Universitat Autònoma de Barcelona

Quantum gravity or the problematic quantization of general relativity: a philosophical analysis

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PhD Thesis

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ABSTRACT

In the recent physics and philosophy literature it has been discussed that spacetime may be emergent from something more fundamental, as motivated by different approaches to quantum gravity. In this thesis I analyze some of these approaches and I argue against the claim that they describe a fundamentally non-spatiotemporal reality. Instead, I argue that these approaches suffer from a series of conceptual and technical difficulties which jeopardize both their justification and their interpretation. These difficulties have their root in the dynamical spatiotemporal structure of general relativity, which clashes with the fixed temporal structure of quantum mechanics when one tries to quantize the theory. This issue has been known for some time for canonical approaches to quantum gravity, where it is known as the problem of time, but here I argue that it also affects covariant approaches and that none of the proposed resolutions is fully satisfactory. In particular, in this thesis I study approaches like quantum geometrodynamics, loop quantum gravity, loop quantum cosmology, spin foam models, and group field theory and argue that they are all affected by some form or another of the problem of time. In this sense, I argue that one can have reservations about these approaches and the way they have been used to argue for a fundamental non-spatiotemporal level of reality.

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CONTENTS

1. INTRODUCTION

This PhD thesis is part of the Proteus research project funded by the European Research Council, which has as one of its goals to analyze the way time is represented and conceptualized by contemporary physics. In particular, one of the questions Proteus aims to answer is in which way time can be considered emergent, as has been suggested in the physics and philosophy of physics literature. For this reason, my own research project, as represented in this thesis, was to analyze the claim that some approaches to quantum gravity lead to a picture in which spacetime is not fundamental.

In this thesis I will analyze the philosophical debate about the emergence of spacetime in quantum gravity, and I will find that if we had good reasons for believing in a theory of quantum gravity based on some non-spatiotemporal entities, then the arguments defending the plausibility of having a non-fundamental spacetime would seem reasonable. In this sense, spacetime would be something non-fundamental in our understanding of the world, just as we consider molecules, tigers or economies to be non-fundamental. However, in this thesis I will analyze several approaches to quantum gravity and I will argue that there is a series of problems with these approaches which should make us cautious when claiming that they describe a non-spatiotemporal reality. These problems have their origin in the dynamical spatiotemporal structure of general relativity, which does not mix well with the fixed temporal structure of quantum mechanics. In particular, this issue affects two major aspects of these approaches. First, the justification for these approaches to quantum gravity is jeopardized, as I will argue that we have reasons to believe that the quantization techniques fail for theories similar to general relativity. Second, even if we ignored these issues, we lack a clear interpretation for the formalisms proposed by these approaches. In this sense, it is unclear how to connect these formalisms with something out there in the world, and this also includes a difficulty to extract empirical predictions out of these approaches. In this sense, I find that the philosophical debate regarding the emergence or not of spacetime lacks a sufficiently well-established theory that could be used as an example of how the world could be non-spatiotemporal and still have the spatiotemporal appearance that our world has.

In this introductory chapter I will start by giving an introduction and motivation for quantum gravity (1.1). Then, in section 1.2 I will study how we conceptualize spacetime in general relativity and the fundamental diffeomorphism invariance of the theory. Next (1.3), I will argue that the temporal structure of general relativity is dynamical, and that this is the source of the conceptual and technical problems for the quantization of the theory, as quantum mechanics is based on a fixed temporal structure. Finally, in section 1.4 I give an overview of the argument and the structure of this thesis.

1.1 The search for quantum gravity

In this section I will give a brief introduction and motivation for quantum gravity and of its philosophical relevance. I refer the interested reader to Callender and Huggett (2001b,a); Butterfield and Isham (2001, 2006); Huggett and Callender (2001); Wüthrich (2006) for more complete accounts of what motivates the search of a theory of quantum gravity and their philosophical implications.

General relativity and quantum field theory (QFT) are our two more successful physical theories. Both of them accurately describe the world in their domain of application. However, they are profoundly different theories and one can even argue that they are incompatible. General relativity describes a world in which matter is classical and spacetime is dynamical, while QFT describes quantum matter in a fixed spacetime. General relativity is our best theory for gravity and it describes gravity not as a force, but as an effect of the curvature of spacetime. In this sense, gravity would be different from the other forces in physics, such as the electromagnetic one, and we can suspect that it cannot be treated in the same way. Indeed, by applying the methods of QFT to the gravitational field of general relativity one quickly runs into trouble[1]. This motivates the search of a theory of quantum gravity, which would be able to account for both quantum and gravitational phenomena and to give a unified framework.

There are some comments that need to be made at this point. First, the definition of a theory of quantum gravity I am giving here is broad: any theory which could unify general relativity and quantum field theory would be considered a theory of quantum gravity, and this does not imply that gravity or spacetime are quantized. That is, if we were able to build a theory on a classical, dynamical spacetime which would explain how quantum matter behaves in the presence of gravity and the way spacetime behaves in the presence of quantum matter, that would be a good enough theory. However, there are some good arguments, even if not conclusive, supporting the conclusion that gravity or spacetime need to be quantized. Indeed, all the theories

[1] In particular I am referring to the fact that general relativity is perturbatively non-renormalizable. This point is discussed for instance in Callender and Huggett (2001b); Butterfield and Isham (2001).

I will be studying in this thesis involve some sort of quantization of general relativity.

Second, one can argue, like Wallace (2021) provocatively does, that the definition of quantum gravity as any theory that combines quantum mechanics and general relativity makes it the case that we already have perfectly working theories of quantum gravity. Wallace is referring to physical situations for which quantum effects and gravitational effects are important and for which physicists use some approximation techniques which allow them to make predictions. For instance, Wallace mentions the physics of neutron stars, for which some semiclassical theories are used, and some methods used for computing quantum fluctuations of the cosmic microwave background which take into account some quantum gravitational fluctuations. In this sense, Wallace is right in arguing that these models constitute theories of quantum gravity, as they give some unified account of phenomena that are both gravitational and quantum.

However, the theories I will focus on this thesis are not the theories Wallace has in mind. That is, the theories I will study are theories which aim to be fundamental theories, or at least more fundamental than the models in Wallace's paper, and they attempt to perform a full quantization of gravity. Insofar as they are more fundamental, these approaches are more exposed to the tensions and contradictions between general relativity and quantum mechanics, and they suggest a number of philosophical implications and debates. In particular, in this thesis I will focus on whether these approaches support the claim that spacetime is not fundamental and the way it has been argued that one can recover the appearance of spacetime from such a non-fundamental entities.

One can see that the search for quantum gravity is interesting from a philosophical point of view, and that it can arguably be motivated by the want of a unified picture of the world. Indeed, this is the main motivation for this search. Other motivations have been claimed but they are more disputable. For instance, one could argue that quantum gravity is needed for empirical reasons, i.e., to predict new phenomena or to explain existing and unexplained ones. However, there is no consensus about which phenomena or in which regimes quantum gravity would be useful. From the point of view of QFT, quantum gravitational corrections to the predictions for the high-energy experiments we carry out in accelerators like the LHC are really tiny, far beyond the precision which is expected to be ever measurable by us. Indeed, the size that a particle collider needs to have in order to be able to observe quantum gravitational corrections is of the order of the size of the solar system. Other potential scenarios where quantum gravity may be empirically useful is for studying black holes and the instants after the big bang, but even this is disputable, as general relativity gives models which are compatible with the evidence we have for such situations and as the approximated models Wallace mentions could be good enough for dealing with some eventual deviation from general relativity.

Other arguments for the need of a theory of quantum gravity have to do with either QFT or general relativity being unsatisfactory, internally inconsistent or presenting some features that are considered unphysical. For instance, the fact that general relativity allows for singular spacetimes or the way the renormalization group techniques are deployed in QFT have been argued to be problematic, and the hope is that a theory of quantum gravity would overcome such alleged issues. However, this kind of motivation is also clearly disputable, as the feature that is problematic for some authors is just a prediction to be embraced for others. In this sense, the clearest reason for seeking for a theory of quantum gravity and the one for which most would agree is the unification one.

Finally, let me mention that there are different approaches and different ways of classifying them have been proposed. In this thesis I will study approaches which are based in more or less direct quantizations of general relativity, that is, they are built in analogous ways to the ways theories like quantum electrodynamics are defined from classical theories like classical electromagnetism. We can classify direct quantizations of general relativity in two groups depending on the types of quantization methods which are applied to general relativity. First we have approaches based on a canonical quantization, such as quantum geometrodynamics or LQG. These approaches have received more philosophical attention, as the problems with the quantization of general relativity appear in a quite straightforward way. In the second group we find covariant approaches, that is, approaches based on a covariant quantization of general relativity. They include models like spin foam models which have been developed more recently and for which not so many philosophical discussions are available. In particular, in this thesis I will argue that covariant approaches formulated in recent years suffer from an analogue version of the problem of time for canonical approaches, which received much attention (see again Callender and Huggett (2001b)) but for which, in my opinion, a satisfactory resolution has not been given.

I am leaving outside of this thesis the approaches which are not based on a more or less direct quantization of general relativity. The most relevant family of approaches in this category is string theory. Roughly speaking, in this approach one studies the quantum behavior of 2-dimensional entities like strings and membranes and by imposing some consistency and symmetry requirements one gets spacetimes which approximately obey the Einstein's equations of general relativity and also spin-2 modes on the strings which would correspond to gravitons, the hypothetical particle which would carry gravitational interactions. In this sense, in string approaches one starts with a quantum theory which does not have anything to do with general relativity and recovers it in an indirect way. For this reason, it will be outside the focus of my thesis, although one can certainly ask interesting philosophical questions about string theory, also regarding the status of spacetime in string approaches.

1.2 Spacetime in general relativity

In this section I will introduce the way spacetime is conceptualized in general relativity. This has two goals. First, as in this thesis I will be discussing the way some approaches to quantum gravity discuss that spacetime is not fundamental, but instead it is something that can be recovered from something more fundamental and non-spatiotemporal, it will thus be good if we start with a more or less clear idea of what we mean by spacetime. Second, the geometrical description of spacetime and the symmetries of general relativity make it the case that there is no preferred way of establishing coordinates in spacetime or distinguish a time variable. This symmetry will be the root of the conceptual and technical problems we will find for quantizing general relativity, and hence it is important to have a good grasp of it.

The formal definition of spacetime is made in the language of differential geometry by means a pair $\langle M, g_{\mu\nu}\rangle$. M is a manifold, that is, a set of points with a topology defined on it which makes it the case that we can assign a dimensionality n to it and define a smooth neighborhood for each point which topologically looks like the Euclidean space $\mathbb{R}^n$. The metric tensor $g_{\mu\nu}$ defined on the manifold defines the rest of geometrical properties of spacetime, that is, it defines a length for any curve in spacetime, an angle for any two intersecting curves, a volume for any spacetime region, and so on. For describing our world we use 4-dimensional manifolds, corresponding with manifolds which have 3 space dimensions and one temporal dimension. The metric we use is Lorentzian, that is, it is strictly speaking a pseudo-metric which at each point defines a light-cone structure. That is, it distinguishes three types of direction according to the sign of the length that the metric assigns to each direction. In this thesis I will be using the convention $\{-,+,+,+\}$ which means that at each point we will define time-like curves to be the ones with negative squared norm tangent vectors, space-like curves to be the ones with positive squared norm tangent vectors and light-like curves to be the ones with null tangent vectors. This defines a notion of causality: two points in the manifold are causally connected if one is inside or on the light-cone of the other, that is, if one can connect the two points by means of a time-like or light-like curve. The length of a time-like curve between two events corresponds to the time elapsed between the two events as a clock traveling along that curve would measure.

Importantly for our discussion, the description of the geometry of spacetime in terms of a pair $\langle M, g_{\mu\nu}\rangle$ is not unique. The reason for this is that starting from one model M_1 one can always build a second model M_2 which describes the same geometry by means of a diffeomorphism transformation, i.e., a mapping between two manifolds which preserves the geometric structure. Under such a map d any point P is mapped to a point dP, and the metric tensor at the point dP corresponds to the push-forward of the metric tensor at the point P. Now, the geometry described by the second model is essentially the same as the one described by the first one, that is, if two points P and Q were joined by a geodesic γ of length l, in the second model we will have the points dP and dQ, the image of P and Q under the diffeomorphism d, joined by a geodesic $d\gamma$ also of length l, as determined by the transformed metric. Similarly, angles, volumes and any other geometric property is preserved under a diffeomorphism in the sense that for every geometric object such as points, curves, surfaces or regions in the original model we find that the geometrical properties of its image under the transformation remain the same.

Let me illustrate this with an example, represented on the left-hand side of figure 1.1. Consider a 2-dimensional flat Euclidean geometry. This can be described by the manifold $\mathbb{R}^2$ and the metric $g_{ab} = \delta_{ab}$. We can define the point P_1 to be the point with coordinates $(1, 0)$ and the point P_2 to be the point with coordinates $(0, 1)$. The geodesic distance between the points P_1 and P_2, as determined by the metric g is $\sqrt{2}$. Now we can find a different model by making use of the following diffeomorphism:

$$\begin{aligned} d &: \mathbb{R}^2 \to \mathbb{R}^2 \\ (x, y) &\to (2x, 2y)\,, \end{aligned}$$

That is, we map every point P in the original space to a different point P' which has coordinates which double that of the original point. Under this diffeomorphism one can show that the new metric is $g'_{ab} = \frac{1}{4}\delta_{ab} = \frac{1}{4}g_{ab}$ and that the points P_1 and P_2 are mapped to the points P'_1 and P'_2 with coordinates $(2, 0)$ and $(0, 2)$ respectively. The geodesic distance between P'_1 and P'_2 according to the metric g' is $\sqrt{2}$, as we were expecting. Now one can convince oneself that is clear that $\langle \mathbb{R}^2, g_{ab} \rangle$ and $\langle \mathbb{R}^2, g'_{ab} \rangle$ can be used for describing the same geometrical facts, as for any geometrical fact in the first model we can use the map d for finding the same fact holding for the transformed model.

However, this leads to some conceptual and metaphysical problems. Notice that there is some sort of indeterminism, as a complete determination of the geometric properties of a spacetime would not suffice for determining the properties that a given point P would have. For instance, in our example knowing that there are two points situated at a distance of $\sqrt{2}$ of each other would not allow one to determine if these points are P_1 and P_2 or P'_1 and P'_2. In other words, the geometric facts do not allow one to distinguish between $\langle \mathbb{R}^2, g_{ab} \rangle$ and $\langle \mathbb{R}^2, g'_{ab} \rangle$. For the case of general relativity that there is this sort of indeterminism is the basis for the famous hole argument[2].

[2] See Hoefer (1996); Norton (2019).

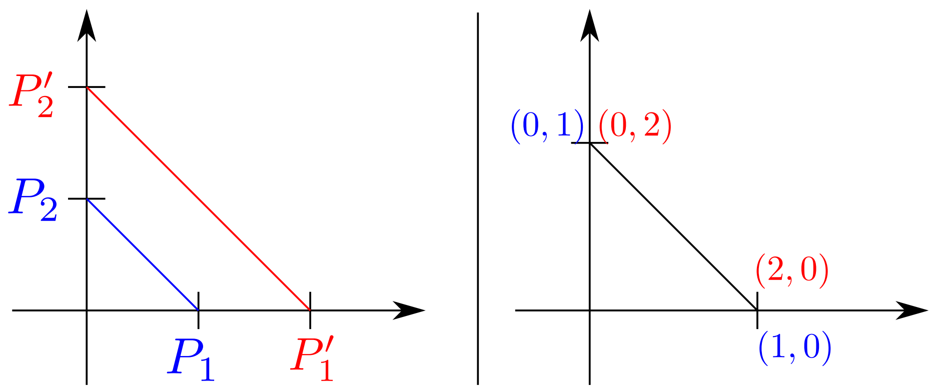

Fig. 1.1: Representation of the diffeomorphism described in the text (left) and of an analogous change of coordinates (right). One kind of transformation can be seen as the active and the other as the passive version of the same transformation. At the end of the day diffeomorphism related models are similar to models expressed in different sets of coordinates, as they represent the same geometry. In this case, the distances along the blue and red lines on the left are the same as the distance along the line on the right.

Nevertheless, this indeterminism can be argued to be similar to the indeterminism of gauge theories, in the sense of it being an indeterminism that affects only mathematical structures in our descriptions of reality and not physical ones. Indeed, one can understand spacetime points to correspond not to just points in the bare manifold, but to be identified by the geometrical relations they have with each other. In this sense, it does not make much physical sense to ask questions about the geometrical properties of points in the bare manifold. Meaningful questions are the ones that are asked about spacetime points understood as defined by their geometrical properties. The formal definition of this is by means of equivalence classes: we can define an spacetime not by means of a pair $\langle M, g\rangle$ but by an equivalence class of them. Similarly a spacetime point is not defined by pointing at a given point P of the bare manifold for a model, but by the equivalence class obtained by transforming P.

An intuitive example of this is the following. Imagine that we built a geometrical model of the surface of the Earth based on a manifold like S^2. The surface of the Earth is not a perfect sphere: it is flatter on the poles and it has mountains, valleys and oceans. If we approximated it as being smooth and differentiable, we could build models $\langle S^2, g\rangle$ that would allow to compute, for instance, the distance between the tip of mount Everest and the North Pole along a given curve, or the length of the equator. Intuitively, the bare points of S^2 do not have any physical or geometrical meaning, but points corresponding to mount Everest, the North Pole or the equator do. In this sense, we have a good grasp of what an Earth-point is and how to identify it in any model of Earth. Formally, once I identify the mathematical point P corresponding to the tip of mount Everest in a model $\langle S^2, g\rangle$[3], I can locate it at any other diffeomorphism-related model if I know what is the diffeomorphism that relates the two models. Similarly, in general relativity one can think of spacetime points as being represented by equivalence classes of models.

A way of formulating the issues with indeterminism is in terms of labels and coordinates. Indeed, for concrete models of general relativity or differential geometry we need to introduce a set of coodinates in order to express objects like the metric tensor as functions of these coordinates. For this reason it is more common to see the metric tensor as a function of the coordinates x than as a function of the manifold point P and one does not usually emphasize the difference between a point as specified by certain coordinates and the point P. Changes of coordinates are different ways of assigning coordinates x to the points of the manifold, and one can check that they are not so different from a diffeomorphism as I have introduced them. Under a diffeomorphism we have already found that the values of $g_{\mu\nu}(x)$ change. Under a change of coordinates we also find that $g_{\mu\nu}(x)$ changes because we have changed the point P to which the coordinates x refer. The difference between diffeomorphisms and changes of coordinates is not very important for practical purposes and only gets relevance at the time of discussing questions about the identity of points. Indeed, one can see diffeomorphisms and changes of coordinates as the active and passive interpretation of the same formal transformation for $g_{\mu\nu}$[4], as I illustrate in figure 1.1 for the simple example above.

Notice that once we consider that the geometrical and physical content of a spacetime model is represented by equivalence classes under diffeomorphisms we can see diffeomorphisms also as a form of change of coordinates or labels: under a diffeomorphism the point which gets the label P changes but the physical or geometrical content of the model is the same, as for every point that was originally labeled as P we find another one dP which has the same properties.

It is also important to notice that general relativity is not the only theory which can be expressed in a generally covariant way, i.e., it is not the only which can represent the geometry of spacetime in the language of differential geometry. Indeed, relativistic physics is commonly expressed by making reference to the Minkowski metric and even Newtonian physics can be expressed in this language. For this reason, the source of the problems for its quantization is not its general covariance,

[3] For symmetric situations there is more than one way to make a model-reality correspondence. That is, there are different ways of deciding which point in the manifold corresponds with a given point out there in the world, and all of them are equally good.

[4] Indeed it is sometimes referred to diffeomorphisms and changes of coordinates as active and passive diffeomorphisms.

or not just that, but lies elsewhere as I will explain in the next section. Before this, let me insist that it is the common interpretation of general relativity to assume that its physical content is contained in an equivalence class of models under diffeomorphisms and that no physical meaning is ascribed neither to bare points P in the manifold nor to points identified by a set of coordinates x. In this sense, general relativity can be treated as a reparametrization invariant theory.

1.3 The problem with time

In this thesis I will explore the problems that we find when applying the standard quantization techniques to the structures of general relativity. Indeed, I will argue that these problems are such that no satisfactory and successful quantization of general relativity is known. One could think that the fact that quantum mechanics and general relativity have very different temporal structures is the source of the troubles that we find when trying to quantize general relativity. However, in this section I will argue that the matter is subtle, as even if the fact that spacetime is curved and relativistic produces some conceptual tensions for quantization, the cause behind the trouble is rather that spacetime is dynamical and not fixed[5].

As I mentioned above, any spacetime structure can be described making use of the language of differential geometry and in a way that is invariant under any diffeomorphism or change of coordinates. In this sense, we can write theories like Newtonian mechanics or special-relativistic electromagnetism in a coordinate-free way. However, we can find satisfactory and successful quantizations for these theories, and hence it is not only the fact that general relativity is expressed in this language which causes the obstacles for the quantization of gravity.

A seemingly more powerful argument says that it is the different temporal structures of quantum mechanics and general relativity which causes the difficulties in quantizing the latter. The formalism of quantum mechanics is based in the same temporal structure as Hamiltonian mechanics: time is understood as a sequence of instants and physical evolution is just the change in the properties of the world/a system for the different instants. Moreover, the standard formalism of quantum mechanics is like Newtonian physics in that there is an absolute time scale. It is therefore natural that the standard interpretations of quantum mechanics, at least in their more direct and naive formulations, have an ontology with an absolute time, like in Newtonian physics.

This is in contrast with the spacetime ontology of relativistic physics, where no absolute distinction between space and time is preferred. Similarly, there is no physically meaningful notion of simultaneity. This can be seen as giving hints that there

[5] This point had been pointed out before in the literature, as for instance in Anderson (2017).

will be some problem when trying to unify relativistic theories and quantum theories. However, it turns out that this is not exactly the case. Relativistic theories can be expressed[6] in the classical paradigm as the evolution of a system evolving in time. For this, one just needs to introduce a foliation of spacetime, that is, a decomposition of spacetime into a series of spacelike leaves corresponding to instants of time. For a general spacetime there is no unique or privileged way of doing so, but it is feasible. Once this is done one can apply some quantization procedure and build a quantum theory.

In this sense, the quantization of relativistic theories and theories on curved backgrounds is perfectly consistent. Indeed, this is the way in which QFT's are formulated, and they constitute some of our most successful theories. A different question to ask is which ontology we should attribute to these theories, and here the tension between the Newtonian and the relativistic paradigm can play a role. However, let me insist that it does not cause any trouble for the formulation of the quantum theory. There can be some other sorts of trouble involved, such as different foliations leading to different and incompatible theories, but it is a fact that considering relativistic spacetimes do not constitute a problem for our quantization procedures.

The problem seems rather to be that in general relativity spacetime itself is dynamical. A popular way of saying this is that general relativity is background independent, that is, it does not rely on any fixed structure such as the time of classical physics or the Minkowski spacetime of special relativity. In other words, in these theories one studies the dynamics of a set of degrees of freedom, but time or spacetime is just given by the theory or the model. That is, one does not have anything like equations of motion for the metric and all the models describe the evolution of the degrees of freedom in the same time or spacetime.

However, one has to be careful when formulating this background independence condition. For instance, if one claims that manifolds do not represent a background structure for general relativity as spacetimes are defined not on a particular manifold but by means of an equivalence class under diffeomorphisms one is falling in the same mistake I have pointed out before, namely, that models with fixed spacetimes can also be represented in the language of manifolds and differential geometry. Despite this, it is true that the invariance under diffeomorphisms plays a crucial role in generating the technical and conceptual problems for the quantization of general relativity. In particular, in chapter 4 I will argue that every theory which is expressed in a way that has an explicit and dynamical[7] reparametrization invariance will suffer from similar technical problems as general relativity when trying to quantize it. However, these

[6] I am ignoring here general relativistic scenarios in which one has non-trivial topologies or closed timelike curves. For such situations one cannot decompose spacetime by using a foliation.

[7] With this I mean that the action has a symmetry associated with reparametrizations.

problems can be overcome for theories with fixed temporal structures, while it will not be the case for general relativity.

Theories with fixed backgrounds can be quantized in a satisfactory way even if they are expressed in a reparametrization or diffeomorphism invariant way, that is even if they are formulated in a way independent of a background manifold or time parameter. For this, one needs to apply a deparametrization, that is, to be able to identify the relevant spatiotemporal structures so that one is able to define a quantum dynamics with respect to them. In the case of a true background independent theory this will not be possible.

I will introduce Jacobi's action as an example of a model which shows a background independence or something very similar to it. One can interpret this model as an example of a model without a metrical time structure at all. In this sense, this is an example of a model with no fixed time structure. If one wants to have a metrical time, there is a way of defining one but just **after** the equations of motion have been solved. In this sense, these models are analogous to general relativity in that there is no way of knowing the time elapsed in between two configurations or two events. This is in contrast with models with an explicit fixed background or models which are deparametrizable, where temporal intervals can be determined without appealing to the dynamics. The quantization of models based on a Jacobi action leads to problems in quantization more similar to the ones of general relativity.

However, there is a subtlety related with these models. As I said, there is a way of defining a preferred temporal metric. If one adopts this metric, the equations of motion become just Newton equations and one could argue that the theory described by Jacobi's action had a very well-defined and fixed temporal structure. In this sense, the similarity with general relativity could be just a product of the formulation chosen. By choosing a different formulation, the one with the standard Lagrangian action, one can avoid the problems that affect general relativity and build a successful quantization. One could resist this kind of move based on philosophical preferences regarding absolute time scales, but the fact is that this move allows to fix the temporal structure and avoid the problems with the quantization of a dynamical temporal structure.

Is there a similar strategy available for general relativity? That is, is there a way of fixing the temporal structure of the theory so that the problems with its quantization are avoided? The answer seems to be negative, the reason being that the geometry of general relativity in four dimensions is far more complicated that the geometry in one dimension, which is just trivial. Let me note however that there are some approaches like unimodular gravity in which some fixation of the temporal structure is tried. In this case is by defining a cosmological time that would play a similar role to Newtonian time. The approach based on unimodular gravity was considered unsuccessful[8], but let me mention that there has been a recent proposal[9] which tries a similar strategy: instead of trying to quantize general relativity it is based on a quantization of shape dynamics, which is claimed to be empirically equivalent to general relativity while having a more fixed temporal structure. One may worry that the objections that were raised against unimodular gravity still apply to this approach, but a deeper analysis of this kind of approach is beyond the scope of this thesis. That is, I will be focusing just on quantizations of standard general relativity with its fully dynamical metric structure, which are, together with string theory, the most extended approaches in the quantum gravity community.

1.4 Overview and structure

As I have explained, the quantization of general relativity is problematic due to the conflict between the fixed temporal structure of quantum theories and the dynamical nature of geometry in general relativity. In this thesis I will argue that not only this quantization is problematic, but that it directly jeopardizes the justification and interpretation of the approaches of quantum gravity based on such quantization. In this sense, one can doubt that these approaches, as they currently stand, constitute examples of how spacetime could emerge from some more fundamental non-spatiotemporal entities. In this section I I will give a brief overview of the structure of this thesis and its contents.

In chapters 2 and 3 I deal with the more philosophical discussions regarding the emergence of spacetime. First, in chapter 2 I will argue for a reductionist view of emergence in general, putting emphasis on the fact that in the philosophy of physics literature and the philosophy of quantum gravity literature we are concerned with how fundamental entities and phenomena explain higher-level ones, and this is generally viewed from a reductionist point of view and the term 'emergence' is used just to express something like novelty or surprise. Second, in chapter 3 I will introduce spacetime functionalism as the strategy for recovering spacetime from non-spatiotemporal entities. I will argue that it seems the way to go if we believed that there are such non-temporal entities, but I will advance that there are reasonable doubts about the theories that support those beliefs.

Next I move to study canonical approaches to the quantization of general relativity. First, in chapter 4 I argue in detail why the above mentioned tension between the

[8] I refer the reader to Brown and York (1989); Henneaux and Teitelboim (1989) for the original papers in which unimodular gravity was formulated and to Kuchař (1992); Isham (1993) for criticisms of these approaches.

[9] See Gryb and Thébault (2014, 2016a,b).

dynamical temporal structure of general relativity and the fixed one of quantum mechanics leads to what is known as a problem of time. I further argue that this problem of time seems not to be solvable and that the candidates for resolutions that have been proposed are not well-justified and do not work for theories like general relativity, as they are resolutions which make sense just for deparametrizable models.

In chapters 5 and 6 I introduce LQG and LQC as the prominent canonical quantizations of general relativity and of some cosmological models and I will argue that they suffer from the problem of time, as any canonical approach. I will further note that the quantization chosen for these models is a peculiar one that allows for some technical improvement with respect to the naive geometrodynamical one. This quantization has as a consequence a particular Hilbert space structure that has been interpreted as implying that space is discrete. I will argue that this interpretation can be disputable and that the very special way this quantization is made raises the worry about whether it is to be preferred to the more intuitive, although technically worse-behaved, geometrodynamical one.

Finally, I will analyze covariant quantizations. In chapter 7 I argue that this quantization method is an alternative way of building the basic structures of quantum theories, namely, a Hilbert space and an evolution operator, which can be represented as a propagator. I will argue that there is a covariant version of the problem of time, that is, that the proposed ways of building propagators for quantizations of general relativity are ill-justified and that they lead to a formalism for which we lack an interpretation. The reason for this is again the dynamical spacetime structure of general relativity. I will also argue that views that defend generalized covariant quantizations, i.e., quantizations which do not define the standard structures of quantum theory but they are based on other structures built by means of path integrals also fail to give a satisfactory interpretation of their formalisms, and hence they do not provide successful candidate theories either.

In chapters 8 and 9 I analyze two covariant approaches, namely spin foam models and group field theories and I argue that they suffer from this covariant version of the problem of time. The structures they define cannot be interpreted as defining a propagator or a clear sense of evolution and they remain obscure. I will argue that these theories are related to LQG and that they appeal to LQG to justify their interpretation as representing discrete spaces or spacetimes. However, as I will argue that this interpretation is dubious for LQG one can suspect the same for these models. Alternatively, one can ignore these worries and define these models directly on discrete structures, but this weakens the justification for these models.

Finally, in chapter 10 I conclude by insisting on the most important conclusions of this thesis.

2. EMERGENCE

In this chapter I will review what one generally means when claiming that there is emergence, I will criticize the notion of emergence used most extensively in the philosophy of quantum gravity literature and I will define the position about emergence I will argue for in this thesis, which will be a reductionist one.

As I will argue, there are several ways in which the term is used in both the philosophical and scientific literature. In general, in order to speak about emergence one needs the presence of a hierarchy of levels and one compares entities, properties or theories associated with different levels. In the case of science, these levels are generally associated with the size of the systems the different sciences or theories study and one considers that the bigger the size, the higher the level is. Smaller-size levels are lower-level in this hierarchy and one says that they are more fundamental. This is associated with the idea that, in science, the small explains the big, e.g., that one can understand a living being by studying its cells or the way a car moves by knowing its parts. The term 'emergent' is sometimes used for the properties in the higher level, and in this chapter I will introduce the different meanings with which it is employed and I will argue that the reductionist methodology of science leads us to generally reject that there is emergence in a strong sense. Therefore, when in the context of quantum gravity we speak about the emergence of spacetime we are highlighting that spacetime may not be fundamental, but this is generally understood in a reductionist way. In chapter 3 I will introduce the debates about the emergence of spacetime in quantum gravity, which turn out to be debates about how spacetime can or cannot be reduced to more fundamental entities. In the rest of chapters of the thesis I will consider different approaches to quantum gravity and the way the structures they define are claimed to be candidates for such fundamental entities.

I will start the chapter by reviewing an important ingredient of the way science develops itself: methodological reductionism. I will explain that even if higher level phenomena can be studied on their own, the way science tries to explain them consists on finding the more fundamental entities and phenomena that cause them. For instance, one can study the life cycle of stars just by studying the observational evidence we have about them, e.g., their luminosity and color, but our knowledge about stars and their life cycles increased when we discovered the elements they are made of and the nuclear reactions happening in their interiors. In this way, phenomena like supernova were explained. In section 2.1 I will argue that this is an important part of science: the search of the fundamental components of the world which explain the phenomena we observe.

In section 2.2 I will introduce the first relevant notions of emergence, which are associated with a failure of reductionism. Importantly, I will distinguish two kinds or families of emergence: epistemological and metaphysical. The latter is a stronger kind, as it implies that there are (scientific) properties in a higher level that cannot be reduced to properties in a more fundamental level. If one holds the position that in the world there are emergent properties of this kind, then one gets in conflict with the generally reductionist spirit of science. On the other hand, claiming epistemological emergence is not such a strong claim and it amounts to saying that there are situations in which, given difficulties like the complexity of studying large systems, we are not able to explain or predict something from the higher level in terms of the fundamental level. This notion of emergence may be rightly used in many areas of science where one lacks the computational capabilities or the access to all the microscopic information that would allow to get this understanding of the higher level from the more fundamental one.

I will argue that in accordance with the reductionist spirit of science it is reasonable to reject metaphysical emergence and accept epistemological emergence for some cases. In particular, I will argue that this is applicable to the case of quantum gravity, and hence that the debates regarding the emergence of spacetime are better understood as debates about whether spacetime could be reduced to something more fundamental.

In scientific practice, however, it is not always clear whether emergence is used in one of those two senses. Instead, it seems that many times terms like 'emergent phenomena' are used to refer to phenomena in a higher level which are in a sense surprising from the point of view of the lower level. For instance, that matter organizes itself in different phases and that it may change between different phases may be seen as a novelty from the point of view of the lower level in which one just has atoms and molecules interacting. This intuition is captured by Butterfield's definition of emergence, which was further developed by Crowther. In section 2.3 I will introduce and analyze these definitions and argue that they capture what is generally meant by the physicists, but also that this notion is more vague and philosophically less interesting. In particular, in the case of the philosophy of quantum gravity it seems true that physicists use the term in this sense, but the interesting philosophical and physical questions, in my opinion, are related with the non-spatiotemporal worlds these theories claim to describe and the way they could or could not contain, in some approximated way, spacetimes.

Finally, in section 2.4 I will comment on a kind of emergence which will also be relevant for this thesis, which is the 'emergence' of the classical world from one which is quantum. There are different understandings of the exact relation between quantum theory and the apparently classical world we live in, which correspond to different interpretations of quantum mechanics and different resolutions of the measurement problem. I will argue that, for realist interpretations of quantum mechanics, such as many-worlds, Bohmian mechanics and collapse models, the way the classical world 'emerges' from the quantum one can be understood in a very similar way to the way macroscopic phenomena emerge from microscopic ones and in a perfectly reductionist way. On the other hand, for more operational interpretations of quantum mechanics it is harder to assess the way the two domains, classical and quantum, are related. This is clear for example in the Copenhagen interpretation, which predicts the probabilities for the different outcomes of a measurement but which does not explain the properties of the measuring devices. I will therefore argue that realist interpretations are clearer in the way the classical world is supposed to emerge. I will also argue that if one is to endorse an operational interpretation of a quantum theory, one should require that there is a clear connection between the formalism and empirical predictions. In later chapters of this thesis I will show that several approaches to quantum gravity suffer precisely from this problem, i.e., they define some of the structures of quantum theory but they do not offer any consistent, well-defined way of connecting them with well-defined empirical claims.

2.1 Methodological reductionism

It is undeniable that much scientific progress has been achieved by the discovery and understanding of the smaller components of the subjects of study of the different sciences. We gained much knowledge about living beings once we understood they were made of cells, chemistry progressed when postulated that matter was composed of atoms and the economy was better understood by studying the collective behavior of many people. In this sense, the sciences tend to be methodologically reductionist, i.e., it is believed that good explanations are given in terms of the smaller constituents.

Methodological reductionism, applied to the sciences, naturally leads to the level structure I mentioned in the introduction: for explaining the human body we can appeal to knowledge about its organs, for explaining the organs we can appeal to cells, for explaining the cells we can appeal to the molecules and so on. This level structure defines a hierarchy of explanations for which we find microphysics as a candidate for the fundamental level, i.e., if we start looking for explanations for a system in terms of its smaller components we will eventually end up in the microscopic level or below. Our current physical theories give us some candidates for the most fundamental, indivisible components of the world which we could take to be the fundamental level which serves as a basis for the rest of scientific explanations. However, the history of science has taught us that there is the possibility that one day we will find out that the entities we believed to be indivisible are indeed composed by something more fundamental. Even more, there is the possibility that there does not exist a fundamental level, i.e., it might be the case that the entities we find at every level are always divisible or that there is always a more fundamental level 'below'. In any case, for our discussion in this section it will not matter whether we correctly identify which level is the fundamental level or whether there actually is a most fundamental level. The point I am making here is that the different sciences provide us with this picture in which the world is layered.

As I said, methodological reductionism in science means that explanations relying on the smaller components of a phenomenon or an entity are regarded as good scientific explanations. This position comes very naturally from an ontological reductionist perspective, i.e., the claim that higher-level entities, properties and phenomena are nothing but collections of lower-level entities, properties and phenomena. For instance, when studying the thermodynamics of a gas one studies its macroscopic properties such as its volume or its pressure. In this case the reductionist position consists in considering that the gas is just the collection of molecules that form it, that its volume is nothing but the volume of the region in which these molecules are present, and that the pressure that the gas exerts on the walls of the recipient that contains it is just the (average) momentum per unit of time and of surface of the molecules of the gas that bounce against the walls.

There is a slightly weaker version of this which is to claim that the higher level supervenes on the lower level. There are different formulations of this concept but let me express it in the following way. A higher-level property supervenes on a set of lower-level properties if and only if it is not possible for two systems to agree on the set of lower-level properties while disagreeing on the higher-level one. For instance, in the case of the gas we may say that pressure supervenes on the momenta of the particles as it is reasonable to claim that if two gases contained the same molecules with equal momenta then they would exert the same pressure. Notice that while reduction claims that higher-level properties are to be identified with (sets of) lower-level ones, supervenience entails a strong dependence relation for the higher-level on the lower-one, but it does not imply that the properties are identical.

In the case of scientific properties and for our discussion, the distinction between supervenience and reduction will not play a very important role. From an empiricist point of view, one may doubt that higher-level properties play any role in our theories once we have the lower-level ones that are, arguably, able to account for the same phenomena and empirical data. For instance, we may attribute to a flower the property of being red and say that this redness supervenes on a set of physical and chemical properties of the plant, like the presence or not of some pigments. But from a strictly empiricist point of view, there is no need to introduce redness, as the physical and chemical properties of the flower are enough to explain that we see it as being red or that the light spectrum it reflects has the properties it has. Despite this difference, either with reduction or with supervenience there is a strong dependence relation between the two levels which implies that the lower level determines what happens in the higher level.

Let me mention that the argument above for preferring reduction over supervenience will not work if one has strong reasons supporting the existence of higher-level entities and properties as different from the lower-level ones. For instance, in the philosophy of mind it is plausible to claim that consciousness supervenes on certain physical properties and processes of the brain while resisting the identification of the mental with the physical. A reason for resisting this identification is not that the explanation of empirical facts like the behavior of people cannot be done in terms of their physical states, but rather that we have a strong intuition that mental properties are radically different from physical ones, and they cannot be easily identified. I will not be assessing this argument here, but I will just note that it loses much force if applied for a case different from the mental. That is, for cases like chemical or biological properties it does not seem as likely that one claims a strong intuition about them.

The arguments in the previous paragraphs have showed examples of an important strategy for science as a reductionist enterprise: functionalism. This consists in identifying the functional and causal roles that a higher-level entity, property or phenomena is playing in our explanations or in the world. In this way, reductionism can proceed by proposing lower-level entities which can satisfy the same roles the higher-level entities were playing. The example of the redness and the chemical and physical properties of the flower is a case in which one can perform such functionalist arguments. In chapter 3 and the rest of the thesis we will see how the functionalist schema has been applied to argue that a spatiotemporal-looking world could be reduced to a world which is fundamentally non-spatiotemporal.

An important remark to make in this discussion is that reductionism and supervenience should not be taken as exact relations, in the sense that there may be a number of approximations and conventions in play. For instance, we have seen that pressure can be defined as an average of the momenta that is exchanged between a gas and its container. Strictly speaking there will be moments of time in which the amount of momenta transferred will be higher than in others and there will be also moments in which the collisions of the particles and the container will not be evenly distributed. For this reason we know that the concept of pressure works fine for time scales and physical scales which are big enough, but that it is only an approximation of the real going-ons, which are not strictly homogeneous in space and in time. In general, when we speak about something in a higher level there will very likely be some approximation in play. In a strict sense, the properties we ascribe to the higher level are not true and we may consider them as useful fictions to apply within some limits.

This is connected with a slightly different notion of reduction which is a relation between theories and not between levels 'in the world'. This notion was developed by Nagel (1961)[1] and it roughly defines that a theory reduces to another if it is possible to derive the former from the latter, together with a set of bridge principles that allow to connect the concepts of both theories and, possibly, with some set of additional assumptions. The example Nagel uses to illustrate this relationship is the thermodynamics of a gas, which can arguably[2] be derived from the laws of Newtonian mechanics together with a bridge principle that relates temperature with the average kinetic energy of the particles of the gas, and some statistical assumption. This approach to reduction is more formal, as it considers logical relations between theories rather than relations between entities or phenomena. As such, it is generally compatible with the notions of reduction and supervenience I have been using so far. However, notice also that this relation does not need of the level structure we were considering. That is, the reducing theory does not need to refer to some smaller scale than the reduced theory. For instance, Nagel considers that Kepler's laws of planetary motion are reduced[3] to Newtonian gravity.

We can see that the notion of reduction as a formal relation between theories might be applied for cases in which a theory replaces another. In this sense we can use the term reduction in debates about the progressiveness of science. For instance, the defendant of the cumulativeness of science may argue for the claim that new theories reduce the older ones in this sense. If this claim were true, then the new theories would contain the old ones and science would be cumulative. As we see, this notion is different from the one that compares properties at different levels. Despite this, it is common to see both notions of reduction entangled, given that it is very natural to consider the relations between the theories that describe the different levels

[1] The basic idea of reduction as derivability was further developed also by Schaffner (1967) and has perdured until our days. An attempt to improve this definition was the 'new wave reduction' developed for instance in Bickle (1996), but as argued in Endicott (1998, 2001) it collapsed to the same definition as Nagel and Schaffner´s. A more modern defense of the Nagel-Schaffner account can be found in Dizadji-Bahmani et al. (2010).

[2] There is some controversy about the extent to which thermodynamics can be reduced to classical mechanics

[3] As pointed out in Needham (2010), once one considers the gravitational interactions between the different planets Newtonian gravity predicts a deviation from Kepler's laws. Therefore, in a strict sense there is not reduction in the sense of derivation. In order to account for this some notion of approximation needs to be introduced, as accepted by Nagel himself (Nagel, 1974).

when studying the relations between the properties and entities at different scales.

With this I conclude this section in which I have introduced the general reductionist methodology of science and the relevant definitions of reduction and supervenience. In the following sections I will introduce the different notions of emergence and I will argue for a reductionist view, in line with the reductionist methodology of science.

2.2 Emergence as antireductionism: metaphysical vs epistemological

In this section I will introduce emergence as a failure of reductionism. I will explain that depending of the notion of reduction or the point where it is alleged to fail we can identify two types of emergence: metaphysical and epistemological emergence. I will argue that in the context of science the former is generally rejected, and that the same will apply in the case of quantum gravity.

Reductionism provided us with a picture in which all the facts of world, at any level, could be derived from the facts about the fundamental level. However, real world science does not proceed by studying the most fundamental level available and then going all the way up to the level that is of interest for some particular situation. Instead, the different sciences are autonomous and self-contained and give successful explanations and predictions on their own particular level. Moreover, it is not just that higher-level scientists prefer explanations in terms of the entities in their level, but also that in general we simply lack the ability to make higher-level predictions from the theories describing the lower-level. For instance, it is practically impossible to keep track of all the particles that form part of every organism in an ecosystem and instead we will describe it in terms of the higher-level facts and laws provided by biology and ecology.

The fact that we are sometimes unable to derive facts about the higher-level can be seen as a failure of reductionism and we can say that there is emergence in this case, as we can say that independent and autonomous sciences emerge. But notice that this emergence is epistemological, i.e., it refers to our knowledge about the different levels and it does not necessarily imply anything about the levels or about the world. This sense of emergence is compatible with a perfectly reductionist view of the world. In the example of the ecosystem one may grant that we have to use the laws of ecology to make predictions but still argue that it is only for practical reasons. That is, if one had perfect knowledge of the positions and velocities of all the particles that form the ecosystem and of the laws that describe their motion, then one would be able to make the same higher-level predictions just by analyzing the lower-level.

Metaphysical emergence is used for referring to stronger claims, like the claim that the properties, entities and phenomena described by a higher-level science are not reducible to properties, entities and phenomena in a lower level. Accepting this kind of emergence is accepting claims like that the whole may be more than the sum of its parts or that chemical properties are different from physical ones. Here we need to distinguish two kinds of emergence: one related with supervenience and a stronger one.

If one believes that there are some properties or entities in the world which supervene on others while not being identical to them, then one believes in a sense of metaphysical emergence. As I commented in the last section, one may doubt that there are these supervenient properties in the world and it is unclear the role they would play in scientific practice. The stronger case for the existence of such property could be made for consciousness[4]. This kind of emergence is compatible with epistemological emergence but does not imply it, as knowing about the fundamental level is enough for knowing about the higher levels. In this sense, this kind of emergence does not pose a challenge to the methodological reductionism of science.

The second type of metaphysical emergence is stronger. Now we have the higher-level properties which are different from the lower-level ones, but they do not supervene on them. That is, if this kind of emergence is possible then we can have two systems which have the same lower-level properties but which nevertheless have different higher-level ones. This position is held by Morrison (Morrison, 2012, 2015), who argues for this position motivated by the phenomenon of universality, the fact that there are many different systems, with very different components, which nevertheless show very similar behaviors. For these systems the claim by Morrison is that, as the microscopic details do not play a crucial role, then the higher-level phenomena are ontologically independent from them. This perspective implies an epistemological emergence, but in this case it is a stronger one, as it means that we cannot know everything about the higher level from knowledge of the lower lever, not even in principle. In our example of the ecosystem this would mean, if we were to believe that this kind of emergence applied in this case, that even if we had knowledge about every single particle of every organism, there would be phenomena that we could not know or predict.

Positions like Morrison's rely on arguments based on universality and on phenomena which occur with some degree of independence of microscopic details. However, these arguments usually dismiss the alternative reductionist explanations which can be given to these phenomena. For this reason, it seems reasonable to argue against the view that this kind of emergence happens in the world and it is not a widely held position neither in science nor in philosophy. In this thesis I will be holding a reductionist view which denies that metaphysical emergence is necessary for understanding the level structure of science in general, and for the case of quantum gravity

[4] For instance, this claim is made in Chalmers (2011), where the term strong emergence is used for referring to this type of metaphysical emergence.

in particular.

Finally, let me also mention that there are other voices which argue for an ontologically pluralistic view in which the concept of reduction does not necessarily apply. For instance, Lombardi and Labarca (Lombardi and Labarca, 2005; Labarca, 2019) treat the case of chemistry and its reduction to physics. They defend an ontological pluralism in which both the ontologies of chemistry and physics can coexist without any hierarchy between them. If one were to hold this position, then one could claim that there is a failure of reductionism, but maybe one should not apply the term emergence, as one does not even have the hierarchical structure any more. Notice that this kind of positions is difficult to reconcile with my presentation in this chapter. The reason for this is that in my presentation I have been assuming a realist position in which I have been assuming that the different theories describe an external, independent world. In the case of Lombardi and Labarca they reject the existence of such a world and instead they claim that both the worlds of chemistry and physics are real. For our discussion of emergence in the case of quantum gravity we will not be dealing with positions like this one, as we will be assuming a realist position about the external world. When we will discuss the way time emerges in quantum gravity, we will be speaking about a single world, its different levels of description, and the properties in these levels.

Let me conclude this section by insisting that metaphysical emergence is generally rejected, and that I will endorse such a rejection for the metaphysics of science in general and for the case of the relation between theories of quantum gravity and spacetime. This reductionist view is also generally held in the quantum gravity and the philosophy of quantum gravity communities and therefore the debates about the emergence of spacetime have more to do with discussing the possibility that a world which is fundamentally non-spatiotemporal could look non-spatiotemporal than with any claim of strong metaphysical emergence. In this sense, the use of the term 'emergence' in these debates is associated with a different notion of emergence, which is associated with the idea of novelty, and which I will analyze in the next section.

2.3 Emergence as novelty

In this section I will analyze two alternative definitions of emergence that aim at the same intuition. The first of these definitions was given by Butterfield and is the basis for the second one by Crowther. Both definitions take emergence to be independent of reduction and associate it with concepts like novelty and autonomy. I will introduce both definitions in detail and I will argue that despite capturing well what the scientists often mean by emergence[5], these definitions are not philosophically interesting

[5] This is the intuition behind the famous paper "More is different" (Anderson, 1972).

beyond the task of describing how the term is used. That is, the relevant question regarding the emergence of spacetime, from the perspective of this thesis, is whether and how an apparently spatiotemporal world could be fundamentally constituted by entities which are not spatiotemporal. Whether one can say that the spatiotemporal world is novel or autonomous seems secondary to this question.

Let me start by analyzing the definition by Butterfield. This definition was given and argued for in Butterfield (2011a) and Butterfield (2011b). Butterfield defines emergent properties and phenomena to be properties or behaviors of a system which are novel and robust with respect to some appropriate comparison class. Usually, and importantly for our discussion here, the comparison class will consist on the subsystems that form the system. For instance, we may claim that thermodynamic behavior is emergent if we consider that when we compare a system formed by a few molecules and one formed by a great number of them, say of the order of Avogadro's number, it is only the latter which behaves in a way that we can call thermodynamic.

To make precise the definition we need to clarify what we mean by novelty and robustness. Butterfield claims that a system shows novel behavior with respect to the systems in the comparison class when this behavior is absent in this class or has different features, perhaps striking ones. By robustness, Butterfield means that this behavior is not very sensitive to the details or choices of assumptions in the comparison class. In the thermodynamics example we may claim that thermodynamic behavior is robust, as we might change the types of molecules in the gas while retaining this behavior. Notice that the idea of robustness fits well with the phenomena of universality that I introduced in the previous section, and that is generally associated with emergent phenomena. The idea of novelty also fits well with the intuitive meaning emergence. However, these notions can be criticized for being vague and subjective, as acknowledged by Butterfield himself in both papers.

Butterfield argues in his papers that emergence, as he defines it, is logically independent from both reduction and supervenience. For showing this he takes the position of Nagel which I have introduced in section 2.1 and which defines reduction to be a relation of derivability between theories. Similarly, Butterfield takes supervenience to be a formal relation between theories in which the supervenient properties of one theory are determined by the properties of another one. To compare with emergence, Butterfield also expresses this concept in terms of theories: one theory is said to emerge from another if it describes systems which show novel and robust behavior with respect to the systems described by another one and with which it is appropriate to compare. With the three definitions in place, Butterfield argues that emergence is independent from reduction and supervenience. That is, that it is possible to have emergence with and without reduction, and similarly for supervenience.

Notice that Butterfield's arguments rely on the definitions which are basically relations between theories, but we can translate them back to relations between systems and properties, as I have been doing in this chapter. Butterfield's notion of reduction is based on Nagel's notion of deducibility, which can be applied to a wider range of cases than just the comparison between levels we are interested in this chapter. Butterfield's arguments also apply for other types of reduction like ontological reduction and hence it is shown that it is logically independent of emergence understood as novel and robust behavior. For instance, in the case of thermodynamics one may argue for a position which is ontologically reductionist while at the same time claim that thermodynamic behavior is novel with respect to the behavior of a few particles.

An important aspect of Butterfield's definition is that it depends on the choice of comparison class. Butterfield mentions two classes. First, as I have commented before, for the case of a composite system the relevant comparison class can consist on subsystems of this class. Second, when the system can be seen as the limit of a sequence of systems, then it is natural to take the comparison class to be this sequence. The example that Butterfield treats in more detail is a system which is in the thermodynamic limit, i.e., in the limit of infinitely many components. In this case the second comparison class is very similar to the first one, but we can think of examples in which the limiting sequence has nothing to do with the limit of infinite components. For instance, we can think of a sequence of special relativistic spacetimes labeled by different values of the speed of light in those spacetimes. In this case we can think of Newtonian spacetime as the spacetime we obtain if we take the limit of an infinite speed of light in the sequence of relativistic spacetimes. As Newtonian spacetime is different enough from Minkowski spacetime we can claim that there is novelty, and if we further argue that the limit is robust, then we can claim that it emerges in the sense of Butterfield.

However, notice that this sense of emergence is somehow different from the senses of emergence and reduction we have been dealing with in this chapter, with the exception of Nagelian reduction. As in that case, the relation seems to be a relation between theories which can be used to answer questions like the way a theory can explain another rather than dealing with the different levels of properties in the world. For instance, the fact that Newtonian spacetime can be recovered as a limit of special relativistic ones can be used for arguing that the use of Newtonian spacetime is justified as an approximation to special relativity in some circumstances. This, arguably, has little to do with the way smaller scales relate to bigger ones. However, as I have commented before, these two notions get many times entangled. Importantly, this is so for the case of interest of this thesis, the emergence of time in quantum gravity. In this case we can intuitively say that there is a level structure: there will be entities in a fundamental level that will compose, in some appropriate sense, spacetime, which will be in a higher level. But we will also find relations between theories in the form of limits and we will be interested in knowing if we can recover general relativity as a suitable approximation to some theory of quantum gravity. In particular, the limits between the different quantum gravity models and general relativity will be studied profusely.

Another point to make is that the limit structure is different from the level structure I have been using in this chapter. In this sense, when I have been speaking about reduction and emergence I have been referring to the properties of a system as seen from different levels, while if we use the definition by Butterfield we will be comparing different systems. This feature further shows the difference between the different notions of emergence and is important to emphasize that Butterfield's sense of emergence is compatible with reductionism. There is a tension between reductionism and emergence which comes from the fact that if we are reductionist about the properties of a system then it seems that there cannot be novelty in its higher level properties, as they are deducible from the lower level ones. However, Butterfield's notion of emergence does not deny this, but instead the claim of emergence in this case refers to behaviors which are present in the system but not in systems formed by fewer components. In this way, reductionism and emergence can be reconciled. Keep in mind however that this notion of emergence is subject to challenge, as it is based in the subjective notion of novelty. That is, it seems plausible to say that, if we are really reductionist, there really is not any novelty in considering systems formed of more particles. For this reason, Butterfield's definition of emergence is not as clear-cut as the ones we have found before. On the other hand, it captures what many scientists mean by terms like 'emergent phenomena'.

Now we can turn to the notion of emergence by Crowther[6]. Crowther reviews other notions of emergence in the literature, basically the ones I have explained in this chapter, and she concludes that claims of emergence generally have two components, which are a claim of dependence and a claim of independence. For instance, the notions of epistemological and metaphysical emergence I have analyzed in the last section consist in a claim of dependence, that higher-level entities are composed of lower-level ones, and one of independence, that we do not have to appeal to micro-level details to explain the macro-level or that we cannot do so. Different notions of emergence therefore consist on different choices for each of these two components.

Having noted this general structure, Crowther builds her own approach by incorporating the intuitions of Butterfield. Crowther also formulates her account in terms of relations between theories rather than referring to systems, properties or phenomena. Similarly to Butterfield, Crowther takes the independence part of the definition of emergence to be that the emergent theory is novel and autonomous with respect to the basis theory. Crowther defines novelty as meaning that the emergent theory is

[6] See Crowther (2015, 2016, 2018, 2021).

strikingly and qualitatively different from the basis theory. By autonomy, she means basically the same as Butterfield with robustness, that the emergent theory is the same for different choices or assumptions about the basis theory[7]. Given the similarity of the approaches, it is subject to my previous criticism, i.e., novelty and autonomy are not objective and hence the whole notion of emergence seems subjective.

Crowther departs from Butterfield in the dependence part of the definition of emergence. Butterfield did not explicitly mention any dependence relation but Crowther introduces it in order to specify the appropriate comparison class which is necessary for Butterfield and which was left to be context dependent. Crowther's dependence claim is that the emergent theory is approximately derivable from the basis one and/or that the former is supervenient on the latter[8]. This is an important difference with Butterfield, as Butterfield claimed that emergence and supervenience are logically independent and that one could have emergence with or without supervenience. Crowther acknowledges this difference and claims that she believes there is supervenience in all the cases she considers and that the counterexamples given by Butterfield do not affect her claims[9]. For the cases of our interest which involve the relation between levels we can generally agree that supervenience holds, but notice that this leaves out some of the strongest claims of ontological emergence which would deny this.

Similarly to Butterfield's definition, the definition by Crowther includes, or is meant to include, relations between theories more general than the relation between different levels. For this reason it is helpful to introduce a further distinction, as explicit in Crowther (2021). Crowther distinguishes two types of emergence: hierarchical emergence and flat emergence. Hierarchical emergence is the one that is closer to the other notions of emergence we have been interested in this chapter. In this kind of emergence one further assumes a level structure and that the basis theory is a more fine-grained description of a system and while the emergent one gives a coarser-grained description. Crowther avoids referring to smaller or bigger systems given that in the context of quantum gravity we may not be able to use metric concepts associated to size. Similarly, Crowther claims that this kind of emergence could be considered as synchronic emergence as we are considering two descriptions of the

[7] In Crowther (2015) autonomy is taken to be more concrete than robustness, as it is explained in terms of ideas of effective field theories. Meanwhile, in Crowther (2021) autonomy is not distinguished from robustness.

[8] As happened for the case of autonomy, we find a difference in the definition Crowther gives in different works. In Crowther (2015) the idea of approximate derivability is made more concrete by stating the dependence relation in terms of effective field theory and renormalization group techniques. In later works (Crowther, 2021) it is spoken about approximate derivability, which is more general and more easily applicable for the cases of our interest.

[9] See footnote 19 in Crowther (2015).

very same system at 'the same time', but that in the context of quantum gravity it is safer to avoid referring to any temporal notion.

Flat emergence, instead of comparing two theories that describe a system at different levels but on the same circumstances, compares the same system, at the same level, but at different circumstances. An example of this kind of emergence would be a phase transition: before and after the phase transition we could describe a system with theories which are certainly different, like a theory describing a solid and the one describing a fluid. For this reason we can think of this kind of emergence as diachronic emergence, but as before, it is preferred to speak about flat emergence to avoid referring to any temporal notion.

In this case, Crowther maintains that the dependence relation is still a supervenience one. In particular, this notion of supervenience, called flat supervenience by Crowther, is different from the usual in philosophy and from the one I have introduced in the previous section. Flat supervenience is introduced just as supervenience, i.e., that there cannot be changes in the properties described by the emergent theory without changes in the properties described by the basis one, but in this case the properties do not refer to an object at the same instant of time or in the same circumstance but at different ones. This relationship could be exemplified for instance by causal relations: in some causal models it is not possible to have the same causes causing different effects. However, this relation can be problematic. For instance, in non-deterministic models we may have a same cause causing different effects with different probabilities. Or in a perfectly deterministic model one could claim that the past supervenes on the future just as much as the other way around, so it could be the case that we have emergence in both directions of time.

The notion of novelty for flat emergence does not change from the one in the general case: there is novelty if the theories are strikingly different. Autonomy is understood as a relation of underdetermination, that is, that there are many initial states as described by the basis theory that could have given rise to the same emergent state. For instance, Crowther claims that this kind of underdetermination applies for some phase transition in which there is a broken symmetry. The paradigmatic example of this is a magnetic material which can be thought of as been made of small magnets. In the symmetric phase each magnet points in a random direction, making the total magnetization of the material 0. There are many microscopic configurations which would satisfy this. When the symmetry is broken, i.e., in the other phase of the system, all the magnets or a great majority are aligned in one direction giving rise to a total magnetization. This means that given a magnetization there are very few microscopic configurations compatible with them. In this sense, one can claim underdetermination for a symmetry-breaking phase transition like this one: there are many possible initial states in the symmetric phase but only a few in the symmetry-broken one.

This intuition of underdetermination in phase transitions can be challenged. Notice that Crowther looks at the microscopic details in order to claim that there is underdetermination, but if what we are comparing are macroscopic phases it seems that we should be comparing macroscopic properties. Surprisingly, one could argue for the opposite conclusion to Crowther's: that the underdetermination goes in the other direction. The reason for this is that in some symmetry-breaking phase transitions there are many ways the symmetry can be broken and hence the symmetric phase does not determine the symmetry-broken one. For the example of the magnetic material, one cannot determine (in the absence of external magnetic fields) which will be the direction of the total magnetization of the material and hence there are many final states that are compatible with the same initial conditions. The same argument seems to apply to the example of the geometrogenesis of quantum graphity[10], which Crowther explicitly studies in section 5.2 of Crowther (2021). In this example one could also argue that the underdetermination goes in the opposite direction of what Crowther claims, as one initial state can give rise to several possible final states. In this sense, one may have reasonable worries that underdetermination plays an important role in our understanding of phase transitions and of flat emergence, if we want to accept such a notion.

The worries above show that the notion of flat emergence is not as sharply defined as the hierarchical one. In any case, Crowther analyzes several models of quantum gravity and claims that there could be flat emergence in some of them. In all of them the flat emergence is meant to apply to a phase transition in our universe that would have occurred at the big bang. In chapter 9 I will analyze a case of alleged emergence of this type, which is the geometrogenesis phase transition of group field theory[11]. For all these cases we can discuss whether the notion of flat emergence as defined by Crowther applies. But, as I have argued, there is some subjectivity in the definition, so one may doubt about the usefulness of defining emergence in this way. That is, maybe the term phase transition was already good enough and more precise for describing this kind of situations. In any case, for the biggest part of this thesis when I will speak about emergence I will be referring to hierarchical sense of emergence and not to the flat conception.

The worry that flat emergence may not be philosophically useful also applies to hierarchical emergence and to emergence as defined by Butterfield. The idea of having emergence associated with novelty and autonomy captures well what is meant by many scientists by emergence and which does not carry strong ontological

[10] See Konopka et al. (2006) for an introduction to this approach.

[11] I refer the reader to Huggett and Wüthrich (2018) for a discussion of other potential examples from quantum gravity.

commitments. But once we accept this meaning of the word, it seems that one is not making a big philosophical statement when claiming emergence of this sort. This is fine, but diminishes the philosophical interest in this use of the term. Of course, this is not to say that phenomena like universality or the autonomy of the different levels are not interesting but in contexts like quantum gravity the focus will be in whether such a hierarchy between a spatiotemporal higher level and a non-spatiotemporal one can be established.

We can compare this conclusion with Crowther's position. She explicitly claims that the interest of her approach to emergence is not to enter in debates in the metaphysics of science or physics, but that it has a different philosophical interest as a relation between theories. In this sense, she advocates for a science-first approach which focus on the way science works and theories are developed and related to each other in contrast with a more metaphysics-first approach which would care more about metaphysical issues. In this thesis, however, I will be more interested in studying the theories of quantum gravity in a realistic way and analyzing what they say about the world. In this sense, approaches like Crowther's will not be very useful, even if accurate from a descriptive point of view.

2.4 Classical-quantum emergence

Finally, in this last section I discuss an important difference that makes it the case that the relation between any putative theory of quantum gravity and general relativity is more complicated than the relation between the classical mechanics of a gas and its thermodynamics. This difference is of course that in the case of quantum gravity we are dealing with a quantum (or postquantum) theory, and this adds interpretative difficulties to the task of clarifying what our tentative more fundamental theory says about the world and its relation with the apparently classical world we experience.

In this context, it is sometimes claimed that a classical world emerges from a quantum one. This kind of claim can be understood roughly in the sense of Crowther: it is usually assumed that there is a sense in which the classical world depends on the quantum one and one can surely argue that the quantum and classical theories are sufficiently different so that the claim of novelty is warranted. However, as I argued in the last section, the interesting philosophical and physical questions come from analyzing the ways one can interpret the quantum formalism as describing our world and from looking for an explanation for why a classical description is approximately successful when the world it describes is in many senses not classical. These questions are the ones that need to be answered in order to understand the relation between the classical and the quantum. In this sense, the interesting question to ask is not whether a classical world emerges but the way it does so.

Answering these questions amounts to solving the measurement problem of quantum mechanics. There are several proposals for solving this. Consider the defense of the many-worlds interpretation by David Wallace in his book 'The emergent multiverse' (Wallace, 2012)[12]. As can be seen from the title, Wallace uses the term emergence for referring to the relation between the fundamental wavefunction and the emergent classical worlds. What he means by this is explained in the second chapter of his book, and we can characterize it in the following way. He adopts a functionalist position about higher-level entities: for instance, he claims that there are tigers just when there are structures which instantiate the roles played by tigers. Similarly, the same applies for semiclassical worlds: there are such worlds just when the functions and roles they play are instantiated, in this case by branches of the wavefunction. As we see, this is perfectly compatible with the methodological reductionism of science and even with ontological reductionism, even if Wallace is not an eliminativist. Wallace also allows for some sorts of epistemological emergence, as he accepts that, given our computational limitations, we need to appeal to higher-level sciences and as he also claims that we cannot dispense with higher-level explanations. In this sense, what is meant by Wallace when appealing to emergent worlds is clear, and we also get a clear metaphysical and physical picture of the interpretation he defends, independently of whether we find it convincing or not.

In the case of the other realist interpretations of quantum mechanics we can see that they also allow for perfectly reductionist accounts, compatible with some sort of epistemological emergence or with emergence in the sense of Butterfield and Crowther. For instance, collapse models are similar in their structure to the Everett interpretation, and hence the arguments that Wallace employs for arguing for the emergence of his many worlds can be used for arguing for the emergence of one world once the collapse mechanism is in place. Consider also Bohmian mechanics. The addition of particles to the ontology of the theory is again compatible with a methodologically reductionist view in which everything is composed of particles and could in principle be explained in terms of particles. A particularly reductionist version of Bohmian mechanics is defended by Michael Esfeld and collaborators and is called super-Humeanism[13]. In this version, every property at every level is reduced to just distance relations between particles. Other versions of Bohmian mechanics allow for a richer ontology.

Each realist interpretation of quantum mechanics gives us a possible picture of the world and an explanation of how the classical and macroscopic domain can be said to 'emerge'. Other interpretations[14], more operational, fail to provide a clear ontology

[12] The same position can be found in Wallace (2013).

[13] See Esfeld et al. (2017).

[14] Let me mention relational quantum mechanics, which has been argued for by Rovelli (1996) and which has a special relevance in the LQG community.

of the world and hence it is difficult, if possible, to assess the relations between the macroscopic and microscopic worlds. For instance, in the Copenhagen interpretation of quantum mechanics one just does not solve the measurement problem and postulates the coexistence of two regimes: a quantum and a classical one. In this case, there are still some claims we can make which relate the macro and the micro, but they are limited. For instance, we could claim that the radiative spectrum of a star can be reduced or that it emerges from the quantum phenomena of the emission and absorption of photons by the atoms in the interior of the star. These phenomena are of quantum nature and some features of them can only be explained by appealing to quantum mechanical properties, like the electronic configuration of some atomic species. It is arguable that a defender of the Copenhagen interpretation could claim that this is a case of emergent behavior which can be accounted by this interpretation: when observing the star we could be said to be measuring the photons emitted by each atom and by measuring all of them together we are observing the higher-level, aggregate phenomenon. This kind of claim is limited as it needs to appeal to concepts like measurement, or to assume a classical regime, even if the details of this regime are explained in terms of quantum properties.

By adopting an operational view of a quantum theory one takes fewer ontological commitments. However, there is a minimum commitment one has to make, which is that one has to identify a part of the formalism with some empirical content. I will argue in this thesis that this will be a problem for some approaches to quantum gravity which are intended to be interpreted operationally. For instance, we will find that some theories define probabilities for measurements which are not spatiotemporally localized or for observables which do not have a clear physical meaning.

For the argument in this thesis, the discussion of the transition from a quantum theory to a classical theory will be crucial. In particular, I will defend that the vast majority of quantum gravity models, not only the ones formulated in the canonical formalism, suffer from some problem of time and that this makes it the case that they present serious shortcomings when interpreting the mathematical structures they define. This will be clearly the case for the standard realist interpretations of the quantum formalism, but it will also affect operationalist views.

It is also worth mentioning that in the physics literature one usually finds claims about the classical limit of a quantum theory. In different contexts what is meant by this varies, and in particular I will analyze what is meant by this in the context of different approaches to quantum gravity. But in any case, I will argue that to give a complete account of how the classical world ‘emerges’ from the quantum one one needs to rely on some interpretation of the quantum formalism and one cannot just appeal to some formal property of the formalism, such as its limit for $\hbar \to 0$. Similarly, in the quantum gravity literature we also find some discussion about the continuum limit of the theory, which arises in models that define discrete structures.

Finally, we can consider the relation between classical theories and quantum theories in the other direction. A quantization is a way of building a quantum theory by starting by a classical one that would represent its classical limit. In this thesis I will analyze the main two quantization schemas, the canonical and covariant one, and the ways they have been applied to general relativity.

Let me conclude this section and this chapter by noticing the similarity between what has been discussed in this section and what will be discussed in this thesis. Physicists in the quantum gravity literature commonly claim that time or spacetime emerge from whatever fundamental structure they propose. Similarly, in quantum mechanics it can be claimed that a classical world emerges from a quantum one. My take on these claims is that they are generally vague and just mean something very similar to what Crowther defines as emergence: that the classical domain, despite being in some way dependent on the quantum one, is different from it. The interesting questions in the philosophy of quantum gravity, just as in the philosophy of quantum mechanics are to evaluate what the theories say about the world and the way it is or not consistent with the apparently classical and spatiotemporal world we seem to live in. Despite the use of the word emergence, the general attitude in the philosophy of quantum gravity is to favor reductionist accounts of the relation between the classical entities like spacetime and the more fundamental ones.

In the next chapter I will consider the philosophical debates about the possibility of having a non-fundamental spacetime constituted by non-spactiotemporal entities, as suggested by quantum gravity. However, I will also argue that we lack any clear example of how this emergence is supposed to work. The reason for this is that the models of quantum gravity used for supporting those claims have serious conceptual shorcomings which raise worries about their justification and interpretation, as I will argue in the rest of this thesis.

3. EMERGENCE OF SPACETIME

In the previous chapter I introduced the different notions of emergence which are generally used in the physical and philosophical literature. I also argued for a reductionist view of physics which nevertheless allowed to speak about emergence in an epistemological sense when there is enough novelty and autonomy in the higher level. In this chapter I will focus on the emergence of spacetime in quantum gravity. In different theories of quantum gravity one finds a level structure in which spacetime is claimed not to be fundamental but, instead, it 'emerges' from some more fundamental structure. In this case the philosophical debate is not about which kind of emergence we have in play (although the reductionist view would be the most extended one), but about whether this emergence is possible at all. That is, the philosophical debate here is whether it is coherent and compatible with our spatiotemporal experience to postulate a physical theory which is not based in space and time[1]. In this chapter I will review the functionalist position which proposes to reduce spacetime to some more fundamental entities and the worries raised against it. I will note that a key point for the success or failure of functionalism, and of the theories of quantum gravity in general, is the way the quantum formalism gets interpreted. Indeed, the examples used in the literature focus just on classical aspects of the theories of quantum gravity, so one could rather say that functionalism has been successfully applied (arguably) to discrete spacetimes, but not to quantum spacetimes. Furthermore, in the rest of the thesis I will argue that, at least for the approaches to quantum gravity I will consider, there are serious conceptual and technical problems which make it the case that we lack a convincing interpretation of the formalisms that these approaches propose and that we should reject considering them satisfactory theories. In this sense, I will argue that the functionalist may be right in that spacetime can be emergent from a non-spatiotemporal theory, but we currently lack any such theory.

I will start in section 3.1 by reviewing the functionalist position defended by Wüthrich and others (Lam and Wüthrich, 2018). From this perspective, if we consider seriously the different theories of quantum gravity that hint in the direction that spacetime is not fundamental, then we should take a functionalist view of spacetime.

[1] This philosophical debate is a different one from the one concerning the background independence of some theories of quantum gravity.

That is, if the entities postulated by the new theories are able to play the same functional roles that are played by spacetime, then it should be coherent to replace spacetime with these new entities. This functionalist argument is very similar to others in the history of science that I mentioned in the last chapter, such as the ones that allowed to replace water by collections of H_2O molecules or the temperature of a gas with facts about the kinetic energy of the molecules that form it. Moreover, I will also mention that functionalism about spacetime has also been discussed in the philosophical literature in the contexts of the foundations of spacetime theories and quantum mechanics. In this sense, spacetime functionalism can be argued not to be so radical, although in the other circumstances where it has been proposed there are alternatives to avoid it, which seem not to be available in the case of quantum gravity.

Next, I will analyze some of the arguments against functionalism applied to spacetime in section 3.2. The most obvious one is that functionalism, as usually understood and in the common examples of science, is phrased in spacetime terms and applies to entities that live in space and time. Therefore one may worry that functionalism is not applicable in this case, given that we are considering it outside its domain of common applicability. Furthermore, that the new fundamental entities are not spatiotemporal is also problematic on its own, given that it clashes with the intuition that to exist is to exist in space and time. Of course, this is related with the way we experience the world, which is usually thought of by means of local entities in space and time, also known in the literature as local beables[2]. I will argue that these issues do not affect the functionalist picture as long as the fundamental entities provide enough structure as to account for local beables at the emergent level. I will also comment on the position that one should require something more than just satisfying some functional and structural roles. For instance, Maudlin (2007) asks for the emergent entities to be physically salient and not just definable from the fundamental ones. However, in my view it is not clear what this requirement amounts to in general and in the case of the emergence of spacetime in particular.

Besides this kind of objections I will analyze two further ones. First, I will comment the intelligibility objection as formulated and rejected by de Haro and de Regt (2020, 2018). I agree with them in that we should not a priori reject that a theory without spacetime could be intelligible. However, I will argue that in current theories of quantum gravity, in particular in those that are studied in this thesis it is not so certain that we have a clear understanding and interpretation of them. In this sense, the intelligibility objection may be worth considering, not because theories without spacetime in general cannot be understood, but because the ones we have can be argued not to be so. Indeed, I will extensively argue for this position in the rest of

[2] This term was first used by Bell (2004) and is commonly used in discussions in the foundations of quantum mechanics.

the thesis. Second, I will analyze the analogy with the hard problem of consciousness in the philosophy of mind as framed in Le Bihan (2021). I will agree that there is a conceptual gap between the spatiotemporal and the non-spatiotemporal but I will reject the analogy with the hard problem, as we may be able to close this gap. I will also consider two claims from Linnemann (2021), namely, that functionalism could not close the gap if at the fundamental level there was not a distinction between space and time, and that in current approaches there is that distinction. I will argue against both of them, as the analysis Linnemann performs of current approaches is superficial and as I will find Linnemann's reasons for claiming that the gap could not be closed not convincing.

Next, in section 3.3, I will introduce the examples which are used in the literature for supporting the functionalist view. These cases are causal set theory and some states in the Hilbert spaces that appear in loop quantum gravity, some spin foam models and group field theory. I will argue that, starting from the discrete structures these theories define, one can formulate a plausible argument for a functionalist reduction of spacetime or space to such structures. However, I will emphasize that such arguments rely only on the classical structure of the theories, and that we are lacking a functionalist story for the complete quantum theories. In this sense, the functionalist seems to be able to argue for the emergence of spacetime from discrete spacetimes or discrete structures, but not from something fully non-spatiotemporal and, in some sense, quantum, which shows a limitation of the approach.

Finally, in section 3.4 I will expand on this criticism and argue that the biggest challenge for the functionalist is to provide a successful interpretation of the theories of quantum gravity, given that to the usual difficulties of interpreting a quantum theory now we have to add two extra sources of complication, namely that now we cannot rely on notions of space and time and that the formalism of the approaches to quantum gravity is not the full quantum formalism. Moreover, I will argue that at least for the approaches I will consider in this thesis, we lack a convincing interpretation. Indeed, in the rest of this thesis I will argue that the diffeomorphism invariance of general relativity makes it the case that its quantizations, canonical or covariant, suffer from some form of the problem of time, which makes it the case that the usual interpretations of quantum mechanics do not apply and which suggests rejecting these approaches. In this sense, I will not argue that functionalism cannot work in principle, but that the approaches I will study[3] are not examples of theories without a spacetime from which a spacetime can be argued to emerge. On the contrary, they fail to constitute satisfactory theories.

[3] In this thesis I study a quite comprehensive list of approaches based on quantizations of general relativity. The most relevant absence are string theory and related approaches, about which I have not performed any analysis.

3.1 Spacetime functionalism

Different approaches to quantum gravity postulate, for different reasons, that spacetime is not fundamental. That is, according to these theories, there is some set of fundamental degrees of freedom which are not spatiotemporal and which constitute what we know as spacetime. In this sense, spacetime is said to be an emergent entity. As I argued in the previous chapter, by claiming that something is emergent in physics one generally means some weak sense of emergence, such as epistemological emergence or some sense of novelty and not a strong sense such as a failure of reduction. In the case of spacetime in quantum gravity the same reductionist spirit is generally assumed, i.e., spacetime would be reduced to the entities in the fundamental level and every fact about it could be derived (probably as an approximation) from facts about the fundamental level. Linnemann and Visser (2017) argue for this reductionist view which is compatible also with the claim of emergence as novelty also for the case of spacetime. Huggett and Wüthrich (2013, 2021) also use expressions like emergence of spacetime implying also derivability while acknowledging the difference between the fundamental and emergent levels. In any case, the relevant question considered is whether it is possible to have a theory without spacetime at its fundamental level.

The answer to this question by the physicists working in some approaches to quantum gravity and by philosophers like Huggett and Wüthrich is affirmative. For recovering spacetime from non-spatiotemporal entities functionalism is appealed to, just as functionalism can arguably help us reduce thermodynamic concepts like temperature and heat to non-thermodynamic concepts like the kinetic energy of the particles of a gas. As I introduced in section 2.1, functionalism consists in identifying an entity with the functions and causal roles that it plays in a given theory. If one is able to find some other entity or set of entities which can fulfill the same roles, then one can argue that one can replace the original entity by the new ones. For instance, by identifying water with the roles it plays (such as being a fluid or being able to dissolve salt) we can provide a reductionist argument that reduces water to H_2O molecules. Similarly, if we are able to identify the roles played by spacetime and find a theory of quantum gravity which postulates some entities which can play these roles, then the emergence of spacetime from these entities would seem plausible. A proposal along these lines can be found for instance in Lam and Wüthrich (2018, 2021).

Of course, functionalism about space and time seems more radical than the usual functionalism of science, which is a functionalism of entities which live in space and time. However, Lam and Wüthrich rightly point out that quantum gravity is not the only context in which functionalism has been applied to spacetime. In realist interpretations of quantum mechanics which take the wavefunction as the primitive ontology of the world, functionalism[4] has been appealed to for recovering 3-dimensional space. The reason for this move is that the wavefunction is mathematically defined as a function in configuration space and not in space. Therefore, in order to explain how the dynamics of such an entity has anything to do with our experience of a 3-dimensional space, one can argue that the wavefunction instantiates the functions and roles that are played by particles living in a three dimensional space.

The other context in which spacetime functionalism has been discussed is in the philosophy of general relativity. Eleanor Knox has proposed (Knox, 2013, 2014, 2019) a different functionalist approach to space and time. Her approach starts[5] from the dynamical perspective to spacetime developed by Brown (2006) and can be seen as an alternative of what she calls the 'orthodox account' (Knox, 2013), i.e., the view that geometry is fundamental in general relativity. The details of this approach and the dispute between the dynamical and the geometrical views of spacetime will not be relevant for our discussion here. Let me just point out that Knox presents her functionalist position as compatible with the view that spacetime is emergent while, for her, the geometric account would have a harder time to accommodate this. However, Lam and Wüthrich (2021) have a different position here: while they reject the functionalist approach of Knox to general relativity and they prefer the geometric view of spacetime, they nevertheless argue for a functionalism in the context of quantum gravity that is able to recover a geometric spacetime.

This divergence between Knox and Lam and Wüthrich causes the two functionalisms to identify different functions as defining a spacetime. Knox remains closer to the dynamic approach and identifies spacetime as whatever what is defining inertial frames in the theory. Meanwhile, Lam and Wüthrich give the following (non-exhaustive) list of spacetime functions to be satisfied: "the determination of spatial distances and temporal durations, and of spatial and temporal relations between physical objects more generally, and so of their relative localisation, i.e., an object's or event's localisation in space and time relative to other, usually nearby, objects and events, which thus furnish a local frame of reference" (Lam and Wüthrich, 2021, p. 2). The functions on this list, despite including a mention to reference frames, are clearly in line with the geometrical interpretation to general relativity. Most of the work in the emergence of spacetime in quantum gravity has been done from the more geometrical perspective, and it has tried to show that the structures defined by quantum gravity are able to play geometric roles like the ones listed by Lam and Wüthrich. For a proponent of the dynamical perspective like Knox this may not

[4] See Albert (2013) for instance.

[5] As argued in Knox (2019), this approach takes the dynamical perspective as a starting point and inspiration, but it is not the only view that can be developed starting from it. For instance, a relationalist project can also be considered heir of Brown's work, while being incompatible with Knox's functionalism.

seem enough, as from this perspective it is only when the dynamics of physical objects is taken into account that it is possible to give a chronometrical significance to the 'geometrical' degrees of freedom. That is, if one takes geometry to be 'measured' by rods and clocks one needs to connect the degrees of freedom of the theory which allegedly define the geometry with the ones which measure it. Of course, from the geometric perspective one can accept that the analysis performed so far of the emergent spacetimes in quantum gravity may be incomplete until one shows that the emergent geometry relates to the emergent matter degrees of freedom in a way which approximates the way in which geometry and matter relate in general relativity, at least in certain limits.

It is also important to point out, as Lam and Wüthrich do, that there is an important difference in the case of functionalism applied to quantum gravity. In the other two cases in which functionalism for space or spacetime is discussed, one has alternatives available which may be preferable to the one of functionalism. In the case of quantum mechanics there are competing interpretations which are based on entities which 'live' in 3-dimensional space, the local beables. Even in the case of an ontology based on the quantum state, like in the many-worlds interpretation, it can be argued that this ontology is based in the 3-dimensional space and not in configuration space[6]. In these cases, one does not need to appeal to space functionalism. Similarly, the spacetime functionalism of Knox competes with the geometric or orthodox view of general relativity. In the case of quantum gravity, if we are considering a given theory which is not spatiotemporal, then it seems that we are forced to use functionalism to try to connect these non-spatiotemporal degrees of freedom with a spacetime.

For this reason, functionalism about spacetime is very natural in the context of theories of quantum gravity which are based on non-spatiotemporal entities. However, functionalism about spacetime seems more radical than functionalism applied to other entities or concepts. In the next section I will analyze some of the objections raised against spacetime functionalism.

3.2 Against spacetime functionalism

In this section I review the arguments against spacetime functionalism that have been raised in the literature. I have divided them in three groups: 1) arguments that put pressure on the way functionalism is intended to work without spacetime, 2) arguments that reject that theories without spacetime are intelligible and 3) arguments that make an analogy with the philosophy of mind and claim that there is a hard

[6] This position has been called spacetime state realism and defended in Wallace and Timpson (2010). Notice that even if no functionalism is used for space, functionalism still plays a role for the emergence of entities and worlds in this view.

problem for spacetime functionalism, just as there is one for functionalism applied to the mind. I will argue that the analysis on some points has been incomplete and that the problem of the intelligibility may aggravate the other two, as without a clear interpretation of a theory of quantum gravity it is less clear the way functionalism is supposed to recover local beables, and also because it makes it difficult to establish whether there is a hard problem or not. Furthermore, the main objection I will raise later on (3.4) against functionalism and about the approaches to quantum gravity that I will consider in this thesis goes in the line of the intelligibility objection: even if I will not argue that it is impossible to have a theory without spacetime, I will argue that the ones I will study do not have satisfactory interpretations and that we have strong reasons for rejecting them.

3.2.1 Functionalism and ontology without spacetime

The first family of objections I will address here are the ones that consider that spacetime functionalism and an ontology not based in space and time may be too radical a proposal and argue for alternatives based on local beables, i.e., on physical structures defined on space and time. Defenders of local beables typically argue that functionalism about space and time is of a different kind from the standard functionalism applied in science and doubt of its feasibility.

For instance, Esfeld (2021) recalls that in the history of the foundations of quantum mechanics many radical proposals have been proposed in order to make sense of the theory, such as changing the way logic works or denying that there are particles following well-defined trajectories in spacetime. These radical proposals turned out not to be strictly necessary once alternative interpretations of quantum mechanics were developed. In particular, Esfeld refers to interpretations like Bohmian mechanics. Esfeld argues that in the same way that in quantum mechanics one does not need to be committed to radical ontologies, in quantum gravity one may also take the same position: to reject non-spatiotemporal interpretations of quantum gravity and wait for a spatiotemporal interpretation to be developed. In fact, Esfeld cites some Bohmian proposals like Vassallo and Esfeld (2014) which try to develop a Bohmian version of LQG which is based on spatiotemporal notions.

Lam and Wüthrich anticipated this move in Lam and Wüthrich (2018) and argued against it. Their argument is that there is a difference between quantum mechanics and quantum gravity and this difference makes it the case that presupposing a spatiotemporal ontology is illegitimate or methodologically inadequate in the case of quantum gravity. This difference is that in the case of quantum mechanics, spacetime is something external to the theory, that is, the theory describes the behavior of quantum systems but it does not describe spacetime. In the case of quantum gravity, however, the candidate theories directly deal with spacetime and, in the most straightforward way of interpreting them, they can be read as directly saying that there is no spacetime in the fundamental ontology. From the realist position Lam and Wüthrich hold, this needs to be taken seriously and this leads them to reject postulating a spatiotemporal ontology. Despite this, Esfeld (2021) still claims that in quantum gravity it is as legitimate as in quantum mechanics to add spatiotemporal entities to the ontology, given that in both cases it is done to connect the quantum formalism with what he calls ‘physical reality’, that is, the world as we experience it. Lam and Wüthrich (2021) reply with a reaffirmation of what already stated in Lam and Wüthrich (2018), that they consider that postulating such an ontology goes against the realist or naturalist view that ontology should be responsive to scientific advances and be able to adapt.

Here I side with Lam and Wüthrich on their view of the relation between ontology and physics and I believe that if scientific theories are pointing towards a change in the ontology, this is an option that has to be taken seriously. At the same time, that there might be a more conservative alternative available like Esfeld proposes seems also possible and it seems as legitimate as Bohmian mechanics, even if someone with a view like Lam and Wüthrich may prefer another alternative. Notice also that someone like Esfeld can deny that the ontology they propose is insensitive to physics. For instance, in Bohmian mechanics there is a non-local dynamics to account for quantum non-locality and in the proposal for a Bohmian version of LQG, space is not ordinary space, but it is made by ‘chunks’ in a way that mimics the structure of the states of LQG. In this sense, the ontologies defended by Esfeld are, to an extent, sensitive to physics and open to some changes in our view of spacetime, while at the same time compatible with the intuition that the world is constituted by entities which live in space and time, i.e., local beables.

Hence we find two positions available depending on how we balance the two intuitions: the realist one that tells us to take the physics seriously as formulated by the physicists and the one that tells us that the world is constituted of entities in space and time. This is the same debate as in the philosophy of quantum mechanics between theories which postulate local beables and those which do not. Importantly, in this case both sides face serious challenges and have work to do. The functionalist has to be able to show that the functional reconstruction of spacetime is really plausible from the non-spatiotemporal ontology they propose. Meanwhile, the defender of local beables[7] has to be able to build a solid alternative able to accommodate the empirical content of the theory of quantum gravity[8]. However, it has been notoriously difficult to extend the ideas of Bohmian mechanics beyond non-relativistic quantum

[7] Here I have mentioned Bohmian mechanics because is the interpretation favored by Esfeld, but there are other alternatives available, like some interpretations of collapse models.

[8] Of course, a controversial point would be to determine what is the empirical content of quantum gravity, or even whether there is any.

mechanics and we should expect the same or even more serious difficulties for defining a working Bohmian quantum gravity[9]. These difficulties seem even greater than the ones faced by the functionalist, given that one needs to add further structure to the theory.

I think it is a fair assessment of the dispute to say that we have two legitimate competing views, but that the local beables one is in disadvantage and is certainly not the preferred one in the quantum gravity community. For this reason the defender of this view needs to argue against the functionalist view in order to make their view preferable. Esfeld (2021) for instance argues that functionalism as usually understood is based on spatiotemporal notions, and that it therefore should not be used as a model for the case in which what we are functionalizing is spacetime itself. This argument does not claim that such a functionalism is a priori impossible but rather that it is of a different kind. Esfeld argues that in the special sciences functionalism allows us to define entities like, say, water or tigers by means of the causal roles they play, that is, something is water if it behaves like water, something is a tiger if it behaves like a tiger, and so on. These roles, according to Esfeld, are spatiotemporal, as they all ultimately refer to the motion of physical entities in space and time. Functionalism about spacetime, in the case of quantum gravity and in the case of the wavefunction monism, however cannot appeal to such spatiotemporal notions, as it is based on a non-spatiotemporal ontology. However, this difference can be accepted by someone like Lam or Wüthrich, whose list of tentative functions to be satisfied contains clearly notions which are not functionalized in the special sciences, like determining temporal durations. Moreover, the examples I will consider in section 3.3 will support the plausibility of the functionalist arguments, even in some limited sense.

A different argument was provided in Maudlin (2007)[10]. In this paper Maudlin argued for approaches to quantum mechanics based on an ontology of local beables. Like Esfeld, he has the intuition that theories based in such entities are more straightforwardly connectable to our experience. He does not deny the possibility of a different ontology which provides local beables by means of some functional approach, but he argues that for a functionalist approach to succeed it is not enough for it to show that its formalism contains enough structure so that the local beable structure can be instantiated. Maudlin claims that the local structure needs not only to be definable, it also needs to be *physically salient*. This argument can also be exported to the case of quantum gravity, and one can similarly ask the functionalist to provide arguments showing that the emergent spacetime is not only definable but that it is also physically salient. The obvious question that we need to answer in order to evaluate this argument is what is meant by the physical salience of a mathematical structure.

[9] Again, see Vassallo and Esfeld (2014) for a proposal for the case of LQG.

[10] See also Maudlin (2013) for further argumentation against configuration space realism.

The notions of local beables and of physical salience are connected with our experience of the world. This can suggest considering questions about the mind-body problem, but it is important to point out, as Huggett and Wüthrich (2013) do, that the problem of explaining consciousness and our experience of local beables in spacetime is a problem for everyone in this debate. That is, as long as one can define a structure rich enough as for containing a 4-dimensional world with local beables and ourselves[11] in it, it does not matter whether these are fundamental or emergent, that the mind-body problem will remain a problem. In this sense, we should reject thinking about the salience of a structure as being related in a very literal sense with our experience.

Huggett and Wüthrich (2013) further analyze the concept of physical salience and conclude that it is unclear. However, as pointed out in Egg (2017) this response fails to answer to the example Maudlin gives to illustrate his point. Maudlin takes the case of the GRW collapse model which can be given at least two different ontologies in terms of local beables: the mass-density ontology and the flash ontology. Maudlin claims that if we are to derive the emergent ontology from a fundamental one based on the wavefunction, we would not be able to tell which one of the two is the true derivative ontology and, hence, really physically salient. Egg points out two different lines of answer to this problem: either one dissolves the difference between the two competing candidate ontologies or one picks one of them in a justified manner. If we want to defend the view that the wavefunction is primitive, then it seems more reasonable to follow the first line of answer.

The way of arguing for such a position is not to say that the two competing views are ontologically equivalent, as the ontologies they propose obviously say different things about the world. Instead, one has to remember that these ontologies, for the primitivist, are just emergent ontologies and as such they have to be taken just as giving approximate versions of what is really going on on the fundamental level. In this sense, both versions of GRW, understood as emergent, can be seen as compatible and the physical salience problem as solved. Notice that for this resolution of the problem we do not need to evaluate whether one of the ontologies is superior to the other.

In the case of the physical salience of spacetime in an ontology without spacetime we may have a similar problem as the one with GRW. If we accept that our theories of quantum gravity provide enough structure so that a functional reconstruction of spacetime and local beables is possible as an emergent ontology, we may nevertheless worry that there are other alternatives definable. However, this should not really worry us, as as long as there is an emergent ontology available with such spatiotemporal structure it does not matter that some other definable entities are weird or

[11] Our physical selves.

do not capture spatiotemporal structure: that there is one description which is spatiotemporal is enough for the functionalist to support their position. As in the case of GRW, if there are different emergent ontologies available, this is not a problem, as they are just different approximations to the fundamental ontology.

Finally, let me also remark an important point in made by Huggett and Wüthrich in their reply. They point out that if one is able to make empirical predictions out of a physical theory then this should count as being able to find the physically salient content of the theory. In the case of quantum gravity, if one were able to apply the functionalist strategy to recover an approximate spacetime and to make new predictions, this would amount to make the theory physically salient. I agree with them that it is difficult to see any sense of physical salience which would make us require more of a theory. The challenge of deriving a spatiotemporal structure out of one which is not is already a hard enough challenge.

A way of phrasing the arguments in Maudlin (2007) is that a theory without local beables risks being empirically incoherent. Barrett (2001) defines empirically incoherent theories as theories which if they happened to be true, their truth would undermine our empirical justification for believing in them. Barrett gives the example of a cartesian evil demon which is able to make us have false experiences. In such a scenario, any evidence we could have that it is the case that there is such a demon would be undermined, as such evidence could have been introduced in our heads by the demon and be false. In the case of quantum mechanics or quantum gravity, the argument trying to show the empirical incoherence of such a theory was formulated clearly in Ney (2015), and the crucial point is that our evidence for quantum mechanics or quantum gravity stems from our experience with the local beables whose experience is denied by the same theory. However, as I have already explained, the functionalist story explains our experience of local beables in a 3-dimensional space, and hence there does not seem to be any risk of empirical incoherence. Huggett and Wüthrich (2013) further argue for this position by giving examples of theories of quantum gravity and the way functionalism could work in them.

Ney points out a different problem that can block the functionalist purpose: the macro-object problem. According to her, functionalism[12] does not explain how 3-dimensional objects could be constituted by the wavefunction. Similarly, we could parallel this issue to the case of quantum gravity and worry that the way spacetime as constituted by whatever fundamental ontology quantum gravity proposes is not explained. Ney gives the example of a hologram: a hologram of a person can be thought a fulfilling many of the functions that a person does. However, the hologram intuitively does not constitute a person, but it just appears to be one. In the case of the local beables that one could derive from a wavefunction, these, according to Ney,

[12] She considers different authors in the context of wavefunction realism.

would be like holograms, they have the appearance of local beables but they would not really be. However, this worry seems to make no harm to the functionalist: from the functionalist perspective there is nothing more to be a spacetime or a beable than to behave like one. If the functionalist can show that the primitive ontology shows the right behavior, then that's it. In other words, the functionalist does not find any difference between the role played by local beables and the role played by their 'holograms'.

To close this subsection, let me conclude that functionalism about spacetime is able to resist the general objections that a defender of local beables may pose. A key point for this is of course to be able to postulate structures that are able to represent local beables in some approximate way.

3.2.2 Understanding without spacetime

Let me also mention a different perspective on the debates mentioned in the last subsection. de Haro and de Regt (2020, 2018) frame the debate in terms of understandability. They have noticed that it is common for the defenders of local beables and fundamental spacetime to make claims like 'it is hard to see how to do physics without spacetime'. Then, one could read the defender of a fundamental local ontology as claiming that a theory without spacetime is a theory that cannot be interpreted or understood. de Haro and de Regt then argue against this claim, arguing that there are different ways in which in scientists get to understand a theory and that not all of them require a spacetime background. Importantly, de Haro and de Regt argue that a precondition for the intelligibility of a theory is that a satisfactory interpretation be available.

They illustrate their claim with some examples of the ways in which different quantum gravity approaches provide understanding despite having no spacetime. However, it can be argued that these examples concern very small parts of the theories in play and that the understanding they provide is very local and fails to give a satisfactory interpretation. For instance, in (de Haro and de Regt, 2018, Sect. 4.3) they deal with the case of loop quantum gravity and spin foams. The ways understanding is gained in their view in this theory is by appeals to similarity with quantum mechanics and quantum field theory and by internal considerations of the theory. However, this understanding is very limited. In chapters 5 and 8 I will argue that LQG and spin foam models suffer from some important interpretational issues, and that we lack a satisfactory interpretation of these models. In this sense, one can reject claims that we have an understanding of LQG and spin foam models and even that they constitute successful theories. Similarly, the analysis of group field theory (GFT) in (de Haro and de Regt, 2018, Sect. 4.4) only shows a very limited understanding, and in chapter 9 I will argue that that the case of GFT is similar to the one of LQG and spin foam models: we have some mathematical structures defined but no successful interpretation of them. In this sense, I will also deny that GFT can be considered a satisfactory theory.

Therefore, even though I agree with de Haro and de Regt in that we should not deny that theories without spacetime could in principle be intelligible, in current approaches to quantum gravity a clear interpretation and a good understanding of these models is lacking, as I will argue in the rest of the thesis. A better case for the intelligibility of theories without spacetime is wavefunction realism, although it is also limited in the sense that it is a theory without space but with time. In this sense, I will not claim that it is conceptually impossible to have a consistent theory without spacetime. However, in section 3.4 I will argue that the current models of quantum gravity face major problems at the time of interpreting the quantum formalism they use and that this is expected to be so in general for approaches to quantum gravity. Indeed, the examples considered in the spacetime functionalism literature do not consider to a great extent the way these theories are quantum and therefore can be said to deal with discrete spacetimes or structures but not with quantum ones. I will show this in section 3.3. Before, I will now analyze the third kind of objection that has been raised against spacetime functionalism.

3.2.3 The 'hard problem' of spacetime

In the philosophy of mind, appeal is also made to functionalism in order to reduce mental phenomena to physical facts. However, this is usually considered as incomplete given what is known as the hard problem: there seems to be an explanatory gap between our experiences and the physical states. For instance, it is intuitive to claim that even if there is a correlation between feeling pain and the activation of certain parts of our nervous system, our experience of pain is not explained just by appealing to these physical facts. In the context of functionalism applied to spacetime it has been proposed that there also might be a hard problem, i.e., that there might be a gap between theories with and without spacetime which functionalism is not able to close. If one believed there is such a unbridgeable conceptual gap between the spatiotemporal and the fundamental, then one has an argument against theories without spacetime.

This idea of comparing the hard problem of consciousness with the problem of the emergence of spacetime was developed in detail in Le Bihan (2021), although the analogy between the two functionalisms had already been made before[13]. Le Bihan argues that even if functionalists like Lam and Wüthrich may be right in denying that there is an explanatory gap between the spatiotemporal and the fundamental

[13] See for instance Knox (2014) and Lam and Wüthrich (2018).

level, functionalists still have to explain the intuitions the skeptics may have that there is such a gap[14]. However, the functionalist reply to this worry seems to be straightforward: while in the case of the mind-body problem a plausible case can be made that the conceptual gap cannot be closed by appealing to functionalism, in the case of spacetime this does not seem to be the case, and once the functionalist argument shows how the non-spatiotemporal can play the roles of the spatiotemporal, the skeptic intuitions can be argued to be wrong. This would be just the same as in successful applications of functionalism in science which have shown that concepts like 'life' or 'Newtonian spacetime' can be replaced once other entities or processes are shown to play the roles those concepts played and to explain why we may have had strong intuitions about them. In this sense, the case of mental phenomena is different given that mental experiences have, arguably, a particular 'what is it like' to experience them, what is known in the philosophy of mind literature as qualia. In the example of pain, the qualia would be the 'what is it like to feel pain' that for many is intuitively not reducible to physical facts about our nervous system. Scientific concepts such as 'life', 'particle' or 'atomic bond' do not seem to have qualia, and hence they are subject to be functionally analyzed and replaced.

It seems to be true that the conceptual gap between spacetime and something non-spatiotemporal is bigger and more challenging than the one we find in other cases in science, e.g. the gap between the pressure of a gas and the properties of the molecules that form it. However, as there is in principle no spacetime qualia, if the functionalist can offer a story about how the functions of spacetime can be realized by non-spatiotemporal entities, then it seems that the gap can be closed. To argue that there is a hard problem in the same sense as in the philosophy of mind one should argue that there is some sort of spacetime qualia, which does not seem to be true and has been rejected in the literature[15]

One intuition that has been analyzed as a candidate for spacetime qualia is the container intuition. According to this intuition, spacetime is like a stage where things happen or like a container which contains all the events in the history of the world. Philosophers who defend substantivalist positions about spacetime tend to share this intuition, while relationalists and antirealists about spacetime reject it and tend to see the talk of containers just as metaphorical. Against this view has argued extensively Knox in defending her approach to spacetime functionalism and this view is subscribed by both Lam and Wüthrich and Le Bihan.

Notice that there are some functions of spacetime that are closely related to the container intuition, such as providing an individuation to physical events and a rela-

[14] This is in analogy with the case of the philosophy of mind, where the physicalist who denies the mind-body gap is usually required to explain the intuition that there is.
[15] See again Knox (2014) and Lam and Wüthrich (2018).

tive localization for them. These functions can be argued to be essential to spacetime and hence we could see how one would be tempted to call them qualia and formulate a hard problem in terms of them. But, as I have argued above, qualia in the philosophy of mind cannot be reduced to their functional roles and if there is nothing more to the container intuition than just the functions of individuation and localization, then it seems wrong to claim that it is something like a spacetime qualia.

Despite this, we can now read the objections to functionalism based on local beables as pointing to a problem related with the container objection: if one believes that in order to have local beables one needs a 'container', functionalism will face a serious problem if the functions of such a container cannot be realized. We should not refer to this problem as the hard problem, but it is anyway a challenge to the functionalist. From this perspective, we can evaluate the plausibility of functionalism for the different approaches to quantum gravity by evaluating how well they accommodate the functions related with the container. For instance, in causal set theory one may argue that there is not a problem in this sense, as models of the theory contain events which can play the role of container[16]. In other approaches like LQG or GFT there are not events in such a straightforward way, and the assessment of whether models of the theory are able to represent the container intuition may be more disputable. The objections I will raise in this thesis about different approaches to quantum gravity are also in part related with this problem. I will argue that the formalism of these approaches does not have a successful interpretation, and this claim could be challenged if one showed how the formalism could be used to individuate and localize physical events. In this sense, the functions associated with the container intuition may not be called qualia, but it is a challenge for the functionalist to show that the structures defined by a given approach to quantum gravity satisfy these functions.

Le Bihan considers the case of disordered locality in LQG (case that I will consider in section 5.2.3) to argue that an ordering of events could be considered a candidate for spacetime quale. However, this seems to be subject to the same criticisms from the functionalist: as long as the microtheory is able to account for such property, there does not seem to be more to explain. Le Bihan says that he is sympathetic to the possibility that candidates to qualia like this one can be successfully addressed by functionalism, but he still claims that framing the debate in terms of a hard problem helps making emphasis in that there is a cognitive dissonance between the spatiotemporal concepts and non-spatiotemporal ones. In this sense, I can agree with Le Bihan in that we can frame the debate in terms of a tension between the two kind of concepts and a gap that the functionalist needs to close while the opponent will deny that it can be closed. Despite this, referring to qualia seems strange in the context of science, where scientific properties are usually understood in functional

[16] This may be true only for the classical version of the theory, as we are lacking a quantum theory.

terms.

There is part of the argument of Le Bihan in Le Bihan (2021) and also in Le Bihan and Linnemann (2019) which relies on metaphysical issues which are a bit foreign from the functionalist perspective. Consider the following quotes:

> We experience a cognitive dissonance when we compare the network of spatio-temporal relations associated with a particular volume of reality at the level of description of GR, with the distinct spin networks or spin-foams, described by LQG. How could it be that these two networks represent the same part of reality? A natural move is to answer that the description by GR is merely approximate and that GR does not capture the richness of the true fundamental physical structure of the world. One might then argue that, as a result, we only need to derive GR from LQG (with mathematical tools and bridge principles between the primitive notions of LQG and GR) in order to explain everything there is to explain. However, and this is the important point, it is not enough to offer this derivation in order to explain everything there is to explain. An ontological picture is still lacking and one wants to hear a story about the (non-)existence of GR spacetime. (Le Bihan, 2021, pp. 383-384)

> [...] note the ambiguity in the claim of derivability of GR from QG. What it means is that in the quantum gravity context, the aim is to obtain GR as a mathematical approximation of QG. However, obtaining GR as a mathematical approximation is not enough to relate facts about the primitive concepts of the two theories, as it would simply relate structural aspects of the two theories (see Le Bihan (2021)). Imagine that we can derive all of the "behaviour" of matter and energy as described by GR, at relevant scales, from a theory of QG. This behaviour corresponds to the structural facts about spacetime according to GR. But now we must explain how some GR structural facts or some GR non-structural facts relate to the theory of QG. (Le Bihan, 2021, p. 382)

I agree that in occasions there is some ambiguity in the notion of derivation. For instance, in section 2.4 I have argued that some formal limits of quantum theories are not enough for explaining the relation between quantum theories and classical theories, and that one need to address the measurement problem in order to show how one can functionally reduce classical phenomena to quantum ones. In this sense, I agree with Le Bihan in that formal derivations may fall short for the task of the functionalist. However, I worry that Le Bihan is asking too much from the functionalist when he is asking for a story about the existence or not of spacetime.

