathrm{rk}(F) \cdot x + \deg(F)$ is the Hilbert polynomial of F with respect to L.

(2) *Let $0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_r = F$ be the HN filtration of a coherent sheaf F. Then the functions of F can be recovered from those of its HN factors:*

$$h^0_{F,L}(x) = \sum_{k=1}^{r} \left(h^0_{F_k/F_{k-1},L}(x) \right) = \sum_{\chi_{F_k/F_{k-1},L}(x) \geq 0} \left(\chi_{F_k/F_{k-1},L}(x) \right)$$

$$h^1_{F,L}(x) = \sum_{k=1}^{r} \left(h^1_{F_k/F_{k-1},L}(x) \right) = \sum_{\chi_{F_k/F_{k-1},L}(x) \leq 0} \left(-\chi_{F_k/F_{k-1},L}(x) \right)$$

(3) *For any $F \in \mathrm{D^b}(E)$ and $i \in \mathbb{Z}$,*

$$h^i_{F,L}(x) = h^0_{\mathcal{H}^i(F),L}(x) + h^1_{\mathcal{H}^{i-1}(F),L}(x).$$

Proof. Item (1) is clear if F is torsion, since in that case $h^0_{F,L}(x) = \chi_{F,L}(x) = \mathrm{length}(F)$ for every $x \in \mathbb{Q}$. Hence we may assume that F is a vector bundle.

Let $x = \frac{a}{b} \in \mathbb{Q}$. If $\chi_{F,L}(x) < 0$ (resp. $\chi_{F,L}(x) > 0$), then we consider a non-decreasing (resp. non-increasing) sequence $\{x_n = \frac{a_n}{b_n}\}_n \subset \mathbb{Q}$ converging to x, such that for every n the multiplication isogeny $\mu_{b_n} : E \to E$ is an étale morphism[2].

We claim that for every degree 1 line bundle M and every $n \in \mathbb{N}$, one has

$$0 = \mathrm{Hom}(M^{-a_n b_n}, \mu^*_{b_n} F) = H^0(\mu^*_{b_n} F \otimes M^{a_n b_n})$$

$$(\text{resp. } 0 = \mathrm{Hom}(\mu^*_{b_n} F, M^{-a_n b_n}) = \mathrm{Ext}^1(M^{-a_n b_n}, \mu^*_{b_n} F)^* = H^1(\mu^*_{b_n} F \otimes M^{a_n b_n})^* \ )$$

Indeed, $\mu^*_{b_n} F$ is μ_L-semistable (we can apply [HL10, Lemma 3.2.2], since μ_{b_n} is a separable isogeny) as well as $M^{-a_n b_n}$. Thus the claim follows from the inequality

$$\mu_L(M^{-a_n b_n}) = -a_n b_n > b_n^2 \mu_L(F) = \mu_L(\mu^*_{b_n} F)$$

$$(\text{resp. } \mu_L(M^{-a_n b_n}) = -a_n b_n < b_n^2 \mu_L(F) = \mu_L(\mu^*_{b_n} F)).$$

Therefore $h^0_{F,L}(x_n) = 0$ (resp. $h^1_{F,L}(x_n) = 0$) for every n, which by Theorem I.2.2.(1) implies $h^0_{F,L}(x) = 0$ (resp. $h^1_{F,L}(x) = 0$). This proves (1).

[2] Of course, if b is not divisible by $\mathrm{char}\, \mathbb{K}$ (e.g., if $\mathrm{char}\, \mathbb{K} = 0$) one can take $x_n = x$ for every n

Item (2) follows by induction on the length r of the HN filtration of F, the initial case being nothing but (1). Let $x = \frac{a}{b} \in \mathbb{Q}$. For the induction step one uses the long exact sequence in cohomology associated to

$$0 \to \mu_b^* F_{r-1} \otimes M^{ab} \to \mu_b^* F_r \otimes M^{ab} \to \mu_b^*(F_r/F_{r-1}) \otimes M^{ab} \to 0$$

for every line bundle M of degree 1, together with the observation that

$$\chi_{F_k/F_{k-1},L}(x) > (<) 0 \iff x > (<) - \mu_L(F_k/F_{k-1})$$

for any $k \in \{1, \ldots, r\}$ and the inequalities $-\mu_L(F_1) < \ldots < -\mu_L(F_r/F_{r-1})$.

For the proof of (3), write $x = \frac{a}{b} \in \mathbb{Q}$ and let M be any line bundle of degree 1. Considering the distinguished triangle in $\mathrm{D^b}(E)$ obtained by truncation of $\mu_b^* F \otimes M^{ab}$

$$\mu_b^*(\tau_{\leq i-1} F) \otimes M^{ab} \to \mu_b^* F \otimes M^{ab} \to \mu_b^*(\tau_{\geq i} F) \otimes M^{ab}$$

and the corresponding long exact sequence of hypercohomology groups

$$\begin{aligned} \ldots \to \mathbb{H}^{i-1}(\mu_b^*(\tau_{\geq i} F) \otimes M^{ab}) \to \mathbb{H}^i(\mu_b^*(\tau_{\leq i-1} F) \otimes M^{ab}) \to \mathbb{H}^i(\mu_b^* F \otimes M^{ab}) \to \\ \to \mathbb{H}^i(\mu_b^*(\tau_{\geq i} F) \otimes M^{ab}) \to \mathbb{H}^{i+1}(\mu_b^*(\tau_{\leq i-1} F) \otimes M^{ab}) \to \ldots \end{aligned}$$

the result becomes a consequence of the following immediate equalities:

$$\begin{aligned} \mathbb{H}^{i-1}(\mu_b^*(\tau_{\geq i} F)) = 0, \quad \mathbb{H}^i(\mu_b^*(\tau_{\leq i-1} F) \otimes M^{ab}) = H^1(\mu_b^*(\mathcal{H}^{i-1} F) \otimes M^{ab}), \\ \mathbb{H}^i(\mu_b^*(\tau_{\geq i} F) \otimes M^{ab}) = H^0(\mu_b^*(\mathcal{H}^i F) \otimes M^{ab}), \quad \mathbb{H}^{i+1}(\mu_b^*(\tau_{\leq i-1} F) \otimes M^{ab}) = 0 \end{aligned}$$

□

Nevertheless, for higher-dimensional abelian varieties not only a general structure for cohomological rank functions is far from being understood, but also the computation of particular examples often becomes a difficult problem.

Consider for instance the functions $h^i_{\mathcal{I}_q,L}$ of the ideal sheaf of a (closed) point $q \in A$; by independence of q, we fix q to be the origin $0 \in A$. It immediately follows from the long exact sequence of cohomology associated to

$$0 \longrightarrow \mathcal{I}_0 \longrightarrow \mathcal{O}_A \longrightarrow \mathcal{O}_0 \longrightarrow 0$$

that the only nonzero functions of $\mathcal{I}_0$ are:

$$h^0_{\mathcal{I}_0,L}(x) = \begin{cases} 0 & x \leq 0 \\ ? & x \geq 0 \end{cases}, \quad h^1_{\mathcal{I}_0,L}(x) = \begin{cases} 1 & x \leq 0 \\ ? & x \geq 0 \end{cases}, \quad h^g_{\mathcal{I}_0,L}(x) = \begin{cases} (-1)^g \chi(L) \cdot x^g & x \leq 0 \\ 0 & x \geq 0 \end{cases}$$

As explained in the introduction, Jiang and Pareschi observed that $h^0_{\mathcal{I}_0,L}$ and $h^1_{\mathcal{I}_0,L}$ encode interesting properties about the basepoint-freeness and the projective normality of the polarization L; subsequent work of Caucci and Ito extended this relation to higher syzygies. In order to state these results, we briefly recall the definition of the property (N_p); for further details, the reader is referred to [Laz04].

Let L be a very ample line bundle on a smooth projective variety X, defining an embedding of X in $\mathbb{P} = \mathbb{P}(H^0(X,L)^\vee)$. Consider the homogeneous coordinate ring $S = \operatorname{Sym} H^0(X,L)$ of $\mathbb{P}$, and the graded S-algebra $R_L = \bigoplus H^0(X, L^m)$.

Definition I.2.4. Let L be an ample line bundle on a smooth projective variety X. We say that L satisfies the *property* (N_p) if it is very ample, and the first $p+1$ steps of the minimal graded free resolution of R_L are of the form

$$\dots \longrightarrow S(-p-1)^{\oplus a_p} \longrightarrow \dots \longrightarrow S(-3)^{\oplus a_2} \longrightarrow S(-2)^{\oplus a_1} \longrightarrow S \longrightarrow R_L \longrightarrow 0$$

Example I.2.5.

(1) (N_0) holds if, and only if, the natural map $S \longrightarrow R_L$ is surjective. In other words, if X is *projectively normal* in $\mathbb{P}$: any divisor of a complete linear system $|L^m|$ is the intersection of X with an hypersurface of degree m in $\mathbb{P}$.

Note that, in such a case, the properties (N_p) $(p \geq 1)$ are conditions on the first p steps of the minimal resolution of the homogeneous ideal $\mathcal{I}_{X/\mathbb{P}} = \ker(S \longrightarrow R_L)$ of X in $\mathbb{P}$.

(2) (N_1) holds if, and only if, X is projectively normal in $\mathbb{P}$ and $\mathcal{I}_{X/\mathbb{P}}$ is generated by quadrics.

(3) (N_2) holds if, and only if, X satisfies (N_1) and the relations between the quadrics generating $\mathcal{I}_{X/\mathbb{P}}$ are generated by linear ones.

Theorem I.2.6 ([JP20, Cau20, Ito21])**.** *For a polarized abelian variety (A, L), the following statements hold:*

(1) *$\mathcal{I}_0\langle L\rangle$ is a GV-sheaf, and it is IT(0) if and only if any ample line bundle representing L is basepoint-free.*

(2) *If $\mathcal{I}_0\langle \frac{1}{2}L\rangle$ is IT(0), then any ample line bundle L is projectively normal.*

(3) *If $\mathcal{I}_0\langle \frac{1}{p+2}L\rangle$ is M-regular for some $p \geq 1$, then any ample line bundle representing L satisfies the property (N_p).*

The reader is referred to [JP20, Section 5] for the definitions of a $\mathbb{Q}$-twisted coherent sheaf $F\langle x_0 L\rangle$ being IT(0), M-regular or a GV-sheaf. In the particular case $F = \mathcal{I}_0$ we will use the following characterization, which is an immediate consequence of [JP20, Proposition 5.3]:

Lemma I.2.7. *Let $x_0 \in \mathbb{Q}$ be a positive rational number.*

(1) *$\mathcal{I}_0\langle x_0 L\rangle$ is a GV-sheaf if and only if $h^1_{\mathcal{I}_0,L}(x_0) = 0$.*

(2) *$\mathcal{I}_0\langle x_0 L\rangle$ is M-regular if and only if $h^1_{\mathcal{I}_0,L}(x_0) = 0$ and $h^1_{\mathcal{I}_0,L}$ is of class $\mathcal{C}^1$ at x_0.*

(3) *$\mathcal{I}_0\langle x_0 L\rangle$ is IT(0) if and only if there is $\epsilon > 0$ such that $h^1_{\mathcal{I}_0,L}(x) = 0$ for all $x \in (x_0 - \epsilon, x_0)$.*

In particular, the *basepoint-freeness threshold*

$$\epsilon_1(L) := \inf \left\{ x \in \mathbb{Q} \mid h^1_{\mathcal{I}_0,L}(x) = 0 \right\},$$

satisfies $\epsilon_1(L) \leq 1$, with equality if and only if the polarization L has base points. In consequence, the functions $h^i_{\mathcal{I}_0,L}$ $(i = 0, 1)$ are unknown only in the interval $(0, 1)$.

I.3 The theta group of an ample line bundle

Let $L \in \mathrm{Pic}(A)$ be an ample line bundle on an abelian variety A. We give a quick review of the representation of the theta group $\mathcal{G}(L)$ on $H^0(A, L)$, explicitly described by Mumford in [Mum66].

Assume that $\mathrm{char}(\mathbb{K})$ does not divide $h^0(L) = \chi(L)$. This guarantees that the polarization isogeny $\varphi_L : A \to \mathrm{Pic}^0(A)$, of degree $\chi(L)^2$, is separable. We will write $K(L) := \ker(\varphi_L)$; for instance, if L is very ample embedding A in $\mathbb{P}(H^0(A, L)^\vee)$, then the points $p \in K(L)$ are those for which the translation t_p on A extends to a projectivity of $\mathbb{P}(H^0(A, L)^\vee)$.

This projective representation comes from the aforementioned representation of the *theta group*

$$\mathcal{G}(L) := \{(x, \varphi) \mid x \in K(L),\ \ \varphi : L \xrightarrow{\cong} t_x^* L\}, \quad (y, \psi) \cdot (x, \varphi) = (x + y, t_x^* \psi \circ \varphi)$$

on $H^0(A, L)$. Note that $\mathcal{G}(L)$ fits into a short exact sequence

$$1 \to \mathbb{K}^* \to \mathcal{G}(L) \to K(L) \to 0,$$

but it is far from being abelian. Indeed, there is a well-defined pairing

$$e^L : K(L) \times K(L) \to \mathbb{K}^*, \quad (x, y) \mapsto \tilde{x} \cdot \tilde{y} \cdot \tilde{x}^{-1} \cdot \tilde{y}^{-1}$$

(here $\tilde{x}, \tilde{y}$ are arbitrary lifts of x, y to $\mathcal{G}(L)$) measuring the noncommutativity of $\mathcal{G}(L)$. This pairing is skew-symmetric and non-degenerate (see [Mum66, Page 293]).

The representation of $\mathcal{G}(L)$ on $H^0(A, L)$ is defined as follows: every $(x, \varphi) \in \mathcal{G}(L)$ induces

$$U_{(x,\varphi)} : H^0(A, L) \to H^0(A, L), \quad s \mapsto t_{-x}^*(\varphi(s))$$

Theorem I.3.1 ([Mum66]). *With the notations above, the following statements hold:*

(1) $K(L) = A(L) \oplus B(L)$, *where* $A(L), B(L) \subset K(L)$ *are maximal totally isotropic subgroups with respect to* e^L. *Moreover, if* L *is of type* $\delta = (d_1, ..., d_g)$, *then* $A(L) \cong \mathbb{Z}/d_1 \oplus ... \oplus \mathbb{Z}/d_g$ *and* $B(L) \cong \widehat{A(L)} = \mathrm{Hom}_{\mathbb{Z}}(A(L), \mathbb{K}^*)$ *via the pairing* e^L.

(2) *As a group,* $\mathcal{G}(L)$ *is isomorphic to* $\mathcal{G}(\delta) := \mathbb{K}^* \times A(L) \times \widehat{A(L)}$ *with the operation*

$$(\alpha, t, l) \cdot (\alpha', t', l') = (\alpha\alpha' \cdot l'(t), t + t', l \cdot l')$$

(3) *The representation of* $\mathcal{G}(L)$ *on* $H^0(A, L)$ *is isomorphic to the representation of* $\mathcal{G}(\delta)$ *on*

$$V(\delta) = \{\mathbb{K}\text{-valued functions on } A(L) = \mathbb{Z}/d_1 \oplus ... \oplus \mathbb{Z}/d_g\}$$

given, for $(\alpha, t, l) \in \mathcal{G}(\delta)$ *and* $f \in V(\delta)$, *as follows:*

$$((\alpha, t, l) \cdot f)(x) = \alpha \cdot l(x) \cdot f(t + x)$$

(4) *Assume that* $\mathrm{char}(\mathbb{K}) \neq 2$ *and* L *is totally symmetric: namely, there exists an isomorphism* $L \cong i^*L$*, acting as* $+1$ *simultaneously on all the fibers* $L(p)$ *of 2-torsion points* $p \in A_2$*. Then the inversion map* $i : A \to A$ *extends to a projectivity of* $\mathbb{P}(H^0(A, L)^\vee)$*; under the isomorphism* $H^0(A, L) \cong V(\delta)$ *of (3), this projectivity is obtained from*

$$\tilde{i} : V(\delta) \to V(\delta), \quad \left(\tilde{i} \cdot f\right)(x) = f(-x)$$

The main advantage of this description is the existence of a canonical basis for $V(\delta)$, which allows an explicit treatment of the endomorphisms $U_{(x,\varphi)}$ and $\tilde{i}$ in coordinates. We will use this approach in section III.3 to obtain lower bounds for the basepoint-freeness threshold $\epsilon_1(L)$.

Chapter II

Chern degree functions

This chapter constitutes the core of this first part of the thesis. We introduce the notion of Chern degree funtions on a polarized surface, and in section II.2 we construct local polynomial expressions for them, which allows to understand their continuity and differentiability (section II.3). After proving the equivalence between Chern degree functions and cohomological rank functions on abelian surfaces (section II.4), in section II.5 we treat the case of (twisted) Gieseker semistable sheaves, for which the Chern degree functions turn out to have a simpler structure.

II.1 Chern degree functions

In the next three sections, (X, L) will be a fixed polarized smooth projective surface over $\mathbb{K}$; in positive characteristic, we will assume that X is neither of general type nor quasi-elliptic with $\kappa(X) = 1$. We present now our main objects of study in this first part of the thesis.

Definition II.1.1. Let $\beta \in \mathbb{Q}$ be a rational number.

(1) If $F \in \mathrm{Coh}^{\beta}(X)$ is an object with HN filtration $0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_r = F$ with respect to $\sigma_{0,\beta}$, we define

$$\mathrm{chd}^0_{F,L}(-\beta) := \sum_{\nu_{0,\beta}(F_i/F_{i-1}) \geq 0} \mathrm{ch}_2^{\beta}(F_i/F_{i-1})$$
$$\mathrm{chd}^1_{F,L}(-\beta) := \sum_{\nu_{0,\beta}(F_i/F_{i-1}) < 0} -\mathrm{ch}_2^{\beta}(F_i/F_{i-1})$$

(2) More generally, for an arbitrary object $E \in \mathrm{D^b}(X)$ and any integer $k \in \mathbb{Z}$, we define the number $\mathrm{chd}^k_{E,L}(-\beta)$ using the cohomologies of E with respect to the heart $\mathrm{Coh}^{\beta}(X)$:

$$\mathrm{chd}^k_{E,L}(-\beta) := \mathrm{chd}^0_{\mathcal{H}^k_{\beta}(E),L}(-\beta) + \mathrm{chd}^1_{\mathcal{H}^{k-1}_{\beta}(E),L}(-\beta)$$

This rule defines, given $E \in \mathrm{D}^{\mathrm{b}}(X)$ and $k \in \mathbb{Z}$, a function $\mathrm{chd}^k_{E,L} : \mathbb{Q} \to \mathbb{Q}_{\geq 0}$ that we will call the *k-th Chern degree function of E*. From the definition, it directly follows that

$$\sum_{k \in \mathbb{Z}} (-1)^k \cdot \mathrm{chd}^k_{E,L}(x) = \mathrm{ch}_2^{-x}(E).$$

so the Chern degree functions are a positive alternate decomposition of the second twisted Chern character.

Remark II.1.2. Thinking of $\sigma_{0,\beta}$ in terms of slicings (see [Bri07] or [MS17, Sections 5.1 and 5.2]), the definition of $\mathrm{chd}^k_{E,L}(-\beta)$ involves all the objects in the HN filtration of $E \in \mathrm{D}^{\mathrm{b}}(X)$ with phase in the interval $\left[\frac{1}{2} - k, \frac{3}{2} - k\right)$.

Example II.1.3. To determine the Chern degree functions of $\mathcal{O}_X$, observe that for every $\beta < 0$ $\mathcal{O}_X \in \mathrm{Coh}^\beta(X)$; moreover, combining Proposition I.1.13 and Theorem I.1.10.(7) we have that $\mathcal{O}_X$ is $\sigma_{\alpha,\beta}$-semistable for every $\alpha > 0$, hence for every $\alpha \geq 0$. A similar situation holds for $\mathcal{O}_X[1]$ when $\beta \geq 0$, so the nonzero Chern degree functions of $\mathcal{O}_X$ are

$$\mathrm{chd}^0_{\mathcal{O}_X,L}(x) = \begin{cases} 0 & x \leq 0 \\ \frac{L^2}{2}x^2 & x \geq 0 \end{cases} \qquad \text{and} \qquad \mathrm{chd}^2_{\mathcal{O}_X,L}(x) = \begin{cases} \frac{L^2}{2}x^2 & x \leq 0 \\ 0 & x \geq 0. \end{cases}$$

Given $F \in \mathrm{Coh}^\beta(X)$, we can also define $\mathrm{chd}^0_{F,L}(-\beta)$ in terms of a unique subobject. Namely, if

$$0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_{s-1} \hookrightarrow F_s \hookrightarrow F_{s+1} \hookrightarrow \ldots \hookrightarrow F_r = F$$

is the HN filtration of F with respect to σ_{0,β_0}, with the inequalities

$$\nu_{0,\beta_0}(F_1) > \ldots > \nu_{0,\beta_0}(F_s/F_{s-1}) > 0 \geq \nu_{0,\beta_0}(F_{s+1}/F_s) > \ldots > \nu_{0,\beta_0}(F/F_{r-1}),$$

then observe that $\mathrm{chd}^0_{F,L}(-\beta) = \mathrm{ch}_2^\beta(F_s)$.

Definition II.1.4. The index $s = s(F)$ is called the *switching index of F* (with respect to $\sigma_{0,\beta}$).

Note that $\nu^-_{0,\beta}(F) > 0$ is equivalent to $F_s = F$, and $\nu^+_{0,\beta}(F) \leq 0$ is equivalent to $F_s = 0$.

These functions satisfy the following properties, analogous to those that are natural from the viewpoint of cohomology:

Proposition II.1.5. *If $E \in \mathrm{D}^{\mathrm{b}}(X)$ and $x \in \mathbb{Q}$, the following properties hold:*

(1) *(Serre vanishing) If $E \in \mathrm{Coh}(X)$, then for $x \gg 0$ one has $\mathrm{chd}^k_{E,L}(x) = 0$ for every $k \neq 0$.*
(2) *(Serre duality) We have $\mathrm{chd}^k_{E,L}(x) = \mathrm{chd}^{2-k}_{E^\vee,L}(-x)$ for every $k \in \mathbb{Z}$.*

Proof. To prove the first item we note that any sheaf E satisfies $E \in \mathrm{Coh}^\beta(X)$ for every $\beta < \mu_-(E)$; in particular, $\mathrm{chd}^k_{E,L}(x) = 0$ for every $x > -\mu_-(E)$ and $k \neq 0, 1$. Thus it only remains to show vanishing for $\mathrm{chd}^1_{E,L}$.

To this end, we consider the HN filtration of E with respect to twisted $(L, -\frac{1}{2}K_X)$-Gieseker stability (if E is not itself torsion-free, we create this filtration using the torsion filtration). By Proposition I.1.13, this is the HN filtration of $E \in \mathrm{Coh}^{\beta}(X)$ with respect to $\nu_{0,\beta}$-stability, for $\beta \ll 0$. Therefore, for $\beta \ll 0$ all the HN factors of E have positive slope $\nu_{0,\beta}$, which proves that $\mathrm{chd}^1_{E,L}(x) = 0$ for $x \gg 0$.

For the second item, we will check the equality assuming that $E \in \mathrm{Coh}^{-x}(X)$; the general statement follows from considering the cohomologies of E with respect to the heart $\mathrm{Coh}^{-x}(X)$.

For simplicity, we write $\beta = -x$. Let

$$0 = E_0 \hookrightarrow E_1 \hookrightarrow \ldots \hookrightarrow E_{r-1} \hookrightarrow E_r = E$$

be the HN filtration of E with respect to $\sigma_{0,\beta}$, where the first HN factor E_1 is assumed to have slope $\nu_{0,\beta} = +\infty$ (if this does not occur, simply write $E_1 = 0$ and let the rest of the filtration be that of E). By definition:

$$\mathrm{chd}^0_{E,L}(-\beta) = \mathrm{ch}^{\beta}_2(E_1) + \mathrm{chd}^0_{E/E_1,L}(-\beta), \quad \mathrm{chd}^1_{E,L}(-\beta) = \mathrm{chd}^1_{E/E_1,L}(-\beta)$$

Moreover, we have a triangle $0 \to (E/E_1)^{\vee}[1] \to E^{\vee}[1] \to E_1^{\vee}[1] \to 0$ in $\mathrm{D^b}(X)$, where:

- $(E/E_1)^{\vee}[1] \in \mathrm{Coh}^{-\beta}(X)$, having $(E/E_{r-1})^{\vee}[1],\ldots,(E_2/E_1)^{\vee}[1]$ as HN factors with respect to $\nu_{0,-\beta}$. This is a consequence of Proposition I.1.16.
- It is not difficult to check that $E_1^{\vee}[2] \in \mathrm{Coh}^{-\beta}(X)$ and it is $\sigma_{0,-\beta}$-semistable (with slope $+\infty$).

It follows that $E^{\vee}[1]$ has two cohomologies with respect to $\mathrm{Coh}^{-\beta}(X)$, namely

$$\mathcal{H}^0_{-\beta}(E^{\vee}[1]) = (E/E_1)^{\vee}[1], \quad \mathcal{H}^1_{-\beta}(E^{\vee}[1]) = E_1^{\vee}[2]$$

whose HN factors are known in terms of those of E. This gives the desired relations

$$\mathrm{chd}^0_{E,L}(-\beta) = \mathrm{chd}^1_{E^{\vee}[1],L}(\beta), \quad \mathrm{chd}^1_{E,L}(-\beta) = \mathrm{chd}^0_{E^{\vee}[1],L}(\beta).$$ □

The following technical lemma, which will be useful later on, is also natural from the same cohomological viewpoint:

Lemma II.1.6. *Let $0 \to E \to F \to Q \to 0$ be a short exact sequence in $\mathrm{Coh}^{\beta}(X)$ ($\beta \in \mathbb{Q}$). If $\mathrm{chd}^0_{Q,L}(-\beta) = 0$ and Q has no subobject $\widetilde{Q} \subset Q$ in $\mathrm{Coh}^{\beta}(X)$ with $\widetilde{Q} \in \ker(Z_{0,\beta})$, then the equality $\mathrm{chd}^0_{E,L}(-\beta) = \mathrm{chd}^0_{F,L}(-\beta)$ holds.*

Proof. Let $0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_s \hookrightarrow F_{s+1} \hookrightarrow \ldots \hookrightarrow F$ be the HN filtration of F with respect to $\sigma_{0,\beta}$, so that $\nu_{0,\beta}(F_s/F_{s-1}) > 0 \geq \nu_{0,\beta}(F_{s+1}/F_s)$. That is, $\mathrm{chd}^0_{F,L}(-\beta) = \mathrm{ch}^{\beta}_2(F_s)$.

By $\mathrm{chd}^0_{Q,L}(-\beta) = 0$ and our extra assumption on $\ker(Z_{0,\beta})$, we know that every subobject of Q has $\nu_{0,\beta_0} \leq 0$. This implies that the morphism $F_s \to F \to Q$ must be 0, and thus $F_s \subset E$.

It turns out that $F_s \subset E$ is the part of the HN filtration of E corresponding to HN factors of positive slope. Therefore, $\mathrm{chd}^0_{E,L}(-\beta) = \mathrm{ch}^{\beta}_2(F_s) = \mathrm{chd}^0_{F,L}(-\beta)$. □

Remark II.1.7. In view of the results in [FLZ21, Section 3] extending the construction of (geometric) Bridgeland stability conditions to a region enlarging the (α,β)-plane, it would be interesting to consider functions defined via weak stability conditions on the "boundary" of this bigger region. More precisely, Fu–Li–Zhao construct a Bridgeland stability condition

$$\widetilde{\sigma}_{a,\beta} = \left(\mathrm{Coh}^{\beta}(X), \widetilde{Z}_{a,\beta} = (-\operatorname{ch}_2^{\beta} + aL^2 \cdot \operatorname{ch}_0) + i(L \cdot \operatorname{ch}_1^{\beta})\right)$$

for every $(a,\beta) \in \mathbb{R}^2$ with $a > \Phi_{X,L}(\beta) - \frac{\beta^2}{2}$, where $\Phi_{X,L}(\beta)$ is the Le Potier function (see [FLZ21, Definition 3.1] and Example I.1.15). In particular, the Bridgeland stability conditions in the (α,β)-plane are recovered as $\widetilde{\sigma}_{\frac{\alpha^2}{2},\beta} = \sigma_{\alpha,\beta}$ for every $\alpha > 0$.

The function $f(x) := \Phi_{X,L}(x) - \frac{x^2}{2}$ being upper-semicontinuous, its discontinuities form a meagre set. Since the complement of a meagre set is dense thanks to the Baire category theorem, it turns out that the points where f is continuous form a dense subset $A_{X,L}$ of $\mathbb{R}$.

Henceforth, one could define functions on $A_{X,L} \cap \mathbb{Q}$ via the HN filtrations with respect to the weak stability conditions $\left\{\widetilde{\sigma}_{f(\beta),\beta} \mid \beta \in A_{X,L} \cap \mathbb{Q}\right\}$. Clearly, in the cases where $f \equiv 0$ (e.g. surfaces with finite Albanese map, as seen in Example I.1.15) this is nothing but our Chern degree functions. In general, to extend these functions to the whole $\mathbb{R}$, one could try to follow the same approach of section II.2; however, while we expect that the existence of *Bridgeland limit filtrations* (i.e. Theorem II.2.8) could be proven following similar arguments, the existence of *weak limit filtrations* (i.e. Theorem II.2.14) seems a much more obscure problem.

Consider for instance a polarized K3 surface (X,L) with $\mathrm{Pic}(X) = \mathbb{Z} \cdot L$ and $L^2 = 2e$. In that case, the existence of spherical objects shows that

$$A_{X,L} \cap \mathbb{Q} = \mathbb{Q} \setminus \left\{\frac{c}{r} \in \mathbb{Q} :\ r | e(c^2+1)\right\}$$

and $f \equiv -\frac{1}{2e}$ on this subset. For $\beta \in A_{X,L} \cap \mathbb{Q}$, one can consider the central charge of $\widetilde{\sigma}_{-\frac{1}{2e},\beta}$, which is given by

$$\widetilde{Z}_{-\frac{1}{2e},\beta} = -v_2^{\beta} + i(L \cdot v_1^{\beta})$$

for $(v_0^{\beta}, v_1^{\beta}, v_2^{\beta}) = v \cdot e^{-\beta L}$ the twisted Mukai vector. Accordingly, if $F \in \mathrm{Coh}^{\beta}(X)$ has HN filtration $0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_r = F$ with respect to $\sigma_{-\frac{1}{2e},\beta}$, we may define for example

$$\mathrm{vdeg}^0_{F,L}(-\beta) := \sum_{\nu_{-\frac{1}{2e},\beta}(F_i/F_{i-1}) \geq 0} v_2^{\beta}(F_i/F_{i-1}).$$

In this particular case we could call such functions *Mukai degree functions*.

Observe that in general one cannot expect to extend the Mukai degree functions to continuous functions in the whole $\mathbb{R}$ mimicking Corollary II.3.3, since discontinuities may arise at certain points of $\mathbb{Q} \setminus A_{X,L}$, as one easily sees with the function of $\mathcal{O}_X$.

In any case, it would be interesting to know if, fixed an object of $\mathrm{D}^{\mathrm{b}}(X)$, there are finitely many such discontinuities and the Mukai degree functions encode information of geometrical or cohomological type.

II.2 Local expressions for the Chern degree functions

This section is devoted to prove that, in a neighborhood of every rational number, the Chern degree functions are piecewise polynomial. The result is analoguous to Theorem I.2.2.(1) for cohomological rank functions, which follows from a transformation formula with respect to the Fourier-Mukai transform. In our case, the proof follows a completely different path, by describing the behaviour of HN filtrations around weak stability conditions σ_{0,β_0} ($\beta_0 \in \mathbb{Q}$). This description may be of independent interest, especially when σ_{0,β_0} lies in the boundary of the (geometric) stability manifold (see Example I.1.15 and Remark II.1.7).

Along this section, we keep fixed a rational number $\beta_0 = \frac{a}{b}$ with a and b coprime integers ($b > 0$).

Bridgeland limit HN filtrations

Our first goal is to control HN filtrations with respect to the Bridgeland stability conditions σ_{α,β_0}, for small values of $\alpha > 0$. In particular, we want to understand whether these filtrations remain constant:

Definition II.2.1. Given $F \in \mathrm{Coh}^{\beta_0}(X)$, if there exists $\alpha_0 > 0$ such that F has the same HN filtration with respect to all the Bridgeland stability conditions σ_{α,β_0} with $\alpha \in (0, \alpha_0)$, we will call this HN filtration the *Bridgeland limit HN filtration* of F at β_0.

In case the Bridgeland limit HN filtration exists, the HN filtration of F at σ_{0,β_0} can be recovered by identifying those limit HN factors with the same tilt slope at $\alpha = 0$.

The first result of this section is that Bridgeland limit HN filtrations exist for σ_{0,β_0}-semistable objects with nonzero tilt slope:

Proposition II.2.2. *Any σ_{0,β_0}-semistable object $F \in \mathrm{Coh}^{\beta_0}(X)$ with $\nu_{0,\beta_0}(F) \neq 0$ admits a Bridgeland limit HN filtration at β_0.*

The case $L \cdot \mathrm{ch}_1^{\beta_0}(F) = 0$ being trivial (in this case F is semistable along the whole line $\beta = \beta_0$), we will assume that $L \cdot \mathrm{ch}_1^{\beta_0}(F) > 0$ (i.e. $\nu_{0,\beta_0}(F) \neq +\infty$). Thanks to the duality functor $\_^{\vee}[1]$ and Proposition I.1.16, we may restrict ourselves to the case $\nu_{0,\beta_0}(F) < 0$.

The proof is then based on two lemmas:

Lemma II.2.3. *Let $F \in \mathrm{Coh}^{\beta_0}(X)$ be σ_{0,β_0}-semistable, with $L \cdot \mathrm{ch}_1^{\beta_0}(F) > 0$ and $\nu_{0,\beta_0}(F) < 0$.*

(1) *The set of subobjects*

$$S_F = \left\{ E \in \mathrm{Coh}^{\beta_0}(X) : E \subseteq F,\ \nu_{0,\beta_0}(E) = \nu_{0,\beta_0}(F),\ \overline{\Delta}(E) \geq 0 \right\}$$

with the same slope and non-negative discriminant is non-empty.

(2) *The expression $\frac{L^2 \cdot \mathrm{ch}_0}{L \cdot \mathrm{ch}_1^{\beta_0}}$ is bounded from below on S_F.*

Before proving this first lemma, we note the following:

Remark II.2.4. By the assumption $\beta_0 = \frac{a}{b}$, every $F \in \mathrm{Coh}^{\beta_0}(X)$ with $\nu^+_{0,\beta_0}(F) < +\infty$ satisfies

$$\left\{L \cdot \mathrm{ch}_1^{\beta_0}(G) : G \subset F \text{ in } \mathrm{Coh}^{\beta_0}(X)\right\} \subseteq \frac{1}{b} \cdot \mathbb{Z}_{>0}, \quad \left\{\nu_{0,\beta_0}(G) : G \subset F \text{ in } \mathrm{Coh}^{\beta_0}(X)\right\} \subseteq \frac{1}{k_F} \cdot \mathbb{Z}$$

where $k_F = 2b\left(bL \cdot \mathrm{ch}_1^{\beta_0}(F)\right)! \in \mathbb{Z}_{>0}$ (it depends only on F). Indeed, since $2b^2\,\mathrm{ch}_2^{\beta_0}(G) \in \mathbb{Z}$ and $bL \cdot \mathrm{ch}_1^{\beta_0}(G) \in (0, bL \cdot \mathrm{ch}_1^{\beta_0}(F)] \cap \mathbb{Z}$, then $\nu_{0,\beta_0}(G) = \frac{2b^2\,\mathrm{ch}_2^{\beta_0}(G)}{2b\left(bL\cdot \mathrm{ch}_1^{\beta_0}(G)\right)} \in \frac{1}{k_F} \cdot \mathbb{Z}$.

Proof of Lemma II.2.3. To prove (1), assume that the set S_F is empty; in particular $\overline{\Delta}(F) < 0$, so F is nonsemistable for every Bridgeland stability condition σ_{α,β_0} with $\alpha > 0$. If $n \in \mathbb{Z}_{>0}$, let G_n be the maximal destabilizing subobject of F with respect to $\sigma_{\frac{1}{n},\beta_0}$.

Observe that $\nu_{0,\beta_0}(G_n) \leq \nu_{0,\beta_0}(F)$, by the σ_{0,β_0}-semistability of F; the emptiness of S_F guarantees a strict inequality. Therefore, $\nu_{0,\beta_0}(G_n) \leq \nu_{0,\beta_0}(F) - \frac{1}{k_F}$ (by Remark II.2.4). Combining with $\nu_{\frac{1}{n},\beta_0}(G_n) > \nu_{\frac{1}{n},\beta_0}(F)$, we get

$$\nu_{0,\beta_0}(F) - \frac{1}{2n^2} \cdot \frac{L^2 \cdot \mathrm{ch}_0(F)}{L \cdot \mathrm{ch}_1^{\beta_0}(F)} = \nu_{\frac{1}{n},\beta_0}(F) < \nu_{\frac{1}{n},\beta_0}(G_n) \leq \nu_{0,\beta_0}(F) - \frac{1}{k_F} - \frac{1}{2n^2} \cdot \frac{L^2 \cdot \mathrm{ch}_0(G_n)}{L \cdot \mathrm{ch}_1^{\beta_0}(G_n)}$$

which gives

$$-\frac{L^2 \cdot \mathrm{ch}_0(G_n)}{L \cdot \mathrm{ch}_1^{\beta_0}(G_n)} > \frac{2}{k_F} n^2 - \frac{L^2 \cdot \mathrm{ch}_0(F)}{L \cdot \mathrm{ch}_1^{\beta_0}(F)} \quad \Longrightarrow \quad -\frac{L^2 \cdot \mathrm{ch}_0(G_n)}{L \cdot \mathrm{ch}_1^{\beta_0}(G_n)} \to +\infty \text{ as } n \to \infty$$

But on the other hand, for every n we have $L \cdot \mathrm{ch}_1^{\beta_0}(G_n) > 0$ and hence

$$0 \leq \overline{\Delta}(G_n) = \left(L \cdot \mathrm{ch}_1^{\beta_0}(G_n)\right)^2 \left(1 - 2\frac{L^2 \cdot \mathrm{ch}_0(G_n)}{L \cdot \mathrm{ch}_1^{\beta_0}(G_n)} \cdot \nu_{0,\beta_0}(G_n)\right),$$

which yields

$$0 \leq 1 - 2\frac{L^2 \cdot \mathrm{ch}_0(G_n)}{L \cdot \mathrm{ch}_1^{\beta_0}(G_n)} \cdot \nu_{0,\beta_0}(G_n) < 1 - 2\frac{L^2 \cdot \mathrm{ch}_0(G_n)}{L \cdot \mathrm{ch}_1^{\beta_0}(G_n)} \cdot \nu_{0,\beta_0}(F)$$

for every n such that $-\frac{L^2\cdot \mathrm{ch}_0(G_n)}{L\cdot \mathrm{ch}_1^{\beta_0}(G_n)} > 0$. Since $\nu_{0,\beta_0}(F) < 0$, this contradicts the limit above and concludes the proof of (1).

To prove (2), let $E \subseteq F$ be a subobject with $\nu_{0,\beta_0}(E) = \nu_{0,\beta_0}(F)$ and $\overline{\Delta}(E) \geq 0$. Then

$$0 \leq \overline{\Delta}(E) = \left(L \cdot \mathrm{ch}_1^{\beta_0}(E)\right)^2 \left(1 - 2\frac{L^2 \cdot \mathrm{ch}_0(E)}{L \cdot \mathrm{ch}_1^{\beta_0}(E)} \cdot \nu_{0,\beta_0}(F)\right)$$

implies, under the assumption $\nu_{0,\beta_0}(F) < 0$, that

$$\frac{L^2 \cdot \mathrm{ch}_0(E)}{L \cdot \mathrm{ch}_1^{\beta_0}(E)} \geq \frac{1}{2\nu_{0,\beta_0}(F)}$$

□

The elements $E \in S_F$ satisfy $\frac{L^2 \cdot \mathrm{ch}_0(E)}{L \cdot \mathrm{ch}_1^{\beta_0}(E)} \in \frac{1}{(bL \cdot \mathrm{ch}_1^{\beta_0}(F))!}\mathbb{Z}$. Hence by Lemma II.2.3.(2), we can consider an element $E \in S_F$ with minimum $\frac{L^2 \cdot \mathrm{ch}_0}{L \cdot \mathrm{ch}_1^{\beta_0}}$ among the objects of S_F; by noetherianity of $\mathrm{Coh}^{\beta_0}(X)$, we may assume as well that E is maximal with this property.

Remark II.2.5. A priori, it is not obvious that E must be unique. This will follow from the proof of Proposition II.2.2, where we will see that E is the first step in the Bridgeland limit HN filtration of F (recall that HN filtrations are unique).

Lemma II.2.6. *There exists $\alpha_0 > 0$ such that E is σ_{α,β_0}-semistable for every $\alpha \in [0, \alpha_0)$.*

Proof. The statement is clear for $\alpha = 0$: σ_{0,β_0}-semistability of E (actually of any element in S_F) trivially follows from that of F.

Now consider $\alpha > 0$ such that E is not σ_{α,β_0}-semistable, and let $G \subsetneq E$ be a maximal destabilizing subobject. Note that $0 < L \cdot \mathrm{ch}_1^{\beta_0}(G)$ (since E is σ_{0,β_0}-semistable) and $L \cdot \mathrm{ch}_1^{\beta_0}(G) < L \cdot \mathrm{ch}_1^{\beta_0}(E)$.

Moreover, $\nu_{0,\beta_0}(G) < \nu_{0,\beta_0}(E)$. Indeed, an equality would imply (since $\nu_{\alpha,\beta_0}(G) > \nu_{\alpha,\beta_0}(E)$) that G has smaller $\frac{L^2 \cdot \mathrm{ch}_0}{L \cdot \mathrm{ch}_1^{\beta_0}}$ than E, contradicting our hypothesis on E. Therefore

$$\mathrm{ch}_2^{\beta_0}(G) < \mathrm{ch}_2^{\beta_0}(E) \cdot \frac{L \cdot \mathrm{ch}_1^{\beta_0}(G)}{L \cdot \mathrm{ch}_1^{\beta_0}(E)} < 0$$

(note that $\mathrm{ch}_2^{\beta_0}(E) < 0$ because $\nu_{0,\beta_0}(E) = \nu_{0,\beta_0}(F) < 0$).

Now, since $\overline{\Delta}(G) \geq 0$ and $\mathrm{ch}_2^{\beta_0}(G) < 0$, we have $\left(L \cdot \mathrm{ch}_1^{\beta_0}(G)\right)^2 \geq 2\left(L^2 \cdot \mathrm{ch}_0(G)\right)\mathrm{ch}_2^{\beta_0}(G)$, so

$$1 \geq \frac{2L^2 \cdot \mathrm{ch}_0(G)}{L \cdot \mathrm{ch}_1^{\beta_0}(G)} \cdot \nu_{0,\beta_0}(G)$$

and

$$\frac{-L^2 \cdot \mathrm{ch}_0(G)}{L \cdot \mathrm{ch}_1^{\beta_0}(G)} \leq \frac{-1}{2\nu_{0,\beta_0}(G)} < \frac{-1}{2\nu_{0,\beta_0}(E)} < -\frac{(L \cdot \mathrm{ch}_1^{\beta_0}(E))^2}{\frac{2}{b}\mathrm{ch}_2^{\beta_0}(E)},$$

where in the last inequality we have used that $L \cdot \mathrm{ch}_1^{\beta_0}(E) > \frac{1}{b}$. Therefore,

$$\begin{aligned}\nu_{0,\beta_0}(E) - \frac{\alpha^2 L^2 \cdot \mathrm{ch}_0(E)}{2L \cdot \mathrm{ch}_1^{\beta_0}(E)} &= \nu_{\alpha,\beta_0}(E) < \nu_{\alpha,\beta_0}(G) = \nu_{0,\beta_0}(G) - \frac{\alpha^2 L^2 \cdot \mathrm{ch}_0(G)}{2L \cdot \mathrm{ch}_1^{\beta_0}(G)} \\ &< \nu_{0,\beta_0}(G) - \frac{\alpha^2 (L \cdot \mathrm{ch}_1^{\beta_0}(E))^2}{\frac{4}{b}\mathrm{ch}_2^{\beta_0}(E)} \leq \nu_{0,\beta_0}(E) - \frac{1}{k_E} - \frac{\alpha^2 (L \cdot \mathrm{ch}_1^{\beta_0}(E))^2}{\frac{4}{b}\mathrm{ch}_2^{\beta_0}(E)},\end{aligned}$$

which gives the inequality

$$\frac{\alpha^2}{2}\left(\frac{(L \cdot \mathrm{ch}_1^{\beta_0}(E))^2}{-\frac{2}{b}\mathrm{ch}_2^{\beta_0}(E)} + \frac{L^2 \cdot \mathrm{ch}_0(E)}{L \cdot \mathrm{ch}_1^{\beta_0}(E)}\right) > \frac{1}{k_E}$$

Using $\overline{\Delta}(E) \geq 0$, one can check that the factor multiplying $\frac{\alpha^2}{2}$ is positive. Since this factor and k_E only depend on E, this yields a lower bound for those α for which E is not σ_{α,β_0}-semistable. □

Proof of Proposition II.2.2. Consider the subobject $E \subseteq F$ defined below the proof of Lemma II.2.3; applying an inductive process (now to F/E, and so on) yields a chain of subobjects of F, which is finite by the noetherianity of $\mathrm{Coh}^{\beta_0}(X)$.

