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Universitat Autònoma de Barcelona

Three Essays in Macrofinance
Cycles, Expectations and Economic Policy

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Three Essays in Macrofinance: Cycles, Expectations and Economic Policy.

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Per als que van a pel tot.

Only a crisis - actual or perceived - produces real change. When that crisis occurs, the actions that are taken depend on the ideas that are lying around. That, I believe, is our basic function: to develop alternatives to existing policies, to keep them alive and available until the politically impossible becomes the politically inevitable.

Milton Friedman.

Agraïments

Esta tesi té un paregut raonable amb la tesi que haguera volgut escriure. No perquè siga cap obra mestra; part del contingut, de fet, està encara a mitges. La raó és menys espectacular: no és -no en sóc conscient, en tot cas- fruit de cap pacte pragmàtic entre desig i realitat. Hi ha faena per fer, però no rebaixes d'objectius. Crec que això haguera provocat un somriure en aquell Pau de fa 6 anys que va abandonar una còmoda vida a Història Econòmica pel que llavors considerava l'obscur món de l'Economia ortodoxa.

L'empresa no ha sigut estrictament individual. Per això, voldria aprofitar estes línies per a agrair en veu alta a totes les personetes que d'una forma o altra, conscients o no, han posat el seu gra d'arena.

L'Albert és un dels responsables més directes. Recorde molt unes paraules de la Yolanda Blasco quan estudiava el màster d'Història Econòmica a la UB: necessites un director a qui intel·lectualment respectes molt i hi confies plenament. Això ho he trobat. Ell m'ha transmès el gust per la rigorositat, i la necessitat d'entendre exactament el que es diu i escriure exactament el que es pensa. A les reunions he trobat avaluacions sinceres; i fora de les reunions, una presència imaginada que m'ha fet empentar més sempre... Què pensaria Albert d'açò? Seré capaç d'explicar-li-ho? Estarà al nivell?

La Marina ha sigut una troballa absolutament contingent però, com les bones contingències, una que fa absolutament inimaginable i indesitjable una altra forma d'haver-ho viscut. Hem fet que mole un muntó lo que per a altres és una etapa dura. Enmig d'equacions, dades i algoritmes, no ha parat de recordar-me lo que està bé i lo que és "asquerós", empentant sempre una versió més atrevida. Això ens va molt bé per a investigar i, en general, per a viure.

La tesi s'ha beneficiat de l'ambient d'investigació de la UAB i la BSE. Sempre he trobat portes obertes i gent disposada a discutir, aprendre i ajudar. He d'agrair en particular a Luis E. Rojas, per les seues sempre bones discussions que, sortosament, ens han portat a col·laborar; i a Jordi Caballé, pels comentaris i l'ajuda en el Job Market. He trobat també companys que treballen i cooperen molt. En particular, l'Adrian Ifrim, amb qui, des de vision diferents del món, hem establert una productiva col·laboració i la Mridula Duggal, per aquelles vesprades infinites cara a Python que tant ens han donat.

Una de les coses més divertides de la tesi va ser la visita al Bank of England. L'Eddie Gerba és responsable de fer que això passara i per això, i tot el que ha vingut després, li estic molt agraït. Londres és sinònim de felicitat també pel Pascal Meichtry, amb les converses de QE i la sambuca, i la Luci Agnoleto, amb qui vam descobrir tants racons.

L'Alfonso Herranz és molt responsable d'este viatge; gràcies per obrir-me els ulls quan encara no era massa tard. El Ramon Ruiz va ser generós i clar quan res ho era massa. Amb el Pablo Cervera vaig començar a agarrar gust a això d'investigar, amb un treballet sobre una crítica de la crítica de Böhm-Bawerk a la teoria del valor de Marx. L'Emilio Benimeli és, en sentit literal, el culpable últim de canalitzar les meues obsessions cap a l'Economia; jutgeu-lo a ell.

Els meus pares són sempre responsables de tot el que em passe. M'han donat molta llibertat, confiança i una sòlida àncora que fa impossible perdre's, no importa quant lluny estiga. Ma uela també ha ajudat, regant cada nit rega les meues arrels. Una tesi implica sacrificar moltes coses, en particular bona part de la vida social. A pesar d'això, molts amics a Vallà, Barcelona, València i Castelló han ajudat a oblidar els sacrificis sempre que ha fet falta. Ho celebrarem com toca amb cada un d'ells.

Hi ha qui escriu una tesi per mera curiositat intel·lectual o per accedir a bons llocs de faena o per prestigi social. No diria que res d'això descriu bé la meua motivació. Quan tenia 14 anys feia pamflets, pancartes i cançons; ara faig sobretot articles científics, que m'han semblat -potser pres de la dissonància cognitiva que sempre em recorda la Miren Etxezarreta- una forma molt més fiable de fer lo mateix. Guanyarem.

Castelló de la Plana, Maig 2023
Pau

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Introduction

This thesis is a compendium of 3 papers about capital markets and economic policies. Capital markets, particularly the stock market, exhibit large fluctuations, creating and destroying large amounts of wealth with potentially far-reaching consequences for the whole society, including the many who do not participate in them. On the other hand, capital markets are also a source of substantial returns that compound the wealth of the wealthy, widening inequality. Stock market cycles, risk premiums, and economic policies to prevent or exploit them are the major themes of this thesis.

Chapter 1 studies the relationship between the Capital Gains Tax (CGT) and stock price cycles. I propose a theory that challenges the mainstream view, hold for instance by Stiglitz, that a higher CGT boosts price fluctuations through the supply-side lock-in effect. Instead, I show that the demand-side capitalization effect is stabilizing because it reduces the elasticity of prices to subjective beliefs, reducing the likelihood of self-fulfilling booms and busts. Capitalization dominates lock-in at empirically realistic levels for the tax elasticity of realization. This theory is derived from a model of learning about prices with portfolio adjustment costs and taxes on realized capital gains that displays the two effects in a tractable way. The theory is applied to the United States, suggesting that the recurrence of asset price cycles in the middle of the Great Moderation, a troubling observation for many macro-finance models, can be partly explained by the observed decline in CGT. Indeed, the structural estimation of the model reveals that CGT cuts account for 25% of the observed rise in stock market volatility. The model also replicates the rise in stock market valuations and a sizable equity premium. Empirical estimates using survey beliefs support the model's prediction of an increase in the sensitivity of prices to subjective expectations due to lower taxes. Finally, I show that optimal policy prescribes a CGT that leans against market expectations, preventing belief-driven business cycles. The optimal policy can be implemented with a CGT = 100% and a subsidy on capital profits.

Chapter 2, joint work with Adrian Ifrim, explores heterogeneous expectations and their relation to stock market cycles. We present a model of expectations that micro-founds the heterogeneous extrapolation and the persistent and procyclical disagreement present in survey data. Extrapolation arises from imperfect knowledge about price formation that pushes agents to learn, in a Bayesian sense, from price news. However, not all agents learn the same way; they have distinct levels of confidence regarding the signal-to-noise content of price news, such that the more confident extrapolate more. Following survey evidence, we associate confidence with optimism. As a result, higher prices lead to higher disagreement, reflecting the data's pro-cyclicality. Besides, persistent disagreement is related to idiosyncratic views about long-run asset growth. The subjective belief system is embedded in an otherwise standard asset pricing framework, which can then quantitatively account for the dynamics of prices and trading. In the model, learning from prices leads to disagreement and trading, which reshuffles the distribution of wealth between lower- and higher-propensity-to-invest agents, affecting aggregate demand and prices. This feedback loop complements the expectations-price spiral typical of models with extrapolation, placing heterogeneity and trading as key drivers of price cycles.

Chapter 3, joint work with Eddie Gerba and Luis E. Rojas, is a theoretical examination of the role of fiscal distortions in shaping the effects of Quantitative Easing (QE). The presence of deadweight losses from taxation makes a fiscally supported QE have effects on prices and allocation since QE gains/losses change the level and volatility of tax distortions. One of the consequences is that the mean-variance profile of future consumption changes. Then, forward-looking agents incorporate this information into their current expectations and adjust current goods and assets demands, affecting prices and risk premiums. In other words, fiscal distortions break Wallace's neutrality. Under some conditions, QE can stimulate aggregate demand, but it simultaneously increases the risk premium. This differs from the standard view that QE stimulates demand precisely by lowering the risk premium due to the relaxation of financial frictions. A Central Bank must then manage QE to find the right balance between the efficiency gains of more QE and the additional risk-taking. Altogether, QE emerges as an alternative way of collecting resources for the State, more efficient but riskier than taxation, which might be read as an efficiency-risk trade-off for public finances. The fiscal channel provides a rationale for targeted QE programs such as the Green Corporate Bond Program or the Transmission Protection Instrument.

Chapter 1

Capital Gains Taxation, Learning and Bubbles

PAU BELDA

Abstract

Why have there been more asset price boom-bust cycles since the 1980s despite the drop in macroeconomic risk? This paper argues that the fall of the Capital Gains Tax (CGT) is one of the reasons. In a model of learning about prices, I show that a lower CGT make prices more responsive to changes in investors' beliefs, thereby elevating the likelihood of self-fulfilling booms and busts. This novel mechanism dominates the more traditional lock-in effect -according to which a lower CGT reduce price fluctuations- whenever the tax elasticity of realization is not too negative. A structural estimation of the model focusing on the US stock market suggests that CGT cuts account for 25% of the increase in Price-Dividend fluctuations. The model also replicates the rise in stock market valuations and a sizable equity premium. Empirical estimates using survey beliefs support the model's prediction of an increase in the sensitivity of prices to subjective expectations due to lower taxes. Finally, I show that optimal policy prescribes a CGT that leans against market expectations, preventing belief-driven business cycles.

First version: July 2020. This version: May 2023. This chapter is a preliminary version of the paper with the same title. The most updated version of the paper can be found at `https://pau-belda.eu/research/`. I thank Albert Marcet for his guidance. Besides, I have benefited from comments from Klaus Adam, Ricardo Reis, Nicola Gennaioli, Dirk Krueger, Jordi Caballé, Luis E. Rojas, Wei Cui, Hugo Rodríguez, Alexander Ludwig, Abhay Abhyankar, Ramon Ruiz and participants in seminars at Universitat Autònoma de Barcelona, Barcelona School of Economics, Universitat de Barcelona, University of Bath, Bocconi University, Universidad Carlos III, CERGE-EI, Federal Reserve Board, Universitat de les Illes Balears, University of Mannheim, University of Oxford, 2022 Winter Meeting of the Econometric Society, SAEe2020, SAEe2022, EconMod2021 conference, ENTER Jamboree.

1.1. Introduction

There was no Great Moderation in the stock or housing markets. While many macroeconomic variables exhibited lower fluctuations, the main asset markets followed the opposite path. Larger booms and busts occurred, such as the Dotcom episode in the late 1990s, the housing bubble in the early 2000s, or the post-Great Recession joint stock and housing price boom.[1]

Is the increase in price fluctuations in the context of lower macroeconomic risk consistent with the theory? Many models would contend otherwise. Following lower consumption growth volatility, models based on macroeconomic fundamentals would predict more stable prices following a less volatile stochastic discount factor (e.g., Campbell and Cochrane, 1999). Besides, theories that explicitly detach prices from fundamentals, as some models of learning, would predict lower belief fluctuations and then more stable prices driven by smaller forecast errors (e.g., Adam, Marcet, and Nicolini, 2016). Even models that link lower macroeconomic risk with higher demand for risky assets can explain part of the stock prices' run-up but not much of their larger swings (e.g., Lettau et al., 2008). Thus, the negative covariance between macroeconomic and asset price fluctuations appears as a troubling observation.[2]

The hypothesis of this study is that the reduction of the Capital Gains Tax (CGT), resulting from a combination of successive tax reforms and a movement of assets to tax-free accounts, explains part of the increase in stock price fluctuations observed during the Great Moderation. However, why would a lower CGT boost price cycles? Two opposite views coexist. The mainstream claims that a lower CGT would ease price fluctuations by reducing the supply-side lock-in effect (e.g., Somers, 1948, Stiglitz, 1983); during a boom, a lower CGT encourages the selling of assets, increasing supply and easing price pressures. An alternative perspective is that a reduction in CGT may exacerbate price fluctuations through the demand-side capitalization effect (Haugen and Heins, 1969, Haugen and Wichern, 1973); in a boom, lower taxes would fuel expected payoffs, leading to an upsurge in stock demand and prices.

I formalize these points using an asset pricing model with learning about prices and taxes on realized capital gains that displays endogenous booms and busts.[3] Its instability engine is a feedback loop between expectations and prices that appears when agents learn about stock prices. Following Adam and Marcet, 2011, this expectations-price spiral

[1] Understanding these events is critical since they appear to be tied to macroeconomic instability, resource misallocation, or wealth inequality (e.g., Hall, 2017, Gopinath et al., 2017, Kuhn et al., 2020).

[2] Using full-information Bayesian techniques, Chen et al., 2019 shows that long-run risks account for less than 25% of the variance of the Price-Dividend ratio and that habit's contribution is negligible. The non-explained residual has been particularly large since the 1990s, showing the difficulty of these models to account for the larger variance of the PD despite more stable aggregate consumption growth.

[3] Realized capital gains allude to the fact that investors only pay taxes when they sell the asset, as it is the case in the United States.

emerges from imperfect knowledge about other agents' expectations. This information friction prevents rational agents from deducing the equilibrium pricing function, forcing them to forecast prices using a statistical model of prices in an otherwise standard Lucas, 1978 setup.[4]

To model a CGT on realization, I resort to two alternative devices. First, I use a model with exogenous realization, where investors face the risk of a catastrophic liquidity shock that forces them to sell all their assets and pay taxes. This first version focuses on the demand-side capitalization effect exclusively. In this case, a CGT is stabilizing because it dampens the expectations-price spiral. Consider a good news shock. Other things equal, investors would become more optimistic, and demand and prices would rise. However, the translation of higher optimism on a higher stock demand depends on the tax level; the higher the tax, the lower the net expected returns and the lower the increase in demand and prices. Thus, taxes reduce the response of demand and prices to a change in beliefs.[5] By Bayesian updating, a lower price increase would lead to a weaker upwards revision of beliefs and a smaller boom. In other words, higher taxes dampen the propagation of shocks through beliefs such that momentum is shorter and mean reversion is faster. As a result, a CGT reduces the beliefs elasticity of prices, anchoring expectations around their fundamental value.

Then, the lock-in effect is included by allowing investors to decide the timing of capital gains realization. In particular, each investor manages a stock of unrealized capital gains G_t facing portfolio management costs in line with Gavin, Kydland, et al., 2007 and Gavin, Keen, et al., 2015. When the realization of capital gains is deferred, the cost function penalizes investors with extra unrealized capital gains, which increase the future tax liability of households. Then, investors face an additional trade-off: realize capital gains and pay taxes today or defer the realization to the future and have an additional tax liability in the future. Through this channel, lower taxes boost the realization of capital gains, counteracting the capitalization effect. Under some conditions, it is shown that the lock-in effect dominates only when the tax elasticity of realization is lower than minus 1, below the empirical estimates.[6]

This theory provides a qualitative mechanism potentially linking lower taxes to larger stock price volatility. However, how much of it can it really explain? To answer

[4]This expectations-prices spiral follows a long tradition in finance (e.g., Keynes, 1936, Minsky, 1976), replicates the extrapolation and underreaction that characterize survey expectations (e.g., Greenwood and Shleifer, 2014, Coibion and Gorodnichenko, 2015) and can account for price excess volatility (e.g., Adam, Marcet, and Beutel, 2017).

[5]This mechanism is consistent with the empirical evidence in Giglio et al., 2021, which points out that the elasticity of stock holdings to beliefs decreases with taxes. Moreover, it also provides a rationale for the finding in Dai, Shackelford, et al., 2013, that document an increase in stock returns volatility following CGT cuts.

[6]In a recent study, Agersnap and Zidar, 2021 estimate an elasticity between -0.3 and -0.5. Besides, the US Congress uses an elasticity of -0.7.

this question, the model is estimated to replicate a list of facts about the US stock market for the 1946-2018 period. The main statistic is the variance of the Price-Dividend (PD) ratio. As customary in the literature, I use the Campbell and Shiller, 1988's equation, extended with capital taxes, to understand the most proximate drivers of this variance. It turns out the direct effect of CGT changes is rather small, with the main driver being a divorce of stock returns from dividend growth, signalling the surge in capital gains. This decomposition suggests that, if any, the role of CGT on volatility must be indirect, in line with the mechanisms outlined above.[7] On top of the increase in the volatility along with the fall in aggregate consumption and dividends growth standard deviation, the list includes a set of standard facts such as the equity premium, the procyclicality of survey expectations and the mean level of the PD ratio. Finally, I also include the increase in the beliefs elasticity of prices, first documented by Adam, Marcet, and Beutel, 2017, which is in line with the model's mechanism.

Computationally, the model is solved using a novel application of the Parameterized Expectations Algorithm with a theory-based approximating function that allows for closed-form solutions. Given the observed path of capital tax cuts and the empirical dividend process, I estimate the remaining structural parameters using the Simulated Method of Moments. Then, I formally test the hypothesis that the model statistics differ from their empirical counterpart.

The central result is that tax cuts account for between 25% and 34% of the increase in the PD variance. These numbers arise from a counterfactual analysis that compares the increase in the variance produced by the model with the one that would have been observed if post-1980s tax cuts had been avoided. If only the capitalization effect is considered, the numbers are much higher, between 38% and 62%. Importantly, the model produces this increase in the variance while changing the elements of the variance decomposition in the right direction and generating at least one-half of the increase in the beliefs elasticity of prices.

In addition, the model matches well the increase in the mean PD ratio, its persistence and its positive correlation with beliefs. Finally, the model delivers a remarkable equity premium along with a low and stable risk-free rate, realistic consumption and dividend growth processes, a non-negative discount factor, and low risk aversion. The reason is twofold. First, the learning model generates high volatility from beliefs, increasing average returns by Jensen's inequality.[8] Additionally, the inclusion of taxes imparts a trend in the

[7]There is a more traditional way of thinking of the role of taxes on asset prices volatility: tax changes induce price changes. Note the theory proposed above emphasizes the role of the tax level; if capitalization dominates, an environment with constant but low taxes would be more volatile than a high tax regime.

[8]The fact that high beliefs volatility helps to get a high stock return was already exploited by Adam, Marcet, and Nicolini, 2016. This volatility coming from subjective beliefs avoids using a too volatile income process or a too high risk aversion.

PD ratio that helps in getting high returns without exaggerating its volatility. Crucially, these two factors do not affect the risk-free rate.

The last part of the paper digs into the normative side of capital gains taxation. In this family of models, asset markets are informationally inefficient (Adam, Marcet, and Beutel, 2017). The extra volatility arising from the learning process can be interpreted as a pecuniary externality.[9] For this pecuniary externality to have significant welfare consequences, excess volatility in asset prices is connected to aggregate consumption fluctuations. I use a tractable two-sector growth model with investment adjustment costs and learning about capital prices. The model links the capital market price to investment decisions, in line with the Q-theory (Tobin, 1969). With subjective beliefs, two feedback loops operate: the first, the one between the stock and price of capital, which is self-correcting; the second, the price-expectations loop, which is self-reinforcing. Their interaction gives rise to large and persistent cycles of over- and under-accumulation of capital.

In such a world, a Social Planner is asked to deliver the best possible competitive equilibrium by choosing a tax on unrealized capital gains and lump-sum transfers. She is endowed with all the relevant information, including investors' beliefs. The optimal policy prescribes using the CGT to counteract too optimistic/pessimistic beliefs about capital gains. In this way, the planner closes the gap between the market price and the shadow price of capital, restoring the First Best allocations.

The optimal CGT is a nonlinear function of the deviations of subjective expectations from its Rational Expectations counterpart (call it β^*). A shortcoming is that the optimal CGT is unbounded, inherits the dynamic properties of subjective beliefs and it is informationally demanding. Since tax volatility might not be desirable and mean subjective beliefs can be difficult to measure, an alternative implementation is suggested.[10] On the one hand, the CGT is set equal to 100% to eliminate the influence of subjective price beliefs (i.e., the source of the externality) on market prices. Since such a high tax depresses the market price a little, a subsidy for capital rents is introduced to restore efficient prices. The subsidy only depends on β^* and then is fairly stable and only requires a notion of fundamental value.

Related literature. The paper speaks to different literature on capital taxation, asset pricing, learning, business cycles and macroprudential policy. In what follows, I highlight the main contributions to each field.

This paper is the first to propose a theory of how a CGT regulates excess price volatility in a general equilibrium model. According to the dominant view, a CGT increases

[9] Since excess volatility emerges from the inability of agents to internalize the equilibrium price formation due to imperfect information about other market participants.

[10] This implementation uses a decomposition between fundamental and non-fundamental price volatility in the spirit of the trading decomposition used by Dávila, 2020.

volatility due to the supply-side lock-in effect (e.g., Somers, 1948, Somers, 1960, Stiglitz, 1983). An alternative viewpoint contending that a CGT stabilizes prices through the demand-side capitalization effect is on Haugen and Heins, 1969 and Haugen and Wichern, 1973.[11] My work builds upon these last two papers; while they use exogenous beliefs, I highlight the critical role of endogenous beliefs.[12]

This novel theory is embedded in a quantitative model to show that tax cuts explain a significant portion of the rise in excess price volatility in the US. This evidence complements the model-based works by McGrattan and Prescott, 2005 and Brun and González, 2017 and the econometric approach of Sialm, 2009, that pointed out that lower taxes can account for the rise is asset valuation levels. Besides, Dai, Shackelford, et al., 2013 document a new statistical negative relationship between taxes and returns volatility exploiting the cross-sectional variations in accrued capital gains and dividend distributions of stocks around the CGT cuts of 1978 and 1997.[13] My paper suggests a theory that can rationalize this evidence and present time-series evidence.

The paper focuses on the excess volatility puzzle, highlighting its time-varying nature. Theoretically, the model deals with the puzzle by bringing up an additional source of variation (beliefs) in line with the Adaptive Learning literature (e.g., Timmermann, 1993, Bullard and Duffy, 2001, Cogley and Sargent, 2008). In particular, I follow the Internal Rationality framework, a microfoundation of learning proposed by Adam and Marcet, 2011. Adam, Marcet, and Nicolini, 2016 and Adam, Marcet, and Beutel, 2017 presented quantitative versions, accounting for many asset pricing facts. The inclusion of taxes in this kind of models solves some of their shortcoming and allows to address new facts as the rise in excess volatility. Empirically, I present a version of the Campbell and Shiller, 1988's PD variance decomposition with taxes. Additionally, I argue that tax cuts help explain the equity premium puzzle. Thus, the paper provides quantitative evidence backing McGrattan and Prescott, 2003.

The model used for optimal policy analysis relates to papers dealing with joint stock market and business cycles (e.g., Boldrin et al., 2001). In particular, recently some papers have considered the impact of learning about capital prices on business cycle through labour demand (Adam and Merkel, 2019), collateral constraints (Winkler, 2020) and wealth effects (Ifrim, 2021). This paper considers another possibility: learning about capital prices directly affect physical investment when there are investment adjustment costs.

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[11] See Dai, Maydew, et al., 2008 for an explanation of the lock-in and capitalization effect and a literature review about their empirical relevance.

[12] In fact, applying Rational Expectations to Haugen and Heins, 1969 and Haugen and Wichern, 1973's model delivers a constant PD ratio so that taxes play no role in excess volatility whatsoever.

[13] Building upon the idea that CGT are a risk-sharing device with the government that affect the level of stock returns (e.g., Lerner, 1943, Stiglitz, 1975, Sikes and Verrecchia, 2012), Dai, Shackelford, et al., 2013 suggests CGT cuts reduce the risk-sharing, rising volatility. This point is comparable to that of Gemmill, 1956's. The theory I propose can accommodate this income effect, but rely primarily on a substitution effect.

While a CGT has been studied as a tool to raise revenue (e.g., Agersnap and Zidar, 2021, Sarin et al., 2022), this paper looks at it from a macroprudential standpoint. The recent literature on macroprudential policy has dealt chiefly with collateral constraints and taxes on borrowing (e.g., Lorenzoni, 2008, Jeanne and Korinek, 2010, Dávila and Korinek, 2018, Jeanne and Korinek, 2019) and nominal rigidities (Farhi and Werning, 2016). This paper shares the emphasis on pecuniary externalities with most of the literature but departs from its origin (i.e., information rather than financial frictions).[14] Besides, while this recent literature has focused on constrained efficiency, Benigno et al., 2019 showed that a superior allocation is attainable with the same instruments. Following them, I study the optimal use of a CGT to restore unconstrained efficiency.

The findings of this study have significant policy implications. Firstly, it highlights the potential of a CGT to serve as a viable alternative to the widely debated Financial Transactions Tax (or "Tobin tax").[15] Secondly, the research demonstrates that, instead of relying solely on monetary policy to regulate asset prices, the implementation of an appropriate CGT can facilitate the disentanglement of interest rate policy from financial stability considerations.[16]

The rest of the paper proceeds as follows. Section 2 explores how Capital Gains Taxes can stabilize asset prices using a model with learning about prices. Section 3 presents a quantitative application of the theory to the US stock market. Section 4 derives an optimal CGT in a two-sector growth model. Section 5 concludes, pointing out some avenues for future research.

1.2. Theory

This section explores theoretically the role of a CGT in determining the level and volatility of stock prices. Section 1.2.1. sets up the basic model, with an exogenous realization of capital gains, focusing on the capitalization effect. In Section 1.2.2., the effects of CGT on both the asset price level and volatility are analyzed. Section 1.2.3. endogeneizes the realization of capital gains, giving rise to the lock-in effect. It shows how this counteracting effect influences the results in Section 1.2.2..

[14] Di Tella, 2019 and Kurlat, 2018 focus on the inefficiencies arising from financial frictions when the planner is informationally constrained. Farhi and Werning, 2020 incorporates extrapolation into the optimal macroprudential analysis, showing that it plays a rather secondary role. In contrast, the analysis in the paper puts it at the forefront.

[15] See, for instance, Buss, Dumas, et al., 2016, Buss and Dumas, 2019 or Dávila, 2020 for theoretical analysis and Umlauf, 1993 or Cappelletti et al., 2017 for empirical results that challenge the ability of the Tobin tax to stabilize asset prices.

[16] Asset pricing targetting has been shown to be appropriate in Nisticò, 2012, Gambacorta and Signoretti, 2014 or Ifrim, 2021.

1.2.1. The model

In this section, the model is set up. Its basic layer is the Lucas, 1978's tree model, extended with a general probability measure, as Adam and Marcet, 2011, and taxes on dividends and realized capital gains.

Demographics. The economy is populated by a unit mass of infinitely living identical investors.

Technology. This is an stochastic exchange economy. There is a single perishable good. There exists a single risky asset, called stock S, in the form of a contract that each period promises D_t units of the good in the form of dividends, following this process

$$\frac{D_t}{D_{t-1}} = \beta^D + \varepsilon_t^d \tag{1.1}$$

with a being the permanent component and $\varepsilon_t^d \sim \mathcal{N}(0, s_d^2)$ an i.i.d. innovation and D_{-1} given. When the time starts, each investor is endowed with one unit of stock ($S_{-1}^i = 1$). Every period, investors face some risk of being hit by a very bad shock. $z_t^i \sim Bernoulli(\pi)$ is a random variable indicating that possibility with π being the probability of that catastrophic event occurring. If the event materializes ($z_t^i = 1$), investor i sells all her stock holdings.

Institutions. There are markets for goods and stocks. Short selling is not allowed, and there is an upper bound on the amount of stock holdings $0 \leq S_t^i \leq \bar{S}$. There is a linear tax $\tau^D \in [0,1)$ on dividends and a tax $\tau^K \in [0,1)$ on realized capital gains.[17,18] Capital gains and losses are treated symmetrically. The revenues from dividend taxes are transferred in a lump-sum way; the revenues from capital gains taxes are transferred back individually, according to each individual contribution. These two terms are gathered in the variable T_t^i, standing for idiosyncratic transfers.

Information. Investors take τ^D, τ^K and T_t^i as given and know the stochastic process followed by D_t and z_t^i. Investors beliefs are not common knowledge. The underlying probability space is given by $(\Omega, \mathcal{B}, \mathcal{P}^i)$ where Ω is the state space with $\omega = \{D_t, z_t, P_t\}_{t=0}^{\infty}$ as a typical element, $\mathcal{B}$ denotes the σ-algebra of Borel subsets of Ω and $\mathcal{P}^i$ agent's i subjective probability measure over $(\Omega, \mathcal{B})$.[19]

Rational behavior. Each investor faces a consumption-savings problem: she chooses sequences of consumption, stock holdings and stock purchases $\{C_t^i, S_t^i, X_t^i\}_{t=0}^{\infty}$ by solving

[17] Note realization is exogenous, driven by the liquidity shock. In section 2.3, I endogeneize π by including portfolio adjustment costs. The introduction of this liquidity shock is equivalent to assume that investors expect that a fixed proportion of the capital gains will be realized as in Sialm, 2009.

[18] In equilibrium, a fraction π of agents sells their assets and pay taxes (let $Z_t = \frac{1}{n}\sum_{i=1}^{n} z_t^i$; by the LLN, $Z_t \xrightarrow{p} \pi$); hence, the effective rate on total capital gains is $\pi\tau^K$.

[19] I follow here the Internal Rationality approach of Adam and Marcet, 2011.

the following optimization program:

$$\max_{\{C_t^i, S_t^i, X_t^i\}_{t=0}^{\infty}} \mathbb{E}_0^{\mathcal{P}^i} \sum_{t=0}^{\infty} \delta^t U(C_t^i) \tag{1.2}$$

subject to the budget constraint

$$C_t^i + P_t X_t^i \le (1 - \tau^D) D_t S_{t-1}^i + z_t^i \left(S_{t-1}^i P_t - \tau^K G_t^i \right) + T_t^i \tag{1.3}$$

the stock holdings law of motion

$$S_t^i = (1 - z_t^i) S_{t-1}^i + X_t^i \tag{1.4}$$

the capital gains law of motion

$$G_t^i = G_{t-1}^i (1 - z_{t-1}^i) + (P_t - P_{t-1}) S_{t-1}^i \tag{1.5}$$

and the stock holdings bounds specified above, given the initial individual stock holdings.

The utility function is a time-separable continuous, increasing in consumption $U'(C_t^i) > 0$ but concave $U''(C_t^i) \le 0$ function. In this section, I assume risk-neutrality.[20] $\delta \in (0, 1)$ is a discount factor. Lower and upper bounds on S_t^i are assumed for convenience: economically, the lower bound rules out short-selling strategies aimed at avoiding taxes; mathematically, these bounds ensure that the feasibility set is compact.

Model Equilibrium. The investor's program consists of maximizing a bounded continuous function over a compact non-empty feasible set.[21] By the Weierstrass extreme value theorem, these are sufficient conditions for the existence of a maximum. Moreover, the convexity of the feasible set implies the first-order conditions are necessary and sufficient to characterize the optimum by the Kuhn-Tucker (KT) theorem.

Investor i's optimality conditions for an interior solution boil down to the following one-period ahead Euler Equation

$$P_t = \delta \mathbb{E}_t^{\mathcal{P}^i} \left[(1 - \tau^D) D_{t+1} + P_{t+1} - z_{t+1}^i \tau^K (P_{t+1} - P_t) \right] \tag{1.6}$$

along with a transversality condition and the sequence of budget constraints and market clearing conditions. Note that agents cannot proceed in a conventional way -iterate forward, use the Law of Iterated Expectations (LIE) and apply a transversality condition- to recover P_t as a discounted sum of dividends. LIE fails to be useful as the probability measure of the marginal agent in future periods $\mathcal{P}^j$ is unknown for i, potentially different

[20] Risk aversion is introduced in Section 3.

[21] See Adam, Marcet, and Beutel, 2017 for a proof in a similar setup.

from $\mathcal{P}^i$. Hence, equilibrium prices must be characterized by equation (1.6). It can be rewritten as

$$P_t = \delta(1-\tau^D)\mathbb{E}_t^{\mathcal{P}^i}\left[\frac{D_{t+1}}{D_t}\right]D_t + \delta\mathbb{E}_t^{\mathcal{P}^i}\left[\frac{P_{t+1}}{P_t}\right]P_t - \delta^2\tau^K\mathbb{E}_t^{\mathcal{P}^i}\left[\frac{P_{t+1}}{P_t}z_{t+1}^i\right]P_t + \delta\tau^K\mathbb{E}_t^{\mathcal{P}^i}\left[z_{t+1}^i\right]P_t \tag{1.7}$$

Solving the previous expression for P_t and dividing both sides by D_t, the PD ratio reads as

$$\frac{P_t}{D_t} = \frac{\delta(1-\tau^D)\mathbb{E}_t^{\mathcal{P}^i}\left[\frac{D_{t+1}}{D_t}\right]}{1-\delta\mathbb{E}_t^{\mathcal{P}^i}\left[\frac{P_{t+1}}{P_t}\right] + \delta\tau^K\left(\mathbb{E}_t^{\mathcal{P}^i}\left[\frac{P_{t+1}}{P_t}z_{t+1}^i\right] - \mathbb{E}_t^{\mathcal{P}^i}\left[z_{t+1}^i\right]\right)} \tag{1.8}$$

Since agents are aware of the true stochastic processes for D_t and z_t^i, $\mathbb{E}_t^{\mathcal{P}^i}\left[\frac{D_{t+1}}{D_t}\right] = \beta^D$ and $\mathbb{E}_t^{\mathcal{P}^i}\left[z_{t+1}^i\right] = \pi$. However, $\mathbb{E}_t^{\mathcal{P}^i}\left[\frac{P_{t+1}}{P_t}\right]$ and $\mathbb{E}_t^{\mathcal{P}^i}\left[\frac{P_{t+1}}{P_t}z_{t+1}^i\right]$ are unknown. To determine them and then obtain the equilibrium PD ratio, I employ two models of price expectations. From here on, the superindex i is removed to save notation.

First, as a benchmark, consider the special case with full information in which investors are perfectly aware of the economic structure, including other investors' beliefs. Full information and rationality give rise to the Rational Expectations Equilibrium whereby prices are equal to

$$\frac{P_t^{RE}}{D_t} = \frac{\delta(1-\tau^D)\beta^D}{1-\delta(1-\pi\tau^K)\beta^D - \delta\pi\tau^K} \tag{1.9}$$

Note that for the PD ratio to be positive $\delta\beta^D < \frac{1-\delta\pi\tau^K}{1-\pi\tau^K} > 1$, where the last inequality follows from $\delta < 1$; this is assumed throughout the paper. From equation (1.9), it is clear that the implicit price model under Rational Expectations is given by

$$\frac{P_t^{RE}}{P_{t-1}^{RE}} = \frac{D_t}{D_{t-1}} = \beta^D + \varepsilon_t^d \tag{1.10}$$

The belief system, captured by $\mathcal{P}$, includes the stochastic processes characterizing D_t and z_t and the price model. Under Rational Expectations, the price model is simply redundant.

Now consider the more general case where beliefs' homogeneity is not common knowledge. Since LIE does not help rational agents to deduce equilibrium prices from their optimality conditions, agents need some model to forecast future prices. The proposed subjective model is

$$\frac{P_t}{P_{t-1}} = b_t + \varepsilon_t^P \tag{1.11}$$

$$b_t = b_{t-1} + \vartheta_t \tag{1.12}$$

with $\varepsilon_t^P \sim i.i.\mathcal{N}(0, s_P^2)$ and $\vartheta_t \sim i.i.\mathcal{N}(0, s_b^2)$. The permanent component of price growth b_t is unobserved and has to be estimated from the history of states. For that purpose, investors use a Kalman filter. The posterior, conditional on the observed price history, is given by $b_t|I_t \sim \mathcal{N}(\beta_t, \sigma^2)$ where σ^2 is the steady state Kalman filter uncertainty and the posterior mean β_t evolves recursively following[22]

$$\beta_t = \beta_{t-1} + g\left(\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}\right) \tag{1.13}$$

where $g = \frac{\sigma^2 + s_b^2}{\sigma^2 + s_b^2 + s_P^2}$ is the steady-state Kalman gain. It follows that subjective price expectations are given by

$$\mathbb{E}_t^{\mathcal{P}}\left[\frac{P_{t+1}}{P_t}\right] = \beta_t \tag{1.14}$$

Thus, with the information friction, price growth beliefs become non-degenerated such that the model of prices is a non-redundant element of the beliefs system.

The reason for choosing this subjective model among the many possible alternatives is threefold. Theoretically, it encompasses RE beliefs as a special case; when $s_b^2 = 0$ and investors' initial prior is $b_0 = \beta^D$ with probability 1, $\beta_t = \beta^D \ \forall t$. Empirically, it gives rise to persistent deviations of price growth from dividend growth in line with the large swings of the empirical PD ratio. Besides, it is consistent with microevidence on investors' beliefs, in particular of extrapolation and under-reaction (Kohlhas and Walther, 2021).

The last object to be determined is $\mathbb{E}_t^{\mathcal{P}}\left[z_{t+1}\frac{P_{t+1}}{P_t}\right]$. Given the subjective model of prices,

$$\mathbb{E}_t^{\mathcal{P}}\left[z_{t+1}\frac{P_{t+1}}{P_t}\right] = \mathbb{E}_t^{\mathcal{P}}\left[z_{t+1}(\beta_t + u_t + \vartheta_{t+1} + \varepsilon_{t+1}^P)\right] = \mathbb{E}_t^{\mathcal{P}}\left[z_{t+1}\right]\beta_t = \pi\beta_t \tag{1.15}$$

Where $u_t \sim i.i.\mathcal{N}(0, \sigma^2)$ is the Kalman forecast error. Plugging the subjective price expectations in the pricing function (1.8), the equilibrium PD ratio with imperfect information and Learning reads as

$$\frac{P_t^L}{D_t} = \frac{\delta(1-\tau^D)\beta^D}{1 - \delta(1-\pi\tau^K)\beta_t - \delta\pi\tau^K} \tag{1.16}$$

[22] As it is standard in the literature, equation (1.13) contains lagged price growth. The reason is that it is assumed agents observe in period t information about the lagged transitory component ε_{t-1}^P. This assumption avoids multiplicity of equilibria and turns out to perform better. See Adam, Marcet, and Beutel, 2017 for a discussion.