For instance, consider the application of the functionalist method to the case of how pressure is understood functionally in terms of bouncing of particles. Surely, there is an ontological component in this case as we believe pressure and gas molecules to be related in some way with phenomena happening in the world. But for the functionalist questions like 'is pressure real' may not be very important or even make too much sense. Similarly, if the worries related to spacetime qualia have to do with whether spacetime is real in some sense that goes beyond its functional characterization, then the functionalist can dismiss them. This is not to say that there is no metaphysical account that can complete the functionalist view and say whether it is meaningful to ascribe reality to higher level entities and how to understand their relation with lower level ones, but there is no reason to think that the case of spacetime or general relativity would be different from other scientific concepts or theories, provided we have a successful derivation that could explain at least the empirical content associated with such a higher level.

In Le Bihan and Linnemann (2019) it is introduced a distinction between an 'ontological hard problem of spacetime' and the more general hard problem of spacetime. The former has to do with the level ontology and questions about their reality, and I have just argued that can be dismissed from the functionalist perspective. And the latter is supposed to affect functionalism even if one dismisses the ontological problem:

> In a nutshell, if one argues that the physical system described by GR is not real in such a way that there is no hard problem of spacetime, the *ontological hard problem of spacetime* will be deflated. Nonetheless, the more general hard problem of spacetime will not be solved because we still have to understand why this approximation (the structure described by GR), rather than another one, describes the world. (Le Bihan and Linnemann, 2019, p. 113)

This formulation of the problem seems similar to the 'physical salience' objection I have studied above. As I argued there, one could argue that given a basic ontology as described by a basic theory, the degree to which an approximation applies to it is just a matter of considering this basic theory. This seems to apply to other cases as why classical mechanics can be seen as a good approximation to Bohmian mechanics or the many-worlds interpretation of quantum mechanics or hydrodynamics to the dynamics of a huge number of molecules.To insist, this sort of account involves something beyond a purely mathematical limit. In this sense, I agree that the functionalist or the quantum gravity theoretician has to to explain how general relativity can be seen as a good approximation to whatever they postulate there is at the fundamental level, but this problem does not seem to be of a different kind to the one that functionalism faces at the time of explaining the relation between other theories.

There are two further arguments that Le Bihan employs in Le Bihan (2021) for supporting the claim that spacetime qualia should be taken seriously. First, he mentions that in the philosophy of mind some physicalists suggest that qualia refer to our concepts of consciousness but not to our experience itself. If we apply this to spacetime qualia we could claim that there are spacetime qualia just if there is some conceptual discrepancy with non-spatiotemporal concepts. But it is not clear what this move achieves, as it seems that everyone in this debate agrees that there is a conceptual gap. Furthermore, the physicalists argue that qualia have to do just with concepts in order to support their functionalist view. In this sense, claiming that there are qualia understood in this way does not seem to be a challenge for the functionalist.

Second, Le Bihan considers that mental qualia apply to certain entities in the philosophy of mind, and that they would therefore have a different origin from the one of spacetime qualia. Despite this, Le Bihan argues that spacetime qualia have to be taken seriously, even if they have a different origin. However, this does not sound really convincing. That is, I agree that the conceptual discrepancy needs to be addressed and the conceptual gap closed and that this has to be independent of the origin of our intuitions about the spatiotemporal and the not spatiotemporal. Nevertheless, the strength of the hard problem was very much related with the fact that our intuitions about qualia have to do directly with our experience. If we accept that the candidates for spacetime qualia are not related in the same way with experience, it seems that the argument loses its force.

In Le Bihan and Linnemann (2019) Le Bihan and Linnemann analyze the conceptual gap between the structres defined in some quantum gravity approaches and spacetime and conclude that in all of these approaches there is a distinction between space and time, or between something quasi-spatial and something quasi-temporal at the fundamental level and hence that the gap may not be as wide as believed. In Linnemann (2021) Linemann goes one step further and claims that if there were not this distinction in the fundamental level, then functionalism would not be able to close the gap. In other words, that there would be a hard problem. I will now argue against both claims.

First, there are reasons to doubt that there really is a distinction between space and time in some approaches to quantum gravity. For instance, Le Bihan and Linnemann mention that in canonical approaches there is a clear distinction between space and time, as these quantizations are built on a foliation of spacetime. However, this analysis is superficial, as these approaches suffer from the problem of time which I will introduce in detail in chapter 4 and which implies that the theory defined in this way does not have an evolution with respect to the time parameter of the foliation. I will argue against the proposed resolutions of this problem, but let me notice that if one takes them seriously there are strong reasons for doubting that there is such a distinction at the fundamental level. In any case, it is wrong to use the foliation to claim that there is a distinction between space and time in these approaches because the foliation plays a role in the derivation, but none at all in the models.

Linnemann and Le Bihan consider also covariant approaches and they argue that include structures that distinguish between space and time. Examples of these approaches are causal set theory, spin foam models or dynamical triangulations and they all define some mathematical structures with a space and time split or in which the Lorentz symmetry plays a role. Despite this being true, it is questionable that these mathematical structures represent the fundamental level of these approaches. The reason for this is that in covariant approaches these structures are similar to Feynman diagrams or paths in a path integral, which, as I will argue in chapter 7, should not be interpreted realistically. We will see that in these approaches one considers sums over different paths and that this means that we should not take any of these paths to be representing something in the world. Interpreting quantum mechanics is controversial, but none of the well-established interpretations takes paths in a path integral to be more than a computational device. In the case of quantum gravity, if we are considering causal sets, spin foams or triangulations of spacetime to be analogues of paths in a path integral, then we cannot infer from them any feature of the fundamental structure that is defined as a 'sum' of them. For instance, it could be the case that we have a 'superposition'[17] of different causal structures, which would make it difficult to claim that there is a well-defined split between space and time. This is just as in the case of standard quantum mechanics, where it would be controversial to infer that a particle follows a well-defined trajectory, given that in the path integral one considers many trajectories. I will argue that covariant approaches to quantum gravity lack a satisfactory interpretation, which means that there is no clear picture of what fundamental level these theories would be describing, if any, and whether there is such a time-space split in them.

Now, we can turn to the second claim, that functionalism could not help if the fundamental level lacked a time-space split. First, let me mention that while in Le Bihan and Linnemann (2019) they simply speak about a difference between 'something quasi-temporal and something quasi-temporal, in Linnemann (2021) this is refined and Linnemann claims that the minimal features for claiming that something is spatiotemporal are: 'that of being some kind of ordering structure and involving some kind of difference between something timelike and something spacelike' (Linnemann, 2021, p. 407). Importantly, the mention of the ordering structure get us closer to

[17] I am writing 'superposition' in scare quotes because superposition in quantum mechanics refers to a system having several or undefined properties at a moment of time and here I refer to the different terms in a sum or path integral.

the container intuition: the ordering gives a way of individuating and localizing physical events, spacetime points or regions. In this sense, if the container intuition is captured by these functions, Linnemann is claiming that the fundamental features of spacetime is captured by the container intuition supplemented by a time-space split. Furthermore, he claims that if these features were not present at the fundamental level, functionalism would not be able to account for how a structure lacking these features could realize them.

Linnemann's argument for such a claim is the following. First, he distinguishes between two sorts of functionalism: one which relies on some sort of diachronicity, i.e., one in which roles and functions are understood as being realized in time and one which does not. Obviously, if we want to explain the emergence of time we cannot rely on a functionalism of the first kind, which leads us to consider a functionalism of the second one. However, Linnemann finds this functionalism also unsatisfactory:

> Spacetime functionalism along these lines (as for instance adhered to by Lam and Wüthrich (2018)) is now equally unsatisfactory since it amounts to nothing else than the claim that 'spacetime can be reduced from many different structures', the claim that GR spacetime is multiple-realisable from different underlying theoretical structures within one approach of quantum gravity. This in fact would however just be a (philosophically-flavoured) reformulation of the claim that GR models are classical, low-energy limits of different models within one approach to quantum gravity, as this limit among other things involves coarse-graining. How such a qualification can in any way settle (an allegedly) deep *conceptual* issue of how spacetime comes out of less spatiotemporal structure—the hard problem (if there is one)—, is not clear: whether a time-split is multiple-realisable or not from a putative non-spatiotemporal structure, would not explain its occurrence. (Linnemann, 2021, pp. 16-17, his emphasis)

This position is somewhat familiar to others I have addressed before. Here it seems that the distinction between the two senses of derivation and reduction can help us clarify Linnemann's position. It seems that when he mentions reduction or multiple-realisability he is thinking in that some purely mathematical limits are not the full story for explaining how a theory is reduced to another. Above I have agreed with this position and I have argued that for a successful account of a functional reduction one has to argue how structures in the world as described by the fundamental theory can approximate structures in the higher level. However, it is not clear to me that a notion of functionalism that does not require diachronicity is limited to the purely mathematical sense of functionalism. In this sense if we understand reduction and multiple-realisability in a stronger sense the objection loses force. Notice also that the final part of this quote can be read as a reformulation of the physical salience objection, which I also addressed above.

Let me also mention that there are radical proposals that deny the importance of the distinction between space and time. For instance, Rovelli proposes a relational dynamics where there is no privileged set of temporal variables and where instead of being concerned with the 'evolution' of some variables in spacetime one is concerned in the evolution of any variable with respect to any other. This view can be seen in the example he gives in (Rovelli, 2004, Sect. 3.2.2) and Colosi and Rovelli (2003) of the timeless double pendulum in which one can express 'evolution' in terms of any of the dynamical variables of the system. I will analyze in more detail this relationalist view in chapter 4 and argue against this particular model, but notice that this further undermines Linnemann's claim that current approaches rely on a time-space split.

Let me also compare this case with the case of the emergence of irreversibility in thermodynamics. We could formulate an objection for this case if we believed that the emergence of irreversible dynamics out of a fundamental dynamics which is reversible supposed a conceptual challenge. However, it is widely agreed that a theory which is irreversible like thermodynamics can be reduced to reversible theories like classical mechanics if we add some sort of special initial conditions[18]. In the case of quantum gravity it may be the case that we are in a similar situation: even if we assume that the fundamental theory lacks a distinction between time and space it may very well be the case that there are some models of the fundamental theory for which this distinction applies at the emergent level. For instance, we can consider a fundamental theory which describes a set of physical events which form a manifold with the $\mathbb{R}^4$ topology but which does not define any time direction and which it does not constrain the possible correlations between those events. As every correlation is allowed, some models of this theory will show the same pattern of correlations that one would expect from a relativistic theory with a well defined light-cone structure. In this case one could say that the time-space split emerges in the same way that ireversibility emerges from a non-reversible basis.

There is a possible objection to this kind of answer which is to complain that by selecting a preferred set of solutions we are introducing 'by hand' the desired spatiotemporality, or reversibility. Another way of putting it would be to say that for these particular models there is not a hard problem. In this case one may refer not to the solution of the problem but to its dissolution. This distinction is not very important for practical reasons, as what is important is that the functionalist can argue that functionalism would be able to explain the relation between the emergent spatiotemporal level and the fundamental one.

A similar objection by Yates (2021) was addressed by Lam and Wüthrich (2021).

[18] This is known as the past hypothesis.

Yates argued that if functionalism is able to account for the spatiotemporal features of the emergent level it is because the fundamental level is already spatiotemporal. To this, Lam and Wüthrich have two objections. First, they insisted that the differences between spacetime in general relativity and the fundamental structures in quantum gravity are so radical that the latter should not be considered spatiotemporal. Second, some of the structures which can adopt a spatiotemporal configuration can also adopt other configurations which are not spatiotemporal, and for this reason one should avoid calling them spacetime. Once we take this into account we see why the functionalist may want to claim that the fundamental level is not spatiotemporal. However, once we accept the possibility of non-spatiotemporal configurations it does not really cause any harm to the functionalist to claim that spacetime was already there at the fundamental level, even if just for some models, just as in other cases of functionalism one can claim that the reduced entities were already in some way in the fundamental level.

With this I close this section with the objections presented against functionalism about spacetime in quantum gravity. I have argued that none of these objections is final, but I have noticed that a key point for the functionalist to be successful is to clarify the way its quantum formalism is to be interpreted. The three objections analyzed in this section are in some way related with the interpretation of quantum mechanics. First, the objections by Esfeld and the ones inspired by Maudlin are directly imported from the debates in the foundations in quantum mechanics. Second, for being able to overcome the intelligibility objection we need an interpretation of the formalism of the theory in play. And third, I have argued in this last subsection that, the gap between our spatiotemporal intuitions and concepts and our candidates for theories of quantum gravity is wider than what represented by Le Bihan and Linnemann, as it is unclear how to interpret such theories and this interpretation may challenge their claim that these theories describe something with a time-space split at the fundamental level. For this reason, I will argue in section 3.4 that the main problem that spacetime functionalism faces in order to recover spacetime from the candidate theories to quantum gravity we currently have is precisely that we lack an interpretation for the formalism of these models. Without this interpretation it is unclear what the non-spatiotemporal entities are supposed to be and how they are able or not to fulfill the spatiotemporal functions. Before that, in the next section I will analyze the examples of spacetime functionalism that are more widely discussed in the literature, and I will conclude that these examples are limited to just some classical or quasi-classical bits of these models, but that they do not consider the interpretation of the full quantum models.

3.3 Examples of spacetime functionalism

In this section I will briefly introduce some examples based in some models of quantum gravity which have been used by functionalists to argue that they illustrate how spacetime can emerge from non-spatiotemporal entities. These examples are the classical version of causal set theory and some states in LQG and GFT. I will argue that the functionalist has a good case for claiming that spacetime can be reduced to a causal set or for claiming that space can be described by some states in LQG. In this sense, some of the objections above can be overcome. However, I will argue that this examples are limited, as in both cases one is considering just some classical elements of the theories. In this sense, what the functionalist is showing is that continuum spacetime can be reduced or approximated by discrete structures, but what would be interesting to study is how spacetime is reducible to quantum ones. In section 3.4 I will further argue that, as we lack a satisfactory interpretation of the candidate theories to quantum gravity, the challenge for the functionalist would be to reconstruct spacetime from such an interpretation.

3.3.1 Causal sets

Causal set theory is based on a theorem by Malament (1977) which shows that a model in general relativity can be determined by a combination of its causal structure and a local volume element. The causal structure of a general relativistic model is a partial order relation which for every two points p, q in the manifold tells us whether p is in the causal past of q, in its future or neither. This partial order relation already determines a good deal of spacetime geometry: it alone determines the topology and conformal geometry of spacetime. For instance, it directly determines the lightcone structure. One only needs to add local information about the volume to recover a full model of a general relativitic spacetime. Schematically, we can describe this model by means of the triple $\langle M, R, V\rangle$: a manifold M[19], a partial order relation R[20] and a local volume element V. This is equivalent to the traditional way of presenting spacetime as the pair $\langle M, g\rangle$, that is, a manifold and a metric tensor defined on it.

Inspired by this way of understanding spacetime, and assuming that spacetime needs to be discrete, causal set theory was proposed in Bombelli et al. (1987). A causal set is defined by the pair $\langle S, R\rangle$: where we have changed the manifold for a

[19] Indeed it is enough if we specify a set with the continuum cardinality, as the manifold structure is induced by R.

[20] In order to have physically well-behaved spacetimes this relation can be restricted. For instance, to avoid closed causal curves to be possible we can forbid a point p to be in both the future and the past of another q. In the case of causal set theory these restrictions are explicit. See for instance (Surya, 2019, Sect. 3).

discrete set of events S, which is a set which contains a countable number of elements and we have dropped the volume element. The reason for this is that in the discrete case there is a natural way of assigning volumes: counting. Intuitively we can think that any element in S has volume 1 (in some natural units) and that if a 'region' is composed by, say, 5 such elements then its volume is 5. With this, a causal set contains exactly the same type of information as a continuum spacetime: causal structure and volume. The only difference is the cardinality: while continuum spacetime is made of uncountably many points, causal sets contain only a countable number of them.

For this reason, it is no surprise that some causal sets are good approximations to relativistic spacetimes and the other way around. Indeed, a way of seeing that a causal set and a spacetime are a good approximation to each other is by embedding the former in the latter. By embedding one means to identify elements of the causal set with points in the spacetime manifold in a way such that the causal relations of the points identified with elements of the causal set agree with the causal relations of the spacetime and in a way that the volume as determined by the metric is approximately the same as the volume obtained by counting elements of the causal set[21]. The approximation is better when one considers regions much greater than the causal set element volume. For two manifolds which differ on regions much smaller than the causal set element volume one cannot find two distinct causal sets which approximate the two manifolds in a way that captures the differences on such small regions. In other words, the causal set volume sets a limit on the resolution of the approximation. Conversely, from the point of view which takes spacetimes to approximate causal sets there are many spacetimes which approximate a causal set (if embeddable[22]), but the difference between them is unphysical, as this difference is only present in some small scale which is considered to be unphysical.

Causal set theory takes precisely this point of view: we do not live in a continuum spacetime but in a causal set. The success of general relativity could in principle be explained by means of a causal set model, given that our tests of general relativity are performed at scales much bigger than the proposed volume of a causal set event, which is something around the Planck scale. I refer the reader to (Surya, 2019, Sect. 4) for a review of how different features of a continuum geometry can be recovered or approximated by properties of causal sets. It therefore seems reasonable to claim that a functionalist argument seems plausible in this case: if we consider the roles of spacetime to be recovered to be its geometric features, causal sets provide a structure

[21] For causal set models to be approximately Lorentz invariant a further condition is usually imposed in the literature. This condition is that the sets considered have to be such that they could have been generated by a particular random Poisson process on a manifold which embeds the causal set. For our discussion here this condition will not be relevant. I refer the reader to (Surya, 2019, Sect. 3) for more details on this.

[22] This clarification is needed, as not all causal sets are embeddable in spacetimes.

able to play those roles.

Notice that this argument applies only at a kinematical level, i.e., it only supports the claim that spacetimes could be approximated by causal sets, but it does not say anything about the dynamics. However, and despite that a complete dynamics has not been built yet, it seems possible to complete the theory in a way which it is able to describe some form of matter satisfying some laws which approximate the laws we have for matter in the continuum and in a way in which the geometry of the causal set is related with the distribution of matter in a way which approximates Einstein equations. If we accept this possibility, then the functionalist argument to reduce spacetime to causal sets seems perfectly valid. In the philosophical literature, such argument has been provided for instance in (Lam and Wüthrich, 2018, Sect. 4).

We can now ask ourselves whether this instance of functionalism is challenged by some of the objections raised in the previous section. First, objections like Esfeld's or Maudlin's were concerned with theories without spacetime or local beables. However, causal sets provide a structure which allows to define local beables. For instance, in a causal set we can have a particle following a trajectory with the only difference that this trajectory would now be discrete. In this sense, causal set theory is not affected by these criticisms because it is based in local beables. Second, the intelligibility objection requires that we do not have or that we will not be able to have an interpretation of the theory, but causal set theory can be readily interpreted as describing a discrete spacetime compatible with our experience. Finally, there is no hard problem in the sense of Le Bihan and Linnemann, as the order relation of a causal set clearly marks a distinction between the temporal and the spatial.

A point to make here is that one could argue that the reason functionalism seems to work for causal sets is because causal sets are spatiotemporal. As I said before, this risks ending up into a semantic discussion, as there are obviously features of continuum spacetimes which are present in causal sets, like causal relations and a volume measure, and there are others which are not or may not be present, like the topology and dimensionality of space. If you consider that the missing features are important enough, then you will claim that causal sets are not spatiotemporal and maybe even that there is emergence in the sense of novelty, as claimed in (Crowther, 2021, Sect. 4.2). If instead you consider that a continuum topology is not indispensable for spacetime, then you will probably find that causal sets are discrete spacetimes, even for cases in which no embedding on a manifold is available[23], and you will find no surprise in the claim that functionalism about spacetime works in this case.

Causal set theory has so far been only formulated as a classical theory. In this

[23] There is a case that has been well-discussed in the literature which is the case of a set ordered in three layers: one layer formed by elements with no ancestors, one by elements with ancestors in the first layer and a third one with ancestors in the second one. This causal set clearly lacks some of the features we normally attribute to spacetime and only lasts for two units of time.

sense, it is not too surprising that functionalism is able to be applied well to this case with no further complication. There are some proposals about how define a quantum dynamics for causal sets[24], all of them relying on a covariant formulation of quantum mechanics. In these formulations one is able to assign an amplitude to each causal set or to processes by which causal sets grow. I will not give more details of these models but I want to point out that these face the characteristic interpretative problems of covariant formulations of quantum mechanics that I will analyze in chapter 7. As we lack a quantum theory of causal sets and an interpretation of such a theory causal set theory cannot be considered to be a successful example of how spacetime can be considered to be emergent from a quantum theory of gravity.

3.3.2 Kinematic sectors of LQG and GFT

Another example that is often discussed as exemplifying spacetime functionalism is the spin network Hilbert space, which is the kinematical Hilbert space of LQG. Spin networks appear in the canonical formulation of loop quantum gravity and similar Hilbert spaces also appear in other approaches like spin foam models and group field theory. I will discuss in detail each of these approaches, the difference between the different spaces employed and their interpretation in chapters 5, 8 and 9, but let me give here a sketch of the way these states are meant to represent or approximate discrete spaces.

In the canonical quantization of a classical theory one starts by quantizing the degrees of freedom of the theory, i.e., one starts by building a Hilbert space which represents these degrees of freedom. In the case of general relativity, these degrees of freedom can be seen to be the geometry of space. The quantization chosen in LQG for these degrees of freedom is pretty peculiar. The basis of the Hilbert space is known as the spin network basis and these states are defined by two elements: a graph Γ composed by a series of nodes which are connected by links, i.e., arbitrary one-dimensional curves in a manifold, and a series of quantum numbers i, j associated to these nodes and links. These states can be shown to be eigenstates of some area and volume observables: the area of any surface in the manifold gets a contribution from every link with crosses the surface, and it depends on the quantum number j associated to it. Similarly, the volume of any region is a sum of a i-dependent contribution of every node in the region. In this sense, we can see that spin network states can be seen to describe a peculiar geometry which is concentrated in the graph. Indeed we can interpret this as describing a distributional geometry. In chapter 5 I will argue that this construction is pretty peculiar and that maybe some other option should be preferred, but let us ignore these worries.

[24] See (Surya, 2019, Sect. 6.3) and references therein.

To account for the diffeomorphism invariance of general relativity one defines its physical content to be in equivalence classes of geometries under diffeomorphism. Similarly, one defines the s-knot Hilbert space to be represented by equivalence classes of spin networks under spatial diffeomorphisms. S-knots are defined on equivalence classes of graphs, which can be characterized by how many points there are in the graph and their connectivity relations but also and importantly by the way the links are knotted around each other and also by some 'angles' that the links form at the nodes[25]. In this sense, I will argue in chapter 5 that the most natural interpretation of s-knots are distributional geometries. However, in the LQG literature and in the philosophy of quantum gravity literature one finds a different interpretation, namely that s-knot states represent discrete spaces made of chunks of space.

The motivation for such an interpretation is that there are s-knots which are dual to tessellations or triangulations of space. Roughly speaking, a tessellation is a way of dividing a space into pieces and we can define a dual network as a way of representing this division by associating to each piece a node, and if two pieces share a boundary we represent this by joining the two nodes with a link. In this sense, we see how networks can be associated to divisions of space, and this provides the interpretation of LQG states as representing a set of 'chunks' of space and the ways they are connected. However, this interpretation is problematic for two reasons. First, if we follow strictly the derivation in the canonical formalism, the s-knot states represent distributional geometries and not discretized ones, as I have argued above. Second, we do not have a nice bijection between states in the s-knot Hilbert space and tessellations of spacetime. For this there are two reasons: s-knot states carry more information than just an abstract graph, as they describe the knotting and some 'angles' and there are graphs which are not dual to triangulations. I will expand on these points in section 5.2.3.

Despite this, the view that s-knots represent something like chunks of space is quite generalized. This is motivated in canonical approaches by restricting the allowed states[26] and in covariant approaches by defining the model directly on a triangulation[27]. Similarly, states in GFT can receive the same interpretation. For discussing the emergence of space from these states it is useful to introduce the distinction made in Dowker and Butterfield (2021) between geometric and combinatorial structures[28], which correspond to two ways of interpreting the sates. First, we have the geometric interpretation, which assumes that each state directly describes some geometry.

[25] More precise definitions will be given in chapter 5.

[26] This is done for instance in Rovelli (2004). I will discuss this move in chapter 5.

[27] This is the case of the spin foam model I will discuss in chapter 8.

[28] In Dowker and Butterfield (2021) they introduce this distinction for discussing simplicial decompositions of Lorentzian manifolds, but it can be extended to any tessellation of any manifold of any signature.

For instance, this view fits naturally with the models associated with triangulations, where it is commonly assumed that the internal geometry of each piece of the tessellation is flat and this allows to build a space out of the data encoded in the state. In this case, the discussion of the emergence of space would be trivial, as we are directly assuming that the state represents a space composed of flat pieces[29]. Alternatively, we can consider that states do not directly represent geometries but that they represent just the combinatorial information of the graph together with the abstract quantum numbers. To discuss the emergence of space in this case it would be necessary to study the ways these states satisfy the functional roles of a 3-geometry. This discussion is not trivial and may require to consider how matter fits into the picture but it seems that there are plausible arguments for supporting this sort of emergence. One of these arguments is that the graph structure gives a way of assigning a location to events or physical structures like fields or particles, which is one of the most basic functions of space, as we have seen above.

There is a different strategy for arguing that states in LQG and GFT can approximate 3-geometries which applies to general s-knot states and not just to states dual to a triangulation. There is a particular class of states, known as weave states, which are such that the graph they have associated can be embedded in a manifold M with metric g in a way such that the values of the geometric observables as defined by the metric agree, up to small corrections and for big regions, with the values that one can assign to those same observables by means of the quantum numbers corresponding to the appropriate nodes and links of the graph. In this sense, we find that the relation between weave states and continuum spaces is similar to the relation between causal sets and spacetimes. The approximation will be better for big scales or regions which contain many nodes and links of the graph just as spacetimes are good approximations to causal sets for big enough regions. As in the case of causal sets, such arguments would be completed if we added matter to the picture and we found that the approximate geometry relates with matter in the way that matter and geometry relate in general relativity.

Generic weave states have a feature that has been considered[30] to be problematic. The approximating metric determines the distance between any two connected nodes and this distance may be arbitrarily high. If we consider that the connectedness relation of the network has to be understood as some kind of fundamental locality relation this means that there is a tension between the two localities, as what is considered local by the fundamental theory is far away according to the approximate description. I will discuss this case in detail in section 5.2.3 and I will argue that it is

[29] There is a problem with this approach, which is that some assignments of quantum numbers do not define proper geometries, but something called twisted geometries. I will discuss this problem in chapter 8.

[30] This has been extensively discussed in the literature. See for instance: Wüthrich (2017).

not so problematic, as it is an example of the tensions that typically occur in a case of emergence. For instance, in the case of the emergence of thermodynamics from Newtonian mechanics we see how the lower-level theory allows for anti-thermodynamic behavior in the same way that LQG would allow for a non-locality that we have not observed in general relativity. In the case of thermodynamics it can be argued that even if the lower-level theory allows for unobserved phenomena it is not problematic because it is considered to be unlikely. Similarly, in the case of LQG the hope is that states with a lot of non-localities are also shown to be unlikely. Notice also that a similar worry can be formulated for causal sets, as there will be models which will not be exactly embeddable, giving rise to some tension.

We see how, despite this problem, the functionalist can make a plausible case for supporting that states in LQG, spin foam models and GFT give an example of how a space can be functionally reduce to some more fundamental structure. In this sense, this example does not seem vulnerable to the objections I have analyzed above. However, it is important to notice that this example only concerns some states in kinematical Hilbert spaces but not a full theory for LQG, spinfoam models or GFT. This is worrisome because in this thesis I will argue precisely that these approaches to of quantum gravity do not define a dynamics successfully and hence that even if we are able to describe discrete spaces using these approaches we lack a story of how these spaces evolve and form something like a spacetime or a quantum spacetime. Indeed, in the next section I will argue that the main challenge for the functionalist is that the current models do not have a satisfactory interpretation and hence that we are not able to give an account of how the formalism of these models represents something which can functionally approximate a spacetime.

3.4 Interpreting the quantum formalism

The discussion in the last section should have made clear that functionalism is able to provide a compelling story about how spacetime or space could be reduced to discrete structures as causal sets or spin networks. In both cases there were features of continuum spacetime that were lost along the way, and if we consider them essential to what it is to be a spacetime then we will claim that spacetime or space emerges from a non-spatiotemporal fundamental structure. However, this discussion has been mostly classical: despite the fact that causal sets and spin networks show some features of what one might expect of a quantum theory, such as discreteness, I have not considered so far full quantum entities in all generality. The standard realist interpretations of quantum mechanics, and even the operationalist ones, rely in concepts of space and time for giving a physical meaning to the quantum formalism, and hence to extend them to a non-spatiotemporal realm is a challenge for the approaches to quantum gravity and for the functionalist. Moreover, current approaches to quantum gravity are not defined in a way in which the whole formalism of quantum mechanics is available, which further complicates their interpretation. Indeed, in this thesis I will argue that we do not have any satisfactory interpretation for the models I will study, and hence that they do not represent complete theories. In this section I will sketch some of the reasons for supporting this claim and I will argue that this constitutes the biggest challenge that the functionalist has to face. In other words, for claiming that spacetime emerges from some non-spatiotemporal entities that are described by some theory of quantum gravity it is necessary to have some clarity about what these entities are.

A first thing to notice is that the objections analyzed in section 3.2 take their strongest form if we formulate them against an uninterpreted quantum formalism. Indeed, if we do not give an interpretation, not even a minimal operational one to the formalism it is unclear at best the way it relates with the appearance of a world containing local beables. The intelligibility objection clearly affects any piece of formalism which is uninterpreted and above I have argued that the conceptual gap between spacetime and quantum gravity is bigger than what described by Le Bihan and Linnemann, as their claim that in quantum gravity there is a distinction between the temporal and the spatial can be challenged on the grounds that this claim is based just on some parts of the formalism and not on full quantum theories.

Notice that the first two objections in 3.2 are directly motivated by objections against interpretations of quantum mechanics which do not rely on local beables. Therefore, it is natural that one would try to import to the case of quantum gravity their favorite strategy and interpretation of quantum mechanics. That is, if we are satisfied with the way a given interpretation like Everettian quantum mechanics or Bohmian mechanics is able to give us understanding and to explain the appearance of local beables, then it is reasonable to try the same interpretations for the formalism of our theories of quantum gravity.

However, standard interpretations rely on some background notions of spacetime, or at least time. The reason for this is that quantum states and/or operators evolve in time and the properties assigned to a quantum system are usually associated to a spacetime region. Thus, if we are to interpret theories of quantum gravity as quantum theories without space and time, we are in trouble, as the way we understand quantum theories is precisely by means of spatiotemporal notions. This affects the three main realist interpretations of quantum mechanics: for an Everettian interpretation of the theory we need to have a quantum state evolving with respect time, for a Bohmian interpretation we need some particles moving in space and for an interpretation based on collapse models we need a stochastic process happening in time. To interpret a quantum theory without space and time would require to modify the already existing interpretations or to provide a new one.

One can doubt about how seriously one should take operationalist interpretations or interpretations like Copenhagen at the time of describing issues of emergence as I have discussed in section 2.4. In any case, even if these approaches are unclear about what they say about the world, they for sure make claims about preparations and measurements in experimental situations. However, even this minimal sense of interpretation is threatened in the context of quantum gravity, as these preparations and measurements are spatiotemporally located.

Furthermore, is not only that the ways we have to understand quantum mechanics rely on spatiotemporal notions, but also that the models of quantum gravity are defined in a way which is problematic for two reasons. First, the models do not define the full set of structures that usually define a quantum theory, and this makes the usual interpretations unavailable. Second, we have strong reasons to think that the quantization methods employed for defining these models fail for the quantization of general relativity, and hence to think that the formalism defined is not the basis for the quantum theory we are looking for. I will argue for these claims later on, but let me sketch some of the arguments supporting them.

In chapter 4 I will introduce the problem of time of canonical approaches to quantum gravity. In a nutshell the way the canonical formalism treats symmetries like the diffeomorphism invariance of general relativity makes it the case that one cannot define a non-trivial temporal evolution for this theory in the same way one would for a theory without the symmetry. Therefore, one is left with states in a Hilbert space but with no evolution in time. There are some proposals for solving this problem, but I will argue against them. Some of my arguments will rely on the comparison with some models for which we also have a problem of time and for which I will argue that these proposals fail. This leads me to conclude that canonical quantizations of general relativity do not lead to satisfactory quantum theories and that, if we ignored this, we still would not have a convincing interpretation of the 'frozen' states they define. These problems affect directly approaches like quantum geometrodynamics, canonical loop quantum gravity and loop quantum cosmology, as I will further develop in chapters 5 and 6.

Furthermore, in chapter 7 I will argue that the problem of time also affects, even if in a more indirect way, approaches to quantum gravity which are based on covariant quantizations of general relativity. These approaches define some 'propagators' which are supposed to encode the physical content of the models, but I will argue that they are not interpretable in any clear way. First, they are not strictly speaking propagators, i.e., they do not define the evolution of a quantum state. Second, they define a series of probabilities but there are problems for giving an interpretation to them, as they do not even have any well-defined operational meaning. Therefore, covariant approaches are also based on some formalism without interpretation and one could even raise the worry that as they stand they do not seem to constitute meaningful theories. This affects approaches like spin foam models, dynamical triangulations or group field theory, which I will study in chapters 8 and 9.

This leaves the functionalist in a hard situation. If the functionalist agrees with my analysis, then the functionalist project of arguing that spacetime emerges from some non-spatiotemporal entities as described by quantum gravity fails, as it is unclear the way the approaches I study in this thesis are supposed to define such non-spatiotemporal entities. Alternatively, even if the functionalist believes that the approaches to quantum gravity studied in this thesis can be given a satisfactory interpretation, the challenge would be not only to find such an interpretation but also to show that the functions of spacetime can be realized. In this sense, my position about spacetime functionalism is not that a theory without spacetime is impossible, but that current approaches to quantum gravity cannot be taken as examples or candidates for such theories without spacetime, as we do not currently have a satisfactory interpretation for them which would support that claim. In the rest of this thesis I will argue for this position.

4. CANONICAL QUANTIZATION AND THE PROBLEM OF TIME

In the previous chapter I have argued that the biggest challenge the functionalist faces at the time of arguing that spacetime emerges from whatever a quantum theory of gravity describes is precisely that we lack a satisfactory quantum theory of gravity. In this chapter I will introduce one of the ways of quantizing a classical theory and I will argue that when applied to reparametrization invariant theories like general relativity it leads to a problem of time. I will further argue that the ways that have been proposed to deal with this problem are not satisfactory, which leads to the conclusion that the canonical quantization schema fails in providing a successful quantization of general relativity.

The root of the problem of time of quantum gravity lies in the diffeomorphism invariance of general relativity. The fact that there is not any preferred coordinate system in general relativity makes it the case that it shares some features with gauge theories, i.e., with theories which in their mathematical formulation include redundant or additional structure to the one strictly necessary to represent what we believe to be the physical content of the theory. However, I will argue that treating and thinking about general relativity as a gauge theory can lead to conceptual problems even at the classical level if one is not careful.

In the first part of the chapter I will have three goals. First, I will introduce the way constrained systems such as gauge theories and general relativity are dealt with in the Hamiltonian formalism, which is the starting point for the canonical quantization program. Second, I will argue against views like Earman (2002) and Montesinos and Rovelli (2001); Montesinos (2001) which claim that time evolution in general relativity is a gauge transformation and that this should make us adopt a new metaphysical picture of time for this theory. Third, I will introduce several toy models which share with general relativity some version of reparametrization invariance, but I will argue that there are important differences in between them which makes it the case that one should just take some class of them as sharing important features with general relativity. In particular, I will argue that the biggest similarity is found for a model of a double harmonic oscillator as described by its Jacobi action. This will lead me to argue that general relativity is not deparametrizable, i.e., that it is not possible to identify some time variable from the degrees of freedom of the phase space of the theory and that there is no other way in which the variables in its configuration or phase space represent time.

The structure of this first part is as follows. In section 4.1 I will introduce the Hamiltonian formalism for constrained systems and gauge theories and I will discuss gauge systems. Then, in section 4.2 I will introduce reparametrization invariant theories, the toy models I have mentioned above, and the way they are treated in the constrained formalism. Here I will argue that the Hamiltonian formalism works fine for these models and that this does not force into changing our metaphysical picture of time and change. Finally, in section 4.3 I will introduce the geometrodynamical formulation of general relativity, i.e., its formulation in the Hamiltonian formalism as the dynamics of a 3-geometry which evolves in time. Here I will also argue for the positions mentioned above.

Next, in the second part of the chapter I will move to the quantum theory. This part also has three goals. First, to introduce the canonical quantization procedure both for non-gauge and gauge systems. Second, to introduce the problem of time as the series of conceptual and technical difficulties that appear when trying to apply the canonical quantization techniques to a reparametrization invariant theory. In particular, the problem lies in that the procedure leads us to a quantum theory with no dynamical equation, i.e., to a theory in which there is no evolution. The third goal of this part is to analyze the proposed ways for dealing with this problem for general reparametrization invariant systems and for general relativity in particular. There are some resolutions which work for deparametrizable theories, but I will argue against applying such proposals to the case of general relativity, as it is not a deparametrizable theory and there are important differences between this kind of theories and theories like general relativity. Furthermore, I will show how these proposals fail for my example of a non-deparametrizable model, the double harmonic oscillator. There are other proposals that I will study, but I will find that they also have some conceptual shortcomings and that they imply some important reinterpretations of the quantum formalism. The conclusion I will therefore reach is that the canonical quantization of general relativity fails in giving a satisfactory theory of quantum gravity.

The structure of this second part mimics the structure of the first one. First I will introduce and discuss the canonical quantization procedure for non-gauge (section 4.4) and gauge systems (section 4.5). Then, in section 4.6 I will study what happens when applying this to reparametrization invariant systems, how this leads to a problem of time and the proposed resolutions for different models. Finally, in section 4.7 I analyze the problem for the specific case of general relativity and analyze the different proposals for dealing with it.

4.1 Gauge theories and constrained systems in the Hamiltonian formalism

As I mentioned in the introduction, the starting point for the canonical quantization of a theory is to express it in the Hamiltonian formalism. Theories with symmetries like gauge theories and general relativity need an special treatment in this formalism, as we will see below that they are constrained systems. In this section I will introduce this formalism in some detail and I will discuss gauge theories.

First, let me give a notion of what I mean by gauge theory here. A gauge theory is commonly understood to be a theory that contains degrees of freedom that are physical, i.e., that are observable, and others that are not, that are only part of the mathematical structures of the theory with no counterpart on the physical world. Importantly, these theories have an important symmetry as we can change the unobservable part of the theory while leaving fixed the observable content. The transformations which achieve this are the gauge transformations of the theory.

Moreover, for a theory to be considered a gauge theory we will require that the equations of motion of the theory do not determine the unobservable parts, i.e., that their solutions depend on arbitrary functions of (space)time. For the observable parts we do not have this dependence and they are fully determined for all times once a set of initial conditions are in place. An example of this is the 4-vector formulation of electromagnetism: Maxwell equations for the 4-potential are not deterministic in the sense that there are several (infinitely many, indeed) solutions for them given a set of initial conditions, but nevertheless they all agree in the values of the electromagnetic field at every spacetime point.

Notice that by taking this definition we are leaving out some theories which may have unobservable content but which do not have equations of motion which allow for such a dependence on arbitrary functions. In other words, we will not consider as gauge theories theories with unobservable degrees of freedom but deterministic equations of motion. For instance classical mechanics will not be considered a gauge theory, as even if it can be seen as having arguably unobservable quantities like absolute position, its equations of motion do not show the freedom we are interested in in this chapter.

We can now proceed to express gauge theories in the Hamiltonian formalism, which is a necessary step for their canonical quantization. For this, we will start from the theory as expressed by a Lagrangian variational principle. A gauge theory will be characterized in the Lagrangian formalism by an action which has some local symmetries, i.e., there are some local transformations (gauge transformations) which leave the action invariant. For such an action we can apply the variational principle to find the Euler-Lagrange equations of the theory:

$$S[q_i] = \int dt L(q_i, \dot{q}_i, t) \tag{4.1}$$

$$\frac{\delta S}{\delta q_i} = 0 \rightarrow \ddot{q}_j \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} = \frac{\partial L}{\partial q_i} - \dot{q}_j \frac{\partial^2 L}{\partial q_j \partial \dot{q}_i} \,. \tag{4.2}$$

Here q_i represent the different variables or fields in the theory and we are using the convention that repeated indices mean that there is a sum over them. This system of differential equations has a unique solution just in case one can invert the Hessian matrix $\frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i}$ and express the accelerations $\ddot{q}_i$ in terms of the positions and velocities. Gauge theories have multiple solutions corresponding to different gauges and hence it is a necessary property of these theories that the Hessian matrix is not invertible[1]. In this case we say that the Lagrangian is singular.

Let me also mention that singular Lagrangians allow not only for indeterminism, but also it can be the case that the set of possible initial conditions is restricted, in the sense that there will be initial conditions for which there does not exist any solution of the Euler-Lagrange equations. For instance, this is the case for electromagnetism: only initial conditions satisfying Gauss law can satisfy the equations of motion. In the case of general relativity we will also find in section 4.3 that not every initial condition is allowed.

Let me explicitly introduce the case of electromagnetism as an example of gauge theory. The physical degrees of freedom of electromagnetism are the electric and magnetic fields which are contained in the Faraday tensor $F^{\mu\nu}$, but the theory can be expressed in terms of the 4-potential A^μ which is related to the Faraday tensor by $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. If two 4-potentials A, A' are related by $A'^\mu = A^\mu + \partial^\mu \Lambda$ for some function Λ, the Faraday tensor for both of them is the same. Therefore, A^μ contains some gauge degrees of freedom together with the physical ones. The equations of motion of electromagnetism can be derived from the action:

$$S[A^\mu] = \int d^4x F^{\mu\nu} F_{\mu\nu} \,. \tag{4.3}$$

Given the gauge invariance of the action, when one applies a variational principle to it what one finds is that of the four equations only three of them are dynamical equations while the fourth one is a constraint equation. The equations obtained by extremizing with respect the spatial components of the field, A^a, contain second-order temporal derivatives of these components, as one would normally expect for a dynamical equation. The equation for A^0 does not contain temporal derivatives of

[1] See Pons et al. (1997); Pons (2005) for discussions of this.

A^0 but, instead, it just imposes the condition that the electric field defined by A^μ satisfies Gauss law ($\vec{\nabla} \cdot \vec{E} = 0$). The fact that one has a constraint equation and not a dynamical equation for A^0 implies that the dynamics for this component is not defined and that one has some freedom in choosing it as long as the constraint is satisfied. In this context one says that A^0 acts like a Lagrange multiplier. The solutions of the equations of motion for the spatial components A^a depend on the (almost) arbitrary function A^0, as we were expecting of a gauge theory. We will find this structure of dynamical equations accompanied by constraint equations in other gauge theories and most importantly in general relativity.

Now we can move to the Hamiltonian formalism. Notice that the fact that we have a singular Lagrangian will require a careful treatment. First, we need to change from the variables q_i, $\dot{q}_i$ on the tangent bundle of configuration space to the variables q_i, p_i on phase space. The momenta p_i are defined as $p_i = \frac{\partial L}{\partial \dot{q}_i}$, and this definition can be shown to be invertible if and only if the Hessian matrix is. As we are dealing with singular Lagrangians, the definition of the momenta is not invertible, and only a subregion of phase space corresponds to the image of $q_i, \dot{q}_i$ under this transformation. This subregion, the constraint surface, is defined as the region where a number of functions ϕ_α, known as primary constraints, vanish. Therefore, any physical evolution can be described by an evolution in the constraint surface and not as an evolution in the whole phase space.

There are two of ways of adapting the formalism to this fact. The first one is to change the usual Hamiltonian mechanics to accommodate the fact that momenta are not independent any more. It can be shown that the Legendre transformation of the Lagrangian defines a canonical Hamiltonian that is only a function of the positions q_i and some of the momenta p_a[2]. Then, one can use this Hamiltonian and treat the remaining momenta p_α as functions of q_i and p_a, as implicitly defined by the constraints[3]. Using this, we can write an action principle and we arrive to a modified version of the Hamilton equations:

$$S[q_i, p_a] = \int dt[p_a \dot{q}_a + p_\alpha(q_i, p_a)\dot{q}_\alpha - H_c(q_i, p_a)] \tag{4.4}$$

$$\dot{p}_i = -\frac{\partial H_c}{\partial q_i} - \dot{q}_\alpha \frac{\partial p_\alpha}{\partial q_i} \tag{4.5}$$

$$\dot{q}_a = \frac{\partial H_c}{\partial p_a} + \dot{q}_\alpha \frac{\partial p_\alpha}{\partial p_a}\,. \tag{4.6}$$

Note that the Hamilton equations for $\dot{q}$ are only for a indices, this means that the

[2] That this is the case is not intuitive at first sight, as the Hamiltonian could have also been dependent of some the velocities $\dot{q}_\alpha$, as not all their relations with the momenta are invertible.

[3] Which momenta we take as independent can be a matter of choice, this would correspond to taking different coordinates in the constraint surface.

$\dot{q}_\alpha$ are left undetermined and this has to do with the gauge freedom. For this to be consistent, we need that the Hamilton equations for $\dot{p}_\alpha$ have to be such that the constraint equations $p_\alpha = p_\alpha(q_i, p_a)$ are always satisfied. This consistency condition for gauge theories may imply some other conditions between the momenta, which are called secondary constraints. It may also be the case that the consistency condition cannot be met, which would correspond to a case in which the Euler-Lagrange equations have no solutions. There is also the possibility that consistency requires that the values of some the 'free' $\dot{q}_\alpha$ are not free anymore and they become functions of the other variables[4]. These constraints are called second-class, and they will be uninteresting for us, as it is the first-class constraints[5] which leave some freedom and they will play the main role in gauge theories and reparametrization invariant theories like general relativity.

The alternative to this is to define a Hamiltonian in full phase space and treat the momenta as independent from each other. Doing so, one is able to use the full machinery of Hamiltonian mechanics as usual. The Hamiltonian one uses in this case is the total Hamiltonian:

$$H_T = H_c + v^\alpha \phi_\alpha \,. \tag{4.7}$$

Where H_c is the canonical Hamiltonian[6] and ϕ_α are the primary constraints[7]. For first-class systems[8] as gauge theories, the v^α are arbitrary functions. The equations of motion for any function in phase space are the usual Hamilton equations for this Hamiltonian:

$$\dot{q}_i = \{q_i, H_T\} = \frac{\partial H_c}{\partial q_i} + v^\alpha \frac{\partial \phi_\alpha}{\partial q_i} \tag{4.8}$$

$$\dot{p}_i = \{p_i, H_T\} = -\frac{\partial H_c}{\partial p_i} - v^\alpha \frac{\partial \phi_\alpha}{\partial p_i} \tag{4.9}$$

$$\dot{f}(q,p,t) = \{f(q,p,t), H_T\} + \frac{\partial f(q,p,t)}{\partial t} \,. \tag{4.10}$$

Where I have introduced the Poisson bracket notation[9] and have used the fact that we are on the constraint surface. It can be shown[10] that the evolution defined by these

[4] For an example of this see (Rothe and Rothe, 2010, p. 37, example 1).

[5] The technical definition of first-class constraints is that they are the constraints that have Poisson brakets that vanish at the constraint surface, i.e., that are linear combinations of the constraints.

[6] In fact it is not a uniquely determined function now, but there is a equivalence class of them.

[7] Let me mention that here and in the literature with the term 'constraints' one refers both to certain functions ϕ^α or operators $\hat{\phi}^\alpha$ and to the condition that these functions vanish, ϕ^α, or that the action of those operators on certain states vanishes $\hat{\phi}^\alpha|\psi\rangle$.

[8] In the case of having second-class constraints the v^α associated with them would not be arbitrary functions but functions that are fixed once we impose that evolution has to preserve the constraints.

[9] $\{f,g\} = \Sigma_i(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i})$

[10] A derivation can be found in (Pons, 2005, Section 2.4).

equations, provided that the initial conditions satisfy the constraints (both primary and secondary), remains always in the constraint surface. Therefore, using the total Hamiltonian we can do business as usual and treat all the momenta as independent, as the formalism will make sure we do not go away from the constraint surface. Moreover, now we have equations for every $\dot{q}_i$, so we have fixed the problem that the relation between $q_i, \dot{q}_i$ and q_i, p_i was not invertible. Different choices of v^α give rise to gauge-equivalent solutions of the equations of motion.

In this formalism there is a relation between constraints and gauge transformations. Each primary first-class constraint has associated a generator of a gauge transformation, as it is intuitive from the fact that there is an arbitrary function v^α associated with each primary first-class constraint. Each generator is a tuned combination of constraints[11] and any gauge transformation can be represented as the exponentiation of an element of the algebra of generators. In gauge theories, physical observables are gauge invariant quantities f, i.e., they are quantities that are not changed by these transformations and they therefore satisfy:

$$\{f, G_i\} = 0 \; \forall \, G_i \,. \tag{4.11}$$

That is, gauge invariant quantities are the ones that have vanishing Poisson brackets with every gauge generator. They are commonly referred to as Dirac observables.

Finally, it has also been defended to use a different formalism which is called the extended formalism. In this formalism the total Hamiltonian is replaced by the extended Hamiltonian, which includes also the secondary constraints:

$$H_E = H_c + v^A \Omega_A \,. \tag{4.12}$$

Ω_A includes all constraints, primary and secondary. Dirac (1964) introduced this Hamiltonian motivated by the fact that primary and secondary constraints are treated differently in the total formalism and by his wrong claim that both primary and secondary first-class constraints generate gauge transformations[12]. This Hamiltonian is **not** equivalent to the total or canonical ones, as the action principle gives different equations of motion. Therefore, the link with the original Lagrangian gauge theory is broken.

Keeping this in mind, one can still use the theory defined by this Hamiltonian. Now it will be the case that we have as many free functions as first-class constraints,

[11] The way to find the generators is to impose that a differential transformation generated by them leaves the action invariant, taking also into account that the v^α will also be transformed. This calculation is done in (Rothe and Rothe, 2010, Sections 5.3, 5.4).

[12] As I have explained in the previous paragraph this is not the case, it is just some particular combinations of them that generate gauge transformation. For a discussion of Dirac's mistake and his propagation see Pitts (2014a); Pons (2005).

then it can be shown that the generators of 'gauge'[13] transformations are the first-class constraints, both primary and secondary. These 'gauge' transformations define a different class of 'gauge' invariant quantities, now they satisfy:

$$\{f, \Omega_A\} = 0 \; \forall \; \Omega_A \,. \tag{4.13}$$

Notice that this is a more restrictive condition than in the total case and that if a quantity is gauge invariant for the extended Hamiltonian, it also is gauge invariant for the total Hamiltonian. Therefore, one could justify the use of the extended formalism if the 'gauge' invariant quantities represented the relevant physical content of the original theory[14]. Despite this, this formalism has been used since Dirac introduced it, and in the quantization process of section 4.5we will find that the gauge condition introduced is equivalent precisely to the one of the extended formalism.

With this we have the three ways a gauge theory can be represented in the Hamiltonian formalism: by using the canonical, the total and the extended Hamiltonian. The expression of a gauge theory in any of these formalisms will be the starting point for its quantization. Let me illustrate how these formalisms are implemented with the example of electromagnetism.

When I introduced the electromagnetic action and its associated equations of motion I found that there was not a dynamical equation for A^0. The reason for this is that in the action there is not any term which contains temporal derivatives of this component. When passing to the canonical formalism this implies that the momentum, π^0, conjugated to A^0 is identically 0. That is, there is a primary constraint $\phi = \pi^0$ which constrains the momenta conjugated to A^a. When we impose that this constraint still holds when we evolve the system we find a secondary constraint $\vec{\nabla} \cdot \vec{\pi}$. When one uses the canonical or the total Hamiltonian, the equations of motion allow to identify $\vec{\pi} = -\vec{E}$, and hence we find that the secondary constraint is just Gauss law, the constraint equation that we had found before for electromagnetism. The other Hamiltonian equations of motion are just the other 3 dynamical equations that we had for electromagnetism.

The case with the extended formalism is more complicated, as I have mentioned before. In this formalism, the relationship between $\vec{\pi}$ and the standard definition of the electric field does not hold any more. Even if the equations of motion for $\vec{\pi}$ are

[13] I will write 'gauge' in quotation marks to distinguish these transformations from the gauge transformations of the original theory.

[14] I refer the reader again to Pitts (2014a). Pitts takes the case of electromagnetism and argues for the inequivalence of the extended and total formalisms and for using the total one. He argues that the physical electric field of the theory is the one represented by some derivatives of the 4-potential A_μ and not by the conjugate momenta $\vec{\pi}$. In the total formalism both fields are the same, but in the extended one only the latter is invariant. As he argues for taking the electric field to be the former, he rejects the extended formalism.

the ones of the electric field one can argue, as Pitts (2014a) does, that $\vec{\pi}$ is not the electric field, as it is not related to A^μ as one would expect. More important for our discussion is the way A^0 is regarded. In the extended formalism one see A^0 as not playing any role at all, as the information about the physical electric and magnetic fields is argued to be contained in A^a and $\vec{\pi}$ and the evolution of these fields depends on A^0 but just in a way that can be reabsorbed by the arbitrary function v introduced by the extended formalism. In this sense one could be tempted to argue, contrary to Pitts, that A^0 does not have any physical meaning.

Given that A^0 acts like a Lagrange multiplier we find that we have a further way of dealing with electromagnetism using the constrained Hamiltonian formalism. This way consists in taking A^0 to be a (fixed) function rather than a phase space variable. Therefore we end up with a Hamiltonian system defined on a smaller phase space and which satisfies only the Gauss constraint. Treating A^0 in this way is compatible with the two positions sketched in the previous paragraph, i.e., with considering it to contain physical information or not. This discussion will become important for the case of general relativity, where we will see that there are some functions (N and N^a) which act like Lagrange multipliers and for which an analogy with the case A^0 can be relevant.

At this point I should make a technical remark. The definitions and expressions in this section has been based in Hamiltonian mechanics for constrained systems with countable degrees of freedom. However, I will be interested in field theories like general relativity, i.e., theories whose degrees of freedom are functions or other geometrical objects defined on a space, usually 3-dimensional space. For such theories, the formalism in this section can be generalized by allowing configuration variables and phase-space variables to be field variables. In this case there can be constraints associated with every spacetime point, like the Gauss constraint above. The Poisson bracket structure for a field theory has a distributional character, and hence it is common to define smeared versions of the constraints and gauge generators to better deal with the technical difficulties of dealing with a field theory. In any case, from a conceptual point of view the introduction of field variables does not make any important difference.

Let me now turn to a couple of aspects of gauge theories that will be important for our discussion. First, notice that the term gauge transformation is used in two different but related ways. The first one, more natural in a Lagrangian context, is that a gauge transformation transforms solutions of the equations of motion into other solutions without changing the physical content described by these solutions, that is, it is a transformation between trajectories in configuration space or phase space. The second one, more natural in Hamiltonian contexts, is to take gauge transformations to apply to instantaneous configurations and therefore relate two descriptions of a same physical state at a given moment of time[15]. This distinction will be important for the case of general relativity, as I will argue that the first notion of gauge invariance applies to this theory, but not the second one.

From a methodological and conceptual point of view, it is important to consider the way one deals with the indeterminism of gauge theories in any of their different formulations. As suggested in (Maudlin, 2002, p. 5) there are basically three ways of working with the indeterminism of these theories.

First, one has the option of doing nothing about it. That is, accept that the dynamics is indeterministic, but as the dynamics for the physical content of the theory is perfectly deterministic, and we know which are these gauge invariant quantities there is nothing to worry about. In the Hamiltonian formalism this means leaving the free functions undetermined.

Second, one can eliminate the indeterminacy by adding extra conditions to the variables of the theory. This procedure is called gauge fixing and makes the representation of each different physical state unique. To see how this works, let me introduce the concept of gauge orbit. A gauge orbit is the region of phase space[16] in which all their points are related by instantaneous gauge transformations. That is, gauge orbits are equivalence classes which consist on sets of points which describe the same physical content. Now, gauge fixing works by adding new constraints to our theory such that for every gauge orbit there is only one state that satisfies the constraints. It can be shown[17] that with this new constraints in play all the constraints become second-class and hence there is no free function left, as expected. In other words, gauge fixing singles out one point from each gauge orbit (see figure 4.1) and the dynamics becomes explicitly deterministic. That is, the mathematical formalism now is such that specifying an initial state uniquely determines a solution of the equations of motion for the gauge-fixed variables which describe the evolution of the system at all times.

And third, indeterminacy can be eliminated by quotienting out the gauge orbits. This means replacing our original phase space by a reduced phase space in which each point represents a gauge orbit. Mathematically, the new space is the quotient of the original by the equivalence classes, i.e., the gauge orbits. In this new space the dynamics is perfectly deterministic. The quotienting process can be mathematically more complicated than gauge fixing and the quotient space may not be smooth, but this process has the advantage over gauge fixing of not needing to search for a condition to single out a single point from each equivalence class.

[15] Note that both kind of gauge transformations, transformations of full trajectories and pointwise transformations are generated by the generators we found before.

[16] Or, equivalently, if one prefers the Lagrangian formalism, the tangent bundle of a configuration space.

[17] (Henneaux and Teitelboim, 1992, Section 4).

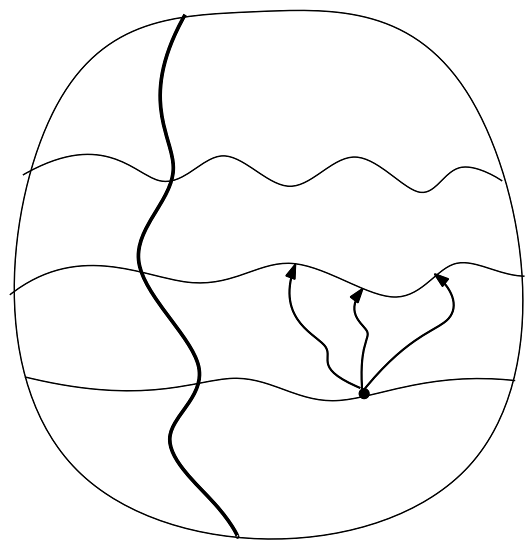

Fig. 4.1: Representation of the phase space of a gauge theory. The horizontal lines represent gauge obits, i.e., points that are physically equivalent. As there is gauge freedom the evolution is not deterministic and from a point of phase space multiple trajectories are allowed, all ending at the same gauge orbit. A few of these possible trajectories are represented. The vertical line represents a gauge fixing: notice that it intersects with every gauge orbit only once, therefore picking one point from each orbit.

Notice that the latter two strategies relied on the instantaneous notion of gauge transformation. When in the next sections I will study reparametrization invariant systems, I will find that this concept will not be as easily applicable to these theories and hence we will have to treat this kind of theories in a slightly different way, even if they are also described by the constrained formalism.

4.2 Reparametrization invariant theories

In this section we will deal with theories that are reparametrization invariant[18], i.e., theories that in the Lagrangian formulation have an action that does not change when changing the way we assign coordinates to the time variable in the action, or, for field theories, to the spacetime points. This symmetry can be seen as a gauge symmetry: for these theories we assume that the parametrization we choose does not have physical meaning and that the physical content is independent of this parametrization. However, if we apply some of the recipes of the last section to reparametrization invariant theories we may run into trouble, specially when the theory we are dealing with is general relativity. For instance, Earman (2002) proposes a new metaphysics of time on the light of the reparametrization invariance of general relativity. Here I will introduce the way these theories are treated in the Hamiltonian formalism and argue against positions like Earman's.

General relativity is an example of a theory of this type, and its successful use makes clear that we are able to extract its physical content. Nevertheless, reparametrization invariance has to be treated carefully and in a way that is different to other gauge symmetries, as I will show in this section. Studying the reparametrization invariance of general relativity and the way it is formalized will be important for understanding the problem of time of the canonical quantization of the theory. For studying this problem and the proposed resolutions it will be important to distinguish between different kinds of reparametrization invariant models, and therefore I will start this section by giving some examples, which will serve not only to illustrate how reparametrization invariant models can be represented in the constrained formalism but also for highlighting their differences. In particular, I will introduce the distinction between deparametrizable and non-deparametrizable models, which will be crucial for evaluating the resolutions of the problem of time in section 4.7. In the second part of this section I discuss in generality reparametrization invariance and the way the definition of observable for gauge theories does not work for models with this symmetry.

[18] Subtleties may arise here in what counts as a reparametrization invariant theory, as any theory that can be expressed in a Lagrangian fashion can be made reparametrization invariant by adding the right auxiliary variables or fields to the theory. In this section when I refer to reparametrization invariant theories I mean theories that are represented by an action with this symmetry and of which we believe that the parametrization does not have physical meaning. Therefore I am including here not only genuinely reparametrization invariant theories like general relativity but also artificially constructed ones as covariant formulations of Newtonian mechanics.

4.2.1 Three examples

Let me start by studying a well-known, simple example of reparametrization invariant theory[19]. Consider the following action that describes the trajectory of a relativistic particle in 1 dimension in a reparametrization invariant manner:

$$S[t,x] = \int d\tau \sqrt{\dot{t}^2 - \dot{x}^2}\,. \tag{4.14}$$

Here the dot represents a derivative with respect to τ. The reparametrization invariance of the action means that if we replace the 'time' parameter τ by any monotonic function of it, $\tau' = \tau'(\tau)$, the action does not change. We can understand this symmetry as a gauge symmetry: changing the way we parametrize a trajectory in Minkowski spacetime is just a formal transformation and not a change of the trajectory $x(t)$. Note than we are taking $x(t)$ to be the physically meaningful, that is, we are perfectly able to make sense of x and t as variables that relate with things an observer could measure[20]. The symmetry of the action means that the Lagrangian is singular and that the solutions of the Euler-Lagrange equations of motion depend on an arbitrary function of τ:

$$t(\tau) = f(\tau) \tag{4.15}$$

$$x(\tau) = Af(\tau) + B = At + B\,. \tag{4.16}$$

A and B are constants to be determined by the initial conditions.We can calculate the momenta and the Hamiltonian:

$$p_t = \frac{\dot{t}}{\sqrt{\dot{t}^2 - \dot{x}^2}} \tag{4.17}$$

$$p_x = -\frac{\dot{x}}{\sqrt{\dot{t}^2 - \dot{x}^2}} \tag{4.18}$$

$$\phi = \frac{1}{2}(p_t^2 - p_x^2 - 1) = 0 \tag{4.19}$$

$$H_c = p_t\dot{t} + p_x\dot{x} - \sqrt{\dot{t}^2 - \dot{x}^2} = 0\,. \tag{4.20}$$

We find that the system has a primary constraint ϕ and that the canonical Hamiltonian vanishes. This implies that there are no secondary constraints and that the

[19] This example, or similar versions of it can be found for instance in Sundermeyer (1982); Henneaux and Teitelboim (1992); Pons and Salisbury (2005).

[20] As we are assuming a theory in Minkowski spacetime this variables relate with the relative positions of synchronized comoving ideal clocks and their readings.

constraint is first-class. With this we can build the total Hamiltonian, which in this case is the same as the extended one:

$$H_t = \frac{\lambda}{2}(p_t^2 - p_x^2 - 1) = \lambda\phi\,. \tag{4.21}$$

Using either the canonical or total Hamiltonian the equations of motion tell us that both p_x and p_t are constants of motion and that the solutions for t and x have the same form[21] and satisfy the same relations we found using the Lagrangian formalism, as we expected. In phase space the trajectories are straight lines in the planes defined by constant p_x and p_t, i.e., straight lines in planes with coordinates x and t. This is illustrated in figure 4.2.

The gauge generator is just the constraint ϕ and, as the canonical Hamiltonian is 0, the constraint is also the generator of time evolution (τ evolution). This gives rise to a number of difficulties. First, gauge transformations are not physical transformations, does this mean that temporal evolution is not physical? Moreover it is easy to check that neither x nor t are gauge invariant quantities (neither of them has vanishing Poisson brackets with ϕ), so we have lost our natural candidates for observables.

An answer to these difficulties comes from adopting the understanding that gauge transformations are transformations between full solutions of the equations of motion. Given a solution $t(\tau), x(\tau)$ we are perfectly able to get its physical content, namely the trajectory in Minkowski spacetime $x(t)$[22]. Notice that we have an understanding of what counts as physical or not that is previous to the formalism and not a consequence of it. Of the three strategies we had for dealing with gauge symmetries we are perfectly able to use the first one, ignoring the indeterminacy, or the second one, fixing gauge[23], and we recover solutions $t(\tau), x(\tau)$ of which we can make perfect sense and no question about the physicality of time evolution or about observability arises.

The difficulties therefore arise only when we consider gauge transformations as relating physically equivalent states at a given instant. But in this case we cannot say that gauge related points represent physically equivalent points: what they in fact represent are different points of a trajectory $x(t)$, which are clearly not physically equivalent points. We should be careful then when interpreting the gauge symmetry

[21] The constant A that appears in the solution 4.16 for $x(\tau)$ is now determined by the ratio p_x/p_t, which at the end of the day has to be determined by the initial conditions.

[22] Remember it is something we can measure with an appropriate set of synchronized, comoving clocks.

[23] The most straightforward way of fixing the gauge is to directly fix the arbitrary relation between t and τ by adding a new constraint $\psi = t - f(\tau)$. The total Hamiltonian now becomes: $H_t = \frac{\dot{f}}{2p_t}(p_t^2 - p_x^2 - 1)$. This gives the deterministic equations of motion for t and x for a given parametrization $t = f(\tau)$. Notice that this gauge fixing condition has to be necessarily τ dependent, that is, for every value of τ it chooses a different element of the gauge orbit.

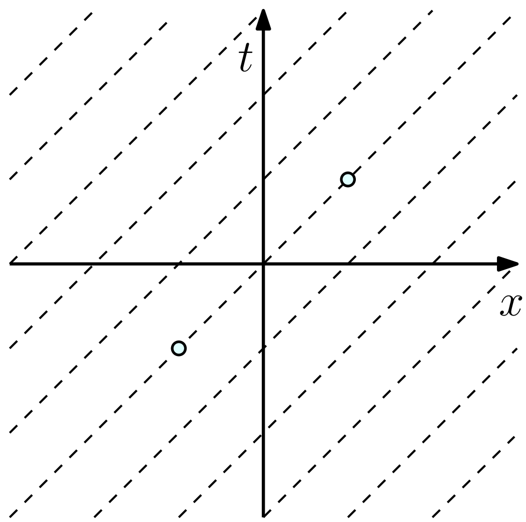

Fig. 4.2: Gauge orbits in a surface of constant p_t and p_x in the phase space of a relativistic particle. The gauge orbits coincide with the trajectories in phase space under Hamiltonian evolution. Two points are represented which, even if they lie in the same gauge orbit, do not represent a same physical state.

restricted to a given moment, as this 'moment' is a moment in τ-time and not in physical time.

What about the observability of x and t? The fact that they do not commute with the gauge generator means that, given the symmetry of the theory, there is no physical meaning to asking what is the position or the time at a given value of τ. In other gauge theories, like electromagnetism, we would take this to imply the unphysicality of the non-commuting quantity; but in this case what we have to reject is the physicality of the parameter τ. In other words, the fact that it is not a sensible answer to ask for the position of the particle at, say, $\tau = 3$ is not because the position of the particle is not a physical quantity, but rather because τ is not a physically meaningful time parameter.

Therefore, we arrive to the conclusion that in a theory with reparametrization invariance in which the parametrization of the theory does not have any physical meaning we should not take the instantaneous view of gauge transformations and we should relax the requirement that observables have to commute with the gauge generators. I will later show that this conclusion extends to any reparametrization invariant theory.

Finally, we can insist to follow the third strategy for dealing with gauge indeterminacy and quotient phase space by the gauge orbits. What we find is that τ-evolution trajectories collapse to a single point, as all the points of the trajectory lie on the same gauge orbit. Each point in the reduced phase space is characterized by the ratio $\frac{p_x}{p_t}$ and a point of the trajectory t_0, x_0, which can be seen as initial conditions[24]. Evolution then 'freezes', for every τ we get the same point of the quotient phase space. Of course this is due to identifying different points in the gauge orbit as physically equivalent, when they represent different physical states like t_1, x_1 and t_2, x_2. We can make a sensible interpretation of this is if we again take the full-trajectory view of gauge transformations. Points in the same gauge orbit are only equivalent in the sense that they generate the same trajectory in Minkowski spacetime when taken as initial conditions. If we take a point of phase space to represent a full solution it is no surprise that gauge orbits collapse to one equivalence class and that there is no τ evolution in the phase space of equivalence classes.

Related with the claim that the dynamics if frozen there is the claim that observables are constants of motion. The latter is supported not only by the former, but also by the fact that the 'observables' defined by the gauge condition $\{f, G\} = 0$ are constants of motion[25] as they automatically have vanishing Poisson brackets with the Hamiltonian. As explained above, we have strong reasons not to require $\{f, G\} = 0$ to our observables when G generates a reparametrization transformation. However, there are some of the observables defined by that requirement, the coincidence observables, which present interesting properties. Strictly speaking these observables are constants of motion and represent the value of a variable when another one (or set of others) takes a given value. In the case of our relativistic particle these observables represent the positions X for a given time T[26]:

$$X_T = x_0 - \frac{p_x}{p_t}(T - t_0) = x - \frac{p_x}{p_t}(T - t)\,. \tag{4.22}$$

In fact I have given two equivalent functions: the first equality gives X_T as a function of the momenta (and of the constants x_0, t_0 and T) and the second as a function of all the phase space variables. It can be easily checked that both have vanishing Poisson brackets with the constraint and therefore are constants of motion. Keep in mind that what is meant with this is that the phase space function X_T does not vary with τ, and not that X_T is a constant for different values of T, which is clearly not the case. Moreover, the first equality gives a function which can be also seen to 'live' in the reduced phase space.

Authors like Earman (2002) propose to take these 'frozen' observables as requiring us to move to a new metaphysics of time. However, I think it is dangerous to run

[24] Of course, we can choose different ways of parametrizing the reduced phase space by picking a different point of the trajectory as initial condition. Therefore the reduced phase space has only two effective coordinates.

[25] As long as they do not have explicit τ dependence.

[26] Equivalently we can build quantities T_X that represent times for given positions.

into metaphysical conclusions from such formal aspects of our theories. That is, one should be careful when interpreting some aspect of the mathematical formalism of our theories to imply metaphysical consequences. In this case, one should note that these observables exist also in non-reparametrization invariant theories and that for these theories no such metaphysical claim about time is made. What these observables are is nothing else but an expression of the fact that in any deterministic theory any quantity can be determined by knowing a set of initial conditions at some arbitrary time and making use of the equations of motion. It is no wonder then that a quantity built in this way remains constant under Hamiltonian evolution, as under this evolution what evolves is the vales of $t(\tau)$ and $x(\tau)$ that play the role of initial conditions, but not the 'prediction' they make for the time $t = T$[27]. Even though these observables exist for any deterministic theory, finding them is by no means trivial, as it requires solving the equations of motion and expressing the solutions as functions of the initial conditions and having a good grasp of which variables represent time. For instance, in our example of the relativistic particle we identified t as a meaningful time and expressed the solutions of the equations of motion for x as a function of t and the initial conditions. For theories which do not explicitly have time as a variable in their configuration space one can also define frozen observables, as I will show for our third example below.

To complete this discussion of 'gauge-invariant' observables it is important to notice that the frozen observables defined above are not the only ones one can define. Indeed, we can define the following observables:

$$X_{T,a,b}(x,t) = x - a - \frac{p_x}{p_t}(T - (t - b))\,. \tag{4.23}$$

These quantities also satisfy that they have vanishing Poisson brackets with the Hamiltonian constraint. However, their physical interpretation is slightly different: they do not represent what the position of the particle will be or was at a given time, but they represent the position the particle would take at time T if its actual position and time at parameter time τ were shifted by a and b. This is a kind of function which is observable in the sense that one can compute it if one observes x and t and knows the equations of motion, but not in any intuitive sense. In this sense, this shows that the class of functions which have vanishing Poisson brackets with the constraints is bigger than the class of coincidence observables. This means that the

[27] Consider the following example to illustrate this. Take the harmonic oscillator with Hamiltonian $H = \frac{1}{2}(x^2+p^2)$. We can build the quantity $X_T = x\cos(T-t)+p\sin(T-t)$ that represents the position of the oscillator at time T. This quantity is a constant of motion: $\frac{dX_T}{dt} = \{X_T, H\} + \frac{\partial X_T}{\partial t} = 0$. That this quantity exists means that at any time the system 'knows', i.e., has enough information to determine its future, or past, state at a time T. But this does not entail anything about the metaphysics of time or about frozen dynamics.

physical interpretation of these functions is not trivial, which is a conclusion I will extrapolate to quantum observables in the second part of this chapter.

Let me now turn to the second example in this section. This example is just a different Lagrangian formulation for the one dimensional relativistic particle based on the following Lagrangian:

$$L(t, x, \dot{t}, \dot{x}, n) = \frac{1}{2n}(\dot{t}^2 - \dot{x}^2) + \frac{n}{2}\,. \tag{4.24}$$

Here, and on-shell[28], n is the lapse function that relates physical time, represented by the interval length s, with the time parameter τ by means of $ds = n d\tau$. A way of showing this is by solving the equation of motion for n, which gives $n d\tau = \sqrt{\dot{t}^2 - \dot{x}^2} d\tau$, which is the Minkowski line element. If we substitute this value of n in the Lagrangian we get back our original action 4.14. Now this action is invariant under a transformation $\tau \to \tau' = f(\tau)$ and $n \to n' = n\frac{d\tau}{d\tau'}$. This transformation is a reparametrization that leaves the line element $n d\tau$ invariant.

In the Hamiltonian formalism we find that there is a primary constraint $\pi = 0$, where π is the canonical momentum conjugate to n, and that the canonical Hamiltonian is:

$$H_c = \frac{n}{2}(p_t^2 - p_x^2 - 1)\,. \tag{4.25}$$

The consistency condition that the value of π remains 0 during the whole evolution imposes a secondary constraint $\phi = \frac{1}{2}(p_t^2 - p_x^2 - 1) = 0$. Notice that it appears explicitly in the Hamiltonian $H_c = n\phi$ and that it is the same constraint we found for the equivalent action 4.14. Now we do not have a directly vanishing canonical Hamiltonian, but nevertheless it vanishes once we apply the secondary constraints. The total Hamiltonian is now:

$$H_T = n\phi + \lambda\pi\,. \tag{4.26}$$

And the gauge generator:

$$G = \epsilon\phi + \dot{\epsilon}\pi\,. \tag{4.27}$$

Notice that the Hamiltonian and the gauge generator generate the same evolution on-shell, as the Hamiltonian equation for n fixes $\dot{n} = \lambda$, which is exactly the same relation than the one between the coefficients of the gauge generator, that is, between ϵ and $\dot{\epsilon}$. Therefore, our previous discussion applies equally well to this action and we have to reject that the local action of the gauge transformation relates physically equivalent points and we get to the same conclusions about gauge transformations, observables and quotienting phase space[29].

[28] On-shell means when the classical equations of motion are satisfied.

[29] We can also point out that for n-independent phase space functions both the Hamiltonian evolution and the gauge transformations are exactly of the same form as if we were using the 4.14 action.

The two examples I have considered shared an important feature: they are deparametrizable. A model is said to be deparametrizable if time, or spacetime coordinates, appear explicitly as configuration or phase space variables or can be identified and separated in some of these spaces by means of some coordinate transformation or canonical transformation. Importantly, in section 4.3 I will argue that general relativity is not deparametrizable and that this will imply that not every intuition we build about how to deal with reparametrization invariance using deparametrizable models will be applicable to general relativity.

For this reason, let me consider a third example, which will be the closest to general relativity and will be very important for my argument, as it is not deparametrizable, its quantization does not lead to a meaningful quantum theory and I will argue that for general relativity the same happens. The model is given by the following action:

$$S[x,y] = 2\int d\tau \sqrt{\frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right)\left(E-\frac{1}{2}(k_x x^2+k_y y^2)\right)}\,. \tag{4.28}$$

This is the Jacobi action for a system of two harmonic oscillators with different couplings. Similar examples were studied in Barbour (1994a) for supporting a Machian perspective of time. In this view, absolute time scale is meaningless and what is really meaningful in mechanics is the succession of configurations. This action exemplifies well this view, as trajectories in τ which minimize it agree in that they describe two oscillators oscillating between two positions, but they disagree in the values of τ they assign to each instant. This example can be generalized to consider a whole universe as described by classical mechanics. From Barbour's perspective, one can make sense of the dynamics of the universe even in the absence of the metric aspect of time, just by retaining the ordering relation it defines. In this sense a history of the universe is described by a trajectory in configuration space, and it does not matter the way we parametrize this trajectory.

Notice however that there is a choice of time parameter which is special in that it makes the equations of motion look simpler. We can define a special time parameter t by imposing $dt = \sqrt{\frac{m(\dot{x}^2+\dot{y}^2)}{2E-k_x x^2-k_y y^2}}d\tau$. With respect to this parameter t what one finds is that the equations of motion become the Newton equations of motion for the harmonic oscillator and that the oscillations of both oscillators are regular with respect to this time parameter, even if each one has its own frequency. In this sense, the factor $\sqrt{\frac{m(\dot{x}^2+\dot{y}^2)}{2E-k_x x^2-k_y y^2}}$ plays the role that the lapse function played in our second example: it relates the arbitrary parameter τ with the physically privileged time t.

As a reparametrization invariant system, in the Hamiltonian formulation this system is a constrained system with vanishing canonical Hamiltonian. That is, evolution

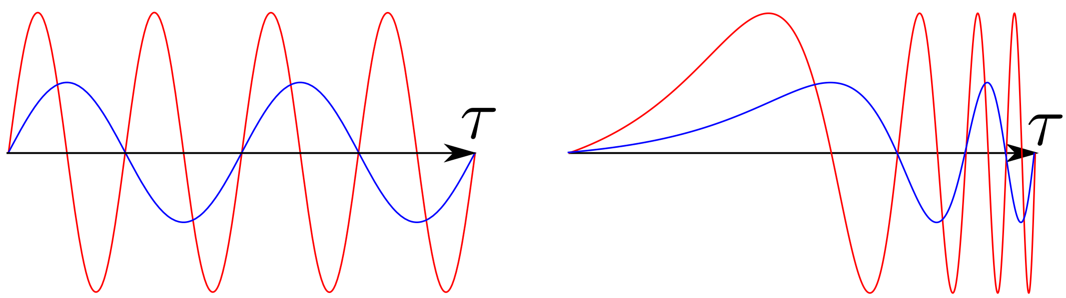

Fig. 4.3: Two equivalent solutions of the equations of motion of the double harmonic oscillator model. They represent the same sequence of oscillations but they differ from each other at the particular values of τ they assign to each moment of time. In this way the parametrization on the left-hand side represents the Newtonian parametrization in which the oscillations are regular and the one in the right-hand side represents a parametrization in which the oscillations become faster as τ passes.

is generated by the primary constraint ϕ:

$$H_T = \lambda\phi = \lambda(\frac{1}{2m}p_x^2 + \frac{1}{2m}p_y^2 + \frac{k_x}{2}x^2 + \frac{k_y}{2}y^2 - E)\,. \tag{4.29}$$

Where λ is a positive function which plays the role of lapse function in the Hamiltonian formalism. Notice that this formulation is very similar to the one of our first example but with the important difference that now time is not one of the variables in the configuration space of the theory. We see that this model is not a deparametrizable one, as both degrees of freedom x and y represent physical degrees of freedom and not a time coordinate.

Let me introduce explicitly the solutions of the Hamilton equations of motion for this system[30]

$$x(\tau) = x_0 \cos(\omega_x t(\tau)) + \frac{1}{m\omega_x} p_{x0} \sin(\omega_x t(\tau)) \quad (4.30)$$

$$p_x(\tau) = -m\omega_x x_0 \sin(\omega_x t(\tau)) + p_{x0} \cos(\omega_x t(\tau)) \quad (4.31)$$

$$y(\tau) = y_0 \cos(\omega_y t(\tau)) + \frac{1}{m\omega_y} p_{y0} \sin(\omega_y t(\tau)) \quad (4.32)$$

$$p_y(\tau) = -m\omega_y y_0 \sin(\omega_y t(\tau)) + p_{y0} \cos(\omega_y t(\tau)) \quad (4.33)$$

$$t(\tau) = \int_{\tau_0}^{\tau} d\tau' \lambda(\tau') \quad (4.34)$$

$$\omega_x = \sqrt{\frac{k_x}{m}} \quad (4.35)$$

$$\omega_y = \sqrt{\frac{k_y}{m}}\,. \quad (4.36)$$

Here x_0, y_0, p_{x0}, p_{y0} are initial conditions at a parameter time τ_0 which satisfy the constraint. One can check that these equations imply that the arbitrary function λ equals $\sqrt{\frac{m(\dot{x}^2+\dot{y}^2)}{2E-k_x x^2-k_y y^2}}$, and hence that the time parameter it defines by means of 4.34 is precisely the Newtonian time for the system. The reparametrization invariance of the model is explicit: different choices of λ give rise to different $x(\tau), y(\tau)$ but they all agree in the succession of configurations they describe, as can be seen if we notice that different $x(\tau), y(\tau)$ are nevertheless the same $x(t), y(t)$ with respect to the Newtonian time t.

Once we know the equation of motion and their solutions, it is easy to build 'frozen constants of motion' like:

$$X_T(x, p_x, \tau) = x \cos(\omega_x t(T) - \omega_x t(\tau)) + \frac{1}{m\omega_x} p_x \sin(\omega_x t(T) - \omega_x t(\tau))\,. \quad (4.37)$$

This quantity is similar to the one I have defined above for the relativistic particle in that it gives the position of one of the oscillators for the τ-time T and in that it is constant along the trajectory defined by the Hamiltonian of the system. However, notice that it is different in that it does not have a vanishing Poisson bracket with the constraint and in that it carries a dependency in the Lagrange multiplier λ and in the parameter time τ. This means that different choices of λ will give us different X_T, so in this sense X_T we can say that they are not gauge invariant. These observables

[30] The following expressions are not of the greatest generality, as I am limiting λ to be a function of τ and not of the phase space functions. However, this restricted case already contains the interesting features of the most general case, which are only technically more involved, as the definition of t is not straightforward for a general λ.

make also explicit that time is not one of the phase space variables x, y, p_x, p_y but that it is represented by τ.

Notice also that one may have wanted to have a frozen observable of the form X_Y, that is, something encoding the position of the first oscillator when the second one is at some given position Y. However, this is in general not possible, as there will not be a unique relationship between the positions of the two oscillators. If the frequencies of the two oscillators are not related by a rational factor it will be the case that every time the second oscillator passes through some given position Y, the first one will be at a different position. Hence, it does not make sense to try to build a function X_Y. From a technical perspective the problem lies in that the relation between Y and T is not invertible, and hence we cannot transform X_T into X_Y.

Moreover, we can explicitly find the observables which have vanishing Poisson brackets with the constraint ϕ. The condition $\{f, \phi\} = 0$ defines a differential equation for f which has as solutions functions of the form $f(k_x x^2 + p_x^2/m, k_y y^2 + p_y^2/m)$[31]. That is, there are two basic quantities which have vanishing Poisson brackets with the constraint, namely, the energy of each oscillator. These observables are conserved in time, but they clearly do not have the interpretation of encoding the position of an oscillator at a later time. In this sense, the 'gauge invariant' content of the model does not support the interpretation that it contains all the physical facts of the theory.

Despite this difference in 'frozen observables' this example shares with the other two that it is a gauge theory from the global point of view and not from the instantaneous one. That is, the constraint defines a gauge generator which has the same form of the Hamiltonian and transforms solutions of the equations of motion to other solutions with different parametrizations. From the instantaneous point of view, these transformations are not gauge transformations, as the positions of the two oscillators change under their action. In this sense, this example further supports the claim that the definition of observables in terms of the instantaneous gauge transformation is not consistent with our intuitions for reparametrization invariant systems. I will next generalize these conclusions for reparametrization invariant models in general and for the case of general relativity in particular.

4.2.2 General reparametrization invariant systems

In the examples above I have considered just temporal reparametrization. More general theories take the form of field theories in which one studies physical properties at different spatiotemporal locations. A field theory is reparametrization invariant if changing the way one assigns coordinates to spacetime points is a symmetry of

[31] If we apply the same method to the constraint for the relativistic particle we find that the 'observables´ are functions of p_x, p_t and the combination $x + p_x/p_t t$. This latter function is the one I've used for defining the family $X_{T,a,b}$.

the theory. As I commented in the previous section, the Hamiltonian formalism for constrained systems can be extended to cover field theories. What we will find when applying this formalism to reparametrization invariant field theories will be analogous to the one-dimensional[32] examples above. Reparametrization invariant theories can be classified into two: theories with homogeneous Lagrangians, which are reparametrization invariant in a straightforward way, and generally covariant theories, which are based on the language of differential geometry. Both types will have similar constraint and gauge structures.

Let me start by introducing theories which have homogeneous, coordinate independent, Lagrangian densities, i.e., they satisfy:

$$S[\phi_i] = \int d^D x L(\phi_i, \partial_\mu \phi_i) \tag{4.38}$$

$$L(\phi_i, \lambda^\mu_\nu \partial_\mu \phi_i) = \det(\lambda^\mu_\nu) L(\phi_i, \partial_\mu \phi_i) \,. \tag{4.39}$$

$d^D x$ is the differential of the D coordinates x^μ, ϕ_i are the fields and λ^μ_ν a matrix. Under a change of coordinates the action is invariant[33].

Two of the examples above, the action 4.14 for the relativistic particle and the action 4.28 for the two harmonic oscillators were one-dimensional[34] cases of this kind of reparametrization invariant theory. Another relevant example of a theory in this category is the Nambu-Goto action which describes the trajectory of a string and which is of relevance for string theory[35]. Finally, any theory can be expressed in the form 4.38 by expanding its configuration space so that the original coordinates X^μ are treated as field of some new arbitrary parameters x^μ. We will see an example of this kind of theory in section 4.6 when I study the quantization of the classical mechanics of a single particle with arbitrary temporal reparametrization.

As a consequence of the Lagrangian being homogeneous, it can be shown[36] that it satisfies $L = \frac{\partial L}{\partial \dot{\phi}_i} \dot{\phi}_i$, where $\dot{\phi}_i$ is the partial derivative with respect the time coordinate, which implies that these theories have a vanishing canonical Hamiltonian density. Moreover, it can also be shown that these systems have D primary, first-class

[32] With one-dimensional I just mean that the examples above can be seen as field theories in one dimension, as all the variables only depend on one parameter τ.

[33] The derivatives of the field transform as $\partial_\mu \phi_i \to \frac{\partial x'^\mu}{\partial x^\nu} \partial_\mu \phi_i$. Using the homogeneity property one gets out the factor $\det(\frac{\partial x'^\mu}{\partial x^\nu})$ which is nothing but the inverse Jacobian of the transformation. It will cancel with Jacobian factor that comes from the transformation of $d^D x$, leaving the action invariant. For a discussion of this see (Sundermeyer, 1982, Appendix D).

[34] Again, one-dimensional in the sense that the variables depend on just one parameter.

[35] The study of string theory is outside the scope of this thesis, but the fact that the actions used in string theory have a reparametrization invariance suggests that it may also be susceptible to some of the conceptual problems that canonical approaches of quantum gravity face.

[36] I refer the reader to (Sundermeyer, 1982, Appendix D) for a discussion of these results.

constraints $\mathcal{H}_\mu$, which intuitively correspond to the reparametrization symmetry of this kind of theory in D dimensions. The total Hamiltonian of these models is a combination of the constraints, just as in the examples above, and we will find a similar problem with gauge transformations, as the Hamiltonian is a particular case of the gauge generator. Before this, let me introduce general covariant theories.

Generally covariant theories are theories formulated on the language of differential geometry. In this language coordinates on their own do not have any geometrical meaning, but instead the geometric properties are encoded in the metric tensor[37] $g_{\mu\nu}$[38]. In this way, the metric tensor defines the geometric structure, such as distances and angles, which gives a geometrical meaning to the points of the bare manifold. Generally covariant theories are invariant under reparametrizations which take into account the geometrical nature of this formalism, that is, under reparametrizations the different geometrical objects have different transformation rules according to their geometrical properties: scalars transform as scalars, tensors as tensors, and so on.

A generally covariant theory[39] can be formulated using an action of the form:

$$S[\phi_i, g_{\mu\nu}] = \int d^D x \sqrt{-g} L(\phi_i, \partial_\mu \phi_i, g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \partial_\rho \partial_\sigma g_{\mu\nu})\,. \tag{4.40}$$

Where besides the metric tensor we allow for a number of other fields ϕ_i, which can be any kind of geometrical objects. For the theory to be generally covariant, we will require that the metric transforms covariantly as $g'_{\mu\nu} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}$ and L transforms as a scalar. Reparametrizations of this kind are also known as diffeomorphisms and they can be defined in a technically more precise way as smooth, invertible maps between manifolds. As I explained in section 1.2 one can further introduce the distinction between active and passive diffeomorphisms to distinguish between a transformation which 'moves' objects around a manifold and another which just relabels the coordinates one assigns to the different points on the manifold. Depending on the context[40] it may matter which type of diffeomorphisms one is referring to, but from a technical perspective both amount to equivalent transformations. In the Hamiltonian context, transformations generated by the appropriate generators are usually understood as

[37] Alternatively, for one-dimensional manifolds one can use the one-form n, as in the example 4.24. This is just a simpler way of encoding the information of the metric tensor for that particular case.

[38] Here I will be using the $\{-,+,+,+,...\}$ sign convention.

[39] Let me mention that any field theory can be formulated in a generally covariant manner. For instance, one can express a Klein-Gordon field theory in a Minkowski spacetime using any arbitrary coordinate system for that spacetime. In this case, the metric tensor is not treated as a dynamical field, which complicates a little the analysis of the diffeomorphism invariance. As we are ultimately interested in general relativity, which treats the metric field as a dynamical field, I will focus on just this kind of theories.

[40] For instance, for the hole argument to work, one needs to appeal to active diffeomorphisms. I refer the reader again to the discussion in section 1.2.

active transformations. As in the case of theories with homogeneous Lagrangians, to specify a diffeomorphism we need to specify D functions, and we will find D primary constraints associated with these transformations.

At this point it is useful to introduce a set of variables which is convenient for representing generally covariant systems in the Hamiltonian formalism. These variables are the variables of the ADM formalism[41] which is a natural way of describing an arbitrary foliation of any spacetime, i.e., a description of spacetime as a space-like surface evolving with respect a time parameter, as represented in figure 4.4. In the use of the formalism there is an implicit assumption that the spacetime one is dealing with has the topology $\mathbb{R} \times \Sigma$, which is equivalent to saying that spacetime can be seen as a space manifold Σ evolving in time. This formalism therefore leaves out some models compatible with general relativity which contain different temporal topologies.

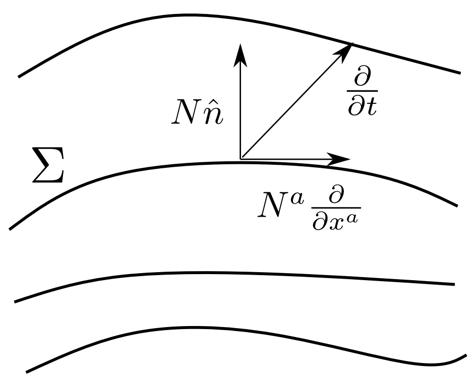

Fig. 4.4: ADM decomposition of a spacetime. Spacetime is described as the evolution of a space Σ. The curves generated by vector field $\frac{\partial}{\partial t}$ connect points with same spatial coordinates in different leaves of the foliation. This vector field is decomposed in two components, one normal and one tangential to the foliation.

Given a foliation of a spacetime, Hamilton equations of motion give us the evolution under the parameter t, which labels the spacelike hypersurfaces which represent different instants of the evolution. The components of the metric are divided into two groups, which will play different roles. First, the spatial components of the metric g_{ab}[42] will describe a metric for the $D-1$ space which is evolving in time. The rest of the components $g_{0\mu}$ get a different interpretation: they can be seen as encoding the information about the vector n normal[43] to the spatial slice at every instant of time. A convenient way of expressing this is by means of the lapse function N and shift

[41] The name comes from its proponents Arnowitt et al. (1962).
[42] I will use the greek indices μ, ν for spatiotemporal indices and the latin ones a, b for spatial ones.
[43] That is, it satisfies $-g_{\mu\nu} n^\mu n^\nu = 1$.

vector N^a[44]. These functions allow to express the vector field ∂_t which describes the foliation in terms of the normal to the foliation and a tangential vector (see figure 4.4):

$$\frac{\partial}{\partial t} = N\hat{n} + N^a \frac{\partial}{\partial x^a} \,. \tag{4.41}$$

Intuitively, the lapse N measures the (physical) time between two time slices t and $t + dt$ following a normal to the slices, and the shift vector N^a measures how much a point with some spatial coordinates in the slice $t + dt$ has shifted with respect to the point at the slice t with the same spatial coordinates. N and N^a play a similar role to the one n played in our second example above. Now, when applying the canonical formalism to a generally covariant theory expressed in these variables we find a canonical Hamiltonian of the form:

$$H_c = \int d^{D-1}x \left(N\mathcal{H}_0 + N^a \mathcal{H}_a \right) \,. \tag{4.42}$$

This form is suggestive, as the parallelism with 4.41 is evident: the canonical Hamiltonian which generates evolution in t is naturally decomposed in two parts: one which generates evolution normal to the foliation and a tangential one, with N and N^a describing precisely how ∂t decomposes in these two components. We also find that generally covariant systems are constrained systems, and that the momenta conjugate to N and N^a, P_0 and P_a, are primary constraints of the theory. Furthermore, $\mathcal{H}_0$ and $\mathcal{H}_a$ are secondary constraints[45], and are known as the Hamiltonian and momentum constraints. The total Hamiltonian of a generally covariant theory can therefore be expressed as:

$$H_T = \int d^{D-1}x \left(N^\mu \mathcal{H}_\mu + \lambda^\mu P_\mu \right) \,, \tag{4.43}$$

where the λ^μ are arbitrary functions, just as the v^α before and I am introducing the more compact notation $N^\mu \mathcal{H}_\mu = N\mathcal{H}_0 + N^a \mathcal{H}_a$[46]. This total Hamiltonian is slightly more complicated than the one for the homogeneous case, but it shares an essential feature for its later quantization, namely that the total Hamiltonian is just a sum of constraints. We will see that this is problematic for the quantization of the theory and will give rise to the problem of time, even if it is unproblematic from the classical perspective.

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[44] In particular, the relation between the metric components and these functions is: $N = \sqrt{-g^{00}}$ and $N^a = -g^{0a}/g^{00}$.

[45] Recall that this means that they are constraints that appear when imposing that the primary constraints are obeyed at every time, that is, when imposing that $\dot{P}_0 = 0$ and $\dot{P}_a = 0$.

[46] Notice that this is useful but not a spacetime notation, as N^μ are not the components of a vector in a coordinate system.

For a general covariant theory one can show that the gauge generators are:

$$G_\xi = \xi^\mu \mathcal{H}_\mu + (\dot{\xi}^\mu + C^\mu_{\nu\rho} N^\rho \xi^\nu) P_\mu \,. \tag{4.44}$$

Here I am using the same notation as in Pons et al. (2010), and everytime there is a sum of repeated indices there is an integration assumed. $C^\mu_{\nu\rho}$ are the structure functions of the Poisson bracket algebra of the secondary constraints[47]. The D components of the vector field ξ^μ are called descriptors, and fixing them fixes the gauge transformation. Indeed, one can show[48] that the transformations generated by G_ξ are diffeomorphisms and that the vector field ξ^μ defines their infinitesimal version, namely:

$$x^\mu \to x^\mu - \epsilon^\mu \tag{4.45}$$

$$\epsilon^\mu = \xi^0 n^\mu + \delta^\mu_a \xi^a \,. \tag{4.46}$$

Notice that the total Hamiltonian is a particular case of gauge generator: setting $\xi^\mu = N^\mu$ and making use of the on-shell identity $\lambda^\mu = \dot{N}^\mu$ the gauge generator becomes the total Hamiltonian[49]. The arguments which rejected identifying two points connected by a transformation generated by the Hamiltonians of the examples above as physically equivalent still apply, and hence one should reject considering time evolution as an instantaneous gauge transformation. Similarly, some other transformations in the gauge group will have to be interpreted in the same way: as generating gauge transformations from the global point of view but not from the instantaneous point of view, i.e., they will map a solution of the equations of motion to an equivalent one, but their action at a given instant produces a physical action.

Let me mention that this discussion applies also for the case of theories with homogeneous Lagrangians. In that case, the constraints are just primary constraints and the gauge generators are simpler, but everything is analogous: we find that gauge transformations are generated by D generators and they correspond to diffeomorphisms which changes the D coordinates. Similarly, the Hamiltonian is a particular case of gauge generator, as it generates transformations which evolve the system in time. Again, the transformation generated by the Hamiltonian cannot be interpreted as a gauge transformation from the instantaneous perspective. I will now study in more detail the action of the gauge generators and I will argue against defining observables just to be the quantities which have vanishing Poisson brackets with the

[47] That is, $\{\mathcal{H}_\mu, \mathcal{H}_\nu\} = C^\rho_{\mu\nu} \mathcal{H}_\rho$.

[48] I refer the reader to Pons et al. (2010) and to Bergmann and Komar (1972) for discussions and proofs of this.

[49] Notice that the term with C vanishes as we contract the antisymmetric $C^\mu_{\nu\rho}$ with the symmetric $N^\nu N^\rho$

constraints. I will continue referring to the gauge generators for the generally covariant case, but the discussion applies also for theories with homogeneous Lagrangians.

For this discussion it is convenient to distinguish between temporal and spatial diffeomorphisms, which are generated by G_0 and G_t, which are obtained restricting the descriptors to correspond to just displacements in the normal or tangential directions to the foliation :

$$G_0 = \xi(\mathcal{H}_0 + C^{\mu}_{0\rho}N^{\rho}P_{\mu}) + \dot{\xi}P_0 \tag{4.47}$$
$$G_t = \epsilon^a(\mathcal{H}_a + C^{\mu}_{a\rho}N^{\rho}P_{\mu}) + \dot{\epsilon}^a P_a\,. \tag{4.48}$$

First, diffeomorphisms generated by G_0 are temporal diffeomorphisms, i.e., they introduce a local temporal reparametrization. As discussed for the examples above, temporal reparametrizations are not gauge transformations from the instantaneous perspective, and one should reject identifying points related by a transformation generated by G_0 as physically equivalent. In a sense, it is the Hamiltonian constraint H_0 which contains the dynamics of the theory. Indeed, the total Hamiltonian depends on N and N^a which act as Lagrange multipliers, but consistency implies that N can never be 0[50], while there is no similar condition for the N^a. In this sense, the dynamical content is always dependent on the Hamiltonian constraint. Any transformation which is generated by G_0, or generated by a combination which contains G_0 will therefore imply some dynamical change and we should apply the same interpretation, i.e., that it only generates gauge transformations in the picture of gauge transformations as transforming solutions of the equations of motion into solutions of the equations of motion and not as something to be applied to an instant of time or to a spacetime point. Similarly as we found in the case of the relativistic particle, one should not impose the condition $\{f, G_0\} = 0$ to the observables of the theory and one should not quotient phase space by the orbits generated by G_0.

Notice that the argument above is enough for me to argue against the view that the analysis of the gauge symmetries of reparametrization invariant theories like general relativity should make us change our metaphysical view of time and to consider that physical observables are frozen. Moreover, this would be enough also for my analysis of the quantum problem of time in the later sections of this chapter. However, I will also comment about the action and gauge nature of the spatial diffeomorphisms. The reason for this analysis is not only completeness, but also that I will argue that even if spatial diffeomorphisms can be seen as instantaneous gauge transformations one should not impose a gauge condition like $\{f, G_t\} = 0$.

Spatial diffeomorphisms are generated by a combination of the spatial generators G_t. These transformations change the spatial coordinates of points but do not change

[50] This is because if N were 0 the vector $\frac{\partial}{\partial t}$ would not be timelike.

their temporal coordinate[51]. In this sense, these transformations preserve the foliation, as they preserve which hypersurfaces are associated to different instants of the time parameter t, even though the way each hypersurface is parametrized changes. From a global point of view, this transformation is a gauge transformation as it maps solutions of the equations of motion into physically equivalent solutions of the equations of motion. From the point of view of instantaneous gauge transformations, it also is a gauge transformation: spaces related by a spatial diffeomorphism are clearly physically equivalent: they just correspond to different ways of labeling their points. From an intuitive point of view, the gauge invariant content of a spacetime theory at a given time is a $D-1$-geometry and the fields defined on it. Spatial diffeomorphism symmetry could in principle be treated as any gauge symmetry: we can define gauge orbits as the equivalence classes of geometries under diffeomorphism, we can introduce a gauge fixing and we can even think in a reduced configuration space in which each point represents a $D-1$-geometry.

However, there is a problem with this proposal. From an intuitive point of view there are a number of geometrical and physical observables that we can assign to a space in a generally covariant theory. For instance, the metric defines distances between points, areas for surfaces and volumes for regions. It also defines a curvature scalar at every point of space. If in addition we have other fields such as a scalar field or the electromagnetic field in our theory it seems intuitive that they should also count as observables. Despite this, the most straightforward definitions for this observables fail to have vanishing Poisson brackets with the spatial diffeomorphism generators, and hence the criterion which was useful for defining observables for other gauge symmetries fails to give us the quantities that we were expecting to be observable. For instance, the Poisson bracket of the metric tensor with the gauge generator is:

$$\{g_{\mu\nu}, G_t\} = \mathcal{L}_\epsilon g_{\mu\nu} \,. \tag{4.49}$$

That is, the action of the gauge generator on the metric tensor gives back the Lie derivative of the metric along the descriptor vector field ϵ which defines the diffeomorphism. After a moment of reflection one can convince oneself that this result was to be expected, as the Lie derivative describes roughly how a geometrical object changes along a vector field. If we want to consider the value of the metric tensor at a point as an observable of the theory, we need to relax the condition on observables. In this direction of argument, Pitts (2018) argues that one should consider as observables of a generally covariant theory all the geometrical objects for which their Poisson brackets with a diffeomorphism generator gives the Lie derivative along the displacement vector field that generates the diffeomorphism[52]. However, this pro-

[51] From the point of view of active transformations, spatial diffeomorphisms move things around the $D-1$ spatial slice.

[52] Pitts actually suggests this condition for every type of diffeomorphism, not just spatial ones.

posal risks being trivial, as it seems natural that if G generates diffeomorphisms, its action on any geometric object gives the Lie derivative of such an object. There are some subtleties involved when considering the action of G in quantities which involve canonical momenta, as their geometrical meaning is sometimes not so straightforward and the action of the G on them can be related with diffeomorphisms sometimes only on-shell. In any case, the point of Pitts is well-taken: the requirement of imposing vanishing Poisson brackets to observables seems too strong.

The fact that the Poisson bracket of objects like the metric tensor with the constraint does not vanish is a consequence of the nature of diffeomorphism. That is, before and after the diffeomorphism the values the metric field takes at a given coordinate point are different from each other. This is just a manifestation that in generally covariant theories it does not make sense to ask questions like 'what is the value of the scalar field at a given coordinate point?'. However, from an intuitive point of view, it does make sense to ask questions like 'what is the value of the field at a given spacetime point?'. This kind of question makes sense for instance if we consider a model of general relativity with a scalar field: in a generic spatial slice for generic initial conditions we will find a distribution of the scalar field in space and geometric observables like the metric or the curvature correlating with it. It this context it seems very natural to speak about the value of the scalar field or the curvature scalar at a spacetime point. For this reason, one would like to define observables in relation with spacetime points, not in relation with coordinate points. However, it seems that in the Hamiltonian formalism (as well as in the Lagrangian one) one lacks the resources for defining observables by relating them to spacetime points and not to coordinate points.

At this point it is worth mentioning that there is some controversy behind the concept of spacetime[53] point and how to define it. Relationalists[54] argue that a spacetime point is defined by the values that the fields take on it, and hence they claim that it does only make sense to speak about the value of a field at a point if one defines the point relative to other fields. For instance, if one has a 3-dimensional manifold, the proposal is to find 3 fields which individuate every point in the manifold and define observables relative to them. That is, find three fields such that there is not any pair of points in the manifold which agree on the value of the three of them and then define the rest of quantities in relation to them. According to this proposal observables are not objects like the value of some field at some given coordinate

[53] This controversy usually refers to the definition of spacetime points in models of general relativity, and, to be precise, above I was referring to the definition of a point in a time-slice. However, from a conceptual point of view these two debates are equivalent, as in both cases we are discussing how to individuate a point from a manifold. My discussion of general relativistic spacetimes in section 1.2 is also related to this.

[54] This kind of positions are argued for instance in Earman (2002); Rovelli (1991a,b); Oriti (2021).

point, but the value of the field when some other fields take some values. Intuitively, if we were able to construct observables in this way, they would be diffeomorphism invariant, as the value of a field at the point where others take some other values is independent of the coordinates one gives to that point.

However, there are several problems with this proposal. First, it may be the case that we have a manifold in which there does not exist any set of variables such that they individuate the points in the manifold. An example of this is just an homogeneous space-like slice of a Minkowski spacetime with no additional field defined on it. In this case the relationalist strategy fails, as in every point of this space the metric and other geometric fields are essentially the same and the relationalist strategy cannot define such fields as observables. Another problem for this approach is that it can work given a particular manifold, but we would be interested in defining relational observables for any manifold with any arbitrary[55] field configuration and not a definition which works just for a particular case. Moreover, if we were able to choose a set of variables which would act as coordinates, they would be treated in a different way from the rest of variables and, in a sense, they would stop being treated as dynamical variables. Finally, there are a number of results in general relativity, like the one in Torre (1992) which show that such relational observables are in general not available.