This chain is a HN filtration for F, for all σ_{α,β_0} with $\alpha > 0$ small enough. Indeed, on the one hand the semistability of the factors is guaranteed by Lemma II.2.6; on the other hand, the inequalities of tilt slopes follow from the properties imposed to the subobjects taken at each step. □

As a first consequence of Proposition II.2.2, we obtain that σ_{0,β_0}-(semi)stability keeps some properties from Bridgeland stability:

Corollary II.2.7. *Let $F \in \mathrm{Coh}^{\beta_0}(X)$ be an object.*

(1) *If F is σ_{0,β_0}-semistable, then $\overline{\Delta}(F) \geq 0$.*
(2) *(Openness of stability) If F is σ_{0,β_0}-stable with $\nu_{0,\beta_0}(F) \neq 0$, then there exists $\alpha_0 > 0$ such that F is σ_{α,β_0}-stable for every $\alpha \in [0, \alpha_0)$.*
(3) *If F is σ_{0,β_0}-semistable with $\nu_{0,\beta_0}(F) \neq 0$, there exists a region of Bridgeland stability conditions in the (α, β)-plane for which F is semistable.*

Proof. Note that property (1) is trivially satisfied when $\mathrm{ch}_2^{\beta_0}(F) = 0$. If $\mathrm{ch}_2^{\beta_0}(F) \neq 0$, according to Proposition II.2.2 the σ_{0,β_0}-semistable object F has a Bridgeland limit HN filtration

$$0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_{r-1} \hookrightarrow F_r = F$$

valid for all σ_{α,β_0} with sufficiently small $\alpha > 0$. Of course each HN factor has $\overline{\Delta}(F_k/F_{k-1}) \geq 0$, and by construction of the filtration the equalities $\nu_{0,\beta_0}(F_k/F_{k-1}) = \nu_{0,\beta_0}(F)$ hold. Hence, the numbers $Z_{0,\beta_0}(F_k/F_{k-1})$ are in a ray of the complex plane and [BMS16, Lemma A.7] gives $\overline{\Delta}(F) \geq \overline{\Delta}(F_{r-1}) \geq \ldots \geq \overline{\Delta}(F_1) \geq 0$.

To prove (2), note that the result is trivial when $L \cdot \mathrm{ch}_1^{\beta_0}(F) = 0$ (in this case, the σ_{0,β_0}-stability of F is equivalent to F being a simple object of $\mathrm{Coh}^{\beta_0}(X)$).

If F is σ_{0,β_0}-stable with $L \cdot \mathrm{ch}_1^{\beta_0}(F) > 0$, no subobject $E \in S_F \setminus \{F\}$ (notation as in Lemma II.2.3) destabilizes F for $\alpha > 0$. Indeed, since F is σ_{0,β_0}-stable we must have $Z_{0,\beta_0}(F/E) = 0$, so

$$\nu_{\alpha,\beta_0}(E) < \nu_{\alpha,\beta_0}(F) < +\infty = \nu_{\alpha,\beta_0}(F/E)$$

for every $\alpha > 0$.

By the construction of Proposition II.2.2, all the potential σ_{α,β_0} destabilizers of F (for α sufficiently small) are in $S_F \setminus \{F\}$; hence the strict inequalities actually imply that F is σ_{α,β_0}-stable (for α sufficiently small), which proves (2).

In (3), there is nothing to prove if F is σ_{α,β_0}-semistable for small values of $\alpha > 0$ (in particular, this covers the case where $L \cdot \mathrm{ch}_1^{\beta_0}(F) = 0$).

For the rest of proof, we assume without loss of generality that $\nu_{0,\beta_0}(F) > 0$ thanks to the duality functor (recall Proposition I.1.16). If $F_1, F_2/F_1, \ldots, F/F_{r-1}$ denote the factors of the Bridgeland limit HN filtration of F at β_0, by construction we have:

$$\nu_{0,\beta_0}(F_1) = \nu_{0,\beta_0}(F_2/F_1) = \ldots = \nu_{0,\beta_0}(F/F_{r-1}) = \nu_{0,\beta_0}(F).$$

Thus by Bertram's Nested Wall Theorem I.1.10, each factor F_i/F_{i-1} defines the same (numerical) wall W for F: this wall is a semicircle whose left intersection point with the line $\alpha = 0$ is $(0, \beta_0)$.

Since the factors $F_1, F_2/F_1, \ldots, F/F_{r-1}$ are σ_{α,β_0}-semistable when $\alpha \geq 0$ is small enough, it turns out that they are Bridgeland semistable along the wall W. Hence F is also semistable along W, since it is a (succesive) extension of semistable objects with the same slope. □

Now we are ready to improve Proposition II.2.2, showing the existence of Bridgeland limit HN filtrations for objects without HN factors of vanishing tilt slope.

Theorem II.2.8. *Let $\beta_0 \in \mathbb{Q}$ and $F \in \mathrm{Coh}^{\beta_0}(X)$ be an object having no HN factor with respect to σ_{0,β_0} of vanishing ν_{0,β_0}. Then, F admits a Bridgeland limit HN filtration at β_0.*

Proof. The result follows from induction on the length of the HN filtration of F with respect to σ_{0,β_0}; the initial case is nothing but Proposition II.2.2.

If $0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_{r-1} \hookrightarrow F_r = F$ is the HN filtration of F with respect to σ_{0,β_0}, by induction hypothesis we may assume that both F_{r-1} and F/F_{r-1} admit a Bridgeland limit HN filtration at β_0.

Then, we can glue these filtrations to form that of F. Indeed, if F_{r-1}/A and B/F_{r-1} respectively denote the last and the first limit HN factors of F_{r-1} and F/F_{r-1}, the inequality

$$\nu_{0,\beta_0}(F_{r-1}/A) = \nu_{0,\beta_0}(F_{r-1}/F_{r-2}) > \nu_{0,\beta_0}(F/F_{r-1}) = \nu_{0,\beta_0}(B/F_{r-1})$$

guarantees this gluing, since we can take α_0 small enough so that $\nu_{\alpha,\beta_0}(F_{r-1}/A) > \nu_{\alpha,\beta_0}(B/F_{r-1})$ for every $\alpha \in (0, \alpha_0)$. □

Weak limit HN filtrations

Even if we are interested in semistability at the line $\alpha = 0$, Bridgeland limit HN filtrations allow us to work in the (α, β)-plane of Bridgeland stability conditions, where the wall-crossing phenomenon is well understood.

Following this strategy, already used in the proof of Corollary II.2.7.(3), now we want to study HN filtrations at $\sigma_{0,\beta}$ for (rational) values of β close to β_0:

Definition II.2.9. Given $F \in \mathrm{Coh}^{\beta_0}(X)$, if there exists $\epsilon > 0$ such that F has the same HN filtration with respect to all the weak stability conditions $\sigma_{0,\beta}$ with $\beta \in (\beta_0, \beta_0 + \epsilon) \cap \mathbb{Q}$, we will call this HN filtration the *right weak limit HN filtration* of F at β_0. Its factors will be called *right weak limit HN factors*.

Analogously we define the *left weak limit HN filtration* of F at β_0.

Lemma II.2.10. *If $F \in \mathrm{Coh}^{\beta_0}(X)$ is an object with $\nu^+_{0,\beta_0}(F) < 0$, then F admits a right weak limit HN filtration at β_0 all of whose right weak limit factors have tilt slope $\nu_{0,\beta_0} < 0$.*

Proof. Denote by

$$0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_{r-1} \hookrightarrow F_r = F$$

the Bridgeland limit HN filtration of F constructed in Theorem II.2.8. We will see that this is the desired right weak limit HN filtration for F. Recall that when $\alpha = 0$ (possibly) some of the HN factors get identified, if they have the same slope.

By assumption $\nu_{0,\beta_0}(F_i/F_{i-1}) < 0$ for every i, so the point $(0, \beta_0)$ lies on the right-hand side of the hyperbolas H_i of the objects F_i/F_{i-1}. In particular, if we study the locus in the (α, β)-plane where these HN factors F_i/F_{i-1} become non-semistable, we find that any such wall contains a segment of the line $\beta = \beta_0$ in its interior (see Figure II.1).

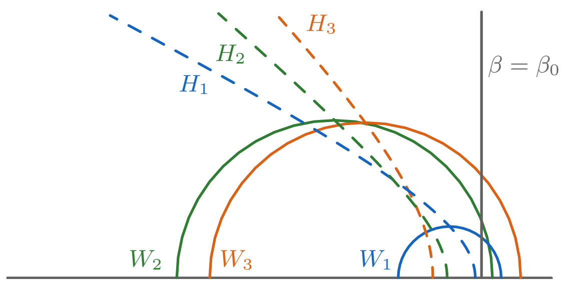


Figure II.1: One hyperbola and at most one semicircular wall for each HN factor F_i/F_{i-1}

This proves that, for all β in a sufficiently small right neighborhood of β_0, the objects F_i/F_{i-1} are $\sigma_{0,\beta}$-semistable. Shrinking if necessary this neighborhood, the inequalities $\nu_{0,\beta}(F_i/F_{i-1}) > \nu_{0,\beta}(F_{i+1}/F_i)$ will be preserved, no matter if these HN factors were merged in the HN filtration for σ_{0,β_0}.

Finally, Remark I.1.7 guarantees that, possibly after another shrinking, the chain of inclusions

$$0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_{r-1} \hookrightarrow F_r = F$$

also holds in $\mathrm{Coh}^{\beta}(X)$, for all β in this neighborhood. Summarizing, we have proved that this chain is the right weak limit HN filtration at β_0. □

When the object is semistable, the same holds for positive tilt slope:

Lemma II.2.11. *If $F \in \mathrm{Coh}^{\beta_0}(X)$ is σ_{0,β_0}-semistable with $\nu_{0,\beta_0}(F) > 0$, then F admits a right weak limit HN filtration at β_0.*

Proof. We start by assuming $L \cdot \mathrm{ch}_1^{\beta_0}(F) > 0$, so that $\beta = \beta_0$ is not a vertical wall for F. According to Corollary II.2.7.(3) and its proof, the object F is Bridgeland semistable along the (uniquely determined) semicircle W' satisfying: W' is centered at the β-axis, its top point lies on the hyperbola H_F and its left intersection point with the β-axis is $(0, \beta_0)$. This is true regardless of whether W' is a numerical wall for F or not.

Let $p = (\overline{\alpha}, \overline{\beta})$ denote the top point of W', i.e. its intersection point with H_F. Local finiteness for Bridgeland stability conditions (see, e.g., [BM11, Proposition 3.3.(b)]) ensures us that, for some $\epsilon' > 0$, the HN filtration of F is constant for all the stability conditions $\sigma_{\overline{\alpha},\beta}$ with $\beta \in (\overline{\beta} - \epsilon', \overline{\beta})$. This implies that, in a sufficiently small annulus inside W', the HN filtration of F stays constant (see Figure II.2).

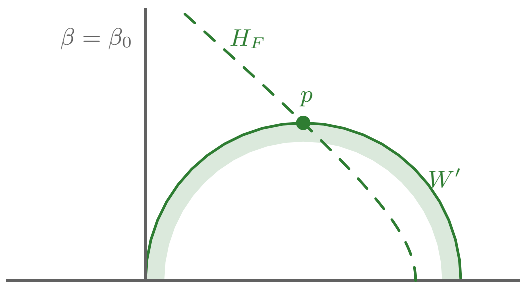


Figure II.2: Semicircle W' and annulus inside it

This constant filtration is the right weak limit HN filtration of F at β_0, as claimed.

It only remains to check the case when $L \cdot \mathrm{ch}_1^{\beta_0}(F) = 0$. The strategy is similar. In this case, F is Bridgeland semistable along the whole vertical wall $\beta = \beta_0$. Local finiteness for Bridgeland stability conditions gives that, inside $\{\alpha > 0, \beta \geq \beta_0\}$, the HN filtration of F remains constant for all the stability conditions in a certain open neighborhood of $\beta = \beta_0$.

Using that objects with $\overline{\Delta} = 0$ never get destabilized, it is not difficult to check that this holds in a "tubular" neighborhood of the form $(0, +\infty) \times [\beta_0, \beta_0 + \epsilon)$, for a certain $\epsilon > 0$. This gives the desired HN filtration of F with respect to $\sigma_{0,\beta}$, when β lies in a right neighborhood of β_0. □

In view of Lemma II.2.10 and Lemma II.2.11, we would like to conclude the existence of right weak limit filtrations at $\beta_0 \in \mathbb{Q}$, at least for objects $F \in \mathrm{Coh}^{\beta_0}(X)$ having no HN factor of tilt slope $\nu_{0,\beta_0} = 0$.

Nevertheless, if our object F has more than one HN factor at σ_{0,β_0} with positive slope, the way of gluing their right weak limit HN filtrations is not trivial. Roughly, this is caused because not all the weak limit HN factors in a limit HN filtration approach the same slope as $\beta \to \beta_0^+$, since σ_{0,β_0} is not a proper stability condition.

In order to solve this problem, we collect some information in the following lemma, which in particular introduces the notion of *core subobject*:

Lemma II.2.12. *Let $F \in \mathrm{Coh}^{\beta_0}(X)$ be a σ_{0,β_0}-semistable object with $\nu_{0,\beta_0}(F) > 0$. Then:*

(1) *There exists a (possibly trivial) subobject $F' \subset F$ in $\mathrm{Coh}^{\beta_0}(X)$ such that $Q = F/F' \in \ker(Z_{0,\beta_0})$. Moreover, F' can be taken minimal satisfying this property. We call it a* core subobject *of F.*
(2) *If $Q \neq 0$, then its tilt slope is discontinuous with respect to β, i.e,*

$$\nu_{0,\beta_0}(Q) = +\infty, \qquad \lim_{\beta \to \beta_0^+} \nu_{0,\beta}(Q) = 0.$$

(3) *If $F' \neq 0$, then the tilt slope $\nu_{0,\beta}$ of every right weak limit HN factor of F' has limit $\nu_{0,\beta_0}(F')$ as $\beta \to \beta_0^+$.*
(4) *The right weak limit HN filtration of F at β_0 consists of the right weak limit HN filtration of F' together with the quotient $Q = F/F'$. In particular, the core subobject F' is unique.*

Proof. The existence of F' is clear, since we allow $F' = F$. If $F' \neq F$, then by Proposition I.1.14 $Q[-1]$ is a twisted $(L, -\frac{1}{2}K_X)$-Gieseker semistable vector bundle with $\mu_L = \beta_0$ and $\overline{\Delta} = 0$. Therefore we can consider F' minimal satisfying $Q = F/F' \in \ker(Z_{0,\beta_0})$, because if

$$\ldots \subset F_2' \subset F_1' \subset F$$

is a chain of subobjects with this property, then one has $0 \leq \ldots < \overline{\Delta}(F_2') < \overline{\Delta}(F_1') < \overline{\Delta}(F)$.

Now, a simple computation shows (2). In order to prove (3), note that Lemma II.2.11 guarantees the existence of a right weak limit HN filtration for F' at β_0.

If $L \cdot \mathrm{ch}_1^{\beta_0}(F') > 0$, by construction this is the HN filtration in an annulus inside the wall for F' passing through $(0, \beta_0)$. It is easy to check, under the assumption of minimality on F', that no right weak limit HN factor of F' has $\beta = \beta_0$ as a vertical wall. Hence every right weak limit HN factor has $L \cdot \mathrm{ch}_1^{\beta_0} > 0$, which shows the continuity of its tilt slope in a neighborhood of $(0, \beta_0)$.

If $L \cdot \mathrm{ch}_1^{\beta_0}(F') = 0$, then every right weak limit HN factor of F' either has tilt slope $\nu_{0,\beta} = +\infty$ for every β, or has $\overline{\Delta} > 0$ and $\beta = \beta_0$ as a vertical wall. In both cases, the tilt slope $\nu_{0,\beta}$ approaches $\nu_{0,\beta_0}(F') = +\infty$ as $\beta \to \beta_0^+$. This finishes the proof of (3).

Finally, observe that $Q \in \mathrm{Coh}^{\beta}(X)$ is semistable for every $\sigma_{\alpha,\beta}$ with $\alpha \geq 0$ and $\beta \geq \beta_0$; this is due to the description of Proposition I.1.14 and Remark I.1.17. Thus (4) becomes a consequence of (3) (all the right weak limit HN factors of F' approach the slope $\nu_{0,\beta_0}(F') = \nu_{0,\beta_0}(F) > 0$ as $\beta \to \beta_0^+$) and (2). □

Note that if the object F' is non-trivial (i.e. $F' \neq 0, F$), then F may simultaneously be σ_{0,β_0}-stable and $\sigma_{0,\beta}$-nonsemistable for every $\beta \to \beta_0^+$. We can visualize the situation in Figure II.3.

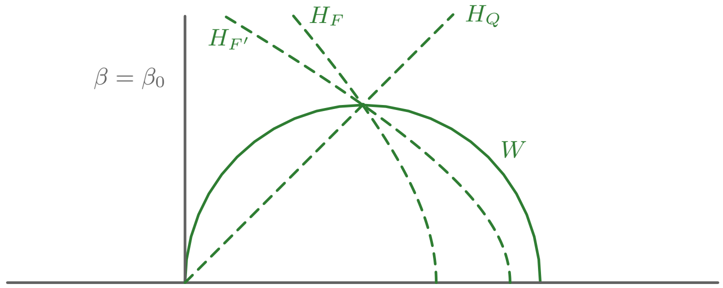


Figure II.3: The wall W is defined by $0 \to F' \to F \to Q \to 0$.

Proposition II.2.13. *If $F \in \mathrm{Coh}^{\beta_0}(X)$ is an object satisfying $\nu^-_{0,\beta_0}(F) > 0$, then F admits a right weak limit HN filtration at β_0. Furthermore, all of its right weak limit HN factors satisfy*

$$\lim_{\beta \to \beta_0^+} \nu_{0,\beta} \geq 0$$

with equality for at most the last weak limit HN factor.

Proof. Let $0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_s = F$ be the HN filtration of F at σ_{0,β_0}. We construct inductively a chain of subobjects $G_1 \subset \ldots \subset G_s$ (with $G_i \subset F_i$) as follows:

- $G_1 = F_1'$ is the core subobject of F_1 (throughout this proof we use the notation of F' to denote the core of F, as defined in Lemma II.2.12.(1)).
- For $i > 1$, we define G_i in such a way that $(F_i/G_{i-1})' = G_i/G_{i-1}$.

We will prove, by induction on s, that F admits a right weak limit HN filtration whose last limit HN factor is F/G_s, and that every weak limit HN factor of G_s has positive limit tilt slope. Note that the case $s = 1$ follows from Lemma II.2.12.

For the induction step, one could assume that the statement is true for F_{s-1}. Nevertheless, it is possible that the right weak limit filtrations of F_{s-1} and F_s/F_{s-1} do not glue directly, since it may happen

$$\lim_{\beta \to \beta_0^+} \nu_{0,\beta}(F_{s-1}/G_{s-1}) = 0 < \lim_{\beta \to \beta_0^+} \nu_{0,\beta}(\text{first limit HN factor of } F_s/F_{s-1}).$$

Then, the strategy consists on replacing F_{s-1} by G_{s-1}. We are allowed to do this because the HN filtration of G_{s-1} with respect to σ_{0,β_0} has length $\leq s-1$. Indeed, G_{s-1} has the same HN polygon as F_{s-1}, as a consequence of the equality $Z_{0,\beta_0}(G_i) = Z_{0,\beta_0}(F_i)$ for every i (see [Bay19, Section 3] or [MS17, Section 4] for the definition and the properties of the HN polygon).

Now, on the one hand by induction hypothesis the right weak limit HN filtration of G_{s-1} exists, and it satisfies

$$\lim_{\beta \to \beta_0^+} \nu_{0,\beta}(\text{last weak limit HN factor of } G_{s-1}) = \nu_{0,\beta_0}(G_{s-1}/G_{s-2})$$

On the other hand, we claim that F_s/G_{s-1} admits a right weak limit HN filtration; this is not obvious at all, since F_s/G_{s-1} is not σ_{0,β_0}-semistable (it contains the subobject F_{s-1}/G_{s-1} with slope $\nu_{0,\beta_0} = +\infty$) and we cannot apply directly Lemma II.2.11.

So to see this, we consider the wall W in the (α, β)-plane defined by the short exact sequence of $\mathrm{Coh}^{\beta_0}(X)$

$$0 \to F_{s-1}/G_{s-1} \to F_s/G_{s-1} \to F_s/F_{s-1} \to 0$$

The left intersection point of W with $\alpha = 0$ is $(0, \beta_0)$; the picture is similar to that of Figure II.3, simply replacing $H_Q, H_F, H_{F'}$ by $H_{F_{s-1}/G_{s-1}}, H_{F_s/G_{s-1}}, H_{F_s/F_{s-1}}$ (respectively).

Note that F_s/G_{s-1} is $\sigma_{\alpha,\beta}$-semistable for all the Bridgeland stability conditions $(\alpha, \beta) \in W$ with $\alpha > 0$, since it is an extension of $\sigma_{\alpha,\beta}$-semistable objects of the same slope. Then, as in the proof of Lemma II.2.11, local finiteness ensures that the HN filtration of F_s/G_{s-1} is constant for all the stability conditions in a small annulus inside W. This gives the desired right weak limit HN filtration for F_s/G_{s-1}.

Every factor of this right weak limit HN filtration for F_s/G_{s-1} satisfies

$$\begin{aligned} 0 < \lim_{\beta \to \beta_0^+} \nu_{0,\beta}(\text{every limit HN factor of } F_s/G_{s-1}) = \nu_{0,\beta_0}(F_s/G_{s-1}) = \nu_{0,\beta_0}(F_s/F_{s-1}) \\ < \nu_{0,\beta_0}(F_{s-1}/F_{s-2}) = \nu_{0,\beta_0}(G_{s-1}/G_{s-2}) = \lim_{\beta \to \beta_0^+} \nu_{0,\beta_0}(\text{last limit HN factor of } G_{s-1}) \end{aligned}$$

with the only possible exception of the last weak limit HN factor for F_s/G_{s-1}, which may have tilt slope of limit 0. Using the induction hypothesis, one easily checks that this last weak limit HN factor for F_s/G_{s-1} is precisely F_s/G_s.

This allows to glue the right weak limit filtrations of G_{s-1} and F_s/G_{s-1} to produce that of F_s, which proves the assertion. □

Now we are ready to prove the main existence result of this subsection, namely the existence of right weak limit HN filtration for objects without HN factors of vanishing tilt slope (see Remark II.2.15 for the left filtrations).

Theorem II.2.14. *Let $\beta_0 \in \mathbb{Q}$ and $F \in \mathrm{Coh}^{\beta_0}(X)$ be an object having no HN factor with respect to σ_{0,β_0} of vanishing ν_{0,β_0}. Then, F admits a right weak limit HN filtration at β_0.*

Proof. Denote by $0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_s \hookrightarrow F_{s+1} \hookrightarrow \ldots \hookrightarrow F_r = F$ the HN filtration of F with respect to σ_{0,β_0}, so that

$$\nu_{0,\beta_0}(F_1) > \ldots > \nu_{0,\beta_0}(F_s/F_{s-1}) > 0 > \nu_{0,\beta_0}(F_{s+1}/F_s) > \ldots > \nu_{0,\beta_0}(F/F_{r-1})$$

On the one hand, by Lemma II.2.10 F/F_s admits a right weak limit HN filtration. It turns out that, if R is a weak limit HN factor of F/F_s (corresponding to a weak limit HN factor of F_i/F_{i-1}

for some $i = s+1, \ldots, r$), then

$$\lim_{\beta \to \beta_0^+} \nu_{0,\beta}(R) = \nu_{0,\beta_0}(F_i/F_{i-1}) < 0$$

On the other hand, according to Proposition II.2.13, F_s has a right weak limit HN filtration, with all its weak limit HN factors satisfying $\lim_{\beta \to \beta_0^+} \nu_{0,\beta} \geq 0$.

A standard glueing of the right weak limit filtrations for F_s and F/F_s finishes the proof. □

Remark II.2.15. Under the same hypothesis on $F \in \mathrm{Coh}^{\beta_0}(X)$, the existence of a left weak limit HN filtration at β_0 follows from the preservation of stability by the derived dual (see Proposition I.1.16).

The main difference is that for $\beta \to \beta_0^-$ the object F may have two nontrivial cohomologies with respect to $\mathrm{Coh}^{\beta}(X)$, namely

$$\mathcal{H}^{-1}_{\beta}(F) = \mathcal{H}^{-1}(F_1), \;\; \mathcal{H}^{0}_{\beta}(F) = F/(\mathcal{H}^{-1}(F_1)[1])$$

where F_1 is a first step in the HN filtration of F with respect to σ_{0,β_0}, satisfying $\nu_{0,\beta_0}(F_1) = +\infty$.

The left weak limit filtration of F is thus obtained by applying $\_^{\vee}[1]$ to the right weak limit filtration at $-\beta_0$ of $\mathcal{H}^0_{-\beta_0}(F^{\vee}[1]) = (F/F_1)^{\vee}[1]$ and $\mathcal{H}^1_{-\beta_0}(F^{\vee}[1]) = F_1^{\vee}[2]$.

Local piecewise polynomial expressions

As an immediate consequence of the right weak limit HN filtrations constructed in Theorem II.2.14, we have:

Corollary II.2.16. *If $F \in \mathrm{Coh}^{\beta_0}(X)$ is an object with all its HN factors with respect to σ_{0,β_0} having slope $\nu_{0,\beta_0} \neq 0$, then there exists $\epsilon > 0$ such that the functions $\mathrm{chd}^0_{F,L}, \mathrm{chd}^1_{F,L}$ are (piecewise) polynomial along the interval $(-\beta_0 - \epsilon, -\beta_0)$.*

More explicitly, let $0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_s \hookrightarrow F_{s+1} \hookrightarrow \ldots \hookrightarrow F_r = F$ be the HN filtration of F with respect to σ_{0,β_0}, so that

$$\nu_{0,\beta_0}(F_1) > \ldots > \nu_{0,\beta_0}(F_s/F_{s-1}) > 0 > \nu_{0,\beta_0}(F_{s+1}/F_s) > \ldots > \nu_{0,\beta_0}(F/F_{r-1})$$

Consider the chain $G_1 \subset \ldots \subset G_s$ in $\mathrm{Coh}^{\beta_0}(X)$ (with $G_i \subset F_i$) inductively defined by the rules $G_1 = F_1'$ and $G_i/G_{i-1} = (F_i/G_{i-1})'$ (where F' denotes the core of F as defined in Lemma II.2.12.(1)). Then, for all (rational) x in a left neighborhood of $-\beta_0$ we have

$$\mathrm{chd}^0_{F,L}(x) = \mathrm{ch}_2^{-x}(G_s), \;\; \mathrm{chd}^1_{F,L}(x) = -\,\mathrm{ch}_2^{-x}(F/G_s).$$

Remark II.2.17. Along the interval where the right weak limit HN filtration of F at β_0 remains constant, the functions $\mathrm{chd}^0_{F,L}, \mathrm{chd}^1_{F,L}$ may still change their polynomial expression. This happens if the last right weak limit HN factor of G_s acquires tilt slope 0.

Observe that the chain $G_1 \subset \ldots \subset G_s$ of the proof of Corollary II.2.16 encodes the Chern degree functions of F in a left neighborhood of $-\beta_0$. For easy reference, we fix the following terminology (adopted for all objects $F \in \mathrm{Coh}^{\beta_0}(X)$, including those with a HN factor of tilt slope 0 as well):

Definition II.2.18. Let $F \in \mathrm{Coh}^{\beta_0}(X)$ and let $F_1, \ldots, F_s/F_{s-1}$ be the HN factors of F having slope $\nu_{0,\beta_0} > 0$, that is, $s = s(F)$ is the switching index of F. The *core filtration* of F at β_0 is the chain $G_1 \subset \ldots \subset G_s$ in $\mathrm{Coh}^{\beta_0}(X)$ (with $G_i \subset F_i$) inductively defined by $G_1 = F_1'$ and $G_i/G_{i-1} = (F_i/G_{i-1})'$, where F' denotes the core of F as defined in Lemma II.2.12.(1).

Now we want to conclude that for every $F \in \mathrm{Coh}^{\beta_0}(X)$ the function $\mathrm{chd}^0_{F,L}$ admits the (piecewise) polynomial expression $\mathrm{ch}_2^{-x}(G_s)$ in a left neighborhood of $-\beta_0$, where G_s is the last object appearing in the core filtration of F at β_0. For this, we need to treat the case of HN factors with slope $\nu_{0,\beta_0} = 0$; we will not find a weak limit HN filtration for them, but the following result will be enough for our purposes.

Proposition II.2.19. *If $F \in \mathrm{Coh}^{\beta_0}(X)$ is σ_{0,β_0}-semistable with $\nu_{0,\beta_0}(F) = 0$, then $\mathrm{chd}^0_{F,L}(-\beta) = 0$ for every rational number $\beta > \beta_0$ such that $F \in \mathrm{Coh}^{\beta}(X)$.*

Proof. Suppose, for the sake of a contradiction, that $\widetilde{\beta} > \beta_0$ is a rational number with $F \in \mathrm{Coh}^{\widetilde{\beta}}(X)$ and $\mathrm{chd}^0_{F,L}(-\widetilde{\beta}) \neq 0$. This means that F has a subobject $E \subset F$ in $\mathrm{Coh}^{\widetilde{\beta}}(X)$ satisfying $\mathrm{ch}_2^{\widetilde{\beta}}(E) > 0$. We may assume that E is the first HN factor F_1 of F with respect to $\sigma_{0,\widetilde{\beta}}$; to see this, we only need to exclude that $\mathrm{ch}_2^{\widetilde{\beta}}(F_1) = 0$. And indeed, since $\nu_{0,\widetilde{\beta}}(F_1) > 0$ the equality $\mathrm{ch}_2^{\widetilde{\beta}}(F_1) = 0$ would imply that $F_1 \in \ker(Z_{0,\widetilde{\beta}})$; thus by Proposition I.1.14 $\mathcal{H}^{-1}(F_1)$ would be a (twisted Gieseker semistable) sheaf of slope $\widetilde{\beta}$. But then the inclusion $\mathcal{H}^{-1}(F_1) \subset \mathcal{H}^{-1}(F)$ would contradict the hypothesis $F \in \mathrm{Coh}^{\beta_0}(X)$ (recall Remark I.1.7).

Replacing E by the first step of its Bridgeland limit HN filtration at $\widetilde{\beta}$ (Theorem II.2.8), we may also assume that E is $\sigma_{\alpha,\widetilde{\beta}}$-semistable for some positive values of α.

Consider the distinguished triangle $E \to F \to Q$ in $\mathrm{D^b}(X)$ inducing the inclusion $E \subset F$ in $\mathrm{Coh}^{\widetilde{\beta}}(X)$. On the one hand, note that $E \in \mathrm{Coh}^{\beta_0}(X)$ as well; indeed, the inequality $\mu_L^+(\mathcal{H}^{-1}(E)) \leq \beta_0$ follows from $\mathcal{H}^{-1}(E) \subset \mathcal{H}^{-1}(F)$ (recall Remark I.1.7). Moreover, by Remark I.1.12 $\nu_{0,\widetilde{\beta}}(E) > 0$ is equivalent to $\widetilde{\beta} < p_E$; hence $\beta_0 < p_E$ holds, which gives $\nu_{0,\beta_0}(E) > 0$.

On the other hand, it is possible that $Q \notin \mathrm{Coh}^{\beta_0}(X)$, so Q may have two nontrivial cohomologies with respect to the heart $\mathrm{Coh}^{\beta_0}(X)$, namely $\mathcal{H}^{-1}_{\beta_0}(Q)$ and $\mathcal{H}^{0}_{\beta_0}(Q)$. In particular, $\mathcal{H}^{-1}_{\beta_0}(Q)$ is a subsheaf of $\mathcal{H}^{-1}(Q)$ such that $\mu_+(\mathcal{H}^{-1}_{\beta_0}(Q)) \in (\beta_0, \widetilde{\beta}]$.

Therefore the distinguished triangle yields an exact sequence

$$0 \to \mathcal{H}^{-1}_{\beta_0}(Q) \to E \to F \to \mathcal{H}^{0}_{\beta_0}(Q) \to 0$$

in $\mathrm{Coh}^{\beta_0}(X)$. Recall that F is σ_{0,β_0}-semistable with $\nu_{0,\beta_0}(F) = 0$, and $\nu_{0,\beta_0}(E) > 0$.

If E is σ_{0,β_0}-semistable, we already have the desired contradiction. In fact, under this assumption we have $\mathrm{Hom}(E,F)=0$ and thus $\mathcal{H}^{-1}_{\beta_0}(Q)=E$; but this would imply $E\notin \mathrm{Coh}^{\widetilde{\beta}}(X)$ by our previous description of $\mathcal{H}^{-1}_{\beta_0}(Q)$.

Therefore, to finish the proof it suffices to check that E may be assumed to be σ_{0,β_0}-semistable. This is essentially due to the support property for Bridgeland stability conditions in the (α,β)-plane.

Recall that E is $\sigma_{\alpha,\widetilde{\beta}}$-semistable for small enough values of $\alpha>0$. Hence, if E is not σ_{0,β_0}-semistable, there exists a short exact sequence $E_1\hookrightarrow E\twoheadrightarrow R_1$ destabilizing E along a wall W_1 in the (α,β)-plane, whose left point $(0,\widetilde{\beta}_1)$ in the β-axis satisfies $\beta_0<\widetilde{\beta}_1<\widetilde{\beta}$:

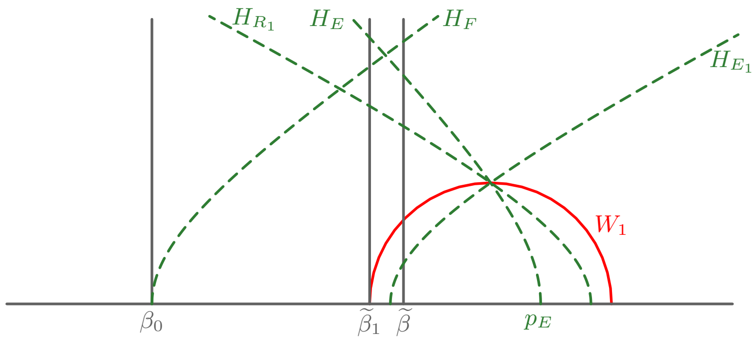


Figure II.4: Wall W_1 along which E destabilizes

We can assume, without loss of generality, that E_1 is the first step of the Bridgeland limit HN filtration of E at $\widetilde{\beta}_1$. Hence E_1 is $\sigma_{\alpha,\widetilde{\beta}_1}$-semistable for small enough values of $\alpha>0$.

At this point, observe that the support property (in the form of Theorem I.1.10.(7)) guarantees $\overline{\Delta}(E_1)+\overline{\Delta}(R_1)<\overline{\Delta}(E)$ (in particular, $\overline{\Delta}(E_1)<\overline{\Delta}(E)$). Moreover, we have an inclusion $E_1\subset F$ in $\mathrm{Coh}^{\widetilde{\beta}}(X)$ (since $E_1\subset E$ holds along the whole wall W_1). Let $E_1\to F\to Q_1$ be the distinguished triangle in $\mathrm{D^b}(X)$ defining this inclusion.

Then, the same arguments as before show that $E_1\in\mathrm{Coh}^{\beta_0}(X)$ and give an exact sequence

$$0\to\mathcal{H}^{-1}_{\beta_0}(Q_1)\to E_1\to F\to\mathcal{H}^{0}_{\beta_0}(Q_1)\to 0$$

in $\mathrm{Coh}^{\beta_0}(X)$. Moreover, $\nu_{0,\beta_0}(E_1)>0$ since $(0,\beta_0)$ lies on the left-hand side of H_{E_1}.

If E_1 were σ_{0,β_0}-semistable, by reasoning as in the case of E σ_{0,β_0}-semistable we would obtain a contradiction. Otherwise, we destabilize E_1 along a wall W_2 via a short exact sequence $0\to E_2\to E_1\to R_2\to 0$ with the same properties.

This finishes the proof, since this process must stop after a finite number of destabilizations thanks to the inequalities

$$0\leq\ldots<\overline{\Delta}(E_2)<\overline{\Delta}(E_1)<\overline{\Delta}(E) \qquad \square$$

Corollary II.2.20. *If $F\in\mathrm{Coh}^{\beta_0}(X)$, there exists $\epsilon>0$ such that the functions $\mathrm{chd}^0_{F,L}$ and $\mathrm{chd}^1_{F,L}$ are (piecewise) polynomial along the interval $(-\beta_0-\epsilon,-\beta_0)$.*

Proof. We assume that F has a HN factor with respect to σ_{0,β_0} of slope $\nu_{0,\beta_0} = 0$, otherwise we are in the situation of Corollary II.2.16. Thus let

$$0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_{s-1} \hookrightarrow F_s \hookrightarrow F_{s+1} \hookrightarrow \ldots \hookrightarrow F_r = F$$

be the HN filtration of F with respect to σ_{0,β_0}, so that $\nu_{0,\beta_0}(F_{s+1}/F_s) = 0$; in particular, $s = s(F)$ is the switching index of F. We claim that $\mathrm{chd}^0_{F/F_s,L}(-\beta) = 0$ for all β in a certain right neighborhood of β_0. To see this, we apply Lemma II.1.6 to the short exact sequence

$$0 \to F_{s+1}/F_s \to F/F_s \to F/F_{s+1} \to 0$$

in $\mathrm{Coh}^\beta(X)$, taking into account:

- That F/F_{s+1} has no subobject belonging to $\ker(Z_{0,\beta})$: otherwise, $\mathcal{H}^{-1}(F/F_{s+1})$ would have a subsheaf of slope β, contradicting $F/F_{s+1} \in \mathrm{Coh}^{\beta_0}(X)$.
- The vanishings $\mathrm{chd}^0_{F_{s+1}/F_s,L}(-\beta) = 0$ (by Proposition II.2.19) and $\mathrm{chd}^0_{F/F_{s+1},L}(-\beta) = 0$ (by Lemma II.2.10).

Now, for all $\beta > \beta_0$ in a certain right neighborhood of β_0, we use the vanishing $\mathrm{chd}^0_{F/F_s,L}(-\beta) = 0$ to apply Lemma II.1.6 again, in this case with the short exact sequence $0 \to F_s \to F \to F/F_s \to 0$. We obtain the equality $\mathrm{chd}^0_{F,L}(-\beta) = \mathrm{chd}^0_{F_s,L}(-\beta)$ in a right neighborhood of β_0.

This proves the result for $\mathrm{chd}^0_{F,L}$, since $\mathrm{chd}^0_{F_s,L}$ is (piecewise) polynomial in (a shrinking of) this neighborhood by Corollary II.2.16. The assertion for $\mathrm{chd}^1_{F,L}$ is simply a consequence of the relation $\mathrm{chd}^1_{F,L}(x) = \mathrm{chd}^0_{F,L}(x) - \mathrm{ch}_2^{-x}(F)$. □

Now we are ready to prove in full generality the existence of left and right polynomial expressions for Chern degree functions:

Theorem II.2.21. *Let $E \in \mathrm{D}^{\mathrm{b}}(X)$ and $k \in \mathbb{Z}$. Then, every rational number $x_0 \in \mathbb{Q}$ admits a left (resp. right) neighborhood where the function $\mathrm{chd}^k_{E,L}$ is given by an explicit polynomial P^- (resp. P^+) depending on x_0, satisfying $P^-(x_0) = \mathrm{chd}^k_{E,L}(x_0) = P^+(x_0)$.*

Proof. The left polynomial expression for $\mathrm{chd}^k_{E,L}$ at x_0 is a consequence of Corollary II.2.20 applied to the cohomology objects $\mathcal{H}^k_{-x_0}(E)$ and $\mathcal{H}^{k-1}_{-x_0}(E)$. The right polynomial expression can be obtained from the left polynomial expression of $\mathrm{chd}^{2-k}_{E^\vee,L}$ at $-x_0$, thanks to the Serre duality of Proposition II.1.5.(2). □

Remark II.2.22. If $\beta_0 \in \mathbb{R} \setminus \mathbb{Q}$ is an irrational number at which F admits a Bridgeland limit filtration, then the same conclusion of Corollary II.2.20 holds. Indeed, under this assumption F admits a HN filtration

$$0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_{s-1} \hookrightarrow F_s \hookrightarrow F_{s+1} \hookrightarrow \ldots \hookrightarrow F_r = F$$

with respect to σ_{0,β_0}, so that $\nu_{0,\beta_0}(F_{s+1}/F_s) = 0$ (we take $F_{s+1} = F_s$ if F has no HN factor of tilt slope 0). Then:

(1) The same arguments of Lemma II.2.10 provide a right weak limit filtration for F/F_{s+1}.
(2) The arguments of Lemma II.2.11 together with induction on s, provide a right weak limit filtration for F_s. Indeed, Z_{0,β_0} is a stability function on $\mathrm{Coh}^{\beta_0}(X)$ since $\beta_0 \notin \mathbb{Q}$; hence all the technical construction in Lemma II.2.12 and Proposition II.2.13 can be avoided. In particular, the core filtration of F at β_0 directly gives $F_1 \subset \ldots \subset F_s$.

Combining these right weak limit filtrations with Proposition II.2.19 (also valid under the assumption $\beta_0 \in \mathbb{R} \setminus \mathbb{Q}$) applied to F_{s+1}/F_s, we obtain the polynomial expression $\mathrm{chd}^0_{F,L}(x) = \mathrm{ch}_2^{-x}(F_s)$ for all x in a left neighborhood of $-\beta_0$.

In all the cases we know, objects have a Bridgeland limit filtration also at irrational numbers (see for instance the examples in section II.5), but we do not know how to prove this in general. This would imply that the Chern degree functions are piecewise polynomial, which cannot be directly deduced from Theorem II.2.21.

II.3 Continuity and differentiability of the functions

Extension as continuous real functions

In this section we extend the Chern degree functions $\mathrm{chd}^k_{F,L}$ to continuous functions of real variable. A similar result first appeared in [BPS20b, section 4] for the *continuous rank functions*, and was later generalized to the study of *cohomological rank functions* in [JP20, section 3].

We essentially follow this second approach, namely: one bounds the derivative of the functions, and then argues by integration. Whereas our arguments to express the functions around rational numbers (section II.2) were much longer than the ones in [JP20], the control of the derivatives for this part is easier in the stability framework.

Following the strategy of section II.2, we first consider the case of objects in the hearts $\mathrm{Coh}^\beta(X)$:

Theorem II.3.1. *Let $F \in \mathrm{Coh}^\beta(X)$ for some $\beta \in \mathbb{R}$. Then, the functions $\mathrm{chd}^0_{F,L}$ and $\mathrm{chd}^1_{F,L}$ extend to continuous functions on the interval $I_F = (-\mu_-(\mathcal{H}^0(F)), -\mu_+(\mathcal{H}^{-1}(F))]$.*

Proof. Notice that I_F is (minus) the interval of Remark I.1.7 delimiting where F belongs to the heart; we have reversed signs for coherence with the definition of the functions.

First of all, we claim that we may restrict ourselves to the case where I_F is bounded (i.e. the numbers $\mu_+(\mathcal{H}^{-1}(F))$ and $\mu_-(\mathcal{H}^0(F))$ are both finite).

This follows from the Serre vanishing of Proposition II.1.5.(1). Indeed, on the one hand, if $I_F = (-\mu_-(\mathcal{H}^0(F)), +\infty)$ is unbounded from the right, then F is a coherent sheaf since $\mathcal{H}^{-1}(F)$ is always torsion-free. By Serre vanishing, there exists $x_0 \in \mathbb{Q}$ so that

$$\mathrm{chd}^1_{F,L}(x) = 0, \quad \mathrm{chd}^0_{F,L}(x) = \mathrm{ch}_2^{-x}(F)$$

for every rational $x \geq x_0$. The extension of the functions is thus clear along the whole $[x_0, +\infty)$, so the problem is reduced to extend along $(-\mu_-(\mathcal{H}^0(F)), x_0]$.

On the other hand, if $I_F = (-\infty, -\mu_+(\mathcal{H}^{-1}(F))]$ is unbounded from the left, then $\mathcal{H}^0(F)$ is torsion (or 0). This implies that the complex $F^\vee$ has at most two cohomology sheaves, which will be its cohomologies with respect to the heart $\mathrm{Coh}^\beta(X)$ for all $\beta \ll 0$:

$$\mathcal{H}^1_\beta(F^\vee) = \mathcal{H}^1(F^\vee), \quad \mathcal{H}^2_\beta(F^\vee) = \mathcal{H}^2(F^\vee).$$

If $x_0 \in \mathbb{Q}$ is a bound ensuring Serre vanishing for the functions of the sheaves $\mathcal{H}^1(F^\vee)$ and $\mathcal{H}^2(F^\vee)$, by Serre duality Proposition II.1.5.(2) we have

$$\mathrm{chd}^i_{F,L}(x) = \mathrm{chd}^{2-i}_{F^\vee,L}(-x) = \mathrm{chd}^0_{\mathcal{H}^{2-i}(F^\vee),L}(-x) = \mathrm{ch}^x_2(\mathcal{H}^{2-i}(F^\vee))$$

for $i = 0, 1$ and every rational $x \leq -x_0$. Hence we only need to extend the functions along $(-x_0, -\mu_+(\mathcal{H}^{-1}(F))]$.

Now take $\beta_0 \in \mathbb{Q}$ such that $F \in \mathrm{Coh}^{\beta_0}(X)$ (equivalently, $-\beta_0 \in I_F \cap \mathbb{Q}$), and let

$$0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_{s-1} \hookrightarrow F_s \hookrightarrow F_{s+1} \hookrightarrow \ldots \hookrightarrow F_r = F$$

be the HN filtration of F with respect to σ_{0,β_0}, where $s = s(F)$ is the switching index of F.

