# Appendix A

# Gradients in tropospheric delay modeling

The derivation of gradients in refractivity was presented in [36] and [40] and we here summarize some aspects to help in the understanding of some concepts here described.

## A.1 Slant delay gradients

Let us write the refractivity in a first-order Taylor approximation at altitude $z$ as

$$N(\vec{\rho}, z, t) = N_0(z, t) + \vec{\psi}(z, t) \cdot \vec{\rho} \tag{A.1}$$

where $\vec{\rho}$ is the horizontal displacement vector. We hence consider that we can separate the dependency in altitude and horizontal, and

$$\psi_i(z, t) = \frac{\partial N(\vec{\rho}, z, t)}{\partial \rho_i}|_{\vec{\rho}=0} \tag{A.2}$$

is the horizontal gradient of refractivity. Then, the delay is:

$$L_t = \int_{s.l.} 10^{-6} N = \int_{s.l.} 10^{-6} N_0(z, t)\ dl + \int_{s.l.} 10^{-6} \vec{\psi}(z, t) \cdot \vec{\rho}\, dl. \tag{A.3}$$

Considering mapping functions that project the magnitudes to the zenith direction, and that the following holds (see Figure A.1):

$$\vec{\rho} = \frac{z}{\tan e} \hat{\rho}(\phi) \tag{A.4}$$

$$dl = dz\ m_o(e) \tag{A.5}$$

then, we can rewrite:

$$L_t = \int_0^\infty 10^{-6} N_0(z, t)\ m_o(e) dz + \int_0^\infty 10^{-6} \vec{\psi}(z, t) \cdot \vec{\rho}\, m_o(e) dz \tag{A.6}$$

$$= m_o(e) \left( \int_0^\infty 10^{-6} N_0(z, t)\ dz + \cot e \int_0^\infty 10^{-6} z \vec{\psi}(z, t) \cdot \hat{\rho}(\phi)\ dz \right) \tag{A.7}$$

$$= m_o(e) \left( L_t^z(t) + \cot e \left( \vec{L}^G(t) \cdot \hat{\rho}(\phi) \right) \right) \tag{A.8}$$

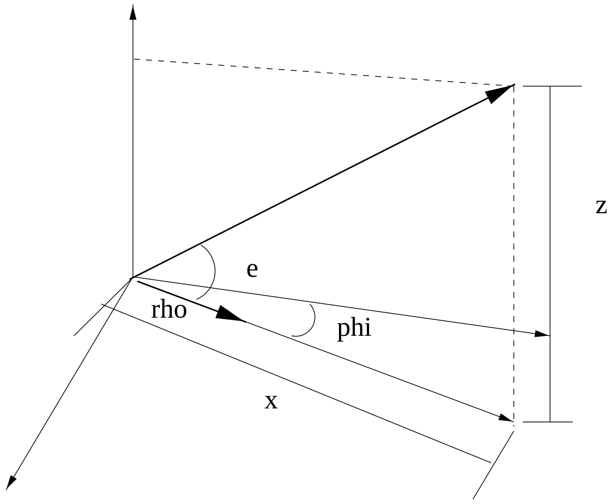


Figure A.1: Representation of an incoming ray and the vectors involved in gradient definition.

and therefore, the delay gradients are defined as:

$$\vec{L}^G(t) = 10^{-6} \int_0^\infty dz z \vec{\psi}(z,t), \tag{A.9}$$

and have units of length (mm).

## A.2 Zenith delay gradients

In its mathematical form, $L^G(t)$ is the first-order momentum of $\psi(z,t)$. We can also compute the zero-order momentum of the gradient, as

$$\vec{G}(t) = 10^{-6} \int_0^\infty dz \vec{\psi}(z,t), \tag{A.10}$$

with units of path delay per unit distance. This is the horizontal gradient computed using the zenith delays over a network. Now, if we consider the refractivity field to have an exponential decay in altitude and a horizontal dependency like ( [85], [72]):

$$N(\vec{\rho}, z, t) = N_s e^{-z/H} \left[1 + \vec{g}_s \cdot \vec{\rho}\right], \tag{A.11}$$

then

$$\psi(z,t) = \nabla_{\vec{\rho}} N(\vec{\rho}, z, t)|_{\vec{\rho}=0} = e^{-z/H} \left(g_{sx}, g_{sy}\right); \tag{A.12}$$

it can be demonstrated that the zero and first-order momentums are related through a constant:

$$m_1(t) = \vec{L}^G(t) = H\vec{G}(t) = Hm_0(t). \tag{A.13}$$

where $H$ is the scale height of the exponential distribution.