This discussion of relationalism will become important in my discussion of the quantum problem of time, where I will argue against strategies which rely on a relationalist view of generally covariant theories and which appeal in some way or another to a deparametrization of the theory. For now, the point I am making here is that in the case of a time-slice, the relationalist is unable to generate observables which have vanishing Poisson brackets with the constraints and that capture our intuitive idea of what an observable in a generally covariant theory is. For this reason we are in a position in which we have a good intuitive idea of the physical content of a generally covariant theory, namely the D-geometry and the fields and objects on top of it, but we are not in a position to express it in terms of mathematical quantities which have the same properties as the observables of other gauge theories. In this sense, the requirement of having vanishing Poisson brackets with the diffeomorphism generators seems too strong.

Let me also comment that there is a class of objects that very intuitively correspond with what one would call observables in a theory like general relativity. For instance, one can define the distance between two points along a certain curve $\mathcal{C}$ of a manifold:

$$D[g] = \int_{\mathcal{C}} \sqrt{g_{ab} dx^a dx^b} = \int_{\mathcal{C}} d\tau \sqrt{g_{ab} \frac{dx^a}{d\tau} \frac{dx^b}{d\tau}} \,. \tag{4.50}$$

[55] Up to the appropriate constraints.

In this expression the coordinates of the points of the curve are given by $x^\mu(\tau)$ and τ is the parameter describing the curve. This quantity is not invariant under the action of a diffeomorphism generated by G_t, as, due to the diffeomorphism, the metric has 'moved' and along the curve $\mathcal{C}$ we find the metric tensor which before was at some different set of points. However, this could be solved if we defined the diffeomorphism transformation in a way such that it also transformed the curve $\mathcal{C}$ to its image $\mathcal{C}'$ under the diffeomorphism. If we were able to do this we would have a diffeomorphism invariant definition of the length along a curve. Unfortunately, without the addition of extra structure it seems that we are unable to construct such observable using the Hamiltonian formalism. In chapter 5, when studying the formulation of general relativity that is used as a basis for LQG, we will find similar objects, the holonomies, and it will be important to study how they change under diffeomorphisms. In the same spirit as here, I will argue that this kind of objects should count as observables of the classical theory, even if they fail to be exactly invariant under transformations generated by G_t. I will argue that the affine information of general relativity can be represented by the set of all holonomies, juts as the metric information can be encoded by the distances along any curve. In this sense, I will argue that the result of the quantization of LQG is counterintuitive, as the result we will obtain is defined just on a finite number of curves and not in all of them.

With this I conclude this section. I have argued that in reparametrization invariant theories one has to be careful when interpreting the diffeomorphism invariance as a gauge transformation. A temporal reparametrization is not a gauge transformation from the instantaneous perspective, i.e., it is not a transformation which leaves the physical content of a system at a given time unchanged, and, on the contrary, the transformation related with temporal reparametrization is a physical evolution of the system. In this sense, I have rejected views like Earman's who argue that reparametrization invariance should make us change our view of time. Finally, I have argued that spatial diffeomorphisms need to be treated also carefully as, even if they are gauge transformations in the instantanous meaning of the term, the definition of observables is also tricky in this case. The conclusions in this section will be important for the quantization of reparametrization invariant systems, as I have showed that they are not exactly gauge systems and that they should not be treated as such. Before, I will introduce the specific case of general relativity and the way the discussion above applies.

4.3 General relativity as a reparametrization invariant theory

In this section I will finally analyze the case of general relativity. First, I will show how general relativity can be expressed in the Hamiltonian formulation and how it is a constrained system as we were expecting for a theory with diffeomorphism invariance. The Hamiltonian version of the theory I am using here is known as geometrodynamics, as it describes the evolution of a 3-dimensional geometry. Second, I will compare geometrodynamics with the examples in the previous section and I will argue that one should not read the configuration space of the theory as being some sort of extended configuration space in which time is encoded in the geometry of space. This will be important in section 4.7 as some resolutions of the problem of time rely on this sort of interpretation. Finally, I will take this and the arguments in the previous section to argue against the view that there is no genuine change in general relativity and that the reparametrization invariance of the theory requires us to reconsider the metaphysics of time associated with this picture. Defenders of this position, like Earman (2002), claim that there is a classical problem of time, in analogy with the problems we will find when quantizing general relativity in the second half of this chapter. The positions that I will argue for in this section will be very important for my assessment of the problem of time in section 4.7, as they will support my rejection of the resolutions which are based on the idea that time is encoded in the configuration space of general relativity.

4.3.1 General relativity in the Hamiltonian formulation

As we have seen in the previous section, we can express a spacetime theory in the Hamiltonian formalism by introducing a ADM decomposition of spacetime. This means limiting the models we will consider to models which are globally hyperbolic, i.e., with spacetime topology $\mathbb{R} \times \Sigma$, which leaves out some models compatible with general relativity. However, these models are usually regarded as unphysical, as they show features like closed timelike curves and their exclusion from Hamiltonian general relativity is usually not considered as problematic. The degrees of freedom of the theory (assuming no matter degrees of freedom for the moment) are the 3-metric g_{ab}, the lapse function N and the shift vector N^a, with which one can recover the metric for the full spacetime $g_{\mu\nu}$.

The dynamics of the theory is contained in Einstein equations:

$$G^{\mu\nu} = 0\,. \tag{4.51}$$

Where $G^{\mu\nu}$ is the Einstein tensor. Given that it is a symmetric tensor, we have 10 equations of motion (in four dimensions). Now, similarly to what happened in the case of electromagnetism, we find that not all of these 10 equations are dynamical equations. Indeed, there are 4 equations (the ones with at least one temporal component) which are constraint equations and not dynamical ones. The other 6 equations are good dynamical equations for g_{ab}. Therefore we find that N and N^a play the same role as a Lagrange multiplier, just as A^0 in electromagnetism and that the equations of motion do not determine them. In this way, the solutions of the equations of motion for g_{ab} carry a dependency on 4 arbitrary functions, provided that the constraint equations are satisfied initially. This dependency is of course associated with the reparametrization invariance of the theory: solutions for different assignments of N and N^a correspond to diffeomorphism related spacetime metrics $g_{\mu\nu}$. In this sense, we find that general relativity is a gauge theory in the global sense, just as one would expect from the discussion in the previous section.

Furthermore, general relativity (at least for vacuum and globally hyperbolic spacetimes) is a deterministic theory, as was proven in Choquet-Bruhat and Geroch (1969). For understanding the theorem by Choquet-Bruhat and Geroch we need to introduce the concept of extrinsic curvature. Given a spatial hypersurface, as any leaf of a foliation of spacetime, the extrinsic curvature is defined as the Lie derivative of the metric in the surface along the direction normal to the surface:

$$K_{ab} = \mathcal{L}_n g_{ab} \,. \tag{4.52}$$

Intuitively, this object gives an idea of the way the spatial metric changes in the direction normal to the surface, or, equivalently, to the way the hypersurface is embedded in the four dimensional spacetime. Notice that this definition is independent of the coordinates used for coordinatizing both the 3-surface and the spacetime. and K_{ab} is well-defined as a tensor on the tangent space of the 3-surface[56]. K_{ab} can be therefore seen as a velocity for g_{ab} with a nice geometrical interpretation. Furthermore, the 4 constraint equations of general relativity can be expressed as relations between the extrinsic and intrinsic curvature of a surface:

$$K_{ab}K^{ab} - K^2 + R[g] = 0 \tag{4.53}$$

$$\nabla_b K^b_a - \nabla_a K^b_b = 0 \,, \tag{4.54}$$

where ∇_b is the 3-dimensional covariant derivative and $R[g]$ the 3-dimensional Ricci scalar. Now, the theorem proves that if one specifies a 3-metric g_{ab} and a extrinsic curvature K_{ab} which satisfy the constraints on a 3-dimensional manifold Σ, then there exists one unique (up to diffeomorphism) spacetime which is a solution of the Einstein equations of motion, which is globally hyperbolic with Cauchy surface Σ, which has g_{ab} and K_{ab} as induced metric and extrinsic curvature on Σ and which is maximal, in the sense that any other spacetime satisfying the other conditions can be isometrically mapped into it. In this sense g_{ab} and K_{ab} at Σ, together with the equations of motion uniquely determine the geometry of the full spacetime and one

[56] For completeness, the definition in terms of coordinates, lapse and shift vector is $K_{ab} = \frac{1}{2N}\left(-\dot{g}_{ab} + \mathcal{L}_{\vec{N}} g_{ab}\right)$, where the dot represents derivative with respect to the parameter t and the Lie derivative is with respect to the shift vector. Using a different time parametrization does not change K_{ab}, and under a different coordinatization of 3-space K_{ab} transforms as a tensor.

can see general relativity as a deterministic gauge theory where the gauge redundancy lies in the different ways of coordinatizing the 4-dimensional spacetime.

Now we can express this in the Hamiltonian formalism. In these variables the Einstein-Hilbert action of general relativity takes the form:

$$S[g_{ab}, N, N^a] = \int dt \int_\Sigma d^3x N |g|^{1/2} \left(K_{ab}K^{ab} - K^2 + R[g] \right) . \qquad (4.55)$$

As we were expecting, the momenta conjugate to g_{ab} are basically functions of the extrinsic curvature:

$$\pi^{ab} = \frac{\delta S}{\delta \dot{g}_{ab}} = -|g|^{1/2} \left(K^{ab} - g^{ab}K \right) . \qquad (4.56)$$

We also find that the momenta conjugate to N and N^a vanish, i.e., they are primary constraints:

$$P_\mu = \frac{\delta S}{\delta \dot{N}^\mu} = 0 \, . \qquad (4.57)$$

Having introduced the momenta, we find the canonical Hamiltonian of general relativity to be of the ADM form:

$$H_c = \int_\Sigma d^3x \left(N\mathcal{H}_0 + N^a \mathcal{H}_a \right) . \qquad (4.58)$$

Where $\mathcal{H}_0$ and $\mathcal{H}_a$ are secondary constraints, the Hamiltonian and momentum constraints, which are just the constraints of general relativity (the four non-dynamical equations of the 10 Einstein equations) expressed in terms of the canonical variables:

$$\mathcal{H}_0 = G_{abcd}\pi^{ab}\pi^{cd} - |g|^{1/2}R[g] \qquad (4.59)$$

$$\mathcal{H}_a = -2\nabla_b \pi_a^b \, . \qquad (4.60)$$

Notice again the similarity with electromagnetism: in both cases we have some variables, A^0 or N^μ, which act as Lagrange multipliers and the associated secondary constraints are not an artifact of the Hamiltonian formalism but instead represent constraints which were present in the original theory. In the expression of the Hamiltonian constraint I have introduced the supermetric G:

$$G_{abcd} = \frac{|g|^{1/2}}{2} \left(g_{ac}g_{bd} + g_{bc}g_{ad} - g_{ab}g_{cd} \right) . \qquad (4.61)$$

The supermetric can be seen as playing the role of a metric in the space of 3-metrics, also known as superspace. The formulation of general relativity in these terms is also known as the geometrodynamics, as it describes how a 3-dimensional geometry evolves in time.

The total Hamiltonian of the theory takes the form:

$$H_T = \int_\Sigma d^3x \left(N^\mu \mathcal{H}_\mu + \lambda^\mu P_\mu \right) \,. \tag{4.62}$$

Where λ^μ again are (quasi) arbitrary functions which represent the gauge freedom of the theory: different choices of λ^μ correspond to different choices for the lapse function and shift vector[57]. This can be seen by noticing that the Hamiltonian equations for the lapse function and shift vector are just $\dot{N}^\mu = \lambda^\mu$.

Let me study the other equations of motion of the system. The equations for $\dot{g}_{ab}$ just give back the relation between g_{ab} and K^{ab}, while the ten remaining equations are the ten Einstein equations. The equations obtained by varying with respect to N^μ are the four constraint equations, while the equations obtain by varying with respect to g_{ab} are the six dynamical equations. In this way we obtain the theory of general relativity we have started with in a way which makes explicit the gauge freedom of the theory.

The analysis performed in the last section about the gauge generators and observables of the theory for a generic ADM decomposition of any spacetime theory still applies to this case. Namely, we find the generators of spatial diffeomorphisms G_t, which can be seen as generating transformations which act at instantaneous states of affairs, i.e., they just transform a 3-geometry into a diffeomorphism related with one and similarly for the extrinsic curvature or momenta π^{ab}. The transformations generated by a generator which contains G_0 involve time evolution, and hence they are not gauge transformations of the instantaneous kind. Again, defining the observables to be the phase space quantities which have vanishing Poisson brackets with the generators or with the constraints goes against our intuitions of what is observable in general relativity and the examples in the previous section showed that this is too restrictive a condition.

Now we can turn to the role of N and N^a. As in the case of electromagnetism, if one uses the extended Hamiltonian formalism the effect of N and N^a in the dynamics can be reabsorbed in the effect of some of the functions v^μ used in this formalism. As happened in the case of electromagnetism, using the extended formalism breaks the link with the original Lagrangian theory. In this case if we take the spacetime metric $g_{\mu\nu}$ taken by combining g_{ab} with N and N^a the resulting spacetime metric will not satisfy Einstein equations, just as using the extended formalism in electromagnetism

[57] The theorem cited above ensures us that given initial conditions g_{ab} and K^{ab} at a given hypersurface Σ there exist a unique spacetime $g_{\mu\nu}$ satisfying Einstein equations. However, the theorem does not tell which choices of N and N^a are appropriate and there are circumstances in which not every choice will be possible. For instance, in (Andersson, 2004, Sect. 2.2) it is commented that even in the case of a foliation of a Minkowski spacetime some choices of N and N^a lead to foliations with singularities.

broke Gauss law[58]. In the case of electromagnetism one could try to argue that the theory in the extended formalism still contained electromagnetism, even if the link with the original theory was broken. For this, one needed to identify $\vec{\pi}$ and not $-\partial_t \vec{A} - \vec{\nabla} A^0$ with the electric field. In the case of general relativity there may be a similar strategy available.

This strategy is based in the thin sandwich conjecture, first formulated in Baierlein et al. (1962). This conjecture states that given a metric g_{ab} on a spacelike surface and its time derivative $\dot{g}_{ab}$ the constraint equations of general relativity determine the lapse function and shift vector. If this holds, given a solution $g_{ab}(t)$ one would be able to reconstruct $N(t)$ and $N^a(t)$ and hence $g_{\mu\nu}$. In the case one uses the extended Hamiltonian formulation, the lapse function and shift vector obtained from the constraint equations will be different from the ones derived from the extended Hamiltonian equations of motion. This was to be expected given that the extended formalism is not equivalent to the original theory. To recover the full spacetime metric one can use the lapse and shift obtained from the constraint equation. In this way, even in the extended formalism one is able to recover the physically meaningful g_{ab} for the cases where the thin sandwich conjecture hold. It has been proven that under certain circumstances the conjecture holds but also that there are some others in which it does not[59]. In this sense, the above strategy might be only of limited scope.

In any case, notice that for recovering the full $g_{\mu\nu}$ we need to use a lapse function and shift vector. In this sense, even if these quantities are gauge dependent, they nevertheless contain physically relevant information from the 4-dimensional point of view. In the quantization of general relativity in section 4.7 we will see that the lapse function and shift vector will not play a role and one could raise the worry of how the physical information they encode is to be recovered.

With this I conclude the discussion of geometrodynamics in the Hamiltonian formalism. This formulation of general relativity will be our starting point for the philosophical discussion in this section and also for its quantization in the section 4.7. However, let me mention that there are other formulations of general relativity based on ADM-like decompositions of spacetime which share the same fundamental structure, and hence that the discussion in this chapter equally applies to them. Examples of these other Hamiltonian formulations of general relativity are the BSW formalism studied in Barbour et al. (2002) and the connection formulation which is the basis for the LQG that I will study in the next chapter. In the rest of this section I will argue for two claims: that general relativity is similar to our third example in the previous section in that its configuration space is not an extended configuration space and that the claims that the Hamiltonian formulation of general relativity should make

[58] Again, see Pitts (2014a).

[59] See Bartnik and Fodor (1993) and Avalos et al. (2017).

us change our metaphysical picture of time in this theory are misguided.

4.3.2 Spatial geometry does not carry information about time

For my analysis of the problem of time in section 4.7 it will be important to distinguish different kinds of reparametrization invariant theories. I will argue there that some of the proposed resolutions of the problem of time only work for deparametrizable models and that they do not apply to general relativity as it is not deparametrizable. Here I will argue that general relativity is similar to the third example in section 4.2, that it is not deparametrizable, and that neither its configuration space nor its phase space can be considered to be carrying information about time in any meaningful way.

Let me start by comparing the configuration space of general relativity with the configuration spaces of the examples in section 4.2. In the first two examples, i.e., the two models for the relativistic particle, the configuration space is an extended configuration space, that is, there explicitly appears a time variable as one of the variables in this space. In the double harmonic oscillator case the configuration space is not an extended one as both variables in it have a physical interpretation as the positions of the two oscillators. In the case of general relativity we have seen that the configuration variables have a physical meaning, namely defining a 3-geometry, and they do not seem to be directly time. In this sense, general relativity seems more similar to the third example.

When we analyze the way temporal information is recovered in the three examples and in general relativity, we also reach the conclusion that general relativity is similar to the double harmonic oscillator case. The time coordinate describing the foliation in general relativity does not have any metrical meaning, it just contains ordering information, i.e., leaves of the foliation with a bigger parameter go after the leaves with a smaller one. In this, it is perfectly analogous to the parameter τ in the three examples I have studied in section 4.2. However, when we want to recover metrical time information, that is, information not about which event goes before another, but information about how much time has elapsed between the two events, what we find is that general relativity is similar to the third example and not to the deparametrizable models. In the case of the two harmonic oscillators time was not a variable in the configuration space and the dynamics was independent of the way we chose to label time, but there was a preferred time we could define which was no other than the Newtonian time. This time was defined by:

$$dt = \sqrt{\frac{m(\dot{x}^2 + \dot{y}^2)}{2E - k_x x^2 - k_y y^2}} d\tau \,. \tag{4.63}$$

This definition can of course be generalized to any Newtonian system, as shown by

Barbour[60]. In general relativity we seem to be in the same situation as the proper time between two events is defined infinitesimally as:

$$ds^2 = -g_{\mu\nu}dx^\mu dx^\nu = N^2 dt^2 - g_{ab}(dx^a + N^a dt)(dx^b + N^b dt)\,. \tag{4.64}$$

Proper time in general relativity and Newtonian time in the example in the previous section are analogous according to Barbour. In both cases we have well-defined dynamical theories which determine a set of physical events with precise ordering relations[61] and, even if we can define a preferred metric time in both cases, it is not necessary to do so[62]. In this sense, Barbour considers Newtonian time in Newtonian physics and proper time in general relativity to be convenient ways of treating time that may simplify our calculations, but he rejects that they have any further meaning as the 'true' scale of time.

In both cases we can define an ideal clock as a system which directly correlates its physical state with the metric time: in the Newtonian case the reading of an ideal clock gives the absolute time, while in the general relativistic case an ideal clock shows the proper time elapsed along its worldline. Of course, real clocks are not ideal and there may not exist any real physical system which ever measures metric time. In any case, it should be clear that in both models metric time and clocks are defined in analogous ways. Therefore, there is a good case supporting that time in general relativity is represented in a similar way to the one chosen by Barbour and not to the one in the deparametrizable models. Notice that this conclusion holds independently of our philosophical position regarding absolute time scales.

As I commented before, general relativity does not look like a deparametrizable theory, as all of these phase space variables have their own physical interpretation and do not seem to represent a time variable. In particular, if we incorporate matter fields, their interpretation is clear, and given that we can consider general relativity for different kinds of matter fields and even for no matter fields at all, if general relativity were a deparametrizable theory it seems that the most natural place to look for spacetime variables is in the geometric degrees of freedom, that is, in g_{ab} and π^{ab}. A way of formulating this is the following. We would be looking for a canonical transformation of the form:

$$g_{ab}, \pi^{ab} \to X^\mu, p_\mu, \phi^A, \pi_A\,. \tag{4.65}$$

[60] I refer the reader to Barbour (1994a) for a discussion of Newtonian systems using Jacobi's principle.

[61] These ordering relations are different in both cases. In the Newtonian case the ordering relation is given by the absolute time, while in the general relativistic case events are partially ordered by the causal structure of spacetime.

[62] In the dynamical approach to general relativity and in the kind of spacetime functionalism defended by Knox that I introduced in section 3.1 the chronogeometric significance of the metric tensor is only a product of the way the metric field couples with the other fields in the model.

That is, we are looking for a transformation that separates the 12 phase space variables (6 components of the metric and 6 conjugate momenta) into two groups. In the first group we would have 4 coordinates X^μ that would be able to identify any spacetime point and 4 momenta p_μ conjugate to them. And the second group would represent the true dynamical degrees of freedom of general relativity, which would be contained in two fields ϕ^A and conjugate momenta π_A. This would be very attractive, as it would allow us to express general relativity in terms of the physical coordinates X^μ, it would give us a clear picture of the physical content of general relativity and would be helpful for the quantization of the theory. Moreover, according to this canonical transformation there would be only two true gravitational degrees of freedom ϕ^A, in accordance with other arguments from general relativity that have led physicists to believe that such is the number of degrees of freedom of general relativity. For instance, it is a well known fact that there are two possible polarization states for gravitational waves.

However attractive this proposal may sound there are two problems with it. First, at a conceptual level we have seen that the variables of general relativity have a physical meaning on their own and none seems to be encoding time. Therefore, it does not look possible that one could build spacetime coordinates from them. Consider again the analogy with the model of the two harmonic oscillators: it seems quite obvious that in the configuration space of the two harmonic oscillators we have just the description of the two oscillators and nothing else. Similarly, in the configuration space of general relativity what we have is the description of a three-geometry (and maybe some matter) and nothing else. Second, there is a major technical difficulty: Torre (1992) showed that the constrained phase space of general relativity cannot be identified with the constrained phase space structure of a deparametrizable model[63], and hence it cannot be treated as such.

This analysis already shows that general relativity is not a deparametrizable theory. Nevertheless, in the quantum gravity literature[64] one can find claims that time is hidden in the configuration space of the theory. In this way, the hope is that general relativity could be something in between the two types of examples above: a non-deparametrizable theory but which still has information about time in its configuration space. This position could be used for supporting certain resolutions of the problem of time, as I will comment in section 4.7. However, the way time is supposed to be hidden in general relativity is unclear and I will now argue against the three

[63] For a deparametrizable model, if we perform a transformation like 4.65, we find that the constraints take the form $p_\mu + h_\mu(X^\mu, \phi^A, \pi_A)$ for some functions h_μ. This can be shown to imply that the constrained space is a manifold, i.e., that it satisfies some properties like being smooth. Torre showed that the constrained space in the case of general relativity is not smooth, and hence that the constraint spaces cannot be identified.

[64] See for instance the quotations below from Kiefer (2012).

arguments most commonly used for supporting that claim.

The first argument comes from the thick sandwich conjecture of general relativity, first stated in the 1962 paper "Three-dimensional Geometry as Carrier of Information about Time" (Baierlein et al., 1962). In this paper it is argued that given the thick sandwich conjecture, one can see three-geometry as carrying some information about time. The conjecture can be formulated as: given an initial and a final 3-geometry the equations of motion of general relativity uniquely determine the full 4-dimensional geometry in between, without needing to specify any information about the time elapsed between the initial and final states. However, even if the conjecture turned out to be true, this does not mean that the 3-geometry carries information about time.

An argument for this is that an analogous version of the sandwich conjecture holds for the example of the two-harmonic oscillators, and in this case we would not claim that the positions of the oscillators carry information about time. We can formulate the conjecture for the harmonic oscillators in a similar way: given the initial and the final positions of the two oscillators, the equations of motion derived from action 4.28 determine the time elapsed between such an initial and a final configurations. In this case, the conjecture holds to some extent: given an initial and a final configuration one can determine trajectories in configuration space which solve the equation of motion. However, these are not necessarily unique, as, depending on the values of the initial and final states and of the couplings of the oscillators, there may be more than one trajectory in configuration space which obeys the equations of motion and satisfies the initial and final condition. Notice that this can be seen as an alternative version of classical mechanics where instead of specifying an initial and a final configuration and the time elapsed between them one specifies the configuration and the energy of the system, as it appears explicitly in the action 4.28. Therefore, there is a very natural sense in which we can say that temporal information is encoded in the energy and in the equations of motion and not in the configuration space. Indeed, the configuration space in this example is identical to the configuration space of the Newtonian formulation of this model, and one would not claim that a Newtonian configuration space carries temporal information. The same can be applied to the case of general relativity to argue that 3-geometries are just 3-geometries, and that it is the Einstein equations which would determine the temporal information, were the conjecture to be true.

The other argument supporting that time is somehow included in the configuration space of general relativity comes from some counting of degrees of freedom. This kind of argument performs an analysis of either the configuration or phase space of general relativity, finds that there are more degrees of freedom than the two physical degrees of freedom that there are believed to be in general relativity and argues that the difference has to be in the temporal information that 3-geometries are supposed to carry. For instance, Kiefer (2012) provides two ways of reaching this conclusion. First, in configuration space

> The three-metric h_{ab}[g_{ab} in my notation] is characterized by six numbers per space point (often symbolically denoted as $6 \times \infty^3$). The diffeomorphism constraints (4.70) [Momentum constraints 4.60] generate coordinate transformations on three-space. These are characterized by three numbers, so $6 - 3 = 3$ numbers per point remain. The constraint (4.69) [Hamiltonian constraint 4.59] corresponds to one variable per space point describing the location of Σ in space-time (since Σ changes under normal deformations). In a sense, this one variable therefore corresponds to 'time', and $2 \times \infty^3$ degrees of freedom remain. (Kiefer, 2012, p. 114)

Second, in phase space:

> [...]the canonical variables $(h_{ab}(x), p^{cd}(y))$ are $12 \times \infty^3$ variables. Due to the presence of the four constraints in phase space, $4 \times \infty^3$ variables have to be subtracted. The remaining $8 \times \infty^3$ variables define the constraint hypersurface Γ_c. Since the constraints generate a four-parameter set of gauge transformations on Γ_c (see Section 3.1.2), $4 \times \infty^3$ degrees of freedom must be subtracted in order to 'fix the gauge'. The remaining $4 \times \infty^3$ variables define the reduced phase space Γ_r and correspond to $2 \times \infty^3$ degrees of freedom in configuration space —in accordance with the counting above. (Kiefer, 2012, pp. 114-115)

This kind of reasoning works well for gauge theories like electromagnetism. However, there are reasons to doubt that it also applies to general relativity. In the first place, I have argued above that reparametrization invariance cannot be treated exactly as a gauge theory, as a reparametrization is a gauge transformation from the global point of view, i.e., it transforms solutions of the equations of motion to physically equivalent solutions of the equations of motion, but not from the instantaneous point of view: given two reparametrization-equivalent models, the physical state at the instant represented by a given parameter time is in general different for both cases. Second, to speak about the degrees of freedom at a spacetime point is tricky in the case of a generally covariant theory: while in the case of a theory like electromagnetism for specifying a spacetime point it is enough with giving the coordinate point, in the case of general relativity to speak about the degrees of freedom at spacetime point is harder because the transformations we care about move things around, i.e., change the coordinate points where physical events happen and also because the concept of spacetime point is harder to define.

To see that the degree of freedom counting is unreliable, we can try to apply it to the examples in the previous sections. The three cases are formally analogous: we have configuration spaces with two variables x, t or x, y, both are described by homogeneous Lagrangians and in both cases evolution and gauge are generated by the Hamiltonian constraint. We can count the degrees of freedom applying Kiefer's method. In configuration space we get that there supposedly is 1 degree of freedom in both theories, that is 2 degrees of freedom - 1 gauge transformation. Similarly, in phase space we start with 4 variables and we have to subtract 1 constraint and 1 gauge transformation, giving the two phase space variables associated with one degree of freedom. For the case of the relativistic particle we are satisfied with the outcome as the system we are describing has one degree of freedom, namely the position of the particle.

However, for the case of the double harmonic oscillator the result of the counting seems wrong. The system we are describing is a system formed by two oscillators, and intuitively it has two physical degrees of freedom. The conclusion of the counting of degrees of freedom argument for this case would lead us to think that of the two oscillators one would be something like a time and the other one the physical degree of freedom. Even if we can use one of the oscillators as a clock, that is, as a device to keep track of time, it does not make sense to consider it as a time and the other as something like the real physical system. One could be more sophisticated and argue that time is not directly one of the configuration variables but some combination of them or that the way time is encoded in the configuration variables is not so straightforward. However, as I have been arguing this does not seem very plausible and it seems more reasonable to state that the configuration variables are just configuration variables and that the only way in which they carry information about time is by means of the equations of motion. Therefore, we should reject the degree of freedom counting argument for this case.

This shows that one should be careful when applying this sort of reasoning. Moreover, given the strong analogy between the double harmonic oscillator example and general relativity I have been arguing for, we have a good case for rejecting the conclusion from Kiefer's arguments that in general relativity time is encoded in three-geometry.

Finally, the third kind of argument which is used in the literature for claiming that time is part of the configuration or phase space of general relativity is by making more or less explicit proposals for this identification[65]. In Kuchař (1992) some of this proposals are explored and found to have some conceptual problems. In particular, let me mention one particularly strong proposal, which is suggested by the form of the Hamiltonian constraint 4.59. The first term in this expression, $G_{abcd}\pi^{ab}\pi^{cd}$ is

[65] I am thankful to Brian Pitts for pointing out this kind of argument to me.

known as the kinetic term and it is formally analogous to the kinetic term one finds in the quantization of a relativistic particle, $\eta_{\mu\nu}p^\mu p^\nu$. In this case, the momenta p^0 are conjugate to the time variable, and one can characterize them because they are time-like with respect to the Minkowski metric η, i.e., they satisfy $\eta_{\mu\nu}p^\mu p^\nu < 0$. In the case of geometrodynamics we find that the supermetric G_{abcd}, as defined in 4.61, is hyperbolic with signature $\{-,+,+,+,+,+\}$. That is, the supermetric is analogous to the Minkowski metric in that it defines a 'time-like' direction in superspace, the space of 3-metrics g_{ab}. This suggests that time is encoded in this 'time-like' direction, just as in the case of the relativitic particle time is 'marked' in the constraints by a negative sign[66].

However, there are several reasons to resist this kind of argument. First, the supermetric is a metric in an abstract infinite-dimensional space and its relation, if there is to be one, with the metric of spacetime is unclear. Kuchař (1992) points this out, and he argues that even for Euclidean spacetimes, i.e., even for spacetimes with no time-like direction, the supermetric would still have the same hyperbolic signature. In this sense, the fact that the supermetric has a 'time-like' direction seems to be just a consequence of the form of the dynamics and not related with the structure of spacetime. Second, the time-like direction of the supermetric is associated with conformal transformations, i.e., with transformations which just (locally) expand or contract space. However, Kuchar also argues that identifying something like a local volume element with the time variable is problematic, as we can conceive of spacetimes that in their evolution expand, contract or remain at a fixed volume. In this sense, from a conceptual point of view the identification of some component or function of the metric tensor with time remains problematic.

Let me also complement those arguments with a comparison with the double harmonic oscillator model. In this model we do not have a kinetic term with negative sign but we can introduce a change to the model to introduce a negative sign. The action would now become:

$$S[x,y] = 2\int d\tau \sqrt{\frac{m}{2}\left(-\dot{x}^2+\dot{y}^2\right)\left(E-\frac{1}{2}(k_x x^2+k_y y^2)\right)}\,. \tag{4.66}$$

The equations of motion for such an action do not represent Newtonian trajectories, and they could be considered unphysical. Despite this, the interpretation of the model remains the same: it represents the evolution of two degrees of freedom with respect to a parameter τ. Now we have an asymmetry between x and y, and we could take the minus sign in the kinetic term for $\dot{x}$ to signal that x has become time. However, this would be wrong, as just changing the form of the dynamics does not change

[66] An example of this kind of reasoning can be found in Teitelboim (1982). For further examples I refer the reader to the cases cited in Kuchař (1992).

the interpretation we make of the variables and what they represent. Moreover, the dynamics of x still allow for x to behave in a not monotonically way: even if solutions to the equations of motion do not correspond with an harmonic oscillator any more, some solutions still describe trajectories with velocities which change sign. This example shows that a negative sign in the action[67] is not necessarily a sign that some variable represents time, as this negative sign can arise naturally in some non-Newtonian models, and it does not imply that that variable will behave monotonically. In this sense, in general relativity it seems plausible that the negative signs that arise are just a consequence of the form of the dynamics and it may be wrong to import our intuitions and interpretations from other models. Furthermore, as I have argued above, to argue that something like a local volume element is time one would need to show that it behaves monotonically. And even in that case, one could still argue against this identification, as from an intuitive point of view this volume element could be argued to be a configuration variable with a physical meaning and not time.

With this I conclude this subsection, where I have argued that general relativity is not reparametrizable and that the arguments supporting that time is encoded in the configuration or phase space of the theory are misleading, as, among other reasons, they would lead to wrong conclusions for the case of the double harmonic oscillator. In this sense, the arguments in this section show the strength of the analogy between general relativity and the double harmonic oscillator example. In the section 4.7 I will comment on the consequences this has for the quantization of general relativity, namely, that this gives strong reasons to think that the problem of time cannot be solved in the ways usually proposed in the literature.

4.3.3 The classical problem of time

Before going to the quantization of general relativity, I will briefly discuss the philosophical debate about the status of time and change in this theory, as motivated by its expression in the Hamiltonian formalism and what has been called the classical problem of time, in analogy with the quantum problem of time, or just the problem of time, which I will study in detail in section 4.7. In the previous section I argued against the positions that there is no change in reparametrization invariant theories and that one should change the view on time after studying the relationship between dynamics and reparametrization invariance in theories like general relativity. For the particular case of general relativity I will defend the same position.

That there is no change in general relativity was first defended in the philosophical literature by Earman (2002, 2006), who was followed by others[68]. The facts that are

[67] This negative sign is also translated to a negative sign in the Hamiltonian constraint: $H_0 = -\frac{1}{2m}p_x^2 + \frac{1}{2m}p_y^2 + \frac{k_x}{2}x^2 + \frac{k_y}{2}y^2 - E$

[68] See Belot (2007); Belot and Earman (2009); Rickles (2006); Huggett et al. (2013).

claimed to support this claim are:

- The Hamiltonian of general relativity is made of constraints, and therefore its value is 0 on shell.
- Dirac observables, quantities f that satisfy $\{f, G\} = 0$ are also constants of motion.
- In the space resulting from quotienting phase space by the gauge orbits the Hamiltonian generates no evolution.

In the previous section I hope to have shown that none of these three is really problematic once we understand the reparametrization symmetry of the theory. First, our examples showed how we can have vanishing Hamiltonians and yet evolution (both in the meaningless coordinates τ or x^μ or of some variables relative to others, as $x(t)$). Second, as the gauge transformation is a diffeomorphism transformation we cannot say that it transforms (locally) states into physically equivalent ones, we have good grounds to relax the requirement of observables having vanishing Poisson brackets with the gauge generators[69]. Moreover, the example of the harmonic oscillators shows that there could be cases of quantities with non-vanishing Poisson brackets with the constraints that we may consider observables. And third, I have argued that different time slices in a gauge orbit are not physically equivalent, and therefore the only sensible interpretation we can make of the reduced phase space is that each point represents a full spacetime and not an instantaneous state.

Therefore, we can argue that there is change in the Hamiltonian formulation of general relativity. This position has also been defended by Maudlin (2002), who directly replies to Earman, and further argued for by others[70]. That there is change in the standard geometrical formulation of general relativity seems quite obvious, and a way it can be argued is by showing that when there are no time-like Killing vectors there is change in any possible parametrization of the theory[71]. Moreover, the formulation of general relativity using differential geometry is explicitly coordinate free and deterministic. If there is no problem with other formulations of general relativity why should we find any when formulating the theory making use of the Hamiltonian formalism?

A possible answer to this is to deny that the Hamiltonian version of general relativity is equivalent to other formulations like the geometric one or the Lagrangian

[69] A similar point was made in Kuchař (1993), where it was argued that 'perennials', i.e., phase space functions with vanishing Poisson brackets with all the constraints are not the only dynamical variables that can be observed.

[70] See (Healey, 2002, 2004; Pons, 2005; Pitts, 2014b, 2017).

[71] This line of argument can be found in Pitts (2014b).

one. Although it is a possible road to go, it seems desirable to maintain the equivalence. Indeed, Earman repeatedly states that his argument is not a mere formal trick of the Hamiltonian formalism and he supports his claim with an argument using standard general relativity. This argument is a variation of the hole argument and Maudlin answer to it seems right: if the hole argument did not convince you of the indeterminism of the theory, then Earman's argument should not convince you that the observables of general relativity do not change.

The defenders of change in general relativity defend the equivalence of the formalisms and on these grounds reject that observables have to have vanishing Poisson brackets with the gauge generator and the full quotienting procedure. In his brief reply in (Maudlin, 2002), Earman accuses Maudlin of 'cherry-picking', that is, to use and interpret the formalism at his convenience rather than in a systematic way. But it is not the case that one has to make an exception for general relativity, what is going on is that there is a class of gauge theories, reparametrization invariant theories, that cannot be treated in the same way as other gauge theories with symmetries that leave the coordinates unchanged. In other words, reparametrization invariant theories and general relativity have to be treated differently because they are not gauge theories from the instantaneous point of view. Other defenders of change like Pitts go one step further and try to get a fix on the formalism so that it can be used for the theory of general relativity and for other generally covariant theories without seeming that one is cherry-picking. For instance, this takes him to look for a new formal definition of observables, as commented in the previous section.

With this I conclude the discussion of classical general relativity and the classical problem of time. I have argued that there is no classical problem of time: once we understand the Hamiltonian formalism any impression of frozen dynamics disappears. In the next half of the chapter I will analyze the canonical quantization process in general and for general relativity and study the quantum version of the problem of time, which I will argue which is a serious problem which should make us consider rejecting canonical approaches to quantum gravity.

4.4 Canonical quantization

In this second half of the chapter I will introduce the canonical quantization program that allows one to define quantum theories starting from theories formulated in the Hamiltonian formalism and I will analyze the serious conceptual problems it leads to when applied to the case of general relativity. I will start in this section by introducing the method in generality for non-gauge systems.

The starting point in the canonical quantization program is a classical theory formulated in the Hamiltonian formalism[72]. We can split the theory in two parts, the kinematics and the dynamics. The kinematics of the theory is given by specifying the corresponding phase space, which is a manifold with a symplectic structure which defines the Poisson brackets of the theory. The Hamiltonian function H defines the dynamics of the theory by means of the Hamilton equations, which for any phase space function can be expressed as:

$$\dot{f}(q,p,t) = \{f(q,p,t), H\} + \frac{\partial f(q,p,t)}{\partial t}\,, \tag{4.67}$$

where I am again using q, p for referring to any set of phase space variables. The canonical quantization of such a theory can also be divided in a kinematics and a dynamics.

First, the kinematic structure of a quantum theory is contained in its algebra of observables, $\mathcal{A}(\mathcal{O})$, which are represented as linear, self-adjoint operators acting on a Hilbert space $\mathcal{H}$. These operators are called observables because they allegedly correspond to physically measurable quantities. For instance, in the case of the quantum mechanics of a single particle, the observables are taken to be the position $\hat{q}_i$ and the momentum coordinates $\hat{p}_i$. As linear operators, they satisfy a certain commutator algebra which mimics the Poisson bracket algebra of the classical theory:

$$[\hat{q}_i, \hat{q}_j] = 0, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}\,. \tag{4.68}$$

The commutation relations play an important role in the dynamics as we will later see, and they also have an important operational meaning: two non-commuting observables cannot be measured simultaneously. A state in classical mechanics is given by specifying a point in the phase space, which uniquely determines the value of all the observable quantities at a time. In quantum mechanics a state is a ray, a unit vector $|\psi\rangle$ on the Hilbert space[73] and it does not specify the values of all the observables. Instead, a ray assigns probabilities for the different possible values that can

[72] Let me mention that any classical theory can be expressed in the Hamiltonian formalism, although they are more naturally represented in the Lagrangian one, as argued for in Curiel (2014). Indeed, the distinction between configuration variables and momenta is not natural from the phase space point of view and this is translated into the quantum theory in that every phase space variable is treated as an operator. However, the physical meaning of momentum variables is derivative from the configuration ones and hence it is unclear how they can be 'measured' or 'observed' if not by measuring how configurations change. Indeed, the Lagrangian way of thinking plays a role in many approaches and interpretations of quantum mechanics, as even if quantum theories are defined on abstract Hilbert spaces, the preferred way of representing states in these spaces is by means of wavefunctions on configuration space. This is the case for covariant approaches to quantum mechanics that I will introduce in chapter 7 and for the Bohmian approach, in which one has an ontology defined on the same configuration space as the one used in the classical theory.

[73] This can be generalized to include density matrices, but this will not be important for our discussion.

be measured and it determines expectation values for all the observables by means of $\langle\psi|\hat{O}|\psi\rangle$.

Ideally, quantizing the kinematic part of a classical theory would therefore imply determining a Hilbert space and a set of operators on it such that one could associate one operator for every classical observable, i.e., for every function on the classical phase space:

$$Q : \mathcal{F}(\mathcal{P}) \quad \to \mathcal{L}(\mathcal{H}) \tag{4.69}$$
$$f(q,p) \quad \to \hat{f}\,. \tag{4.70}$$

The algebra that the observables have to obey would be determined by the Poisson bracket algebra of the classical theory[74]:

$$[\hat{f},\hat{g}] = i\hbar\widehat{\{f,g\}}\,. \tag{4.71}$$

However, there does not exist any quantization map Q able to satisfy this condition for every pair of functions on phase space, as was proved by Groenewold (1946). Therefore, the quantization map has to be limited to a given subalgebra of observables and it cannot be rigorously defined for every phase space function. For instance, in the case of the single particle one maps q_i and p_i to $\hat{q}_i$ and $\hat{p}_i$ which satisfy 4.68, but for more complicated functions, like monomials of q_i and p_i, there is not a unique quantum operator which corresponds to them. Indeed, if we consider the function $q_1^2p_1^2$ all the operators $\hat{q}_1^2\hat{p}_1^2, \hat{p}_1^2\hat{q}_1^2, \hat{q}_1\hat{p}_1\hat{q}_1\hat{p}_1, \hat{p}_1\hat{q}_1\hat{p}_1\hat{q}_1, \hat{p}_1\hat{q}_1^2\hat{p}_1$ and $\hat{q}_1\hat{p}_1^2\hat{q}_1$ could potentially be the quantum counterpart of it, and there is no reason for preferring any of them, as Groenewold showed that none of them is part of a quantization which satisfies 4.71 for any arbitrary pair of monomials. In this sense, we find that there is an ordering ambiguity in passing from a classical function to a quantum operator. This ambiguity will be important at the time of defining the dynamics of the theory.

The first choice one has to make for defining a quantum theory starting from a classical one is therefore the choice of the subalgebra of observables that will be mapped to operators. Generally, the natural choice of quantizing the phase space coordinates q_i, p_i leads to the right quantum theories, and this is the procedure followed in the majority of theories of quantum mechanics and quantum field theory. But notice that there are other choices available, which will play a role in the development of different theories of quantum gravity, as we will see in this thesis. In particular, the quantization of loop quantum gravity is a case of polymer quantization, one of such alternative quantizations. In chapter 6 I will contrast two possible quantizations of a cosmological model: the Schrödinger quantization which directly quantizes the basic

[74] The reason for this is that defining the kinematics in this way will be necessary for obtaining the right dynamics, as I will show below.

variables q and p and the polymer one, for which there is an exponentiated version of q as an operator, but not of q itself. Even if we restricted ourselves to converting directly phase space coordinates into operators there would still be some ambiguity, as there are different ways of coordinatizing phase space related by canonical transformations, and choosing different coordinates leads to formulating different quantum theories.

Once the algebra that is going to be quantized is chosen, the next step is to find the Hilbert space on which the operators will be defined. Here there are several choices available too. For instance, the algebra defined by 4.68 is usually represented in the following two ways. First, in the position representation the Hilbert space is the space of square integrable functions on space $L^2[\mathbb{R}^3]$ and the position and momentum operators act on states in this representation in the following way:

$$\hat{q}_j\psi(q_i) = q_j\psi(q_i) \quad \hat{p}_j\psi(q_i) = -i\hbar\frac{\partial}{\partial q_j}\psi(q_i)\,. \tag{4.72}$$

Second, the momentum representation is based also on the Hilbert space $L^2[\mathbb{R}^3]$, which is now interpreted as the space of functions defined on momentum space, and the action of the operators is given by:

$$\hat{q}_j\psi(p_i) = i\hbar\frac{\partial}{\partial p_j}\psi(p_i) \quad \hat{p}_j\psi(p_i) = p_j\psi(p_i)\,. \tag{4.73}$$

These two representations are physically equivalent, as there is a unitary map that maps from one representation to the other, mapping states in one Hilbert space to states in the other, operators to operators, while preserving the inner product between different states and the expectation values for any observable and state. Indeed, the Stone-Von Neumann theorem assures us that the irreducible representations of the algebra of any system with any finite number of degrees of freedom are unitarily equivalent, and hence the physical content of this theory is contained in the algebraic structure of its observables in a way that is independent of the particular Hilbert space representation one chooses for representing this algebra.

In the case of theories with an infinite number of degrees of freedom like field theories, the theorem does not apply and one may have different representations which are not unitarily equivalent. In this case, different representations may in principle represent different physical situations and hence the choice of representation may have physical consequences. Nevertheless, it is possible to argue, like in Wallace (2006), that the ambiguity implied by the multiplicity of representations available is not a worrisome one. Wallace notices that in a QFT the different representations available are a product of the two types of infinities present in a continuum theory: the ultraviolet infinities, which correspond to having at each finite volume an infinite number of degrees of freedom, and the infrared infinities, which are due to the theory being defined on an infinite space. Wallace argues for a view of QFTs in which the degrees of freedom of arbitrarily small scales are suppressed by means of some cut-off or similar construction. In this view it is natural to disregard the representation ambiguity associated with the ultraviolet degrees of freedom as an artifact of a theory which defines more degrees of freedom than the really existent. Meanwhile, the choice of representation associated with the infrared degrees of freedom is physical and corresponds to different asymptotic properties of the states on a given representation. Wallace gives some examples of such representations and notes that choosing one of them is equivalent to choosing a particular set of boundary conditions at infinity in classical physics. In this sense, even in field theories the physically most relevant choice at the time of quantizing a theory is the choice of the algebra of observables and not the Hilbert space chosen for representing it.

Now we can turn to the dynamics of the theory. The quantum dynamics is closely related to the classical dynamics and it is defined once we specify a Hamiltonian operator $\hat{H}$. Indeed, the evolution equation for the expectation value of any operator is given by:

$$\frac{d}{dt}\langle\hat{O}\rangle = \frac{1}{i\hbar}\langle[\hat{O},\hat{H}]\rangle + \langle\frac{\partial\hat{O}}{\partial t}\rangle\,. \tag{4.74}$$

Notice the strong parallelism with the classical dynamics as expressed by equation 4.67. The quantum Hamiltonian $\hat{H}$ is usually taken to be the quantization of the classical Hamiltonian function. As I have discussed above there may be some operator ordering ambiguity in the quantization of this function, and different orderings may give rise to different theories. For field theories one may also need to introduce some regularization in order to obtain a well-defined operator.

This dynamics can be given by means of different equivalent Hilbert space representations. In the Schrödinger picture states evolve in time while operators remain constant, while in the Heisenberg picture it is the other way around. In both cases the dynamics is given by a dynamical equation equivalent to 4.74: the Schrödinger equation is used for describing the evolution of states in the first case and the Heisenberg equation evolves operators in the second one. There are intermediate options, like the interaction picture in QFT, which have both states and operators evolving in time. All of these representations are equivalent, as there are unitary transformations that map any of these representations to any of the others.

Let me comment further on equation 4.74 and its relation with the classical theory. For simplicity let me consider a quantum system formed by one particle moving in one dimension and with classical Hamiltonian $H = \frac{p^2}{2m} + V(q)$. In this case, the equations of motion for the quantum system are:

$$\frac{d}{dt}\langle\hat{q}\rangle = \frac{1}{m}\langle\hat{p}\rangle \tag{4.75}$$

$$\frac{d}{dt}\langle\hat{p}\rangle = -\langle V'(\hat{q})\rangle = \langle F(\hat{q})\rangle \,. \tag{4.76}$$

Where by $F = -V'$ I represent the force function, i.e., minus the derivative of the potential $V(q)$ with respect the position variable q. Notice that in this case there was not any ordering ambiguity given that the Hamiltonian is neatly separated into a kinetic part which depends on momentum and a potential one which depends on position. These equations are known as Ehrenfest theorem, and they are very similar to the classical equations of motion of a single particle. Indeed, if we are able to make the approximation $\langle F(\hat{q})\rangle \approx F(\langle\hat{q}\rangle)$ the equations become:

$$\frac{d}{dt}\langle\hat{q}\rangle = \frac{1}{m}\langle\hat{p}\rangle \tag{4.77}$$

$$\frac{d}{dt}\langle\hat{p}\rangle = F(\langle\hat{q}\rangle) \,. \tag{4.78}$$

Or equivalently:

$$m\frac{d^2}{dt^2}\langle\hat{q}\rangle = F(\langle\hat{q}\rangle) \,. \tag{4.79}$$

Which is precisely Netwon's second law but for the expectation value of the position $\langle\hat{q}\rangle$ instead of for position itself. This result is important as it allows to bridge between the quantum and the classical theories. Imagine that we have a quantum state which we can represent as a wavepacket which has as a sharply defined position and momentum, i.e., a state such that the expectation value for the deviation with respect to the average position is small and similarly for the momentum[75]. As long as this condition holds, we are justified in approximating $\langle F(\hat{q})\rangle \approx F(\langle\hat{q}\rangle)$, and our quantum state will be that of a wavepacket following a classical trajectory.

From the point of view of several interpretations of quantum mechanics, the generalization of this result explains why even if the world is quantum it can be successfully described by a classical theory. That is, if we are able to connect in some way the quantum state with things in the world and the state is such that is peaked in a wavepacket, then the world behaves in a way which satisfies some classical equations. In this sense, a way of testing that the quantum theory one has constructed has the right 'classical limit' is to check whether the dynamical equations 4.74 for such a quantum theory return the classical ones one started with if applied to peaked states. Different realist interpretations will tell a different story about how the quantum state

[75] Concretely, that $\langle(\hat{q} - \langle\hat{q}\rangle)^2\rangle < \epsilon$ and that $\langle(\hat{p} - \langle\hat{p}\rangle)^2\rangle < \epsilon'$ for some small ϵ, ϵ'.

relates with the world and about what mechanism is there which makes it the case that we are allowed to approximate the quantum state of the world with a peaked state, but all of them can appeal to results like the above to argue that their theory explains the classical behavior of quantum systems in certain situations. In the case of psi-epistemic interpretations of quantum mechanics the question of how the quantum state and the world relate to each other is harder to answer, and hence it is more difficult to use these results for justifying that a theory has the right classical behavior. I refer the reader back to chapter 2 for a discussion of the different interpretations of quantum mechanics and the way they claim a classical world emerges from a quantum theory.

Let me also mention that when we will study the covariant formulation of quantum mechanics in chapter 7 we will discuss the importance of the limit $\hbar \to 0$ in determining the classical limit of a quantum theory. In the canonical setting this limit has not entered explicitly in our discussion. However, to be able to approximate the quantum dynamics 4.74 with a dynamics analogue to the classical dynamics 4.67 we have needed to use a state $|q_i, p_i\rangle$ peaked around some coordinates q_i, p_i of phase space. In quantum mechanics, as a consequence of the canonical commutation relations, we cannot have states which are arbitrarily peaked, and the limit on how much a state can be peaked around certain values q_i, p_i is essentially given by $\hbar$. In this sense, we can think of the limit $\hbar \to 0$ as the limit of peaked states, which justifies having a classical dynamics. Similarly, we can think that each Hamiltonian operator defines position and momentum scales for which the classical approximation is a good approximation. If the scale defined by $\hbar$ is smaller than these scales, we can have states which behave in a classical way. In covariant approaches I will argue in chapter 7 that the $\hbar \to 0$ limit is discussed in a more heuristic way.

Finally, I have left out of the definition of the quantum theory any mention to the collapse of the wavefunction. In some interpretations it may make sense to speak about such a process as a physical process and as part of the dynamics of a quantum theory, but here I am providing just the basic definition of a quantum theory, which can be further modified to include such processes.

With this I finish this brief introduction to canonical quantization. In table 4.1 I summarize the relations between the classical and the quantum theory. The quantization process can be summarized as:

1. Start with a classical theory defined on a phase space.

2. Choose a subalgebra of functions on phase space and quantize them, i.e., build an algebra of operators on a Hilbert space $\mathcal{H}$ such that the commutator algebra is defined by the Poisson algebra of the classical functions (eq. 4.71).

3. Build a Hamiltonian operator which is a quantization of the classical one. The

dynamics of the theory is contained in equation 4.74 or in some equivalent form.

As I have argued, this has to be taken as a guideline for building a quantum theory and there are several choices and mathematical subtleties involved at each step. Luckily, for the majority of quantum theories that physicists use there are natural choices which lead to the appropriate quantum theories. In the next sections I will comment on how this formalism can be adapted to incorporate theories with gauge symmetries and reparametrization invariant theories.

	Classical Theory	Quantum Theory
Basic space	Phase space $\mathcal{P}$	Hilbert space $\mathcal{H}$
Observables	Functions on $\mathcal{P}$	Operators on $\mathcal{H}$
Algebra	Poisson algebra	Commutator algebra
Dynamics	$\dot{f}(q,p,t) = \{f(q,p,t), H\} + \frac{\partial f(q,p,t)}{\partial t}$	$\frac{d}{dt}\langle\hat{O}\rangle = \frac{1}{i\hbar}\langle[\hat{O},\hat{H}]\rangle + \langle\frac{\partial\hat{O}}{\partial t}\rangle$

Tab. 4.1: Correspondence between the elements of a classical and a quantum theory.

4.5 Canonical quantization of gauge theories

In this section I will introduce how gauge theories can be treated in the canonical formalism. In particular I will introduce Dirac quantization, which is the same schema that will be later on applied to the canonical quantizations of reparametrization invariant theories and general relativity. We will see how imposing the constraints will automatically imply that states are gauge invariant, which will be problematic for reparametrization invariant systems.

The first thing to notice is that if we have a gauge theory like the ones introduced in section 4.1 the most straightforward way to proceed is to directly quantize the gauge reduced version of the theory, i.e., to first formulate the theory in a form which only has physical and not gauge degrees of freedom and then quantize it. This can be achieved by gauge fixing the theory or by quotienting out the gauge orbits. In both cases, this produces a reduced phase space which can work as a starting point of the quantization process.

Alternatively, one can proceed in the inverse order: start with the full gauge theory, quantize it, and then deal with the gauge symmetry. This option is equivalent to quantization in a gauge reduced version of the theory, and in some cases one may choose one option or the other just for convenience. In fact, in (Perez, 2004, Footnote k) it is argued that in the case of the loop quantization of general relativity this strategy is preferred given that the gauge symmetry of the theory is better dealt with in the quantum version of the theory. This procedure is also known as Dirac quantization, as it was first formulated by Dirac (1964).

Here I will outline the procedure for constrained systems with first-class constraints only. Recall from section 4.1 that first-class constraints are associated with a gauge freedom of the system, while second-class constraints are not associated with a gauge freedom. Second-class constraints can also be accommodated in the Dirac quantization formalism, but they will not be relevant for our discussion, as the systems we are interested in here are just first-class systems.

The first step for Dirac quantization is just as in a non-gauge system: select a subalgebra of observables on the phase space and find a set of operators on a Hilbert space such that their commutator algebra mimics the Poisson bracket algebra of the phase space. This Hilbert space contains observables which are not gauge invariant, and hence it is an intermediate space which is bigger than the space that interests us. For this reason, this space is called the kinematical Hilbert space, $\mathcal{H}_{kin}$.

Recall that gauge theories are constrained systems, which means that only a subspace of the phase space, the constraint surface, is physically relevant, and it is in this subspace where all physical trajectories happen. This space is determined by the constraints of the theory, both primary and secondary, Ω_A. In building the quantum theory we will want similar conditions to hold. The way the Dirac quantization imposes these conditions is by requiring that physical states are eigenstates of the constraints with eigenvalue 0:

$$\hat{\Omega}_A|\psi\rangle = 0 \quad \forall A\,. \tag{4.80}$$

Notice that there is some ambiguity in this step, as the quantization of the constraints may be subject to the same operator ordering ambiguity we found for defining the Hamiltonian operator as the quantization of the Hamiltonian function. Besides, depending of the algebra of observables chosen as a basis for the quantization, it may be the case that $\hat{\Omega}_A$ may not be straightforwardly definable. For instance, in LQG we will find (see chapt. 5) that some constraints of the theory are not expressed in terms of the basic algebra of the quantum theory, the holonomy-flux algebra, and its definition as an operator is mathematically subtle and requires of some regularization.

For arbitrary operators, conditions like 4.80 are not always possible to be satisfied. In particular, consider two constraints $\hat{\Omega}_A$ and $\hat{\Omega}_B$ which are to be satisfied by the same state $|\psi\rangle$. Consistency requires that the following equations hold:

$$\hat{\Omega}_A\hat{\Omega}_B|\psi\rangle = 0 \tag{4.81}$$

$$\hat{\Omega}_B\hat{\Omega}_A|\psi\rangle = 0 \tag{4.82}$$

$$\left[\hat{\Omega}_A, \hat{\Omega}_B\right]|\psi\rangle = 0\,. \tag{4.83}$$

The first two equations are obtained just by multiplying the eigenvalue equations 4.80 for operators $\hat{\Omega}_A$ and $\hat{\Omega}_B$ with the other operator. The bottom line equation is obtained by substracting the second equation from the first one, and it is the interesting one, as it is of the form:

$$\hat{O}|\psi\rangle = 0\,. \tag{4.84}$$

This equation is again of the form of 4.80, and hence it looks like this equation implies that an additional constraint needs to be satisfied by the physical states. This is a result we want to avoid, as we want the quantum theory to be as constrained as the classical one and no more. The only way this consistency condition does not represent an additional constraint is if it is already implied by the other constraints. This happens when the commutator takes takes the following form:

$$[\hat{\Omega}_A, \hat{\Omega}_B] = \hat{c}^C_{AB}\hat{\Omega}_C \quad \forall A, B\,. \tag{4.85}$$

Where $\hat{c}^C_{AB}$ are some operators. Notice that this consistency condition is the quantum analogue of the condition in the classical theory that all the constraints are first-class, i.e., that the Poisson brackets of any two constraints can be expressed as a linear combination of other constraints. However, there is an important difference here and it is that in the quantum case the order in which the factors appear matters, and the condition requires that the commutator of any two constraints is a linear combination of other constraints where the constraints appear on the right. Importantly, notice again that Poisson bracket relations do not automatically transform into their quantum counterparts, so we are not assured that the condition 4.85 will hold. Sometimes, one can use the factor ordering ambiguity in defining $\hat{\Omega}_A$ for finding a definition which satisfies 4.85, but for some theories this will not be possible. When we have a quantization where the algebra of constraints does not follow the classical algebra, we say that the quantization is anomalous.

Provided there is no anomaly, the physically interesting states are the ones which satisfy the constraint equations 4.80. The space spanned by these states is called the physical Hilbert space of the theory, $\mathcal{H}_{phys}$. For constraints with discrete spectra, the physical Hilbert space is just a subspace of the kinematical space while in the case of constraints with continuum spectra the case is mathematically subtler. Indeed, strictly speaking equations 4.80 do not have a solution in the kinematical Hilbert space but are similar to distributions in this space. For instance, consider the kinematical Hilbert space $L^2[\mathbb{R}]$ and the constraint $\hat{x}$. The constraint equation in this case is:

$$x\psi(x) = 0\,. \tag{4.86}$$

There is no square integrable function $\psi(x)$ which satisfies this equation, but there is a distribution which does, the Dirac's delta $\delta(x)$. The delta distribution is not an element of $L^2[\mathbb{R}]$, but of the space of distributions on this Hilbert space. For a general kinematical Hilbert space and set of constraints, the physical Hilbert space is the subspace of the space of distributions which satisfy the constraints.

States in the physical Hilbert space have automatically an important property: they are gauge invariant. The reason for this is that the generators of gauge transformations are linear combinations of the constraints, as we saw in section 4.1 for the classical theory. In the quantum theory, one can quantize the classical generators and use them for generating a transformation. As always in the quantization process, one has to be careful to make sure that the quantization ambiguities do not spoil the correspondence between the quantum and the classical. Provided that we are able to find quantum generators which generate gauge transformations which are analogous to the classical ones, we can now consider their action on physical states. A generic gauge generator is of the form $v^A\hat{\Omega}_A$ and its action on a physical state is:

$$v^A\hat{\Omega}_A|\psi\rangle = 0\,. \tag{4.87}$$

This is of course a consequence of the constraint conditions 4.80 and implies that states in the physical Hilbert space are invariant under gauge transformations. Notice that this is different from the situation in the classical theory: in the classical theory the imposition of the constraints does not imply that states are gauge invariant, i.e., the action of the gauge transformation on a point in the constraint surface is to move to another point, while in the quantum theory gauge transformations leave physical states invariant.

Recall from section 4.1 that one of the ways one can deal with a classical gauge theory is by ignoring the gauge freedom, that is, we could work with gauge dependent states and equations and only worry about the physical content of the theory at the time of making predictions. However, this option is not available for quantum theories in the Dirac formalism, as imposing the constraints automatically implies that states are gauge invariant. This is not a problem for gauge theories, as the formalism still gives a theory with the right predictions. In the case of reparametrization invariant theories we will see in the next section that the situation is different as we have argued that in this case the transformations generated by the constraints are not gauge transformations but correspond to physical transformations. In this sense we would like to have the imposition of the constraints without this implying invariance under certain 'gauge' transformations, but the quantum formalism makes it the case that these two always go together.

Notice also that the gauge transformations generated by the operators $\hat{\Omega}_A$ also define a set of gauge invariant observables by means of:

$$[\hat{O},\hat{\Omega}_A] = \hat{c}_A^B\hat{\Omega}_B \quad \forall A\,. \tag{4.88}$$

Any operator $\hat{O}$ satisfying this condition will be well-defined for physical states. This condition is the quantum version of the requirement that physical observables in a classical theory are invariant under transformations generated by first-class constraints, so one may expect that classically invariant observables get translated into quantum invariant observables, provided that the quantization allows for an ordering such that the classical condition gets translated into the quantum one. In the quantum case we are defining the observables as the observables which are invariant under transformations generated by any first-class constraint, while in the classical case gauge transformations were generated just by certain combinations of constraints, as we saw in section 4.1 and was argued in Pitts (2014a). In this sense, the set of observables in the quantum theory is smaller than in the classical theory, which could be argued to be problematic for this quantization.

For instance, in the case of electromagnetism one finds that the physical Hilbert space is independent of A^0 and that there is not any operator associated with it. However, we have well-defined operators for A^a and $\vec{\pi}$ which we can argue that contain all the physical content of electromagnetism. Notice that here the situation is completely analogous to the one with the extended Hamiltonian formalism, where from an Lagrangian perspective one could argue like Pitts against this formalism. However, in the quantum case if we follow the Dirac quantization procedure there seems that we do not have an alternative more respectful with the original Lagrangian formulation of the theory. This problem will be present for every theory which has secondary constraints, like general relativity.

Let me also point out that giving a complete definition of the physical Hilbert space may also not be straightforward and there could be some choices to make or even some issues to solve at the time of defining the inner product of such a state. For this reason there are several techniques available for defining the physical Hilbert space. In particular, let me mention group averaging, which will be used in both LQG and LQC (chapters 5 and 6), and which uses the fact that the imposition of the constraints is related with a set of states invariant under transformations generated by these constraints. In this sense, group averaging goes in the opposite direction of what I have presented here: by finding invariant states it finds the states that satisfy the constraints.

The last step in the definition of the quantum theory is to define the dynamics. As for the non-gauge theory this consists in finding a Hamiltonian operator which defines how states or operators evolve according to 4.74. But in section 4.1 we saw that there are several Hamiltonian functions which were argued to produce the right classical dynamics[76]. However, all these Hamiltonians generate the same quantum dynamics on states in the physical Hilbert space, as they only differ on a sum of

[76] As Pitts (2014a) argues, the extended Hamiltonian formalism leads to a dynamics that can be considered incorrect. This problem will not affect our discussion for the quantum case..

constraints. For instance, consider the action of the extended Hamiltonian:

$$\hat{H}_E|\psi\rangle = \hat{H}_c|\psi\rangle + v^A\hat{\Omega}_A|\psi\rangle = \hat{H}_c|\psi\rangle\,. \tag{4.89}$$

The last equality follows of course from the constraint equations 4.80. The fact that the dynamics is generated just by the canonical Hamiltonian was to be expected, as the other terms in the total and extended Hamiltonian generated gauge transformations[77] and the physical states are invariant under these transformations. In the definition of the dynamics we have the familiar ordering ambiguity, and in order to have the result that all three Hamiltonians generate the same quantum dynamics we need to have the constraint operators on the right, i.e., that if the v^A in 4.89 represent operators and not numbers they need to be on the left.

In order to have a satisfactory dynamics, it has to be consistent with the constraints. That is, the evolution of a quantum state should not take it away from the physical Hilbert space. Imposing this condition on equation 4.74 leads to:

$$[\hat{\Omega}_A, \hat{H}]|\psi\rangle = 0 \quad \forall A\,. \tag{4.90}$$

The same argument we used for deriving the consistency condition 4.85 for the constraints leads us to:

$$[\hat{\Omega}_A, \hat{H}] = \hat{d}_A^B\hat{\Omega}_B \quad \forall A\,. \tag{4.91}$$

This condition is the quantum analogue of the relation between the constraints and the Hamiltonian for a first class system. Again, that the analogue condition holds in the classical setting does not imply that it will hold in the quantum one, and one may have to choose carefully the way the operators are defined in order to obtain such a condition. If this condition is not satisfied we also obtain an anomalous quantization.

Let me finish this section by giving a brief summary of the Dirac quantization process for gauge theories:

1. Start with a classical gauge theory defined on a phase space.

2. Choose a subalgebra of functions on phase space and quantize them, i.e., build an algebra of operators on a kinematical Hilbert space $\mathcal{H}_{kin}$ such that their commutator algebra is defined by the Poisson algebra of the classical functions (eq. 4.71).

3. Impose the constraints. That is, define the physical Hilbert space $\mathcal{H}_{phys}$ as the space of the states which satisfy $\hat{\Omega}_A|\psi\rangle = 0$. The states in this space are automatically gauge invariant.

[77] Again, in the case of the extended Hamiltonian the class of transformations generated is bigger than the gauge transformations in the Lagrangian formulation of the theory, as argued in Pitts (2014a).

4. Build a Hamiltonian operator which is a quantization of one of the Hamiltonians that generate the constrained dynamics in the classical system. The dynamics of the theory is contained in equation 4.74 or in some equivalent form.

As I have argued in this section, this process is mathematically subtler and more complex than the same process for a non-gauge theory. In particular, there is the possibility of having an anomalous quantization which makes it the case that the quantized theory does not respect the constraints and gauge symmetries of the original classical theory. Important for our discussion is the fact that by imposing the constraints we automatically get gauge invariance, which will be problematic for reparametrization invariant models and the source of the problem of time for the case of general relativity.

4.6 Canonical quantization of reparametrization invariant theories

In the first part of the chapter I argued that one has to be careful when treating reparametrization invariant theories as gauge theories. The reason for this is that the 'gauge' transformations of reparametrization invariant theories in the Hamiltonian formalism represent a physical transformation, namely the time evolution, and not a gauge transformation from the instantaneous perspective. In other words, these transformations are not gauge transformations in the sense that they do not connect two states which represent the same physical situation at a time but instead they connect the states of affairs of a system at two different times. From a practical perspective, we can use the constrained formalism just as for gauge theories for describing reparametrization invariant models, but we will reject identifying the physical content of the model to be invariant under transformations generated by the constraints. However, when we move to quantum mechanics we are in trouble, as the formalism of quantum mechanics is such that one cannot impose the constraints without requiring a gauge invariance. For the case of reparametrization invariant models it means that temporal evolution becomes trivial, giving raise to the problem of time.

In this section I will first (4.6.1) introduce the problem of time for a simple example and I will show how it can be overcome, in a way that motivates some proposals of resolutions for all reparametrization invariant models in general and general relativity in particular. However, I will then (4.6.2) study the case for the quantization of the three examples in section 4.2 and I will argue that the problem of time can be solved for the models describing the relativistic particle but that it cannot be solved for the case of the double harmonic oscillator. I will further argue that the reason for this difference is that the double harmonic oscillator model is non-deparametrizable, which is also the case for general relativity, as I will study in section 4.7. Finally (4.6.3), I will also consider the approach to the problem of time by Colosi and Rovelli, who argue that even if the problem of time does not give us the standard structures of quantum theory one can still make sense of the formalism. I will argue against this view, as it does not offer any sensible interpretation of the states and probabilities it defines.

4.6.1 The problem of time in a simple case

Let me start by introducing a simple case, the reparametrization invariant version of a classical particle, and let me apply the quantization schema introduced in the previous section to it. This simple case is important, as it is commonly[78] used as a model for the quantization of general relativity. We start with the Newtonian action

$$S[x] = \int dt[\frac{1}{2}m(\frac{dx}{dt})^2 - V(x)]\,, \tag{4.92}$$

and we introduce a reparametrization invariance in the theory by introducing an arbitrary parameter τ such that the physical time t is now a configuration variable which depends on τ[79]:

$$S[x,t] = \int d\tau[\frac{1}{2}m\frac{\dot{x}^2}{\dot{t}} - \dot{t}V(x)]\,. \tag{4.93}$$

One can check that this reparametrization invariant system leads to Newton equations in the x, t variables. This model has a homogenous Lagrangian and therefore one can show that it has a vanishing canonical Hamiltonian and that it satisfies the Hamiltonian constraint:

$$\mathcal{H}_0 = p_t + \frac{1}{2m}p_x^2 + V(x) = p_t + H(x, p_x) = 0\,, \tag{4.94}$$

where I have introduced the Hamiltonian function H, which is the Hamiltonian function of the system once the reparametrization invariance is removed. This model is a one-dimensional model as it only depends on one parameter, in this case τ. This model is clearly deparametrizable, as we are able to identify the variable t as the time variable and given any solution of the equations of motion of the model we are able to invert the relation $t(\tau)$ and express $x(\tau)$ as $x(t)$. Let me mention that in general, we can introduce an artificial temporal or spatiotemporal reparametrization invariance to any Hamiltonian theory to obtain a reparametrization invariant theory

[78] For instance, this case is discussed in Isham (1993); Kuchař (1992); Dittrich (2005); Rovelli and Vidotto (2015).

[79] There is a sense in which t is not treated as an ordinary configuration variable, as it is imposed that is monotonic in τ. This implies that for every value of t there is a unique value of τ.

with homogeneous Lagrangian[80] by adding some extra parameters. Obviously, these models are deparametrizable, as we can always eliminate these extra parameters.

We can apply the quantization schema introduced in the last section to this system. The natural kinematical Hilbert space for this system is $\mathcal{H}_{kin} = L^2[\mathbb{R}^2, dxdt]$, that is, the space of square integrable functions both in position x and time t. In this representation, the constraint condition for physical states takes the form:

$$\hat{\phi}\psi(x,t) = -i\hbar\frac{\partial}{\partial t}\psi(x,t) + \hat{H}\psi(x,t) = 0\,. \tag{4.95}$$

This equation is nothing but the familiar Schrödinger equation of quantum mechanics. Solutions to this equation are distributional, in the sense that they do not belong to the kinematical Hilbert space. This can be seen by computing the norm of such a function on this space:

$$\langle\psi|\psi\rangle = \int dt \int dx|\psi(x,t)|^2 = \int dtC = \infty\,. \tag{4.96}$$

Here I have used the fact that the spatial norm of a function satisfying Schrödinger equation is conserved in time. As the norm of any function satisfying 4.95 diverges, these functions are not square integrable functions in both space and time and hence they are not elements of the kinematical Hilbert space. The physical Hilbert space $\mathcal{H}_{phys}$ is defined by specifying a vector space, which is the space of functions satisfying 4.95, but also one needs to specify an inner product for this space. In this case we have the natural candidate:

$$\langle\psi_1|\psi_2\rangle = \int dx\psi_1^*(x,t_0)\psi_2(x,t_0)\,. \tag{4.97}$$

Which is the familiar inner product used in quantum mechanics. The parameter t_0 is an arbitrary time parameter, given that the unitarity property of Schrödinger evolution makes it the case that the value of the inner product is independent of t_0.

The last step in the quantization process described in the previous section was to define a dynamics. In this case this would introduce a dynamics in which states in $\mathcal{H}_{phys}$ evolve with respect to the parameter τ. However, the canonical Hamiltonian of this theory is identically 0, and the total Hamiltonian is just the constraint (multiplied by an arbitrary function), so in any case we find that the action of the Hamiltonian on physical states vanishes.

$$\hat{H}_T|\psi\rangle = 0\,. \tag{4.98}$$

This leads to the conclusion that there is no evolution with respect to the parameter τ. This is a difference with respect the classical case: in the classical theory we could

[80] See for instance (Sundermeyer, 1982, pp. 291-294).

describe the system as a system evolving in time τ, while in the quantum theory this is not possible. We may say that time τ is missing or that evolution is frozen in τ. This is also known as the problem of time of reparametrization invariant theories.

Despite this, this case does not seem very problematic. After step 3 we obtained functions which satisfy the Schrödinger equation and it is straightforward to reinterpret them as the usual wavefunctions of standard quantum mechanics. Crucial for this is that we are able to identify the configuration variable t as a time variable and to, consequently, treat it differently from the position variable x. By doing this we recover the standard quantum mechanics of a non-relativistic particle.

Notice that this solves the problem of time in that by inspecting the states in the physical Hilbert space we were able to recover a time and a unitary time evolution with respect to it. However, the time evolution we were originally looking for, that is, evolution in τ is not recovered. This is arguably not problematic in this case, as one can argue that t is the physical time and that physical time evolution is evolution in t. But in the case of general relativity this will not be so, as time is not a variable in the configuration space of the theory.

The problem of time appears for any reparametrization invariant model. The reason for this is that the total Hamiltonian of any reparametrization invariant model is a sum of constraints, and hence the total Hamiltonian does not define a non-trivial dynamics on the physical Hilbert space. For the case of the model I just studied for the non-relativistic model we could overcome the problem by identifying states in the physical Hilbert space with solutions of the Schrödinger equation. We may want to apply this resolution of the problem of time to other reparametrization invariant systems. For these systems we therefore would have that the canonical quantization process consists on the following steps:

1. Start with a classical reparametrization invariant theory defined on an extended phase space.

2. Choose a subalgebra of functions on phase space and quantize them, i.e., build an algebra of operators on a kinematical Hilbert space $\mathcal{H}_{kin}$ such that their commutator algebra is defined by the Poisson algebra of the classical functions (eq. 4.71).

3. Impose the constraints. That is, define the physical Hilbert space $\mathcal{H}_{phys}$ as the space of the states which satisfy $\hat{\Omega}_A|\psi\rangle = 0$. The states in this space are interpreted as already containing all the dynamical information of the quantum theory.

This quantization process is taken as a model for the canonical quantization of general relativity, as we will see in the next subsection and in the following chapters. However, there are some issues with this quantization process and the physical interpretation of the physical Hilbert space. In particular, I will argue next that this quantization process seems to apply in a meaningful way at most to deparametrizable models, as I will show next by considering the quantization of the models in section 4.2.

4.6.2 The problem of time in our examples

Let me consider the quantization of the relativistic particle as described in section 4.2, but introducing a factor of m in front of the action[81].The kinematical Hilbert space for this case is the same one as for the non-relativistic particle, but the constraint equation now becomes:

$$(\partial_t^2 - \partial_x^2 + \frac{m^2}{\hbar^2})\psi(x,t) = 0\,. \tag{4.99}$$

This equation is the Klein-Gordon equation. This equation is sometimes introduced as the relativistic version of the Schrödinger equation but there is an important difference that makes it the case that one cannot interpret the Klein-Gordon field as a wave function. This difference is that for every solution of the Schrödinger equation the inner product is conserved in time, while in the case of the Klein-Gordon equation this is not the case. This means that one cannot interpret the Klein-Gordon field as defining a probability density of finding the particle at some position of space, as this probability would not be conserved. That is, even if we started with a normalized field which defined a well-behaved probability at an initial time, it is possible that at later times the total probability is greater than 1, less than 1 or even 0! This makes it the case that general solutions of Klein-Gordon equation are not interpreted as wavefunctions and that other alternatives are used for describing relativistic particles quantum mechanically.

One possible way of solving this problem is by imposing further conditions on ψ so that the inner product is well-defined and conserved in time. In the case of the Klein-Gordon equation this can be achieved in the following way. Any solution of 4.99 can be written as a sum of two terms:

$$\psi(x,t) = \psi_+(x,t) + \psi_-(x,t)\,. \tag{4.100}$$

These two terms are called the positive and negative parts of ψ, as they satisfy the following Schrödinger equation with different signs:

$$\mp i\hbar\partial_t\psi_\pm(x,t) = \sqrt{-\hbar^2\partial_x^2 + m^2}\,\psi_\pm(x,t)\,. \tag{4.101}$$

[81] This factor does not affect the discussion on that section but allows for a dimensionaly consistent quantization. By setting $m = 1$ one can recover the action of section 4.2. Also, have been implicitly using spacetime coordinates such that $c = 1$. In some other presentations of the Klein-Gordon equation this convention is not followed and some factors of c appear.

If we restrict ψ to be just its positive frequency part ψ_+ then we can define a conserved inner product, given the fact that ψ_+ satisfies a Schrödinger-like equation ensures that the same interpretation is available. Similarly, if we restrict to negative-frequency functions we obtain the same result, i.e., a set of functions satisfying the Klein-Gordon equation and for which we can define a conserved inner product.

The quantum theory defined this way is a consistent quantum theory with the same formal structure as other quantum theories. However it has a problematic feature: the evolution defined by 4.101 is non-local. For this reason, this quantum theory is not generally used for describing relativistic particles and one relies on QFT. In any case, this example shows that for defining a quantum theory of the form of a standard quantum theory one may need to impose further conditions on states on the physical Hilbert space. We will see an example of this in chapter 6, where, in order to obtain a quantum theory in a cosmological model, a positive frequency condition is imposed.

Notice that we are picking one of the variables of the theory and we are identifying it as a time. However, this corresponds only to one of the possible ways there are to foliate the Minkowski spacetime in which the theory is defined. If we pick a different foliation we will have a different quantum theory, and this gives rise to a multiple choice problem. In the case of the Klein-Gordon case one is lucky, as the symmetries of the background spacetime make it the case that the different quantum theories are compatible[82]. With this I mean that if we have two foliations between an initial and a final time slices and we use the quantum theories associated with both to evolve a quantum state from the initial to the final slice, we will find the same quantum state. For a general case without these symmetries, like the case of the configuration space of general relativity, choosing different time variables leads to incompatible quantum theories.

Let me also mention the quantization of the relativistic particle as described by our second example in section 4.2. Here we have the primary constraint $\pi = 0$ which can be straightforwardly applied:

$$\hat{\pi}\psi(n, x, t) = \partial_n \psi(n, x, t) = 0 \,. \tag{4.102}$$

In other words, this constraint just means that the quantum state is independent of the auxiliary lapse function of the theory. Now imposing the secondary constraint just gives us back the Klein-Gordon case.

Notice the similarity with the case of general relativity. If we take the lapse function and shift vector as part of the configuration space, the constraint equations immediately tell us that the physical state is independent of them. This at first

[82] This was discussed and proved in Kuchař (1991).

sounds natural, as they obviously depend on the foliation of spacetime, and are gauge-dependent. However, notice that there is an important difference between the example just considered and general relativity: in the example the lapse function is just an auxiliary field which basically contains redundant information, as there is a time variable in the extended configuration space. In the case of general relativity I have argued in section 4.3 that N and N^a have an important physical meaning as they are needed for reconstructing the 4-metric $g_{\mu\nu}$. Therefore, the elimination of these 'Lagrange' multipliers can be more problematic in the general relativistic case.

Important for solving the quantum problem of time for these two examples was the fact that we are able to identify a time variable t and interpret a sate in $\mathcal{H}_{phy}$ as representing a state which evolves with respect to this variable. In other words, we are able to deparametrize the theory. By doing so, we recover the standard quantum formalism and we are free to use our favorite interpretation of quantum mechanics to such a state. However, from section 4.3 we know that there are some results which show that general relativity cannot be deparametrized. This forces one to take some different strategies for interpreting states in the physical Hilbert space which I will analyze in the next section. Moreover, I have argued that general relativity is more similar to the third example in section 4.2, and hence it is interesting to study the quantization of such a system.

The kinematical Hilbert space of the system of two harmonic oscillators as defined in action 4.28 is just the space of square integrable functions on the real plane $L^2[\mathbb{R}^2]$, and the constraint equation is:

$$\left(\frac{1}{2m}\hat{p_x}^2 + \frac{1}{2m}\hat{p_y}^2 + \frac{k_x}{2}\hat{x}^2 + \frac{k_y}{2}\hat{y}^2 - E\right)|\psi\rangle = 0 \to \hat{H}|\psi\rangle = E|\psi\rangle\,. \tag{4.103}$$

This is just the time-independent Schrödinger equation for the system of two harmonic oscillators. Obviously, if we were expecting to get the standard quantum theory for two harmonic oscillators, the conclusion we reach is that this quantization procedure has failed. In this sense, the quantization of this example, that remember I have argued is the most similar to general relativity, casts doubts on the viability of the quantization schema proposed above for a subset of reparametrization invariant theories.

Notice that in the case of the non-relativistic particle we could lose evolution in τ as long as we could define evolution in t. In the case of the double harmonic oscillator it seems more harmful to lose evolution in τ, as I have argued above that temporal evolution is a succession of configurations of both x and y and not definable by relative evolution as $x(t)$ was in the case of the deparametrizable models for the non-relativistic and the relativistic particles. In other words, as the configuration space of this example is a proper configuration space and not an extended one we do not have any time variable left if we lose τ, while in the deparametrizable models we still had t.

Looking more carefully at what is going on in this case we find an interesting feature. In action 4.28 there explicitly appears E, which is the energy of the system. In this sense, one can read this as saying that it is part of the dynamics, or if you want, of the laws that describe the system. This is a difference with the standard formulation of mechanics, in which the total energy is a conserved quantity but it is not fixed by the action. Therefore, it is not so surprising that the quantization which stems from action 4.28 leads to just the quantum way of stating that the system has a fixed energy E. As in quantum mechanics for having non-trivial time evolution one needs a superposition of energies, this has as a consequence the problem of time.

This example shows that for non-deparametrizable theories the problem of time looks more serious than for non-deparametrizable ones and that one should consider rejecting their canonical quantization. I will extrapolate this to the case of general relativity in the next section. Before this, in the following subsection I will consider the approach by Colosi and Rovelli, who do not take the problem of time to be an obstacle and instead propose a way one can define some probabilities out of the structures in the physical Hilbert space. However, I will argue against their position.

4.6.3 A different approach to the problem of time

In the view defended in Colosi and Rovelli (2003), states define transition amplitudes which define the probabilities for certain physical transitions to happen. Colosi and Rovelli argue in basis of an example defined by the following Hamiltonian constraint:

$$H = a^2 + b^2 + p_a^2 + p_b^2 - 2\hbar(N+1)\,. \tag{4.104}$$

This is the same constraint as for the previous case for the special case $m = k_x = k_y = 1$ and with a, b playing the role of x, y. Applying the steps defined above we find that the natural kinematical Hilbert space is the space of square integrable functions on a and b and the physical Hilbert space is the subspace of states of the form:

$$\psi(a,b) = \sum_{n_a+n_b=N} c_{n_a n_b}\psi_{n_a}(a)\psi_{n_b}(b)\,. \tag{4.105}$$

Where N is required to be a non-negative integer and $\psi_{n_a}(a)$, $\psi_{n_b}(b)$ are the eigenfunctions of the harmonic oscillator with non-negative eigenvalues n_a and n_b and $c_{n_a n_b}$ are constant coefficients. If the above algorithm is sensible, we should be able to give a physical interpretation of any such state. We may try to proceed as in the case of the Schrödinger or Klein-Gordon particles and try to identify one variable which plays the role of time. But we find that the two variables a and b play identical roles both in the constraint and in the definition of the inner product. In this sense, there is nothing which makes a or b special for any state in $\mathcal{H}_{phys}$. If we focus on a particular state, it will generally not be symmetric in a and b, and hence one could try to argue that in some states a or b plays the role of time given some of the properties of the state.

Even if we were somehow able to determine that a and not b, or the other way around, plays the role of time, our troubles are not over, as the states $\psi(a, b)$ are not so easily interpretable as describing a time evolution. First, notice that for $a \to \pm\infty$ the value of ψ goes to zero, which signals that the 'evolution' in a is not unitary, i.e., that using the natural inner product measure, the probability density on b defined by ψ is not conserved in 'time' a. One may try to mimic the strategy followed for the Klein-Gordon case and find a subset of states for which a unitary evolution in a or b could be defined but this strategy has to face technical difficulties as the fact that the 'square-rooted' version of the Hamiltonian constraint, i.e., the equivalent of 4.101 for the present case, may not be well-defined as an Hermitian operator. In this sense, trying to identify or pick one variable as a time variable such that we are able to recover the Schrödinger evolution picture seems difficult, and one may expect that for more complicated systems, like general relativity, the situation will be even worse.

For this reason, Colosi and Rovelli, among others, do not try to look for a time variable in between the available, but propose a different interpretation. Notice that proposals of this kind necessarily imply a modification of the way quantum theory is defined and interpreted. Colosi and Rovelli follow Reisenberger and Rovelli (2002) and postulate that one can make a probabilistic interpretation of states in $\mathcal{H}_{phys}$. For this we will define the states $|\mathcal{R}\rangle$ on the kinematical Hilbert space which are associated with a small region $\mathcal{R}$ in the a, b space:

$$|\mathcal{R}\rangle = C_{\mathcal{R}} \int_{\mathcal{R}} da db |ab\rangle \,. \tag{4.106}$$

Where $C_{\mathcal{R}}$ is a normalization constant and $|ab\rangle$ represents the (improper) state with eigenstates a and b. Given a physical state ψ, the probability of measuring a and b in the small region $\mathcal{R}$ is defined to be:

$$P\left((a, b) \in \mathcal{R}\right) = |\langle \mathcal{R}|\psi\rangle|^2 \,. \tag{4.107}$$

If we further assume that ψ is constant in the region and we introduce $V_{\mathcal{R}}$ as the volume of the region we can approximate this probability as:

$$P\left((a, b) \in \mathcal{R}\right) \approx V_{\mathcal{R}}^2 C_{\mathcal{R}}^2 |\psi(a_0, b_0)|^2 \,. \tag{4.108}$$

Furthermore, in certain limits and choosing certain convenient regions, the probability defined this way approximates the probability one would expect for standard quantum mechanics if we chose a as a time parameter and b as an observable or the other way around. That is, the probability of measuring a value of b in an interval of width Δb around a given value b_0 at a time a_0 is approximately $|\psi(a_0, b_0)|^2 \Delta b$, as one would expect from standard quantum mechanics.

A way in which the physical state ψ is proposed to be defined is in the following operational way. Suppose we have 'measured' a and b to be in some small region $\mathcal{R}'$. As before, we have the state $|\mathcal{R}'\rangle$ in the kinematical space which is naturally associated with such a region. Now we can postulate that the physical state following the 'measurement' collapses to a state associated with the region. However, $|\mathcal{R}'\rangle$ is not a state in the physical Hilbert space, and hence we postulate that the collapse is to the projection of this state into the physical Hilbert space. That is, after measuring the system in the region R' we postulate that the physical state of the system is $\hat{P}|\mathcal{R}'\rangle$[83], where $\hat{P}$ is the projector of states in $\mathcal{H}_{kin}$ into $\mathcal{H}_{phys}$. If one postulates this and applies the definition of the probability 4.107 one finds:

$$P\left((a, b) \in \mathcal{R} |\, (a, b) \in \mathcal{R}'\right) = |\langle \mathcal{R} | \hat{P} | \mathcal{R}' \rangle|^2 \,. \tag{4.109}$$

This takes the form of a conditional probability and can be read as giving the probability of measuring some outcome given that an initial state was prepared. Thus, we find that the approach that Colosi and Rovelli are defining has a strong operational flavor.

Let me also mention that the way Reisenberger and Rovelli found expression 4.107 was by applying the standard Copenhagen machinery to a system formed by a non-relativistic particle and a detector which is configured in such a way that it detects the particle if it goes through a small spacetime region $\mathcal{R}$. In this sense these probabilities strictly correspond to the probability of the detector detecting the particle or not. In the case of our system it seems harder to imagine what would it mean to have a detector in a region in the (a, b) space, given that so far we have not given any spatiotemporal meaning to a and b. Furthermore, the operational talk can be useful in a case in which we can think of an observer external to the system but in this toy model, and more importantly, in quantum gravity, this external observer is not available so it is difficult to make sense of the probabilities defined by expressions like 4.107 or to get a picture of what the quantum theory is saying about the system, or, in the case of quantum gravity, the world. Notice also that the operational characterization in terms of an external observer introduces some temporal notions as measuring processes happening in time, collapses of the quantum state happening or the idea of an initial preparation of a state and a later observation[84].

[83] The normalization constants $C_{\mathcal{R}}$ are defined such that these states are normalized.

[84] In expression 4.107 the order of preparation and observation can be reversed, but the point I am making is that the operational picture defines correlations between different observations, which can be considered as different events or different moments in time.

Finally notice also that while in the case of the particle studied in Reisenberger and Rovelli (2002) we are justified in using 4.107 given some approximations and by making use of the standard rules of quantum mechanics, for other systems like the one in our example the usage of this rule is simply postulated. This means that the only justification for the rule is the analogy with the simple case, where it could be justified as relying on standard quantum mechanics. For those with realist inclinations, the usage of the probability rule can be justified using some realist interpretation of quantum mechanics for the single particle, but it is not clearly so for the generalization to the example we are considering, as the structure defined is different from the usual one. And for those with operationalist inclinations, we have seen above that it is difficult to give an operational meaning to probabilities like 4.107.

Let me make a last remark about this example. The classical equations of motion derived from the constraint 4.104 describe ellipses on the a, b plane parametrized by the arbitrary parameter τ and are just a particular case of our third example in section 4.2. Alternatively, the same trajectories can be determined by means of the following Hamiltonian, which does not have the same reparametrization invariance:

$$H = \frac{1}{2}(p_a^2 + p_b^2 + a^2 + b^2)\,. \tag{4.110}$$

The trajectory generated by Hamilton equations for this Hamiltonian, for some initial conditions, is the same ellipse generated by the constrained system but it is now described in terms of a unique time parameter t. Now, the quantization of this alternative classical formulation leads to a standard quantum theory: a theory of wavefunctions defined as square-integrable functions on the a, b plane which evolve with respect to the time parameter t. This is just the same as we found for the case of the harmonic oscillators. However, Colosi and Rovelli (2003) have a different interpretation already in the classical level. They reject seeing the system as a succession of instants of time but they claim that the content of the classical theory is just that it defines a set of possible correlations between the two observables. This view is related with the view defended by Earman (2002) and against which I have argued in section 4.3.

All in all, it seems that the approach by Colosi and Rovelli is unsatisfactory. Indeed, they define some probabilities using some of the structures I have defined in this chapter, but it is completely unclear the way these probabilities and measurements are to be interpreted and the justification for this framework is also wanting. The same verdict will apply when I analyze how this is meant to apply for the case of quantum gravity in the following section and at other points of this thesis.

Let me finish this section by insisting on what we have learnt about the quantization of reparametrization invariant systems. We have seen that the procedure followed for the quantization of gauge systems had to be modified, as there is a problem of time, i.e., we lack a dynamics with respect to a time parameter as usually defined in quantum mechanics. For this reason, one tries to interpret states in the physical Hilbert space as containing all the physical information of the quantum theory. In some cases, like the case of the non-relativistic particle, we are able to immediately recover a quantum theory of the standard form by identifying a time variable in between the variables of the theory while in some others this requires to overcome some technical difficulties which in some cases, like the relativistic particle case, can be solved by imposing further conditions to physical states. However, these cases are cases of deparametrizable models. For non-deparametrizable models we can suspect that the problem of time cannot be solved in the same way, as was well illustrated by the example of the double harmonic oscillator. Alternatively, we have seen how authors like Rovelli and Colosi propose a new interpretation of the quantum formalism in order to be able to derive some probabilities from the states in the physical Hilbert space of the theory. However, I have argued that these new proposals are unattractive, as they do not have any clear and well-justified interpretation. In the next section I will finally study the case of the quantization of general relativity, and we will find ourselves in a similar situation: as general relativity is non-deparametrizable as the double harmonic oscillator it therefore seems that the problem of time is serious and that none of the proposed resolutions is really satisfactory, as I will argue.

4.7 Canonical quantization for GR: the Wheeler-deWitt equation and the problem of time

Finally we are in position to analyze what happens when one applies the canonical quantization techniques to general relativity. Depending on the exact formulation of general relativity one uses as a starting point for the quantization procedure one is lead to different Hilbert spaces, but one finds a problem of time for any formulation which respects the diffeomorphism invariance of the theory. In this section I will take as a starting point the geometrodynamic formulation of general relativity that I introduced in section 4.3. The reason for choosing this one is that it is more intuitive and that it is the one that was developed first and on which the conceptual problems I am interested in were clearly formulated. I will argue that there is a problem of time and that we have good reasons for rejecting the proposed resolutions. In this sense, I will argue that the canonical quantization of general relativity does not lead to a satisfactory theory of quantum gravity as the example of the double harmonic oscillator shows how the quantization of a non-deparametrizable model does not give the right quantum theory. Even if we ignored this, the proposed resolutions do not give quantum theories which can be satisfactorily interpreted and I will argue that we should reject them. Later on in this thesis I will extend this conclusion not only to approaches to quantum gravity and cosmology based on the canonical formulation, but also to approaches based on the covariant one.

I will start this section by introducing the quantization of geometrodynamics and the problem of time in 4.7.1. Then I will study the proposed resolutions and I will argue that none of them is satisfactory. In particular, I will study strategies based in the analogy with the non-relativistic and relativistic particle models in 4.7.2, strategies which interpret states in the physical Hilbert space to be giving probabilities for 3 or 4-geometries in 4.7.3, the frozen observables strategy in 4.7.4 and the transition amplitudes strategy in 4.7.5. Finally, in 4.7.6 I conclude that the problem of time has not been solved for general relativity and that we can expect it not to be solvable. I also consider some alternative views, one which is based on a Bohmian approach and other based on a quantization not of general relativity but of the empirically equivalent shape dynamics.

4.7.1 Wheeler-deWitt equation and the problem of time

As we saw in section 4.3, the geometrodynamic formulation of general relativity describes spacetime as the temporal evolution of a 3-geometry. In this formulation the basic phase-space functions are the components of 3-metric g_{ab}, their conjugate momenta π^{ab}, which was related with the extrinsic curvature, the lapse function N, the shift vector N^a and their conjugate momenta P_μ. As N and N^a act as Lagrange multipliers it is usual to work in a phase space which does not include them, in the same way that we have seen that A^0 can be ignored in electromagnetism. If we keep them as phase space variables, the primary constraints $P_\mu = 0$ imply that the quantum state is independent of them, so we reach the same Hilbert space. As I have argued in section 4.3, N and N^a have a physical meaning, as they contribute to the definition of the 4-metric $g_{\mu\nu}$ and hence the fact that N and N^a do not play a role in the quantum theory could be problematic.

The direct quantization[85] of the remaining variables of the geometrodynamical phase space leads us to a kinematical Hilbert space in which each state is a wavefunctional on superspace, a space in which each point represents a configuration of g_{ab}. Intuitively, the wavefunctional defines something like a probability density in this space in the same way that the wavefunction of a particle defines a probability density for its position. In this case the probability defined would be the probability of finding different geometries of space.

The next step in our quantization process is the imposition of the constraints to define the physical Hilbert space. First, we have the momentum constraints $\mathcal{H}_a$. Intuitively, they are related with spatial diffeomorphisms, just as in the classical case,

[85] Leaving aside for the moment the mathematical subtleties that appear each time a field theory is quantized.

and their imposition should leave us with a Hilbert space in which each state is a wavefunctional of diffeomorphism-invariant 3-geometries and not of configurations of the 3-metric g_{ab}. However, in section 4.2 I argued that the quantities that one would intuitively think of as observables are hard to define and I argued against the claim that observables had to had vanishing Poisson brackets with the diffeomorphism generators. In the quantum case this seems to lead to problems, as the only observables that are well-defined in the Hilbert space of functionals which satisfy the momentum constraints necessarily have vanishing commutators with the diffeomorphism generators. Therefore, in the same way we had problems for defining observables in the classical case, we also will have them in the quantum one. In this section I will not expand more on this difficulty, but this is a problem to take into account and which also affects approaches like LQG as we will see in chapter 5, where I will argue that we do not have a nice diffeomorphism invariant definition of geometrical observables like area and volume operators, although we have an intuitive grasp of them

The constraint equation for the geometrodynamical formulation of general relativity is also known as the Wheeler-deWitt equation and takes the following form when expressed in terms of operators:

$$\left(G_{abcd}(x;\hat{g})\hat{\pi}^{ab}\hat{\pi}^{cd} - |\hat{g}|^{1/2}R(x;\hat{g})\right)|\psi\rangle = 0\,. \tag{4.111}$$

Alternatively, we can also write it as a functional differential equation:

$$G_{abcd}(x;g)\frac{\delta^2\psi[g]}{\delta h_{ab}\delta h_{cd}} - |g|^{1/2}R(x;g)\psi[g] = 0\,. \tag{4.112}$$

Here G_{abcd} is the supermetric introduced in section 4.3 and R is the Ricci scalar for the 3-metric. Here I am writing the equation just as it was first published in DeWitt (1967a) and as it is found in references like Kiefer (2012), but notice that there is an operator ordering ambiguity and other orderings would also be a priori acceptable quantizations of the Hamiltonian constraint. Notice also that this equation is not mathematically well-defined and that some regularization is needed, given that we are taking two functional derivatives at the same point.

For having a complete definition of the physical Hilbert space of the theory one also needs to define an inner product on the space of solutions to this equation. The Wheeler-deWitt equation is a second order functional differential equation, and in this sense it is more similar to the Klein-Gordon equation than to the Schrödinger equation. As happened with the Klein-Gordon equation, one has to be careful with the definition of the inner product, or restrict the set of solutions of the equation to some particular set in order to have a well-behaved inner product. The difficulty in defining a consistent and well-behaved inner product is known in the literature as the Hilbert space problem[86].

[86] See Isham (1993); Kuchař (1992).

Ignoring for the moment such technical difficulties, we are now in the situation described in the previous section. We have followed the steps of the quantization process of a gauge theory and we have found a physical Hilbert space but no non-trivial time evolution can be defined, as the Hamiltonian of general relativity is made of constraints. That is, we have a problem of time. As I explained in the previous section, the optimist reading of this is as implying that all the physical content of the theory is encoded in this physical Hilbert space and that we need a clever way of recovering it. However, the double oscillator example shows that this attitude may be too optimistic, as there may be the case that the physical Hilbert space does not describe a dynamically meaningful theory but just a system which obeys some constraints. In this sense, the pessimistic reading would lead us to reject the canonical quantization of general relativity.

Next, I will analyze the ways the problem of time has been proposed to be solved, and I will argue for the pessimistic view, i.e., that these resolutions are ill-motivated and that they do not work. I will build on the reviews Isham (1993) and Kuchař (1992), where one can find a complete survey and criticism of these different approaches as they stood in the nineties, and I will complement it with the developments in the field and one proposal of resolution, the transition amplitude strategy, that had not been formulated at that time. However, my position with respect the feasibility of the resolutions they studied will be negative and based on the claim I argued in section 4.3 that general relativity is not deparametrizable and that time is not somehow encoded in the variables of the configuration or phase spaces of the theory, while Isham and Kuchar seem to find some problems with the resolutions proposed, but they do not explicitly reject the possibility of time being an internal variable of the theory. I have grouped the proposed resolutions in four groups, which I move to analyze now.

4.7.2 Internal time resolutions

First, there is a family of approaches which tries to mimic the strategy followed in the previous section for the classical parametrized particle, i.e., to interpret states in the physical Hilbert space as solutions to some dynamical equation in which there is evolution with respect to some time variable which was already present in the phase space of the theory. In other words, this would imply deparametrizing general relativity to identify a time variable, in either the gravitational or the matter degrees of freedom and express the dynamics, either before or after quantization with respect that variable to recover a Schrödinger-like equation.

My main objection to this kind of approach is that general relativity is not deparametrizable and that time is not represented by the variables in the phase space of the theory, as I argued in 4.3. From this point of view, trying to deparametrize general relativity or identify a time variable is just wrong, just as it would be wrong for the double harmonic oscillator case to claim that one of the oscillators is time. Similarly, interpreting the Wheeler-deWitt equation as a Schrödinger equation seems wrong, as it would more naturally be interpreted as the time-independent Schrödinger equation we found for the double harmonic oscillator. This conceptual objection is enough for me to reject this kind of approach, but let me mention some of the other objections raised against it.

First, Isham and Kuchar do not challenge the view that the 6 degrees of freedom of general relativity represent the two real degrees of freedom plus four spacetime coordinates. However, they point out at results like the ones in Torre (1992) I have mentioned above to argue that even if time were encoded by the variables of general relativity, one may not be able to separate it, neither before or after quantization. In this sense, they worry that the identification of time fails even at a technical level.

Second, if we were able to overcome this worry or if we simply ignored it, we could have the opposite problem, the multiple choice problem. That is, if no variable stands as a preferred time variable, any variable, or some set of variables, could work as a time variable and we would not know which one to choose. For this, we would need to be able to define some suitable inner product for such choices of 'time' variables. Assuming we were able to do so, we would be left with a variety of candidate theories which may not be easy to compare, or even worse, as some of these theories lead to incompatible predictions. In the case of the Klein-Gordon case above I commented that the fact that spacetime had a Killing vector allowed one to build consistent quantum theories for different foliations, i.e., for different time variables, but this will not be the case for the case of general relativity and one should expect different quantum theories for different choices of time to be incompatible[87].

Notice also that in this approach the degree of freedom that is chosen as a time variable is treated differently from the rest of variables, just as time is treated differently from the rest of variables in the Schrödinger equation. For instance, it will be considered to be defining the instants of time while the other variables are the ones that can be observed at the different instants with probabilities defined by the wavefunction. In this sense, the variable chosen as a time variable is special and less quantum[88]. This is worrisome if we consider the analogy with the double harmonic oscillator to hold: in the same way that we want a quantum theory which treats both oscillators on the same footing and which does not treat one classically and the other one in a quantum way, the same seems legitimate to ask for a quantum theory of general relativity. That is, we do not want any component of the metric, of the conjugate

[87] For a discussion of this point I refer the reader again to Kuchař (1991).

[88] There is a sense in which it can have quantum properties as being discrete (see chapter 6), but it is not quantum in the same sense as time is classical in quantum mechanics.

momenta or of the matter fields to be treated as a classical variable, and we should expect quantum phenomena like superposition and entanglement to be happening to all of them.

The technical difficulties in defining an unitary evolution with respect to one parameter in general relativity can be ignored in simplified models such as the ones used in quantum cosmology and a Schrödinger-like equation can be found. In chapter 6 I will introduce some of these models, the models of Loop Quantum Cosmology, which follow this interpretative strategy for the problem of time. However, I will argue that one has to be careful with taking seriously these models, as the fact that one can easily write a Schrödinger equation for these cases and that some choices of time may seem natural in this context are just a consequence of the simplicity of the models and it does not solve the above mentioned issues.

Finally we find refinement of this family of approaches which is the semiclassical approach[89]. In this approach, not every solution to the Wheeler-deWitt equation can be interpreted using concepts like time or probability, but only some of them can be given such an interpretation after a semiclassical interpretation is in place. For these states one is able to approximate a Schrödinger equation with respect to some suitable parameter. This approach is susceptible to the criticisms above against this kind of relational evolution, but on top of that it also faces some additional challenges. For instance, it is reasonable to ask about the interpretation of states which are not semiclassical or about how to interpret states where the approximation breaks.

Besides this, the typical strategy to justify the approximations in place is to appeal to decoherence, but this strategy has some conceptual and technical issues. Decoherence in this context[90] is understood in the following way. First, one splits the degrees of freedom of the system into two groups: the relevant ones and the irrelevant or environmental ones. If one just focuses on the relevant degrees of freedom, the interaction with the environment will typically make the state for these degrees of freedom look classical, and hence some semiclassical approximation could be justified. One big problem in the context of quantum gravity is of course that we have appealed to the dynamics of the system and its interaction with an environment, but if what we want to justify is precisely the existence of a time variable and a dynamics the argument risks being circular. Moreover, the split between system and environment seems question begging in the context of quantum gravity. For these and other reasons, this appeal to decoherence[91] was not considered successful by authors like

[89] For works in this approach see for instance Kiefer (2012) and Anderson (2007a,b). See also Bojowald et al. (2011b,a) for an elaborated approach which explicitly deals with non-deparametrizable models. This approach seems to be vulnerable to my conceptual objections here, although a more detailed analysis is beyond the scope of this thesis.