Consider the subobject $G_s \subset F_s$, where $G_1 \subset \ldots \subset G_s$ is the core filtration of F at β_0. Then, for all $x \in \mathbb{Q}$ in a left neighborhood of $-\beta_0$ the function $\mathrm{chd}^0_{F,L}$ is polynomially expressed as

$$\mathrm{chd}^0_{F,L}(x) = \mathrm{ch}^{-x}_2(G_s).$$

Therefore, $L \cdot \mathrm{ch}^{\beta_0}_1(G_s)$ is the left derivative of $\mathrm{chd}^0_{F,L}$ at $-\beta_0$. Since this is the rank of G_s in the heart $\mathrm{Coh}^{\beta_0}(X)$, it turns out that $0 \leq D^- \,\mathrm{chd}^0_{F,L}(-\beta_0) \leq L \cdot \mathrm{ch}^{\beta_0}_1(F)$.

Summing up, the function $\mathrm{chd}^0_{F,L}$ has a left (and right) derivative at every $x \in I_F \cap \mathbb{Q}$ (actually at every $x \in I_F \cap U$, where $U \subset \mathbb{R}$ is an open subset containing $\mathbb{Q}$), and both derivatives coincide almost everywhere. Moreover, these derivatives are non-negative and bounded from above. By integration, it follows that $\mathrm{chd}^0_{F,L}$ extends to a continuous function on the whole interval I_F.

The result for $\mathrm{chd}^1_{F,L}$ follows directly by defining $\mathrm{chd}^1_{F,L}(x) = \mathrm{chd}^0_{F,L}(x) - \mathrm{ch}^{-x}_2(F)$ for $x \in I_F$. □

Remark II.3.2. It follows from the proof that the function $\mathrm{chd}^0_{F,L}$ (resp. $\mathrm{chd}^1_{F,L}$) is non-decreasing (resp. non-increasing) along the interval I_F. This (a posteriori) explains Proposition II.2.19.

From this basic case we can easily obtain the extension as continuous real functions for arbitrary objects of $\mathrm{D^b}(X)$:

Corollary II.3.3. *Let $E \in \mathrm{D^b}(X)$ and $k \in \mathbb{Z}$. Then, the function $\mathrm{chd}^k_{E,L}$ extends to a continuous real function of real variable.*

Proof. Using the definition of $\mathrm{chd}^k_{E,L}$ this becomes an immediate consequence of Theorem II.3.1, provided that $\mathrm{chd}^k_{E,L}$ is continuous at the (finitely many) points $x = -\beta$ where the cohomologies $\mathcal{H}^i_\beta(E)$ change.

And this continuity is guaranteed by Theorem II.2.21, since such points of change are rational (they correspond to μ_L-slopes of Harder-Narasimhan factors of cohomology sheaves of E). □

Critical points

Let $\beta_0 \in \mathbb{Q}$. Now we want to study when the functions $\mathrm{chd}^k_{F,L}$ attached to an object $F \in \mathrm{D^b}(X)$ are not of class $\mathcal{C}^\infty$ at $-\beta_0$; following the terminology of Jiang–Pareschi ([JP20, Section 4]), we will say that $-\beta_0$ is a *critical point*.

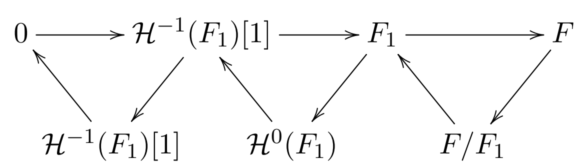
For the sake of simplicity, we will assume $F \in \mathrm{Coh}^{\beta_0}(X)$; in the general case one has to consider all the cohomologies of F with respect to the heart $\mathrm{Coh}^{\beta_0}(X)$.

Let $0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_r = F$ be the HN filtration of F with respect to σ_{0,β_0}, satisfying

$$+\infty = \nu_{0,\beta_0}(F_1) > \ldots > \nu_{0,\beta_0}(F_s/F_{s-1}) > 0 = \nu_{0,\beta_0}(F_{s+1}/F_s) > \ldots > \nu_{0,\beta_0}(F/F_{r-1}),$$

that is, $s = s(F)$ is the switching index of F. We write $F_1 = 0$ (resp. $F_s = F_{s+1}$) if F has no HN factor of tilt slope $+\infty$ (resp. tilt slope 0). Then:

- As dictated by Remark I.1.7, $F \in \mathrm{Coh}^\beta(X)$ for all small enough $\beta > \beta_0$. This means that F will only have two nonzero functions in a left neighborhood of $-\beta_0$, namely $\mathrm{chd}^0_{F,L}$ and $\mathrm{chd}^1_{F,L}$.
- For big enough $\beta < \beta_0$, F may have two nontrivial cohomologies in $\mathrm{Coh}^\beta(X)$. More precisely, recall that by Proposition I.1.14 $\mathcal{H}^{-1}(F_1)$ is a μ_L-semistable sheaf of slope β_0, and $\mathcal{H}^0(F_1)$ is a 0-dimensional sheaf. Therefore, we have distinguished triangles



  with $\mathcal{H}^{-1}(F_1), \mathcal{H}^0(F_1), F/F_1 \in \mathrm{Coh}^\beta(X)$ for $\beta < \beta_0$. This tells us that $\mathcal{H}^{-1}_\beta(F) = \mathcal{H}^{-1}(F_1)$ and $\mathcal{H}^0_\beta(F) = F/(\mathcal{H}^{-1}(F_1)[1])$ (where this quotient is taken in $\mathrm{Coh}^{\beta_0}(X)$).

  Moreover, note that the only nonzero Chern degree functions of F in a right neighborhood of $-\beta_0$ may be $\mathrm{chd}^{-1}_{F,L}$, $\mathrm{chd}^0_{F,L}$ and $\mathrm{chd}^1_{F,L}$.

Now we consider the polynomial expressions for the functions in a left and a right neighborhood of $-\beta_0$, that we found in section II.2. Explicitly, there exists $\epsilon > 0$ such that

$$\operatorname{chd}^{-1}_{F,L}(x) = \begin{cases} 0 & -\beta_0 - \epsilon \le x \le -\beta_0 \\ -\operatorname{ch}_2^x\left(\mathcal{H}^{-1}(F_1)^\vee[1]/P\right) & -\beta_0 \le x \le -\beta_0 + \epsilon \end{cases}$$
$$\operatorname{chd}^{0}_{F,L}(x) = \begin{cases} \operatorname{ch}_2^{-x}(G_s) & -\beta_0 - \epsilon \le x \le -\beta_0 \\ \operatorname{ch}_2^x(P) - \operatorname{ch}_2^x\left((F/F_1)^\vee[1]/R_{s+1}\right) + \operatorname{length}(\mathcal{H}^0(F_1)) & -\beta_0 \le x \le -\beta_0 + \epsilon \end{cases} \tag{II.3.1}$$
$$\operatorname{chd}^{1}_{F,L}(x) = \begin{cases} -\operatorname{ch}_2^{-x}(F/G_s) & -\beta_0 - \epsilon \le x \le -\beta_0 \\ \operatorname{ch}_2^x(R_{s+1}) & -\beta_0 \le x \le -\beta_0 + \epsilon \end{cases}$$

where:

- The chain $G_1 \subset \ldots \subset G_s$ (with $G_i \subset F_i$) is the core filtration of F at β_0.
- $P = \left(\mathcal{H}^{-1}(F_1)^\vee[1]\right)'$ is the core subobject of $\mathcal{H}^{-1}(F_1)^\vee[1]$ at $-\beta_0$.
- The chain $R_{r-1} \subset \ldots \subset R_{s+1}$ in $\operatorname{Coh}^{-\beta_0}(X)$ (with $R_i \subset (F/F_i)^\vee[1]$) is the core filtration of $(F/F_1)^\vee[1]$ at $-\beta_0$. It is inductively constructed by letting $R_{r-1} = ((F/F_{r-1})^\vee[1])'$ and $R_i/R_{i+1} = ((F/F_i)^\vee[1]/R_{i+1})'$.

A critical point for $\operatorname{chd}^{-1}_{F,L}$ arises whenever the object $\mathcal{H}^{-1}(F_1)^\vee[1]/P$ is nonzero. Since the regularity of $\operatorname{chd}^{0}_{F,L}$ can be deduced from that of $\operatorname{chd}^{-1}_{F,L}$ and $\operatorname{chd}^{1}_{F,L}$, we are left to compare the polynomial expressions for $\operatorname{chd}^{1}_{F,L}$. To this end we will use that $\operatorname{ch}_2^x(R_{s+1}) = -\operatorname{ch}_2^{-x}(R_{s+1}^\vee[1])$, and we will exhibit a chain of morphisms connecting $R_{s+1}^\vee[1]$ with F/G_s.

By construction, there is a short exact sequence

$$0 \to R_{s+1} \to (F/F_{s+1})^\vee[1] \to Q \to 0$$

in $\operatorname{Coh}^{-\beta_0}(X)$ with $Q \in \ker(Z_{0,-\beta_0})$; hence the object Q is of the form $Q = S[1]$ for a μ_L-semistable vector bundle S with $\mu_L(S) = -\beta_0$ and $\overline{\Delta}(S) = 0$. Dualizing, this yields a short exact sequence

$$0 \to S^\vee \to F/F_{s+1} \to R_{s+1}^\vee[1] \to 0$$

in $\operatorname{Coh}^\beta(X)$, for big enough values of $\beta < \beta_0$. Consequently, we have obtained a sequence of morphisms

$$F/G_s \to F/F_s \to F/F_{s+1} \to R_{s+1}^\vee[1]$$

for which there exists $\epsilon' > 0$ satisfying: $F/G_s \to F/F_s$ is a surjection in $\operatorname{Coh}^\beta(X)$ for $\beta \in [\beta_0, \beta_0+\epsilon')$, $F/F_s \to F/F_{s+1}$ is a surjection in $\operatorname{Coh}^\beta(X)$ for $\beta \in (\beta_0 - \epsilon', \beta_0 + \epsilon')$ and $F/F_{s+1} \to R_{s+1}^\vee[1]$ is a surjection in $\operatorname{Coh}^\beta(X)$ for $\beta \in (\beta_0 - \epsilon', \beta_0)$.

With all this information, we get the following result:

Proposition II.3.4. *Let $F \in \operatorname{Coh}^{\beta_0}(X)$ and we keep the notation of (II.3.1).*

(1) *The function $\operatorname{chd}^{-1}_{F,L}$ has a critical point at $x = -\beta_0$ if and only if $P \subsetneq \mathcal{H}^{-1}(F_1)^\vee[1]$ in $\operatorname{Coh}^{-\beta_0}(X)$. In such a case, the function $\operatorname{chd}^{-1}_{F,L}$ is of class $\mathcal{C}^1$ at $x = -\beta_0$.*

(2) *The function* $\operatorname{chd}^1_{F,L}$ *has a critical point at* $x = -\beta_0$ *if and only if* $F/G_s \neq R^\vee_{s+1}[1]$*. This is equivalent to one of the following conditions holding:*

   (a) $F_s \neq F_{s+1}$*, namely* F *has a HN factor (with respect to* σ_{0,β_0}*) of slope* $\nu_{0,\beta_0} = 0$*.*

   (b) $G_s \neq F_s$

   (c) $F/F_{s+1} \neq R^\vee_{s+1}[1]$

   Furthermore, $\operatorname{chd}^1_{F,L}$ *is of class* $\mathcal{C}^1$ *at* $x = -\beta_0$ *unless condition (a) holds.*

(3) *The function* $\operatorname{chd}^0_{F,L}$ *has a critical point at* $x = -\beta_0$ *if and only if* $\operatorname{chd}^{-1}_{F,L}$ *or* $\operatorname{chd}^1_{F,L}$ *has a critical point. In such a case,* $\operatorname{chd}^0_{F,L}$ *is of class* $\mathcal{C}^1$ *at* $x = -\beta_0$ *unless (a) holds.*

Remark II.3.5. Intuitively, we may think that the functions have critical points at $x = -\beta_0$ if and only if the hyperbola of one of its (possibly weak limit) HN factors at β_0 passes through the point $(0, \beta_0)$ of the (α, β)-plane.

For instance, condition (b) is equivalent to F_s having a left weak limit HN factor with tilt slope $\nu_{0,\beta}$ tending to 0 as $\beta \to \beta_0^+$.

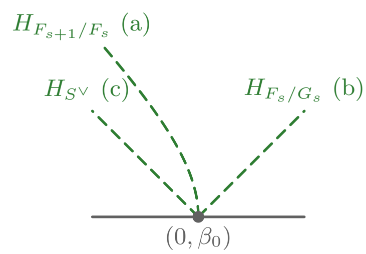


Figure II.5: Hyperbolas through $(0, \beta_0)$ producing a critical point for $\operatorname{chd}^0_{F,L}$ and $\operatorname{chd}^1_{F,L}$.

(1) The case (1) is certainly exceptional, in the sense that it requires a (nontrivial) condition on one of the (finitely many) points where the cohomologies of F with respect to the hearts $\operatorname{Coh}^\beta(X)$ change.

   On the other hand, whereas the case (a) is naturally described in terms of σ_{0,β_0}-stability, condition (b) (resp. condition (c)) requires the nontriviality of the core filtration of F_s (resp. the core filtration of $(F/F_{s+1})^\vee[1]$) at β_0 (resp. $-\beta_0$). In particular, (b) and (c) require that σ_{0,β_0} is not a Bridgeland stability condition.

(2) The possibilities (a), (b) and (c) producing critical points in the function $\operatorname{chd}^1_{F,L}$ may not be mutually exclusive. Indeed, we will see in chapter III an example (Example III.1.5) where (a) and (c) hold simultaneously. It would be interesting to know whether conditions (b) and (c) may hold at the same time or not.

We point out that the same study can be applied when β_0 is an irrational number, at which F admits a Bridgeland limit HN filtration (recall Remark II.2.22).

In such a case, $F \in \mathrm{Coh}^{\beta}(X)$ for some values $\beta < \beta_0$ as well, so $\mathrm{chd}^{-1}_{F,L}$ is identically 0 in an open neighborhood of $-\beta_0$. Moreover, possibilities (b) and (c) must be excluded since Z_{0,β_0} (resp. $Z_{0,-\beta_0}$) is a stability function on $\mathrm{Coh}^{\beta_0}(X)$ (resp. $\mathrm{Coh}^{-\beta_0}(X)$) when $\beta_0 \notin \mathbb{Q}$. Therefore, the only possibility for a critical point is (a), which gives a point where the functions $\mathrm{chd}^0_{F,L}$ and $\mathrm{chd}^1_{F,L}$ are not differentiable.

II.4 The case of abelian surfaces

In this part we will prove that the Chern degree functions $\mathrm{chd}^k_{F,L}$ attached to any object $F \in \mathrm{D^b}(X)$ on a polarized surface (X, L) recover, in the case where X is an abelian surface, the cohomological rank functions $h^k_{F,L}$ of Jiang and Pareschi. By abuse of notation, we will also denote by L an ample line bundle on X representing the polarization.

The key point of the proof are the following two lemmas. The first one is the analogue to the fact that a coherent sheaf on an elliptic curve only may have h^0 and h^1 as nonzero functions.

Lemma II.4.1. *If $F \in \mathrm{Coh}^{\beta}(X)$ for a number $\beta \in \mathbb{Q}$, then $h^i_{F,L}(-\beta) = 0$ for every $i \neq 0, 1$.*

Proof. Since F is a complex with at most two nontrivial cohomology sheaves (namely $\mathcal{H}^{-1}(F)$ and $\mathcal{H}^0(F)$), it turns out that $h^i_{F,L}(-\beta) = 0$ for every $i \notin \{-1, 0, 1, 2\}$. Moreover, we have

$$h^{-1}_{F,L}(-\beta) = h^0_{\mathcal{H}^{-1}(F),L}(-\beta), \quad h^2_{F,L}(-\beta) = h^2_{\mathcal{H}^0(F),L}(-\beta)$$

so it suffices to check that $h^2_{E,L}(-\beta) = 0$ whenever $E \in \mathcal{T}_\beta$, and $h^0_{G,L}(-\beta) = 0$ whenever $G \in \mathcal{F}_\beta$.

To prove the first vanishing, we consider a non-decreasing sequence of rational numbers $\beta_n = \frac{a_n}{b_n}$ converging to β, such that μ_{b_n} is a separable isogeny for every n. Let $0 = E_0 \hookrightarrow E_1 \hookrightarrow \ldots \hookrightarrow E_r = E$ be the HN filtration of E with respect to μ_L-stability. Since torsion sheaves are always μ_L-semistable (they have slope $\mu_L = +\infty$), it follows from [HL10, Lemma 3.2.2] that

$$0 = \mu_{b_n}^* E_0 \hookrightarrow \mu_{b_n}^* E_1 \hookrightarrow \ldots \hookrightarrow \mu_{b_n}^* E_r = \mu_{b_n}^* E$$

is a HN filtration for $\mu_{b_n}^* E$ with respect to μ_L-stability, for every n. Observe that we have

$$\mu_L(\mu_{b_n}^*(E_i/E_{i-1})) = b_n^2 \mu_L(E_i/E_{i-1}) > b_n^2 \cdot \beta_n = a_n b_n$$

for every $i \in \{1, \ldots, r\}$, thanks to the condition $E \in \mathcal{T}_{\beta_n}$ inherited from $E \in \mathcal{T}_\beta$. Therefore

$$0 = \mathrm{Hom}(\mu_{b_n}^*(E_i/E_{i-1}), L^{a_n b_n}) = \mathrm{Ext}^2(L^{a_n b_n}, \mu_{b_n}^*(E_i/E_{i-1}))^* = H^2(\mu_{b_n}^*(E_i/E_{i-1}) \otimes L^{-a_n b_n})^*$$

for every i, and the equality $h^2_{E,L}(-\beta_n) = 0$ follows from an easy induction on the length r of the HN filtration. This is enough to prove $h^2_{E,L}(-\beta) = 0$, thanks to Theorem I.2.2.(1).

For the second vanishing, we will check that $h^0_{G,L}(x) = 0$ for every rational number $x = \frac{c}{d}$ with $x < -\beta$. Again, this is more than enough for our purposes thanks to Theorem I.2.2.(1).

We approach x by a non-decreasing sequence $x_n = \frac{c_n}{d_n}$ of rational numbers such that the multiplication μ_{d_n} is étale. As before, the HN filtration $0 = G_0 \hookrightarrow G_1 \hookrightarrow \ldots \hookrightarrow G_r = G$ of G in μ_L-stability induces the HN filtration $0 = \mu_{d_n}^* G_0 \hookrightarrow \mu_{d_n}^* G_1 \hookrightarrow \ldots \hookrightarrow \mu_{d_n}^* G_r = \mu_{d_n}^* G$ of $\mu_{d_n}^* G$, for every n.

The hypothesis $G \in \mathcal{F}_\beta$ says that

$$\mu_L(\mu_{d_n}^*(G_i/G_{i-1})) = d_n^2 \mu_L(G_i/G_{i-1}) \leq d_n^2 \beta < -d_n^2 x_n = -c_n d_n$$

for every i, which implies

$$0 = \operatorname{Hom}(L^{-c_n d_n}, \mu_{d_n}^*(G_i/G_{i-1})) = H^0(\mu_{d_n}^*(G_i/G_{i-1}) \otimes L^{c_n d_n})$$

The equality $h^0_{G,L}(x_n) = 0$ is again obtained by induction on r, and then $h^0_{G,L}(x) = 0$ follows. □

The second lemma is the analogue of Proposition I.2.3.(1), namely that at a fixed point, at most one function is nonzero for a semistable sheaf on an elliptic curve:

Lemma II.4.2. *If $F \in \operatorname{Coh}^\beta(X)$ ($\beta \in \mathbb{Q}$) is $\sigma_{0,\beta}$-semistable, then $h^1_{F,L}(-\beta) = 0$ (resp. $h^0_{F,L}(-\beta) = 0$) if $\nu_{0,\beta}(F) \geq 0$ (resp. if $\nu_{0,\beta}(F) \leq 0$).*

Proof. First of all, observe that the same arguments of [BMS16, Proposition 6.1] yield that $\mu_b^* F \in \operatorname{Coh}^{b^2\beta}(X)$ and it is a $\sigma_{0,b^2\beta}$-semistable object, for every $b \in \mathbb{Z}_{>0}$ such that μ_b is a separable isogeny[1]. Moreover, we have

$$\nu_{0,b^2\beta}(\mu_b^* F) = b^2 \nu_{0,\beta}(F)$$

as follows from $\operatorname{ch}(\mu_b^* F) = (\operatorname{ch}_0(F), b^2 \operatorname{ch}_1(F), b^4 \operatorname{ch}_2(F))$.

If $\nu_{0,\beta}(F) \geq 0$, we will prove that $h^1_{F,L}(x) = 0$ for every $x \in \mathbb{Q}$ with $x > -\beta$; then, $h^1_{F,L}(-\beta) = 0$ will follow again from Theorem I.2.2.(1). To this end, we approach x by a non-increasing sequence $x_n = \frac{c_n}{d_n}$ such that μ_{d_n} is separable.

Observe that the condition $\frac{c_n}{d_n} > -\beta$ reads $-c_n d_n < d_n^2 \beta$. Therefore, $L^{-c_n d_n}[1]$ is $\sigma_{0,d_n^2\beta}$-semistable with $\nu_{0,d_n^2\beta}(L^{-c_n d_n}[1]) < 0 \leq \nu_{0,d_n^2\beta}(\mu_{d_n}^* F)$, which gives

$$0 = \operatorname{Hom}(\mu_{d_n}^* F, L^{-c_n d_n}[1]) = \operatorname{Ext}^1(\mu_{d_n}^* F, L^{-c_n d_n}) = \operatorname{Ext}^1(L^{-c_n d_n}, \mu_{d_n}^* F)^* = H^1(\mu_{d_n}^* F \otimes L^{c_n d_n})^*$$

and thus $h^1_{F,L}(x_n) = 0$. It follows that $h^1_{F,L}(x) = 0$, as desired.

If $\nu_{0,\beta}(F) \leq 0$, following the same strategy it suffices to check that $h^0_{F,L}(x) = 0$ for every rational $x = \frac{c}{d}$ with $x < -\beta$ and μ_d étale. And indeed, $\frac{c}{d} < -\beta$ reads $-cd > d^2\beta$, so L^{-cd} is $\sigma_{0,d^2\beta}$-semistable with $\nu_{0,d^2\beta}(L^{-cd}) > 0 \geq \nu_{0,d^2\beta}(\mu_d^* F)$. This implies

$$0 = \operatorname{Hom}(L^{-cd}, \mu_d^* F) = H^0(\mu_d^* F \otimes L^{cd})$$

and therefore $h^0_{F,L}(x) = 0$. □

[1] The condition $\beta \in \mathbb{Q}$ is required at this point, to ensure the existence of HN filtrations with respect to $\sigma_{0,\beta}$ and $\sigma_{0,d^2\beta}$.

We are now ready to prove that, on abelian surfaces, cohomological rank functions are recovered by Chern degree functions:

Theorem II.4.3. *Let (X, L) be a a polarized abelian surface. Then, the Chern degree function $\mathrm{chd}^k_{F,L}$ equals the cohomological rank function $h^k_{F,L}$ for every object $F \in \mathrm{D^b}(X)$ and $k \in \mathbb{Z}$.*

Proof. It suffices to prove the equality $\mathrm{chd}^k_{F,L}(-\beta) = h^k_{F,L}(-\beta)$ for every $\beta \in \mathbb{Q}$.

We start with the basic case where $F \in \mathrm{Coh}^\beta(X)$ is $\sigma_{0,\beta}$-semistable. Assume that $\nu_{0,\beta}(F) \geq 0$. On the one hand, by definition of the functions $\mathrm{chd}^k_{F,L}$, we have $\mathrm{chd}^k_{F,L}(-\beta) = 0$ for every $k \neq 0$ and $\mathrm{chd}^0_{F,L}(-\beta) = \mathrm{ch}^\beta_2(F)$. On the other hand, Lemma II.4.1 and Lemma II.4.2 give $h^k_{F,L}(-\beta) = 0$ for every $k \neq 0$, and thus $h^0_{F,L}(-\beta) = \chi_{F,L}(-\beta)$.

Since $\mathrm{ch}^{-x}_2(F)$ equals $\chi_{F,L}(x)$ as a polynomial (in x), it follows that $\mathrm{chd}^k_{F,L}(-\beta) = h^k_{F,L}(-\beta)$ for every k.

If $F \in \mathrm{Coh}^\beta(X)$ is $\sigma_{0,\beta}$-semistable with $\nu_{0,\beta}(F) \leq 0$, then the same arguments yield $\mathrm{chd}^k_{F,L}(-\beta) = 0 = h^k_{F,L}(-\beta)$ for $k \neq 1$ and $\mathrm{chd}^1_{F,L}(-\beta) = -\mathrm{ch}^\beta_2(F) = h^1_{F,L}(-\beta)$.

When $F \in \mathrm{Coh}^\beta(X)$ is an arbitrary object (not necessarily $\sigma_{0,\beta}$-semistable), the result follows by induction on the length of the HN filtration of F with respect to $\sigma_{0,\beta}$, arguing similarly to the proof of Proposition I.2.3.(2).

To prove the result for a general $F \in \mathrm{D^b}(X)$, we approach $-\beta$ by a non-decreasing sequence $-\beta_n = \frac{a_n}{b_n}$ of rational numbers such that the multiplication maps μ_{b_n} are étale. If $\tau^{\beta_n}_{\leq k-1}$, $\tau^{\beta_n}_{\geq k}$ denote the truncation functors of the bounded t-structure defined by $\mathrm{Coh}^{\beta_n}(X)$, then one immediately checks (again, using [BMS16, Proposition 6.1.a]) that

$$\tau^{b_n^2\beta_n}_{\leq k-1} \circ \mu^*_{b_n} = \mu^*_{b_n} \circ \tau^{\beta_n}_{\leq k-1}, \quad \tau^{b_n^2\beta_n}_{\geq k} \circ \mu^*_{b_n} = \mu^*_{b_n} \circ \tau^{\beta_n}_{\geq k}$$

for every n. Therefore, we have a distinguished triangle in $\mathrm{D^b}(X)$

$$\mu^*_{b_n}(\tau^{\beta_n}_{\leq k-1}F) \otimes L^{a_n b_n} \to \mu^*_{b_n}F \otimes L^{a_n b_n} \to \mu^*_{b_n}(\tau^{\beta_n}_{\geq k}F) \otimes L^{a_n b_n}$$

Arguing with its associated long exact sequence of hypercohomology groups as in the proof of Proposition I.2.3.(3) (and using that the assertion has already been proved for objects of $\mathrm{Coh}^{\beta_n}(X)$), we obtain $\mathrm{chd}^k_{F,L}(-\beta_n) = h^k_{F,L}(-\beta_n)$ for every n. Then the equality $\mathrm{chd}^k_{F,L}(-\beta) = h^k_{F,L}(-\beta)$ is a consequence of Theorem II.2.21 and Theorem I.2.2.(2). □

This description of cohomological rank functions on abelian surfaces establishes a clear analogy with the case of elliptic curves. In particular, the proof shows that the cohomological rank functions of an object $F \in \mathrm{D^b}(X)$ at $x = -\beta$ split into simpler pieces, corresponding to its HN factors with respect to $\sigma_{0,\beta}$. The main difference is that, in the case of elliptic curves, the study via μ_L-stability is actually global and proves that cohomological rank functions are piecewise polynomial, with all their critical points being rational. In dimension 2 the study is strictly local as we saw in section II.2, which makes the situation much richer.

To finish this section, it is worth mentioning the following immediate consequence of this new presentation in terms of Chern degree functions, which is a refinement of [JP20, Lemma 6.1] for the case of abelian surfaces:

Corollary II.4.4. *If (X, L) is a polarized abelian surface and $F \in \mathrm{D}^{\mathrm{b}}(X)$, any local polynomial expression of the cohomological rank function $h^k_{F,L}$ has integral coefficients.*

II.5 Chern degree functions of Gieseker semistable sheaves

In order to illustrate with examples the previous account, in this section we discuss briefly some properties of Chern degree functions of ($(L, -\frac{1}{2}K_X)$-twisted) Gieseker semistable sheaves. With a view towards the examples of chapter III, we will mainly focus on Gieseker semistable sheaves on abelian surfaces, where these properties will become properties of the corresponding cohomological rank functions.

So we fix a polarized surface (X, L) and a torsion-free Gieseker semistable sheaf F. Note that $F \in \mathrm{Coh}^{\beta}(X)$ (resp. $F[1] \in \mathrm{Coh}^{\beta}(X)$) for every $\beta < \mu_L(F)$ (resp. $\beta \geq \mu_L(F)$), hence $\mathrm{chd}^2_{F,L}(x) = 0$ (resp. $\mathrm{chd}^0_{F,L}(x) = 0$) for $x \geq -\mu_L(F)$ (resp. $x \leq -\mu_L(F)$).

Moreover, if $\beta < \mu_L(F)$ then F is $\sigma_{\alpha,\beta}$-semistable for all $\alpha \gg 0$ (Proposition I.1.13), so the problem of computing $\mathrm{chd}^0_{F,L}$ and $\mathrm{chd}^1_{F,L}$ in $(-\mu_L(F), +\infty)$ consists of studying how the trivial HN filtration of F in the Gieseker chamber varies as we reach the line $\alpha = 0$.

Trivial Chern degree functions

We will call a Chern degree function *trivial* if its support is disjoint with the support of all the other functions attached to the same object. In terms of stability, this reads as the simplest possible situation:

Proposition II.5.1. *The function $\mathrm{chd}^0_{F,L}$ is trivial if, and only if, F is $\sigma_{\alpha,\beta}$-semistable for every $\alpha > 0$ and $\beta < \mu_L(F)$. In such a case,*

$$\mathrm{chd}^0_{F,L}(x) = \begin{cases} 0 & x \leq -p_F \\ \mathrm{ch}_2^{-x}(F) & x \geq -p_F \end{cases}$$

where $-p_F$ is the largest root of the Chern degree polynomial $\mathrm{ch}_2^{-x}(F)$ (recall also Remark I.1.12).

Proof. If the semistability assumption on F is fulfilled, F is $\sigma_{0,\beta}$-semistable for every $\beta < \mu_L(F)$ and thus $\mathrm{chd}^0_{F,L}(-\beta)$ is simply defined according to the sign of the tilt slope $\nu_{0,\beta}(F)$.

Conversely, assume that the function $\mathrm{chd}^0_{F,L}$ is trivial. If F is not $\sigma_{\alpha,\beta}$-semistable for every $\alpha > 0$ and $\beta < \mu_L(F)$, there exists an actual wall W (intersecting H_F at its top point) along which F destabilizes. Let $0 \to E \to F \to Q \to 0$ define this wall.

Then, for every $\beta \in (p_F, p_E)$ E is a subobject of F in $\mathrm{Coh}^{\beta}(X)$ with $\nu_{0,\beta}(E) > 0$, which gives $\mathrm{chd}^0_{F,L}(x) > 0$ for $x \in (-p_E, -p_F)$. This is a contradiction to the triviality of $\mathrm{chd}^0_{F,L}$, since for every $-\mu_L(F) < x < -p_F$ we have $\mathrm{chd}^1_{F,L}(x) > 0$ as a consequence of the relation $\mathrm{chd}^0_{F,L}(x) - \mathrm{chd}^1_{F,L}(x) = \mathrm{ch}_2^{-x}(F) < 0$. □

Example II.5.2. Let us give some examples of trivial Chern degree function $\mathrm{chd}^0_{F,L}$.

(1) Gieseker semistable sheaves with $\overline{\Delta}(F) = 0$. These are the only examples of Gieseker semistable sheaves where the function $\mathrm{chd}^0_{F,L}$ is trivial and of class $\mathcal{C}^1$ at their critical point $-p_F$, according to Proposition II.3.4 (see also Proposition C).

(2) Gieseker semistable sheaves with $\Delta(F) + C_L(L \cdot \mathrm{ch}_1(F)) = 0$, where $\Delta = (\mathrm{ch}_1)^2 - 2\,\mathrm{ch}_0 \cdot \mathrm{ch}_2$ and C_L is the constant of [BMS16, Lemma 3.3].

These objects are more general than the objects considered in (1). For example, for abelian surfaces (where one can choose C_L to be zero) the objects in (1) only recover semihomogeneous vector bundles with determinant proportional to L, while here we are considering all semihomogeneous vector bundles.

(3) The ideal sheaf $\mathcal{I}_q$ of a point q on a principally polarized abelian surface is well known to have a trivial h^0 function, according to the analysis in [JP20, section 8]. Conversely, $\mathcal{I}_q$ has no actual wall for $\beta < 0$ as observed in [Mea12, section 4.1].

(4) Let $i : C \hookrightarrow X$ be an Abel-Jacobi embedding of a smooth curve C of genus 2 inside its (principally polarized) Jacobian $X = JC$, and let $F = i_*M$ for a line bundle M of odd degree on C.

In this case F is torsion, but the same analysis works as in the torsion-free case (see [BL17, Proposition 3.1]). As explained in [JP20, Example 4.3], the function $h^0_{F,L}$ is trivial; hence it follows from Proposition II.5.1 that F is semistable on the whole (α, β)-plane.

Chern degree functions of semistable sheaves of low discriminant

There are other situations in which $\mathrm{chd}^0_{F,L}$, even if not trivial, can be explicitly described. To this end we consider the *minimal discriminant*, defined as the positive generator m of the ideal of $\mathbb{Z}$ generated by $\{\overline{\Delta}(v) \mid v \in \Lambda\}$.

Note that the quantity m essentially depends on the intersection pairing of $\mathrm{NS}(X)$. For instance, if $\mathrm{NS}(X) = \mathbb{Z} \cdot L$ (or more generally, if $L^2 | D \cdot L$ for every divisor class D) then m is a multiple of L^2. Of course, in full generality one can only assert $m \in \mathbb{Z}_{\geq 1}$.

Let us assume now that $\overline{\Delta}(F) = m$. Then by Theorem I.1.10.(7) either the function $\mathrm{chd}^0_{F,L}$ is trivial (as happens in Example II.5.2.(3)-(4)), or F destabilizes along an actual wall W defined by an exact sequence $0 \to E \to F \to Q \to 0$ with $\overline{\Delta}(E) = 0 = \overline{\Delta}(Q)$ (see Figure II.6).

In the latter case, E and Q can only be destabilized at their vertical walls $\beta = p_E$ and $\beta = p_Q$, so it follows that E and Q are the $\sigma_{0,\beta}$-HN factors of F for all $\beta \in (p_Q, p_E)$. Clearly F is $\sigma_{0,\beta}$-semistable

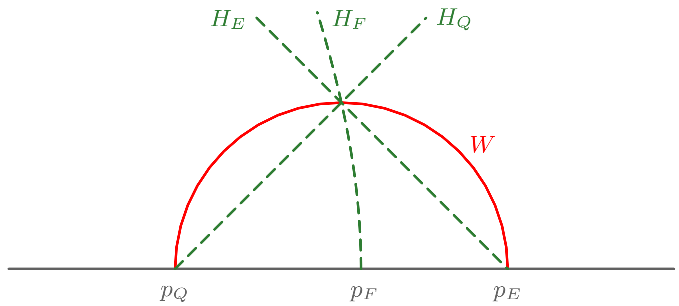


Figure II.6: Actual wall W defined by $0 \to E \to F \to Q \to 0$, for F of minimal discriminant

for $\beta \leq p_Q$ and $p_E \leq \beta < \mu_L(F)$, so whenever nontrivial the function $\mathrm{chd}^0_{F,L}$ reads

$$\mathrm{chd}^0_{F,L}(x) = \begin{cases} 0 & x \leq -p_E \\ \mathrm{ch}_2^{-x}(E) & -p_E \leq x \leq -p_Q \\ \mathrm{ch}_2^{-x}(F) & x \geq -p_Q \end{cases}$$

and, according to the description of Proposition II.3.4, the function is $\mathcal{C}^1$ at $-p_E$ and $-p_Q$.

Example II.5.3. Let (X, L) be a $(1,2)$-polarized abelian surface with $\mathrm{NS}(X) = \mathbb{Z} \cdot L$. Under these assumptions the linear system $|L|$ has exactly four base points, that are identified under the isogeny $\varphi_L : X \to \mathrm{Pic}^0(X)$. In other words, for every $q \in X$ there exists a unique $\alpha \in \mathrm{Pic}^0(X)$ such that $H^1(\mathcal{I}_q \otimes L \otimes \alpha) \neq 0$. This allows us to destabilize $\mathcal{I}_q$, which has minimal discriminant.

Indeed, by Serre duality there is a non-trivial extension $0 \to L^{-1} \otimes \alpha^{-1} \to E \to \mathcal{I}_q \to 0$. Rotation of the triangle yields a destabilizing short exact sequence for $\mathcal{I}_q$, so that $h^0_{\mathcal{I}_q,L}$ reads

$$h^0_{\mathcal{I}_q,L}(x) = \begin{cases} 0 & x \leq \frac{1}{2} \\ \chi_{E,L}(x) = 4x^2 - 4x + 1 & \frac{1}{2} \leq x \leq 1 \\ \chi_{\mathcal{I}_q,L}(x) = 2x^2 - 1 & x \geq 1. \end{cases}$$

Now assume $\overline{\Delta}(F) = 2m$. If $\mathrm{chd}^0_{F,L}$ is not trivial, then F destabilizes along a wall W. We can choose a destabilizing sequence $0 \to E \to F \to Q \to 0$ defining the HN filtration of F in a small annulus just below W. We have to distinguish several possibilities.

If $\overline{\Delta}(E) = \overline{\Delta}(Q) = 0$, then E and Q are semistable for all the stability conditions in the interior of W. Thus the function $\mathrm{chd}^0_{F,L}$ admits the same description as the one given in the case $\overline{\Delta}(F) = m$, i.e. it has critical points $-p_E$ and $-p_Q$ (see section III.1 for an example).

The other possibility is that either E or Q has positive discriminant, say $\overline{\Delta}(E) = m$ and $\overline{\Delta}(Q) = 0$. Note that Q is semistable in the whole interior of W. If this is the case for E as well, then $\mathrm{chd}^0_{F,L}$ again has $-p_E$ and $-p_Q$ as its critical points (the function being not differentiable at $-p_E$).

Otherwise, E destabilizes along a wall W_E inside W, defined by a sequence $0 \to E_1 \to E \to E_2 \to 0$ with $\overline{\Delta}(E_1) = 0 = \overline{\Delta}(E_2)$:

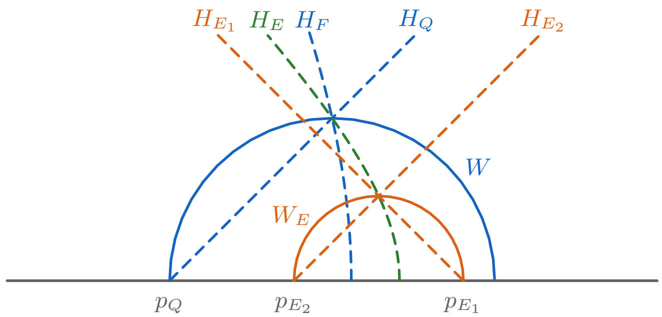


Figure II.7: Successive destabilizations for F of twice the minimal discriminant

Clearly, both E_1 and E_2 are semistable in the whole interior of W_E. With this information, it is easy to describe the HN filtrations of F for all the $\sigma_{0,\beta}$ with $\beta < \mu_L(F)$, and one obtains

$$\mathrm{chd}^0_{F,L}(x) = \begin{cases} 0 & x \leq -p_{E_1} \\ \mathrm{ch}_2^{-x}(E_1) & -p_{E_1} \leq x \leq -p_{E_2} \\ \mathrm{ch}_2^{-x}(E) & -p_{E_2} \leq x \leq -p_Q \\ \mathrm{ch}_2^{-x}(F) & x \geq -p_Q \end{cases}$$

(with the function being differentiable at its three critical points).

More generally, one may try to apply this philosophy for an arbitrarily big $\overline{\Delta}(F)$ as follows. If $\mathrm{chd}^0_{F,L}$ is not trivial, then F destabilizes along a wall W. We keep track of its HN filtration

$$0 = F_0 \hookrightarrow F_1 \hookrightarrow \ldots \hookrightarrow F_{r-1} \hookrightarrow F_r = F$$

for Bridgeland stability conditions in a (sufficiently small) annulus just below W, which necessarily satisfies $\overline{\Delta}(F_1) + \overline{\Delta}(F_2/F_1) + \ldots + \overline{\Delta}(F/F_{r-1}) < \overline{\Delta}(F)$.

Now, it is possible that some HN factors F_i/F_{i-1} are not semistable in the whole region inside W, so they destabilize along a wall W_i. For each such F_i/F_{i-1}, again we keep track of its HN filtration for stability conditions just below W_i

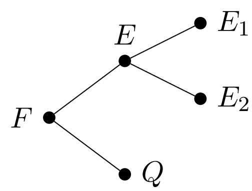
$$0 \hookrightarrow F_{i,1}/F_{i-1} \hookrightarrow \ldots \hookrightarrow F_{i,r_i-1}/F_{i-1} \hookrightarrow F_{i,r_i}/F_{i-1} = F_i/F_{i-1}$$

which satisfies $\overline{\Delta}(F_{i,1}/F_{i-1}) + \overline{\Delta}(F_{i,2}/F_{i,1}) + \ldots + \overline{\Delta}(F_{i,r_i}/F_{i,r_i-1}) < \overline{\Delta}(F_i/F_{i-1})$. Proceeding inductively, we must finish in a finite number of steps thanks to the strict inequalities on discriminants.

The process yields a tree, in which the final vertices correspond to objects G that are semistable in an open neighborhood of $(0, p_G)$ in the (α, β)-plane. By construction, we can consider a lexicographical order on the final vertices:

Definition II.5.4. We say that the tree is *well ordered* if $G_1 < G_2$ implies $p_{G_1} \geq p_{G_2}$ for any two final vertices G_1 and G_2.

Example II.5.5. Assume that $\overline{\Delta}(F) = 2m$. In the stability situation of Figure II.7 F has the following tree, which is well ordered since $p_{E_1} > p_{E_2} > p_Q$:



If the tree of F is well ordered with final vertices $G_1 < ... < G_k$, it is not difficult to recover the Bridgeland limit filtration (and thus the $\sigma_{0,\beta}$-HN filtration) of F at every $\beta < \mu_L(F)$. The corresponding function $\mathrm{chd}^0_{F,L}$ is piecewise polynomial, and their critical points are the points $-p_{G_i}$ for the final vertices G_i of the tree. More precisely,

$$\mathrm{chd}^0_{F,L}(x) = \begin{cases} 0 & x \leq -p_{G_1} \\ \displaystyle\sum_{j=1}^{i} \left(\mathrm{ch}_2^{-x}(G_j)\right) & -p_{G_i} \leq x \leq -p_{G_{i+1}} \ \ (1 \leq i \leq k-1) \\ \mathrm{ch}_2^{-x}(F) & x \geq -p_{G_k} \end{cases}$$

Furthermore, since $\overline{\Delta}(G_i)$ is the discriminant of the Chern degree polynomial $\mathrm{ch}_2^{-x}(G_i)$, some interesting properties about the critical points $-p_{G_i}$ can be read in terms of $\overline{\Delta}(G_i)$:

- **Rationality:** The critical point $-p_{G_i}$ is rational if and only if $\overline{\Delta}(G_i)$ is a perfect square.
- **Differentiability:** The function $\mathrm{chd}^0_{F,L}$ is differentiable at the critical point $-p_{G_i}$ if and only if $\overline{\Delta}(G_i) = 0$.

Even if we do not expect every Gieseker semistable sheaf to have a well-ordered tree, in many concrete situations this is the case, and thus the previous description of Chern degree functions applies. The examples in the next chapter (especially those of section III.1) illustrate this phenomenon for cohomological rank functions on abelian surfaces.

Chapter III

Examples of functions on abelian surfaces

This chapter is devoted to the computation of two examples of cohomological rank functions on abelian surfaces. On the one hand, ideal sheaves of finite subschemes on principally polarized abelian surfaces illustrate many of the phenomena explained in chapter II (especially the method of section II.5). On the other hand, the ideal sheaf of one point is geometrically interesting in its own right, and has direct implications on the syzygies of abelian surfaces.

III.1 Finite subschemes on principally polarized abelian surfaces

Along this section, (X, L) will be a principally polarized complex abelian surface with $\mathrm{NS}(X) = \mathbb{Z}\cdot L$. By abuse of notation, L will also denote a symmetric line bundle representing the polarization.

Under these assumptions, the minimal discriminant is $m = 4$ and X is the Jacobian of a genus 2 curve C; after embedding C in X by means of one of its Weierstrass points, we fix $C \in |L|$ as a symmetric theta divisor.

We want to compute $h^0_{\mathcal{I}_T,L}$ for ideal sheaves $\mathcal{I}_T$ of 0-dimensional subschemes T on X. If $n = h^0(\mathcal{O}_T)$ denotes the length of T, we have: $\mathrm{ch}(\mathcal{I}_T) = (1, 0, -n)$, $\chi_{\mathcal{I}_T,L}(x) = x^2 - n$ (in particular, $p_{\mathcal{I}_T} = -\sqrt{n}$) and $\overline{\Delta}(\mathcal{I}_T) = 4n$ (i.e. $\overline{\Delta}(\mathcal{I}_T)$ is n times the minimal discriminant).

Case $n = 2$

In order to understand geometrically the destabilization of the ideal sheaf of a length two 0-dimensional subscheme T, we first need to control the translates of C that contain T. This is the (scheme-theoretic) support of the sheaf $\varphi_L^* R^2\Phi_{\mathcal{P}^\vee}((\mathcal{I}_T \otimes L)^\vee)$, which in the language of Jiang–Pareschi controls $h^0_{\mathcal{I}_T,L}$ in a right neighborhood of $x = 1$ (recall Theorem I.2.2.(1)).