The dynamics of this variable are completely determined by $\{\beta_t\}$. Combining equilibrium prices (1.8) with the beliefs updating equation (1.13), it can be shown that $\{\beta_t\}$ is determined by the following second-order nonlinear difference equation:

$$\beta_t = \beta_{t-1}(1-g) + g\left(\frac{1-\delta(1-\pi\tau^K)\beta_{t-2} - \delta\pi\tau^K}{1-\delta(1-\pi\tau^K)\beta_{t-1} - \delta\pi\tau^K}\right)(\beta^D + \varepsilon^D_{t-1}) \tag{1.17}$$

The steady state of the equation corresponds to the Rational Expectations ($\beta_t = \beta^D$ for all t). Out of the steady state, beliefs orbit around the RE equilibrium, giving rise to beliefs booms and busts (see Adam, Marcet, and Nicolini, 2016). In the next section, I show the implication of taxes for beliefs and price dynamics.

1.2.2. The effect of CGT on the price level and volatility

In this section, the impact of capital taxes on the PD ratio is explored. The literature has conceptualized two opposite effects of CGT on the price level (see Dai, Maydew, et al., 2008). On the one hand, a capitalization effect lowers the price since buyers discount future tax liabilities, reducing demand. On the other hand, a lock-in effect increases the price since the value option of selling goes down, reducing supply. I derive its consequences for the price volatility too. In this section, I focus on the capitalization effect; the lock-in is introduced in the next section.

Let's start with RE. Given the pricing formula for RE (1.9),

$$\frac{dP^{RE}_t/D_t}{d\tau^K} = -\frac{\delta(1-\tau^D)\beta^D\delta\pi(\beta^D-1)}{(1-\delta(1-\pi\tau^K)\beta^D - \delta\pi\tau^K)^2} < 0 \tag{1.18}$$

if $\beta^D > 1$. Thus, the model predicts a negative relationship between capital taxes and stock market valuations under RE, in line with McGrattan and Prescott, 2005 or Sialm, 2009.[23]

As for learning, τ^K has an extra effect through subjective beliefs dynamics. Thus,

$$\frac{dP^L_t/D_t}{d\tau^K} = \frac{P^L_t/D_t}{\tau^K} + \underbrace{\frac{P^L_t/D_t}{\beta_t}\frac{d\beta_t}{d\tau^K}}_{\text{Learning Amplification}} \tag{1.19}$$

Like under RE, $\frac{P^L_t/D_t}{\tau^K} < 0$ when agents expect positive capital gains (i.e., $\beta_t > 1$). Moreover,

$$\frac{d\beta_t}{d\tau^K} = \beta^D\varepsilon^D_{t-1}\frac{-\delta\pi\Delta\beta_{t-1}(1-\delta(1-\pi\tau^K)\beta_{t-1} - \delta\pi\tau^K) - \delta(1-\pi\tau^K)\Delta\beta_{t-1}\delta\pi(\beta_{t-1}-1)}{(1-\delta(1-\pi\tau^K)\beta_{t-1} - \delta\pi\tau^K)^2} \tag{1.20}$$

[23]This contrast with the results in Sialm, 2006, according to which the tax level is irrelevant to the PD ratio level under CRRA utility. A consumption tax on the purchasing of new stocks drives his result. On the contrary, capital income taxes deliver results more aligned with the empirical observations.

which turns out to be negative provided $\Delta\beta_{t-1} > 0$ and $\beta_{t-1} > 1$. Since $\frac{P_t^L/D_t}{\beta_t} > 0$, τ^K dampens the PD ratio during booms. Hence, with learning, the negative effect of τ^K on the PD ratio level is reinforced through its effect on capital gains beliefs.[24]

What about the PD variance? Under Rational Expectations, the PD ratio is a constant and then taxes play no role.[25] However, with information frictions and learning, a CGT dampens price fluctuations coming from beliefs; it depresses prices during booms and increases prices during bursts. The following proposition explores this ability formally. For notational convenience, I denote $P_t/D_t = P_t^L/D_t$ from now on.

Assumption 1. *Assume: i)* $g \in (0, \bar{g})$ *with* $\bar{g} = \frac{1-\delta(1-\pi\tau^K)\beta^D - \delta\pi\tau^K}{\delta(1-\pi\tau^K)\beta^D}$*; ii)* $\pi\tau^K < \bar{\tau} = \frac{2\delta\beta^D-1}{2\delta\beta^D-\delta}$.

The first condition says that agents update their expectations in the direction of the difference between current price growth and expectations but the learning process is sluggish.[26] The second condition puts an upper bound on effective taxes, which is close to 1 for δ close 1, as typically assumed, and then it is a rather lax condition. Under this reasonable assumption, the following proposition holds:

Proposition. *Up to a linear approximation around Rational Expectations, the variance of the PD ratio is decreasing on the CGT level, that is,*

$$\frac{d\mathbb{V}ar[P_t/D_t]}{d\tau^K} < 0 \tag{1.21}$$

Proof. Appendix B.

The proposition shows that when subjective beliefs are close to Rational Expectations, the variance of the PD ratio is given by the sensitivity of prices to beliefs times the variance of subjective beliefs

$$\mathbb{V}ar\left[\frac{P_t}{D_t}\right] \approx \omega^2 \times \mathbb{V}ar(\beta_t) \tag{1.22}$$

[24]This suggests that capital gains taxes are particularly important to explain the rise in stock market valuations, complementing the role of dividend taxes pointed out by McGrattan and Prescott, 2005.

[25]This is the result of the i.i.d. dividends growth assumption. In Appendix C I explore a case with persistent growth, in the spirit of Bansal and Yaron, 2004. Then, a CGT affects negatively the variance of the PD ratio only by reducing ω (i.e., it has no effect on the variance of the growth process that originates the movements in the PD ratio). Thus, learning is sufficient but not necessary for a CGT to reduce the PD variance; it is an amplification mechanism.

[26]A small g is consistent with the empirical evidence on underreaction pointed out by Coibion and Gorodnichenko, 2015.

with $\omega = \frac{P_t/D_t}{\beta_t}\Big|_{\beta_t=\beta^D}$. It turns out that a CGT reduces both the transmission of belief fluctuations to prices (ω) and the variance of beliefs:

$$\frac{d\mathbb{V}ar[P_t/D_t]}{d\tau^K} \approx \underbrace{\frac{\mathbb{V}ar[P_t/D_t]}{\omega^2}}_{>0} \underbrace{\frac{d\omega^2}{d\tau^K}}_{<0} + \underbrace{\frac{\mathbb{V}ar[P_t/D_t]}{\mathbb{V}ar[\beta_t]}}_{>0} \underbrace{\frac{d\mathbb{V}ar[\beta_t]}{d\tau^K}}_{<0} < 0 \tag{1.23}$$

The proposition can be illustrated by plotting the belief dynamics, that fully characterize the PD dynamics, at different tax levels. The second-order difference equation (1.17) can be represented in a two-dimensional phase diagram on the (β_{t-1}, β_t) plane (keeping the dividend shock at its mean values). Starting from the Rational Expectations Equilibrium (call it β^*), figure 1.1 shows the dynamic response of beliefs to a shock (to dividends). When a positive news shock hits the market, prices become higher than expected, and investors review their beliefs upward following Bayes' Law. If this revision is strong enough, prices would rise further, feeding into even higher beliefs. Thus, for some periods, there is momentum in the sense of a rise in optimism that is self-reinforcing. At some point, prices do not grow as much as expected, so investors start correcting their beliefs downward; this is a bust. It is through a sequence of booms and busts that beliefs revert to their fundamental value. In line with the proposition, these oscillations around β^* are smaller the higher the CGT. In other words, momentum is shorter and mean reversion faster the higher the CGT such that a CGT anchors expectations closer to its fundamental value.

1.2.3. The lock-in effect

In this section, the realization of capital gains is endogeneized by introducing portfolio adjustment costs. This endogenous realization gives rise to the lock-in effect. An old proposition is that the lock-in effect makes CGT a destabilizing tool, restricting supply during booms due to the lower value of selling and increasing supply during bursts, to realize losses and receive a subsidy (e.g., Somers, 1948, Somers, 1960, Stiglitz, 1983). Now, I include this effect and study the conditions under which it dominates the stabilizing capitalization effect.

The liquidity shock z_t^i is replaced by a choice of the timing of realization. Following Gavin, Kydland, et al., 2007 and Gavin, Keen, et al., 2015, each investor manages a stock of unrealized capital gains G_t facing portfolio management costs. G follows this law of motion

$$G_{t+1} = G_t + (P_t - P_{t-1})S_{t-1} - g_t + AC_t \tag{1.24}$$

with g_t are the realized capital gains, and AC_t stands for adjustment costs. It is assumed $AC_t = g_t - G_t - \phi(\bar{\pi}_t)G_t$ for $\bar{\pi}_t \equiv g_t/G_t$, with $\phi'(\cdot) > 0$, $\phi''(\cdot) < 0$. It follows that when the realization of capital gains is deferred, the cost function penalizes investors with extra unrealized capital gains, increasing the future tax liability of households. With these

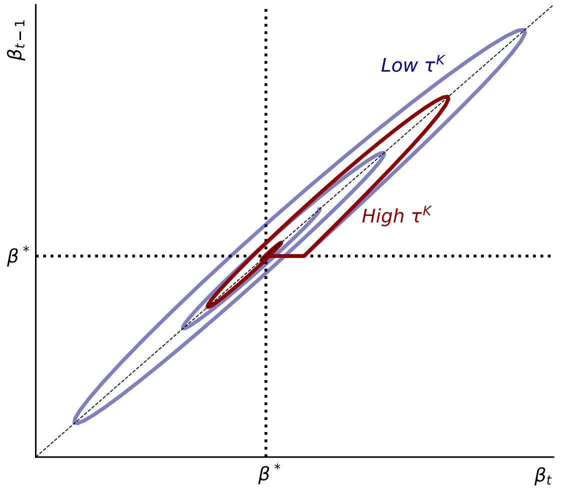

Figure 1.1. **Shock propagation at different levels of the Capital Gains Tax with the capitalization effect.** The graph illustrates the dynamics of subjective capital gains expectations (β) given by the second-order difference equation (1.17) when shocked by a one-off dividend shock. It considers $\Delta\beta_{t+1} = f(\beta_t, \beta_{t-1})$ with $\{\varepsilon^D_{t-1}\} = \{0.01, 0, 0, ...\}$. $\beta^* = \beta^D$, the Rational Expectations Equilibrium.

adjustment costs, investors face an additional trade-off: realize g_t capital gains and pay taxes $\tau^K g_t$ today or defer the realization and face an extra tax liability in the future.

Altogether, the investor's problem consists now in choosing sequences of consumption, stock holdings, stock purchases, realized and unrealized gains $\{C_t, S_t, X_t, g_t, G_t\}_{t=0}^{\infty}$ to maximize lifetime welfare (1.2), subject to the stock bounds, the stocks law of motion (1.4), the unrealized capital gains law of motion (1.24) and the budget constraint

$$C_t + P_t X_t \leq (1 - \tau^D) D_t S_{t-1} - \tau^K g_t + T_t \tag{1.25}$$

Let the cost function be $\phi(\bar{\pi}_t) = (\tau^K)^{1+\xi} ln(\bar{\pi}_t)$. Keeping the assumption of risk neutrality, the first-order conditions boil down to

$$P_t = \delta \mathbb{E}^{\mathcal{P}}_t \left[(1 - \tau^D) D_{t+1} + P_{t+1} - \mu_{t+1}(P_{t+1} - P_t) \right] \tag{1.26}$$

$$\bar{\pi}_t = \mu_t (\tau^K)^{\xi} \tag{1.27}$$

$$\mu_t = (\tau^K)^{1+\xi} \mathbb{E}^{\mathcal{P}}_t \left[\delta \mu_{t+1} \Big(1 - ln(\bar{\pi}_{t+1}) \Big) \right] \tag{1.28}$$

Equation (1.26) replaces equation (1.6). They differ only in one detail: the liquidity shock that determined the payment of taxes $z_{t+1}\tau^K$ is replaced by the Lagrange multiplier associated with the capital gains accumulation equation (1.24). The presence of μ points out that the additional burden unrealized capital gains represent diminishes the one-period ahead payoffs. In turn, according to equation (1.27), the optimal realization of capital fraction $\bar{\pi}_t$ depends on two terms: positively on the shadow price of unrealized gains μ_t indicating that the more costly non-realizing gains is the more agents realize them; and on the tax level τ^K. If $\xi < 0$, a higher τ^K lowers the optimal realization, expressing the lock-in effect. From this equation is clear that ξ is the tax elasticity of realization

$$\frac{\bar{\pi}_t}{\tau^K}\frac{\tau^K}{\bar{\pi}_t} = \mu_t \xi (\tau^K)^{\xi-1} \frac{\tau^K}{\mu_t (\tau^K)^{\xi}} = \xi \tag{1.29}$$

which is a structural parameter of the model that can be directly estimated from the data. Finally, equation (1.28) is a first-order stochastic difference equation determining the shadow price of unrealized capital gains. Shifting it one-period ahead:

$$\mu_{t+1} = (\tau^K)^{1+\xi} \mathbb{E}^{\mathcal{P}}_{t+1}\left[\delta \mu_{t+2}\Big(1 - ln\big(\mu_{t+2}(\tau^K)^{\xi}\big)\Big)\right] = (\tau^K)^{1+\xi} M_{t+1} \tag{1.30}$$

Then, opening up and operating on the Euler Equation (1.26), it becomes

$$P_t = \delta(1-\tau^D)\mathbb{E}^{\mathcal{P}}_t\left[\frac{D_{t+1}}{D_t}\right]D_t + \delta\mathbb{E}^{\mathcal{P}}_t\left[\frac{P_{t+1}}{P_t}(1-(\tau^K)^{1+\xi}M_{t+1})\right]P_t + \delta(\tau^K)^{1+\xi}P_t\mathbb{E}^{\mathcal{P}}_t\Big[M_{t+1}\Big] \tag{1.31}$$

Let $m_t \equiv \mathbb{E}^{\mathcal{P}}_t\Big[M_{t+1}\Big]$. Using the subjective model of prices,

$$\mathbb{E}^{\mathcal{P}}_t\left[\frac{P_{t+1}}{P_t}M_{t+1}\right] = \mathbb{E}^{\mathcal{P}}_t\Big[(\beta_t + u_t + \vartheta_{t+1} + \varepsilon^P_{t+1})M_{t+1}\Big] \approx \beta_t m_t \tag{1.32}$$

under the assumption that $\mathbb{C}ov(M_{t+1}, x_t) \approx 0$ for $x_t = (u_t, \vartheta_{t+1}, \varepsilon^P_{t+1})$ where u_t is the Kalman prediction error. Under that approximation, the PD ratio with endogenous capital gains realization satisfies

$$\frac{P_t}{D_t} = \frac{\delta(1-\tau^D)\beta^D}{1 - \delta\beta_t(1-(\tau^K)^{1+\xi}m_t) - \delta(\tau^K)^{1+\xi}m_t} \tag{1.33}$$

This formula embeds the one under exogenous realization (1.16) for $m_t = \pi$ and $\xi = 0$. Consider m_t as given for a moment. Then, this formula says that the lock-in effect dominates when $\xi < -1$, that is

$$\xi \lesseqgtr -1 \quad \Rightarrow \quad \frac{P_t/D_t}{\tau^K} \gtreqless 0 \tag{1.34}$$

However, for equation (1.33) to be an equilibrium equation, m_t must be characterized. Equation (1.28) does not have an analytical solution but can be computed numerically.[27]

[27]The algorithm to do so combines the Parameterized Expectations Algorithm with numerical integration. See Appendix X.

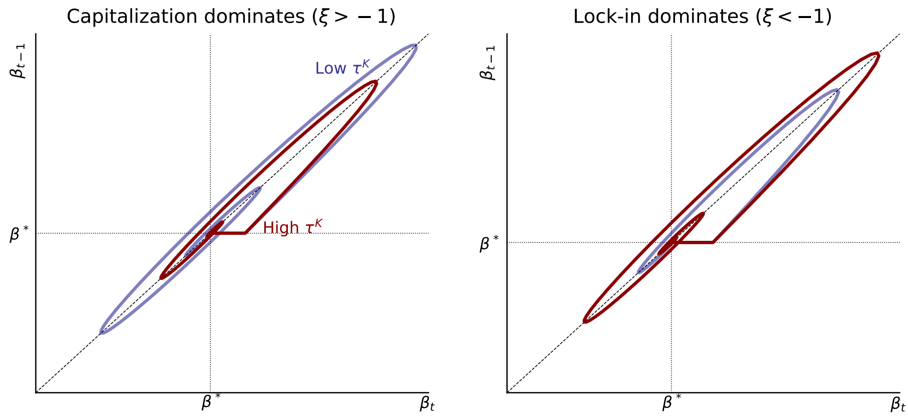

Figure 1.2. **Shock propagation at different tax levels with both the capitalization and the lock-in effect.** The graph illustrates the dynamics of subjective capital gains expectations (β) when shocked by a one-off dividend shock. The blue (red) line uses $\tau^K = 0.1 (= 0.4)$. The LHS (RHS) graph uses $\xi = -0.5$ ($\xi = -1.2$).

Since m_t is a function of the state variables, including P_t/D_t, now there is no closed form for equilibrium prices. Hence, the effects of taxes on the PD level and variance with the lock-in must be explored via simulations.

Figure 1.2 plots the propagation of a one-off dividend shock, as figure 1.1 but with the lock-in effect. When $\xi > -1$, the capitalization effect dominates and thereby, a higher τ^K reduces the fluctuations, as proven analytically in Section 2.2 for $\xi = 0$, $m_t = 1$. Nonetheless, $\xi < -1$ makes the lock-in effect to dominate resulting in a destabilizing effect of a higher τ^K in line with Stiglitz, 1983.[28]

1.3. Quantitative Analysis

This section uses the theory to understand the increase in the US stock market aggregate fluctuations in the middle of the Great Moderation, relating them to the decline in capital taxes. In Section 1.3.1., asset pricing facts for the 1946-2018 period are presented, including a novel version of the Campbell and Shiller, 1988 PD variance decomposition with capital taxes. Section 1.3.2. adds some ingredients to Section 2's models and introduces a novel application of the Parameterized Expectations Algorithm to solve it. Section 1.3.3. describes the parameterization procedure, which involves structural estimation via the Simulated Method of Moments. Section 1.3.4. and Section 1.3.5. reports the estimation results and

[28] When numerically computing m_t, it turns out that the influence of P_t/D_t on it is small. Thus, the intuition of the dominance of the lock-in when $\xi < -1$ appeared in expression (1.34) is approximately correct.

their robustness to several alternative choices. Finally, Section 1.3.6. examines why the model can generate a significant equity premium.

1.3.1. Facts

This section documents asset pricing facts using US data from 1946 to 2018. It splits the observations into two halves to highlight the changes that have occurred since the 1980s. I report standard statistics involving excess volatility, the equity premium, and macroeconomic risk. Besides, I include statistics related to the theory, such as capital taxes, an extended version of the Campbell and Shiller, 1988 variance decomposition that includes capital taxes, and the sensitivity of prices to survey beliefs.

Fact 1: Decline in capital taxes. Personal taxes on investment income went down in the last decades (McGrattan and Prescott, 2005, Sialm, 2009). Investment income is affected by taxes on dividends, capital gains and interests. As it is customary in the literature, I measure them using effective average marginal rates, that is, a value-weighted mean of the marginal tax rates of investors in the various income brackets once adjusting for the features of the tax code (as maximum and minimum taxes, partial inclusion of social security or phaseouts of the standard deduction).[29] Thus, the dividends tax rate is

$$\tau_t^D = \tau_t^d(1 - \eta_t) \tag{1.35}$$

the capital gains tax

$$\tau_t^K = (\phi\tau_t^{skg} + (1 - \phi)\tau_t^{lkg})(1 - \eta_t) \tag{1.36}$$

and finally, the interest tax

$$\tau_t^B = \tau_t^b(1 - \eta_t) \tag{1.37}$$

In the previous expressions, τ_t^d, τ_t^{skg}, τ_t^{lkg} and τ_t^b are the effective average marginal rates on dividends, short, long capital gains and interest income respectively; ϕ is the average weight of short capital gains on total capital gains; η_t is the non-taxable share. Data sources are in Appendix A; computation details on the non-taxable share are in Appendix C. As illustrated in figure 1.3, taxes exhibited a substantial decline which, although with different timing, represented a movement towards a generally lower tax environment. This overall tax decline was the result of the joint action of tax reforms along with regulatory changes involving pensions savings vehicles that led to a massive change in asset holdings

[29] These rates are provided by the TAXSIM program of the NBER and can be accessed on his website. Before 1960, τ_t^d, τ_t^{skg} and τ_t^{lkg} rates are taken from Sialm, 2009. See Appendix A for details.

from taxable to non-taxable accounts (see McGrattan and Prescott, 2005).[30,31]

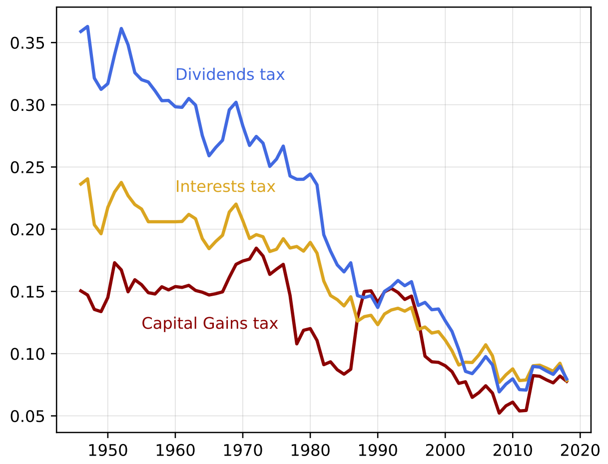

Figure 1.3. **Capital taxes rates along the postwar period.** The graph plots the capital taxes on dividends (blue), interest income (yellow) and capital gains (red) as defined by equation 1.35, 1.36 and 1.37. Annual series 1946-2018. See Appendix A for data sources and Appendix C for details on the computations.

Fact 2: Rise in stock valuations. The mean PD ratio almost doubled after the 1980s. This fact has been extensively documented in the literature (e.g., Shiller, 2000, McGrattan and Prescott, 2005, Brun and González, 2017) and is illustrated in 1.4. The accounting reason is that the increase in price growth (from a quarterly average of 0.48% to 1.48%) has exceeded by far a slightly higher dividend growth (from 0.49% to 0.75%). Thus, a higher PD ratio is a result of the sharp rise in capital gains. A related observation is that mean returns have mildly decreased, giving rise to some reduction in the equity premium.

Fact 3: Rise in the variance of the PD ratio.[32] The PD ratio standard deviation became two and a half times higher after 1982 than before. The Campbell and Shiller,

[30] Important reforms were the reduction of capital gains by Carter in 1978 and Clinton in 1997, partially counteracted by Reagan in 1986. When it comes to dividends, Reagan 1982 and Bush 2001 and 2003 represented substantial tax cuts.

[31] According to my estimates, the share of equity income paying taxes drop from 87% in 1946 to just 30% in 2018. This sharp decline is in line with the literature estimations (McGrattan and Prescott, 2005, Sialm, 2009, Rosenthal and Austin, 2016)

[32] In the paper, I focus on the unconditional variance. However, the rise in volatility is also observed for the conditional variance. See Appendix H.

1988's "dynamic accounting equation" provides the standard framework to understand the most proximate drivers of its variance. It reads as

$$p_t - d_t \approx \text{constant} + \sum_{j=1}^{\infty} \rho^{j-1} \Delta d_{t+j} - \sum_{j=1}^{\infty} \rho^{j-1} r_{t+j} \tag{1.38}$$

where lowercase letters mean log-variables ($x_t = lnX_t$), $\rho = PD/(1+PD)$, with PD being the mean PD ratio in the sample. Thus, the accounting fact is that the log PD ratio is approximately equal to the difference between the discounted sum of future dividend growth and stock returns escalated by a constant; a higher PD ratio must be due to higher dividends or lower returns in the future.

Taxes can be included in this equation by decomposing pre-tax returns into net returns and taxes. Following a standard procedure, the version of the Campbell-Shiller equation with taxes is

$$\begin{aligned} p_t - d_t \approx & \text{constant} + \overbrace{\sum_{j=1}^{\infty} \rho^{j-1} (\Delta d_{t+j})}^{\equiv \bar{d}_t} - \overbrace{\sum_{j=1}^{\infty} \rho^{j-1} (\hat{r}_{t+j})}^{\equiv \bar{r}_t} \\ & + \underbrace{\sum_{j=1}^{\infty} \rho^{j-1} \rho \ln(1 - \upsilon \tau^K_{t+j})}_{\equiv \bar{\tau}^K_t} + \underbrace{\sum_{j=1}^{\infty} \rho^{j-1} \tilde{\rho} \ln(1 - \tau^D_{t+j})}_{\equiv \bar{\tau}^D_t} \end{aligned} \tag{1.39}$$

with

$$\hat{r}_{t+1} = (1 - \upsilon \tau^K_{t+1}) \left(\frac{P_{t+1} - P_t}{P_t} \right) + (1 - \tau^D_{t+1}) \frac{D_{t+1}}{P_t} \tag{1.40}$$

$\rho = \frac{(1-\upsilon\tau^K)\frac{P}{D}}{(1-\upsilon\tau^K)\frac{P}{D}+1-\tau^D}$, $\tilde{\rho} = \rho \frac{1-\tau^D}{(1-\upsilon\tau^K)\frac{P}{D}}$ with all the variables being evaluated at their means and υ standing for the share of short capital gains expected to be realized next period.[33] Then, a higher PD ratio must come from higher dividends, lower after-tax returns or lower taxes in the future. Following a standard computation, the variance of the log PD ratio can be expressed as follows:

$$\begin{aligned} \mathbb{V}ar(p_t - d_t) \approx \quad & \mathbb{C}ov\left(p_t - d_t, \bar{d}_t\right) - \mathbb{C}ov\left(p_t - d_t, \bar{r}_t\right) \\ & + \mathbb{C}ov\left(p_t - d_t, \bar{\tau}_t{}^K\right) + \mathbb{C}ov\left(p_t - d_t, \bar{\tau}_t{}^D\right) \end{aligned} \tag{1.41}$$

[33] For the computations, $\upsilon = 0.016$ computed using data from the IRS and the US Financial Accounts. The estimation uses the synthetic capital gains tax as defined by (1.36), to be consistent with the model specification. Using τ_t^{skg} barely changes the numbers.

Table 1.1 shows the estimation of this equation.[34] An initial observation is that about 1/4 of the PD ratio variance typically attributed to movements in discount rates seems rather related to changes in capital taxes, especially dividend tax cuts. But the main utility of this decomposition is to figure out the sources of the increase in the PD ratio volatility. Basically, it is due to four factors: i) an increase in the covariance between the PD ratio and future returns; ii) a change from negative to positive covariance between the PD ratio and future dividends, signalling that returns and dividend growth varied in opposite directions since the 1980s; iii) tax cuts;[35] iv) a greater discount factor due, in part, to lower taxes.

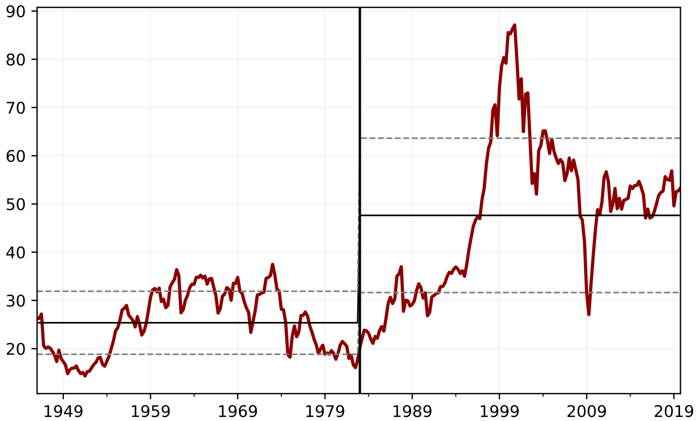

Figure 1.4. **Change in the mean and standard deviation of the Price/Dividend ratio.** The graph plots the evolution of the PD ratio in the 1946:I-2018:IV period. The continuous lines plot the mean of each subperiod 1946:I-1982:II and 1982:III - 2018:IV. The dotted bands shows ± one standard deviation.

Fact 4: Rise in the sensitivity of prices to expectations. An important link between low taxes and high volatility is the sensitivity of prices to beliefs changes. It was shown that the model with liquidity shocks predicts an increase in this sensitivity following a fall in CGT. One way of capturing this change is by running the following

[34] To empirically implement the previous equation, one has to deal with infinite sums, which are not observable. To that end, I follow the VAR approach first outlined by Campbell and Shiller, 1988 with short-run restrictions, with the variable ordering being dividends, taxes, returns and the PD ratio.

[35] Notice it does not mean that investors anticipated tax cuts necessarily. The present value of taxes is a projection from past data through the lens of the VAR. It turns out the correlation between the present value of taxes at time t $\bar{\tau}^K{}_t$ and time t taxes τ_t^K is -0.91 and -0.92 for each sample, respectively; for dividends, it is -0.99 in both samples. In other words, the present value of taxes approximately follows the path of current taxes.

Table 1.1. **Variance Decomposition of the Price-Dividend ratio.** The table reports $\mathbb{C}ov\left(p_t - d_t, \tilde{x}_t\right)$ with $\tilde{x}$ being the present value of dividend growth, stock returns, a capital gains tax factor and a dividend tax factor as specified in equation (1.41). The smaller gray values shows $\frac{\mathbb{C}ov(p_t - d_t, \tilde{x}_t)}{\mathbb{V}ar(p_t - d_t)} \times 100$ for the same variables. Present values are computed using a VAR, estimated separately for each subsample; see the main text for more details.

	1946-2018		1946-1982		1982-2018	
Returns	-18.89	-13.28	-8.53	-9.19	-13.18	-11.43
	98.85%	69.49%	118.54%	127.79%	93.67%	81.21%
Dividend growth	1.58	2.77	-1.01	-2.00	1.65	2.35
	8.27%	14.51%	-13.98%	-27.94%	11.73%	16.85%
Capital Gains tax	-	0.7	-	-0.06	-	0.35
	-	3.87%	-	-0.82%	-	2.47%
Dividend tax	-	3.92	-	0.36	-	1.01
	-	20.53%	-	5.00%	-	7.15%
Total Approximation	20.47	20.72	7.52	7.49	14.83	15.15
	107.12%	108.40%	104.56%	104.03%	105.40%	107.67%
$\mathbb{V}ar(p_t - d_t)$	19.11		7.20		14.07	
Discount factor ρ	0.9784		0.9729		0.9816	

regression for each subperiod

$$PD_t = \alpha + \zeta \ln \beta_t^s + \varepsilon_t \tag{1.42}$$

where β_t^s are survey expectations about capital gains.[36] As shown by Adam, Marcet, and Beutel, 2017, ζ goes up by a factor of 3 after the 1980s and the test of equality of pre- and post-1982 coefficients is strongly rejected.[37,38] Figure 1.5 illustrates the change towards a

[36] Following Adam, Marcet, and Beutel, 2017, I extend the UBS Gallup survey for the whole period using an updating equation in line with the implied by the Kalman filter.

[37] The null of $\zeta_{post} - \zeta_{pre} = 0$ is clearly rejected when using this statistic $t = \frac{\hat{\zeta}_{post} - \hat{\zeta}_{pre}}{(\hat{\sigma}_{pre}^2 + \hat{\sigma}_{post}^2)^{0.5}}$ with $\hat{\sigma}$ being Newey-West standard errors, which yields a value of 8.04.

[38] To the best of my knowledge, Adam, Marcet, and Beutel, 2017 were the first to report this fact, focusing on the pre- and post-2000s and relating it to lower interest rates. I stress that the change can be placed earlier and be related to lower taxes.

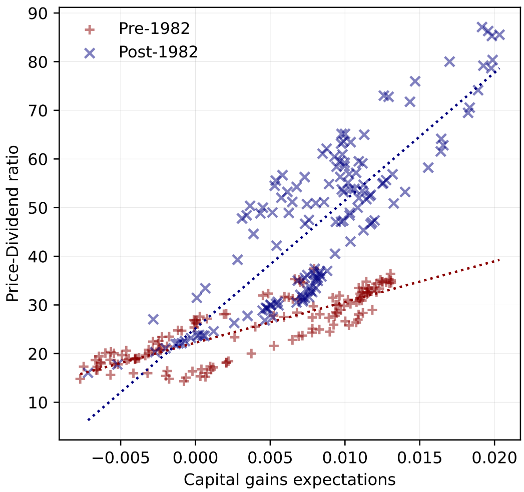

Figure 1.5. **PD ratio against capital gains survey expectations, pre and post-1982.** The graph plots the quarterly PD ratio of the SP500 against the log real mean price growth expectations implied by an extended version of the UBS Gallup survey. The time periods are 1946:I - 1982:II and 1982:III-2018:IV.

steeper relationship between beliefs and prices.

Relation to the theory. Can the theory exposed in Section 2 account for these facts altogether? The model predicts that lower taxes would increase the price level, as expected payoffs would soar, linking Fact 1 and Fact 2. This is in line with McGrattan and Prescott, 2005 and Brun and González, 2017. Additionally, it also predicts an increase in the PD variance following CGT cuts if the capitalization effect dominates over the lock-in effect, potentially linking Fact 1 and Fact 3, although noting the counteracting effects of both lower macro-risk and the relaxation of the lock-in effect. The key mechanism behind this outcome is the increase in the beliefs elasticity of prices, which is in line with Fact 4. By raising expected payoffs and their influence on prices, lower taxes would boost capital gains, reducing the correlation between returns and dividends, in line with the variance decomposition reported as part of Fact 3. The next section explores how much this hypothesis can account for the facts.

1.3.2. Extended model and a new solution algorithm

This section extends the model set up in Section 2 to better equip it to replicate the stylized facts reported in the previous section. Moreover, it introduces a new algorithm to solve

Table 1.2. **Facts. US Stock Market changes: 1946-1982 vs. 1982-2018.**
This table reports U.S. stock market moments using the data sources described in Appendix A. Growth rates and returns are annualized.

		1946-1982	1982-2018
		Fact 1: Decline in capital taxes	
Capital Gains tax	$\mathbb{E}(\tau_t^K)$	0.15	0.09
Dividends tax	$\mathbb{E}(\tau_t^D)$	0.29	0.12
Received Interest tax	$\mathbb{E}(\tau_t^B)$	0.20	0.11
		Fact 2: Rise in asset price levels	
PD level	$\mathbb{E}(PD_t)$	25.48	47.09
Dividend growth	$\mathbb{E}(D_t/D_{t-1} - 1)$	0.49	0.75
Stock price growth	$\mathbb{E}(P_t/P_{t-1} - 1)$	0.48	1.84
Quarterly real bond returns	$\mathbb{E}(r_t^b)$	0.42	0.38
Quarterly real stock returns	$\mathbb{E}(r_t^s)$	4.73	4.34
		Fact 3: Rise in PD volatility	
PD volatility	$\mathbb{V}ar(p_t - d_t)$	7.20	14.07
Comovement PD - dividends	$\mathbb{C}ov(p_t - d_t, \bar{d}_t)$	-2.00	2.35
Comovement PD - returns	$\mathbb{C}ov(p_t - d_t, \bar{r}_t)$	-9.19	-11.43
Comovement PD - Capital Gains tax	$\mathbb{C}ov(p_t - d_t, \bar{\tau}_t^K)$	-0.06	0.34
Comovement PD - Dividends tax	$\mathbb{C}ov(p_t - d_t, \bar{\tau}_t^D)$	0.36	1.00
Stock returns volatility	$\sigma(r_t^s)$	7.87	7.41
Dividend growth volatility	$\sigma(D_t/D_{t-1} - 1)$	2.52	1.97
Consumption growth volatility	$\sigma(C_t/C_{t-1} - 1)$	1.13	0.60
		Fact 4: Higher sensitivity of prices to beliefs	
Sensitivity of prices to beliefs	ζ	0.84	2.63

the model based on the Parameterized Expectations Algorithm.

The two versions of the model, either with liquidity shocks are with portfolio adjustment costs, are modified along four dimensions. First, the assumption of risk neutrality is abandoned. In this version, investors are allowed to dislike risk in a Constant Relative Risk Aversion (CRRA) sense, with γ regulating its risk aversion level. Second, an additional source of exogenous income is introduced to avoid a too-high correlation between dividends and consumption at odds with the data. Following Adam, Marcet, and Beutel, 2017, it is assumed agents get a wage endowment W_t each period, following this process:

$$\ln\left(1+\frac{W_t}{D_t}\right) = (1-p)\ln(1+\rho) + p\ln\left(1+\frac{W_{t-1}}{D_{t-1}}\right) + \ln\varepsilon_t^w \qquad (1.43)$$

D_t are aggregate dividends, $1+\rho$ is the average wage-dividend ratio and $p \in [0,1)$ is its quarterly persistence. The innovations are jointly distributed with dividend shocks following

$$\begin{pmatrix}\ln\varepsilon_t^D \\ \ln\varepsilon_t^W\end{pmatrix} \sim ii\mathcal{N}\left(-\frac{1}{2}\begin{pmatrix}\sigma_D^2 \\ \sigma_W^2\end{pmatrix}, \begin{pmatrix}\sigma_D^2 & \sigma_{DW} \\ \sigma_{DW} & \sigma_W^2\end{pmatrix}\right)$$

Third, stochastic taxes are introduced. It is assumed that each of the taxes on investment income follows a unit root process, that is

$$\tau_t^j = \tau_{t-1}^j + {}_t^{\tau^j} \qquad (1.44)$$

where ${}_t^{\tau_j} \sim ii\mathcal{N}(0, s_{\tau_j}^2)$, for j = K, D, B.[39] Tax shocks are assumed to be orthogonal to dividend and consumption shocks. Finally, risk-free bonds and taxes on bond interest are introduced in order to deal with the equity premium. The informational assumptions are as before, such that prices cannot be deduced from individual optimal computations, and investors use the subjective price model determined by equation (2.38) and (2.39), with the updating equation (1.13).

State variables. The state space is made of income sources and taxes/transfers, the stock of unrealized capital gains, previous stock holdings and current aggregate stock supply and, due to information incompleteness, current price and price growth beliefs, that is, $X_t = (D_t, W_t, \tau_t, T_t, G_t, S_{t-1}, P_t, \beta_t)$, with τ being the vector of dividends, capital gains and interest taxes. For the model with liquidity shocks, it also includes z_t. Given the homogeneity property of the CRRA function, the state vector can be reduced to $X_t = (\frac{W_t}{D_t}, \tau_t^K, \tau_t^D, \tau_t^B, \frac{T_t}{D_t}, S_{t-1}, \frac{G_t}{D_t}\frac{P_t}{D_t}, \beta_t)$. In this way, the model is rewritten in terms of non-explosive ratios.