# Appendix B

# Basics of Atmospheric Physics

In this appendix we include some basic expressions to clarify some meteorological concepts used in some sections of this thesis. We do not intend to give a thorough analysis of the physics of the atmosphere; for a compact and complete discussion on the subject refer to [74] and [86].

## B.1 Hydrostatic equation

In the presence of the gravitational field, the density of an atmosphere decreases with altitude. Considering that vertical motions are orders of magnitudes smaller than horizontal motions, it can be assumed a static equilibrium. Then, we can write

$$dP = -g\rho dz, \tag{B.1}$$

where $P$ is the pressure, $\rho$ is the density and $z$ is the height; $g$ is the gravitational acceleration that can be considered to remain constant throughout the atmosphere. Using the equation of state for a perfect gas, with molecular mass $M_r$,

$$\rho = \frac{M_r P}{RT}, \tag{B.2}$$

($R$ is the gas constant), integration gives

$$P = P_0 \exp(-z/H), \tag{B.3}$$

where $H = RT/M_r g$, which is the scale height (using $T = 290$ K, then H=8.5 km for dry air).

## B.2 Temperature lapse rate

Let us first consider an atmosphere transparent to all radiation, with no liquid particles in it and in hydrostatic equilibrium. We want to determine the vertical motion of an air parcel with volume $V$ at pressure $P$ and temperature $T$. In such a case, we can write the first law of thermo-dynamics applied to unit mass as:

$$dq = c_v dT + PdV, \tag{B.4}$$

where $c_v$ is the specific heat at constant volume and $q$ is the heat. For an adiabatic motion, $dq = 0$ and combining the above with the differentiation of the equation of state:

$$PdV + VdP = (R/M_r)dT = (c_p - c_v)dT, \tag{B.5}$$

where $c_p$ is the specific heat at constant pressure and $R/M_r = (c_p - c_v)$, results

$$c_p dT = VdP \tag{B.6}$$

and using the hydrostatic equation,

$$\frac{dT}{dz} = -\frac{g}{c_p} = \Gamma_d. \tag{B.7}$$

Therefore, under these assumptions, the atmosphere is heated by contact with the surface and the gradient is uniform with altitude; $\Gamma_d$ is the *adiabatic lapse rate* for dry atmosphere, which, for the earth's atmosphere is $\Gamma_d \cong 10$ K km$^{-1}$. As a consequence, given a temperature gradient, it is stable if $-dT/dz \leq \Gamma_d$ and unstable otherwise.

## B.3 Potential Temperature

We can derive the expression of entropy by substituting $dq$ with $TdS$ and using the equation of state to substitute $V$ and then integrate to yield:

$$S = c_p \ln T - RM_r^{-1} \ln P + const \tag{B.8}$$

and then define the *potential temperature*, $\theta$ of air at temperature $T$ and pressure $P$, as the temperature that reaches if an air parcel is brought from $T$ and $P$ with constant $S$ to a reference pressure $P_0$:

$$S = c_p \ln T - RM_r^{-1} \ln P + const = c_p \ln \theta - RM_r^{-1} \ln P_0 + const \tag{B.9}$$

$$\theta = T \left(\frac{P_0}{P}\right)^{\kappa}, \tag{B.10}$$

where $\kappa = (c_p - c_v)/c_p = 0.288$ for dry air in the earth's atmosphere.