Lemma III.1.1. *If $T \subset X$ is a 0-dimensional subscheme of length 2, then the locus*

$$\left\{\alpha \in \mathrm{Pic}^0(X) \mid h^0(\mathcal{I}_T \otimes L \otimes \alpha) > 0\right\}$$

parametrizing translates of C containing T, with its natural scheme structure as support of the sheaf $R^2\Phi_{\mathcal{P}^\vee}((\mathcal{I}_T \otimes L)^\vee)$, is a 0-dimensional subscheme $\Gamma \subset \mathrm{Pic}^0(X)$ of length 2.

Proof. When X consists of two distinct points p and q, the translates of C passing through p and q are parametrized by the (scheme-theoretic) intersection $(C+p) \cap (C+q)$. If this intersection is transverse (which happens for a generic pair (p,q)), there are exactly two translates of C containing X. In case of non-transverse intersection, there is only one (proper) translate $\widetilde{C}$ of C containing p and q: moreover, the tangent direction to $\widetilde{C}$ at p and q is the same (under the canonical identifications of tangent spaces via translations).

If T is non-reduced consisting of a point $p \in S$ and a tangent direction $w \in \mathbb{P}(T_pX)$, the translates of C containing T are parametrized by the locus of points in $C+p$ having w as tangent direction. If we denote the Gauss map of $C+p$ by

$$G : C + p \longrightarrow \mathbb{P}^1 = \mathbb{P}((T_0X)^*)$$

the locus we are looking for is the preimage $G^{-1}(w')$, where $w' \in \mathbb{P}((T_0X)^*)$ is the element corresponding to w. Since the Gauss map can be identified with the canonical map of $C+p$, the result follows; clearly, Γ will be reduced or not depending on whether w' is a branch point of G or not. □

To illustrate the strategy outlined in section II.5, we consider the first possible destabilizing wall W for the Chern character $(1,0,-2)$, which has center $-\frac{3}{2}$ and radius $\frac{1}{2}$. Indeed, it can be defined by the following combinations of Chern characters[1]:

$$(1,-L,1) \hookrightarrow (1,0,-2) \twoheadrightarrow (0,L,-3), \quad (2,-2L,2) \hookrightarrow (1,0,-2) \twoheadrightarrow (-1,2L,-4).$$

It is an actual wall for $\mathcal{I}_T$, and the first step in the HN filtration of $\mathcal{I}_T$ after crossing W has Chern character $(2,-2L,2)$. Let us see how to construct this subobject.

Let $\Gamma \subset \mathrm{Pic}^0(X)$ be the subscheme of Proposition VI.4.5, and let $\pi : X \times \mathrm{Pic}^0(X) \longrightarrow X$ and $\sigma : X \times (-\Gamma) \longrightarrow X$ denote the first projection maps. Then $E = \sigma_*(\pi^*(L^{-1}) \otimes \mathcal{P}_{|S\times(-\Gamma)})$ is a semihomogeneous vector bundle of rank 2 on S with $\mathrm{ch}(E) = (2,-2L,2)$, coming with a natural epimorphism of sheaves $E \twoheadrightarrow \mathcal{I}_T$.

For instance, if Γ is reduced (i.e. there are two distinct translates C_1, C_2 of C containing T), then the short exact sequence attached to $E \twoheadrightarrow \mathcal{I}_T$ is nothing but the Koszul complex of the complete intersection $T = C_1 \cap C_2$.

In any case, taking the short exact sequence of sheaves and rotating the triangle, we obtain a short exact sequence in $\mathrm{Coh}^{-\sqrt{2}}(X)$ defining the HN filtration of $\mathcal{I}_T$ just below W. Since both HN factors

[1] One can use Schmidt's implementation [Sch20, Appendix] to find the potential walls.

have discriminant 0, they are semistable in the whole interior of W and our tree is well ordered. Thus we obtain the following cohomological rank function for $\mathcal{I}_T$:

$$h^0_{\mathcal{I}_T,L}(x) = \begin{cases} 0 & x \leq 1 \\ \chi_{E,L}(x) = 2(x-1)^2 & 1 \leq x \leq 2 \\ x^2 - 2 & x \geq 2. \end{cases}$$

Case $n = 4$

For the cases $n \geq 3$, we will use the following result in [Mea12] describing the stability of $\mathcal{I}_T$ along the vertical line $\beta = -2$:

Lemma III.1.2 ([Mea12, Lemma 3.3.6]). *Let $T \in \mathrm{Hilb}^n(X)$.*

(1) *If $n \neq 5$: the object $\mathcal{I}_T$ is destabilized at the vertical line $\beta = -2$ if, and only if, T contains a collinear (i.e. contained in a translate of C) subscheme of colength m, for some $0 \leq m < \frac{n-2}{2}$.*

   In such a case, the destabilizing subobject in $\mathrm{Coh}^{-2}(X)$ is $L^{-1} \otimes \mathcal{I}_{T'} \otimes \alpha$, for some $T' \in \mathrm{Hilb}^m(X)$ and $\alpha \in \mathrm{Pic}^0(X)$.

(2) *If $n = 5$: $\mathcal{I}_T$ can also be destabilized at $\beta = -2$ by K, where K is a slope-stable locally free sheaf with $\mathrm{ch}(K) = (2, -3L, 4)$.*

   This destabilization takes place if, and only if, every subscheme of T with length 4 is non-collinear and contains a unique collinear subscheme of length 3.

Now assume that T is a length four 0-dimensional subscheme. We consider two subcases.

If no translate of C contains T, then $\mathcal{I}_T$ has trivial cohomological rank function:

$$h^0_{\mathcal{I}_T,L}(x) = \begin{cases} 0 & x \leq 2 \\ x^2 - 4 & x \geq 2. \end{cases}$$

Indeed, by Lemma III.1.2 $\mathcal{I}_T$ remains semistable along the vertical line $\beta = -2$. Since $p_{\mathcal{I}_T} = -2$, this means that $\mathcal{I}_T$ is semistable in the whole region $\beta < 0$.

Now assume that T is a collinear subscheme (i.e. it is contained in a translate C_1 of C). To simplify the notation and illustrate the strategy, we restrict ourselves to the case where $T = \{p, q, r, s\}$ is reduced.

Consider any two points of T: there is another translate of C containing them, unless C_1 has the same tangent direction at these points (see the proof of Proposition VI.4.5). Since the Gauss map of C_1 has degree 2, it follows that (possibly after reordering the points of X) there are two distinct translates of C passing through p and q (resp. through r and s) simultaneously; being C_1 one of them, we denote by C_2 (resp. C_3) the other one.

We take also $\alpha, \beta, \gamma \in \mathrm{Pic}^0(X)$ such that $C_1 \in |L \otimes \alpha|$, $C_2 \in |L \otimes \beta|$ and $C_3 \in |L \otimes \gamma|$.

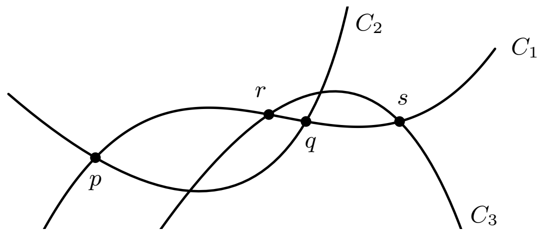


Figure III.1: Translates C_1, C_2 and C_3 of C

The Chern character $(1,0,-4)$ has a unique possible wall W, of center $-\frac{5}{2}$ and radius $\frac{3}{2}$. Since T is collinear, we can destabilize $\mathcal{I}_T$ (as predicted by Lemma III.1.2) via the exact sequence

$$0 \longrightarrow L^{-1} \otimes \alpha^{-1} \xrightarrow{\cdot s_1} \mathcal{I}_T \longrightarrow \mathcal{O}_{C_1}(-p-q-r-s) \longrightarrow 0$$

where $s_1 \in H^0(L \otimes \alpha)$ defines C_1. This sequence gives the HN filtration of $\mathcal{I}_T$ just below W.

The subobject $E = L^{-1} \otimes \alpha^{-1}$ is everywhere semistable since $\overline{\Delta}(E) = 0$. The quotient $Q = \mathcal{O}_{C_1}(-p-q-r-s)$ has $\overline{\Delta}(Q) = 4$, and destabilizes along a wall inside W defined by a sequence

$$0 \longrightarrow L^{-2} \otimes \beta^{-1} \otimes \gamma^{-1} \xrightarrow{\cdot s_2 s_3|_{C_1}} \mathcal{O}_{C_1}(-p-q-r-s) \longrightarrow (L^{-3} \otimes \alpha^{-1} \otimes \beta^{-1} \otimes \gamma^{-1})[1] \longrightarrow 0$$

where s_2 and s_3 are sections defining C_2 and C_3.

Both $Q_1 = L^{-2} \otimes \beta^{-1} \otimes \gamma^{-1}$ and $Q_2 = (L^{-3} \otimes \alpha^{-1} \otimes \beta^{-1} \otimes \gamma^{-1})[1]$ have $\overline{\Delta} = 0$, so it is not necessary to study further destabilizations. We obtain a well-ordered tree with final vertices E, Q_1 and Q_2, which gives the following cohomological rank function:

$$h^0_{\mathcal{I}_T,L}(x) = \begin{cases} 0 & x \leq 1 \\ \chi_{E,L}(x) = (x-1)^2 & 1 \leq x \leq 2 \\ \chi_{E,L}(x) + \chi_{Q_1,L}(x) = (x-1)^2 + (x-2)^2 & 2 \leq x \leq 3 \\ x^2 - 4 & x \geq 3 \end{cases}$$

Remark III.1.3. For T collinear and nonreduced we obtain the same function, as easily follows from semicontinuity arguments. Nevertheless, in that case the second destabilization does not admit such a simple presentation.

Case $n = 3$

If T is a length three 0-dimensional subscheme, we consider first the case where T is collinear, where the cohomological rank function of $\mathcal{I}_T$ is

$$h^0_{\mathcal{I}_T,L}(x) = \begin{cases} 0 & x \leq 1 \\ (x-1)^2 & 1 \leq x \leq 2 \\ x^2 - 3 & x \geq 2. \end{cases}$$

Indeed, if $C_1 \in |L \otimes \alpha|$ is the translate of C containing T, then $\mathcal{I}_T$ destabilizes along the wall W defined by the short exact sequence

$$0 \longrightarrow L^{-1} \otimes \alpha^{-1} \xrightarrow{\cdot s} \mathcal{I}_T \longrightarrow \mathcal{O}_{C_1}(-T) \longrightarrow 0$$

where the section $s \in H^0(L \otimes \alpha)$ defines C_1. This was already predicted by Lemma III.1.2.

The subobject $L^{-1} \otimes \alpha^{-1}$ is obviously semistable everywhere inside W; the same happens for the quotient $\mathcal{O}_{C_1}(-T)$, since it is a line bundle of odd degree on a genus 2 Abel-Jacobi curve (recall Example II.5.2.(4)). This completes the tree and we obtain the desired cohomological rank function.

If T is not collinear, then according to Lemma III.1.2 $\mathcal{I}_T$ remains semistable along the whole line $\beta = -2$. The next possible wall W' has center $-\frac{7}{4}$ and radius $\frac{1}{4}$; let us describe the destabilization of $\mathcal{I}_T$ along this wall, under the assumption that $T = \{p, q, r\}$ is reduced and every pair of points on T is contained in two distinct translates of C.

Let $C_1 = C + t_1 \in |L \otimes \alpha|$ and $\widetilde{C_1} = C + \widetilde{t_1} \in |L \otimes \widetilde{\alpha}|$ be the two translates of C passing through the points p and q. Among all the translates of C passing through r, we take two curves $C_2 = C + t_2$ and $\widetilde{C_2} = C + \widetilde{t_2}$ such that $t_2 - \widetilde{t_2} = t_1 - \widetilde{t_1}$. This is possible because the subtraction morphism $C \times C \longrightarrow S$ is surjective (the curve C being non-degenerate).

Writing $C_2 \in |L \otimes \beta|$ and $\widetilde{C_2} \in |L \otimes \beta \otimes \widetilde{\alpha} \otimes \alpha^{-1}|$ (according to the condition $t_2 - \widetilde{t_2} = t_1 - \widetilde{t_1}$), it is easy to check that $h^0(\mathcal{I}_X \otimes L^2 \otimes \beta \otimes \widetilde{\alpha}) \geq 2$ and thus $h^1(\mathcal{I}_X \otimes L^2 \otimes \beta \otimes \widetilde{\alpha}) \geq 1$. By Serre duality, the last condition reads $\mathrm{Ext}^1(\mathcal{I}_X, L^{-2} \otimes \beta^{-1} \otimes \widetilde{\alpha}^{-1}) \neq 0$. A (rotated) nontrivial extension gives a short exact sequence, destabilizing $\mathcal{I}_T$ along W'.

By repeating this process (starting with the two curves through p and r, and the two curves through q and r) and taking direct sum, it is possible to destabilize $\mathcal{I}_T$ via a short exact sequence $0 \to E \to \mathcal{I}_T \to Q \to 0$ corresponding to the HN filtration of $\mathcal{I}_T$ just below W'.

Since $\mathrm{ch}(E) = (4, -6L, 9)$ and $\mathrm{ch}(Q) = (-3, 6L, -12)$, both E and Q have $\overline{\Delta} = 0$ and thus the construction of the tree has no more steps. The corresponding function $h^0_{\mathcal{I}_T, L}$ is

$$h^0_{\mathcal{I}_T,L}(x) = \begin{cases} 0 & x \leq \frac{3}{2} \\ 4x^2 - 12x + 9 & \frac{3}{2} \leq x \leq 2 \\ x^2 - 3 & x \geq 2. \end{cases}$$

Case $n = 5$

The Chern character $(1, 0, -5)$ has three possible walls intersecting the vertical line $\beta = -2$, namely:

- A wall W_1 of center -3 and radius 2.
- A wall W_2 of center $-\frac{5}{2}$ and radius $\frac{1}{2}\sqrt{5}$.
- A wall W_3 of center $-\frac{7}{3}$ and radius $\frac{2}{3}$.

Let us see how each of these three walls corresponds to one of the special geometric situations described in Lemma III.1.2. In these three special situations, the usual method constructs the well-ordered tree of $\mathcal{I}_T$ and hence recovers $h^0_{\mathcal{I}_T,L}$.

Example III.1.4 (Wall W_1)**.** If T is collinear and $C_1 \in |L \otimes \alpha|$ is a translate of C containing it, then there is a short exact sequence

$$0 \longrightarrow L^{-1} \otimes \alpha^{-1} \xrightarrow{\cdot s_1} \mathcal{I}_T \longrightarrow \mathcal{O}_{C_1}(-T) \longrightarrow 0$$

destabilizing $\mathcal{I}_T$ along the wall W_1, where s_1 is a section defining C_1.

The subobject $E = L^{-1} \otimes \alpha^{-1}$ does not destabilize, and the same happens for the quotient $Q = \mathcal{O}_{C_1}(-T)$: again, we have a line bundle of odd degree supported on an Abel-Jacobi curve. This allows us to recover the cohomological rank function as

$$h^0_{\mathcal{I}_T,L}(x) = \begin{cases} 0 & x \leq 1 \\ \chi_{E,L}(x) = (x-1)^2 & 1 \leq x \leq 3 \\ x^2 - 5 & x \geq 3. \end{cases}$$

Example III.1.5 (Wall W_2)**.** Assume that T itself is not collinear, but contains a collinear subscheme Y of length 4. In this case $\mathcal{I}_T$ gets destabilized along the wall W_2, and the function $h^0_{\mathcal{I}_T,L}$ presents a remarkable peculiarity which illustrates a special situation in our characterization of critical points (Proposition II.3.4).

Assume for simplicity that $T = \{p_1, p_2, p_3, p_4, p_5\}$ is reduced. If $Y = \{p_1, p_2, p_3, p_4\}$ and $C_1 \in |L \otimes \alpha|$ is the translate of C containing Y, then

$$0 \longrightarrow \mathcal{I}_{p_5} \otimes L^{-1} \otimes \alpha^{-1} \xrightarrow{\cdot s_1} \mathcal{I}_T \longrightarrow \mathcal{O}_{C_1}(-p_1 - p_2 - p_3 - p_4) \longrightarrow 0$$

is the exact sequence destabilizing $\mathcal{I}_T$ along the wall W_2. It is also the HN filtration of $\mathcal{I}_T$ for stability conditions just below W_2. Since both HN factors have $\overline{\Delta} = 4$, we have to study possible further destabilizations:

- $E = \mathcal{I}_{p_5} \otimes L^{-1} \otimes \alpha^{-1}$ is semistable everywhere inside W_2, according to Example II.5.2.(3).
- The sheaf $Q = \mathcal{O}_{C_1}(-p_1 - p_2 - p_3 - p_4)$ destabilizes as described in the case of 4 collinear points (see section III.1). This destabilization takes place along a wall W_Q of center $-\frac{5}{2}$ and radius $\frac{1}{2}$. Both the subobject Q_1 and the quotient Q_2 inducing this wall have $\overline{\Delta} = 0$, so we are done.

Figure III.2 shows the position of walls and hyperbolas in the (α, β)-plane:

Again our tree (with final vertices E, Q_1, Q_2) is well ordered, but in this case we have an overlap $p_E = p_{Q_1} = -2$. In the terminology of Proposition II.3.4, this implies that $x = 2$ is a critical point where conditions (a) and (c) are simultaneously fulfilled.

The cohomological rank function $h^0_{\mathcal{I}_T,L}$ can be recovered as

$$h^0_{\mathcal{I}_T,L}(x) = \begin{cases} 0 & x \leq 2 \\ \chi_{E,L}(x) + \chi_{Q_1,L}(x) = 2(x-1)(x-2) & 2 \leq x \leq 3 \\ x^2 - 5 & x \geq 3. \end{cases}$$

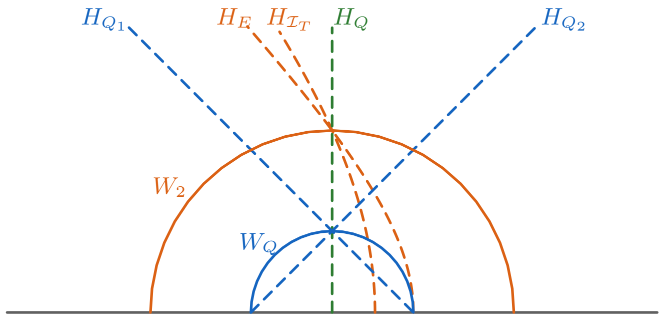


Figure III.2: Walls and hyperbolas involved in the tree of $\mathcal{I}_T$

Example III.1.6 (Wall W_3)**.** Assume that every subscheme of T with length 4 is non-collinear and contains a unique collinear subscheme of length 3.

Under this assumption, T must be non-reduced (otherwise T would contain too many length 4 subschemes to satisfy the hypothesis). Moreover, it is easy to check that such configurations are possible for subschemes T supported at four or less distinct points:

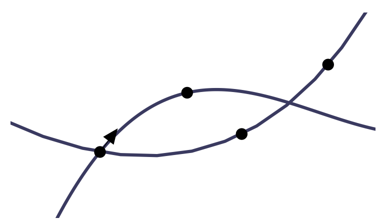


Figure III.3: Possible configuration for a subscheme T supported at four distinct points

According to [Mea12], $\mathcal{I}_T$ destabilizes along the wall W_3 via a distinguished triangle

$$0 \longrightarrow K \longrightarrow \mathcal{I}_T \longrightarrow Q \longrightarrow 0$$

where K is a slope-stable locally free sheaf with $\mathrm{ch}(K) = (2, -3L, 4)$. Both the subobject and the quotient are semistable everywhere:

- Even if $\overline{\Delta}(K) = 4$, the Chern character of K has no actual walls.
- The quotient Q, with Chern character $\mathrm{ch}(Q) = (-1, 3L, -9)$, has $\overline{\Delta} = 0$.

Therefore, we obtain the cohomological rank function

$$h^0_{\mathcal{I}_T,L}(x) = \begin{cases} 0 & x \leq 2 \\ \chi_{K,L}(x) = 2(x-1)(x-2) & 2 \leq x \leq 3 \\ x^2 - 5 & x \geq 3 \end{cases}$$

which is exactly the same as that of Example III.1.5. Hence it is worth noting that stability distinguishes between two configurations of points that cannot be distinguished via cohomological rank functions.

Finally, let us consider the case where T satisfies none of the special geometric conditions studied in the previous examples. The next possible wall W for $\mathcal{I}_T$ has center $-\frac{9}{4}$ and radius $\frac{1}{4}$. According to [Mac12, Theorem 10.2], the locus

$$\widetilde{\Gamma} = \{\alpha \in \operatorname{Pic}^0(X) \mid h^0(\mathcal{I}_T \otimes L^2 \otimes \alpha) > 0\}$$

(with its natural scheme structure as support of the sheaf $R^2\Phi_{\mathcal{P}^\vee}((\mathcal{I}_T \otimes L^2)^\vee)$) is a 0-dimensional subscheme of length 5. In other words, among all the translated linear systems $|L^2 \otimes \alpha|$ there exist exactly five curves (counted with multiplicities) that contain T.

Similarly to the case $n = 2$, one can use these five curves to destabilize $\mathcal{I}_T$ along W. The tree of $\mathcal{I}_T$ consists of this single step and the cohomological rank function reads

$$h^0_{\mathcal{I}_T,L}(x) = \begin{cases} 0 & x \leq 2 \\ 5(x-2)^2 & 2 \leq x \leq \frac{5}{2} \\ x^2 - 5 & x \geq \frac{5}{2}. \end{cases}$$

III.2 The basepoint-freeness threshold (I): Upper bounds

In this section, we will perform explicit computations of the function $h^0_{\mathcal{I}_0,L}$ (and hence of $h^1_{\mathcal{I}_0,L}$), as an another instance of the techniques described in section II.5. As explained in section I.2, this is relevant with regard to syzygies of abelian surfaces (a problem that is not formulated in terms of derived categories). Indeed, we will obtain new effective results in this direction (Corollary III.2.5), as a combination of our computations with the criteria of Jiang-Pareschi, Caucci and Ito.

In the rest of this chapter, we will work again over an arbitrary algebraically closed field $\mathbb{K}$. Let (X, L) be a $(1, d)$-polarized abelian surface, such that L^2 divides $D \cdot L$ for every divisor class D. Note that this includes the case where X has Picard rank 1.

First of all, observe that $\mathcal{I}_0$ is a Gieseker semistable sheaf with $v(\mathcal{I}_0) = (L^2, 0, -1)$ (in particular, $\mu_L(\mathcal{I}_0) = 0$), $\chi_{\mathcal{I}_0,L}(x) = dx^2 - 1$ and $p_{\mathcal{I}_0} = -\frac{\sqrt{d}}{d}$.

Moreover, $\overline{\Delta}(\mathcal{I}_0) = 2L^2 (= 4d$, by Riemann-Roch) takes the minimal positive value; indeed, by our assumptions on (X, L) we have $4d | \overline{\Delta}(v(E))$ for every $E \in \mathrm{D^b}(X)$. Recall that this is a strong constraint, which guarantees either that $\mathcal{I}_0$ is $\sigma_{\alpha,\beta}$-semistable for every $\beta < 0$ and $\alpha > 0$ (in which case $h^0_{\mathcal{I}_0,L}$ is trivial), or $\mathcal{I}_0$ destabilizes along a wall W defined by a short exact sequence $0 \to E \to \mathcal{I}_0 \to Q \to 0$ in $\operatorname{Coh}^{-\frac{\sqrt{d}}{d}}(X)$, with $\overline{\Delta}(E) = 0 = \overline{\Delta}(Q)$.

Lemma III.2.1. *Let $0 \to E \to \mathcal{I}_0 \to Q \to 0$ be a destabilizing short exact sequence as above. Then $v(E) = (d(\widetilde{x}+1), -2d\widetilde{y}, \frac{\widetilde{x}-1}{2})$ and $v(Q) = ((1-\widetilde{x})d, 2d\widetilde{y}, -\frac{\widetilde{x}+1}{2})$, where $(\widetilde{x}, \widetilde{y})$ is a positive nontrivial solution to Pell's equation $x^2 - 4d \cdot y^2 = 1$.*

Proof. By the assumption $2d = L^2 | D \cdot L$ for every divisor class D, we may write $v(E) = (2dr, 2dc, \chi)$

and $v(Q) = (2d(1-r), -2dc, -1-\chi)$ for certain integers r, c and χ. The condition $\overline{\Delta}(E) = \overline{\Delta}(Q)$ is easily checked to read as $r = \chi + 1$.

Imposing now $\overline{\Delta}(E) = 0$ gives $\chi(\chi+1) - dc^2 = 0$, which after multiplying by 4 and adding 1 at both sides, becomes $(2\chi+1)^2 - 4d \cdot c^2 = 1$. Therefore, $(2\chi+1, c)$ is a solution to the equation $x^2 - 4d \cdot y^2 = 1$. Note that this solution must be non-trivial: otherwise, either E or Q would have class $v = (0, 0, -1)$, which is impossible.

Finally, we have to determine the signs of the solution $(2\chi+1, c)$ to Pell's equation. On the one hand, since E is a subobject of the torsion-free sheaf $\mathcal{I}_0$ in the category $\mathrm{Coh}^{-\frac{\sqrt{d}}{d}}(X)$, it follows that E is a sheaf with $r = \mathrm{ch}_0(E) > 0$ (hence $2\chi + 1 > 0$).

On the other hand, the right intersection point of W with the β-axis equals $p_E = \mu_L(E) = \frac{c}{r}$ by the condition $\overline{\Delta}(E) = 0$; since W is an actual wall for $\mathcal{I}_0$, it lies entirely in the region with $\beta < 0$, which gives $c < 0$ and finishes the proof. □

This explicit description of the potential destabilizing walls of $\mathcal{I}_0$ leads to the following expressions for the function $h^0_{\mathcal{I}_0,L}$:

Theorem III.2.2. *Let (X, L) be a $(1, d)$-polarized abelian surface such that $D \cdot L$ is a multiple of L^2 for every divisor class D.*

(1) *If d is a perfect square, then the cohomological rank function $h^0_{\mathcal{I}_0,L}$ reads*

$$h^0_{\mathcal{I}_0,L}(x) = \begin{cases} 0 & x \leq \frac{\sqrt{d}}{d} \\ dx^2 - 1 & x \geq \frac{\sqrt{d}}{d} \end{cases} \qquad \text{(III.2.1)}$$

In particular, $\epsilon_1(L) = \frac{\sqrt{d}}{d}$ and $h^0_{\mathcal{I}_0,L}$ is not differentiable at $\frac{\sqrt{d}}{d}$.

(2) *If d is not a perfect square, then the cohomological rank function $h^0_{\mathcal{I}_0,L}$ is either* (III.2.1) *or*

$$h^0_{\mathcal{I}_0,L}(x) = \begin{cases} 0 & x \leq \frac{2\tilde{y}}{\tilde{x}+1} \\ \frac{d(\tilde{x}+1)}{2}x^2 - 2d\tilde{y} \cdot x + \frac{\tilde{x}-1}{2} & \frac{2\tilde{y}}{\tilde{x}+1} \leq x \leq \frac{2\tilde{y}}{\tilde{x}-1} \\ dx^2 - 1 & x \geq \frac{2\tilde{y}}{\tilde{x}-1} \end{cases}$$

where $(\tilde{x}, \tilde{y})$ is a nontrivial positive solution to Pell's equation $x^2 - 4d \cdot y^2 = 1$. In particular, if (x_0, y_0) is the minimal positive solution to this equation, then $\epsilon_1(L) \leq \frac{2y_0}{x_0 - 1}$.

Proof. If d is a perfect square (equivalently, $4d$ is a perfect square), then Pell's equation involved in Lemma III.2.1 admits only trivial solutions, so $\mathcal{I}_0$ is $\sigma_{\alpha,\beta}$-semistable along the whole region $\beta < 0$. Therefore, the function $h^0_{\mathcal{I}_0,L}$ is trivial by Proposition II.5.1, which leads to the expression (III.2.1).

Now assume that d is not a perfect square. If $\mathcal{I}_0$ destabilizes (equivalently, $h^0_{\mathcal{I}_0,L}$ is not trivial), then by Lemma III.2.1 the destabilizing wall corresponds to a positive nontrivial solution $(\widetilde{x}, \widetilde{y})$ of $x^2 - 4d \cdot y^2 = 1$, for which the classes $v(E)$ and $v(Q)$ are known. Once more, we are in the situation of a well-ordered tree for $\mathcal{I}_0$, which gives the stated expression for $h^0_{\mathcal{I}_0,L}$.

Finally, observe that in the same way as the quotients $\frac{\widetilde{y}}{\widetilde{x}}$ converge to $\frac{\sqrt{d}}{2d}$, the potential destabilizing walls for $\mathcal{I}_0$ accumulate towards the point $p_{\mathcal{I}_0} = -\frac{\sqrt{d}}{d}$ in the β-axis:

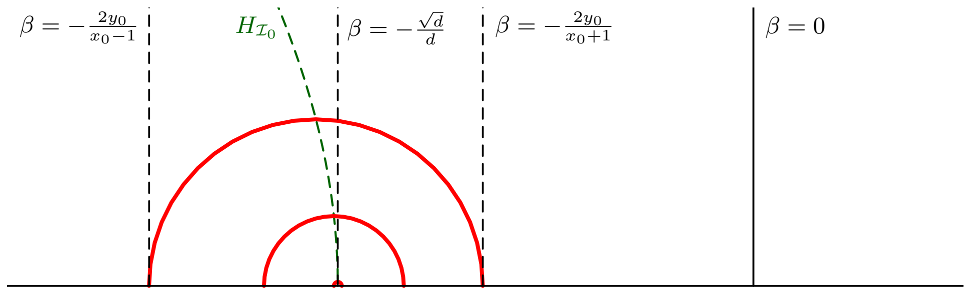


Figure III.4: Possible walls for $\mathcal{I}_0$ parametrized by solutions to Pell's equation

It follows that the largest possible wall is associated to the minimal solution (x_0, y_0), and hence the inequality $\epsilon_1(L) \leq \frac{2y_0}{x_0-1}$ holds. □

Remark III.2.3. Upper bounds for $\epsilon_1(L)$ have been given by Ito for general abelian surfaces over $\mathbb{C}$, using completely different techniques (see [Ito20b, Proposition 4.4]). When d is a perfect square, he already obtained the equality $\epsilon_1(L) = \frac{\sqrt{d}}{d}$ (and thus the expression for $h^0_{\mathcal{I}_0,L}$).

On the other hand, for d not a perfect square our upper bound refines the one given by Ito. Indeed, both bounds coincide for several values of d, but in general the inequality $\epsilon_1(L) \leq \frac{2y_0}{x_0-1}$ is stronger as one can check in the following table of upper bounds (see $d = 7, 11, 13, 19, 21, 22$). The equalities denote exact values for $\epsilon_1(L)$, obtained by Ito via lower bounds; we will come back to this question in section III.3.

d	2	3	5	6	7	8	10	11	12	13	14	15	17	18	19	20	21	22
Ito	$=1$	$=\frac{2}{3}$	$=\frac{1}{2}$	$=\frac{1}{2}$	$\frac{3}{7}$	$\frac{3}{8}$	$\frac{1}{3}$	$\frac{1}{3}$	$=\frac{1}{3}$	$\frac{4}{13}$	$=\frac{2}{7}$	$=\frac{4}{15}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{21}$	$\frac{5}{22}$
$\frac{2y_0}{x_0-1}$	1	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{8}{21}$	$\frac{3}{8}$	$\frac{1}{3}$	$\frac{10}{33}$	$\frac{1}{3}$	$\frac{5}{18}$	$\frac{2}{7}$	$\frac{4}{15}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{170}{741}$	$\frac{1}{4}$	$\frac{2}{9}$	$\frac{3}{14}$

Table III.1: Comparison of bounds for $\epsilon_1(L)$ when d is not a perfect square

One of the advantages of our approach is that it controls the differentiability of the function $h^0_{\mathcal{I}_0,L}$, which is meaningful in terms of M-regularity. Indeed, as an immediate consequence of Theorem III.2.2 and Lemma I.2.7 we directly obtain:

Corollary III.2.4. *Let (X, L) be a $(1, d)$-polarized abelian surface such that $D \cdot L$ is a multiple of L^2 for every divisor class D.*

(1) *If d is a perfect square, then $\mathcal{I}_0\langle\frac{\sqrt{d}}{d}L\rangle$ is a GV-sheaf which is not M-regular.*

(2) *If d is not a perfect square, then $\mathcal{I}_0\langle \frac{2y_0}{x_0-1} L\rangle$ is M-regular.*

In particular, for $m \in \mathbb{Z}_{>0}$ $\mathcal{I}_0\langle \frac{1}{m} L\rangle$ is M-regular if and only if $m < \sqrt{d}$ (i.e. $m^2 < d$).

We point out that this gives an affirmative answer, in the case of abelian surfaces, to a question posed by Ito ([Ito21, Remark 6.4]). By means of Corollary III.2.4, we prove the following result on the syzygies of primitively polarized abelian surfaces:

Corollary III.2.5. *Let (X, L) be a $(1, d)$-polarized abelian surface such that $D \cdot L$ is a multiple of L^2 for every divisor class D.*

(1) *If $d \geq 7$, then any ample line bundle representing L is projective normal.*
(2) *If $d > (p+2)^2$ for $p \geq 1$, then any ample line bundle representing L satisfies the property (N_p).*

Proof. Under the assumptions on (X, L), $\mathcal{I}_0\langle \frac{1}{2} L\rangle$ is $IT(0)$ for every $d \geq 7$, as an immediate application of Lemma I.2.7 and the upper bounds for $\epsilon_1(L)$. Thus the first assertion follows from Theorem I.2.6.(2).

If $d > (p+2)^2$ for some $p \geq 1$, then $\mathcal{I}_0\langle \frac{1}{p+2} L\rangle$ is M-regular by Corollary III.2.4. Hence Theorem I.2.6.(3) guarantees the property (N_p) for representatives of L. □

III.3 The basepoint-freeness threshold (II): Lower bounds

Let d be a positive integer which is not a perfect square, and let (x_0, y_0) be the minimal positive solution to $x^2 - 4d \cdot y^2 = 1$. In the sequel, we will assume that $\mathrm{char}(\mathbb{K})$ divides neither x_0^2 nor $x_0^2 - 1$ (in particular, $\mathrm{char}(\mathbb{K}) \neq 2$).

This section is devoted to prove that only two of the potential functions described in Theorem III.2.2.(2) may happen: those corresponding to the two smallest positive solutions of Pell's equation. In particular, we will obtain lower bounds for $\epsilon_1(L)$.

Our approach is based on the following result (which is actually valid without the hypothesis of Theorem III.2.2):

Proposition III.3.1. *Let (X, L) is a $(1, d)$-polarized abelian surface, and let L also denote a symmetric representative of it. Then $h^0(X, \mu_{x_0}^* \mathcal{I}_0 \otimes L^{2x_0y_0}) \geq x_0^2$; in other words, the linear system of curves $|L^{2x_0y_0}|$ has at least x_0^2 independent elements that contain all the x_0-torsion points of X.*

Proof. Since the subgroup $T \cong (\mathbb{Z}/x_0)^4$ of x_0-torsion points is contained in

$$K(L^{2x_0y_0}) \cong (\mathbb{Z}/2x_0y_0 \oplus \mathbb{Z}/2dx_0y_0) \times (\widehat{\mathbb{Z}/2x_0y_0 \oplus \mathbb{Z}}/2dx_0y_0),$$

we will use the representation of the theta group $\mathcal{G}(L^{2x_0y_0})$ on $H^0(X, L^{2x_0y_0})$ to understand how translation by points of T acts on the linear system $|L^{2x_0y_0}|$.

We consider the isomorphism of Theorem I.3.1.(3), which in particular identifies $\mathcal{G}(L^{2x_0y_0})$ with $\mathbb{K}^* \times K(L^{2x_0y_0})$ (with a noncommutative group operation), and $H^0(X, L^{2x_0y_0})$ with

$$V(2x_0y_0, 2dx_0y_0) = \{\mathbb{K}\text{-valued functions on } \mathbb{Z}/_{2x_0y_0} \oplus \mathbb{Z}/_{2dx_0y_0}\}.$$

Denote by $\{\delta_{j,k} \mid (j,k) \in \mathbb{Z}/_{2x_0y_0} \oplus \mathbb{Z}/_{2dx_0y_0}\}$ the canonical basis of $V(2x_0y_0, 2dx_0y_0)$, that is: $\delta_{j,k}(l,m) = 1$ if $(j,k) = (l,m)$, and $\delta_{j,k}(l,m) = 0$ otherwise.

Moreover, let $\{a_1, a_2, a_3, a_4\}$ be the following basis of T inside $K(L^{2x_0y_0})$:

- $a_1 = (2y_0, 0)$, $a_2 = (0, 2dy_0)$ in $\mathbb{Z}/_{2x_0y_0} \oplus \mathbb{Z}/_{2dx_0y_0}$.
- $a_3, a_4 \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/_{2x_0y_0} \oplus \mathbb{Z}/_{2dx_0y_0}, \mathbb{K}^*)$ are the homomorphisms given by

  $$a_3(1,0) = \xi,\ a_3(0,1) = 1,\ a_4(1,0) = 1,\ a_4(0,1) = \xi,$$

  where ξ is a primitive x_0-th root of 1.

Consider the lifts $(1, a_i) \in \mathcal{G}(L^{2x_0y_0})$ of a_i (for $i = 1,2,3,4$) to the theta group. According to the representation described in Theorem I.3.1.(3), they induce the endomorphisms

$$\widetilde{a}_1 : \delta_{j,k} \mapsto \delta_{j-2y_0,k}\ , \qquad \widetilde{a}_2 : \delta_{j,k} \mapsto \delta_{j,k-2dy_0}\ , \qquad \widetilde{a}_3 : \delta_{j,k} \mapsto \xi^j \delta_{j,k}\ , \qquad \widetilde{a}_4 : \delta_{j,k} \mapsto \xi^k \delta_{j,k}$$

on $H^0(X, L^{2x_0y_0})$. Recall that the projectivization of $\widetilde{a}_i$ on the linear system $|L^{2x_0y_0}|$ corresponds to (the dual of) the projectivity $t_{a_i} : \mathbb{P}(H^0(X,L)^\vee) \to \mathbb{P}(H^0(X,L)^\vee)$ extending the translation $t_{a_i} : X \to X$.

Observe that $\widetilde{a}_3, \widetilde{a}_4$ are diagonalizable endomorphisms that commute (as corresponds to a_3, a_4 generating a totally isotropic subgroup of $K(L^{2x_0y_0})$). This implies that every eigenspace of $\widetilde{a}_3$ is an invariant subspace for $\widetilde{a}_4$, and conversely.

Therefore, we can find a decomposition

$$H^0(X, L^{2x_0y_0}) = \bigoplus_{l,m \in \{0,\ldots,x_0-1\}} E_{(l,m)},$$

where $E_{(l,m)}$ is a subspace of eigenvectors for both $\widetilde{a}_3$ and $\widetilde{a}_4$ (of eigenvalue ξ^l for $\widetilde{a}_3$, and eigenvalue ξ^m for $\widetilde{a}_4$). Explicitly, we have

$$E_{(l,m)} = \langle \delta_{j,k} \mid j \equiv l \text{ and } k \equiv m \pmod{x_0} \rangle,$$

so every subspace $E_{(l,m)}$ has dimension $2y_0 \cdot 2dy_0 = 4dy_0^2 = x_0^2 - 1$.

The projectivization of $E_{(l,m)}$ represents a $(x_0^2 - 2)$-dimensional linear system $\mathcal{L}_{l,m} \subset |L^{2x_0y_0}|$, formed by curves which remain invariant under translation by points of the subgroup $\langle a_3, a_4 \rangle \subset T$. In particular, any curve of $\mathcal{L}_{l,m}$ containing $\langle a_1, a_2 \rangle \subset T$ will automatically contain all of T.

Moreover, since $\gcd(x_0, 2dy_0) = 1$, it follows from the above description of $\widetilde{a}_1, \widetilde{a}_2$ that the subgroup $\langle (1,a_1), (1,a_2) \rangle \cong (\mathbb{Z}/_{x_0})^2 \subset \mathcal{G}(L^{2x_0y_0})$ acts transitively on the set $\{E_{(l,m)}\}$. Thus for our purposes it suffices to find a curve $C \in \mathcal{L}_{0,0}$ containing the x_0^2 points of $\langle a_1, a_2\rangle \subset T$. Indeed, the set of x_0^2 curves will be formed by one curve in each $\mathcal{L}_{l,m}$, obtained from C by translation with the corresponding point of $\langle a_1, a_2\rangle$.

Since $L^{2x_0y_0}$ is totally symmetric, we may consider the involution of $H^0(X, L^{2x_0y_0})$

$$\widetilde{i} : \delta_{j,k} \mapsto \delta_{-j,-k},$$

whose projectivization extends the inversion $i : X \to X$ to a projectivity of $\mathbb{P}(H^0(X, L^{2x_0y_0})^\vee)$.

The subspace $E_{(0,0)}$ is clearly invariant by this endomorphism, and the restriction $\widetilde{i}_{|E_{(0,0)}}$ satisfies:

- The subspace $E^1_{(0,0)} \subset E_{(0,0)}$ of eigenvectors of eigenvalue 1 has dimension $2dy_0^2 + 2 = \frac{x_0^2-1}{2} + 2$. Explicitly, a basis of $E^1_{(0,0)}$ is given by

$$\delta_{sx_0,tx_0} + \delta_{(2y_0-s)x_0,(2dy_0-t)x_0}$$

  for $s \in \{0, ..., y_0\}$, and $t \in \{0, ..., 2dy_0 - 1\}$ (if $s \neq 0, y_0$) or $t \in \{0, ..., dy_0\}$ (if $s = 0, y_0$).
- The eigenspace $E^{-1}_{(0,0)} \subset E_{(0,0)}$ of eigenvalue -1 has dimension $2dy_0^2 - 2$, with basis

$$\delta_{sx_0,tx_0} - \delta_{(2y_0-s)x_0,(2dy_0-t)x_0}$$

  for $s \in \{0, ..., y_0\}$, and $t \in \{0, ..., 2dy_0 - 1\}$ (if $s \neq 0, y_0$) or $t \in \{1, ..., dy_0 - 1\}$ (if $s = 0, y_0$).

The projectivization of $E^1_{(0,0)}$ defines a $(\frac{x_0^2-1}{2} + 1)$-dimensional linear system $\mathcal{L}^1_{0,0} \subset \mathcal{L}_{0,0}$, formed by symmetric curves that remain invariant under translation by points of $\langle a_3, a_4\rangle \subset T$.

Since x_0 is odd, the only 2-torsion point of $\langle a_1, a_2\rangle \cong (\mathbb{Z}/x_0)^2$ is the origin of X; accordingly, points of $\langle a_1, a_2\rangle$ impose at most $\frac{x_0^2-1}{2} + 1$ independent conditions on $\mathcal{L}^1_{0,0}$. It is thus possible to find a curve of $\mathcal{L}^1_{0,0}$ containing all the points of $\langle a_1, a_2\rangle \subset T$, which finishes the proof. □

Corollary III.3.2. *Let (X, L) be a $(1, d)$-polarized abelian surface satisfying the hypothesis of Theorem III.2.2, where d is not a perfect square.*

Then the expression for $h^0_{\mathcal{I}_0,L}$ is the one corresponding to either the minimal solution (x_0, y_0) or to the second smallest positive solution (x_1, y_1), providing that $\mathrm{char}(\mathbb{K})$ divides neither x_0^2 nor $x_0^2 - 1$. In particular, $\epsilon_1(L) \in \{\frac{2y_0}{x_0-1}, \frac{2y_1}{x_1-1}\}$.

Proof. Proposition III.3.1 shows (via Serre duality and cohomology and base change) that the sheaf $R^2\Phi_{\mathcal{P}^\vee}((\mu^*_{x_0}\mathcal{I}_0 \otimes L^{2x_0y_0})^\vee)$ is nonzero. In virtue of the explicit expression for $h^0_{\mathcal{I}_0,L}$ given in Theorem I.2.2.(1), this implies that $h^0_{\mathcal{I}_0,L}(x)$ is positive for $x > \frac{2y_0}{x_0}$.

On the other hand, since $x_1 = x_0^2 + 4dy_0^2$ and $y_1 = 2x_0y_0$, the equality $\frac{2y_0}{x_0} = \frac{2y_1}{x_1+1}$ holds.

Therefore, by Theorem III.2.2.(2) we conclude that only two expressions for $h^0_{\mathcal{I}_0,L}$ are possible (those corresponding to the solutions (x_0, y_0) and (x_1, y_1)). In particular, $\epsilon_1(L) \in \{\frac{2y_0}{x_0-1}, \frac{2y_1}{x_1-1}\}$. □

Remark III.3.3.

(1) It follows, at least when $\mathrm{char}(\mathbb{K}) = 0$, that $\epsilon_1(L)$ is rational under the assumption of Theorem III.2.2. It would be interesting to know whether this holds true for every polarized abelian surface (or more generally, for every polarized abelian variety).

(2) There are several examples of non-perfect squares d where $\epsilon_1(L)$ is known for a general $(1,d)$-polarized (complex) abelian surface (X, L) (see [Ito20b, Example 5.11] and Table III.1). For all of them there is an equality $\epsilon_1(L) = \frac{2y_0}{x_0-1}$, so it seems reasonable to expect this for every non-perfect square d.

Assume the equality $\epsilon_1(L) = \frac{2y_1}{x_1-1}$ holds. According to the expression for $h^0_{\mathcal{I}_0,L}$ given by Theorem III.2.2, for every $x > \frac{2y_1}{x_1+1}$ small enough we have

$$h^0_{\mathcal{I}_0,L}(x) = \frac{d(x_1+1)}{2}x^2 - 2dy_1 \cdot x + \frac{x_1-1}{2} = dx_0^2\left(x - \frac{2y_0}{x_0}\right)^2.$$

Then, an elementary manipulation of the expression given in Theorem I.2.2.(1) shows that $R^2\Phi_{\mathcal{P}^\vee}((\mu_{x_0}^*\mathcal{I}_0 \otimes L^{2x_0y_0})^\vee)$ is a 0-dimensional sheaf of length x_0^2.