Recursive Solution via the Parameterized Expectations Algorithm. A recursive solution boils down to a time-invariant stock demand function $S_t = S(X_t)$.[40] The main difficulty in deriving such an invariant function is that optimality conditions include an unknown subjective conditional expectation. For the exposition, I focus on the version with liquidity shocks. The solution to the version with portfolio adjustment costs follow

[39] When the observed tax time series is fit into an AR(1) model, the estimated coefficients are not statistically different from 0 (intercept) and 1 (slope). Thus, the unit root process constitutes a realistic representation of the tax process. Moreover, their residuals behave as Gaussian white noise. Normality has been tested via the Shapiro-Wilk Normality test.

[40] See Adam, Marcet, and Beutel, 2017 for a proof of the existence of a recursive equilibrium in the same model without taxes. It continues to hold with taxes.

similar steps, extended along the lines described in Section 2.3. The optimal Consumption-Dividends ratio must satisfy

$$\frac{C_t}{D_t} = \left\{\delta\mathbb{E}_t^{\mathcal{P}}\left[\left(\frac{P_{t+1}}{D_{t+1}}+1-\tau_{t+1}^{D}-z_{t+1}\tau_{t+1}^{K}\left(\frac{P_{t+1}}{D_{t+1}}-\frac{P_t}{D_t}\frac{D_t}{D_{t+1}}\right)\right)\frac{D_t}{P_t}\left(\frac{C_{t+1}}{D_{t+1}}\right)^{-\gamma}\left(\frac{D_{t+1}}{D_t}\right)^{1-\gamma}\right]\right\}^{-1/\gamma} \tag{1.45}$$

In equilibrium, the previous conditional expectation is a function $\mathcal{E}$ of the state variables, hence

$$\frac{C_t}{D_t} = \left(\delta\mathcal{E}(X_t)\right)^{-\frac{1}{\gamma}} \equiv \bar{\mathcal{E}}(X_t) \tag{1.46}$$

To solve the model, $\bar{\mathcal{E}}(X_t)$ must be computed somehow. The Parameterized Expectations Algorithm (PEA), initially proposed by Marcet, 1988, is one of the alternatives. PEA consists of replacing the conditional expectation $\mathcal{E}(X_t)$ with some parametric function ψ. ψ is not unique; popular possibilities are polynomials, splines or neural networks. In this model, there is no practical difference between approximating the conditional expectation $\mathcal{E}(X_t)$ and approximating the policy function $\bar{\mathcal{E}}(X_t)$. Exploiting that, I propose an approximating function rooted in economic theory. The idea is that of homotopy: start with a version of the model that has an analytical solution and keep the structure of the policy function as an approximating function.[41] For a model with exogenous i.i.d. returns, Hakansson, 1970 proved that the policy function consisted of consuming a constant fraction of wealth, with the constant propensity to consume given by one minus the expected utility of returns. Keeping that structure, I propose the following ψ:

$$\frac{C_t^*}{D_t} = \bar{\mathcal{E}}(X_t) \approx \psi(X_t;\chi) = c_t^y Y_t + c_t^w \frac{P_t}{D_t} S_{t-1} \tag{1.47}$$

where $c_t^y \equiv 1 - \chi\delta(1-\tau_t^D)\beta^D$ is the time-varying propensity to consume out of income, Y_t collects all the income sources (wages, dividends, net transfers) normalized by dividends, $c_t^w \equiv 1-\chi\delta(1-\tau_t^K)\beta_t$ is the propensity to consume out of wealth, and χ is a parameter of ψ to be estimated. The stock policy function can be obtained using the budget constraint, such that:

$$S_t^* = S(X_t) \approx \left((1-c_t^y)Y_t + (1-c_t^w)\frac{P_t}{D_t}S_{t-1}\right)\frac{D_t}{P_t} \tag{1.48}$$

This demand function indicates that investors save a time-varying fraction of their current resources, driven by discounted expectations. Thus, a rise in optimism would increase demand, but such an increase would be smaller the higher the tax. This mechanism

[41] Homotopy has already been applied to PEA problems. The difference here is that the form of the policy function is maintained instead of using arbitrary order polynomials or other functions.

is aligned with the empirical evidence reported by Giglio et al., 2021. As a result, the magnitude of the price change decreases with the tax level. This is an illustration of the Section's 2 *Proposition* that explicitly uses investors' demand as a mediator. Figure 1.6 illustrates these mechanics.

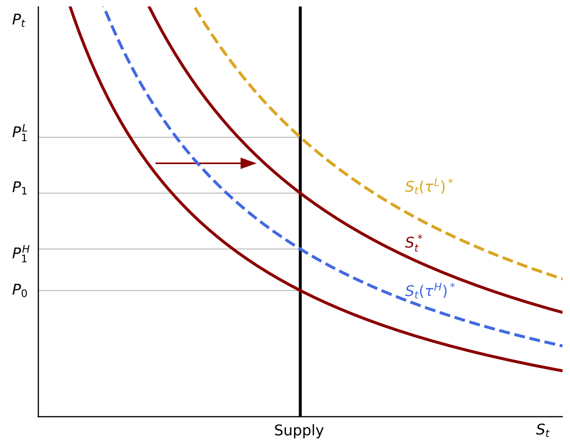

Figure 1.6. **Response of stock demand to an increase in optimism at different tax levels.** The graph plots the stock policy function (equation (1.48)), keeping everything constant except prices. Then, as β_t increases, the curve moves rightward. This displacement is shown at three different tax levels: low (blue), moderate (the baseline, in red) and high (yellow).

Finally, equilibrium prices can be obtained using good market clearing and the equity market clearing condition:

$$\sum_i S_i^* \left(\frac{P_t}{D_t}, \cdot \right) = \bar{S} = 1 \tag{1.49}$$

where the index i was momentarily reintroduced to emphasize the equilibrium condition of aggregate demand matching aggregate supply $\bar{S}$. Due to the handy policy function, this equation can be analytically solved for prices such that prices such that

$$\frac{P_t}{D_t} = \frac{\chi\delta(1-\tau_t^D)\beta^D}{1-\chi\delta(1-\pi\tau_t^K)\beta_t}\left(1+\frac{W_t}{D_t}\right) \tag{1.50}$$

Altogether, the use of this simple approximating function has some advantages. First, we are left with a single parameter to estimate as opposed to the potentially large number of parameters of alternative approximating functions. As a result, multicollinearity problems typically associated with PEA are avoided. Moreover, the procedure delivers a closed-form solution for equilibrium prices. Of course, a potential cost is that the approximating function ψ is not very flexible, as compared with arbitrary order polynomials or neural networks; however, it turns out to perform very well, with Euler Equation errors equivalent to \$1 out of a million. See Appendix D for a detailed explanation of the algorithm and its accuracy and the Appendix "Solving Asset Pricing Models with Learning using PEA" in Chapter 2's appendix for a more detailed explanation.

1.3.3. SMM estimation

This section explains the simulation strategy. It has to deal with two issues: a discontinuity in the pricing formula and the parameterization of the model. The former is solved by introducing a projection facility; the latter follows a mixed strategy, with some parameters calibrated from the US data and the rest being estimated via the Simulated Method of Moments.

As is standard in the learning literature, I employ a projection facility that restricts beliefs to ensure non-negative and non-explosive prices. Following Adam, Marcet, and Nicolini, 2016, the projection facility starts to dampen belief coefficients that imply a PD ratio equal to PD^L and sets an effective upper bound at PD^U. It can be understood as an approximate implementation of a Bayesian updating scheme where agents have a truncated prior that puts probability zero on beliefs that imply a too-high PD ratio. Appendix D contains the details.

On the other hand, the parameterization strategy is twofold. A subset of parameters related to the income processes, the vector $\widetilde{\theta} = \{\beta^D, \sigma_D, \sigma_W, \sigma_{WD}, p\}$ is picked directly from US data. I calibrate $\beta^D, \sigma_D, \sigma_W, \sigma_{WD}$ distinguishing between the two studied subperiods to capture the reduction in macroeconomic volatility. Parameter values are specified in panel a) of Table 1.3 and data sources are reported in Appendix A. Additionally, for the model with portfolio adjustment costs, the tax elasticity of realization ξ is set equal to the lower bound value of the recent estimations in Agersnap and Zidar, 2021, which represents the less favourable case for the stabilizing effect.

The remaining parameters, collected in the vector $\theta = \{\delta, g, \gamma, \pi, \rho, PD^L, PD^U\}$, are estimated via an extension of the Simulated Method of Moments, following Adam, Marcet, and Nicolini, 2016. Aiming at testing the power of taxes to explain the various observed changes, estimated parameters are kept fixed throughout the sample. Hence, a total of n=7 parameters are estimated to match a subset of M moments from the ones reported in table 1.2. The vector θ is chosen to minimize the distance between model $\widetilde{\mathcal{S}}(\theta)$ and data $\hat{\mathcal{S}}$ statistics, that is,

$$\hat{\theta} =_{\theta\in\Theta} \left[\hat{\mathcal{S}} - \widetilde{\mathcal{S}}(\theta)\right]' \hat{\Sigma}_{\mathcal{S}}^{-1} \left[\hat{\mathcal{S}} - \widetilde{\mathcal{S}}(\theta)\right] \tag{1.51}$$

where $\hat{\mathcal{S}}$ an $\widetilde{\mathcal{S}}(\theta)$ are Mx1 vectors and $\hat{\Sigma}_{\mathcal{S}}$ is a MxM weighting matrix, which determines the relative importance of each statistic deviation from its target. A diagonal weighting matrix whose diagonal is composed of the inverse of the estimated variances of the data statistics is used. Model-implied statistics are generated through a Montecarlo experiment with 1000 realizations. I formally test the hypothesis that any of the individual model statistics differ from its empirical counterpart. Finally, the model is fed with the empirical time series for capital taxes and dividend growth.[42]

Table 1.3. **Values of the model's parameters.** Panel a) table reports the values of the parameters calibrated directly from US data, using various data sources specified in Appendix A. Panel b) reports the estimated parameters when M=22 or M=8 statistics are included in the SMM estimation.

a) Calibrated parameters		1946-1982	1982-2018
Mean dividend growth	β^D	1.0049	1.0075
Dividends growth standard deviation	σ_D	0.0252	0.0197
Wage shocks standard deviation	σ_W	0.0261	0.0196
Covariance (wage, dividend)	σ_{WD}	-0.0006	-0.0004
Persistence wage-dividend ratio	p	0.99	
Tax elasticity of realization	ξ	-0.5	
b) Estimated parameters		**M=22**	**M=8**
Discount factor	δ	1.00	1.00
Kalman gain	g	0.0233	0.0234
Risk aversion	γ	0.55	0.66
Probability of a liquidity shock	π	0.0395	0.0382
Average wage-dividend ratio	ρ	5.51	5.49
Projection facility starting value	PD^L	551.04	532.45
Projection facility upper bound	PD^U	796.22	533.20

[42] In this way, the simulated series can potentially exhibit the same trajectories and trends as the observed ones. Instead, if I use a constant tax on each subsample, trends appearing in the PD ratio would not show up, distorting then the comparison between the observed and simulated data. In other words, by introducing the empirical time series, I compute the possible transition from high to low taxes instead of just simulating two long-lasting regimes.

1.3.4. Estimation results

In this section, the estimation results are reported. The version of the model with liquidity shocks is estimated using two different subsets of M statistics. On the one hand, M=22 statistics are included, that is, all the statistics reported in table 1.2 except the ones related to income processes, which are calibrated. A second estimation with the following M=7 statistics

$$\left\{\mathbb{E}(PD_t), \mathbb{E}(P_t/P_{t-1}), \mathbb{E}(r_t^s), \mathbb{E}(r_t^b), \mathbb{V}ar(p_t - d_t), \sigma(r_t^s), \hat{\zeta}\right\} \tag{1.52}$$

only for the first subsample is also run. The idea is to obtain the parameters that match the first subsample levels and variances right without any constraint in terms of the model's ability to match the changes in levels and volatility happening in the second subsample. In this second estimation, all the weight of replicating the observed changes lies upon the exogenous processes and the model's mechanisms. These two estimations, whose estimated parameter vector $\hat{\theta}$ is reported in panel b) of table 1.3, are then used to explore the model that incorporates the lock-in effect, which eventually will serve to compute the net effect of CGT cuts on the variance of the PD ratio.

Table 1.4 contains the statistics from the US data and the baseline estimation, which includes M=22 statistics, collected in the upper half of the table. Standard errors and t-statistics testing the null of equality between data and model statistics are also shown. The model statistics pass many of the t-tests. The increase in the mean PD ratio, although not as strong as observed, is substantive, driven by the model's ability to produce an increase in mean capital gains above the exogenous increase in dividend growth. The mean stock return is high enough at the same time that the risk-free rate is pretty low, delivering a remarkable equity premium. Matching the risk-free rate requires a very low risk aversion, which would reduce stock returns, other things equal. This force is counteracted by extra volatility coming through subjective beliefs (in terms of the parameters particularly observed in the high bounds for the projection facility). The equity premium issue will be discussed further in Section 3.6.

In terms of the variance, the model delivers an increase equivalent to 45% of the observed. A small part of the increase is due to tax changes, as in the data, but most of it comes from an excessive decline in the covariance between the PD ratio and future after-tax returns. An associated observation is that returns volatility goes up a bit in the second period, at odds with the data. Importantly, the model misses the role of the covariance between the PD and future dividends to explain the higher variance. Altogether, the model generates a non-negligible part of the higher PD variance but misses some factors and exaggerates others. Besides, there is a sizable increase in the elasticity, as pointed out in the theoretical analysis. Finally, the model also replicates well the persistence of the PD ratio and the high procyclicality of survey beliefs.

Table 1.5 shows the results when the lock-in effect is considered. The theoretical analysis showed that lock-in could counteract the effect of tax cuts by increasing the

Table 1.4. **Baseline estimation results; model with the capitalization effect only.** This table reports moments for the model with Bernoulli liquidity shocks with only the capitalization effect. The first four columns report the observed statistics along with their Newey-West standard error for the US data. The next four columns report model-implied statistics and their t-statistics. The model uses the parameterization described in Table 1.3 for M=22. Rates of growth, variance and covariance have been multiplied by 100. The top panel reports moments included in the SMM estimation; the bottom panel non-included moments.

	US data				Model			
	1946-1982		1982-2018		1946-1982		1982-2018	
	$\hat{S}_j$	$\hat{\sigma}_{\hat{S}_j}$	$\hat{S}_j$	$\hat{\sigma}_{\hat{S}_j}$	$\widetilde{S}_j(\hat{\theta})$	t-stat	$\widetilde{S}_j(\hat{\theta})$	t-stat
	Included in the SMM estimation							
$\mathbb{E}(PD_t)$	25.48	1.55	47.09	4.04	27.71	-1.43	40.29	1.68
$\mathbb{E}(P_t/P_{t-1}-1)$	0.48	0.64	1.84	0.63	0.87	-0.62	1.46	0.60
$\mathbb{E}(r_t^s)$	4.73	0.76	4.33	0.76	4.96	-0.31	4.38	-0.06
$\mathbb{E}(r_t^b)$	0.42	0.02	0.38	0.03	0.32	5.33	0.48	-4.15
$\mathbb{V}ar(p_t-d_t)$	7.20	1.35	14.07	3.56	8.34	-0.88	11.93	0.57
$\mathbb{C}ov(p_t-d_t,\bar{d}_t)$	-2.00	0.60	2.35	0.55	-0.38	-2.68	-0.80	5.71
$\mathbb{C}ov(p_t-d_t,\bar{r}_t)$	-9.13	1.47	-11.35	3.24	-8.88	-0.17	-11.80	0.14
$\mathbb{C}ov(p_t-d_t,\bar{\tau}_t^K)$	-0.06	0.06	0.34	0.06	-0.03	-0.50	0.20	2.39
$\mathbb{C}ov(p_t-d_t,\bar{\tau}_t^D)$	0.36	0.33	1.00	0.33	0.08	0.83	0.58	1.27
$\sigma(r_t^s)$	7.87	0.72	7.41	0.81	7.09	1.08	8.15	-0.91
$\hat{\zeta}$	0.84	0.06	2.63	0.21	0.92	-1.27	2.09	2.53
	Non-included in the SMM estimation							
$\mathbb{E}(D_t/D_{t-1}-1)$	0.49	0.35	0.75	0.34	0.49	0.00	0.75	0.00
$\sigma(D_t/D_{t-1})$	2.52	0.46	1.97	0.32	2.52	0.00	1.97	0.00
$corr(PD_t, PD_{t-1})$	0.96	0.13	0.98	0.07	0.97	-0.06	0.95	0.37
$corr(PD_t, \beta_t)$	0.84	0.11	0.84	0.22	0.83	0.16	0.83	0.04

realization of capital gains and by reducing the volatility of expected future tax liabilities. The results are in line with this analysis. The increase in the mean PD ratio is lower than when only capitalization plays out, and, especially, the increase in the PD variance is considerably smaller. Perhaps remarkably, the changes in the elements of the variance decomposition go all in the right direction, and the volatility of returns goes down a bit, as in the data.

Table 1.5. **Model statistics with the lock-in effect.** This table reports moments for the model with portfolio adjustment costs, which has both the capitalization and lock-in effect. The first four columns report the observed statistics along with their Newey-West standard error for the US data. The next four columns report model-implied statistics and their t-statistics. The model uses the parameterization described in Table 1.3 for M=22. Rates of growth, variance and covariance have been multiplied by 100. The top panel reports moments included in the SMM estimation; the bottom panel non-included moments.

	US data		Model			
	1946-1982	1982-2018	1946-1982		1982-2018	
	$\hat{\mathcal{S}}_i$	$\hat{\mathcal{S}}_i$	$\widetilde{\mathcal{S}}_i(\hat{\theta})$	t-stat	$\widetilde{\mathcal{S}}_i(\hat{\theta})$	t-stat
$\mathbb{E}(PD_t)$	25.48	47.09	27.37	-1.22	38.23	2.19
$\mathbb{E}(P_t/P_{t-1}-1)$	0.48	1.84	0.86	-0.60	1.35	0.78
$\mathbb{E}(r_t^s)$	4.73	4.33	5.00	-0.36	4.34	-0.00
$\mathbb{E}(r_t^b)$	0.42	0.38	0.33	5.18	0.48	-4.21
$\mathbb{V}ar(p_t-d_t)$	7.20	14.07	8.26	-0.83	9.43	1.28
$\mathbb{C}ov(p_t-d_t,\bar{d}_t)$	-2.00	2.35	-0.48	-2.49	-0.06	4.36
$\mathbb{C}ov(p_t-d_t,\bar{r}_t)$	-9.13	-11.35	-8.92	-0.14	-8.93	-0.75
$\mathbb{C}ov(p_t-d_t,\bar{\tau}_t^K)$	-0.06	0.34	-0.04	-0.37	0.15	3.23
$\mathbb{C}ov(p_t-d_t,\bar{\tau}_t^D)$	0.36	1.00	0.05	0.93	0.56	1.34
$\sigma(r_t^s)$	7.87	7.41	7.08	1.09	6.86	0.68
$\hat{\zeta}$	0.84	2.63	0.91	-1.15	1.78	3.97
$\mathbb{E}(D_t/D_{t-1}-1)$	0.49	0.75	0.49	0.00	0.75	0.00
$\sigma(D_t/D_{t-1})$	2.52	1.97	2.52	0.00	1.97	0.00
$corr(PD_t,PD_{t-1})$	0.96	0.98	0.97	-0.06	0.96	0.20
$corr(PD_t,\beta_t)$	0.84	0.84	0.83	0.16	0.87	-0.12

Table 1.6 reports the results when only M=7 statistics are included in the SMM estimation. In this case, all the included statistics pass their individual t-test. The second subsample statistics move all in the right direction, delivering a higher PD mean; higher capital gains; lower stock returns; a higher PD variance with a positive covariance between PD and dividends, more negative ; lower returns volatility; higher beliefs elasticity. This signals that the model's mechanisms produce outcomes that resemble the data indeed. Quantitatively, all these changes turn out to be too small, pointing out that there are other important forces that are not in the model but were important to produce the data.

Table 1.6. **Estimation results for M=7.** This table reports model-implied statistics when only a subset of moments is targeted. The first two columns report the observed statistics for the US data. The next four columns report model-implied statistics and their t-statistics for the model with liquidity shocks. The last four columns report statistics with the model augmented with portfolio adjustment costs. The model uses the parameterization described in Table 1.3 for M=7. Rates of growth, variance and covariance have been multiplied by 100. The top panel reports moments included in the SMM estimation; the bottom panel non-included moments.

	US data		Capitalization				+Lock-in			
	1946-1982	1982-2018	1946-1982		1982-2018		1946-1982		1982-2018	
	$\hat{S}_j$	$\hat{S}_j$	$\widetilde{S}_j(\hat{\theta})$	t-stat	$\widetilde{S}_j(\hat{\theta})$	t-stat	$\widetilde{S}_j(\hat{\theta})$	t-stat	$\widetilde{S}_j(\hat{\theta})$	t-stat
Included in the SMM estimation (only 1946-1982)										
$\mathbb{E}(PD_t)$	25.48	47.09	27.50	-1.30	39.03	2.00	26.99	-0.97	37.33	2.42
$\mathbb{E}(P_t/P_{t-1}-1)$	0.48	1.84	0.80	-0.51	1.30	0.85	0.78	-0.48	1.27	0.90
$\mathbb{E}(r_t^s)$	4.73	4.33	4.87	-0.19	4.20	0.19	4.90	-0.23	4.25	0.11
$\mathbb{E}(r_t^b)$	0.42	0.38	0.40	0.67	0.59	-7.54	0.40	0.48	0.59	-7.52
$\mathbb{V}ar(p_t-d_t)$	7.20	14.07	7.40	-0.19	8.90	1.42	7.03	0.09	7.61	1.79
$\sigma(r_t^s)$	7.87	7.41	6.58	1.78	6.34	1.32	6.45	1.95	5.78	2.00
$\hat{\zeta}$	0.84	2.63	0.88	-0.71	1.75	4.11	0.85	-0.17	1.56	4.99
Non-included in the SMM estimation										
$\mathbb{E}(D_t/D_{t-1}-1)$	0.49	0.75	0.49	0.00	0.75	0.00	0.49	0.00	0.75	0.00
$\sigma(D_t/D_{t-1})$	2.52	1.97	2.52	0.00	1.97	0.00	2.52	0.00	1.97	0.00
$\mathbb{C}ov(p_t-d_t,\bar{d}_t)$	-2.00	2.35	-0.42	-2.62	0.08	4.10	-0.51	-2.45	-0.11	4.45
$\mathbb{C}ov(p_t-d_t,\bar{r}_t)$	-9.13	-11.35	-7.99	-0.78	-8.13	-0.99	-7.73	-0.95	-7.19	-1.29
$\mathbb{C}ov(p_t-d_t,\bar{\tau}_t^K)$	-0.06	0.34	-0.03	-0.48	0.20	2.29	-0.04	-0.38	0.14	3.38
$\mathbb{C}ov(p_t-d_t,\bar{\tau}_t^D)$	0.36	1.00	0.08	0.82	0.59	1.25	0.061	0.89	0.56	1.34
$corr(PD_t, PD_{t-1})$	0.96	0.98	0.97	-0.11	0.97	0.06	0.97	-0.11	0.98	0.01
$corr(PD_t, \beta_t)$	0.84	0.84	0.85	-0.01	0.89	-0.22	0.84	0.00	0.90	-0.27

With this set of estimations, a natural question is: What is then the marginal effect of CGT cuts on PD volatility? The answer is not immediate as, apart from CGT cuts, the model includes other tax cuts, a reduction in macroeconomic volatility and an increase in mean dividend growth, all of them influencing prices. To answer the question, then, I run a counterfactual analysis. Each of the previous models is simulated again, keeping $\tau_t^K = 0.15$ for the whole 1982-2018 period, which is the mean of the first subsample and also the level in 1989, before starting a continuous decline until reaching the bottom 5% in 2011. In other words, instead of falling, taxes are kept constant at a reasonably high level in historical terms.

These counterfactual simulations deliver an alternative $\mathbb{V}ar(p_t - d_t)$ for the period 1982-2018, called $\mathbb{V}_1^c$, with the superindex c standing for "counterfactual". This number is then compared with the model-implied variance when the true tax time series are used, called $\mathbb{V}_1$. The difference between the two is a measure of the absolute marginal effect of tax cuts in terms of variance points. That measure might be difficult to interpret, though. Then, it is normalized by the variance level in the first subsample, called $\mathbb{V}_0$, which delivers the percentual increase in the variance caused by CGT cuts in the model. This number is then compared with the percentual increase in the PD variance observed in the data. The ratio of the two is a measure of the CGT cuts marginal effect on the volatility of the PD ratio.

Table 1.7 collects all these numbers for the two versions of the model and the two estimated parameter vectors. The model featuring only the capitalization effect and using the parameterization that includes all the statistics yields a marginal effect of 62%, meaning that CGT cuts would be responsible for 62% of the observed increase in PD volatility. That effect drops to 34% when the lock-in effect is considered. If, instead, the parameters obtained for M=7 are used, the marginal effect is 38% when considering only capitalization and 25% when also including the lock-in effect.

Table 1.7. **Estimations of the marginal effect of CGT cuts on Price-Dividend volatility.** The table reports model-implied variance of the PD ratio under different periods and scenarios. $\mathbb{V}$ stands for $\mathbb{V}ar(p_t - d_t)$. The subindex 0 (1) indicates the first (second) subperiod 1946-1982 (1982-2018); $\mathbb{V}_1^c$ is the counterfactual variance if $\tau_t^K = 15\%$ for all t since 1982. $\mathbb{V}_1 - \mathbb{V}_1^c$ shows the effects of post-1982 tax cuts on the absolute level of $\mathbb{V}$, being a measure of an absolute marginal effect. $\frac{(\mathbb{V}_1 - \mathbb{V}_1^c)/\mathbb{V}_0}{(\mathbb{V}_1^{us} - \mathbb{V}_0^{us})/\mathbb{V}_0^{us}}$ compares the counterfactual increase in volatility with respect to the observed increase, being a measure of a percentual marginal effect. M

		$\mathbb{V}_0$	$\mathbb{V}_1$	$\mathbb{V}_1^c$	$\mathbb{V}_1 - \mathbb{V}_1^c$	$\frac{(\mathbb{V}_1 - \mathbb{V}_1^c)/\mathbb{V}_0}{(\mathbb{V}_1^{us} - \mathbb{V}_0^{us})/\mathbb{V}_0^{us}}$
M=22	Capitalization	8.34	11.93	6.92	5.01	0.62
	+ Lock-in	8.26	9.43	6.72	2.71	0.34
M=7	Capitalization	7.40	8.90	6.19	2.71	0.38
	+ Lock-in	7.03	7.61	5.91	1.70	0.25

1.3.5. Alternative assumptions about expectations and taxes

This section explores the robustness of the baseline results along some dimensions. On the one hand, the model is simulated under the assumption of Rational Expectations to have a traditional benchmark. Moreover, I simulate the model with tax foresight and top statutory rates, respectively.

Table 1.8 reports the statistics using the baseline parameterization in the same model but with Rational Expectations. This version fails in many dimensions. As already pointed out by Adam, Marcet, and Beutel, 2017, RE is capable of generating neither enough volatility nor the equity premium. Moreover, despite delivering a sizable increase in the mean PD ratio, its volatility actually goes down. This last fact emphasizes that higher prices are not automatically associated with higher volatility and that, indeed, the role of taxes on the feedback loop between prices and beliefs is crucial to explain the observed patterns.

Under tax foresight, agents could include future taxes in their subjective model of prices to forecast future prices better. Consider the following modified subjective model of prices for the model with liquidity shocks, where equation (2.38) is replaced by

$$\frac{P_t}{P_{t-1}} = \frac{1 - \pi\tau_t^K}{1 - \pi\tau_{t-1}^K} b_t + \varepsilon_t^P \tag{1.53}$$

Agents understand that the liquidity shock will hit a fraction π of agents and that tax rates will change, incorporating that into their price model. Consequently, subjective beliefs on capital gains become $\mathbb{E}_t^{\mathcal{P}}\left[\frac{P_{t+1}}{P_t}\right] = \frac{1-\pi\tau_{t+1}^K}{1-\pi\tau_t^K}\beta_t$. The fraction would cancel out if τ_{t+1}^K is uncertain as $\mathbb{E}_t^{\mathcal{P}}\left[\tau_{t+1}^K\right] = \tau_t^K$, but not now. Table 1.9's columns 4-7 show the model statistics with this alternative model. Results are remarkably similar to the baseline except for the higher increase in volatility. The larger volatility is related. In other words, including tax changes in the beliefs model adds volatility to expectations and prices, but not for the reasons pointed out by the empirical variance decomposition.

Finally, table 1.9's columns 8-11 show the statistics when the model is simulated using top statutory rather than average marginal effective rates. Top statutory rates are generally higher and are not affected by the drop of equities in taxable accounts. As a result, mean prices are lower, the volatility is also lower, and its increase is very modest, although more than enough to offset the stabilizing effect of the Great Moderation.

1.3.6. The Equity Premium

This section explores the reasons behind the relatively good equity premium generated by the learning model using realistic consumption and dividend growth processes, a positive discount factor and low risk aversion. First, it analyzes the drivers behind mean stock returns. Second, it explores its relation to the drivers of the risk-free rate.

Table 1.8. **Rational Expectations statistics for the model with only the capitalization effect.** This table reports statistics for the model with liquidity shocks under Rational Expectations. The first two columns report the observed statistics for the US data. The next four columns report model-implied statistics and their t-statistics. The model uses the parameterization described in Table 1.3 for M=22, baseline column. Rates of growth, variance and covariance have been multiplied by 100.

	US data		Model			
	1946-1982	1982-2018	1946-1982		1982-2018	
	$\hat{\mathcal{S}}_i$	$\hat{\mathcal{S}}_i$	$\widetilde{\mathcal{S}}_i(\hat{\theta})$	t-stat	$\widetilde{\mathcal{S}}_i(\hat{\theta})$	t-stat
$\mathbb{E}(PD_t)$	25.48	47.09	64.31	-25.04	81.94	-8.63
$\mathbb{E}(P_t/P_{t-1}-1)$	0.48	1.84	0.61	-0.20	0.90	1.48
$\mathbb{E}(r_t^s)$	4.73	4.33	2.61	2.80	2.49	2.44
$\mathbb{E}(r_t^b)$	0.42	0.38	0.28	7.80	0.44	-2.78
$\mathbb{V}ar(p_t-d_t)$	7.20	14.07	1.75	4.00	1.49	3.50
$\mathbb{C}ov(p_t-d_t,\bar{d}_t)$	-2.00	2.35	0.98	-4.93	0.77	2.87
$\mathbb{C}ov(p_t-d_t,\bar{r}_t)$	-9.13	-11.35	-0.31	-6.00	-0.49	-3.35
$\mathbb{C}ov(p_t-d_t,\bar{\tau}_t^K)$	-0.06	0.34	0.23	-4.92	0.13	3.53
$\mathbb{C}ov(p_t-d_t,\bar{\tau}_t^D)$	0.36	1.00	0.07	0.88	0.23	2.35
$\sigma(r_t^s)$	7.87	7.41	3.85	5.56	3.29	5.08
$\hat{\zeta}$	0.84	2.63	-	-	-	-
$\mathbb{E}(D_t/D_{t-1}-1)$	0.49	0.75	0.49	0.00	0.75	0.00
$\sigma(D_t/D_{t-1})$	2.52	1.97	2.52	0.00	1.97	0.00
$corr(PD_t,PD_{t-1})$	0.96	0.98	0.96	-0.03	0.96	0.20
$corr(PD_t,\beta_t)$	0.84	0.84	-	-	-	-

To articulate the discussion, I use the following decomposition of the stock return geometric mean[43]

$$\left(\prod_{t=1}^{N}\frac{P_t+D_t}{P_{t-1}}\right)^{\frac{1}{N}}=\underbrace{\left(\prod_{t=1}^{N}\frac{D_t}{D_{t-1}}\right)^{\frac{1}{N}}}_{R_1}\underbrace{\left(\frac{PD_N+1}{PD_0}\right)^{\frac{1}{N}}}_{R_2}\underbrace{\left(\prod_{t=1}^{N-1}\frac{PD_t+1}{PD_t}\right)^{\frac{1}{N}}}_{R_3} \tag{1.54}$$

[43]It was first suggested by Adam, Marcet, and Nicolini, 2016.

Table 1.9. **Model statistics under alternative assumptions about capital taxes for the model with only the capitalization effect.** This table reports moments for the model with portfolio adjustments costs when there is tax foresight and when statutory rather than effective rates are used. The first four columns report the observed statistics along with their Newey-West standard error for the US data. The next four columns report model-implied statistics and their t-statistics. The model uses the parameterization described in Table 1.3 for M=22. Rates of growth, variance and covariance have been multiplied by 100. The top panel reports moments included in the SMM estimation; the bottom panel non-included moments.

	US data		Tax Foresight				Statutory Rates			
	1946-1982	1982-2018	1946-1982		1982-2018		1946-1982		1982-2018	
	$\hat{\mathcal{S}}_j$	$\hat{\mathcal{S}}_j$	$\widetilde{\mathcal{S}}_j(\hat{\theta})$	t-stat	$\widetilde{\mathcal{S}}_j(\hat{\theta})$	t-stat	$\widetilde{\mathcal{S}}_j(\hat{\theta})$	t-stat	$\widetilde{\mathcal{S}}_j(\hat{\theta})$	t-stat
$\mathbb{E}(PD_t)$	25.48	47.09	27.91	-1.56	40.54	1.62	22.93	1.65	33.52	3.36
$\mathbb{E}(P_t/P_{t-1}-1)$	0.48	1.84	0.90	-0.66	1.47	0.58	0.66	-0.30	1.17	1.05
$\mathbb{E}(r_t^s)$	4.73	4.33	4.96	-0.31	4.42	-0.11	5.40	-0.89	4.41	-0.10
$\mathbb{E}(r_t^b)$	0.42	0.38	0.33	4.87	0.48	-3.96	0.31	5.82	0.47	-3.82
$\mathbb{V}ar(p_t-d_t)$	7.20	14.07	8.32	-0.87	13.17	0.23	4.96	1.62	5.21	2.46
$\mathbb{C}ov(p_t-d_t,\bar{d}_t)$	-2.00	2.35	-0.25	-2.89	0.08	4.11	0.04	-3.37	0.01	4.23
$\mathbb{C}ov(p_t-d_t,\bar{r}_t)$	-9.13	-11.35	-8.68	-0.31	-12.38	0.32	-5.06	-2.77	-4.74	-2.04
$\mathbb{C}ov(p_t-d_t,\bar{\tau}_t^K)$	-0.06	0.34	-0.02	-0.72	0.21	2.23	0.05	-1.88	0.15	3.18
$\mathbb{C}ov(p_t-d_t,\bar{\tau}_t^D)$	0.36	1.00	0.20	0.49	0.50	1.54	0.14	0.65	0.51	1.48
$\sigma(r_t^s)$	7.87	7.41	6.96	1.25	9.22	-2.23	5.86	2.78	4.80	3.22
$\hat{\zeta}$	0.84	2.63	0.92	-1.26	1.95	3.20	0.59	3.83	1.22	6.60
$\mathbb{E}(D_t/D_{t-1}-1)$	0.49	0.75	0.49	0.00	0.75	0.00	0.49	0.00	0.75	0.00
$\sigma(D_t/D_{t-1})$	2.52	1.97	2.52	0.00	1.97	0.00	2.52	0.00	1.97	0.00
$corr(PD_t, PD_{t-1})$	0.96	0.98	0.97	-0.08	0.94	0.58	0.98	-0.15	0.98	-0.06
$corr(PD_t, \beta_t)$	0.84	0.84	0.84	0.08	0.79	0.22	0.84	0.05	0.88	-0.15

Thus, the mean gross return can be understood as the product of three elements. The first term (R_1) is the mean dividend growth. The second term (R_2) is the ratio of the terminal over the initial PD ratio value, which might be related to the existence of a time trend. Finally, the last term (R_3) is a convex function of period t PD ratio. It increases with the volatility of the PD time series, but decreases with its mean.

Table 1.10 reports the decomposition using empirical and simulated data. Since the dividend growth process has been parameterized directly from the data, the models exactly replicate R_1. Regarding R_2, both models show a certain increase in the PD ratio in both periods, although weaker than the observed. This is a mismatch of the model, partly due to the pre and post-1982 division; the CGT falls temporarily at the end of the first subsample and beginning of the second subsample, bringing the PD ratio up during the final quarter of the 1st subsample and the initial quarters of the 2nd subperiod. As a result, R_2 gets too high (low) in 1946-1982 (1982-2018). R_3 is reasonably close to the observed one in the first subsample, because both models get the PD mean and variance correctly. In the second subsample, though, both exaggerate R_3 due to the fact they produce a too-low PD mean.

Table 1.10. **Decomposition of the stock return geometric mean.** The table shows the stock returns mean decomposition according to expression (1.54). The first column uses U.S. data; the second, simulated data using the learning model; the third, simulated data using the RE model. The last row is the stock return geometric mean. Simulated data uses the parameterization shown in table 1.3.

	US data		Capitalization		+Lock-in	
	1946-1982	1982-2018	1946-1982	1982-2018	1946-1982	1982-2018
R_1	0.46	0.74	0.46	0.74	0.46	0.74
R_2	-0.27	0.88	0.28	0.44	0.26	0.40
R_3	4.10	2.37	4.35	2.97	4.40	3.02
$\mathbb{E}(r^s)$	4.30	4.03	5.12	4.18	5.15	4.20

The second part of the equity premium is the risk-free rate. Although it has no closed-form solution in the quantitative model, it does so for RE when $p = 1$. This benchmark is useful to understand why it is not too high. It is given by

$$r_t^b = \left(\delta^{-1}\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right]^{-1} - 1\right)\frac{1}{1-\tau_t^b} \tag{1.55}$$

where $\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right] = a^{-\gamma}exp\left\{\gamma(\sigma_W^2 + \sigma_D^2)(1+\gamma)/2\right\}exp\{\sigma_{DW}\gamma^2\}$.

Thus, r_t^b depends essentially on the mean and volatility of the income processes and the level of risk aversion. As a result, getting a high stock return and a low bond rate is complicated in many models; the reason is that either high risk aversion or income volatility is needed. However, the required levels appear unrealistic (Mehra and Prescott, 1985), plus high risk aversion would also lead to a too high risk-free rate (Weil, 1989). Contrarily, this paper resorts to alternative forces that make returns high enough. The main driver is non-fundamental volatility coming from beliefs, which makes compatible realistic income processes with high and volatile stock returns. However, belief volatility is unable to do all the job (Adam, Marcet, and Nicolini, 2016, Adam, Marcet, and Beutel, 2017). The second driver is the increase in the PD ratio brought about by stronger income growth and lower taxes. Thus, $R_3 > 1$ helps to increase mean stock returns. In other words, relying on beliefs alone would either be insufficient (as in Adam, Marcet, and Beutel, 2017) or require a too-high beliefs volatility while introducing a trend in the PD ratio (as the one coming from taxes) helps sort this problem out.