## B.4 Water vapor in the atmosphere

Let us now consider a parcel of volume $V$ with water vapor, but still unsaturated (saturation meaning that water vapor is in chemical equilibrium with condensed water). Then, the air will rise, as we have seen, and cool at a rate of $\Gamma_d = 10$ K km$^{-1}$, maintaining the mixing ratio (or ratio of mass of water vapor to mass of air) while increasing the relative humidity until water vapor condensates. If we consider 1 g of air and $m$ g of water vapor in equilibrium, the air remains saturated and condensation and evaporation occur to cancel each other. Then, the entropy is given by the above expression plus the excess of liquid water $\zeta - m$ and the additional entropy $Lm/T$ required to convert $m$ g of liquid water to water vapor:

$$S = c_p \ln T - RM_r^{-1} \ln P + \zeta c \ln T + Lm/T + const \qquad \text{(B.11)}$$

which can be differentiated and combined with the hydrostatic equation for wet air, to yield a lapse rate for saturated air $\Gamma_s \approx 6$ K km$^{-1} < \Gamma_d$. Again, the stability of saturated air depends on the lapse rate.

The parameters that characterize the water vapor present in the air are the *saturation water vapor pressure* ($e_c$) and the *saturation mixing ratio* ($m$):

$$e_c = \frac{\rho_v RT}{M_{rv}}, \qquad \text{(B.12)}$$

$$m = \frac{\rho_v}{\rho_a} = \frac{e_c(T)\epsilon}{P - (1-\epsilon)e_c(T)} \cong \frac{e_c(T)\epsilon}{P} \qquad \text{(B.13)}$$

where $\epsilon = M_w/M_a = 0.622$ kg/kg.

These magnitudes are defined for saturated air. In unsaturated air, they are referred as *water vapor pressure* ($P_w$) and *mixing ratio* ($r$) (in Equations 2.17 and 5.5) and related to $e_c$ and $m$ via the *relative humidity.* The magnitude $r$ represents the abundance of water vapor and only change when there is a transition of phase.

The *specific humidity* ($q$) is defined as the ratio of water vapor density to total air density (including water vapor), as opposed to $r$ that considers water vapor density to dry air density. However, because the former is usually much smaller than the latter, both magnitudes are considered to be equivalent $r \approx q$.

The *dew point*, $T_d$ or the temperature to which the system at temperature $T$ must be cooled isobarically to achieve saturation. The *dew point spread* is the difference $T - T_d$. The larger the dew point, the smaller the spread and the larger the abundance of water vapor. This measures how far is the system from being saturated.

The saturation water vapor pressure ($e_c$) is a function of temperature and is given by the

Clasius-Clapeyron expression

$$\ln e_c = 21.65374 - \frac{5.420 \cdot 10^3}{T} \tag{B.14}$$

or in a more recent and accurate expression, by the Tetens formula ( [87], [84])

$$\ln e_c = 19.0789548 - \frac{4.098026 \cdot 10^3}{T - 35.86} \Rightarrow e_c = 6.1078 \exp\left(17.2693882 \frac{T}{T - 35.86}\right) \tag{B.15}$$

where $T$ is given in K.

Note that because $T_d$ is a temperature at which the air saturates, keeping the same pressure, and there is no transition of phase, the saturation mixing ratio for $T_d$ and $P$ is equal to the mixing ratio for $T$ and $P$, and we can therefore write:

$$r(T, P) = m(P, T_d) = \frac{e_c(T_d)\epsilon}{P - (1 - \epsilon)e_c(T_d)} \tag{B.16}$$

## B.5 Skew-T plots

A usual representation of radiosonde data is the skew-T plots or tephigrams. In such plots, one represents entropy as a function of temperature. However, instead of representing entropy one can represent the logarithm of the potential temperature ($\ln \theta$). Contours of other quantities are also plotted to help the visualization of meteorological magnitudes. These quantities are:

- Pressure: isobars are straight lines and the plot is rotated so that they fall parallel to the x-axis (we then have $P$ in the y-axis).
- Adiabatic lines for dry air (constant $\theta$).
- Adiabatic lines for saturated air, showing the entropy change of the saturated air.
- Saturation mixing ratio ($m$) as a function of $P$ and $T$. A plot of $m$ against $P$ for unsaturated air is equivalent to that of dew point against $P$.