But note that Proposition III.3.1 precisely shows that, if $R^2\Phi_{\mathcal{P}^\vee}((\mu_{x_0}^*\mathcal{I}_0 \otimes L^{2x_0y_0})^\vee)$ is 0-dimensional, then it has length $\geq x_0^2$. Hence a slightly stronger version of Proposition III.3.1 (with x_0^2+1 independent curves on $|L^{2x_0y_0}|$, or with a curve in a translated linear system $|L^{2x_0y_0} \otimes \alpha|$ containing also T) would yield a contradiction, leading to a proof of $\epsilon_1(L) = \frac{2y_0}{x_0-1}$.

Prym semicanonical pencils

Chapter IV

Further preliminaries

This chapter establishes the preliminaries for our study of Prym semicanonical pencils. Throughout this second part of the thesis, we will work over the field of the complex numbers.

By a *very general* point in a variety we will mean a point lying outside a union of countably many nontrivial Zariski-closed subsets. Given an abelian variety A, A_2 will denote its subgroup of 2-torsion points. The *genus* of a curve will refer to its arithmetic genus.

IV.1 The moduli spaces $\overline{\mathcal{R}}_g$ and $\widetilde{\mathcal{R}}_g$

In this section we give a brief review of Beauville's partial compactification $\widetilde{\mathcal{R}}_g$ by admissible covers, the Deligne-Mumford compactification $\overline{\mathcal{R}}_g$ and its boundary divisors. We essentially follow the presentation of [FL10, Section 1], where the reader is referred for further details.

Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g, and let $\overline{\mathcal{M}}_g$ be its Deligne-Mumford compactification by stable curves. Following the standard notations, we denote by Δ_i ($i = 0, \dots, \lfloor g/2 \rfloor$) the irreducible divisors forming the boundary $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$, that is:

- The general point of Δ_0 is an irreducible curve with a single node.
- The general point of Δ_i (for $i \geq 1$) is the union of two smooth curves of genus i and $g-i$, which intersect transversely at a point.

The classes δ_i of the divisors Δ_i, together with the Hodge class λ, are well known to form a basis of the rational Picard group $\mathrm{Pic}(\overline{\mathcal{M}}_g)_{\mathbb{Q}}$ (see for instance [AC87, Section 2]).

We will denote by $\mathcal{R}_g$ the moduli space of double étale irreducible covers of smooth curves of genus g. In other words, $\mathcal{R}_g$ parametrizes isomorphism classes of pairs (C, η), where C is smooth of genus g and $\eta \in JC_2 \setminus \{\mathcal{O}_C\}$. It comes with a natural forgetful map $\pi : \mathcal{R}_g \to \mathcal{M}_g$ which is étale of degree $2^{2g} - 1$. Then, the Deligne-Mumford compactification $\overline{\mathcal{R}}_g$ is obtained as the normalization of $\overline{\mathcal{M}}_g$

in the function field of $\mathcal{R}_g$. This gives a commutative diagram

$$\begin{array}{ccc} \mathcal{R}_g & \longrightarrow & \overline{\mathcal{R}}_g \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ \mathcal{M}_g & \longrightarrow & \overline{\mathcal{M}}_g \end{array}$$

where $\overline{\mathcal{R}}_g$ is normal and the morphism $\overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$ (that we will denote by π as well) is finite. The variety $\overline{\mathcal{R}}_g$ parametrizes isomorphism classes of *Prym curves* of genus g:

Definition IV.1.1. A *Prym curve* of genus g is a triple (X, η, β), where:

(1) X is a quasi-stable curve of genus g, i.e. X is semistable and any two of its exceptional components are disjoint[1].

(2) $\eta \in \mathrm{Pic}^0(X)$ is a nontrivial line bundle of total degree 0, such that $\eta\big|_E = \mathcal{O}_E(1)$ for every exceptional component $E \subset X$.

(3) $\beta : \eta^{\otimes 2} \to \mathcal{O}_X$ is generically nonzero over each non-exceptional component of X.

Then the morphism $\pi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$ sends (the class of) (X, η, β) to (the class of) the *stable model* $\mathrm{st}(X)$, obtained by contraction of the exceptional components of X.

Remark IV.1.2. In case that β is clear from the context, by abuse of notation the Prym curve (X, η, β) will be often denoted simply by (X, η).

The moduli space $\overline{\mathcal{R}}_g$ contains the partial compactification of $\mathcal{R}_g$ by admissible covers, which was introduced by Beauville ([Bea77a]) in order to compactify the Prym map (see section IV.3).

Definition IV.1.3. Let $\widetilde{C}$ be a stable curve of genus $2g-1$ endowed with an involution $\tau : \widetilde{C} \to \widetilde{C}$. We say that $\widetilde{C} \to C := \widetilde{C}/\langle\tau\rangle$ is an *admissible cover* if the following conditions are fulfilled:

(1) All the fixed points of τ are singular points of $\widetilde{C}$, where the branches are not exchanged.

(2) The number of nodes exchanged under τ equals the number of irreducible components exchanged under τ.

Under these assumptions, the curve C is stable of genus g. We will denote by $\widetilde{\mathcal{R}}_g$ the moduli space of admissible covers, which admits a natural inclusion into $\overline{\mathcal{R}}_g$.

Example IV.1.4.

(1) (*Wirtinger covers*) Let X be a smooth curve of genus $g-1$, and let $p, q \in X$ be two distinct points. We consider the cover of the nodal curve $C := X_{p\sim q}$ constructed as

$$\widetilde{C} := (X_1 \sqcup X_2)_{p_1\sim q_2, p_2\sim q_1}$$

[1] Recall that a smooth rational component $E \subset X$ is called *exceptional* if $\#E \cap \overline{X \setminus E} = 2$, namely if it intersects the rest of the curve in exactly two points.

where X_1, X_2 are isomorphic copies of X. This cover is admissible, since the corresponding involution on $\widetilde{C}$ has no fixed point, exchanges the two irreducible components of $\widetilde{C}$ and also exchanges its two nodes.

(2) (*Non-admissible covers*) Let $(\widetilde{X}, X) \in \mathcal{R}_{g-1}$ a double étale cover of a smooth curve X of genus $g-1$. For two distinct points $p, q \in X$ we consider p_i, q_i $(i = 1, 2)$ their preimages in $\widetilde{X}$. Then the curves

$$\widetilde{C} := \widetilde{X}_{p_1 \sim q_1, p_2 \sim q_2}, \quad C := X_{p \sim q}$$

define a cover which is not admissible, since the curve $\widetilde{C}$ is irreducible and the corresponding involution exchanges the two nodes.

(3) (*Beauville covers*) Let $\widetilde{X} \to X$ be a double cover of a smooth curve X of genus $g-1$, branched at two points $p, q \in X$. If $\tilde{p}, \tilde{q} \in \widetilde{X}$ are the ramification points, then

$$\widetilde{C} := \widetilde{X}_{\tilde{p} \sim \tilde{q}}, \quad C := X_{p \sim q}$$

define an admissible cover. Indeed $\widetilde{C}$ is irreducible, its node is the unique fixed point of the involution and branches are not exchanged there.

Using pullbacks of the boundary divisors of $\overline{\mathcal{M}}_g$, the boundary $\overline{\mathcal{R}}_g \setminus \mathcal{R}_g$ of the Deligne-Mumford compactification admits the following description (see [FL10, Examples 1.3 and 1.4]):

(1) Let (X, η, β) be a Prym curve, such that $\mathrm{st}(X)$ is the union of two smooth curves C_i and C_{g-i} (of respective genus i and $g-i$) intersecting transversely at a point P. In such a case $X = \mathrm{st}(X)$, and giving a 2-torsion line bundle $\eta \in \mathrm{Pic}^0(X)_2 \setminus \{\mathcal{O}_X\}$ is the same as giving a pair $(\eta_i, \eta_{g-i}) \in (JC_i)_2 \times (JC_{g-i})_2$ with $\eta_i \neq \mathcal{O}_{C_i}$ or $\eta_{g-i} \neq \mathcal{O}_{C_{g-i}}$.

   Then the preimage $\pi^{-1}(\Delta_i) \subset \overline{\mathcal{R}}_g$ decomposes as the union of three irreducible divisors (denoted by Δ_i, Δ_{g-i} and $\Delta_{i:g-i}$), which are distinguished by the behaviour of the 2-torsion bundle. More concretely, their general point is a Prym curve (X, η), where $X = C_i \cup_P C_{g-i}$ is a reducible curve as above and the pair $\eta = (\eta_i, \eta_{g-i})$ satisfies:

   - $\eta_{g-i} = \mathcal{O}_{C_{g-i}}$, in the case of Δ_i.
   - $\eta_i = \mathcal{O}_{C_i}$, in the case of Δ_{g-i}.
   - $\eta_i \neq \mathcal{O}_{C_i}$ and $\eta_{g-i} \neq \mathcal{O}_{C_{g-i}}$, in the case of $\Delta_{i:g-i}$.

(2) Let (X, η, β) be a Prym curve, such that $\mathrm{st}(X) = C_{p \sim q}$ is the irreducible nodal curve obtained by identification of two points p, q on a smooth curve C of genus $g-1$.

   If $X = \mathrm{st}(X)$ and $\nu : C \to X$ denotes the normalization, then the short exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathrm{Pic}^0(X) \xrightarrow{\nu^*} JC \longrightarrow 0$$

   tells us that $\eta \in \mathrm{Pic}^0(X)_2$ is determined by the choice of $\eta_C = \nu^*(\eta) \in JC_2$ and an identification of the fibers $\eta_C(p)$ and $\eta_C(q)$.

- If $\eta_C = \mathcal{O}_C$, there is only one possible identification of $\mathcal{O}_C(p)$ and $\mathcal{O}_C(q)$ giving a nontrivial $\eta \in \mathrm{Pic}^0(X)_2$, which is identification by -1. In other words, we have

$$\Gamma(U,\eta) = \{f \in \Gamma(\nu^{-1}(U), \mathcal{O}_C) \mid f(p) = -f(q)\}$$

  for every open subset $U \subset X$. The corresponding element (X,η) is a Wirtinger cover of X.
- If $\eta_C \neq \mathcal{O}_C$, for each of the $2^{2g-2}-1$ choices of η_C there are two possible identifications of $\mathcal{O}_C(p)$ and $\mathcal{O}_C(q)$. The $2(2^{2g-2}-1)$ corresponding Prym curves (X,η) are non-admissible covers of X.

If $X \neq \mathrm{st}(X)$, then X is the union of C with an exceptional component E passing through p and q. The line bundle $\eta \in \mathrm{Pic}^0(X)$ must satisfy $\eta\big|_E = \mathcal{O}_E(1)$ and $\eta\big|_C^{\otimes 2} = \mathcal{O}_C(-p-q)$, which gives 2^{2g-2} possibilities. The corresponding Prym curves (X,η) give Beauville covers of $\mathrm{st}(X)$.

It follows that $\pi^{-1}(\Delta_0) = \Delta_0' \cup \Delta_0'' \cup \Delta_0^{ram}$, where Δ_0' (resp. Δ_0'', resp. Δ_0^{ram}) is an irreducible divisor whose general point is a non-admissible (resp. Wirtinger, resp. Beauville) cover. Moreover, Δ_0^{ram} is the ramification locus of π.

In terms of rational divisor classes, we have the equalities

$$\pi^*(\delta_i) = \delta_i + \delta_{g-i} + \delta_{i:g-i} \ (1 \leq i \leq \lfloor g/2 \rfloor), \qquad \pi^*(\delta_0) = \delta_0' + \delta_0'' + 2\delta_0^{ram}$$

where $\delta_i, \delta_{g-i}, \delta_{i:g-i}$ and $\delta_0', \delta_0'', \delta_0^{ram}$ are the classes of the boundary divisors of $\overline{\mathcal{R}}_g$. These boundary classes, together with the pullback (also denoted by λ) of the Hodge class of $\overline{\mathcal{M}}_g$, form a basis of the rational Picard group $\mathrm{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}$ for $g \geq 5$ (see [MP21, Remark 2.1.23] for an explanation).

Remark IV.1.5. By abuse of terminology, the class of a divisor in $\mathrm{Pic}(\overline{\mathcal{M}}_g)_{\mathbb{Q}}$ or $\mathrm{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}$ will often be called its *cohomology class.*

IV.2 Divisors of Prym semicanonical pencils

If C is a smooth curve of genus g, by a *semicanonical pencil* on C we mean an even, effective theta-characteristic; in the literature, this is also frequently referred to as a *vanishing theta-null.* By *dimension* of a theta-characteristic L we mean the (projective) dimension $h^0(C,L)-1$ of the linear system $|L|$.

The locus of smooth curves admitting a semicanonical pencil is known to be a divisor $\mathcal{T}_g \subset \mathcal{M}_g$, whose general element C admits a unique semicanonical pencil L and satisfies $h^0(C,L) = 2$ (see [TiB87, Theorem 2.17]). The irreducibility of $\mathcal{T}_g$ was proved in [TiB88, Theorem 2.4]; in the same paper, the cohomology class of its closure in $\overline{\mathcal{M}}_g$ was also computed.

Since the parity of theta-characteristics remains constant in families ([Mum71]), the pullback of $\mathcal{T}_g$ by the forgetful map $\pi : \mathcal{R}_g \to \mathcal{M}_g$ decomposes as $\pi^{-1}(\mathcal{T}_g) = \mathcal{T}_g^e \cup \mathcal{T}_g^o$, where

$$\mathcal{T}_g^e = \big\{(C,\eta) \in \mathcal{R}_g \mid C \text{ has a semicanonical pencil } L \text{ with } h^0(C, L \otimes \eta) \text{ even}\big\}$$
$$\mathcal{T}_g^o = \big\{(C,\eta) \in \mathcal{R}_g \mid C \text{ has a semicanonical pencil } L \text{ with } h^0(C, L \otimes \eta) \text{ odd}\big\}$$

Note that both $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$ have pure codimension 1 in $\mathcal{R}_g$, since their union is the pullback by a finite map of an irreducible divisor. Furthermore, the restriction

$$\pi\big|_{\mathcal{T}_g^e} : \mathcal{T}_g^e \longrightarrow \mathcal{T}_g \quad (\text{resp. } \pi\big|_{\mathcal{T}_g^o} : \mathcal{T}_g^o \longrightarrow \mathcal{T}_g)$$

is surjective and generically finite of degree $2^{g-1}(2^g+1)-1$ (resp. of degree $2^{g-1}(2^g-1)$). This follows from the fact that a general element of $\mathcal{T}_g$ has a unique semicanonical pencil, as well as from the number of even and odd theta-characteristics on a smooth curve.

Remark IV.2.1. Abusing of notation, we will use interchangeably the notation $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$ for the divisors in $\mathcal{R}_g$ and also for their closures in the Deligne-Mumford compactification $\overline{\mathcal{R}}_g$ or in the partial compactification $\widetilde{\mathcal{R}}_g$ by admissible covers. Similarly, $\mathcal{T}_g$ will also denote the closure of this divisor in $\overline{\mathcal{M}}_g$.

Even if it will be clear from the context what of the notions is being considered at every moment, for the sake of clarity we point out that:

- Along chapter V, $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$ (resp. $\mathcal{T}_g$) will denote the closures in $\overline{\mathcal{R}}_g$ (resp. $\overline{\mathcal{M}}_g$).
- Along chapter VI, $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$ are in general considered in $\mathcal{R}_g$ for $g \neq 5$, whereas for $g=5$ (section VI.4) we will work in $\widetilde{\mathcal{R}}_5$. The only exception to this rule is the use of cohomological arguments, for which we take closures in $\overline{\mathcal{R}}_g$.

Example IV.2.2. When $g=3$, a semicanonical pencil is the same as a g_2^1, and thus the divisor $\mathcal{T}_3 \subset \mathcal{M}_3$ equals the hyperelliptic locus $\mathcal{H}_3$. Of course, the semicanonical pencil on every smooth curve $C \in \mathcal{T}_3$ is unique. The 63 non-trivial elements of JC_2 can be represented by linear combinations of the Weierstrass points $R_1, \dots, R_8$ as follows:

- Those represented as a difference of two Weierstrass points, $\eta = \mathcal{O}_C(R_i - R_j)$, form a set of $\binom{8}{2} = 28$ elements. Observe that in this case the theta-characteristic $g_2^1 \otimes \eta = \mathcal{O}_C(2R_j + R_i - R_j) = \mathcal{O}_C(R_i + R_j)$ is odd.
- Those expressed as a linear combination of four distinct Weierstrass points, $\eta = \mathcal{O}_C(R_i + R_j - R_k - R_l)$, form a set of $\frac{\binom{8}{4}}{2} = 35$ elements[2]. According to the number of odd and even theta-characteristics on a genus 3 curve, in this case $g_2^1 \otimes \eta$ is even.

Hence we obtain

$$\mathcal{T}_3^o = \{(C,\eta) \in \mathcal{R}_3 \mid C \text{ hyperelliptic}, \, \eta = \mathcal{O}_C(R_i - R_j)\}$$
$$\mathcal{T}_3^e = \{(C,\eta) \in \mathcal{R}_3 \mid C \text{ hyperelliptic}, \, \eta = \mathcal{O}_C(R_i + R_j - R_k - R_l)\}$$

and, since monodromy on hyperelliptic curves acts transitively on Weierstrass points, it turns out that both divisors $\mathcal{T}_3^o$ and $\mathcal{T}_3^e$ are irreducible.

[2]Division by 2 comes from the fact that any two complementary sets of four Weierstrass points induce the same two-torsion line bundle.

IV.3 Prym varieties

The Prym map

We recall some basic facts on Prym varieties (most of them coming from the seminal work [Mum74]) and the Prym map.

Given a smooth Prym curve $(C,\eta) \in \mathcal{R}_g$, denote by $f : \widetilde{C} \longrightarrow C$ the étale, smooth, irreducible double cover associated to η. The kernel of the norm map $\mathrm{Nm}_f : J\widetilde{C} \longrightarrow JC$ breaks into two connected components; the *Prym variety* $P = P(C,\eta)$ of (C,η) is the component containing the origin of $J\widetilde{C}$.

Equivalently, if $\sigma : \widetilde{C} \to \widetilde{C}$ denotes the involution exchanging sheets of f (as well as the induced involution on $J\widetilde{C}$), one may define $P := \mathrm{Im}(\mathrm{Id} - \sigma) = \ker(\mathrm{Id} + \sigma)^0$.

The following properties will be used several times in chapter VI:

Lemma IV.3.1 ([Mum74, Section 3])**.**

(1) *The 2-torsion subgroup P_2 of the Prym variety fits into a short exact sequence*

$$0 \longrightarrow \langle\eta\rangle \longrightarrow \langle\eta\rangle^{\perp} \xrightarrow{f^*} P_2 \longrightarrow 0\ ,$$

where $\langle\eta\rangle^{\perp} \subset JC_2$ denotes the orthogonal of $\langle\eta\rangle$ with respect to the Weil pairing on JC_2.

(2) *The equality $f^*(JC) = \ker(Id - \sigma)$ holds.*

Furthermore, the principal polarization on $J\widetilde{C}$ restricts to twice a principal polarization on P, giving rise to the so-called *Prym map*

$$\mathcal{P}_g : \mathcal{R}_g \longrightarrow \mathcal{A}_{g-1}.$$

This map was extended by Beauville ([Bea77a]) to a proper map

$$\widetilde{\mathcal{P}}_g : \widetilde{\mathcal{R}}_g \longrightarrow \mathcal{A}_{g-1}$$

by considering Prym varieties of admissible covers. The idea behind this construction is that $\mathcal{P}_g$ is extended to a rational map

$$\overline{\mathcal{P}}_g : \overline{\mathcal{R}}_g \dashrightarrow \overline{\mathcal{A}}_{g-1}$$

(where $\overline{\mathcal{A}}_{g-1}$ is a toroidal compactification of $\mathcal{A}_{g-1}$), and then the moduli space of admissible covers $\widetilde{\mathcal{R}}_g \subset \overline{\mathcal{R}}_g$ is the open subset of covers whose Prym variety lies in $\mathcal{A}_{g-1}$.

It has been known since work of Friedman and Smith ([FS86]) that $\overline{\mathcal{P}}_g$ does not extend to a morphism defined on the whole $\overline{\mathcal{R}}_g$, for any toroidal compactification $\overline{\mathcal{A}}_{g-1}$ (see for instance [CMGHL17] for recent progress in the study of the indeterminacy locus). In any case, $\overline{\mathcal{P}}_g$ is a morphism on the open subset of $\overline{\mathcal{R}}_g$ which lies over the locus in $\overline{\mathcal{M}}_g$ of stable curves with at most one node; furthermore, this open subset is mapped to the semiabelian varieties of torus rank ≤ 1. This information will be more than enough for our purposes.

Prym varieties and the Andreotti-Mayer locus

As before, let $(C, \eta) \in \mathcal{R}_g$ be a smooth Prym curve with associated double étale cover $f : \widetilde{C} \longrightarrow C$. Mumford ([Mum74, Section 6]) described the singularities of a theta divisor Ξ representing the principal polarization of the Prym variety as follows. Let

$$P^+ = \{M \in \mathrm{Pic}^{2g-2}(\widetilde{C}) \mid \mathrm{Nm}_f(M) = \omega_C \text{ and } h^0(\widetilde{C}, M) \text{ is even}\}$$

be a "canonical presentation" of the Prym variety P in $\mathrm{Pic}^{2g-2}(\widetilde{C}) = \mathrm{Pic}^{g(\widetilde{C})-1}(\widetilde{C})$, and let $\Theta_{\widetilde{C}} \subset \mathrm{Pic}^{2g-2}(\widetilde{C})$ denote the canonical theta divisor of $J\widetilde{C}$. Then $P^+ \cdot \Theta_{\widetilde{C}} = 2\Xi^+$, where

$$\Xi^+ = \{M \in P^+ \mid h^0(\widetilde{C}, M) \geq 2\}$$

is a canonical presentation of the theta divisor of P, and singularities of Ξ^+ may arise in two different situations:

- Points of P^+ which have a high multiplicity in $\Theta_{\widetilde{C}}$, namely $M \in \Xi^+$ with $h^0(\widetilde{C}, M) \geq 4$. Such singularities are usually called *stable*.
- Points M at which P^+ and $\Theta_{\widetilde{C}}$ are tangent, namely the tangent space $T_M(P^+)$ is contained in the tangent cone $TC_M(\Theta_{\widetilde{C}})$. Such singularities are of the form $M = f^*L \otimes A \in \Xi^+$, with $h^0(C, L) \geq 2$ and $h^0(\widetilde{C}, A) > 0$. They are usually called *exceptional*: indeed, they do not occur for $(C, \eta) \in \mathcal{R}_g$ general.

An elementary example of exceptional singularity is given by f^*L, where L is a semicanonical pencil on C such that $h^0(\widetilde{C}, f^*L) = h^0(C, L) + h^0(C, L \otimes \eta)$ is even (i.e. $h^0(C, L \otimes \eta)$ is even). Therefore $\mathcal{P}_g(\mathcal{T}_g^e) \subset \mathcal{N}_0$, where $\mathcal{N}_0 \subset \mathcal{A}_{g-1}$ denotes the *Andreotti-Mayer locus* of principally polarized abelian varieties (*ppav*'s in the sequel) whose theta divisor has singularities.

More precisely one has $\mathcal{P}_g(\mathcal{T}_g^e) \subset \theta_{null}$, where $\theta_{null} \subset \mathcal{A}_{g-1}$ is the divisor of ppav's whose (symmetric) theta divisor contains a singular 2-torsion point. This follows from the fact that the symmetric models of the theta divisor in $P \subset \mathrm{Pic}^0(\widetilde{C})$ are obtained when (P^+, Ξ^+) is translated by a theta-characteristic lying in P^+; in particular, the 2-torsion points of P in the canonical model P^+ are the theta-characteristics of $\widetilde{C}$ lying in P^+.

Note that $\mathcal{N}_0 = \theta_{null} = \mathcal{A}_1 \times \mathcal{A}_1$ in $\mathcal{A}_2$, and $\mathcal{N}_0 = \theta_{null} \subset \mathcal{A}_3$ is the divisor of hyperelliptic Jacobians. For $g \geq 5$, the Andreotti-Mayer locus of $\mathcal{A}_{g-1}$ is the union of two irreducible divisors ([Mum83, Deb92]):

$$\mathcal{N}_0 = \theta_{null} \cup \mathcal{N}_0'.$$

Whereas the theta divisor of the general element of θ_{null} has a unique singular point (which is 2-torsion), the theta divisor of a general element of $\mathcal{N}_0'$ has exactly two singular (opposite) points. Using this fact, Mumford computed the multiplicity of each component (see [Mum83]), proving the following equality as cycles: $\mathcal{N}_0 = \theta_{null} + 2\mathcal{N}_0'$.

For $g=5$ this was already proved in [Bea77a], where Beauville used the surjective, proper map $\widetilde{\mathcal{P}}_5 : \widetilde{\mathcal{R}}_5 \longrightarrow \mathcal{A}_4$ to show that $\mathcal{N}_0 \subset \mathcal{A}_4$ has two irreducible components: the Jacobian locus and the divisor θ_{null}.

Brill-Noether loci on Prym varieties

In contrast to the even case, an odd semicanonical pencil L for a smooth Prym curve (C,η) does not provide singularities in the canonical theta divisor Ξ^+, since the pullback f^*L lands in the other component of $\mathrm{Nm}_f^{-1}(\omega_C)$:

$$P^- = \left\{ M \in \mathrm{Pic}^{2g-2}(\widetilde{C}) \mid \mathrm{Nm}_f(M) = \omega_C \text{ and } h^0(\widetilde{C},M) \text{ is odd} \right\}.$$

To understand the situation, following Welters [Wel85] we consider the *Brill-Noether-Prym loci*

$$V^r(C,\eta) := \left\{ M \in \mathrm{Pic}^{2g-2}(\widetilde{C}) \mid \mathrm{Nm}_f(M) = \omega_C,\ h^0(\widetilde{C},M) \geq r+1,\ h^0(\widetilde{C},M) \equiv r+1 (\mathrm{mod}\ 2) \right\}$$

with the scheme structure defined by $P^+ \cap W^r_{2g-2}(\widetilde{C})$ (r odd) or $P^- \cap W^r_{2g-2}(\widetilde{C})$ (r even).

Example IV.3.2. The first cases of Brill-Noether-Prym loci are:

(1) $V^{-1}(C,\eta) = P^+$ and $V^0(C,\eta) = P^-$.

(2) $V^1(C,\eta) = \Xi^+$ and $V^3(C,\eta) \subset \Xi^+$ is the locus of stable singularities.

(3) Assume that C is not hyperelliptic and fix a point $x_0 \in \widetilde{C}$. Then the *Abel-Prym map*

$$j_{x_0} : \widetilde{C} \longrightarrow P, \quad x \mapsto \mathcal{O}_{\widetilde{C}}(x - x_0 - \sigma(x) - \sigma(x_0))$$

is an embedding ([BL04, Corollary 12.5.6]), and for $g \geq 4$ the scheme-theoretic equality $V^2(C,\eta) = T(\widetilde{C})$ holds ([LN13, Theorem A]). Here the *theta-dual* $T(\widetilde{C})$, equipped with an appropriate scheme structure, parametrizes the translates of $\widetilde{C} \subset P$ contained in the theta divisor (see [PP08, Section 4]).

Note that the embedding of $\widetilde{C}$ in P is not canonical, and moreover $T(\widetilde{C}) \subset P$ and $V^2(C,\eta) \subset P^-$ lie in different models of P. Hence the equality $V^2(C,\eta) = T(\widetilde{C})$ must be understood up to translation.

Observe that for $(C,\eta) \in \mathcal{T}_g^o$ with an odd semicanonical pencil L, we have $f^*L \in V^2(C,\eta)$. Moreover, f^*L is a singular point of $V^2(C,\eta)$ thanks to the following result, which is essentially an application of [Hoe12, Lemma 3.1]:

Lemma IV.3.3. *Let $(C,\eta) \in \mathcal{R}_g$ be a non-hyperelliptic Prym curve of genus $g \geq 5$. If $M \in V^2(C,\eta) \setminus V^4(C,\eta)$, then M is a singular point of $V^2(C,\eta)$ if and only if $M = f^*L \otimes A$, for line bundles L and A satisfying $h^0(C,L) \geq 2$ and $h^0(\widetilde{C},A) > 0$.*

Proof. Since $V^2(C,\eta)$ has pure dimension $g-4$ (see [LN13, Lemma 4.1]), the "only if" part is exactly the statement of [Hoe12, Lemma 3.1] for $r=2$. Following that proof the converse statement is obtained as well, if one uses that every element of $\bigwedge^2 H^0(\widetilde{C},M)$ is decomposable by the assumption $h^0(\widetilde{C},M) = 3$. □

Trigonal and tetragonal constructions

We close this chapter of preliminaries with a quick review of the trigonal and the tetragonal construction, which are instances of the *polygonal constructions* detailed by Donagi in [Don92, Section 2]. We will restrict ourselves to the case of double étale covers of smooth curves, even if both constructions can be applied under more general assumptions.

The *trigonal construction* was studied by Recillas ([Rec74]). It associates to every tower

$$\widetilde{C} \xrightarrow{f} C \xrightarrow{3:1} \mathbb{P}^1$$

(where f is a double étale cover of a genus g smooth curve C) a smooth curve X of genus $g-1$ with a morphism $X \xrightarrow{4:1} \mathbb{P}^1$, such that $P(\widetilde{C}, C) \cong JX$ as ppav's. More precisely, one has

$$\begin{array}{ccc} (f^{(3)})^{-1}(\mathbb{P}^1) = X \sqcup X & \longrightarrow & \widetilde{C}^{(3)} \\ \downarrow & & \downarrow f^{(3)} \\ \mathbb{P}^1 = g^1_3 & \longrightarrow & C^{(3)} \end{array}$$

where the two connected components of $(f^{(3)})^{-1}(\mathbb{P}^1)$ are exchanged by the natural involution $\sigma^{(3)}$ of $\widetilde{C}^{(3)}$ (induced by the involution σ of $\widetilde{C}$ associated to the double cover f).

Assuming that the g^1_4 on X contains no divisor of the form $2p + 2q$ or $4p$, this construction can be reversed; we will deal explicitly with this inverse construction in section VI.3.

The *tetragonal construction* is due to Donagi ([Don81]), and associates to every tower

$$\widetilde{C} \xrightarrow{f} C \xrightarrow{4:1} \mathbb{P}^1$$

two further towers $\widetilde{C}_i \xrightarrow{f_i} C_i \xrightarrow{4:1} \mathbb{P}^1$ $(i = 0, 1)$ with the property $P(\widetilde{C}, C) \cong P(\widetilde{C}_0, C_0) \cong P(\widetilde{C}_1, C_1)$ as ppav's. This allowed Donagi to prove the non-injectivity of the Prym map in every genus; he actually conjectured that all the non-injectivity of the Prym map comes from the tetragonal construction, but this was disproved by Izadi and Lange ([IL12]).

The construction is obtained from the diagram

$$\begin{array}{ccc} (f^{(4)})^{-1}(\mathbb{P}^1) = \widetilde{C}_0 \sqcup \widetilde{C}_1 & \longrightarrow & \widetilde{C}^{(4)} \\ \downarrow & & \\ C_0 \sqcup C_1 & & \downarrow f^{(4)} \\ \downarrow & & \\ \mathbb{P}^1 = g^1_4 & \longrightarrow & C^{(4)} \end{array}$$

where the two covers $\widetilde{C}_i \longrightarrow C_i$ $(i = 0, 1)$ are defined by the involution $\sigma^{(4)}$ of $\widetilde{C}^{(4)}$ (which leaves invariant $\widetilde{C}_0$ and $\widetilde{C}_1$). This construction is a triality: for instance, when applied to $\widetilde{C}_0 \xrightarrow{f_0} C_0 \longrightarrow \mathbb{P}^1$ it returns the towers $\widetilde{C} \xrightarrow{f} C \longrightarrow \mathbb{P}^1$ and $\widetilde{C}_1 \xrightarrow{f_1} C_1 \longrightarrow \mathbb{P}^1$.

Lemma IV.3.4. *There is a bijection between the following two sets of data:*

(1) *Triples* (R, L, W), *where* $R \in \mathcal{M}_{g+1}$ *is a trigonal curve,* L *is a* g^1_3 *on* R *and* $W = \{0, \mu_1, \mu_2, \mu_3\} \subset JR_2$ *is a totally isotropic subgroup with respect to the Weil pairing.*

(2) *A tetragonally related triple* (C_i, η_i, M_i) $(i = 1, 2, 3)$ *with* $(C_i, \eta_i) \in \mathcal{R}_g$ *and* M_i *a* g^1_4 *on* C_i.

For a proof of this result, the reader may consult [Don92, Lemma 5.5]. It establishes a relation between the trigonal and the tetragonal construction, since the bijection is explicitly given as follows:

- Every element (R, L, μ_i) corresponds to (C_i, M_i) under Recillas' trigonal construction.
- The 2-torsion point $\eta_i \in (JC_i)_2$ is defined by $\mu_j \in \langle \mu_i \rangle^{\perp} \subset JR_2$ for $j \neq i$ (recall Lemma IV.3.1.(1)).

Chapter V

Study of the divisors of Prym semicanonical pencils

In this chapter we focus on the following two aspects of the divisors of Prym semicanonical pencils: their cohomology classes in $\overline{\mathcal{R}}_g$ (section V.1) and their irreducibility (section V.2).

V.1 Cohomology classes

This section is entirely devoted to prove the following result:

Theorem V.1.1. *Let $g \geq 5$ and let $[\mathcal{T}_g^e], [\mathcal{T}_g^o] \in \mathrm{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}$ denote the cohomology classes in $\overline{\mathcal{R}}_g$ of (the closures of) the divisors $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$. Then, the following equalities hold:*

$$[\mathcal{T}_g^e] = a\lambda - b_0'\delta_0' - b_0''\delta_0'' - b_0^{ram}\delta_0^{ram} - \sum_{i=1}^{\lfloor g/2 \rfloor}(b_i\delta_i + b_{g-i}\delta_{g-i} + b_{i:g-i}\delta_{i:g-i}),$$

$$[\mathcal{T}_g^o] = c\lambda - d_0'\delta_0' - d_0''\delta_0'' - d_0^{ram}\delta_0^{ram} - \sum_{i=1}^{\lfloor g/2 \rfloor}(d_i\delta_i + d_{g-i}\delta_{g-i} + d_{i:g-i}\delta_{i:g-i}),$$

where

$$\begin{aligned}
a &= 2^{g-3}(2^{g-1}+1), & c &= 2^{2g-4},\\
b_0' &= 2^{2g-7}, & d_0' &= 2^{2g-7},\\
b_0'' &= 0, & d_0'' &= 2^{2g-6},\\
b_0^{ram} &= 2^{g-5}(2^{g-1}+1), & d_0^{ram} &= 2^{g-5}(2^{g-1}-1),\\
b_i &= 2^{g-3}(2^{g-i}-1)(2^{i-1}-1), & d_i &= 2^{g+i-4}(2^{g-i}-1),\\
b_{g-i} &= 2^{g-3}(2^{g-i-1}-1)(2^{i}-1), & d_{g-i} &= 2^{2g-i-4}(2^{i}-1),\\
b_{i:g-i} &= 2^{g-3}(2^{g-1}-2^{i-1}-2^{g-i-1}+1), & d_{i:g-i} &= 2^{g-3}(2^{g-1}-2^{g-i-1}-2^{i-1}).
\end{aligned}$$

First of all, observe that the pullback of the class $[\mathcal{T}_g] \in \mathrm{Pic}(\overline{\mathcal{M}}_g)_{\mathbb{Q}}$ (computed in [TiB88, Proposition 3.1]) expresses $[\mathcal{T}_g^e] + [\mathcal{T}_g^o]$ as

$$\pi^*[\mathcal{T}_g] = 2^{g-3}\left((2^g+1)\lambda - 2^{g-3}(\delta_0' + \delta_0'' + 2\delta_0^{ram}) - \sum_{i=1}^{\lfloor g/2 \rfloor}(2^{g-i}-1)(2^i-1)(\delta_i + \delta_{g-i} + \delta_{i:g-i})\right).$$

This relation, together with the linear independence of the basic classes considered in $\overline{\mathcal{R}}_g$, simplifies the computations: if we know a coefficient for one of the divisors, then we also know the coefficient corresponding to the same basic class for the other divisor. Keeping this in mind, the coefficients of Theorem V.1.1 can be determined by essentially following three steps:

(1) The pushforward $\pi_*[\mathcal{T}_g^e]$ easily gives the coefficient a (hence c), as well as a relation between b_0', b_0'' and b_0^{ram} (hence between d_0', d_0'' and d_0^{ram}).
(2) We adapt an argument of Teixidor [TiB88] to compute the coefficients b_i, b_{g-i} and $b_{i:g-i}$ for every $i \geq 1$: first we describe the intersection of $\mathcal{T}_g^e$ with the boundary divisors Δ_i, Δ_{g-i} and $\Delta_{i:g-i}$, and then we intersect $\mathcal{T}_g^e$ with certain test curves.
(3) Finally, d_0' and d_0'' are obtained intersecting $\mathcal{T}_g^o$ with test curves contained inside Δ_0' and Δ_0'' respectively. The relation obtained in (1) determines d_0^{ram} as well.

For step (1), note that on the one hand

$$\pi_*[\mathcal{T}_g^e] = \deg(\mathcal{T}_g^e \to \mathcal{T}_g)\cdot[\mathcal{T}_g] = (2^{g-1}(2^g+1)-1)2^{g-3}\left((2^g+1)\lambda - 2^{g-3}\delta_0 - \dots\right)$$

where $\dots$ is a expression involving only the classes $\delta_1, \dots, \delta_{\lfloor g/2 \rfloor}$. On the other hand

$$\pi_*[\mathcal{T}_g^e] = a\pi_*\lambda - b_0'\pi_*\delta_0' - b_0''\pi_*\delta_0'' - b_0^{ram}\pi_*\delta_0^{ram} - \sum_{i=1}^{\lfloor g/2 \rfloor}(b_i\pi_*\delta_i + b_{g-i}\pi_*\delta_{g-i} + b_{i:g-i}\pi_*\delta_{i:g-i})$$

and, since $\pi_*\lambda = \pi_*(\pi^*\lambda) = \deg\pi\cdot\lambda$ and the divisors Δ_0', Δ_0'' and Δ_0^{ram} of $\overline{\mathcal{R}}_g$ have respective degrees $2(2^{2g-2}-1), 1$ and 2^{2g-2} over $\Delta_0 \subset \overline{\mathcal{M}}_g$, we obtain

$$\pi_*[\mathcal{T}_g^e] = a(2^{2g}-1)\lambda - (2(2^{2g-2}-1)b_0' + b_0'' + 2^{2g-2}b_0^{ram})\delta_0 + \dots$$

where $\dots$ again denotes a linear combination of $\delta_1, \dots, \delta_{\lfloor g/2 \rfloor}$.

Using that $\lambda, \delta_0, \dots \delta_{\lfloor g/2 \rfloor} \in \mathrm{Pic}(\overline{\mathcal{M}}_g)_{\mathbb{Q}}$ are linearly independent, we can compare the coefficients of λ and δ_0. Comparison for λ yields

$$a = \frac{(2^{g-1}(2^g+1)-1)2^{g-3}(2^g+1)}{2^{2g}-1} = 2^{g-3}(2^{g-1}+1),$$

therefore $c = 2^{2g-4}$ due to the relation $a + c = 2^{g-3}(2^g+1)$.

Comparison for δ_0 gives

$$(2^{2g-1}-2)b_0' + b_0'' + 2^{2g-2}b_0^{ram} = 2^{2g-6}(2^{g-1}(2^g+1)-1),$$

or equivalently

$$(2^{2g-1}-2)d_0' + d_0'' + 2^{2g-2}d_0^{ram} = 2^{3g-7}(2^g-1).$$

In step (2), the key point is the following description of the intersection of $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$ with the preimages $\pi^{-1}(\Delta_i)$. It is nothing but an adaptation of [TiB88, Proposition 1.2]:

Proposition V.1.2. *For $i \geq 1$, the general point of the intersection $\mathcal{T}_g^e \cap \pi^{-1}(\Delta_i)$ (resp. $\mathcal{T}_g^o \cap \pi^{-1}(\Delta_i)$) is a pair (C, η) where:*

(1) *The curve C is the union at a point P of two smooth curves C_i and C_{g-i} of respective genera i and $g-i$, and satisfies one of these four conditions ($j = i, g-i$):*

   α_j) *C_j has a 1-dimensional (even) theta-characteristic L_j. In this case, the 1-dimensional limit theta-characteristics on C are determined by the aspects $|L_j| + (g-j)P$ on C_j and $|L_{g-j} + 2P| + (j-2)P$ on C_{g-j}, where L_{g-j} is any even theta-characteristic on C_{g-j}.*

   β_j) *P is in the support of an effective (0-dimensional) theta-characteristic L_j on C_j. The aspects of the 1-dimensional limit theta-characteristics on C are $|L_j + P| + (g-j-1)P$ on C_j and $|L_{g-j} + 2P| + (j-2)P$ on C_{g-j}, where L_{g-j} is any odd theta-characteristic on C_{g-j}.*

(2) *$\eta = (\eta_i, \eta_{g-i})$ is a non-trivial 2-torsion line bundle on C, such that the numbers $h^0(C_i, L_i \otimes \eta_i)$ and $h^0(C_{g-i}, L_{g-i} \otimes \eta_{g-i})$ have the same (resp. opposite) parity.*

Proof. First of all, note that item (1) describes the general element of the intersection $\mathcal{T}_g \cap \Delta_i$ in $\overline{\mathcal{M}}_g$: this is exactly [TiB88, Proposition 1.2].

Moreover, if $(C, \eta) \in \mathcal{T}_g^e \cap \pi^{-1}(\Delta_i)$ (resp. $(C, \eta) \in \mathcal{T}_g^o \cap \pi^{-1}(\Delta_i)$) is general, then there exists (a germ of) a 1-dimensional family $(\mathcal{C} \to S, H, \mathcal{L})$ of Prym curves $(\mathcal{C}_s, H_s)$ endowed with a 1-dimensional theta-characteristic $\mathcal{L}_s$, such that:

(1) For every $s \neq 0$, $(\mathcal{C}_s, H_s)$ is a smooth Prym curve such that $\mathcal{L}_s \otimes H_s$ is an even (resp. odd) theta-characteristic on $\mathcal{C}_s$.
(2) The family $(\mathcal{C} \to S, H)$ specializes to $(C, \eta) = (\mathcal{C}_0, H_0)$.

The possible aspects of the 1-dimensional limit series of $\mathcal{L}$ on $C = \mathcal{C}_0$ are described by item (1). Now the result follows from the fact that, on the one hand, the aspects of the limit series of $\mathcal{L} \otimes H$ on $C = \mathcal{C}_0$ are the same aspects as the limit of $\mathcal{L}$, but twisted by $\eta = H_0$; and on the other hand, the parity of a theta-characteristic on the reducible curve C is the product of the parities of the theta-characteristics induced on C_i and C_{g-i}, by Mayer-Vietoris. □

Remark V.1.3. Fixed a general element C of the intersection $\mathcal{T}_g \cap \Delta_i$ (i.e. a curve C satisfying the condition (1) above), the number of $\eta = (\eta_i, \eta_{g-i})$ such that $(C, \eta) \in \mathcal{T}_g^e$ can be easily computed.

Indeed, the number of η giving parities (even,even) is the product of the number of even theta-characteristics on C_i and the number of even theta-characteristics on C_{g-i}:

$$2^{i-1}(2^i+1)2^{g-i-1}(2^{g-i}+1) = 2^{g-2}(2^i+1)(2^{g-i}+1).$$

Similarly, the number of η giving parities (odd,odd) is

$$2^{i-1}(2^i-1)2^{g-i-1}(2^{g-i}-1) = 2^{g-2}(2^i+1)(2^{g-i}-1).$$

From all these, we have to discard the trivial bundle $(\mathcal{O}_{C_i}, \mathcal{O}_{C_{g-i}})$. Hence the number of η is

$$2^{g-2}(2^i+1)(2^{g-i}+1) + 2^{g-2}(2^i+1)(2^{g-i}-1) - 1 = 2^{g-1}(2^g+1) - 1,$$

which indeed coincides with the degree of $\mathcal{T}_g^e$ over $\mathcal{T}_g$. Of course the configuration of the fiber $\pi|_{\mathcal{T}_g^e}{}^{-1}(C)$ along the divisors Δ_i, Δ_{g-i} and $\Delta_{i:g-i}$ will depend on whether C satisfies α_j) or β_j).

Lemma V.1.4. *If C is a smooth curve of genus g and $\eta \in JC_2$ is a non-trivial 2-torsion line bundle, then there are exactly $2^{g-1}(2^{g-1}-1)$ odd theta-characteristics L on C such that $L \otimes \eta$ is also odd.*

Proof. This can be checked using the theory of syzygetic triads (see for instance [Dol12, Section 5.4.1]). Note that three odd theta-characteristics L, M, N form a syzygetic triad if and only if the theta-characteristic $L \otimes M \otimes N^{-1}$ is odd.

Fix an odd theta-characteristic N such that $M = N \otimes \eta$ is also odd (i.e. N is a fixed solution to our problem); then the required odd theta-characteristics L are those extending the pair M, N to a syzygetic triad. The number of such L (different from M and N) is

$$2(2^{g-1}+1)(2^{g-2}-1) = 2^{g-1}(2^{g-1}-1) - 2$$

(see [Dol12, Proposition 5.4.3]); to this number we add, of course, the two solutions M and N. □

Now, given an integer $i \geq 1$, we proceed to compute the coefficients b_i, b_{g-i} and $b_{i:g-i}$ of the cohomology class $[\mathcal{T}_g^e]$. We follow the argument in [TiB88, Proposition 3.1].