[90] See again the reviews Kuchař (1992) and Isham (1993).

[91] As a side remark let me mention that there is a better account of decoherence that does not

Isham and Kuchar[92]. Despite this, authors like Kiefer (2012) have kept appealing to decoherence for justifying semiclassical interpretations of quantum gravity and they have met with similar criticisms[93].

Let me mention that the semiclassical interpretation of quantum gravity connects with the discussion about spacetime functionalism in chapter 3. Indeed, approximating solutions of the Wheeler-deWitt equation to solutions of a Schrödinger equation can be seen as a form of a functional argument for recovering a time parameter. However, if the only states that get interpreted are the semiclassical ones (assuming that the technical difficulties can be overcome), it seems that rather than explaining how a spatiotemporal structure emerges from a non-spatiotemporal one just has a physical theory for the cases for which the semiclassical argument applies. A different way of putting this is to say that we lack any theory about an hypothetical fundamental level which would be described by a solution of the Wheeler-deWitt equation and that would be essentially non-spatiotemporal. In this sense, the problem would not be with functionalism, as, if we ignored the criticisms which make the approach unattractive, we have a clear idea of the way the time parameter is defining an ordering with respect evolution is defined. Instead, the problem is with the fundamental level, for which we do not really have an interpretation.

With this I close my criticisms to these resolutions which are based on looking for an internal time. To insist, the fact that general relativity is not deparametrizable and that time is not encoded in its basic variables makes these approaches untenable, as the comparison with the example of the double harmonic oscillator showed.

4.7.3 Probabilistic resolutions

Let me now move to a different family of resolutions of the problem of time. In this family, solutions of the Wheeler-deWitt equation are taken to represent the wavefunction of the universe and to encode a probability of finding the universe in certain ways. Different approaches differ in exactly the way these probabilities are encoded and in the way these probabilities should be interpreted.

First, Julian Barbour (1994a,b, 1999) directly follows one of the possible routes I sketched in the previous section when considering the quantization of the system of

need to distinguish between different kind of variables which is the notion used in Wallace (2012) for defending the Everett interpretation of quantum mechanics. However, even appealing to this notion of decoherence would not avoid the conceptual problem of appealing to a dynamical process for justifying the existence of time and a dynamics.

[92] Besides this conceptual worry with decoherence, they also raise several more technical worries related to some problems which arise from the fact that the Wheeler-deWitt equation does not involve imaginary factors and with the fact that superpositions seem to spoil the semiclassical approximation.

[93] See Chua and Callender (2021).

harmonic oscillators: he argues that the quantization of general relativity leads one to a formalism without dynamics, but instead of rejecting it, he tries to make sense of it. The way he does so is by taking the wavefunction to directly define a probability density in the space of 3-geometries, which is interpreted as the probability density that a given 3-geometry is actualized. This of course raises two important questions: what does it mean that a 3-geometry is actualized and how should one interpret this probability. Barbour claims that in this context the difference between the Copenhagen interpretation and the Everett interpretation dissolves, and that they should be seen as offering just an external and an internal account of quantum systems. From the external perspective, Barbour claims that we should be interpreting the probability as if one were picking at random one 3-geometry from a 'heap of actualities' which contained a number of 3-geometries in the proportion described by such probability density. This interpretation is clearly problematic, as the picture of picking up one state of the universe at random from a heap is hardly to be understood literally.

For understanding the internal perspective and what agents in Barbour's instants experience we need to say a few words about Barbour's view on time on the light of the Wheeler-deWitt equation. Barbour claims that time, understood as a succession of instants of time or configurations of the universe, does not exist. Accordingly, these instants of time exist or are actualized on their own and the idea of the universe having a history is something like an illusion. Barbour introduces the concept of time capsules for explaining this illusion: a time capsule is a physical configuration which looks like if there has been a past. For instance, the instant I am presently living on has plenty of time capsules: the letters on my screen seem to be records of me typing them, the scar on my finger reminds me of that time when I cut my finger, and so on. In Barbour's view, what we take to be records or memories are just these time capsules which actualized just with everything else in the instant we are living in and not any evidence that there is something like a succession of instants. It is interesting to see this case as an example of spacetime functionalism as introduced in chapter 3: Barbour does not recover a spacetime, but his argument aims to explain some of the features we usually ascribe to time, even if just as an appearance or illusion.

This particular view on time makes it the case that it seems difficult to treat Barbour's probabilities in the same way that probabilities are treated in some Everettian approaches like the one in Wallace (2012). In these approaches probabilities are associated with the credences a rational agent would form. That is, according to Wallace a rational agent who lives in a branching universe would use Born rule to 'place their bets' about a quantum experiment they are about to perform. However, this implies some diachronicity, as agents are forming credences regarding later experimental outcomes. In the case of Barbour's view, agents just live for one instant and there is nothing like a flow of time so the agent-theoretic strategy for recovering probability is really unlikely to work in this case. Moreover, it is hard to see how typically quantum phenomena like superposition, interference or entanglement would be represented in this view.

In summary, Barbour's interpretation of probability is wanting from both the Copenhagen/operational point of view and the many worlds point of view, as I have argued that both the 'heap of actualities' picture and the internal agent picture are problematic. I will now consider other authors which have taken the wavefunction to encode probabilities.

Hawking and collaborators[94] have defended the view that the wavefunction encodes the probability to find the universe at a certain state. This view has been called as the naive Schrödinger interpretation[95] and in as far as similar to Barbour's view, it is subject to a similar criticism[96]. A possible difference is that in the work by Hawking and Hartle, the probability is taken to be the probability for a universe with a given 3-geometry to have appeared from nothing. We can take this to mean something different from what Barbour had in mind, but in any case it is difficult to make sense of such claims. Even if we were satisfied with this picture for something like the 'creation' of a 3-geometry, we would still be lacking a theory of how this geometry evolves.

Isham (1993) takes a different interpretation of what the naive Schrödinger account means, as he takes the approach to define probabilities for finding whole spacetimes and not just spatial configurations. This approach is also problematic from the conceptual point of view for two reasons. First, one may have reservations about having a probability defined for a whole universe and not for possible evolutions given an initial state. As a universe is a one-time event one can raise question about the epistemic access we may have to this probability and the role it plays in our theory. Second, if we leave aside these worries, what one is able to obtain according to Isham is a probability for a classical spacetime which is arguably not what we were looking for in a theory of quantum gravity. In other words, it seems that we do not have quantum phenomena like superposition, interference or entanglement if we just have a probability distribution over classical spacetimes.

Approaches like this also face a number of technical difficulties that make them definitely unattractive. Isham and Kuchar also consider some refinements of these approaches, like conditional probabilities approaches and the consistent histories approach, and find that them also face similar technical and conceptual difficulties. The consistent histories approach is related also with the covariant formulation of quantum mechanics and I will come back to it in chapter 7 to discuss it as an inter-

[94] See for instance Hartle and Hawking (1983).
[95] Name coined in Unruh and Wald (1989).
[96] The naive Schrödinger interpretation is previous to the work of Barbour.

pretation of the covariant formalism and to argue that it is likely to collapse to either the Copenhagen or the Everett interpretation if pushed on some points. I will argue that covariant approaches do not escape the different difficulties associated with the problem of time and that interpretations like the consistent histories one do not help in solving them.

I can conclude by insisting in that probabilistic resolutions for the problem of time are not convincing, as they define some probabilities of finding either a 3-space or a spacetime but they do not seem to be satisfactory theories. In particular, the interpretation of these approaches is problematic and there is a risk that no quantum phenomenon is represented by them.

4.7.4 Frozen observables

Next, there are two different approaches I want to comment on. Interestingly, both of the have been defended by Carlo Rovelli at different moments of his career. The first view is called the evolving constants of motion or frozen time formalism and was developed in the early nineties. This approach takes as a starting point the view I have argued against in section 4.3 that the physical content of general relativity is contained in the functions of phase space which have vanishing Poisson brackets with the constraints. In a nutshell, in the quantum version of this view, the physical content of quantum gravity is argued to be encoded in the quantum observables which correspond to quantizations of the classical 'observables'.

A first thing to notice about this approach is that from a technical perspective it is really challenging to find phase space functions in general relativity which have vanishing Poisson brackets with the Hamiltonian and diffeomorphism constraints. See for instance Dittrich (2005) for a technical review in which the technical difficulty for finding such quantities is recognized and it is also argued that such observables would correspond to non-local functions. Even worse, if these quantities were to be found in the classical theory, their quantization might be challenging, as there could be non-polynomial functions and operator ordering ambiguities. In the reviews by Isham and Kuchar this issue was already spotted[97], together with other technical difficulties related with the problems we have found in other approaches, such as the lack of a global time function or the difficulty to find a suitable inner product.

A further criticism comes from the analysis I have performed in sections 4.2 and 4.3. There, in agreement with Maudlin and Kuchar, I have argued against the view that one should only consider quantities which have vanishing Poisson brackets as observables. If we translate this to the quantum formalism, it seems that in principle one should reconsider what one takes to be observable. However, there is a prob-

[97] See also the criticism in Anderson (1995).

lem with this, as even if one wanted to consider observables which do not commute with the constraints, in the quantum formalism this is not possible, as the action of corresponding operators would take the state outside of the physical Hilbert space. This is another consequence of the double role that constraints play in quantizing a constrained theory: while in phase space we can think on a gauge-dependent state which satisfies the constraints, this is not possible in the quantum theory because a state which obeys the constraints is automatically invariant under a transformation generated by these. In this sense, we find the version of the problem of time expressed in terms of observables: in the same way that we would like to have states which were evolved using the Hamiltonian but we cannot get them given that it also plays the role of a constraint, we are not able to keep the observables we would like to have because of this double role of the Hamiltonian.

We can raise a different kind of conceptual worry. The frozen observables are meant to represent something like the observable X_T represented for the example of the relativistic particle, i.e., the position of the particle at the time T. Similarly in general relativity we would be seeking for an observable of the form $O_{\phi_1\phi_2\phi_3\phi_4}$, which would represent the value of an observable O when 4 other function take some set of values. For this to be weel-defined we would have to require that the set of 4 functions uniquely determine the value of O, which for a set of non-trivial observables O implies that the four functions work as spacetime coordinates. But then what we are effectively doing is deparametrizing general relativity, which is something I have argued against. Therefore, one can reasonably reject the frozen observables strategy from this perspective.

This becomes even clearer if we compare with the case of the double harmonic oscillator. As I argued in section 4.2, one cannot define an observable X_Y or Y_X which would represent the position of one of the oscillators when the other one is at a given position. The reason for this was of course that the relation between the position of any of the oscillators and time is not invertible and that we cannot take the position of the oscillator to represent a time variable. In other words, the system is not deparametrizable. In the same way, we can expect that one cannot build the operators $O_{\phi_1\phi_2\phi_3\phi_4}$ for the case of general relativity. Moreover, I also argued that the quantities with vanishing Poisson brackets with the constraint in the case of the double harmonic oscillator are functions of the form $f(E_1, E_2)$, where E_i are the energies of the oscillators, and that one cannot interpret them as representing something like we would expect X_Y to represent. Similarly, we can reasonably expect that if one were able to build an operator which commutes with the quantum constraints in the case of general relativity, it would not represent any relational or frozen observable.

From a more conceptual point of view, Rovelli (1991a) proposed to interpret this formalism as a generalization of the Heisenberg picture of quantum mechanics.

Indeed, the frozen observables X_T for the relativistic particle can be analyzed from this perspective, even in the classical case. In the same way that in the Schrödinger picture of quantum mechanics we have a state that evolves, the classical 'Schrödinger picture' is the usual way we think about evolution in classical mechanics, that is, as a system moving around phase space. One can see the evolving constants of motion X_T to be the classical equivalent of the Heisenberg picture: even if we do not allow the system to evolve in phase space we can get the physical predictions of the theory by analyzing the observables X_T and how they evolve with respect the parameter T. In this sense, by looking for a resolution of the problem of time in terms of observables one proposes a resolution which is based on a Heisenberg picture of quantum mechanics.

However, shifting to talking about the Heisenberg picture of quantum mechanics does not solve the above issues and Rovelli argued for a generalization of the Heisenberg picture based on some of them. In the usual Heisenberg picture we have a quantum state and a series of operators or observables $\hat{O}_t$ which are labeled with a time parameter. Making use of the rules of quantum mechanics the state defines certain probabilities and expectation values for measurements of the observables occurring at time t. Rovelli proposes to generalize this so that the set of observables one considers is not necessarily labeled by a time parameter and which a priori does not have to satisfy relations like the relations that operators in the usual Heisenberg picture have to obey in order to preserve unitarity and conservation of probability. In this way, the observables in the physical Hilbert space, together with a state in this space define a series of probabilities and expectation values, and this would be taken to represent the physical content of the theory.

There are two main objections to this view. First, as I have pointed out above, it seems that no coincidence observable analogous to X_T can be found for the case of general relativity, both for conceptual and technical reasons. As in the case of the double harmonic oscillator, the operators which can be defined on the physical Hilbert space of the theory do not seem to have any nice interpretation. If we applied Rovelli's proposal to the Hilbert space of the double harmonic oscillator we would have to recover evolution from observables which are just functions of the energies, which does not seem possible. Rovelli's proposal clearly does not solve the problem of time for the double harmonic oscillator and we can expect it to be equally unsuccessful for the case of general relativity, given the analogy between both models.

The second objection is that we lack any interpretation for such a generalized Heisenberg picture. Quantum mechanics in its Heisenberg picture is just equivalent to any other formulation of the theory, and it can be understood using any of the realist interpretations of quantum mechanics or even some more operational interpretation as the Copenhagen one. However, this generalized picture does not get easily interpreted.

As operators are no longer functions of a time variable we cannot represent evolution by means of a wavefunction in configuration space[98] changing in time and we have trouble for applying any realist interpretation. Rovelli's approach has certainly an operationalist flavor, but even from this perspective it is unclear what it would mean to measure an 'observable' $\hat{O}$ in the physical Hilbert space of the theory.

Notice that these objections exemplify the general objection I anticipated in section 3.4 and which will apply to most approaches I will study in this thesis. That is, in this case, as will be the case in general, I find that the proposed approaches or resolutions for certain problems are not well justified or directly do not work and even if this could be left aside we find ourselves with some piece of formalism for which no clear interpretation is available.

4.7.5 Transition amplitudes

Let us turn to the other way in which Rovelli deals with problem of time of quantum gravity, which is basically the application of same approach I have analyzed in section 4.6.3 but applied to the case of general relativity. This approach has become specially important in the recent years, as the quantum gravity community working on spin foam models is highly influenced by the views by Rovelli. Later on, in chapters 7 and 8 I will come back to study the relationship between this view and covariant quantizations of general relativity.

In this approach, instead of asking about the probability of measuring a certain observable as defined by an operator one considers the probability of finding a final state given that one started with an initial one. For this, we need to define a little bit of mathematical structure. First, we can define the map η[99] which maps states in the kinematical Hilbert space to states in the physical Hilbert space:

$$\begin{aligned} \eta : \mathcal{H}_{kin} &\rightarrow \mathcal{H}_{phys} \\ |\psi\rangle &\rightarrow \eta(|\psi\rangle) = (\eta(\psi)| \, . \end{aligned}$$

I have introduced the notation $(...|$ to make clear the distinction between states in the kinematical Hilbert space and states in the physical Hilbert space, which from the point of view of the kinematical Hilbert space they are distributions[100]. The action of the distribution can be used for defining an inner product in the physical Hilbert

[98] Even if this picture is usually associated with the Schrödinger picture the wavefunction can be defined for either picture: $\psi(q,t) = \langle q|\psi, t\rangle = \langle q, t|\psi\rangle$. The first equality gives the definition in the Schrödinger picture while the second one gives it in the Heisenberg one.

[99] Sometimes referred to as the projector, even if strictly speaking only is a projector for constraints with discrete spectra, as explained in section 4.4.

[100] This is the case for Hamiltonians with continuum spectra.

space:

$$(\eta(\psi)|\eta(\phi)) = (\eta(\psi)|\phi\rangle\,. \tag{4.113}$$

In the case of the parametrized particle the map η maps (square integrable) functions of position and time into solutions of the Schrödinger equation and the inner product defined in this way gives us the standard inner product of Schrödiger quantum mechanics. The map η in a way contains all the dynamics of Schrödinger equation. A way of seeing this is by considering two (improper) quantum states in the kinematical Hilbert space of this theory $|x_1, t_1\rangle$ and $|x_2, t_2\rangle$, that is, two states concentrated around two well-defined positions and times, and apply the machinery above to find that the inner product of their counterparts in the physical Hilbert space is:

$$(\eta(x_1, t_1)|\eta(x_2, t_2)) = \langle x_1|e^{\frac{-i(t_1-t_2)}{\hbar}\hat{H}}|x_2\rangle = K(x_1, t_1; x_2, t_2)\,. \tag{4.114}$$

On the right hand side what we find is an expression in the standard Schrödinger picture which has a straightforward meaning: it is the probability density amplitude for observing the particle at position x_1 at time t_1 given that it has been observed at position x_2 at time t_2. More importantly, this transition amplitude is the propagator K, which allows one to build solutions to the Schrödinger equation starting from any initial state. I will give a more detailed definition of the propagator in chapter 7.

Rovelli proposes generalizing this to the case of general relativity, i.e., to interpret the physical inner product of the 'projection' of two states in the kinematical Hilbert space as be giving the probability of measuring one given that the other has been measured. This proposal is the same as in the toy model in Reisenberger and Rovelli (2002) that I have analyzed in section 4.6.3 and one finds the same conceptual problems for understanding it. That is, the way these probabilities are to be interpreted is completely mysterious.

Notice that in the case of the parametrized particle we found the propagator of the Schrödinger equation as a 'transition amplitude' but that this will not be the case for the quantization of general relativity. The reason for this is that for interpreting $(\eta(x_1, t_1)|\eta(x_2, t_2))$ as a propagator we need to identify the variable t as a time variable and that certain technical conditions are satisfied. Again we find that a deparametrization is in place and again I argue that as time is not a variable in the configuration space of general relativity, this interpretation is not available. In this sense, we cannot use the transition amplitude strategy for building a propagator and recovering a wavefunction evolving in time. Therefore, we are left with just some uninterpreted probabilities like the probability of finding a final geometry given that one observed an initial one. As I commented in section 4.6.3 the interpretation of these probabilities is unclear, and I will expand on this criticism when I study how this transition amplitude strategy is supposed to work in other approaches.

Furthermore, here we can again compare with the case of the double harmonic oscillator. In this case, we find that the transition amplitude $(\eta(x_1, y_1)|\eta(x_2, y_2))$ is just an inner product between energy eigenstates, and it neither has any probabilistic interpretation nor defines a propagator. In this sense, for general relativity one can doubt that what Rovelli is defining has anything to do with physical transitions.

This resolution of the problem of time will be present in LQG and it will also play a role in interpreting spin foam models and other covariant approaches to quantum gravity. Therefore, I will later on come back to it and apply my criticisms to those particular settings. To insist, the transition amplitudes strategy does not define a propagator from which to recover a Schrödinger-like evolution, seems to be ill-justified for non-deparametrizable models and defines a set of probabilities with no clear interpretation.

4.7.6 Conclusions and alternatives

My conclusion after analyzing the proposed resolutions of the problem of time is that most of them are based on some sort of analogy with deparametrizable models, for which these resolutions can work. However, as general relativity is not deparametrizable, the situation is like the case of the double harmonic oscillator, for which the above resolutions do not work and for which we reach the conclusion that the canonical quantization formalism does not give a meaningful quantization. The conclusion for general relativity may very well be just the same. I have argued that if one wants to insist on some of the resolutions proposed, and is able to deal with the technical complications, one nevertheless faces a great challenge at the time of giving an interpretation to the formalism. In the rest of this thesis I will expand this conclusion to LQG and LQC and also to covariant approaches.

Before this, let me finish this section by mentioning two alternative views which agree that the problem of time is not solved by any of the resolutions studied above, but propose different approaches, starting from the fact that the physical Hilbert space of general relativity does not define a time evolution.

First, there is room for a Bohmian resolution of the problem of time, as explored in Vassallo and Esfeld (2014) for a version based in LQG. As in Bohmian mechanics the dynamics one ultimately cares about is the dynamics of the Bohmian particles, one can perfectly make sense of the theory even in the case that the wavefunction is static in time. For the case of general relativity one would have to define the equivalent of the particles, which would probably be 3-geometries, and define a dynamics for it. This proposal seems able to overcome the conceptual difficulties raised by the problem of time, although one can worry about how is the guidance equation to be constructed or the fact that solutions to the Wheeler-deWitt equation are just real functionals, and standard guidance equations need of complex wavefunctions in order to get evolution for the particles.

Finally, let me also mention the work by Thebault and Gryb[101]. Their analysis of classical general relativity agrees with the position I have defended in section 4.3 that the transformation generated by the Hamiltonian constraint should not be treated as a gauge transformation. Similarly, they criticize the relational strategies for interpreting the physical Hilbert space that one gets by applying the canonical quantization techniques to general relativity in its ADM formulation. Moreover, Thebault and Gryb propose an alternative which is able to avoid the problem of time. Their alternative consists in instead of quantizing general relativity, quantizing shape dynamics. Shape dynamics is a theory which is claimed to be physically equivalent to general relativity under certain conditions. Shape dynamics and general relativity however disagree on the symmetries that they instantiate: while shape dynamics does not have the familiar space-time diffeomorphism invariance of general relativity, it has some extra conformal invariance which means that it is only shapes and no absolute scales which matter. Shape dynamics has a different ontology: it describes the evolution of 3-spaces with respect to a time parameter. Although this theory still has a temporal reparametrization invariance, Thebault and Gryb propose to deal with it in a similar way to the way one can solve the problem of time for the double harmonic oscillator, i.e., by relaxing the constraint equation and letting it define a dynamical equation[102]. This construction can be criticized[103], but further study and comment are beyond the scope of this thesis.

Let me finish by giving an overview of this chapter. In the first part I have introduced the constrained formalism for reparametrization invariant theories and I have argued that this symmetry in the case of general relativity is not a gauge transformation in the sense that it does not transform between physically equivalent states of affairs at a given time. I also argued that as a consequence of this, proposals to change the metaphysical picture one has about space and time or to search for a different notion of observable are misled. I have also noticed that one can distinguish between reparametrization invariant theories which are deparametrizable and theories which are not. I have argued that general relativity is of the latter kind and that none of the variables in its configuration or phase spaces represents time. In the second part of the chapter I have introduced the canonical quantization procedure for constrained systems and the problem of time for reparametrization invariant systems. Here I have

[101] See Gryb and Thébault (2014, 2016a,b).

[102] Indeed, the way this is suggested to be done is by introducing explicitly the time parameter in the phase space of the theory and defining a new constraint which would replace the original Hamiltonian constraint while allowing for evolution.

[103] In particular one may wonder for the justification for replacing one constraint with another, and the relation of the approach to approaches like unimodular gravity, that were criticised and rejected in the nineties (see Kuchař (1992); Isham (1993)), may be seen as worrisome.

argued that the distinction between theories defined on extended configuration spaces and theories defined just of configuration spaces plays an important role as for theories of the first kind one can apply some deparametrization procedure to obtain a quantum theory while in the second case this seems artificial and unjustified. For this reason, I have criticized the attempts of resolution of the problem which are based in some way or another in trying to follow such a deparametrization or relational strategy. I have also studied a number of other proposals which implied a reinterpretation of the quantum formalism, and I have argued that these interpretations have to deal with some conceptual issues in order to appear as solid alternatives. In this sense, I have reached the conclusion that the problem of time should make us consider rejecting the canonical quantization of general relativity as a route to quantum gravity.

In this sense, the analysis in this chapter supports what I argued for in chapter 3, namely that the approaches to quantum gravity I am considering, as they currently stand, do not lead to satisfactory quantum theories and that the claims of emergence of spacetime are unjustified, as it is unclear the way these approaches can be interpreted and the way they are to describe a non-spatiotemporal reality. In the following two chapters I will study LQG and LQC, which are two canonical approaches for which there is a problem of time and for which one can extend my analysis in this chapter. After this, I will move to consider covariant approaches and I will find that the diffeomorphism invariance of general relativity will lead to similar problems and to the same negative conclusion.

5. LOOP QUANTUM GRAVITY

In the previous chapter I introduced the canonical quantization program as a 'recipe' or guideline to construct a quantum theory starting from its classical counterpart. For the case that interests us, general relativity, there is a whole family of theories and research programs that were born by applying this program to one or another formulation of general relativity. This family of theories ranges from the original Wheeler's quantum geometrodynamics to the current, more sophisticated loop quantum gravity (LQG). In this chapter I give an overview of the canonical version of this theory, analyze its philosophical implications and argue for two main claims. First, at a kinematical level, the quantization chosen is quite peculiar and a few objections and worries will be raised about it. This justifies having a skeptic position about the Hilbert space structure of the theory and its kinematical predictions, such as the discreteness of the spectra of some geometric operators. Second, as a canonical quantization of general relativity, it suffers from the problem of time, and as I argued in chapter 4, this problem should make us reject the theory or leaves us with some formalism that, to my judgment, has not been given a plausible physical interpretation.

The original quantum geometrodynamics developed in the 1960's had several formal problems (in addition to the conceptual ones) that motivated the search for an alternative. In the 1980's general relativity was expressed in a new set of variables, the connection variables[1], which turned out to allow for a quantization process that avoids some technical difficuties of the Wheeler-DeWitt quantization and lead to the original formulations of LQG.

LQG originally was developed by applying the canonical quantization program to general relativity expressed in connection variables. In particular, early formulations used a particular set of variables, loop variables, which gave name to the theory. Current formulations of the theory have somehow emancipated from the original canonical approach and instead are formulated in a different way, known as the covariant formulation and they will be analyzed in the last part of this thesis.

In this chapter we will deal with the canonical approach, not only for historical reasons, but also because it will allow for a natural transition from the previous

[1] First introduced in Ashtekar (1986).

chapter and because much of the machinery developed in the canonical formalism will be later used in the covariant one. Let me group the steps in the development of the theory following the canonical quantization program in four blocks[2]:

1. Formulation of the classical theory, general relativity, in terms of convenient variables, the connection variables. Definition of the classical Poisson algebra for the relevant quantities, i.e., the holonomy-flux algebra.

2. Definition of a kinematical Hilbert space in which to represent the quantum counterpart of the holonomy-flux algebra. This is the Hilbert space of square integrable functionals of the connection with the Ashtekar-Lewandoski measure.

3. Imposition of the 'kinematical' constraints: Gauss and spatial diffeomorphism constraints. This defines the s-knot Hilbert space.

4. Imposition of the 'dynamical' constraint, that is, the Hamiltonian constraint. This defines the physical Hilbert space and the final quantum theory.

Notice that I have separated the imposition of the Hamiltonian constraint from the rest of constraints, while in the schema in the previous chapter the imposition of all the constraints was made in one step. This is of course because it is the one that defines the dynamics and the one that is associated to the problem of time and deserves a separated discussion. In this chapter I will explain how these four steps are taken in canonical LQG and along the way a number of philosophical and interpretational issues will arise. Let me briefly mention them ordered by the steps of the quantization program they will arise at.

1) When formulating general relativity in connection variables the following issues will arise:

- The formulation of general relativity in terms of the connection variables introduces gauge degrees of freedom in the theory. I will argue that this should be interpreted from the classical point of view as any other gauge theory, that is, one has to consider the gauge structures just as mathematical tools that are convenient, but the physical content of the theory lies just in the gauge invariant part. I will also note that this may be revised once we consider quantum phenomena and the presence of fermions.

[2] These correspond to the three steps outlined in the previous chapter, where I have divided the step related with the imposition of the constraints into two: one for the 'kinematical' constraints and another for the 'dynamical' or Hamiltonian one.

- The new formulation of general relativity is equivalent to the standard one if we ignore the gauge symmetry. I argue then, that the interpretation of the theory should not change from one formulation to the other. In this sense, what discussed in chapter 4 about the interpretation of general relativity still holds. Importantly, our expectations for what to expect when quantizing the theory or what to consider a successful quantization should remain the same.

- I will also discuss a truncation of the degrees of freedom of general relativity to a theory in which space is divided into cells and the physical degrees of freedom can be represented by a graph. I will argue that general relativity can be seen as a limit of more and more refined graphs and that we one can even claim that general relativistic space emerges (in the reductionistic sense defended in chapters 2 and 3) from this structure. This will be also important when we study the quantum theory, as a similar graph structure will be present.

2) The second step of the program is the definition of the kinematical Hilbert space and the following will be discussed:

- The most relevant ingredient for the definition of the kinematical Hilbert space is the choice of inner product. In this case the Ashtekar-Lewandoski measure was chosen because the inner product it defines is well-behaved under gauge and diffeomorphism transformations. However, it is a peculiar choice that is very different from more standard quantizations, like the Schrödinger quantization. In particular, it is a case of polymer quantization, a family of quantizations which have the property of inducing some sort of discreteness and that leaves some operators undefined. This last feature complicates the definition of the Hamiltonian constraint and the physical Hilbert space.

- The relevant states in the kinematical Hilbert space are networks of a finite number of edges. This choice leaves out two potentially interesting types of states to consider. First, we could have preferred to have infinite networks which would be analogous to the infinite spaces sometimes considered in cosmology. And second, we could also consider infinitely fine networks. Intuitively, these networks would collect smaller-scale information and leaving them out implies having something similar to a cut-off which complicates defining a continuum limit of the theory.

- Another topic that I will discuss is the separability of the Hilbert space. The kinematical Hilbert space is not separable, which means that there is no countable basis for it. This makes the Hilbert space much bigger than what is usual in quantum mechanics and quantum field theory and means that many of the theorems and techniques physicists employ would not be valid for this theory. For this reason, and mathematical tractability, non-separability is seen as a unwanted feature. However, I will note that non-separability is not a surprising feature of a Hilbert space that arises from a quantization of a field theory and that this feature disappears when the constraints are imposed.

3) In the third step we define the s-knot space, which is the result of imposing the momentum and the Gauss constraints related with the spatial diffeomorphism and the internal gauge symmetries of the connection formulation of general relativity. I will discuss the following:

- First, states in the s-knot space are described by knot classes and superpositions of them. A knot class is defined by a number of vertices and links between them and the way these links are knotted around each other. This information is diffeomorphism invariant. Links and vertices carry some quantum numbers, called spin and intertwiners. These states are argued to be eigenvectors of area and volume operators, and I will discuss their possible physical interpretation as representing discrete spaces. In the covariant formulation similar states will be present and the same interpretation will apply to them (see chapter 8).

- I will show how some information like the way knots are knotted does not have a physical meaning in this picture and complicate the interpretation. For this reason, in some recent formulations of the theory knots are replaced by abstract graphs. Nevertheless I will argue that this is an ad hoc move that takes us away from the strict canonical formulation.

- I will explore the relation between the discrete space encoded in the s-knot states and the continuum space of general relativity. This case is one of the most discussed examples in the philosophical literature about the emergence of spacetime and spacetime functionalism, as I discussed in section 3.3.2. I will expand the discussion and I will argue that a functionalist account can be successful in explaining the way relativistic spaces can be reduced to some selected spin network states.

- I will also comment on some criticisms to the theory which say that the discreteness of space of the theory is a consequence of the gauge group chosen, SU(2), being compact, and that the gauge group should instead be SO(3,1), the gauge group of full general relativity. If this were so, the spectrum of the area and volume operators would be continuous and not discrete.

The issues raised in the first three steps of the quantization process lead me to the conclusion that there are strong reasons to worry about the quantization of general relativity carried out in LQG at this kinematical level. It may be a consistent quantization, but one has to be aware that it is a quantization that is very different from the intuition we had from geometrodynamics. The states it defines can be seen as describing some sort of distributional geometry, which is conceptually troubling. Therefore, there are also reasons for doubting about this particular quantization and about the main prediction and interpretation of LQG at this kinematical level, namely that space is discrete or even made of atoms.

4) Finally the last point of the program is the imposition of the Hamiltonian constraint and the definition of the physical Hilbert space. At this point I will comment first on the technical difficulties that have impeded this last stage to be completed satisfactorily. Then, I will argue that there is a problem of time for this theory, just as there was one for geometrodynamics. In the same way I argued in the previous chapter that the attempts to solve the problem of time for geometrodynamics are misguided as they are inspired by resolutions which work for deparametrizable models, the same applies for the case of LQG. In this sense, I will argue that these strategies fail to give us satisfactory interpretations of the formalism of LQG. In particular, I will argue for this conclusion for the two most common resolutions of the problem of time in LQG: the evolving constants of motion approach and the transition amplitudes approach.

All these issues will be discussed in the different sections of this chapter. These sections contain more technical subsections in which the mathematical details are explored and more philosophical ones in which the topics above are treated. In particular, section 5.1 deals with step 1, section 5.2, with steps 2 and 3 and section 5.3, with the final step 4.

5.1 Reformulation of general relativity: connection variables

In this section I will introduce and discuss the formulation of general relativity that is used as the starting point for applying the canonical quantization program in LQG. This formulation is based in connection variables instead of using the metric ones of the ADM formalism. I will show how these variables are defined and how they lead to a formulation of the theory as a Yang-Mills theory. LQG applies the techniques developed for the Yang-Mills theory, such as the Wilson loops, in a way that will prove useful for its later quantization. Before moving to quantization I will also introduce the concept of graph, that will be useful and can already be defined at a purely classical level. Finally, in the last subsection I make some philosophical remarks about the reformulation of the theory.

5.1.1 Tetradic formulation of general relativity

The first step in this construction is introducing the tetrad formulation of general relativity. To define the tetrad field we start with a solder form over the manifold:

$$e^I(x) = e^I_\mu(x) dx^\mu \,. \tag{5.1}$$

Here, and throughout this chapter I will use greek indices $\mu, \nu ... = 0, 1, 2, 3$ for manifold or spacetime coordinates, capital latin indices $I, J ... = 0, 1, 2, 3$ for internal Minkowski indices and lower-case latin indices $i, j = 1, 2, 3$ for these indices restricted to just the spatial part. In this chapter, unless stated otherwise, we will restrict ourselves to the 4-dimensional theory. The e^I take vectors from the tangent space of the manifold and carry them to a vector space V with inner product defined using the Minkowski metric η_{IJ}:

$$\begin{aligned} e^I : T_p M \to V \\ v^\mu \to v^I = e^I_\mu v^\nu \,. \end{aligned}$$

The inner product in the space V, together with the form e^I also induce a inner product on the tangent space of the original manifold and hence a metric tensor on the manifold:

$$\begin{aligned} \langle u, v \rangle = \eta_{IJ} u^I v^J =& \eta_{IJ} e^I_\mu u^\mu e^J_\mu v^\mu = g_{\mu\nu} u^\mu v^\nu \\ g_{\mu\nu} =& \eta_{IJ} e^I_\mu e^J_\mu \,. \end{aligned}$$

Therefore, one can take the e^I as fundamental and defining the metric tensor and all metric notions on the manifold. As usual, we will use the metric $g_{\mu\nu}$ and its inverse to raise and lower spacetime indices. But now we will also use η_{IJ} for the internal indices and e^I_μ and its inverse e^μ_I to change between spacetime and internal indices.

Moreover, the inverse e^μ_I is a set of four vector fields on the manifold that at each point define a set of four orthonormal vectors[3] and is called the tetrad field. The tetrads define reference frames that are locally Minkowskian.

A key feature of this formulation is that there is gauge freedom to choose the tetrad field. This can be seen by noting that one can always rotate the tetrads at each point to get another set of four orthonormal vectors. This is nothing else but a change of orthonormal basis in the internal space V. More concretely, the group of transformations of this gauge symmetry is the group of local Lorentz transformations in the internal space, i.e., local SO(3,1) transformations. See figure 5.1 for an illustration of this in three dimensions. Notice also that this gauge symmetry is not present in the standard formulation of general relativity and it is different from a

[3] It is trivial to check that $g_{\mu\nu} e^\mu_I e^\nu_J = \eta_{IJ}$

choice of coordinates. Indeed, the solder forms e^I and the tetrad vector fields e_I are tensorial objects and as such they are independent of the coordinate system used for describing them.

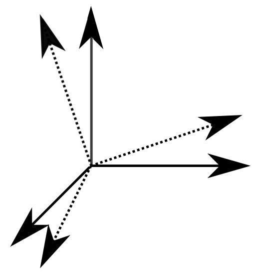

Fig. 5.1: Two gauge equivalent sets of triads in three dimensions. They are related by a gauge transformation, that is, a rotation in 3 dimensions. In four dimensions one has four vectors, a tetrad, and the group of rotations is bigger.

As in the standard formulations of general relativity, one also needs to define an affine connection that allows to connect the tangent spaces at different points. In the tetrad formalism this is encoded in the one-form:

$$\omega^I_J(x) = \omega^I_{\mu J}(x) dx^\mu \,. \tag{5.2}$$

This is a one-form with values in the Lie algebra of the Lorentz group SO(3,1) and hence satisfies $\omega^{IJ} = -\omega^{JI}$[4]. This allows to define covariant derivatives both in the spacetime (μ) and in the internal (I) indices. The torsion and the curvature two-forms are defined[5] by:

$$T^I = de^I + \omega^I_J \wedge e^J \tag{5.3}$$

$$R^I_J = d\omega^I_J + \omega^I_K \wedge \omega^K_J \,. \tag{5.4}$$

If we impose that the torsion vanishes, $T^I = 0$, we get:

$$de^I + \omega^I_J \wedge e^J = 0 \,, \tag{5.5}$$

[4] The Lie algebras $\mathfrak{so}(3,1)$ and $\mathfrak{so}(3)$ are algebras of anti-symmetric matrices. Therefore, one can give an element of the algebra by giving its matrix components, ω^{IJ}. But one can also define a basis for the algebra, i.e. the anti-symmetric matrices τ^I, and any element of the algebra is specified by giving its components in this basis. Both ways define uniquely an element of the algebra, and for the standard basis we can go from one convention to the other by using the three-indexed Levi-Civita symbol ϵ^I_{JK} in some appropriate form. Elements of these algebras are both matrices and vectors and depending on the context it will be more useful to express them in one form or another.

[5] Here and in the following by '$\wedge$' I denote the external product of differential forms and by d the exterior derivative.

which is an equation for ω^I_J as a function of e^I. The ω^I_J that satisfies this condition is called the spin connection and the covariant derivative it defines is the standard one in general relativity, that is, the one using the Levi-Civita connection.

The curvature two-form is related to the Riemann curvature tensor, and one can write Einstein vacuum equations in terms of it:

$$\epsilon_{IJKL}(e^I \wedge R^{JK} - \frac{2}{3}\Lambda e^I \wedge e^J \wedge e^K) = 0\,, \tag{5.6}$$

where ϵ_{IJKL} is the four dimensional Levi-Civita symbol and Λ the cosmological constant. These equations can be derived from the Palatini action:

$$S[e,\omega] = \frac{1}{16\pi G}\int \epsilon_{IJKL}(\frac{1}{4}e^I \wedge e^J \wedge R^{KL}[\omega] - \frac{1}{12}\Lambda e^I \wedge e^J \wedge e^K \wedge e^L)\,. \tag{5.7}$$

This action is the familiar Einstein-Hilbert action expressed in terms of e and ω and G is Newton's constant. Notice that the Palatini action is generally written as a functional of both the tetrad field and the spin connection, instead of just a functional of the tetrad field. Recall that 5.5 determines ω for a given e, so one could replace ω by its expression in terms of e to get another expression for the action. These two action principles are known as the first and second order formalisms[6], being the first order formalism the one which treats e and ω independently and the second order formalism the one that expresses ω in terms of e and in which second derivatives of e naturally appear. The two formalisms are equivalent in the absence of fermions: varying the action with respect to e in any of the formalisms gives the Einstein equations 5.6 and varying with respect to ω in the first order formalism gives the relation 5.5 between ω and e.

The Palatini action allows to add an extra term known as the Holst term[7]:

$$\frac{1}{8\pi G\gamma}\int e_I \wedge e_J \wedge R^{IJ}\,, \tag{5.8}$$

where the coupling constant γ is the Barbero-Immirzi parameter. This addition does not affect the classical equations of motion. This follows from the fact that varying this term with respect the connection gives again condition 5.5, and when this condition is applied the Holst term identically vanishes. Even if this term does not have any effect in the classical equations of motion, this term will play a role in certain quantum models that I will study in chapter 8.

[6] This distinction also applies to other formulations of general relativity or modified gravity: we call first order formalisms the ones in which metric and connection variables are treated independently and second order in which some relation between the two of them is imposed as for instance that the connection is the Levi-Civita connection.

[7] This was introduced in Holst (1996)

Finally, let me say that there are some reasons in favor of using this formalism besides formulating LQG. The most important one is that it allows to include fermion fields in a covariant way. In this introduction I have not dealt with the matter degrees of freedom of the theory. The tetrad formalism allows for expanding its action to include matter terms, which give the dynamics of the matter degrees of freedom and their effect in gravity. The explicit form of these terms, for all the known kinds of matter, in the tetrad formalism can be found in (Rovelli, 2004, Sect. 2.1.2). In particular, the novelty of the tetrad formulation with respect to the metric one is that these terms include spinor fields which represent fermions. The reason for this is that spinor indices can be chosen to match the internal indices of the tetrads, allowing for a covariant formulation, while in the metric formulation such a construction is not available. However, let me refer the reader to (Pitts, 2012) for a review of an alternative view in which spinors are defined in a different way, such that the tetrad formalism becomes unnecessary. This alternative comes at some cost, as giving up coordinate independence. For this reason let me just say that the presence of fermions motivates the choice of the tetrad formalism, but not because it is the only alternative available to represent them, but because it is a well-known, well-behaved formalism which respects the basic principles of general relativity.

Let me close this introduction to the tetrad formulation of general relativity by pointing out some key ideas. In this formulation we have replaced the metric and connection of differential geometry by the tetrad fields and the spin connection, which carry the same information with an extra of gauge redundant information that corresponds to the freedom one has in choosing a local Lorentz frame. In other words, in this formulation of general relativity there is an additional, local SO(3,1) gauge symmetry.

5.1.2 Hamiltonian formulation

Now we are in a position to move on to the Hamiltonian formalism. Recall from the previous chapter that in the ADM formulation of general relativity one introduces a foliation and has as canonical variables the metric g_{ab} of the three-dimensional time slices and a momentum that is basically a function of the extrinsic curvature K_{ab} on the slices. By solving the Hamiltonian equations of motion one can recover the full evolution of g_{ab} and K_{ab} and reconstruct the full four dimensional geometry $g_{\mu\nu}$.

In the tetrad formalism we can proceed in the same way and introduce a foliation. In this case, there is one gauge that comes in a natural way. This consists in aligning the time-like tetrad field e_0^μ with the vector $\hat{n}$, which is normal to the foliation. By doing this we achieve two things. First, we are able to express the components of e_0^μ in terms of the lapse function N and shift vector N^a of the foliation. And second and more important, the other three tetrad fields become a natural triad for the three-dimensional space, as can be seen from the fact that their 0th component vanishes $e_i^0 = 0$. In other words, in this gauge the four-dimensional vectors become meaningful quantities for the three-dimensional geometry.

In this gauge we can use the triad for the three-dimensional space in the same way we used the tetrad in the four-dimensional space. We can express the three-dimensional metric in terms of the triad:

$$g_{ab} = \delta_{ij} e_a^i e_b^j \,. \tag{5.9}$$

I will be using lower-case latin indices starting from the beginning of the alphabet $a, b... = 1, 2, 3$ for spatial coordinates in the three-dimensional time-slice and lower-case latin indices starting from the middle $i, j... = 1, 2, 3$ for coordinates in the three-dimensional internal space. Notice that now the internal space is a three-dimensional euclidean vector space.

In this gauge the components of the connection ω^{IJ} also split and have direct geometrical interpretation from the three-dimensional point of view. First, we have the 3 ω^{ij} one-forms from which we can distinguish the spatial and the temporal components. The spatial components ω_a^{ij} are the three-dimensional spin connection for the three-dimensional space, that is they represent the intrinsic connection and the intrinsic curvature can be derived from them. The temporal components ω_0^{ij} can be interpreted as Lagrange multipliers related to how the triads change between different leaves of the foliation, in the same way that N and N^i can be interpreted as the Lagrange multipliers associated with the diffeomorphism invariance. Finally, the spatial components[8] of ω^{0i} represent the extrinsic curvature:

$$\omega_a^{0i} = K_{ab} e^{bi} = K_a^i \,. \tag{5.10}$$

The triad K_a^i will play a role of canonical conjugate momenta to e_i^a in the same way that K_{ab} did to g_{ab} in the ADM formalism.

Notice that by fixing this gauge we have not exhausted all the gauge symmetry: there is still freedom to internally rotate the triad in the internal three-dimensional space. The local group of symmetry that describes this symmetry is the group of rotations in three dimensions SO(3), which is a subgroup of the original Lorentz group SO(3,1). This group has the same local structure as SU(2), and most formulations of the LQG take the Lie algebra-valued objects to be elements of $\mathfrak{su}(2)$. Formulations using either algebra and associated group are equivalent.

With this in place we can rewrite Palatini action 5.7 using these variables and carry out a Hamiltonian analysis[9] to find that e_i^a and K_a^i are canonically conjugate

[8] The temporal component ω_0^{0i} is a function of the other quantities described and will not play an important role for our analysis.

[9] For a clear and detailed Hamiltonian analysis of the tetrad formulation of general relativity see Pons et al. (2000b).

variables, in the same way that g_{ab} and K_{ab} were canonically conjugate variables in the ADM formalism:

$$\{e^a_i(x), K^j_b(x')\} \sim \delta^a_b \delta^j_i \delta^3(x-x') \,, \tag{5.11}$$

where the exact proportionality constant will depend on the constants in front of the action. The canonical Hamiltonian (density) for this formulation of general relativity takes the form:

$$H_c = N\mathcal{H}_0 + N^a \mathcal{H}_a + \lambda^i \mathcal{G}_i \,. \tag{5.12}$$

As in the case of the ADM Hamiltonian, this canonical Hamiltonian is basically a combination of secondary constraints. Associated to each constraint we have a gauge symmetry of the theory. $\mathcal{H}_0$ and $\mathcal{H}_a$ are the Hamiltonian and momentum constraints expressed in tetrad variables and they are associated with the diffeomorphism transformations and $\mathcal{G}_i$ are the Gauss constraints which are associated to the $SO(3)$ gauge symmetry. N, N^a and $\lambda^i = \frac{1}{2}\epsilon^i_{jk}\omega^{jk}_0$ are the lapse function, shift vector and the temporal components of the spin connection and they act as Lagrange multipliers or non-dynamical variables. I refer the reader back to chapter 4 for a more detailed discussion of the ADM Hamiltonian and its interpretation.

5.1.3 Connection variables

Finally, in this subsection I introduce the variables that are used in LQG. These variables are usually known as the connection variables or Ashtekar variables. First, we introduce the densitized triad:

$$E^a_i = \frac{1}{2}\epsilon_{ijk}\epsilon^{abc} e^j_b e^k_c = (\det e) e^a_i \,. \tag{5.13}$$

This quantity is also known as Ashtekar electric field and essentially contains the same information as the triad field. The connection A^i_a is:

$$A^i_a = \frac{1}{2}\epsilon^i_{jk}\omega^{jk}_a + \beta K^i_a \,, \tag{5.14}$$

where β is a for now an unspecified complex parameter. This connection is a complex one-form on the three-dimensional manifold that combines the intrinsic spin connection ω^{jk}_a with the extrinsic curvature K^i_a[10].

It can be shown[11], and it is not a trivial result, that these variables have the following commutation relations under the Poisson bracket structure of general relativity

[10] This definition can be extended to include also the temporal dimension of the spatiotemporal index: $A^i_0 = \frac{1}{2}\epsilon^i_{jk}\omega^{jk}_0 + \beta\omega^{0i}_0$. This quantity is not very important for our analysis, as in the canonical approach ω^{jk}_0 play the role of Lagrangian multipliers and ω^{0i}_0 is a function of other quantities.

[11] For a detailed Hamiltonian analysis of general relativity expressed in terms of Ashtekar variables refer to Pons et al. (2000a).

defined above (5.11):

$$\{A_a^i(x), A_b^j(x')\} = 0 \tag{5.15}$$
$$\{E_i^a(x), E_j^b(x')\} = 0 \tag{5.16}$$
$$\{A_a^i(x), E_j^b(x')\} = \beta 8\pi G \delta_b^a \delta_i^j \delta^3(x - x')\,. \tag{5.17}$$

As A is a connection it defines a parallel transport, covariant derivative, and a curvature two-form:

$$F^i = dA^i + \epsilon^i_{jk} A^j A^k\,. \tag{5.18}$$

And with all these definitions we can rewrite the constraints[12] in the new variables:

$$\mathcal{H} = E_i^a E_j^b (\epsilon_k^{ij} F_{ab}^k - 2(1+\beta^2) K^i_{[a} K^j_{b]} + \frac{\Lambda}{3} \epsilon_{abc} \epsilon^{ijk} E_k^c) \tag{5.19}$$
$$\mathcal{H}_a = F_{ab}^i E_i^b \tag{5.20}$$
$$\mathcal{G}_i = D_a E_i^a\,. \tag{5.21}$$

Notice that if we choose the parameter β to be i, the Hamiltonian constraint in absence of cosmological constant simplifies to:

$$\mathcal{H} = E_i^a E_j^b \epsilon_k^{ij} F_{ab}^k\,. \tag{5.22}$$

This expression is polynomial in E and A and therefore much simpler and easier to deal with from a technical point of view than the Hamiltonian constraint in the standard ADM formalism or for a different choice of β. This was first noticed by Ashtekar (Ashtekar, 1986) and the connection 5.14 with the choice $\beta = i$ is known as the Ashtekar connection[13].

An important feature of the Ashtekar connection is that it is complex valued. Therefore, we have expanded our phase space to complex E and A. In order for this to be physically meaningful, that is, in order for our theory to be equivalent to general relativity and not to a complex-valued extension of it, we have to impose the reality conditions:

$$A_a^i + \bar{A}_a^i = \epsilon^i_{jk} \omega_a^{jk}[E] \tag{5.23}$$
$$E_a - \bar{E}_a = 0\,. \tag{5.24}$$

[12] In the definition of the constraints there is sometimes an extra factor of $\sqrt{g}$. In the classical theory choosing one set of constraints or the other leads to just a redefinition of the lapse function, but for the quantum theory this choice may have effects. See Thiemann (2007a) for a discussion of this point.

[13] The Ashtekar connection has a geometrical meaning from the 4-dimensional point of view. It can be shown that it is the pullback of the self-dual part of the 4-dimensional spin connection.

The fact of including complex variables, and that one needs to enforce these conditions opens the door to a number of complications in both the classical and quantum theory. In particular, ω_a^{jk} is a non-polynomial function of E, which seems to reintroduce the complexity one has removed from the Hamiltonian constraint, and the definition of quantum operators for complex-valued variables is convoluted[14].

The most prominent alternative to the Ashtekar connection is the Barbero connection[15]. This is the connection for real β. In particular, the value of β is taken to be the Barbero-Immrizi paremeter γ we have introduced before as the coefficient for the Holst term.

With this choice we lose the simplicity of the Hamiltonian constraint that first attracted Ashtekar to use his connection but on the other hand the variables we are using are always real-valued functions, so we avoid some of the complications that come from using complex variables. In the recent formulations of LQG, such as (Rovelli and Vidotto, 2015), the Barbero connection is preferred, and it is the one I will be using in the rest of the chapter unless stated otherwise.

Even if the simplicity promised at first sight by the Ashtekar variables did not turn out to be so simple, there is still a strong reason to be attracted to a connection formulation of general relativity. This reason is that the theory now takes the form of a Yang-Mills gauge theory, and this kind of theory has been widely studied in quantum field theory. In the next subsection I introduce some of the variables and techniques that are imported to LQG and that are essential for its formulation.

5.1.4 Holonomies, fluxes and loops

As we have formulated general relativity as a Yang-Mills theory, and as it plays a crucial role in the definition of LQG, let me start by quickly reviewing what a Yang-Mills theory is. This theory is the generalization of Maxwell theory to any gauge group. In electromagnetism the physics is invariant under local U(1) transformations that affect the vector potential A_μ, which can be seen as an element in the $\mathfrak{u}(1)$ algebra. In Yang-Mills theories this is expanded to have gauge transformations generated by any Lie group G. The gauge potentials A_μ^i[16] are one-forms with values in the Lie algebra $\mathfrak{g}$ associated with G.

Yang-Mills theories can be formulated in the geometrical language of principal bundles, where gauge potentials can be seen as connections that define gauge covariant derivatives D_μ, in the same way that the Levi-Civita connection defines the covariant

[14] For more details on this point see the discussion in (Thiemann, 2007a, Sect. 4.2).

[15] This alternative was formulated in Barbero G. (1995).

[16] Notice that I am following the notation used in the previous sections in which μ and a denote spacetime indices and i internal indices. It is standard in introductions to Yang-Mills theories to use an opposite convention where a's are internal indices and i's are space indices.

derivative ∇ in general relativity. The curvature two-form is defined also in a way analogous to the one followed in general relativity.

Canonical analysis of Yang-Mills theories shows that the variables conjugate to the gauge potentials A_a^i are the generalized electric fields E_i^a, which are derived from the A_μ^i. The gauge symmetry of the theory is reflected in the fact that the system is a constrained system and that the 0th components of the gauge potentials act as Lagrange multipliers. Gauge transformations are related[17] with a series of constraints, the Gauss constraints, that generalize the Gauss law of electromagnetism:

$$\mathcal{G}_i = D_a E_i^a \,. \tag{5.25}$$

This is is the same expression we have found before (5.21) for the connection formulation of general relativity but for any group G and generalized electric field E_i^a.

In order to build a quantum Yang-Mills theory it is useful to introduce quantities which are well-behaved under gauge transformations. A useful one is the Wilson loop:

$$W_\gamma[A] = \mathrm{Tr}(\mathcal{P}\exp \oint_\gamma \mathbf{A}_a(x) dx^a) \,. \tag{5.26}$$

Where $\mathcal{P}\exp$ is the path-ordered exponential along the closed curve γ. Here by $\mathbf{A}_a = A_a^i \tau_i$ I denote the connection in its matrix form, and not in its vector form[18]. As elements of the algebra are generators of the group, their exponential gives an element of the group. Finally, by taking the trace the result is fully gauge invariant.

Wilson loops are useful because it can be shown[19] that knowing all the Wilson loops is equivalent to determining A_a^i up to gauge transformations. Therefore, Wilson loops can be used as an alternative formulation of any Yang-Mills theory.

In particular, LQG was inspired by this idea and some of its first formulations were formulated in terms of loops, which gave the name to the theory. In more modern accounts LQG is formulated in terms of a similar quantity, the holonomy. We define it to be:

$$h_e[A] = \mathcal{P}\exp \int_e \mathbf{A}_a(x) dx^a \,. \tag{5.27}$$

That is, an holonomy is the path-ordered exponential of the connection along a path e in the manifold, not necessarily closed. Notice that we will use the letter e to denote a path or an edge and from the context it should not be confused with the triad fields. Holonomies are group elements[20] and they are not gauge invariant but

[17] Recall from last chapter that even though it is many times claimed that gauge transformations are generated by constraints there are some subtleties involved.

[18] This is explained in footnote 4 for the gauge group and algebra of general relativity, and here the same applies for any generic Lie algebra and Lie algebra basis τ_i.

[19] See Giles (1981).

[20] Pertaining to SO(3) or equivalently SU(2).

just gauge covariant. That is, gauge transformations only affect holonomies by their action on the starting and ending points:

$$A \to A' = gAg^{-1} - dgg^{-1}$$
$$h_e[A] \to h_e[A'] = g_s h_e[A] g_t^{-1} \,,$$

where g is the gauge transformation and by g_s and g_t I denote its value at the starting and ending points of the edge. We see that holonomies have a regular behavior under gauge transformations, which is a wanted feature for our fundamental variables. Moreover, holonomies are invariant under reparametrizations of the edge e, and therefore they also behave nicely under diffeomorphisms.

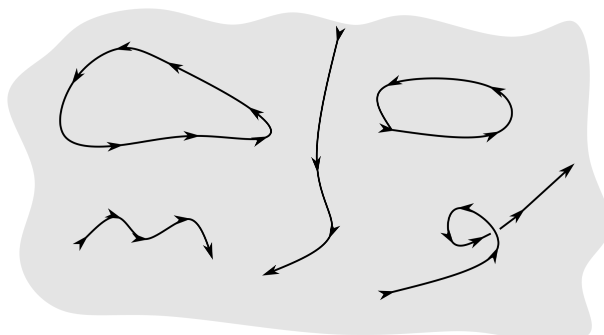

Fig. 5.2: Open and closed (loops) edges on a manifold. The basic variables of the theory, the holonomies, are defined on the edges.

For formulating the theory in terms of holonomies we need to find their canonical conjugate variables. As A_a^i and E_i^a are canonically conjugate it is not surprising that the conjugate of smearing A_a^i along a path is a smearing of E_i^a, called flux. We first define the dual two-form[21]:

$$E_i = \epsilon_{abc} E_i^a dx^b \wedge dx^c = \frac{1}{2}\epsilon_{ijk} e^j \wedge e^k \,. \tag{5.28}$$

The flux $X_i[S]$ is sometimes defined[22] as the integral of this two-form on a surface S:

$$X_i[S] = \int_S E_i \,. \tag{5.29}$$

[21] Indeed this two-form is the three dimensional version of the 4-dimensional Plebanski 2-form for this gauge. The Plebanski 2-form will play a role in chapter 8, where I will introduce covariant versions of LQG.

[22] Later on we will see that there are some subtleties involved in the definition of fluxes, but for now let me take this definition.

Let me point out that the flux is a vector in three dimensions, which can be seen as an element of the algebra associated with the rotation group. Indeed, this was to be expected, as in general the conjugate variables to group variables like the holonomies are elements of the associated algebra. This algebraic point will be relevant later on. There are a couple of reasons for defining the flux as the smearing of E_i on a surface. First, E_i is a two-form, so from the geometric perspective this smearing is natural. And second, this smearing makes the Poisson bracket with the holonomies to be well behaved as I will show now. Indeed, the Poisson bracket between an holonomy that intersects the surface S at just one point is just:

$$\{X_i[S], h_e[A]\} = \pm 8\pi G\gamma h_{e_1}[A]\tau_i h_{e_2}[A]\,, \tag{5.30}$$

where $h_{e_1}[A]$ is the holonomy from the starting point of e to the intersection point of e and S and $h_{e_2}[A]$ the holonomy from this point to the end of e with $e = e_1 \circ e_2$. The sign depends on the orientations of S and e. We can say then that the action of the Poisson bracket with the flux is to split the holonomy $h_e[A]$ by introducing the matrix τ_i[23] at the intersection point.

It can also be shown that if the edge does not intersect S, or is tangential to it, the bracket of $X_i[S]$ with the holonomy is 0. For any countable number of intersection points, 5.30 generalizes to:

$$\{X_i[S], h_e[A]\} = \sum_{P\in e\cap S} \pm 8\pi G\gamma h_{e_1^P}[A]\tau_i h_{e_2^P}[A]\,, \tag{5.31}$$

where e_1^P and e_2^P are like e_1 and e_2 before but for every intersection point P. That is, the bracket of the flux with a holonomy results in the splitting of the holonomy at every intersection point.

It can also be shown that the Poisson bracket between two holonomies is 0, and that the bracket between two fluxes is:

$$\{X_i[S], X_j[S]\} = -\epsilon_{ij}^k X_k[S]\,. \tag{5.32}$$

This is a bit surprising, as one would naively think that this bracket should vanish. The mathematical reasons of this result are complex, let me just say that the connection and triad are defined as three dimensional functions and as we are smearing them on one and two dimensional submanifolds this smearing is, in a sense, distributional and this opens the door to mathematical subtleties that give this counterintuitive result[24].

[23] Recall that this is the basis of the Lie algebra $\mathfrak{su}(2)$.

[24] For more detail and mathematical rigor I refer the reader to (Ashtekar et al., 1998) and (Thiemann, 2001, 2007b).

With these basic brackets we can define a Poisson algebra, the holonomy-flux algebra. This algebra consists in functions that depend on a finite number of holonomies and fluxes and for any two such functions the brackets can be computed by making use of the chain rule and the basic brackets derived above. This algebra is closed: given two functions which depend on finite numbers of holonomies and fluxes their bracket is also a function of this kind. Recall from chapter 4 that this is a requisite we ask for an algebra to be a basis for our quantization. Let me also point out that there is a particular subspace of this algebra which will play a special role in the quantum theory. This is the space of cylindrical functions which are functions which depend just on a finite number of holonomies.

The holonomy-flux algebra will be the starting point of the quantization procedure. I will now show that fluxes are of physical relevance, as they are related with geometric quantities as the area of a surface. In particular the area of a surface S can be expressed as:

$$A[S] = \int_S d^2\sigma \sqrt{n_a n_b E^a_i E^b_j \delta^{ij}} = \int_S |E| \, . \tag{5.33}$$

Here σ are coordinates on S and n_a is the normal. The first equality can be shown to be equivalent to the standard definitions of area by using the relations between E^a_i and the triads and of these and the metric tensor. The second equality involves the norm of the two-form E_i, something that is not mathematically well defined[25], and has to be read as meaning just the content of the first expression, which certainly has some resemblance with a norm.

Now consider the norm of the flux defined before:

$$|X[S]| = \sqrt{X_i X_j \delta^{ij}} = \sqrt{\sum_i \left(\int_S E_i \right)^2} \, . \tag{5.34}$$

This expression and 5.33 are different, but in the limit of very small S they agree. For any S we can now divide it into N smaller pieces, S_n and compute the quantity:

$$\sum_n^N \sqrt{X_i[S_n] X^i[S_n]} \, . \tag{5.35}$$

Now, if we take the limit of very small pieces the area of each piece and the norm of the flux are the same, and therefore in the limit this is an expression for the area:

$$A[S] = \lim_{N\to\infty} \sum_n^N \sqrt{X_i[S_n] X^i[S_n]} \, . \tag{5.36}$$

[25] Notice that a norm involves taking a square, and the square of a two-form is not well defined.

In this way, we have shown that areas can be expressed in terms of fluxes, which will be fundamental for the quantum theory we will construct. Furthermore, the set of all holonomies and fluxes contains essentially the same information as the phase space of connections and triads, up to gauge transformations. Holonomies and fluxes, and their Poisson algebra are naturally represented by graphs, so let me in the next subsection discuss the formulation and truncation of the theory in terms of graphs.

5.1.5 Graph phase space

Given a manifold we can define an embedded graph Γ on it as a series of vertices $v \in V_\Gamma$ joined by a series of edges, paths on the manifold $e \in E_\Gamma$[26] joining different vertices, see figure 5.3. Given a graph one can divide a manifold into a cellular decomposition that is said to be dual to it. For a mathematical definition of cellular decomposition dual to a graph I refer the reader to (Freidel et al., 2013), but for our purposes it will be enough to say that the decomposition in three dimensions consists on dividing the space into three-dimensional regions or cells bounded by the union of two-dimensional faces. For the decomposition to be dual to the graph it has to satisfy that there is only one vertex inside each region and that there is also a one-to-one relation between edges e and faces F_e: each face intersects with one and only one edge and each edge intersects only with one face. For each graph there is not a unique cellular decomposition dual to it, as the boundaries can be moved around and their shapes changed.

Now, for a given graph and decomposition we can define the following quantities:

$$h_e = \mathcal{P} \exp \int_e \mathbf{A}_a(x) dx^a \tag{5.37}$$

$$X_{i(F_e,\pi_e)} = \int_{F_e} h_{\pi_e}(x) E_i(x) h_{\pi_e}(x)^{-1} \,. \tag{5.38}$$

That is, we define a flux and an holonomy for each edge. The definition of the holonomy is the same as in the last subsection and depends only of the values of A along the edge. The definition of the flux instead is slightly different from the basic one we introduced in the last section[27] and it has a dependence not only on the edge and the tetrad field, but also on a choice of face and paths π, and in the connection[28].

[26] In this context be careful again with the notation and do not confuse edges e with triad fields e^i_μ nor the set of edges E_Γ with the densitized triads E^i_μ

[27] This quantity has a better behaviour under gauge transformations and its Poisson bracket is more convenient.

[28] Again, for the mathematical details refer to (Freidel et al., 2013) or (Thiemann, 2001).

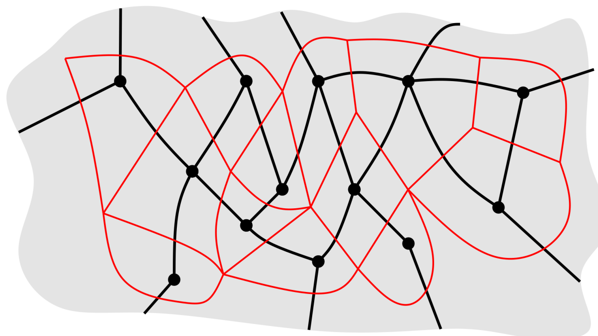

Fig. 5.3: An example of a graph embedded on a manifold and its dual cellular decomposition. The black dots are the vertices of the graph and the black lines the edges of it. The red lines represent the faces (in three dimensions these are not lines but surfaces) dual to the edges and they delimit the cells dual to each vertex. Each edge, each black line crosses just one red line, one face, and each cell contains just one vertex.

These variables have the following Poisson bracket structure:

$$\begin{aligned}&\{h_e, h_{e'}\} = 0\\&\{X^i_{(F_e,\pi_e)}, h_{e'}\} = -\tau^i h_e \delta_{ee'} + \tau^i h_e^{-1}\delta_{ee'^{-1}}\\&\{X^i_{(F_e,\pi_e)}, X^j_{(F_{e'},\pi_{e'})}\} = \delta_{ee'}\epsilon^{ij}_k X^k_{(F_e,\pi_e)}\,.\end{aligned}$$

Here again τ^i as the basis of the $\mathfrak{su}(2)$ algebra. Notice that the bracket between the holonomy and the flux now does not result in the holonomy get split, which makes the Poisson algebra for a given graph closed[29]. Now we can consider a phase space consisting only in these variables. That is, we can define a truncation from the original phase space to a graph phase space:

$$\begin{aligned}&I : \mathcal{P} \to \mathcal{P}_\Gamma\\&(A, E) \to (h_e, X_{(F_e,\pi_e)})\,.\end{aligned}$$

This phase space is the space of couples (h_e, X_e) consisting of a group element and an algebra element. This is expressed mathematically as: $\mathcal{P}_\Gamma = \underset{e}{\times}\,(\mathrm{SU}(2)\times\mathfrak{su}(2))_e = \underset{e}{\times} T^*\mathrm{SU}(2)\,_e$. The second equality just says that the space $\mathrm{SU}(2)\times\mathfrak{su}(2)$ is the tangent space of the group SU(2). Here I am following Freidel et al. (2013) in using $\mathfrak{su}(2)$, but

[29] This is a consequence of the particular definition of flux, and it is the reason for which that definition was taken in Freidel et al. (2013).

a similar structure arises if we formulate the theory using the $\mathfrak{so}(3)$ algebra[30]. The symplectic structure on the graph phase space is given by the brackets given above.

Indeed, this procedure allows for a definition for a family of graph phase spaces associated with different graphs. This family has a natural partial order relation corresponding with finer-grained graphs. That is, when all the nodes and edges of a graph Γ_1 are also nodes and edges of Γ_2 we can say that Γ_2 contains Γ_1 or that it is finer-grained. This relation also applies to the associated phase spaces: there is a natural projection from Γ_2 to Γ_1. The notion of partial order allows one to construct finer and finer graphs and to define the limit of infinitely fine graphs. It can be shown, treating the mathematical details carefully and defining the limit in a rigorous way[31], that the limit is $\mathcal{P}$, the phase space of general relativity.

The motivation for introducing this phase space comes from the fact that in the quantum theory we will find similar structures. However, it is interesting to point out that a truncation of general relativity can be defined previous to quantization. The truncation so far has been made just at the phase space level, but already allows one to ask questions about the relation between phase spaces. For instance, the limit structure allows for speaking about emergence in a Butterfieldian sense. Importantly, in this discussion we are not dealing with of the dynamics, so we are not saying anything about how a space represented by a graph evolves nor about the relation of a truncated dynamics with the dynamics of general relativity.

We have seen the truncation that takes one from a manifold to a graph. The relation between the manifold phase-space and the graph phase-space is many-to-one: one point in the graph phase-space corresponds to many possible ways of embedding the graph in the manifold and with many choices of the faces delimiting the cells. Importantly, not all of these different embeddings are related by diffeomorphism. We see that the mapping is not invertible and one cannot associate one graph with just one geometry. Instead one can define an equivalence class of geometries associated to a set of h_e, X_e. Some work in this direction was done in (Freidel et al., 2013).

Before moving on, let me mention that of all the possible embeddings of a graph in a manifold there is one that will be specially relevant for us in chapters 8 and 9. In this embedding we consider a manifold which is made by joining different flat polyhedra by their faces[32]. In this case we can assign a node of the graph to each polyhedron and a link to each shared face. The particular case in which all the nodes are four-valent corresponds to a manifold made of tetrahedra, which is called

[30] In more generality, the theory can be extended to any number of dimensions and a graph truncation gives a phase space that is the product of $G \times \mathfrak{g}$, for a group and algebra appropriate for the dimensions of the space. I refer the reader to (Thiemann, 2001) for more details.

[31] See (Thiemann, 2001; Dittrich and Geiller, 2015) for discussions of this.

[32] Strictly speaking this is not a manifold, as at the edges of the polyhedra this manifold is not differentiable and smooth.

simplicial manifold and will be fundamental for defining some spin foam models and group field theory models in 8 and 9.

With this I conclude the review of the formulation of general relativity in the variables that will be used for the quantization in LQG. In the next subsection I include some philosophical comments that can be made in the light of this formulation.

5.1.6 Philosophical remarks

In this section I have introduced the tetrad and connection formulations of general relativity, as they will be the basis for the quantization carried out in LQG. Before getting into such a quantization, in this subsection I will make some brief philosophical remarks motivated by the reformulations of general relativity.

In the first place, notice that the tetrad and the connection formulations of general relativity are just expansions of the theory by the addition of a local gauge degree of freedom, namely the SO(3,1) freedom in choosing the tetrad fields. We can ask ourselves whether this new formulation is equivalent to the original one. As in other gauge theories like electromagnetism, it seems that one can always say that the physical content of the theory lies in its gauge invariant part and claim that the gauge extension is not saying anything new about the theory.

However, even when one has a new formulation of a theory, which is taken as equivalent, it may be the case that it highlights some aspect of the theory that may motivate giving a different interpretation to it. In this case, the new formulation was technically useful, and it will be useful in the quantum case, but it did not seem to highlight any aspect of general relativity that was especially relevant for the foundations of the theory.

If we said that the physical content of the theory also included the gauge fields and not only the gauge invariant quantities, it would arguably suggest a change of the metaphysics of the theory. There are two arguments for regarding the triad fields as physical: i) As I said it has been argued[33] that the tetrad formalism is favored if one wants to include fermions in the theory, and ii) In other gauge theories, gauge fields are unobservable at a classical level, but they may seem necessary for explaining some quantum phenomena like the Aharonov-Bohm effect. I believe neither of the two arguments are definitive and I will not argue for or against them here. I will just say that as long we remain in the classical theory, general relativity can be safely and reasonably interpreted from the metric perspective ignoring the gauge fields. In other words, the new formulation does not favor or disfavor any position in the traditional debates in the foundations of general relativity.

[33] I refer the reader again to (Rovelli, 2004, Sect. 2.1.2) and Pitts (2012) for two different views on this point.

In particular, we may also translate to this new formulation the argument about the classical problem of time. The same I argued in sections 4.2 and 4.3 applies also to this formulation of general relativity. That is, diffeomorphisms with a temporal component, i.e., diffeomorphisms that change the time coordinates, are gauge transformations just from the global perspective, as they transform solutions of the equations of motion of the theory to other equivalent solutions of the equations of motion. From an instantaneous perspective, these diffeomorphisms are not gauge transformations, as they transform how things are at a leaf of the foliation to the way they will be at some other leaf, and these are clearly not equivalent in general.

Notice also that I have not discussed the action of the diffeomorphism constraint on the holonomies and fluxes. This discussion is also absent in the LQG literature, but it is important to consider it also at a classical level. As I commented in section 4.2, imposing that observables have to have vanishing Poisson brackets with the spatial diffeomorphism generators was too strong a condition. This point will be important for the definition of the Hilbert space of LQG.

Finally, let me also point out that the graph phase spaces discussed in the previous subsection may be taken as a basis for a possible argument saying that general relativity is emergent from a classical theory formulated in terms of graphs, that could work as an intermediate step in the relation between LQG and general relativity. However, one would need to specify the dynamics of this missing intermediate link before fully analyzing its relation with general relativity.

5.2 Canonical quantization I: kinematics and spin networks

In this section I explain the second and third steps of the canonical quantization program of LQG as I have outlined it in the introduction. As I have explained in the introduction, these consist in the definition of a kinematical Hilbert space in which the algebra of the relevant quantities is represented and the imposition of the 'kinematical' constraints, that is, all the constraints but the Hamiltonian one, which defines the dynamics. In particular, the holonomy-flux algebra gets represented in the kinematical Hilbert space in a peculiar way analogous to the polymer quantization, which in this case is implemented by the choice of the Ashtekar-Lewandoski measure. We will see that the states after the imposition of the kinematical constraints can be interpreted to represent something like a discrete space.

This section is divided in four parts. In the first one I describe the derivation of the kinematical Hilbert space and the imposition of the constraints, that leads to a particular basis, the spin network or s-knot basis, which I characterize. Next, I comment on the area and volume operators that can be defined at this kinematical level, and on the fact that they are found to have discrete spectra. In the third part

I analyze the physical interpretation that can be given to these states. In particular, they are argued to represent discrete spaces from which space could emerge. Finally, I also give some technical details and criticisms about the way the quantization has been carried, which seems to be subject to a number of choices which could have been made in different ways, giving rise to theories in which some of the central claims of LQG, such as the discreteness of area, do not hold.

5.2.1 Kinematical Hilbert space and spin networks

In this subsection I will introduce the kinematical Hilbert space of LQG and I will comment on how the Gauss constraint and the spatial diffeomorphism constraint are imposed to get the s-knot Hilbert space, which is where the Hamiltonian constraint should be imposed to reach the physical Hilbert space. The kinematical Hilbert space is chosen to represent the algebra of observables of the classical theory, the holonomy-flux algebra I introduced in the last section. In this section I will be following Ashtekar and Lewandowski (2004) and Rovelli (2004) and I refer the mathematically oriented reader to Thiemann (2007b) for a rigorous algebraic construction of the Hilbert space.

In standard quantum mechanics the wavefunction is a square integrable function on the configuration space. In LQG the classical configuration space is the space of connections $\mathcal{A}$ over a spatial manifold Σ. Now one can intuitively think of the wavefunction as a 'square integrable' functional on this space, but as now we are dealing with an infinite-dimensional system (notice that for a field theory there are degrees of freedom at every point of space) one has to be careful with how to define such a functional and a measure to give the notion of inner product. In LQG the way this is done is by the introduction of cylindrical functions, as I explain next.

To 'reduce' the number of degrees of freedom of the field what one does is to probe it at a finite number of regions. In particular, for the connection the way we choose to probe the field is by using the holonomies h_e and graphs introduced in the previous section. Holonomies are probes in the sense that they represent a smearing of the connection along an edge e, and therefore they provide some information about the field along that edge. As we have seen in the last section, holonomies are very natural quantities in connection theories with good transformation properties and for defining LQG we will choose holonomies as our fundamental variables, as part of the holonomy-flux algebra. For a given graph Γ with n edges one can define a wavefunction as:

$$\Psi[A] = \psi(h_{e_1}[A], \dots h_{e_n}[A])\,, \tag{5.39}$$

where ψ is a function of the n holonomies of the graph and, as I will explain in a moment, it can be treated as a wavefunction for a system with a finite number of degrees of freedom. But notice that Ψ can be seen as a functional on the space of connections. Of course it is a very special one, as in general a functional could depend on the infinite degrees of freedom of the field, and Ψ here only depends on a finite number of them. These functions are cylindrical functions, as introduced in last section.

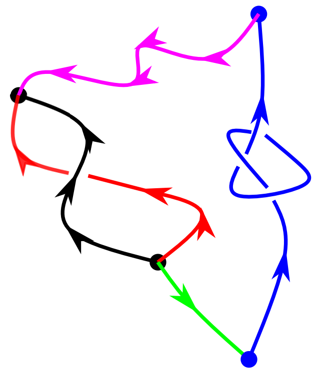

Fig. 5.4: An example of an oriented graph Γ on a 3-dimensional manifold.

One can restrict to a given graph Γ and the functions defined on its edges. These functions form a linear space, Cyl_Γ, and we can define an inner product on this space:

$$\langle \Psi_1, \Psi_2 \rangle = \int_{G^n} d\mu_H^n \bar{\psi}_1(h_{e_1}, ...)\psi_2(h_{e_1}, ...) \,, \tag{5.40}$$

where the integral is over the n copies of the group G corresponding to the n holonomies of the graph Γ and $d\mu_H$ is the Haar measure of the group. With this we have the structure of a Hilbert space $\mathcal{H}_\Gamma = (L^2[G, d\mu_H])^{\otimes n}$ for every graph Γ.

Now one can consider the space of all cylindrical functions, Cyl, and extend the inner product we have just defined to any pair of cylindrical functions. The way of doing it is by noticing that given any two cylindrical functions ψ_1 and ψ_2 on graphs Γ_1 and Γ_2 there exists a bigger graph Γ_3 that contains both Γ_1 and Γ_2 and on which one can use the definition of inner product on a given graph. It can be shown that this inner product is independent of the choice of Γ_3. Therefore, the inner product is well defined for the whole space Cyl and we get a well defined Hilbert space structure. This Hilbert space is the Cauchy completion of Cyl for this inner product. Given that we are defining our states to be dependent just on the holonomies and not directly on the connection, the states in this Hilbert space are indeed defined as functionals on an extension $\bar{\mathcal{A}}$ of the original configuration space. In the same way that a point in $\mathcal{A}$ defined a smooth connection with values at every point of space a point in $\bar{\mathcal{A}}$ defines an holonomy for every possible edge. This extended space also includes discontinuous configurations of A and in this sense is a bigger space, but importantly it can be shown that $\bar{\mathcal{A}}$ is dense in $\mathcal{A}$, that is, given any generalized connection, there is one smooth connection that is arbitrarily close to it. Even so, notice that generalized connections define holonomies and not connections properly and for this reason we will find that there is no connection operator in the Hilbert space we are defining. This is a consequence of the way we are defining our Hilbert space, which is known as polymer quantization and differs from the more common Schrödinger quantization[34].

The Hilbert space we have defined is $\mathcal{H} = L^2[\bar{\mathcal{A}}, d\mu^0]$, that is, the space of square integrable functionals on the space of generalized connections. The measure μ^0 is known as the Ashtekar-Lewandowski measure and it is the Borel measure induced by the family of Haar measures on the cylindrical functions[35]. This Hilbert space can also be defined as a projective limit of the graph Hilbert spaces $\mathcal{H}_\Gamma$, that is, it is the limit of considering a sequence of Hilbert spaces associated with more and more refined graphs.

The holonomies and graphs are not only useful for defining the Hilbert space, they also give an orthonormal basis. For a given graph Γ we have seen that one can associate a Hilbert space $\mathcal{H}_\Gamma = (L^2[G, d\mu_H])^{\otimes n}$ which is a space of square integrable functions on n copies of the group. For a compact group, like the $SU(2)$ or $SO(3)$ groups we are interested in here, the Peter-Weyl theorem assures us that this Hilbert space has a basis that corresponds to the different irreducible representations of the group. The matrix elements $\pi^j_{\alpha\beta}(h)$ of the representations of the group can be seen as functions of the group, and the theorem shows that they form a basis of $L^2[G, d\mu_H]$. These functions are labelled by three numbers: the 'spin' j, a non-negative half-integer that identifies the representation of G, and the pair of 'magnetic numbers' α and β that are also half-integers which identify the matrix element. For a graph with n edges we have an orthonormal basis of $\mathcal{H}_\Gamma$ indexed by the set of n triples of numbers j_e, α_e, β_e.

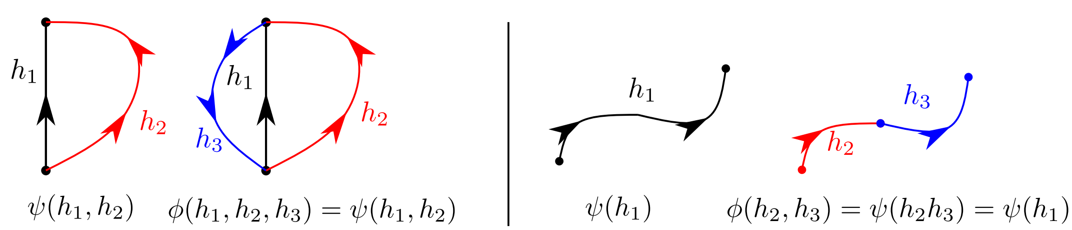

Fig. 5.5: Two examples of different graphs that represent the same functions.

However, the basis of $\mathcal{H}$ is not just the union of all such states for all given graphs.

[34] I refer the reader to Ashtekar et al. (2003) and Corichi et al. (2007) for discussions about polymer quantization.

[35] This was introduced in Ashtekar and Lewandowski (1993).

The reason for this is that there is overlap between states of different $\mathcal{H}_\Gamma$. There are two situations in which this can happen as illustrated in figure 5.5. First, having a link which carries the trivial representation, that is $j_e = 0$, and not having this link is completely equivalent. That is, consider two graphs Γ and Γ', such that Γ' is built by adding one edge to Γ. Now consider a state ψ on Γ and a state ψ' on Γ' that is identical to ψ for all the coincident edges and that has $j = 0$ for the extra one. Then, it can be shown that ψ and ψ' are indeed the same state. Second, there can also be overlap between two graphs that differ in that one graph has a bivalent node, a vertex that joins two edges e_1, e_2, while the other contains an edge that is the union of these two, $e = e_1 \circ e_2$. Given this overlap between the spaces $\mathcal{H}_\Gamma$ and to avoid overcounting, one builds the spaces $\mathcal{H}'_\Gamma$ which exclude states with $j = 0$ and states with problematic bivalent edges. These spaces are disjoint, and they decompose the Hilbert space:

$$\mathcal{H} = \bigoplus_\Gamma \mathcal{H}'_\Gamma = \bigoplus_\Gamma \bigoplus_{j_e} \mathcal{H}'_{\Gamma,j_e}\,, \tag{5.41}$$

where $\mathcal{H}'_{\Gamma,j_e}$ is the subspace of $\mathcal{H}'_\Gamma$ for the set of representations j_e. The first direct sum sums over all graphs, including the empty one that corresponds to the wave-functional $\Psi[A] = 1$. An analogy can be made here with the Fock space of quantum field theory. The 'primed' spaces $\mathcal{H}'_\Gamma$ are equivalent to n-particle spaces, while the spaces $\mathcal{H}_\Gamma$ are equivalent to the subspace of the Fock space of states with at most n particles. With this decomposition of the LQG Hilbert space we have a basis formed by the states:

$$|\Gamma(x) j_e \alpha_e \beta_e\rangle\,. \tag{5.42}$$

That is, we have a basis indexed by graphs and quantum numbers. This basis shows that $\mathcal{H}$ is not separable, i.e., that there is no countable basis for this space. Separability is a feature that one would like for the Hilbert space of a quantum theory, and it has been argued that the fact that the kinematical Hilbert space of LQG is not separable is a drawback of this theory. However, it is argued that it the dynamical Hilbert space which should be separable, so let me postpone this discussion for the time being.

Now that we have characterized the Hilbert space we can start to impose the constraints to this space. The first constraint we will impose is the Gauss constraint. Remember that holonomies are gauge covariant under gauge transformations, i.e., that gauge transformations only affect holonomies by their action on the initial and final points of the edges. That is, gauge transformations act only on the vertices. For the representations π of G, that are the basis of our Hilbert space, we have the same transformation rule:

$$\begin{aligned} h_e &\rightarrow h'_e = g_s h_e g_t^{-1} \\ \pi^j_{\alpha\beta}(h_e) &\rightarrow \pi^j_{\alpha\beta}(h'_e) = \pi^j_{\alpha\gamma}(g_s)\pi^j_{\gamma\delta}(h_e)\pi^j_{\delta\beta}(g_t)\,, \end{aligned}$$

where g_s and g_t are the gauge transformations at the starting and ending points of the edge, that is, at the source and target vertices, and we have used the convention that repeated indices are summed over. Here it is explicit that the two indices α and β are related with the left and right group action of the gauge group, and that each of them can be associated naturally with a vertex: the quantum number α is associated with the source vertex and the number β with the target vertex. This association illustrates the fact that each graph Hilbert space for a given set of representations j_e can be decomposed as a tensor product of Hilbert spaces associated with the vertices:

$$\mathcal{H'}_{\Gamma,j} = \bigotimes_{v\in\Gamma} \mathcal{H'}_{\Gamma,j,v}\,. \tag{5.43}$$

Intuitively, each $\mathcal{H'}_{\Gamma,j,v}$ contains the α degrees of freedom of the edges that are outgoing from v and the β degrees of freedom of the edges that are incoming to v. Now, it is in these spaces where the gauge transformations act. The physically interesting states are of course the gauge invariant ones, which can be shown[36] to be given by:

$$|\Gamma(x) j_e i_v\rangle\,, \tag{5.44}$$

where i_v represent the degrees of freedom that are left at each vertex once we apply the gauge invariance requirement. The i_v are known as intertwiners, and they are defined as the different tensors[37] that 'mix' the degrees of freedom of the edges on the vertices where they meet. As there usually are different ways of doing this 'mixing', there are different choices of intertwiners and hence there are these remaining degrees of freedom. Notice also that not for every combination of j_e's incoming or outgoing in a vertex there is a gauge invariant combination, and hence imposing gauge invariance also implies a restriction on the choices of j_e. The states $|\Gamma(x) j_e i_v\rangle$ are known as spin network states, and they are characterized by a graph, a spin j_e associated to each edge and the intertwiners i_v associated to each vertex.

The Hilbert space spanned by the spin network states is $\mathcal{H}^G$, which is the gauge invariant part of the kinematical Hilbert space $\mathcal{H} = L^2[\bar{\mathcal{A}}, d\mu^0]$. This same space can be reached by reducing the gauge before quantizing. Namely, instead of defining a quantum theory of functions on $\bar{\mathcal{A}}$ and then applying the Gauss constraint one can start from the gauge reduced configuration space, $\bar{\mathcal{A}}/\bar{\mathcal{G}}$, and then define the quantum theory. The space $\bar{\mathcal{A}}/\bar{\mathcal{G}}$ is the quotient between the space of connections and the gauge group, i.e., each function in this space is gauge invariant. If we define a Hilbert space of square integrable functionals on this space using the Ashtekar-Lewandoswki

[36] One can show that these states are directly gauge invariant (Giesel, 2017, Sect. 5.1) or one can equivalently show (Ashtekar and Lewandowski, 2004, Sect. VI A) that they are eigenstates of the Gauss constraint with eigenvalue 0.

[37] For the definition of the intertwiner tensors I refer the reader to (Rovelli, 2004, Sect. 5.3.6).

metric we find the space $\mathcal{H} = L^2[\bar{\mathcal{A}}/\bar{\mathcal{G}}, d\mu^0]$ which is precisely the space $\mathcal{H}^G$. This Hilbert space can also be defined as a projective limit of gauge invariant graph spaces, $\mathcal{H}^G_\Gamma = L^2[G^n/G^{n_v}, d\mu_H]$. The space G^n/G^{n_v} contains the n copies of SU(2) or SO(3) corresponding to the n edges of the graph but quotiented by the gauge action of the group on the n_v vertices of the graph.

After having imposed the Gauss constraint we are in a position to impose the momentum constraints, which is intuitively related with the idea that states have to be invariant under spatial diffeomorphisms. The imposition of this symmetry is both technically and conceptually more complicated. As explained in the previous chapter[38], when one tries to impose a constraint which has a continuum spectrum, the states that satisfy the constraint have a distributional character and are not states on the Hilbert space but instead they are part of an extension of the Hilbert space. In the case of LQG this means that the physical states are not states on $\mathcal{H}^G$ but on a distributional extension of it. As usual in the imposition of constraints, one needs to define an inner product for the physical Hilbert space as the inner product of the kinematical Hilbert space is not defined for the physical one.

Diffeomorphisms can have two effects on a spin network, as illustrated in figure 5.6. First, they can move and deform the graph on the manifold, i.e., they can change the place where the graph is placed and change its shape. Second, diffeomorphisms can map a graph to itself while changing the order and orientation of the edges. These transformations are known as graph symmetries. Therefore, when we impose diffeomorphism invariance we require states to be independent of the exact shape and location of the graph and of its symmetries.

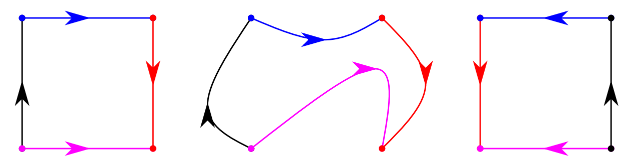

Fig. 5.6: Illustration of the two effects diffeomorphisms can have on a network. We can get the second network by deforming the first one and the third one is related to the first one by a graph symmetry (an inversion or 180° rotation with respect to the central axis).

A way diffeomorphism invariant states can be built is by group averaging. I will give an intuitive picture of this procedure and will refer the reader to the literature for mathematical detail[39]. The idea of group averaging is that gauge invariant states,

[38] See section 4.5.
[39] See (Thiemann, 2007b, Sect. 9.2) and (Ashtekar and Lewandowski, 2004, Sect. VI B).

for any gauge group, can be built as equal-weight superpositions of all the elements of a gauge orbit. For instance imagine a quantum system whose degrees of freedom can be represented by a direction on the plane. And imagine that gauge transformations are rotations of multiples of $\pi/2$, represented by the rotation matrices $R(\frac{k\pi}{2})$ with integer k. If we start with the state $|\uparrow\rangle$ we can build the following state:

$$|\eta(\uparrow)\rangle = \frac{1}{4}\sum_{k=0}^{3} R(\frac{k\pi}{2})|\uparrow\rangle = \frac{1}{4}(|\uparrow\rangle + |\rightarrow\rangle + |\downarrow\rangle + |\leftarrow\rangle)\,. \tag{5.45}$$

The state $|\eta(\uparrow)\rangle$ is gauge invariant, as can be seen by applying any $R(\frac{k\pi}{2})$ to it. In general we can build a gauge invariant state starting with an arbitrary state $|\psi\rangle$ and applying the same projector to it:

$$|\eta(\psi)\rangle = (\frac{1}{4}\sum_{k=0}^{3} R(\frac{k\pi}{2}))|\psi\rangle\,. \tag{5.46}$$

The operator we are applying is just an average of the elements of the gauge group. For the case of diffeomorphisms we would like to apply the same idea but it is difficult to define an average over the group of diffeomorphisms, as this group has uncountably infinite elements and defining an average over the group has not been achieved. Instead, we can define the 'group averaged' state $\eta(\psi)$, for a cylindrical function ψ_Γ defined on Γ, by giving its action on another cylindrical function $\psi_{\Gamma'}$ on Γ':

$$(\eta(\psi_\Gamma)|\psi_{\Gamma'}\rangle = \sum_{\phi\in \mathrm{Diff}/\mathrm{Diff}_\Gamma} \langle U_\phi \psi_\Gamma|\psi_{\Gamma'}\rangle\,. \tag{5.47}$$

Here $\langle U_\phi\psi|$ is the state resulting of applying the diffeomorphism ϕ to the state ψ. The sum runs over diffeomorphisms ϕ in $\mathrm{Diff}/\mathrm{Diff}_\Gamma$, where Diff_Γ is the space of diffeomorphisms that leave Γ invariant. Intuitively, each element in this quotient space transforms Γ into another $\phi(\Gamma)$ in a one-to-one way: for each Γ to $\phi(\Gamma)$ there is only one ϕ that represents this transformation, that is, ϕ represents the equivalence class of diffeomorphisms that have the same effect on Γ. Therefore at most only a finite number of terms contribute in the sum, the ones with $\phi(\Gamma) = \Gamma'$ and the ones in which $\phi(\Gamma)$ and Γ' are related by graph symmetries. In this way, $\eta(\psi_\Gamma)$ is well defined, as even if the sum is a sum over uncountably infinitely many terms, only a few contribute.

$(\eta(\psi_\Gamma)|$ is not an element of $\mathcal{H}^G$, but an element of Cyl*, the space of distributions on cylindrical functions. That is why I am now[40] using the notation $|...)$ to distinguish

[40] Later on, when the distinction is not necessary, I will come back the notation $|...\rangle$ for any state independently of the space it lives in.

these states from states $|...\rangle$ on $\mathcal{H}^G$. The action of $\eta(\psi_\Gamma)$ is diffeomorphism invariant, i.e., its action over a state $|\psi_{\Gamma'}\rangle$ is the same than its action over the diffeomorphism-transformed $|U_\phi\psi_{\Gamma'}\rangle$. The definition also makes it the case that the group average states of two diffeomorphism related states ψ_Γ and $U_\phi\psi_\Gamma$ are the same: $\eta(\psi_\Gamma) = \eta(U_\phi\psi_\Gamma)$. For the group averaged states it is natural to define the following inner product:

$$(\eta(\psi_\Gamma)|\eta(\psi_{\Gamma'})) = (\eta(\psi_\Gamma)|\psi_{\Gamma'}\rangle \,. \tag{5.48}$$

This inner product is well defined and independent of the choice of $\psi_{\Gamma'}$ on the right hand side because of the diffemorphism invariance of the action of $(\eta(\psi_\Gamma)|$.

The map $\eta : \mathrm{Cyl} \to \mathrm{Cyl}*$ provides solutions to the diffeomorphism constraint, that is, it gives us states that are invariant under diffeomorphisms. We can define the space $\mathcal{H}_{diff}$ as the Cauchy completion of $\eta(\mathrm{Cyl}\cap\mathcal{H}^G)$, that is, the Cauchy completion of the space all diffeomorphism invariant states built from the cylindrical functions that satisfy the Gauss constraint. The Hilbert space $\mathcal{H}_{diff}$ is the space of gauge and diffeomorphism invariant states.

The basis of $\mathcal{H}_{diff}$ is given by:

$$|[\Gamma]j_e i_v\rangle \tag{5.49}$$

These states are known as spin-knot states or as s-knot states. However, it is common to refer to them also as spin network states in contexts where the distinction between the diffeomorphism dependent and invariant states is not relevant. $[\Gamma]$ is the equivalence class of all graphs diffeomorphic to Γ and j_e and i_v are the same quantum numbers we had for spin-networks. Notice that equivalence classes contain less information than spin networks: in the process of imposing diffeomorphism invariance we have got rid of the information about the position and shape of the network. Still, there is some information left that makes it the case that the equivalence classes are not abstract graphs, i.e. they are more than a set of connectedness relations between points. Some of what is missing in an abstract graph is the description of how the graph is knotted. For instance, an edge can be wrapped around another one. A knot class is the set of all graphs that are knotted in the same way. Notice also that knot classes depend on the manifold they are defined on, that is, the topological properties of the base manifold affect the ways there are of constructing graphs and knotting them. Therefore, knots cannot be understood as abstract graphs independent of the manifold.

The other diffeomorphism invariant information that abstract graphs fail to represent is information about 'angles' at vertices of high valence. When one has two spin networks of the same knot class with nodes of valence 3 or less one can always find a diffeomorphism between them. For nodes with higher valence this does not hold, as there are some 'angles', called moduli, that are conserved under diffeomorphisms (see figure 5.7). Therefore, equivalence classes of networks $[\Gamma]$ are given by a knot class K and a set of moduli $\{\theta_i\}$. There are countably many knot classes, but infinitely uncountably many possible moduli, which means that the space $\mathcal{H}_{diff}$ is not separable.

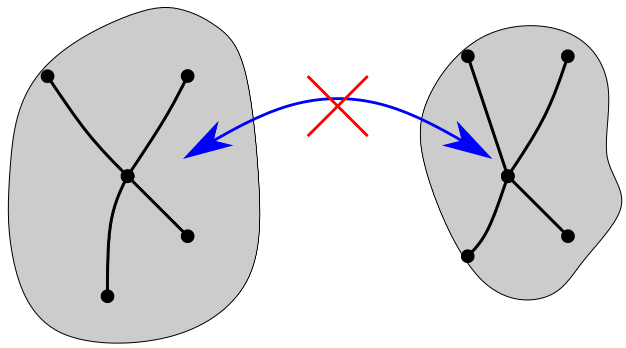

Fig. 5.7: For two nodes of valence 4 or higher, in general there is not a diffeomorphism that maps one to the other in three dimensions. The reason for this is that the transformation at the node has to be differentiable and therefore it is not completely free. In particular, this implies that diffeomorphisms cannot affect some structures, the moduli.

It has been proposed[41] to enlarge the symmetry group to include generalized diffeomorphisms, transformations that are allowed not to be smooth at a countable set of points. In this way, there are no angles conserved at the vertices and one can map any graph into any other of its same knot class. Therefore, states would be given by the knot class and quantum numbers j_e, i_v and the Hilbert space would be separable. However, it seems that there is no other motivation to enlarge the symmetry group in this way. And it is argued[42] that the physical Hilbert space, the one resulting from applying also the Hamiltonian constraint, will be separable even if one has regular diffeomorphisms and not generalized ones as symmetry transformations.

The emphasis I have just made to remark that knots are not abstract graphs is motivated by more recent formulations of LQG, in particular covariant formulations[43] in which the basis of Hilbert space is given by abstract graphs and not by knots. The reason for dropping these degrees of freedom, as explained in (Rovelli, 2011a,b), is that since the formulation of LQG no physical meaning was found for them[44]. In any

[41] See for instance (Rovelli, 2004, Sect. 6.2 and 6.4).
[42] In Thiemann (2007a) for instance.
[43] I will discuss them in chapter 8
[44] There are proposals like the one in (Bilson-Thompson et al., 2007) to take the knot structure to

case, just note that canonical quantization leads us to knots and not to graphs.

The next step in the canonical program is to apply the Hamiltonian constraint to finally arrive to the physical Hilbert space where the physically meaningful states of the theory live. Before analyzing how this is implemented and interpreted in the next section, in the next subsection I discuss the physical interpretation one can give to spin network and s-knot states.

But let me summarize what we have achieved in this section. We chose to take the connection as our basic configuration variable and built a Hilbert space, $\mathcal{H}$ of square-integrable functionals on this configuration space. For this we chose the Ashtekar-Lewandoski measure, which is adequate for the gauge and diffeomorphism symmetries of the theory. Next, we have imposed the gauge and diffeomorphism invariance of the states to find the space $\mathcal{H}_{diff}$. A basis of this space is given by the s-knot basis, which is labelled by an equivalence class of graphs and a set of quantum numbers on its edges and vertices.

5.2.2 Area and volume operators: discreteness of space

In the last subsection I have introduced the spin-network Hilbert space as the space where the functions in the classical holonomy-flux algebra are represented as linear operators with commutation relations that obey the quantum version of the algebra. In this subsection I will explicitly give the form of the basic operators and I will build geometric operators as area and volume operators. With these operators defined, we will be able to give a physical interpretation of the states of both the spin network and s-knot Hilbert spaces.

Recall that the kinematical Hilbert space of LQG is the space of square integrable cylindrical functions. The basic operators in this space are the holonomies and the fluxes[45]. The holonomy operator acts by multiplication:

$$\hat{h}_e[A]\psi[A] = h_e[A]\psi[A]\,. \tag{5.50}$$

The flux operator acts in the following way:

$$\hat{X}_i[S]\psi[A] = i\hbar\{X_i[S], \psi[A]\}\,, \tag{5.51}$$

encode not gravitational nor geometric information, but matter degrees of freedom like the standard model. Even if this were possible, it would still be true that the knotting would not play a role as a gravitational or geometric degree of freedom.

[45] Recall that generalized connections do not define values for the connection at every point. Therefore, there is no well-defined connection operator in this space. Nevertheless, in some works in LQG, as Rovelli (2004), one can find these operators defined. This has to be taken heuristically, as these operators do not strictly speaking exist, but they lead to holonomy and flux operators with the right properties.

where the brackets are the classical Poisson brackets introduced in 5.1.4 and, as ψ is a cylindrical function, they are well-defined and give back another cylindrical function. With the operators defined in this way, their commutation relations correctly represent the classical holonomy-flux algebra.

Let me illustrate the action of the flux operator on a simple case in which the quantum state is just (a component of) a holonomy . Using the brackets 5.30 we obtain:

$$\hat{X}_i[S]h_e[A] = \sum_{P\in e\cap S} \pm i\hbar 8\pi G\gamma h_{e_1^P}[A]\tau_i h_{e_2^P}[A]\,. \tag{5.52}$$

The result of applying the flux operator to a holonomy is very similar to the result of computing the bracket between them: it consists in splitting the holonomy at the intersection points and introducing a matrix τ_i at those points. Again, segments of e that are contained in S do not contribute. For a general state in the spin network basis the effect is the same: every edge e, carrying a representation j, that intersects the surface gets split into two, and a matrix $\pi^j(\tau_i)$ is inserted between the resulting edges.

With this in place, we can now define geometric operators. For instance, in section 5.1 we introduced an expression for the area in terms of fluxes. Recall that we could compute the classical area of a surface S by dividing it into cells S_n and adding up the areas of all the cells. In the limit of very small cells, $N \to \infty$, the area of each cell equals the norm of the flux on the cell, and therefore its sum gives the total area:

$$A[S] = \lim_{N\to\infty} \sum_n^N \sqrt{X_i[S_n]X^i[S_n]}\,. \tag{5.53}$$

Now, we can take this classical expression and turn it into a quantum one by promoting the fluxes into operators. That this operator is well-defined came as a surprise[46], as we are taking two functional derivatives at one point and taking a square root of that.

To see how this operator acts, let us start by studying how $\hat{X}_i[S_n]\hat{X}^i[S_n]$ acts on a spin network state that has only one edge that intersects S_n and it only does so once. In the spin network basis, each $\hat{X}_i[S]$ introduces the j representation of the matrix τ_i, $\pi^j(\tau_i)$ at the intersection point, where j is the representation that the state assigns to the edge that intersects with S. The operator $\hat{X}_i[S_n]\hat{X}^i[S_n]$ introduces therefore two matrices and contracts them. This contraction $-\pi^j(\tau_i)\pi^j(\tau^i)$ is the Casimir element of SU(2), $j(j+1)\mathbb{I}$, and therefore the result simplifies to:

$$\hat{X}_i[S_n]\hat{X}^i[S_n]|\Gamma j_e\alpha_e\beta_e\rangle = (\hbar 8\pi G\gamma)^2 j(j+1)|\Gamma j_e\alpha_e\beta_e\rangle\,. \tag{5.54}$$

[46] See for instance the derivation and discussion in Ashtekar and Lewandowski (1997a).

We find that elements in the spin network basis, that only intersect once with S_n are eigenstates of the $\hat{X}_i[S_n]\hat{X}^i[S_n]$ operator, and even more, in a gauge invariant way. If instead of intersecting once, we had multiple intersections we would not have such a simple result. The reason for this is that each operator would insert multiple matrices $\pi^j(\tau_i)$ at the different intersection points and no simplification like in the one intersection point case would occur when contracting the matrices. However, this is not a problem for our definition of area operator, because we are taking the limit of very small cells, and we can construct this limit in a way such that for any spin network and any surface S, we always reach a moment in which there is at most one edge intersecting each cell, and from then on this still holds. This is possible because we are always dealing with a finite number of analytic edges[47].

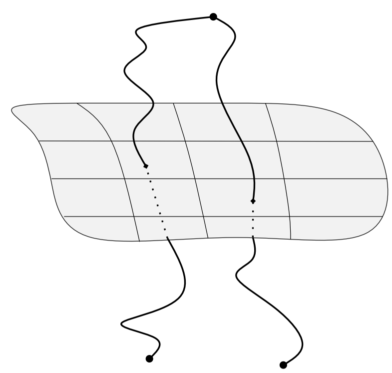

Fig. 5.8: Only the two cells which are intersected by the spin network contribute to the value of the area operator when applied to the spin network state. For any finer division of the grey surface we will find that only two cells are intersected by the network and hence we will find the same value of the area.

We can imagine a cell S_n such that from one point in the limiting process always intersects just one edge e, the effect of the operator $\hat{X}_i[S_n]\hat{X}^i[S_n]$ on the spin network would be the same for every refinement of the partition, and hence in the limit. Therefore the action of the area operator on a spin network state $|s\rangle$ that intersects the surface S at a series of points on its edges is:

$$\hat{A}[S]|s\rangle = \hbar 8\pi G\gamma \sum_{P\in\Gamma\cap S} \sqrt{j_P(j_p+1)}|s\rangle\,. \tag{5.55}$$

This is a remarkable result. We have found that spin network states are eigenvectors of the area operator and that the eigenvalues depend just on the representation carried

[47] If the number of edges were infinite it would be possible that there is no way to partition S into cells such that there is only one intersection per cell. If the edges were not analytic it could be the case that they intersect the surface S at an infinite number of points too.

by the edges that intersect the surface S and hence this operator has the same action in both gauge invariant and variant states. Moreover, we have found that the area operator has a discrete spectrum, therefore the area is quantized. In this sense, it is said that edges on a spin network carry quanta of area. This result also implies that there is a minimum value for the area, that depends on the fundamental constants $\hbar$ and G, and also in γ, the Barbero-Immirzi parameter[48]. This minimum area scale is of the order of the Planck scale.