The previous reasoning explains why the model does a decent job matching the equity premium level. Additionally, its decline is captured by the model too. In reality as well as in the model, the fall in stock returns is mostly due to the reduction in R_3 as a result of a higher PD ratio, which overcomes the opposite effect via R_2. Besides, the mean risk-free rate is also declining, mostly due to the fall in τ^b.[44] In other words, the fact the model produces an increase in the PD ratio helps to explain both the level and trajectory of the equity premium.

1.4. Optimal Capital Gains Taxation

In this section, the normative use of a tax on unrealized capital gains for macrofinancial stability purposes is studied. Thus, whereas Section 2 pointed out the particular role of capital gains taxation and Section 3 analyzed the ability of taxes, as historically given, to explain certain transformations in the US stock market, this section asks: What is the appropriate capital gains tax to get the best of imperfect capital markets?

The problem in hand is the excess volatility in capital prices which can be read as a pecuniary externality. The reason is that excess volatility emerges from the inability of agents to internalize the equilibrium price formation due to information frictions.[45] Thus, the lack of knowledge of the true determinants of prices pushes agents to make decisions using forecasts derived from their subjective models that ignore the effect of their own forecasts on market prices and everybody else's predictions. In short, rational individuals trying to make the best prediction about future prices but missing general equilibrium effects end up causing excessive volatility.

A precondition to exploring an optimal tax is establishing a connection between asset prices and consumption fluctuations that is missing from the previous endowment model. To that end, I set up a tractable two-sector growth model with investment adjustment costs and learning about capital prices. The model links the capital market price to investment decisions, in line with the Q-theory. As a result, cycles of over- and under-accumulation of capital emerge, driven by excessive asset price fluctuations.

[44] In this case, there is no feedback loop affecting the bond price due to its one-period maturity. Hence, τ^b level is neutral and only tax changes have an impact on bond prices. This impact is very small, so the risk-free rate is very stable.

[45] This inability is due to the fact investors ignore other investors' characteristics such that the standard derivation of equilibrium prices combining individual and aggregate optimality conditions is not possible.

The section is structured as follows. Section 1.4.1. sets up a centralized two-sector growth model with investment adjustment costs and a CGT on an accrual basis. Section 1.4.2. decentralizes the economy by introducing efficient capital markets. Contrarily, Section 1.4.3. decentralized the economy when investors have imperfect market knowledge and learn about prices. Section 1.4.4. studies an optimal taxation problem. Finally, Section 1.4.5. proposes an alternative implementation of the optimal policy that avoids too volatile taxes.

1.4.1. The First Best economy

In this section, a model with endogenous consumption is introduced. It consists of a two-sector growth model with investment adjustment costs. The model is highly simplified, reduced to the minimum ingredients needed to connect capital prices to output. The model structure is described next.

Demographics. The economy is populated by a continuum of measure 1 of infinitely living identical agents.

Goods. There is a perishable consumption good (or simply "good") and a non-perishable capital good (or simply "capital") that depreciates at a constant rate d each period. Goods deliver utility whereas capital is used to produce goods.

Production technology. There is a goods production function that uses capital K with an inelastically supplied 1 unit of labour in a particular technological environment given by Z to deliver goods $F : (Z, K) \rightarrow \mathbb{R}_{+}$. F has neoclassical properties; the technology level Z is exogenous and stochastic. In addition, capital is produced via a linear function that converts $I_t + G(I_t)$ units of goods into I_t units of capital; $G(I_t)$ represents investment adjustment costs, a convex function, symmetric, with $G(0) = 0$, $G'(0) = 0$ and $G''(\cdot) > 0$[46].

Welfare. The utility function U is time-separable, continuous, at least twice-differentiable function with $U'(C_t) > 0$ and $U''(C_t) < 0$, with Inada properties.

Social Planner's problem. In the previous economy, the Social Planner faces a dynamic allocation problem consisting on distribute goods between investment and consumption to maximize the lifetime social welfare:

$$\max_{\{C_t, I_t, K_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \delta^t U(C_t) \tag{1.56}$$

s.t. i) Consumption-goods resource constraint:

$$C_t + I_t + G(I_t) \leq F(Z_t, K_{t-1}) \tag{1.57}$$

[46]This specification implicitly assumes diminishing returns to scale in adjustment costs. In this way, Hayashi, 1982's theorem does not hold. The violation of the theorem can be avoided by assuming G also depends negatively on K and it is homogeneous of degree one. However, it complicates the analysis a bit without adding any crucial insight to the question in hand. See Romer and Romer, 2010 for a discussion.

ii) Capital-goods resource constraint:

$$K_t \leq I_t + (1-d)K_{t-1} \tag{1.58}$$

iii) Non-negative consumption:

$$C_t \geq 0 \tag{1.59}$$

First Best (optimal growth path). Given initial capital K_{-1} and an exogenous productivity process $\{Z_t\}_{t=0}^{\infty}$, the Social Planner equilibrium consists of sequences of allocations $\{C_t, I_t, K_t\}_{t=0}^{\infty}$ such that:

1. Resource constraints (1.57)-(1.58) are satisfied.

2. First order conditions:

$$u_t^c = \lambda_t \tag{1.60}$$

$$1 + G_t^I = q_t/\lambda_t \tag{1.61}$$

$$\frac{q_t}{\lambda_t} = \delta \mathbb{E}_t\left[\frac{\lambda_{t+1}}{\lambda_t}\left(F_{t+1}^k + (1-d)\frac{q_{t+1}}{\lambda_{t+1}}\right)\right] \tag{1.62}$$

   where $G_t^I = \frac{G(I_t)}{I_t}$, $u_t^c = \frac{U(C_t)}{C_t}$, $F_{t+1}^k = \frac{F(Z_{t+1},K_t)}{K_t}$; λ_t is the Lagrange multiplier of the goods resource constraint, reflecting the marginal value of goods; q_t is the Lagrange multiplier of the capital resource constraint and then, q_t/λ_t reflects the marginal value of capital (in terms of goods).

3. A transversality condition.

$$\lim_{j\to\infty} \delta^j \mathbb{E}_t\left[\frac{u_{t+j}^c}{u_t^c}\frac{q_{t+j}}{\lambda_{t+j}}K_{t+j}\right] = 0 \tag{1.63}$$

Altogether, the model equilibrium is characterized by choices about capital accumulation. It is determined by the intersection of two functions that relate capital stock to its shadow price. First, combining equations (1.58) and (1.61), a positive relationship between K and $\bar{q} \equiv q/\lambda$ arises. Besides, by the properties of F, the Euler Equation (1.62) gives rise to a negative $K - \bar{q}$ relationship. These two curves pin down a unique equilibrium, from which investment, consumption and output follows. As it is well-known, in dynamic terms the model is described by two difference equations characterizing the evolution of K and $\bar{q}$.[47]

[47] See Romer and Romer, 2010 for a textbook treatment.

1.4.2. Efficient Markets

In this section, investment is decentralized. Thus, on top of the previous elements, markets are introduced and with them the information atomistic investors possess is specified.

Markets. The economy consists of two markets for capital and goods. In the former, capital producers and capital users meet to sell and buy new and old capital at price Q_t[48]. In the latter, capital producers acquire the inputs they need to produce new capital and households meet their consumption demand. Goods price acts as the unit of account of the economy and as such, it is normalized to 1. Markets are competitive.

Information set. Agents have all the structural knowledge about the economy. In particular, homogeneity is common knowledge and households are aware of firms' problem.

Then, we must characterize the problems of the two group of agents: capital producers and producing households.

Capital producers. They maximize profits by choosing investment on new capital. Then, they acquire goods to produce capital (facing the adjustment costs $G(I_t)$) that will be sold at price Q_t in capital markets as to maximize their profits Π_t. Their static problem can be stated as

$$\max_{\{I_t\}_{t=0}^{\infty}} \Pi_t = Q_t I_t - I_t - G(I_t) \tag{1.65}$$

Producing households. In this economy, households buy goods to satisfy their consumption demand and capital to produce goods. Each of them supply a unit of labour inelastically. Hence, their problem can be written as

$$\max_{\{C_t, K_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \delta^t U(C_t) \tag{1.66}$$

s.t. i) Budget constraint:

$$C_t + Q_t K_t \leq F(Z_t, K_{t-1}) + (1-d) Q_t K_{t-1} + \Pi_t \tag{1.67}$$

[48] A mapping between capital and stock price can be established along Adam and Merkel, 2019's lines. Assume that capital K_t can be securitized via equities S_t without any cost. In equilibrium, arbitrage is not possible and then, the ex-dividend equity price must be equal to the market value of capital net of dividends. Thus, consider that a fraction $x \in (0,1)$ of profits is distributed such that dividends $D_t = x K_{t-1} F_t^k$. Assume that the rest is reinvested in new capital $(1-x) K_{t-1} F_t^k / Q_t$. Hence, the market value of capital per share after dividends payments is $P_t = Q_t((1-d)K_{t-1} + (1-x)K_{t-1} F_t^k Q_t)$. It follows that the PD ratio is given by

$$\frac{P_t}{D_t} = \frac{(1-d)}{x} \frac{Q_t}{F_t^k} + \frac{1-x}{x} \tag{1.64}$$

For reasonable x (not too small), the PD is basically a proportion of the Capital-Rent ratio. Therefore, the connection with the stock market model is that learning about stock prices would be an implicit way of learning about the market value of capital.

ii) Non-negative consumption:

$$C_t \geq 0 \tag{1.68}$$

Competitive Equilibrium. Given K_{-1}, a Competitive Equilibrium consists of sequences of allocations $\{C_t, I_t, K_t\}_{t=0}^{\infty}$ and prices $\{Q_t\}_{t=0}^{\infty}$ such that:

1. Capital producers behave optimally, satisfying

$$Q_t = 1 + G_t^I \tag{1.69}$$

2. Households behave optimally, satisfying:

   a) The sequence of budget constraints (1.67).

   b) The sequence of Euler Equation

$$Q_t = \delta \mathbb{E}_t \left[\frac{u_{t+1}^c}{u_t^c} \left(F_{t+1}^k + (1-d) Q_{t+1} \right) \right] \tag{1.70}$$

   c) A transversality condition

$$\lim_{j \to \infty} \delta^j \mathbb{E}_t \left[\frac{u_{t+j}^c}{u_t^c} Q_{t+j} K_{t+j} \right] = 0 \tag{1.71}$$

3. Markets clear:

$$\text{Goods:} \;\; C_t + I_t + G(I_t) = F(Z_t, K_{t-1}) \tag{1.72}$$

$$\text{Capital:} \;\; K_t = I_t + (1-d) K_{t-1} \tag{1.73}$$

First Welfare Theorem. It is clear that both institutions, the planner and markets, have to satisfy the same aggregate resource constraints. Besides, in equilibrium, market and planner's Euler Equation reads exactly the same

$$1 + G_t^I = \delta \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} \left(F_{t+1}^k + (1-d)(1 + G_{t+1}^I) \right) \right] \tag{1.74}$$

which implies

$$Q_t = \bar{q}_t \tag{1.75}$$

Thus, the market capital price is equal to its shadow price. By the arguments in the previous section, it follows that quantities will be those of the First Best.

1.4.3. Inefficient Markets

In this section, the full information assumption is relaxed. This departure from Rational Expectations gives rise to an additional uncertainty source, price formation, that adds new dynamics to the model. First, the new information set is specified:

Information set. Households have structural knowledge about the economy except they ignore they all are equal. This incomplete information makes them unable to derive current capital prices from their optimality conditions since they cannot either use market clearing conditions ex-ante nor apply the Law of Iterated Expectations. This friction is formalized by introducing a subjective probability measure $\mathcal{P}^i$ that reflects investors' views about productivity, capital and prices.Thus, the underlying probability space is given by $(\Omega, \mathcal{B}, \mathcal{P}^i)$ with $\mathcal{B}$ denoting the corresponding σ-algebra of Borel subsets of Ω and $\mathcal{P}^i$ agent's i subjective probability measure over $(\Omega, \mathcal{B})$. For generality, we include prices in the the state space Ω, with $\omega = \{Z_t, K_t, P_t\}_{t=0}^{\infty}$ as a typical element.

In this world, the problems agents face are the same as in the efficient market case except now households use their subjective probability measure, that is

$$\max_{\{C_t,K_t\}_{t=0}^{\infty}} \mathbb{E}_0^{\mathcal{P}} \sum_{t=0}^{\infty} \delta^t U(C_t) \tag{1.76}$$

Hence, the Euler Equation reads as

$$Q_t = \delta \mathbb{E}_t^{\mathcal{P}} \left[\frac{u_{t+1}^c}{u_t^c} \left(F_{t+1}^k + (1-d) Q_{t+1} \right) \right] \tag{1.77}$$

To fully characterize equilibrium, the following subjective price model is assumed:

$$\frac{Q_{t+1}}{Q_t} \frac{u_{t+1}^c}{u_t^c} = \theta_t + \varepsilon_t^P \tag{1.78}$$

$$\theta_t = \theta_{t-1} + \nu_t \tag{1.79}$$

with i.i.d. normally distributed innovations. The posterior of the unobserved component θ follows a Normal distribution

$$\theta_t \sim \mathcal{N}(\beta_t, \sigma_\theta^2)$$

where σ_θ^2 is the steady state Kalman estimate uncertainty and the posterior mean evolves recursively following

$$\beta_{t+1} = \beta_t + g\left(\frac{Q_t}{Q_{t-1}} \frac{u_t^c}{u_{t-1}^c} - \beta_t \right) \tag{1.80}$$

Hence, $\mathbb{E}_t^{\mathcal{P}} \left(\frac{Q_{t+1}}{Q_t} \frac{u_{t+1}^c}{u_t^c} \right) = \beta_t$.

Investors use this model to forecast capital gains and learn from new information, responding to their uncertainty about equilibrium price formation. This learning process adds an additional source of fluctuations to the model. In particular, the model equilibrium dynamics are now described by three difference equations: the capital law of motion (1.73), the Euler Equation (1.77) and the expectations updating equation (1.80). Then, two feedback loops operate in learning markets. First, the one between the stock and price of capital, which is self-correcting. Second, the price-expectations loop described throughout the paper, which is reinforcing and can drive the economy in waves of over and under capital accumulation.

The expectations loop amplifies the dynamics emerging from the efficient model. To illustrate it, figure 1.7 plots the response of both the capital stock and price to a transitory productivity shock in the (K_t, Q_t) diagram, starting from the steady state. With efficient markets, an increase in productivity would move the price and stock of capital up for one period, surprising the agents. However, since the displacement is known to be temporary, they find no reason to revise expectations so that the only force at play are lower returns from a higher stock of capital that brings prices down; then, with prices below and the capital stock above their steady state levels, the economy enters a path of gradual disinvestment until reaching the steady state. With learning, the initial price surprise leads agents to review their forecast upwards which in turn, raises prices and capital feeding back into a new upward revision. However, there is a counteracting force: as the price boom leads to accumulate capital, returns decline which pushes prices downwards. Eventually, declining capital rents overcome the effect of more optimistic expectations, which are defeated. At that point, the process revert in the form of a bust. It is throughout a sequence of boom and busts, rather than following an smooth saddle path, that the economy goes back to the steady state.

1.4.4. A capital gains tax to stabilize inefficient markets

In this section, a tax on unrealized capital gains is introduced and an optimal taxation problem is analyzed. In line with Benigno et al., 2019, the optimal tax is derived to implement the First Best[49].

Capital gains taxation in a production economy. I first modify the household's budget constraint by introducing taxes on capital gains (τ^K):

$$C_t + Q_t K_t \leq F(Z_t, K_{t-1}) + (1-d)Q_t K_{t-1} + \Pi_t - \tau_t^K (Q_t - Q_{t-1})(1-d)K_{t-1} + T_t \quad (1.81)$$

Besides, it is assumed the government simply transfers the revenues back in a lump-sum manner, that is,

$$\tau_t^K (Q_t - Q_{t-1})(1-d)K_{t-1} = T_t \quad (1.82)$$

[49] Benigno et al., 2019 argue that while the literature on pecuniary externalities focuses on setting the right taxes to implement constrained efficiency, it is possible to use better these instruments and implement the First Best.

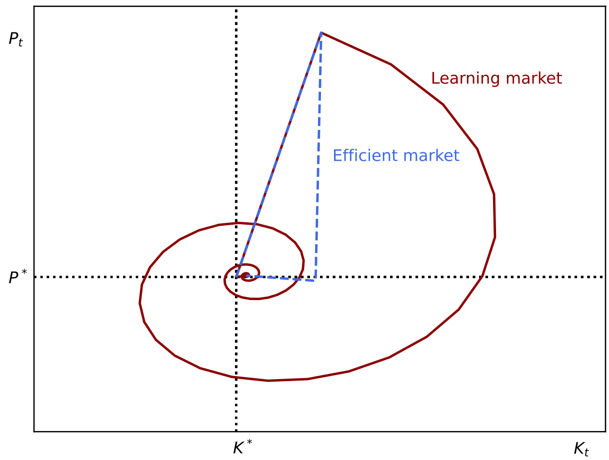

Figure 1.7. **Response of capital and capital price to a transitory productivity shock under Rational Expectations and Learning.** The graph uses a plane with the capital stock on the x-axis and the capital price on the y-axis. Starting in the steady state, the economy is perturbed by a one-off productivity shock. The blue line shows the response of capital stock and price under efficient pricing (Rational Expectations). The red line shows that response when agents learn.

Then, the Euler Equations becomes

$$Q_t = \delta \mathbb{E}_t^{\mathcal{P}} \left[\frac{u_{t+1}^c}{u_t^c} \left(F_{t+1}^k + (1 - \tau_{t+1}^K)(1-d)Q_{t+1} + \tau_{t+1}^K (1-d) Q_t \right) \right] \tag{1.83}$$

The tax distort the intertemporal incentives by influencing the present value of future payoffs and then the capital price and equilibrium allocations.

Optimal taxation problem. Given K_{-1} and an exogenous productivity process $\{Z_t\}_{t=0}^{\infty}$, the paternalistic planner's problem is to choose both capital gains and lump-sum taxes to deliver the best competitive equilibrium with learning, that is,

$$\max_{\{\tau_t^K, T_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \delta^t U(C_t)$$

s.t. households budget constraint (1.81); the government budget constraint (1.82); the capital producers' profits equation; goods and capital market clearing conditions (1.72, 1.73); the investment function (1.69); the households' Euler Equation (1.83); and the beliefs updating equation (1.80).

Solution. To replicate the efficient allocations, it is sufficient for the planner to set taxes as to equalize the Euler Equation under Rational Expectations and learning and to transfer the proceeds back to households in a lump-sum manner. If that is possible, prices in the learning world would be the same as under Rational Expectations. In turn, lump-sum taxes would undo the income effect triggered by the capital gains tax, leaving the budget constraint unchanged. Altogether, with prices at the right level and unchanged resources, the allocations will be the efficient ones as the remainder optimality conditions are exactly the same in both worlds. In other words, the only difference with efficient markets is that now there is learning to respond to deal with limited information so that the planner would like to use taxes to undo the effects of that friction. Agents will continue to have imperfect knowledge and learn, but that process would not generate excess price volatility anymore because taxes would avoid the transmission of beliefs deviations from RE to prices and quantities.

The Rational Expectations' Euler Equation can be rewritten as:

$$Q_t^{RE} = \frac{\delta \mathbb{E}_t\left[\frac{u^c_{t+1}}{u^c_t} F^k_{t+1}\right]}{1 - \delta(1-d)\beta^*_t} \tag{1.84}$$

where $\beta^*_t \equiv \mathbb{E}_t\left[\frac{u^c_{t+1}}{u^c_t}\frac{Q_{t+1}}{Q_t}\right]$. The learning counterpart with taxes reads as

$$Q_t^L = \frac{\delta \mathbb{E}_t^{\mathcal{P}}\left[\frac{u^c_{t+1}}{u^c_t} F^k_{t+1}\right]}{1 - \tau^K_{t+1}\delta(1-d)\mathbb{E}_t^{\mathcal{P}}\left[\frac{u^c_{t+1}}{u^c_t}\right] - (1-\tau^K_{t+1})\delta(1-d)\beta_t} \tag{1.85}$$

Then, the market inefficiency under learning, call it X, boils down to the distance between the efficient and the learning price, that is, $X_t = Q_t^{RE} - Q_t^L(\tau^K_{t+1})$. The optimal taxation problem amounts to find the root of X. In other words, a tax level τ^* is optimal if and only if

$$X_t(\tau^*) = 0 \tag{1.86}$$

The root of X_t can be written as

$$\tau^*_{t+1} = 1 - \frac{\mathbb{E}_t\left[u^c_{t+1}\left(P_{t+1} - P_t\right)\right]}{\mathbb{E}_t^{\mathcal{P}}\left[u^c_{t+1}\left(P_{t+1} - P_t\right)\right]} \tag{1.87}$$

Thus, the optimal tax is a nonlinear function of the deviation of subjective from objective expectations about capital gains (adjusted by wealth's marginal value). Consider the limit case with vanishing risk aversion to derive clear intuition. Then, the previous formula simplifies to

$$\tau^*_{t+1} = 1 - \frac{\beta^*_t - 1}{\beta_t - 1} \tag{1.88}$$

There are two limit cases that can be derived. First, when subjective beliefs tend toward the objective ones, the optimal tax is zero:

$$\lim_{\beta_t \to \beta^*_t} \tau^*_{t+1} = 0$$

Second, when objective expectations tend to 1, the optimal tax is simply one:

$$\lim_{\beta^*_t \to 1} \tau^*_{t+1} = 1$$

Beyond these cases, the sign of the tax can be defined by parts:[50]

$$\tau^*_{t+1} = \begin{cases} > 0 \text{ if } \begin{cases} \beta_t > \beta^*_t \mid \beta_t > 1 \ (A) \\ \beta_t < \beta^*_t \mid \beta_t < 1 \ (B) \end{cases} \\ < 0 \text{ if } \begin{cases} \beta_t > \beta^*_t \mid \beta_t < 1 \ (C) \\ \beta_t < \beta^*_t \mid \beta_t > 1 \ (D) \end{cases} \end{cases} \tag{1.89}$$

Intuitively, case A shows that when investors are too optimistic, meaning they expect prices to rise more than justified by fundamentals, capital gains should be taxed. That can be the situation in a typical a boom. Taxes should also be positive when investors are too pessimistic, meaning they expect prices to decrease more than justified by fundamentals (case B). In this case, typical of a burst, taxes on negative capital gains are actually subsidizing capital losses. Hence, in A (B), taxes dampen the upwards (downwards) hike in beliefs.

The formula recommends a negative tax in two scenarios. In case C, investors are not optimistic enough, meaning they expect only a moderate increase in price growth, below what would be reasonable based on fundamentals. Then, investors would be actually subsidized to boost their optimism. In case D, investors are not pessimistic enough, meaning they expect only a soft reduction in price growth, below what Rational Expectations investors would forecast. Then, a negative tax on negative expected capital losses would take resources from investors, aiming at making them expecting more losses until anchoring their beliefs at their fundamental value. Figure 1.8 illustrates these four cases.

[50] Cases in which $\beta^*_t = 1$ or $\beta_t = 1$ are ignored; the first because leads to a tax equal to 1 as already pointed out; the second because it yields an undefined fraction.

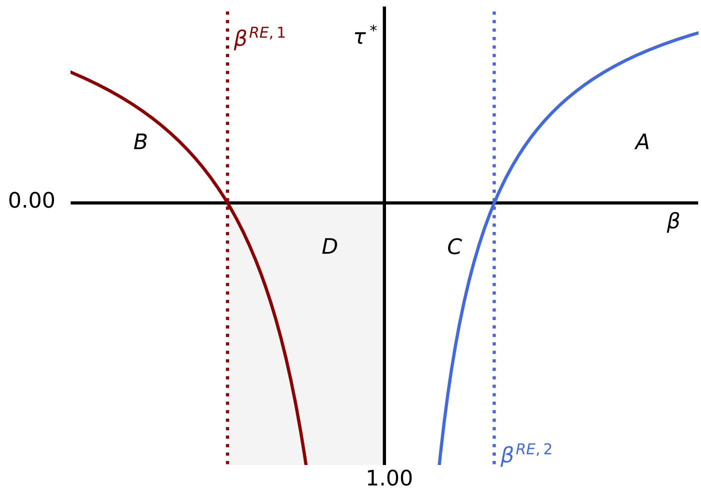

Figure 1.8. **Optimal capital gains tax.** The figure shows the optimal capital gains tax τ^* as a function of subjective expectations β_t for two different values of objective expectations β^{RE}, one above and one below 1. Letters signal the 4 four cases highlighted in expression (1.89). Shadowed areas sets out the cases in which investors expect to pay taxes, when expected capital gains are positive (negative) and the tax is positive (negative).

The optimal tax inherits the subjective expectations dynamics. By the learning updating rule, $\beta_t = \beta(\beta_{t-1}, \beta_{t-2}, \tau_{t-1}, \tau_{t-2}, \cdot)$ shows high serial correlation (for small gains). In turn, β_t^* is a function of the states (Z_t, K_{t-1}), both obeying AR(1) process. It follows that optimal taxes would display high serial correlation. Yet, subjective beliefs β_t deviates from RE quite substantially which may generate big movements in taxes. In fact, the optimal tax is unbounded and then, in some cases, the tax might reach values well beyond ± 1. From a policy standpoint, that is probably an important shortcoming; next section deals with it.

1.4.5. An alternative implementation

In this section, an alternative implementation of the optimal policy is presented. It uses a CGT to eradicate the influence of subjective beliefs on prices and a subsidy on capital rents to avoid chronic under-investment. To a great extent, this combination avoids tax volatility.

A CGT equal to 100% can eliminate the influence of beliefs on prices. To illustrate why, I use a decomposition of total volatility between fundamental and non-fundamental. Following the procedure in Section 3, the variance of the capital price can be approximated by

$$\mathbb{V}ar(Q_t) \approx \underbrace{z^2\mathbb{V}ar(Z_t) + k^2\mathbb{V}ar(K_{t-1})}_{\text{Fundamental}} \quad + \quad \underbrace{b^2\mathbb{V}ar(\beta_t)}_{\text{Non-Fundamental Volatility} \equiv \mathcal{V}} \tag{1.90}$$

where $x = Q_t/X_t$ evaluated at the approximation point for $x = z, k, b$. Then, the optimal tax must satisfy

$$\tau^* \iff \mathcal{V}(\tau^*) = 0 \tag{1.91}$$

Note that the two objects in $b\mathbb{V}ar(\beta_t)$ depend on taxes. Then, finding a τ that makes $b = 0$ would be a sufficient condition. Since

$$b = \frac{Q_t}{\beta_t}(Z^*, K^*, 1) = \frac{\delta^2 F(Z^*, K^*)(1-d)(1-\tau^K_{t+1})}{(1-\delta(1-d)\tau^K_{t+1} - \delta(1-d)(1-\tau^K_{t+1}))^2} \tag{1.92}$$

with $f(Z^*, K^*)$ being $\mathbb{E}_t(F^k_{t+1})$ evaluated at the approximation point, it turns out that a tax equal to 1 eliminate the externality

$$\tau^* = 1 \iff b = 0 \implies \mathcal{V} = 0$$

When $\tau^K_{t+1} = 1$, the equilibrium capital price becomes

$$Q^L_t = \frac{\delta\mathbb{E}_t\left[\frac{u^c_{t+1}}{u^c_t}F^k_{t+1}\right]}{1-\delta(1-d)} \tag{1.93}$$

which is exactly the price under Rational Expectations when $\beta^*_t = 1$. In other words, this derivation reaches the same conclusion as before but in the other direction: an optimal tax equal to 1 generates a price equivalent to the efficient when no capital gains are expected.

Importantly, $\tau^* = 1$ does not imply a trivial solution consisting of correcting non-fundamental volatility by killing also fundamental volatility. Thus,

$$\lim_{\tau^K \to 1} x = \tilde{x} > 0$$

for $x = z, k$. Put it differently, the volatility implementation of the optimal tax is in the spirit of the so-called "Principle of Targetting" of Pigouvian taxation (see Dixit, 1985), according to which a corrective tool has to tax directly the source of the externality. In this case, the direct source of the externality is the excessive volatility of capital gains expectations and thus, a tax on capital gains is directly related to it.

The main shortcoming of this approximation is that it might deliver a too low capital price and then, chronic sub-investment. The question is whether this can be compensated by a new instrument, since lump-sum taxes cannot affect the capital price. A tax on capital rents might be a natural alternative. Thus, suppose the government can tax capital profits with τ^r. With this new instrument, equation (1.93) becomes

$$Q_t = \frac{\delta(1-\tau^r_{t+1})\mathbb{E}_t\left[\frac{u^c_{t+1}}{u^c_t}F^k_{t+1}\right]}{1-\delta(1-d)} \tag{1.94}$$

Hence, by setting

$$(\tau^r_{t+1})^* = 1 - \frac{1-\delta(1-d)}{1-\delta(1-d)\beta^*_t} \tag{1.95}$$

Thus, if RE implies an almost constant capital gains expectations, $(\tau^r_{t+1})^*$ would be almost constant and then, the First Best can be implemented by a constant τ^K and a not-too volatile τ^r along with lump-sum taxes.[51] Altogether, the alternative implementation offers a way of stabilizing capital markets avoiding excessive volatility in taxes and relaxing the informational requirements.

1.5. Conclusions

This paper has analyzed how a Capital Gains Tax influences asset price cycles. I propose a theory that challenges the mainstream view, held, for instance, by Stiglitz, 1983, that a higher CGT boosts price fluctuations through the supply-side lock-in effect. Instead, I show that the demand-side capitalization effect is stabilizing because it reduces the elasticity of prices to subjective beliefs, reducing the likelihood of self-fulfilling booms and busts. This theory is derived from a model of learning about prices with portfolio adjustment costs and taxes on realized capital gains that displays the two effects in a tractable way.

The theory is applied to the United States, suggesting that the recurrence of asset price cycles in the middle of the Great Moderation, a troubling observation for many macro-finance models, can be partly explained by the observed decline in CGT. Indeed, the structural estimation of the model reveals that CGT cuts account for 25% of the observed rise in stock market volatility. The model also replicates the rise in stock market valuations and a sizable equity premium. Empirical estimates using survey beliefs support the model's prediction of an increase in the sensitivity of prices to subjective expectations due to lower taxes.

Furthermore, the last part of the paper has explored the usage of taxes to correct excess price volatility stemming from investors' information limitations. While subjective beliefs are crucial in explaining stock market volatility, the excess volatility they cause can be seen as a pecuniary externality that can cause undesirable real fluctuations. In such

[51] $(\tau^r_{t+1})^*$ would be less volatile than τ^*_{t+1} as long as β^*_t is more stable than β_t.

a case, a tax on unrealized capital gains that corrects too optimistic/pessimistic beliefs proves able to restore the First Best.

Altogether, the arguments developed in the paper suggest that a CGT can be an effective tool to prevent asset price booms and the financial and macroeconomic fluctuations associated with them. Thus, while the ability of a Financial Transaction Tax to prevent excessive price volatility has been widely questioned, a CGT emerges as a sound alternative.

The research has left some issues opened. On the empirical side, the analysis has focused on the US aggregate stock market leaving cross-sectional analysis unexplored. Although the effect of capital gains tax on the cross-section of stocks was analyzed by Dai, Shackelford, et al., 2013 for two tax reforms, it would be interesting to expand the analysis using larger time windows. Moreover, the decline in capital taxes since the 1980s was a global phenomenon. An international analysis of its effects on capital and real markets and its interaction with financial deregulation and capital flows liberalization appears as an interesting research avenue.

The paper has abstracted from many other potential drivers of the larger financial volatility, such as the decline in interest rates or the rise in stock repurchases. Besides, I have taken payout policies as given, ignoring the possible reaction of firms to tax changes and its impacts on investment, productivity or employment. I leave them for future research.

From a broader standpoint, the optimal capital taxation literature has not considered the use of capital gains taxes so far due to their focus on one-sector models.[52] Thus, the optimal use of capital gains taxes to fund government spending is to be explored. Finally, it is well known that capital gains have important redistributive implications.[53] The analysis in this paper would suggest that a CGT could help not only ex-post (i.e., redistributing capital gains) but even ex-ante (i.e. avoiding part of the wealth inequality that comes from asset price dynamics).

[52] Not even the recent work of Chari et al., 2020 that includes a rich tax system with taxes on dividends, capital rents or wealth.

[53] See Fagereng et al., 2022 for a recent analysis.

1.A Appendix

Appendix A: Data Sources

Stock market data. Stock prices, dividends and CPI inflation comes from Robert Shiller database. They can be downloaded here: `http://www.econ.yale.edu/~shiller/data.htm`. The risk-free rate is the 90 days T-Bill, from the FRED database `https://fred.stlouisfed.org/series/TB3MS`.

The data has been transformed into quarterly frequency by taking the last month of the considered quarter. Besides, the nominal variables have been transformed to real terms using Shiller's CPI inflation index. Finally, as is standard in the literature, I have deseasonalize dividends (by taking the average over the current and past 3 quarters) to compute the price-dividend ratio.

Macroeconomic data. Consumption data is the BEA real quarterly personal consumption expenditures series. Wages are the BEA compensation of employees. When computing the Wage-Dividends ratio, I use the Net Dividends from the BEA (Corporate Profits after tax with IVA and CCAdj: Net Dividends).

Capital tax rates. The base effective average marginal rates on dividends, short and long capital gains and interests are supplied by the TAXSIM program of the National Bureau of Economic Research (NBER). See Feenberg and Coutts, 1993 for a description of the program. They can be found here `https://taxsim.nber.org/marginal-tax-rates/`. These rates are offered on an annual basis from 1960 to 2018 at federal level and from 1979 to 2008 at state level. I took the rates computed using 1984 national data for each state and year.

Following Sialm, 2009, I adjusted for state and local taxes before 1979 and after 2008 as well as for the distinction between qualified and non-qualified dividends from 2003 on to get a complete series for the 1960-2018 period. Before 1960, τ_t^d, τ_t^{skg}, τ_t^{lkg} and rates are taken from Sialm, 2009. τ_t^B are interpolated.

The weights for the convex combination are computed using the dividend, short and long capital gains yields offered by Sialm, 2009. They are averaged over the 1954-2006 period. Letting them vary barely change the synthetic rate. For details on the taxable share, see Appendix C.

Capital gains. The total realized capital gains are a 5 year moving average on the capital gains reported in the adjusted gross income, coming from the IRS. As for total capital gains, I use a 5 year moving average of the nominal taxable gains, obtained from the Financial Accounts. I am grateful to Jacob Robbins for providing these data, coming from his paper **robbins2019capital**. The portion of capital gains coming from equities is obtained from the US Financial Accounts, covering the 1951-2018 on a quarterly basis. Finally, the portion of realized capital gains coming from equities is computed using data from the IRS for the year 1985 and 1997-2012.

Survey expectations. For the test of the tax indirect effect, I have used the UBS survey is the UBS Index of Investor Optimism. The quantitative question on stock market expectations has been surveyed over the period Q2:1998-Q4:2007 with 702 responses per month on average. To make the data consistent with the model, I have run some adjustment. First, the series have been deflated by using inflation expectations from the Michigan Surveys of Consumers, available at `https://data.sca.isr.umich.edu/data-archive/mine.php`. Second, I transformed real returns expectations into capital gains expectations by subtracting the mean dividend growth along the period over each period price-dividend ratio.

Appendix B: Proof of the Proposition.

Take a linear approximation of equation (1.16) around the Rational Expectations value (i.e., $\beta_t = \beta^D$):

$$\frac{P_t}{D_t} \approx \frac{P_t^{RE}}{D_t} + \omega(\beta_t - \beta^D) \tag{1.96}$$

with $\frac{P_t^{RE}}{D_t} = \frac{\delta(1-\tau^D)\beta^D}{1-\delta(1-\pi\tau^K)\beta^D - \delta\pi\tau^K}$ being a constant and ω evaluated at the approximation point. Taking the variance of both sides

$$\mathbb{V}ar\left[\frac{P_t}{D_t}\right] \approx \omega^2 \times \mathbb{V}ar(\beta_t) \tag{1.97}$$

as claimed in point i). Assume τ^D and $\pi\tau^K$ are both within the [0,1) interval. Then,

$$\omega = \frac{P_t/D_t}{\beta_t}\bigg|_{\beta_t=\beta^D} = \frac{\delta^2(1-\tau^D)\beta^D(1-\pi\tau^K)}{(1-\delta(1-\pi\tau^K)\beta^D - \delta\pi\tau^K)^2} > 0 \tag{1.98}$$

Differentiating (1.98) with respect to τ^K

$$\frac{d\omega}{d\tau^K} = \frac{-\delta^2(1-\tau^D)\beta^D\pi(1-\delta(1-\pi\tau^K)\beta^D - \delta\pi\tau^K) - \delta^2(1-\tau^D)\beta^D(1-\pi\tau^K)\delta\pi(\beta^D-1)}{(1-\delta(1-\pi\tau^K)\beta^D - \delta\pi\tau^K)^2} \tag{1.99}$$

The numerator boils down to $-\delta^2(1-\tau^D)\beta^D\pi(1-\delta)$. Since $\delta < 1$, the numerator is negative, $\frac{d\omega}{d\tau^K} < 0$ and $\frac{d\omega^2}{d\tau^K} < 0$ as claimed.

Applying the chain rule to expression (1.97),

$$\frac{d\mathbb{V}ar[P_t/D_t]}{d\tau^K} \approx \frac{\mathbb{V}ar[P_t/D_t]}{\mathbb{V}ar[\beta_t]}\frac{d\mathbb{V}ar[\beta_t]}{d\tau^K} + \frac{\mathbb{V}ar[P_t/D_t]}{\omega^2}\frac{d\omega^2}{d\tau^K}$$

Note $\frac{\mathbb{V}ar[P_t/D_t]}{\mathbb{V}ar[\beta_t]} = \omega^2 > 0$ and $\frac{\mathbb{V}ar[P_t/D_t]}{\omega^2} = \mathbb{V}ar[\beta_t] > 0$, provided the latter exists. Given, $\frac{d\omega^2}{d\tau^K} < 0$, the only thing left is to characterize $\mathbb{V}ar[\beta_t]$ and prove $\frac{d\mathbb{V}ar[\beta_t]}{d\tau^K} < 0$.