Fix two smooth curves C_i and C_{g-i} of respective genera i and $g-i$ having no theta-characteristic of positive dimension, as a well as a point $p \in C_i$ lying in the support of no effective theta-characteristic. We denote by F the curve (isomorphic to C_{g-i} itself) in $\Delta_i \subset \overline{\mathcal{M}}_g$, obtained by identifying p with a variable point $q \in C_{g-i}$. This curve has the following intersection numbers with the basic divisor classes of $\overline{\mathcal{M}}_g$:

$$F \cdot \lambda = 0,\ F \cdot \delta_j = 0 \text{ for } j \neq i,\ F \cdot \delta_i = -2(g-i-1)$$

(for a justification of these intersection numbers, see [HM82, page 81]).

Since the curve $F \subset \overline{\mathcal{M}}_g$ does not intersect the branch locus of the morphism π, it follows that the preimage $\pi^{-1}(F)$ has $2^{2g}-1$ connected components; each of them is isomporhic to F, and corresponds to the choice of a pair $\eta = (\eta_i, \eta_{g-i})$ of 2-torsion line bundles on C_i and C_{g-i} being not simultaneously trivial.

Let $\widetilde{F}_i$ be one of the components of $\pi^{-1}(F)$ contained in the divisor Δ_i of $\overline{\mathcal{R}}_g$; it is attached to an element $\eta = (\eta_i, \mathcal{O}_{C_{g-i}})$, for a fixed non-trivial $\eta_i \in (JC_i)_2$.

On the one hand, clearly δ_i is the only basic divisor class of $\overline{\mathcal{R}}_g$ that intersects $\widetilde{F}_i$. The projection formula then says that the number $\widetilde{F}_i \cdot \delta_i$ in $\overline{\mathcal{R}}_g$ equals the intersection $F \cdot \delta_i = -2(g-i-1)$ in $\overline{\mathcal{M}}_g$. Therefore,

$$\widetilde{F}_i \cdot [\mathcal{T}_g^e] = \widetilde{F}_i \cdot (a\lambda - b_0'\delta_0' - \dots) = 2(g-i-1)b_i.$$

On the other hand, according to Proposition V.1.2 an element $(C, \eta) \in \widetilde{F}_i$ belongs to $\mathcal{T}_g^e$ if and only if the two following conditions are satisfied:

- The point $q \in C_{g-i}$ that is identified with p lies in the support of an effective theta-characteristic. That is, C satisfies β_{g-i}).
- The odd theta-characteristic L_i of C_i, when twisted by η_i, remains odd.

This gives the intersection number

$$\widetilde{F}_i \cdot [\mathcal{T}_g^e] = (g-i-1)2^{g-i-1}(2^{g-i}-1)2^{i-1}(2^{i-1}-1),$$

where we use Lemma V.1.4 to count the possible theta-characteristics L_i.

Comparing both expressions for $\widetilde{F}_i \cdot [\mathcal{T}_g^e]$, it follows that $b_i = 2^{g-3}(2^{g-i}-1)(2^{i-1}-1)$.

With a similar argument (considering a connected component of $\pi^{-1}(F)$ contained in Δ_{g-i} or $\Delta_{i:g-i}$), one can find the numbers

$$b_{g-i} = 2^{g-3}(2^{g-i-1}-1)(2^i-1), \quad b_{i:g-i} = 2^{g-3}(2^{g-1}-2^{i-1}-2^{g-i-1}+1).$$

Now we proceed with step (3). We will determine the constants d_0', d_0'', d_0^{ram} of the class $[\mathcal{T}_g^o]$ by using the test curve of [HM98, Example 3.137].

Fix a general smooth curve D of genus $g-1$, with a fixed general point $p \in D$. Identifying p with a moving point $q \in D$, we get a curve G (isomorphic to D) which lies in $\Delta_0 \subset \overline{\mathcal{M}}_g$. As explained in [HM98], the following equalities hold:

$$G \cdot \lambda = 0, G \cdot \delta_0 = 2-2g, G \cdot \delta_1 = 1, G \cdot \delta_i = 0 \text{ for } i \geq 2,$$

where the intersection of G and Δ_1 occurs when q approaches p; in that case the curve becomes reducible, having D and a rational nodal curve as components.

Combining this information with the known class $[\mathcal{T}_g]$ in $\overline{\mathcal{M}}_g$, we have

$$G \cdot [\mathcal{T}_g] = 2^{g-3}((g-3) \cdot 2^{g-2}+1).$$

In order to compute d_0'', let $\widetilde{G}''$ be the connected component of $\pi^{-1}(G)$ obtained by attaching to every curve $C = D_{p\sim q}$ the 2-torsion line bundle $e = (\mathcal{O}_D)_{-1}$ (i.e. $\mathcal{O}_D$ glued by -1 at the points p, q). Indeed e is well defined along the family G, so $\widetilde{G}''$ makes sense and is isomorphic to G.

Then:

- By the projection formula, $\widetilde{G}'' \cdot \lambda = 0$.
- Again by projection, $\widetilde{G}'' \cdot (\pi^*\delta_0) = 2-2g$. Actually, since $\widetilde{G}'' \subset \Delta_0''$ and $\widetilde{G}''$ intersects neither Δ_0' nor Δ_0^{ram}, the following equalities hold:

$$\widetilde{G}'' \cdot \delta_0'' = 2-2g, \quad \widetilde{G}'' \cdot \delta_0' = 0 = \widetilde{G}'' \cdot \delta_0^{ram}.$$

- We have $\widetilde{G}'' \cdot (\pi^*\delta_1) = 1$, with $\widetilde{G}'' \cdot \delta_1 = 1$ and $\widetilde{G}'' \cdot \delta_{g-1} = 0 = \widetilde{G}'' \cdot \delta_{1:g-1}$.

  Indeed, the intersection $G \cap \Delta_1$ occurs when $p = q$; for that curve, the 2-torsion that we consider is trivial on D but not on the rational component. Hence the lift to $\widetilde{G}''$ of the intersection point $G \cap \Delta_1$ gives a point in $\widetilde{G}'' \cap \Delta_1$.
- It is clear that $\widetilde{G}'' \cdot \delta_i = \widetilde{G}'' \cdot \delta_{g-i} = \widetilde{G}'' \cdot \delta_{i:g-i} = 0$ for $i \geq 2$.
- Since twisting by e changes the parity of any theta-characteristic in any curve of the family G by [Har82, Theorems 2.12 and 2.14], it follows that all the intersection points of G and $\mathcal{T}_g$ lift to points of $\widetilde{G}'' \cap \mathcal{T}_g^o$.

All in all, we have

$$2^{g-3}((g-3)\cdot 2^{g-2}+1) = \widetilde{G}'' \cdot [\mathcal{T}_g^o] = (2g-2)d_0'' - 2^{g-3}(2^{g-1}-1)$$

and solving the equation we obtain $d_0'' = 2^{2g-6}$.

For the computation of d_0', we consider $\widetilde{G}' = \pi^{-1}(G) \cap \Delta_0'$ in $\overline{\mathcal{R}}_g$. Note that for an element $(C = D_{pq}, \eta) \in \widetilde{G}'$, η is obtained by gluing a nontrivial 2-torsion line bundle on D at the points p, q. Then:

- $\widetilde{G}' \cdot \lambda = 0$ by the projection formula.
- Again by projection, $\widetilde{G}' \cdot (\pi^*\delta_0) = \deg(\widetilde{G}' \to G)(G \cdot \delta_0) = 2(2-2g)(2^{2g-2}-1)$. Moreover, since $\widetilde{G}' \subset \Delta_0'$ intersects neither Δ_0'' nor Δ_0^{ram} it follows that

$$\widetilde{G}' \cdot \delta_0' = 2(2-2g)(2^{2g-2}-1), \quad \widetilde{G}' \cdot \delta_0'' = 0 = \widetilde{G}' \cdot \delta_0^{ram}.$$

- $\widetilde{G}' \cdot (\pi^*\delta_1) = \deg(\widetilde{G}' \to G)(G \cdot \delta_1) = 2(2^{2g-2}-1)$. We claim that $\widetilde{G}' \cdot \delta_1 = 0$ and $\widetilde{G}' \cdot \delta_{g-1} = 2^{2g-2} - 1 = \widetilde{G}' \cdot \delta_{1:g-1}$.

  Indeed, $G \cap \Delta_1$ occurs when $p = q$; when such a point is lifted to $\widetilde{G}'$, the 2-torsion is nontrivial on D (by construction). This gives $\widetilde{G}' \cdot \delta_1 = 0$.

  Moreover, triviality on the rational nodal component will depend on which of the two possible gluings of the 2-torsion on D we are taking; in any case, since $\widetilde{G}' = \pi^{-1}(G) \cap \Delta_0'$ considers simultaneously all possible gluings of all possible non-trivial 2-torsion line bundles on D, we have $\widetilde{G}' \cdot \delta_{g-1} = \widetilde{G}' \cdot \delta_{1:g-1}$. This proves the claim.

- Of course, $\widetilde{G}' \cdot (\pi^*\delta_i) = \widetilde{G}' \cdot \delta_{g-i} = \widetilde{G}' \cdot \delta_{i:g-i} = 0$ whenever $i \geq 2$.
- Finally, we use again that the parity of a theta-characteristic on a nodal curve of the family G is changed when twisted by $e = (\mathcal{O}_D)_{-1}$. Since the two possible gluings of a non-trivial 2-torsion bundle on D precisely differ by e, the sets $\widetilde{G}' \cap \mathcal{T}_g^e$ and $\widetilde{G}' \cap \mathcal{T}_g^o$ will have the same number of points, with the union of both giving the lift of $G \cap \mathcal{T}_g$ to $\widetilde{G}'$. That is,

$$\widetilde{G}' \cdot [\mathcal{T}_g^e] = \widetilde{G}' \cdot [\mathcal{T}_g^o] = (2^{2g-2} - 1) \cdot 2^{g-3}((g-3) \cdot 2^{g-2} + 1).$$

Putting this together with the coefficients $d_{g-1} = 2^{2g-5}$ and $d_{1:g-1} = 2^{g-3}(2^{g-2} - 1)$ obtained in step (2), we get

$$\begin{aligned}(2^{2g-2} - 1) \cdot 2^{g-3}((g-3) \cdot 2^{g-2} + 1) = \widetilde{G}' \cdot [\mathcal{T}_g^o] = \\ = 2(2g-2)(2^{2g-2} - 1)d_0' - 2^{2g-5}(2^{2g-2} - 1) - 2^{g-3}(2^{g-2} - 1)(2^{2g-2} - 1)\end{aligned}$$

and therefore $d_0' = 2^{2g-7}$.

Finally, to compute d_0^{ram} we simply combine the relation

$$(2^{2g-1} - 2)d_0' + d_0'' + 2^{2g-2}d_0^{ram} = 2^{g-1}(2^g - 1)2^{2g-6}$$

obtained in step (1) with the coefficients d_0', d_0'' just found, to obtain $d_0^{ram} = 2^{g-5}(2^{g-1} - 1)$. This concludes step (3) and hence the proof of Theorem V.1.1.

V.2 Irreducibility of $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$

In this section we study the irreducibility of the divisors $\mathcal{T}_g^o$ and $\mathcal{T}_g^e$. We make no claim of originality about the arguments: essentially we adapt those of Teixidor in [TiB88, Section 2], used to prove the irreducibility of $\mathcal{T}_g$ in $\overline{\mathcal{M}}_g$. Our main result of this part is:

Theorem V.2.1. *For every $g \neq 4$ the divisors $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$ are irreducible.*

The result is valid also for $g = 4$: in that case, the irreducibility of $\mathcal{T}_4^o$ and $\mathcal{T}_4^e$ will be obtained from an analysis of the Prym map $\mathcal{P}_4$ restricted to these divisors (see Corollary VI.3.6 and Corollary VI.3.8).

When $g = 3$, we already saw in Example IV.2.2 that the divisors $\mathcal{T}_3^o$ and $\mathcal{T}_3^e$ are irreducible. For the general case ($g \geq 5$), we are going to intersect $\mathcal{T}_g^o$ and $\mathcal{T}_g^e$ with the boundary divisor $\Delta_1 \subset \overline{\mathcal{R}}_g$ (this intersection being described by Proposition V.1.2). Before that, we need some previous considerations:

Remark V.2.2. In a neighborhood of a given point, the irreducibility of $\mathcal{T}_g^o$ (resp. $\mathcal{T}_g^e$) is implied by the irreducibility of the scheme X^o (resp. X^e) parametrizing pairs $((C, \eta), L)$, where (C, η) is a Prym curve and L is a semicanonical pencil on C such that $L \otimes \eta$ is odd (resp. even). This follows from the surjectivity of the forgetful map $X^o \to \mathcal{T}_g^o$ (resp. $X^e \to \mathcal{T}_g^e$).

Lemma V.2.3. *Let $\mathcal{D} \subset \mathcal{R}_g$ be a divisor, where $g \geq 5$. Then the closure $\overline{\mathcal{D}} \subset \overline{\mathcal{R}}_g$ intersects Δ_1 and Δ_{g-1}.*

Proof. We borrow the construction from [MNP16, Section 4], where (a stronger version of) the corresponding result for divisors in $\mathcal{M}_g$ is proved.

Fix a complete integral curve $B \subset \mathcal{M}_{g-2}$, two elliptic curves E_1, E_2 and a certain 2-torsion element $\eta \in JE_1 \setminus \{0\}$. If Γ_b denotes the smooth curve of genus $g-2$ corresponding to $b \in B$, one defines a family of Prym curves parametrized by Γ_b^2 as follows.

If $(p_1, p_2) \in \Gamma_b^2$ is a pair of distinct points, glue to Γ_b the curves E_1 and E_2 at the respective points p_1 and p_2 (this is independent of the chosen point on the elliptic curves). To this curve attach a 2-torsion bundle being trivial on Γ_b and E_2, and restricting to η on E_1.

To an element $(p, p) \in \Delta_{\Gamma_b} \subset \Gamma_b^2$, we attach the curve obtained by gluing a $\mathbb{P}^1$ to Γ_b at the point p, and then E_1, E_2 are glued to two other points in $\mathbb{P}^1$. Of course, the 2-torsion bundle restricts to η on E_1, and is trivial on the remaining components.

Moving b in B, this construction gives a complete threefold $T = \bigcup_{b\in B} \Gamma_b^2$ contained in $\Delta_1 \cap \Delta_{g-1}$. Let also $S = \bigcup_{b \in B} \Delta_{\Gamma_b^2}$ be the surface in T given by the union of all the diagonals; it is the intersection of T with Δ_2. Then, the following statements hold:

(1) $\delta_{1|S} = 0$ and $\delta_{g-1|S} = 0$ (the proof of [MNP16, Lemma 4.2] is easily translated to our setting).
(2) $\lambda\big|_{\Delta_{\Gamma_b^2}} = 0$ for every $b \in B$, since all the curves in $\Delta_{\Gamma_b^2}$ have the same Hodge structure.
(3) If $a \in \mathbb{Q}$ is the coefficient of λ for the cohomology class $[\overline{\mathcal{D}}] \in \mathrm{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}$, then $a \neq 0$. Indeed, $2^{2g-1}a \in \mathbb{Q}$ is the coefficient of λ for the cohomology class $[\overline{\pi(\mathcal{D})}] \in \mathrm{Pic}(\overline{\mathcal{M}}_g)_{\mathbb{Q}}$; then [MNP16, Remark 4.1] proves the claim.

These are the key ingredients in the original proof of [MNP16, Proposition 4.5]. The same arguments there work verbatim in our case and yield the analogous result: $[\overline{\mathcal{D}}]\big|_T \neq m \cdot S$ for every $m \in \mathbb{Q}$.

In particular, the intersection $\overline{\mathcal{D}} \cap T$ is non-empty (and not entirely contained in S). □

Remark V.2.4. Let C be a smooth hyperelliptic curve of genus g, with Weierstrass points $R_1, \ldots, R_{2g+2}$.

Then, it is well-known that the theta-characteristics on C have the form $r \cdot g_2^1 + S$, r being its dimension (with $-1 \leq r \leq [\frac{g-1}{2}]$) and S being the fixed part of the linear system (which consists of $g-1-2r$ distinct Weierstrass points).

In addition, we will use the following observation: given a 2-torsion line bundle of the form $\eta = \mathcal{O}_C(R_i - R_j)$, the theta-characteristics changing their parity when twisted by η are exactly those for which $R_i, R_j \in S$ (the dimension increases by 1) or $R_i, R_j \notin S$ (the dimension decreases by 1).

Proposition V.2.5. *For $g \geq 5$, the divisor $\mathcal{T}_g^o$ is irreducible.*

Proof. According to Proposition V.1.2, the intersection $\mathcal{T}_g^o \cap \Delta_1$ consists of two pieces α and β. The general point of each of these pieces is the union at a point P of a Prym elliptic curve (E, η) and a smooth curve C_{g-1} (with trivial line bundle) of genus $g-1$, such that:

- In the case of α, the curve C_{g-1} has a 1-dimensional theta-characteristic, i.e, $C_{g-1} \in \mathcal{T}_{g-1}$ in $\overline{\mathcal{M}}_{g-1}$. There is no assumption on (E, η): η will be the theta-characteristic on E induced by the 1-dimensional limit theta-characteristic on $C_{g-1} \cup_P E$. By irreducibility of $\mathcal{T}_{g-1}$, we may assume that α is irreducible.
- In the case of β, P is in the support of a 0-dimensional theta-characteristic on C_{g-1}. Again, there is no condition on (E, η): the induced theta-characteristic on E is $\mathcal{O}_E$.

Now we consider a reducible Prym curve $(C, \eta) \in \Delta_1$ constructed as follows: C is the join of an elliptic curve E and a general smooth hyperelliptic curve C' of genus $g-1$ at a Weierstrass point $P \in C'$, whereas the line bundle η is trivial on C'. Note that (C, η) is the general point of the intersection $\widetilde{\mathcal{H}}_g \cap \Delta_1$, where $\widetilde{\mathcal{H}}_g \subset \mathcal{T}_g^o$ is the locus of pairs formed by an hyperelliptic curve and a difference of two Weierstrass points on it.

Of course (C, η) belongs to α and β; we claim that it actually belongs to any component of β.

Indeed, the rational map between a component of β and $\mathcal{M}_{g-1}$ is generically surjective. As argued in [TiB88, Remark 1.3], the reason is that the locus in $\mathcal{M}_{g-1}$ of curves with an odd theta-characteristic of dimension ≥ 2 has codimension 3.

Thus we can assume that every component of β contains a Prym curve which is the union of C' (with trivial 2-torsion) and a Prym elliptic curve (E', η') at a Weierstrass point $Q \in C'$. Since the monodromy on hyperelliptic curves acts transitively on the set of Weierstrass points, we may replace Q by our original point P without changing the component of β. Using that $\overline{\mathcal{R}}_1$ is connected we can also replace (E', η') by (E, η). This proves the claim.

Now, to prove the irreducibility of $\mathcal{T}_g^o$ we argue as follows: since $\mathcal{T}_g^o$ has pure codimension 1, we know by Lemma V.2.3 that each of its components intersects Δ_1. As our point (C, η) belongs to all the irreducible components of $\mathcal{T}_g^o \cap \Delta_1$, it suffices to check the irreducibility of $\mathcal{T}_g^o$ in a neighborhood of (C, η).

To achieve this, in view of Remark V.2.2 we will check the irreducibility of the scheme X^o. In other words, we need to study the *limit semicanonical pencils on C changing parity when twisted by η*. We do this in the rest of the proof.

Let R_1, R_2, R_3 be the points on E differing from P by 2-torsion, and let $R_4, \dots, R_{2g+2}$ be the Weierstrass points on C' that are different from P: reordering if necessary, we assume $\eta\big|_E = \mathcal{O}_E(R_1 - R_2)$. Note that $R_1, \dots, R_{2g+2}$ are the limits on C of Weierstrass points on nearby smooth hyperelliptic curves, since they are the ramification points of the limit g^1_2 on C.

With this notation, arguing as in the proof of Proposition V.1.2, the possible aspects on E of a *limit semicanonical pencil changing parity on* (C, η) are:

- Those of type α have aspect on E differing from the even theta-characteristic η by $(g-1)P$, hence $\mathcal{O}_E(R_3 + (g-2)P) = \mathcal{O}_E(R_1 + R_2 + (g-3)P)$.
- Those of type β have aspect differing from the odd theta-characteristic $\mathcal{O}_E$ by $(g-1)P$, hence $\mathcal{O}_E((g-1)P) = \mathcal{O}_E(R_1 + R_2 + R_3 + (g-4)P)$.

Given a family of *semicanonical pencils changing parity* on nearby smooth curves of $\widetilde{\mathcal{H}}_g$, we can distinguish the type of its limit on C by knowing how many of the $g-1-2r$ fixed Weierstrass points in the moving theta-characteristic specialize to E. If this number is 0 or 3 (resp. 1 or 2) our limit is of type β (resp. of type α).

Hence, after using monodromy on smooth hyperelliptic curves to interchange the (limit) Weierstrass point R_3 with an appropriate (limit) Weierstrass point on C', we obtain that monodromy on $\widetilde{\mathcal{H}}_g \subset \mathcal{T}_g^o$ interchanges any *limit semicanonical pencil changing parity* of type β with one of type α. The only possible exception is a limit of $\frac{g-1}{2} \cdot g_2^1$ when $g \equiv 3 (\mathrm{mod}\ 4)$, since in that case there are no fixed points to interchange with R_3.

By irreducibility of α, monodromy on α acts transitively on the set of *limit semicanonical pencils changing parity* of type α. Therefore to conclude the proof of the irreducibility of X near (C, η) it only remains to show that, if $g \equiv 3 \pmod 4$, the monodromy on $\mathcal{T}_g^o$ interchanges the limit of $\frac{g-1}{2} \cdot g_2^1$ with a limit of theta-characteristics of lower dimension.

This can be achieved exactly with the same family of limit theta-characteristics as in [TiB88, Proposition 2.4] for certain reducible Prym curves C_X (which in this case, have non-trivial 2-torsion only on the component E). This moves the limit of $\frac{g-1}{2} \cdot g_2^1$ as desired. □

Remark V.2.6. In the case of $\mathcal{T}_g^e$, the intersection $\mathcal{T}_g^e \cap \Delta_1$ consists only of the piece α and the irreducibility of $\mathcal{T}_g^e$ follows in a much simpler way.

Chapter VI

The Prym map on $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$

In this chapter we study the Prym maps restricted to $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$. This connects the geometry of curves, which appeared in chapter V, with the geometry of abelian varieties (and cubic threefolds).

First we specify the covers whose Prym variety lies in $\theta_{null} \subset \mathcal{A}_{g-1}$ and their relation to $\mathcal{T}_g^e$ (section VI.1). Then we concentrate on the rich geometry of the odd cases $\mathcal{T}_g^o$ for $3 \leq g \leq 5$; each genus occupies a section. Finally, in section VI.5 we study the Prym map on $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$ for $g \geq 6$, with a special view towards $\mathcal{T}_6^o$.

VI.1 Even semicanonical pencils and the theta-null divisor

In this first section we describe the preimage of $\theta_{null} \subset \mathcal{A}_{g-1}$ under $\mathcal{P}_g : \mathcal{R}_g \to \mathcal{A}_{g-1}$. In general, this preimage will consist of $\mathcal{T}_g^e$ and other irreducible components of higher codimension, for which we include a few comments.

Proposition VI.1.1. *For every $g \geq 3$,*

$$\mathcal{P}_g^{-1}(\theta_{null}) = \mathcal{T}_g^e \cup \left\{ (C,\eta) \in \mathcal{R}_g \;\middle|\; \begin{array}{l} C \textit{ has an odd theta-characteristic } L \textit{ such that} \\ h^0(C,L) \geq 3 \textit{ and } L \otimes \eta \textit{ is also odd} \end{array} \right\}.$$

Proof. Let $\widetilde{C} \xrightarrow{f} C$ be the double étale cover defined by $(C,\eta) \in \mathcal{R}_g$, and let $\sigma : \widetilde{C} \to \widetilde{C}$ be the induced involution exchanging sheets. Throughout this proof, we consider the canonical presentation (P^+, Ξ^+) in $\mathrm{Pic}^{2g-2}(\widetilde{C})$ of the Prym variety; recall that the 2-torsion points of P correspond to theta-characteristics of $\widetilde{C}$ lying in P^+.

We already know that the inclusion $\mathcal{T}_g^e \subset \mathcal{P}_g^{-1}(\theta_{null})$ holds. Moreover, if C has an odd theta-characteristic L with $h^0(C,L) \geq 3$ and $L \otimes \eta$ odd, then

$$h^0(\widetilde{C}, f^*L) = h^0(C,L) + h^0(C, L \otimes \eta) \geq 4$$

and is even, hence f^*L is a theta-characteristic on $\widetilde{C}$ defining a stable singularity of Ξ^+.

Therefore, to finish the proof it suffices to check that if $(C,\eta) \in \mathcal{P}_g^{-1}(\theta_{null}) \setminus \mathcal{T}_g^e$, then C has an odd theta-characteristic L with $h^0(C,L) \geq 3$ and $L \otimes \eta$ odd. So let $M \in \Xi^+$ be a singular point, corresponding to a theta-characteristic on $\widetilde{C}$.

If $h^0(\widetilde{C},M) = 2$, then the singularity M is exceptional with $M = f^*L \otimes A$ and $h^0(C,L) = 2$, $h^0(\widetilde{C},A) > 0$. Let us check that this cannot happen under the assumption $(C,\eta) \notin \mathcal{T}_g^e$.

Indeed, if $\deg L = g-1$ then $A = \mathcal{O}_{\widetilde{C}}$ and $M = f^*L$. Since $\omega_C = \mathrm{Nm}_f(M) = L^2$, it follows that L is an even semicanonical pencil for the cover f, which is a contradiction. If $\deg L < g-1$, then under the assumption $M^2 = \omega_{\widetilde{C}}$ we have

$$f^*L^2 \otimes A \otimes \sigma(A) = f^*(\mathrm{Nm}_f(M)) = f^*\omega_C = \omega_{\widetilde{C}} = M^2 = f^*L^2 \otimes A^2$$

and therefore A is invariant by the action of the involution σ. In virtue of Lemma IV.3.1.(2), this implies that we can express $M = f^*(L')$ for a line bundle L' of degree $g-1$, which again leads to a contradiction.

Now assume that $M \in \Xi^+$ is defining a stable singularity, namely $h^0(\widetilde{C},M) \geq 4$. There is a chain of equalitites

$$M^2 = \omega_{\widetilde{C}} = f^*\omega_C = f^*\,\mathrm{Nm}_f(M) = M \otimes \sigma(M)$$

giving $M = \sigma(M)$, hence $M = f^*L$ for a line bundle L of degree $g-1$ on C. Moreover, the condition $\mathrm{Nm}_f(M) = \omega_C$ reads as L being a theta-characteristic on C, for which

$$4 \leq h^0(\widetilde{C}, f^*L) = h^0(C,L) + h^0(C, L\otimes\eta).$$

By the assumption $M \in \Xi^+$ both summands must have the same parity, and cannot be even since $(C,\eta) \notin \mathcal{T}_g^e$. It follows that the summands must be odd, which finishes the proof. □

Corollary VI.1.2. *The divisor of even semicanonical pencils $\mathcal{T}_g^e$ satisfies:*

(1) *For every $3 \leq g \leq 5$, the equality $\mathcal{T}_g^e = \mathcal{P}_g^{-1}(\theta_{null})$ holds.*
(2) *For every $g \geq 6$, $\mathcal{T}_g^e$ is the divisorial component of $\mathcal{P}_g^{-1}(\theta_{null})$. Any other irreducible component of $\mathcal{P}_g^{-1}(\theta_{null})$ has codimension 3 in $\mathcal{R}_g$.*

Proof. A smooth curve C of genus $g \leq 4$ has no theta-characteristic L with $h^0(C,L) \geq 3$ (otherwise it would contradict Clifford's theorem). When $g = 5$, such a theta-characteristic is necessarily a g^2_4, so C must be hyperelliptic (it has Clifford index 0). This proves (1), since covers of hyperelliptic curves are contained in $\mathcal{T}_g^e$ for $g \geq 4$.

Item (2) is a direct consequence of Proposition VI.1.1 and [TiB87, Theorem 2.17]. □

Remark VI.1.3. Let $(C,\eta) \in \mathcal{R}_g$ be a general point of any codimension 3 component of $\mathcal{P}_g^{-1}(\theta_{null})$. Again by [TiB87, Theorem 2.17], there exists a unique odd theta-characteristic L on C with $h^0(C,L) \geq 3$, which satisfies $h^0(C,L) = 3$ and $h^0(C, L\otimes\eta) = 1$. Let $\widetilde{C} \xrightarrow{f} C$ denote the double étale cover associated to (C,η). Then:

- On the one hand, the 2-torsion point $M = f^*L \in \Xi^+$ is a stable singularity since $h^0(\widetilde{C}, M) = 4$ (i.e. M has multiplicity 4 in the canonical theta divisor $\Theta_{\widetilde{C}}$ of $J\widetilde{C}$).
- On the other hand, following the analysis in [Mum74, Section 6] it is easy to check that the tangent space to P^+ at M is contained in the tangent cone to $\Theta_{\widetilde{C}}$ at M (one can take a basis $s_1, ..., s_4$ of $H^0(\widetilde{C}, M)$ pulled back from bases of $H^0(C, L)$ and $H^0(C, L \otimes \eta)$, so that all the differentials $\omega_{ij} = \langle s_i, s_j \rangle \in H^0(\widetilde{C}, \omega_{\widetilde{C}})$ will be symmetric).

Combining these two facts, we obtain that M is a point of multiplicity ≥ 3 in Ξ^+.

Let us illustrate this phenomenon of 2-torsion points of high multiplicity, by describing a codimension 3 component of $\mathcal{P}_6^{-1}(\theta_{null})$:

Example VI.1.4. Let JV be the intermediate Jacobian of a smooth cubic threefold $V \subset \mathbb{P}^4$. Recall that its (canonical) theta divisor has multiplicity 3 at the origin, which is its unique singularity (see [CG72] and [Bea82]). Hence we can consider inside $\theta_{null} \subset \mathcal{A}_5$ the 10-dimensional locus of intermediate Jacobians of smooth cubic threefolds.

The fiber $\mathcal{P}_6^{-1}(JV)$ under the Prym map $\mathcal{P}_6$ is 2-dimensional, given by (an open subset of) the Fano surface of lines on V ([DS81, Part V]). Indeed, for a line $l \subset V$ one fixes a supplementary $\mathbb{P}^2$ in the ambient space $\mathbb{P}^4$, and considers the conic bundle structure given by the projection $\mathrm{Bl}_l V \longrightarrow \mathbb{P}^2$ from l. The discriminant curve is a quintic $Q_l \subset \mathbb{P}^2$, coming with a natural double cover $(Q_l, \eta) \in \mathcal{R}_6$ such that $h^0(Q_l, \mathcal{O}_{Q_l}(1) \otimes \eta)$ is odd (conversely, the Prym of any cover of a smooth quintic with this parity condition is the intermediate Jacobian of a cubic threefold).

It follows that the preimage $\mathcal{P}_6^{-1}(\theta_{null})$ contains

$$\mathcal{RQ}^- = \left\{(Q, \eta) \in \mathcal{R}_6 \mid Q \text{ is a smooth plane quintic, } h^0(Q, \mathcal{O}_Q(1) \otimes \eta) \text{ is odd}\right\}$$

as a 3-codimensional irreducible component, since $\mathcal{RQ}^-$ is not contained in $\mathcal{T}_6^e$ (a general quintic admits no semicanonical pencil).

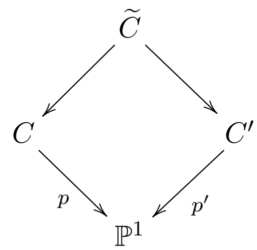
VI.2 Genus 3 and hyperelliptic Prym curves

This section is the devoted to study the restricted map $\mathcal{P}_3\,|_{\mathcal{T}_3^o}$. Recall that the divisors $\mathcal{T}_3^e$ and $\mathcal{T}_3^o$ in $\mathcal{R}_3$ are disjoint, and their union are the double étale covers of smooth hyperelliptic curves; in addition, $\mathcal{T}_3^e$ and $\mathcal{T}_3^o$ are easily described in terms of the number of Weierstrass points needed to express the 2-torsion line bundle, as we saw in Example IV.2.2.

The Prym map (on smooth covers) $\mathcal{P}_3 : \mathcal{R}_3 \to \mathcal{A}_2$ is surjective. When we consider its restriction to $\mathcal{T}_3^e$ and $\mathcal{T}_3^o$, two distinct behaviours arise. On the one hand, $\mathcal{T}_3^e = \mathcal{P}_3^{-1}(\theta_{null})$ as we saw in Corollary VI.1.2.(1), where $\theta_{null} = \mathcal{A}_1 \times \mathcal{A}_1 \subset \mathcal{A}_2$ is the locus formed by products of elliptic curves. On the other hand $\mathcal{P}_3\,|_{\mathcal{T}_3^o}$ is dominant, and its general fiber can be described as follows:

Theorem VI.2.1. *The Prym map $\mathcal{P}_3|_{\mathcal{T}_3^o}$ is dominant, and its general fiber is isomorphic to the complement in the projective plane of six lines and a smooth conic.*

Proof. Let C' be a smooth curve of genus 2. Since C' is hyperelliptic, by [Mum74, page 346] expressing JC' as the Prym of a cover of a genus 3 hyperelliptic curve C is equivalent to the construction of a diagram



where: the double cover p is branched at the six branch points of p' and two extra points, and $\widetilde{C}$ is the normalization of $C \times_{\mathbb{P}^1} C'$. This proves the dominance of $\mathcal{P}_3|_{\mathcal{T}_3^o}$.

In order to determine the fiber, we consider a curve C' which is general in the following sense: if $p_1, \ldots, p_6$ are the branch points of the double cover $p' : C' \to \mathbb{P}^1$, then there is no nontrivial projectivity of $\mathbb{P}^1$ mapping four points of $\{p_1, \ldots, p_6\}$ to (possibly other) four points of $\{p_1, \ldots, p_6\}$. Under this assumption, the fiber $\mathcal{P}_3|_{\mathcal{T}_3^o}^{-1}(JC')$ parametrizes all the possible choices of two non-repeated points in $\mathbb{P}^1 \setminus \{p_1, \ldots, p_6\}$, according to the previous description.

Consider the natural isomorphism $(\mathbb{P}^1)^{(2)} \cong (\mathbb{P}^2)^*$ identifying a pair of points on a smooth plane conic with the line joining them. Under this identification $(\mathbb{P}^1 \setminus \{p_1, \ldots, p_6\})^{(2)}$ is isomorphic to the complement in $(\mathbb{P}^2)^*$ of six lines (no three of them concurring), since we are considering lines passing through none of the six marked points of the conic.

To finish the description of the general fiber $\mathcal{P}_3|_{\mathcal{T}_3^o}^{-1}(JC')$, simply note that we are avoiding lines that are tangent to the conic as well, since we are considering pairs in $(\mathbb{P}^1 \setminus \{p_1, \ldots, p_6\})^{(2)}$ formed by two distinct points. $\square$

Remark VI.2.2. Since the the divisors $\mathcal{T}_3^e$ and $\mathcal{T}_3^o$ are disjoint in $\mathcal{R}_3$ and the equality $\mathcal{P}_3^{-1}(\theta_{null}) = \mathcal{T}_3^e$ holds, it follows that $\mathcal{P}_3|_{\mathcal{T}_3^o}$ is dominant but not surjective. Indeed, $\mathcal{P}_3(\mathcal{T}_3^o) = \mathcal{A}_2 \setminus \theta_{null}$.

Furthermore, since a variety which dominates a rationally connected variety with rationally connected generic fibers is rationally connected ([GHS03]), from Theorem VI.2.1 we obtain:

Corollary VI.2.3. *The divisor $\mathcal{T}_3^o$ is rationally connected.*

Remark VI.2.4. Alternatively, one can directly prove that both divisors $\mathcal{T}_3^e$ and $\mathcal{T}_3^o$ are rationally connected using their description in terms of Weierstrass points. Indeed, any two smooth Prym curves of $\mathcal{T}_3^e$ (resp. $\mathcal{T}_3^o$) can be connected by a chain of (at most five) rational curves contained in $\mathcal{T}_3^e \subset \overline{\mathcal{R}}_3$ (resp. $\mathcal{T}_3^o \subset \overline{\mathcal{R}}_3$); basically, each rational curve parametrizes hyperelliptic Prym curves with all but one of its branch points remaining constant.

VI.3 Genus 4 and Recillas' trigonal construction

In this section, we carry out an analysis of the case of genus 4. More precisely, we prove that $\mathcal{P}_4|_{\mathcal{T}_4^o}$ is dominant; we determine its general fiber, and also that of $\mathcal{P}_4|_{\mathcal{T}_4^e}$. This enables us to show the irreducibility of $\mathcal{T}_4^e$ and $\mathcal{T}_4^o$, which were the only pending cases after Theorem V.2.1.

Roughly, the idea for our arguments is the following. Recillas' trigonal construction provides an isomorphism between two moduli spaces, each of them equipped with a natural involution. Then we will exploit the fact that these involutions are compatible with Recillas' construction, and the fact that the divisors $\mathcal{T}_4^o$ and $\mathcal{T}_4^e$ are contained in the locus of fixed points of one of these involutions.

First of all, recall that a smooth hyperelliptic curve C of genus 4 has ten distinct semicanonical pencils, corresponding to the sum of the g^1_2 as movable part with a Weierstrass base point. A non-hyperelliptic C is embedded by the canonical map $C \to \mathbb{P}^3$ as the complete intersection of a quadric Q and a cubic surface S.

If Q is smooth, then the curve C has exactly two g^1_3, which parametrize the intersection of S with the lines in each of the rulings of Q. Observe that the sum of both g^1_3 is the canonical divisor of C and the curve C has no semicanonical pencil. Instead, if the quadric Q is singular, then C has a unique g^1_3 which is a semicanonical pencil; moreover, the g^1_3 is given by the intersections of S with the system of lines in Q containing the singular point.

It follows that $\mathcal{H}_4 \subset \mathcal{T}_4 \subset \mathcal{M}_4$ (for $\mathcal{H}_4$ the hyperelliptic locus), and $\mathcal{T}_4$ is the closure of the locus of non-hyperelliptic curves whose canonical model is contained in a singular quadric. Moreover, since the semicanonical pencil of a non-hyperelliptic curve of $\mathcal{T}_4$ is unique, we have $\mathcal{T}_4^e \cap \mathcal{T}_4^o = \mathcal{RH}_4$, where $\mathcal{RH}_4 = \pi^{-1}(\mathcal{H}_4) \subset \mathcal{R}_4$ are the hyperelliptic Prym curves.

Now we address the problem of understanding the restriction of the Prym map $\mathcal{P}_4 : \mathcal{R}_4 \to \mathcal{A}_3$ to the divisors $\mathcal{T}_4^e$ and $\mathcal{T}_4^o$. Consider the following moduli spaces:

$$\mathcal{RG}^{\,1}_{4,3} = \{(C,\eta,M) \mid (C,\eta) \in \mathcal{R}_4 \setminus \mathcal{RH}_4 \text{ and } M \text{ is a } g^1_3 \text{ on } C\}/\cong$$
$$\mathcal{G}^{\,1}_{3,4} = \{(X,L) \mid X \in \mathcal{M}_3 \text{ and } L \text{ is a (not necessarily complete) base-point-free } g^1_4 \text{ on } X\}/\cong$$

That is, $\mathcal{RG}^{\,1}_{4,3}$ parametrizes (isomorphism classes of) covers of non-hyperelliptic genus 4 curves endowed with a g^1_3, and $\mathcal{G}^{\,1}_{3,4}$ parametrizes genus 3 curves endowed with a base-point-free g^1_4. Both moduli spaces have projection maps forgetting the linear series:

- The projection $\mathcal{RG}^{\,1}_{4,3} \xrightarrow{\varphi} \mathcal{R}_4 \setminus \mathcal{RH}_4$ is generically finite of degree 2. Moreover, $\mathcal{RG}^{\,1}_{4,3}$ carries a natural involution σ defined by
$$\sigma(C,\eta,M) = (C,\eta,\omega_C \otimes M^{-1}), \tag{VI.3.1}$$
which exchanges the two sheets of the open subset of $\mathcal{RG}^{\,1}_{4,3}$ where φ is finite.
- Let us study the fiber of the projection $\mathcal{G}^{\,1}_{3,4} \xrightarrow{\psi} \mathcal{M}_3$ over a curve $X \in \mathcal{M}_3$. First of all, note that the scheme $\mathcal{G}^1_4(X)$ of g^1_4 linear series on X is easily identified with the blow-up of $\mathrm{Pic}^4(X)$ at the

canonical sheaf ω_X. This scheme carries a natural involution, given by $L \mapsto \omega_X^2 \otimes L^{-1}$; indeed, this involution, defined outside the exceptional divisor of $\mathcal{G}_4^1(X)$, extends as the identity on the exceptional divisor as proved in [FNS20, Proposition 6.1].

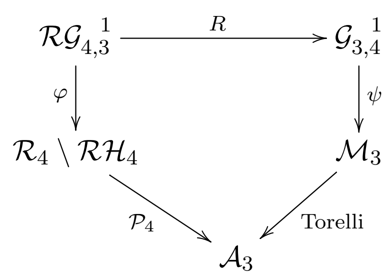
If X is non-hyperelliptic and we regard it as a quartic plane curve, the g_4^1's on X with base points are exactly those given by pencils of lines through points of X. Linear series with base points are thus parametrized by X, and are contained in the exceptional divisor of $\mathcal{G}_4^1(X)$.

If X is hyperelliptic, tetragonal series with base points are those of the form $g_2^1 + p + q$ and hence the open subset of $\mathcal{G}_4^1(X)$ parametrizing series without base points is the complement of a copy of $X^{(2)}$.

According to this description, it follows that for any smooth curve X the open subset of $\mathcal{G}_4^1(X)$ formed by base-point-free tetragonal series remains invariant by the involution. Since this involution is compatible with the automorphisms of X as well, we obtain an involution $\tau : \mathcal{G}_{3,4}^{\;1} \to \mathcal{G}_{3,4}^{\;1}$ defined by

$$\tau(X, L) = (X, \omega_X^2 \otimes L^{-1}). \tag{VI.3.2}$$

In this situation, Recillas' trigonal construction yields a morphism $R : \mathcal{RG}_{4,3}^{\;1} \longrightarrow \mathcal{G}_{3,4}^{\;1}$ making commutative the following diagram:



More precisely, R provides and isomorphism of $\mathcal{RG}_{4,3}^{\;1}$ with the open set $(\mathcal{G}_{3,4}^{\;1})^o \subset \mathcal{G}_{3,4}^{\;1}$ of tetragonal pairs (X, L) with the property that the g_4^1 L contains no divisor of the form $2p + 2q$ or $4p$ (see [Don92, Theorem 2.9]). This open set dominates $\mathcal{M}_3$.

Note that $\mathcal{G}_{3,4}^{\;1}$ is clearly irreducible, since $\mathcal{M}_3$ and all the fibers of the projection $\mathcal{G}_{3,4}^{\;1} \to \mathcal{M}_3$ are so. Therefore $(\mathcal{G}_{3,4}^{\;1})^o$ and $\mathcal{RG}_{4,3}^{\;1}$ are also irreducible.

Now our purpose is to prove that Recillas' construction commutes with the natural involutions σ and τ (defined in (VI.3.1) and (VI.3.2)), namely:

Proposition VI.3.1. *The equality $R \circ \sigma = \tau \circ R$ holds.*

By irreducibility, it is enough to check that $\sigma \circ R^{-1} = R^{-1} \circ \tau$ on an open set U. We define U to be the intersection of $(\mathcal{G}_{3,4}^{\;1})^o$ with the open set of pairs (X, L) where X is non-hyperelliptic and L is not contained in the canonical bundle of X.

Hence let $X \in \mathcal{M}_3$ be non-hyperelliptic, regarded as a quartic plane curve, and consider $L \in \mathrm{Pic}^4(X) \setminus \{\omega_X\}$ a complete g^1_4 on X such that the linear system $|L|$ contains no divisor of the form $2p+2q$ or $4p$; this is true for $L \in \mathrm{Pic}^4(X) \setminus \{\omega_X\}$ general.

The element $R^{-1}(X,L) \in \mathcal{RG}^1_{4,3}$ can be explicitly described as follows: the curve

$$C'_L = \{p+q \in X^{(2)} \mid h^0(L(-p-q)) \neq 0\}$$

is smooth of genus 7, with a fixed-point-free involution i_L sending $p+q$ to the unique divisor $r+s \in |L(-p-q)|$. The quotient $C_L = C'_L/\langle i_L \rangle$ has genus 4 and naturally comes with a degree 3 map to $\mathbb{P}^1$, corresponding to a g^1_3 that we denote by M_L. Then

$$R^{-1}(X,L) = (C_L, \eta_L, M_L)$$

where $\eta_L \in (JC_L)_2$ defines the étale cover $C'_L \to C_L$.