There is one possibility I have not mentioned, and it is that we have some of the nodes lying on the surface S. This case is more complicated, as spin network states $|\Gamma j_e i_v\rangle$, for some arbitrary intertwiner basis, are not in general eigenvectors of the area operator. Nevertheless, for a fixed graph, j_e, and surface S we can find a basis of the intertwiner degrees of freedom such that the area operator is diagonal. As happened in the simpler case, we find a discrete spectrum, and hence the conclusion that area is quantized still holds. But notice that the area operators of two different surfaces that intersect on the same node cannot be in general diagonalized at the same time, i.e., there are no states with well defined areas for all the surfaces that go through a node. In other words, area operators do not commute[49].

Similarly, we can think of finding an operator corresponding to the volume of a region. This turns out to be more complicated than in the case of the area operator, and there are at least two different definitions, depending on the different ways in which the limit is taken and on the different choices in the process. I refer the reader to (Thiemann, 2007b, Sect. 13.3) for a discussion of the form of the volume operator and a comparison of the proposals by Rovelli and Smolin (1995) and Ashtekar and Lewandowski (1997b). In this chapter Thiemann gives reasons for preferring the latter proposal of volume operator. The fact that the volume operator depends on different choices in the definition and prescription of the limit can be seen as unsatisfactory, as pointed out in Nicolai et al. (2005). The criticism is not that the definition is not consistent, but that it may have an unwanted dependence on the regularization procedure, i.e., in the way the different limits are defined and taken.

In either case, it seems to be true that the volume operator also has a discrete spectrum and that vertices carry quanta of volume, in a similar way as edges carry quanta of area. There is a basis of the intertwiner degrees of freedom that makes the volume operator diagonal. States in this basis are eigenstates of the volume operator, and the volume of a region depends only on the intertwiners of the vertices in the region.

Notice that geometric operators defined for spin network states are quite different

[48] In case this construction is done using the complex Ashtekar variables the result is the same but replacing γ by one.

[49] This opens the door to relations with a branch of mathematics called non-commutative geometry.

from what our intuitions from geometrodynamics would have told us. In a geometrodynamic setting, an eigenstate of the geometric observables would be a state which would define a metric field g_{ab} in every point in the manifold and geometric quantities would be defined in a straightforward way. In the LQG quantization, geometry has become sort of distributional, i.e., instead of being defined by a nicely spread field in space, it is concentrated in 0-dimensional (vertices) and 1-dimensional (edges) regions. This implies that most of space (a set of measure 1) is devoid of geometrical properties and that the usual intuitions from Riemannian geometry do not apply. For instance, in Riemannian geometry if a region contains another one, the volume of the former is necessarily bigger than the volume of the latter. In the case of LQG states, this is not true, as, if it is the case that both regions contain the same nodes, the volume of the two will be the same. Therefore, the way the geometry described by spin network states is different from what one would expect goes beyond the fact that observables have discrete spectra.

The area and volume operators are defined relative to surfaces and regions on the manifold. Therefore they are not diffeomorphism invariant. It is easy to see that as a diffeomorphism moves a spin network around on the manifold, it will change which edges intersect a surface and which vertices lie on a region. Therefore, they are not operators on $\mathcal{H}_{diff}$. Nevertheless, we can intuitively extrapolate their properties from the diffeomorphism dependent description to the diffeomorphism invariant one in the same way that in geometrodynamics we formed the intuitive idea that observables should be related with the geometrical properties of 3-geometries. Let me briefly discuss how to think about this extrapolation.

In geometrodynamics, before imposing spatial diffeomorphism invariance, we can define geometric observables such as lengths, areas and volumes on the manifold relative to a metric tensor $g_{ab}(x)$. However, when we apply a diffeomorphism, we 'move' the metric tensor and the values of the geometric properties of some particular region the manifold change. Given the diffeomorphism invariance of general relativity, and of LQG, 'bare' points of the manifold do not have any physical meaning, but the geometric properties we may ascribe to them are present in any diffeomorphism related description. For instance, if a metric tensor g on a manifold describes two points of curvature R_1 and R_2 separated by a geodesic distance d, another diffeomorphism related tensor g' will also describe two points with the same curvatures separated by the same distance. Therefore, all the metric tensors related by diffeomorphisms describe essentially the same geometry. We call a 3-geometry the geometry defined by the equivalence class of metric tensors on a 3-dimensional manifold which are related via diffeomorphisms.

We can extrapolate these learnings to the case of LQG. If we interpreted a particular spin network as saying that the surface that was intersected by an edge had some area, and the region that contained a vertex had some volume; we can say that every diffeomorphism transformed version of the network describes the same geometry: the edges give surfaces of the same area and the vertices give regions of the same volume. An equivalence class under diffeomorphisms defines an s-knot state, and we can ascribe to it the same geometric properties that were ascribed to each spin network: edges give areas of surfaces and vertices give volumes of regions. In this way, an s-knot state can be seen as containing some information about the 3-geometry. As we have defined it, a s-knot state describes an equivalence classes of the distributional geometry described by a spin network state. However, we will see below that they have been given a different interpretation.

As I commented in chapter 4, the intuitive picture of equivalence classes of 3-geometries faces some technical problems at the time of getting formalized. For instance, I argued that it is difficult to define geometrical observables that have vanishing Poisson brackets with the momentum constraints. Moreover, I even argued that that would be too strong a requirement. In this case we are in the same situation: given the definition and interpretation of spin network states and their geometrical properties, we have an intuitive idea of how they relate with their diffeomorphism invariant counterparts, the s-knots, but we lack a formal definition of the geometrical observables in terms of diffeomorphism invariant observables.

In the next subsection I will comment more on the physical interpretation of spin network and s-knot states. In order to do that it is useful to study how these states, eigenstates of area and volume operators, may correspond to areas and volumes defined classically. We would expect to recover classicallity in a limit of very dense networks or, equivalently, for scales much bigger than the fundamental geometric scale of the theory, i.e., the Planck scale. There is a particular class of states, known as weave states, which show classical behaviour precisely for scales l much bigger than the Planck scale l_P. Namely, we can give an approximate metric g such that for regions $\mathcal{R}$ much bigger than this scale, the geometrical properties of the region and the eigenvalues of the geometric operator agree up to small corrections:

$$\hat{A}[\partial\mathcal{R}]|S\rangle = \left(A[\partial\mathcal{R}] + \mathcal{O}(l/l_P)\right)|S\rangle \tag{5.56}$$

$$\hat{V}[\mathcal{R}]|S\rangle = \left(V[\mathcal{R}] + \mathcal{O}(l/l_P)\right)|S\rangle\,. \tag{5.57}$$

Let me make an analogy here. Consider a quantization of a matter field that is also distributional, in the sense that it describes matter concentrated just in a series of points and not spread in space. However, for some states, some scales, and some situations we can certainly approximate these states by some states which describe a continuously spread distribution of matter. The way weave states approximate a 3-geometry is analogous to this case, and the fact that we are able to construct states with properties that can be seen as good approximations to the properties of a 3-geometry is a necessary condition if we will later want to claim that the theory has something to do with the macroscopic world that we know and can describe by means of 3-geometries.

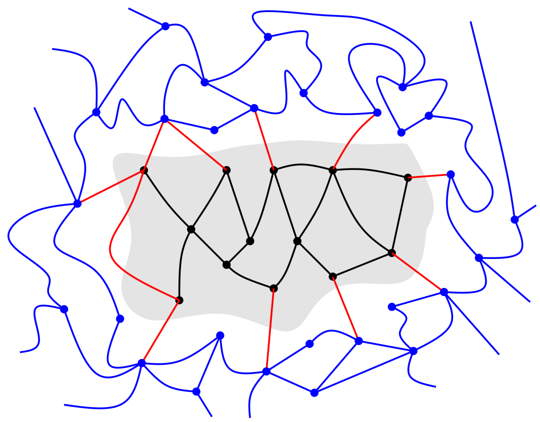

Fig. 5.9: Example of how to define regions on a network. The grey zone defines a region: its volume will be given by the intertwiners of the nodes contained in the region (in black) and the surface of the boundary by the representations of the edges that cross it (in red). When these areas and volumes can be approximated by a metric on a manifold we have a weave state.

With this we end this subsection about the geometric operators not only in the kinematical Hilbert space but also in the spaces $\mathcal{H}^G$ and $\mathcal{H}_{diff}$ that satisfy some of the constraints. So far the discussion has been mostly technical: I have defined and characterized the relevant Hilbert spaces, states and operators, but I have not entered into which physical interpretation should be made of them. In the next subsection, I comment the different interpretations that are available for the structures defined so far.

5.2.3 Interpretation of states in the s-knot Hilbert space

Before getting into the interpretation of the theory, notice that the discussion has been at the level of the Hilbert spaces $\mathcal{H}$, $\mathcal{H}^{\mathcal{G}}$ and $\mathcal{H}_{diff}$ and not at the level of the physical Hilbert space. The final answer about the structure of the Hilbert space and the properties of the operators depends on that final space, of which we will discuss in the next section. Therefore, the interpretations I will give in this subsection may have to be corrected once the full dynamics is taken into account. However, it is plausible that much of the structure defined at this level and its interpretation will remain valid. In particular, it is an extended view that the characterization of states in terms of s-knots with discrete values for the areas and volumes will remain true. There are some reasonable doubts about these claims, as raised for instance in Dittrich and Thiemann (2009)[50]. In any case, the interpretations given at this level should be taken with a bit of caution. This subsection complements and expands the discussion of subsection 3.3.2, where I argued that from some states in LQG one could say that space emerges in a reductionist way that appeals to functionalism.

As I mentioned above, there is a tendency in the LQG community to interpret s-knot states as abstract graphs and not as 1-dimensional regions on a manifold. This tendency makes it the case that the most extended interpretation of these states is as describing a discretized space as I will explain below. But notice that what the outcome of the canonical formalism is that s-knots are equivalence classes of distributional geometries, and the most straightforward way to discuss the emergence of space is by considering weave states. Distributional geometries have not been considered in the philosophical literature as a genuine interpretation of s-knot states as far as I am aware, but they should if one is considering the canonical quantization procedure seriously. A feature of this view is that one preserves the underlying manifold, which can be problematic as it entails having those points devoid of geometrical properties. But it can also be a virtue, as the most direct way of thinking about matter fields is as fields defined on the manifold. Moreover, the discrete space picture suffers from some tensions, as I will explain next.

The picture of s-knots describing a discrete space is motivated by the discrete graph structure of s-knot states and the fact that they are eigenstates of area and volume operators. Space would be formed by 'atoms', quanta of volume, which can be 'adjacent' to each other, that is, they can share a boundary. As we have seen, these relations are encoded by the graph and the quantum numbers give the volumes and areas. Therefore, one can interpret these states as describing a discrete space, or equivalently, a cellular decomposition of a space as the one we have introduced in 5.1.5. As happened then, cellular decompositions compatible with a graph are not unique[51].

In the philosophical literature about LQG this picture is taken seriously to discuss the ontology of spacetime in quantum gravity. For instance, see (Wüthrich, 2006, 2013, 2017, 2019a,b) and (Vassallo and Esfeld, 2014). However, a word of caution has to be taken before embracing this picture.

Notice that knots are not abstract graphs, and they contain more information than just the connectivity relations. As I mentioned before, more recent formulations of LQG, like the covariant formulations tend to use some graph Hilbert space instead

[50] See also an answer to this in Rovelli (2007).

[51] In the theories we will study in chapters 8 and 9, spin foams and group field theories, there is a preference for simplicial or polyhedral cellular complexes. The connection between the s-knot Hilbert space and polhedra is explained in Bianchi et al. (2011).

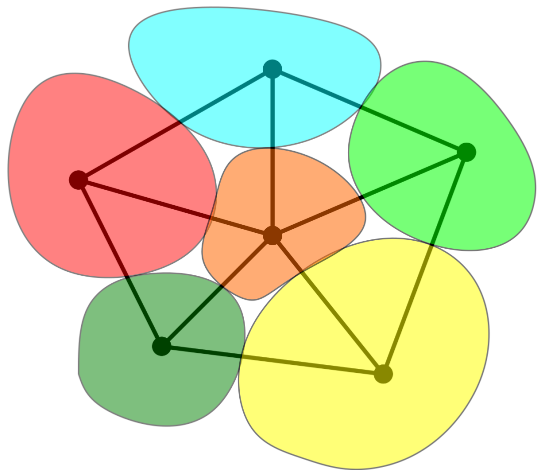

Fig. 5.10: Each node of the spin network can be interpreted as an atom or a chunk of space and the links represent the adjacency relations between them.

of the knot Hilbert space originally found by the canonical program. In this section we will stick to the original s-knot states, and we will see that the interpretation of them as showing some abstract adjacency relations finds some problems that are not there if one just takes the graph Hilbert space.

First, as we have seen before, knots carry more information than which point is connected with which. Therefore, if we 'simplify' the information carried in a knot state to just a simple relation we may be throwing away information that may be physically relevant. Consider a case like the one in figure 5.11 in which two knots, both dual to the same cellular complex, exemplify the same connectivity relations. It is possible to interpret both situations as physically equivalent, as they share adjacency relations, areas and volumes, but they are represented by different states and may turn out to have different physical properties[52].

Second, not every knot state can be associated in a straightforward way with a cellular decomposition of the space. See for instance the state represented in figure 5.12. In this case we could interpret the state as two sets of independent 'bubbles' of space which do not cover all of space. But notice that these two sets have a relation between them: they are knotted in a way that they 'go through' each other. This relation cannot be captured by the simple abstract graph structure. Another case that cannot be interpreted as a cellular decomposition is the case of having a loop: what would it mean for an atom to have a boundary with itself? Moreover, as we

[52] A different piece of information that we could be throwing out are the moduli in vertices of high valence as explained in the last subsection.

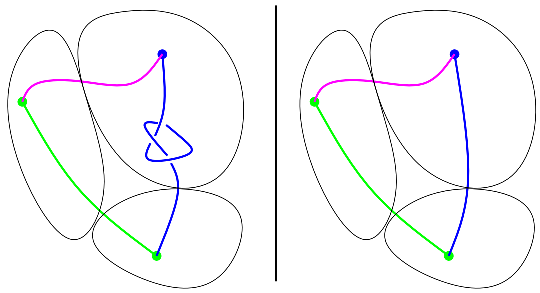

Fig. 5.11: Here I show two knots, which are identical in their connectivity, i.e., in which node is connected to which other nodes and in quantum numbers but differ in the way they are knotted. In this case it is the blue link that has a knot on the left graph and none in the right one. If we consider the two graphs to be equivalent, we would be saying that the information of the knot is not physical.

complicate the graph structure it is expected that we have more complicated graphs that do not correspond to cellular decompositions of a manifold. Therefore, if we restrict ourselves to a picture in which spin network states represent a space divided into cells with the adjacency relations represented by a graph, it seems that we are leaving some states out.

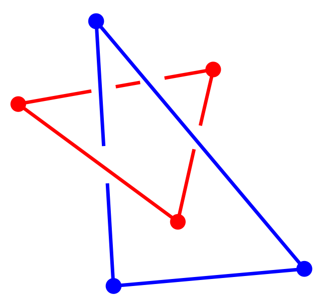

Fig. 5.12: These two interlocked triangles cannot be given an interpretation of a cellular decomposition that covers all of space (nor at least all of the surface of either the blue or red triangles) and satisfies that only the cells represented by nodes of the same color are adjacent to each other. If we stick to the idea of atoms of space we have two disconnected regions that somehow are interlocked while not adjacent.

If we nevertheless wanted to push further the idea of 'atoms' of space we would have to be cautious about how to understand them. An option could be to interpret them as literally chunks of space with volumes and areas given by the quantum numbers. But if we take into account the fact that area operators for surfaces with different orientations do not always commute at the vertices, it seems that a standard geometry cannot be assigned to the atoms. Then, if we want to keep the talk of 'atoms' it is better that we do not take literally the idea of chunks of space but consider atoms to be fundamental, indivisible entities such that when taken many together they behave as space[53].

In any case, the question of how general relativistic spaces are recovered from LQG states is a key question that needs to be answered. The best answer so far is to rely on the weave states I have introduced in the previous subsection. As weave states can be approximated on large scales by metrics on a manifold, one can claim that these states give us the right spaces. Hugget and Wüthrich (Wüthrich, 2019b; Huggett and Wüthrich, 2013) claim that the relation between spin networks and relativistic spaces is a relation of emergence compatible with reduction, as there are approximations and limits that can be taken from one description to the other. As explained in chapter 2, this use of the term emergence is close to what physicists have in mind, but philosophically vague. Therefore, I would just say that a reductionist view of relativistic spaces from spin networks is fully plausible, at least for weave states and scales much bigger than the Planck scales.

Wüthrich (Wüthrich, 2017) points out to a potential tension for the emergence of relativistic spaces and spin networks as understood from the 'atomic' space view. In a weave state it can be the case that two nodes that are connected in the graph are at a great distance apart according to the 'emergent' metric. This creates a conflict between the notion of locality of the emergent space and the connectedness relation of the graph, which, given the graph structure, would be the most natural candidate for playing the role of spacetime locality in the fundamental theory. Roughly speaking, we could characterize the notion of locality in physics by saying that something is local if it is only directly influenced by its surroundings and it can only influence directly on them. Modern physics, in particular field theories and general relativity, describe the physical interactions of objects in the world to be local and it is defended as a principle in theory building[54]. Therefore, if one is to take the connectedness relation of the graph to be defining a new notion of locality, it would be desirable for this new notion to agree as much as possible with the old spatiotemporal one. The tension pointed out by Wüthrich is that it can be the case that there is not such an agreement and that, from the spatiotemporal point of view the emergent theory has non-localities: regions of space which are far away from each other and which can nevertheless influence each other. This would go against our fundamental theories

[53] See (Vassallo and Esfeld, 2014) for a proposal in which this kind of atoms form the basic ontology of a Bohmian version of LQG.

[54] In quantum mechanics this may be slightly different, as the phenomenon of entanglement is arguably non-local.

of physics as we understand them now and against empirical evidence. The strategy Wüthrich follows to try to solve this conflict is to claim that non-localities are not so common in weave states, and hence in most cases the two notions of locality agree and from a big scale perspective non-local effects are negligible. However, there is a second line of answer one can take to solve the issue of non-localities.

This other option is to deny that the adjacency relations of the spin network define fundamental 'locality' relations, and the most natural way of doing so is by rejecting the atomic picture of space and embracing the distributional geometry one. The fact that two nodes are connected was interpreted to mean that they were adjacent to each other, that they shared a boundary. But recall that, when we introduced holonomies and cylindrical functions, they were introduced for arbitrary edges on the manifold which could be arbitrarily long and connecting any two arbitrary regions. To claim that this relation defines a strong sense of locality seems a bit too much at the kinematical level. If the dynamics of the theory ends up showing that it is connectedness on the graph what determines what is 'causally' connected to what, then we could be in a position to argue that conectedness defines a strong notion of locality. But if the dynamics of the theory is local in the topology defined by the underlying manifold, then the connections on the graph cannot be said to define a meaningful locality notion.

Therefore, we find here two different positions available. One is to consider graphs as fundamental and defining all relevant notions and the second one is to remain closer to the derivation process of the theory and keep the manifold structure as fundamental. In any case, it is when the dynamics of the theory is in play that we will be able to say whether connectedness on the graph translates to some kind of non-locality.

Finally, there is a further point to raise. The problem that we have discussed so far is how spin networks and weave states could form a discrete space and its relation with the relativistic space. But generic states in LQG will be superpositions of these spin networks and hence the geometric interpretation is not straightforward. The way we interpret these states will be connected to the way we approach the measurement problem in LQG.

Before closing this subsection let me remark again that so far we have seen states that represent the degrees of freedom of a three dimensional space and we are still missing the temporal dimension. Therefore, the interpretation of a discrete space so far has said nothing about the properties of time. As happened in geometrodynamics and any canonical approach, space and time are separated and independent, contrary to the relativistic idea of spacetime.

5.2.4 Comments on the quantization schema

In this section we have described how the Hilbert space of LQG is defined starting from the connection formulation of general relativity. We have seen that we are led to the spin network basis which is diagonal for some area and volume operators, which suggests interpreting these sates as describing a discrete space or something like a distributional geometry. In this final subsection I give some critical comments about the construction of the theory.

First, let me insist again that the Ashtekar-Lewandoski representation of the holonomy-flux algebra is a case of a polymer representation. This differs significantly from the standard Schrödinger representation in quantum mechanics and the Fock space representation in quantum field theory. The polymer quantization implies that the Hilbert space is in a sense discrete and that not every classical observable has a quantum counterpart. For instance, there is a polymer representation of the quantum mechanics of one particle in which wavefunctions become discrete: they are not smooth functions over a continuum of positions, but they become just superpositions of a finite number of positions. These positions could be any number on the real line but just a finite number of them. In this representation, there does not exist a momentum operator. We have seen that in the case of LQG we have a similar structure: states are finite superpositions of networks and the connection operator is not defined.

Notice that this discrete structure was later key to impose the diffeomorphism constraint and reach the s-knot space. In this sense notice that the discrete structure of s-knots, and of space, is directly a consequence of the choice of the Ashtekar-Lewandoski representation and of the imposition of the diffeomorphism constraint. Being such an important ingredient for LQG it is important that we first acknowledge it and, then, that we question whether this choice is sensible or not.

A first thing to notice is that from a conservative point of view perspective polymer representations are suspicious, not only because they are not part of our successful theories but also because they carry a discrete structure which is not common in physics, much less in the case of smooth field theories. Moreover, polymer representations imply that some operators are not defined, possibly and most relevantly this may be the case for the Hamiltonian. This leads to the need of introducing some regulators to define the Hamiltonian and the dynamics of the theory, which is a further complication. This, as we will see in the next section, will also be the case of the Hamiltonian constraint of LQG.

From a conceptual point of view, the polymer quantization carried out leads to the picture of a distributional geometry, which we may be tempted to reject. A reason for this is that while we could consider seriously having point-like distributions of matter in space, it is certainly conceptually more challenging to consider such a distribution for geometry. In particular, it is unclear how to interpret points of the base manifold once they have been removed some of their geometrical properties. One the one hand they retain their topological properties and one can still use this manifold to define some other fields on it. On the other, it is usually assumed that it is the distance relations between points what properly defines a 3-geometry. A different way of putting it is that we are treating the metric tensor as if it were a matter field[55] on the manifold, while it is part of what defines space, and hence it may strike as wrong to treat separately the metric and the manifold.

The problems just raised can be partially met by noticing that in the case of LQG there are a couple of reasons that make the Ashtekar-Lewandoski representation privileged. First, we have seen that the inner product in this representation is invariant under gauge and diffeomorphism transformations, which has made possible the imposition of the constraints. And second, there is a uniqueness theorem known as the LOST theorem[56] that assures us that under some assumptions, like being diffeomorphism invariant, the Ashtekar-Lewandowski representation is unique[57].

Another technicality I have mentioned before is that as field theories seem to describe systems with infinite degrees of freedom, at least one per spatial point, this leads to mathematical difficulties. In particular, a naive view of quantum field theory leads to a formulation in terms of non-separable Hilbert spaces, which are spaces which do not have a countable basis. However, using a Fock space representation for the quantization of field theory one gets separable Hilbert spaces. For the case of a quantum theory of gravity, if we want to use all the machinery and techniques of quantum mechanics and quantum field theory, it is then desirable to have also a separable Hilbert space. As we have seen, it is expected that the physical Hilbert space of LQG is separable, even if $\mathcal{H}_{diff}$ is separable only if we enlarge the group of symmetries.

Notice that in LQG what allows for the Hilbert space to be separable is the diffeomorphism invariance of the theory. It was by applying the group averaging procedure that we radically reduced the size of the Hilbert space. Notice also that for a field theory without diffeomorphism invariance, like Yang-Mills theory, the quantization procedure of LQG would not give a theory formulated in a separable space. It is then in virtue of the way one deals with diffeomorphism invariance and the momentum

[55] By this I am not implying that for the quantization of matter fields one necessarily needs to use quantizations like the loop quantization.

[56] See Sahlmann (2011a,b); Lewandowski et al. (2006).

[57] There are proposals, like (Dittrich and Geiller, 2015), for different representations which explicitly do not satisfy the assumptions of the LOST theorem. But in this case the representation obtained is also polymeric, with some properties similar to the Ashtekar-Lewandoski representation but in which the role of holonomies and fluxes is exchanged. This representation leads to a different Hilbert space and a theory different from LQG.

constraints in LQG that one gets a separable and treatable Hilbert space. Later on I will raise some doubts about the way diffeomorphism invariance is treated in LQG, but let me comment before a couple of points regarding the adequacy of this separable Hilbert space.

It will be enlightening here to compare with the case of QFT. In (Wallace, 2006, sect. 4) it is discussed the ambiguity present in the choice of representation in QFT, that is, when restricting to a separable Fock space in QFT one is choosing between uncountably many inequivalent representations that would correspond to sectors of the non-separable Hilbert space that one would get by naively dealing with the field theory as a theory with independent degrees of freedom at each spacetime point. Wallace argues that the choice of a representation is harmless for the case of QFT, given that the choice of representation has two sources that one can deal with, namely that one deals with an infinite system and that at every point there are degrees of freedom. In the case of LQG the causes of the non-separability of the kinematic Hilbert space are essentially the same, but one can worry that in going to the diffeomorphism invariant Hilbert space we are losing some interesting features. There will be features associated with both the infinitely small, i.e., the problem of having degrees of freedom at every point of space, and the infinitely big, i.e., the problem of having a potentially infinite space.

Related with the infinitely small we can consider the case of 'infinitely' dense graphs, i.e., graphs which have an infinite number of edges in a given finite region. By constructing the diffeomorphism invariant Hilbert space only out of 'finitely' dense graphs one is leaving these structures out of the theory. There are two senses in which leaving these out could be problematic. First, in a naive view, these graphs could be useful if we wanted to consider something like the continuum limit of the theory in a similar way to the one discussed section 5.1.5 for classical discretizations of general relativity: there, classical phase spaces were associated with graphs, and we could take the limit of infinitely dense graphs to get back to the phase space of general relativity. In the case of the Hilbert space of LQG such a limit is not available, which may make difficult the connection with the continuum theory. Second, and more worrisome is the fact that the area and volume operators we have defined are not well-defined when the graphs become infinitely dense. This leaves open the possibility that had we chosen a different quantization which included these states, and had we chosen a different definition for the geometric operators, maybe it would have been the case that the geometric operators would have had different properties, as for instance, continuum spectra.

Related with the infinitely big we find a different kind of ambiguity. In the case of QFT the ambiguity comes from the fact that different representations correspond to different asymptotic properties of the field. For instance, one can have two different representations, a Fock space one which describes excitations over an empty space and another one which describes excitations over a space with a well-defined non-zero mass density[58]. Similarly, in the case of LQG we consider always finite graphs, but we could have considered graphs with an infinite number of nodes and links which go all the way to infinity (for the case of open manifolds, of course). For instance, we can imagine a manifold diffeomorphic to $\mathbb{R}^3$ and a network forming a cubic lattice on it (figure 5.13). This kind of graphs are also excluded from the diffeomorphism invariant Hilbert space, and this can be seen as problematic for two reasons. First, an infinite network which covers the manifold would be much closer to our intuitions of space than a network formed by a finite and small number, and even 0 of elements. In this sense, it would make also easier the connection with the continuum theory. Second, for doing cosmology infinite networks seem to be much more appropriate than just finite ones, as it is common in cosmology that one considers infinite, open universes.

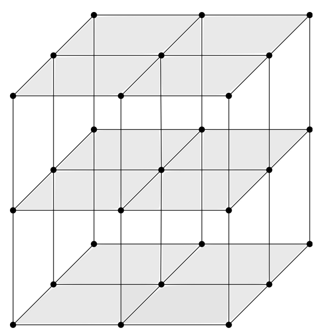

Fig. 5.13: An example of a cubic lattice that could expand infinitely. There is no spin network state associated to such a graph.

A different problem of LQG has also been pointed out[59] and it is that the gauge group of LQG is SU(2) (or equivalently SO(3)) and not the full Lorentz group in four dimensions SO(3,1). Recall that this is a consequence of the gauge fixing we introduced in section 5.1.2: we fixed one of the elements of the tetrad, which limited the gauge transformations to just the group of rotations of the other three, instead of the original group of rotations and boosts. From a classical perspective this choice is harmless as one can always choose to fix or not some gauge degrees of freedom. The problem is that when quantizing the theory the result may be anomalous, i.e., the resulting quantum algebra may not respect the algebra of diffeomorphisms. A reason for suspecting this is that Barbero's connection is not a geometric quantity

[58] See (Wallace, 2006, sect. 4) for a detailed discussion.
[59] See (Samuel, 2000; Alexandrov and Roche, 2011; McCabe, 2018).

from the 4-dimensional point of view, and hence its transformation behaviour under diffeomorphisms is not straightforward[60].

As an alternative, there have been some attempts to define a quantum theory for a connection that respects the Lorentz symmetry and is not anomalous using the techniques of LQG[61]. This is technically more involved, as one has more degrees of freedom to deal with and as one has to work with the group SO(3,1), which is not compact and which is mathematically less tractable. For this reason, no alternative has been developed in the same mathematically consistent way that canonical LQG has been developed.

However, there is an important feature to point out from these attempts, and it is that in these theories area operators are defined in an analogous way to the way they are defined in LQG but one finds that their spectra are continuous! Even if these theories have difficulties, this is a serious challenge to one of the main claims of LQG. The reason for taking seriously this challenge is that discreteness of geometry in LQG was a consequence of the compactness of SU(2). If one considers that the gauge group of the theory has to be the non-compact SO(3,1), there is good reason to doubt that areas and volumes are discrete.

Let me finish this section by insisting in that the quantization chosen in LQG is a technical improvement over the geometrodynamical one in that it defines a mathematically tractable Hilbert space, but that from a conceptual point of view there are reasons to doubt about whether it is an appropriate quantization. As I have explained above, the functions considered and the choice of measure makes it the case that states do not describe 3-geometries, but instead they describe some distributional version of them. They have been interpreted as states describing some discrete space, but this interpretation faces some troubles as I explained above (5.2.3). In this sense, one may worry that the quantization chosen may be consistent, but it may have nothing to do with the real world and one has to be careful when considering the predictions which derive from it.

5.3 Canonical quantization II: dynamics and problem of time

We are now in a position to analyze the last step of the canonical quantization program for LQG, i.e., the imposition of the Hamiltonian constraint and the dynamics of the theory. In order to do this, I divide this section in two parts. The first one is more technical and in it I sketch the definition of the constraint and the physical Hilbert

[60] Ashtekar's connection can be given a more geometrical reading as a pull-back of the 4-dimensional spin connection, but as currently LQG is formulated in terms of Barbero's connection it is subject to the difficulties we are describing.

[61] See Alexandrov and Roche (2011) and references therein.

space and point out some problems in this construction. The second one makes the connection with the previous chapter and analyzes the interpretational issues related with the problem of time and the solutions to it that have been proposed in the context of LQG.

5.3.1 Hamiltonian constraint and physical Hilbert space

In the previous section we have defined the kinematical Hilbert space of LQG and we have imposed both the gauge and the spatial diffeomorphism constraints. The final step we need for completing the quantization program and having a formally defined theory is to define the physical Hilbert space which also satisfies the Hamiltonian constraint and to define an inner product in this space. In this subsection I explain the progress made in this direction in LQG, which leads to a theory that is mathematically better defined than the original quantum geometrodynamics or any other attempt to canonically quantize gravity. Despite this, quantization is not complete, as there remain some difficulties to have a satisfactory definition of the physical Hilbert space.

First, note that the group average technique used to impose the diffeomorphism constraint is not available for the Hamiltonian constraint. The technical reason for this is that the constraint algebra is an open algebra and not an honest Lie algebra[62]. Therefore, the constraint has to be directly imposed. For this, we need to express the constraint in terms of the basic variables of the theory, i.e., holonomies and fluxes. Recall that what at first was attractive about Ashtekar's complex connection was precisely that it seemed to allow for a simple polynomial expression of the constraint. Using Barbero's real connection, which is the one in which the theory is currently formulated, the constraint does not take a simple form, and, hence, expressing it as a quantum operator on the kinematical Hilbert space is a formidable challenge that was first achieved by Thiemann (Thiemann, 1996, 1998).

Here I will not enter into the details of Thiemann's construction and will just point to some relevant aspects of it. First, as happened with the area and volume operators, the way the operator is defined is by introducing some partitioning of the bare manifold into cells, defining the action of the operator on those cells and then taking the limit of the size of the cells going to 0. This limit is well-defined, giving a well-defined operator. But as happened in the case of the area and the volume operators we want the limit to be as independent as possible from the regularization procedure. That is, it is desired that the operator forgets as much as possible about the way the cells were defined and the limit taken. If the operator remembers about the regularization procedure we may have different operators for different regularizations. In the case of the Hamiltonian constraint in Thiemann's proposal, it has some dependency on

[62] See (Thiemann, 2007a, Chapt. 10)

the regularization, and hence there are several different alternatives. However, it is argued (Thiemann, 2007b) that this choice is more natural than the alternatives and that in any case, once diffeomorphism invariance is taken into account, there are only a finite number of them.

Second, there are some other ambiguities in the definition of the operator. Some of these ambiguities are the familiar factor ordering ambiguities of any quantum theory and some others are more specific to LQG, like ambiguities that have their origin in the choice of the representation used for the holonomies. Again, Thiemann (Thiemann, 2007b) argues that his definition is the most natural one and that not every choice is possible: for instance, different factor orderings lead to ill-defined operators. The conclusion is that, all ambiguities considered, there are only a few possibilities, of which Thiemann's definition is preferred. Notice that this is not very problematic, as what would be problematic would be to have too many alternatives or to have none. To have some well defined operator, even if it is not unique, is indeed significant progress with respect to geometrodynamics, where this was not achieved.

Third, the action of the operator on a spin network state is roughly to create and eliminate edges in a small neighborhood of each node. When one takes the limit of the regulator going to zero, the size of the neighborhood around each node which is affected by the operator also shrinks to zero. In this sense, the action of the Hamiltonian constraint is said to be ultralocal, as it only affects very locally the nodes of the network. This has been argued Smolin (1996) to be problematic: if one is to recover the dynamics general relativity as some sort of classical limit, it seems reasonable to expect the dynamics to be able to represent some perturbation propagating, as happens in lattice gauge theory. However, as the action of the constraint only affects the neighborhood of the nodes, there is no propagation possible. Thiemann (Thiemann, 2007b) replies to this possible drawback of his formulation that it is not conclusive, as one should be careful when importing expectations from such a different theory as QCD in its lattice formulation. In other words, Thiemann believes that it is possible that his constraint represents the correct dynamics despite its ultralocal action.

As happened with the diffeomorphism constraint, the states that satisfy the Hamiltonian constraint are not states in the kinematical Hilbert space $\mathcal{H}$ but they are distributional and 'live' in a different space. However, it is not clear which precise distribution space this is and, moreover, an inner product for this Hilbert space is lacking. For constructing such a product we would like to have a map similar to the one we had for the diffeomorphism constraint:

$$\begin{aligned} \eta : \mathcal{H} &\to \mathcal{H}_{phys} \\ |\psi\rangle &\to \eta(|\psi\rangle) = (\eta(\psi)| \,, \end{aligned}$$

where again I am using the notation $(...|$ to refer to distributional states that satisfy the constraints. With such a map available one can define the inner product in $\mathcal{H}_{phys}$

by:

$$(\eta(\psi)|\eta(\phi)) = (\eta(\psi)|\phi\rangle\,. \tag{5.58}$$

This map is sometimes referred to in the literature[63] as the projector, even if it is not strictly speaking a projection. The reason for this terminology is that in the case of constraints with discrete spectra, like the Gauss constraint, the mapping is indeed a projection. This mapping will be specially important for the transition amplitude resolution of the problem of time I introduced in section 4.7 and that I will analyze for the particular case of LQG in the next subsection. In any case, such a mapping has not been defined in a satisfactory way and LQG currently lacks a well-defined physical Hilbert space and inner product.

The other great challenge that the theory faces at this level is to define some sort of classical or semiclassical limit in which some of the features of general relativity are recovered. As I have mentioned, even if we have followed the canonical quantization program, along the way we have made a series of choices that make it possible that something has gone wrong. Moreover, and as explained in section 5.2.4, there is the possibility that the quantization of LQG is anomalous[64]. In any case, having a classical limit is a consistency check that is still missing from LQG.

Finally, let me say that the difficulties in finding the physical Hilbert space and inner product were part of the motivation for the LQG community to shift from the canonical formulation of the theory to the covariant one. Nevertheless, let me insist that the canonical formulation of LQG gets much further than any other canonical quantization of general relativity and that it provided with the basic ingredients needed for the covariant formulation. We will deal with the covariant formulation later on, but I will first introduce the conceptual and interpretational difficulties that are present in the canonical formulation.

5.3.2 Interpreting canonical LQG

As we saw in chapter 4, theories of quantum gravity formulated following the canonical quantization program suffer from a series of conceptual difficulties related with the problem of time. As I explained there, the quantum constraint equations for reparametrization invariant imply that there is no non-trivial dynamical equation describing an evolution of states in the physical Hilbert space. For having a satisfactory resolution of the problem of time we would like to be able to give a physical interpretation of the formalism obtained following the canonical procedure. This involves giving answers to questions related with the nature of time, the interpretation of

[63] For instance in Rovelli (2004).

[64] See (Alexandrov and Roche, 2011) and (Thiemann, 2007b) for arguments supporting and rejecting that LQG is an anomalous quantization.

quantum mechanics and the role of probability. In chapter 4 I argued that there are strong reasons to doubt that the resolutions proposed for solving the problem of time for quantizations of general relativity are successful for this theory, and here I will argue that the same applies to LQG.

In section 4.7 I argued that the key feature that allowed to solve the problem of time for some simple models like the reparametrization invariant model for a non-relativistic model is that these models are deparametrizable, i.e., that we are able to identify the variables in their configuration spaces which play the role of a time variable or of spacetime variables. In this sense, we say that these models are defined on extended configuration spaces and not proper configuration spaces. The different resolutions of the problem of time for these models rely on this fact to define a relational evolution with respect to the time variable in this configuration space and not with respect to the arbitrary evolution parameter used in the reparametrization invariant formalism. In chapter 4 I argued that these resolutions do not seem to be applicable to general relativity, as it is not a deparametrizable theory, it is not defined in an extended configuration space and its configuration space does not seem to encode temporal information in any other way.

For the case of LQG the same arguments apply. The connection formulation of general relativity I have introduced and discussed in section 5.1 is just a gauge extension of the ADM formulation we studied in chapter 4. That is, the theory is essentially the same, as the variables in the connection formulation have the same physical content as the ADM variables, together with some extra gauge degrees of freedom. In this sense, it follows that if general relativity was not deparametrizable and its configuration space was not an extended configuration space and it did not carry information about time in any other way, then exactly the same holds for its connection formulation. This implies that the proposed resolutions of the problem of time in the case of LQG are as misguided as I argued they are in the geometrodynamical formulation.

Let me mention two common ways in which the problem of time has been addressed by the LQG gravity community. The first one is by means of the Dirac observables strategy that I outlined in section 4.7.4. For instance, Thiemann (2007b) advocated for this strategy. The observables that Thiemann aims to define are coincidence observables which describe the value that a physical quantity takes when others (4 in the case of general relativity) take some values, just like X_T in described the position of the particle at the time T in the example I used in section 4.2. However, as I argued in chapter 4, this strategy does not seem to work, not only for technical but also, and more importantly, for conceptual reasons, if the theory we are considering is not deparametrizable, just as in the case of general relativity. In this sense, my arguments above lead one to reject a formulation of LQG based on the

Dirac observables strategy like Thiemann's.

More recently, the LQG community has shifted to the transition amplitudes strategy and to formulations of the theory which rely on covariant quantizations like the ones used in spin foam models. This shift is well-represented by Rovelli's book (Rovelli, 2004), where it is explicit[65] that the goal of the theory is the definition of 'transition amplitudes' in an analogous way to the transition amplitudes I defined in equation 4.114 in chapter 4, that is, the transition amplitude is defined using states in the kinematical Hilbert space, and the projector map η to the physical Hilbert space I have defined above. In covariant formulations and spin foam models one can define transition amplitudes by means of some analog of a path integral without referring to the canonical formalism, but notice that it is usually the case that some equivalence between the formalisms is expected[66]. Even if that were not the case, some of the conceptual worries that I raised in section 4.7.5 about the transition amplitude still apply to transition amplitudes defined in ways different from the canonical formalism. In chapter 8 I will introduce covariant formulations of LQG and spin foam models and I will expand on this point.

Let me recall my analysis of section 4.7.5. The transition amplitudes strategy worked well for the case of the deparametrizable model of the non-relativistic particle as the transition amplitudes it defined are nothing but the propagator of the theory, which allowed one to recover the standard formulation of quantum mechanics. For non-deparametrizable models I argued that the quantity defined as a transition amplitude does not satisfy the properties we would expect of a propagator and hence it cannot be given that interpretation[67]. Therefore, if one wants to insist that the quantities defined as transition amplitudes have some physical meaning and interpretation one cannot appeal to the notions and intuitions that applied to the case of the deparametrizable model. In this sense, the interpretation of these quantities as probabilities remains unclear, and the standard interpretations of quantum mechanics are not available in this case, as we lack the usual structure of a quantum state evolving in a Hilbert space[68]. Moreover, in the double harmonic oscillator example

[65] See for instance sections 5.4 and 7.4 in Rovelli (2004).

[66] Consider for instance the textbook Rovelli and Vidotto (2015). The transition amplitudes defined in this book (see for instance chapter 7) are defined by means of a spin foam model and not directly from the canonical formalism, but they are nevertheless considered to be some sort of approximation to the 'true' transition amplitudes which would agree with the ones defined by the canonical formalism (see the discussion in sections 2.4 and 8.3.2).

[67] Interestingly, for the case of LQG the same point is raised in (Thiemann, 2007b, p. 96), where it is argued that one should refer to these inner products as inner products and not as transition amplitudes.

[68] I refer the reader again to Colosi and Rovelli (2003), where it is argued for such a probabilistic interpretation of 'transition amplitudes' defined by means of the inner product of the physical Hilbert space and my criticism to this approach in section 4.6.

in section 4.6 it was clear that the inner product did not have any straightforward interpretation as a transition amplitude and that it was signaling that the canonical quantization program simply failed to give a successful quantization of the theory. In the case of LQG this is also a possibility one should also consider.

An alternative strategy to approach the problem of time is the Bohmian one. For a Bohmian, having a wavefunction or quantum state with no time parameter is not fatal, as the guidance equation may still have a time parameter and therefore one is able to have some basic ontology evolving with time. Some first steps in the construction of a Bohmian loop quantum gravity are sketched in Vassallo and Esfeld (2014). In this case the basis beables of the theory would not be smooth 3-geometries but discrete ones described by spin networks. As happened with geometrodynamics, this strategy has some drawbacks as breaking covariance, as it clearly separates space and time, against the spirit of relativity. Moreover, one can worry about how is the guidance equation which defines the dynamics to be derived or postulated.

In chapters 7 and 8 I will introduce covariant formulations of quantum gravity in general and LQG in particular and I will argue that they also suffer from interpretative issues related with the problem of time, as I have just suggested. Before, in chapter 6 I will introduce loop quantum cosmology, a family of cosmological models which uses some of the techniques of LQG and which suffers from some of the same shortcomings. In particular, the way the problem of time is dealt with in this approach is also vulnerable to my criticisms in this section and in chapter 4.

Let me finish this section and this chapter by insisting on the two main problems I have found in the LQG approach to quantum gravity. First, at a kinematical level the quantization chosen is very peculiar and I have raised several worries regarding it. In this sense one may wonder if this quantization is really an improvement over the geometrodynamical one and one should be careful when considering the predictions of discrete areas and volumes. Second, the most important issue with LQG, as with any canonical approach to quantum gravity is the problem of time. I have argued in this section, as well as in chapter 4, that the problem of time is a serious one and the proposed resolutions face serious conceptual challenges, as they work for deparametrizable models but they do not seem to work for theories like general relativity. If one wants to insist that these proposals represent a solution of the problem of time one has to accept that the interpretation of the quantities they propose as fundamental, like the transition amplitudes, cannot be subject to the interpretation they usually receive in a standard quantum theory, because we are lacking some of their usual properties and some of the usual structures of a quantum theory. It is in this sense that I argued in chapter 3 that for discussing whether spacetime can be said to emerge from LQG we lack clear understanding and interpretations of the formalism, which makes it difficult, if not impossible to apply the functionalist strategy to the theory.

6. LOOP QUANTUM COSMOLOGY

The loop quantization methods I have introduced in the previous chapter can also be applied to cosmological models. This gave rise to a new direction of research, known as loop quantum cosmology (LQC), which studies the models that are obtained by loop quantization of simple cosmological models. From a philosophical perspective, the interest of LQC is twofold. First, its cosmological models are of physical interest on their own, as we will see that these models usually have interesting properties and predictions, such as replacing the big bang singularity with a bounce. Second, given the symmetries and simplifying assumptions of LQC it is a much more simple and controlled theory than the full LQG and therefore LQC can be seen as a conceptual lab where to experiment with the ideas of LQG for gaining knowledge and guidance for the development of the full theory. However, I will argue that LQC suffers from the problem of time just as LQG, and furthermore, that the relational resolutions applied for solving it are inappropriate, just as I rejected similar resolutions in chapters 4 and 5.

LQC is **not** directly derived by imposing some symmetries to LQG models. Instead, it is derived by applying the loop quantization methods to some symmetric models familiar from cosmology, like the FLRW models. Therefore, even if the LQC community expects LQG to contain LQC and its predictions as some particular sector of the theory or some approximation to it, we are not guaranteed that this will be so. However, as LQG models get more complex and realistic, the confidence in the validity of the general results of the theory increases between its practitioners. In particular, LQC is getting to a level in which it is able to make some testable predictions[1]. Nevertheless, in this chapter I will raise some doubts about its foundations, as I will argue that the way the problem of time is addressed in not adequate.

In this chapter I will introduce and study a basic LQC model, a FLRW spacetime with an scalar field. I will start by reviewing the classical model (subsect. 6.1) and by applying the loop quantization to it to define a kinematical Hilbert space and the Hamiltonian constraint (subsect. 6.2). Next, I will comment on two relational resolutions of the problem of time for this model and their physical predictions: first, the one that takes the geometric degrees of freedom as the internal clock or time

[1] See for instance Ashtekar et al. (2020).

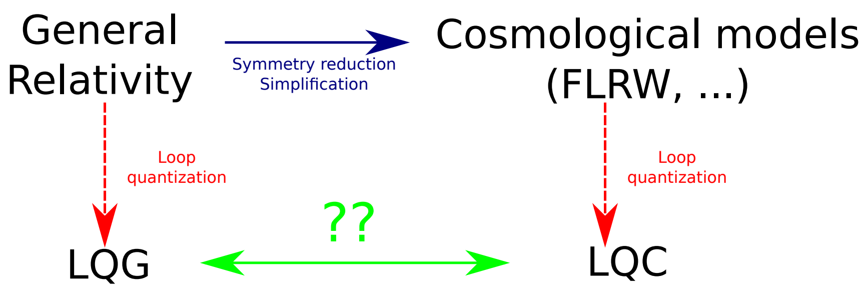

Fig. 6.1: Diagram showing the relations between general relativity, LQG, the cosmological models used for LQC and LQC itself. The relation between LQG and LQC is not clear.

variable (6.3) and then the ones that take the matter degrees of freedom as the clock or time variable (6.4). I will next compare both approaches and evaluate the use of the relational resolution of the problem of time (6.5). I will argue that this resolution seems to give an intelligible quantum theory, but I will argue that we should reject this resolution, essentially for the same reasons outlined in chapter 4 and also because the intelligibility of this case is a product of the simplicity of the model. Finally, in the last subsection 6.6 I will comment on the present state of the art of canonical LQC and will introduce effective cosmological dynamics suggested by LQC. I will argue that the criticisms outlined in this chapter for the simple LQC models apply also for generic LQC models and that therefore there are serious reasons to doubt about the foundations of LQC in general.

6.1 Classical FLRW model

The model I will be using in this section is a flat Friedmann-Lemetre-Robertson-Walker (FLRW) spacetime in which there is only one matter scalar field ϕ. The flat FLRW spacetime is a spacetime such that there is system of coordinates in which the metric takes the form:

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)\,. \tag{6.1}$$

That is, in these coordinates spacetime is described as a succession of homogeneous and isotropic spaces, with sizes given by the function $a(t)$, known as the scale factor. We have fixed the lapse function $N(t)$ to be 1, hence the time coordinate t has the physical meaning of being the proper time of observers who are at rest with respect to this coordinate system. Models of this type are known as mini-superspace models, as recall from chapter 4 that superspace was the configuration space of general relativity, i.e., the space of 3-dimensional geometries. Mini-superspace is therefore the restriction of this space to just a space with one degree of freedom, the scale factor.

Thanks to the homogeneity of space, by studying what happens in a coordinate region, that we will call the fiductial cell $\mathcal{V}$, we learn what happens in the whole universe. By restricting our integrals to $\mathcal{V}$ we avoid the mathematical complications of dealing with a potentially divergent quantities. The fidutial metric δ_{ij} gives the coordinate distance between any two spatial points in the comoving coordinates we are using. The scale factor a relates the physical distance with the coordinate one. If the fidutial cell has a coordinate volume V_0, its physical volume at some moment of time is a^3V_0. The scale factor therefore tells us how the physical volume of the cell, and of the universe too, change, i.e., whether it expands or contracts. Of course, physical predictions of a theory of gravity, classical or quantum, should be independent of the choice of cell.

Given the symmetry of this spacetime, for describing it we no longer need a field theory, as the degrees of freedom have lost their spatial dependence due to the homogeneity of space. Recall that in the ADM formulation of general relativity, the configuration space is the space of 3-metrics and it is also known as superspace. For the restricted case we will study here, we will consider only 3-geometries of the FLRW form, and the configuration space is known as minisuperspace. The dynamics of general relativity reduce to the Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \,. \tag{6.2}$$

Where ρ is the energy density of matter and the dot represents a time derivative. There is a further differential equation in the dynamics, the Raychaudhuri equation, but it can be derived from the Friedmann equation and the continuity equation of matter:

$$\dot{\rho} + \frac{3\dot{a}}{a}(\rho + P) = 0 \,, \tag{6.3}$$

where P is the pressure of matter.

We can formulate the dynamics in Hamiltonian form, and due to the temporal reparametrization invariance we get a constrained system, as happened for full general relativity. The Hamiltonian constraint can be written[2] as:

$$\mathcal{H} = -\left(\frac{4\pi G}{3}p_a\right)^2\frac{1}{a} + \frac{8\pi G}{3}a^3\rho \,, \tag{6.4}$$

where p_a is the momentum conjugate to a:

$$p_a = -\frac{3}{8\pi G N}a\dot{a} \tag{6.5}$$

[2] See for instance the derivation in (Calcagni, 2017, sect. 10.1).

and ρ has to be understood a function of the canonical variables of the matter fields. Imposing the constraint and choosing proper time as the time parameter gives us back the Friedmann equation.

I will now follow Ashtekar and Singh (2011) to formulate this models using connection variables similar to the ones introduced in sect. 5.1 in the previous chapter. We choose the connection and triad to be:

$$A_a^i = cV_0^{-1/3}\delta_a^i \tag{6.6}$$

$$E_i^a = pV_0^{-2/3}\delta_i^a\,. \tag{6.7}$$

Here we are using the fidutial triad and cotriad, δ_i^a and δ_a^i, which we have gauge-fixed to align with the fidutial coordinates. The variables c and p are related to the scale factor by the following relations:

$$a = \sqrt{|p|}V_0^{-2/3} \tag{6.8}$$

$$c = \gamma\dot{a}V_0^{1/3}\,. \tag{6.9}$$

Here γ is again the real-valued Barbero-Immirzi parameter. Notice that p can take positive and negative values, depending on the orientation of the physical triad with respect to the fidutial one. If it were 0 we would have a degenerate metric, which is a situation which classically would not be allowed. However, in building a quantum theory one may consider the case of flipping orientations and degenerate metrics. Notice also that p has the direct interpretation of giving the physical volume of the cell $\mathcal{V}$[3] and c is its conjugate momentum:

$$\{c, p\} = \frac{8\pi G\gamma}{3}\,. \tag{6.10}$$

In these variables the Hamiltonian constraint takes the form:

$$\mathcal{H} = \mathcal{H}_{grav} + \mathcal{H}_{matter} = -\frac{3}{8\pi G\gamma^2}c^2\sqrt{|p|} + \mathcal{H}_{matter}\,, \tag{6.11}$$

where $\mathcal{H}_{grav}$ is the gravitational Hamiltonian and $\mathcal{H}_{matter}$ is the matter Hamiltonian. As I said, in this section I will be looking at the simplest case possible, the case in which there is only one massless scalar field. In this case the constraint in terms of the field $\phi$and its conjugate momentum p_ϕ is:

$$\mathcal{H} = -\frac{3}{8\pi G\gamma^2}c^2\sqrt{|p|} + \frac{1}{2}|p|^{-3/2}p_\phi^2\,. \tag{6.12}$$

[3] As I said the physical volume is given by a^3V_0, and by replacing the relation between a and p we get that the volume is $|p|^{3/2}$.

Let me point out that this model contains basically two degrees of freedom, a (equivalently, p) and ϕ and that what we are interested in here is in their evolution in time, $a(t)$ and $\phi(t)$. In this sense, this model is very similar to the double harmonic oscillator model I introduced in section 4.2. That is, both models describe system with two degrees of freedom which evolve in an arbitrary parameter τ but for both we can define a privileged way of coordinatizing time: in the case of the double harmonic oscillator it was the Newtonian time and in this case it is the proper time for an observer at rest. In chapter 4 I argued that the problem of time for the double harmonic oscillator was fatal as it did not lead to the quantum theory for two harmonic oscillators and attempting some of the standard resolutions of the problem of time to this model did not work. In the case of LQC I will argue for the same conclusion. However, there is a difference between both cases which makes it the case that the problem of time for this model does not look so problematic. The problem is that this model is so simple that solutions to it have that both ϕ and a behave monotonically in time[4], and hence they can be used as clocks. I will argue that this leads to confusing them with time variables, which is conceptually wrong. In the next section I apply the canonical quantization procedure to this model.

6.2 Loop quantization of the model

Before explaining the loop quantization of the FLRW model with a massless scalar field let me remind you that loop quantization is the alternative to the Wheeler-deWitt quantization we saw in chapter 4. Hence it will later be interesting to compare the two quantum cosmologies that arise from applying the two quantization schemes to the same FLRW model. As happened for the full theory, the Wheeler-deWitt quantization is more conservative in the sense that it sticks closer to the Schrödinger picture, while the loop quantization uses its own measures, analogously to the Ashtekar-Lewandowski measure for the full theory, but results in a mathematically better-behaved theory.

As in the case of LQG, the algebra of LQC is formed by holonomies and fluxes. Given the homogeneity and isotropy of the model, it suffices to consider holonomies just on straight edges following the fidutial directions. For an holonomy along an edge of length μV_0, where μ is an arbitrary real number, we get:

$$h_i^{\mu}(c) = e^{\mu c \tau_i} = \cos(\frac{\mu c}{2}) I + \sin(\frac{\mu c}{2}) \tau_i \,, \tag{6.13}$$

where τ is again the basis of the $\mathfrak{su}(2)$ algebra. Therefore, we see that the information

[4] More precisely solutions to the Friedman equations describe a monotonically expanding/contracting universe from/to a singularity, with monotonic change in the field.

of the holonomies is essentially contained in the quantities:

$$N_\mu(c) = e^{\frac{i\mu c}{2}} . \tag{6.14}$$

Given the symmetries of the model, fluxes also get simplified. The flux along any surface on the cell is just p multiplied by a factor proportional to the fidutial area of the surface. Therefore, the basic elements of the holonomy-flux algebra are just $N_\mu(c)$ and p, and their Poisson bracket structure is:

$$\{N_\mu(c), p\} = i\frac{4\pi G\gamma}{3}\mu N_\mu(c) . \tag{6.15}$$

Now, for building the quantum theory we need to choose a Hilbert space and a representation of the holonomy-flux algebra. As happened in the case of LQG, in LQC one does not take the Schrödinger representation, which would lead to a theory equivalent to Wheeler-DeWitt's[5]. Instead, we choose a peculiar representation with special properties, namely a discrete topology. In particular, the Hilbert space is $L^2[\mathbb{R}_B, d\mu_H]$, this is, the Hilbert space of square integrable functions on Bohr's compactification[6] of the real line $\mathbb{R}_B$ under the Haar measure $d\mu_H$. Let me introduce this space and its properties.

This Hilbert space is the Cauchy completion of the space of almost periodic functions[7], i.e., functions that can be expressed in the following way:

$$\psi(c) = \sum_j a_j e^{\frac{i\mu_j c}{2}} = \sum_j a_j N_{\mu_j}(c) , \tag{6.16}$$

where the sum over the index j contains only a finite number of terms, each with a different real-valued μ_j and coefficient a_j. The functions $N_{\mu_j}(c)$ are a basis of this space, that I will represent by $|\mu\rangle$ and the inner product between them is:

$$\langle\mu|\mu'\rangle = \delta_{\mu\mu'} , \tag{6.17}$$

where the δ is a Kronecker delta, not a Dirac one. For any two elements of the space the inner product is defined by linearity:

$$\langle\psi|\psi'\rangle = \sum_{j,j'} a_j^* b_{j'} \delta_{\mu_j\mu_{j'}} . \tag{6.18}$$

[5] For concretion, this representation leads to the Hilbert space of square summable functions of c, $L^2[\mathbb{R}, dc]$, under the canonical Lebesgue measure.

[6] This space was discovered by the mathematician Harald Bohr, brother of Niels Bohr and founder of the field of almost periodic functions.

[7] The reason for this name is that these functions are not strictly speaking periodic, but they are so to any degree of accuracy.

This inner product is necessarily finite, as the sum runs over a finite number of indices j, j'. Notice here the resemblance with the Ashtekar-Lewandowski inner product and the spin-network Hilbert space. Our basis elements $|\mu\rangle$ play the same role as the spin network states, and general states are only finite superpositions of them, as happened in LQG. In the same way that the Ashtekar-Lewandowski product imposed a discreteness on the spin-network Hilbert space because of the Kronecker delta between the networks, this inner product is also related with the discrete topology of the space of almost periodic functions. As happened in LQG, this choice makes some operators ill-defined, and we will see that some regularization procedure is needed in order to build operators such as the Hamiltonian.

There is no rigorously well-defined operator for c, in the same way that the connection operator in LQG was ill-defined. The operator associated with p, is well defined as a derivative operator, and the basis $|\mu\rangle$ is its eigenbasis:

$$\hat{p}|\mu\rangle = \frac{8\pi\gamma l_P^2}{6}\mu|\mu\rangle\,, \tag{6.19}$$

Where l_P is the Planck length. The sign of μ determines the orientation of the triad and, knowing that the relation between p and the volume of the fidutial cell is $V = |p|^{3/2}$, its absolute value gives us the volume:

$$\hat{V}|\mu\rangle = (\frac{8\pi\gamma|\mu|}{6})^{3/2} l_P^3|\mu\rangle\,. \tag{6.20}$$

Therefore, we have a clear physical interpretation for this basis of states. Finally, notice also that even if the Hilbert space has a discrete topology, its basis is non-separable, as there are as many basis elements as real numbers.

The last piece for defining the theory is to express the Hamiltonian constraint 6.12 using the holonomy-flux operators, which a needs regularization procedure, and to impose this constraint to define the physical Hilbert space. In full LQG we have seen that this has not been achieved in a satisfactory way, but, given that LQC is much simpler, this last step is possible to achieve for this theory. During the development of LQC different regularizations have been proposed, and here I will comment on a couple of them as later on I will be interested in analyzing the cosmological picture they motivated.

The classical expression for the Hamiltonian constraint contains powers of c, which does not have a well-defined quantum counterpart. Therefore the regularization procedure works by replacing the classical constraint $\mathcal{H}$ by an expression $\mathcal{H}_\lambda$ that contains some holonomies of size λ that form a small loop. When the limit of $\lambda \to 0$ is taken, that is, when we shrink the loop to a point we recover the original $\mathcal{H}$. Even if there was not a quantum operator associated with $\mathcal{H}$, there is one (up to factor ordering) associated with the regularized expression $\hat{\mathcal{H}}_\lambda$. However, the limit $\lambda \to 0$ for these operators does not exist. This is taken as a signal of the fact that quantum geometry implies that there is a minimal scale, the Planck scale.

The first regularization I will study consist in fixing a constant value μ_0 for λ and taking $\hat{\mathcal{H}}_{\mu_0}$ as the Hamiltonian constraint. This is the regularization used in the earlier LQC models[8]. The value of μ_0 can be fixed on different grounds: in Bojowald and Morales-Técotl (2004) it is taken to be unity while in Ashtekar et al. (2006a) it is fixed so that the area associated with the holonomy $h^\mu(c)$ is the area gap from LQG. In any case, the constraint equation for this regularization takes the form:

$$\hat{H}_{matter}\psi_\mu = f'_+(\mu)\psi_{\mu+4\mu_0} + f'_0(\mu)\psi_\mu + f'_-(\mu)\psi_{\mu-4\mu_0}\,, \tag{6.21}$$

where f'_+, f'_0 and f'_- are functions of μ and μ_0. Notice that due to the discrete character of the representation we have chosen, the right hand side has the aspect of a difference equation, not a differential one. I will shortly comment on this.

Alternatively, more recent formulations[9] of canonical LQC use a different regularization. In this case λ is not fixed to be a constant value but instead it is a function $\bar{\mu}$ of the phase space variables chosen in such a way that the physical area of the loops used in the regularization correspond to the area gap of LQG. This regularization is independent of the fidutial cell chosen, and hence is better motivated. For a discussion of the choice of regularization I refer the reader to Corichi and Singh (2008). Given the dependence of $\bar{\mu}$ on p, the Hamiltonian constraint takes a slightly different form, and it is convenient to express it using a different basis. For this let me introduce the variable ν:

$$\nu = K\mathrm{sign}(\mu)|\mu|^{2/3}\,. \tag{6.22}$$

Here K is a numerical constant that is argued to be $2\sqrt{2}/3\sqrt{3\sqrt{3}}$[10]. Other choices are possible here[11], but it is a matter of convention. ν is related with the volume operator, while μ was related with the area. Now it is convenient to express states in terms of their volume eigenvalues, and the constraint equation has the following form:

$$\hat{H}_{matter}\psi_\nu = f_+(\nu)\psi_{\nu+4} + f_0(\nu)\psi_\nu + f_-(\nu)\psi_{\nu-4}\,. \tag{6.23}$$

Now f_+, f_0 and f_- are functions of ν. For this regularization we have found also a difference equation but now the spacing is different: now the equation relates the

[8] See for instance Bojowald and Morales-Técotl (2004); Ashtekar and Lewandowski (2004); Ashtekar et al. (2006a).

[9] See for instance Ashtekar et al. (2006b); Ashtekar and Singh (2011); Agullo and Singh (2016).

[10] See Ashtekar et al. (2006b).

[11] See for instance Ashtekar and Singh (2011).

quantum states for equally separated volumes ν, while equation 6.21 related the quantum states for equally separated areas μ. From now on I will focus on this equation and regularization as, as I said before, is physically well motivated. But keep in mind that given the structural similarity much of what I will say will apply for both regularizations (changing the spacing).

These difference equations have the peculiarity that they decompose the kinematical Hilbert space in different sectors. A way of seeing this is the following, take as 'initial' conditions the functions $\psi_{\epsilon-4}$ and ψ_{ϵ}, for some value of ϵ. Using the constraint equation 6.23 these functions determine the function $\psi_{\epsilon+4}$. Now, this function, together with ψ_{ϵ} and using again the constraint equation determine $\psi_{\epsilon+8}$, and we can go on to determine $\psi_{\epsilon+4n}$ for any natural n. Similarly, our initial functions also determine $\psi_{\epsilon-8}$, and this in turn determines any $\psi_{\epsilon-4n}$. Therefore, we see that the 'evolution' dictated by the constraint only relates the states separated by a multiple of four. In this sense we can split the kinematical Hilbert space in sectors:

$$\mathcal{H} = \bigoplus_{\epsilon\in(0,4]} \mathcal{H}_{\epsilon}\,, \tag{6.24}$$

where $\mathcal{H}_{\epsilon}$ is the sector of the Hilbert space that contains only the states with $\nu = \epsilon + 4n$, with integer n. We have therefore separated the Hilbert space in as many independent sectors as real numbers between 0 and 4. The gravitational part of $\mathcal{H}_{\epsilon}$ has therefore a countable basis, and we avoid the problems related with non-separable Hilbert spaces (recall the discussion in section 5.2).

Notice that, in each sector, there are functions for both positive and negative ν. In this sense, the constraint equation 'goes through' $\nu = 0$, which would classically correspond with the big bang (or big crunch) singularity. For any sector with $\epsilon \in (0, 4)$ the functions with $\nu = 0$ are not part of the sector, and the constraint equation is perfectly well-defined and relates the positive and negative ν functions. There is one sector, the sector which contains the $\nu = 0$, which needs a more careful study, as it contains a state that could need of a different treatment.

Here, the different regularizations we have introduced lead to different situations. In the μ_0 regularization, the difference equations are such that the function ψ_0 does not appear in any of them! Therefore ψ_0 decouples and can be taken to be independent of the other functions, while the rest of functions $\psi_{4n\mu_0}$ are related by the constraint equations. In the $\bar{\mu}$ regularization however, it seems that no such decoupling happens, and ψ_0 is just another function. As we prefer this regularization, should we worry that we are running into some trouble?

It turns out this is not the case. In either regularization schema, independently if there is decoupling or not, that the wavefunction does not vanish for a 0-size of the universe does not lead to singular behaviour. Classically, the two quantities that diverge at the big bang singularity are the spacetime curvature and the matter density. It is a remarkable result of LQC, in both regularizations, that neither of them shows divergent behavior for the 0-volume state. I refer the reader to Ashtekar and Singh (2011) for more details on how these operators are bounded in LQC. From a conceptual point of view a state that describes the universe as having no size may be as puzzling, but from the point of view of mathematical consistency this is an improvement with respect to general relativity.

Finally, to have a well-defined Hilbert structure one needs to define an inner product. We will see that depending on the interpretation one wants to make of the states that satisfy the Hamiltonian constraint there will be different inner products that will be more convenient. As I said above, in LQC it is extended the use a relational strategy to solve the problem of time which consists in identifying one of the variables as the clock variable and describe the evolution of the quantum state with respect to that variable. In LQC models there are two alternative choices for clock variable, either one chooses some of the geometric degrees of freedom to keep track of time or either one chooses some matter degrees of freedom. In this simplified model this amounts to taking the scale factor or the scalar field as clocks. In the following let me explore both possibilities, their advantages and drawbacks and the consequences of both approaches as argued from the LQC community. As I have argued above and more extensively in chapter 4, I believe this resolution of the problem of time is inappropriate, but I leave the discussion of this point to section 6.5.

6.3 Geometry as a clock

The first choice of clock variable we will study are the geometrical variables. This option was the preferred one in the earlier works in LQC[12] and there are some considerations that at first lead to it. In the classical version of a simple scalar field model we are considering here, both ϕ and a are monotonic functions of t, and if we are interested just in the correlation between these two variables it is indistinct if we study $\phi(a)$ or $a(\phi)$. But one of the aims of LQC is to generalize to more general and realistic matter Hamiltonians and in this case it is very likely that the fields do not have such a nice and monotonic behaviour. Therefore, choosing the geometric degrees of freedom as the clock variable allows for an easier generalization.

Moreover, in a universe like ours it seems that the scale factor of the universe is a monotonic function of time, that is, in most of our cosmological models by specifying the scale factor at a given spacetime point one uniquely specifies the time of that event. With time here I do not refer to a meaningless coordinate, but to a physical coordinate that specifies the proper time since some cosmological event, like the CMB release. Thus, our cosmological observations so far indicate that geometry could work

[12] See Bojowald and Morales-Técotl (2004).

as a well-behaved clock. Of course, if our universe happened to have an expansive phase followed by a contraction, this argument for preferring geometry would lose its force.

Let us consider what are the consequences of taking the geometric degrees of freedom as our time variable. As we have seen, these degrees of freedom, ν[13], are organized in sectors and take discrete values. If we consider just one sector, what we find is that time is discrete. That is, we can interpret the state $\psi_{\epsilon+4n}(\phi, ...)$ as a quantum state for the matter degrees of freedom at different instants of time. These instants do not 'flow' in a smooth and continuous manner, but instead we have a countable succession of them. As we have seen before, the state is well-defined also for negative values of ν. One can read this as saying that there is nothing stopping, nor starting, the evolution at the zero volume state. If we start from a state at a given time, lets say $\psi_\epsilon(\phi, ...)$, and run the evolution backwards what we find is that as we run into the past the size of the universe is each time smaller, until we reach a state of negative ν. Remember that these states only differed from the positive ones in their orientation, and that it is the absolute value of $|\nu|$ which gave the volume. Therefore, as we go 'through' the singularity, volume just starts growing again, with an unobservable change of orientation. This model is a bounce model: it describes a universe with no beginning nor end, that starts in a contracting phase until it reaches a minimal size, and then reexpands.

Now, let us consider what happens when we consider a state on the physical Hilbert space with support on two or several different sectors. As we said, each sector is dynamically independent from the rest and therefore they evolve independently from each other. Then, this state would be like having several causally disconnected universes. For instance consider the sequences $\psi_1(\phi, ...), \psi_5(\phi, ...), ...$ and $\psi_2(\phi, ...), \psi_6(\phi, ...), ...$ Nothing of what happens in the first sequence influences the second, as the constraint equations 6.23 only relates elements of a same sequence. Even if the universe was such that it contained several sequences, for an observer in one of them it is as if the rest do not exist and he or she can forget about them.

A further requirement for saying that the evolution in ν is temporal or physical is to ask for this evolution to be unitary. For this we need first a notion of inner product. But notice that this is not difficult to achieve, as the wavefunction at a time ν is just a function of the fields, and we can choose the standard inner product for such fields. For the case of the scalar field this is as simple as the inner product in the Schrödinger representation, $L^2[\mathbb{R}, d\phi]$:

$$\langle\psi|\psi'\rangle = \int d\phi \psi^*_\nu(\phi)\psi'_\nu(\phi)\,. \tag{6.25}$$

[13] This analysis is also true for the variable μ.

This inner product is defined at a fixed time ν. To ask for unitarity is to ask that for any two solutions ψ, ψ' of the Hamiltonian constraint this product is independent of the moment of time at which it is calculated. This is important because it implies that probabilities are conserved.

A first thing to notice is that if the wavefunctions have support in more than one sector, and as the evolution is independent for each sector, the inner product on each sector is also independent from the inner product on the rest of sectors. Therefore, it only makes sense to speak about the unitarity or not of the evolution inside one sector, and we will restrict to wavefunctions with support on one sector. I am not aware of any result in the literature showing that the evolution in ν dictated by the Hamiltonian constraint is (or is not) unitary.

Finally, there is a further issue with the picture just given. Classically, the sign of ν does not have any physical meaning, unless, as we explained in section 5.1, the coupling with the fermion fields is dependent on such orientation. In any case, a good case can be made for considering it just as gauge. Therefore, we are led to identify the states $|\nu\rangle$ and $|-\nu\rangle$ as physically equivalent and one restricts to states that satisfy $\psi_\nu = \pm\psi_{-\nu}$, i.e., the states which assign the same squared amplitude to both $\pm\nu$[14]. But if we are willing to accept this restriction we find that interpreting ν as time is not so straightforward as before.

We will restrict now to the sector of symmetric wavefunctions, $\psi_\nu = \psi_{-\nu}$. This sector is more studied in the literature, as the antisymmetric one does not contain the interesting case of $\nu = 0$. Imposing this symmetry implies that the division in independent sectors of the Hilbert space changes a little. Now, if a state has support on some value ϵ of ν, not only the constraint equation forces it to have support on every value $\epsilon + 4n$, but due to the symmetry condition, it will also have to have support in the symmetric values $-\epsilon + 4n$. Therefore, the sectors are now given by wavefunctions with support in $\pm|\epsilon| + 4n$, for integer n. Each sector is given by a value of ϵ in the interval $[0, 2]$.

Alternatively, we can label these states by the absolute value of ν, which we are now taking to be physical. Can we take $|\nu|$ to be a time parameter? For each sector, for each positive ϵ between 0 and 2 the wavefunction has support in the following values of $|\nu|$: $\epsilon, 4-\epsilon, 4+\epsilon, 8-\epsilon, 8+\epsilon....$ Therefore there is a possible interpretation of $|\nu|$ as a time parameter following such a sequence in increasing order. This gives the picture of a discrete time which has a beginning (or an end), contrary to the bounce picture we had before.

However, there is something puzzling with this interpretation once we look at the constraint equations. We have seen that these equations relate states at values of ν

[14] Notice that the constraint equations 6.21 and 6.23 do not involve the imaginary unit i, and we can restrict states in the physical Hilbert space to be given by real-valued functions.

separated by a distance of 4. For instance, what happens at $\nu = 11$ is determined by the states at $\nu = 7$ and $\nu = 3$. Equally we could have said, as the equations do not carry any asymmetry, that it is determined by the states at $\nu = 15$ and $\nu = 19$. The point is that the fundamental dynamical distance is four. But in the sequences of $|\nu|$'s for each of these sectors includes values that are not separated by a distance of four. In our example the sequence would be: $\{1, 3, 5, 7, 9, 11...\}$. The close-by times according to this ordering are at a distance of 2, i.e., the instant of time that goes before 11 would be 9 and the one that goes after it would be 13. But we have seen that from the point of view of the equations of motion, the instant that goes just before 11, the latest instant that dynamically influences 11, is 7 and not 9. Similarly, the instant that goes dynamically after 11 is 15 and not 13. In other words, the natural order relation in the real numbers that we have postulated to be the temporal order of events does not fit well with the dynamics.

Therefore, it seems more natural to take the order dictated by the dynamics to be the temporal ordering of the events. In this case the values of $|\nu|$ get ordered in the following way: $\{...8 - \epsilon, 4 - \epsilon, \epsilon, 4 + \epsilon, ...\}$. In our example this is the sequence $\{...9, 5, 1, 3, 7, 11...\}$. What we have found, is that in this more natural ordering, we get our bounce back: we get a universe that starts contracting and then reexpands again.

Before moving on to the option of taking the scalar field as a clock let me summarize the situation if we choose the geometry as a clock. The Hamiltonian constraint understood as an evolution equation in ν leads to the picture of a bouncing cosmology. This is independent of whether we consider the sign of ν gauge or not, provided that we use the dynamically preferred ordering. The weakest point of this construction (beyond the criticisms related with the problem of time) is that the evolution for the matter sector has not been given explicitly nor shown unitary.

6.4 Scalar field as a clock

Let us turn to the alternative option of choosing the matter degrees of freedom as the clock variables. From the relationalist point of view there is something natural about this choice; after all physical clocks, at least the ones we are used to utilizing, are material objects made of matter. Moreover, I will consider the case of the simplest matter field we have been studying, the massless scalar field. I will show that choosing the field as the time variable allows to overcome some of the difficulties that were present when we chose the geometry degrees of freedom. Nevertheless, it does so at the price of not being a realistic model.

As we have seen before, in the classical theory, the field ϕ is a monotonic function of time. Classically, this makes it a good candidate for a relational clock: a value of ϕ uniquely determines the instant of time at which it was measured. However, in any more realistic model which includes more matter fields, which in general will not be massless, it will rarely be the case that in the classical theory one of the fields, or one combination of them is a monotonic function of time and hence a good clock variable. In a realistic model this 'choice of clock' is therefore rather complicated, but let us use this simple model to bypass these difficulties and study its possible physical consequences.

We can now rewrite the difference equation 6.23 replacing the generic matter Hamiltonian by the scalar field Hamiltonian to get:

$$\partial_\phi^2\psi(\nu,\phi) = f''_+(\nu)\psi(\nu+4,\phi) + f''_0(\nu)\psi(\nu,\phi) + f''_-(\nu)\psi(\nu-4,\phi) = -\Theta\psi(\nu,\phi)\,. \quad (6.26)$$

The last equality is just the definition of the operator Θ. This equation has just the same form as Klein-Gordon equation, $(\partial_t^2 - \nabla^2)\psi = 0$. It is straightforward to see that ψ is playing the role of time in the Klein-Gordon equation and it can be shown that the operator Θ is positive definite, just as $-\nabla^2$. Therefore, we apply some of the ideas I mentioned in section 4.6 when I introduced the Klein-Gordon equation as a quantization of a model describing a relativistic model.

Solutions of the Klein-Gordon equation are of the form:

$$\psi(t,x) = \int dk\left(\psi_+(k)e_k(x)e^{i\omega t} + \psi_-(k)e_k(x)e^{-i\omega t}\right)\,, \quad (6.27)$$

where $e_k(x)$ are eigenfunctions of $-\nabla^2$, i.e., $-\nabla^2 e_k(x) = k^2 e_k(x)$ and ω is related with k by $\omega = |k|$. The field has two 'components' one with positive frequency, ψ_+ and one with negative frequency, ψ_-. Each of them, obeys a Schrödinger-like equation:

$$\mp i\partial_t\psi_\pm = \sqrt{-\nabla^2}\psi_\pm\,. \quad (6.28)$$

This equation gives a unitary evolution for each of the components of the Klein-Gordon field. Therefore, if we consider only states with just positive or just negative frequency we obtain a well-defined unitary evolution.

This can be also applied to our case. We will restrict to solutions of the Hamiltonian constraint 6.26 of positive frequency[15], which are given by:

$$\psi(\nu,\phi) = \int dk\psi(k)e_k(\nu)e^{i\omega\phi}\,, \quad (6.29)$$

where $e_k(\nu)$ are now eigenfunctions of Θ, $\Theta e_k(\nu) = k^2 e_k(\nu)$ and ω is still $|k|$. The evolution equation for this sector can be written in the Schrödinger form:

$$-i\partial_\phi\psi(\nu,\phi) = \sqrt{\Theta}\psi(\nu,\phi)\,. \quad (6.30)$$

[15] The choice of negative frequency leads to the same results.

Of course all these expressions have to be managed making use of the facts that Θ is a difference operator and not a differential one and that we are restricting ourselves to just one sector[16]. This evolution can be shown to be unitary for the following scalar product:

$$\langle\psi|\psi'\rangle = \sum_{\nu\,\in\,\{\pm|\epsilon|+4n, n\in\mathbb{Z}\}} \psi^*(\nu,\phi)\psi'(\nu,\phi)|\nu|^{-1}\,. \tag{6.31}$$

This precise form of the scalar product can be found by using the group average technique, in the same way that in section 5.2 allowed us to define an inner product for s-knot states starting from the spin network's kinematical space. The inner product in this case is essentially the same as in the gravitational kinematical Hilbert space, just modulated by a factor $|\nu|^{-1}$[17]. The technical reason for this factor is to make some operators self-adjoint.

An important point to make is that the group average technique used to find the scalar product and physical Hilbert space does not make use of the fact that we are choosing ϕ to be the clock variable. Instead, the group average technique seems to be suggesting it: as the application of the group averaging technique leads to an inner product which is conserved in ϕ it is natural to take ϕ to be the time variable.

In this respect, we see how interpreting ϕ as the clock variable has a great advantage over taking ν or any other geometrical variable as the clock. Having a well-defined inner product and a dynamics which is unitary is a requisite for a quantum theory that, as far as I am aware, has only been achieved for the first choice. Moreover, the choice of inner product we are taking is suggested by the group averaging technique. If one were able to define a different inner product such that evolution is unitary in ν, this result would be in conflict with what we have obtained by application of the the group average technique. Now we can turn to study what cosmological picture is suggested by this dynamics.

One useful technique used in LQC, and in quantum cosmology in general, is to study semiclassical states, that is, states which are narrow around a classical configuration and which many times follow a trajectory similar to the classical trajectory. For instance, in the Wheeler-DeWitt quantization of simple models like ours what one usually finds is that semiclassical states remain peaked around a classical trajectory for their whole evolution and that they run into the same singularities and difficulties as their classical counterparts. In LQC we can also study this kind of states, but what we find is something different. Suppose that we start with a state for a late ϕ that starts peaked around a big volume ν, i.e., we start with a state

[16] Here I am following Ashtekar et al. (2006b); Agullo and Singh (2016) and imposing symmetry in ν. The results in this interpretation do not depend in any essential way on this.

[17] Sometimes in the literature one finds a different factor $B(\nu)$. This difference does not affect our discussion.

that describes a universe like ours, and we evolve it to the past, to smaller values of ϕ. What we find now is that the state stays peaked around a well-defined trajectory, but it now diverges from the classical trajectory when it is approaching small volumes. What happens is that, instead of following the classical trajectory to the zero volume state, the trajectory slows down the contraction rate, reaches a minimum volume, and starts reexpanding again, reaching for very early times a semiclassical state which corresponds to a contracting universe. This picture is supported both analytically and numerically for a variety of semiclassical states[18]. In other words, the analysis of semiclassical states from the perspective of using ϕ as a clock gives also the same bounce we found using geometry as a clock.

Moreover, recent works[19] have expanded these results to include more general states, not only semiclassical ones. These works show that bounces are generic in LQC with a massless scalar field, that is, it is shown that quantum evolution never reaches a singularity and that observables such as the energy density are always bounded. That the sigular behaviour typical from general relativity is replaced by a smooth evolution is a feature expected of LQC and of theories of quantum gravity in general.

Finally, let me point out that the analysis of the dynamics can also be approached from the perspective of defining and studying operators that correspond to Dirac observables. For instance, one can define the operator $\hat{V}_{\phi_0}$, which is the quantum version of the phase space function V_{ϕ_0} which gives the volume for a given value of ϕ. For any state one can study the evolution in ϕ_0 of the expectation value of this operator and find similar results to what presented before[20]. As I argued in chapter 4, for deparametrizable systems the frozen observable strategy is a equivalent to other forms of the relational strategy, and given that we are treating the system as if it were deparametrizable it is no surprise that we are getting to the same results. In the next section I will argue against this resolution of the problem of time for this model.

6.5 The relational strategy in LQC

In this section I will analyze how the relational strategy has been deployed in LQC and the extent to which it has produced an intelligible quantum theory. As I argued in chapter 4, we have good reasons for rejecting this resolution for the case of quantizations of general relativity, and I will argue here that the same holds for this model. Finally, I will comment on some claims of emergence done in the LQC context and I

[18] See Ashtekar et al. (2006b).

[19] See Diener et al. (2014b,a); Agullo and Singh (2016).

[20] See Ashtekar et al. (2006a,b).

will relate them with the discussions in chapters 2 and 3.

Let me start by noticing that the relationalist claims that what matters is the relations between physical observables, such as $a(\phi)$ in our case. In the relational resolutions I have presented above, and for the two choices of clock variable, this is precisely what we have obtained, but in a quantum way: now we have a wavefunction for a that evolves in ϕ, or the other way around. But in this model a good case can be made for arguing that there is more to the classical model and to its physical content of the theory than just the relations between a and ϕ.

Indeed, in the classical theory we have the time parameter t which has a clear physical meaning. Remember that t was the proper time between time slices; t is the time a comoving observer would measure. In the classical theory it was meaningful and interesting to study the relations $a(t)$ and $\phi(t)$. For instance, imagine that we use this model to study the big bang, it is certainly physically meaningful to ask how much time has elapsed since the universe had a given density. Therefore, the relationalist quantum theory, which lacks any information about t, is missing some key physical content. This is in analogy with the case of the double harmonic oscillator I introduced in section 4.2, for which I argued that the relations $x(y)$ or $y(x)$ did not contain all the physical content of the model.

There are a couple of answers the relationalist can give to this. First, she can insist that it is only the relation $a(\phi)$ that is physically meaningful, maybe by arguing that in such an homogenous universe with such a simple matter content that is the only relation that matters. However, as we are interested in using such a model as an approximation for our universe, it seems unlikely that we would happily let go the temporal information.

Alternatively, the relationalist could acknowledge that the time t corresponds to something physical, like the position of the hand of a mechanical clock. But then, the relationalist answer would go, in order for the relational picture to include this mechanical clock, one would need to explicitly include it into the model: define its configuration space, its Hamiltonian, etc.

There are two problems for the relationalist if she accepts that t represents something physical. First, if one accepts that time is missing from the model with just a and ϕ, then it still seems the case that the conclusions or predictions we took from this model could be put into doubt. Second, my discussion of the example of the double harmonic oscillator supported the conclusion that it is wrong to conflate the concepts of clock and time: a physical system can be used to keep track of time but it is still something different from a time variable. This becomes particularly important at the time of quantization, where we expect clock degrees of freedom to be quantized as any other physical degree of freedom and not to behave as a time variable, i.e., not to be defining an ordering relation between the quantum state of the rest of the system at different instants of time.

Another way of putting the criticism to relationalism is the following. Relationalism assumes that evolution is something 'internal' to a system, i.e., that it amounts to relations between the degrees of freedom. As in the physical Hilbert space of LQC one only finds the degrees of freedom of the field and of the scale factor, evolution has to be the relations between these two. To include something else from this perspective, we would need to enlarge the model to make it also internal. The problem is that this internal perspective does not fit well with models like the one we are studying here, where, as I have argued, the time t can be considered to be physical even if it is not treated as a configuration variable.

Let me mention that this criticism to relationalism is independent of any position we may hold about absolute time scales. In chapter 4 when I studied the double harmonic oscillator motivated by Barbour's work, it was clear that there is a minimal role that time plays that it seems essential to our physical models, namely, that it defines an ordering between different configurations of physical systems. This ordering relation may be captured by some physical degree of freedom in some circumstance and model, but in general it will not be the case that this is true for a whole evolution of the system and it is important to distinguish time from physical systems used for keeping track of it.

Now, proper time in cosmology has also a metric element attached to it. Its physical meaning in general relativity is usually given by means of the clock hypothesis, that is, that proper time is what an ideal clock would measure. The relationalist may be right in pointing out that ideal clocks do not exist and that at the end of the day it is real physical systems which we use for defining this metric component of time. However, this does not invalidate my argument above that relationalism conflates the ordering relation of time with some physical degree of freedom in a problematic way, specially for quantization. Moreover, there are good reasons for keeping the metric aspect of time, even if one considers it as just an approximation. In the models consider by Barbour it is clear that Newtonian time was a convenient parametrization of time which simplified calculation and the same can be said to be true of proper time for general relativity. Besides, in the real world there are physical systems, like atomic clocks, whose physical readings is very close to what one would expect from an ideal clock and it is useful to keep proper time as an idealization of what these systems are measuring.

In this sense, it seems completely legitimate that cosmology preserves proper time as a physically meaningful variable and that we reject the relational view that all the physical content of our cosmological model is in the relation $a(\phi)$ or $\phi(a)$. In this sense it seems that claims like 'The big bang singularity happened 13.7 billions of years ago' are completely meaningful and something we want to preserve in our models. When moving to a quantum model of cosmology, there is no reason a priori to assume that this perspective will not be valid. Hence, the loss of physical content in the quantum version of the theory is not satisfactory.

Related with this, there is the role that matter fields play. Notice that if no matter field were present we would be left with no possible relational evolution, as there would be just one degree of freedom. Again, this is unsatisfactory, as in general relativity, or in other theories of spacetime, we can make perfect sense of a universe that evolves, or that remains static, even if it does not contain any matter. Moreover, if we consider the scalar field to be our clock, it seems as it is introduced in an ad hoc manner just to conveniently have the right properties to have an evolution as the one one would expect. Of course, if we included more realistic matter fields they would not probably lead to such a neat picture of unitary evolution.

For more complicated models, we have seen that relations between variables become more complex and it may be the case that there is no single variable that behaves monotonically and which can be argued to play the role of a clock variable or a time variable. This more realistic situation is more similar to the double harmonic oscillator example and it makes it clear that the relational strategy mixes the concepts of time and clocks and that this can only get worse as we approximate real-world models or fully general relativistic models. In this sense, the fact that a Schrödinger-like equation, or an equivalent evolving constants picture, was available was only a product of the simplicity of the model and not a hint that the problem of time was correctly addressed.

Let me comment that in this section I have been saying that one can choose either ν or ϕ to be the clock variable. From the relationalist perspective, in the classical theory there is no difference in understanding ν as a function of ϕ or ϕ as a function of ν. From the quantum perspective, at least in standard quantum mechanics, the roles that the time variable and the configuration variable play are quite different. In the case of LQC, the group averaging technique, which does not know about the interpretational choices we are making, gave rise to an inner product which is defined at constant ϕ and is unitary under ϕ-evolution. This favors taking ϕ to be the time variable and ν to be the configuration variable. However, notice that this goes against our initial intuition that both ν and ϕ are degrees of freedom of the theory which are to be treated equally, that is, naively we would have expected that both variables to be able to show quantum phenomena like being able to be in a superposition and so on. This is another of the drawbacks of the relational resolution of the problem of time, as I also argued in chapter 4.

In the discussion of the LQC models some authors use the term 'emergent time' for the relational time[21]. This allows us to make the connection with chapters 2 and

[21] For instance, it is used in Ashtekar et al. (2006a,b).

3. From a functionalist perspective one could say that the roles of time (describing an ordering relation with an absolute scale) are played by some other quantities and therefore one may want to argue that time emerges from these quantities. However notice that as I have been arguing this is because in the relational strategy the role of time is usurped by either the scalar field or the scale factor. In this sense it is not that some temporal properties are found to be emergent from some non-temporal entities, but rather that these temporal properties have been wrongly imposed to the scalar field or the scale factor. In any way, let me insist that even if we considered this model to be successful and a case of emergence of spacetime, the way we use the term emergence here is to express some novelty or surprise, but in a way that is completely compatible with reductionism.

Let me close this section by insisting in that the way the problem of time is addressed in the LQC literature is inappropriate for the reasons addressed above and by referring the reader to 4 for a more detailed discussion of this. This should make us skeptical of the approach in general and its predictions in particular.

6.6 Modern LQC and effective dynamics

In this last section I will briefly mention more recent advances of LQC to give a more complete picture of it. I will also comment on some effective classical dynamics which is inspired by LQC, which gives an effective spacetime description. I will analyze this case from the perspective of the debates on the emergence of spacetime and I will conclude that we cannot claim that this is an example of an emergence of spacetime from something more fundamental, because the problem of time affects LQC and we cannot consider it a well-formulated theory, as I have been arguing in this chapter.

Cosmological models in LQC go beyond what I have exposed so far in this section. More modern models are able to include some degree of anisotropies and inhomogeneities and to include and study primordial fluctuations related to inflation. In this sense, LQC has gone beyond its earlier, oversimplified models and is able to offer more complete models comparable to more standard cosmological models. For some of the latest works and reviews of LQC I refer the reader to (Agullo et al., 2013; Agullo and Singh, 2016; Ashtekar and Gupt, 2017; Ashtekar et al., 2020). An important feature of these models is that the bounce is present in all of them, supporting that it is something generic in LQC and not an artifact of the very symmetric conditions of the initial models. However, these approaches deal with the problem of time in similar ways to the simple model considered in this chapter, and we are justified in doubting about this approach to LQC.

In this section I want to mention also that there are some effective classical spacetime models that are motivated by LQC. That is, we are able to write down some equations that are considered to be good approximations to the effective behavior of spacetime. For instance, Friedmann equation is modified to be:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_{max}}\right). \tag{6.32}$$

The equation has acquired a new term which depends on ρ_{max} and which has the effect of giving an upper bound for the energy density. For densities ρ much smaller than ρ_{max}, the new term is negligible and we get back the standard Friedmann dynamics, while for densities of the order of ρ_{max} the new term becomes relevant and goes to zero. This implies that solutions to this equation may have growing densities and contracting universes, but they reach a minimal size and a maximal density and then reexpand again. In other words, this equation captures the bouncing dynamics we have seen in the full quantum dynamics. Moreover, it can be shown that solutions to this equation describe the trajectory of the peak of the semiclassical states we have studied in the quantum case.

This equation has to be taken to be an effective description while the fundamental physics is quantum gravitational. Notice that it can be argued that there is no problem with the spacetime description, not even at very early times. This is contrary to the expectation that at the big bang the spacetime description of the universe we are used to using would in some sense break down and stop being valid. The effective dynamics of LQC does not find any singularity and it offers a spacetime picture that does not seem to break down. Of course, this is true as long as the effective dynamics is valid, and this will ultimately depend on quantum gravity. In any case, this is in contrast with the picture that is given in some other approaches to quantum gravity in which it is contemplated that the spacetime description is also an effective description of our universe, but in these models the effective description stops being valid at certain point. For instance, this is what is postulated to happen in the group field theory geometrogenesis, which I will study in chapter 9.

Could we say that there is emergence in this case? Similarly to the layered structure introduced in chapter 2, we have a system and different levels of description of it. In this case, we would have the effective description of the modified Friedmann equation, then the more fundamental LQC and finally LQG, which would be the most fundamental theory. LQC is an effective description that tries to capture the essential of LQG in some specific situation, and the same can be said about the effective spacetime dynamics with respect to LQG. I have argued that LQG and LQC do not solve the problem of time in a successful way, and hence that they cannot be considered as fully developed theories. But if this problem were solved, it seems that we would be able to apply the functionalist strategy outlined in chapter 3 and claim emergence in the sense of novelty or surprise but which preserves reductionism.

However, as we have this problem with our theories the claim of emergence remains unclear.

As a consequence of the problem of time, there is a sense in which the effective LQC dynamics is richer than the LQC dynamics. This is because the effective equations are equations in time t, which as I have explained had disappeared from the quantum theory. In this sense, the effective dynamics avoids the interpretational issues of the quantum theory and has more content than LQC, as it may answer questions like how long has it been since the bounce happened. In my opinion, this does not imply something like a failure of reductionism or that time is emergent in a strong sense. Rather, what it shows is the failure of LQC to capture all the relevant physical information, i.e., that LQC does not represent time in any way due to the problem of time. This just supports my argument in section 6.5 that the problem of time in LQC, as it was the case for full general relativity, is not satisfactory solved.

This concludes this chapter and this part of the thesis in which I have dealt with canonical quantizations of gravity. As I have argued, these suffer from the problem of time and the proposed resolutions of this problem are problematic. In the first place, it seems justified to reject them as they seem to rely on some false premises like that time is somehow encoded in the configuration space of general relativity or that some sort of deparametrization is possible so that a relational view is tenable. If we were nevertheless to ignore these serious issues, we would find that these proposed resolutions lead to some formalisms of unclear physical interpretation for which not only the usual interpretations of quantum mechanics do not seem to apply, but also for which the more operational interpretations that have been proposed are not satisfactory, in my opinion. In the rest of the thesis I will consider covariant quantizations of general relativity, which is a different family of approaches to general relativity. However, I will also argue that these approaches also suffer from similar problems which put into risk their justification and their interpretation.

7. COVARIANT QUANTIZATION

In the previous three chapters I have explained how the canonical quantization procedure for building a quantum theory for gravity leads to a number of technical and conceptual difficulties, which are known as the problem of time. In particular, this affected one of the most prominent candidate theories for quantum gravity, LQG, and the related cosmological models of LQC. However, more recent formulations of LQG are not built by following the canonical quantization procedure but instead take a different route, the route of covariant quantization. Other approaches to quantum gravity, like quantum Regge calculus, dynamical triangulations or group field theory, to name a few, are also formulated by means of covariant quantization. Therefore, to understand these theories and to interpret them one should analyze this alternative method of quantization. In this chapter I will introduce covaraint quantization and I will argue that the technical and conceptual difficulties associated with the problem of time of the canonical quantization persist in this alternative formulation.

Covariant quantization was born after Richard Feynman (Feynman, 1948, 1950) discovered that in standard quantum mechanics the probability amplitude for finding a particle at some given position, given that one knows where it started, could be expressed in a very suggestive form: as a path integral. A path integral is a sum over all the possible trajectories, or paths, that a system can take between an initial and a final state. Each path is assigned a weight, given by (the exponential of) its classical action, and the sum gives the probability amplitude. This way of expressing the amplitude looks more covariant, that is, closer to the spirit of relativity for two reasons. First, each element in the sum represents a trajectory, and trajectories are more natural from a relativistic perspective than positions at a given time. And second, each trajectory gets weighted by its associated Lagrangian action, and hence everything is expressed in terms of the Lagrangian formalism instead of the Hamiltonian one. The Hamiltonian formalism explicitly breaks covariance, as it splits spacetime into space and time, hence, the Lagrangian formalism is preferred if one tries to preserve covariance.

The realization that propagators could be written in terms of path integrals led to a further insight: instead of defining quantum theories by means of the Hamiltonian formalism and the canonical quantization procedure, one can reach the same result by carefully defining a space of paths and an action for them. Moreover, the formulation of quantum mechanics using path integrals allows for the use of pertubative techniques, like Feynman diagrams, and this made their use dominant in quantum field theories. Nevertheless, the covariant formulation of a quantum theory is formally equivalent to the canonical one: the path integral defines a wavefunction which evolves unitarily in a Hilbert space just in the same way that a Hamiltonian together with a Schrödinger equation defines the same unitary evolution in Hilbert space. From an interpretational point of view the covariant formalism may suggest new interpretations of quantum mechanics, such as the consistent histories approach, but from a formal perspective there is not a difference between both approaches, as they are just two different ways of defining the evolution of a quantum state.

Given that covariant quantization looks more relativity-friendly, it is natural that attempts to formulate a quantum theory of gravity rely on this quantization method. However, I will argue that the technical and conceptual difficulties that the diffeomorphism invariance of general relativity produces in the canonical quantization of the theory will also be present in the covariant quantization of the theory. I will argue for this in the first four sections of this chapter, following the same structure as the last four sections in chapter 4. That is, I will start by introducing covariant quantization in general. Then, I will study the way it can be applied to gauge symmetries and the way it can be applied to some reparametrization invariant models before finally analyzing the case for general relativity.

In section 7.1 I first show how the dynamics of the quantum theory describing a non-relativistic particle can be expressed in terms of path integrals and then I present covariant quantization as the quantization method which uses similar structures for defining a quantum theory. I discuss the importance that this quantization may have for the foundations of quantum mechanics and I argue that covariant quantization is fully equivalent to canonical quantization. In particular, I discuss the consistent histories interpretation of quantum mechanics, and I argue that it is likely to collapse to either the Copenhagen or many-worlds interpretation. I also argue against interpreting the paths in the path integral as real physical processes.

In section 7.2 I will introduce Hamiltonian path integrals, which will allow me to connect better the covariant quantization with the Hamiltonian formalism in general and with the constrained formalism of gauge theories and general relativity. I will analyze the way a gauge theory can be covariantly quantized and I will show that covariant quantizations of gauge theories rely, as their canonical quantizations did, in being able to identify the physical degrees of freedom of the theory. Therefore, the path integral for a gauge theory relies on paths on a gauge-reduced space or that are gauge-fixed.

In the same way that in section 4.5 I argued that the problem of time of deparametrizable models could be solved, the aim of section 7.3 is to show that these theories can also be formulated in the covariant formalism. For this I discuss in some extension the covariant formulation of the quantum mechanics of a relativistic particle. I will argue that the quantization of deparametrizable systems is not so different from the quantization of a gauge theory, as the quantization relies in distinguishing the variables in the configuration space which are 'physical' and the ones which represent time or spacetime coordinates. In this sense, path integrals can be formulated directly in a gauge-reduced way, i.e., as paths in physical time, or in a gauge-fixed way, i.e., using reparametrization invariant actions and extended configuration spaces but by introducing a gauge fixing condition that fixes the relation between physical time and the arbitrary parameter τ. I will also discuss the physical meaning of some of the objects that appear in the quantization, such as backwards in time trajectories or proper time propagators, and I will argue that only the propagator has physical meaning.

In section 7.4 I finally turn to the case of general relativity. I first notice that the techniques that were valid for covariantly quantizing deparametrizable models cannot be applied to theories like general relativity or the double harmonic oscillator I discussed in chapter 4. This is in perfect parallelism with the fact that the problem of time could be solved for deparametrizable models and not for general relativity or the double harmonic oscillator model. I analyze then the approach by Reisenberger and Rovelli (1997) to the covariant quantization of gravity, and which contains some of the main ideas that later approaches follow. I argue that their approach is not satisfactory for at least three reasons. First, it is an approach which is connected with the canonical formalism and the transition amplitudes resolution. As in section 4.7.5 I argued that this resolution is not satisfactory and the problem of time is not solved, then, when we reexpress things in the covariant formalism, these objections are still present. Second, the object it defines is not a propagator in any straightforward sense, and it cannot be interpreted as a mathematical object which contains the dynamics of a quantum state and hence it is not connected with the standard formalism and interpretations. Third, the way in which it is proposed to be interpreted is problematic, as it is by means of some probabilities which cannot even been given an operational interpretation.

Finally, in sections 7.5 and 7.6 I introduce some intended generalizations of the covariant quantization formalism. I will argue that the generalizations are problematic for defining a quantum theory, that they have a strong operational character and that they do not solve the conceptual and technical difficulties generated by the diffeomorphism invariance of general relativity. However, I introduce them because they are of extended use in the quantum gravity literature as it will be clear when I analyze concrete covariant approaches to quantum gravity in chapters 8 and 9.

7.1 Path integrals

In this section I will introduce the covariant formulation of quantum mechanics as a reformulation of its canonical version. The covariant formulation reexpresses the unitarily evolving wavefunction in terms of path integrals, which are objects which are more covariant-looking. First, I will sketch how the quantum mechanics of a non-relativistic particle can be expressed in terms of path integrals and then I will discuss its generalization to all kinds of quantum systems. Then, I will introduce covariant quantization as a way to define a quantum theory inspired by the path integral reformulation. I will show that this alternative procedure is equivalent to canonical quantization. Finally, I will discuss the foundational significance that the reformulation of quantum mechanics in terms of path integrals could have. I will argue that the covariant formulation can be a useful technique, but that it remains equivalent to the canonical one and therefore that it has to face the same interpretational issues. In particular, I will argue that the consistent histories approach to quantum mechanics inspired by the path integrals formulation risks collapsing to some interpretation already available in the canonical formalism, namely the Copenhagen and many-worlds interpretations.

7.1.1 Definition of the path integral

Let me start by defining the path integral for a simple quantum system, a non-relativistic particle, and then generalize to any arbitrary system. As I introduced in section 4.6, the quantum state $|\psi\rangle$ of a non-relativistic particle in one dimension is given by its wavefunction, that is, a normalized complex-valued function over space. In other words, $|\psi\rangle$ is an element of $L^2[\mathbb{R}, dx]$, the Hilbert space of square-integrable functions on the real line. In this space, physical quantities like position and momentum are represented by linear operators. The Hamiltonian operator defines the dynamics of the system via the Schrödinger equation[1]:

$$\hat{H}|\psi(t)\rangle = i\hbar \frac{d}{dt}|\psi(t)\rangle \, . \tag{7.1}$$

A way of obtaining solutions of this equation is by applying the time evolution operator $\hat{U}(t, t')$ to any initial state at time t', $|\psi(t')\rangle$. Explicitly, for a time-independent

[1] In chapter 4 I mentioned that there are different equivalent representations of the dynamics. In this chapter it will be more convenient to work in the Schrödinger picture, as it makes the connection between covariant and canonical quantizations more straightforward.

Hamiltonian[2] this operator is given by:

$$\hat{U}(t-t') = e^{-\frac{i(t-t')}{\hbar}\hat{H}}\,. \tag{7.2}$$

For working directly with wavefunctions $\psi(x)$ it is convenient to introduce the propagator, which is defined as:

$$K(x,t;x',t') = \langle x|\hat{U}(t-t')|x'\rangle = \langle x|e^{-\frac{i(t-t')}{\hbar}\hat{H}}|x'\rangle\,. \tag{7.3}$$

In terms of the propagator, the solution of the Schrödinger equation for some initial conditions $\psi(x',t')$ at time t' can be expressed as:

$$\psi(x,t) = \int dx' K(x,t;x',t')\psi(x',t')\,. \tag{7.4}$$

In other words, the propagator is the mathematical object which allows us to know what the wavefunction will be at a time t given that we know its state at a time t'. In this sense, it describes the propagation of the wave according to the Schrödinger equation. Notice that this expression is valid both for $t > t'$ and $t < t'$, that is, we can use it for propagating the wave both forwards and backwards in time.

The propagator contains the dynamics of the quantum system, and this is generalizable to any quantum system: the dynamics of any wavefunction that follows a Schrödinger evolution for some Hamitonian can be expressed in terms of an appropriate propagator. Finding the propagator is thus equivalent to solving the wave equation in full generality.

Moreover, the propagator can be given the following interpretation: $K(x,t;x',t')$ is the probability amplitude density of finding the particle at position x at time t given that at time t' it was at position x'. It is easy to see that this has a direct operational meaning: K directly gives a probability distribution for an experimental set-up in which a particle is prepared at a given position and measured a time afterwards. In the case of QFT, the propagator is related with the S-matrix, which also has this direct operational significance: the S-matrix gives the probabilities of finding some particles with some momenta after some other particles have collided. For this reason, someone with antirealist or operationalist inclinations will be tempted to take the propagator to contain all the empirical content of a quantum theory and disregard the rest of the theory. Of course, the realist position will consider that there is more to quantum theory than just propagating probabilities between the preparation and measurement stages of an experiment.

[2] For a time dependent Hamiltonian the time evolution operator is formally more complicated, consisting of a time-ordered exponential and not a simple one. The generalization to time dependent Hamiltonians can be carried out without affecting the general picture I am giving here.