Start with the characterization of $\mathbb{V}ar[\beta_t]$. Equation (1.17) can be linearly approximated as an AR(2) process around $(\beta_{t-1}, \beta_{t-2}, \varepsilon^D_{t-1}) = (\beta^D, \beta^D, 1)$:

$$\beta_t \approx \beta^D + \mathcal{A}(\beta_{t-1} - \beta^D) + \mathcal{B}(\beta_{t-2} - \beta^D) + \mathcal{C}(\varepsilon^D_{t-1} - 1) \tag{1.100}$$

with

$$\mathcal{A} = \frac{\beta_t}{\beta_{t-1}}\Bigg|_{\substack{\beta_{t-1}=\beta^D \\ \beta_{t-2}=\beta^D \\ \varepsilon^D_{t-1}=1}} = 1 - g + \frac{g\delta\beta^D(1-\pi\tau^K)}{1-\delta(1-\pi\tau^K)\beta^D - \delta\pi\tau^K} \tag{1.101}$$

$$\mathcal{B} = \frac{\beta_t}{\beta_{t-2}}\Bigg|_{\substack{\beta_{t-1}=\beta^D \\ \beta_{t-2}=\beta^D \\ \varepsilon^D_{t-1}=1}} = -\frac{g\delta\beta^D(1-\pi\tau^K)}{1-\delta(1-\pi\tau^K)\beta^D - \delta\pi\tau^K} \tag{1.102}$$

$$\mathcal{C} = \frac{\beta_t}{\varepsilon^D_t}\Bigg|_{\substack{\beta_{t-1}=\beta^D \\ \beta_{t-2}=\beta^D \\ \varepsilon^D_{t-1}=1}} = g\beta^D \tag{1.103}$$

If $\{\beta_t\}$ is stationary, it would have the following variance

$$\mathbb{V}ar(\beta_t) \approx \frac{(1-\mathcal{B})\mathcal{C}^2\sigma^2_D}{(1+\mathcal{B})(1-\mathcal{A}-\mathcal{B})(1+\mathcal{A}-\mathcal{B})} \tag{1.104}$$

Now I verify the conditions that ensure $\{\beta_t\}$ is stationary. It is known that for the process to be stationary, parameters $\mathcal{A}$, $\mathcal{B}$ must lie within the region $-1 < \mathcal{B} < 1$, $\mathcal{A} + \mathcal{B} < 1$, $\mathcal{B} - \mathcal{A} < 1$. I proceed to verify these conditions. Note $\mathcal{B} < 0$ provided $\pi\tau^K < 1$. $\mathcal{B}$ must also satisfy $\mathcal{B} = -\frac{g\delta(1-\pi\tau^K)\beta^D}{1-\delta(1-\pi\tau^K)\beta^D-\delta\pi\tau^K} > -1$. Rearranging the terms, $1-\delta(1-\pi\tau^K)\beta^D - \delta\pi\tau^K > g\delta(1-\pi\tau^K)\beta^D$. The left hand side is positive by assumption

A. The right hand side is also positive given $\pi\tau^K < 1$. Then, a positive but small enough gain is sufficient for the inequality to hold. In particular,

$$g < \frac{1 - \delta(1 - \pi\tau^K)\beta^D - \delta\pi\tau^K}{\delta(1 - \pi\tau^K)\beta^D} \equiv \bar{g} \tag{1.105}$$

The next condition is $\mathcal{A} + \mathcal{B} < 1$. Using expression (1.101) and (1.102), the condition boils down to $\mathcal{A} + \mathcal{B} = 1 - g + \frac{g\delta\beta^D(1-\pi\tau^K)}{1-\delta(1-\pi\tau^K)\beta^D - \delta\pi\tau^K} - \frac{g\delta\beta^D(1-\pi\tau^K)}{1-\delta(1-\pi\tau^K)\beta^D - \delta\pi\tau^K} = 1 - g$. Thus, $\mathcal{A} + \mathcal{B} < 1$ holds if

$$g > 0 \tag{1.106}$$

Conditions (1.105) and (1.106) requires there is some learning, but not too much. The final condition is $\mathcal{B} - \mathcal{A} < 1$. Since $\mathcal{B} < 0$, proving $\mathcal{A} > 0$ is enough. It requires $1 > g\left(1 - \frac{\delta\beta^D(1-\pi\tau^K)}{1-\delta\beta^D(1-\pi\tau^K) - \delta\pi\tau^K}\right)$. Since $g > 0$, $\frac{\delta\beta^D(1-\pi\tau^K)}{1-\delta\beta^D(1-\pi\tau^K) - \delta\pi\tau^K} > 1$ is sufficient such that the element within the parenthesis is non-positive. Rearranging the previous inequality, $0 \geq 1 - 2\delta\beta^D(1 - \pi\tau^K) - \delta\pi\tau^K$. The following expression ensures the last inequality to hold

$$\pi\tau^K \leq \frac{2\delta\beta^D - 1}{2\delta\beta^D - \delta} \equiv \bar{\tau} \tag{1.107}$$

Since $\delta < 1$, $\bar{\tau} < 1$, compatible with $\pi\tau^K < 1$. Thus, a not too high effective tax is enough to ensure $\mathcal{A} > 0$.

The next step is to prove $\mathbb{V}ar[\beta_t]$ is decreasing on τ^K. Using the definition of $\mathcal{A}$, $\mathcal{B}$ and the chain rule it turns out

$$\frac{d\mathbb{V}ar(\beta_t)}{d\tau^K} = \frac{\mathbb{V}ar(\beta_t)}{\mathcal{A}}\frac{d\mathcal{A}}{d\tau^K} + \frac{\mathbb{V}ar(\beta_t)}{\mathcal{B}}\frac{d\mathcal{B}}{d\tau^K} \tag{1.108}$$

Now I show $\frac{d\mathbb{V}ar(\beta_t)}{d\tau^K} < 0$ holds. First,

$$\frac{\mathbb{V}ar(\beta_t)}{\mathcal{A}} = -\frac{2(\mathcal{B}-1)C^2\sigma_D^2\mathcal{A}}{(\mathcal{B}+1)(\mathcal{A}-\mathcal{B}+1)^2(\mathcal{A}+\mathcal{B}-1)^2} \tag{1.109}$$

Given $-1 < \mathcal{B} < 1$ holds, $(\mathcal{B} - 1) < 0$ and $\mathcal{B} + 1 > 0$. $\mathcal{A}$ is also positive. Then, this derivative has a positive sign.

Now, note $\mathcal{A}$ is decreasing on τ^K

$$\frac{d\mathcal{A}}{d\tau^K} = \frac{-g\delta\beta^D\pi(1 - \delta(1 - \pi\tau^K)\beta^D - \delta\pi\tau^K) - g\delta\beta^D\pi(1 - \pi\tau^K)\delta(\beta^D - 1)}{(1 - \delta(1 - \pi\tau^K)\beta^D - \delta\pi\tau^K)^2} < 0 \tag{1.110}$$

since the numerator boils down to $-g\delta\beta^D\pi(1 - \delta)$ and $\delta < 1$.

Now, check the effects of τ^K through $\mathcal{B}$. The derivative of the variance with respect to $\mathcal{B}$ is given by

$$\frac{\mathbb{V}ar(\beta_t)}{\mathcal{B}} = \frac{2C^2\sigma_D^2(\mathcal{B}^3 - 2\mathcal{B}^2 + \mathcal{B} + \mathcal{A}^2)}{(\mathcal{B}+1)^2(\mathcal{A} - \mathcal{B} + 1)^2(\mathcal{A} + \mathcal{B} - 1)^2} \tag{I.111}$$

Since the denominator and $2C^2\sigma_D^2$ are both positive, the sign is determined by $(\mathcal{B}^3 - 2\mathcal{B}^2 + \mathcal{B} + \mathcal{A}^2)$. Since $\mathcal{B} - \mathcal{A} < 1$, $(\mathcal{B}-1)^2 < \mathcal{A}^2$. Then, $(\mathcal{B}^3 - 2\mathcal{B}^2 + \mathcal{B} + (\mathcal{B}-1)^2) > 0$ implies $(\mathcal{B}^3 - 2\mathcal{B}^2 + \mathcal{B} + \mathcal{A}^2) > 0$. Note $(\mathcal{B}^3 - 2\mathcal{B}^2 + \mathcal{B} + (\mathcal{B}-1)^2) > 0$ is equivalent to $\mathcal{B}^3 - \mathcal{B}^2 - \mathcal{B} + 1 > 0$ which holds for $-1 < \mathcal{B} < 1$ and $\mathcal{B} > 1$. Hence, within the stationarity region $-1 < \mathcal{B} < 1$, the expression is positive which makes the derivative positive as well.

The effect of τ^K on $\mathcal{B}$ is given by

$$\frac{d\mathcal{B}}{d\tau^K} = \frac{g\delta\beta^D\pi(1 - \delta(1 - \pi\tau^K)\beta^D - \delta\pi\tau^K) + g\delta\beta^D\pi(1 - \pi\tau^K)\delta(\beta^D - 1)}{(1 - \delta(1 - \pi\tau^K)\beta^D - \delta\pi\tau^K)^2} \tag{I.112}$$

which is positive for the same reasons that $\frac{d\mathcal{A}}{d\tau^K} < 0$.

Altogether, for $\frac{d\mathbb{V}ar(\beta_t)}{d\tau^K} < 0$ it must be

$$\underbrace{\frac{\mathbb{V}ar(\beta_t)}{\mathcal{A}}}_{>0}\underbrace{\frac{d\mathcal{A}}{d\tau^K}}_{<0} + \underbrace{\frac{\mathbb{V}ar(\beta_t)}{\mathcal{B}}}_{>0}\underbrace{\frac{d\mathcal{B}}{d\tau^K}}_{>0} < 0 \tag{I.113}$$

Since $\frac{\mathcal{A}}{\tau^K} = -\frac{\mathcal{B}}{\tau^K}$, inequality (I.113) boils down to $\frac{\mathbb{V}ar(\beta_t)}{\mathcal{A}} > \frac{\mathbb{V}ar(\beta_t)}{\mathcal{B}}$, that is,

$$-\frac{2(\mathcal{B}-1)C^2\sigma_D^2\mathcal{A}}{(\mathcal{B}+1)(\mathcal{A} - \mathcal{B} + 1)^2(\mathcal{A} + \mathcal{B} - 1)^2} > \frac{2C^2\sigma_D^2(\mathcal{B}^3 - 2\mathcal{B}^2 + \mathcal{B} + \mathcal{A}^2)}{(\mathcal{B}+1)^2(\mathcal{A} - \mathcal{B} + 1)^2(\mathcal{A} + \mathcal{B} - 1)^2}$$

Simplifying the previous expression, one gets to $-(\mathcal{B}-1)\mathcal{A} > frac\mathcal{B}^3 - 2\mathcal{B}^2 + \mathcal{B} + \mathcal{A}^2\mathcal{B} + 1$. It can be shown this inequality holds for $-1 < \mathcal{B} < 1$ and $\mathcal{B} - \mathcal{B}^2 < \mathcal{A} < 1 - \mathcal{B}$. Note $-1 < \mathcal{B} < 1$ and $\mathcal{A} < 1 - \mathcal{B}$ are stationary conditions already proven; $\mathcal{B} - \mathcal{B}^2 < \mathcal{A}$ must be proven. Using the definition of $\mathcal{A}$, $\mathcal{B}$, it turns out $\mathcal{A} = 1 - g - \mathcal{B}$. Using this equality, $\mathcal{B} - \mathcal{B}^2 < 1 - g - \mathcal{B}$ or $0 < 1 - g + \mathcal{B}^2 - 2\mathcal{B}$. Intuitively, $\mathcal{B}^2$ and $-2\mathcal{B}$ are positive so that, the inequality would hold if g is not too high. In particular, given $g > 0$, the inequality holds if $\mathcal{B} < 1 - g^{0.5}$ which given $\mathcal{B} < 0$ is true if $g^{0.5} < 1$. To check this last inequality, use the upper bound $\bar{g} > g$, that is, check if $\bar{g}^{0.5} = \left(\frac{1-\delta(1-\pi\tau^K)\beta^D - \delta\pi\tau^K}{\delta(1-\pi\tau^K)\beta^D}\right)^{0.5} < 1$. It boils down to $\delta(1 - \pi\tau^K)\beta^D > 1 - \delta(1 - \pi\tau^K)\beta^D - \delta\pi\tau^K$ which has been already shown to hold provided $\pi\tau^K < \bar{\tau}$. Hence, $\frac{d\mathbb{V}ar(\beta_t)}{d\tau^K} < 0$ as claimed in the proposition.

Appendix C: Capital Gains Tax stabilization properties under RE

Assume that dividends growth contains a persistent component x_t in the spirit of Bansal and Yaron, 2004:

$$\ln D_t = \ln\beta^D + \ln D_{t-1} + \varphi \ln x_t + \ln\varepsilon_t^D \tag{1.114}$$

$$\ln x_t = \ln x_{t-1} + \ln\vartheta_t^x \tag{1.115}$$

with $\ln\varepsilon_t^D \sim i.i.\mathcal{N}(-\frac{\sigma_D^2}{2}, \sigma_D^2)$ and $\ln\vartheta_t^x \sim i.i.\mathcal{N}(-\frac{\varphi s_x^2}{2}, s_x^2)$. In this setup, the following proposition holds:

Proposition: ***A Capital Gains Tax can stabilize the PD ratio under Rational Expectations.*** *Assume that investors have perfect information, including the dividends stochastic process (1.114) and (1.115), and agents' homogeneity. Then, the variance of the PD ratio is decreasing on the CGT level, that is,*

$$\frac{d\mathbb{V}ar\left[P_t^{RE}/D_t\right]}{d\tau^K} < 0 \tag{1.116}$$

In this setup, forward iteration on the Euler Equation (1.6), the Law of Iterated Expectations and a transversality condition delivers the following equilibrium PD ratio

$$\frac{P_t^{RE}}{D_t} = \frac{\delta(1-\tau^D)\beta^D x_t^{\varphi}}{1-\delta(1-\pi\tau^K)\beta^D x_t^{\varphi} - \delta\pi\tau^K} \tag{1.117}$$

since $\mathbb{E}_t\left[\frac{D_{t+j}}{D_t}\right] = (\beta^D x_t^{\varphi})^j$. Following the same steps as in the proposition in the main text, the unconditional variance of the PD ratio can by approximated around $x_t = 1$ as

$$\text{v} = \mathbb{V}ar\left[\frac{P_t^{RE}}{D_t}\right] \approx \omega^2 \times \mathbb{V}ar(x_t) \tag{1.118}$$

It turns out

$$\frac{d\mathbb{V}ar\left[P_t^{RE}/D_t\right]}{d\tau^K} = \frac{\mathbb{V}ar\left[P_t^{RE}/D_t\right]}{\omega^2}\frac{d\omega^2}{d\tau^K} < 0$$

since $\frac{\mathbb{V}ar\left[P_t^{RE}/D_t\right]}{\omega^2} = \mathbb{V}ar(x_t) > 0$ and $\frac{d\omega^2}{d\tau^K} < 0$.

Appendix D: Computing the non-taxable share

The evolution of the effective capital tax rates depends essentially on two factors: statutory rates and regulations. Legal regulations are accounted for by the NBER TaxSim rates. The important exception is the amount of capital income accruing to non-taxable units, as pension funds, IRAs or non-profit institutions. The Financial Accounts of the United States, run by the Fed, report the household share of corporate equity. Some takes that as a proxy for the taxable share of ownership, but that overestimate it given the inclusion of IRAs (see Rosenthal and Austin, 2016 for a critical review of the different measures). Therefore, the goal is to get an estimate of the fraction of equities hold by households in taxable accounts. I follow Rosenthal and Austin, 2016.

Table 1.11 reports the steps followed to compute the taxable share. Essentially, it amounts to an adjustment of the Fed's households equity share, considering IRAs, indirect holdings and so on. Here I detail the abbreviations dictionary: CE = corporate equities; HHNPI = households and nonprofit institutions; RoW = rest of the world; ETF = exchange traded fund; CEF = closed-end fund; REIT = real estate investment trust; C-CE = C corporations CE; MF = mutual funds; IRA = investment retirement accounts. The variables comes from the Federal's Reserve Financial Accounts of the United States, except for those variables whose construction is explained in the table. Besides, as in Rosenthal and Austin, 2016, the stock held in self- directed IRAs is based on data from the Investment Company Institute. Calculations files are available upon request.

Figure 1.9 plots the estimated taxable share from 1951:IV to 2018:IV. As observed, it displays a steady decline until the early 2000s, when stabilizes around 30%. In other words, there was a big structural change in the stock ownership, moving it away from taxable units.

Appendix E: Projection facility

The equilibrium PD ratio given by 1.117 faces a discontinuity. For this reason, simulation requires to set up the following modified belief updating equation to ensure non-negative prices

$$\beta_{t+1} = w\left(exp\left\{ln\beta_t(1-g) + gln\frac{P_t}{P_{t-1}}\right\}\right) \tag{1.119}$$

where

$$w(x) = \begin{cases} x & \text{if} \quad x \leq \beta_t^L \\ \beta_t^L + \frac{x-\beta_t^L}{x+\beta_t^U-2\beta_t^L}(\beta_t^U - \beta_t^L) & \text{if} \quad x > \beta_t^L \end{cases} \tag{1.120}$$

	Total CE HHNPI	
1.- Substract foreign equities	- RoW x (Total CE HHNPI /Total CE All Sectors)	
	= HHNPI domestic CE	
2.- Substract the stocks issued by the passthrough entities	-S Corporations – (ETF + CEF + REITs) x HH share of Mutual Funds	
S corporations, ETFs, CEF and REITs	= HHNPI domestic C-CE	
3.- Substract NPI holdings	-NPI domestic C-CE	NPI domestic C-CE = (NPI CE+MF stocks) x NNHPI [CE / (CE+MF)] (NPI CE+MF) given by the Fed after 1987 (CE+MF together). Before: NPI CE+MF = (HHNPI CE + MF) x $(NPI CE + MF)_{1987}$/ $(HHNPI CE + MF)_{1987}$
	= HH domestic C-CE	
4.- Substract IRAs and 529 savings plans holdings	- IRA C-CE - 529 C-CE	IRA C-CE = CE IRA x C-CE Fraction CE IRA = IRA Other Assets x 0.75 [a] 529 C-CE = College Savings Plans Assets x 0.5 [b] C-CE fraction = All sectors C-CE/(All sectors C-CE + ETF + CEF + REITs) All sectors C-CE = All sectors domestic CE – S corp – ETF - CEF – REITs
	= Direct HH domestic C-CE	
5.- Add Indirect Holdings of C Corporation Equity	+ Indirect HH domestic C-CE	Indirect HH C-CE = (CE MF + (CE ETF + CE CEF) x HH share of MF) x Direct HH domestic CE / Total HHNPI CE
	= HH Taxable CE	
6.- Divide by the total C corporation equity	/ All sector C-CE	
	= Taxable share	

Figure 1.9. **Taxable share evolution.** The graph plots the taxable share of equity income, estimated following the procedure explained above. It uses data from 1951:IV to 2018:IV.

and

$$\beta_t^q = PD^q \left\{ PD^q \xi\delta(1-\pi\tau_{t+1}^K)^2 + \chi\delta(1-\pi\tau_{t+1}^K)\left(\frac{W_{t+1}}{D_{t+1}} + 1 - \tau_{t+1}^D + \pi\tau_{t+1}^K \frac{P_t}{D_t}\frac{D_t}{D_{t+1}}\right)\right\}^{-1}$$

(1.121)

for $q = L, U$. Thus, this projection facility starts to dampen belief coefficients that imply a price-dividend ratio equal to PD^L and sets an effective upper bound at PD^U. Projection facilities are usual devices in this sort of algorithms (see Ljung, 1977); particularly, (1.120) is similar to the one used by Adam, Marcet, and Nicolini, 2016. It can be understood in a Bayesian sense, so that agents attach zero probability to beliefs coefficients implying a PD ratio higher than PD^U.

Appendix F: Parameterized Expectations Algorithm

In the spirit of Hakansson, 1970, the proposed approximating function ψ is

$$\frac{C_t^*}{D_t} = \bar{\mathcal{E}}(X_t) \approx \psi(X_t;\chi) = c_t^y Y_t + c_t^w \frac{P_t}{D_t} S_{t-1} \quad (1.122)$$

where $c_t^y \equiv 1 - \chi\delta(1 - \tau_t^D)\beta^D$ is the time-varying propensity to consume out of income, Y_t collects all the income sources (wages, dividends, net transfers) normalized by dividends, $c_t^w \equiv 1 - \chi\delta(1 - \tau_t^K)\beta_t$ is the propensity to consume out of wealth, and χ is a parameter of ψ to be estimated. To evaluate the performance of this approximating function, χ must be estimated. To do so, I resort to simulation and Montecarlo integration. The algorithm involves the following steps:

1. Draw a series of the exogenous processes for a large T.

2. For a given $\chi \in \mathbb{R}^n$, recursively compute the series of the endogenous variables.

3. Minimize the Euler Equation square residuals using non-linear least squares

$$G(\chi) = \underset{\xi \in \mathbb{R}^n}{\operatorname{argmin}} \frac{1}{(T - \underline{T})} \sum_{t=\underline{T}}^{T} \left[\phi\left(z_{t+1}^{\mathcal{P}}(\chi), \varepsilon_{t+1}, z_t(\chi)\right) - \frac{\psi(X_t(\chi); \xi)^{-\gamma}}{\delta} \right]^2$$

with $\underline{T}$ are some initial periods burned. ϕ is the interior of the conditional expectation $\bar{\mathcal{E}}(X_t)$, z are the endogenous variables and the exogenous shocks.

Note the interior of the expectation must be computed according to investor's beliefs. Since investors know the process for dividends and wage-dividends, the only problematic objects are next period prices and next period consumption. Using agents subjective price model

$$\beta_{t+1}^{\mathcal{P}} = \beta_t^i \nu_{t+1} \Rightarrow \left(\frac{P_{t+1}}{P_t}\right)^{\mathcal{P}} = \beta_t^i \nu_{t+1} \varepsilon_{t+1}^p \Rightarrow \left(\frac{P_{t+1}}{D_{t+1}}\right)^{\mathcal{P}} = \left(\frac{P_{t+1}}{P_t}\right)^{\mathcal{P}} \frac{D_t}{D_{t+1}} \frac{P_t}{D_t}$$

In turn, expected consumption reads

$$\frac{C_{t+1}^{\mathcal{P}}}{D_{t+1}} = (1 - \chi\delta(1 - \pi\tau_{t+1}^K)\beta_{t+1}^{\mathcal{P}}) \left(\left(\frac{P_{t+1}}{D_{t+1}}\right)^{\mathcal{P}} + 1 - \tau_{t+1}^D + \frac{W_{t+1}}{D_{t+1}} - \pi\tau_{t+1}^K \left[\left(\frac{P_{t+1}}{D_{t+1}}\right)^{\mathcal{P}} - \frac{P_t}{D_t} \frac{D_t}{D_{t+1}} \right] \right) \tag{1.123}$$

4. Find a fixed point $\chi = G(\chi)$. For that, update χ following

$$\chi^{j+1} = \chi^j + d(G(\chi^j) - \chi^j) \tag{1.124}$$

where j iteration number and d the dampening parameter.

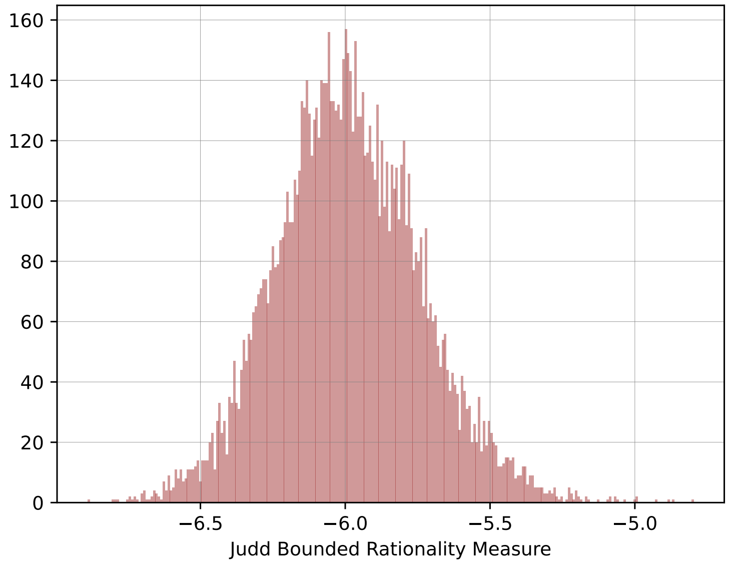

Figure 1.10. **Histogram of the Judd Bounded Rationality Measure.** The histogram plots the Judd criterion as defined by equation (1.125) resulting from 10.000 simulations of the model.

To evaluate how good is the approximation, I explore the size of the errors. The approximating errors are given by

$$u_{t+1} = \phi\Big(z_{t+1}, \varepsilon_{t+1}, z_t\Big) - \frac{\psi(\chi; x_t)^{-\gamma}}{\delta}$$

The criterion to determine the degree of accuracy is the Bounded Rationality Measure (Judd, 1992):

$$J = \log_{10}\left(\mathbb{E}_t \left|\frac{u_{t+1}}{\frac{C_t}{D_t}}\right|\right) \tag{1.125}$$

being J a dimension-free quantity expressing that error as a fraction of current consumption, which expresses the results in economic terms. For the baseline model, J = -5.99. It is equivalent to a mistake of $1 out of a million. The Mean Square Error is 5.71e-06. Figure 1.10 plots the histogram of J for 10.000 simulations of the model.

Solving the model with the lock-in effect

Algorithm to compute m_t:

1. Approximate the conditional expectation determining μ_{t+1} via a function $\Psi(X_t)$, where X_t is a vector of state variables, that is:

$$\mu_{t+1} = (\tau^K)^{1+\xi} M_{t+1} = (\tau^K)^{1+\xi} \mathcal{E}(X_{t+1}) \approx (\tau^K)^{1+\xi} \Psi(X_{t+1}) \tag{1.126}$$

In particular, I use this linear polynomial:

$$\Psi(X_t) = \alpha_0 + \alpha_1 \beta_t + \alpha_2 G_t + \alpha_3 \frac{P_t}{D_t} \tag{1.127}$$

2. For a given vector α, compute the expectation of μ_t conditional on the information at time t, that is

$$m_t = \mathbb{E}_t^{\mathcal{P}}(\mu_{t+1}) = \mathbb{E}_t^{\mathcal{P}}(\mathcal{E}(X_{t+1})) \approx \mathbb{E}_t^{\mathcal{P}}(\Psi(X_{t+1})) \tag{1.128}$$

   a) Future state variables depend on four shocks:

      i. β_{t+1} is predetermined: $\beta_{t+1} = \beta_t(1-g) + g\left(\frac{P_t}{D_t}\frac{D_t}{D_{t-1}}\frac{D_{t-1}}{P_{t-1}}\right)$

      ii. G_{t+1}/D_{t+1} depends on ε^D_{t+1}:

$$\frac{G_{t+1}}{D_{t+1}} = \frac{P_t/D_t - P_{t-1}/D_{t-1}D_{t-1}/D_t - \phi(\bar{\pi}_t)G_t/D_t}{\beta^D + \varepsilon^D_{t+1}}$$

      iii. P_{t+1}/D_{t+1} depends on (according to the subjective price model):

$$\frac{P_{t+1}}{D_{t+1}} = \frac{P_{t+1}}{P_t}\frac{P_t}{D_t}\frac{D_t}{D_{t+1}} = \frac{P_t}{D_t}\frac{\beta_t + u_t + \nu_{t+1} + \varepsilon^P_{t+1}}{\beta^D + \varepsilon^D_{t+1}}$$

   b) Use the Gauss-Hermite quadrature rule extended to the multidimensional case:

      i. Let $\{q_i, \omega_i\}_{i=1}^I$ be a set of Hermite nodes and weights.

      ii. Note $\Psi(X_{t+1}) = \Psi(X_t, \varepsilon^D_{t+1}, u_t, \nu_{t+1}, \varepsilon^P_{t+1})$.

      iii. Using the quadrature, each shock ε^x_{t+1} is replaced by $\sqrt{2}\sigma_x q_h$ (shocks are zero-mean Normally distributed) such that the expectation is computed as:

$$m_t \approx \mathbb{E}_t^{\mathcal{P}}(\Psi(X_{t+1})) = \pi^{-N/2} \sum_i \sum_j \sum_k \sum_l \Psi(X_t, q_i, q_j, q_k, q_l)\omega_i\omega_j\omega_k\omega_l$$

      where N is the number of shocks.

3. Note that $m_t = m(P_t/D_t, \cdot)$ such that

$$\frac{P_t}{D_t} = \frac{\delta(1-\tau^D)\beta^D}{1 - \delta\beta_t(1-(\tau^K)^{1+\xi}m(P_t/D_t,\cdot)) - \delta(\tau^K)^{1+\xi}m(P_t/D_t,\cdot)} \tag{1.129}$$

That is a nonlinear equation that can be solved numerically.

4. Estimate α via PEA.

Appendix G: Alternative measures of the increase in volatility

In this appendix, I present alternative measures of the increase in the volatility of the Price-Dividend ratio. In the main text, I focus on the unconditional variance. In this appendix, I show that the the conditional variance also went up. Since taxes are a relatively slow-moving variable, I concentrate on the long run component of the conditional variance. First, I present long-run volatility measures based on a GARCH model variant. Second, I include taxes in the long run component. Finally, I show volatility measures for daily price growth and stock returns.

G1.- The AR-GARCH-MIDAS model

The long run price-dividend volatility is the permanent component of the conditional variance. The procedure to obtained it builds upon the GARCH-MIDAS model outlined by Engle, Ghysels, et al., 2013. They decompose the conditional variance between a permanent and a transitory component, where the former is a filter over a number of lags of the realized volatility. However, they assume a constant conditional mean that is suitable for stock returns (with a mean close to zero and i.i.d. deviations from the mean) but not for the PD ratio (which is highly persistent, with time varying mean and, as a result, serially correlated deviations from the constant long run mean). To adapt the model for the PD ratio, I introduce an AR(1) model for the conditional mean, in which case the AR(1) residuals become serially uncorrelated and then the GARCH-MIDAS procedure can be applied for the variance.

The model boils down to the following list of equations. Unexpected changes in the price-dividend ratio in quarter q of year t are uncorrelated and normally distributed

$$\frac{P_{q,t}}{D_{q,t}} - \mathbb{E}_{t-1}\left(\frac{P_{q,t}}{D_{q,t}}\right) = {}_{q,t}, \quad {}_{q,t} \sim ii\mathcal{N}(0, \sigma^2_{q,t}) \tag{1.130}$$

with the conditional expectation given by an AR(1) process

$$\mathbb{E}_{t-1}\left(\frac{P_{q,t}}{D_{q,t}}\right) = \mu + \rho \frac{P_{q-1,t}}{D_{q-1,t}}$$

The conditional variance model hypothesizes that there is a short run (or transitory) $\mathrm{g}^2_{q,t}$ and a long run (or permanent) variance v^2_t. The permanent component captures an underlying state which makes equivalent surprises in the PD ratio have different effects. For instance, better than expected dividends might have a different impact in stock prices in high or low capital tax environments. As stated by Engle and Rangel, 2008, the long-memory component can be interpreted as a trend around which the conditional variance fluctuates. All in all, the errors standard deviation is the product of the short and long run components

$$\sigma_{q,t} = \mathrm{v}_t g_{q,t} \tag{1.131}$$

It is assumed that the transitory component follows a GARCH(1,1)

$$\mathrm{g}^2_{q,t} = 1 - \alpha_0 - \alpha_1 + \alpha_0 \frac{{}^{2}_{q-1,t}}{\mathrm{v}^2_t} + \alpha_1 \mathrm{g}^2_{q-1,t} \tag{1.132}$$

In turn, the long run volatility is a MIDAS filter over K past realized volatility

$$\mathrm{v}^2_t = \phi_0 + \phi_1 \sum_{k=1}^{K} \varphi_k(w) RV^Q_{t-k} \tag{1.133}$$

with the realized volatility defined as a moving variance over a fixed window of Q quarters

$$RV^Q_t = \frac{1}{Q-1} \sum_{q=1}^{Q} \left(\frac{P_{q,t}}{D_{q,t}} - \frac{1}{Q} \sum_{q=1}^{Q} \frac{P_{q,t}}{D_{q,t}} \right)^2 \tag{1.134}$$

and the weighting scheme given by a beta lag structure, which yields a monotonically decreasing sequence determined by a single parameter

$$\varphi_k(w) = \frac{(1 - k/K)^{w-1}}{\sum_{j=1}^{K} (1 - j/K)^{w-1}} \tag{1.135}$$

Altogether, the parameter vector θ contains a total of 7 parameters $\theta = \{\mu, \rho, \alpha_0, \alpha_1, \phi_0, \phi_1, w\}$, jointly estimated through Quasi-Maximum Likelihood[54].

For the baseline application, I use an intra-annual standard deviation as the measure of realized volatility (i.e., Q=4) and 10 lags of this realized volatility to compute the long run component v (i.e., K=10). The financial literature usually works with high frequency data (often daily variables) and regard the long run component as the underlying monthly or quarterly trend (e.g., Schwert, 1989, Engle, Ghysels, et al., 2013). Differently, here I have adopted a low frequency approach closer to macroeconomics, where the long run frequency tend to go beyond the business cycle. As a result, I have regarded the long run as a 10-year trend. The reason is that taxes are a much slower evolving variable than prices

[54] It is well known that the QML estimator is consistent and asymptotically normal for GARCH(1,1), provided that the innovation distribution has a finite fourth moment, even if the true distribution is far from Gaussian (e.g., see Lumsdaine, 1996). This is the case here indeed: residuals are non-Gaussian (due to fat tails) but exhibit an empirical kurtosis of 5.98 so that quasi-maximum likelihood estimators are asymptotically Normally distributed.

and dividends. In other words, potentially taxes reflect a long-lasting structure or regime, which determines the business conditions for long periods.

Table 2.9 shows the estimation results. All the coefficients are significant at usual confidence levels. As observed in figure 1.11, this estimation yields a long run volatility with a steep increase since the 1990s. The variable peaks in the aftermath of the Great Recession but the post-GR volatility is still way above its historical mean. The persistence of higher volatility turns out to be robust across alternative specifications: the number of the realized volatility lags considered for the long run filter; the fixed vs. rolling window specification for realized volatility.

Table 1.12. **AR-GARCH-MIDAS model estimation results.** The table shows the QML estimation of all the parameters of the AR-GARCH-MIDAS model for the fixed window realized volatility with Q=4 and K=10. The data used for the estimation covers the 1940:I-2018:IV period.

	Coefficient	Std. Errors	t-stat	p-val
μ	2.91	0.58	5.04	0.00
ρ	0.96	0.01	139.88	0.00
α_0	0.26	0.06	4.11	0.00
α_1	0.67	0.067	10.08	0.00
ϕ_0	10.17	2.60	3.90	0.00
ϕ_1	0.66	0.33	1.99	0.05
w	1.03	0.18	5.63	0.00

The left hand side plot of figure 1.11 plots the estimation with the fixed window for the realized volatility and different number of lags. As expected, the higher the lags considered, the smoother is the trajectory, without modifying the baseline result. The case of K=5 results in a volatile series, resembling the one for the conditional variance. That illustrates two things: probably 5 year are not enough to capture of long-lasting permanent component; the persistence of volatility is mostly due to the echoes of the Great Recession (i.e., the short run volatility after the GR has not been particularly high).

The right hand side graph of figure 1.11 compare the fixed to the rolling specification for the realized volatility. The former transforms quarter data into annual long run volatility; the latter keeps the long run volatility at quarter frequency. Both uses 10 years of data to produce the long run measure. As observed, the gray and the red line display the same qualitative trajectory. However, the rolling measure fluctuates way more, reaching a higher peak and reversion.

Finally, to check for the potential model-dependency of the result, I have computed alternative measures. On the one hand, the black line (R2) plots a simple 10 year rolling window standard deviation of the price-dividends deviations from their conditional mean (i.e., the residuals from equation 1.130). This measure is the one that resembles the most to the baseline one. On the other hand, the dotted black line (R3) shows a 10 year rolling window standard deviation of the Hodrick-Prescott cyclical component of the price-dividend quarterly data. This one follows the rolling window AR-GARCH-MIDAS closely. The correlation matrix among them gives a quantitative view of this comparison (the baseline measure displays correlations above 0.9 with all the alternative measures, except for the medium-term measure).

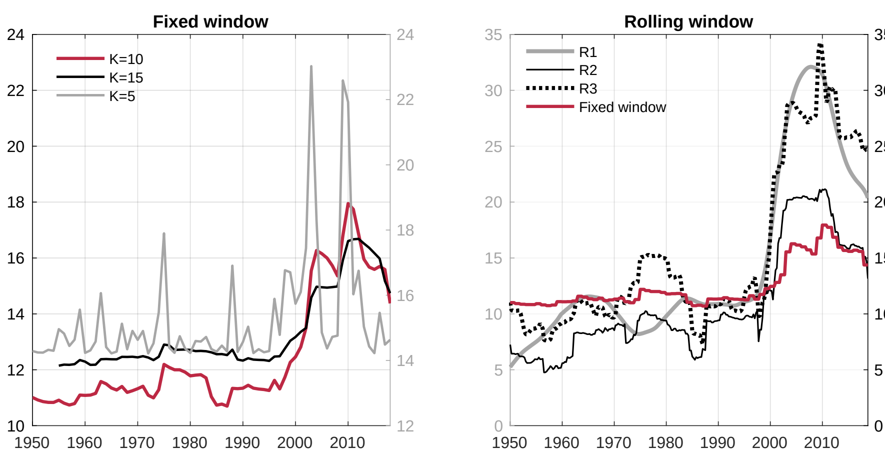

Figure 1.11. **Time-varying volatility measures.** The left hand side graph plot the baseline model with a Q-fixed window realized volatility (for Q=4) and changes the number of annual lags used for computing the long volatility (equation 1.133). The red line is the baseline measure (K=10). The right hand side graph plots the baseline fixed window measure (K=10) against a number of rolling window based measures: R1 is the AR-GARCH-MIDAS estimate for the case of a 10-year rolling window for the realized volatility; R2 is a 10-year rolling window standard deviation over the AR(1) residuals of equation 1.130; R3 is a 10-year rolling window standard deviation over the HP cyclical component of the quarterly price-dividend ratio. The data used for the estimation covers the 1940:I-2018:IV period.

G2.- Taxes as the long run component

One of the contributions of the Engle, Ghysels, et al., 2013's GARCH-MIDAS model is allowing for the introduction of macroeconomic variables directly in the long term

Table 1.13. **Volatility measures correlation matrix.** The table shows the contemporaneous correlation among all the long run volatility measures. The labels are as in figure 1.11. All the correlations are significant at 99% confidence level. The data used for the estimation covers the 1940:I-2018:IV period.