In such diagrams, dry temperature and dew point temperature are usually represented, to show the dew point spread (difference between $T_d$ and $T$) to show how far the air at each altitude is from saturation point; it is a qualitative way to visualize the wet refractivity.

## B.6 Atmospheric Radiation

The atmosphere absorbes radiation in two distinct bands: shortwave (SW), due to the solar radiation, with a peak in the visible at $\lambda \approx 0.5\mu$m and extending from UV ($\lambda < 0.3\mu$m) to the near-infrared region ($\lambda > 0.7\mu$m); and longwave (LW), due to earth's surface emission, with the peak in the IR at $\lambda \approx 10\mu$m and extending from $\lambda \approx 5\mu$m into the microwave region. The atmosphere looses energy in LW emission to the Earth and to space.

For the sake of simplicity, let us assume a *grey* atmosphere, in which the absorption coefficient ($k$) of radiation is non-zero but uniform in the emitting infrared region and independent of pressure and temperature. Then we can obtain the conditions for radiative equilibrium: when radiation of intensity $I$ traverses a parallel slab of atmosphere ($dz$), the absorption is proportional to the intensity and to the mass of absorber per unit section ($\rho dz$):

$$dI = -Ik\rho dz \;\Rightarrow\; I = I_0 \exp\left(-\int k\rho dz\right) \tag{B.17}$$

The emission, on the other hand may be approximated by the black-body radiation at temperature $T$:

$$dI_{bb} = Bk\rho dz \tag{B.18}$$

where $B$ is the black-body emission per unit solid angle per unit area. Then the radiative transfer is given by

$$dI = -Ik\rho dz + Bk\rho dz \tag{B.19}$$

$$\frac{dI}{d\chi} = I - B \tag{B.20}$$

where $\chi = \int k\rho dz$ is called optical depth. When the radiative transfer is applied to a parallel atmosphere and we impose conditions of thermal equilibrium $dT/dt = 0$, it follows that the net flux is constant throughout $z$. The radiative transfer equation from above can be integrated over the solid angle to obtain the fluxes and the equations can be rewritten to yield the value of $B$, in turn related with temperature by the Stefan-Boltzmann law:

$$B = \frac{\sigma T^4}{\pi} \tag{B.21}$$

It can be shown (see Figure B.1) that such thermal equilibrium leads to a temperature lapse rate greater than the mean adiabatic lapse rate (for dry and wet), and, therefore, the gradient is unstable and vertical motion or convection develops to cancel the instability caused by radiative transfer. This is true as long as the radiative equilibrium curve is below the uniform gradient with $\Gamma_s$ as the lapse rate. This point is generally at 10 km and is the tropopause. Below that level, convection is the dominant mechanism of vertical heat transfer (*troposphere*); and above the tropopause, radiative transfer is the dominant mechanism (*stratosphere*).

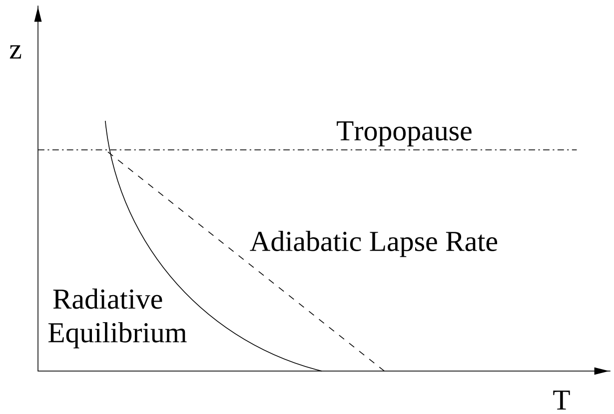


Figure B.1: Radiative transfer equilibrium temperature as a function of height. In dashed line is the temperature lapse rate required for an adiabatic movement. This model leads to a troposphere dominated by convection and a stratosphere (above the tropopause) dominated by radiative transfer.