Let me now introduce the path integral formalism for computing the propagator. This formalism was first introduced by Feynman in Feynman (1948) for the case of the quantum mechanics of a non-relativistic particle. Later on[3] it was generalized to all kinds of quantum systems. Let me now sketch how the propagator 7.3 of the non-relativistic particle can be expressed as a path integral. First, we can divide the evolution operator U between t' and t into N operators corresponding to smaller time intervals $\Delta t = \frac{t-t'}{N}$:

$$\hat{U}(t-t') = e^{-\frac{i(t-t')}{\hbar}\hat{H}} = e^{-\frac{i\Delta t}{\hbar}\hat{H}} e^{-\frac{i\Delta t}{\hbar}\hat{H}} ... e^{-\frac{i\Delta t}{\hbar}\hat{H}} \,. \tag{7.5}$$

Now, for computing this expression we do the following trick. One can always introduce a identity matrix multiplying in an expression without affecting it. The identity matrix can be expressed as $\mathbb{I} = \int dx_i |x_i \times x_i|$, that is, as the sum of all the projectors of the position eigenstates, and we will introduce identity matrices expressed in this form in our expression for the projector to get:

$$K(x,t;x',t') = \int dx_1 dx_2 ... dx_{N-1} \langle x| e^{-\frac{i\Delta t}{\hbar}\hat{H}} |x_{N-1} \times x_{N-1}| e^{-\frac{i\Delta t}{\hbar}\hat{H}} |x_{N-2}\rangle ... \langle x_1 | e^{-\frac{i\Delta t}{\hbar}\hat{H}} |x'\rangle \,. \tag{7.6}$$

Let us take a look at this expression. The integrand is a product of propagators between a position x_i and the next x_{i+1}. In this sense, the set $\{x', x_1, x_2, ... x_{N-1}, x\}$ defines a discrete path and the integrand represents the probability amplitude associated with this path, that is, the probability amplitude for a particle in position x' to travel to x given that at each time step t_i it is at the position x_i. The integral over all the x_i's means that we are summing over all possible discrete paths. This is the main insight of the path integral approach to quantum mechanics, that propagators can be expressed as the sums of the contributions of each possible path. In figure 7.1 we represent some of these paths. The next step in our process is to take the limit of big N, that is, the limit of very short time intervals, which will mean that our trajectories will stop being discrete and we can consider them continuous[4].

For a Hamiltonian of the form $H = \frac{p^2}{2m} + V(x)$ the amplitude for each step can be shown to be in the limit of very small Δt:

$$\langle x_{i+1} | e^{-\frac{i\Delta t}{\hbar}(\frac{\hat{p}^2}{2m} + \hat{V}(x))} |x_i\rangle \approx \langle x_{i+1} | e^{-\frac{i\Delta t}{\hbar}\frac{\hat{p}^2}{2m}} e^{-\frac{i\Delta t}{\hbar}\hat{V}(x)} |x_i\rangle \,. \tag{7.7}$$

[3] In subsequent works like Feynman (1949, 1950) Feynman expanded his ideas to other systems. In his 1965 book (Feynman and Hibbs, 1965) the ideas developed during those years were presented in a more pedagogical manner.

[4] A word of caution has to be taken here, as in the context of QFT it is proved that the majority of paths obtained by taking this limit are not smooth, but instead are spiky and even discontinuous. However, this point will not be relevant for our discussion.

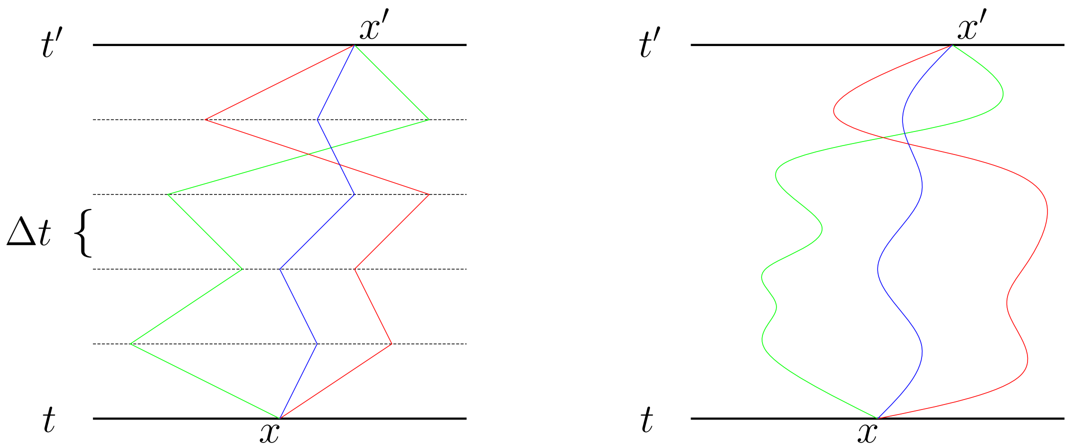

Fig. 7.1: Illustration of how the way we are computing the propagator. On the left hand side I represent different trajectories which take different positions at each moment of time, which are separated by an interval Δt. On the right hand side I have taken the limit $\Delta t \to 0$ and the trajectories are not discrete any more.

That this equality holds in the limit $\Delta t \to 0$ is a consequence of the Trotter product formula[5]. Notice that we need to appeal to this formula to 'split' the exponential because it is an exponential of a sum of non-commuting operators and not an exponential of real numbers. Having made this split we can repeat the trick we have performed before and introduce a resolution of the identity in between the two exponentials. This time we will introduce the identity expressed in terms of the momentum eigenbasis and the amplitude takes the form:

$$\begin{aligned}\langle x_{i+1}|e^{-\frac{i\Delta t}{\hbar}\frac{\hat{p}^2}{2m}}e^{-\frac{i\Delta t}{\hbar}\hat{V}(x)}|x_i\rangle &= \int dp\langle x_{i+1}|e^{-\frac{i\Delta t}{\hbar}\frac{\hat{p}^2}{2m}}|p\times p|e^{-\frac{i\Delta t}{\hbar}\hat{V}(x)}|x_i\rangle = \\ = \int dp\langle x_{i+1}|p\times p|x_i\rangle e^{-\frac{i\Delta t}{\hbar}\frac{p^2}{2m}+V(x_i)} &= \frac{1}{2\pi\hbar}\int dp e^{\frac{i\Delta t}{\hbar}(p\frac{x_{i+1}-x_i}{\Delta t}-\frac{p^2}{2m}-V(x_i))}\,. \end{aligned} \tag{7.8}$$

The next step in our derivation is to perform the Gaussian integral on p. Once we perform the integral we get:

$$\langle x_{i+1}|e^{-\frac{i\Delta t}{\hbar}(\frac{\hat{p}^2}{2m}+\hat{V}(x))}|x_i\rangle \approx \sqrt{\frac{m}{2\pi\hbar\Delta t}}e^{\frac{i\Delta t}{\hbar}(\frac{1}{2}m(\frac{x_{i+1}-x_i}{\Delta t})^2-V(x_i))}\,. \tag{7.9}$$

[5] I refer the reader to (Hall, 2013, Chap. 20) and (Takhtadzhian, 2008, Chap. 5 & 6) for the mathematical details.

Now we can multiply the amplitudes for all steps to get the propagator:

$$K(x,t;x',t') \approx (\frac{m}{2\pi\hbar\Delta t})^{N/2} \int dx_1 dx_2 ... dx_{N-1} \exp\left[\frac{i\Delta t}{\hbar} \sum_{\substack{i\in[0,N-1] \\ x_0=x' \\ x_N=x}} \left(\frac{1}{2}m(\frac{x_{i+1}-x_i}{\Delta t})^2 - V(x_i)\right)\right]. \tag{7.10}$$

This approximate equation becomes exact in the limit $N \to \infty$ ($\Delta t \to 0$). As I have said before, we can interpret this limit as the limit in which the discrete paths become continuous trajectories. In this limit, the quantity $\frac{x_{i+1}-x_i}{\Delta t}$ is nothing but the derivative of position with respect to time, that is, the velocity. The sum over positions becomes an integral and hence what we find in the exponent is nothing but the classical action:

$$\lim_{N\to\infty} \frac{i\Delta t}{\hbar} \sum_{\substack{i\in[0,N-1] \\ x_0=x' \\ x_N=x}} \left(\frac{1}{2}m(\frac{x_{i+1}-x_i}{\Delta t})^2 - V(x_i)\right) = \frac{i}{\hbar}\int_{x(t')=x'}^{x(t)=x} dt \left(\frac{1}{2}m\dot{x}^2 - V(x)\right) = \frac{iS[x]}{\hbar}. \tag{7.11}$$

Finally, the product of integration measures (and the factors of $\frac{m}{2\pi\hbar\Delta t}$) can be seen as defining a measure $\mathcal{D}x$ in the space of paths:

$$\lim_{N\to\infty} (\frac{m}{2\pi\hbar\Delta t})^{N/2} \int dx_1 dx_2 ... dx_{N-1} = \int \mathcal{D}x\,. \tag{7.12}$$

Notice that the factors that enter in the definition of the measure depend on m, and hence on the quantum theory we started with. Putting together the last expressions we arrive to:

$$K(x,t;x',t') = \int_{\substack{x(t')=x' \\ x(t)=x}} \mathcal{D}x e^{\frac{iS[x]}{\hbar}}\,. \tag{7.13}$$

As promised, we have expressed the propagator as a sum of the amplitudes of all the possible paths between the initial and final state. The exact mathematical meaning of the last expression is given by the limit above, that is to say, the integral over the space of paths has to be understood just as a formal artifact and not as an integral with a well-defined measure over the space of paths[6].

This result is generalizable to other quantum systems. For a general system with configuration variables q^a and momenta p^a the same derivation holds if the Hamiltonian operator is of the form $\hat{H}(q^a,p^a) = \hat{K}(p^a) + \hat{V}(q^a)$ where the kinetic energy

[6] Again, I refer the reader to (Hall, 2013, Chap. 20) and (Takhtadzhian, 2008, Chap. 5 & 6) for precise mathematical discussions.

operator $\hat{K}$ is quadratic in the momenta. I will show this in the next section, but for now notice that in the Lagrangian formalism this corresponds to Lagrangians with kinetic terms quadratic in the velocities. These requirements are not too restrictive and they are met by a wide variety of systems which include field theories, where the covariant formalism is most applied.

For a system with configuration variables q^a which satisfies the above stated requirements we can formally express the propagator between an initial and final states as:

$$K(q^a, t; q'^a, t') = \int_{\substack{q^a(t')=q'^a \\ q^a(t)=q^a}} \mathcal{D}q^a e^{\frac{iS[q^a]}{\hbar}} . \tag{7.14}$$

Again, this expression is formal and its exact meaning will be only defined by some appropriate limit. Furthermore, even if at first sight this sum over paths does not seem much of a simplification, in this formalism one can use some powerful perturbative techniques to identify the paths that contribute most to the propagator, and compute the propagator to a good degree of approximation just by taking into account these paths. These techniques are commonly used in quantum field theory, in which 'paths' between an initial set of particles and a final set of particles can be represented by a Feynman diagram[7].

7.1.2 Covariant quantization

I have just shown that under certain conditions we can express the dynamics of quantum theories formulated in terms of the canonical formalism in terms of the covariant formalism. We can go one step further and define the quantum theory directly in terms of the structures defined above, without need to explicitly mentioning a phase space, an operator algebra or a Hamiltonian. This is known as covariant quantization and it is usually presented as a better way of understanding quantum theories. However, I will argue that the deep structure of quantum theory is nevertheless present in covariantly quantized theories, i.e., that even if one tries avoiding making reference to Hilbert spaces, observables or similar structures, these structures are nevertheless there.

Let me follow the same structure I introduced in section 4.4 for the canonical quantization program to sketch the covariant quantization program as:

1. Start with a classical theory defined on a configuration space.

[7] To be sure, perturbation techniques and Feynman diagrams may be used and may arise in the context of Hamiltonian quantum mechanics, but even in this context at the end of the day what one finds is that amplitudes can be expressed as sums of the amplitudes of a series of processes or histories.

2. Define a measure on configuration space. This allows for computing probabilities and implicitly defines the Hilbert space structure of the theory.

3. Define the propagator as a path integral. For this, specify a measure over trajectories in configuration space. This defines the dynamics of the theory.

In the same way as the canonical quantization program had a step defining the kinematic structure of the theory and another one defining the dynamic one, covariant quantization follows similar steps, which I comment next.

Let me start by commenting the last step. As it should be clear from the discussion in the previous subsection, the dynamical content of a quantum theory can be represented by a propagator, and under certain conditions it can be expressed in the form of a path integral of the exponential of the action. We can directly define the propagator as a path integral by specifying a measure $\mathcal{D}q^a$ in the space of trajectories in configuration space and a classical action $S[q^a]$. This defines a dynamics for the quantum system just in the same way that a Hamiltonian operator and a Schrödinger or Heisenberg equation do[8].

The kinematic step of the quantization usually receives less attention, but it is an important step, as it implicitly defines the Hilbert space structure of the theory. A measure in configuration space allows one to connect a wavefunction with a probability density. In the case of the non-relativistic particle the measure was just the Lebesgue measure in $\mathbb{R}$ and for instance it defined the probability of finding the particle in a given interval, given a wavefunction as:

$$P[x \in (x_1, x_2)] = \int_{x_1}^{x_2} dx |\psi(x)|^2 \tag{7.15}$$

Similarly, for a generic theory defined on any configuration space the measure $d\mu(q^a)$ connects wavefunctions, i.e., complex-valued functions on configuration space, with a probability:

$$P[q^a \in \mathcal{R}] = \int_{\mathcal{R}} d\mu(q^a) |\psi(q^a)|^2 \tag{7.16}$$

This definition not only allows to connect the wavefunctions with probabilities, but it also defines an inner product by means of:

$$\langle \psi_1 | \psi_2 \rangle = \int d\mu(q^a) \psi_1^*(q^a) \psi_2(q^a) \,. \tag{7.17}$$

[8] The Schrödinger equation can be recovered by considering the propagator for an infinitesimal time elapsed between the initial and final state. A different way of putting it is to say that the propagator gives the evolution for finite intervals of time and the Hamiltonian and the Schrödinger equation gives its differential form.

The measure and inner product allows one to define the Hilbert space of the theory as the space of square integrable functions on the configuration space under the measure $d\mu$. This is many times not made explicit in presentations of covariant quantization as there are some choices of measure, like the Lebesgue measure, which are just assumed. However, the previous two chapters should have made it clear that there are different choices of measures available and that this choice can have important consequences for the theory. For instance, in chapter 5 I showed that the choice of the Ashtekar-Lewandoski measure lead to a theory with a strongly discrete character. For the case of field theories, the choice of measure becomes more complicated.

Let me also mention that once we have a measure and a Hilbert space we can also define all the kinematical structures we had for a canonical theory, i.e., we can define operators, commutation relations and so on. In this sense, at a kinematical level canonical quantization and covariant quantization are perfectly analogous. Finally, the definition of the measure is also indispensable for the dynamics, as the dynamics defined by the propagator will be unitary or not depending on the measure.

Let me comment on the equivalence between the canonical and the covariant quantization schemas. Even if the dynamics of a covariantly quantized theory is formulated in terms of a propagator one can find out the Hamiltonian operator which would give rise to such a dynamics. Now one can wonder about the relation between this Hamiltonian and one obtained by means of the canonical quantization procedure. It is no surprise that if one chooses measures which are, in a sense, inspired by the one we have derived before in 7.12 for actions quadratic in velocities one recovers the canonical Hamiltonian for such theories. That is, for a wide span of theories, which include most of the relevant field theories, covariant quantization is fully equivalent to canonical quantization given that one chooses some 'natural' integration measure. A note to make is that, as analyzed in chapter 4, the canonical quantization procedure does not uniquely determine a Hamiltonian operator, as there could be operator ordering ambiguities. What we find when we express the propagator in terms of path integrals is that each inequivalent $\hat{H}$ leads to a different action, of which at most one will be the classical Lagrangian action we started with. Therefore, in the covariant formalism the order ambiguity is also present but it is encoded in the choice of action. I will expand a bit on this point in the next section.

For the case of theories which are not quadratic in velocities one may also try to define a covariant quantization using a path integral for a relevant action. However, care should be taken in such a definition, as I have argued before that we are not assured that the canonical quantizations of such theories can be expressed in terms of a Lagrangian path integral. In this sense, if we define a quantum theory following the covariant quantization schema, it is possible that we define a consistent quantum theory, but its connection with the canonically quantized theory may be broken.

Moreover, there may be some issues with the classical limit of such a theory, as I will comment in the next subsection.

To sum up, covariant quantization allows one to define quantum theories with their usual structure in a slightly different way. For most of our theories and for reasonable measures this quantization process gives us quantum theories that are perfectly equivalent to the ones obtained by canonical quantization. For other theories or other measures one may obtain different theories, which could nevertheless be given in terms of the usual structures of quantum theory. In the following, I discuss some foundational issues suggested by path integrals and covariant quantization.

7.1.3 Foundational significance of the path integral

From a foundational perspective, the form of the path integral may suggest new interpretations of the quantum formalism. In particular, it is tempting to interpret the different histories in the integrals in a realistic way, that is, as real physical processes occurring in space and time. However, this interpretation rapidly becomes problematic for several reasons. Not only because in quantum mechanics to speak about well-defined trajectories is tricky (the uncertainty principle seems to forbid it) but also because in the path integral one has a plurality of histories and not just a single one. For these reasons, the paths in the path integrals and the Feynman diagrams are usually regarded as computational tools and not interpreted literally[9]. This point will be relevant in the case of quantum gravity, where the temptation of interpreting paths as real processes will reappear.

More generally, as the path integral formulation of quantum mechanics is equivalent to one in which one has all the typical structure of the theory, that is, the wavefunction, the Hilbert space and so on, any interpretation inspired by this formulation would have to answer to the same foundational problems and paradoxes that any other approach to quantum mechanics faces. Similarly, any insight on quantum mechanics that the covariant formulation may provide can be translated back to the canonical one.

One approach to quantum mechanics that has been inspired, at least for some of its proponents[10], by the path integral formalism is the consistent or decoherent histories approach. Different authors have worked on the approach and have given different emphasis to their formulations and here I will just give some general notions and comments. The key idea is to take histories as the basic ingredient of the formalism. In this approach one can define some coarse-grained histories for which probabilities can be defined consistently. The exact details on how these coarse-grained histories

[9] For detailed discussions about the role of the Feynman diagrams I refer the reader to Meynell (2008); Brown (2018); Passon (2019) and to Kaiser (1997) for a more historical account.

[10] This is explicit in Hartle (1993).

and probabilities are defined depend on the specific proposal, but let me use the case of the non-relativistic particle as an example. A fine-grained history for this system would be a complete specification of the trajectory of the particle, that is, $x(t)$. However, in quantum mechanics this is not well-defined and no probability can be assigned to it. One can argue that what we usually observe is a coarse-grained version of the trajectory: we observe the particle at a series of points at a series of instants, moreover our observations carry some uncertainty. In this sense one can define the coarse-grained history of the system as the set of all trajectories which are compatible with our observations. In the decoherent histories approach it is claimed that for some of these coarse-grained histories, quantum mechanics allows for a consistent assignation of probabilities.

Now the relevant question is: which histories are the ones for which one can give probabilities? That is, how can one explain that there is some preferred coarse-graining that selects the possible histories? There are several possible lines of answer to this question. First, one can say that the situations in which there is such selection are the ones in which there is a measurement happening. In our example, if we measure the position of the particle at times t_1, t_2 and so on we will get a coarse grained history $\{x_1, x_2, ...\}$. For such a situation we can certainly use the standard formalism of quantum mechanics and get a probability for all possible histories. Nevertheless, if we do not say any more about what we mean with measurement we would have just a reformulation of Copenhagen quantum mechanics.

Indeed, one of the motivations of the approach is to be applicable to closed quantum systems, like the universe, so that no special role can be played by any external observer or device. Therefore, proponents of consistent histories usually appeal to decoherence for explaining how probabilities can be assigned to histories. Nevertheless, it is well-known that decoherence alone does not solve the measurement problem[11], and it is even recognized by some of the proponents of consistent histories [12]. However, other defendants of consistent histories, like Hartle and Gell-Mann in Gell-Mann and Hartle (1989), insist that decoherence explains the emergence of quasiclassical domains[13]. It is unclear in which way this is achieved, and by doing so their approach may be interpreted in an Everettian way. Consider for instance that in the same paper the authors claim that it is more convenient to think about their approach as a 'many-histories' approach than about a 'many-worlds' approach. Despite this, it remains unclear whether Gell-Mann and Hartle's approach is a many-world or single-world one[14]. In this sense, it seems that the consistent histories approach tries to be a middle-ground between the Copenhagen and Everett interpretations, with different

[11] An argument for this can be found in Adler (2003).
[12] See Griffiths (2014).
[13] A more recent version of their approach is explained in Hartle (2011).
[14] The same conclusion is argued for in Kent (2010).

authors being closer to one or the other. Indeed, it is likely that once one starts putting pressure to this approach it would collapse to one of the two.

Related with the histories approach we find a couple of more radical interpretations. We have seen that we can assign probabilities to some sets of coarse-grained histories. For finer-grained histories one can assign some quantities that, even if they do not behave like probabilities, they do so when considering coarse grainings. Then, it has been suggested[15] to consider these quantities as a generalization of probabilities. Similarly, it has been suggested[16] to replace classical logic by a new quantum logic. These ideas, interesting as they could be, remain just as sketches of interpretations as long as these new, revolutionary redefinitions remain unclear. That is, as long as we cannot give a sensible interpretation of what these generalized probabilities or quantum logic are, it is obscure to try to interpret quantum mechanics appealing to them.

A last point to be made about the path integral formulation of quantum mechanics is that it motivates some claims about the classical limit of the theory. The path integral is an integral of an exponential of imaginary argument, which is a function which oscillates. These integrals get the most important contribution from the regions in which there are less oscillations, i.e. from the regions in which the argument of the exponential changes less, that is, from its stationary points. The argument of the exponential is the classical action of the theory and its stationary points give us trajectories which satisfy the classical equations of motion. Moreover, the oscillatory behaviour of $\exp \frac{iS[q]}{\hbar}$ depends on the value of $\hbar$: the smaller the value of $\hbar$ the more rapid the oscillations of the exponential are. In the limit of $\hbar \to 0$ the only relevant contributions are the ones which correspond to the stationary points. Therefore, there is a sense in which by taking the limit $\hbar \to 0$ one is taking a sort of classical limit and recovering the classical theory. But nevertheless this is just a formal relation, and for relating a quantum theory with a classical description one needs to give an interpretation of the quantum formalism and tell a story about how this relates with the macroscopic, classical world.

In this sense, if one really wants to address the way the classical world emerges from a quantum one as I discussed in section 2.4 it is not enough to study the $\hbar \to 0$ limit of the path integral. One needs to appeal to an interpretation of the formalism and some strategy like the one outlined in section 4.4, that is, a combination of Ehrenfest theorem with decoherence or with physical collapse of the quantum state.

With this I conclude this introduction to the covariant formalism of quantum mechanics. To reemphasize, the formalism is equivalent to the standard one which uses operators and states on a Hilbert space and the Schrödinger equation[17] and

[15] See Sorkin (1994, 1997); Mozota Frauca and Sorkin (2017).

[16] See Isham (1994).

[17] Of course it is also equivalent to any other formalism equivalent to Schrödinger's, like Heisen-

from an interpretational perspective the same questions and difficulties that were present in the canonical formalism are still present in the covariant one. The term covariant makes reference to the fact that this formalism is more covariant-looking: one deals with trajectories in spacetime and with the Lagrangian formalism rather than with states at a time and the Hamiltonian one. But in essence, this formalism is as non-covariant as the canonical one, we still have a quantum state which evolves in time or a propagator between two states at different times. I have argued that the interpretations that are inspired by this formalism are either ambiguous and unclear or likely to collapse to some of the well-established interpretations. This also applies to the relation of the quantum theory with the classical one: at the end of the day we need to give an interpretation of an equivalent formalism, and the well-established ways of making this connection for the canonical formalism work as well in the covariant one.

After this first section in which I have introduced covariant quantization, in the rest of the chapter I turn to the way covariant quantization and its generalizations are applied to quantum gravity. In the next section I will start by introducing the way the formalism can be generalized to study systems with constrained and gauge theories, which as I argued in chapter 4, are very important to study the quantization of gravity, as general relativity is a constrained system.

7.2 Hamiltonian path integrals and gauge theories

In this section I will complement the previous one with the discussion of Hamiltonian path integrals and of gauge theories in the covariant formalism. The discussion in this section will allow me to better connect the discussion in chapter 4 with the covariant formalism. Recall that in that chapter I compared general relativity and reparametrization invariant theories with gauge theories, as both are represented as constrained systems in the Hamiltonian formalism. The discussion in this section will allow me to discuss an example of a covariant quantization of a reparametrization invariant model in the following one.

Let me start by introducing the phase space path integral. The path integrals used in the covariant quantization are integrals of paths in the configuration space. For this reason, it is common to associate the Lagrangian formalism, and not the Hamiltonian one, with paths integrals. Nevertheless, I will show that propagators can also be expressed in terms of phase space path integrals. Indeed, they arise in a more natural way from the canonically quantized theories.

Recall the general process I followed in the last section for deriving the configuration space path integral. For a generic system, with configuration variables q^a and

berg's.

Hamiltonian $\hat{H}(q^a, p^a)$, we can follow the same process and find an expression like 7.6 but for the states $|q^a\rangle$ and for the Hamiltonian $\hat{H}(q^a, p^a)$. Now, we need to evaluate expressions like:

$$\langle q_i^a | e^{-\frac{i\Delta t}{\hbar}\hat{H}} | q_{i-1}^a \rangle \tag{7.18}$$

These expressions can be computed by introducing a resolution of the identity as we did in equation 7.8:

$$\begin{aligned}\langle q_i^a | e^{-\frac{i\Delta t}{\hbar}\hat{H}} | q_{i-1}^a \rangle &= \int dp^a \langle q_i^a | p^a \times p^a | e^{-\frac{i\Delta t}{\hbar}\hat{H}} | q_{i-1}^a \rangle \\ \approx \int dp^a \langle q_i^a | p^a \times p^a | \mathbb{I} - \frac{i\Delta t}{\hbar}\hat{H} | q_{i-1}^a \rangle &= \int dp^a \langle q_i^a | p^a \times p^a | q_{i-1}^a \rangle (1 - \frac{i\Delta t}{\hbar} h(q_{i-1}^a, p^a)) \\ &\approx \frac{1}{2\pi\hbar}\int dp^a \exp \frac{i\Delta t}{\hbar}(p^a \frac{q_i^a - q_{i-1}^a}{\Delta t} - h(q_{i-1}^a, p^a))\,.\end{aligned} \tag{7.19}$$

The $\approx$ means that these expressions will be valid just in the limit $\Delta t \to 0$. I have introduced the function $h(q_{i-1}^a, p^a) = \frac{\langle p^a | \hat{H} | q_{i-1}^a \rangle}{\langle p^a | q_{i-1}^a \rangle}$ which looks like a Hamiltonian. For instance, in the case of the non-relativistic particle this function was $\frac{p^2}{2m} + V(x_{n-1})$, as can be seen in 7.8. But nevertheless, notice that it is not exactly the Hamiltonian, as it is a function that depends on the operator ordering. For instance, consider a classical Hamiltonian $H = pq$ and the operators $\hat{H}_1 = \hat{q}\hat{p}$ and $\hat{H}_2 = \hat{p}\hat{q}$. When computing the function h for these operators we find $h_1(q,p) = pq + i\hbar$ and $h_2(q,p) = pq$. This is the reason why different operator orderings lead to different actions. Note also that I have chosen to introduce the identity before the evolution operator, but I could have perfectly chosen to introduce it after it. In this case the result would have been equivalent, but with a different 'Hamiltonian'[18].

At this point, in the derivation of the Lagrangian path integral, I performed the integral in momentum, but it was not necessary. We can leave the integral and multiply all the amplitudes to obtain an expression for the propagator in terms of a series of integrals, now both in 'positions' q^a and momenta p^a. As before, we can represent this in the limit of $\Delta t \to 0$ with a path integral:

$$K(q^a, t; q'^a, t') = \int \mathcal{D}q^a \mathcal{D}p^a e^{\frac{i}{\hbar}\int_{q^a(t')=q'^a}^{q^a(t)=q^a} dt (p^a \dot{q}^a - h(q^a, p^a))}\,. \tag{7.20}$$

This expression is analogous to 7.14 but in a Hamiltonian context. The action in this path integral is not a Lagrangian action $S[q] = \int dq L(q, \dot{q}, t)$ but a Hamiltonian

[18] Indeed with the one defined by: $h(q_i^a, p^a) = \frac{\langle q_i^a | \hat{H} | p^a \rangle}{\langle q_i^a | p^a \rangle}$. In this case the value of q^a involved is the one on the left q_i^a instead of the one on the right q_{i-1}^a.

one $S[q,p] = \int dt(p\dot{q} - H(q,p,t))$. The integration measure is also different from the one in the Lagrangian path integral. As I commented in the previous section, the measure in the Lagrangian path integral has some dependence on the theory. This showed up in our example as there were some factors that contained the mass m in the definition of the measure 7.12. In the Hamiltonian path integral the measure is completely independent of the theory, as the limit defining the measure only involves the factors of $2\pi\hbar$[19]:

$$\lim_{N\to\infty} (\frac{1}{2\pi\hbar})^N \int dx_1 dp_1 dx_2 dp_2 ... dx_{N-1} dp_{N-1} = \int \mathcal{D}x\mathcal{D}p\,. \tag{7.21}$$

The paths we are integrating over are paths in phase space and not in configuration space. Notice that q^a and p^a are varied independently, and therefore the momenta p^a are independent of the velocities q^a. This is a problem if one tries to interpret paths in phase space as real processes: as the physical meaning of momenta in classical mechanics is tied to velocities, a new meaning for these variables would be needed if they were to be considered physical. As I argued in the last section, trying to interpret paths in a path integral as representing real physical processes is problematic. In the case of phase space path integrals it is even more problematic.

The conceptual advantage of phase space path integrals is that they can be readily formulated for all quantum systems. The function $h(q,p)$ in the action will correspond to the classical Hamiltonian, or related to it by an ordering ambiguity relation. Now, from the phase space path integral we can think of going to the configuration space path integral by performing the integration in the momenta. This is conceptually possible, but for a generic theory this integral is not necessarily easily solvable and the result may not have anything to do with the classical Lagrangian. The one case for which we surely know how to perform the integral is the case of quadratic Hamiltonians, which lead to a Gaussian integration in momenta[20]. In this case, as mentioned in the previous section, the result in the integration in momenta is the Lagrangian path integral with the classical action.

As I said in the previous section, the majority of theories of physical interest are quadratic in momenta and do not present any problem for performing the integrals in momenta. However, models like the relativistic particle I will study in the next section are not quadratic in momenta, and we will have to be careful with such models. The other piece we need to add to the picture in order to be able to represent the quantum version of that theory in the way the reparametrization invariance is going to be treated. For this, let me discuss first the way gauge symmetries are represented in the covariant formalism.

[19] Recall that these factors come just from the factor in the product $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$

[20] This is an integral of the form: $\int_{-\infty}^{\infty} e^{-ax^2+bx} dx$. This is precisely the form we found for the non-relativistic case in eqn. 7.8.

The formalism that is most extended in the quantum field theory community to deal with gauge systems in the path integral formulation of quantum mechanics is the Faddeev-Popov formalism. Here I will just explain the basic idea of the formalism, and I will leave aside the technical details and other related approaches to the quantization of gauge systems such as the BRST formalism[21]. This formalism delivers a quantization that is equivalent to the canonical quantization of a gauge theory that I introduced in section 4.5. In the Faddeev-Popov formalism the propagator is defined by means of a path integral which is defined in the full configuration space of the gauge theory, that is, in the configuration space which contains the gauge redundant degrees of freedom. If we defined naively the path integral in this space we would be overcounting, as we would be summing over physically equivalent trajectories. To avoid this, what one does is to introduce a gauge fixing condition that makes it the case that we are considering every physically distinct trajectory just once. The path integral then takes the form:

$$\int \mathcal{D}q\,[|\Delta_f|\delta(f_i)]\,e^{\frac{iS[q]}{\hbar}}\,. \tag{7.22}$$

Here, and for the rest of the chapter by the brackets [...] I will mean that the factors in between the brackets are repeated for every time step in the construction of the path integral. In this particular case, this means that there is one delta function for each time slice. The effect of the delta functions is to impose the gauge conditions $f_i[q] = 0$, i.e. only the histories that satisfy $f_i[q] = 0$ will contribute to the integral. Of course, not every set of f_i will be a good set of gauge fixing conditions and it has to be chosen carefully so that just one element of each gauge orbit satisfies the conditions. The factor $|\Delta_f|$ is called the Faddeev-Popov determinant, and its role is to make the integral independent of the choice of gauge fixing conditions, as it should be. That is, if we replace $|\Delta_f|\delta(f)$ by $|\Delta'_f|\delta(f')$ for a different gauge fixing condition the result of the integral does not change. Its role is therefore analogous to the one of a Jacobian matrix: in the same way that the Jacobian matrix describes how the integration measure changes under a change of coordinates, the $|\Delta_f|$ contains the information about the way f changes under a gauge transformation. The explicit form of $|\Delta_f|$ will not be relevant for us, but it is essentially built from 'derivatives' of f with respect to the gauge transformation.

This is the standard textbook presentation of the Faddeev-Popov determinant for the covariant quantization of a field theory. But let me point out that this expression was derived in Faddeev (1969) by making use of the phase space path integrals and

[21] I refer the reader to Faddeev (2009) for an introduction to Faddev-Popov formalism and to Feynman (1963); DeWitt (1967b); Faddeev and Popov (1967); Faddeev (1969) for the historical papers in which the techniques were developed. For a review of the BRST formalism, see van Holten (2004).

then by integrating out the momenta. In phase space the form of the path integral is even more intuitive and the connection with the constrained formalism is explicit:

$$\int \mathcal{D}q\mathcal{D}p \, [|\det\{\phi_\alpha, \chi_\beta\}|\delta(\phi_\alpha)\delta(\chi_\beta)] \, e^{\frac{iS[q,p]}{\hbar}} \,. \tag{7.23}$$

Now we have two sets of delta functions: the delta functions $\delta(\phi_\alpha)$ impose the constraints of the Hamiltonian theory and the delta functions $\delta(\chi_\beta)$ are the gauge fixing conditions, that is, they play the same role as $\delta(f)$ before. The factor $|\det\{\phi_\alpha, \chi_\beta\}|$ is the Faddeev-Popov determinant for the phase space: it is a factor that makes sure that the integral is independent of the choice of χ_β, as was also proved in Faddeev (1969). A necessary condition for the χ_β to be good gauge fixing conditions is precisely that the determinant of their Poisson brackets with the constraints does not vanish[22], which make it well-behaved for good gauge fixing conditions. When the integration in momenta can be computed[23], we recover the expression 7.22.

The action in 7.23 is the Hamiltonian action for the canonical Hamiltonian. We can also perform a little trick so that action in the path integral is the total action and we get something closer to the constrained formalism for gauge systems I introduced in section 4.1. We will use the following representation of the delta function:

$$\delta(\phi_\alpha) = \frac{1}{2\pi} \int d\lambda \epsilon e^{\epsilon\lambda\phi_\alpha} \,. \tag{7.24}$$

Where ϵ is an arbitrary, non-zero real number that we will adjust so that our final expression has the form:

$$\int \mathcal{D}q\mathcal{D}p\mathcal{D}\lambda^\alpha \, [|\det\{\phi_\alpha, \chi_\beta\}|\delta(\chi_\beta)] \, e^{\frac{i}{\hbar}\int dt(p\dot{q} - H(q,p) - \lambda^\alpha\phi_\alpha)} \,. \tag{7.25}$$

This expression is a path integral in an extended phase space which also includes the Lagrange multipliers λ^α. As I have commented before, the paths that appear in phase space path integrals are difficult to interpreted as representing real processes, and in this case even less, as we have extended our phase space to include some multipliers, which in principle do not have any physical meaning neither in the classical theory nor in the quantum one. There will be some theories for which the integration in momenta can be performed, leading to a path integral in a configuration space extended by the multipliers. We will see an example of this form of path integral in the next section.

For our discussion this level of detail will be enough. For completeness let me mention that the expression 7.22 can be written in the form of a path integral with no

[22] This is because the introduction of the gauge-fixing constraints has to be such that the resulting system is a second-class system, as discussed in section 4.1.

[23] Again, the majority of systems of physical interest are quadratic and there is no problem at this step.

prefactors, that is, with no explicit delta functions. For this, one introduces additional fields, known as ghost fields. Of course, these fields are used only for computational reasons and are understood as convenient tools, and hence the name. This is in the line of what I have been arguing in this chapter: path integrals can be suggestive, but one has to be careful when inferring any physical or metaphysical content from them. This lesson will also be important when we deal with the covariant quantization of systems like the relativistic particle or general relativity.

There is an alternative way of dealing with gauge systems that consists in working directly in the reduced phase space of the theory, that is, to consider just the gauge invariant variables of the theory. If such variables are known and mathematically tractable, one can define the propagator of the theory in terms of a path integral in such a reduced phase, and maybe even in a reduced configuration space. Of course, for the case of a gauge field theory, like Yang-Mills theory, working in the reduced phase space is mathematically involved, and that is why using the Faddeev-Popov or related techniques is preferred. In either way, the quantum theory formulated via covariant quantization is equivalent to the one constructed via canonical quantization.

Let me emphasize that any of the covariant quantizations introduced in this section are equivalent and that all share an important feature with the canonical one. In the canonical quantization of a gauge system we imposed the constraints of the theory and this implied the gauge invariance of the physical Hilbert space and of the dynamics. Similarly, the covariant formulation of the dynamics is equivalent and there is no gauge symmetry left, as even if the expressions may be defined on spaces with some gauge symmetries, at the end of the day all gauge degrees of freedom are fixed. This point is important to make, as in section 4.6 I argued that it is precisely the fact that imposing the constraints implies gauge invariance which caused the problem of time for reparametrization invariant theories.

For analyzing the case of reparametrization invariant theories and general relativity it will be useful to sketch the covariant quantization process of a gauge theory as:

1. Start with a gauge theory defined on a configuration space. Identify the gauge degrees of freedom and the physical degrees of freedom.

2. Define a measure on the gauge-reduced or on the gauge-fixed configuration space. This allows for computing probabilities and implicitly defines the (physical) Hilbert space structure of the theory.

3. Define the propagator as a path integral. There are different options available depending on the variables one chooses to use. For gauge-reduced variables the definition is as for a non-gauge theory. For gauge-dependent or extended variables the definition of the path integral has to include the corresponding

delta functions and Faddeev-Popov factors to fix a gauge and ensure that the result is independent of the gauge chosen. This defines the dynamics of the theory.

In the next section we will see that for deparametrizable models one can follow a schema similar to this to successfully quantize the theory. Then, in section 7.4 I will argue that this is not the case for theories like general relativity, in consistency of what I argued in chapter 4. That is, in the same way that we could solve the problem of time for deparametrizable models by treating them in a similar way to gauge systems in the canonical formalism and this did not work for general relativity, we will find a similar situation for the covariant quantization of reparametrization invariant theories.

7.3 Covariant quantization of deparametrizable models: the relativistic particle revisited

In section 4.6 I argued that theories with reparametrization invariance suffer from a problem of time and that for a class of models, the deparametrizable models, this problem could be solved. In this section I describe how the covariant quantization model can be applied to this kind of model by considering the covariant quantization of one of the examples I discussed at length in chapter 4. I will argue that the covariant quantization of deparametrizable models can be treated in a very similar way to the one for gauge theories I have discussed in the previous section. For other reparametrization invariant theories, I argued that the problem of time was not solvable and I will discuss their possible covariant quantization in the next section. I will also discuss some interpretational issues which arise when considering models like the relativistic particle. These issues arise from interpreting some of the paths considered in the path integral realistically or when trying to give some physical meaning to intermediate objects which appear in the computation of the propagator, like what is known as the 'proper time propagator'. I will argue that one should reject interpreting paths in the path integral or intermediate steps as physical objects, conclusion that we will extrapolate for covariant approaches to the quantization of gravity

In section 4.6 I showed that the imposition of the quantum counterpart of the Hamiltonian constraint for a relativistic particle is the Klein-Gordon equation:

$$\Box\psi + m^2\psi = 0\,. \tag{7.26}$$

In this equation and in this section I am using the convention of using units such that $\hbar = 1$. This will simplify our expressions and make it easier to compare with the expressions in Hartle and Kuchař (1986), which will be important in this section.

As I argued in chapter 4, not every solution of the Klein-Gordon equation can be directly interpreted as a wavefunction, as there is not a conserved, positive-definite inner product. For being able to interpret ψ as a wavefunction, we will restrict ψ to be a positive frequency field[24]. For such fields, the Klein-Gordon equation is equivalent to a Schrödinger equation with the Klein-Gordon Hamiltonian:

$$\hat{H} = \sqrt{\hat{p}^2 + m^2} \,. \tag{7.27}$$

As I commented in section 4.6, the quantum theory defined by this Hamiltonian has some aspects which are problematic. For instance, one can show that evolution is non-local. Nevertheless, the quantum theory is a consistent quantum theory and it will be illustrative to consider how the reparametrization invariance of the theory can be represented in the covariant formalism. The propagator for this Hamiltonian is the Newton-Wigner propagator, which in three dimensions takes the form:

$$K(x,t;x',t') = \int \frac{dp^3}{(2\pi)^3} \exp i(p(x-x') - \sqrt{p^2+m^2}(t-t')) \,. \tag{7.28}$$

Now I will show how this expression can be cast in the form of path integrals by making use of the techniques developed in the last section. This will be necessary and illustrative for two reasons: first, the Klein-Gordon Hamiltonian is not quadratic in momenta, and second, the action of the relativistic particle was reparametrization invariant, which can be treated very similarly to gauge invariance. We will see that the ideas described in the last section for gauge symmetry apply also in a similar way to reparametrization invariance. That is, we will be able to express the propagator in terms of path integrals over parametrization-dependent trajectories, just as path integrals for gauge systems were defined on the full configuration or phase space with the gauge redundancy. For casting the propagator in its path integral forms I will follow Hartle and Kuchař (1986). The first path integral we can construct is the phase space path integral, and we are able to build it as the quantum theory is already expressed in terms of the physical variables x evolving in physical time t:

$$K(x,t;x',t') = \int \mathcal{D}^3x \mathcal{D}^3p e^{i\int dt (p\dot{x} - \sqrt{p^2+m^2})} \,. \tag{7.29}$$

The action in this expression is the Hamiltonian action for the relativistic particle for the gauge $\tau = t$[25]. As shown in Hartle and Kuchař (1986), the integration in p cannot be computed and hence this expression cannot readily be expressed as a

[24] Similarly, we could have chosen to restrict the ψ to the negative frequency sector.

[25] This can be obtained from action 4.14 by setting $\tau = t$ and by including the mass factor m in front of the action that was absent in that section for simplicity reasons.

Lagrangian path integral. Now, we can transform this expression into one in which paths are taken in an extended phase space by using some techniques very similar to the ones of Faddeev and Popov. First, at each instant of time we can introduce two extra integrals of delta functions, one in time and other one in the 0th component of momentum. It allows us to express the path integral in terms of new variables τ and p^0 without changing the value of the integral. Explicitly, I will introduce

$$1 = \int dx^0 \delta(\tau - f(x^0))|\frac{\partial f}{\partial x^0}| = \int dx^0 \delta(F)|\frac{\partial F}{\partial x^0}|\,. \tag{7.30}$$

This introduces a reparametrization of time. The relation between the original time x^0 or t and the new one is encoded in $F = \tau - f(x^0)$, where f is required to be a monotonic function. The factor $|\frac{\partial F}{\partial x^0}|$ appears in order to have a reparametrization invariant integral, that is, to be independent of the choice of f. In order to have a momentum p^0 conjugate to time we introduce:

$$1 = m^{-1} \int dp^0 \delta H \theta(p^0)|p^0|\,. \tag{7.31}$$

Where $H = \frac{1}{2m}(p^\mu p^\nu \eta_{\mu\nu} + m^2)$ is a form of the Hamiltonian constraint 4.19 that we found in chapter 4[26]. The theta function $\theta(p^0)$ is there to select just the positive value of p^0 and the rest of factors are there for normalization reasons. We can write these two integrals together:

$$1 = \int dx^0 dp^0 |\{F, H\}| \theta(p^0)\delta(F)\delta(H)\,, \tag{7.32}$$

where the factor $\{F, H\} = \frac{p^0}{m}\frac{\partial F}{\partial x^0}$ includes all the factors that appeared in the previous integrals. Notice that 7.32 is precisely the Faddeev-Popov determinant in phase space for the constraint H and the gauge fixing condition F. Now we can introduce this at every time step of the path integral and the propagator takes the form:

$$K(x, t; x', t') = \int \mathcal{D}^4 x \mathcal{D}^4 p \left[|\{F, H\}| \theta(p^0)\delta(F)\delta(H) \right] e^{i\int d\tau p_\mu \dot{x}^\mu}\,, \tag{7.33}$$

where the dot $\dot{x}^\mu$ represents a derivative with respect to the parameter time τ. This shows that the Faddeev-Popov techniques can also be applied to the reparametrization invariant case. As for the case of a gauge theory, we have obtained a Hamiltonian action for the canonical Hamiltonian, which is in this case vanishes, as I showed in equation 4.20.

[26] Again, the difference in between both expressions lies in the appearance of the factors of m, which were omitted in chapter 4 for simplicity

Now we can perform the same trick as in the last section and express the delta function of the constraints as an exponential[27]. This allows to write the path integral as an integral of paths in an even more extended phase space and the exponential of the canonical action becomes the exponential of the total action (that is, the one in which the Hamiltonian is the one I found in expression 4.21.):

$$K(x,t;x',t') = \int \mathcal{D}^4x\mathcal{D}^4p\mathcal{D}N\,[|\{F,H\}|\theta(N)\delta(F)]\,e^{i\int d\tau(p_\mu \dot{x}^\mu - NH)}\,. \tag{7.34}$$

This path integral now is over paths that also allow for the variable N to change and we can interpret N as a Lagrange multiplier. The integration in momenta, unlike in expression 7.32[28], can be performed and it leads to the following Lagrangian expression:

$$K(x,t;x',t') = \int \mathcal{D}^4x\mathcal{D}N\left[\theta(N)\delta(F)|\frac{\partial F}{\partial x^0}|\right]e^{\frac{im}{2}\int d\tau(N^{-1}\dot{x}^\mu\dot{x}_\mu - N)}\,. \tag{7.35}$$

This is a path integral in an extended configuration space that includes time and the variable N. The action now corresponds to the Lagrangian 4.24, which included a redundant lapse function. Notice that in this expression, and in the rest of the section as well, I have been sloppy about the prefactors appearing in the integrals, absorbing them in the integration measures. The exact form of the prefactors is by no means trivial but will not be very important for us. Notice that this illustrates the fact that, in general, in covariant quantization the choice of the integration measure is not straightforward if one wants to keep equivalence with the canonical quantization. In other words, if we were trying to construct the theory by covariant quantization we would have to be really careful or really lucky to have chosen such a measure.

A similar remark can be made about the domains of integration: the different θ functions that restrict the path integrals to just positive p^0 or N did not appear in the standard path integral. Moreover, notice that not every form of the action is allowed, as we have not found any form of the propagator containing the basic action $S = m\int d\tau\sqrt{\dot{t}^2 - \dot{x}^2}$ (this is equation 4.14). All of this supports the conclusion of Hartle and Kuchar (Hartle and Kuchař, 1986) that there may exist a systematic prescription for defining covariant quantizations of reparametrization invariant theories, but it is not trivial and it implies restrictions in the form of the action, the integration measure

[27] Indeed, the representation of the delta function as an exponential is not unique, and different representation lead to different but equivalent path integrals. In Hartle and Kuchař (1986) two alternatives are presented, of which I only show one which is the one that allows for performing the integration in momenta and expressing the propagator as a cofiguration space path integral.

[28] This is also shown in Hartle and Kuchař (1986).

and the terms that ensure the invariance. The fact that we want the covariantly quantized theory to be equivalent to the canonically quantized one contributes to this restriction.

Let me now turn to the interpretation that is sometimes done of path integrals like the ones I have just found. As I argued in the previous section, one has to be careful in giving a physical interpretation to the momenta in this integrals and the multiplier N. The latter is specially relevant because in the classical theory with action $\int d\tau(N^{-1}\dot{x}^\mu\dot{x}_\mu - N)$ we said that, on shell, N was a lapse function that related the parameter τ with the proper time of the particle. Here we one might be tempted to say something like 'we sum over all lapse functions, or, equivalently, over all proper times'. But this is wrong, as the value of N in the integral is completely unrelated with the other variables and hence its value is independent of the proper time of the particle. Similarly, if we ignore the delta function $\delta(F)$ and we look just at the path integrals we may think that we are summing over paths in spacetime which evolve with respect to the parameter τ. That would include trajectories of particles that are allowed to travel backwards in time, that is, we could have trajectories that for a while in τ-time travel forwards in time, then turn to travel backwards and finally forwards again. This kind of trajectory is explicitly forbidden by the delta function, so at the end of the day the trajectories we are summing over are just normal trajectories which move forward in time.

In the early days of quantum electrodynamics some backwards-in-time trajectories were discussed. Indeed, trajectories like the one just described were discussed and were given a nice interpretation. Particles travelling backwards in time were interpreted as antiparticles and the turning point at which a particle stopped going forwards and started going backwards would just be a point in which a particle and an antiparticle meet and annihilate. Similarly, the other possible turning point, that is, from backwards to forwards in time was interpreted as a point in which a particle-antiparticle pair was created. These trajectories appeared in the computation of some 'propagators', like the Feynman propagator. These quantities are Green's functions for the Klein-Gordon equation, such as the Newton-Wigner propagator we have been working with in this section, but they could not be given the interpretation of being the propagator of a quantum state with all the required probabilistic requirements. For this reason, in modern QFT one does not interpret Klein-Gordon fields as wavefunctions but as classical fields, which are then quantized. Particles and antiparticles then arise from such quantization, with no backward-in-time travel involved. Feynman propagators and similar objects play a role in the formalism of quantum field theory, but they do not have the same interpretation as the propators of a wavefunction and, if they are expressed in a form that involves integrals over histories that can travel backward in time, those histories are not interpreted to represent any real process.

This point needs to be made, because in interpreting quantum theories like the one for the relativistic particle we are dealing with, some quantum theory of gravity, or some other reparametrization invariant theory one might be falling in the same wrong interpretations. For instance, in Reisenberger and Rovelli (1997) they follow Feynman (1950) and Nambu (1950) in defining a 'proper time propagator'. This propagator can be found by noticing that we can express the propagator 7.35 as:

$$K(x,t;x',t') = \int \mathcal{D}N\,[\theta(N)]\,K_N(x,t;x',t')\,. \tag{7.36}$$

That is, we can define the 'fixed-lapse propagator' $K_N(x,t;x',t')$ for a fixed N such that the physical propagator is just the path integral of this object. Now we can define the proper time T to be:

$$T = \int d\tau N\,. \tag{7.37}$$

Notice that this naming is not really appropriate, as T could take any positive value in a way that is unrelated with the physical position, with the physical time and even with the real proper time. Now we can define the proper time propagator as the path integral of the fixed-lapse propagator over all the lapse functions which determine the same proper time. That is:

$$K_T(x,t;x',t') = \int_{\int d\tau N = T} \mathcal{D}N\,[\theta(N)]\,K_N(x,t;x',t')\,. \tag{7.38}$$

The physical propagator can be found from the proper time propagator by performing the integral on the last degree of freedom left in the path integral, i.e., the proper time:

$$K(x,t;x',t') = \int dT K_T(x,t;x',t')\,. \tag{7.39}$$

That is, we can recover the physical propagator by performing the remaining integral from the path integral. The physical propagator has a clear physical meaning, as discussed in section 7.1. Meanwhile, the proper time propagator lacks this meaning, as it is dependent of one of the arbitrary parameters of integration we have introduced just for convenience. Nevertheless, in Feynman (1950); Nambu (1950) and also in Reisenberger and Rovelli (1997) the parameter T is interpreted physically as something like an internal clock that the particle is carrying. Therefore, the proper time propagator is interpreted as the probability for the particle to travel from the spacetime point x', t' to x, t in a proper time T. Again, this interpretation has not any physical support, as the parameter T is just a parameter we have introduced in the formalism for convenience without any physical meaning. Similarly, we should reject assigning any physical meaning to the fixed-lapse propagator K_N. Reisenberger and Rovelli in the same paper also tried to extend this interpretation for the case of the covariant quantization of gravity. In the next section I will argue against their view for the case of gravity, too.

The other main claim I want to make in this section is that what the example of the relativistic particle illustrates can in principle be extended to any deparametrizable model. In section 4.6 I argued that deparametrizable models have a problem of time which in principle can be solved by deparametrizing the model. That is, as we are able to find a time variable in the configuration space of the theory we are able to avoid the problem of time by defining evolution with respect to that time variable. The example in this section shows how the quantum theories for such systems can be defined in the same way as for any gauge theory in terms of path integrals over gauge-dependent histories. However, as in gauge theories, these integrals are restricted, and at the end of the day one is the considering only gauge-fixed trajectories in configuration space. Let me sketch the covariant quantization schema for a deparametrizable model in an analogous way to the one I introduced in the previous section for a gauge system:

1. Start with a deparametrizable model defined on an extended configuration space. Identify the 'coordinate' degrees of freedom and the physical degrees of freedom.

2. Define a measure on the physical configuration space, i.e., the configuration space once we have removed the redundant degrees of freedom which are used just to parametrize the dynamics. This allows for computing probabilities and implicitly defines the Hilbert space structure of the theory.

3. Define the propagator as a path integral. There are different options available depending on the variables one chooses to use. For the deparametrized variables the definition is as for a non-gauge theory. For parametrized variables, i.e., for variables in the extended configuration space, the definition of the path integral has to include the corresponding delta functions and Faddeev-Popov factors to fix a parametrization and ensure that the result is independent of the parametrization chosen. This defines the dynamics of the theory.

In this sense, the problem of time is avoided in the same way as in the canonical version. Notice that we are accomplishing two things when we identify one of the variables as a time variable. First, at a kinematical level it allows us to define the Hilbert space of the theory as the space of wavefunctions over the rest of configuration variables, which indeed are the true configuration variables. Second, at a dynamical level it allows to define the paths integral as trajectories with respect to this time variable, or to define them in the extended configuration space or phase space, but with the pertinent delta function factors that identify this variable as the time variable. This strategy will not be available for theories like general relativity, as one could have expected from my discussion in section 4.7 in which I argued that the problem of time was much more serious for general relativity than for the case of deparametrizable theories. In the next section I analyze the case for non-deparametrizable theories, focusing in general relativity.

7.4 Covariant approaches and the problem of time

In this section I will argue that problem of time of quantum gravity is a serious issue to consider even if the way one intends to quantize gravity is based in the covariant formalism. I will first argue that the techniques that we could apply to solve this problem for the deparametrizable models do not apply for non-deparametrizable models. Then, I will introduce the approach by Reisenberger and Rovelli, which works as an example and which is many times referred to in the quantum gravity literature. I will argue that in this approach the covariant version of the problem of time presents itself in that the 'propagators' that are defined covariantly are equivalent to the ones defined canonically, and they cannot be given a straightforward interpretation neither as true propagators nor as defining some meaningful probabilities.

In the previous section I have showed how one can deal with the reparametrization invariance of deparametrizable models in the covariant formalism. In this sense, in the same way that the problem of time could be solved for canonical quantizations of these models, their reparametrization invariance does not suppose a major obstacle for their covariant quantization. However, for the case of non-deparametrizable models like the double harmonic oscillator I introduced in section 4.2 or general relativity this will not be the case. As I said above, what allows to apply the covariant quantization methods to deparametrizable models is that one can identify a time variable in the extended configuration space of the theory. In chapter 4 I argued extensively that this is not the case for models like the double harmonic oscillator and general relativity. Therefore, the way we can covariantly quantize a deparametrizable model will not be applicable to these models.

The reasons for this conclusion are basically the same reasons that I argued for in chapter 4, and I refer the reader back to sections 4.2 and 4.3 for a longer and more careful discussion. Here I will just sketch some of those reasons for the example of the double harmonic oscillator, which can be extrapolated for the case of general relativity. I have argued that it did not make sense to separate one of the two oscillators as the 'physical' degree of freedom and the other one as the time degree of freedom. Even if for short periods of time (i.e., for less than one oscillation) one can use one of the oscillators as a clock for the other, both degrees of freedom stand on the same grounds and we should expect the formalism to treat them similarly. This should make us reject both the kinematical and dynamical steps of the quantization outlined above. The kinematical one because we would like to have a Hilbert space that represents both degrees of freedom, in which both oscillators could be in a superposition of states and in which we could have entanglement. We should also reject the dynamical step because by fixing one oscillator as parametrizing the trajectory of the other we are giving it the role of time variable.

As I mentioned in chapter 4, the problem of time for the double harmonic oscillator can be solved if we choose to quantize the standard Newtonian action for this system. Similarly, this theory can be covariantly quantized with no further difficulty. However, in the case of general relativity this option is not available, as there is no physically preferred parametrization of the theory available. In other words, as long as the reparametrization invariance of general relativity is unavoidable, we will find the problem of time or a similar issue for defining a quantum version of the theory.

Let me now turn to the way general relativity is covariantly quantized in the quantum gravity literature. I will introduce the approach by Reisenberger and Rovelli (1997), which will be a paradigmatic example which is in the basis of some spin foam models like the ones I will study in chapter 8. Reisenberger and Rovelli (1997) start by defining a proper time propagator similar to the one I have introduced in the previous section for the dynamics of a particle. They use the canonical Hamiltonian of general relativity:

$$H_c = \int d^3x N^\mu \mathcal{H}_\mu \,, \tag{7.40}$$

where N^μ represent the lapse function and shift vector and $\mathcal{H}_\mu$ the Hamiltonian and momentum constraints of the theory. This allows them to define formally a 'propagator' of the form:

$$K_{N^\mu}(g_{ab}, t; g'_{ab}, t') = \langle g_{ab}| \exp\left(-\frac{i}{\hbar}\int_{t'}^{t} dt \hat{H}_c\right)|g'_{ab}\rangle \,. \tag{7.41}$$

This is in analogy with the definition of the propagator for any Hamiltonian theory, and in particular with the propagator 7.3. From a naive perspective, this propagator would represent the evolution operator for a quantization of general relativity in which the foliation was fixed, that is, it represents the evolution of a 3-geometry along a foliation defined by the fields N^μ. In particular, K_{N^μ} would represent something like the probability amplitude (density) of finding at time t the 3-geometry g_{ab} given that at time t' the geometry was g'_{ab}. This of course leaves at a side the subtleties of general relativity being a constrained theory. This propagator is the general relativistic analogue of the fixed-lapse propagator I defined above for the relativistic particle.

Reisenberger and Rovelli define the multi-finger proper time $T(x)$ to be the proper time along a geodesic that starts 'at rest' at the first time slice at a position x. I refer the reader to Reisenberger and Rovelli (1997) for the details of this construction, but an important aspect to consider is that this proper time is just a functional of N^μ. That is, in the same way that the lapse function in the case of the particle defines a proper time, in the case of general relativity we find that the lapse function and shift vector also define proper times, but now they define a proper time per space point, that's why it's called multi-fingered time. Following the analogy with the particle model, Reisenberger and Rovelli define the multi-fingered proper time propagator as:

$$K_{T(x)}(g_{ab}, t; g'_{ab}, t') = \int_{\text{Fixed } T(x)} \mathcal{D}N^\mu K_{N^\mu}(g_{ab}, t; g'_{ab}, t')\,, \tag{7.42}$$

where the integration is performed only over lapse functions and shift vectors which define a multi-fingered proper time $T(x)$. Finally, Reisenberger and Rovelli define the 'physical' propagator as:

$$K(g_{ab}, t; g'_{ab}, t') = \int \mathcal{D}T K_{T(x)}(g_{ab}, t; g'_{ab}, t')\,, \tag{7.43}$$

There is an analogy here with the particle case: in both cases, we can use fixed-lapse propagators or proper-time propagators for defining the physically meaningful propagator. As I argued in section 7.3, those intermediate propagators are useful computational objects, but neither them nor the variables they use have any physical meaning. For the case of general relativity the same conclusion can be reached.

Notice however, that the discussion for the relativistic particle model started from one of the forms of the propagator for the quantum theory of the model and that for the case of general relativity we have just started from the definition of the fixed-N^μ propagator. Therefore, the physical meaning of the 'physical propagator' is so far unclear. We can express it by using all the definitions above as:

$$K(g_{ab}, t; g'_{ab}, t') = \int \mathcal{D}N^\mu \langle g_{ab}| \exp\left(-\frac{i}{\hbar}\int_{t'}^{t} d^4x N^\mu \mathcal{H}_\mu\right)|g'_{ab}\rangle\,. \tag{7.44}$$

Now notice that we can formally perform the integration in N^μ by using the relation between delta functions and exponentials that I have introduced in equation 7.24 and used also for the case of the relativistic particle. We can express the physical propagator (in a non-rigorous way) as:

$$K(g_{ab}, t; g'_{ab}, t') = \langle g_{ab}|\delta(\hat{H}_\mu)|g'_{ab}\rangle\,. \tag{7.45}$$

The delta functions in this expression are delta functions of the constraints, so we can read them as mapping the states $|g_{ab}\rangle$ in the kinematical Hilbert space to states in the physical Hilbert space. In other words, what we have found is that the physical propagator Reisenberg and Rovelli have defined corresponds to a propagator as discussed in section 4.7. Recall that in the canonical formalism we can define the map η which maps states from the kinematical Hilbert space, in this case some quantization of superspace, to the physical Hilbert space of the theory, i.e., to states that satisfy the constraints. And this map allows to define an inner product and a propagator as:

$$K(g_{ab}, g'_{ab}) = (\eta(g_{ab})|\eta(g'_{ab})) \tag{7.46}$$

The informal derivation of the expression above in terms of delta functions suggests that the object that Reisenberger and Rovelli define is precisely this propagator, as indeed they claim. In this sense, we have connected the canonical and the covariant formalism.

Here it is important to discuss the dependence of the propagator on the time coordinate t. During the derivation I have been carrying the dependence on t, but given the diffeomorphism invariance of general relativity[29], the result is independent of the time variable used to coordinatize the foliation of spacetime. This is consistent with the interpretation of the propagator as the inner product between two states in the physical Hilbert space, given that this definition does not carry any dependence on any time parameter. In this sense, in many covariant approaches to quantum gravity the propagators defined are independent on any time elapsed between the initial and final slices. This is the first symptom that there is a problem of time for covariant approaches: the time dependence has dropped from the dynamics, just as in canonical approaches we could only define trivial time evolution.

Given this connection, the criticism I made in chapter 4 applies to this approach. The object we have defined is naturally interpreted as an inner product between states in the physical Hilbert space, and not like a propagator. That is, even if for simple models like the reparametrization invariant model of the non-relativistic particle the object defined in this way (see equation (4.114)) is actually the propagator that allows to recover the dynamical equations of the quantum theory, for non-deparametrizable models this was not the case. For instance, in the case of the double harmonic oscillator I argued that the 'propagator' is just the inner product for states in the subspace in the Hilbert of the two oscillators which have a fixed energy E.

For interpreting the inner product as a propagator in the case of a deparametrizable model one needs to identify one variable as a time variable and then, if the inner product satisfies some requirements like allowing for unitarity, we can interpret it as a propagator. In the case of the double harmonic oscillator and in general relativity this is not the case, and such an interpretation is not possible. However, in the case

[29] This invariance has to be respected by the integration measure $\mathcal{D}N^\mu$. This is not trivial, but for this kind of formal argument one assumes it.

of general relativity it is extended the claim that the configuration variables carry information about time. I argued against this claim in section 4.3, and hence I reject that the propagator defined above defines an evolution with respect to a time that is encoded in the 3-geometry.

One can accept that the propagator defined is not a propagator in the usual sense, and nevertheless claim that it defines something like 'the probabilty amplitude (density) of finding a final 3-geometry g_{ab} given that at some initial moment it was observed to be g'_{ab}'. This kind of interpretation has been defended by Rovelli[30] but it faces some difficulties, as I already mentioned in chapter 4. First, this approach cannot be connected with the usual quantum dynamics and interpretation which rely on a quantum state evolving on a Hilbert space. Second, as this approach has an operationalist flavour, we would like to have a clear way to connect the objects it defines with probabilities for certain outcomes in some experimental setting. However, we are lacking this, as it is not clear how to make sense of 'the probability of finding a 3-geometry' if this is not accompanied by more information. For instance, in the quantum theory defined for a fixed foliation we could give a physical meaning to the time parameter t and we could easily think that the fixed-foliation propagator defined the probability of measuring a final geometry at a time t which can have an operational meaning[31]. But when we have the 'physical propagator' we have just a definition of a probability, without any extra information that would allow to operationalize or localize where and how the 'measurements' are done.

In the next two chapters I will introduce some covariant approaches to quantum gravity. In those chapters I will expand my arguments in this section to apply to those approaches. The same three general objections I have outlined in this section will also apply to them. To insist, let me recap these three objections. First, I took the problem of time of general relativity to be unsolved and this does not change in the covariant formalism. Second, the 'propagators' defined are not strictly speaking propagators, which disconnects the approach from the usual structures and interpretations of quantum mechanics. And third, the probabilities defined in these approaches are not easily interpreted, not even in an operational way.

In this sense, the conclusion I reach is that the covariant quantiation of gravity does not solve the serious conceptual and technical issues that I found in the canonical quantization. In the next two sections I explore some ways in which the covariant quantization formalism has been generalized and I argue that the problems I have outlined in this section persist.

[30] See for instance Rovelli (2004).

[31] This could be possible if we had a clock which measured t, and by repeating the experiment several times we could check if the experimental frequencies agree with the probabilities.

7.5 General boundary formulation

In this section I will introduce the general boundary approach to quantum mechanics, which is explicitly adopted in some approaches to quantum gravity. This approach is based on a generalization of quantum mechanics. The motivation from this generalization comes from field theories. The classical dynamics of a field theory can be expressed using the standard structures of classical mechanics which treat the field theory as a system evolving with respect to a time parameter. For this we need to decompose spacetime into a space in which the fields are defined and a time with respect to which we can say these fields evolve. Of course, this foliation of spacetime is not unique for relativistic spacetimes, which suggests other ways of considering the dynamics. In particular, one can define dynamics in more local ways, and the idea of the general boundary approach to quantum mechanics is to generalize quantum mechanics so that its dynamics relies on more local structures and not on foliations of spacetime. In this section I will first expand on the relation between foliations and quantum mechanics and then I will introduce and analyze the general boundary approach to quantum mechanics. I will argue for two conclusions. First, that the best way of understanding it is as an operationalist approach which is not compatible with other interpretations of quantum mechanics. And second, that the introduction of this formalism for the case of general relativity does not solve the problem of time for its quantization.

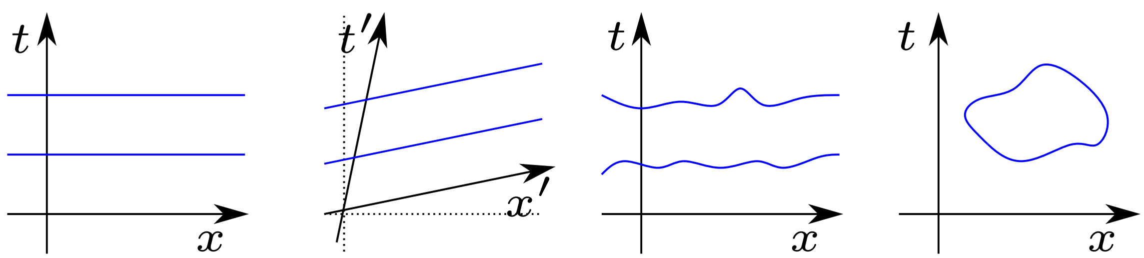

Fig. 7.2: Here I represent four possible propagators for a spacetime of dimension 2. The spacetime coordinates are given by the axis in black and the propagators are associated with the surfaces in blue. The propagators represented are, in order: the propagator in Newtonian time of standard quantum mechanics, a propagator in a Lorentz frame in QFT, a propagator for arbitrary spacelike boundaries and a generalized propagator for a closed boundary.

The reason for seeking for a generalization of the quantum formalism is that it contains a rigid temporal structure which goes against the principles of relativity. This structure is clear in the canonical formulation of quantum mechanics: we have a quantum state in a Hilbert space which evolves with respect to some external absolute time parameter. When covariantly quantizing a classical theory this absolute structure remains: from the first step in the quantization we need a notion of instant such that configuration space is the space that represents the ways our system is at a moment of time. For field theories and other theories defined on a spacetime it is therefore necessary to define a foliation of spacetime, i.e., a way of splitting spacetime into a sequence of instants of time. For any such foliation, we can define the dynamics of the theory by means of a propagator of the form $K(q^a, t; q'^a, t')$.

In this sense, the natural temporal structure of quantum mechanics is the temporal structure of Newtonian mechanics. That is, both theories have an absolute time parameter with respect to which the dynamics unfolds. For this reason, it is for non-relativistic theories and regimes that quantum mechanics is more natural and leads to fewer conceptual troubles. In figure 7.2 I represent the different propagators I will consider in this section, being the one for an absolute Newtonian time the first on the left.

When we consider the quantization of relativistic theories we will find some tension between the structure of the relativistic theory and the absolute time structure of quantum mechanics. Let me start by considering specially relativistic theories, i.e., theories defined on Minkowski spacetime. Examples of these theories are quantum field theories and the quantum mechanics of the relativistic particle. It is well-known that the simplest and physically more meaningful coordinate system for this spacetime is the system of Lorentzian coordinates or frames. This system of coordinates is not unique and different Lorentzian frames are related by simple coordinate transformations, which are known as Lorentz transformations. Any of these frames defines a foliation of spacetime, and we can naturally define a quantization of any specially relativistic theory by picking one of these foliations. This is represented by the second picture in figure 7.2. In this sense, the structure of special relativity forces us to make a choice between the different Lorentz foliations available and select it as priviledged for the quantization of the theory.

Despite this tension between the temporal structures of specially relativistic theories and quantum theories let me mention that it is not regarded as problematic. For all practical purposes, in quantum field theories one is usually interested in computing asymptotic quantities for which the choice of Lorentz frame does not have an impact. For instance, one important quantity in QFT is the cross-section for different transitions that can happen in a particle collider. These quantities encode the probabilities that a given distribution of particles is produced given that certain others collide at a certain energy. The information encoded in these cross-sections can be expressed using any reference frame, and they give the same predictions which moreover agree with the observed frequencies. In this sense, the choice of Lorentz frame does not have any practical impact. In other words, if nature picks up a preferred Lorentz frame at a quantum level, so far no difference has been noticed.

In a special-relativistic setting a case can be made that Lorentz foliations are priviledged even if there are other possible foliations available. However, when we consider general-relativistic settings and curved spacetimes in general it is no longer the case that there is some preferred way to separate spacetime into space and time. That is, any decomposition of spacetime like the ADM decompositions I introduced in section 4.2 could work as the foliation we need to quantize any theory defined on this spacetime[32]. I represent this situation in the third picture on figure 7.2.

One could hope that this multiple choice is not problematic, in the same way that this issue can be ignored for practical purposes in special relativistic setting. However, this is not the case, as it has been shown and argued for in Kuchař (1991). Kuchar showed that quantum theories defined using different foliations of spacetime lead to different predictions. For instance, Kuchar showed that the propagator for the dynamics of a particle moving between an initial and a final time-slices at a generic spacetime depends on the foliation used for computing this propagator. That is, different foliations lead to different predictions about the probability of finding the particle at certain locations. In other words, different foliations of spacetime lead to incompatible quantizations. Moreover, while there are just a few parameters that determine a Lorentz frame in special relativity, there would be infinitely many that would be necessary to specify a generic foliation in a spacetime. This means that there is a serious problem of underdetermination for the quantization of a theory defined on a curved spacetime.

Things get even worse when we consider general relativity itself. The underdetermination problem I have just described is that unless there are some symmetries we would not be able to choose from a set of possible quantum theories. But in the case of general relativity I have argued that we may not even be in a situation in which a quantum theory can be defined, as there is the problem of time. A way of expressing the difference between a theory in a curved spacetime and general relativity is that in the former the theory is defined with respect to a background structure, spacetime, which allows to define a foliation and a quantum theory. Meanwhile, for the case of general relativity, its structure is not defined with respect to any background structure. At most we can separate spacetime as a bare manifold and a metric tensor defined on top of it, but as the bare manifold does not have any physical meaning we are led to the problem of time.

As we see, different theories with different spacetime structures lead to different foliations of spacetime and different quantizations. When spacetime is dynamical we face the problem of time, as the techniques that we used for the quantization of any

[32] Indeed, this is a way quantum field theories on curved backgrounds are defined, as explained in Ford (1997); Parker and Toms (2009).

other theory relied on a foliation independent of the theory being quantized. Now, the idea of a foliation can be useful for thinking about dynamics, but for field theories this is not the only alternative. In particular, field theories are local, which means that the value of a field at a point is determined just by the values of the field on neighboring spacetime points. In this sense it is more natural to define dynamics by referring to spacetime regions than by referring to instants of time. In particular, in a field theory we can say that the dynamics of the theory is contained in the equations of motion that allow one to determine the values of the physical fields in a spacetime region M given that we know the values of the fields in the boundary δM of that region. This is the natural generalization of the classical mechanics view of equations of motion determining the state of the system at every time between an initial and final times for which we know the state of the system. The general boundary approach to quantum mechanics proposes to generalize it to include this field-theoretic view of dynamics. That is, instead of considering propagators to be defined on an initial and final time-slices, it defines them on the boundaries of arbitrary spacetime regions. This is represented by the fourth picture in figure 7.2.

The general boundary approach was introduced by Robert Oeckl[33] and it has been imported to quantum gravity by Rovelli[34]. In this approach, one assigns a Hilbert space, $\mathcal{H}_{\partial M}$, to the boundary of a spacetime region. A state in this boundary Hilbert space corresponds to a configuration of the physical fields of the theory in this boundary. One defines the generalized propagator as a map that assigns a complex number to each state in this Hilbert space:

$$\begin{aligned} K : \mathcal{H}_{\partial M} \to & \mathbb{C} \\ |\psi\rangle \to & K(\psi, \partial M) \,. \end{aligned}$$

When we consider the case of a manifold region bounded by two spacelike surfaces Σ_1 and Σ_2 the boundary Hilbert space is just the product $\mathcal{H}_{\Sigma_1} \otimes \mathcal{H}^*_{\Sigma_2}$, that is the product of the Hilbert spaces for the two surfaces. In other words, the Hilbert space is the product space of the spaces at the initial and final times. In this sense, the propagator is interpreted in operational terms as I explained above, that is, as giving the probability of finding a given configuration at the final time given that it was observed at some other configuration at the initial time. This can be thought in terms of preparation and measurement. For more general boundaries, Oeckl and Rovelli also interpret states in the boundary Hilbert space to represent possible outcomes of measurements or possible observations/preparations and the propagator to give probabilities for such measurements and observations. Moreover, Oeckl and Rovelli

[33] See Oeckl (2003) for its first introduction and Oeckl (2008) for a more formalized version.

[34] That Rovelli follows this approach to quantum mechanics can be seen for instance in (Rovelli, 2004, Chapt. 5) and (Rovelli and Vidotto, 2015, Sect. 2.4.2).

argue that the distinction between preparation and measurement is not fundamental and they consider both of them to correspond to just observations in the boundary. An argument for this is that the propagator of quantum mechanics, $K(x,t;x',t')$, is time symmetric up to complex conjugation, i.e., from a formal point of view there is no difference between the initial and final state. In this sense we can interpret $K(x,t;x',t')$ as giving the probability amplitude for a measurement at time t of a state prepared at time t' or the other way around, i.e., as the probability amplitude for a measurement at time t' of a sate that will be prepared at time t. In the case that we consider a general boundary, the distinction between measurement and preparation is even less clear-cut, as there are many ways in which we could divide the boundary into two regions. For this reason, the boundary Hilbert space gets interpreted as representing the information in the boundary, without distinguishing between measurement or preparation.

The way the propagator gets interpreted is slightly different in the presentations by Oeckl and Rovelli. Oeckl claims that the transition amplitude for a given configuration cannot be given a straightforward interpretation as a probability, but instead it can be taken together with the transition amplitudes for other configurations to compute conditional probabilities. For instance, imagine that we know that we have some information about the state in the boundary, say that know the configuration of the fields in some part of the region. In this case we could ask ourselves the probability that in some other part of the boundary some other configuration is observed. This can be formulated as the conditional probability of finding the state $|\psi\rangle$, corresponding to both observations, given that the state is an element of the subspace $\mathcal{H}_0$ compatible with the configuration we know for sure. In this case, Oeckl defines the conditional probability to be:

$$P(\psi|\mathcal{H}_i) = \frac{|K(\psi,\partial M)|^2}{\sum\limits_{\mathrm{Span}\{|\phi_j\rangle\}=\mathcal{H}_0} |K(\phi_j,\partial M)|^2}\,, \tag{7.47}$$

where $\{|\phi_j\rangle\}$ is a basis of the subspace $\mathcal{H}_0$. This conditional probability is a well-defined probability, i.e for any state and subspace of the Hilbert space it is a real number between 0 and 1.

As a consistency check, we can use this formula to recover the conditional probabilities of standard quantum mechanics. Let's consider the boundary Hilbert space $\mathcal{H}_{\Sigma_t \bigcup \Sigma_{t'}} = L^2[\mathbb{R}, dx] \otimes L^2[\mathbb{R}, dx']$, that is, the Hilbert space for the position of a particle at an initial time t' and at a final time t. We can define the propagator so that it agrees with the standard propagator I introduced in section 7.1:

$$\begin{aligned} K : \mathcal{H}_{\Sigma_t} \otimes \mathcal{H}_{\Sigma_{t'}} \to& \mathbb{C} \\ |x\rangle \otimes |x'\rangle \to& K(x,t;x',t') = \langle x| e^{-\frac{i(t-t')}{\hbar}\hat{H}} |x'\rangle\,. \end{aligned}$$

Given this definition for the propagator, we can use Oeckl's definition of conditional probability. Imagine we observe at time t' the particle to be at position x' and we want to compute the probability of finding the particle at some final position x at time t. That is, we want to know the probability of finding the state $|xx'\rangle$ knowing that the state is of the form $|\psi\rangle \otimes |x'\rangle$. To apply Oeckl's expression with some mathematical rigor we cannot use the improper states $|x\rangle$, so let me introduce some proper states ψ_x which will be square integrable wavefunctions peacked around a given value x and then, after using Oeckl's formula we can take the limit to infinitely peaked states. Therefore, we will first use Oeckl's formula for computing the probability of finding ψ_x at time t given that $\psi_{x'}$ was observed at time t':

$$P(\psi_x, t|\psi_{x'}, t') = \frac{|K(\psi_x, t; \psi_{x'}, t')|^2}{\sum_j |K(\psi_j, t; \psi_{x'}, t')|^2} = |K(\psi_x, t; \psi_{x'}, t')|^2 \,, \tag{7.48}$$

where I have introduced $\{\psi_j\}$ as a basis of $L^2[\mathbb{R}, dx]$. The denominator is equal to 1 given the properties of the propagator of quantum mechanics. By taking the limit of peaked states we get the desired result:

$$P(x, t|x', t') = |K(x, t; x', t')|^2 \,. \tag{7.49}$$

In Oeckl's approach it is only by means of the conditional probabilities that $|K(\psi, \partial M)|^2$ has a physical interpretation. Rovelli instead claims that the propagator squared can be directly interpreted as a probability. However, that this interpretation is reasonable is not clear. Compare it with the conditional probabilities defined by Oeckl. The conditional probabilities were well-defined probabilities, i.e., they associated a number between 0 and 1 to every possible state $|\psi\rangle$ in the subspace $\mathcal{H}_0$. If we choose a basis of $\mathcal{H}_0$, say $\{|\phi_j\rangle\}$[35], expression 7.47 assigns a probability to each of these states, and the sum of all of this probabilities is one. That is, in the definition by Oeckl we have a clear sample space, that is, a space of possibilities, and a coherent assignment of probabilities. Meanwhile, if we take $|K(\psi, \partial M)|^2$ it is not clear to me what the sample space is, i.e., it is not clear with which other possibilities are we comparing the possibility of $|\psi\rangle$ to happen. A related problem is that $|K(\psi, \partial M)|^2$ is not necessarily normalized, that is, if we sum the 'probabilities' for all the states of a given basis of $\mathcal{H}_{\partial M}$ the result could be different from one. For instance, if we take the propagator of quantum mechanics $K(x, t; x', t')$ and integrate over all x and x' the

[35] This choice of basis can be thought as selecting the kind of measurement/preparation in the boundary. For instance if the Hilbert space is the one particle Hilbert space, choosing the position basis corresponds to a measurement in which position is measured. Alternatively, for computing probabilities for momentum measurements we would use the momentum basis.

result diverges[36]. For all these reasons, one has to be careful when interpreting the propagator for a general boundary directly as a probability and not as a conditional probability or as an intermediate step used to compute a conditional probability.

An important aspect to comment is that the approach by Oeckl and Rovelli has an important operational character. I have shown that the generalized propagator is used to define probability amplitudes that are associated with certain operations in the boundary, such as preparations, observations or measurements. As I commented in section 7.1, the propagator in quantum mechanics can also be given such an operational interpretation, but it also allowed us to define a wavefunction and a temporal evolution, which is necessary for having a realist interpretation of the theory. Therefore it is interesting to ask whether we can define some similar object in the general boundary formalism. First, we can notice that if we look at the propagator not as a function on the Hilbert space, but just as a function on the basis of this space, it is a complex-valued functional of a configuration space. That is, it is not so different from a wavefunction. There are some important differences though. First, a point in this configuration space does not represent the way a system is at a moment of time, but instead it represents a configuration of fields on the boundary. Second, we have seen that this propagator is not normalized, and hence we cannot directly interpret it as encoding the probability of certain configurations.

To further discuss the meaning of the propagator, it will be useful to introduce a concrete definition of it. Rovelli (2004) defines the propagator for a general boundary in analogy with the covariant definition of the propagator of quantum mechanics as:

$$K[\phi, \delta M] = \int D\phi e^{-\frac{i}{\hbar}S_M[\phi]} \,. \tag{7.50}$$

That is, the generalized propagator is defined just by generalizing the propagator of standard quantum mechanics to consider 'paths', that is, field configurations in M which agree in the values of those fields in the boundary. This definition implies that the propagators for different regions are related by some relations encoded in some sort of generalized Schrödinger equation which describe how the propagator changes when the surface we are considering is deformed. For instance, in the case of two spacelike surfaces we can consider deformations which consist in moving just one of the slices, and the evolution of the propagator is argued[37] to be just the field equivalent of Schrödinger equation. For this case we can recover the usual dynamics and wavefunction, as it was to be expected. However, let me raise the worry that these definitions and arguments are not mathematically precise, and that it is likely that

[36] A way of seeing this is by taking the integrations one at a time. The integration over x of the propagator is 1 due to unitarity and the integration over x' of a constant diverges. Even if we repeated this computation using proper states the result would still be divergent.

[37] See the discussion in Conrady et al. (2004).

when one defines in a mathematically consistent way the propagator one will have to face some difficulties like the ones pointed out by Kuchar regarding inequivalent quantizations for different foliations.

When we consider the case of arbitrary boundaries, it does not seem that we can take the generalized Schrödinger equation to be giving something like a dynamics. That is, even if we consider a sequence of surfaces which can be thought as a deformation of each other, like the boundaries of a sequence of spheres of decreasing radius, this sequence still cannot be interpreted as a wavefunction evolving. In this sense, the propagator that Rovelli defines seems to be defining just conditional probabilities for any arbitrary boundary and not anything similar to the wavefunction we need for having a realistic interpretation of the formalism. That is, only for the particular case of having two spacelike boundaries we are in a situation in which can interpret this formalism by appealing to approaches like many-worlds, Bohmian mechanics or collapse models.

For this reason, it is not only that the generalized boundary formalism is presented in operational terms, it is also very difficult to interpret using the standard realist interpretations, as it does not rely on the wavefunction as a fundamental object. As happens for other interpretations that we could label as operationalist like Copenhagen or quantum Bayesianism, it is difficult to get a clear idea of what they say about the world, if we try to understand this theories as saying something about the world. A minimal thing that can be said is that the Hilbert space structure used for describing the boundary state describes the possible outcomes of measurements. In the case of gravity for instance, if we use the LQG Hilbert space for describing the boundary and we measure geometric observables as areas or volumes, the structure of the Hilbert space will imply that the results of these measurements are discrete.

A further criticism that can be made to this formalism is that one motivation for adopting it is the locality of field theories, but it is not very clear that this goal is achieved, at least not in a way that improves the case of standard quantum mechanics. The probabilities we have defined are probabilities for measurements in the boundary of a spacetime region and in a sense these probabilities depend on the boundary or on the whole region. Meanwhile, in quantum field theory we can define operators corresponding to field measurements at every point of spacetime. These operators and probabilities certainly look more local than the objects defined using the generalized boundaries formalism, although one can object that for computing the probabilities one still needs to appeal to a quantum state, which will be associated with a foliation of spacetime. In any case, it is disputable that the generalized boundary formalism achieves clearly a greater degree of locality than the standard formulation of quantum mechanics.

The generalized boundary approach is therefore, in my opinion, a generalization of quantum mechanics which is problematic, specially for someone with realist inclinations. I will now consider its application to theories with a problem of time, and to general relativity in particular. The generalized boundary approach can be connected with the transition amplitudes resolution of the problem of time of canonical quantizations, which I introduced in chapter 4. Recall, that in this resolution the physical content of the theory is believed to be encoded in the map η that maps states in the kinematical Hilbert space to the physical Hilbert space and that allows to define a 'transition amplitude' or 'propagator' by means of the inner product in the physical space:

$$K : \mathcal{H}_{kin} \otimes \mathcal{H}_{kin} \to \mathbb{C}$$
$$|\psi\rangle \otimes \ |\psi'\rangle \to K(\psi, \psi') = (\eta(\psi)|\eta(\psi'))\,.$$

For the case of the non-relativistic particle this construction gives back the propagator for that theory, as I showed in section 4.7.5:

$$K : L^2[\mathbb{R}^2, dxdt] \otimes L^2[\mathbb{R}^2, dx'dt'] \to \mathbb{C}$$
$$|x, t\rangle \otimes \ |x', t'\rangle \to K(x, t; x', t') = (\eta(x, t)|\eta(x', t')) = \langle x|e^{\frac{-i(t-t')}{\hbar}\hat{H}}|x'\rangle\,.$$

Notice the strong analogy with the definition above of the propagator using the boundary Hilbert space. However, there is an important difference between both formalisms: the Hilbert spaces they rely on are different Hilbert spaces with different interpretations. In the boundary formalism we are using the boundary Hilbert space, which describes the different configurations we can find at the boundary, while in the transition amplitude case we are using the kinematical Hilbert space, which for theories like the non-relativistic particle is a Hilbert space over an extended configuration space. In this sense, a sate in the kinematical Hilbert space can be a superposition of states corresponding to two different times. Consider for instance the state $|x, t_1\rangle + |x, t_2\rangle$ which would represent a superposition of two times at which the particle could be. If we take some of these states and compute the transition amplitude for them, this transition amplitude does not have the form of a propagator between two times. In this sense, for the transition amplitude strategy to work we need to restrict the definition to states with well-defined times. Similarly, for interpreting the propagator probabilistically, we have to treat in different ways the configuration variables x and the time variables t. In this sense, the transition amplitude strategy for solving the problem of time has to be treated carefully and it is only by identifying the time variables in the extended configuration space that we are able to successfully solve the problem of time and connect the transition amplitude with the propagator we are able to give a physical meaning.

The transition amplitudes strategy, if it is to be associated to the boundary formulation it is clearly better adapted to represent the standard propagator of quantum mechanics, as the inner product takes two arguments, and this invites to consider a propagator between two disconnected boundaries. In the case of having just a single boundary, as it is generally the case for the general boundary formulation it is unclear how we could use the structures of the canonical formalism. We can still think in associating a kinematical Hilbert space to a generalized boundary, but for defining the propagator using the map η we need to compute the inner product with some other state, which we would be lacking. In (Rovelli, 2004, Sect. 7.4), for the case of LQG, it is proposed to use the empty state, which is interpreted as representing a no-space, to define the propagator for generalized boundaries. I will discuss this proposal in some detail in section 8.3 but let me just say that there is no obvious motivation for this choice and that it seems ad hoc. In this sense, the connection of the general boundary formulation of quantum mechanics and the transition amplitude resolution of the problem of time seems to be limited to the case in which we restrict to standard quantum mechanics.

Now we can consider explicitly the case of general relativity. In chapter 4 I have argued that the kinematical Hilbert space of the canonical quantization of the theory is not a Hilbert space defined on an extended configuration space in which the time variable or spacetime coordinates are included. In this sense, I argued that a state in the kinematical Hilbert space of a quantization of general relativity is a wavefunction over 3-geometries, i.e., something representing the geometry or a superposition of geometries of a time slice. For the boundary formulation we can import the same structure and define the boundary Hilbert space as a space of wavefunctionals of 3-geometries, but in this case not of time slices but of arbitrary boundaries. There is an issue with this definition, as general boundaries can contain time-like and null directions and the Hilbert spaces of geometrodynamics and LQG only represent spatial 3-geometries. This problem is largely ignored in the literature, and I will also leave it at a side now and come back to it in the next chapter. Notice that unlike the case of theories defined on extended configuration space, we have no problem in interpreting the kinematical Hilbert space as a boundary space.

The problem with the quantization of general relativity using the general boundary formalism lies in the dynamics. The general boundary formalism allows to define a propagator for general relativity by means of a path integral. This propagator can be thought to be an object of the form $K[g_{ab}]$, that is, just dependent on the 3-geometry on a surface. The propagator should not carry an explicit dependence on the surface given the diffeomorphism invariant of general relativity. Therefore, the way of precisely define the path integral can be technically involved. However, if we were able to define it we could take it as a starting point for the interpretation of the theory. I will argue that this approach is not convincing for two kinds of reasons. First, there the general objections I have stated above against the general boundary formalism. Second, the objections I have exposed in section 7.4 against some covariant quantizations of gravity are largely still valid if we use the general boundary formalism, as I will briefly discuss now.

My first objection was that the problem of time is not addressed in covariant formulations. For the transition amplitude strategy this meant that the transition amplitudes defined by means of the map η or a path integral are best interpreted as just an inner product. This was clear for the case of the double harmonic oscillator and I argued that it seemed to be true for general relativity too. It is unclear at best the way this worry dissolves if we adopt the general boundary formalism. Related with this was the second objection, which was that the 'propagator' is not strictly speaking a propagator which can be used for evolving a wavefunction in time. I have argued that this is even more worrisome in the case of using the general boundary formalism, as even for theories with no problem of time we lack a dynamical interpretation of the 'propagator' which allows to reconstruct a wavefunction and the quantum dynamics in the usual sense. And third, I objected that the probabilities defined by means of covariant quantization were not easily interpretable. For the case of the general boundary case this problem persists, as it is not clear, not even from an operational point of view, how to interpret something like 'the probability of measuring some 3-geometry at the boundary'. Again, for making sense of the theory we would need to know how to locate 'the boundary' and how to identify a measurement in it.

Let me conclude this section by giving an overview. I have introduced the general boundary formulation of quantum mechanics as an attempt to formulate quantum mechanics in a way which does not rely on the rigid temporal structure of the theory. However, I have argued that this approach is problematic from a realistic perspective, as it is of a clear operationalist character. Moreover, when we use this formalism to address the quantization of general relativity, the problems that appeared for the covariant quantization of the theory are still there. This means that this generalization of the covariant formalism is not an improvement for solving the problems related with the diffeomorphism invariance of general relativity and that in the canonical formalism showed up as the problem of time.

7.6 Other 'covariant quantizations'

In this last section I will comment about some other ways the dynamics of a quantum theory are claimed to be defined. For instance, sometimes in the physics literature one finds claims like that the partition function of a theory defines its quantum dynamics[38]. However, there are different objects that are referred to as the partition

[38] I will analyze some of these claims in chapter 9, as in the group field theory approach to quantum gravity it is claimed that the dynamics of the theory is defined by its partition function.

function of the theory or that are defined in similar ways, so in this section I will mention some of them and I will analyze the extent to which they define a quantum theory. I will argue that some of these objects are useful as intermediate steps for defining the propagator of the theory and others may encode some dynamical information, but that on their own these objects do not define a quantum theory. In particular, some of the 'expectation values' they define do not correspond in any straightforward way to expectation values of experiments. In this sense, it is only covariant quantization as I have presented it in this chapter that defines rigorously a quantum theory. I will also argue that even if we claimed that a 'partition function' defined a quantum theory of gravity it seems that this does not solve the conceptual problems related with the diffeomorphism invariance of the theory.

In table 7.1 I collect all the different functions and mathematical objects which are defined in covariant quantizations, in attempts of covariant quantizations or for other purposes like studying the statistical mechanics of quantum theories. All of them can be given definitions in terms of path integrals, although their constructions differ from one another in some ways. The first one in the list is of course the propagator I have defined in section 7.1 and the rest can be constructed by using similar techniques and by applying some generalizations. First, some of these quantities are not defined on Lorentzian manifolds, but they are defined on Euclidean ones. This difference makes it the case that the objects they define are real numbers and related with standard probabilities, like the ones used in thermal physics, rather than complex numbers and associated with quantum amplitudes. Second, some of these objects are defined on compact spaces, that is, instead of being defined on a boundary of some spacetime region, they are defined as a function of that compact space. This of course makes it impossible to interpret them as propagating some degree of freedom from one instant of spacetime to another. From a technical perspective, the way of computing this kind of path integral is equivalent to imposing periodic boundary conditions to the paths. Finally, some of this objects are generating functionals or similar objects which allow to define some 'expectation values'. I will now make a few comments about each of these generalizations.

The first generalization of the paths integrals is motivated by thermal physics. A way of computing thermal states is by means of an Euclidean path integral. An Euclidean path integral differs from the Lorentzian one, i.e., the one we have been using in this chapter, in two ways that are represented in the following schematic definition:

$$\int \mathcal{D}q^a e^{-\frac{1}{\hbar}S_E[q^a]} \,. \tag{7.51}$$

The first difference is that the amplitude associated to each path is a positive real number, which can be seen in that the imaginary unit i is not present in the exponential. This makes it the case that the amplitudes now look like probabilities.

Function	E/L	PBC	Meaning
Propagator	L	N	Defines evolution of the wavefunction
Generating functional	L	N	Generates 'expectation values'
Euclidean propagator	E	N	Defines a probability amplitude
Partition function	E	Y	Defines a thermal state
Lorentzian partition function	L	Y	Generates 'expectation values'

Tab. 7.1: List of mathematical objects that can be defined by means of path integrals. For each of them it is specified whether the function is defined using a Lorentzian or an Euclidean path integral, whether periodic boundary conditions (PBC) are imposed and its physical meaning.

The second difference is that the action S_E is not the classical action of the theory defined on a Lorentzian space, but an action defined on a different space which is Euclidean. As I said, Euclidean path integrals arise naturally in the context of statistical mechanics, and their interpretation does not have anything to do with quantum mechanics. Alternatively, Euclidean path integrals are also frequently used as a proxy for computing Lorentzian path integrals. The reason for this is that is in principle possible to transform one type of integral to the other by means of a transformation called Wick rotation. This transformation replaces a time parameter t with an imaginary parameter $\tau = it$, which transforms one kind of path integral into the other and which transforms a Lorentzian metric into an Euclidean one and viceversa.

In the context of quantum gravity, it is common to find Euclidean path integrals. For instance, in Loll (1998) several models of quantum gravity are introduced by means of some covariant quantization which relies only on Euclidean path integrals. However, it is a necessary condition for having a quantum theory that the paths integrals used are Lorentzian. In this sense, these approaches are limited, and as long as one is not able to Wick-rotate to Lorentzian spacetime the structures that have been defined in these models do not define a quantum theory. This limitation is acknowledged and discussed in Loll (1998).

The second generalization I have mentioned is that some path integrals are defined on compact manifolds, or, equivalently by imposing some periodic boundary conditions. The partition function of statistical mechanics is an example of this kind of object, it is a case of Euclidean path integral with periodic boundary conditions and its interpretation has nothing to do with quantum mechanics. Alternatively, one can define a 'Lorentzian partition function' which is a Lorentzian path integral, like the propagator of quantum mechanics, but defined on a compact manifold, i.e., defined on a space with no boundaries. If the generalization in section 7.5 was a generalization of the quantum formalism to consider any arbitrary boundary, this generalization can be seen as a generalization to consider no boundaries. This type of generalization is present in some approaches to quantum gravity, as can be seen in the review Freidel and Louapre (2004), and its interpretation is problematic. In the first place, the manifolds in which it is defined are closed manifolds, which means that time in these manifolds is circular in the sense that there will be closed time-like curves. This is clearest if we think on the periodic boundary conditions: when we reach the 'future' limit of the integral we are back to the 'past' limit. One may reject this approach on the grounds that we do not believe our spacetime to be like this. Moreover, the 'Lorentzian partition function' does not define any wavefunction on the boundary nor any probability for measurements on it, as there is no boundary. For its interpretation I need to introduce the third generalization I mentioned above.

The last generalization I mentioned above is that path integrals can also be used for defining some objects which look like the expectation values of quantum mechanics. For instance, consider the following object defined for the quantum mechanics of a single particle:

$$\int_{\substack{x(t)=x\\x(t')=x'}} \mathcal{D}x\, x(t_1) e^{\frac{i}{\hbar}S[x]} \,. \tag{7.52}$$

This is the same path integral as for the propagator of the theory but with one factor $x(t_1)$ inserted on it. It can be shown that this path integral can be expressed in terms of states and operators in the canonical formalism, both in the Schrödinger and Heisenberg pictures:

$$\langle x| e^{-\frac{i(t-t_1)}{\hbar}\hat{H}} \hat{x} e^{-\frac{i(t_1-t')}{\hbar}\hat{H}} |x'\rangle = \langle x, t|\hat{x}(t_1)|x', t'\rangle \,. \tag{7.53}$$

This quantity is a matrix element and it is sometimes referred to as 'expectation value'. However, even if it looks like an expectation value, they are not expectation values in a strict sense. In quantum mechanics, if we know the quantum state ψ of a quantum system the expectation value of any observable O at a time t is given by $\langle\psi|O(t)|\psi\rangle$[39], which has a direct experimental meaning. One may be tempted to interpret $\langle x, t|x(t_1)|x', t'\rangle$ as an expectation value corresponding to an experiment in which the particle is prepared at time t' in position x', its position measured at time t_1 and finally found at time t in position x. However, it is easy to see that the expectation value that the rules of quantum mechanics give for such an experiment is different from $\langle x, t|x(t_1)|x', t'\rangle$. For these reason, one has to be careful and keep in mind that quantities like $\langle x, t|x(t_1)|x', t'\rangle$ are not true expectation values for experimental situations.

[39] Here I am using the Heisenberg picture in consistency with the rest of the section, but the expression in the Schrödinger picture is equivalent. Here I am assuming the standard projective measurement of quantum mechanics, leaving aside generalized measurements and POVMs, which would not affect the point I am making.

This can be generalized to any quantum theory and to multiple operators:

$$\langle\psi, t|\mathcal{T}(\hat{O}(t_1)...\hat{O}(t_n))|\psi', t'\rangle = \int_{\substack{x(t)=x \\ x(t')=x'}} \mathcal{D}q^a O(t_1)...O(t_n)e^{\frac{i}{\hbar}S[q^a]}\,. \tag{7.54}$$

That is, for any quantum theory we can define by means of path integrals the 'expectation value' of any time-ordered product of operators[40]. As I have argued for the case of the single particle, these 'expectation values' are not expectation values corresponding to experimental situations. However, in quantum field theories these quantities are useful to compute, as there are some theorems that allow to compute physical transition amplitudes by knowing them.

The definition of the 'expectation values' can be extended to include Euclidean path integrals and/or periodic boundary conditions. In the case of thermal physics some expectation values can be defined and they may have a physical meaning. For the case of quantum gravity, however, the physical meaning of the 'expectation values' is less clear. For instance, in the context of the 'Lorentzian partition function' one defines 'expectation values' by inserting functions in the path integral, as it is done in Freidel and Louapre (2004). If these quantities were real expectation values one could think this would be a way of giving an interpretation to the formalism. However, it is disputable that this defines an expectation value, just as the quantities for the single particle case do not represent an expectation value. In any case, notice that we would be in a situation similar to the one in section 7.5: we have some definitions of something with an intended operational meaning but with no clear connection with the usual structures of quantum mechanics and for which none of the realistic interpretations of the theory apply. Notice that in the case of the general boundary formulation we had the works of Oeckl and Rovelli in which they discussed the foundations of their approach, while for the case with no boundary I am not aware of any discussion of the foundations of such approach nor of any recognition that it is a deviation from the standard formalism.

I can conclude that the three generalizations (and their combinations) I have explored in this section can play two different roles in the formulation of a theory of quantum gravity. First, definitions using Euclidean path integrals and even periodic boundary conditions can be seen as intermediate steps necessary to develop a final, Lorentzian quantum theory, and hence they represent some auxiliary structures that are expect to help us formulate the real quantum theory, just as in the case of quantum field theory many computations are performed in Euclideanized versions of the

[40] There are some technical details that I am leaving aside here such as why the product has to be of time-ordered operators or how to associate a function O to every operator $\hat{O}$ acting on the Hilbert space. Those technicalities are not important for my discussion here and can be found in many introductions to the path integral formalism such as Skinner (2017).

models. Second, we can take some objects like the 'Lorentzian partition function' and the associated 'expectation values' to define themselves a quantum theory. I have argued against this option, as it is unclear the way of interpreting such theory.

Finally, we can consider the application of these path integrals to the case of gravity. None of the generalizations considered in this chapter had nothing to do with general relativity, coordinates nor diffeormorphism invariance. In this sense, it is to be expected that the same conceptual problems that the diffeomorphism invariance of the theory produce in the canonical formalism and the more standard covariant formalism will also be present in these generalizations of the covariant formalism. Therefore, we find that the problem of time is not addressed by these approaches.

Let me conclude this chapter by giving an overview of my argument in it. I have introduced the covariant formulation procedure as a way of quantizing a classical theory which is essentially equivalent to the canonical formulation. I have also introduced some generalizations of this formalism, like the general boundary formalism and the boundary-less formalism, but I have argued that they do not really introduce any conceptual improvement. I have analyzed the way gauge symmetries and reparametrizations are treated in the covariant formalism and I have reached the same conclusion as in the canonical one: for theories like general relativity we find a problem of time, i.e., we are not able to define a quantum theory with a well-defined dynamics. I have discussed the proposed resolution of the problem which consists in interpreting the inner product in the physical Hilbert space as a propagator and I have rejected it on the grounds of three objections. In the next chapters I will discuss concrete theories of quantum gravity and I will extend my conclusion to these theories too. In this sense, this conclusion supports my argument in chapter 3 that in quantum gravity it is common to find some formalism for which no interpretation is provided, or just some disputable one, and that this makes it very hard, if not impossible, to discuss issues like a possible emergence of time in those theories.

8. SPIN FOAMS

In the previous chapter I introduced covariant quantization as an alternative to canonical quantization. In this approach one computes propagators and amplitudes by means of path integrals, i.e., by assigning to each possible history a weight which depends on the classical action and by summing the weights of all the possible histories. I argued that the covariant quantization of general relativity faces some technical and conceptual difficulties, which are the covariant version of the problem of time of the canonical quantization of the theory, that is, these problems arise because of the diffeomorphism invariance of the theory which means that we are considering the quantization of spacetime itself. In this chapter I will introduce spin foam models, by following the construction by Rovelli and Vidotto (2015). I will give some criticisms about how these models are built, and I will insist that the covariant problem of time needs to be addressed in order for the models to be considered acceptable candidates for theories of quantum gravity.

A spin foam is, roughly speaking, a spacetime version of a spin network[1]. Recall that spin networks can be represented by graphs embedded in space, even if the details of the embedding are irrelevant. Now we want to represent a spin network which changes in time, let's say between an initial and a final state at different times. The spacetime diagram we would obtain for such evolution is like the one we see on the right hand side of figure 8.1. We represent the temporal dimension in the vertical direction, and the spatial dimensions in the horizontal directions of the diagram. As the spin network evolves in time, as it moves upwards in the diagram, the links of the graph sweep surfaces which connect the links of the initial and final graphs. The geometric object formed by these surfaces is called 2-complex and the spin foam is the assignment of geometric information to it, in the same way that spin network states in LQG are defined by assigning quantum numbers to their links and edges.

A spin foam model is a covariant quantization which assigns a quantum amplitude to each spin foam just as for the non-relativistic particle we assigned an amplitude to each possible trajectory of the particle. There are different spin foam models that differ in the ways this assignment is performed, or in the types of spin foams considered. There are a couple of reasons for this. First, spin foam quantum gravity

[1] This view of spin foams is well-represented in Baez (1998, 2000).