	K=10	K=15	K=5	R1	R2	R3
K=10	1.00					
K=15	0.97	1.00				
K=5	0.50	0.42	1.00			
R1	0.94	0.89	0.50	1.00		
R2	0.94	0.88	0.53	0.96	1.00	
R3	0.97	0.93	0.52	0.94	0.96	1.00

component. That would avoid a two-step approach (as Schwert, 1989) consisting of measuring volatility and estimate VAR models with volatility proxies and macrovariables. The problem with the 2-step approach is the measurement error in RV that would bias the coefficient capturing the effect of past volatility on current volatility and on the macroeconomic variables (see Engle, Ghysels, et al., 2013). On the contrary, a single-step procedure would circumvent this problem by adding the macrovariable directly to the variance model. That is one of the contributions of the GARCH-MIDAS approach.

In this paper, though, the impact of taxes on volatility is analyzed through a VAR, that is, via a 2-step approach. The potential cost is the bias effect coming from measurement errors, indeed. However, by assuming that potential cost, I can compare the time series, which opens the door to a much richer analysis than a coefficient significant test (for instance, I can explore the dynamic effects of tax cuts via IRFs). Since that seems to be a substantial gain, the potential measurement error problem is addressed in the VAR context itself (following Forni et al., 2020). On top of that, the VAR is only used for descriptive purposes and the causal analysis is left for the structural DSGE model.

Nonetheless, for the sake of completeness, in this appendix I report the results of the Engle, Ghysels, et al., 2013's single-step approach. That amounts to modify the long run component as follows:

$$\mathrm{v}_t^2 = \phi_0 + \phi_1 \sum_{k=1}^{K} \varphi_k(w)\tau_{t-k} \tag{1.136}$$

where τ is the capital income tax defined in the paper. Table 1.14 reports the estimation results. Notice that all three coefficients get a decent t-stat value for different number of lags (for K=5, all of them are significant at standard confidence levels).

Table 1.14. **AR-GARCH-MIDAS model with taxes as the long run component.** The table shows the QML estimation of all the parameters of the model when replacing equation 1.133 by 1.136, for the fixed window realized volatility with Q=4. t-statistics in parenthesis. The data used for the estimation covers the 1940:I-2018:IV period.

	K=5	K=10	K=15
μ	2.19	3.64	4.62
	(3.51)	(6.91)	(9.94)
ρ	0.99	0.98	0.97
	(136.20)	(189.46)	(238.36)
α_0	0.25	0.28	0.28
	(2.98)	(3.12)	(3.46)
α_1	0.63	0.63	0.63
	(7.33)	(7.46)	(7.49)
ϕ_0	21.29	20.87	22.36
	(2.16)	(1.51)	(1.66)
ϕ_1	-3.78	-3.51	-3.93
	(-1.83)	(-1.17)	(-1.31)
w	1.16	3.62	1.02
	(3.01)	(0.94)	(3.81)

G3.- Absolute and Relative Volatility

The baseline volatility measure uses the absolute deviations from the price-dividend conditional mean as the main ingredient (see equation 1.130). In this sense, it can be regarded as a measure of absolute volatility. That contrasts with **schwert2011stock**'s view, that regard volatility as a relative measure (dealing with price percentage changes or rate of returns). In this section, I comment on the relevance of using price-dividends absolute deviations as well as on the robustness of the rise in volatility when using high-frequency relative measures.

One of the concerns of using absolute measures is that they may "exaggerate" the degree of volatility when the variable level is in high heights. That concern has been repeatedly expressed by Schwert (Schwert, 1989, **schwert2011stock**). In this regard, it is convenient to remember that larger deviations from the mean are not a mechanical consequence of a higher mean at all; in other words, they carry some useful information. It suffices to point out the strong positive correlation between absolute price-dividend fluctuations and investment cycles (Cochrane, 2017, Adam and Merkel, 2019). The fact that high-mean times tend to go in hand with high-variance times is not a trivial coincidence but something to be explained. The hypothesis of the paper is that the decline in capital taxes is an important driver behind not only the rising trend in stock market valuations (in line with McGrattan and Prescott, 2005 and others) but also the larger swings around the trend.

For the particular case of price-dividend ratios, it is worthy it to remember that the variable is a ratio such that pure time-trend effects are accounted for[55]. Besides, the variance of the price-dividend is an important object in the price excess volatility literature, starting with Shiller, 1981[56].

Finally, the rise in long run volatility is also observed when using high frequency price percentage changes or rate of returns. Figure 1.12 plots a non-parametric measure of it, a 10-year rolling standard deviation over the daily series[57]. In both cases, the long run measure reveals a gradual increase in volatility since the early 1970s.

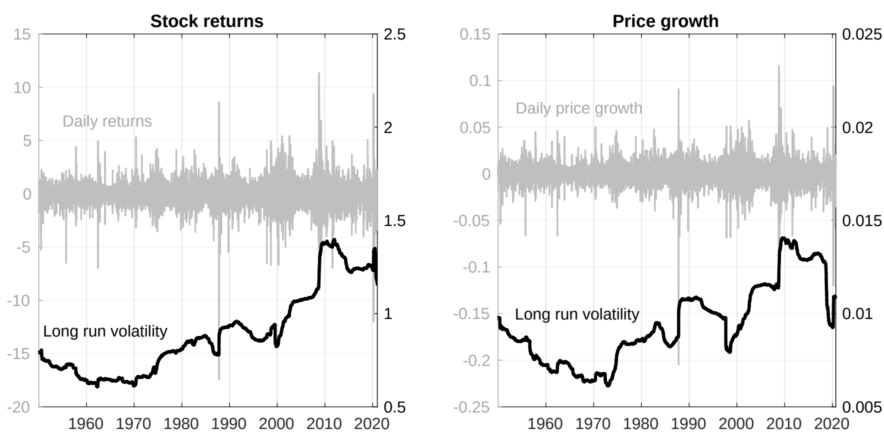

Figure 1.12. **Long run volatility of relative variables.** The left hand side graph pictures daily stock returns from French's website;the right hand side graph pictures daily price growth from Yahoo's historical data. Both graphs includes the long run volatility, as the 10-year rolling window standard deviation of the series. Data is from 1950 to 2020.

[55]The results hold when using a detrended price-dividend ratio, no matter the detrending method.

[56]Some of the papers uses the log(PD) and then, focus on percentage changes. However, that is more a requirement of the log-linearization approach to derive the price-dividend variance decomposition than a claim in favour of the relevance of the percentage changes.

[57]Results are robust to detrend the variables.

The rising volatility phenomenon is masked when using relative variables (as returns, price growth and even price-dividend growth) at lower frequency. That signals the existence of some highly volatile but short-lived events, which when aggregated over a month or quarter they partly offset each other. All in all, the rise in volatility showed consistently by the different measures is just a statistical way of capturing the increase in the frequency and magnitude of well known stock market booms since the 1980s (the late 80s crashes, the Dot-com bubble and the Great Recession and its aftermath).

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Chapter 2

Heterogeneous Expectations and Stock Market Cycles

PAU BELDA & ADRIAN IFRIM

Abstract

We present a model of expectations that micro-founds the heterogeneous extrapolation and the persistent and procyclical disagreement present in survey data. Extrapolation arises from imperfect knowledge about price formation that pushes agents to learn, in a Bayesian sense, from price news. However, optimists extrapolate more since they are more confident about the signal-to-noise content of price news. This makes disagreement procyclical. Besides, agents hold idiosyncratic views about long-run asset growth, which generates persistent disagreement. The subjective belief system is embedded in an otherwise standard asset pricing framework, which can then quantitatively account for the dynamics of prices and trading. In the model, learning from prices leads to disagreement and trading, which reshuffles the distribution of wealth between lower- and higher-propensity-to-invest agents, affecting aggregate demand and prices. This feedback loop complements the expectations-price spiral typical of models with extrapolation, placing heterogeneity and trading as key drivers of price cycles.

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2.1. Introduction

The purpose of this paper is to provide an asset pricing model with heterogeneous beliefs that can replicate basic facts about survey beliefs along with aggregate dynamics involving prices and trading. This framework allows us to shed light on the expectation formation process at the individual level and how it shapes the aggregate dynamics of the stock market.

An increasing amount of the recent asset-pricing literature has emphasized the importance of understanding how investors form beliefs and the implications for asset pricing. One of the reasons for this focus on expectation formation is the evidence coming from survey data that shows significant departures from the Rational Expectations (RE) hypothesis.[1] The opening quote is taken from the latest NBER asset pricing program agenda for future research, which clearly points out the importance of incorporating realistic belief systems in asset pricing models. We seek to contribute to this enterprise by presenting empirical facts about survey beliefs, proposing a model of expectations that replicates them and exploring their implications for asset pricing.

Two deviations from Rational Expectations have been extensively documented: people tend to extrapolate from recent events (Greenwood and Shleifer, 2014); consensus beliefs under-react to new information (Coibion and Gorodnichenko, 2015), but individual agents over-react (Bordalo et al., 2020). Recently, Giglio et al., 2021 added a third dimension: investors' subjective expectations are characterized by persistent heterogeneity across agents ("individual fixed effects"), which cannot be explained by observables such as wealth, age, gender or past returns. Thus, the expectations coordination implied by RE is strongly rejected.

Based on this evidence, we use the cross-section of individuals from the UBS Gallup survey to build sentiment groups that replicate this persistent heterogeneity and document several facts. First, all agents extrapolate, but the optimists do it much more. Second, disagreement is always high without large variations, which we refer to as "perpetual disagreement". However, it exhibits meaningful dynamics: it comoves positively with prices and trading (as shown in early research, for instance, Vissing-Jorgensen, 2003, Adam, Beutel, et al., 2015) and is mostly driven by optimists.

We propose a model of expectations that is consistent with these facts. We conjecture that agents have imperfect knowledge about price formation, in line with the Internal Rationality literature (Adam and Marcet, 2011). They cope with this imperfect information by using a statistical model of prices that generalizes the RE model. They use this model to form price expectations and, as Bayesian learners, update them when new information about prices comes up.

Agents differ in two dimensions. First, they hold different views on mean price growth, which we interpret as beliefs about the long-run asset's value. Thus, beliefs

[1] See Adam, Nagel, et al., 2022 for a review.

fluctuate around this long-run value with the short-run dynamics arising from learning about prices. This subjective long-run growth is a micro-foundation of the statistical fixed-effect documented by Giglio et al., 2021 and the perpetual nature of the disagreement. Besides, investors differ in their speed of learning, reflecting a different way of processing public information; some are more confident, believing that that information has a high signal-to-noise ratio, and others are more skeptical about it. This heterogeneity in the processing of information can be related to two empirical observations: the different degrees of extrapolation that we document and the comovement between disagreement and prices.[2]

We embed this expectation formation process into an otherwise standard Lucas, 1978 model. Apart from the price-expectations spiral typical of models of learning about prices that generates recurrent price booms and busts, the model features an additional mechanism: a feedback loop between prices and trading. Price news provoke more disagreement, as agents process the information in a heterogeneous way. This generates trading, since investors who value stocks relatively more after the price news will buy them from investors who value them less. Trading triggers a redistribution of wealth between investors with different propensities to invest, affecting aggregate demand and prices. Hence, learning connects prices to expectations; heterogeneous learning connects prices to disagreement and trading; trading changes the distribution of stocks, influencing aggregate demand and prices. Altogether, trading emerges as a key driver of asset prices, breaking with the mainstream theory that explains asset pricing without any reference to trading dynamics.

An example of boom dynamics would be as follows. It starts with an aggregate exogenous factor (e.g. goods news, extraordinary incomes) that makes some investors more willing to invest in the stock market. This generates a rise in prices which turns all investors more optimistic, raising demand and prices further over time in a reinforcing manner. Nevertheless, not all investors react equally to the rise in prices; some are more conservative than others, interpreting the news as containing more noise, and forgoing the wave of optimism. This heterogeneous reaction implies an increase in disagreement, which leads to trading: optimists will buy from pessimists. Trading moves resources from lower- to higher-propensity-to-invest agents, which raises aggregate demand and prices, restarting the process.

Our framework also allows us to investigate the contribution of different sentiment groups to booms and busts in price cycles. Through the lens of our model, the positive correlation between disagreement and prices that we observe in booms is driven by optimists becoming more optimistic and not pessimistic agents adjusting their beliefs upward. In this regard, managing capital gain expectations for the most optimistic agents is crucial

[2]It turns out optimists are more confident such that, when prices are high, they become even more optimistic in relation to other groups, widening the disagreement.

for leaning against the wind policies in reducing the inefficiencies created by belief-driven cycles.[3]

To the best of our knowledge, this is the first paper to jointly replicate quantitatively the distribution of subjective beliefs along with price and trading dynamics in the context of the stock market. The literature on belief heterogeneity and asset pricing is vast. Nevertheless, most of the literature has not provided a realistic quantitative evaluation yet. Atmaz and Basak, 2018 is an example of a theoretical model of heterogeneous beliefs that is able to replicate several of the stylized facts observed in the data. In contrast to that framework, in which agents possess beliefs about fundamentals (dividends), we work with expectations on expected return, which allows us to compare the model directly with survey data and evaluate the quantitative performance of the model. On a similar note, WR Martin and Papadimitriou, 2022 develop a model with heterogeneous beliefs about probabilities of good/bad news in which sentiment is another source of risk fully internalized by agents and which stimulates speculation and volatility. See Simsek, 2021 for a comprehensive review of the literature on heterogeneous beliefs about asset prices.

The rest of the paper is organized as follows. Section 2 presents several stylized facts regarding the empirical survey's distribution of beliefs. Section 3 lays out a model of expectations in line with the evidence embedded in a theoretical asset pricing model. Section 4 shows the quantitative performance and the mechanism through which heterogeneous beliefs drive asset price cycles. Section 5 concludes.

2.2. Stylized Facts about Heterogeneous Expectations

We use the Gallup survey on future stock market return expectations of individual investors for the period 1998Q2-2007Q4. We choose this survey because it includes the most respondents per period (around 700), which should bring more reliability in capturing the heterogeneous dynamics of expectations.[4] We first split the distribution of beliefs into sentiment groups based on the level of optimism/pessimism of individual investors regarding future returns. Specifically, we order the distribution of beliefs across agents at each point in time in three subgroups ranked by their level of optimism and compute averages for each group. Although our data are not a panel, the evidence from Giglio et al., 2021 shows that beliefs are persistent over time, meaning that optimists remain optimists and pessimists remain pessimists without interchanging, which is robust to other surveys,

[3] Belief-driven asset price cycles can impact the real economy through multiple channels: see Ifrim, 2021 for demand side inefficient wealth effects, Winkler, 2020 for supply side with financial frictions or Belda, 2023 for supply side due to investment adjustment costs.

[4] An alternative option is to work with the RAND dataset, which is a panel. One of the shortcomings is that RAND responses are coded in categories and need some assumptions to convert the answer to a continuous variable. We are currently working with the RAND dataset to test the robustness of the facts.

as the RAND panel. Given this fact, we argue that the mean of each sentiment group captures reasonably well the heterogeneity of expectations of each group and proceed with this caveat in mind.

First, we study the features of these group-level expectations. Figure 2.1 presents the evolution over time of the sentiment groups, with S_1 being the most pessimistic, S_2 the average investors and S_3 representing the sentiment group of agents with the most optimistic beliefs. At the top of the dot-com bubble, optimists were expecting as high as 30% yearly returns, while pessimists only expected 7%. Sentiment groups' beliefs are highly correlated with each other (0.8-0.95).

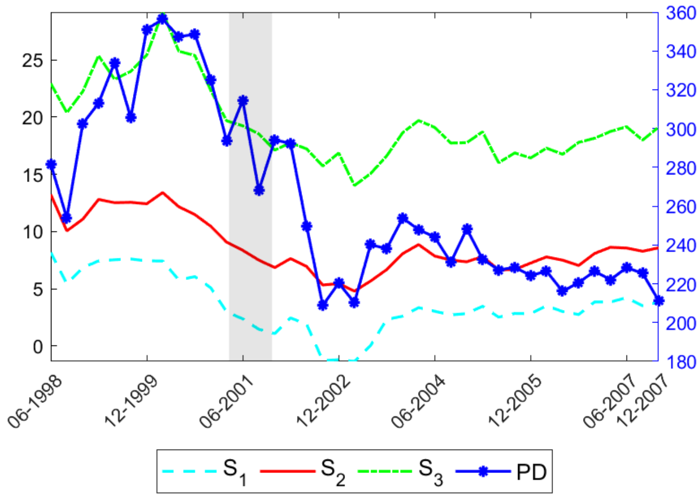


Figure 2.1. **Return expectations by sentiment Groups.** Each sentiment group represents the average return expectation at each point in time across agents depending on the position in the distribution (eg. S_1 represents the average of the beliefs between 0 and $\frac{1}{3}$ percentiles); shaded bars denote NBER recessions.

Extrapolation. At first sight, figure 2.1 suggests a positive comovement between survey expectations and prices: investors are more optimistic during the boom and more pessimistic at the bust. This eyeball test suggests the existence of extrapolation, possibly to a different degree for each group. To formally test this possibility, we run the RE test proposed in Adam, Marcet, and Beutel, 2017 for each group.[5] The test implies running

[5] This test is similar to the extrapolation test used by Kohlhas and Walther, 2021. They collapse the two equations into a single one by subtracting the first from the second line and studying the sign of $c - c$. We use the version with two equations as it delivers more information.

the following two regressions

$$\begin{aligned} \mathbb{E}_t^s[R_{t,t+n}] &= a + c\ PD_t + u_t + \mu_t \\ R_{t,t+n} &= a + c\ PD_t + _t \end{aligned} \tag{2.1}$$

where $\mathbb{E}_t^s$ represents survey expectations regarding future returns at time t, PD_t is the Price Dividend ratio and $R_{t,t+n}$ is the realized return between t and $t + n$. Moreover, u_t and $_t$ represent variations in survey expectations and returns due to factors other than the PD ratio and μ_t captures measurement error in survey expectations, which is assumed to be uncorrelated with the previous two exogenous variations. The RE test is basically a test of equality between c and c. Results from table 2.8 indicate that the RE hypothesis with respect to survey expectations on capital gains is rejected at the 1% significance level for each one of the three sentiment groups.[6]

	c	c	*p-value* H_0: c= c
p_{0-33}	0.0576***	-0.2423 ***	0.0000
p_{33-66}	0.0545***	-0.2423***	0.0000
p_{66-100}	0.0809***	-0.2423***	0.0000

Table 2.1. **RE Tests across different sentiment groups:** *p_{0-33} denotes the sentiment group whose expectations lie between the 0 and 1/3 percentile. The data in each group is aggregated by taking the average of that particular group at each point in time. Data used for this particular test is the Gallup UBS survey data for all individuals' expected stock market return. Estimates are based on asymptotic theory and have been adjusted for small sample bias. *** denotes significance at the 1% level.*

However, the point estimates indicate important differences, especially for optimists. To check this we run a test of equality among the coefficients, c^s with $s = 1 : 3$, among different sentiment groups and present the *p-value* in the following table. Results indicate that the sensitivity of expectations to the PD ratio for pessimists and moderates is statistically identical. Nevertheless, optimists exhibit a higher coefficient, c^3, that is significantly different from the other two groups, suggesting a higher degree of extrapolation.

H_0:	$c^1 = c^2$	$c^1 = c^3$	$c^2 = c^3$
p-value	0.3630	0.0160**	0.0000***

Table 2.2. **Equality tests for coefficients c across sentiment groups:** See footnote for table 1 for additional details. *** denotes significance at the 1% level and ** at the 5% level

Disagreement dynamics. Our preferred measure of disagreement/dispersion of beliefs is defined by the difference between the beliefs held by the most optimistic/ pessimistic groups.[7] For three sentiment groups, this measure is defined as $DI_{33}^{33} = S_3 - S_1$.

[6] The results are unchanged if, instead of three sentiment groups, we consider two or four, see Appendix 1 for results on RE tests based on different partitions of the distribution of subjective returns.

[7] A similar measure has been used by Giacoletti et al., 2018 to measure disagreement in bond markets.

Figure 2.2 presents the evolution of disagreement together with the PD ratio. Disagreement about future stock returns tends to be high near the top of the price cycle and highly correlated with the PD ratio (0.7). Moreover, subjective beliefs are characterized by persistent positive disagreement with a mean of approximately 16%, in line with the evidence from Giglio et al., 2021 on the existence of individual fixed effects in the cross-section of beliefs.

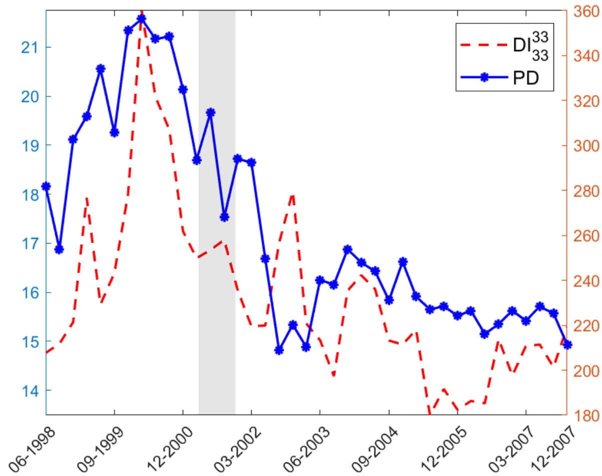


Figure 2.2. **Disagreement and PD ratio.**

The next figure shows disagreement computed both as the inter-group standard deviation and as the difference between the 90th and 10th percentile (DI_{10}^{10}). These measures behave very similarly to our benchmark specification with correlation coefficients higher than 0.9. This suggests that the dynamics of disagreement is not sensitive on the exact measure used but instead is fundamentally rooted into the data. Table 2.3 collects several stylized facts about the heterogeneity of beliefs and their interaction with aggregate variables.

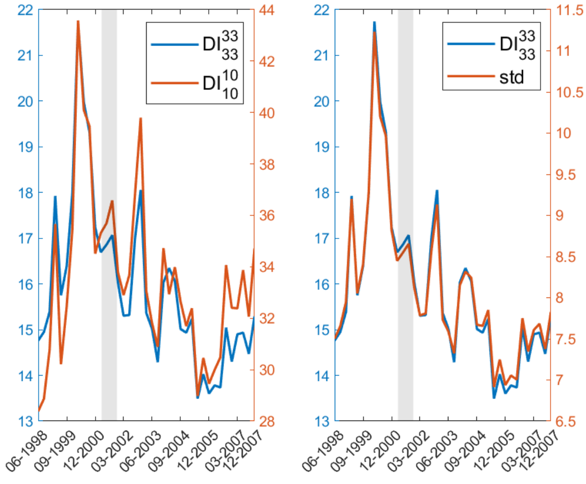


Figure 2.3. **Alternative measures of disagreement.**

Fact	Statistic	Value
1. Persistence of expectations	ρ	0.90
2. Extrapolation of mean capital gains expectations	$corr(\beta_t, PD)$	0.82
3. Heterogeneous extrapolation	c^1	0.0576
	c^2	0.0545
	c^3	0.0809
4. Perpetual disagreement	$\mathbb{E}(DI)$	0.04
	$\sigma(DI_t)$	0.0044
5. Disagreement led by i) optimists	$corr(DI_t, S_t^3)$	0.73
ii) pesimists	$corr(DI_t, S_t^1)$	0.36
6. Disagreement procyclicality	$corr(DI_t, PD_t)$	0.72
7. Comovement disagreement-trading	$corr(DI_t, TV_t)$	0.41
8. Correlation among sentiment groups	$corr(S^1, S^2)$	0.95
	$corr(S^1, S^3)$	0.87
	$corr(S^2, S^3)$	0.95

Table 2.3. **Facts on Subjective Expectations.**

2.3. A model with heterogeneous expectations

In this section, we present an asset pricing model with heterogeneous beliefs consistent with the empirical evidence from the previous section. We begin by suggesting a model of expectations that replicates the previous facts. Then, we embed this piece in an asset pricing model à la Lucas, 1978, with Internal Rationality following Adam, Marcet, and Beutel, 2017.

2.3.1. A model of expectations

In light of the previous evidence, we conjecture a model of learning about prices with different layers of heterogeneity. Investors from the sentiment group i possess the following subjective model about stock prices

$$\begin{aligned} lnP_t &= lnP_{t-1} + b_t^i + ln\varepsilon_t^{P,i} \\ b_t^i &= (1-\rho_i)\bar{\beta}_i + \rho_i b_{t-1}^i + ln\nu_t \end{aligned} \tag{2.2}$$

where b_t^i represents the permanent price growth component and $\varepsilon_t^{P,i}$ a transitory innovation. The permanent component, b_t^i, follows an auto-regressive process with persistence ρ_i and mean $\bar{\beta}_i$. The latter represents the perceived long-term mean of stock price return of sentiment group i. We interpret it as a subjective view of the perceived long-term growth of the asset value. Innovations $ln\varepsilon_t^P$ and $ln\nu_t$ are jointly normal but uncorrelated. The noisy price component is comprised of two independent components

$$ln\varepsilon_t^{P,i} = ln\varepsilon_{t+1}^{P1,i} + ln\varepsilon_t^{P2,i}. \tag{2.3}$$

where $ln\varepsilon_t^{Pj,i} \sim \mathcal{N}\left(\frac{-\sigma^{Pj}}{2}, \left(\sigma^{Pj}\right)^2\right)$ with $j = 1, 2$. We assume further that only $ln\varepsilon_t^{P1,i}$ is observed at time t. The permanent price growth component, b_t, is unobserved and is estimated optimally using the available information from price signals. Given their belief system from equation 2.2, the optimal posterior distribution of the permanent component of prices is

$$b_t^i \sim \mathcal{N}(\beta_t^i, (\sigma^i)^2) \tag{2.4}$$

where σ^2 is the steady state variance of the posterior, and β_t^i is the conditional mean. The latter is evolving according to the Kalman updating equation

$$\beta_t^i = (1-\rho_i)(1-g^i)\bar{\beta}_i + \rho_i\beta_{t-1}^i + g^i(lnP_{t-1} - lnP_{t-2} - \rho_i\beta_{t-1}^i) + g^i ln\varepsilon_t^{P1,i} \tag{2.5}$$

where g^i represents the steady state Kalman gain, entailing different views on the signal-to-noise ratio of the price signals. The shock $ln\varepsilon_t^{P1,i}$ will be interpreted as an information shock to the beliefs of agents from group i.

Qualitatively, equation (2.5) contains elements that might replicate the key observations from surveys: the heterogeneous long-run views about the fundamental value of the asset can be linked to the individual fixed-effects and the perpetual disagreement; the different views about the signal-to-noise ratio of the price signals can lead to different degrees of extrapolation; the persistence parameter can be directly linked to the persistence from the survey; the fact that all agents use the same price information would generate a high comovement between sentiment groups. To quantitatively test whether this equation is a reasonable description of the survey evidence, we estimate it for each sentiment group.[8] We estimate the parameters by NLS for each sentiment band individually and present the results in the following table.[9]

Sentiment group i	1	2	3
g^i	0.0139	0.0204	0.0301
	(0.0025)	(0.0006)	(0.007)
ρ_i	0.90	0.90	0.91
	(0.0013)	(4.4e-5)	(0.0013)
$\bar{\beta}^i$ (in %)	-0.50	1.01	4.79
	(0.14)	(0.11)	(0.5)

Table 2.4. **Estimated Learning Parameters.** Parameters have been estimated by non-linear least squares; bootstrap standard errors in parentheses calculated by a sieve bootstrap method over 1000 simulations using AR(p) innovations with order p chosen by the AIC criterion.

Table 2.4 shows that the speed of learning (g^i) is increasing with the sentiment band, with optimists (S^3) having the highest learning parameter.[10] On the other hand, the persistence is similar among these groups and the measure of long-term heterogeneity increases in optimism as expected. Figure 2.4 shows the fit for each sentiment band.

Altogether, the different heterogeneity layers on the expectations formation process allow for capturing salient features of surveys. First, different long-run views β^i give rise to a perpetual disagreement: optimistic investors are always more optimistic than pessimists. This is a way of micro-found the statistical fixed-effect reported by Giglio et al., 2021, that respects the observation that this parameter is unrelated to investors' profile. Second, investors extrapolate news at different intensities g^i: some react faster, and others are more conservative. This difference is in line with heterogeneous extrapolation and relates

[8]We transform the UBS survey return expectations into price growth using the following identity: $R_{t+1} = \frac{P_{t+1}}{P_t} + \beta^d \frac{D_t}{P_t}$ where β^d is the expected quarterly dividend growth which we set equal to 1.0048. The resulting nominal capital gain data is transformed into real series by subtracting SPF inflation forecasts.

[9]Appendix 1 presents the bootstrap distributions of these estimated parameters.

[10]Using the same survey data as us, Adam, Beutel, et al., 2015 show that the constant gain parameter is inversely related to investors experience of investors with low experience investors having the largest parameter. According to this evidence, the optimist investors are mostly characterized by low experience while the reverse is true for pessimists.

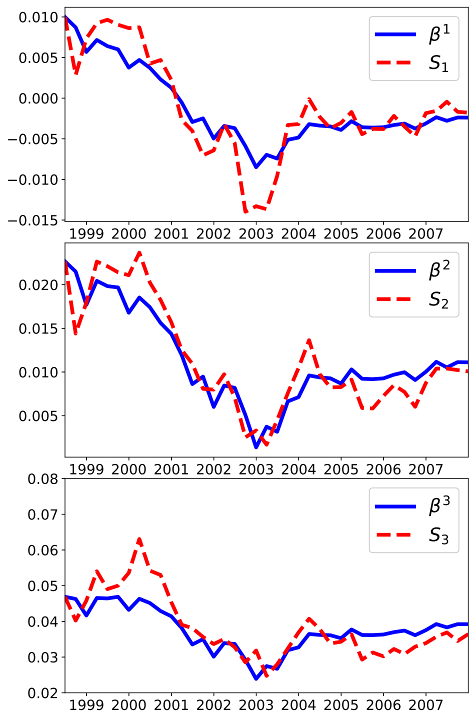


Figure 2.4. **Model fit from equation 2.5.** The equation has been estimated by non-linear least squares by minimizing for each sentiment group $\sum(S_i - \beta^i)^2$.

disagreement to price dynamics (see Section 4.2.). relates disagreement to price dynamics; in good times, disagreement will tend to rise, in line with the procyclicality observed in the data.

2.3.2. An asset pricing model

Consider an endowment economy populated by M types of agents, $i \in [1, M]$, who solve the following utility maximization problem

$$\max_{\{C_t^i, S_t^i\}_{t=0}^{\infty}} \mathbb{E}_0^{\mathcal{P}_i} \sum_{t=0}^{\infty} \delta^t \frac{(C_t^i)^{1-\gamma}}{1-\gamma}$$

$$s.t. \tag{2.6}$$

$$C_t^i + P_t S_t^i \leq (P_t + D_t) S_{t-1}^i + W_t^i$$

$$\underline{S} \leq S_t^i \leq \bar{S}$$

where C denotes consumption, W income (wages) that agents receive, S the amount of stock holdings in the risky asset with price P that pays exogenous dividend D. $\mathcal{P}_i$ represents the probability measure of agents of type i. We assume that the risky asset, which we interpret as stocks, is in fixed supply $S^s > 0$. The share of each agent in the population is equal to μ_i with $\sum_{i=1}^{M} \mu_i = 1$.

Exogenous processes. Following Adam, Marcet, and Beutel, 2017, we specify in a similar way the exogenous processes for dividend growth and wage-dividend ratio to obtain empirical plausible processes for dividends, consumption and consumption to dividend ratio.

1. Dividends: grow at a constant rate a with *iid* growth innovations $ln\varepsilon_t^D$ to be described further below

$$lnD_t = lna + lnD_{t-1} + ln\varepsilon_t^D. \tag{2.7}$$

2. Wage-dividend ratio: follow an AR(1) process with persistence p, mean $1 + WD$ and innovation $ln\varepsilon_t^W$

$$ln\left(1 + \frac{W_t^i}{D_t}\right) = (1-p)ln(1 + WD) + p\ ln\left(1 + \frac{W_{t-1}^i}{D_{t-1}}\right) + ln\varepsilon_t^{W,i}. \tag{2.8}$$

where innovations are given by the following exogenous processes

$$\begin{pmatrix} ln\varepsilon_t^D \\ ln\varepsilon_t^{W,i} \end{pmatrix} \sim \mathcal{N}\left(-\frac{1}{2}\begin{pmatrix} \sigma_D^2 \\ \sigma_W^2 \end{pmatrix}, \begin{pmatrix} \sigma_D^2 & \sigma_{DW} \\ \sigma_{DW} & \sigma_W^2 \end{pmatrix}\right), \tag{2.9}$$

$$\begin{pmatrix} ln\varepsilon_t^{W,i} \\ ln\varepsilon_t^{W,-i} \end{pmatrix} \sim \mathcal{N}\left(-\frac{1}{2}\begin{pmatrix} \sigma_W^2 \\ \sigma_W^2 \end{pmatrix}, \begin{pmatrix} \sigma_D^2 & \sigma_{WW} \\ \sigma_{WW} & \sigma_W^2 \end{pmatrix}\right). \tag{2.10}$$

Agents' Belief System. Agents are endowed with full knowledge of the law of motions for dividends and wages given by equations (2.7) and (2.8). However, we endow agents with imperfect knowledge regarding how stock prices evolve and the exact mapping from fundamentals to prices. To forecast prices, they use the price model (2.2) with subjective mean beliefs evolving according to (2.5).

Equilibrium. It consists of sequences of prices $\{P_t\}_{t=0}^{\infty}$ and allocations $\{C_t, S_t\}_{t=0}^{\infty}$ such that:

1. Given their belief system and exogenous processes, agents optimally solve their optimization problem 2.6.

2. Markets clear

   - Goods market: $\sum_{i=1}^{M} \mu_i C_t^i = D_t S^s + \sum_{i=1}^{M} \mu_i W_t^i$
   - Stock market: $\sum_{i=1}^{M} \mu_i S_t^i = S^s$

Recursive Solution via the Parameterized Expectations Algorithm: A recursive solution boils down to a time-invariant stock demand function $S_t = S(X_t)$.[11] We solve the model using the PEA approach first proposed by Belda, 2023 and extended in the appendix of this chapter. The idea is to numerically approximate the stock policy function via a function grounded on economic theory. Following the solution for exogenous i.i.d. returns derived in Hakansson, 1970, we propose the following approximation function for the stock demand function:

$$S_t^i \approx \chi^i \beta_t^i \frac{\left(WD_t^i + (PD_t + 1)S_{t-1}^i\right)}{PD_t} = \chi^i \beta_t^i Z_t^i, \tag{2.11}$$

where χ is the unique parameter of the approximating function to be estimated. This function says that stock demand is the product of two elements: $\chi^i \beta_t^i$, which can be read as a marginal propensity to invest, and Z_t^i, which are the resources of the agent i. Appendix 2 contains a detailed explanation of this approach to solving models with learning.

One of the advantages of this approach is that the stock market clearing condition

$$\sum_i \mu^i S_t^i \left(\frac{P_t}{D_t}, \cdot\right) = \bar{S} \tag{2.12}$$

can be solved for the P/D ratio in closed-form. Equilibrium prices read as

$$\frac{P_t}{D_t} = \frac{\sum_{i=1}^{M} \mu^i \chi^i \beta_t^i \left(\frac{W_t^i}{D_t} + S_{t-1}^i\right)}{\bar{S} - \sum_{i=1}^{M} \mu^i \chi^i \beta_t^i S_{t-1}^i}, \tag{2.13}$$

where χ^i is the only parameter of the approximation function. Thus, equilibrium prices depend on the distribution of expectations and wealth across agents. Of course, a potential cost is that the approximating function is not very flexible, as compared with arbitrary order polynomials or neural networks; however, it turns out to perform very well, with Euler Equation errors equivalent to \$1 out of a million.

[11] Adam, Marcet, and Beutel, 2017 proved the existence of a recursive equilibrium in the same model with homogeneous expectations. We assume it continues to hold in this setup.

Connections to demand-system asset pricing. A recent approach in quantitative asset pricing, pursued by Koijen and Yogo, 2019, is to estimate characteristic demand functions for different types of investors while allowing for heterogeneity in beliefs. Specifically, the authors estimate the following equation for each type of investor

$$\delta_{i,t}(n) = exp\left(\beta_{0,i,t}ME_t(n) + \sum_{k=1}^{K-1}\beta_{k,i,t}x_{k,t}(n) + \beta_{k,i,t}\right)_{i,t}(n) \tag{2.14}$$

where $\delta_{i,t}(n)$ is the demand or portfolio share of investor i in stock n, ME denotes market equity and $x_{k,t}$ is an individual characteristic of the stock among $K-1$ total characteristics (e.g. book value). The last term from the equation, $_{i,t}(n)$, is interpreted by the authors as latent demand related to heterogeneous beliefs of each individual investor i. They show that this last term explains over 80% of the variance of stock returns.

Returning to our asset pricing framework, equation (2.11) can be rewritten as

$$S_t^i = exp(z_t)\beta_t^i\chi^i \tag{2.15}$$

where lower-case variables denote variables in logs. Two observations are in place. First, in our case, the latent demand is exactly given by the marginal propensity to invest, which is a scaled version of capital gains expectations. Secondly, the fundamental demand is determined by the wealth of each investor. One important difference between our approach and the one in Koijen and Yogo, 2019 is that while the latter focuses on the portfolio choice among a universe of assets, we focus here on the aggregate stock market. Nevertheless, the aggregate demand of stocks exhibits a similar functional form in which latent demand or beliefs multiply fundamental demand.

2.4. Quantitative Analysis

In this section, we evaluate the quantitative performance of the model in replicating the stylized facts about the heterogeneity of beliefs and stock market cycles and then, use the model to examine the role of heterogeneity in driving the cycles.

2.4.1. Model performance

We start by calibrating the model parameters. We assume that there are three types of agents in our model, $M = 3$ and set their share μ_i equal to $\frac{1}{3}$. Since our model is an extension of the one from Adam, Marcet, and Beutel, 2017 we approach the calibration of most of the parameters in a similar way except for the parameters concerning the dynamics of the three sentiment groups (ρ^i, g^i and β^i), which are set according to the empirical evidence presented in the previous section. We calibrate the stock supply of stocks, S^s, such that to obtain a reasonable average price-dividend ratio while the parameter for the covariance of income shocks, σ_{WW}, implies a correlation of around 0.3 among these shocks. Table 2.5 gathers the calibrated parameters in our model.