Now we denote $\widetilde{L} = \omega_X^2 \otimes L^{-1}$, so that $\tau(X,L) = (X, \widetilde{L})$. Then Proposition VI.3.1 boils down to prove that

$$(C_L, \eta_L, \omega_{C_L} \otimes M_L^{-1}) = \sigma(R^{-1}(X,L)) = R^{-1}(\tau(X,L)) = (C_{\widetilde{L}}, \eta_{\widetilde{L}}, M_{\widetilde{L}}),$$

which will be a consequence of the following two lemmas:

Lemma VI.3.2. *There is an isomorphism $C'_L \xrightarrow{\rho} C'_{\widetilde{L}}$ such that $\rho \circ i_L = i_{\widetilde{L}} \circ \rho$.*

Proof. For a point $p+q \in C'_L$, we let $\rho(p+q) \in X^{(2)}$ be the residual intersection of X with the line pq (this line being the tangent line to X at p, if $p=q$). Writing $r+s = i_L(p+q)$, then the eight points obtained by intersection of X with the lines pq and rs give a divisor in $|\omega_X^2|$. This implies $|\rho(p+q)+\rho(r+s)| \in |\widetilde{L}|$, which gives $\rho(p+q) \in C'_{\widetilde{L}}$ (i.e. ρ is well defined) and $i_{\widetilde{L}}(\rho(p+q)) = \rho(r+s) = \rho(i_L(p+q))$.

To finish the proof, simply note that ρ is an isomorphism since it has an obvious inverse. □

It follows that $(C_L, \eta_L) = (C_{\widetilde{L}}, \eta_{\widetilde{L}})$ as elements of $\mathcal{R}_4$. To finish the proof of $\sigma(R^{-1}(X,L)) = R^{-1}(\tau(X,L))$, we only need to check that the isomorphism $C_L \to C_{\widetilde{L}}$ induced by ρ (that we denote by ρ as well, abusing of notation) sends $\omega_{C_L} \otimes M_L^{-1}$ to $M_{\widetilde{L}}$.

Lemma VI.3.3. $\rho_*(\omega_{C_L} \otimes M_L^{-1}) = M_{\widetilde{L}}$.

Proof. Since C_L (hence $C_{\widetilde{L}}$) is a non-hyperelliptic curve of genus 4, by the discussion at the beginning of this subsection it turns out that C_L has at most two g^1_3 (namely M_L and $\omega_{C_L} \otimes M_L^{-1}$). Therefore, it suffices to check that $\rho_* M_L \neq M_{\widetilde{L}}$.

Take $D = p_1+p_2+p_3+p_4 \in |L|$. Then, M_L is the line bundle on C_L represented by the divisor

$$\{p_1+p_2, p_3+p_4\} + \{p_1+p_3, p_2+p_4\} + \{p_1+p_4, p_2+p_3\}$$

and therefore $\rho_* M_L$ on $C_{\widetilde{L}}$ is represented by the divisor

$$\{a_{12}+b_{12}, a_{34}+b_{34}\} + \{a_{13}+b_{13}, a_{24}+b_{24}\} + \{a_{14}+b_{14}, a_{23}+b_{23}\}$$

where for $i, j \in \{1,2,3,4\}$ the points a_{ij}, b_{ij} are the residual intersection of X with the line $p_i p_j$. We may take D with p_1, p_2, p_3, p_4 distinct, and such that for every $i \neq j$ we have $a_{ij} \neq b_{ij}$.

If the equality $\rho_* M_L = M_{\widetilde{L}}$ were true, then we would have equalities in $X^{(4)}$

$$a_{12}+b_{12}+a_{34}+b_{34} = a_{13}+b_{13}+a_{24}+b_{24} = a_{14}+b_{14}+a_{23}+b_{23}$$

with this divisor representing the line bundle $\widetilde{L}$. But these equalities are easily seen to imply that the points p_1, p_2, p_3, p_4 are aligned, which is a contradiction since $L \neq \omega_X$. □

This finishes the proof of Proposition VI.3.1. As a consequence of it, we deduce that τ leaves invariant the image of R, and the fixed points of both involutions correspond by R.

First, let us study the fixed points of σ. If $(C, \eta, M) \in \mathcal{RG}^{\,1}_{4,3}$ with C non-hyperelliptic, then $M \cong \omega_C \otimes M^{-1}$ if and only if M is the unique g^1_3 on C, namely $C \in \mathcal{T}_4$ and M is the semicanonical pencil of C. Therefore, the locus of fixed points of σ consists of two pieces:

(A_1) Triples $(C, \eta, M) \in \mathcal{RG}^{\,1}_{4,3}$ with $(C, \eta) \in (\mathcal{T}_4^e \cup \mathcal{T}_4^o) \setminus \mathcal{RH}_4$ and M the semicanonical pencil on the curve C.

(A_2) The set of $(C, \eta, M) \in \mathcal{RG}^{\,1}_{4,3}$ with $(C, \eta) \notin \mathcal{T}_4^e \cup \mathcal{T}_4^o$ having a nontrivial automorphism exchanging the two g^1_3's on C.

The locus of fixed points of τ is formed by the following pieces:

(B_1) The union of the exceptional divisors in $\mathcal{G}^1_4(X)$ moving X in $\mathcal{M}_3$, that is:

$$\bigcup_{X \in \mathcal{M}_3} (|\omega_X|^* \setminus X) \,/\, \mathrm{Aut}(X)$$

(B_2) Pairs $(X, L) \in \mathcal{G}^{\,1}_{3,4}$ with X hyperelliptic. Indeed, the hyperelliptic involution on X exchanges any $L \in \mathrm{Pic}^4(X)$ with $\omega_X^2 \otimes L^{-1}$.

(B_3) The set of pairs $(X, \omega_X \otimes \eta)$ with $X \in \mathcal{M}_3$ and $\eta \in JX_2 \setminus \{\mathcal{O}_X\}$. This set is naturally identified with $\mathcal{R}_3$.

(B_4) The (closure of) the set of pairs $(X, L) \in \mathcal{G}^{\,1}_{3,4}$ with X non-hyperelliptic, having a nontrivial automorphism sending L to $\omega_X^2 \otimes L^{-1}$.

Essentially, we will prove that the part $\varphi^{-1}(\mathcal{T}_4^o)$ of (A_1) corresponds (under Recillas' construction) to the piece (B_1), and the part $\varphi^{-1}(\mathcal{T}_4^e)$ of (A_1) corresponds to the piece (B_2).

Remark VI.3.4. The piece (B_3) corresponds, under R^{-1}, to the irreducible component of (A_2) formed by covers of bielliptic curves of genus 4.

Indeed, consider (X, L) with X a quartic plane curve and $L = \omega_X \otimes \eta$, $\eta \in JX_2 \setminus \{\mathcal{O}_X\}$. We can express $L = \theta_1 \otimes \theta_2$ for two distinct odd theta-characteristics θ_1 and θ_2; namely, $|L|$ has a divisor given by the contact points of two distinct bitangent lines. Moreover, using the theory of syzygetic triads ([Dol12, Section 5.4.1]) it is easy to check that $|L|$ exactly contains six such "distinguished" divisors (i.e. formed by contact points of two bitangent lines).

Then, the curves C_L and $C_{\widetilde{L}}$ are equal by definition. Moreover, the natural isomorphism $\rho : C_L \to C_{\widetilde{L}} = C_L$ of Lemma VI.3.2 is an involution with exactly six fixed points (which lie over the six distinguished divisors of $|L|$). It follows that C_L is a bielliptic curve, and the bielliptic involution ρ exchanges the two g^1_3's on C_L by Lemma VI.3.3.

Conversely, Recillas' trigonal construction applied to a cover of a bielliptic curve is well known to give an element of the piece (B_3), see [Dol08, Section 3].

Keeping all this construction in mind, we are now ready to prove that $\mathcal{P}_4|_{\mathcal{T}_4^o}$ is surjective and describe its general fiber:

Theorem VI.3.5. *The Prym map $\mathcal{P}_4|_{\mathcal{T}_4^o} : \mathcal{T}_4^o \to \mathcal{A}_3$ is surjective, and the fiber of a general Jacobian $JX \in \mathcal{A}_3$ is the complement in the projective plane of the union of the canonical model of X and the 28 lines that are bitangent to it.*

Proof. We will first prove that $\mathcal{P}_4|_{\mathcal{T}_4^o}$ is dominant, by describing the fiber of a general Jacobian $JX \in \mathcal{A}_3$ (in particular, showing its non-emptiness). To this end, we take a non-hyperelliptic curve $X \in \mathcal{M}_3$ without automorphisms, and denote $\mathcal{G}_4^{1,o}(X) = \psi^{-1}(X) \cap \left(\mathcal{G}^{\,1}_{3,4}\right)^o$. That is, $\mathcal{G}_4^{1,o}(X)$ parametrizes g^1_4 linear series on X with no divisor of the form $2p + 2q$ or $4p$.

Observe that, since the whole fiber $\mathcal{P}_4^{-1}(JX)$ is contained in $\mathcal{R}_4 \setminus \mathcal{RH}_4$, according to Recillas' diagram the fiber $\mathcal{P}_4|_{\mathcal{T}_4^o}^{-1}(JX)$ equals $\varphi\big(R^{-1}(\mathcal{G}_4^{1,o}(X))\big) \cap \mathcal{T}_4^o$. The latter is isomorphic to $R^{-1}\big(\mathcal{G}_4^{1,o}(X)\big) \cap \varphi^{-1}(\mathcal{T}_4^o)$, since the restriction $\varphi|_{\varphi^{-1}(\mathcal{T}_4^o)}$ is an isomorphism; note that the intersection $R^{-1}\big(\mathcal{G}_4^{1,o}(X)\big) \cap \varphi^{-1}(\mathcal{T}_4^o)$ lies in the piece (A_1) of the locus of fixed points of σ.

On the other hand, by our assumptions on X, the intersection of $\mathcal{G}_4^{1,o}(X)$ with the locus of fixed points of τ consists of a 2-dimensional irreducible component (intersection with the piece (B_1)) and finitely many points (intersection with the piece (B_3)). Therefore, the intersection of $R^{-1}\big(\mathcal{G}_4^{1,o}(X)\big)$ with the locus of fixed points of σ consists of finitely many points of the piece (A_2) and a 2-dimensional component.

We claim that this 2-dimensional component must be $R^{-1}\big(\mathcal{G}_4^{1,o}(X)\big) \cap \varphi^{-1}(\mathcal{T}_4^o)$. Indeed, if this were not the case then the piece (B_1) would correspond to (A_2); hence (A_2) would be 8-dimensional, which is absurd since the locus of (non-hyperelliptic) curves of genus 4 with automorphisms is well known to have lower dimension (see [Cor87]).

All in all, we obtain that the fiber is isomorphic to the intersection of $\mathcal{G}_4^{1,o}(X)$ with the locus (B_1) of the fixed locus of τ. This intersection reads as the set of all the non-complete, base-point-free

g^1_4 on X containing no divisor of the form $2p+2q$ or $4p$. Such a g^1_4 is defined by the pencil of lines through a fixed point of $\mathbb{P}^2$, outside the curve X and lying in no bitangent line to X.

Finally, we proceed to prove that the map $\mathcal{P}_4|_{\mathcal{T}_4^o}: \mathcal{T}_4^o \to \mathcal{A}_3$ is not only dominant, but also surjective. For this, note that we have shown that $\varphi^{-1}(\mathcal{T}_4^o)$ is mapped via R to the locus (B_1); it follows that Jacobians $JX \in \mathcal{A}_3$ of non-hyperelliptic curves X with automorphisms lie in the image of $\mathcal{P}_4|_{\mathcal{T}_4^o}$ as well. In addition, every element of θ_{null} (i.e. an hyperelliptic Jacobian or product of Jacobians in $\mathcal{A}_3$) can be obtained as the Prym variety of a cover in $\mathcal{RH}_4 \subset \mathcal{T}_4^o$; this follows from Mumford's description on Prym varieties of covers of hyperelliptic curves ([Mum74, Page 346]), that we already used in section VI.1. □

As a consequence of this description, we get the irreducibility of $\mathcal{T}_4^o$:

Corollary VI.3.6. *The divisor $\mathcal{T}_4^o$ is irreducible and rationally connected.*

Proof. In Theorem VI.3.5 we have proved that $\mathcal{P}_4|_{\mathcal{T}_4^o}: \mathcal{T}_4^o \to \mathcal{A}_3$ is surjective, with all the fibers of elements in $\mathcal{A}_3 \setminus \theta_{null}$ being irreducible of the same dimension. Moreover, since $\mathcal{P}_4^{-1}(\theta_{null}) = \mathcal{T}_4^e$ by Corollary VI.1.2.(1), we have $\mathcal{P}_4|_{\mathcal{T}_4^o}^{-1}(\theta_{null}) = \mathcal{T}_4^o \cap \mathcal{T}_4^e = \mathcal{RH}_4$.

Thus if $\mathcal{T}_4^o$ were not irreducible, it would have $\mathcal{RH}_4$ as an irreducible component, contradicting the equidimensionality of $\mathcal{T}_4^o$.

Finally, the rational connectedness of $\mathcal{T}_4^o$ follows again from the results in [GHS03]. □

We finish this section by determining the fiber $\mathcal{P}_4^{-1}(JX) \subset \mathcal{T}_4^e$ of a general hyperelliptic Jacobian $JX \in \theta_{null}$. As a consequence, we obtain that $\mathcal{T}_4^e$ is irreducible.

Theorem VI.3.7. *The fiber $\mathcal{P}_4^{-1}(JX) \subset \mathcal{T}_4^e$ of a general hyperelliptic Jacobian $JX \in \theta_{null} \subset \mathcal{A}_3$ is birationally equivalent to its Kummer variety.*

Proof. Take a general genus 3 hyperelliptic curve X (in particular, having the hyperelliptic involution as its only nontrivial automorphism). The intersection $\mathcal{P}_4^{-1}(JX) \cap \mathcal{RH}_4$ can be described following Mumford's trick for covers of hyperelliptic curves, as we did in the case of genus 3: this intersection is the complement in $\mathbb{P}^2$ of the union of eight lines and a conic.

Now, we proceed to describe the "non-hyperelliptic" part of the fiber $\mathcal{P}_4^{-1}(JX)$. As usual, denote by $\mathcal{G}_4^{1,o}(X) = \psi^{-1}(X) \cap (\mathcal{G}^{\,1}_{3,4})^o$; due to the action of the hyperelliptic involution on $\mathcal{G}_4^1(X)$, $\mathcal{G}_4^{1,o}(X)$ is birationally equivalent to the Kummer variety of JX.

According to the commutative diagram given by Recillas' construction, $\mathcal{P}_4^{-1}(JX) \setminus \mathcal{RH}_4$ equals $\varphi(R^{-1}(\mathcal{G}_4^{1,o}(X)))$, which is isomorphic to $R^{-1}(\mathcal{G}_4^{1,o}(X))$ since the restriction $\varphi|_{\varphi^{-1}(\mathcal{T}_4^e)}$ is an isomorphism. It follows that $\mathcal{P}_4^{-1}(JX) \setminus \mathcal{RH}_4$ is birational to the Kummer variety of JX.

Finally, since $\mathcal{P}_4^{-1}(JX) \setminus \mathcal{RH}_4$ is 3-dimensional and $\mathcal{P}_4^{-1}(JX) \cap \mathcal{RH}_4$ is 2-dimensional, the whole fiber $\mathcal{P}_4^{-1}(JX)$ must be irreducible (otherwise, $\mathcal{T}_4^e$ would not be equidimensional). □

Corollary VI.3.8. *The divisor $\mathcal{T}_4^e$ is irreducible.*

VI.4 Genus 5 and cubic threefolds

Similarly to the cases $g = 3$ and $g = 4$, the behaviour of the Prym map on $\mathcal{T}_5^o$ is quite different from the behaviour on $\mathcal{T}_5^e$; indeed, $\mathcal{T}_5^o$ dominates $\mathcal{A}_4$ as already observed by Izadi in [Iza95, Proof of Theorem 6.14]. In this section we first give a brief different proof of this fact, by means of the cohomology classes of $\mathcal{T}_5^o$ and $\mathcal{T}_5^e$ (Proposition VI.4.1); then we study in more detail the general fiber of $\widetilde{\mathcal{P}}_5|_{\mathcal{T}_5^o}$, whose geometry reveals enumerative properties of cubic threefolds.

We start by recalling Donagi's description [Don92, Section 5] of the general fiber of the (proper, surjective) Prym map $\widetilde{\mathcal{P}}_5 : \widetilde{\mathcal{R}}_5 \to \mathcal{A}_4$. There is a birational map

$$\kappa : \mathcal{A}_4 \dashrightarrow \mathcal{RC}^+$$

where $\mathcal{RC}^+$ denotes the moduli space of pairs (V, δ) with $V \subset \mathbb{P}^4$ a smooth cubic threefold and $\delta \in JV_2$ an even 2-torsion point (i.e. $\delta \notin \Theta_V$ for the canonical choice of the theta divisor $\Theta_V \subset JV$). In [Iza95] Izadi gave an explicit geometric realization of this birational map, and described an open subset of $\mathcal{A}_4$ on which κ is an isomorphism.

Then the fiber of $\kappa \circ \widetilde{\mathcal{P}}_5$ over a general (V, δ) is isomorphic to the double étale cover $\widetilde{F(V)}$ of the Fano surface of lines $F(V)$ defined by δ (recall that $\mathrm{Pic}^0(F(V)) \cong JV$).

Proposition VI.4.1. *The restricted Prym map $\widetilde{\mathcal{P}}_5|_{\mathcal{T}_5^o} : \mathcal{T}_5^o \to \mathcal{A}_4$ is dominant.*

Proof. For $A \in \mathcal{A}_4$ general, we write $(V, \delta) = \kappa(A)$ and let $\widetilde{F(V)} = (\widetilde{\mathcal{P}}_5)^{-1}(A)$ be its fiber by $\widetilde{\mathcal{P}}_5$. If $\iota : \widetilde{F(V)} \hookrightarrow \widetilde{\mathcal{R}}_5 \hookrightarrow \overline{\mathcal{R}}_5$ denotes the inclusion, then the pullback map

$$\iota^* : \mathrm{Pic}(\overline{\mathcal{R}}_5)_{\mathbb{Q}} \to \mathrm{Pic}(\widetilde{F(V)})_{\mathbb{Q}}$$

annihilates the classes $\delta_0', \delta_0'', \delta_1, \delta_2, \delta_3, \delta_4, \delta_{1:4}$ and $\delta_{2:3}$.

Indeed, the general element of the divisors Δ_i $(i = 1, \ldots, 4)$ and $\Delta_{i:5-i}$ $(i = 1, 2)$ is an admissible cover whose Prym variety is a decomposable ppav; hence $\widetilde{\mathcal{P}}_5(\Delta_i \cap \widetilde{\mathcal{R}}_5)$ and $\widetilde{\mathcal{P}}_5(\Delta_{i:5-i} \cap \widetilde{\mathcal{R}}_5)$ have positive codimension in $\mathcal{A}_4$, so $\Delta_i \cap \widetilde{\mathcal{R}}_5$ and $\Delta_{i:5-i} \cap \widetilde{\mathcal{R}}_5$ are disjoint with the general fiber $\widetilde{F(V)}$. On the other hand Δ_0'' is formed by Wirtinger covers, so $\widetilde{\mathcal{P}}_5(\Delta_0'' \cap \widetilde{\mathcal{R}}_5)$ is contained in the Jacobian locus, which is a divisor in $\mathcal{A}_4$; this again implies that $\iota^*(\delta_0'') = 0$. Finally, $\iota^*(\delta_0') = 0$ since $\widetilde{F(V)} \subset \widetilde{\mathcal{R}}_5$ and the general element of Δ_0' is a non-admissible cover.

Since $\mathcal{T}_5^e$ is mapped by $\widetilde{\mathcal{P}}_5$ to $\theta_{null} \subset \mathcal{A}_4$, the divisor $\mathcal{T}_5^e$ does not intersect the generic fiber $\widetilde{F(V)}$. Hence, by Theorem V.1.1 we obtain

$$0 = \iota^*[\mathcal{T}_5^e] = 68\iota^*\lambda - 17\iota^*\delta_0^{ram}$$

which implies the relation $\iota^*\delta_0^{ram} = 4\iota^*\lambda$.

Now, the restriction of $\mathcal{T}_5^o$ to the fiber $\widetilde{F(V)}$ has class

$$\iota^*[\mathcal{T}_5^o] = 64\iota^*\lambda - 15\iota^*\delta_0^{ram} = 4\iota^*\lambda,$$

which is clearly nonzero since the Hodge structure cannot remain constant along the (open) subset of $\widetilde{F(V)}$ formed by smooth covers. Therefore, the restriction of $\mathcal{T}_5^o$ to the general fiber is not trivial and $\mathcal{T}_5^o$ dominates $\mathcal{A}_4$. □

In the previous proof, observe that the class $\iota^*[\mathcal{T}_5^o]$ equals $\iota^*\delta_0^{ram}$. This is consistent with the fact that the involution $j : \widetilde{F(V)} \to \widetilde{F(V)}$ (induced by the double étale cover) exchanges $\mathcal{T}_5^o \cap \widetilde{F(V)}$ and $\Delta_0^{ram} \cap \widetilde{F(V)}$. In order to understand this, we need to recall the geometric description of j given also by Donagi in [Don92, Section 5].

Given a smooth cover $(C, \eta) \in \widetilde{F(V)}$, there exists a unique representation $JC = \widetilde{\mathcal{P}}_6(Q, \nu)$ as the Prym of a cover of a plane quintic Q (the theta-characteristic $\mathcal{O}_Q(1) \otimes \nu$ on Q being even). More explicitly, (Q, ν) is the double cover induced by the involution -1_{JC} on the symmetric curve $W_4^1(C) \subset JC$.

Consider the short exact sequence

$$0 \to \langle \nu \rangle \to \langle \nu \rangle^\perp \to JC_2 \to 0$$

of Lemma IV.3.1.(1); here $\langle \nu \rangle^\perp \subset JQ_2$ is the orthogonal for the Weil pairing on JQ_2. Then the preimage of $\langle \eta \rangle \subset JC_2$ is a totally isotropic subgroup of four elements $0, \nu, \sigma$ and $\sigma \otimes \nu$; moreover, the theta-characteristics $\mathcal{O}_Q(1) \otimes \sigma$ and $\mathcal{O}_Q(1) \otimes \sigma \otimes \nu$ on Q have opposite parities.

Say $\mathcal{O}_Q(1) \otimes \sigma \otimes \nu$ is even. Then $\widetilde{\mathcal{P}}_6(Q, \sigma \otimes \nu)$ is the Jacobian of a genus 5 curve C', and $\nu \in \langle \sigma \otimes \nu \rangle^\perp \subset JQ_2$ induces a nonzero element $\eta' \in (JC')_2$; one has $j(C, \eta) = (C', \eta')$.

This picture beautifully closes with the observation that $\widetilde{\mathcal{P}}_6(Q, \sigma) \cong JV$ as ppav's and the even 2-torsion point $\delta \in JV_2$ is induced by $\nu \in \langle \sigma \rangle^\perp \subset JQ_2$. In particular, the double cover $\widetilde{F(V)} \to F(V)$ sends (C, η) to the line $l \in F(V)$ having Q as discriminant curve for the conic bundle structure defined by l.

Remark VI.4.2. In [Iza95, Section 3], Izadi gave an alternative description of the involution j. Given a smooth cover $(\widetilde{C}, C) = (C, \eta) \in \mathcal{R}_5$, the theta-dual $T(\widetilde{C}) = V^2(C, \eta) \subset P^-(C, \eta)$ is a symmetric curve (when properly translated to $P(C, \eta)$). If C' is the quotient of $T(\widetilde{C})$ by -1, then the cover $(T(\widetilde{C}), C')$ corresponds to $(\widetilde{C}, C)$ under the involution j.

For any smooth cubic threefold $V \subset \mathbb{P}^4$, we consider the set

$$\Gamma = \{l \in F(V) \mid \exists \text{ a 2-plane } \pi \text{ and a line } r \in F(V) \text{ with } V \cdot \pi = l + 2r\}$$

parametrizing the lines $l \in F(V)$ whose conic bundle structure has a singular discriminant curve (see [Bea77b, Proposition 1.2]). In other words, Γ parametrizes presentations of JV as the Prym variety of an (admissible) cover of a singular (plane quintic) curve.

The set Γ is well known to have pure dimension 1 and it is irreducible for a general cubic threefold V; assuming the irreducibility of Γ (which will be proven in Proposition VI.4.5), let us determine the behaviour of the general fiber of $\widetilde{\mathcal{P}}_5|_{\mathcal{T}_5^o}$ under the involution j of $\widetilde{F(V)}$.

Proposition VI.4.3. *For a general $(V, \delta) \in \mathcal{RC}^+$, the preimage $\widetilde{\Gamma} \subset \widetilde{F(V)}$ of Γ by the double étale cover consists of two irreducible components, namely $\mathcal{T}_5^o \cap \widetilde{F(V)}$ and $\Delta^0_{ram} \cap \widetilde{F(V)}$. These two components are exchanged by the involution j.*

Proof. By irreducibility of Γ, $\widetilde{\Gamma}$ has at most two irreducible components. Thus for the first statement it suffices to check that both $\mathcal{T}_5^o \cap \widetilde{F(V)}$ and $\Delta^0_{ram} \cap \widetilde{F(V)}$ are contained in $\widetilde{\Gamma}$.

On the one hand, if $(C, \eta) \in \mathcal{T}_5^o \cap \widetilde{F(V)}$ with C smooth, then the associated quintic $Q = W^1_4(C)/\langle \pm 1 \rangle$ is singular; indeed, the semicanonical pencil on C is a point of $W^1_4(C)$ fixed by the involution.

On the other hand, given a general element $(C, \eta) \in \Delta^0_{ram}$ the expression of the (non-abelian) variety JC as a Prym variety necessarily comes from a non-admissible cover (Q, σ). In particular, Q is singular as well.

The fact that j exchanges the components of $\widetilde{\Gamma}$ is nothing but [Iza95, Lemma 3.14]. This is also immediately observed from Remark VI.4.2, and the fact that for smooth covers $(\widetilde{C}, C) = (C, \eta) \in \mathcal{T}_5^o$ the theta-dual $T(\widetilde{C}) = V^2(C, \eta)$ is singular by Lemma IV.3.3. □

Corollary VI.4.4. *The fiber of $\widetilde{\mathcal{P}}_5|_{\mathcal{T}_5^o}$ at a general $(V, \delta) \in \mathcal{RC}^+$ is a curve dominating Γ.*

In the rest of this section, we concentrate on the geometry of both curves Γ and $\mathcal{T}_5^o \cap \widetilde{F(V)}$; in particular, in Corollary VI.4.9 we will be more precise, by proving that $\mathcal{T}_5^o \cap \widetilde{F(V)}$ is a partial desingularization of Γ.

To this end, we consider for any smooth cubic threefold $V \subset \mathbb{P}^4$ the curve

$$\Gamma' = \{r \in F(V) \mid \exists \text{ a 2-plane } \pi \text{ and a line } l \in F(V) \text{ with } V \cdot \pi = l + 2r\}$$

formed by *lines of second type*. In contrast to Γ, this curve has received considerable attention in the literature. For instance:

(1) ([CG72, Proposition 10.21]) Γ' has pure dimension 1 and, as a divisor in the Fano surface $F(V)$, is linearly equivalent to twice the canonical divisor $K_{F(V)}$.

(2) In [Mur72, Corollary 1.9], it is stated that Γ' is smooth for every V. Nonetheless, what Murre's local computations really show is that singularities of Γ' correspond to lines $r \in F(V)$ for which there exists a 2-plane π satisfying $V \cdot \pi = 3r$.[1]

    An easy count of parameters shows that, for a general V, such lines do not exist (namely the curve Γ' is smooth).

[1] For the interested reader, the linear polynomial l appearing in equation (13) of [Mur72, Page 167] not only depends on the variables u and v, but also on the variable x.

For every $r \in \Gamma'$, there exist a unique 2-plane π and a unique $l \in F(V)$ such that $V \cdot \pi = l + 2r$ (see [NO19, Lemma 2.4]). This naturally defines a surjective morphism $\Gamma' \to \Gamma$. Even if one would be tempted to think that it defines an isomorphism between Γ' and Γ, this is not the case[2]:

Proposition VI.4.5. *For a general smooth cubic threefold V, the following hold:*

(1) *Γ' is smooth and irreducible.*

(2) *Γ is irreducible and singular, with only nodes as singularities.*

(3) *The map $\Gamma' \to \Gamma$ is birational (i.e. Γ' is the normalization of Γ).*

Proof. The smoothness of Γ' being known by the discussion above, for the irreducibility of Γ' (and hence that of Γ) one argues as in the proof of [NO19, Proposition 2.6]. In particular, the irreducibility of Γ completes the proof of Proposition VI.4.3.

In order to prove that $\Gamma' \to \Gamma$ is birational, we need to prove that a general $l \in \Gamma$ has a unique preimage in Γ'. Note that the preimages of a line $l \in \Gamma$ correspond to nodes on the discriminant (plane quintic) curve Q_l of the conic bundle structure defined by l.

Fix an even 2-torsion point $\delta \in JV_2$ (such that the pair $(V, \delta) \in \mathcal{RC}^+$ is general), and denote by $\varphi : \widetilde{F(V)} \to F(V)$ the double étale cover defined by δ. If $(C, \eta) \in \widetilde{\mathcal{R}}_5$ denotes a Prym curve lying in both $\mathcal{T}_5^o$ and $\varphi^{-1}(l)$, then by Proposition VI.4.3 the nodes of Q_l are also in correspondence with the semicanonical pencils on the curve $C \in \mathcal{T}_5$. Since the general curve of $\mathcal{T}_5$ has a unique semicanonical pencil, the birationality of $\Gamma' \to \Gamma$ follows.

Now we proceed to prove that the curve Γ is always singular. For this, it suffices to check that there exist points of Γ with (at least) two preimages in Γ'. Namely, that there exist lines $l \subset V$ such that there are two distinct 2-planes π_1, π_2 and lines $r_1, r_2 \subset V$ with the property $V \cdot \pi_i = l + 2r_i$ $(i = 1, 2)$.

Take a reference system in $\mathbb{P}^4$ so that $l \cap r_1 = [0 : 0 : 0 : 1 : 0]$, $l \cap r_2 = [0 : 0 : 0 : 0 : 1]$, $[0 : 1 : 0 : 0 : 0] \in r_1$ and $[0 : 0 : 1 : 0 : 0] \in r_2$. Denoting by x, y, z, u, v the homogeneous coordinates in this reference system, a cubic threefold V will satisfy $V \cdot \pi_i = l + 2r_i$ $(i = 1, 2)$ if and only if it admits an equation of the form

$$F(x, y, z, u, v) = x \cdot Q(x, y, z, u, v) + \lambda_{15} y^2 z + \lambda_{16} y z^2 + \lambda_{17} yzu + \lambda_{18} yzv + \lambda_{19} yv^2 + \lambda_{20} zu^2$$

with Q a quadratic polynomial. This family of equations forms a 20-dimensional linear variety in the projective space $\mathbb{P}^{34}$ of cubic equations in five variables (up to constant).

On the other hand, the projectivities leaving invariant the lines l, r_1, r_2 depend on ten projective parameters. Therefore, the moduli space of cubic threefolds V for which there exist lines $l, r_1, r_2 \subset V$ and 2-planes π_1, π_2 as asserted is 10-dimensional. In other words, every smooth cubic threefold V admits such a configuration.

[2] In particular, this fixes an inaccuracy in the original published version of [NO19, Proposition 2.6], already corrected in the arXiv version arXiv:1708.06512.v3.

A similar parameter count shows that for a general cubic threefold V:

- There are no lines $l \in \Gamma$ admitting three or more preimages in Γ'.
- There are no lines $l \in \Gamma$ admitting two distinct preimages $r_1, r_2 \in \Gamma'$ with the property $l \cap r_1 = l \cap r_2$.

This shows that, for a general cubic threefold V, the curve Γ is singular and its singular points are of multiplicity 2. Hence to finish the proof, it only remains to check that such singular points are ordinary. We will prove the following: for any singular point $l \in \Gamma$, the tangent directions to Γ' at the two preimages $r_1, r_2 \in \Gamma'$ of l are mapped to independent directions in the tangent space to $F(V)$ at l. We will use the local analysis of Γ' performed by Murre ([Mur72, Section 1A]).

According to our discussion, singularities of Γ correspond to lines $l \subset V$ for which there exist two distinct 2-planes π_1, π_2 and (disjoint) lines $r_1, r_2 \subset V$ such that $V \cdot \pi_i = l + 2r_i$ $(i = 1, 2)$. Taking coordinates as before, we let

$$F(x,y,z,u,v) = \lambda_0 x^3 + \lambda_1 x^2 y + \lambda_2 x^2 z + \lambda_3 x^2 u + \lambda_4 x^2 v + \lambda_5 xy^2 + \lambda_6 xyz + \lambda_7 xyu + \lambda_8 xyv + \lambda_9 xz^2$$
$$+\lambda_{10} xzu + \lambda_{11} xzv + \lambda_{12} xu^2 + \lambda_{13} xuv + \lambda_{14} xv^2 + \lambda_{15} y^2 z + \lambda_{16} yz^2 + \lambda_{17} yzu + \lambda_{18} yzv + \lambda_{19} yv^2 + \lambda_{20} zu^2$$

be the equation defining V. Observe that $\lambda_{19} \neq 0$ and $\lambda_{20} \neq 0$, otherwise V would contain one of the 2-planes $\pi_1 : x = z = 0$, $\pi_2 : x = y = 0$ and hence V would be singular.

In the Grassmannian $\mathbb{G}(1,4)$ of lines in $\mathbb{P}^4$, we take local coordinates $x', x'', y', y'', z', z''$ for lines l' around l, given by

$$l' \cap \{v = 0\} = [x' : y' : z' : 1 : 0], \;\; l' \cap \{u = 0\} = [x'' : y'' : z'' : 0 : 1].$$

Since F can be written as $F = x \cdot f(x,y,z,u,v) + y \cdot g(x,y,z,u,v) + z \cdot h(x.y,z,u,v)$ with

$$\begin{aligned} f(x,y,z,u,v) &= \lambda_{12} u^2 + \lambda_{13} uv + \lambda_{14} v^2 + \text{terms of lower degree in } u, v, \\ g(x,y,z,u,v) &= \lambda_{19} v^2 + \text{terms of lower degree in } u, v, \\ h(x,y,z,u,v) &= \lambda_{20} u^2 + \text{terms of lower degree in } u, v, \end{aligned}$$

following [Mur72, Section 1A] the tangent plane $T_l F(V)$ to $F(V)$ at l is described by the four independent equations

$$\lambda_{12} x' + \lambda_{20} z' = \lambda_{13} x' + \lambda_{12} x'' + \lambda_{20} z'' = \lambda_{14} x' + \lambda_{13} x'' + \lambda_{19} y' = \lambda_{14} x'' + \lambda_{19} y'' = 0.$$

Since $\lambda_{19}, \lambda_{20} \neq 0$, observe that the coordinates x', x'' are independent in this tangent plane.

Similarly, we take local coordinates $a', a'', b', b'', c', c''$ for lines r' around r_1 in $\mathbb{G}(1,4)$, where

$$r' \cap \{u = 0\} = [a' : 1 : b' : 0 : c'], \;\; r' \cap \{y = 0\} = [a'' : 0 : b'' : 1 : c''].$$

Following again [Mur72, Section 1A], $T_{r_1} F(V)$ is described by the independent equations

$$\lambda_5 a' + \lambda_{15} b' = \lambda_7 a' + \lambda_5 a'' + \lambda_{17} b' + \lambda_{15} b'' = \lambda_{12} a' + \lambda_7 a'' + \lambda_{20} b' + \lambda_{17} b'' = \lambda_{12} a'' + \lambda_{20} b'' = 0,$$

which are equivalent to $a' = a'' = b' = b'' = 0$. Therefore, we may take c', c'' as coordinates in the tangent plane $T_{r_1}F(V)$, which is naturally identified with the set of lines contained in the 2-plane π_1 (and avoiding the point $[0:0:0:0:1] \in \pi_1$).

Under our assumptions of generality on V, the analysis in [Mur72, Section 1A] shows that Γ' is smooth at r_1, with tangent line

$$T_{r_1}(\Gamma') : (\lambda_7\lambda_{20} - \lambda_{12}\lambda_{17})c' + (\lambda_{12}\lambda_{15} - \lambda_5\lambda_{20})c'' = 0$$

(again, $\lambda_7\lambda_{20} - \lambda_{12}\lambda_{17}$ and $\lambda_{12}\lambda_{15} - \lambda_5\lambda_{20}$ are not simultaneously zero by smoothness of V).

Let us assume that $\lambda_7\lambda_{20} - \lambda_{12}\lambda_{17} \neq 0$. Given $c'' \in \mathbb{C}$, we denote by $r_{1,c''} \in T_{r_1}(\Gamma')$ the line

$$r_{1,c''} = \left[0:1:0:0:\frac{\lambda_5\lambda_{20} - \lambda_{12}\lambda_{15}}{\lambda_7\lambda_{20} - \lambda_{12}\lambda_{17}}c''\right] \vee [0:0:0:1:c''].$$

Using the description in [Mur72, 1.3], elementary (but tedious) calculations show that the first order deformation $r_{1,c''}$ of r_1 along Γ' yields a first order deformation of the 2-plane π_1 given by

$$\pi_{1,c''} = r_{1,c''} \vee \left[\frac{\lambda_{19}\lambda_{20}}{\lambda_7\lambda_{20} - \lambda_{12}\lambda_{17}}c'' : 0 : -\frac{\lambda_{12}\lambda_{19}}{\lambda_7\lambda_{20} - \lambda_{12}\lambda_{17}}c'' : 0 : 1\right],$$

and thus it yields a first order deformation $l_{c''}$ of l along Γ given by the following local coordinates around l:

$$x'(l_{c''}) = 0, \quad y'(l_{c''}) = -\frac{\lambda_{13}\lambda_{20}}{\lambda_7\lambda_{20} - \lambda_{12}\lambda_{17}}c'', \quad z'(l_{c''}) = 0,$$
$$x''(l_{c''}) = \frac{\lambda_{19}\lambda_{20}}{\lambda_7\lambda_{20} - \lambda_{12}\lambda_{17}}c'', \quad y''(l_{c''}) = -\frac{\lambda_{14}\lambda_{20}}{\lambda_7\lambda_{20} - \lambda_{12}\lambda_{17}}c'', \quad z''(l_{c''}) = -\frac{\lambda_{12}\lambda_{19}}{\lambda_7\lambda_{20} - \lambda_{12}\lambda_{17}}c''.$$

In other words, the first order deformation of r_1 along Γ' defines the tangent direction $x' = 0$ to Γ at l. A similar analysis shows that the first order deformation of r_2 along Γ' induces the tangent direction $x'' = 0$ to Γ at l. Since both directions are distinct, it turns out that l is a node of the curve Γ, which finishes the proof. □

In view of the existence of singularities of Γ and their geometric significance, it seems natural to ask about the number of nodes of the curve Γ for a general cubic threefold V.

Let us recall that for V very general, the Fano surface $F(V)$ has Picard rank 1 (see [Rou11, Section 1.3]). The (numerical) Néron-Severi group $\mathrm{NS}(F(V)) \cong \mathbb{Z}$ is generated by the class L of the incidence divisor

$$C_s = (\text{closure of}) \ \{l \in F(V) \mid l \cap s \neq \emptyset,\ l \neq s\} \subset F(V)$$

of lines intersecting a given line $s \in F(V)$ (note that $C_s \subset JV$ is the Abel-Prym curve for the Prym presentation of JV using the conic bundle structure defined by s).

We have $L^2 = 5$, as this is the number of lines on a smooth cubic surface intersecting two given skew lines on it. Moreover, $K_{F(V)} = 3L$ in $\mathrm{NS}(F(V))$ and Γ' is linearly equivalent to $2K_{F(V)}$ (see [CG72, Section 10]).

Theorem VI.4.6. *For every smooth cubic threefold V, the curve Γ is numerically equivalent to $8K_{F(V)}$ in $F(V)$.*

Proof. Since both Γ and $K_{F(V)}$ are the restriction to V of divisors in the universal Fano variety of lines in cubic threefolds, it is enough to prove the result when V is very general.

Pick an even 2-torsion point $\delta \in JV_2$, and consider the double étale cover $\varphi : \widetilde{F(V)} \to F(V)$ defined by δ. Recall from Proposition VI.4.3 that $\widetilde{\Gamma} = \varphi^{-1}(\Gamma)$ has $\mathcal{T}_5^o \cap \widetilde{F(V)}$ and $\Delta_0^{ram} \cap \widetilde{F(V)}$ as irreducible components (exchanged by the natural involution on $\widetilde{F(V)}$).

Let us write mL for the class of Γ in $\mathrm{NS}(F(V))$. Recall from the proof of Proposition VI.4.1 that, if $\iota^* : \mathrm{Pic}(\overline{\mathcal{R}}_5)_{\mathbb{Q}} \to \mathrm{Pic}(\widetilde{F(V)})_{\mathbb{Q}}$ is the natural pullback map, then $\iota^*([\mathcal{T}_5^o]) = \iota^*(\delta^0_{ram}) = 4\iota^*\lambda$ and ι^* annihilates any other basic divisor class of $\mathrm{Pic}(\overline{\mathcal{R}}_5)_{\mathbb{Q}}$. Therefore, we have an equality

$$m\varphi^* L = \varphi^*\Gamma = 8\iota^*\lambda$$

in $\mathrm{NS}(\widetilde{F(V)})_{\mathbb{Q}}$. Now we will compare the classes φ^*L and $\iota^*\lambda$ by means of the canonical divisor $K_{\widetilde{F(V)}}$ of $\widetilde{F(V)}$.

On the one hand, note that $\widetilde{F(V)} \subset \widetilde{\mathcal{R}}_5$ is the general fiber of the rational map $\overline{\mathcal{P}}_5 : \overline{\mathcal{R}}_5 \dashrightarrow \overline{\mathcal{A}}_4$ (here $\overline{\mathcal{A}}_4$ denotes any toroidal compactification of $\mathcal{A}_4$). It follows that $K_{\widetilde{F(V)}} = \iota^*(K_{\overline{\mathcal{R}}_5})$ in $\mathrm{Pic}(\widetilde{F(V)})_{\mathbb{Q}}$. Using the expression for the canonical class $K_{\overline{\mathcal{R}}_5}$ given in [FL10, Theorem 1.5], we obtain

$$K_{\widetilde{F(V)}} = \iota^*\big(13\lambda - 2(\delta_0' + \delta_0'') - 3\delta_0^{ram} - \dots\big) = 13\iota^*\lambda - 3\iota^*\delta_0^{ram} = \iota^*\lambda.$$

On the other hand, applying the Hurwitz formula to the double étale cover φ yields an equality

$$K_{\widetilde{F(V)}} = \varphi^* K_{F(V)} = 3\varphi^* L$$

in $\mathrm{NS}((\widetilde{F(V)})$. Comparing both expressions for $K_{\widetilde{F(V)}}$, we find the equality $\iota^*\lambda = 3\varphi^*L$ in $\mathrm{NS}((\widetilde{F(V)})_{\mathbb{Q}}$, and hence

$$m\varphi^* L = \varphi^*\Gamma = 8\iota^*\lambda = 24\varphi^* L$$

from which we deduce that $m = 24$. □

Now we can answer the question above, namely, count the number of nodes in the curve Γ for a general cubic threefold V. Indeed, this number arises as the difference between the arithmetic genus and the geometric genus of Γ.

The geometric genus $g(\Gamma)$ of Γ is that of its normalization Γ'. Since $\omega_{\Gamma'} = \mathcal{O}_{\Gamma'}(\Gamma' + K_{F(V)})$ by adjunction, we have

$$2g(\Gamma') - 2 = 6L \cdot (6L + 3L) = 54L^2 = 270$$

from which the equality $g(\Gamma') = 136$ follows.

On the other hand, since $\Gamma = 24L$ in $\mathrm{NS}(F(V))$ by Theorem VI.4.6, again by adjunction the arithmetic genus of Γ satisfies

$$2p_a(\Gamma) - 2 = 24L \cdot (24L + 3L) = 648L^2 = 3240$$

and thus $p_a(\Gamma) = 1621$. Therefore, Γ has exactly 1485 nodes.

Geometrically these nodes translate as follows:

Corollary VI.4.7. *A general smooth cubic threefold $V \subset \mathbb{P}^4$ contains exactly 1485 lines $l \subset V$ for which there exist 2-planes $\pi_1, \pi_2 \subset \mathbb{P}^4$ and lines $r_1, r_2 \subset V$ satisfying $V \cdot \pi_i = l + 2r_i$.*

Another consequence of Theorem VI.4.6 is a good control of the intersection of Γ and Γ', which reads geometrically as:

Corollary VI.4.8. *A general smooth cubic threefold $V \subset \mathbb{P}^4$ contains exactly 720 lines $l \subset V$ with the following property: there exist 2-planes $\pi_1, \pi_2 \subset \mathbb{P}^4$ and lines $r_1, r_2 \subset V$ such that $V \cdot \pi_1 = l + 2r_1$ and $V \cdot \pi_2 = 2l + r_2$.*

Coming back to our description of the general fiber of the restricted Prym map $\widetilde{\mathcal{P}}_5|_{\mathcal{T}_5^o}$, we find:

Corollary VI.4.9. *For a general $(V, \delta) \in \mathcal{RC}^+$, the following hold:*

(1) *$\mathcal{T}_5^o \cap \widetilde{F(V)}$ is a partial desingularization of Γ, with exactly 765 nodes.*

(2) *The intersection of $\mathcal{T}_5^o \cap \widetilde{F(V)}$ and $\Delta_0^{ram} \cap \widetilde{F(V)}$ is transverse, and consists of 1440 points.*

Proof. As usual, let us denote by $\varphi : \widetilde{F(V)} \to F(V)$ the double étale cover induced by δ, whose associated involution exchanges the two components $\mathcal{T}_5^o \cap \widetilde{F(V)}$ and $\Delta_0^{ram} \cap \widetilde{F(V)}$ of $\widetilde{\Gamma} = \varphi^{-1}(\Gamma)$.

Since the morphism $\varphi : \widetilde{\Gamma} \to \Gamma$ is étale and Γ has only nodes as singularities, it follows that $\mathcal{T}_5^o \cap \widetilde{F(V)}$ and $\Delta_0^{ram} \cap \widetilde{F(V)}$ have only nodes as singularities, and intersect transversely.