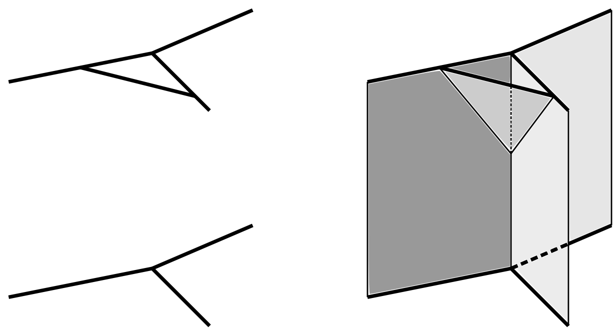

Fig. 8.1: The spin foam on the right represents a possible evolution that connects the two spin networks on the left.

is a work in progress, and during its history models have been modified in order to solve the different problems that the models presented. Second, spin foams arise in a few approaches to quantum gravity, each of them with its own perspective and peculiarities. The fact that different approaches to quantum gravity converge to spin foams is remarkable, but implies that there will be some variety of models and attitudes with respect to their physical meaning.

In this chapter I will mention the different models and the ways they relate with each other, but I will explicitly analyze the version of the EPRL spin foam model discussed in Rovelli and Vidotto (2015). Rovelli and Vidotto's approach to spin foams is related to discretizations of spacetime, i.e., the spin foams in their model are seen as representing a spacetime constituted by 'chunks'. In particular, their approach is related to truncations of general relativity like Regge calculus, which are theories in which the continuous and smooth spacetime of general relativity is triangulated, that is, replaced by a spacetime made of small, flat pieces. I will start in section 8.1 by introducing in more detail how spacetime can be approximated by a triangulation and how Regge calculus approximates general relativity. I will also comment on quantizations based on Regge calculus and I will briefly argue that they suffer from the covariant problem of time. Finally, I will also introduce the 2-complex dual to a Regge triangulation, which will allow to connect to the EPRL spin foam model.

In section 8.2 I will introduce the EPRL[2] spin foam model as introduced by Rovelli and Vidotto. The starting point for the quantization is a truncated version of a constrained BF model, which is a way of truncating general relativity and express it in terms of some variables which are similar to the connection variables I introduced in chapter 5 for the quantization of LQG. These variables are defined for the edges and

[2] The model is named after the four physicists who developed it in Engle et al. (2008a): Engle, Pereira, Rovelli and Livine.

faces of a 2-complex dual to a triangulation of spacetime and I introduce them and the truncated BF theory in more detail in subsection 8.2.1. Rovelli and Vidotto follow the general boundary formalism I introduced in section 7.5 for the quantization, and hence their model has two ingredients: a boundary Hilbert space and a generalized propagator. Given the variables chosen, the boundary Hilbert space is the LQG Hilbert space for the graph in the boundary of the 2-complex, as I discuss in 8.2.2. The definition of the propagator is not straightforward: first it is introduced the 3-dimensional version of it (8.2.3) and then it is extrapolated to the 4-dimensional case (8.2.4). In this section I raise some worries about the way this model is defined, I also mention some possible generalizations in subsection 8.2.5 but I leave the more philosophical discussion out of it.

It is in section 8.3 that I will deal with the more philosophical and interpretational issues of the model. There are three aspects that are I will consider. First, I will argue that the covariant version of the problem of time affects this model, which means that the propagator defined in the model cannot be interpreted as a propagator in a straightforward way and that the proposed interpretations are not satisfactory. Second, I consider the relation with canonical LQG and I argue that even if there are some heuristic arguments relating spin foam models and canonical LQG, in general we cannot say they are equivalent. This is specially true for the case of the model presented by Rovelli and Vidotto, which is directly related to a triangulation of spacetime, and, as I argued in chapter 5, the interpretation of LQG states as representing discrete spaces is questionable and not applicable to all the states of the theory. Finally, spin foam models are an interesting case to study intertheoretic relations and the way one could claim spacetime emerges from something more fundamental.

Finally, in section 8.4 I will analyze the applications of spin foam models to cosmology. First, I will analyze how the propagators of LQC can be cast in a form very similar to a spin foam expansion. That this form of the propagator is available is just a formal nicety which does not change the physical interpretation of the theory. Second, I will mention cosmological models which are based on spin foam models. These models have the same interpretational problems as any spin foam model, i.e. the covariant version of the problem of time, and hence they illustrate the shortcomings of the approach.

8.1 Simplicial manifolds and Regge calculus

In this section I will introduce Regge calculus. It is a discretization of general relativity which can be used to motivate the EPRL spinfoam model that I will be studying in detail in this chapter. Besides, I will also introduce two approaches to quantum gravity: quantum Regge calculus and dynamical triangulations and I will argue that

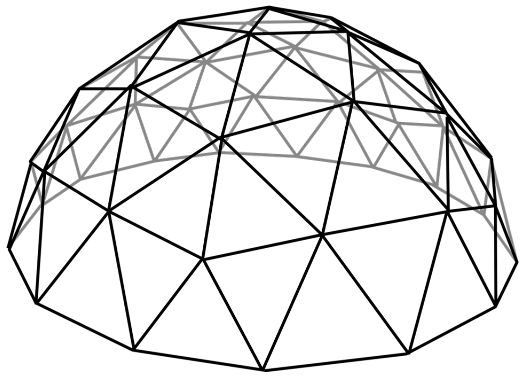

Fig. 8.2: A geodesic dome is a structure used in architecture to approximate spheres. We can see that domes are actually made of small, flat triangles.

they suffer from the conceptual shortcomings which are associated with the diffeomorphism invariance of general relativity and that I will argue that will also be present in the spin foam models. Finally, I will show that one can define a 2-complex dual to any Regge triangulation. It is precisely on those 2-complexes that the EPRL model I will study in this chapter is defined.

Regge calculus is based on a very simple idea: we can approximate any curved manifold by a set of small, flat pieces joined together. For instance, geodesic domes, like the one in figure 8.2, are used in architecture to approximate curved surfaces like spherical surfaces. Regge calculus is precisely the application of the very same idea to the spacetimes of general relativity, i.e., to approximate complicated curved spacetimes by a collection of flat and simple pieces.

Technically speaking, the pieces one uses in Regge calculus are called simplices and the collection of simplices is known as simplicial manifold. A simplex is the generalization of the triangle to any number of dimensions, and it is the simplest unit[3] with which it is possible to triangulate a manifold, that is, to divide it into regions. In the case of Regge calculus for 4-dimensional general relativity, this basic piece is called 4-simplex. If we were interested in studying general relativity in lower dimensions, we would be considering collections of (irregular) tetrahedra or triangles. Notice also that one chooses simplices because of their simplicity, but for the aim of approximating a manifold one could have chosen more complicated figures made of more points and faces.

Despite being made of flat pieces, simplicial manifolds achieve an effective curvature. This is possible because simplicial manifolds are not manifolds in a strict sense,

[3] More technically, the simplex is the polytope, the generalization of the polygon to any dimension, with lower number of vertices in any dimension.

as the places where different simplices are joined together are not smooth. Indeed, it is in the union points, called hinges, where the effective curvature lies. In language of differential geometry we could say that in these points the curvature tensor diverges. For instance, consider the case of a two-dimensional simplicial manifold like the one in figure 8.3. As usual in differential geometry, we can detect the curvature of a manifold by parallel transporting a vector along a closed circuit. If the manifold has curvature, the vector will at the end of the circuit be different from the vector at the beginning. In the case of the circuit in figure 8.3 we see that the simplicial manifold is effectively curved, as the vector we started with is different from the one with which we arrived. The angle between these two vectors is the deficit angle, and measures the effective curvature around the vertex. This idea can be generalized to more dimensions, and in general we will say that the curvature of a simplicial manifold is captured by a series of deficit angles around some appropriate hinges. In the case of 4-dimensional simplicial manifolds, the hinges are the faces of the triangles where 4-simplices meet.

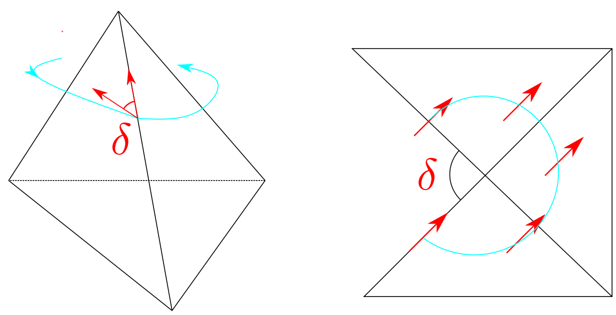

Fig. 8.3: Illustration of the concept of deficit angle for a 2-dimensional simplicial manifold, which consists in the four triangles that form a tetrahedron. If we start with a vector parallel to one of the edges of the tetrahedron and we parallel transport it around the vertex (blue line), the vector we end up with is no longer parallel to the edge and it differs by an angle δ. This angle is known as deficit angle. The same can be seen in the flat development on the right.

The geometry of a simplicial manifold, including all the deficit angles, is given by specifying the length of every segment of every simplex. But, in the same way that in general relativity we are not interested in any spacetime but just in the ones that satisfy Einstein equations, in Regge calculus we want to impose some dynamics that selects the relevant simplicial manifolds. Doing so, we will find the simplicial manifolds which approximate the right spacetimes. For this, we define the Regge action as:

$$S_{Regge} = \sum_h A_h(l_i)\delta_h(l_i)\,. \tag{8.1}$$

This action is just a sum over all hinges of the product of the area of the hinge times the deficit angle associated with this hinge[4]. Both the areas and the deficit angles can be expressed as functions of the lengths l_i of the segments that constitute the simplicial manifold. Regge action can be shown to approximate the Einstein-Hilbert action of general relativity:

$$S = \int d^4x \sqrt{-g} R \,. \tag{8.2}$$

An intuition of why is this so is given by the following. The curvature scalar R contains the information of how a vector which is parallel transported around a closed loop around a point in a manifold changes, exactly as depicted in figure 8.3 but for a smooth manifold. When we take the limit of a very small circuit around a point the deficit angle is proportional to the curvature scalar and the area of the circuit. Indeed, R is given by:

$$R = \lim_{A \to 0} \frac{\delta_A}{A} \,. \tag{8.3}$$

That is, R it is the quotient between the deficit angle and the area of the circuit in the limit of circuits of vanishing area A. In the simplicial manifold case, curvature is not smoothly distributed, but it instead lies in the hinges. Moreover, the deficit angle around a hinge is independent of the circuit chosen. This can be seen for the two-dimensional case in figure 8.3: no matter how small the circuit around the tip of the tetrahedron is, the deficit angle does not change. For this reason, when we take the limit in the definition of the curvature for a circuit around a hinge, the curvature diverges as the area gets smaller and smaller. We say that the curvature is distributional and given by $\delta_h \delta^2(x - x_h)$. The Dirac delta functions $\delta^2(x - x_h)$ represent the divergence of the curvature at the hinge, and there are two of them to match the dimensions of inverse area. When we integrate this curvature in the Einstein-Hilbert action 8.2 we find that the deficit angle is a constant of integration and that the integration of the delta function gives just the volume element of the $d-2$-dimensional hinge. In four dimensions this is just the area A_h of the triangle, which means that for each hinge the action is just $A_h \delta_h$. Summing for all hinges we find Regge action.

By varying the action with respect to the lengths we obtain the Regge equations:

$$\sum_h \frac{\partial A_h(l_i)}{\partial l_j} \delta_h(l_i) = 0 \,. \tag{8.4}$$

Notice that there is one equation per segment and that despite the sum being over every hinge, only the terms of the hinges which contain that segment will contribute.

[4] This is the expression for the 4-dimensional case. For a theory in dimesion d, the hinges are $d-2$-dimensional and A_h has to be replaced by the volume in $d-2$ dimensions. For instance, in three dimensions, in the action angles are multiplied by the lengths of the hinges, which are segments.

It can be shown[5] that simplicial manifolds that satisfy these equations approximate differential manifolds which satisfy Einstein equations. This is expected, as Regge action is a simplicial approximation of the action of general relativity.

Let me also remark that Regge calculus can be applied to approximate both Euclidean and Lorentzian manifolds. In the case of Lorentzian manifolds we will find that segments can be spacelike or timelike[6], which will mean that we will represent some of them as real numbers and some other ones as imaginary ones. Areas and angles can also be imaginary, but it can be shown that when the area of the hinge is imaginary, its associated deficit angle is imaginary too, and their product and hence the action remain real numbers[7]. Once we take this into account we can use Regge calculus to approximate the Lorentzian spacetimes of general relativity.

The intuitive idea that simplicial manifolds approximate differentiable manifolds and that Regge calculus approximates general relativity can be made more formal. Indeed, one can define limiting procedures which support the claim that general relativity is the continuum limit of Regge calculus. For instance, we can imagine a series of simplicial manifolds Δ_1, Δ_2, Δ_3... which are different approximations to the same differentiable manifold of increasing degree of fineness of triangulation, i.e., each triangulation is made of more and smaller pieces than its predecessor. Intuitively, as we take the limit of increasingly fine triangulations we get something closer to the differentiable manifold. Moreover, if we compute the Regge action for the elements in this series, its limit is precisely the Einstein-Hilbert action of the differentiable manifold.

The idea of continuum limit makes a connection with the discussion about levels, approximations and emergence in chapter 2. For instance, one can use Butterfield's definition of the term 'emergence' to claim that general relativity emerges as the continuum limit of Regge calculus, meaning that the properties of general relativistic models such as smoothness are novel and robust with respect to the properties of the simplicial manifolds of Regge calculus. But these two theories exemplify a more interesting relationship between theories, which is that of truncation. A truncation of a theory can be defined as a reduction or simplification of the degrees of freedom of a theory in order to build a simpler one. In this sense, Regge calculus is a truncation of general relativity as the infinite degrees of freedom of the latter are reduced to a countable number of degrees of freedom in the former. Some spin foam models are based on a truncation of general relativity, and hence I will come back to study this relation in section 8.3.3 where I will analyze more carefully the relations between spin

[5] I refer the reader to (Misner et al., 1973, Chapt. 42) for a discussion in detail of Regge calculus, including this point.

[6] Lightlike segments are more difficult to deal with and they are excluded from the formalism without affecting its ability to approximate general relativity.

[7] Again, I refer the reader to (Misner et al., 1973, Chapt. 42) for the technical details.

foam models and general relativity and the senses in which spacetime can be argued to emerge from spin foam models.

An important difference between Regge calculus and general relativity is that the diffeomorphism invariance of general relativity does not play a role in Regge calculus. This is because the theory is defined on the 'rigid' structure of the simplicial manifold which did not require any notion of coordinate or diffeomorphism to be defined. Of course we can introduce different coordinate systems in our simplicial manifold, and they will be related by diffeomorphism transformations, but the fact is that the physical variables which define the geometry of the simplicial manifold, i.e., the lengths of the edges of the triangulation are independent of any such coordinate system. The fact that diffeomorphism invariance does not play a role simplifies the model (for instance we did not have to discuss any gauge invariance in the action) and avoids the conceptual difficulties associated with this symmetry.

With this I conclude the introduction of classical Regge calculus. It has been used as the starting point for defining different theories of quantum gravity, such as quantum Regge calculus and dynamical triangulations[8]. Quantum Regge calculus is defined by means of path integrals on a fixed triangulation. The possible histories here correspond to the different legths the segments of the manifold could have. This defines path integrals of the form:

$$\int \prod_i \mathcal{D}l_i e^{\frac{i}{\hbar} S_{Regge}[l_i]} . \tag{8.5}$$

The idea of dynamical triangulations is the opposite: one fixes the length of all the segments to be a length scale a[9] and sums over all possible triangulations Δ_i:

$$\sum_i \frac{1}{C(\Delta_i)} e^{\frac{i}{\hbar} S_{Regge}(\Delta_i)} , \tag{8.6}$$

where $C(\Delta_i)$ is a factor which depends on the symmetry properties of the triangulation.

These approaches suffer from some of the difficulties that I argued in chapter 7 that are common in covariant quantizations of gravity. For instance, the dynamic triangulation models analyzed in Loll (1998) follow the boundary-less formalism I introduced in section 7.6 to define expectation values for triangulations of compact spaces. Other problem is that much of the work on these models is done in Euclidean

[8] See Roček and Williams (1981); Williams and Tuckey (1992); Williams (2009) for some references of the former and Loll (1998); Ambjørn et al. (2012); Loll (2020) for some of the latter.

[9] The spirit in the dynamical triangulation community is to take the value of a as a parameter to take to 0 for recovering a continuum limit in some appropriate manner, probably making use of renormalization group techniques. See Ambjørn et al. (2012).

versions of them, and it is not always clear how to translate them to the (hopefully) physically meaningful Lorentzian ones.

But, as I argued in the previous chapter, the biggest issue with covariant quantizations of general relativity comes from the way one deals with the diffeomorphism invariance of the theory, which makes it the case that one cannot apply the standard formalisms and interpretations of quantum mechanics to these models. In the case of Regge calculus and dynamical triangulations we can define 'propagators' of the form $K(final, initial)$, where $final$ and $initial$ represent the final and initial discrete geometries that one can consider in each model. These objects are as difficult to interpret as $K(g_{ab}, g'_{ab})$ in section 7.4. I will expand this criticism for the case of spin foam models in section 8.3.

A different quantum theory inspired by Regge calculus is the Ponzano-Regge model, which is a model for quantum gravity in three dimensions. This model is very similar to quantum Regge calculus with the only difference that now we restrict the possible lengths we are summing over to be half-integers. That is, length is quantized and given by a quantum spin number $j \in \{0, 1/2, 1, 3/2...\}$. When Ponzano and Regge built this model, this assumption was just a guess. Later on, in Rovelli (1993) it was shown that this quantization of length could be related with the quantization of length in LQG in 3 dimensions. This already points out to a relation between triangulations and spin foams.

Rovelli and Vidotto (2015) give two more reasons for moving from a theory defined in a simplicial manifold to one in terms of spin foams. The first one is that Regge calculus is based in lengths, which are metric variables, and as I explained in chapter 5, connection variables are arguably better fitted to include fermions in the theory. The second one is that lengths in Regge calculus cannot be chosen freely, as they have to satisfy triangular inequalities. These inequalities just represent the condition that in flat space one cannot have a triangle such that one of its sides is longer than the length of the other two together. This technical constraint makes the variables of Regge calculus inconvenient for quantization.

Given this motivation, let me explain how to get from simplicial manifolds or triangulations to the 2-complexes which will be the basis of our spin foam model. 2-complexes are related with simplicial manifolds by a relation of duality[10]. To build a 2-complex which is dual to a triangulation one chooses an internal point to every simplex, which we will call vertex. Now, if two simplices share a boundary we join the vertices associated with both of them with a segment, that we will call edge. Notice that the edges limit a series of surfaces, that we will call faces. The set of all

[10] This is not very different from the duality relation I introduced in section 5.1.5 between cellular decompositions and graphs. Indeed, simplicial triangulations are a kind of cellular decomposition and 2-complexes are the graphs dual to such decompositions together with the faces closed in between the links of the graph.

vertices, edges and faces forms the 2-complex. In figure 8.4, I represent this for the 3-dimensional case. In this case we have a tetrahedron associated with each vertex, a triangle of the tetrahedron associated with each edge and a side of the triangle associated with each face. Notice that every vertex has 4 edges, just because every tetrahedron is made of four triangles. We are interested in the 4-dimensional case, which unfortunately cannot be easily represented. In any case, the same rules apply and the 2-complex will consist on a set of vertices representing 4-simplices, a set of edges representing tetrahedra and a set of faces representing triangles. In this case, there are 5 edges per vertex, as there are 5 tetrahedra bounding each 4-simplex. The correspondence between elements of a triangulation and a 2-complex are summarized in table 8.1.

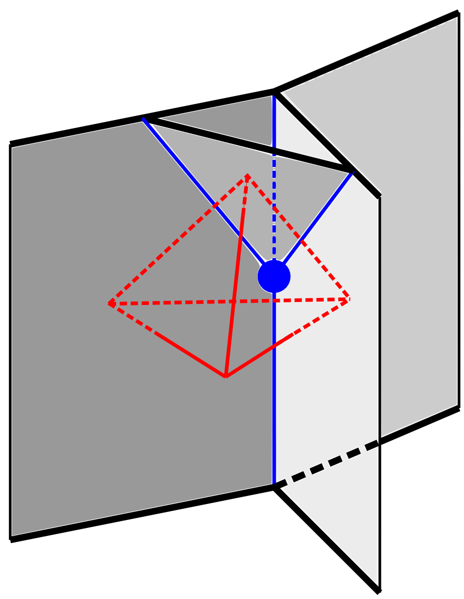

Fig. 8.4: Section of a 2-complex which is dual to a tetrahedron (in red). We see that in the center of the tetrahedron there is one vertex (blue dot) and that there are four edges (in blue) going out from it, which are dual to the four triangles of the tetrahedron. The 2-complex has 6 faces (in different shades of grey), which are dual to the 6 segments of the tetrahedron. Notice that each edge of the 2-complex intersects just one triangle and that each segment of the tetrahedron intersects just one face of the 2-complex.

As I said in the introduction, spin foams can be seen as the time evolution of a spin network. Similarly to spin networks, spin foams are diffeormorphism invariant in the sense that a diffeomorphism can move around a graph or a 2-complex, and it can deform those objects, but their basic properties are preserved. For the case of the spin network these correspond to the connectivity between nodes, the quantum numbers or colors and also the knotting class and some moduli. For the case of the spin foam diffeomorphisms preserve the relations between vertices, edges and faces and the quantum numbers we will assign to them. One can also define knotting classes and moduli for general 2-complexes, but for complexes dual to a triangulation there will not be any knot nor any moduli involved. I will consider more general spin foam models when considering generalizations of the EPRL model in section 8.2.5.

For either the formalism of quantum mechanics or the generalized boundary formalism we need to consider the boundary of the simplicial manifold. The boundary $\partial\Delta$ of a simplicial manifold Δ, if there is a boundary, is another simplicial manifold, connected or not[11], of one dimension less. For instance, a collection of joined tetrahedra is bounded by a collection of triangles and a 4-dimensional triangulation is bounded by tetrahedra. On this boundary, we can also construct a dual graph by assigning a point to each simplex of this boundary manifold and joining the adjacent ones. Here and in this chapter I will adopt the notation used in Rovelli and Vidotto (2015) and will call the vertices of such a graph nodes and the edges, links or external edges. Notice also that this graph is the boundary of the 2-complex. To see this, let me consider the three-dimensional case. Each tetrahedron is represented by a vertex with 4 edges, which connect the tetrahedron with the tetrahedra with which it shares a triangle. For tetrahedra in the boundary, there will be some triangles that are not shared by two tetrahedra. We can nevertheless associate an edge to these triangles. In this case, the edge does not connect two vertices, but just a vertex, i.e. the interior of a tetrahedron, with the triangle. The end point of this edge, that is, the intersection between the edge and the boundary of the triangulation is precisely the node. In a similar way, the links of this boundary graph are associated with faces of the 2-complex. This holds in any number of dimensions. Notice also that the number of links per node in the graph is also determined by the triangulation, just as the number of edges per vertex is. For instance, the boundary graph of a 3-dimensional triangulation is made of three-valent nodes, that is, by nodes that represent triangles and links that represent their sides.

With this, I have shown that one can associate a 2-complex to a given triangulation, and a graph to its boundary. This is important because it will allow to interpret the spin foam model of the next section as related with a triangulation of spacetime. But notice that this only holds for this particular case of spin foams. That is, not every 2-complex is dual to a simplicial manifold or to a triangulation of a spacetime.

[11] This remark is important for the case of the standard formulation of quantum mechanics, in which we have a manifold bounded by two regions that we can think of as the initial and final instants of time for a time evolution.

2-complex	Triangulation (3D)	Triangulation (4D)
Vertex	Tetrahedron	4-simplex
Edge	Triangle	Tetrahedron
Face	Segment	Triangle
Node (boundary)	Triangle	Tetrahedron
Link (boundary)	Segment	Triangle

Tab. 8.1: Correspondence between elements of a 2-complex and its boundary and dual triangulations in 3 and 4 dimensions.

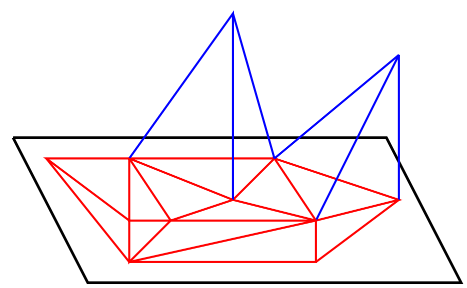

Fig. 8.5: A three-dimensional triangulation is made of tetrahedra (a couple are represented in blue). The boundary of this triangulation is a triangulation in 2 dimensions (in red).

This is analogous to the discussion in section 5.2.3 about not every graph being dual to a triangulation of space, which complicated the interpretation of some states of canonical LQG. I will leave the discussion of the possible generalization of the spin foam model to more general 2-complexes and their interpretation for section 8.2.5, after I have introduced the EPRL spin foam model in section 8.2.

In this section I have introduced Regge calculus as a truncation of general relativity. I have introduced some models of quantum gravity based on such a truncation and I have argued that they suffer from the covariant version of the problem of time. I have shown that associated to a Regge triangulation there is a 2-complex, which will be the starting point for defining the EPRL model in the next section.

8.2 EPRL spin foam model

In this section I will introduce the EPRL model as defined in Rovelli and Vidotto (2015). Roughly speaking, we can say that the relation of this model with quantum Regge calculus is like the relation of LQG with geometrodynamics: it is a quantization which is based on connection variables rather than on metric variables and at the end of the day this implies, according to its proponents, some spacetime discreteness. I will leave the philosophical comments and the interpretation of the theory for the next section, but I will raise three worries about the construction of the model. First, the way gauge is treated is not rigorous, and one wonders how much of the formalism and of the interpretation by Rovelli and Vidotto depends on this. Second, there is a limitation in the formalism which makes it the case that the triangles one considers, both in the boundary and the bulk of the manifold, are spacelike, for which there is no physical motivation. And third, the definition of the amplitude associated to each spin foam is not defined directly by means of a path integral for the case of the 4-dimensional theory. Instead, it is defined by means of an analogy with the Euclidean 3-dimensional case. Therefore, there are several steps in the construction which lead me to conclude that this model has to be taken carefully just as a tentative approach to quantum gravity which is subject to improvement.

This section is structured in five subsections. In the first one, 8.2.1, I introduce the classical model which is the basis for our quantization, which is nothing but general relativity expressed as a constrained BF theory and then truncated to just a few degrees of freedom associated with a triangulation and 2-complex. Then, I introduce the quantization of the model in the next three subsections. In subsection 8.2.2 I discuss the boundary Hilbert space of the theory, and I compare it with the Hilbert space of LQG. The propagator of the model is defined by analogy with the theory in 3 dimensions and for this reason I introduce the propagator for the 3-dimensional model in 8.2.3 before introducing the propagator of the 4-dimensional EPRL model in 8.2.4. Finally, in subsection 8.2.5 I explain how this model relates to others and how it can be generalized.

8.2.1 Classical model

The EPRL spin foam model is a quantization of a model similar to the Regge theory but instead of being based in metric variables, it is based in tetrad variables. Recall from section 5.1 that general relativity could be expressed in terms of a tetrad field e^I_μ and a connection ω^{IJ}. Both of them are one-forms which contain the two pieces of information of general relativity: e^I_μ describes the metric properties of the manifold and ω^{IJ}, the affine ones. In this formulation of the theory there are additional $SO(3,1)$ gauge degrees of freedom related with an internal rotation which does not change the physical content of e^I_μ and ω^{IJ}. As I explained in chapter 5, we can write an action for general relativity in these variables which takes the form of a Palatini action with

Holst term[12]

$$S[e,\omega] = \frac{1}{2}\int \epsilon_{IJKL} e^I \wedge e^J \wedge R^{KL}[\omega] + \frac{1}{\gamma}\int e_I \wedge e_J \wedge R^{IJ}[\omega]\,. \tag{8.7}$$

where $R^{IJ}[\omega]$ is the curvature of the connection, as defined in chapter 5. Variation with respect to e and ω leads to the Einstein equations and to ω being the Levi-Civita connection. The second term is the Holst term, which does not affect the equations of motion of the classical theory. We can rewrite this expression as:

$$S[B,\omega] = \int B_{IJ} \wedge R^{IJ}[\omega]\,, \tag{8.8}$$

where I have grouped all the factors containing e in the two form B:

$$B_{IJ} = \frac{1}{2}\epsilon_{IJKL} e^K \wedge e^L + \frac{1}{\gamma} e_I \wedge e_J = (\Sigma^*)_{IJ} + \frac{1}{\gamma}\Sigma_{IJ}\,. \tag{8.9}$$

In the last equation I have introduced the Plebanski 2-form $\Sigma^{IJ} = e^I \wedge e^J$ and its dual $(\Sigma^*)_{IJ} = \frac{1}{2}\epsilon_{IJKL} e^K \wedge e^L$. The model defined by the action 8.8 is known as the BF model.

We will take the two form B to be our basic variable instead of the tetrad variables e. This is allowed as long as we impose that B is related with e by the relation just given. In other words, when we express the theory in terms of B and ω we have to impose an additional constraint, called the simplicity constraint[13], which imposes the relation between B and e. This constraint is an additional technical complication to the theory, and I will argue that the way it is imposed in Rovelli and Vidotto (2015) is problematic.

Now we are in a position to introduce the variables that we will use in the discretized version of the theory. The definition of these variables is very similar to the definitions of fluxes and holonomies for the case of canonical LQG that I introduced in section 5.1.4. The greatest difference between the two situations is that now the variables we will be interested in and that will play a role will be defined for the different parts of the 2-complex dual to the triangulation, while the variables in section 5.1.4 were not constrained to a particular set of paths and surfaces. In particular, we define holonomies along the edges e of the 2-complex:

$$U_e[\omega] = \mathcal{P}\exp\int_e \omega\,. \tag{8.10}$$

[12] I am taking this action from Rovelli and Vidotto (2015). It differs from the expression 5.7 used in chapter 5 by an unessential multiplicative factor, the absence of the cosmological constant term and by the fact that the convention is chosen so that $8\pi G = 1$.

[13] The name of the constraint comes from the fact that B is constrained to a have a simple form. Notice also that this constraint would be different if we had not included the Holst term in the action. The different choices of simplicity constraints are discussed in Kamiński et al. (2010).

As in chapter 5, the holonomy is the path-ordered exponential of the connection, and hence an element of the Lorentz group $SO(3,1)$ or of its covering $SL(2,\mathbb{C})$. Intuitively we can think of this group element as containing the information of how the tetrad field rotates along an edge, that is, when translated from one 4-simplex to its neighbour. Now we can take the holonomies of all the edges that close a face f of the 2-complex and multiply them in order to get U_f, which is the group element corresponding to a closed circuit around a triangle. U_f is analogous to the deficit angle in Regge calculus, as it captures how the tetrad field changes when parallel transported around a triangle, i.e., around the hinge which has the effective curvature.

The other set of basic variables are associated with the faces of the 2-complex and correspond to the flux of the 2-form B along the triangle t_f dual to a face f:

$$B_f^{IJ} = \int_{t_f} B^{IJ} \, . \tag{8.11}$$

This flux is a bivector associated with the triangle t and can be seen as element of the algebra $\mathfrak{so}(3,1)$ or, equivalently as an element of $\mathfrak{sl}(2,\mathbb{C})$. In the same way that the flux in canonical LQG is related with the area of a surface, this flux is also related with the area of the triangle. However, while in section 5.1.4 we specified this relation in a precise way by equation 5.36 which defined the area of any surface as a limit of an expression containing fluxes, if we follow the presentation in Rovelli and Vidotto (2015) the relation between fluxes and areas is only valid for a given gauge and by assuming that the triangles are part of a simplicial decomposition, i.e., that the metric can be defined to be flat in them. In this sense, we can see the definition and interpretation in Rovelli and Vidotto (2015) just as tentative or trying to give a more intuitive picture[14].

For a spacelike triangle we can choose a coordinate system such that it lies in a spatial hypersurface of constant time coordinate. Now, assuming that the triangle is part of a simplicial decomposition we can fix a gauge such that $e^I_\mu = \delta^I_\mu$. In this case the spatial components of B_f are:

$$B_f^{ij} = \frac{1}{\gamma} \int_t dx^i \wedge dx^j \, . \tag{8.12}$$

This is (proportional to) the three-dimensional area bivector associated with the triangle. The geometric meaning of this bivector is more transparent when expressed

[14] Rovelli and Vidotto are aware of this limitation in their presentation, as can be seen in two footnotes in Rovelli and Vidotto (2015): footnote 2 of chapter 4 and footnote 2 of chapter 7. However, they do not discuss this point in detail and stick to the definition and interpretation I am giving.

in terms of its hodge dual:

$$L_t^i = \frac{1}{2}\epsilon^i_{jk} B_f^{jk} = \frac{1}{2\gamma}\epsilon^i_{jk}\int_t dx^j \wedge dx^k \,. \tag{8.13}$$

This vector can be shown to have the following properties: it is orthogonal to the triangle t and its norm is given by:

$$|L_t| = \frac{1}{\gamma} A_t \,. \tag{8.14}$$

Where A_t is the area of the triangle. The temporal components B_f^{0i} of the flux on the triangle also define a vector $\vec{K}_t$ which turns out to be proportional to $\vec{L}_t$ with the Barbero-Immirzi constant as the ratio between the two, i.e., $\vec{K}_t = \gamma \vec{L}_t$. This is just the form that the simplicity constraint takes for the discrete fluxes. Therefore, we see that the algebra element B_f contains two times the information about the area of the triangle when expressed in the correct gauge. This is represented in figure 8.6.

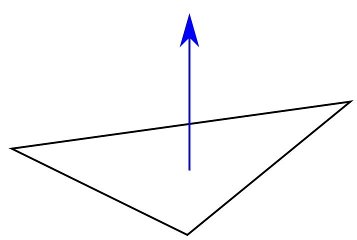

Fig. 8.6: Triangle and vector normal to it. In the truncation of general relativity we are defining on a 2-complex dual to a triangulation of 4-dimensional spacetimes the fluxes associated with each face B_f, when defined in the right gauge, contain the information about this vector for the corresponding triangle in the triangulation.

An important point to make, and that is missing in the discussion by Rovelli and Vidotto, is that in any triangulation like the ones we are interested in there could be triangles with timelike components. For example, in a Minkowski spacetime we can think of a triangle contained in the $x - t$ plane. In this case we can perform an analysis similar to the one for the spacelike triangle and define a flux B_f for this triangle. Before, we chose a frame and distinguished between the timelike and spacelike components of B_f, which defined the vectors $\vec{L}_t$ and $\vec{K}_t$. In the same way, we now choose a direction (given by the one-form n_I) orthogonal to the triangle and define the vectors $K_t^I = n_J B_f^{JI}$ and $L_t^I = n_J (B_f^*)^{JI}$. They are orthogonal to n, and hence we can write them as vectors in a three-dimensional space. Given the simplicity constraint they are proportional to each other $\vec{K}_t = \gamma \vec{L}_t$. Let us take L_t^i, which is analogous to the vector we defined for the spacelike triangle. The difference now is that the space in which the vector is defined is not Euclidean, as there is a timelike direction. For this reason, the symmetry group for the internal indices i is no longer SU(2), closely related to group of rotations in 3 dimensions SO(3), but the Lorentz group for 1 timelike and 2 spacelike directions, SO(2,1), or equivalently SU(1,1). Given the Lorentzian metric, and the sign convention used, the norm of the vector L_t is imaginary. Just as one would expect given the experience with Lorentzian Regge calculus, the areas of the triangles with a timelike direction are imaginary. The fact that Rovelli and Vidotto do not discuss this point when introducing the B variables will be important, as we will see that in both the quantum kinematics and dynamics some of their arguments will rely on reducing SL(2,$\mathbb{C}$) variables to SU(2) ones, in a similar way to the way in which we have gone from B_f to L_t^i in the case of the spacelike triangle. The timelike triangle shows how the reduction is sometimes needed to be to a different group like SO(2,1) or SU(1,1), which has certain properties, like non-compactness, which could be problematic. Hence, we will see that the model by Rovelli and Vidotto is limited to spacelike triangles and tetrahedra, even if it is not clearly stated in this way[15].

The holonomies U_e and fluxes B_f are the two sets of variables that will 'colour' the 2-complex. We can define an action for a given 2-complex in terms of these variables:

$$S[U_e, B_f] = \sum_f \mathrm{tr}(B_f U_f)\,. \tag{8.15}$$

Notice the similarity of this action with Regge action 8.1. In both cases we have a sum over the hinges, that is, over the triangles of the triangulation or over the faces of the 2-complex. For each triangle in the Regge sum we had a term that was a product of the area of the hinge and its associated deficit angle, which captures the curvature. Now we have analogous products of B_f, which is related with the area of the triangle, with U_f, which is related with the curvature. Hence, one would be right in calling this action the Regge action in tetrad variables. Indeed, in (Engle and Pereira, 2008, Sect. 3.3) it is shown that in the regime of low curvature, or equivalently, in the regime of fine triangulation, both actions are equivalent. Moreover, in the same section it is also shown that this action is a truncation of the BF action 8.8 and in the following section that the classical equations of motion associated with this action approximate the tetradic Einstein equations.

[15] This limitation is not explicitly stated when constructing the model (Rovelli and Vidotto, 2015, Chapt. 7) but when applications of the model are discussed this limitation seems to be present. For instance, only spacelike boundaries are considered in several passages of chapters 8, 10 and 12 of Rovelli and Vidotto (2015).

8.2.2 Boundary Hilbert space

Now we can turn to the quantum version of the theory. In this subsection I will be dealing with the definition of the boundary Hilbert space of the EPRL model. I will first introduce the boundary Hilbert space as defined in Rovelli and Vidotto (2015), which turns out to be the Hilbert space of LQG for a fixed graph. Then, I will argue that there are some problems with this definition. First, as in the case of LQG, the gauge group of the model is reduced from $\mathrm{SL}(2,\mathbb{C})$ to $\mathrm{SU}(2)$, and one can raise the worry that this reduction is unjustified and doubt about the predictions that follow from it, namely the discreteness of areas and volumes. Second, the way the simplicity constraint is imposed seems to apply only for spacelike surfaces, while in the generalized boundary formalism the surfaces considered could also be timelike. Finally, I will comment on the way in which states in the Hilbert space represent a geometry and on the way these states are interpreted and the differences with canonical LQG.

In the generalized boundary formalism followed by Rovelli and Vidotto, as well as in the standard formalism of quantum mechanics, we have a Hilbert space associated with a boundary of a spacetime region. In the case of a 2-complex dual to a triangulation, this boundary is given by a graph. The classical variables we have defined for the bulk of the 2-complex get translated into the graph. First, each link l of the graph can be seen as an external edge of the 2-complex, and hence we can define an holonomy U_l for each link. Second, each link is the boundary of a face of the 2-complex, therefore we can associate with each link the flux corresponding to the face it bounds: $B_l = B_f$. This means that the complete classical description of the boundary is given by the L couples (U_l, B_l) associated with the L links of the boundary graph. In more technical terms, we can associate this with the graph phase space $\left(\mathrm{SL}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})\right)^L = T^*\mathrm{SL}(2,\mathbb{C})^L$. This phase space is the same graph phase space we introduced in section 5.1.5 but for a larger group, as in the case of LQG the group was $\mathrm{SU}(2)$.

Given the similarity of this space with the canonical LQG phase space it is no surprise that its quantization leads to the Hilbert space $L^2[\mathrm{SL}(2,\mathbb{C})^L]$, that is, the space of square integrable functions of L copies of $\mathrm{SL}(2,\mathbb{C})$. A state in this space is a spin network state for the boundary graph. This Hilbert space is a kinematical Hilbert space, to which one imposes the simplicity and Gauss constraints. Notice that the Hamiltonian constraint is not imposed on the basis that it defines the dynamics and because of all the conceptual difficulties that are associated to it, but a case could be made arguing the opposite. The reason for this is that the Hamiltonian and momentum constraints in general relativity are four of the Einstein equations, and not requiring them to be satisfied by the initial and final states seems similar to not imposing the Gauss constraint to the electromagnetic field. In any case, Rovelli and

Vidotto proceed without considering this constraint for the boundary Hilbert space and only impose the other two constraints.

The simplicity constraint is imposed by Rovelli and Vidotto in the following way. First, we can construct the Hilbert space $L^2[SL(2,\mathbb{C})]$ from unitary irreducible representations of $SL(2,\mathbb{C})$. These representations are labeled by the numbers p, k and define the vector spaces $V^{p,k}$. Recall that the group $SL(2,\mathbb{C})$ is essentially the Lorentz group and can be generated from 6 algebra elements which correspond to 3 rotations, L^i, and 3 boosts K^i. Notice that the rotations generate a SU(2) subgroup, but that the distinction between rotations and boosts depend on a choice of frame. Given the group $SL(2,\mathbb{C})$, we can arbitrarily choose a frame and select a preferred SU(2) subgroup. Now, we can further decompose the vector spaces $V^{p,k}$ into representations of this arbitrarily chosen $SU(2)$ subgroup: $V^{p,k} = \bigoplus_j V_j^{p,k}$. In other words, a basis of this space is formed by vectors given by $|p,k,j,m\rangle$, where j, m are the familiar SU(2) quantum numbers. Given this, one can define the boost $\hat{K}^i$ and angular momentum $\hat{L}^i$ operators on these states. Having defined this Hilbert space and these operators, Rovelli and Vidotto impose the condition $\hat{K}^i = \gamma \hat{L}^i$ on these states and conclude that only states of the form $|\gamma j, j, j, m\rangle$ satisfy the constraint[16]. These states form a SU(2) subspace, and hence we can define a map Y_γ which relates vectors in representations of SU(2) with vectors in representations of SL(2,(C)) which satisfy the constraint:

$$\begin{aligned} Y_\gamma : V_j &\to V_j^{p,k} \\ |j,m\rangle &\to |\gamma j, j, j, m\rangle \,. \end{aligned}$$

This map transforms the original Hilbert space, $L^2[\mathrm{SL}(2,\mathbb{C})^L]$, to the familiar Hilbert space of LQG, $L^2[\mathrm{SU}(2)^L]$.

The next constraint we have to impose is the SL(2,$\mathbb{C}$) gauge invariance, that is, that states are invariant under a rotation of the tetrad. When one imposes the simplicity constraint in the boundary in the way of (Rovelli and Vidotto, 2015, Sect. 7.2.1), one is choosing a preferred frame for the boundary. For this preferred frame, the SL(2,$\mathbb{C}$) degrees of freedom were reduced to SU(2) degrees of freedom, which implies that the gauge group is also reduced to SU(2) gauge transformations at the nodes of the graph[17]. Hence, the constraint reduces to the same Gauss constraint that we imposed in section 5.2.1 for the case of canonical LQG. The constraint has a nice geometrical interpretation, as it can be written as:

$$\vec{L}_{l_1} + \vec{L}_{l_2} + \vec{L}_{l_3} + \vec{L}_{l_4} = 0\,. \tag{8.16}$$

[16] I refer to (Rovelli and Vidotto, 2015, Sect. 7.2) for the details in this. The way in which the constraint is satisfied is not exact, but just holds approximately for large values of j.

[17] That gauge transformations are limited to transformations at the nodes is true for the holonomy variables. The fluxes as we have defined them do not transform in such a nice way. However, Rovelli and Vidotto restrict the gauge transformations to transformations acting on the nodes.

Where $\vec{L}_{l_i}$ are the vectors dual, just as in 8.13, to the $\mathfrak{su}(2)$ algebra elements B_{l_i}[18] which correspond to the four links that share a node. As I said above, for the definition of Rovelli and Vidotto and in the right gauge, these vectors can be interpreted as vectors tangent to the manifold in the directions orthogonal to the triangles and with a norm proportional to their area. The equation 8.16 implies that the four triangles associated with the four links that share a node close, i.e., that they form a tetrahedron. For this reason this constraint is also known as the closure constraint. In figure 8.7 I illustrate this point for our three-dimensional case and also for the two-dimensional one. Finally, notice that for a general gauge, $\vec{L}_{l_i}$ are not spacetime vectors, but vectors in the internal gauge space. The closure constraint can be read as saying that no matter how we rotate these vectors in this space we have to do it in a way such that they always close.

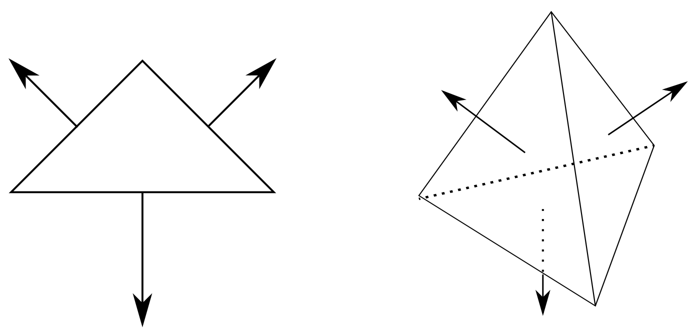

Fig. 8.7: Illustration of the closure constraint for a triangle and a tetrahedron. Each side/face is described by a vector orthogonal to it and of length proportional to the length of the side or to the area of the face. The fact that the three sides close to form a triangle and that the four faces close to form a tetrahedron implies that the sum of the vectors has to be zero, as intuitive from the picture. This is generalizable to any number of dimensions.

The Hilbert space resulting from imposing the N closure constraints associated with the N nodes of the graph is $L^2[\mathrm{SU}(2)^L/\mathrm{SU}(2)^N]$. A basis of this space is given by: $|\Gamma, j_l, v_n\rangle$. This is just the same as in the canonical LQG case: we have a quantum state which is associated with a graph and it carries a quantum number j_l associated with each of its links and another one v_n associated with each node. This last quantum number reflects the intertwiner degrees of freedom in the nodes. The physical interpretation we can make of these states is similar to the one of a state in canonical LQG. The way of doing this is by noticing that as the vectors $\vec{L}_t$ carry geometric

[18] Of course by B_{l_i} I mean the $\mathfrak{su}(2)$ restriction of the B_{l_i} defined before, that were elements of $\mathfrak{sl}(2, \mathbb{C})$

information, the corresponding quantum operators will be the ones with which one can build geometric observables. In this sense, the vectors $\vec{L}_t$ are analogous to the flux of canonical LQG. First, the quantum number associated to each link gives the area of the triangle dual to it by the relation:

$$A_t = 8\pi G\hbar\gamma\sqrt{j_l(j_l+1)}\,. \tag{8.17}$$

This is the same expression we found in the canonical LQG case. Similarly, the quantum number v_n is the eigenvalue of the volume operator. The details of the definition of such operators is given in (Rovelli and Vidotto, 2015, Sect. 7.2.2). As I said, they are fundamentally the same we obtained in section 5.2.2, the only difference being that now these operators are defined for particular surfaces (the triangles) and regions (the tetrahedra) and not for arbitrary ones.

We therefore recover one of the main kinematical results of LQG: that areas and volumes are discrete in the sense of only taking values from a discrete set. As we commented in chapter 5, the root of this discreteness lies in the fact that SU(2) is a compact group, and compact groups lead to discrete spectra, as happens in the case of angular momentum. I commented in section 5.2.4 that one could doubt about the discreteness result if one argues that spin network states should be constructed using the larger and non-compact group SL(2,$\mathbb{C}$). In that case the argument was based on the fact that when building canonical LQG there is a moment in which one fixes the gauge and reduces the original gauge group to SU(2). A similar argument can be rehearsed in the model I am analyzing in this section, and one could criticise that the way the simplicity constraint was imposed was too strong, and that our quantum theory should reflect the SL(2,$\mathbb{C}$) gauge symmetry that it intuitively has.

Indeed, there is a further worry that we may have related with this. In the previous subsection I have explained how triangles with a timelike direction could be described by elements of $\mathfrak{so}(2,1)$ and that the convention mandates that their area is given by an imaginary number. However, the Hilbert space I have introduced is based on $SU(2)$ variables and the spectrum of the area operator is real. Therefore it seems that the boundary geometry described by this Hilbert space can only be spacelike. This is not a problem for the situation in which we have a process bounded by two spacelike boundaries, but it is a problem for the generalized boundary formalism for closed boundaries. The idea of introducing the generalized propagator was precisely to generalize the idea of propagator to generic boundaries, which generically include timelike regions. In other words, there is a tension between the idea we had for the boundary before defining the quantum theory and the geometry we have obtained for it.

The origin of this problem is the way the simplicity constraint is imposed in (Rovelli and Vidotto, 2015, Sect. 7.2). As I said before, the simplicity constraint, in the form we found for the triangles, $\vec{K} = \gamma \vec{L}$, is imposed as an operator condition $\hat{K}^i = \gamma \hat{L}^i$. However, I will now argue that the identification of these vectors and operators is only valid for spacelike triangles. When I discussed the meaning of the B variables I introduced the vectors $\vec{K}$ and $\vec{L}$ and indeed found that the simplicity constraint meant that they are proportional to each other. However, when I discussed the case of having a triangle with a timelike direction, I argued that these vectors had to be understood as $\mathfrak{so}(2,1)$ vectors rather than $\mathfrak{so}(3)$ or $\mathfrak{su}(2)$ vectors. For this reason, it seems wrong to identify the operator $\hat{L}$, associated with a rotation algebra $\mathfrak{su}(2)$, with the vector $\vec{L}$ we have defined before, which could be associated with that algebra but only in the case that the triangle is spacelike. Therefore, the way the constraint is defined and implemented for the operators $\hat{K}$, $\hat{L}$ does not seem justified for boundaries with timelike components, which explains why Rovelli and Vidotto's boundary Hilbert space does not reflect the intuitions we had for it. For this reason, it seems reasonable to restrict the application of this Hilbert space to just spacelike boundaries. Rovelli and Vidotto do not comment on this point when introducing the boundary Hilbert space in (Rovelli and Vidotto, 2015, Sect. 7.2), but then when they discuss applications of their model they restrict themselves to spacelike boundaries. In subsection 8.2.5 I will discuss some possible generalization of this spin foam model to include timelike triangles.

Leaving aside these criticisms, let me also mention that in this Hilbert space we can study the geometry of each of the tetrahedra on its own. For each tetrahedron we have associated a vertex with four edges, so we can take its 'quantum geometry' to be given by a state $|j_1 j_2 j_3 j_4 v\rangle$, that is by the four areas of its faces and its volume. Notice that in order to define a tetrahedron one needs to specify 6 numbers, for instance one can give the lengths of its six edges or some of these lengths and some angles. The quantum state gives only 5 numbers, and therefore does not specify uniquely the shape of the tetrahedron. In (Rovelli and Vidotto, 2015, Sect. 1.3.1) this case is studied, and it is concluded that a quantum tetrahedron has a 'fuzzy' geometry, meaning that not all its classical properties are sharply defined. This case is compared with the case of angular momentum or spin: a quantum state representing these quantities cannot describe all their components sharply. For instance, one can have a quantum state which describes an angular momentum by giving its norm $|L|$ and its z component, but not its x and y components. In this sense, quantum states for angular momentum are always fuzzy, as there is at least some component that cannot be given sharply. Rovelli and Vidotto argue that in the case of the geometry of the tetrahedron we are in the same situation, as we can only specify 5 quantum numbers of the 6 that would be necessary for fully specifying the geometry of the tetrahedron. When in chapter 9 I introduce group field theory we will also find states that correspond to quantum tetrahedra.

Can we export this lesson to the quantum states defined on the network? That is, is the geometry they define equally fuzzy? The passage I have just mentioned is from an introductory chapter from Rovelli and Vidotto (2015), and in there they conclude that quantum states of geometry, in general, will be fuzzy just as the state of the tetrahedron is. When they later on introduce the boundary Hilbert space (Rovelli and Vidotto, 2015, Sect. 7.2.3) they insist on assuring that geometry is fuzzy for these states. However, their argument for a single tetrahedron does not seem to extend for the case of a boundary state. The reason is that tetrahedra in the boundary of a triangulation are not alone and they share faces. Therefore, the 6 edges of a tetrahedron are also edges of other tetrahedra, and hence their lengths can also be related with the areas and volumes of such tetrahedra. For instance, consider the case of two tetrahedra that share a face. In this case, for fully determining the geometry of both tetrahedra we would need 9 numbers, which correspond to the 6 edges of the first tetrahedron and the 3 edges of the second which are not shared with the first one. In this case we would have 9 quantum numbers: the 5 quantum numbers of the first tetrahedron plus three quantum numbers of the faces of the second tetrahedron that are not shared and the volume of the second tetrahedron. In this case it seems that we have enough information for fully reconstructing the geometry of both tetrahedra. Therefore, it seems that for the boundary states there is not this problem of underdetermination.

Indeed, we could be facing the opposite problem, that of overdetermination. For example, consider the simplest triangulation possible of a 4-dimensional spacetime: just one 4-simplex. The boundary of this triangulation is the boundary of the 4 simplex which consists in 5 tetrahedra (see figure 8.8). These 5 tetrahedra share many of their edges and faces: they are made by just 10 faces and 10 edges. In this case the geometry of the boundary would be given by determining the 10 lengths of the edges. However, if we count the quantum numbers associated with the boundary graph, we find that there are 15 of them, corresponding to the volumes of the 5 tetrahedra and the areas of the 10 faces. In this case, there will be assignments of quantum numbers that will not correspond to a boundary of a 4-simplex. To avoid this, one should impose some consistency conditions. For a generic triangulation this will also be true: not every assignment of quantum numbers to a graph will correspond to a well-defined Regge geometry. In Freidel and Speziale (2010) the term 'twisted geometry' was introduced for referring to the 'geometries' which are defined by assignments of quantum numbers which do not correspond to Regge geometries. Twisted geometries may play a role in some approaches based in LQG, but if we want to restrict ourselves to triangulations of spacetime we will need to impose some consistency conditions. In the model we are studying, i.e., the version of the EPRL model presented by Rovelli and Vidotto one does not impose any such constraint on the states considered in the boundary Hilbert space.

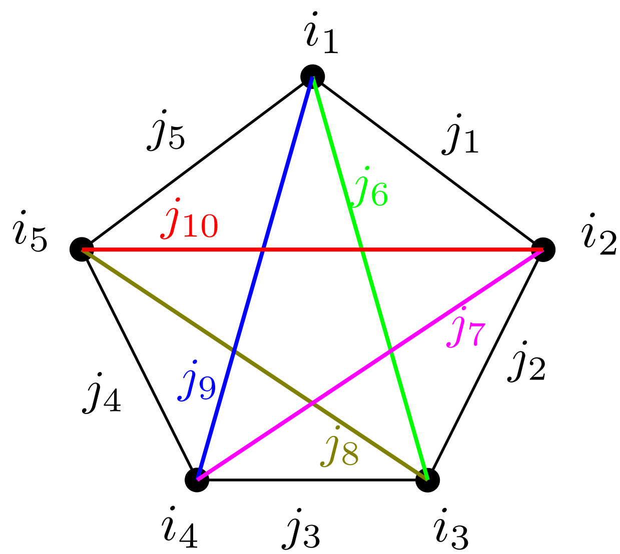

Fig. 8.8: . Boundary graph for a 4-simplex. Each node represents one of the 5 tetrahedra that bound a 4-simplex. The links of the graph represent the shared faces between tetrahedra. There are 15 quantum numbers associated with this graph: 10 spins j, one per link, which represent the area of the shared faces and 5 intertwiners, one per node, which represent the volume of each tetrahedron.

According to what I have just discussed, a state in the $|\Gamma, j_l, v_n\rangle$ basis which satisfies such consistency conditions describes precisely a simplicial geometry for the boundary, contrary to the fuzzy picture given by Rovelli and Vidotto. Notice however, that as the areas and volumes are quantized, they cannot take arbitrary values. In particular, there are minimal values that they can take, which implies that one cannot take boundaries to be constituted by a collection of arbitrarily small pieces. This is in contrast to the classical picture we started with and with the case in quantum Regge calculus: in the classical picture we thought of Regge calculus as an arbitrarily fine truncation which allowed us to approximate a differentiable manifold, and in quantum Regge calculus there was not any constraint in the boundary geometry. The difference between both quantizations is somewhat analogous to the difference between the quantization of position and of angular momentum in standard quantum mechanics: while angular momentum is constrained to have a value from a discrete set, with a minimal non-zero value, position is allowed to take any real value. The quantization carried out in this section is therefore more like the quantization of angular momentum, while the one of quantum Regge calculus is more similar to the one of position[19]. Recall that in the classical case we could recover general relativity by taking the limit of finer and finer triangulations. In this case, we will have to be careful when taking such a limit, as there is a limit to how small triangles and tetrahedra can be. I will comment more on the continuum limit of the theory in section 8.3 after defining its dynamics.

Finally, there is a last point which is important to raise, which is the question of how literally should one interpret these boundary states. An option is to interpret these states literally as saying that the geometry of the boundary is that of a simplicial manifold. For instance, if we consider the case in which the boundary consists in two disconnected, initial and final parts, the literal interpretation would be to say that space, as represented by the boundary states for two instants, consists in these tetrahedra joined together. We therefore get the picture of space being made of chunks of space. However, Rovelli and Vidotto advocate for a not too literal interpretation. That is, they claim that quanta of space should not be taken as chunks of space but as 'modes of interaction'. By this, what they mean is that if there is some physical interaction whose effect is proportional to an area, the fact that area is quantized will be reflected in this interaction. The reason they give for this is that in general one has to be careful when interpreting a quantum state. There are a couple of remarks that can be made to this position. First, it is true that the interpretation of quantum mechanical states in a straightforward way may be misleading, but there are interpretations of quantum mechanics in which one can directly translate from some features of the quantum state to physical properties that are instantiated in the world. This is arguably the case of the Everett interpretation and for Bohmian mechanics. Second, it is not clear what the talk of modes of interaction tries to achieve. It seems a trivial statement to claim that the physical properties of a system show up by their interaction with other systems. For the case of space, we have the further complication of the debate between substantivalist and relationalist positions about the nature of spacetime. But it seems that both the substantivalist and the relationalist would agree in that the discreteness of space would be reflected in the interactions between physical systems.

It is interesting to compare the interpretation of the Hilbert spaces of canonical LQG and the spin foam model in this section, which formally are the same[20]. Recall from the discussion in chapter 4 that the more straightforward way of interpreting states in LQG is as equivalence classes of distributional geometries. It was only by

[19] Indeed, the same comparison can be made between quantum geometrodynamics and canonical LQG.

[20] Let me remark that even if the area operator in the graph Hilbert space is the same, the way it is arrived to is different, as in LQG it is derived by means of a careful definition in terms of limits while in the spin foam model it is directly defined as proportional to the norm of some components of the flux in some gauge.

restricting the types of graphs and by making some interpretative assumptions that one could associate states to cellular decompositions of space. The boundary Hilbert space defined in this section is proposed to be interpreted in this second way, for which there is a gap with the pure canonical LQG and its interpretation. I will say more about the relation of spin foams and LQG in section 8.3.

Before that, there is the second piece of the EPRL model that we need to introduce: the propagator. As in any covariant approach, we will express the propagator as a sum over histories, but the route we will take to define it is not straightforward. Indeed, the way transition amplitudes are defined in Rovelli and Vidotto (2015) is not by directly postulating a sum over all histories weighed by the action 8.15, but by studying the 3-dimensional case and establishing the analogy with the 4-dimensional case, which is the one of our interest. The reason for this is that it allows for a definition of well-defined and finite transition amplitude, while a more direct definition of the transition amplitude faces some divergences. For this reason, I will introduce in the next subsection the definition of the transition amplitudes for the 3-dimensional case and then generalize them to the 4-dimensional one in the following subsection.

8.2.3 Transition amplitudes: 3D

The 3-dimensional EPRL model introduced in Rovelli and Vidotto (2015) is a model defined on a triangulation of an Euclidean manifold and its basic variables are holonomies U_e associated with the edges, and algebra elements L_f which give the length of the segments of the triangulation, just like B gives areas of triangles in 4 dimensions. We can define the propagator by:

$$K(U_l) = \mathcal{N} \int dU_e \int dL_f e^{\frac{i}{\hbar} \sum_f \mathrm{tr}(L_f U_f)} \,. \tag{8.18}$$

This is a sum over all the possible colourings of the 2-complex, that is, over all the possible values that the holonomies and fluxes can take. The factor $\mathcal{N}$ is just an appropriate normalization factor. The weight of each path is given by the exponential of the action, which is the three-dimensional analogue of 8.15. Notice that we are integrating over the two sets of variables even though they are conjugate variables like position and momenta in quantum mechanics. The reason for this is that, as I explained in section 5.1.1, in the first order formalism for gravity we treat conjugate variables as if they were independent. Alternatively, one can think of the path integral just defined to be something like a phase space path integral of the kind I introduced in section 7.2.

There is a close relation between the model studied here and lattice gauge theory which is the restriction of gauge theories like Yang-Mills to a lattice. Indeed, the connection formulation of general relativity that is used in canonical LQG and the models studied in this chapter is formally that of a gauge theory, as I showed in chapter 5. It is not surprising then that the truncation general relativity to a given lattice is similar to lattice gauge theory. I will discuss the importance of this analogy in section 8.3, but I bring it forward now because of the following. In lattice gauge theories, transition amplitudes are defined as sums over all possible assignments of holonomies to the lattice. This is done without including any gauge fixing term or using any of the Faddeev-Popov techniques we introduced in chapter 7 to deal with the gauge symmetry of the theory. Recall that the reason for introducing the gauge fixing terms was to avoid counting more than once the gauge equivalent stories, which can lead to the wrong quantum theory and to divergent path integrals. But in the case of lattice gauge theories it can be shown that it is not the case[21]. One fact that makes this possible is that integrations are over group variables for compact groups, which implies that the integration over gauge degrees of freedom is finite.

In the model defined by 8.18, we have not introduced any gauge fixing term either. A justification for this can be based in the analogy with the lattice gauge theory case and the fact that the group considered, SU(2), is compact. But notice that there is a difference between our simplicial model and lattice gauge theory, and it is that in the simplicial model we are also integrating over the conjugate degrees of freedom, i.e. the algebra elements L_f, which could in principle contain gauge degrees of freedom and make some of the integrals diverge. In Rovelli and Vidotto (2015) the propagator 8.18 is introduced without any discussion about gauge fixing. This point is relevant for having a rigorously defined path integral and for having a better justification for the propagators defined, both in the 3 and 4 dimensional cases. Unfortunately, as happened in the case of the definition of the flux variables, Rovelli and Vidotto do not pay much attention to this issue. Notice also that in the 4-dimensional case there are two differences which makes it more complicated. First, the gauge group is $SL(2, \mathbb{C})$, which is not compact and may lead to divergences. And second, in the 4-dimensional case we would have to deal with one constraint more, the simplicity constraint.

Having said this, we can take expression 8.18 and write it in a way that will be useful for the theory in 4 dimensions. First, the integration over L_f gives just a delta function for U_f[22].

$$K(U_l) = \mathcal{N} \int dU_e \int dL_f e^{\frac{i}{\hbar} \sum_f \operatorname{tr}(L_f U_f)} = \mathcal{N}' \int dU_e \prod_f \delta(U_f) \,. \tag{8.19}$$

Second, we will introduce a set of more convenient variables, which are represented in figure 8.9. We can split each holonomy in two parts: $U_e = g_{ve} g_{ev'}$. That is, for an

[21] I refer the reader to Creutz (2015) and the references therein.

[22] This is just the group analogous of other representations of the delta function which we have found in other places in this thesis. For instance, in section 7.2 I used that a delta function for a constraint could be written as an exponential. That is: $\delta(\phi) = \int d\lambda e^{i\lambda\phi}$.

edge e joining two vertices v and v' we can choose a point P[23] in between the vertices and define g_{ve} and $g_{ev'}$ to be the holonomies from v to P and from P to v'. The product of these two holonomies gives the holonomy for the full edge. We can replace the integration over U_e with two integrations over g_{ve} and $g_{v'e}$[24]. This introduction of an extra degree of freedom is like an introduction of a gauge variable, and it is again thanks to the compactness of $SU(2)$ that it is possible.

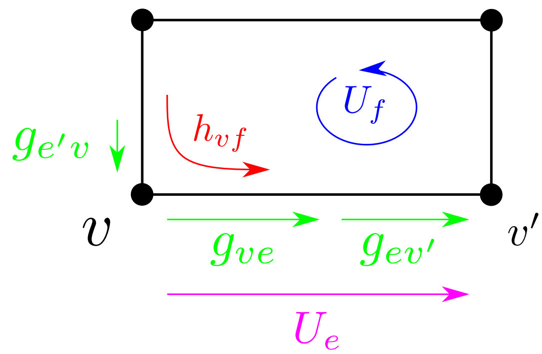

Fig. 8.9: Representation of the relevant variables for the path integral.

Now that we have split the holonomies we can group them in a different way. For each face and each vertex on this face we can define $h_{vf} = g_{ev}g_{ve'}$, i.e., the holonomy representing the parallel transport from a point P in the edge e to the vertex v and then to the point P' in the edge e'. In other words, by introducing h_{vf} we can think of the holonomies as being associated to the vertices. In this sense, can write the holonomy around a face, U_f, as a product of the holonomies for all the vertices: $U_f = \prod_{v\in f} h_{vf} = h_f$. I have introduced the notation $h_f = U_f$ for the holonomy around the face just for maintaining the same notation as Rovelli and Vidotto (2015). For introducing integrations on the h_{vf} variables one also needs to introduce a series of corresponding Dirac deltas, which can be arranged in a convenient way. Once this is done the propagator takes the form:

$$K(U_l) = \mathcal{N}' \int dh_{vf} \prod_f \delta(h_f) \prod_v A(h_{vf}) \,, \tag{8.20}$$

[23] Here I am following (Rovelli and Vidotto, 2015, Sect. 5.4.2) where nothing is said about the choice of P, and hence it can be read as a free choice. In other works like Engle et al. (2008b) the point P is chosen to be in the triangle dual to the edge and the holonomy $g_v e$ gets the interpretation of describing the parallel transport from the center v of a tetrahedron to one of the triangles that bound it.

[24] The holonomy $g_{v'e}$ is just the inverse of $g_{ev'}$. As $g_{ev'}$ represents the holonomy from P to v', $g_{v'e}$ is just the holonomy for the inverse path, i.e., from v' to P.

where we have introduced the vertex amplitude $A(h_{vf})$ which is a function of all the holonomies associated with the vertex v. As in three dimensions there are 4 edges per vertex, there are 6 faces that share a vertex, and therefore A is a function of 6 holonomies. In particular, it takes the form:

$$A(h_{vf}) = \int dg_{ve} \prod_{f|v\in f} \delta(g_{e'v} g_{ve} h_{vf})\,. \tag{8.21}$$

For each vertex consists in 4 group integrals, one per edge, of six delta functions, each of them associated with one of the faces. The integration gives just some deltas, which imposes some relations between the $h_v f$ that share a vertex. Importantly, it can be shown that the last integration is trivial, as the result of performing three integrations is independent of the fourth variable. Given the compactness of $SU(2)$ this last integral does not affect the result and we can even rewrite the vertex amplitude as:

$$A(h_{vf}) = \int d' g_{ve} \prod_{f|v\in f} \delta(g_{e'v} g_{ve} h_{vf})\,, \tag{8.22}$$

where the prime in the measure means that we will perform just three of the four integrals and ignore the last one. This redefinition can be done without further problem here as the function defined is the same.

The vertex amplitude A has a further property which will be relevant for us. Given the $SU(2)$ invariance of the integration measure and the delta function it can be shown that A is invariant under $SU(2)^4$ transformations, which intuitively correspond to the $SU(2)$ gauge transformations that one can associate with each edge. In this sense the vertex amplitude is not only a function on $SU(2)^6$, the space of functions ordered sets of six elements of $SU(2)$, but also a function on $SU(2)^6/SU(2)^4$, the subspace of $SU(2)^4$ invariant functions. This space should sound familiar to the reader, as it is the familiar spin network Hilbert space[25] for a graph with 6 links and 4 nodes. Indeed, we can represent this graph by drawing a small spherical surface around a vertex in the 2-complex (see figure 8.10). The intersection of the 4 edges with this surface gives the 4 nodes and the intersection of the faces with the surface gives the links between these nodes. We can think of the vertex amplitude as a function for this spin network. A thing one can notice is that the graph has the same topology as a tetrahedron, but it is important to notice that this graph is not the tetrahedron of the triangulation (indeed it is in a sense dual to it).

[25] Notice that it is mathematically a Hilbert space, but we are not associating to it any of the properties or interpretations that are done to the Hilbert spaces of quantum theories. That is, we are not identifying this Hilbert space with any wavefunction or with any measurement.

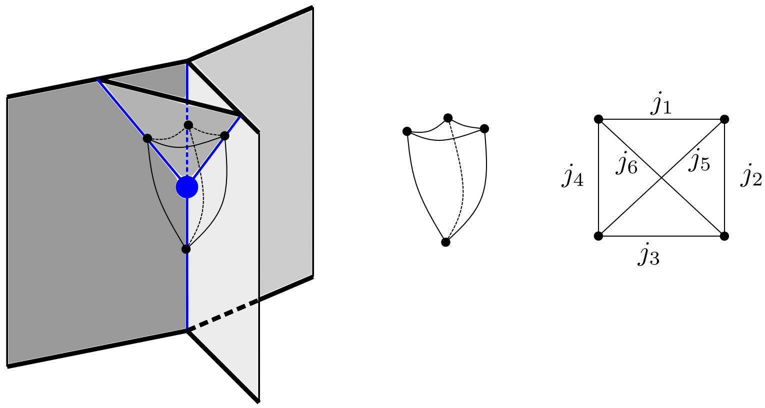

Fig. 8.10: Representations of the spin network associated with the vertex amplitude. Given a spinfoam vertex we can imagine we draw a small suface around it. The intersection of this surface with the 2-complex is a graph: the nodes of the graph are the intersection with the edges and the links, the intersection with the faces. To these links we can associate the holonomies h_{vf} or the representations j_f. The graph in three dimensions has the connectivity of a tetrahedron.

Finally, it will be useful to express the delta function as an expansion in the representations:

$$A(h_{vf}) = \sum_{j_f} \int d' g_{ve} \prod_{f|v \in f} (2j_f + 1) \mathrm{tr}_{j_f} [g_{e'v} g_{ve} h_{vf}] \,, \tag{8.23}$$

where tr_{j_f} represents the trace of a group element in the representation j_f. This expression for the vertex amplitude will be our starting point for the four dimensional case. Finally, the propagator we have found (8.20) can be expressed in terms of representations instead of group variables and it takes the form:

$$K(j_l) = \mathcal{N}_\Delta \sum_{j_f} \prod_f (-1)^{j_f} d_{j_f} \prod_v (-1)^{J_v} \{6j\}_v \,. \tag{8.24}$$

This is a sum over spin[26] assignments to faces, J_v is the sum of all the spins on a vertex and by $\{6j\}_v$ I represent the $6j$-symbol, which is a function of the 6 spins associated with a vertex, that is, with the representations of $SU(2)$ assigned to the 6 faces that meet at a vertex. The explicit form of this function can be found in (Rovelli and Vidotto, 2015, Sect. 5.3). The normalization factor $\mathcal{N}_\Delta$ is dependent on the triangulation. This form of the amplitude is worth mentioning for several

[26] Recall that by spin we just mean a representation j of $SU(2)$. That is, a positive half-integer.

reasons. First, this is precisely the same amplitude that was found by Ponzano and Regge by constraining quantum Regge calculus to just half spin lengths and therefore the relation with Regge calculus is explicit. In other words, we have found that the propagator is just a sum over possible (discrete) length assignments to the edges of the triangulation. Second, expressed in this form the propagator is explicitly gauge invariant, as it represents sums over gauge invariant assignments of spin to the faces of the 2-complex. Third, this expression is useful for considering the 'classical limit' of the theory, $\hbar \to 0$, which for the EPRL model is equivalent to the limit $j \to \infty$, as I will discuss in section 8.3.3. Finally, we will find a similar expression again in chapter 9, when I will study the relation between spin foam models and group field theories.

8.2.4 Transition amplitudes: 4D

Now we are in a position to generalize the model from 3 to 4 dimensions. Rovelli and Vidotto directly define the propagator by analogy with the 3-dimensional case as:

$$K(U_l) = \mathcal{N} \int_{SU(2)} dh_{vf} \prod_f \delta(h_f) \prod_v A(h_{vf})\,. \tag{8.25}$$

The variables we are integrating over in the propagator are holonomies associated with each vertex and face, just as in the 3 dimensional case. As we now are in four dimensions, and in a Lorentzian setting, we would expect the holonomies to be elements of SL(2,$\mathbb{C}$), which is the relevant gauge group here. However, the holonomies are defined to be SU(2) variables. The reason for this[27] is that one is imposing the simplicity constraint in the same way it is imposed in the boundary Hilbert space, and hence the degrees of freedom of the theory are reduced to SU(2) degrees of freedom. Notice that this is a difference with the 3-dimensional case, where there is no simplicity constraint. There is another difference with the 3-dimensional model considered above and it is that that model was a model for an Euclidean manifold, while now we are interested in a Lorentzian manifold. Above I have argued that the way the simplicity constraint is implemented by Rovelli and Vidotto does not seem to work for triangles with timelike directions, and the same criticism seems to apply here. Therefore, the choice of variables of the EPLR model we are studying here is in tension with what they were intended to represent: a generic triangulation of a Lorentzian manifold. With this choice in play only the triangulations formed by spacelike triangles would be allowed[28].

[27] This reason is not explicitly stated by Rovelli and Vidotto when presenting the propagator but can be found in other presentations of the EPRL model like (Perez, 2013, Sect. 8.2).

[28] Despite that at first sight the claim that one can triangulate a Lorentzian manifold using just spacelike traingles may seem counter-intuitive, it is a true one.

The vertex amplitude is defined to be:

$$A(h_{vf}) = \sum_{j_f} \int_{SL(2,\mathbb{C})} d' g_{ve} \prod_{f|v\in f} (2j_f+1)\mathrm{tr}_{j_f}[Y_\gamma^\dagger g_{e'v} g_{ve} Y_\gamma h_{vf}] \,. \tag{8.26}$$

This expression is analogous to the three dimensional case, and now there are 5 edges per vertex and 10 faces associated to each vertex, and the vertex amplitude is a function of the 10 holonomies associated to each vertex. Notice that the vertex amplitude is defined by an integration of 4 SL(2,ℂ) variables g_{ve} which correspond to variables associated to 4 out of the 5 edges at the vertex, as indicated by the prime, in a similar way to the notation in three dimensions, where we could choose to perform 3 out of the 4 integrations. In this case, this definition is necessary in order to avoid divergences. The form of this vertex amplitude can be justified by analogy with the three dimensional case, which would justify the expression in terms of the SL(2,ℂ) variables. Indeed, in the expression above appears Y_γ, the map which maps vectors in representations of SU(2) into vectors in representations of SL(2,ℂ). This allows to relate the two different kinds of variables. Notice also that in the same way that the vertex amplitude in three dimensions was a function in $L^2[SU(2)^6/SU(2)^4]$ and could be represented by a spin network with the connectivity of a tetrahedron, the vertex amplitude in four dimensions is a function in $L^2[SU(2)^{10}/SU(2)^5]$ and can be represented by a graph with the same connectivity of a 4-simplex, just like the graph in figure 8.8.

Expressions 8.25 and 8.26 define the EPRL model introduced by Rovelli and Vidotto. As for the 3-dimensional case, we can also rewrite these expressions in terms of representation variables instead of group variables by expanding in terms of representations and performing the group integrals to get[29]:

$$K(j_l) = \mathcal{N}_\Delta \sum_{j_f} \sum_{i_e} \prod_f (1+\gamma^2) j_f^2 \prod_v A(j_f, i_e) \,. \tag{8.27}$$

Notice that the expansion in terms of representations is a sum over representations j_f associated with the faces of the 2-complex but also over intertwiners i_e associated with the edges. The vertex amplitude in this expression is a function of 10 spins j_f and 5 intertwiners i_v. This is what we would expect, as the vertex amplitude is a function in $L^2[SU(2)^{10}/SU(2)^5]$ and a basis for this space is given by the spin network basis labeled precisely by these 15 numbers. This expression is in terms of the same variables that we used for the boundary Hilbert space: representations j that describe areas of triangles and intertwiners i that represent the volume of the tetrahedra.

For the boundary Hilbert space I have made the remark that not every possible assignment of quantum numbers to the graph dual to a triangulation defines a Regge

[29] This expression is taken from Perez (2013).

geometry, that is, that they define in general a 'twisted geometry' which is a more general structure than a simplicial manifold. For the bulk variables we are in the same situation, we are assigning quantum numbers to triangulations of spacetime, but in general they will define a twisted geometry for spacetime. If one believed that we should be summing just over standard geometries and not over twisted ones then one should restrict the sum over j's and i's so that twisted geometries are excluded from the sum. In Rovelli and Vidotto (2015) this restriction is not made, but is argued (Sect. 8.3.1) that the sum is dominated by the terms which represent standard geometries, specially in the classical limit.

With this I conclude the presentation of the EPRL model as described by Rovelli and Vidotto. As we have seen this model is defined on a triangulation and for each assignment of boundary data, i.e., for each specification of a geometry for the boundary, the generalized propagator defined in the model gives a quantum amplitude. This quantum amplitude is defined by means of a sum over all possible internal assignments of variables compatible with the boundary. There are some unsatisfactory aspects of this construction. First, as I have mentioned, the triangulations considered are not the more general possible, as we are leaving out the possibility of having time-like triangles. Second, the way gauge is treated is not rigorous. The interpretation of the variables B^{IJ} seems only valid in a particular gauge, and a discussion about how gauge is treated in the path integral is missing. This is particularly important for the 4-dimensional case, where the gauge group is non-compact and one can expect a greater relevance of this issue. And third, the way the EPRL path integral for the 4-dimensional case was defined is also not rigorous, as it is simply defined by making an analogy with the 3-dimensional case. One would have expected it to be defined from a path integral weighted by an exponential of the action 8.8, and the relation of the proposed model and the classical theory therefore remains unclear.

8.2.5 Generalizations and relations with other models

In this last subsection I will mention some possible generalizations of the EPRL model before moving to its interpretation in the following section. This will give a more general picture of spin foam models that is not limited to just one model which as we have seen is a work in process and about which one can raise reasonable doubts.

The first generalization one can make is to extend the model to any 2-complex, that is, not only for the ones dual to a triangulation. Some work in this direction is presented in Kamiński et al. (2010) and Ding et al. (2011). The definition of the model for a generalized 2-complex starts from the same action 8.15 as before and one gets to an expression like 8.27 but with a generalized vertex amplitude that accounts for the fact that now vertices are not restricted to be 5-valent. That is, the spin foam is defined by taking an arbitrary 2-complex, i.e., a set of vertices, edges and faces and assigning spins j to each face and intertwiners i to each edge. The amplitude for each such spin foam will be given by an expression similar to 8.27.

There are two geometrical interpretations available for such generalizations. First, in Ding et al. (2011) it is argued that the generalization they carry out just means that the 2-complexes they work with are dual to more general cellular decompositions of spacetime. That is, that now one does not have a space decomposed into tetrahedra and a spacetime decomposed into 4-simplices, but instead the decomposition is made in terms of polyhedra of arbitrary number of faces. This is contrast to Kamiński et al. (2010), where the generalization performed allows for spinfoam histories with knotting and links. Recall from section 5.2 that when we considered spin networks with knots and links, their interpretation as dual to a cellular decomposition was more difficult. In this sense, the proposal of Kamiński et al. (2010) allows to recover more of the structure present in the canonical formalism of LQG, and, as happened then, the geometric interpretation is not so straightforward. In the case of the canonical approach I argued that preserving knots and links would be the more conservative option which would not throw away part of the structure we had arrived to by following the quantization procedure. In the foam case it seems that one could chose either of the two attitudes by following similar arguments as the ones discussed then.

In the context of the EPRL spin foam model we find a novel argument for the position that claims that knotting degrees of freedom and the linking of the graphs and foams is irrelevant. In Bahr (2011) it is shown that, using a vertex amplitude which generalizes the EPRL vertex amplitude for vertices of all valences, the amplitude assigned to a spinfoam is the same that one would assign to a knotted version or a version with a different linking of the graph. The conclusion defended in this paper is that these degrees of freedom present in the kinematical structure of LQG are irrelevant once one moves to the dynamics of the theory. However, it is also explained that one could modify the EPRL vertex amplitude so that it becomes sensitive to such degrees of freedom. Therefore, we can conclude that the spin foam model can be defined for all kinds of 2-complexes, but that the spin foam models defined so far are insensitive to knotting and links.

The second generalization one can make to the model is to consider different vertex amplitudes besides the EPRL vertex amplitude. The other most prominent spin foam model is the Freidel-Krasnov model or FK model. This model was first formulated in Freidel and Krasnov (2008) and is similar to EPRL in that it is a covariant quantization based on the BF action 8.8, but it differs in the way this quantization is carried out. The resulting model is similar to the EPRL model but with a different function vertex amplitude A_v. However, there is a range of values of the Barbero-Immirzi parameter γ for which the two vertex amplitudes agree. Therefore, the details about how to interpret the EPRL spin foam model will also apply to the FK model, and the discussion in this chapter will remain of wide applicability, despite being focused on the EPRL model.

In the last section I have argued that the the EPRL model as it is introduced by Rovelli and Vidotto only seems to consider a triangulation which only contains spacelike triangles and tetrahedra. This could be considered a limitation of the approach, which only gets worse if we consider that some results, such as the discreteness of the area and volume spectra depend on the compactness of the gauge group SU(2) associated with the fact that the triangle is spacelike. Therefore, generalizing the triangulation to allow it to contain both spacelike and timelike triangles is worth considering. In Conrady and Hnybida (2010) it is argued that this generalization is possible and that one obtains that the area of timelike triangles also have a discrete spectrum. In this case the reason for this result is that the simplicity constraint limits the allowed representations of the relevant group SU(1,1) to form just a discrete set and not a continuous one.

Another consideration to make is that the EPRL model is divergent, i.e., the sum we have defined in equation 8.27 is generically infinite. The reason for this is that we allow the variables of the sum to take arbitrarily large quantum numbers. This is in a sense equivalent to the divergences which occur typically in QFT, when one has integrals in internal variables which are allowed to take arbitrarily big and arbitrarily small values which lead to two kind of divergences: infrared divergences and ultraviolet divergences. In the case of the EPRL model we only have infrared divergences, as the quantization of spin implies that there is no problem for small spins. A way of solving this is by slightly modifying the theory by changing the gauge group by a 'quantum' gauge group which is just a group regulated by a parameter q. With this modification, the propagator becomes finite. Moreover, the introduction of this modified group adds a new term to the theory which acts like a cosmological constant term. This modified model is inspired in the Turaev-Viro model in three dimensions Turaev and Viro (1992) and was formulated in Fairbairn and Meusburger (2012).

Finally, it also worth mentioning that the EPRL spin foam model is also inspired by the Barret-Crane model, which can be considered as the first spin foam model (Barrett and Crane, 1998). This first model had some shortcomings such as not having an appropriate classical limit, but it served for opening the way to the more modern EPRL and FK models.

The generalizations mentioned in this subsection can be considered just variations in a same family of models. However, in the next section I will argue that the EPRL model suffers from the interpretational difficulties that affect any covariant approach to the quantization of gravity, and all the variants consider in this subsection are similarly affected by them.

8.3 Interpretation

In the last section I have introduced in detail the version of EPRL spin foam model described by Rovelli and Vidotto in their book Rovelli and Vidotto (2015). Essentially, the model provides a definition of a generalized propagator for a discrete structure (the triangulation or the spinfoam) that, with some caveats, we can associate to something like a discretized version of a spacetime. In this section I analyze the interpretation that can be done of this model. In particular, I analyze three aspects. First, in subsection 8.3.1 I argue that, according to the analysis I performed in chapter 7, the EPRL spin foam model, as any covariant quantization suffers from a covariant version of the problem of time, that is, one defines some transition amplitudes or propagators which cannot be interpreted as the propagator of standard quantum mechanics and, in my opinion, we are missing a satisfactory interpretation for them. I will notice that for models defined on lattices one can avoid this at the price of giving physical meaning to the lattice, but I will argue against this possibility, in agreement with the attitude of the quantum gravity community.

Second, in subsection 8.3.2 I will analyze the relation between spin foam models like the EPRL spin foam model and canonical LQG as I introduced it in chapter 5. I will argue that there are some heuristic arguments supporting the equivalence of the two of them but not a rigorous proof. Moreover, I will argue that the equivalence seems more likely if we consider general 2-complexes and not just the ones which are dual to a triangulation. The idea of a triangulation and quantum gravity describing spacetime as made of chunks may be appealing, but the fact remains that there is an important gap between that interpretation and the LQG formalism. I will also argue that the connection with the LQG formalism is more dubious for the case we consider generalized boundaries, as the canonical formalism is better associated with transitions between two states. Finally, I will argue that if we insist in the equivalence with the canonical formalism, my argument in subsection 8.3.1 becomes stronger, as then the formalism can also be directly criticized on the grounds of the canonical problem of time and my arguments in chapters 4 and 5 directly apply.

Finally, in subsection 8.3.3 I analyze the ways one can define classical and continuum limits for these models. Even if I have considered the EPRL to be insatisfactory, it is interesting to consider the ways it is supposed to be related with general relativity. For instance, if the theory were successful, this kind of discussion is what one would need to consider for analyzing the issues discussed in chapters 2 and 3. Moreover, the discussion of the model by Rovelli and Vidotto is interesting also because it considers the EPRL model to be not a complete theory of quantum gravity but an approximation to it.

8.3.1 Interpreting the propagator: the covariant problem of time

In section 7.4 I argued that the diffeomorphism invariance of general relativity makes it the case that the 'propagators' one would define by following the covariant quantization methods leads to a covariant version of the problem of time. That is, in the same way that in chapter 4 I argued that it was wrong to take the 'propagator' defined by canonical means for the double harmonic oscillator or for general relativity as a meaningful propagator which can be used for evolving a wavefunction in time or be given an operational probabilistic interpretation, the same applies to the covariantly defined 'propagators' for general relativity. In this subsection I will argue that my analysis applies to the EPRL model.

The EPRL model defines a propagator of the form $K(s_1, s_2)$ or $K(s)$ depending on whether we are using the standard formalism of quantum mechanics or the generalized boundary formalism. In section 7.5 I argued against this second approach and I will not insist on my objections here but I will just insist in that the criticisms to the meaning and interpretation of the propagator of the standard form also apply to the generalized propagator, and even more strongly, as the formalism is one step further away from the formalism of quantum mechanics, which makes its interpretation more difficult. In Rovelli and Vidotto (2015) it is argued that the triangulation should not play any role and hence that the propagator has to be understood as a function just of the geometry on the boundaries and not of the triangulation. I will for now assume the same, but I will later comment on the possibility of considering a triangulation-dependent propagator.

My first argument in section 7.4 came directly from comparing the covariant and the canonical quantizations and considering that it is wrong to interpret an inner product in a physical Hilbert space as a propagator if there is no variable in this physical Hilbert space that plays the role of time. Here we can insist in that there is no variable neither in the configuration space of general relativity nor in its truncation that plays that role and that our intuitions from the attempt of canonical quantization are still valid. However, I will leave this aside and come back to the relation with the canonical formalism in the next subsection.

My second argument was that the 'propagator' defined is not strictly speaking a propagator. That is, one cannot use the propagator to define the evolution of a wavefunction. In the case of deparametrizable models one could identify one variable as time and then, if some technical conditions[30] apply, one could interpret the propagator as a representation of the time evolution operator which allows to evolve with respect to the time variable the quantum state for the rest of variables. For non-deparametrizable theories I argued that this is not applicable, and in particular I argued the same for the case of general relativity. In the case of the EPRL model the

[30] For instance, that the evolution defined is unitary.

same applies, as none of its variables can play that role of time. The 'propagator' defined by the EPRL model cannot be used for defining a wavefunction evolving in time, which makes the covariant quantization not translatable to the standard structures and interpretations of quantum mechanics.

Finally, my third argument was directed against interpretations like the one by Rovelli which is also argued for in Rovelli and Vidotto (2015). In this view, the meaning of the 'propagator' is that it defines some probabilities. For instance, it describes the probability of measuring a final geometry given that an initial one was observed. I argued that the interpretation of these probabilities is, at best, unclear. As the standard formalism of quantum mechanics does not apply, we cannot understand these probabilities in the way we understand them using some of the standard realist interpretations of quantum mechanics. Therefore, the most natural way of understanding these probabilities is in an operational way, but even in this sense they are not interpreted easily. For instance, how to measure a geometry in the boundary seems completely mysterious if there is no physical or operational way of defining the boundary, which is just formally defined by some arbitrary coordinates. In the EPRL model this criticism also applies and it is completely unclear the physical or operational meaning that one could give to the probabilities one could define from objects like $K(s_1, s_2)$.

Therefore, I conclude that the covariant version of the problem of time affects the EPRL model, and spin foam models in generality, just as it affects other covariant approaches to quantum gravity. In this sense, the definition of a consistent and interpretable dynamics remains a challenge also for spin foam models. Let me complement this section by studying the possibility that we take the triangulation to be something physical and not an artifact of the theory. In the case of spin foam models this possibility is not considered, and there are good grounds for this, as I will show next and in section 8.3.3.

If we take the triangulation to be fixed and meaningful we could avoid the covariant problem of time in the following way[31]. Imagine a triangulation of spacetime which is layered like the one in figure 8.11, that is, imagine a triangulation for which we can define a series of layers or instants of time. In this case one could define a wavefunction $\psi(j, i, t)$ for such a structure, i.e., we could define a wavefunction which defines a probability for the geometric observables, here represented by j and i, for each layer of the triangulation, labeled by the parameter t. Therefore, we would recover quantum mechanics as usual, with the difference that the time parameter would be discrete and not continuous.

However, one can raise a worry against this kind of approach. For the approach to make sense we need the lattice to be considered a physical background structure which

[31] This strategy is applicable also for other models like quantum Regge calculus.

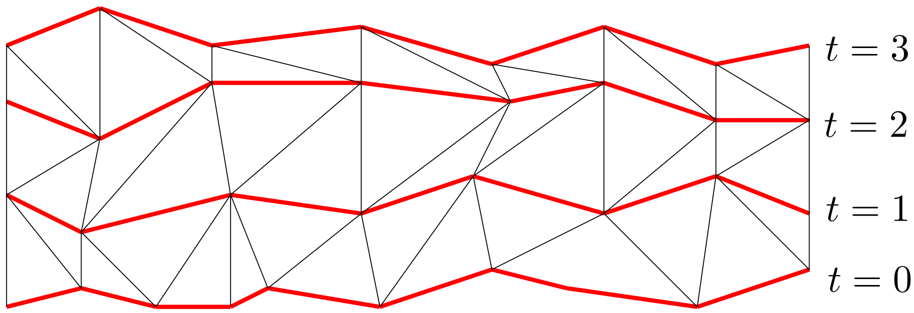

Fig. 8.11: For some triangulations one can define a 'time' parameter which specifies a layer of the triangulation. In principle, we could define a wavefunction which evolves with respect to this parameter.

defines, at least, the time parameter t between the different layers. This assumption is very disputable, to say the least. To begin with, one could raise the worry that the parameter t does not seem to have any physical meaning. The parameter t is in a sense very similar to a time coordinate in a general relativistic manifold, as for us it is only a label for the layers of an arbitrary triangulation of a manifold, which has as little physical meaning as an arbitrary time parameter labelling a foliation of spacetime. Fixing a triangulation and defining an evolution with respect to the parameter time is equivalent to fixing a foliation of spacetime, and every reason we had to reject that move seems to be applicable in this case. A different way of putting this is that we would be avoiding the problems originated by the diffeomorphism invariance of general relativity precisely by killing this invariance and choosing a preferred (even if discrete) set of coordinates.

8.3.2 Relation with canonical LQG

In this subsection I will analyze the relation of spin foam models like the EPRL model with the canonical LQG. In the literature it is argued that spin foam models are the covariant version of LQG. I will argue that this relation is generally just heuristic, i.e., that it the equivalence seems reasonable but it is not fully proven[32]. I will argue that this equivalence holds better if one considers general 2-complexes and not just the ones dual to a triangulation, which makes it the case that the intuitive picture of the EPRL model by Rovelli and Vidotto is less likely to be true if we consider the canonical approach to be fundamental. I will also argue that if the connection is established, my argument that there is a problem of time becomes stronger.

[32] I refer the reader to Alexandrov et al. (2012) for a review in which it is argued for the complementary of the approaches but where the differences are acknowledged.

Let me start by recalling how the canonical formalism is intended to work for LQG. The states which are supposed to be physically meaningful in LQG are the ones in the physical Hilbert space, a space which satisfies all the constraints of the theory. This space can be constructed, in principle, by imposing the constraints to the states in the kinematical Hilbert space. In the case of LQG this kinematical Hilbert space is the space of spin network states[33], and that we can represent an element of the basis of this space by $|s\rangle$, where s represents both the graph and its coloring. To impose the dynamics of the theory is to find a map from this kinematical Hilbert space to the physical one:

$$\begin{aligned}\eta : \mathcal{H} &\to \mathcal{H}_{phys} \\ |s\rangle &\to \eta(|s\rangle) = (\eta(s)|\,.\end{aligned}$$

Where I am using the notation $(...|$ to refer to distributional states[34] that satisfy the constraints. With such a map available one can define the inner product in $\mathcal{H}_{phys}$ by:

$$(\eta(s)|\eta(s')) = (\eta(s)|s'\rangle\,. \tag{8.28}$$

As I argued in chapters 4 and 5 the problem of time of canonical quantum gravity is that we are not able to define a dynamics for this physical Hilbert space. There are different ways to approach this problem, and I argued that all have important shortcomings. In this section I will focus in the transition amplitude strategy which is defended by a part of the quantum gravity community[35] and which allows to make the connection with spin foam models. In this approach, the inner product between states in the physical Hilbert space is interpreted as a propagator. That is, one defines $K(s,s')$ as:

$$K(s,s') = (\eta(s)|\eta(s'))\,, \tag{8.29}$$

and then one tries to interpret it in the same way one would interpret the propagator in any quantum theory. This strategy works for the quantization of deparametrizable systems, as I explained in chapter 4. For instance, for a non-relativistic particle we can define a kinematical Hilbert space with basis $|x,t\rangle$ and the map η maps states in this space to solutions of the Schrödinger equation. And one can show that the physical inner product defined by this map is indeed the propagator of the Schrödinger equation:

$$K(x,t;x',t') = (\eta(x,t)|\eta(x',t'))\,. \tag{8.30}$$

[33] Recall from chapter 5 that we introduced the distinction between spin networks and s-knots and that the Hilbert space of LQG is based on the latter but that when the distinction is not relevant we can just refer to s-knots as spin networks.

[34] In chapter 5 I showed that the physical Hilbert space is not a subspace of the kinematical Hilbert space but a space of distributions on this space.

[35] See for instance (Rovelli, 2004, Chapt. 5) and (Rovelli and Vidotto, 2015, Chapt. 2).

However, in chapter 4 I argued that this strategy is problematic for the quantization of non-deparametrizable models like general relativity. As I argued, the difference between both cases is that states in the kinematical Hilbert space do not carry any temporal information in the case of general relativity or the double harmonic oscillator, while they do so in cases like the non-relativistic particle. This is because the configuration space we are taking for the quantization of the non-relativistic particle is an extended configuration space, i.e., it contains a time variable t together with the configuration variable x. In the case of general relativity, the configuration space we take as starting point for the quantization of the theory is just the configuration space of the theory[36], and not a extension containing any temporal variable. For this reason, I have argued against interpreting the physical inner product as a propagator in the case of general relativity and, hence, against this resolution of the problem of time.

In any case, given the formal similitude, one may wonder if the propagator defined using the canonical formalism and one defined by means of a spin foam model like the EPRL spin foam model I have introduced in this chapter are identical, that is, that the canonical and the covariant formalisms were only equivalent ways of defining the same theory. However, this equivalence has not been proved and one may doubt that it will be proved so, given that the canonical formulation is not fully satisfactorily constructed and that there are several inequivalent spin foam models. Despite this, there are some heuristic reasons for believing that the physical inner product of the canonical theory can be expressed in terms of spin foams.

One such heuristic reasoning can be seen in Reisenberger and Rovelli (1997), which was one of the first works in introducing spin foams. The way this is done is by expressing 8.28 in the following way:

$$(\eta(s)|\eta(s')) = \lim_{T\to\infty} \langle s|e^{-i\int_0^T dtdx\hat{H}(x)}|s'\rangle\,. \tag{8.31}$$

Where one introduces the limit of the exponential as a way of expressing a delta function, that is, as a way of imposing the Hamiltonian constraint $\hat{H}(x)$. This is the same reasoning I explained in section 7.4 Now one can transform the expression into a path integral in a similar fashion to the ways I have used several times in chapter 7. For this, we can divide the integration interval and introduce resolutions of the identity:

$$\langle s|e^{-i\int_0^T dtdx\hat{H}}|s'\rangle = \sum_{s_1,s_2...s_N} \langle s|e^{-i\int_0^{\Delta T} dtdx\hat{H}(x)}|s_N \times s_{N-1}|....\langle s_1|e^{-i\int_0^{\Delta T} dtdx\hat{H}(x)}|s'\rangle\,. \tag{8.32}$$

[36] As I have explained previously, different quantizations of general relativity differ in the exact form of the configuration space chosen. This leads to different quantizations like geometrodynamis or LQG.

Now for small ΔT one can expand each exponential as $e^{-i\int_0^{\Delta T} dtdx\hat{H}(x)} \approx \mathbb{I} + \Delta T\tilde{H}$. The action of the identity matrix is to leave the spin network state unchanged, while the action of $\tilde{H}$ will change the network to a different one. Therefore we can group the histories we are summing over by the number of changes that happen in it, that is, we can sum first the histories with no changes, then the histories with one intermediate step, then with two steps and so on. Roughly we can write:

$$\langle s|s'\rangle + \sum_{s_1} A(s, s_1, s') + \sum_{s_1,s_2} A(s, s_2, s_1, s') + \dots . \tag{8.33}$$

Where by A I represent the amplitude corresponding to any given sequence of spin network states. But now we can realize that each sequence $s, s_N, s_{N-1}...s_1, s'$ can be represented as a spin foam! In other words, we have found that the inner product can be written as:

$$(\eta(s)|\eta(s')) = \sum_{\mathcal{F}:s'\to s} A(\mathcal{F})\, . \tag{8.34}$$

Where by $\mathcal{F} : s' \to s$ I represent a spin foam bounded by s and s', like the ones in figure 8.12. The amplitude $A(\mathcal{F})$ is to be determined by the perturbative expansion. This sort of expansion should ring a bell for anyone familiar with QFT and particle physics, as it is the same kind of expansion from which one derives Feynman diagrams. In the case of QFT the expansion gives a sum over all the possible diagrams which agree in the initial and final states. In the case of QFT the states are given by a set of particles, which is represented by a set of points. Feynman graphs are just the possible ways of joining the initial and final points, and hence are made of one-dimensional lines. In the case of LQG, states are made of one dimensional networks, therefore it is natural that the equivalent of Feynman diagrams for spin networks are based in 2-complexes. In a Feynman diagram the vertices in the graph represent an interaction between particles, as it is the place where they meet, and we can roughly say that they are the place where changes happen. In the case of a spin foam this role is played by the vertices of the 2-complex, as it is the place where different faces and edges meet and as the spin networks before and after a vertex are different.

Therefore, we have found that it seems intuitively possible that the physical inner product of spin network states is given by a spin foam expansion. Despite this, notice that what we have found is a sum over all possible 2-complexes, while the EPRL spin foam model was defined for just one given spin foam. This is an important difference that complicates bridging between both approaches and which motivates seen spin foam models just as approximations to the full theory of quantum gravity. I will expand on this point in the next subsection.

From the perspective of the canonical formalism spin foams can be interpreted as Feynman diagrams, that is, as useful tools for computing the physical inner product

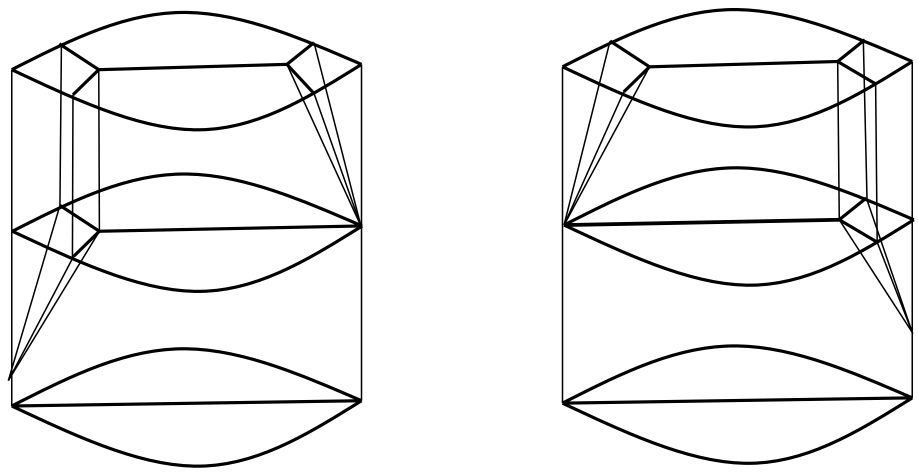

Fig. 8.12: A couple of spin foams which are bounded by the same initial and final spin networks. For finding the transition amplitude between these two spin networks one would have to sum over all foams $\mathcal{F}$ like the ones represented here. These two diagrams have two vertices, while generic diagrams in the sum will have more vertices.

but without necessarily assigning any physical meaning to them. Feynman diagrams are similar to path integrals in the sense that both are ways of computing propagators by computing a sum over possible processes. However, in chapter 7 I argued that paths in a path integrals should not be interpreted as physical processes, the same applies to the processes represented by Feynman diagrams and the same conclusion can be extended to the case of spin foams. Notice also that in the case of Feynman diagrams the graphs are diagrams in spacetime, while in the case of the spinfoams in expansion 8.34 it is less clear that one could give that interpretation. The reason for this is that these foams 'evolve' with respect to the parameter T, which is just an artifact of the representation chosen for η and not a physical parameter. In section 7.4 I argued against interpreting parameters like T as representing any real time, as the physical content of the theories is supposed to be in the propagators and not in the way we compute them.

In contrast, from the perspective of the spin foam model introduced in this chapter the spin foam does not represent an abstract transition between two spin networks, but it is instead very closely related with a triangulation of spacetime. For this reason, from the covariant perspective one is more likely to interpret spin foams as linked with spacetime or representing spacetime. But even if in these models the triangulation is fixed, we are nevertheless summing over all possible geometries that can be associated to this triangulation and hence one should be careful and avoid interpreting these spacetime geometries as being the real geometry of spacetime. This is just the application of what I argued in chapter 7 but for the case of the spin foams: paths in a path integral should not be taken to represent real processes.

Let me also mention something which I have already commented before, namely, that the picture of the triangulation of spacetime is not really compatible with canonical LQG. The expression of the inner product in terms of path integrals supports this conclusion, as in LQG both the states in the boundary and the states in the intermediate steps in the spin foam expansion are defined on arbitrary graphs and not on graphs dual to a triangulation. This makes it the case that the 2-complexes in the expansion are, in their vast majority, not dual to a triangulation of spacetime. In this sense, canonical LQG and the EPRL model based on a triangulation are clearly not equivalent theories.

Finally, let me also mention that there is a suggestion by Rovelli to connect also the generalized boundary formalism with the canonical formalism. This proposal appears in (Rovelli, 2004, Sect. 7.4) and it consists in identifying the propagator for a general boundary $K(s)$ with the physical inner product between the state associated with the spin network state of that boundary and the empty state $|\emptyset\rangle$ that contains no spin network:

$$K(s) = (\eta(\emptyset)|\eta(s)) \,. \tag{8.35}$$

This identification is formally consistent in the sense that the physical inner product defines an amplitude for the boundary state s. But from a conceptual point of view there are reasons for doubting of this construction, which is not really justified in Rovelli (2004). Remember that the canonical formalism was based in a foliation of spacetime and hence states should, in principle, be associated with a leaf of such foliation. The general boundary on which the state s is defined is not, in general, a leaf of such foliation and hence it seems that one may be abusing of the canonical formalism, as we are applying it outside its natural scope. Moreover, it seems that we are identifying $K(s)$ with the transition from no space to a given space coded by s, while when we first introduced generalized propagators we interpreted them as giving an amplitude for a process happening inside the region of spacetime they bound. Therefore it seem that we are conflating two different things. For these reasons one should be cautious when trying to connect the covariant formalism with general boundaries with the canonical formalism.

To conclude, let me just insist that spin foam models and canonical LQG are not, in general, equivalent but they share some formal aspects and also some conceptual difficulties. The biggest of these difficulties is to give a physical interpretation of the models, which are affected by the problem of time. A difference in the definition of both approaches is that the spin foam model is defined in terms of a triangulation of spacetime while in the canonical formalism the 2-complex arises as computational device. In the next subsection I will analyze in more detail the role of the triangulation, the continuum and classical limits of spin foam models and the way they are considered just as approximations and not a full theory of quantum gravity.

8.3.3 Role of the truncation and limits

In this last subsection on the interpretation of spin foam models I want to focus on an interesting aspect of the way spin foam models are treated, namely, the way they are supposed to be related with general relativity. This discussion is interesting even if I have argued that spin foam models are not satisfactory, as it allows to connect with the discussion about the emergence of spacetime in chapters 2 and 3.

A key aspect to consider of the spin foam models I have introduced is that they are defined on a truncation of general relativity. More precisely, this truncation reduced the field degrees of freedom of general relativity to a set of variables B, U defined on a 2-complex, which, for the main model considered, is dual to a triangulation of spacetime. For considering the relation between the spin foam models and general relativity, there are two limits that will play an essential role, which are the classical and the continuum limit.