Parameter	Symbol	Value
Discount factor	δ	0.995
Mean dividend growth	a	1.0048
Dividends growth standard deviation	σ_D	0.0167
Wage-dividends shocks standard deviation	σ_W	0.0167
Covariance (wage-dividend, dividend)	σ_{WD}	0.000351
Covariance wage-dividends agents	σ_{WW}	0.009
Persistence wage-dividend process	p	0.96
Average consumption-dividend ratio	$1+WD$	23
Std of transitory component	$\sigma^{P1} = \sigma^{P2}$	0.04
Risk aversion parameter	γ	2
Stock Supply	S^s	3.3
Expectations persistence	ρ^i	Table 2.4
Learning speed	g^i	Table 2.4
Long run view on asset long-run fundamental growth	β^i	Table 2.4

Table 2.5. **Benchmark calibration.** This table reports the values of the model parameters used for the quantitative analysis.

We introduce the quantitative performance in table 2.6 for three specifications of the model. The first one (column 4) represents our benchmark calibration with heterogeneous income and information shocks, in the second one (column 5) we shut off information shocks ($ln\varepsilon_t^{P1,i} = 0$), while the third specification (column 6) assumes homogeneous wages ($\varepsilon_t^{W,i} = \varepsilon_t^{W}$). On top of the statistics regarding the heterogeneity of expectations from table 2.3 we also present stylized facts about the trading behaviour (panel III) and aggregate stock market behaviour (panel IV).

			Model		
Fact	Statistic	US data	Benchmark	$\ln\varepsilon_t^{P1,i} = 0$	$W_t^i = W_t$
I. Expectation Heterogenity					
Expectations persistence	$corr(\beta_t, \beta_{t-1})$	0.90	0.91	0.91	0.88
Correlation among sentiment groups	$corr(\beta_t^1, \beta_t^2)$	0.96	0.87	1	0.49
	$corr(\beta_t^1, \beta_t^3)$	0.87	0.86	1	0.46
	$corr(\beta_t^2, \beta_t^3)$	0.95	0.87	1	0.46
Expectations procyclicality	$corr(PD_t, \beta_t)$	0.82	0.66	0.66	0.41
	$corr(PD_t, \beta_t^3)$	0.86	0.66	0.66	0.31
	$corr(PD_t, \beta_t^1)$	0.7	0.66	0.66	0.34
II. Disagreement					
Disagreement driven by beliefs	$corr(DI_t, \beta_t^3)$	0.73	0.94	0.99	0.88
	$corr(DI_t, \beta_t^1)$	0.36	0.63	0.99	0
Perpetual disagreement	$\mathbb{E}(DI_t)$	0.04	0.04	0.04	0.04
	$\sigma(DI_t)$	0.0044	0.0047	0.0037	0.0032
Disagreement procyclicality	$corr(DI_t, PD_t)$	0.72	0.53	0.39	0.54
III. Trading					
Comovement disagreement-trading	$corr(DI_t, TV_t)$	0.41	0.24	0.26	0.36
Trading driven by beliefs	$\hat{\beta}(\lvert\Delta S_t^1\rvert, \lvert\Delta\beta_t^1\rvert)$	0.2*	0.2	0.15	0.04
	$\hat{\beta}(\lvert\Delta S_t^2\rvert, \lvert\Delta\beta_t^2\rvert)$	0.2	0.012	-0.01	0.14
	$\hat{\beta}(\lvert\Delta S_t^3\rvert, \lvert\Delta\beta_t^3\rvert)$	0.2	0.047	0.02	0.25
IV. Stock Prices					
Mean Price-Dividend	$\mathbb{E}(PD_t)$	154.86	173	173	159
Price-Dividend volatility	$\sigma(PD_t)$	64.42	55	55	13
Price-Dividend persistence	$\rho(PD_t, PD_{t-1})$	0.98	0.96	0.96	0.96
Mean returns	$\mathbb{E}(r_t)$	1.89	1.015	1.015	1.01
Returns volatility	$\sigma(r_t)$	7.70	9.2	9.1	3.8

Table 2.6. **Model quantitative performance.** This table reports the statistics of the model together with the US data for the period 1973:I-2019:IV for prices and returns and 1998:II-2007:IV for expectations-related and trading statistics. Model implied statistics are obtained via a long simulation with T=10.000 periods; $\hat{\beta}(Y, X)$ denotes the OLS regression coefficient between Y and X; *estimate from Giglio et al., 2021

The benchmark calibration captures well all of the stylized facts, including the heterogeneity of expectations, the nature of the disagreement, trading behaviour and the excess volatility of the stock price cycles. Our model produces highly correlated beliefs among sentiment groups and positive co-movement between expectations and prices. Expectations shocks contribute to reducing the co-movement between beliefs, as can be seen when comparing with the calibration excluding sentiment shocks (column 5). The mean and volatility of disagreement match exactly those observed in the data and reproduce the positive correlation with prices.

Moreover, similarly to the data, the expectations of optimists exhibit a stronger correlation with disagreement compared to the pessimist group. As argued in the next section, the positive co-movement between prices and disagreement is driven largely by optimists becoming more optimistic, increasing trading, prices and disagreement. Panel III shows that disagreement is also positively related to trading and that changes in beliefs do not lead to trading, consistent with the empirical evidence presented in Giglio et al., 2021. Finally, panel IV documents that our model replicates closely aggregate stock market volatility and persistence.

We also run the same RE test on a simulated sample from the model under the baseline calibration. Table 2.7 which is directly comparable with table 2.1, reveals that the model is able to replicate the stylized facts from Section 2: high prices predicts negative future returns but positive beliefs about them; this extrapolation is stronger for the most optimists (c^3 is significantly different from c^2 and c^1).

	c	c	*p-value* H_0: c= c
p_{0-33}	0.0147***	-0.1726 ***	0.0000
p_{33-66}	0.0225***	-0.1726***	0.0000
p_{66-100}	0.0329***	-0.1726***	0.0000

Table 2.7. **RE Tests on model simulated data across different sentiment groups.** The sample consists of 7000 observations simulated from the model under the baseline calibration; *** denotes significance at the 1% level.

Figure 2.5 plots one simulation arising from the calibrated model. Notice that although different sentiment groups have persistently different beliefs, stock holdings vary across agents, and there is not only one group holding the largest/smallest amount of stocks. Instead, agents with the largest/smallest equity holdings alternate among sentiment groups over time.[12]

2.4.2. Dissecting stock market dynamics

In this section, we highlight the key mechanisms behind to joint evolution of prices, trading and expectations. The cycle starts with an exogenous factor (e.g. particular news (the "expectations shock") or extraordinary incomes (the "wage shock")) that make some investors more willing to invest in the stock market. This generates a rise in prices which turns all investors more optimistic, creating amplification over time. Nevertheless, not all investors react equally to the rise in prices due to their different expectation formation processes; some are more conservative than others, interpreting the news as containing

[12] This is an observation from the UBS dataset that also matches Giglio et al., 2021: there is no clear mapping between the distribution of wealth and the distribution of expectations. Our model features that.

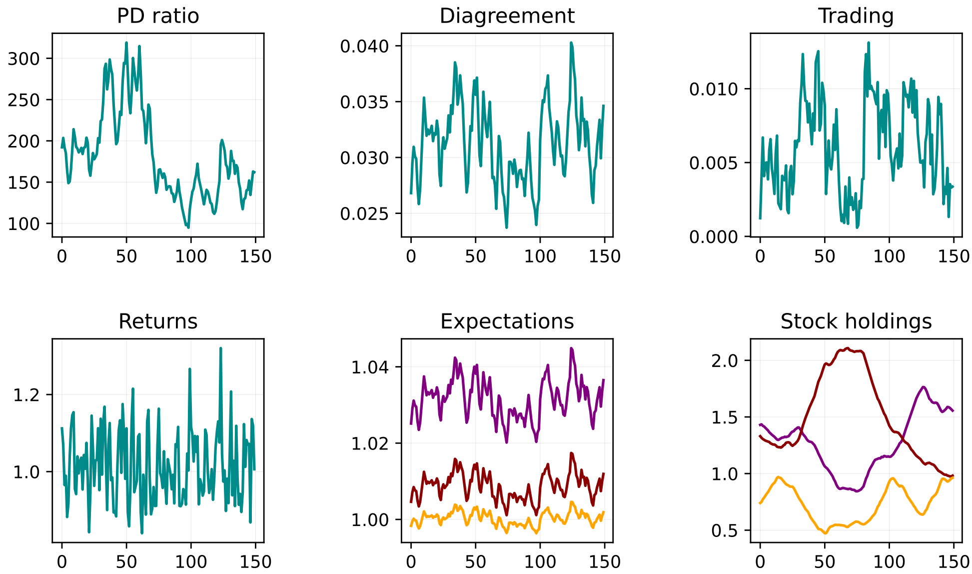


Figure 2.5. **Simulation of 150 periods based on the benchmark model.**
In the last two graphs, purple lines are for optimists, red for moderates and orange for pessimists.

more noise and then updating their expectations less. Thus, the heterogeneous reaction of expectations to prices increases disagreement and trading. Trading reshuffles the wealth distribution, moving resources from low to high propensity to invest agents, raising aggregate demand and prices. We first highlight the mechanisms at play and then resort to simulates to illustrate them.

2.4.2.1. Mechanisms

Three mechanisms intervene in these dynamics.

Mechanism 1: expectations-price spiral. From the equilibrium P/D ratio (equation 2.13), it follows

$$\frac{P_{t-1}}{P_{t-2}} = f_1\left(\{\beta^i_{t-1}, \beta^i_{t-2}\}_{i=1}^{M}, \cdot\right) \tag{2.16}$$

and from the expectations law of motion (equation 2.5) it is clear that

$$\beta^i_t = f_2\left(\frac{P_{t-1}}{P_{t-2}}, \cdot\right). \tag{2.17}$$

Other things equal, these two equations constitute a feedback loop that produces endogenous price cycles as a result of self-fulfilling prophecies. An increase in optimism would raise stock demand and prices which would confirm the initial optimistic expectations (or even overcome them, rescaling the process upwards). This feedback loop is a mechanism capable of replicating the high observed volatility of stock prices. This mechanism has been exploited in the learning about prices literature, mostly focusing on the homogeneous beliefs case (see Adam, Marcet, and Nicolini, 2016).

Mechanism 2: heterogeneous expectations and disagreement. Based on survey evidence, we introduce idiosyncratic long-run expectations, which are characterized by two parameters: the long-run view β^i and its weight on current expectations ρ^i. However, according to survey data, only β^i is significantly different among investors, and therefore we focus here on it. Imposing $\rho^i = \rho$ and $g^i = g$ and the same initial conditions $\beta_0^i = \beta_0$, the expectations law of motion can be rewritten as

$$\beta_t^i = (1-\rho)(1-g)\beta^i \sum_{j=0}^{t-1} \tilde{\rho}^j + g \sum_{j=0}^{t-1} \tilde{\rho}^j ln \frac{P_{t-j}}{P_{t-1-j}} + \tilde{\rho}^{t-1}\beta_0 \tag{2.18}$$

where $\tilde{\rho} = \rho(1-g\rho)$. It follows that

$$\beta_t^i - \beta_t^m = (\beta^i - \beta^m)(1-\rho)(1-g)\frac{1-\tilde{\rho}^t}{1-\tilde{\rho}}, \tag{2.19}$$

where β_t^m represents the beliefs of agent $m \neq i$. Since $\tilde{\rho} < 0, \tilde{\rho}^t$ goes to zero relatively quickly. Thus, disagreement among investors i and m would be almost constant, reflecting their perpetual differences in long-run views up to a scale. Altogether, heterogeneous long-run expectations produce perpetual disagreement, as the one documented in surveys.

However, this idiosyncratic β^i does not explain the dynamics of disagreement. In particular, in the data, we observe a positive covariance between prices and disagreement. To explain these non-random movements in disagreement, we need additional heterogeneity in the expectations formation process. As in the data, consider the case of heterogeneous learning speed g^i. In this case, the disagreement between investor i and m can be written as:

$$\begin{aligned}
\beta_t^i - \beta_t^m =& (1-\rho)\left(\beta^i \frac{(1-\rho^t(1-g^i\rho)^t)(1-g^i)}{1-\rho(1-g^i\rho)} - \beta^m \frac{(1-\rho^t(1-g^m\rho)^t)(1-g^j)}{1-\rho(1-g^m\rho)}\right) \\
&+ \sum_{j=0}^{t-1} ln \frac{P_{t-j}}{P_{t-1-j}} \rho^j \left(g^i(1-g^i\rho)^j - g^m(1-g^m\rho)^j\right) \\
&+ ln\beta_0(\rho^{t-1}(1-g^i\rho)^{t-1} - \rho^{t-1}(1-g^m\rho)^{t-1}) \\
&\approx c(\beta^i - \beta^m) + \sum_{j=0}^{t-1} ln \frac{P_{t-j}}{P_{t-1-j}} \rho^j \left(g^i(1-g^i\rho)^j - g^m(1-g^m\rho)^j\right)
\end{aligned} \tag{2.20}$$

where $c(\beta^i - \beta^m)$ is a constant, increasing on the difference of long run views. Hence, the element determining the sign of the comovement between disagreement and price growth is the parenthesis of the last line summation. It turns out that

$$\frac{g^i(1-g^i\rho)^j}{g^i} \begin{cases} > 0 & \text{if } j < 1/g\rho - 1 \\ \leq 0 & \text{otherwise} \end{cases}$$

In other words, for relatively recent periods (low j), the higher the learning speed, the larger the disagreement. That would reverse for higher j, but at that point, ρ^j becomes very close to zero, almost cancelling this effect. Hence,

$$g^i > g^m \Rightarrow \beta_t^i - \beta_t^m \approx f\Big(\underset{(+)}{ln\frac{P_{t-j}}{P_{t-1-j}}}, \cdot\Big).$$

Returning to the quantitative model and noting that optimistic investors have higher learning speeds than pessimistic investors ($g^3 > g^1$), an exogenous increase in the beliefs of the optimists (β^3) would imply an increase in price and, via the above equation, in disagreement producing a positive co-movement among these variables. The impulse response analysis from figure 2.6 illustrates this mechanism. Notice that an increase in pessimists' expectations increases prices but generates a negative co-movement with disagreement. In Appendix C, we report an equivalent shock to disagreement coming from different sources: a positive information shock to optimists and a negative shock to pessimists. In both cases, cases disagreement widens. However, the effects on aggregate prices are the opposite: when optimists become more optimistic, mean expectations and prices go up; when pessimists become more pessimist (driving up disagreement), mean expectations and prices decrease. The effect of heterogeneous ρ is similar to that of heterogeneous g.

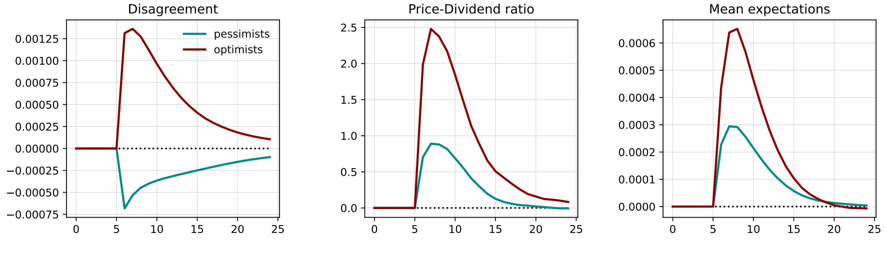


Figure 2.6. **Responses to a positive information shock.** The graph shows the GIRFs of different variables to a positive information shock hitting either the optimists or the pessimists.

Mechanism 3: price-trading feedback loop. Trading is an aggregate property of the model that requires a time-varying heterogeneity among agents.[13] The model includes three idiosyncratic features: wage shocks, information shocks and expectation formation parameters.[14] Thus, agents trade in the stock market to insure against income risk (fundamental motive) or because of their different views about the future evolution of stock prices (speculative motive).

Differently from income or information shocks, heterogeneity in expectation formation is a mechanism that endogenously produces disagreement and trading. Consider a price shock that surprises agents positively. Investors will tend to get more optimistic in general, but to a different degree; some will interpret it as truly fundamental change updating their beliefs more, while others would think of it mostly as noise, not changing their beliefs much. Due to this different processing of new information, disagreement will widen. Other things equal, investors believing the news would buy stocks from the more skeptical investors. Thus, through the heterogeneous expectation formation, a change in prices leads to disagreement and trading. On the other direction, trading implies a change in the wealth distribution; in the previous example, from pessimists to optimists. Thus, the market share of optimists is increased, which makes the market look more optimistic on average; since more optimistic agents demand more stocks, trading implies an increase in the total demand for stocks, moving prices up. Altogether, heterogeneous learning connects prices to trading, which redistributes wealth, influencing the aggregate stock demand and prices.

2.4.2.2. Simulated Impulse Response Functions

We resort to simulation to illustrate mechanisms 2 and 3, which emerge from the heterogeneous beliefs model. We show four experiments to explore the role of disagreement and trading.

A permanent disagreement shock. Figure 2.7 shows that a permanent increase in the optimist's long-run expectations implies a permanent rise in the level of their expectations. Following this burst of optimism, prices (and mean expectations) go up and, via learning, that optimism spills over the expectations of the other groups, reinforcing their effect on prices. However, the effect across groups is unequal: the impact on optimists' expectations is much larger, and their propensity to invest out of wealth increases at a faster rate compared to the ones of the other two groups. This also explains why stock holdings of pessimists and moderates decrease although their return expectations increase:

[13] Notice that a constant heterogeneity (for instance, in terms of long run views) would generate inequality (other things equal, the most optimist would hold more stocks) but not trading.

[14] The distribution of stock holdings is time-varying, capturing nothing but the joint dynamics of the three aforementioned variables.

since prices go up (driven by optimists' expectations), their wealth increases sufficiently rapidly to counterbalance the desire to accumulate more equity. Optimists, on the other hand, experience a rapid increase in expectations (driving up disagreement) and acquire more stocks, reducing their consumption along the way. Hence, trading increases to accommodate the stock holdings in line with the expectations distribution. Finally, the rise in prices gives rise to a temporary spike in returns due to capital gains.

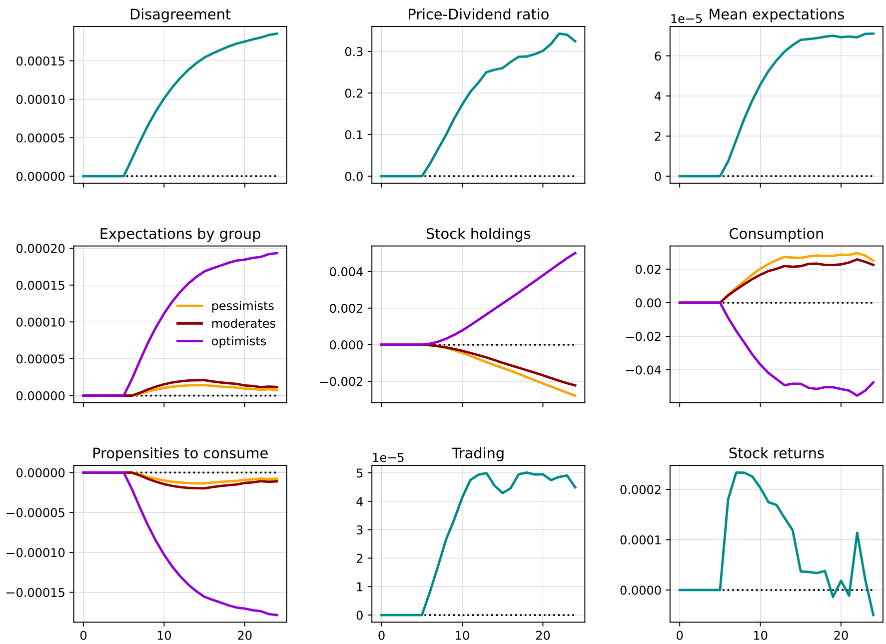


Figure 2.7. **Responses to a long run optimism shock to optimists.** The graph show the GIRFs of different variables to a permanent increase in β^3 in period 5. Periods are quarters. IRFs are computed following these steps: i) simulate the model T =10.000 periods across U=100 different shock realizations; ii) introduce a shock to the variable/parameter in a particular period p and compute new TxU series; iii) repeat ii) at different P=5 points, to tackle possible nonlinearities; iv) compute the differences between shocked and unshocked series at each P and U; v) average the differences across points and realizations.

A transitory disagreement shock. Define a pure disagreement shock as a shock that increases disagreement but does not affect mean beliefs on impact. This can be implemented as a joint shock to optimists' and pessimists' beliefs of the same magnitude in absolute value but different signs. Specifically, we define an $x\%$ positive pure disagreement shock, ${}^{DI}_{t}$, as

$$
{}^{DI}_{t} \equiv \begin{cases} {}^{3}_{t} = \frac{x}{2} > 0, \\ {}^{2}_{t} = 0, \\ {}^{1}_{t} = -\frac{x}{2}. \end{cases} \tag{2.21}
$$

We consider the dynamic effects of an *i.i.d* pure disagreement shock for the case in which other information shocks (^{P}) are absent. These results are illustrated in figure 2.8. It produces a sharp increase of 1% in disagreement, which exhibits high persistence over time, remaining positive even after five years. The effects on the other variables are different compared to the previously analyzed shocks: mean expectations are almost constant while the expectations of optimists and pessimists have opposite signs and manifest high persistence over time. Although average sentiment does not move significantly, the PD ratio jumps on impact and continues to increase for 3 quarters, remaining positive for the whole horizon considered. These dynamic effects generate a positive co-movement between disagreement and the PD ratio of approximately 0.8 helping in matching the high positive correlation between these two variables seen in the data.

A trading shock. To explore the effect of trading on prices, we run the following experiment. When market-clearing prices are already set up, shock the equilibrium stock holdings:

$$
S_t \equiv \begin{cases} \Delta S_t^3 = \frac{x}{2} > 0, \\ \Delta S_t^2 = 0, \\ \Delta S_t^1 = -\frac{x}{2}. \end{cases} \tag{2.22}
$$

This is a pure redistributive shock that moves assets from pessimists to optimists. What are the effects? The weight of optimists in the market goes up, which increases mean beliefs and aggregate demand as they have a larger propensity to invest. Hence, prices go up. This is the first-round effect. Due to learning, all the agents become more optimistic, but with different intensities as they process information differently: optimists will become more optimistic than pessimists. Disagreement goes up, leading to trading; pessimists will sell assets to optimists, restarting the process. Thus, a transitory shock is propagated for a while.

An income shock. Consider now a transitory shock to optimists' wages. As with the previous redistributive shock, it represents an inflow of resources for optimists. However, now the distribution of stocks is unchanged. The dynamics resemble the ones of a wealth shock, but responses are notably less persistent. The main difference is that consumption for pessimists does not go down, as the wage shock represents a net inflow of resources into the economy while the trading shock redistributes, keeping aggregate resources unchanged. Appendix C shows the IRFs of an aggregate wage shock. The dynamics are very similar, except that, initially, that shock raises the stock market participation of pessimists

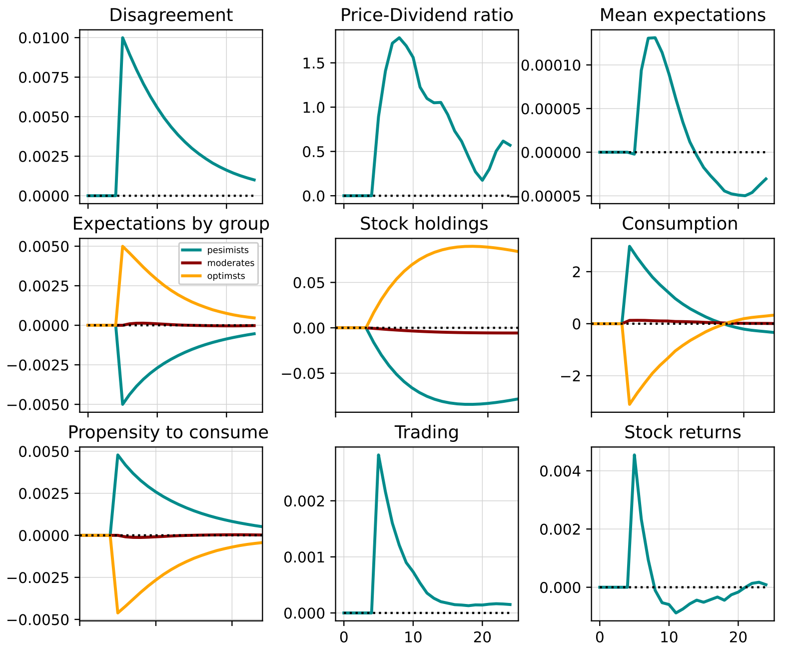


Figure 2.8. **Responses to a pure disagreement shocks** ^{DI}**.** Periods are quarters and the simulation does not include other information shocks that are present in the baseline calibration. IRFs are computed as in figure 2.7

and moderates who end up increasing their stock holdings by trading with optimists.

2.4.2.3. An agent with Rational Expectations

While the evidence on the cyclical properties individual investors' beliefs points out clear departures from Rational Expectations, the evidence for institutional investors is much less clear (Adam, Nagel, et al., 2022). Despite the recent surge in retail trading, the market is still clearly dominated by institutional investors. This individual-institutional investor composition opens a question about the interaction between extrapolative and non-extrapolative agents or, in other words, whether investors who make forecast using wrong models will be kicked out of the market by agents using better models, a prediction associated with Friedman.

When the persistence of the wage process is close to 1 and all the agents hold Rational Expectations, the equilibrium PD ratio is a constant and the price growth expectations

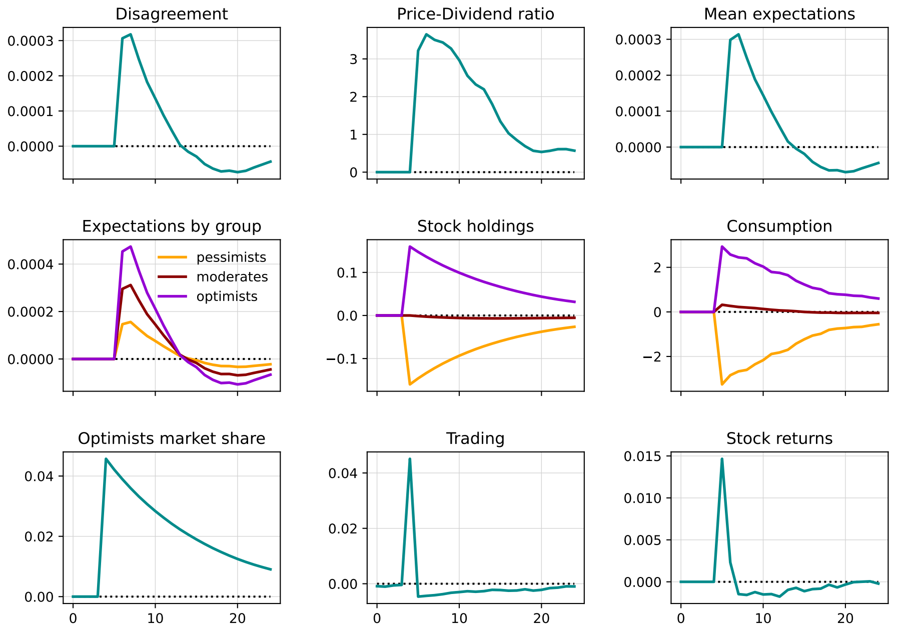


Figure 2.9. **Responses to a trading shock that redistributes stocks from pessimists to optimists.** The graph show the GIRFs of different variables to a trading shock. Periods are quarters. IRFs are computed as in figure 2.7

.

are equal to dividends growth beliefs, which boils down to a constant. However, if expectations coordination does not take place, an RE agent has to acknowledge the existence of non-RE agents. Thus, RE is the fixed point of the mapping from perceived to actual expectations

$$
\begin{aligned}
\beta_t^{RE} \equiv & \mathbb{E}_t\left(\frac{P_{t+1}}{P_t}\right) = \\
= & \mathbb{E}_t\left[\frac{\bar{S} - \sum_{i=1}^{M-1} \mu^i \chi^i \beta_t^i S_{t-1}^i - \mu^{RE}\chi^{RE}\beta_t^{RE} S_{t-1}^{RE}}{\bar{S} - \sum_{i=1}^{M-1} \mu^i \chi^i \beta_{t+1}^i S_t^i - \mu^{RE}\chi^{RE}\beta_{t+1}^{RE} S_t^{RE}}\right. \\
& \left. \times \frac{\sum_{i=1}^{M-1} \mu^i \chi^i \beta_{t+1}^i (W_{t+1}^i / D_{t+1} + S_t^i) + \mu^{RE}\chi^{RE}\beta_{t+1}^{RE}(W_{t+1}^{RE}/D_{t+1} + S_t^{RE})}{\sum_{i=1}^{M-1} \mu^i \chi^i \beta_t^i (W_t^i / D_t + S_{t-1}^i) + \mu^{RE}\chi^{RE}\beta_t^{RE}(W_t^{RE}/D_t + S_{t-1}^{RE})} \frac{D_{t+1}}{D_t}\right]
\end{aligned}
\tag{2.23}
$$

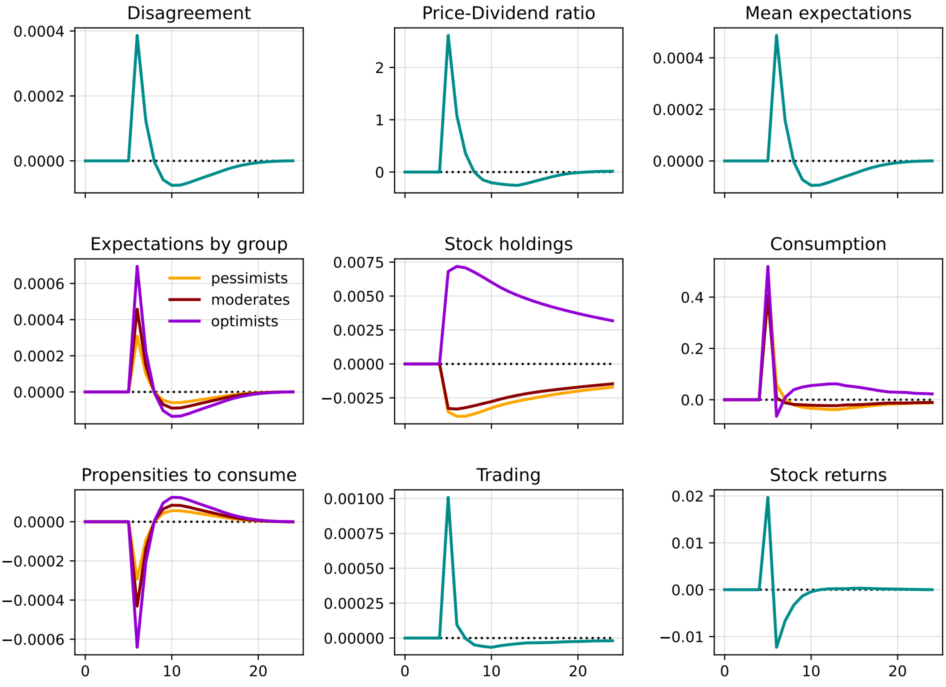


Figure 2.10. **Responses to a shock to optimists wages $ln\varepsilon_t^{w,3}$.** The graph show the GIRFs of different variables to a wage shock to group 3. Periods are quarters. IRFs are computed as in figure 2.7

.

While solving the previous equation is difficult, it is well-known in the learning literature that OLS learning converges to RE under certain conditions. Exploiting that convergence, we conjecture that the RE agents update their expectations following

$$\hat{\beta}_t^{RE} = \hat{\beta}_{t-1}^{RE} + \frac{1}{t-1}\left(\frac{P_t}{P_{t-1}} - \hat{\beta}_{t-1}^{RE}\right) \tag{2.24}$$

Figure 2.11 plots expectations and stock holdings of the previous 3 sentiments groups and the added RE investor after 5000 periods, implying $\hat{\beta}_t^{RE} \approx \hat{\beta}_{t-1}^{RE} \approx \beta_t^{RE}$. As expected, RE beliefs are much more stable than extrapolative beliefs. However, that does not imply they will take over the whole market. In fact, it turns out that OLS learners are outperformed in terms of average forecast errors by the moderate extrapolators. This illustrates that RE might be the best strategy when there is belief coordination but not otherwise. In this case, in terms of Guesnerie, 2011 , RE is a Nash Equilibrium but not a dominant strategy.

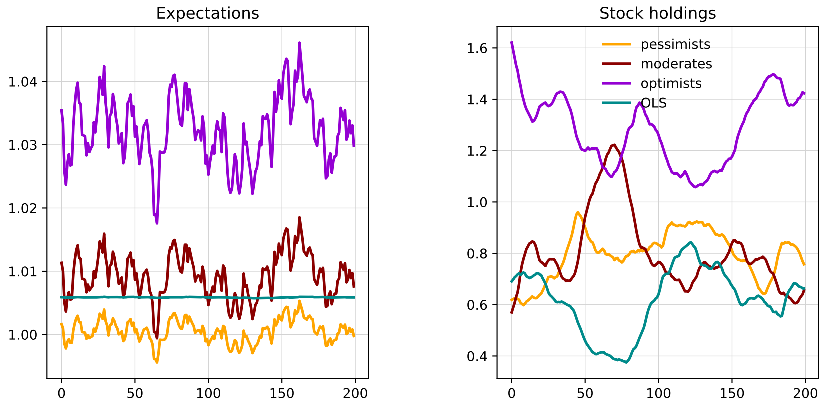


Figure 2.11. **Model simulations for expectations and stock holdings with a Rational Expectations agent.**

2.5. Conclusions

In this paper, we present a quantitative model that jointly replicates the empirical dynamics of stock prices, trading, and the heterogeneity of expectations. We place our emphasis on a model of expectations that allows for different layers of heterogeneity. In particular, we point out the role of heterogeneous long-run expectations and the signal-to-noise perceptions that determine the speed at which agents adapt their expectations to new information. This model captures salient features of recent survey evidence, such as high and permanent disagreement and the pro-cyclical nature of both individual expectations and disagreement, which we first document using available survey data on expected returns.

We show that an otherwise asset pricing framework endowed with this model of beliefs delivers a remarkable quantitative performance across a wide variety of stylized facts regarding the joint dynamics of prices, heterogeneous expectations, and trading patterns. The good quantitative performance legitimizes the use of the model to shed some additional light on the mechanics of stock market booms. In particular, we point out that disagreement and trading emerge as key drivers of asset price dynamics, as they shape the distribution of beliefs and wealth that determines aggregate demand and prices. This contrasts with mainstream asset pricing, where trading plays a marginal role.

Finally, we point out some shortcomings. First, the data analysis needs to be extended by using more surveys and including tests on under-reaction. Second, the model of expectations has to be compared with existing alternatives to RE, to make clear the points of continuity and divergence, with an eye on the ability to replicate the survey evidence.

Finally, although the model replicates the joint movement of expectations and trading, it is completely unable to generate a level of trading similar to that of the real world. We conjecture that it is related to the type of agents we are modeling ("retail investors"), characterized by infrequent trading that accounts for a rather small fraction of total trading volume. The inclusion of institutional investors, perhaps with different mandates than households, might help in that direction.

2.A Appendix

Appendix A: Additional results

	c	c	*p-value* H_0: c= c
		2 Sentiment groups	
p_{0-50}	0.0546 ***	-0.2421***	0.0000
p_{50-100}	0.0744 ***	-0.2419 ***	0.0000
		3 Sentiment groups	
p_{0-33}	0.0576***	-0.2421 ***	0.0000
p_{33-66}	0.0545***	-0.2415***	0.0000
p_{66-100}	0.0809***	-0.2423***	0.0000
		4 Sentiment groups	
p_{0-25}	0.0591	-0.2421	0.0000
p_{25-50}	0.0501	-0.2422	0.0000
p_{50-75}	0.0621	-0.2420	0.0000
p_{75-100}	0.0867	-0.2421	0.0000

Table 2.8. **RE Tests across different sentiment groups.** p_{0-50} denotes the sentiment group which expectations lies between between the 0 and 50th percentile. The data in each group is aggregated by taking the average of that particular group. Data used for this particular test is the Gallup UBS survey data for expected stock market return of all individuals. Estimates have are based on asymptotic theory and have been adjusted for small sample bias

Appendix B: Responses to aggregate shocks

In this section we report the responses of the model main variables to simultaneous equivalent shocks on investors wages (figure 2.13) and transitory information (figure 2.14).

Appendix C: Solving asset pricing models with learning and heterogeneous agents with PEA

In this note, we propose a new way of solving asset pricing models with learning under a range of setups. We apply the Parameterized Expectations Algorithm using an approximation function rooted in economic theory. We use a one asset representative agent model

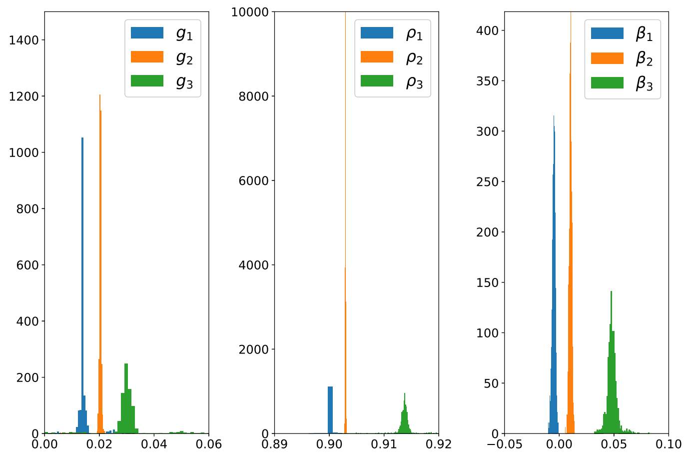


Figure 2.12. Bootstrap densities of estimated parameters from equation 2.5

as a baseline. Then, we solve that model for heterogeneous investors. Finally, we include multiple assets.

C.1. Representative Agent Economy

Model structure

Demographics. The economy is populated by an arbitrary number of infinitely living identical investors with weight μ^i (such that $\sum_i \mu^i = 1$).