Conversely, the preimage of a node of Γ must consist of:

- Either a node of $\mathcal{T}_5^o \cap \widetilde{F(V)}$ and a node of $\Delta_0^{ram} \cap \widetilde{F(V)}$.
- Or two intersection points of $\mathcal{T}_5^o \cap \widetilde{F(V)}$ and $\Delta_0^{ram} \cap \widetilde{F(V)}$ (where the intersection is transverse).

Therefore the proof is reduced to compute the intersection number $[\mathcal{T}_5^o \cdot \Delta_{ram}^0]_{\widetilde{F(V)}}$. Using the projection formula and Theorem VI.4.6, we deduce that

$$2[\mathcal{T}_5^o \cdot \Delta_{ram}^0]_{\widetilde{F(V)}} = [\mathcal{T}_5^o \cdot \Delta_{ram}^0]_{\widetilde{F(V)}} + [\mathcal{T}_5^o \cdot \mathcal{T}_5^o]_{\widetilde{F(V)}} = [\mathcal{T}_5^o \cdot \varphi^*\Gamma]_{\widetilde{F(V)}} =$$
$$= [\varphi_*(\mathcal{T}_5^o \cdot \varphi^*\Gamma)]_{F(V)} = [\Gamma \cdot \Gamma]_{F(V)} = 2880$$

(the first equality follows from the fact hat $\mathcal{T}_5^o$ and Δ_{ram}^0 have the same class in $\widetilde{F(V)}$).

It turns out that $[\mathcal{T}_5^o \cdot \Delta_{ram}^0]_{\widetilde{F(V)}} = 1440$. According to the previous description, these 1440 intersection points form the preimage by φ of 720 nodes of Γ. The remaining 765 nodes of Γ lift to nodes of $\mathcal{T}_5^o \cap \widetilde{F(V)}$ and $\Delta_0^{ram} \cap \widetilde{F(V)}$. ∎

VI.5 Genus at least 6

In this final section we examine the restricted maps $\mathcal{P}_g\,|_{\mathcal{T}_g^e}$ and $\mathcal{P}_g\,|_{\mathcal{T}_g^o}$ for $g \geq 6$. First part we prove that they are generically finite onto their image, and we pay special attention to their degrees for $g = 6$. In the second part, we propose a geometric characterization of $\mathcal{P}_6(\mathcal{T}_6^o)$ as a divisor in $\mathcal{A}_5$.

Generic finiteness on $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$

Our first purpose is to prove that, for $g \geq 6$, the restrictions $\mathcal{P}_g\,|_{\mathcal{T}_g^e}$ and $\mathcal{P}_g\,|_{\mathcal{T}_g^o}$ are generically finite onto their image. This result is actually valid for restrictions to arbitrary divisors when $g \geq 8$, whereas in the cases $g = 6, 7$ the use of specific cohomology classes is required in our approach:

Theorem VI.5.1. *For every $g \geq 6$ the restricted Prym maps $\mathcal{P}_g\,|_{\mathcal{T}_g^e}$ and $\mathcal{P}_g\,|_{\mathcal{T}_g^o}$ are generically finite onto their image.*

Proof. It is well known (see the proof of the main theorem and the remark in [Nar96]) that, if $(C, \eta) \in \mathcal{R}_g$ is a point where the differential $d\mathcal{P}_g$ fails to be injective, then $\mathrm{Cliff}(C) \leq 2$. According to classical results of Martens ([Mar80, Beispiel 7 and 8]), this means that either C has a g^1_4 or C is a plane sextic; of course the latter may only happen if $g = 10$.

If $g \geq 8$, the locus in $\mathcal{M}_g$ of tetragonal curves has codimension at least 2, whereas the locus of plane sextics in $\mathcal{M}_{10}$ has codimension 8. It turns out that, for $g \geq 8$, $d\mathcal{P}_g$ is injective at the general point of any divisor $D \subset \mathcal{R}_g$ and hence $\mathcal{P}_g\,|_D$ is generically finite onto its image.

If $g = 7$, the Brill-Noether number $\rho(7, 1, 4)$ equals -1. This implies (see [EH89]) that the Brill-Noether locus $\mathcal{M}^1_{7,4}$ of tetragonal curves in $\mathcal{M}_7$ is an irreducible divisor, whose class in $\mathrm{Pic}(\overline{\mathcal{M}}_7)_{\mathbb{Q}}$ is known up to scalar:

$$[\mathcal{M}^1_{7,4}] = c\big(10\lambda - \frac{4}{3}\delta_0 - 6\delta_1 - 10\delta_2 - 12\delta_3\big),$$

for some $c \in \mathbb{Q}$. On the other hand, the class of $\mathcal{T}_7 \subset \overline{\mathcal{M}}_7$ is

$$[\mathcal{T}_7] = 16(129\lambda - 16\delta_0 - 63\delta_1 - 93\delta_2 - 105\delta_3)$$

(see [TiB88]). Comparing both expressions we conclude that the general elements of $\mathcal{T}_7^e$ and $\mathcal{T}_7^o$ are not covers of tetragonal curves.

For the case $g = 6$, we use [Far12, Section 8]. There, by an analysis of the syzygies of Prym-canonical curves, the locus in $\mathcal{R}_6$ where the infinitesimal Prym-Torelli theorem fails is characterized as a divisor $\mathcal{U}_{6,0} \subset \mathcal{R}_6$[3], which is nothing but the ramification divisor of the generically finite map $\mathcal{P}_6$.

[3] The divisor $\mathcal{U}_{6,0}$ is a particular case of the loci $\mathcal{U}_{2i+6,i} \subset \mathcal{R}_{2i+6}$ of Prym curves whose Prym-canonical model has a nonlinear i-th syzygy. According to the *Prym-Green conjecture*, every $\mathcal{U}_{2i+6,i}$ is expected to be a divisor in $\mathcal{R}_{2i+6}$; see [Far12, Section 8] for more details.

The cohomology class of $\mathcal{U}_{6,0}$ in $\mathrm{Pic}(\overline{\mathcal{R}}_6)_{\mathbb{Q}}$ is given by the following formula (see [FL10, Theorem 0.6] and [Far12, Theorem 8.6]):

$$[\mathcal{U}_{6,0}] = 7\lambda - \frac{3}{2}\delta^{ram} - \delta_0' - \dots$$

Furthermore, the divisor $\mathcal{U}_{6,0}$ is irreducible (see [FGSMV14, Theorem 0.4], where this ramification divisor is denoted by $\mathcal{Q}$). By comparison against the classes of Theorem V.1.1, it follows that the supports of $\mathcal{T}_6^e$ and $\mathcal{T}_6^o$ are different from that of $\mathcal{U}_{6,0}$, which finishes the proof. □

In the sequel, we will focus on the case $g = 6$. Recall that the Prym map $\mathcal{P}_6 : \mathcal{R}_6 \to \mathcal{A}_5$ is dominant and generically finite of degree 27 ([DS81]). Moreover, the correspondence induced on a general fiber by the tetragonal construction is isomorphic to the incidence correspondence on the 27 lines of a smooth cubic surface ([Don92, Section 4]).

Therefore, from Corollary VI.1.2 and Theorem VI.5.1 one immediately deduces that $\mathcal{P}_6|_{\mathcal{T}_6^e}$ is generically finite of degree 27; in particular, this indicates that $\mathcal{T}_6^e$ remains invariant under the tetragonal construction. And in fact, we have:

Proposition VI.5.2. *Let (C_i, η_i, M_i) $(i = 1, 2, 3)$ be a tetragonally related triple of smooth Prym curves $(C_i, \eta_i) \in \mathcal{R}_6$ with a g_4^1 M_i on C_i.*

(1) *If $(C_1, \eta_1) \in \mathcal{T}_6^e$ is general, then $(C_2, \eta_2), (C_3, \eta_3) \in \mathcal{T}_6^e$ as well.*
(2) *If $(C_1, \eta_1) \in \mathcal{T}_6^o$ is general, then $JC_2, JC_3 \in \mathcal{P}_7(\mathcal{T}_7^o)$.*

Proof. According to Lemma IV.3.4, giving the tetragonally related triple (C_i, η_i, M_i) is equivalent to giving a trigonal curve $R \in \mathcal{M}_7$ and a subgroup $W = \{0, \mu_1, \mu_2, \mu_3\} \subset JR_2$ (totally isotropic with respect to the Weil pairing), in such a way that:

- (R, μ_i) corresponds to (C_i, M_i) under Recillas' trigonal construction.
- The 2-torsion point $\eta_i \in (JC_i)_2$ is defined by $\mu_j \in \langle \mu_i \rangle^\perp$ $(j \neq i)$.

Fix a general $(C_1, \eta_1) \in \mathcal{T}_6^e$ (resp. a general $(C_1, \eta_1) \in \mathcal{T}_6^o$), and consider any g_4^1 M_1 on C_1. By generality, C_1 admits a unique theta-characteristic L_1 with $h^0(C_1, L_1) \geq 2$, which is a semicanonical pencil with $h^0(C_1, L_1) = 2$ and $L_1 \otimes \eta_1$ even (resp. odd).

Let $R \in \mathcal{M}_7$ be the trigonal curve and let $W \subset JR_2$ be the totally isotropic subgroup defining the tetragonally related triple. Since $\mathcal{P}_7(R, \mu_1) = JC_1 \in \theta_{null} \subset \mathcal{A}_6$, it follows from Corollary VI.1.2 and Remark VI.1.3 that $(R, \mu_1) \in \mathcal{T}_7^e$, so R has a semicanonical pencil L_R such that $h^0(R, L_R \otimes \mu_1)$ is even. Moreover, if $f_1 : R_1 \to R$ is the cover determined by $(R, \mu_1) \in \mathcal{R}_7$, then the 2-torsion singular point L_1 of the canonical theta divisor $\Theta_{C_1} \subset \mathrm{Pic}^5(C_1)$ corresponds to $f_1^* L_R$ under the identification of $(P^+(R, \mu_1), \Xi^+(R, \mu_1))$ with $\left(\mathrm{Pic}^5(C_1), \Theta_{C_1}\right)$.

Now we want to determine the parity of the theta-characteristics $L_R \otimes \mu_2$ and $L_R \otimes \mu_3$. Observe that both parities are equal, since the Riemann-Mumford relation (see [Mum71] or [Har82, Theorem 1.13]) reads as follows:

$$h^0(R, L_R) + h^0(R, L_R \otimes \mu_1) + h^0(R, L_R \otimes \mu_2) + h^0(R, L_R \otimes \mu_3) \equiv \langle \mu_1, \mu_2 \rangle \equiv 0 \pmod 2,$$

(here we use that $\mu_3 = \mu_1 \otimes \mu_2$).

Following [Don92, Theorem 1.5] and the notations therein, this means that for $i \in \{2,3\}$ we have $L_R \otimes \mu_i \in (\mu_1)^{\perp'}$, and hence:

$$\begin{aligned} h^0(R, L_R \otimes \mu_i) &\equiv q_{JR}(L_R \otimes \mu_i) \equiv q_{P(R,\mu_1)}(f_1^*(L_R \otimes \mu_i)) \\ &\equiv q_{JC_1}(L_1 \otimes \eta_1) \equiv h^0(C_1, L_1 \otimes \eta_1) \pmod 2. \end{aligned}$$

If $(C_1, \eta_1) \in \mathcal{T}_6^o$, then $L_1 \otimes \eta_1$ is an odd theta-characteristic on C_1, and hence we obtain $(R, \mu_i) \in \mathcal{T}_7^o$ for $i \in \{2,3\}$. Therefore $JC_i = \mathcal{P}_7(R, \mu_i) \in \mathcal{P}_7(\mathcal{T}_7^o)$, which proves (2).

If $(C_1, \eta_1) \in \mathcal{T}_6^e$, then $L_1 \otimes \eta_1$ is an even theta-characteristic on C_1 and thus $(R, \mu_2), (R, \mu_3) \in \mathcal{T}_7^e$. For $i \in \{2,3\}$ this gives $JC_i = \mathcal{P}_7(R, \mu_i) \in \theta_{null} \subset \mathcal{A}_6$, namely $C_i \in \mathcal{T}_6$ admits a (unique, by generality) semicanonical pencil L_i.

Therefore, to finish the proof of (1) we only have to check that $(C_i, \eta_i) \in \mathcal{T}_6^e$, namely that $L_i \otimes \eta_i$ is even. This is again a consequence of [Don92, Theorem 1.5]:

$$h^0(R, L_R \otimes \mu_1) \equiv q_{JR}(L_R \otimes \mu_1) \equiv q_{JC_i}(L_i \otimes \eta_i) \equiv h^0(C_i, L_i \otimes \eta_i) \pmod 2. \qquad \square$$

At present, we lack an interpretation for the Jacobian of a curve $C \in \mathcal{M}_6$ being the Prym variety of a trigonal cover in $\mathcal{T}_7^o$. This prevents us to completely understand the tetragonal construction applied to elements of $\mathcal{T}_6^o$, and hence to describe the (divisorial) components of $\mathcal{P}_6^{-1}(\mathcal{P}_6(\mathcal{T}_6^o))$. Another natural question would be to find the degree of the map $\mathcal{P}_6 |_{\mathcal{T}_6^o}$.

In this direction, partial information is obtained from cohomology classes. This reveals once again differences between the odd and the even case:

Proposition VI.5.3. $\mathcal{P}_6^{-1}(\mathcal{P}_6(\mathcal{T}_6^o))$ *contains other divisorial components apart from* $\mathcal{T}_6^o$*. In particular, the degree of the generically finite map* $\mathcal{P}_6 |_{\mathcal{T}_6^o}$ *is strictly smaller than* 27.

Proof. Let us denote by $\overline{\mathcal{P}} : \overline{\mathcal{R}}_6 \dashrightarrow \overline{\mathcal{A}}_5$ the rational Prym map in genus 6. Here $\overline{\mathcal{A}}_5$ stands for the perfect cone compactification of $\mathcal{A}_5$, whose rational Picard group $\mathrm{Pic}(\overline{\mathcal{A}}_5)_{\mathbb{Q}}$ is generated by the Hodge class L and the class D of the irreducible boundary divisor.

According to [FGSMV14, Theorem 7.4], the pushforwards of the basic divisor classes of $\overline{\mathcal{R}}_6$ are:

$$\begin{aligned} \overline{\mathcal{P}}_*\lambda = 18\cdot 27L - 57D, \;\; \overline{\mathcal{P}}_*\delta_0^{ram} = 4(17\cdot 27L - 57D), \;\; \overline{\mathcal{P}}_*\delta_0' = 27D, \\ \overline{\mathcal{P}}_*\delta_0'' = \overline{\mathcal{P}}_*\delta_i = \overline{\mathcal{P}}_*\delta_{g-i} = \overline{\mathcal{P}}_*\delta_{i:g-i} = 0 \;\text{ for } 1 \le i \le [g/2] \end{aligned}$$

On the other hand, the pullback map $\overline{\mathcal{P}}^* : \mathrm{Pic}(\overline{\mathcal{A}}_5)_{\mathbb{Q}} \to \mathrm{Pic}(\overline{\mathcal{R}}_6)_{\mathbb{Q}}$ satisfies

$$\overline{\mathcal{P}}^* L = \lambda - \frac{1}{4}\delta_0^{ram}, \qquad \overline{\mathcal{P}}^* D = \delta_0'$$

(see [GM14, Theorem 5]). The boundary divisors $\delta_0'', \delta_i, \delta_{g-i}, \delta_{i:g-i}$ do not appear since they are contracted by $\overline{\mathcal{P}}$.

Using the cohomology class $[\mathcal{T}_6^o] \in \mathrm{Pic}(\overline{\mathcal{R}}_6)_{\mathbb{Q}}$ dictated by Theorem V.1.1, we have

$$\overline{\mathcal{P}}_*[\mathcal{T}_6^o] = 10584L - 1320D$$

Observe that this class equals $d \cdot [\mathcal{P}_6(\mathcal{T}_6^o)]$, where $d = \deg(\mathcal{P}_6 |_{\mathcal{T}_6^o})$ and $[\mathcal{P}_6(\mathcal{T}_6^o)] \in \mathrm{Pic}(\overline{\mathcal{A}}_5)_{\mathbb{Q}}$ is the class of (the closure in $\overline{\mathcal{A}}_5$ of) $\mathcal{P}_6(\mathcal{T}_6^o)$. Pulling back we obtain

$$\overline{\mathcal{P}}^*\overline{\mathcal{P}}_*[\mathcal{T}_6^o] = 10584\lambda - 1320\delta_0' - 2646\delta_0^{ram}$$

Since these coefficients are not proportional to the corresponding ones in $[\mathcal{T}_6^o]$, it follows that $\mathcal{T}_6^o$ cannot be the unique divisorial component of $\mathcal{P}_6^{-1}(\mathcal{P}_6(\mathcal{T}_6^o))$. □

$\mathcal{T}_6^o$ and singular surfaces of twice the minimal class

In this final subsection we give the first steps towards an intrinsic description of the locus $\mathcal{P}_6(\mathcal{T}_6^o)$ in $\mathcal{A}_5$, with the help of Brill-Noether loci on Prym varieties. In order to be consistent with the notation in the proof of Theorem VI.5.1, we denote by $\mathcal{U}_{6,0} \subset \mathcal{R}_6$ the ramification divisor of $\mathcal{P}_6$.

Recall that the Andreotti-Mayer locus $\mathcal{N}_0$ in $\mathcal{A}_5$ is the union of two irreducible divisors θ_{null} and $\mathcal{N}_0'$. The theta divisor of a general element of θ_{null} has a unique singular point (which is 2-torsion), whereas the theta divisor of a general element of $\mathcal{N}_0'$ has exactly two singular (opposite) points.

The relation between $\mathcal{P}_6$ and the component $\mathcal{N}_0'$ of the Andreotti-Mayer locus in $\mathcal{A}_5$ is described in [FGSMV14, Sections 6 and 7]. In particular, the following statements hold:

(1) The divisor $\mathcal{N}_0' \subset \mathcal{A}_5$ is the branch divisor of $\mathcal{P}_6$.

(2) The preimage $\mathcal{P}_6^{-1}(\mathcal{N}_0')$ has two divisorial components: the ramification divisor $\mathcal{U}_{6,0}$ and an *antiramification divisor* $\mathcal{U}$. As cycles, there is an equality

$$\mathcal{P}_6^*\mathcal{N}_0' = 2\mathcal{U}_{6,0} + \mathcal{U}.$$

(3) $\mathcal{U}_{6,0}$ is the set of $(C,\eta) \in \mathcal{R}_6$ for which $V^3(C,\eta) \neq \emptyset$ (i.e. the theta divisor of $P(C,\eta)$ has a stable singularity), and is mapped six-to-one to $\mathcal{N}_0'$ (see [Don81, Corollary 2.3]).

(4) $\mathcal{U} = \pi^*(\mathcal{GP}_{6,4}^1)$ is the pullback to $\mathcal{R}_6$ of the Gieseker-Petri locus

$$\mathcal{GP}_{6,4}^1 = \left\{ C \in \mathcal{M}_6 \mid \exists L \in W_4^1(C) \text{ such that the Petri map } \mu_{0,L} \text{ is not injective} \right\}$$

and is mapped fifteen-to-one to $\mathcal{N}_0'$.

As usual, for $(C,\eta) \in \mathcal{R}_6$ let us denote by $f : \widetilde{C} \to C$ the corresponding double étale cover, and by $\sigma : \widetilde{C} \to \widetilde{C}$ the involution exchanging sheets.

Proposition VI.5.4. *If $(C,\eta) \in \mathcal{R}_6$ is a non-hyperelliptic Prym curve with $V^4(C,\eta) = \emptyset$, then $V^2(C,\eta)$ is singular if and only if $(C,\eta) \in \mathcal{U} \cup \mathcal{T}_6^o$.*

Proof. According to Lemma IV.3.3, singular points $M \in V^2(C,\eta)$ are exactly those of the form

$$M = f^*L \otimes A$$

with $h^0(C,L) \geq 2$ and $h^0(\widetilde{C},A) > 0$. In order to prove the statement, we distinguish the possible values of $d = \deg L$ allowing the existence of such an M.

For $d \leq 4$, this condition is equivalent to the existence of $L \in W^1_d(C)$ and A effective satisfying $h^0(\widetilde{C}, f^*L \otimes A) = 3$ and $\omega_C = L^2 \otimes \mathrm{Nm}_f(A)$. This happens if and only if there exists $L \in W^1_d(C)$ with $\omega_C \otimes L^{-2}$ effective: the "only if" part being clear, if $\omega_C \otimes L^{-2}$ is effective then Mumford's parity trick ([Mum83, bottom of page 186]) allows to find A effective with $\mathrm{Nm}_f(A) = \omega_C \otimes L^{-2}$ and $h^0(\widetilde{C}, f^*L \otimes A) = 3$.

If there exists $L \in W^1_3(C)$, then one immediately checks that $\omega_C \otimes L^{-2}$ is effective. Moreover, take $x \in C$ such that $\omega_C \otimes L^{-2}(-x)$ is effective. Since $\mathrm{Cliff}(C) \geq 1$ by assumption, one has $h^0(C, L(x)) = 2$ and the kernel of the Petri map

$$\mu_{0,L(x)} : H^0(C, L(x)) \otimes H^0(C, \omega_C \otimes L^{-1}(-x)) \longrightarrow H^0(C, \omega_C)$$

is $\ker(\mu_{0,L(x)}) \cong H^0(C, \omega_C \otimes L^{-2}(-x)) \neq 0$ by the base-point-free pencil trick ([ACGH85, page 126]). In other words, both statements $\mathrm{Sing}\, V^2(C,\eta) \neq \emptyset$ and $(C,\eta) \in \mathcal{U}$ hold whenever C is trigonal.

Now assume that C is not trigonal. We claim that the existence of $L \in W^1_4(C)$ with $\omega_C \otimes L^{-2}$ effective is equivalent to $C \in \mathcal{GP}^1_{6,4}$, namely to $(C,\eta) \in \mathcal{U}$. Indeed, if one can write $\omega_C = L^2(a+b)$ for points $x, y \in C$, then $\omega_C \otimes L^{-1} = L(x+y)$ and thus the Petri map $\mu_{0,L}$ fails to be injective. Conversely, if the Petri map

$$\mu_{0,L} : H^0(C,L) \otimes H^0(C, \omega_C \otimes L^{-1}) \longrightarrow H^0(C,\omega_C)$$

of a certain $L \in W^1_4(C)$ has nonzero kernel, then the line bundle $\omega_C \otimes L^{-2}$ is effective since $\ker(\mu_{0,L}) \cong H^0(C, \omega_C \otimes L^{-2})$ by the base-point-free pencil trick.

Now we can assume that there exists no $L \in W^1_4(C)$ with $\omega_C \otimes L^{-2}$ effective. It only remains to check the case $d = 5$: the condition reads $f^*L \in V^2(C,\eta)$, which is equivalent to the existence of a theta-characteristic L on C with

$$h^0(C,L) + h^0(C, L \otimes \eta) = 3.$$

Under our assumption on $W^1_4(C)$, this is equivalent to either L or $L \otimes \eta$ being an odd semicanonical pencil for the cover f, namely to $(C,\eta) \in \mathcal{T}^o_6$. □

Remark VI.5.5.

(1) If $V^4(C,\eta) \neq \emptyset$, then $V^2(C,\eta)$ is automatically singular at points $M \in V^4(C,\eta)$ (this is an immediate application of [ACGH85, Proposition IV.4.2]).

On the other hand, for $M \in V^4(C,\eta)$ one immediately deduces from Mumford's parity trick that $M(x - \sigma(x)) \in V^3(C,\eta)$ for every $x \in \widetilde{C}$. As a consequence, $V^3(C,\eta)$ is at least 1-dimensional whenever $V^4(C,\eta) \neq \emptyset$ (in particular, $(C,\eta) \in \mathcal{U}_{6,0}$).

(2) Let $C \in \mathcal{GP}^1_{6,4}$ be general, so that there is a unique $L \in W^1_4(C)$ and unique $x, y \in C$ satisfying $L^2(x+y) = \omega_C$. If $\widetilde{x}, \sigma(\widetilde{x})$ (resp. $\widetilde{y}, \sigma(\widetilde{y})$) are the two points of $\widetilde{C}$ lying over x (resp. over y), then the four candidates to define a singularity of $V^2(C,\eta)$ are:

$$f^*L(\widetilde{a}+\widetilde{b}),\;\; f^*L(\sigma(\widetilde{a})+\widetilde{b}),\;\; f^*L(\widetilde{a}+\sigma(\widetilde{b})),\;\; f^*L(\sigma(\widetilde{a})+\sigma(\widetilde{b})).$$

By Mumford's parity trick, these line bundles can be divided into two pairs (namely $f^*L(\widetilde{a}+\widetilde{b}), f^*L(\sigma(\widetilde{a})+\sigma(\widetilde{b}))$ and $f^*L(\sigma(\widetilde{a})+\widetilde{b}), f^*L(\widetilde{a}+\sigma(\widetilde{b}))$) according to the component of $\mathrm{Nm}^{-1}(\omega_C) = P^+ \cup P^-$ in which they live.

Since $(C,\eta) \notin \mathcal{U}_{6,0}$ by genericity (and thus $V^3(C,\eta) = \emptyset$), it follows that two of them satisfy $h^0 = 2$ and the other two satisfy $h^0 = 3$. In other words, for a general $(C,\eta) \in \mathcal{U}$ the theta divisor Ξ^+ has two exceptional singularities (as corresponds to $P(C,\eta) \in \mathcal{N}_0'$) and the Brill-Noether locus $V^2(C,\eta)$ has two singular points.

Let us recall that, for a non-hyperelliptic Prym curve $(C,\eta) = (\widetilde{C}, C) \in \mathcal{R}_g$, the Brill-Noether locus $V^2(C,\eta)$ (when properly translated to $P(C,\eta)$) is a subvariety of twice the minimal class (see [DCP95, Theorem 9] and [LN13, Corollary 4.4]). Moreover $V^2(C,\eta)$ is symmetric in $P(C,\eta)$, if the translation is performed with a theta-characteristic of $\widetilde{C}$ lying in P^-.

Combining this observation with Proposition VI.5.4 and Remark VI.5.5, it is tempting to propose the following analogue of the decomposition of the Andreotti-Mayer locus:

Question VI.5.6. *Let $\mathcal{V} \subset \mathcal{A}_5$ be (the closure of) the locus of ppav's (A,Θ) containing an integral surface S with the following properties:*

(1) *The cohomology class of S is twice the minimal class: $[S] = 2\frac{[\Theta]^3}{6}$ in $H^6(A,\mathbb{Z})$.*
(2) *S is symmetric.*
(3) *S has singular points.*

Then $\mathcal{V}$ decomposes as the union of two irreducible divisors: the closure of $\mathcal{P}_6(\mathcal{T}_6^o)$ (whose general element contains at least one such surface S with a unique singular point, which is 2-torsion) and $\mathcal{N}_0'$ (whose general element contains fifteen such surfaces S, with two singular opposite points each).

Bibliography

[AB13] Daniele Arcara and Aaron Bertram. Bridgeland-stable moduli spaces for K-trivial surfaces. *J. Eur. Math. Soc. (JEMS)*, 15(1):1–38, 2013. With an appendix by Max Lieblich. arXiv:0708.2247. iv, x, 2, 17, 19, 20

[AC87] Enrico Arbarello and Maurizio Cornalba. The Picard groups of the moduli spaces of curves. *Topology*, 26(2):153–171, 1987. 79

[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves. Vol. I*, volume 267 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985. 121

[Bay19] Arend Bayer. A short proof of the deformation property of Bridgeland stability conditions. *Math. Ann.*, 375(3-4):1597–1613, 2019. 18, 43

[Bea77a] Arnaud Beauville. Prym varieties and the Schottky problem. *Invent. Math.*, 41(2):149–196, 1977. vi, xii, 8, 80, 84, 86

[Bea77b] Arnaud Beauville. Variétés de Prym et jacobiennes intermédiaires. *Ann. Sci. École Norm. Sup. (4)*, 10(3):309–391, 1977. 110

[Bea82] Arnaud Beauville. Les singularités du diviseur Θ de la jacobienne intermédiaire de l'hypersurface cubique dans $\mathbf{P}^4$. In *Algebraic threefolds (Varenna, 1981)*, volume 947 of *Lecture Notes in Math.*, pages 190–208. Springer, Berlin-New York, 1982. 101

[BL04] Christina Birkenhake and Herbert Lange. *Complex abelian varieties*, volume 302 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2004. 86

[BL17] Arend Bayer and Chunyi Li. Brill-Noether theory for curves on generic abelian surfaces. *Pure Appl. Math. Q.*, 13:49–76, 2017. 58

[BLMS17] Arend Bayer, Martí Lahoz, Emanuele Macrì, and Paolo Stellari. Stability conditions on Kuznetsov components, 2017. arXiv:1703.10839. 23

[BLNP12] Miguel Angel Barja, Martí Lahoz, Juan Carlos Naranjo, and Giuseppe Pareschi. On the bicanonical map of irregular varieties. *J. Algebraic Geom.*, 21(3):445–471, 2012. iv, x

[BM11] Arend Bayer and Emanuele Macrì. The space of stability conditions on the local projective plane. *Duke Math. J.*, 160(2):263–322, 2011. arXiv:0912.0043. 18, 41

[BMS16] Arend Bayer, Emanuele Macrì, and Paolo Stellari. The space of stability conditions on abelian threefolds, and on some Calabi-Yau threefolds. *Invent. Math.*, pages 1–65, 2016. 20, 38, 55, 56, 58

[BPS20a] Miguel Ángel Barja, Rita Pardini, and Lidia Stoppino. Higher-dimensional Clifford-Severi equalities. *Commun. Contemp. Math.*, 22(8):1950079, 15, 2020. 1

[BPS20b] Miguel Ángel Barja, Rita Pardini, and Lidia Stoppino. Linear systems on irregular varieties. *J. Inst. Math. Jussieu*, 19(6):2087–2125, 2020. arXiv:1606.03290. iii, ix, 1, 49

[Bri07] Tom Bridgeland. Stability conditions on triangulated categories. *Ann. of Math. (2)*, 166(2):317–345, 2007. iv, x, 17, 18, 23, 32

[Bri08] Tom Bridgeland. Stability conditions on $K3$ surfaces. *Duke Math. J.*, 141(2):241–291, 2008. iv, x, 2, 17, 19, 20, 21

[Cau20] Federico Caucci. The basepoint-freeness threshold and syzygies of abelian varieties. *Algebra Number Theory*, 14(4):947–960, 2020. v, xi, 5, 24, 27

[CG72] C. Herbert Clemens and Phillip A. Griffiths. The intermediate Jacobian of the cubic threefold. *Ann. of Math. (2)*, 95:281–356, 1972. vii, xiii, 11, 101, 111, 114

[CJ18] Jungkai Alfred Chen and Zhi Jiang. Positivity in varieties of maximal Albanese dimension. *J. Reine Angew. Math.*, 736:225–253, 2018. iv, x

[CMGHL17] Sebastian Casalaina-Martin, Samuel Grushevsky, Klaus Hulek, and Radu Laza. Extending the Prym map to toroidal compactifications of the moduli space of abelian varieties. *J. Eur. Math. Soc. (JEMS)*, 19(3):659–723, 2017. With an appendix by Mathieu Dutour Sikirić. 84

[Cor87] Maurizio Cornalba. On the locus of curves with automorphisms. *Ann. Mat. Pura Appl. (4)*, 149:135–151, 1987. 107

[DCP95] Corrado De Concini and Piotr Pragacz. On the class of Brill-Noether loci for Prym varieties. *Math. Ann.*, 302(4):687–697, 1995. 122

[Deb92] Olivier Debarre. Le lieu des variétés abéliennes dont le diviseur thêta est singulier a deux composantes. *Ann. Sci. École Norm. Sup. (4)*, 25(6):687–707, 1992. 85

[Dol08] Igor V. Dolgachev. Rationality of $\mathcal{R}_2$ and $\mathcal{R}_3$. *Pure Appl. Math. Q.*, 4(2, Special Issue: In honor of Fedor Bogomolov. Part 1):501–508, 2008. 107

[Dol12] Igor V. Dolgachev. *Classical algebraic geometry*. Cambridge University Press, Cambridge, 2012. A modern view. 92, 107

[Don81] Ron Donagi. The tetragonal construction. *Bull. Amer. Math. Soc. (N.S.)*, 4(2):181–185, 1981. 87, 120

[Don92] Ron Donagi. The fibers of the Prym map. In *Curves, Jacobians, and abelian varieties (Amherst, MA, 1990)*, volume 136 of *Contemp. Math.*, pages 55–125. Amer. Math. Soc., Providence, RI, 1992. vii, xiii, 10, 87, 88, 104, 109, 110, 118, 119

[DS81] Ron Donagi and Roy C. Smith. The structure of the Prym map. *Acta Math.*, 146(1-2):25–102, 1981. 101, 118

[EH89] David Eisenbud and Joe Harris. Irreducibility of some families of linear series with Brill-Noether number -1. *Ann. Sci. École Norm. Sup. (4)*, 22(1):33–53, 1989. 117

[Far09] Gavril Farkas. Birational aspects of the geometry of $\overline{M}_g$. In *Surveys in differential geometry. Vol. XIV. Geometry of Riemann surfaces and their moduli spaces*, volume 14 of *Surv. Differ. Geom.*, pages 57–110. Int. Press, Somerville, MA, 2009. vii, xiii

[Far12] Gavril Farkas. Prym varieties and their moduli. In *Contributions to algebraic geometry*, EMS Ser. Congr. Rep., pages 215–255. Eur. Math. Soc., Zürich, 2012. 117, 118

[FGSMV14] G. Farkas, S. Grushevsky, R. Salvati Manni, and A. Verra. Singularities of theta divisors and the geometry of $\mathcal{A}_5$. *J. Eur. Math. Soc. (JEMS)*, 16(9):1817–1848, 2014. 118, 119, 120

[FL10] Gavril Farkas and Katharina Ludwig. The Kodaira dimension of the moduli space of Prym varieties. *J. Eur. Math. Soc. (JEMS)*, 12(3):755–795, 2010. vii, xiii, 79, 81, 115, 118

[FLZ21] Lie Fu, Chunyi Li, and Xiaolei Zhao. Stability manifolds of varieties with finite Albanese morphisms, 2021. arXiv:2103.07728. 2, 4, 22, 23, 34

[FNS20] Paola Frediani, Juan Carlos Naranjo, and Irene Spelta. The fibers of the ramified Prym map, 2020. To appear in Commun. Contemp. Math. arXiv:2007.02068. 104

[FS86] Robert Friedman and Roy Smith. Degenerations of Prym varieties and intersections of three quadrics. *Invent. Math.*, 85(3):615–635, 1986. 84

[GHS03] Tom Graber, Joe Harris, and Jason Starr. Families of rationally connected varieties. *J. Amer. Math. Soc.*, 16(1):57–67, 2003. 102, 108

[GM14] Samuel Grushevsky and Riccardo Salvati Manni. The Prym map on divisors, and the slope of $\mathcal{A}_5$. *Int. Math. Res. Not. IMRN*, 2014(24):6645–6660, 2014. With an appendix by Klaus Hulek. 119

[GP98] Mark Gross and Sorin Popescu. Equations of $(1, d)$-polarized abelian surfaces. *Math. Ann.*, 310(2):333–377, 1998. 7

[Gre84] Mark L. Green. Koszul cohomology and the geometry of projective varieties. *J. Differential Geom.*, 19(1):125–171, 1984. v, xi

[Har82] Joe Harris. Theta-characteristics on algebraic curves. *Trans. Amer. Math. Soc.*, 271(2):611–638, 1982. 94, 118

[HL10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves.* Cambridge University Press, Cambridge, 2010. 22, 25, 54

[HM82] Joe Harris and David Mumford. On the Kodaira dimension of the moduli space of curves. *Invent. Math.*, 67(1):23–88, 1982. With an appendix by William Fulton. vii, xiii, 92

[HM98] Joe Harris and Ian Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998. 93

[Hoe12] Andreas Hoering. Geometry of Brill-Noether loci on Prym varieties. *Michigan Math. J.*, 61(4):785–806, 2012. 86

[HRS96] Dieter Happel, Idun Reiten, and Sverre O. Smalø. Tilting in abelian categories and quasitilted algebras. *Mem. Amer. Math. Soc.*, 120(575):viii+ 88, 1996. 19

[IL12] E. Izadi and H. Lange. Counter-examples of high Clifford index to Prym-Torelli. *J. Algebraic Geom.*, 21(4):769–787, 2012. 87

[Ito20a] Atsushi Ito. Basepoint-freeness thresholds and higher syzygies on abelian threefolds, 2020. arXiv:2008.10272. 5

[Ito20b] Atsushi Ito. Higher syzygies on general polarized abelian varieties of type $(1, ..., 1, d)$, 2020. arXiv:2011.09687. 5, 6, 7, 72, 76

[Ito21] Atsushi Ito. M-regularity of $\mathbb{Q}$-twisted sheaves and its application to linear systems on abelian varieties, 2021. arXiv:2102.12175. v, xi, 5, 27, 73

[Iye99] Jaya N. Iyer. Projective normality of abelian surfaces given by primitive line bundles. *Manuscripta Math.*, 98(2):139–153, 1999. 6

[Iza95] E. Izadi. The geometric structure of $\mathcal{A}_4$, the structure of the Prym map, double solids and Γ_{00}-divisors. *J. Reine Angew. Math.*, 462:93–158, 1995. vii, xiii, 8, 109, 110, 111

[Jia20] Zhi Jiang. Cohomological rank functions and syzygies of abelian varieties, 2020. To appear in Math Z. arXiv:2010.10053. 5

[Jia21] Zhi Jiang. On Severi type inequalities. *Math. Ann.*, 379(1-2):133–158, 2021. 1

[JLT13] Zhi Jiang, Martí Lahoz, and Sofia Tirabassi. On the Iitaka fibration of varieties of maximal Albanese dimension. *Int. Math. Res. Not. IMRN*, (13):2984–3005, 2013. iv, x

[JP20] Zhi Jiang and Giuseppe Pareschi. Cohomological rank functions on abelian varieties. *Ann. Sci. Éc. Norm. Supér. (4)*, 53(4):815–846, 2020. iii, ix, 1, 2, 3, 4, 5, 23, 24, 27, 49, 51, 57, 58

[Kob87] Shoshichi Kobayashi. *Differential geometry of complex vector bundles.* Iwanami Shoten Publishers and Princeton University Press, Princeton, 1987. 22, 23

[Kos20] Naoki Koseki. On the Bogomolov-Gieseker inequality in positive characteristic, 2020. arXiv:2008.09800. 2, 20

[Lan16] Adrian Langer. The Bogomolov-Miyaoka-Yau inequality for logarithmic surfaces in positive characteristic. *Duke Math. J.*, 165(14):2737–2769, 2016. 2, 19

[Laz90] Robert Lazarsfeld. Projectivité normale des surfaces abéliennes. Redige par O. debarre. Prepublication no. 14. *Europroj-C.I.M.P.A.*, Nice, 1990. 6

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].* Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series. 26

[Li19] Chunyi Li. On stability conditions for the quintic threefold. *Invent. Math.*, 218(1):301–340, 2019. 2

[LN13] Martí Lahoz and Juan Carlos Naranjo. Theta-duality on Prym varieties and a Torelli theorem. *Trans. Amer. Math. Soc.*, 365(10):5051–5069, 2013. 86, 122

[LNR21] Martí Lahoz, Juan Carlos Naranjo, and Andrés Rojas. Geometry of Prym semicanonical pencils and an application to cubic threefolds, 2021. arXiv:2106.08683. 8

[LR21] Martí Lahoz and Andrés Rojas. Chern degree functions, 2021. arXiv:2105.03263. 1

[Mac12] Antony Maciocia. A Fourier-Mukai approach to the enumerative geometry of principally polarized abelian surfaces. *Math. Nachr.*, 285(16):1981–1998, 2012. 70

[Mac14] Antony Maciocia. Computing the walls associated to Bridgeland stability conditions on projective surfaces. *Asian J. Math.*, 18(2):263–279, 2014. 20

[Mar80] G. Martens. Funktionen von vorgegebener Ordnung auf komplexen Kurven. *J. Reine Angew. Math.*, 320:68–85, 1980. 117

[Mea12] Ciaran Meachan. Moduli of Bridgeland-stable objects, 2012. PhD Thesis, University of Edinburgh. 20, 58, 65, 69

[MNP16] Valeria Marcucci, Juan Carlos Naranjo, and Gian Pietro Pirola. Isogenies of Jacobians. *Algebr. Geom.*, 3(4):424–440, 2016. 96

[MP12] Valeria Ornella Marcucci and Gian Pietro Pirola. Generic Torelli theorem for Prym varieties of ramified coverings. *Compos. Math.*, 148(4):1147–1170, 2012. 8

[MP21] Carlos Maestro Pérez. Aspects of the geometry of Prym varieties and their moduli, 2021. PhD Thesis, Humboldt-Universität zu Berlin. 9, 82

[MPR21] Carlos Maestro Pérez and Andrés Rojas. The divisors of Prym semicanonical pencils, 2021. arXiv:2103.01687. 8, 9

[MS17] Emanuele Macrì and Benjamin Schmidt. Lectures on Bridgeland stability. In *Moduli of curves*, volume 21 of *Lect. Notes Unione Mat. Ital.*, pages 139–211. Springer, Cham, 2017. 17, 19, 21, 23, 32, 43

[Muk78] Shigeru Mukai. Semi-homogeneous vector bundles on an abelian variety. *Kyoto J. Math.*, 18(2):239–272, 1978. 22

[Mum66] David Mumford. On the equations defining abelian varieties. I. *Invent. Math.*, 1:287–354, 1966. 7, 28

[Mum71] David Mumford. Theta characteristics of an algebraic curve. *Ann. Sci. École Norm. Sup. (4)*, 4:181–192, 1971. 82, 118

[Mum74] David Mumford. Prym varieties. I. In *Contributions to analysis (a collection of papers dedicated to Lipman Bers)*, pages 325–350. Academic Press, New York, 1974. v, xi, 9, 84, 85, 101, 102, 108

[Mum83] David Mumford. On the Kodaira dimension of the Siegel modular variety. In *Algebraic geometry—open problems (Ravello, 1982)*, volume 997 of *Lecture Notes in Math.*, pages 348–375. Springer, Berlin, 1983. 85, 121

[Mur72] J. P. Murre. Algebraic equivalence modulo rational equivalence on a cubic threefold. *Compositio Math.*, 25:161–206, 1972. 111, 113, 114

[Nar96] Juan Carlos Naranjo. The positive-dimensional fibres of the Prym map. *Pacific J. Math.*, 172(1):223–226, 1996. 117

[NO19] Juan Carlos Naranjo and Angela Ortega. Generic injectivity of the Prym map for double ramified coverings. *Trans. Amer. Math. Soc.*, 371(5):3627–3646, 2019. With an appendix by Alessandro Verra. 8, 112

[Par00] Giuseppe Pareschi. Syzygies of abelian varieties. *J. Amer. Math. Soc.*, 13(3):651–664, 2000. v, xi, 5

[PP03] Giuseppe Pareschi and Mihnea Popa. Regularity on abelian varieties. I. *J. Amer. Math. Soc.*, 16(2):285–302, 2003. iii, x

[PP04] Giuseppe Pareschi and Mihnea Popa. Regularity on abelian varieties. II. Basic results on linear series and defining equations. *J. Algebraic Geom.*, 13(1):167–193, 2004. iv, x

[PP08] Giuseppe Pareschi and Mihnea Popa. Generic vanishing and minimal cohomology classes on abelian varieties. *Math. Ann.*, 340(1):209–222, 2008. iv, x, 86

[PP11] Giuseppe Pareschi and Mihnea Popa. GV-sheaves, Fourier-Mukai transform, and generic vanishing. *Amer. J. Math.*, 133(1):235–271, 2011. iii, x

[PS14] Mihnea Popa and Christian Schnell. On direct images of pluricanonical bundles. *Algebra Number Theory*, 8(9):2273–2295, 2014. iv, x

[Rec74] Sevin Recillas. Jacobians of curves with g^1_4's are the Prym's of trigonal curves. *Bol. Soc. Mat. Mexicana (2)*, 19(1):9–13, 1974. 87

[Roj21] Andrés Rojas. The basepoint-freeness threshold of a very general abelian surface, 2021. arXiv:2107.01896. 1

[Rou11] Xavier Roulleau. Fano surfaces with 12 or 30 elliptic curves. *Michigan Math. J.*, 60(2):313–329, 2011. 114

[Sch20] Benjamin Schmidt. Bridgeland stability on threefolds: some wall crossings. *J. Algebraic Geom.*, 29(2):247–283, 2020. 20, 64

[TiB87] Montserrat Teixidor i Bigas. Half-canonical series on algebraic curves. *Trans. Amer. Math. Soc.*, 302(1):99–115, 1987. 9, 82, 100

[TiB88] Montserrat Teixidor i Bigas. The divisor of curves with a vanishing theta-null. *Compositio Math.*, 66(1):15–22, 1988. vi, xii, 8, 82, 90, 91, 92, 95, 97, 98, 117

[Voi02] Claire Voisin. Green's generic syzygy conjecture for curves of even genus lying on a $K3$ surface. *J. Eur. Math. Soc. (JEMS)*, 4(4):363–404, 2002. v, xi

[Voi05] Claire Voisin. Green's canonical syzygy conjecture for generic curves of odd genus. *Compos. Math.*, 141(5):1163–1190, 2005. v, xi

[Wel85] Gerald E. Welters. A theorem of Gieseker-Petri type for Prym varieties. *Ann. Sci. École Norm. Sup. (4)*, 18(4):671–683, 1985. 86

[Yan89] Jae-Hyun Yang. Holomorphic vector bundles over complex tori. *J. Korean Math. Soc.*, 26(1):117–142, 1989. 22