Schematically, the derivation process of the EPRL spin foam model we have followed in this chapter is:

$$\text{General Relativity} \underset{\text{Truncation}}{\longrightarrow} \text{BF on a 2-complex} \underset{\text{Quantization}}{\longrightarrow} \text{Spin foam model}$$

We are now interested in the relation between the theories in the opposite direction, i.e., we would like to know if general relativity, the theory that we believe describes our world at big scales, can be 'derived' or approximmated by the spin foam model in an appropriate way. In this sense, as we discussed in chapters 2 and 3 we could claim that general relativity and the classical continuum spacetime 'emerge' from our model of quantum gravity, in this case the spin foam model. In this thesis I am defending a reductionist position and I am not using the word emergence in any strong sense. In other words, if we believed that the spin foam model described our world and its conceptual problems could be solved, I would argue that the relation of emergence is just that general relativity is an appropriate approximated description for our world for some regime of the spin foam model, which by assumption is considered true. Having said this, we can split this process of emergence in two steps:

$$\text{Spin foam model} \underset{\substack{\text{Classical}\\ \text{limit}}}{\longrightarrow} \text{BF on a 2-complex} \underset{\substack{\text{Continuum}\\ \text{limit}}}{\longrightarrow} \text{General relativity}$$

We have already discussed a continuum limit in this chapter when comparing Regge calculus with general relativity. Given the similarity of BF theory on a 2-complex with Regge calculus it is well established that one can define a continuum limit of the theory considering a sequence of 2-complexes representing finer and finer triangulations. In this sense, that general relativity is the continuum limit of BF theory is not a controversial statement. More problematic is the first step of this emergence process, which will even jeopardize the second one.

In the discussion in Rovelli and Vidotto (2015) about the classical limit of the theory they consider taking the limit of big spins, that is $j \to \infty$. This mathematical limit is formally equivalent to taking the limit of small $\hbar$, which is the limit I discussed in chapter 7 for general path integrals. I argued then that the fact that the path integral is dominated for classical trajectories in the limit $\hbar \to 0$ is not a sufficient condition for claiming that a classical world 'emerges' from a quantum theory. For claiming this one needs to address the measurement problem and explain the relation between the formalism of quantum mechanics and the real world. As I have argued in section 8.3.1, the interpretation of the 'propagator' for spin foam models is at best unclear, which clearly represents a major obstacle to claiming that there is a classical limit to the theory. This is in consonance with what I argued in section 3.4, namely that the biggest obstacle for considering the classical limit of a wide class of theories of quantum gravity and the way spacetime relates with the structures they postulate is precisely that it is unclear the way one can reasonably interpret the formalism they propose, if possible at all.

Leaving this aside, one can still consider the limit $j \to \infty$ of the model as a consistency check. This limit is considered for the EPRL model in (Rovelli and Vidotto, 2015, Sect. 8.3). In this section it is analyzed how the vertex amplitude 8.26 behaves for large j. Recall that the variables j, i were associated to the geometrical properties of the tetrahedra that form the boundary of the 4-simplex dual to the vertex. The analysis by Rovelli and Vidotto finds that the amplitude is suppressed for twisted geometries, that is, if the geometric information encoded by j, i does not correspond to the geometry a flat 4-simplex could have, the amplitude for such j, i is very close to 0. In this sense, twisted geometries do not play a relevant role in the large j limit. Moreover, for j, i defining a proper geometry the vertex amplitude for large j behaves like:

$$A(j,i) \sim c e^{iS_{Regge}(j,i)} + c' e^{-iS_{Regge}(j,i)} \,. \tag{8.36}$$

This means that the amplitude is dominated by two terms which correspond to the classical Regge action for the 4-simplex geometry defined by j, i. There are two terms because of the two possible orientations one can give to the 4-simplex. Therefore, we see that the vertex amplitude of the model behaves in an appropriate way for large j. The same behaviour was found for the Ponzano-Regge model in three dimensions. In this sense, even if this model has a serious problem at the the time of getting interpreted, the mathematical functions it defines satisfy some minimal consistency conditions.

Roughly speaking, and ignoring the problems in the quantum-classical transition, we can think of the process from the spin foam model to general relativity as the combination of taking the big j limit and then the limit of fine triangulations, that we can represent by $\Delta \to \infty$. However, these two limits may be in tension with each other as when one goes from a triangulation to a finer one the lengths and areas intuitively decrease, meaning that j also decreases. Remember that there is a relation between curvature and fineness of triangulation: for approximating a region of great curvature one needs a very fine triangulation. Therefore, one has to find an appropriate regime such that j is big enough to effectively suppress quantum effects while still be able to approximate well enough a manifold with certain curvature. If we assign a length scale L to a spin foam model and we want to claim that it approximates a classical manifold with a curvature scale $L_{Curvature}$ then we will require L to satisfy:

$$L_{Planck} \ll L \ll L_{Curvature} \,. \tag{8.37}$$

This regime was defined in more detail in Han (2014) and was given the name of the Einstein sector of the theory. Notice that in order to have a consistent approximation we cannot have an arbitrarily high curvature, as this would imply an arbitrarily low $L_{Curvature}$. In other words, with the spin foam model defined one cannot have a model that approximates a classical spacetime with regions of arbitrarily high curvature. Furthermore, even if we ignored the restriction imposed by the classical limit, the quantization of j implies that there is a minimum value it can take, and hence one cannot approximate arbitrarily high curvatures in small regions.

That there is a regime of validity for the approximation of the model by a general relativistic model instead of being defined by a strict limit is not a problem and indeed is a feature of many 'emergent' theories. For instance, in fluid dynamics one performs a coarse graining in order to approximate the motion of a fluid made of molecules. By coarse graining we represent the fluid as smooth, which is an approximation which holds as long as we are considering a scale L much bigger than the scale at which we can distinguish the molecules $L_{molecule}$. In this sense one may say that the fluid description emerges in the big scale limit, that is, $L \to \infty$. But notice that by this limit what we really mean is $L_{molecule} \ll L$. Indeed if the coarse graining scale is too big we may lose relevant information. For instance, consider that our fluid is moving through a slit of size L_{slit}. If we want to describe this process in a correct way we'd better have a coarse graining scale which is smaller than this scale. Therefore, we obtain that our smooth fluid description is a good description for the range $L_{molecule} \ll L \ll L_{slit}$. This is in perfect analogy with the case of the spin foam and the approximate smooth geometry.

With this we have introduced a key notion for understanding the attitude of Rovelli and Vidotto towards the triangulation, which is the notion of approximation.

Rovelli and Vidotto argue that in the same way that the triangulation was introduced in the classical theory as a truncation which approximates the complete theory, in the quantum theory we should have the same attitude. In this sense, the complete quantum theory of gravity is not the spin foam model introduced, but it is instead its limit when considering finer and finer triangulations. In figure 8.13 I represent the 4 relevant theories for our discussion and their relations according to Rovelli and Vidotto.

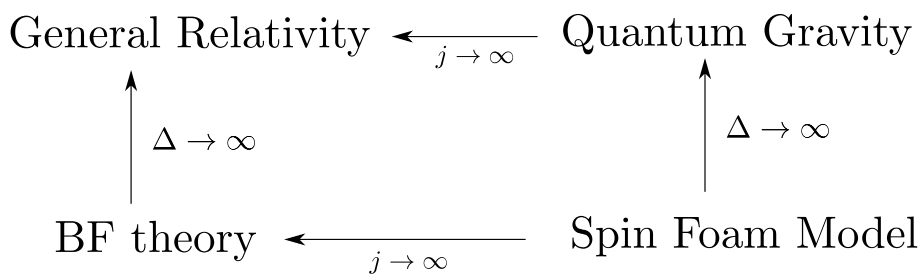

Fig. 8.13: The four relevant theories for our discussion and the formal relations between them. This figure is inspired in table 8.1 of Rovelli and Vidotto (2015).

Let me comment on this 'continuum' limit between the spin foam models and the quantum gravity theory. I will consider the two ingredients of the quantum theory, the Hilbert space and the propagator in turn. First, let me consider the boundary Hilbert space. As we make the triangulation finer and finer the Hilbert space becomes the Hilbert space of larger and larger spin networks, in the sense of containing of more and more links and nodes. By taking the projective limit, what we obtain is the Hilbert space of LQG, just as we found in section 5.2.1. Therefore, one recovers the same Hilbert space as in canonical LQG, with the difference that now these states are meant to represent boundary states and not states in the kinematical Hilbert space. In other words, states in the boundary Hilbert space are given by a graph Γ and the colourings j_l for the links and i_n for the nodes. There can be difference here between the Hilbert space of canonical LQG and this covariant theory, which is that in the canonical theory there was not any restriction on this states while in the covariant one we may impose that the nodes are four-valent so that every state represents a triangulation or one may even impose further conditions to avoid fuzzy geometries. This is a consequence of what I commented in subsection 8.3.2 that the model based on a triangulation does not correspond to the full LQG.

As I commented in chapter 5, this Hilbert space structure is counterintuitive: what one would naively expect for a state in the complete Hilbert space is that it is able to represent an arbitrary geometry, and that finer networks represent geometries in more detail. However, the quantization of geometric observables implies that this is not the case, as we are not able to describe arbitrarily small areas and volumes. In chapter 5 I argued that this is a consequence of the variables and representation chosen for the canonical quantization. In this case, it is also the choice of flux variables for the quantization which implies that we cannot consider the limit as describing arbitrarily fine triangulations made of arbitrarily small areas or volumes. In this sense one can wonder if this feature of the quantization is something to be accepted or rejected, given the clear difference with the continuum limit in the classical theory.

The Hilbert space of LQG and spin foam models is similar to the Fock space of particle physics in that they both can be defined as a projective limit. In the case of Fock space, the N-particle Hilbert space describes a system of up to N particles and the Fock space is the limiting space when one allows N to be arbitrarily large. The LQG Hilbert space is analogous: we can define a Hilbert space for graphs with N nodes and L links and then by allowing N and L to be arbitrarily large we obtain the space of interest for the theory. In this sense, the spin network basis of LQG is very similar to the particle basis of quantum field theories.

Rovelli and Vidotto insist on the analogy between LQG and QFT when considering the dynamics of the theory. They notice that the propagator of a QFT can generally be expanded as a sum over Feynman diagrams, and that, for practical purposes, in real life physicists only consider a few of these diagrams. That is, when computing a propagator one can obtain a good enough approximation by truncating the series. Rovelli and Vidotto claim that the same attitude has to be taken towards the spin foam models like the EPRL spin foam model they provide: the propagator 8.27 has to be taken as an approximation to the true propagator. It is hoped that the finer the triangulation the better the approximation will be, just as in QFT the approximation is better when one includes diagrams with more vertices. However, one can raise some worries with this analogy. First, in QFT we get better and better approximations when we sum over more and more diagrams. That is, it is not that the finer diagrams are better approximations, but that we get a closer result to the true one when we consider more diagrams. Moreover, in QFT contexts it is usually the case that the diagrams with a few vertices are already good approximations to the propagator, while the diagrams with more vertices contain just small corrections. In this sense, just by looking at the parallelism between Feynman diagrams in QFT and expansions like 8.34 one can doubt about the intuition defended by Rovelli and Vidotto that the finer the 2-complex the better the approximation to the dynamics is.

Notice also that there is a methodological difference between the approximation done in QFT and the one in LQG. In the case of a QFT one generally knows the full quantum theory and then truncates it for having a reasonable approximation while in the case of the spin foam model one inverts the order and first truncates and then quantizes. One may doubt that the processes of truncation and quantization commute and that we can rely on the way the theory is built for the spin foam. For instance, it seems reasonable to say that there is more control over an approximation or truncation when it is explicitly considered from the more fundamental theory than when considered from the point of view of the emergent theory.

There is another similarity with a different kind of approximation in quantum field theory which is that of a lattice gauge theory. In a lattice gauge theory one approximates a field theory with degrees of freedom at every point of space with degrees of freedom associated with a lattice. Indeed, the degrees of freedom of a gauge theory are a connection field and in its lattice version these degrees of freedom are captured by the holonomy of the connection. Therefore, we again find that there is a strong similarity between LQG and gauge theories. The difference is that in the lattice gauge theory the lattice represents a lattice in spacetime, i.e., a set of points separated from each other by a distance a, while in the foam model the lattice itself, the 2-complex, defines spacetime. That is, if we consider just the 2-complex without the variables B, U, it does not carry any geometric information, while the lattice in lattice gauge theory has an associated scale. The reason behind this difference is that the gauge theory is defined on a background spacetime while the theory of gravity is defined in a diffeomorphism invariant context.

This difference has an important consequence when we compare the continuum limit of both theories. In the case of the lattice theory we can take the limit of finer lattices, which is equivalent to take the limit of the lattice spacing a going to 0. In the foam models, the fineness of the lattice and its physical size go apart, and, moreover, we cannot take the limit of very small physical sizes for the triangulation as it is encoded by the quantum numbers j, i which have minimum values due to their quantization. For this reason we find that the continuum limit of both theories is very different despite both theories being theories defined on discrete structures: while we take the limit of the lattice theory to be a field theory, the limit of the spin foam has still a discrete character. I am unaware of any rigorous definition of the gauge theory as a limit, projective or otherwise, of the theory in the lattice and this shows what I was commenting before: the ideas of limit and approximation are usually much clearer from the perspective of the complete theory than from the perspective of the approximating theories. This comparison also shows how counterintuitive and different the quantization of LQG is: while the standard quantization of field theories gives theories which still rely on the concept of field, the loop quantization gives a theory defined on a different structure. I emphasized this difference in chapter 5, where I raised the possibility that a quantization like geometrodynamics may be more intuitively acceptable.

In a sense, we could say that the lattice of spin foam models looks more the lattice of lattice field theory than the Feynman diagram of a QFT. However Rovelli and Vidotto argue (Rovelli and Vidotto, 2015, Sect. 7.3.2) that from the point of view of a quantum theory of gravity these two objects are not so different. Indeed, they can correspond to two different approximations of a same theory. Furthermore, if we consider something like LQG to be true then space is made of discrete chunks and the physical degrees of freedom would have to captured by a discrete structure like a lattice. The interactions of this physical lattice can be interpreted as a kind of Feynman diagram. We can apply this idea to the case of the spin foam model and interpret a given spin foam as a Feynman diagram for interacting quanta of space. The view of spin foams as Feynman diagrams is explored in group field theory, which I will analyze in chapter 9.

Finally, let me mention that Rovelli and Vidotto do not discuss in detail what the 'untruncated' theory of quantum gravity is. As I said, they define it as a limit from the EPRL model for finer and finer 2-complexes. This can be seen as a good enough definition, or one can look for a different definition, like the one provided by group field theory or by the canonical version of LQG.

Let me conclude this section by giving a brief summary of what has been discussed. First, I have argued that spin foam models like the EPRL model are not satisfactory theories, as they suffer from the covariant version of the problem of time, which means that the 'propagators' they define are not successfully interpreted. Second, I have argued that one can connect the canonical formalism with spin foam models, but that the picture of a triangulation is not natural if we make that connection, just as in chapter 5 I argued that the picture of LQG describing a discrete space is not justified if we consider all the states of the theory. Finally, I have discussed the inter-theoretic relations that can be motivated by spin foam models. In particular, spin foam models are regarded just as approximations to quantum gravity, but I have argued that there are some problems with this picture. If one could overcome the problem of time, to study the way spacetime could emerge from spin foams one would need to study both the classical and the continuum limits of the theory.

8.4 Spin foams and cosmology

In this section I will analyze how spin foam models are applied in cosmology. First, I will show that the LQC models that I analyzed in chapter 6 can be written in a form which looks very similar to spin foam models. I will argue that this is particular expansion is just a formal reexpresion of the same model, and that this does not modify the interpretation of these models, in particular, this means that they still suffer from a problem of time. In this sense, the fact that LQC models can be recast in the form of a spin foam expansion does not mean that spin foams have some deep meaning in this kind of model. Finally, I will take a look at some models directly defined in the foam paradigm. These models are not so well-developed as models in LQC, as they are more recent. I will argue that these models fail to provide a clear cosmological picture.

In chapter 6 I introduced one model of LQC in which quantum states are given by wavefunctions of the form $\psi(\nu, \phi)$, where ν represents the volume of the universe and ϕ is a scalar field. A peculiarity of the model is that the volume ν is quantized in the sense that it is only allowed to take discrete values. Physical states satisfied a Schrödinger-like equation:

$$-i\partial_\phi \psi(\nu, \phi) = \sqrt{\Theta}\psi(\nu, \phi)\,. \tag{8.38}$$

Where Θ is an operator which acts only on the ν part of the state. A possible interpretation of the states satisfying this equation is that they represent a universe which evolve with respect to the parameter ϕ. In chapter 6 I argued against this interpretation, as I argued that we should not identify time with physical fields, even if they can act as clocks. However, let me explore in this section how this model can be cast in the spin foam paradigm. Here, I will follow the construction in Ashtekar (2009); Ashtekar et al. (2010). Without loss of generality, we can write the propagator of the LQC model as:

$$K(\nu_f, \phi; \nu_i, 0) = \langle \nu_f | e^{i\phi\sqrt{\Theta}} | \nu_i \rangle\,. \tag{8.39}$$

This propagator allows one to compute the wavefunction for the universe at a 'time' ϕ after an initial state, i.e., if we were to accept the relational resolution, it would allow us to compute the probability of finding a given value for the volume of the universe knowing the state it started at. This propagator can be expressed by means of a sum over paths in 'time', just as I explained in section 7.3 one can do for a deparametrizable system (even if I have argued that this interpretation is wrong!). We can split the time interval ϕ in N pieces to get an expression of the form:

$$K(\nu_f, \phi; \nu_i, 0) = \sum_{\nu_1, \ldots \nu_{N-1}} A_N(\nu_i, \nu_1, \ldots \nu_f)\,. \tag{8.40}$$

This is just the a expansion similar to the expansion by Feynman which allowed us to define the propagator for the non-relativistic particle in section 7.1. By $A_N(\nu_i, \nu_1, \ldots \nu_f)$ I represent the amplitude assigned to a path in which the volume of the universe takes the values $\nu_i, \nu_1, \ldots \nu_f$. If now we took the limit for $N \to \infty$ we would recover the path integral expression for the theory. Instead, we will take a different route. For a given N we can classify the space of paths by the number of transitions M that happen in the path. That is, there are histories in which the volume of the universe does not change, there are others in which it changes 1 time, 2 times, and all the way up to N times, which are the histories for which the volume has not remained constant for two consecutive instants. In each of these categories we will find histories which agree on the different volumes the universe has but which disagree in the times at which the universe grows or shrinks. For instance, for $N = 3$ and 1 change we will find the histories: $\{\nu_i, \nu_i, \nu_i, \nu_f\}$, $\{\nu_i, \nu_i, \nu_f, \nu_f\}$ and $\{\nu_i, \nu_f, \nu_f, \nu_f\}$. This is represented on the left hand side of figure 8.14. We can define the amplitude A_N^M as the sum of all the amplitudes that agree on the sequence of volumes the universe has but disagree on what are the times at which there is a change. For the previous example we have:

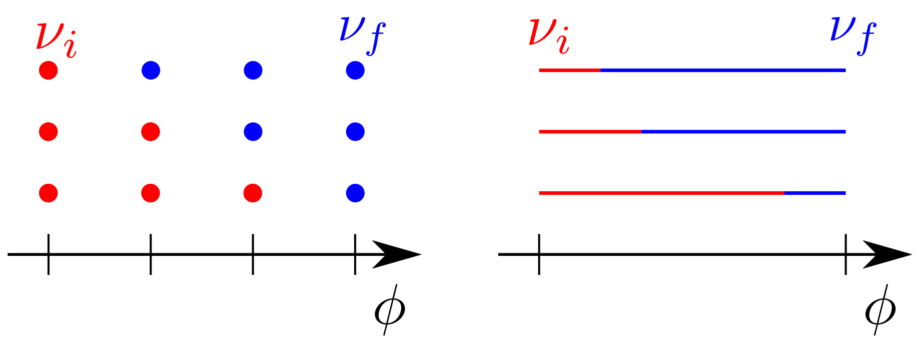

Fig. 8.14: On the left I represent the 3 possible sequences for $N = 3$ and one change between an initial and final volume. On the right we take the $N \to \infty$ limit. The transition can now happen at any moment of 'time' ϕ. Notice that these three are diffeomorphic to one another in the sense explained in the text .

$$A_3^1(\nu_i, \nu_f) = A_3(\nu_i, \nu_i, \nu_i, \nu_f) + A_3(\nu_i, \nu_i, \nu_f, \nu_f) + A_3(\nu_i, \nu_f, \nu_f, \nu_f)\,. \tag{8.41}$$

With the introduction of this amplitude we can rearrange the terms in the sum to be:

$$K(\nu_f, \phi; \nu_i, 0) = \sum_{M=0}^{N} \left[\sum_{\substack{\nu_1, \ldots \nu_{M-1} \\ \nu_i \neq \nu_{i-1}}} A_N^M(\nu_i, \nu_1, \ldots \nu_{M-1}, \nu_f) \right] . \tag{8.42}$$

In this way we have a sum over different number of transitions M that depend on the sequences of volume rather than on the exact moment where this transitions take place. Now one can take the limit $N \to \infty$ to get:

$$K(\nu_f, \phi; \nu_i, 0) = \sum_{M=0}^{\infty} \left[\sum_{\substack{\nu_1, \ldots \nu_{M-1} \\ \nu_i \neq \nu_{i-1}}} A^M(\nu_i, \nu_1, \ldots \nu_{M-1}, \nu_f) \right] . \tag{8.43}$$

Notice that in this expression I have replaced the amplitudes A_N^M by their limits A^M. That these limits are well-defined and that the sum is well-behaved are non-trivial facts proved in Ashtekar (2009); Ashtekar et al. (2010). This expression for the propagator takes the form of a spin foam expansion like 8.34 rather than the form of a path integral like the ones in chapter 7.

Notice that one of the similarities with the spin foam models is that the amplitudes $A^M(\nu_i, \nu_1, ...\nu_{M-1}, \nu_f)$ are 'diffeomorphism invariant' in the sense that these amplitudes do not depend on the possible ways of deciding the precise time at which the transition happens. That is, I can choose to represent the sequence as having changes at times $\phi_1, \phi_2...\phi_M$ or at a diffeomorphic set of times $\phi'_1, \phi'_2, ...\phi'_M$. This is represented on the right hand side of figure 8.14. This is the same as choosing two 2-complexes for representing the same spin foam model which are related by a diffeomorphism transformation.

This result was taken in the LQG and LQC communities to get the models of LQC closer to the spin foam paradigms[37]. But it is important to realize that the origins of both diffeomorphism invariances are different. In the case of LQC the fact that A^M does not depend on the times of the transitions has nothing to do with time or coordinates being physical or not in the same way that coordinates are not physical in general relativity. Instead, this invariance comes from the fact that for constructing this quantity we have summed over all transition times, and hence the result cannot know about it. This example makes clear that one has to be careful when interpreting the terms over which one sums in a path integral as representing real processes or as capturing symmetries of the theory under consideration. For interpreting LQC one should rather look at the quantum theory it defines than at possible spin-foam-like expansions of the theory. The same lesson can be extrapolated to LQG: it may be tempting to interpret the spin foams that appear in some expansions as representing a spacetime, but one has to be cautious in making such interpretative moves.

Again, comparison with Feynman diagrams in particle physics can be enlightening. A transition amplitude may be expressed as a countable sum of terms represented by different diagrams and these terms and diagrams may show interesting symmetries or behaviors. But as these are just computational devices one should look for the physical properties and phenomena in the transition amplitudes themselves.

Let me comment also that there is one approach to cosmology genuinely based on spin foams[38]. While canonical LQC takes cosmological models like FRLW and quantizes them using the loop techniques, the spin foam approach directly looks for states in the full theory which can be used for representing cosmologically relevant geometries. As in cosmology one is interested in very few degrees of freedom, the

[37] An example of this attitude can be found in Rovelli and Vidotto (2010).

[38] An overview of the approach can be found in (Rovelli and Vidotto, 2015, Sect. 11.3).

states used are states defined on very simple and regular graphs. For representing a closed, homogeneous and isotropic universe one takes a regular graph which is dual to a cellular decomposition of a 3-sphere and assigns to all the links the same quantum numbers. For instance, a graph that can be used is the graph Γ_5, which is the graph in figure 8.8 but assigning the same quantum numbers to every link. This graph represents a compact space formed by gluing together 5 equal tetrahedra.

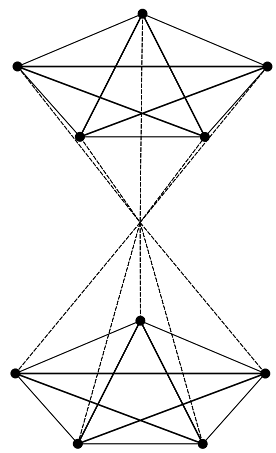

Fig. 8.15: The spin foam used in spin foam cosmology. This spin foam allows to compute the one-vertex approximation for the propagator of two homogeneous and isotropic geometries.

Now one can define a propagator between two different states on Γ_5. That is, a propagator $K(h_l, h'_l)$ which gives the probability of finding some final geometry, i.e. size of the universe, given an initial one. This propagator was found in Borja et al. (2010); Vidotto (2011) using the simplest spin foam approximation, which consists in a spin foam with one vertex only like the one in figure 8.15. Notice that this foam is not dual in any straightforward way to a triangulation of spacetime. Indeed, the vertex in such an expansion is not dual to any 4-simplex, nor even to any 4-dimensional polyhedra. Despite this, the propagator is found to have a 'classical' limit, a large spin limit like the one discussed in section 8.3.3, which agrees with what would be expected for the action of a FRLW model.

However, the physical interpretation of objects like $K(h_l, h'_l)$ suffers from the same interpretational shortcomings that spinfoam models suffer from, as I have argued in this chapter. In particular, we can express the problem of time for this model in the following way. The propagator defines a probability of finding a final size of the universe given an initial size, with no reference to how much time elapses between the initial and final moments. Notice that the relational strategy will not work for such a simple model: as we are only considering one variable we cannot think that the propagator encodes the evolution of one variable with respect to another, as there simply is not any other variable. Similarly, it seems completely mysterious how to interpret the probability of finding a volume without specifying something like a moment for ‘measuring’ or an operational ‘instant’.

The case of spin foam cosmology exemplifies well the worries I have raised in this chapter about spin foam models in general, and it is therefore a good example for closing the chapter. Spin foam models have the virtue to provide mathematically well defined transition amplitudes and can be seen as an improvement building up on the basis of canonical LQG. However, at the time of giving a physical interpretation of spin foam models our position is not any better than in the case of LQG: we have neither a realist interpretation available nor clear operational predictions.

9. GROUP FIELD THEORY

After having studied spin foam models in the previous chapter, in this chapter I will study a closely related approach to quantum gravity, group field theory (GFT). This builds on spin foams and it has allowed to apply some concepts from quantum field theory and condensed matter theory to build cosmological models. In particular, some works in GFT[1] incorporate the concept of phase transition and speculate that the big bang can be treated as a phase transition between a disordered, non-spatiotemporal phase and the universe as we know it. In this chapter I will introduce GFT and analyze both its foundations and the cosmological predictions it makes. I will argue that as LQG and spin foam models, this approach has serious shortcomings at the time of giving physical meaning to the quantities it defines.

I will start in section 9.1 by introducing the GFT partition function as a computationally useful object that allows one to compute the transition amplitudes of several spin foam models like the ones I introduced in the previous chapter. We will see that the reason for this connection is that the Feynman diagrams that one finds in perturbative expansions of GFT naturally correspond to spin foams. In this sense, one can see GFTs as a useful computational technique for finding important quantities for spin foam models. However, the GFT community defends that GFT is more than a tool for computing quantities in a spin foam model and that there is a quantum defined the partition function. I will argue against this view, as it is not clear at all what this theory would amount to.

In section 9.2 I will discuss an alternative approach to GFT which is based on a canonical formalism. The kinematical Hilbert space of the model is postulated to be a Fock space which is very similar to the spin network Hilbert space of LQG and the dynamics is defined by imposing a constraint which is derived from an action principle, with the hope to make a connection with the covariant version of GFT. However, I will argue that the connection between both models is unclear. Furthermore, I will also argue that in this canonical version, GFT suffers from a problem of time and that its proposed resolutions are relational resolutions similar to the ones I introduced in chapter 6, and that they are equally unsatisfactory.

Finally, I will analyze the two cosmological applications of GFT, which they both

[1] See for instance Oriti (2014, 2017).

rely on ideas from condensed matter theory. First, in section 9.3 I will review condensate models which lead to the prediction of a bounce similar to the one predicted by LQC. However, I will argue that one should be skeptical about this model and its predictions, just as in chapter 6 I argued the same about LQC models.

Second, in section 9.4 I will introduce the idea of geometrogenesis, i.e., the idea that spacetime began after the big bang as a result of a phase transition between a non-geometric GFT state and a geometric one. However, I will argue that even if the idea is interesting, it is more of a suggestion than a prediction of the theory and that for implementing this idea in GFT it would require a different formulation of the theory or the incorporation of an additional and unwanted time variable. In this sense, GFT does not provide the structure to speak about geometrogenesis in a consistent way.

9.1 Covariant GFT

One way in which current GFTs were first formulated was by realizing that some spin foam models had a form which reminded of the Feynman diagrams of quantum field theory[2]. In the first part of this section I analyze how spin foams can be understood as Feynman diagrams of covariant GFTs, while in the second part I argue that covariant GFTs can be understood as useful mathematical tools for computing spin foam amplitudes but that interpreting them as a theory of their own is problematic. The reasons for this are the ones I pointed out in section 7.6 when I argued that sometimes in the quantum gravity literature it is not clear that the structures that are defined correspond to a quantum theory and that one can give them a satisfactory interpretation. Moreover, in the case of GFT, these objects are defined not with respect to a spacetime manifold, but with respect to a group manifold which is not interpreted as a spacetime.

9.1.1 GFT and spin foams

In chapter 8 we found that spin foam propagators took a suggestive form that reminded us of Feynman diagrams. For instance, we found expressions of the form:

$$K(U_l) = \mathcal{N}' \int dh_{vf} \prod_f \delta(h_f) \prod_v A(h_{vf}) \,. \tag{9.1}$$

[2] Here I am presenting GFT by introducing its relation to spin foams. However, the history of GFT is richer and it is certainly also related with other models of quantum gravity called matrix models. For our purposes it will be enough to present GFTs without entering in such an historical detail. I refer the reader to Freidel (2005); Oriti (2012) for more details on the relation between GFTs and matrix models and to De Pietri and Petronio (2000); Reisenberger and Rovelli (2001) for some of the first works relating spin foam models and GFT.

This expression is very similar to the one that gives the amplitude associated with a given diagram in a quantum field theory, which is usually given by a product of some amplitudes associated to each interaction vertex and some propagators which join the different vertices. In the case of the spin foam we find the same structure and it is tempting to interpret the spin foam as a Feynman diagram. And if Feynman diagrams arise in perturbative expansions of QFTs, then spin foams should also arise from expansions to some theory. Group field theories are precisely the mathematical structures which are such that when treated perturbatively give rise to spin foam amplitudes.

In the amplitudes we studied in the previous chapter for spin foams the fundamental variables were the holonomies h. In a Feynman diagram the basic variables are the positions and momenta of a set of particles, which are seen as excitations of a field. That the excitations of the field are characterized by positions is because the field itself is a field which is defined over position space[3]. Similarly, if we think of holonomies as excitations of some field, it needs to be a field which is defined on some group manifold, since holonomies are group elements. Thus, we arrive to the conclusion that if spin foams are Feynman diagrams of an underlying field theory, this field theory needs to be a group field theory.

For instance, let's consider the Boulatov model (Boulatov, 1992), which I will show is related with gravity in 3 spacetime dimensions. The field φ in this model is defined on a group manifold consisting on three copies of SU(2). In other words, φ is a function which takes three elements of SU(2) and gives a real number. The action for such a field is given by:

$$S_B[\varphi] = \frac{1}{2}\int \mathrm{d}^3 g \varphi^2(g_1, g_2, g_3) - \frac{\lambda}{4!}\int \mathrm{d}^6 g \varphi(g_1, g_2, g_3)\varphi(g_1, g_4, g_5)\varphi(g_2, g_5, g_6)\varphi(g_3, g_6, g_4)\,. \tag{9.2}$$

We can define a 'partition function' as a path integral for such an action and expand perturbatively on the interaction parameter λ to get:

$$Z = \int \mathcal{D}\varphi e^{-S_B[\varphi]} = \sum_\Gamma \lambda^{V(\Gamma)} \sum_{j_f} \prod_f d_{j_f} \prod_v \{6j\}_v = \sum_\Gamma \lambda^{V(\Gamma)} Z_{PR}(\Gamma)\,. \tag{9.3}$$

The expansion in an interaction parameter in a QFT gives rise to a sum which can be represented by means of Feynman diagrams: each diagram represents a term in the sum and is multiplied by as many powers of the interaction parameter as interaction vertices in the diagram. In the expression for the Boulatov model the expansion can also be represented diagramatically as in figures 9.1 and 9.2. Each

[3] Similarly, we can think of the field as defined in momentum space and the particles to be characterized by their momenta.

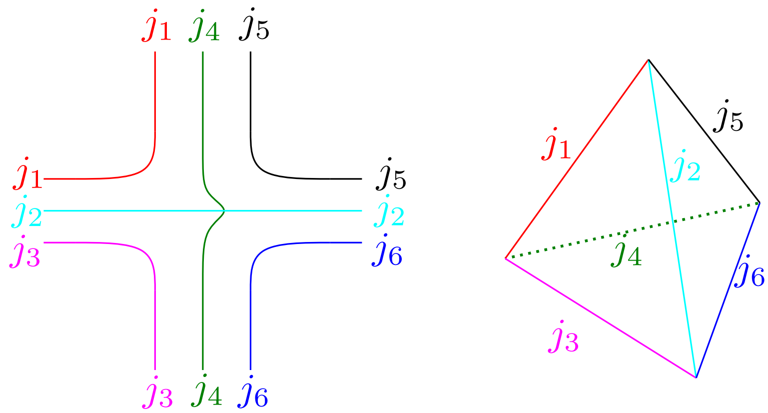

Fig. 9.1: Interaction vertex of the Boulatov model in the j representation and the tetrahedron which can be interpreted to be represented by such an interaction vertex.

term in the sum is labelled by a graph Γ which is formed by triple strands which meet in groups of four at vertices[4] and form a closed graph. Now, we can see each loop in this graph as defining a surface, and this, together with the structure[5] of the diagram defines a 2-complex, which is dual to a simplicial decomposition of a closed three dimensional space. A way of seeing this is by imagining each strand as associated with an edge of the triangulation, each triple of strands therefore is dual to a triangle and each vertex is the place where 4 triangles meet, i.e., a tetrahedron. In table 9.1 I summarize the relations between the parts of the diagrams, the 2-complex, and the dual triangulation.

Feynman diagram	2-complex	Triangulation
Vertex	Vertex	Tetrahedron
Triple strand	Edge	Triangle
Single strand (closed loop)	Face	Edge

Tab. 9.1: Correspondence between the elements of the Feynman diagrams of the Boulatov model and the different parts of the 2-complex and dual triangulation. The relation between the diagram and the triangulation should be clear from the representation in figures 9.1 and 9.2.

[4] The reason for this representation is that fields have three arguments, and hence the triple strand, and the interaction term is a product of four fields, and hence four excitations meet at each vertex and combine following the same pattern of the interaction term $\varphi(g_1, g_2, g_3)\varphi(g_1, g_4, g_5)\varphi(g_2, g_5, g_6)\varphi(g_3, g_6, g_4)$ as can be seen in figure 9.1.

[5] By this I mean that the three surfaces associated with a triple strand share an edge, and each vertex of the diagram is a vertex of the 2-complex, as it is the place where 6 surfaces meet.

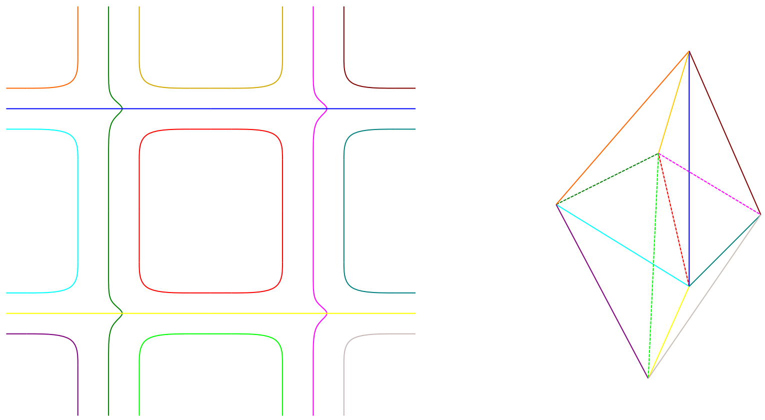

Fig. 9.2: Portion of a Feynman diagram of the Boulatov model which corresponds to four tetrahedra glued together to form an octahedron. The closed red line in the diagram corresponds to the dual of the red line in the octahedron, i.e., the line (not drawn) which goes around the red line joining the four tetrahedra.

Moreover, these diagrams carry irreducible representations j and we find that the amplitude associated with each of these graphs is essentially a product of $6j$ symbols, just as in the Ponzano-Regge model[6] I introduced in the previous chapter and which is the analogue of the EPRL spin foam model in three dimensions. In this way we find what I anticipated: that the terms that appear in perturbative expansions of GFTs correspond to the amplitudes associated with some spin foam models.

This duality can be explored further and for instance we can build quantities like the following[7]:

$$K[f] = \int \mathcal{D}\varphi f(\varphi) e^{-S_B[\varphi]}\,, \tag{9.4}$$

where f is any function of the field which is invariant under SU(2) gauge transformations of the form $\varphi(g_1, g_2, g_3) \to \varphi(hg_1, hg_2, hg_3)$ for any $h \in \mathrm{SU}(2)$. The reason why this gauge restriction is imposed to the GFT is that it will get translated to the gauge restriction of the dual spin foam model. In particular, a basis in momentum

[6] Notice that there is a discrepancy in the sign factors between the expression here and the one that appeared in equation 8.24. I am taking here the expression from Carrozza et al. (2020) which is the one discussed in the context of GFT. In Barrett and Naish-Guzman (2009) it is explained that the right expression for the Ponzano-Regge action is the one with the sign factors and points out that many times in the literature the wrong expression is given. For our discussion here this will not matter, as this example illustrates the point that in a perturbative expansion of a GFT there appear amplitudes of spin foam models.

[7] Here I am following the construction in (Rovelli, 2004, Sect. 9.3.3) but adapted to the case of the Boulatov model

space for the space of functions which satisfy this condition is no other that a set of spin network functions! After a moment of reflection this should not be so surprising, as spin network functions, as we introduced them in chapter 5, are precisely gauge invariant functions. We can label these functions by a graph Γ and an assignment of representations j to edges of the network[8]. The field φ takes three arguments, which implies that the spin network functions in this case are restricted to graphs with nodes of valence 3. For these functions we can perturbatively expand the quantity I have just defined to find:

$$K(s) = \int \mathcal{D}\varphi f_s(\varphi) e^{-S_B[\varphi]} = \sum_{\Gamma_s} \lambda^{V(\Gamma_s)} \sum_{j_f} \prod_f d_{j_f} \prod_v \{6j\}_v = \sum_{\Gamma_s} \lambda^{V(\Gamma_s)} K_{PR}(\Gamma_s)\,. \tag{9.5}$$

This expansion is the same we found for the partition function but with one important difference. Now the sum is over graphs Γ_s which correspond to 2-complexes which have the spin network s as a boundary. K_{PR} is therefore[9] the propagator we found in the previous chapter for the Ponzano-Regge model. Recall that the point of view of authors like Rovelli and Vidotto was that the 'true' propagator of full quantum gravity had to be precisely a sum of terms coming from different spin foams. In this sense, the propagator we have just defined from GFT is a candidate for such a quantity. Moreover, from a mathematical point of view this quantity is better defined than some of the heuristic expressions we used in chapter 8, which allows for a more systematic study of its properties.

This example can be generalized to other spin foam models and in more spacetime dimensions by considering different GFTs, that is, by considering field theories defined on different group manifolds and with different actions[10]. For instance, generalizations of the Boulatov model like the Ooguri model[11] give rise to diagrams like the one in figure 9.3. In this case the strands are quadruple, each single line represents a triangle and each set of four lines, a tetrahedron. I express the correspondences between the diagrams of this model, the 2-complex and dual triangulation in table 9.2.

As we have just seen, GFTs can be seen as a way of defining and computing in a mathematically controlled way propagators like $K(s)$ for different spin foam models.

[8] For models in higher dimensions spin network functions are also labeled by intertwinners at the nodes of the graph. Notice also that the graph here is purely combinatorial, as it arises from the contractions of polynomials of φ. In this sense, this spin network space is smaller than the space we found in chapter 5, which included networks which allowed for knotting and that included other diffeomorphism invariant information, the moduli.

[9] Up to some sign corrections, see footnote 6.

[10] This relation between GFTs and spin foam models was established in Reisenberger and Rovelli (2001).

[11] See for instance (Baratin and Oriti, 2012a, Sect. 2.1) for an introduction to this model and the diagrams it produces. The original formulation of the model can be found in Ooguri (1992).

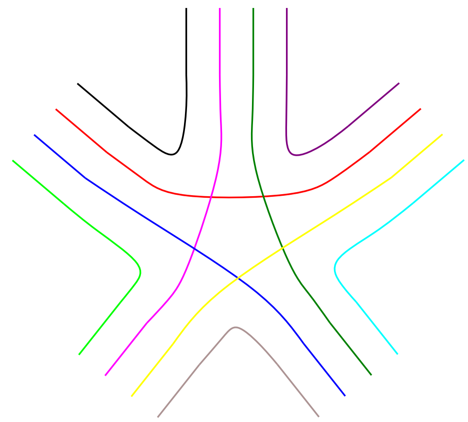

Fig. 9.3: Interaction vertex of the Ooguri model. This vertex is dual to a 4-simplex in four dimensions, as can be seen by comparing with the graph in figure 8.8.

Feynman diagram	2-complex	Triangulation
Vertex	Vertex	4-simplex
Quadruple strand	Edge	Tetrahedron
Single strand (closed loop)	Face	Triangle

Tab. 9.2: Correspondence between the elements of the Feynman diagrams of the Ooguri model and the different parts of the 2-complex and dual triangulation. Notice that the relation between the elements of the diagram and the 2-complex has not changed, while the elements of the triangulation are one dimension higher.

However, GFT is considered by its practitioners as a theory on its own and not just as a useful computational tool. In the rest of the chapter I will analyze the way GFT has been interpreted as an independent theory which also describes an emergent spacetime.

There are two ways in which GFT has been interpreted as a theory on its own. The first one is the covariant one which takes the theory to be defined by means of a partition function like the one we have introduced before, while the second one uses the canonical formalism of quantum mechanics. These two ways are presented in the literature as equivalent[12], but I will argue against this claim. I will further argue that neither of them is satisfactory. The canonical approach will suffer from a problem of time, while the covariant approach is even more problematic: the spinfoam transition amplitudes it defines suffer from the covariant version of the problem of time, and

[12] The distinction between the two is not commonly made explicit. I refer the reader to (Gielen and Sindoni, 2016, Sect. 2.2) for an example of how the canonical version of GFT is introduced as a reformulation of the covariant theory.

the interpretation of the rest of structures in the model is also unclear. In the next subsection I analyze the covariant approach, leaving the canonical one to section 9.2.

9.1.2 Definition of the GFT by means of a 'partition function'

GFTs are claimed to be defined by a set of mathematical structures like the ones I have defined in the examples above. These structures are a 'partition function' and some 'expectation values', and for defining them we need to specify the following: a group manifold on which the field lives, the gauge symmetries it satisfies, and an action. With this we can define:

$$Z = \int \mathcal{D}\varphi e^{-S[\varphi]} \tag{9.6}$$

$$\langle f \rangle = \int \mathcal{D}\varphi f(\varphi) e^{-S_B[\varphi]} \,. \tag{9.7}$$

In the GFT community this is taken to define a quantum theory. However, in section 7.6 I argued the similar view which defined similar 'partition functions' and 'expectation values' but for field theories defined on spacetime manifolds. The criticisms in that section also apply to this case, and even more strongly, as the manifold on which the fields are defined in this case is a group manifold and not a spacetime manifold, which further complicates their interpretation.

Let me repeat the worries I raised in section 7.6. First, these definitions are not equivalent in any straightforward way to the standard formalism of quantum mechanics, i.e., a quantum state evolving on a Hilbert space. This makes it impossible to apply any of the realist interpretations of quantum mechanics. Second, one may try to give an operational meaning to expectation values of functions of the field $\langle f \rangle$, which is again problematic, as it is not clear how one could, if possible, measure such a function and whether such a measurement could be repeatable so that the expectation value could be given an statistical meaning. The best operational interpretation would probably be the one I explained in the previous section, which related expectation values of spin network functions with the propagators of spin foam models. Nevertheless, there are two problems with this. First, it would be reducing GFT to just a way of computing the propagators of spin foam models, and second, that these objects are not easily interpreted either, as I argued in chapter 8 that they suffer from a version of the problem of time.

To these worries, we can add another layer related to the fact that GFTs are not defined on spacetime manifolds but on some group manifold. The definitions above clearly assign to such a group manifold the same role that spacetime plays in a quantum field theory. However, this further complicates the interpretation of the formalism. In a QFT, 'expectation values' of the form $\langle \varphi(x) \rangle$ were interpreted (even if sometimes wrongly) as the expectation value for what we would observe if we were to measure the field φ at the spacetime point x. By analogy, in a GFT $\langle\varphi(h)\rangle$ would represent an expectation value for the field at the point h of the group manifold. But the proponents of GFT certainly do not want to (and indeed they do not) claim that the group manifold represents something like a spacetime and that the expectation values correspond to measurements on such a manifold. Furthermore, in the case of a QFT defined by a 'partition function' one can recover the standard picture of quantum mechanics by defining a propagator in between two spacelike boundaries of the manifold, but this move will not be available for the case of the GFT if we reject interpreting the group manifold as a spacetime, as this would imply rejecting any view in which slices in such a manifold represent instants of time.

The reasons for which one can reject interpreting the base manifold as a spacetime is not only its dissimilarity or that spacetime is supposed to play a different role, but also that considering it to be spacetime is also problematic. For instance, if we tried to define an evolution in this space we would find a series of challenges as identifying which direction represents time or how to deal with the non-locality of the action. This problem would prevent someone to define a Schrödinger-like evolution, as this non-locality is a non-locality not only in space but also in time. This blocks not only the application of quantum mechanics, but even the classical Lagrangian and Hamiltonian mechanics require to have a temporally local dynamics. In this sense, if one wanted to interpret the group manifold as a spacetime, one would have to accept that it would be certainly different to the one we are used to.

Finally, let me mention that in GFT spacetime as we know it is claimed to be emergent. As I have argued that the we are lacking a convincing interpretation of the formalism of GFT, it remains unclear the way it is supposed to represent a fundamentally non-spatiotemporal reality out of which spacetime would emerge. The best way for supporting the claim of emergence is that the transition amplitudes for spin network functions allow to make a connection with spin foam models. However, this is of course problematic as in section 8.3.3 I have rejected that spin foam models give a successful account of a possible emergence of spacetime.

Let me finish this section by insisting in that the formal definitions given above do not seem enough for providing a full-fledged theory with its own interpretation. In the next section I will analyze how this theory is claimed to be expressed in the canonical formalism, although I will argue against this equivalency.

9.2 Canonical GFT

In this section I introduce and analyze the quantum theories that can be constructed inspired by some of the ideas of the covariant GFTs introduced in the previous section but using the canonical formalism of quantum mechanics. This theory is defined in two steps: a definition of a kinematical Hilbert space and the imposition of some constraint which defines the dynamics of the theory. In subsection 9.2.1 I analyze the kinematical Hilbert space proposed, which is a Fock space for the GFT field and which has some similarities with the Hilbert space of LQG. Then, in section 9.2.2 I analyze the way the dynamics is defined. I argue that the theory defined is in no clear way equivalent to the covariant models I introduced in the previous section and that the theory suffers from a problem of time. In this sense, I conclude that the canonical version of GFT has, at least, the same shortcomings as the canonical approaches I have studied in this thesis.

9.2.1 The kinematical Hilbert space

In chapter 4 I introduced canonical quantization as a process that allows to construct a quantum theory from a classical one. In GFT this procedure is not followed, but instead one postulates some structures that imitate the ones we have found for theories like LQG: one starts by postulating a kinematic Hilbert space and then one defines the physical Hilbert space as the space which satisfy some appropriate constraints. Notice that one cannot define the canonical GFT by directly applying the canonical quantization methods to some of the classical actions in the previous section. The reason for this is that these actions define classical field models just in the sense that by means of an action principle one could postulate that only fields which minimize those actions would be physical. However, these actions are non-local, and so are their associated equations of motion and hence they cannot be expressed as a classical theory on a phase space, which is the starting point for the canonical quantization of quantum mechanics. In this sense, one cannot canonically quantize a classical GFT model in a strict sense, and instead one defines a quantum theory just inspired by this model. In this subsection I will analyze the kinematical Hilbert space proposed.

The Hilbert space postulated for GFT is the Fock space for the field defined on the group manifold of the GFT we are considering. This is motivated by considering the field as what plays a fundamental role in the theory, but there is no further justification for this given that we are not following any quantization procedure but just directly postulating a model. The Fock space for the GFT model is however an interesting one, which I will comment on next.

Recall that Fock spaces are the spaces used in QFT for quantizing a field and allow for a particle interpretation, since one can decompose a Fock space as a sum over subspaces which correspond to a different number of field excitations or particles:

$$\mathcal{F} = \bigoplus_{n} S(\mathcal{H}^{\otimes n})\,. \tag{9.8}$$

Here $\mathcal{H}$ represents the Hilbert space of one such particle or excitation. In the case of a QFT like a scalar field theory this corresponds to the space $L^2[\mathbb{R}^3, dx]$, that is, the Hilbert space used for describing the quantum mechanics of one particle in three-dimensional space. The space $\mathcal{H}^{\otimes n}$ is the product of n copies of this space, and hence represents a state which describes n particles living in space.

To completely define a Fock space one also needs to specify the symmetry conditions that will be satisfied by the states of the theory, which is imposed by means of the symmetry operator S, which restricts the symmetry properties of the subspaces of the space. In the case of GFT it is generally chosen[13] to have symmetric states, which allow for the bosonic statistics needed for using condensate states which will be useful later on.

In the case of GFT, the space in which particles live is not ordinary space, but the group manifold of the theory. In this sense, the role that position played in characterizing a QFT state is now played by the group variables of the theory. In the case of GFTs aiming to recover a 4-dimensional spacetime one common option is to choose the field to be defined on 4 copies of SU(2). This means that the single particle Hilbert space is chosen to be $L^2[\mathrm{SU}(2)^4]$, which has as a basis the states $|g_1 g_2 g_3 g_4\rangle$. For a state representing n particles, the Hilbert space is $L^2[\mathrm{SU}(2)^{4n}]$. Recall that the kinematical Hilbert space of LQG for a graph, as I introduced it in section 5.2.1, was precisely $L^2[\mathrm{SU}(2)^L]$, where L is the number of links in a graph. This suggests that there is a relation between the GFT and the LQG Hilbert spaces. Indeed, one particle states in GFT are considered to represent something equivalent to a node of a spin network but disconnected, and by gluing different GFT particles one can build a spin network[14]. That is, we can define a subspace of the Hilbert space of GFT which corresponds to states that look like spin network states, i.e., we can represent them by means of a network which joins the particles that are glued together and by an assignment of group elements to the links of the resulting graph.

A couple of comments need to be made here. First, the inner product in the GFT Hilbert space when restricted to the class of spin network states is **not** the same as the inner product of the LQG Hilbert space[15]. Therefore, the Hilbert space of GFT does not strictly contain the space of LQG. Second, notice that nodes in GFT are restricted to be four-valent, as the one particle space was defined on four copies of SU(2), while in LQG we have nodes with arbitrary number of nodes. Besides, networks in

[13] In the case of a QFT defined on spacetime the spin-statistics theorem makes it the case that depending of the spin of the particle one symmetry condition or the other needs to be applied, leading to the two different kinds of particles: bosons and fermions. As GFTs are not defined on a spacetime there is more freedom at the time of defining the theory in the most convenient way.

[14] The details of how this gluing can be defined can be found in Chirco et al. (2019).

[15] The reason for this is that in the GFT Hilbert space there may be overlap between states with the same number of nodes but different graphs, while in LQG the inner product between two such states is necessarily 0.

GFT are purely combinatorial, while in LQG the networks one finds by applying the canonical quantization procedure have additional structure, as knotting classes and moduli. In chapter 5 I commented that in LQG there are authors who defend striping this additional structure from the states of the theory while others would argue for retaining them. In the case of GFT, graphs are necessarily combinatorial.

In the case of LQG the theory had a gauge invariance that implied that we had to apply a further constraint at every vertex. In GFT, if one wants to mimic this one can impose that the field is required to obey some symmetry condition like being invariant under transformations of the form $\varphi(g_1, g_2, g_3, g_4) \to \varphi(hg_1, hg_2, hg_3, hg_4)$ for any $h \in \mathrm{SU}(2)$. This implies that a basis of the one-particle Hilbert space is given by states of the form $|j_1, j_2, j_3, j_4, v\rangle$, and their gluing leads to gauge invariant spin network states. Indeed, this is precisely the form of the states that I discussed in section 8.2.2 when discussing the boundary Hilbert space of the EPRL model. There, I argued that a state of the form $|j_1, j_2, j_3, j_4, v\rangle$ could be interpreted as representing a quantum tetrahedron, where the quantum numbers j would carry the information about the areas of the faces of the tetrahedron and v about its volume. This suggest interpreting the GFT quanta as atoms of space which can be combined to formed a simplicial decomposition of a space and relates this Hilbert space more with the Hilbert space of spin foam models than with the general Hilbert space of LQG.

However, notice that in the case of the states we considered for spin foam models in chapter 8 not every assignation of quantum numbers represented a simplicial decomposition of a space, as it is the case that one cannot always determine in a consistent way the lengths of all the edges of all the tetrahedra so that they satisfy that the areas and volumes are given by the quantum numbers. In the case of GFT we also have the additional problem that the way the atoms are connected can be such that it is not compatible with a simplicial manifold. In this sense, only some states in the Hilbert space of GFT can be associated with discrete spaces.

Moreover, even for those states we should be cautious when claiming that they form a space. If we assume that atoms are literally tetrahedra, then we are already assuming a lot of geometric concepts and it is not surprising that a space emerges from this description, as we are literally assuming that the basic ingredients for our description are chunks of space. If instead we assume that atoms are entities with algebraic properties, the way they relate with the emergent geometrical description seems more complicated. For instance, it does not seem justified to impose a flat geometry inside each tetrahedron, as this would be assuming more structure than the one that is provided by the albegraic and combinatorial information of the GFT states. For this reason Dowker and Butterfield (2021) make emphasis on the distinction between geometric and combinatorial simplicial complexes: while geometric simplicial complexes are geometric, combinatorial ones are purely algebraic, and further argument is needed if one is to claim that the former emerges from the latter. Schematically, we can describe the approximation process as a series of steps:

> GFT state → Combinatorial simplicial complex → Geometric simplicial complex → Smooth manifold (space)

The first and third steps are uncontroversial, i.e., it is clear the way some GFT states have the same structure as a combinatorial simplicial complex and it is clear the way geometric simplicial complexes are triangulations which approximate manifolds. As I was commenting, the non-trivial step is to go from the purely algebraic and combinatorial to the geometric, which in the GFT literature seems to be just assumed.

We can take the approximation process sketched above as a schema that the spacetime functionalist could use for arguing that the roles of space can be played by GFT states, as I discussed in section 3.3.2. The functionalist argument in this case is not very different from the one for LQG states in section 5.2.3 or the one for the boundary Hilbert space of the EPRL spin foam model in section 8.2.2. The functionalist argument seems plausible, although it would be good to strengthen the step from the combinatorial to the geometric by further studying how specific functions of space, such as localization, could be represented in a combinatorial simplicial complex. Also notice that in the case of canonical LQG one could try a functional argument without assuming that states are related with a triangulation by studying weave states. This is clearly not the way chosen in the spin foams or GFT literature, but it seems to be available.

Finally, let me also point out that while the Hilbert space of LQG was found by following the canonical quantization procedure for general relativity in one particular form, this Hilbert space structure seems to be less justified, as the arguments for using it rely on some way or another on spaces and models of LQG and spin foam models. That is, we can introduce GFT inspired by those other models, but it certainly takes us one step further away from general relativity. In this sense, the worries one may had about the relation between LQG and general relativity are worse in the case of GFT.

9.2.2 GFT dynamics

Now we can consider the way the dynamics is defined in the canonical version of GFT. As I commented above, the theory is not defined by following a canonical quantization procedure but it is just directly postulated. Therefore, it could have just defined a Hamiltonian operator which would have defined the evolution of states, but instead it was chosen to define the dynamics in the way it is defined for theories with a temporal reparametrization invariance and a problem of time, i.e., it is argued that states which satisfy some constraint contain the dynamical information of the theory.

For the case of quantizations of gravity I have argued against this claim[16], and for the case of the canonical version of GFT I will argue for the same conclusion.

To make a connection with covariant GFT, one defines the constraint, and hence 'physical' states, in relation to one of the actions introduced in the previous section. In particular, the constraint equation which is imposed is the following:

$$\frac{\delta S[\hat{\varphi}]}{\delta \hat{\varphi}(g)}|\psi\rangle = 0\,, \tag{9.9}$$

where we have quantized the classical action by replacing the fields by field operators[17]. In Gielen and Sindoni (2016) it is argued that this choice of dynamical equation is a natural one: by varying the action with respect to the field one obtains the classical equation of motion, by translating this equation to one expressed using quantum operators[18] we obtain a natural candidate for the constraint. Another alternative is to select the states such that they satisfy a tower of equations known as the Schwinger-Dyson equations:

$$\langle\psi|\hat{O}\frac{\delta S[\hat{\varphi}]}{\delta \hat{\varphi}(g)}|\psi\rangle = \langle\psi|\frac{\delta \hat{O}}{\delta \hat{\varphi}(g)}S[\hat{\varphi}]|\psi|\rangle\,. \tag{9.10}$$

These equations have to be satisfied for every operator $\hat{O}$ in the GFT Hilbert space. In particular, notice that a state satisfying 9.9 automatically satisfies the Schwinger-Dyson equation for $\hat{O} = \mathbb{I}$. In this sense both choices are approximately equivalent. The option of selecting states which satisfy Schwinger-Dyson equations is of extended use in the GFT literature[19].

In standard QFTs Schwinger-Dyson equations like the ones I have presented arise naturally from a covariant approach in which they are derived as a consequence of functional calculus applied to path integrals. In the case of GFT a similar motivation has been given[20] by using path integrals like the ones defined by 9.6 and 9.7. However, as I have argued in the previous section, these expressions are not clearly associated with a Hilbert space structure, and hence it is not possible to identify expressions of the form $\langle\psi|\hat{O}|\psi\rangle$ (like the ones in 9.10) with expressions of the form $\langle\hat{O}\rangle$ as defined by the path integral 9.7. In this sense, the connection with the covariant formalism is only heuristic and the imposition of equations 9.10 is a reasonable choice but not imposed from any principle.

[16] See again the discussion in sections 4.6 and 4.7.

[17] As every time we have introduced a quantization there may be some ordering ambiguity in this step.

[18] As emphasized at several points of this thesis this process is not trivial, as there may be ordering ambiguities.

[19] See for instance Gielen et al. (2013b); Oriti (2017).

[20] See for instance (Gielen et al., 2013b, Sect. 5).

A justification for these dynamical equations, beyond the fact that the diagrammatic expansion of a GFT partition function gives rise to spin foams, was provided in Oriti (2016). There, it was shown that one can derive an equation like 9.9 by starting with the Hamiltonian constraint of LQG and translating it into a constraint in the Hilbert space of GFT, which can be done given the relation between both spaces explained above. In this sense we find that the connection of GFT with general relativity is again mediated by means of LQG.

For any way of constraining the set of physical states we find that there is a problem of time for GFT in the same way that there was one for geometrodynamics or LQG. That is, the physical sector of GFT consists in superpositions of (discretized and with some caveats) three-dimensional geometries, and a priori none of the variables describing these states is a time variable with respect to which one could define an evolution. In the case of the quantizations of general relativity, I argued in chapter 4 that the most reasonable reading of the situation is that states in the physical Hilbert space do not represent anything like an dynamics and that the proposed resolutions of the problem fail, given that they are based in the misleading example of deparametrizable models. In the case of GFT, it is common to apply a relational strategy very similar to the one I studied in chapter 6, and which I will argue fails to be a satisfactory resolution.

The way this strategy is implemented is by expanding the theory by including 4 additional degrees of freedom which are interpreted as scalar fields and which are used as 'clocks and rods', i.e., with the aim of representing relational observables which single out spacetime points. In Li et al. (2017) it is argued that scalar fields can be represented in GFT by making the GFT field dependent not only on the group variables but also in the scalar fields and by modifying the action to include a dependence on these variables. In this way, in the expansion of the partition function one gets Feynman diagrams which are dual to spin foams which are 'dressed', that is, spin foams which carry information about the scalar field at each of their vertices.

The addition of the scalar fields implies modifying the kinematical Hilbert space of the theory. But notice that there are several options available. For instance, the simplest would be to leave one scalar field, let me call it ϕ^0, unquantized and take it as time parameter with respect to which the quantum state evolves. In this case, the imposition of 9.9 would lead to a Schrödinger-like equation with respect to the parameter ϕ^0 and the problem of time would be answered in a similar way to the one of LQC, which I criticized in chapter 6.

Alternatively, one can build a bigger Fock space, one in which each particle carries information about both the group and field variables. The straightforward interpretation of such a Hilbert space would be just the same as before but with the addition that now the discrete geometries carry more information. As we have seen for other cases, for the relational resolution of the problem of time one needs to identify a clock variable that allows to describe evolution with respect to it. In this case, we would like this clock variable to be the field ϕ^0 [21] but we find the problem that ϕ^0 is not directly an observable on this Hilbert space. Instead, each particle has associated its own value of ϕ^0, so it is like every particle is carrying its own clock. In this sense, if we insist on interpreting ϕ^0 as measuring time, generic states describe particles which are at different 'times' and cannot generally be decomposed in terms of a nice temporal evolution. That this Hilbert space is problematic for defining relational dynamics is recognized and discussed in more detail in Marchetti and Oriti (2021).

In cosmological applications of GFT, the states considered are restricted to a very specific subset, for which a relational dynamics can be defined, as I will explain in the following section. For the rest of states it seems that the case of GFT is not different from the one of other theories with a problem of time: once we consider realistic situations with some variety of variables involved it becomes hard to define a relational dynamics in a way that allows to formally overcome the problem of time. And I say formally because from a conceptual point of view I rejected those resolutions[22].

Also notice that for most cosmological applications one just adds one scalar field and no 4 based on arguments of symmetry and one is able to define the evolution of a (discrete) three-dimensional space in with respect some 'time'. In this sense, it seems that the addition of so many scalar fields was not really necessary, specially because the kinematic Hilbert space was already able to describe three-dimensional geometries without need of introducing fields to 'coordinatize' space. And we can even raise a question also about how necessary the addition of ϕ^0 is, given that in LQG, according to its proponents, it is not necessary to add any additional field to solve the problem of time.

Another point to make is that a priori any of the four fields introduced could have played the role of time, and hence it seems that there would be a hard problem of spacetime functionalism as defined by Linnemann and Le Bihan and as I introduced in chapter 3. Recall that they claimed that when the structure of the quantum theory of gravity does not define a difference between spatial and temporal degrees of freedom, then there is a problem for the emergence of spacetime, as the emergent structure has this difference. Other relational resolutions of the problem of time also faced this problem, which I argued in chapter 6 that was a potential shortcoming of the relational strategy in general.

Finally, we have to consider the physical interpretation of the fields ϕ. If we consider them to be physical fields we have a serious problem of time and we should

[21] Or any of the other 3.

[22] I refer the reader back to the discussions in chapters 4 and 6.

reject the relational resolution. That is, a state in the kinematical Hilbert space would represent something like a discrete space with some fields defined on it, which would be analogous to the kinematical Hilbert space for a quantization of general relativity with some matter fields. In the same way we claimed that it was wrong to interpret some of the degrees of freedom as a time, the same would apply to GFT and the relational resolution would be insatisfactory and we should reject the theory. There is the option of thinking about ϕ_0 as a time variable and not as a scalar field. But we see that this move is a bit ad hoc, in the sense that we would be solving the problem by just adding the time variable we are missing. Of course, this strategy was not available in the case of general relativity as there is no motivated expansion of the theory to include such variable.

In conclusion, the canonical version of GFT suffers from a problem of time which makes it vulnerable to similar criticisms to the ones I have rehearsed for geometrodynamics, LQG and LQC. In this sense, it seems that we should consider the canonical version of GFT not as a complete, satisfactory theory, just as happened for those approaches. Let me also mention that the theory we have defined is not in any straightforward way equivalent to the covariant version of section 9.1. For quantum theories defined following the canonical and the covariant formalism as I have introduced them in chapters 4 and 7 I argued that one can show the equivalence in certain circumstances. In the case of GFT, neither the covariant nor the canonical versions were formulated following the quantization procedures, and hence one cannot expect to extrapolate the equivalency to this case. Moreover, the two approaches define completely different structures (partition function and expectation values and states on a physical Hilbert space) and hence it is not clear the way the equivalence is supposed to go. In any case, even if the equivalence were shown, I have argued that both approaches suffer from major conceptual shortcomings. In the following sections I will study the applications of GFT to cosmology. Even if the skeptic conclusion about GFT in general will also be applicable to these models, from a conceptual point of view they are interesting, as they introduce some notions from condensed matter which could in principle suppose an improvement over LQC and LQG models and which suggest considering the big bang as a phase transition.

9.3 GFT condensates and cosmology

As we saw in the other chapters of the thesis, the extent to which LQG is applied to cosmology is limited. First, models of LQC are not derived from LQG but are just believed to an approximation based on some heuristic reasoning. Second, models using the full theory of loop quantum gravity or some spin foam models become computationally complex when one wants to consider a big universe containing a great number of nodes and links. For this reason, it is interesting to consider alternatives[23] which allow to deal with a great number of excitations. Such an alternative is available in GFT, given that its Fock space structure is similar to the one used in condensed matter theory for representing systems with a great number of particles. In particular, the states that are used for cosmological applications of GFT are condensate states like the ones used for modeling Bose-Einstein condensates. In this section I will start by introducing such states, I will then analyze the way they are applied in the case of GFT cosmology and I will argue that GFT models suffer from the problem of time, and hence that there are reasons to believe that we lack a solid justification for them and that their interpretation is an ad-hoc one which can be challenged.

A condensate state is a state for a bosonic system in which all or nearly all particles are on the ground state[24]. In particular, we can approximate the N-body wavefunction as a product state of N identical states:

$$|\Psi\rangle = |\psi\rangle \otimes |\psi\rangle \otimes .... \otimes |\psi\rangle\,. \tag{9.11}$$

This approximation is a great simplification which allows to describe the quantum state of a system of many particles in terms of just one one-particle wavefunction $\psi(x)$. The equation which describes the dynamics of ψ is the Gross-Pitaevskii equation[25], which looks similar to the Schrödinger equation but which includes terms that account for the interactions between the different particles[26]:

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} + g|\psi|^2\right)\psi\,. \tag{9.12}$$

We can express any solution of this equation as:

$$\psi(x,t) = \sqrt{\rho(x,t)}e^{i\theta(x,t)}\,. \tag{9.13}$$

Where, making use of the Born rule, $\rho(x,t)$ would represent a probability density[27]. However the interpretation that is given to this function is a different one: instead of interpreting ρ as some probability it is directly interpreted as the actual density of condensate particles. In this sense, ρ would describe the macroscopic properties of

[23] Another alternative was explored in Bojowald et al. (2012), where they present a model which is closer in spirit to LQC.

[24] For the case of fermionic systems this is not possible, as the exclusion principle forbids having two particles with the same state.

[25] See for instance Rogel-Salazar (2013) for a derivation of this equation. In this case the equation is found by means of thermodynamic arguments: it is shown that a state satisfying the Gross-Pitaevskii equation minimizes the free energy, once one considers a number of approximations.

[26] This can be seen as a mean field approximation.

[27] In the case of the Gross-Pitaevskii equation this is usually normalized to N.

the condensate. A heuristic justification for this is that for thermodynamic systems like the condensates we are in the limit of large N, and here we expect frequencies to match probabilities. In other words, we take ψ to describe the overall distribution of particles in the condensate instead of being a property of each particle. In this sense, one could say that ψ has become a classical field[28], and the predictions made using this classical field interpretation agree with the experimental results[29]. From a rigorous point of view, the identification of ρ as a macroscopic classical variable is not fully justified unless one deals with the interpretative issues of quantum mechanics and the measurement problem. But for practical applications, this identification is a useful one.

Condensates can also be described using the second quantization formalism and the Fock space structure of quantum field theory. The use of the field theory techniques has been proven useful for the study of these systems, and it allows for a direct application for the case of GFT. Instead of considering a N-particle state like 9.11 one considers a coherent state. The coherent states used for describing condensates are a good approximation to N-particle states for large N, in the sense that the expectation value of the number operator is N and it has only small fluctuations. Coherent states are eigenstates of the field operator, and hence they are characterized by their eigenvalue ψ, which plays the same role as in the case of the N-state case: besides describing the quantum state it is also argued to represent the classical, macroscopic properties of the system.

In applications of GFT to cosmology, condensate states are used as a powerful approximation. Condensate states allow to describe systems of many particles in a simple way. A condensate state in the GFT Hilbert space is a state representing a great number of GFT quanta in the same state. As such, it does not represent the gluing of different quanta to form a space, but just a collection of disconnected atoms of space. Even though the connectivity of the atoms is an important feature for the emergence of a space from GFT, condensate states are believed to represent correctly other features of cosmological importance, like the expectation value for the volume of the universe[30].

As in the case of a standard Bose-Einstein condensate we can take φ, the eigenvalue of the field operator to represent a classical distribution of particles. In this case each particle would be represented by a tetrahedra and the value of $|\varphi|^2$ for some values j[31] tells us how many tetrahedra with the geometry as described by j there

[28] The field ρ is now interpreted as a density field and θ is a field that will describe the velocity of the condensate.

[29] These results include phenomena like superfluidity and superconductivity.

[30] Indeed, this is the main cosmological observable studied in GFT cosmological models as Oriti et al. (2016); de Cesare and Sakellariadou (2017); Gielen (2018, 2019).

[31] Here I am taking φ in its representation basis, which is the one which can be translated into

are. This is just the same as in the case of the Bose-Einstein condensate case, where $|\psi(x)|^2$ is interpreted as a density function which tells how many particles are there at position x. By adding up[32] the volume of all the atoms we obtain a value for the total volume of the universe, which is the observable of cosmological importance.

As we saw in the previous section, the dynamics of GFT is defined by a equation or set of equations of the form of 9.9 or 9.10. In the case of cosmology the same equations are imposed, and the problem of time is addressed by means of the relational resolution: a scalar field ϕ^0 is introduced and evolution is defined with respect to this field. In this sense we can read $|\varphi(j, \phi^0)|^2$ as giving the number of tetrahedra with geometry j at 'time' ϕ^0. Given the simplicity of the condensate states one can avoid considering the technical difficulties mentioned in the last section for the definition of a relational time.

The cosmological models obtained in this way are very similar to the ones obtained in the case of LQC and analyzed in chapter 6. They are similar in that they predict the same qualitative behavior, i.e., both kinds of models describe bouncing cosmologies, even if they can differ in the exact expressions[33]. And more importantly, the GFT cosmological models are subject to some of the same criticisms that I raised against LQC models. Namely, the way the problem of time is dealt with is unsatisfactory. For instance, in the case of LQC we would have wanted to have a quantum theory able to treat both the volume and the scalar field as quantum observables subject to all quantum phenomena like superposition or entanglement, but choosing the scalar field as a time variable makes it become classical. Similarly, the same applies in GFT. As I have argued in this thesis and in the previous section, we should be careful with the models of quantum gravity or cosmology which suffer from a problem of time, as both their justification and interpretation can be rightfully challenged.

The conclusion of the analysis of the GFT cosmological models is that they offer an interesting idea, the use of condensates, for improving on the cosmological models of LQG and LQC, but that they are subject to the same major conceptual problems. In the next section I will discuss the other idea from condensed matter theory that has been imported to cosmology in the context of GFT: phase transitions. In GFT, it has been suggested that the big bang was a phase transition between two different phases the GFT quanta could be organized in. I will argue that there are some issues (beyond the general worries about GFT itself) in the way this idea is supposed to be

the geometric properties of tetrahedra.

[32] Alternatively, one can also determine this volume by computing the expectation value of the volume operator that can be defined for the theory. Taking this alternative is similar in that it does not address the measurement problem, but just claims that this expectation value is physically meaningful.

[33] See for instance the models in Oriti et al. (2016); de Cesare and Sakellariadou (2017); Gielen (2018, 2019); Gabbanelli and De Bianchi (2020).

implemented in the case of GFT.

9.4 Phase transitions and geometrogenesis

In this last section I will analyze how GFT includes the possibility of having different phases and transitions between them. The basic idea would be that for different values in the action, the dynamics would select different types of states. In particular, there would be some that would approximate a spacetime as we usually think about it, i.e., the GFT quanta would glue to each other nicely so that we obtain something like a simplicial decomposition of spacetime, while some other states would not correspond to spacetimes because, for instance, the gluing relations are not the appropriate. There is obviously a problem with this picture according with what I have argued in this chapter about the problem of time. Namely, I have argued against the view that states in the physical Hilbert space of GFT approximate a spacetime, as they are just superpositions of potential 3-geometries. Leaving this problem aside, let us nevertheless assume that GFT states can represent something which approximates a spacetime for the sake of the discussion. The possibility of having a transition from one kind of state to the other and the connection between GFT and condensed matter systems used in cosmology suggests thinking about this possibility in terms of phase transitions. In particular, it is speculated that such a phase transition is possible, and that this is what could have happened at the big bang, in a process that is called geometrogenesis[34]. In this section I will first review the general notion of phase transition (9.4.1) and then I analyze the way this is supposed to be implemented in the context of GFT (9.4.2). I will argue that the idea is interesting but, as it stands, it is either incompatible with GFT as discussed and introduced in the rest of the chapter or it needs of an additional and unwanted temporal dimension that is nevertheless unable to identify the phase transition with the big bang singularity.

9.4.1 Phase transitions in thermodynamics

Let me briefly review the way the concepts of phase and phase transition as they are employed in some branches of physics. In thermodynamics, a system is said to have different phases if for different values of its thermodynamic properties, e.g. for different temperatures, it behaves qualitatively differently. The most common

[34] The term geometrogenesis has been also used in different approaches to quantum gravity. In GFT, and in this section we will use the term geometrogenesis to refer to a physical process by which 'geometry is created', a process that transforms a pregeometric state of the universe into a geometric one. Keep in mind that in some other approaches the term geometrogenesis may be used with different meanings. For instance, I refer the reader to Markopoulou (2009) for an example of how geometrogenesis is conceived of in quantum graphity.

example is water: depending on its temperature it can behave as a solid, as a liquid or as a gas. This can be represented in a phase diagram: this is a diagram which represents the different values the thermodynamic properties of a system can have and it is divided into different regions, which correspond to the different phases.

Notice that in thermodynamics we deal with equilibrium states, so there is no temporal evolution associated with the points of the phase diagram. Now, the thermodynamic properties of a system can be modified[35] and it can be the case that the system that started in one given phase ends up in a different one. An example of this process is what happens when we heat up ice until it melts. This process is called phase transition and is represented in the phase diagram by a flow connecting two points in different regions, in different phases.

Phase transitions have a close relation with the renormalization group[36] which it is important to mention because of the role it will play in GFT. Roughly speaking, when we apply the renormalization group to a condensed matter system, we zoom out to get a coarser grained picture of the system, that is, we blur the short scale details to focus on the large scale. The renormalization group is therefore a useful tool to study the system from different scales. It can be shown that the scaling properties of a system that is undergoing a phase transition are special, and, in particular, at a critical point[37] systems show the same behaviour at all scales, and therefore it is invariant under the action of the renormalization group. In other words, no matter how much we zoom out that if the system is at a critical point it will always look the same. When a system is in some different point on the phase diagram and we zoom out, the properties of the zoomed out version will look different. For example, for a gas we would say that the zoomed out version has a different temperature and pressure. The renormalization group then defines a flow on phase space: it connects each point with others that can be seen as representing zoomed out versions of the system represented by the original point. Now, it has been shown that all the points in a same phase show the same behaviour under this flow, that is, they all flow towards the same points. Therefore, studying the renormalization group flow allows one to identify different phases.

There is a different version of the renormalization group flow which is the one that is used in high-energy physics. In this context, the flow is a flow in a space of theories rather than a flow in a phase space which describes systems with different thermodynamic properties. The flow in this case connects theories which have different cut-offs, i.e., different regimes of validity, but which agree in their predictions for

[35] This can be seen as a process produced by us adjusting some knobs in a lab, or as natural processes happening when systems are not isolated and interact with an environment that changes.

[36] For technical discussion of this see Kumar (1993); Cardy (2015).

[37] A critical point in a phase diagram indicates the last point where the coexistence of different phases is possible.

the phenomena in the regions of validity they share. For instance, in quantum electrodynamics the renormalization group flow relates two different theories, one with a high-energy cut-off and one with a lower one, and both theories can be used for describing the same phenomena, e.g. the interactions of photons and electrons, at low energies. In this sense, the renormalization group flow does not represent any physical process but it is just a flow in an abstract space of theories.

We have therefore to distinguish the renormalization group flow in the two situations: it can either represent a coarse-graining process or it can be relating different versions of some theories with cut-offs. It is in the first case in which it contains information about the different thermodynamic phases of a system. In any case, to insist, the flow is an abstract flow which does not correspond to any physical process. Physical phase transitions correspond to changes of the thermodynamic parameters of a system and these changes, as all changes between thermodynamic equilibrium states, are driven from outside the system. Moreover, the way these parameters change depends on the environment of the system and it is completely independent of the renormalization group flow.

9.4.2 Geometrogenesis in GFT

After having reviewed these basic notions of phase transitions, we can turn to Oriti's proposal of geometrogenesis in the context of GFT as presented in Oriti (2014, 2021). As I have explained in the previous section, in GFT condensate states are believed to be able to represent cosmological spacetimes. Oriti (2014) proposes two further hypothesis, that we conceive of a condensation process by which these condensate states are formed and that we identify this process with the big bang singularity:

> That is, identify the process of quantum spacetime condensation with a known, even if not understood, physical process: the big bang singularity. Better, we identify the coming of the universe, that is of space and time, into being with the physical condensation of the "spacetime atoms". There was no space and not time before this condensation happened. Therefore, we could call the spacetime condensation *geometrogenesis*. (Oriti, 2014, p. 194, his emphasis)

Notice that if we take seriously that there was no space and time 'before' the condensation happened, we find some tension, as we would naively think of the condensation process as a process happening in time. Oriti adds a footnote to this passage precisely on this point:

> The "tensed" wording is inevitable and can only refer to some internal time variable, which started running monotonically from the condensation onwards, just like the corresponding tensed statements about the

> evolution of the universe in (quantum) cosmology; for example, this variable could be a hydrodynamic variable corresponding to the volume of the universe. (Oriti, 2014, Footnote 11, p. 194)

Here we see that Oriti is appealing to the notion of internal time (something like the relational time ϕ_0 in my discussion above), that allowed to define an evolution for a given state. However, as Oriti notes, this internal variable is only available from the condensation onwards, and hence it cannot work as a time variable for the transition itself. In this sense, this footnote does not clarify how we should take tensed language when referring to the transition and it is so far unclear the way in which this transition is physical. Later on, Oriti is more explicit about the way this transition is implemented:

> We have seen that GFTs, just as the field theories describing the fundamental atoms in condensed matter systems, are defined usually in perturbative expansion around the Fock vacuum. In this approximation, they describe the interaction of quantized simplices and spin networks, in terms of spin foam models and simplicial gravity. The true ground state of the system, however, for non-zero couplings and for generic choices of the macroscopic parameters, will not be the Fock vacuum. The interacting system will organize itself around a new, non-trivial state, as we have seen in the case of standard Bose condensates. The relevant ground states (which, due to diffeomorphism invariance, cannot correspond to minima of an energy functional) for different values of the parameters (couplings, etc.) will correspond to the different macroscopic, continuum phases of the theory, with the dynamical transitions from one to the other being indeed phase transitions of the physical system we call spacetime. (Oriti, 2014, p. 195)

In this passage Oriti makes the analogy with condensed matter systems which may have different phases for different values of the parameters which describe them. A key insight used in statistical mechanics to determine the macroscopic properties of a system is that equilibirum thermodynamic states minimize or maximize certain thermodynamic potentials. For different values of the thermodynamic parameters, the thermodynamic potentials change, and this makes it the case that for different values of the parameters we may have different phases, which correspond to different extrema of the potentials. In the case of Bose condensates that Oriti mentions[38], the thermodynamic potential that is minimized is directly the energy functional of the field theory associated with the condensate, and hence one is interested on the ground

[38] See his presentation on section 5 of Oriti (2014).

state of the system. A change in the parameters of the system may imply a change in the ground state of the system, leading to a phase transition. For a thermodynamic system, these parameters may be fixed, e.g., the masses of the molecules that form a gas, or may be externally changed, e.g., the temperature or pressure of the gas.

This analogy does not fit well with the case of GFT. First, in thermodynamics each state represents the way a system is at a given time, and we can represent a process in which this state changes as a process in time. However, in the case of GFT each condensate state is interpreted to represent spacetime and not a state of a system at a time. If the system is spacetime as Oriti claims, a phase transition would require to introduce an additional evolution parameter that would connect the different spacetimes or GFT states. This would amount to introduce a metatime or fifth dimension, which would be an undesired consequence. That is, evolution in this parameter would connect different spacetimes, and possibly also different GFT states which do not represent spacetimes. But we lack any evidence of any such fifth dimension, and it is not clearly related with the big bang singularity. Hence we end up with some additional unwanted structure which does not do the job it was supposed to do. Second, in thermodynamics we can account for the change of the thermodynamic properties as changes external to the system, but in the case of GFT we seem to be lacking a story of how and why the parameters in the theory change and give rise to different phases. In this sense, the GFT phase transition, as a physical process, is unclear and somewhat mysterious as described in Oriti (2014).

Oriti (2021) addresses some of these issues. First, he rejects understanding the phase transition as a temporal process:

> The main difficulty is the immediate temptation to interpret a cosmological phase transition not only as physical but also as a *temporal* process. This is also a problem with the very language we use to characterize physical *processes*. A phase transition is pictured as the outcome of 'evolution' in the phase diagram of the theory, or of a 'flow' of its coupling constants; we say we 'move' towards the cosmological, geometric phase from the non-geometric, non-spatiotemporal phase, or viceversa. However, we are dealing with a system which is already described at level 2: there is no continuum space, no continuum time, no geometry in the usual sense; and it is also not characterized by features which are just 'one approximation away' from time and space. (Oriti, 2021, p. 30, his emphasis)

In this paper Oriti uses levels to speak about the way possible theories relate with continuum spacetime: the higher the level, the further away one is from the spacetime of general relativity. In this case, with level 2 he is referring to the phase which is described a GFT state which cannot be given a geometric interpretation in terms of space and time. As at this level there is no notion of time, he rejects using temporal notions for describing the transition. However, if we are to take the analogy with the phase transitions of condensed matter systems seriously, we need to distinguish between the system, the GFT state/spacetime, and the transition itself. That the GFT state may not be spatiotemporal does not imply anything about the phase transition, of which we do not know much yet. In the 5-dimensional picture I was alluding to before, it seems plausible to ascribe a temporal meaning to the fifth dimension, even though there may already be some other notion of time in the model.

In any case, Oriti continues:

> So, first, we need to have a background-independent and non-spatiotemporal notion of 'evolution' in the space of quantum gravity coupling constants, i.e. in the 'theory space' characterizing the quantum gravity formalism at hand. Notice that such evolution will relate different continuum theories, in particular different macroscopic effective dynamics, for the same fundamental quantum entities. This notion of evolution in theory space is what specific renormalization group (RG) schemes in various quantum gravity formalism will provide. (Oriti, 2021, pp. 30-31)

In this paragraph it is clear that for each set of parameters (coupling constants like λ in equation 9.2) one can associate a 'theory', and hence a spacetime if we restrict ourselves to condensate states and interpret these states as spacetimes. Here Oriti introduces the flow of the renormalization group as providing a notion of evolution. As I have said above, the renormalization group flow is generally considered not as a physical flow associated with any physical process. In this passage it may seem that Oriti has the same picture in mind, given that he speaks about theory space and writes evolution in scare quotes. However, we know that the interpretation he wants to make of the flow and of transitions is to take them as physical processes. Oriti recognizes that there is a conceptual problem with the analogy with the condensed matter system:

> The reason why we have no particular conceptual issue in understanding the flow in theory space and the approach to phase transitions in temporal terms, despite the fact that they refer to a change in the time-independent coupling constants of the system, is that we can easily imagine an external observer (the experimental physicist in the lab) tuning such coupling constants towards their critical values, and thus pushing the system towards the relevant phase transition. Needless to say, no such external observer is available in quantum gravity. (Oriti, 2021, p. 31)

The phrasing here may suggest that the problem has to do with the notion of external observers, but the key point is that in the case of a condensed matter system we can think of the system as embedded in a spatiotemporal structure in which the thermodynamic parameters change, while in the case of the theory of quantum gravity the theory itself defines a spatiotemporal structure, and introducing a further process of change seems to introduce additional metatemporal dimension. Despite recognizing this, Oriti still claims that:

> Any notion of time or, better, 'proto-time' that could be associated to such flow across the quantum gravity phase diagram would in any case deserve such name only in the sense that, once used to parametrize the flow across a non-geometric phase towards a geometrogenesis phase transition, it ends up matching some spatiotemporal observable that can be used as a time variable within the geometric phase. Viceversa, it would correspond to what is left of some geometric variable used to define a notion of time in such phase, and used as well as a notion of RG scale for the quantum gravity system, once the same system flows across a geometrogenesis phase transition into a non-spatiotemporal phase. (Oriti, 2021, p. 31)

In this passage Oriti proposes that, in order to avoid having two times, one can identify time with proto-time, which is the way he calls the parameter which describes the renormalization group flow. However, time and proto-time cannot be identified in any straightforward way: for any 'instant' of proto-time we have a value for the coupling constants and, possibly, a full spacetime, with its whole range of possible times. In other words, the picture we have is clearly 5-dimensional and the two time parameters have nothing to do the one with the other and there is no principled way to identify one with the other. Also note that the transition happens in proto-time but is not localized spatiotemporally, which means that, contrary to what we wanted, there is nothing associating the big bang singularity, which is localized in spacetime, with the phase transition.

In this sense, Oriti fails to provide a satisfactory answer to the challenge posed by the fact that adding an additional time parameter or proto-time makes our system become equivalent to a 5-dimensional one. Moreover, that the dynamics is given by the renormalization group flow seems ad-hoc and it would be positive for the proposal to have it not only better justified but made more concrete by defining a particular flow.

The natural way of implementing the idea of geometrogenesis would be in a theory that has a 3+1 split. In this case one can speak about transitions in a natural way, as there is a time parameter available. In the case of GFT, this option does not seem available given the way the formalism is interpreted and the resolution that is given to the problem of time. If instead we took states in the Hilbert space of GFT to represent spaces and not spacetimes, as it is more natural, and we provided a time parameter and a time evolution with respect to which both the states of the system and the couplings of the model evolved, then the idea of geometrogenesis could be implemented in a conceptually clear way. We can imagine a situation in which we start with a GFT state that does not look like a nice space (maybe because the connectivity of the atoms is not the right one) and that this state evolves to reach one which is a nice simplicial decomposition of a space. In such a case we could have a geometrogenesis nicely realized in a way that is similar to the way it was first formulated in other approaches[39]. Of course, for this it is crucial that we have the time parameter, and hence one could argue that it is not a geometrogenesis of the most radical kind, as time would still be there before the transition.

Let me compare the idea of geometrogenesis as presented by Oriti with the way the possibility that a non-spatiotemporal 'region' happened before the big bang is studied by Huggett and Wüthrich (2018). They consider some model with a signature change, that is, the model is based on a manifold which has some regions with Lorentzian signature and some regions with Euclidean signature. Arguably, one can claim that there is only time in the regions with Lorentzian signature, and that in 'going' from some region to another time appears or disappears. This would have happened at the big bang: there was a non-spatiotemporal region of the universe 'contiguous' to our spatiotemporal region[40]. This kind of picture seems to be what Oriti aimed for, but my criticism is that, instead, what he obtained is a 5-dimensional picture. To insist, as long as the renormalization group flow applies to the parameters defining a spacetime and not to different 'regions' or parts of the quantum gravity stuff, the picture it will have associated is the one in 5 dimensions and not something similar to what Huggett and Wüthrich discuss.

Finally, let me also mention that both Oriti (2021) and Huggett and Wüthrich (2018) associate geometrogenesis with emergence. In chapter 2 I mentioned the distinction between hierarchical and flat emergence introduced by Crowther (2021). In this case it seems clear that the idea of emergence that best applies to this case is a flat one and not a hierarchical one. As I argued in chapter 2, this use of the term emergence can reflect the novelty and the distinctness of the different stages of a transition, but does not reflect any conceptual novelty that was not already present on the idea of phase transition.

Let me close this chapter by insisting that GFT is an approach inspired by LQG and spin foam models which is in some aspects similar to these approaches and which faces similar interpretative problems and shortcomings. The covariant formulation of GFT lacks conceptual clarity and it just defines some probabilities but it does

[39] See for instance Konopka et al. (2006).

[40] To clarify: Huggett and Wüthrich's discussion is clearest using the signature change example in which one has an underlying manifold, but they allow for other possibilities. That is, there can be something like a geometrogenesis as long as there are some parts or regions of the fundamental quantum gravity stuff, whatever it is, which approximate a spacetime and some that do not.

not give a meaning to them. The canonical one has a problem of time just as the canonical version of LQG had, and I have argued that its relational solution is not fully satisfactory. Therefore, I have argued that neither approach can be said to be able to overcome the difficulties in the justification and interpretation that we have also found for other approaches to quantum gravity. Finally, the cosmological applications of the theory provide some interesting ideas, but I have argued that they still lack a solid foundation.

10. CONCLUSION

In this last chapter I give an overview of the main conclusions of this thesis. First, in section 10.1 I comment on the status of discreteness. I will argue that one can raise a doubt about the extent to which some sort of discreteness of space or spacetime is predicted by the models I have studied in this thesis. Second, in section 10.2 I will insist in that the tension between the dynamical spacetime structure of general relativity and the fixed temporal structure of the quantum formalism leads to some version of the problem of time when trying to quantize general relativity, and that this also affects covariant approaches. In this sense, I will briefly recap why I think that there are serious issues at the time of justifying and interpreting the outcome of the quantization procedures as the theory of quantum gravity we were looking for. Finally, in section 10.3 I speculate about the alternatives that are left if we are convinced by my analysis that the quantization of general relativity fails.

10.1 Discreteness?

A first thing to comment is that almost every approach I have mentioned in this thesis described some kind of discrete space or spacetime. That space or spacetime could turn out to be discrete is certainly a possibility for our world, and one can even argue that it is a likely one when one considers quantum gravity. However, in this section I want to emphasize that, for the approaches I have considered, this is not a necessary consequence of the quantization of gravity. Instead, it is, at best, one of the interpretations which are available, or, at worst, an assumption of the model.

First, let me mention that in approaches like causal set theory, quantum Regge calculus or causal dynamical triangulations (if we do not take the vanishing lattice size limit) discreteness is directly and explicitly an assumption in their models. In these approaches discreteness is just an assumption which can be motivated by some heuristic reasoning or adopted for convenience reasons.

It is in the family of approaches related to LQG that the status of discreteness as prediction or as an assumption is more disputable. Consider for instance the case of canonical LQG, as I discussed it in chapter 5. There are two aspects of the way discreteness is 'deduced' in this model which I have criticized, namely, the justification for the Hilbert space of the theory and the interpretation of states in this space.

Regarding the construction of the s-knot Hilbert space I have raised several doubts. First, the classical theory we are aiming to quantize is just general relativity expressed in the connection formulation and what one would naively expect from the quantization of such theory is something much more similar to the picture from geometrogenesis. That is, a Hilbert space in which each state can be read as assigning probabilities to each possible 3-geometry or, alternatively, to each possible configuration of the connection field on space. Second, the quantization chosen already carries a great deal of discreteness, as it defines a quite discrete measure, the Ashtekar-Lewandoski measure. The same applies for the case of LQC as I introduced it in chapter 6, where the temporal discreteness was just a consequence of the measure chosen. In this sense, part of the discreteness is a consequence of the particular way the quantization is carried out.

To claim that this discreteness is a necessary feature of the quantization of general relativity one would need to argue that this quantization is the only possible. In the case of LQC it is clear that one could have defined a different measure which leads to a picture in which time is continuous. Similarly, for the case of full general relativity one could have insisted on preferring a geometrodynamical quantization of the theory, where states in the Hilbert space are wavefunctionals on superspace. A precise definition of this Hilbert space is lacking, and hence the s-knot Hilbert space is far more tractable and well-behaved from a mathematical point of view. Nevertheless, the fact that choosing a Hilbert space which is defined on discrete structures is better behaved is not surprising and one can still wonder if it represents the right quantization, as I have argued that from an intuitive point of view the geometrodynamical quantization seems more plausible. In this case it seems that at the time of defining a Hilbert space there is a trade-off between mathematical tractability and intuition[1].

Finally, there is the further worry that the gauge restriction in the building of the Hilbert space is unjustified. Recall that this step was important because it forced the gauge group to be the compact SU(2), which implied the discreteness of the area and volume operators. This is another instance of a choice between a more tractable quantization and a more intuitive one. It is because in LQG one prefers the tractable route that the discreteness in the spectra is obtained. If one chooses to take the more intuitive and mathematically worse-behaved geometrodynamical option, it seems that continuum space can be maintained[2].

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[1] A somewhat analogous case can be found in the foundations of QFT, where 'Lagrangian QFT' is a more intuitive approach while the algebraic or axiomatic approach is defined in a mathematically rigorous way but seems unable to accommodate the predictions of Lagrangian QFT. See Wallace (2006); Halvorson (2007); Fraser (2011); Wallace (2011) for the relevant literature in this debate.

[2] Of course, in the quantum theory we would expect to have superpositions and other quantum

The other point I have criticized in chapter 5 about the way LQG 'predicts' a discrete space is the interpretation that the states in the Hilbert space of the theory are given. There are two different features of the s-knot Hilbert space which supported the interpretation that space is discrete, namely the graph structure of the Hilbert space and the fact that area and volume operators have discrete spectra. This motivated the picture in which s-knot states are associated with tessellations of space by interpreting each node as representing a chunk of space, each link, an adjacency relation, and the quantum numbers, the volume of each chunk and the area of each boundary. However, I challenged this interpretation in sections 5.2.3 and 5.2.4. First, the direct interpretation of the states in the LQG Hilbert space, at least attending to their derivation, is as representing a distributional geometry, that is, a geometry which is not described by a smooth metric field but which is concentrated at particular points and curves. Second, not every state in the s-knot Hilbert space is dual to a tesselation, i.e., one cannot associate a decomposition of a 3-space to every state. In this sense, there are states for which the interpretation in terms of chunks does not work. And third, the structure of the s-knots states is richer than the structure of dressed abstract graphs, as s-knots also encode the way the edges are knotted and linked and they also contain moduli. These pieces of information had a physical meaning in the classical theory, and one may then worry that it is lost if we restrict ourselves to abstract graphs. For these reasons, I argued that the view of LQG states as representing discrete spaces is disputable. Furthermore, if one restricts the set of states that one is considering to just the ones which are dual to a tesselation one is moving away from what the canonical quantization procedure had produced in a way that is not justified and that may leave out physically relevant states.

In this sense, I have argued that the picture of an atomistic space is not a necessary outcome of the canonical quantization and that, in fact, represents a departure from it. For this reason, I have argued that one may have reasons for endorsing such a discrete picture, but it is more an assumption or an interpretative choice than a prediction derived from the quantization of general relativity.

This analysis applies also to the covariant approaches I considered in this thesis. In chapter 8 I showed that there are different motivations available for spin foam models. If one takes the picture in which spin foams are the evolution of spin networks, the arguments that applied to canonical LQG directly apply to this picture. That is, if the justification for adopting a spin foam model comes from LQG, the worry that discreteness in LQG is more an assumption than a necessary feature will also affect the spin foam model. If the picture one is embracing is the one in which spin foams

phenomena affecting space and this would imply a departure from classical space. However, while in LQG it is argued that one has superpositions of discrete spaces, in geometrodynamics it seems that the spaces are continuous.

are dual to triangulations, then it is even clearer that discreteness is just assumed in this picture.

Finally, the motivation for GFT, as I argued in chapter 9, comes directly from spin foam models. In this sense, discreteness is directly an assumption of the model inspired by the idea that spin foam models are Feynman diagrams of the GFT. If one wanted to argue that this discreteness is a consequence of the quantization of gravity one would have to argue for the necessity of the loop quantization, of the link between LQG and spin foam models and finally between these models and GFTs.

Let me mention also that the LQG community has claimed that LQG, in its canonical or spin foam form, correctly predicts the thermodynamic properties of black holes, and that the discrete structure of the theory plays an important role for this derivation. It is beyond the scope of this thesis to analyze the way this prediction is done and the extent to which discreteness is necessary for explaining the results that link black holes with thermodynamic behavior. I will just note that there is a debate in the philosophy of physics community[3] about the extent to which black holes can be considered thermodynamic objects and the validity of expressions like the Benkenstein-Hawking entropy expression and that the derivations by the LQG community[4] rely on several assumtions and interpretations linking the LQG formalism with statistical physics, entanglement entropy, black hole physics and thermodynamics which can be questioned.

All in all, I have argued that in the approaches I have studied in this thesis the picture of a discrete space or spacetime is an assumption, or is motivated by some particular interpretation of the s-knot Hilbert space of LQG. I have argued that there are some choices in the construction of this Hilbert space that are disputable, and that even if one accepted this Hilbert space as being 'the right quantization' for general relativity one can still put pressure on the interpretation that it describes an atomistic space. Again, the reasons for accepting that Hilbert space were associated with some technical improvement with respect to the naive geometrodynamical approach. However, these improvements were not enough, according to my analysis, to satisfactorily solve the problem of time, as I will comment in next section. In this sense, I conclude that it is an open question whether the quantization of gravity leads to a picture in which spacetime is discrete.

[3] See for instance Dougherty and Callender (2016) for a critical position with respect to black hole thermodynamics and statistical mechanics and Wallace (2019) for a more positive one.

[4] See (Rovelli and Vidotto, 2015, Chap. 10) and the references therein.

10.2 The problems of time

The main conclusion of this thesis is that the quantizations of general relativity I have studied and the related approaches do not describe a non-spatiotemporal quantum world out of which spacetime emerges, but they instead suffer from serious technical and conceptual problems which make it the case that we should take that picture with caution. That is, by applying some quantization technique or another we do not reach a formalism for which an interpretation which would connect it with the world, or with some possible or conceivable world has been provided or seems to be forthcoming. Moreover, I have argued that we have reasons to think that the quantization procedures studied fail in the sense that it is not that these procedures give a formalism which is challenging to interpret as a theory, but rather that this formalism may not even constitute a theory, as exemplified by the canonical quantization of the double harmonic oscillator. Here I will briefly insist on my main arguments for this.

The reason why the quantization of general relativity fails is that the spacetime structure of general relativity is dynamical, and this is represented by the diffeomorphism invariance of the theory, which seems to be ineliminable. This invariance causes the problem of time in the canonical quantization of general relativity and similar troubles in covariant approaches. The problem of time affects the quantization of every reparametrization invariant model in that the way one usually defines the dynamics fails. In the canonical formalism, one cannot define a Schrödinger equation which describes evolution with respect to the arbitrary time parameter of the reparametrization invariant model. In the covariant formalism, one gets propagators which do not depend on that time parameter either. For the theories defined on a fixed background I argued that this problem can be solved, while for the case of general relativity it is not the case.

$$\Psi(q,t) \rightarrow \Psi(q,\xcancel{t})$$
$$K(q,t;q',t') \rightarrow K(q,\xcancel{t};q',\xcancel{t'})$$

Fig. 10.1: Schematic representation of the problem of time in its canonical and covariant versions. In both cases we do not obtain the time dependence of a standard quantum theory.

In section 4.6.1 I analyzed the problem of time for a reparametrization invariant formulation of the dynamics of a non-relativistic particle and argued that one is able to interpret states in the physical Hilbert space as solutions to the Schrödinger equation with respect to the physical time t. Similarly, in section 7.3, I argued that the covariant quantization of the reparametrization invariant model of the relativistic particle was not problematic either, as the propagator one defines is not a propagator that defines an evolution with respect to the arbitrary parameter τ, but can be seen as the propagator of a Schrödinger-like equation by identifying the variable t as a time variable. In this sense, I argued that when the classical theory is deparametrizable, i.e., when time can be identified as one of the configuration variables of the theory, the problem of time can be, in principle, solved. That is, for this class of theories one is able to build a quantum theory with the usual time evolution.

In sections 4.6.2 and 4.7 I argued that when one has a non-deparametrizable model, such as the Jacobi action for the double harmonic oscillator or general relativity one cannot solve the canonical problem of time by applying the resolutions which work for the deparametrizable models. The quantization of the double harmonic oscillator model clearly failed: instead of giving us the quantum theory for the system as we know from its Newtonian formulation, it just gave us a Hilbert space of energy eigenstates. I argued against interpreting these states as containing any dynamical information, and that the same can be applied to the case of general relativity. In particular, in section 4.7 I surveyed the resolutions that have been proposed for the problem of time for quantum gravity and found them wanting, as they are either based on the wrong intuitions from deparametrizable models or on some unclear probabilistic interpretation, which I argued failed for the double harmonic oscillator case.

The transition amplitude strategy deserves special mention, as it allows to connect with the covariant quantization and the covariant version of the problem of time. As I commented, a significant part of the LQG community has shifted to covariant approaches such as spin foam models and therefore it seems to be fair to say that the currently preferred way of dealing with the problem of time and of allegedly extracting physical information from an approach to quantum gravity is by studying a 'propagator' of the form $K(q; q')$. However, I have argued against this view, as I have found that we do not have a convincing interpretation for such objects.

There are several reasons I have given for supporting this claim. First, the analogy with the double harmonic oscillator case and the fact that general relativity is not deparametrizable, i.e., that its phase space does not contain any time variable, make it the case that the interpretation of K as a propagator is unjustified. Second, K cannot be interpreted as a propagator like the standard propagator of quantum mechanics, as it, in general, cannot be used for obtaining the standard structures of quantum mechanics. Therefore, the standard interpretations and ways of thinking about quantum mechanics cease to be applicable. Finally, I have argued that even if we were willing to depart from standard quantum mechanics, the interpretation of K remains unclear. It supposedly represents the probability of 'measuring' a final 3-geometry given that an initial one was 'observed'. This is clearly not easily interpretable from a realist perspective, but it is even challenging to interpret from an operationalist one. In an operationalist reading of standard quantum mechanics, one is able to relate the formalism with some experiments or operations carried out in space and time, but in this case it is completely mysterious how would one locate or identify an initial and a final observation. In this sense, we are lacking a way of connecting the formalism with the world, or at least with some possible world.

For these reasons I have argued for the conclusion that the models I have studied in this thesis are not examples of how the quantization of general relativity leads to a non-spatiotemporal picture which suggests some interesting philosophical issues about the nature of space and time, but rather cases of a failure of the quantization. As I have argued, the formalism of quantum mechanics needs of some fixed spatiotemporal structure which is not available in general relativity and this makes it the case that the standard quantization procedures fail to build a satisfactory quantum theory which would have general relativity as a classical limit.

Finally, let me also mention that I have also studied the alleged cosmological implications of these models of quantum gravity. As I have argued that these models are also affected by some form of the problem of time, I have also argued for taking the cosmological models related with these approaches with skepticism.

10.3 Closing remarks

If the analysis carried out in this thesis is consistent, the direct quantization of general relativity is not the right way to build a quantum theory of gravity. Again, the reason for this is the conflict between the dynamical spatiotemporal structures of general relativity and the fixed temporal structure of quantum mechanics. In this last section I will finish by saying a few words about how one may try to avoid this problem when looking for a theory of quantum gravity.

A first way to go would be to try to somehow fix the temporal structure of general relativity and then apply some quantization procedure. In chapter 4 I mentioned the approach based on shape dynamics, which goes along this direction, although one may worry that the temporal structure in this theory is still not fully fixed. Similarly, I have also mentioned Bohmian approaches which, if feasible at all, would also involve some sort of fixation of the spatiotemporal structure. This kind of approach however is counterintuitive if one takes the dynamical spatiotemporal structure of general relativity to be an essential feature of both the theory and our world.

Alternatively, one could try to modify quantum mechanics so that it does not rely on a fixed temporal structure. That is, instead of quantizing general relativity, make quantum mechanics general relativistic. Of course, this faces the problem that quantum mechanics and the well-established interpretations of the theory generally rely on an absolute temporal structure. However, one can read the Lorentz invariance of quantum field theory as hinting towards a more general way of thinking about quantum mechanics and there is some work in the foundations and philosophy of physics studying the ways in which quantum mechanics could be generalized or thought about such that it is compatible with relativity. In chapter 7 I commented that in some covariant approaches some moves in this direction have been done and it is sometimes argued for new interpretations and generalizations of the formalism. However, I argued that many of these approaches face serious difficulties and that many of them are likely to collapse to just some of the standard, well-established interpretations. In this sense, this illustrates the challenge that the construction of a theory of quantum gravity represents, both technically and conceptually.

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