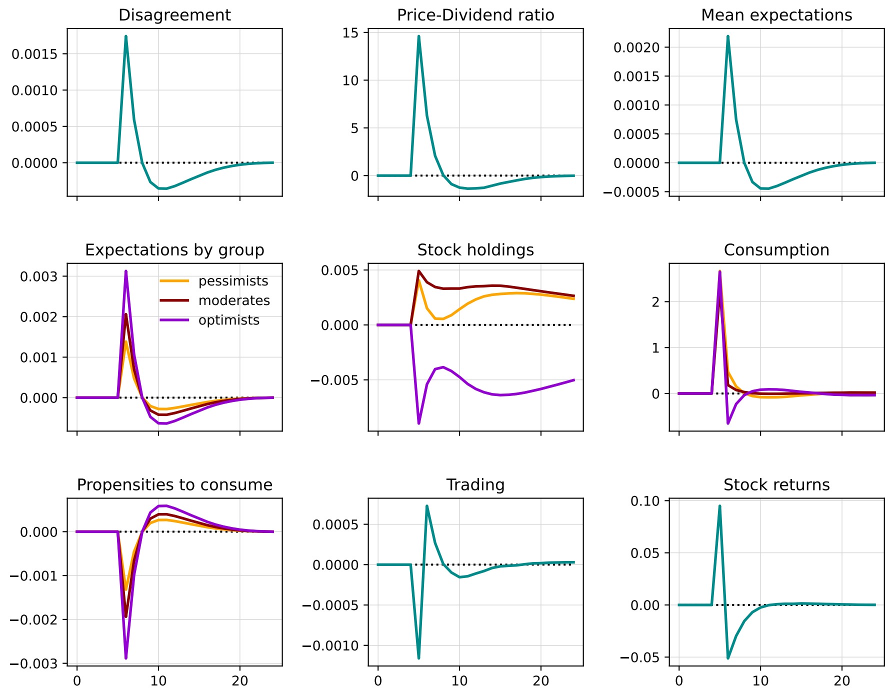
Goods and assets. There is a single perishable good in the economy. Besides, there exist a single risky asset S in the form of a contract that delivers dividends each period and is marketable at an uncertain future price, giving rise to capital gains and losses.

Resource processes. This is a pure exchange economy. When the time starts, each investor is endowed with one unit of stock ($S^i_{-1} = 1$). Dividends D are exogenous, obeying a random walk with drift process

$$\ln D_t = \ln a + \ln D_{t-1} + \ln {}^d_t \tag{2.25}$$

with $a \geq 1$ being the permanent component and ${}^d_t \sim \log \mathcal{N}(1, e^{s_d^2} - 1)$ the random unpredictable shock. Shocks are independent and identically distributed. Capital gains are endogenously determined.

Figure 2.13. **Responses to a general wage shock.** The graph show the GIRFs of different variables to an equivalent wage shock enjoyed by all investors. group 3. Periods are quarters.



Markets. Financial markets are competitive but incomplete[15]. A negative amount of stocks is allowed up to some point (specified below)[16]. Goods market behaves also competitively.

Investors' information set. Investors do not know that they all are identical. They make decisions using their subjective probability measure $\mathcal{P}^i$. It is a primitive of the model that sets agents' perceptions about the processes beyond their control: dividends and prices. It is assumed they know the stochastic processes for dividends. However, they are not endowed with knowledge about the pricing function. As a result, there is an additional uncertainty source that enlarges the sample space Ω, being $\omega = \{D_t, P_t\}_{t=0}^{\infty}$ a typical element of Ω. Therefore, the underlying probability space is given by $(\Omega, \mathcal{B}, \mathcal{P})$ with $\mathcal{B}$ denoting the corresponding σ-algebra of Borel subsets of Ω and $\mathcal{P}$ the agent's subjective probability measure over $(\Omega, \mathcal{B})$.

[15] Because of the existence of constraints on stock holdings and the nonexistence of contingent claims markets.

[16] In this case, a negative position would be equivalent to the so-called covered short position, at which an investor borrows shares and pays a borrowing rate during the time the short position is hold.

Figure 2.14. **Responses to a general information shock.** The graph show the GIRFs of different variables to an information equally received by all investors. Periods are quarters.

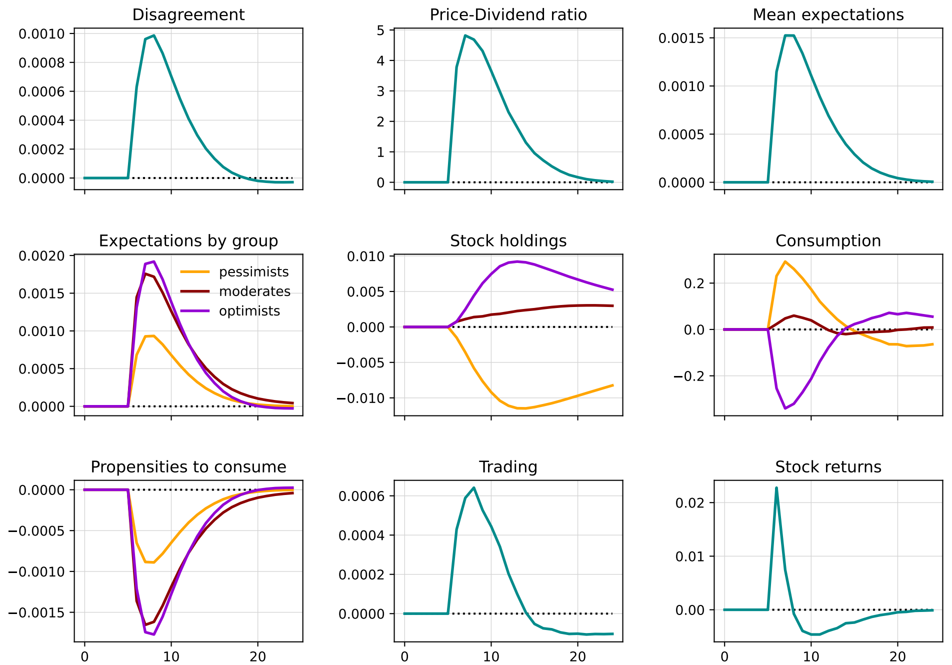
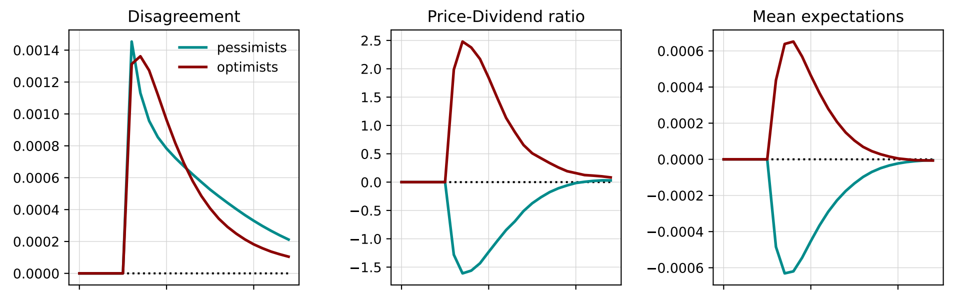


Figure 2.15. **Responses to a disagreement shock.** The graph show the GIRFs of different variables to a positive information shock hitting the optimists and a negative shock hitting the pessimists.



Investors' program. Each investor faces a consumption-saving decision, choosing sequences of consumption and stock holdings $\{C_t^i, S_t^i\}_{t=0}^{\infty}$ to maximize their lifetime expected utility with respect to their subjective probability measure $\mathcal{P}^i$:

$$\max_{\substack{\{C_t^i, S_t^i\}_{t=0}^{\infty} \in \Gamma \\ S_{-1}^i = 1}} \mathbb{E}_0^{\mathcal{P}^i} \sum_{t=0}^{\infty} \delta^t \frac{(C_t^i)^{1-\gamma}}{1-\gamma} \tag{2.26}$$

where

$$\Gamma = \left\{ C_t^i, S_t^i \;\mid\; C_t^i + P_t S_t^i \leq (P_t + D_t) S_{t-1}^i; \;\; \underline{S} \leq S_t^i \leq \bar{S}; \;\; 0 \leq C_t^i \right\} \tag{2.27}$$

The welfare function $U(C_t^i) = \frac{(C_t^i)^{1-\gamma}}{1-\gamma}$ is a time-separable continuous, increasing in consumption $U'(C_t^i) > 0$ but strictly concave $U''(C_t^i) < 0$ function. Inada conditions hold. This parametric specification of U represents a risk averse investor, being γ her risk aversion level. Γ sets up the feasible set, determine by the budget constraint and some bounds. Lower and upper bounds on S_t^i are assumed for convenience; mathematically, these bounds ensure that the feasibility set is compact; economically, the lower bound rules out Ponzi schemes, which are out of interest here.

Equilibrium Since the model outlined in the previous section has some non-standard elements (subjective expectations), we start by pointing out that the problem is well defined. The investors program consists of maximizing a bounded continuous function over a compact non-empty feasible set[17]. By the Weierstrass extreme value theorem, these are sufficient conditions for the existence of a maximum. Besides, since the objective function is strictly concave, the maximum is unique. Moreover, given the feasible set is convex, the first order conditions are necessary and sufficient for the optimum by the Karush-Kuhn-Tucker theorem[18]. Let us define now the Competitive Equilibrium in sequential form.

Competitive Equilibrium. Given initial endowments, the income process and the probability measure $\{\mathcal{P}_i\}_{i=1}^{I}$, a Competitive Equilibrium consists of sequences of allocations $\{\{C_t^i, S_t^i\}_{t=0}^{\infty}\}_{i=1}^{I}$ and prices $\{P_t\}_{t=0}^{\infty}$ such that:

1. Investors behave optimally, satisfying:

[17] For this setup, it can be shown that the objective function is bounded, following Adam, Marcet, and Beutel, 2017. Besides, a no-trading choice is always an interior feasible possibility

[18] Standard transversality conditions are also required.

a) KKT First Order Conditions. They boil down to the following Euler Equation[19]

$$(C_t^i)^{-\gamma} = \delta \mathbb{E}_t^{\mathcal{P}_i}\left(\frac{P_{t+1} + D_{t+1}}{P_t}(C_{t+1}^i)^{-\gamma}\right) \tag{2.28}$$

and the budget constraint.

b) A Transversality condition:

$$\lim_{j\to\infty} \delta^j \mathbb{E}_t^{\mathcal{P}_i}\left[\frac{D_{t+j}}{D_t} P_{t+j} S_{t-1+j}\right] = 0 \tag{2.29}$$

2. Markets clear:

$$\text{Equities:} \quad \sum_{i=1}^{n} \mu^i S_t^i = \sum_{i=1}^{n} S_{-1}^i \equiv \bar{S} = 1 \tag{2.30}$$

$$\text{Goods:} \quad \sum_{i=1}^{n} \mu^i C_t^i = D_t \tag{2.31}$$

Rational Expectations Equilibrium.

In this section, I assume agents possess all the information about the economy, knowing they all are equal and how dividends map into prices at any contingency. To derive the CE under RE, I follow the standard procedure: use the market clearing condition to pin down the allocation path and then, go to the Euler Equation to recover prices. Thus, to clear the markets $C_t^i = D_t$ and $S_t^i = \bar{S}$ must hold in every period. Then, iterating forward on P_{t+1}, using the Law of Iterated Expectations and applying a transversality condition, the standard pricing formula comes up:

$$P_t = \mathbb{E}_t\left[\sum_{j=1}^{\infty} \delta^j \left(\frac{D_{t+j}}{D_t}\right)^{-\gamma} D_{t+j}\right] \tag{2.32}$$

Given the knowledge of the dividends process, this formula can be solved to derive a more particular pricing function

$$P_t = \frac{\delta a^{1-\gamma}\rho}{1 - \delta a^{1-\gamma}\rho} D_t \tag{2.33}$$

where $\rho \equiv \mathbb{E}_t(\varepsilon_{t+n}^{1-\gamma}) = exp\{-\gamma(1-\gamma)s_d^2/2\}$.

[19] Since Inada conditions hold, we can ignore consumption lower corner. By concavity, the budget constraint will always bind. Assets lower and upper bounds are large enough to never bind.

To have a benchmark for the approximating function used later, I specify the implicit optimal policies in this model. Let $Z_t \equiv (P_t + D_t)S_{t-1}$, then the optimal consumption rule is

$$C_t^i = cZ_t \tag{2.34}$$

with $c \equiv 1 - \delta a^{1-\gamma}\rho$. In other words, it is optimal to consume a constant fraction of the investor's resources. This is in line with Hakansson, 1970 results, for a problem with exogenous i.i.d. returns and it was also pointed out by Williams, 2012. I proceed to verify that the policy (2.34) is indeed a solution. First, it is consistent with market clearing. Note that (given $S_{t-1} = 1$)

$$C_t^i = cZ_t = (1 - \delta a^{1-\gamma}\rho)(P_t + D_t) = (1 - \delta a^{1-\gamma}\rho)\left(\frac{\delta a^{1-\gamma}\rho}{1 - \delta a^{1-\gamma}\rho} D_t + D_t\right) = D_t$$

Second, plugging (2.34) in the budget constraint, the stock policy can be derived

$$S_t^i = (1 - c)\frac{Z_t}{P_t} \tag{2.35}$$

Equilibrium prices are recovered using this demand in the stock market clearing condition,

$$(1 - c)\frac{Z_t}{P_t} = 1 \Rightarrow P_t = (1 - c)(P_t + D_t) \Rightarrow P_t = \frac{1 - c}{c} D_t = \frac{\delta a^{1-\gamma}\rho}{1 - \delta a^{1-\gamma}\rho} D_t$$

Finally, using the consumption policy and the prices implied by it in the Euler Equation:

$$\begin{aligned}
&(C_t^i)^{-\gamma} P_t = \delta \mathbb{E}_t\left[(C_{t+1}^i)^{-\gamma}(P_{t+1} + D_{t+1})\right];\\
&\left[c\left(\frac{1 - c}{c} + 1\right)D_t\right]^{-\gamma}\left(\frac{1 - c}{c}\right)D_t = \delta \mathbb{E}_t\left[\left[c\left(\frac{1 - c}{c} + 1\right)D_{t+1}\right]^{-\gamma}\left(\frac{1 - c}{c} + 1\right)D_{t+1}\right];\\
&(1 - c)D_t^{1-\gamma} = \delta \mathbb{E}_t\left[D_{t+1}^{1-\gamma}\right];\\
&(1 - c) = \delta \mathbb{E}_t\left(a^{1-\gamma}\varepsilon_{t+1}^{1-\gamma}\right) = \delta a^{1-\gamma}\rho = 1 - c
\end{aligned}$$

where the 2nd line uses the policy and equilibrium prices ($P_t = (1 - c)D_t/c$).

Notice the policies are a function of the beginning of period portfolio S_{t-1}^i, the relevant information about fundamentals D_t and prices P_t, as pointed out by Lucas, 1978. Of course, if agents know the pricing function, prices would turn informationally redundant.

Equilibrium with imperfect information.

Consider now a world with some information frictions. On the one hand, homogeneity is not common knowledge. As a result, investors cannot mechanically use the goods market clearing condition because $\text{Prob}(C^i_{t+j} \neq D_{t+j}) > 0$. On the other hand, there is an additional layer of uncertainty: price formation. In particular, investors do not know the pricing function. As showed by Adam and Marcet, 2011, that extra uncertainty leads to an extended state space **Ω** that include price history (with $\omega = \{D_t, P_t\}_{t=0}^{\infty}$ being a typical element of Ω)[20].

The consequence of these information frictions is that market clearing cannot be mechanically imposed so that equilibrium prices cannot be deduced from the Euler Equation. In general, agents cannot iterate forward because there exists the possibility that the marginal agent pricing the asset changes over time (and with her, the probability measure used to discount future payoffs). Then, the Law of Iterated Expectations cannot be applied. But even if expectations homogeneity is assumed to be known[21], forward iteration on the Euler Equation yields

$$P_t = \mathbb{E}_t^{\mathcal{P}_i}\left[\sum_{j=1}^{\infty} \delta^j \left(\frac{C^i_{t+j}}{C^i_t}\right)^{-\gamma} D_{t+j}\right] = \mathbb{E}_t^{\mathcal{P}_i}\left[\sum_{j=1}^{\infty} \delta^j \left(\frac{C(P_{t+j},\cdot)}{C(P_t,\cdot)}\right)^{-\gamma} D_{t+j}\right] \tag{2.36}$$

In words: current prices are an unknown function of future expected prices[22]. Therefore, decision rules must be computed such that equilibrium prices will equate those to aggregate supplies. The next steps are: i) set up the subjective probability measure $\mathcal{P}$; ii) solve the model dealing with the conditional expectation.

Agents' subjective price model. In this environment, $\mathcal{P}$ is a primitive of the model that must be set up. To do so, the RE implicit price model is taken as a benchmark. From the RE pricing equation (2.33), it follows that prices mimics the dynamics of dividends:

$$\ln P_t = \ln a + \ln P_{t-1} + \ln \varepsilon_t^d \tag{2.37}$$

The proposed subjective price model generalizes that, by allowing price growth to differ from dividends growth:

$$\ln P_t = \ln \beta_t + \ln P_{t-1} + \ln \varepsilon_t^P \tag{2.38}$$

$$\ln \beta_t = \ln \beta_{t-1} + \ln \eta_t \tag{2.39}$$

with i.i.d. normally distributed disturbances with known parameters. The persistent component of price growth β_t is unobserved and has to be estimated from price signals.

[20] Under RE, Ω is just made of the dividends history, being $\omega = \{D_t\}_{t=0}^{\infty}$ a typical element.

[21] Agents could still differ in other dimensions (risk aversion, discount factors, initial endowments) such that they still cannot know $C^i_t = D_t$.

[22] The two information frictions are needed. If homogeneity is unknown but agents know how prices are formed, future consumption would not depend on an endogenous variable (prices) but just on the exogenous state (dividends) such that it could be predetermined. If prices are uncertain but investors know they all are equal, future consumption is predetermined (just equal to the aggregate endowment).

For pursuing that task, investors use a Kalman filter. Their prior is centered around the RE value

$$\ln\beta_0^i \sim \mathcal{N}(\ln a, (\sigma^i)^2) \tag{2.40}$$

where $(\sigma^i)^2$ is the steady state Kalman variance. Its posterior (conditional on the observed history) is given by

$$\ln\beta_t^i \sim \mathcal{N}(\ln m_t^i, (\sigma^i)^2) \tag{2.41}$$

The expected log price growth $\ln m_t^i$ evolves recursively (with an information lag[23]):

$$\ln m_t^i = \ln m_{t-1}^i + g^i\left(\ln\frac{P_{t-1}}{P_{t-2}} - \ln m_{t-1}^i\right) + g^i \ln\varepsilon_t^P \tag{2.42}$$

where $\ln\varepsilon_t^P \sim \mathcal{N}\left(\frac{-(\sigma_\varepsilon)^2}{2}, (\sigma_\varepsilon)^2\right)$ is an innovation to agents information set that captures the extra knowledge agents receive about past transitory growth component[24]. In this version with homogeneous agents, $\sigma^i = \sigma \;\; \forall i$ so that $g^i = g$ and $m_t^i = m_t \;\; \forall i$.

Solution Strategy via PEA

We start by pointing out the vector of state variables X of the problem. Under RE, dividends shocks are the only state $X_t^i = D_t$. With information frictions, the state space is larger $X_t^i = (D_t, P_t, m_t^i)$. The inclusion of prices was discussed before (they are exogenous random variable from the investor's point of view). The inclusion of m_t follows from the subjective price model (it is a variable that summarizes investors' view of the future).

The difficulty of the solution is that optimality conditions includes an unknown conditional expectation. The one-period ahead Euler Equation can be rewritten as:

$$C_t^i = \left\{\delta\mathbb{E}_t^{\mathcal{P}_i}\left(\frac{P_{t+1} + D_{t+1}}{P_t}(C_{t+1}^i)^{-\gamma}\right)\right\}^{-1/\gamma} \tag{2.43}$$

To solve the model, it must be computed somehow. The Parameterized Expectations Algorithm (PEA)[25] is one of the alternatives[26]. In general, the conditional expectation is a

---

[23]To impart recursiveness to the expectations-price relationship (avoiding multiplicity of equilibria) I follow Adam, Marcet, and Beutel, 2017 and assume agents observe in period t information about the lagged transitory component $\ln\varepsilon_{t-1}^P$.

[24]In the model quantitative analysis, I always set $\ln\varepsilon_t^P = 0$ for all agents and assets.

[25]I

[26]The first use of this approach was due to Wright and Williams, 1982a, Wright and Williams, 1982b, Wright and Williams, 1984. My application builds on the version outlined by Marcet, 1988.

function of the states X_t^i

$$\begin{aligned}\mathbb{E}_t^{\mathcal{P}_i}\left(\frac{P_{t+1}+D_{t+1}}{P_t}(C_{t+1}^i)^{-\gamma}\right) &= \mathbb{E}_t^{\mathcal{P}_i}\left[\phi\left(z_{t+1}^i, \varepsilon_{t+1}^i, z_t^i\right)\right] \\ &= \mathbb{E}^{\mathcal{P}_i}\left[\phi\left(z_{t+1}^i, \varepsilon_{t+1}^i, z_t^i\right)|X_t^i\right] \\ &= \mathcal{E}(X_t^i)\end{aligned}$$

where z are the model endogenous variables (prices and consumption in this case) and the exogenous shocks (dividend shocks here). Therefore, the consumption decision rule must be

$$C_t^i = \mathcal{C}(X_t^i) = \left(\delta\mathcal{E}(X_t^i)\right)^{-\frac{1}{\gamma}} \equiv \bar{\mathcal{E}}(X_t^i) \tag{2.44}$$

PEA consists of replacing the conditional expectation $\mathcal{E}(X_t^i)$ by some parametric function ψ. The choice of the approximating functions ψ is not obvious and not unique. Popular possibilities are polynomials, splines, neural networks, etc. In this model (indeed, in models without occasionally binding constraints), there is no practical difference between approximating the conditional expectation $\mathcal{E}(X_t^i)$ and approximating the policy function $\bar{\mathcal{E}}(X_t^i)$. Exploiting that, I propose a novel approach: approximating functions rooted in economic theory. The idea is that of homotopy: start with a version of the model that has analytical solution (in this case, the RE version) and keep the structure of the policy function as an approximating function. I illustrate this idea for the previous model. Consider this version of the above RE rules as the approximating function:

$$C_t^i = \mathcal{C}(X_t^i;\chi) = (1-\chi\zeta(m_t))(P_t + D_t)S_{t-1}^i \tag{2.45}$$

where ζ is some function of the expected permanent capital gains. Plugging it in the budget constraint, the stock policy reads as

$$S_t^i = \mathcal{C}(X_t^i;\chi) = \chi\zeta(m_t)\frac{(P_t + D_t)S_{t-1}^i}{P_t} \tag{2.46}$$

Hence, applying the stock market clearing condition $S_t^i = \tilde{S}_t = 1$, equilibrium prices reads as

$$P_t = \frac{\chi\zeta(m_t)}{1-\chi\zeta(m_t)}D_t \tag{2.47}$$

Consider $\zeta(m_t) = m_t$. Since under RE $m_t = a^{1-\gamma}\rho$, it would recover the exact pricing function for $\chi = \delta$. That provides a reasonable range of starting values for χ.

Note the advantages of the approach: we are left with a single parameter to estimate (χ) as opposed to the potentially large number of parameters of traditional approximating function; this rules out any multicollinearity problem typically associated with PEA[27]; we have a closed-form solution for equilibrium prices; the approximating function yields a policy function with economically interpretable properties (for instance, it shows that the price-elasticity of stock demand is function of expectations. See Belda, 2023 for an economic analysis).

To evaluate the performance of this approximating function, χ must be estimated. To do so, I resort to simulation and Montecarlo integration. To simulate the model, first I perform some normalization to ensure the variables are stationary. In particular, I normalize the problem by dividends, such that the states become $X_t = \left(\frac{P_t}{D_t}, m_t\right)$. Second, I need to impose a projection facility to ensure that beliefs fall in an acceptable range (avoiding both negative and explosive prices). Note that at $m_t = 1/\chi$ prices faces a discontinuity. Then, the expectations updating equation is modified as:

$$\ln m_t = w\left[\ln m_{t-1} + g\left(\ln\frac{P_{t-1}}{P_{t-2}} - \ln m_{t-1}\right)\right] \tag{2.48}$$

with

$$w(s_t) = \begin{cases} s_t \ \text{ if } \ m_t \leq \bar{m} \\ s_t = s_{t-1} \ \text{ if } \ m_t > \bar{m} \end{cases} \tag{2.49}$$

where $\bar{m} = (1-\underline{c})/\chi$, where $\underline{c}$ is the minimum propensity to consume. In the application, I set $\underline{c} = 0.01$.

Third, let the interior of the conditional expectation be

$$\phi\left(z_{t+1}(\chi), \varepsilon_{t+1}, z_t(\chi)\right) = \frac{\frac{P_{t+1}}{D_{t+1}}(\chi) + 1}{\frac{P_t}{D_t}(\chi)}\left(\frac{C_{t+1}}{D_{t+1}}(\chi)\right)^{-\gamma}$$

Finally, the next-period endogenous variables should be computed with respect to the subjective probability measure. The following box summarizes the algorithm.

PEA implementation

1. Draw a series of the exogenous processes $\left\{\varepsilon_t^d\right\}_{t=1}^T$, for a large T.
2. For a given $\chi \in \mathbb{R}^n$, recursively compute the series of the endogenous variables $\{\{C_t^i(\chi), S_t^i(\chi)\}_{i=1}^n, P_t(\chi)\}_{t=1}^T$.
3. Minimize the prediction error.

[27] Although that can also be solved in other ways (e.g., using Chebyshev polynomials. See Christiano and Fisher, 2000).

a) $\phi()$ must be consistent with agents' expectations $\mathcal{P}$. Therefore:
i) Expected prices:

$$m_{t+1}^{\mathcal{P}} = m_t v_{t+1} \Rightarrow \left(\frac{P_{t+1}}{P_t}\right)^{\mathcal{P}} = m_t v_{t+1} \varepsilon_{t+1}^{p} \Rightarrow \left(\frac{P_{t+1}}{D_{t+1}}\right)^{\mathcal{P}} = \left(\frac{P_{t+1}}{P_t}\right)^{\mathcal{P}} \frac{D_t}{D_{t+1}} \frac{P_t}{D_t}$$

ii) Expected consumption:

$$\left(\frac{C_{t+1}}{D_{t+1}}\right)^{\mathcal{P}} = (1 - \chi m_{t+1}^{\mathcal{P}}) \left(\left(\frac{P_{t+1}}{D_{t+1}}\right)^{\mathcal{P}} + 1\right) S_t$$

b) Nonlinear least square regression

$$G(\chi) = \underset{\xi \in \mathbb{R}^n}{\operatorname{argmin}} \frac{1}{(T - \underline{T})} \sum_{t=\underline{T}}^{T} \left[\phi\left(z_{t+1}^{\mathcal{P}}(\chi), \varepsilon_{t+1}, z_t(\chi)\right) - \frac{\psi(\xi; X_t(\chi)^{-\gamma}}{\delta} \right]^2$$

with $\underline{T}$ are some initial periods burned.

4. Update χ

$$\chi^{j+1} = \chi^j + d(G(\chi^j) - \chi^j)$$

until reaching a fixed point $\chi_f = G(\chi_f)$.

Solution accuracy

To evaluate how good is the approximation, I explore the errors size. Recall approximating errors are given by

$$u_{t+1} = \phi\left(z_{t+1}, \varepsilon_{t+1}, z_t\right) - \frac{\psi(\chi; x_t)^{-\gamma}}{\delta}$$

Then, we need a criterion to determine the degree of accuracy. One of the most popular one is the Bounded Rationality Measure (Judd, 1992):

$$J = \log_{10}\left(\mathbb{E}_t \left| \frac{u_{t+1}}{\frac{C_t}{D_t}} \right| \right) \tag{2.50}$$

being J a dimension-free quantity expressing that error as a fraction of current consumption, which expresses the results in economic terms. The following table contains some statistics of interest from the PEA estimation.

Table 2.9. PEA estimation results.

T	$\hat{\chi}$	J	J_{max}	J_{min}	MSE
100.000	0.9691	-3.33	-3.11	-3.61	4.67e-04

The model was simulated for a very long horizon (T=100.000 periods) as a safety horizon (the model relies on a single long simulation to compute the expectations, exploiting ergodicity). Anyway, if one looks for speed, even for T=10.000 the estimation delivers equal $\hat{\chi}$ up the 8th decimal when using different sequences of random shocks in the simulations, which more than fulfils Creel, 2005's criterion[28]. The estimated coefficient $\hat{\chi}$ is 0.969, indeed relatively close to the discount factor δ, which was calibrated at 0.995. Turning to accuracy, Judd's measure shows a good performance, with an average mistake of \$0.47 out of \$1.000 (and a maximum error of \$0.78 out of \$1.000). Figure 2.16 shows the histogram of Judd's measure. The use of other metrics just confirms the good accuracy of the approximation (for instance, the mean square error turns out to be negligible).

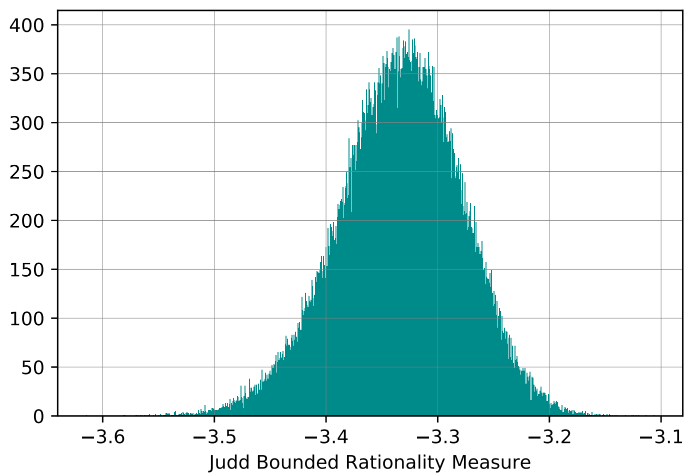


Figure 2.16. Histogram of the Judd's Bounded Rationality Measure for $\hat{\chi}$.

C.2. Endogenous production

We modify the economic environment to allow for endogenous production. We keep the dividends process exogenous, as before. To endogenous this we need to add monopolistic competition with time-varying markups, an extension we consider later on.

[28]Creel suggests that "the coefficients of the function that approximates expectations should be the same at the second decimal place across all replications of the PEA solution that use different sequences of random numbers in the simulations". For T=10.000 and good starting values, the estimation takes no longer than 20min. In addition, $\hat{\chi}$ is robust to different starting values χ_0 up to 3 decimals.

Consumers

$$\max_{\substack{\{C_t^i, S_t^i\}_{t=0}^{\infty} \in \Gamma \\ S_{-1}^i = 1}} \mathbb{E}_0^{\mathcal{P}^i} \sum_{t=0}^{\infty} \delta^t \frac{(C_t^i)^{1-\gamma}}{1-\gamma} - \lambda \frac{(L_t^i)^{1+\phi}}{1+\phi} \tag{2.51}$$

where

$$\Gamma = \left\{ C_t^i, S_t^i \;\mid\; C_t^i + P_t S_t^i \leq (P_t + D_t) S_{t-1}^i + w_t L_t^i;\;\; \underline{S} \leq S_t^i \leq \bar{S};\;\; 0 \leq C_t^i \right\} \tag{2.52}$$

where λ controls the magnitude of the disutility of labour, L_t^i and w_t is the wage for supplying 1 unit of labour and ϕ the inverse of the Frisch elasticy of labour supply.

On top of the Euler equation for stock, the intra-temporal optimality condition for labour supply is given by

$$\begin{aligned} (CD_t^i)^{-\gamma} &= \delta \mathbb{E}_t^{\mathcal{P}} \left(\left(\frac{D_{t+1}}{D_t} \right)^{1-\gamma} \frac{(PD_{t+1} + 1)}{PD_t} (CD_{t+1}^i)^{-\gamma} \right) = \Psi(X_t^i; \chi)^{-\gamma} \\ \lambda L_t^{\phi} &= w_t C_t^{-\gamma} \end{aligned} \tag{2.53}$$

where we have already imposed that the labour market clears.

We assume for now also that the individual agent has perfect information about the production technology and the problem of the firm to which we turn next.

Firm

Assume the firms have production technology that depends only on labour, $Y_t = A_t L_t^{\alpha}$ where A_t is the productivity assumed to be exogenous. The problem of the firm is to maximize profits

$$\max_{L_t} A_t L_t^{\alpha} - w_t L_t \tag{2.54}$$

with FOC

$$w_t = \alpha A_t L_t^{\alpha - 1} \tag{2.55}$$

PEA algorithm. We adapt the previous algorithm to the case of endogenous labour supply and production. First, we need to modify the functional form used for the parameterized expectations since we have an additional state variable: technology shocks. This problem does not allow close form solution or at least I could not find one but a technology shock should increase (in RBC framework) both wages, labour and consumption and therefore can be seen as a shock in wealth.

In this case:

$$CD_t = \Psi_c = \Psi_c(S_{t-1}, PD_t, A_t, \beta_t^i) = B(\beta_t)\Big((PD_t + 1)S_{t-1} + \chi_2 \frac{A_t}{D_t}\Big) \tag{2.56}$$

where

$$B_t = B(\beta_t) = 1 - \chi_1 \beta_t. \tag{2.57}$$

Combining the wage equation 2.55 with the optimality condition on wages 2.53, we obtain

$$L_t = \left(\frac{\alpha}{\lambda} A_t \Psi_c D_t^{-\gamma}\right)^{\frac{1}{1+\phi-\alpha}}. \tag{2.58}$$

The consumption policy function at $t+1$ under subjective beliefs reads

$$CD(\chi)_{t+1}^{i,\mathcal{P}} = (1-\chi_1)\beta_{t+1}^{\mathcal{P}}) \left[\left(\left(\frac{P_{t+1}}{D_{t+1}}\right)^{\mathcal{P}} + 1\right) S_t + \chi_2 A_{t+1}\right] \tag{2.59}$$

and the stock holding can be recovered from the budget constraint

$$S_t = \frac{(1-B_t)\Big((PD_t + 1)S_{t-1}^i\Big) + \frac{w_t L_t - B(\beta_t)\chi_2 A_t}{D_t}}{PD_t}. \tag{2.60}$$

Given a positive technology shock coupled with high subjective capital gains expectations we should expect a higher demand for stock.

The equilibrium *PD* becomes

$$\frac{P_t}{D_t} = \frac{(1-B_t)}{B_t} + \frac{w_t L_t - B(\beta_t)\chi_2 A_t}{D_t B_t} \tag{2.61}$$

D.3. Heterogeneous Agents

Consider the economy described in the main body of Chapter 2.

Solution algorithm for one asset.

The concavity of the objective function and the convexity set guarantee the sufficiency of the first-order conditions for an interior optimal plan. The optimal condition for the household plan is given by the Euler equation:

$$(CD_t^i)^{-\gamma} = \delta \mathbb{E}_t^{\mathcal{P}} \left(\left(\frac{D_{t+1}}{D_t}\right)^{1-\gamma} \frac{(PD_{t+1}+1)}{PD_t} (CD_{t+1}^i)^{-\gamma}\right) = \delta \mathcal{E}(X_t^i) \tag{2.62}$$

where X_t^i are the state variables. The problem is that this Euler Equation includes an unknown conditional expectation. To solve the model, it must be computed somehow. The Parameterized Expectations Algorithm (PEA) is one of the alternatives. PEA consists of replacing the conditional expectation $\mathcal{E}(X_t^i)$ by some parametric function ψ (Marcet, 1988). The choice of the approximating functions ψ is not obvious and not unique. Popular possibilities are polynomials, splines, neural networks, etc. We follow the approach outlined by Belda, 2022: use approximating functions rooted in economic theory. Among the advantages of that approach is the possibility of getting closed-form solutions. Altogether, we follow the next steps

1. Approximate the consumption policy using a

$$CD_t^i = CD(S_{t-1}^i, PD_t, WD_t, \beta_t^i) = B(\beta_t^i)\Big(WD_t + (PD_t + 1)S_{t-1}^i\Big) \tag{2.63}$$

$$B_t^i = B(\beta_t^i) = 1 - \chi^i \beta_t^i \tag{2.64}$$

   where χ^i is an unknown parameter which will be estimated via PEA to be discussed below. The consumption policy function is linear in wealth and the propensity to consume depends negatively on expectations.

2. Obtain the stock holdings policy function by plugging the consumption policy in the budget constraint:

$$S_t^i = (1 - B_t^i)\frac{\Big(WD_t^i + (PD_t + 1)S_{t-1}^i\Big)}{PD_t}. \tag{2.65}$$

3. Determine market-clearing prices by adding individual demands, equating them to the aggregate supply and solving for prices. In this case,

$$\frac{P_t}{D_t} = \frac{\sum_{i=1}^M \mu_i (1 - B_t^i)(S_{t-1}^i + \frac{W_t^i}{D_t})}{S^s - \mu_i \sum_{i=1}^M (1 - B_t^i) S_{t-1}^i}. \tag{2.66}$$

The only unknown at this point is the parameter χ^i from equation 2.73. To obtain this parameter we make use of PEA on the first order condition of the agent which we rewrite as

$$(CD_t^i)^{-\gamma}\delta^{-1} = \mathbb{E}_t^{\mathcal{P}}\left(\left(\frac{D_{t+1}}{D_t}\right)^{1-\gamma}\frac{(PD_{t+1} + 1)}{PD_t}(CD_{t+1}^i)^{-\gamma}\right). \tag{2.67}$$

Then, we run the previous PEA algorithm to estimate χ.

Solution algorithm for two assets.

We consider now the case in which there is another asset, a risk free bond on top of the risky asset. We provide below an efficient algorithm to solve for the equilibrium in the economy with two assets. Let q_t be the price of a bond that pays 1 unit of consumption with certainty at time $t+1$. Letting $BD_t = \frac{B_t}{D_t}$, the budget constraint of the households becomes (after normalizing by dividends, D_t)

$$CD_t + q_t BD_t^i + PD_t S_t = PD_t(S_{t-1}+1) + BD_{t-1}^i \Delta D_t + WD_t \tag{2.68}$$

The optimal condition for the households' consumption plan is as before :

$$(CD_t^i)^{-\gamma} = \delta \mathbb{E}_t^{\mathcal{P}}\left(\left(\frac{D_{t+1}}{D_t}\right)^{1-\gamma}\frac{(PD_{t+1}+1)}{PD_t}(CD_{t+1}^i)^{-\gamma}\right) = \delta\mathcal{E}(X_t^i) \tag{2.69}$$

where X_t^i are the state variables. On top of this we have the optimal condition for bond holdings

$$q_t = \delta \mathbb{E}_t^{\mathcal{P}}\left(\left(\frac{CD_{t+1}^i}{CD_t^i}\Delta D_{t+1}\right)^{-\gamma}\right) \tag{2.70}$$

Notice that this equation must hold true for each individual agent and if we would parameterize directly this equation the system would be over-determined and we could not solve for prices and bond portfolios.

The bond Euler equation can be re-written as

$$(CD_t^i)^{-\gamma} = \delta\frac{\mathbb{E}_t^{\mathcal{P}}\left(\left(CD_{t+1}^i \Delta D_{t+1}\right)^{-\gamma}\right)}{q_t}. \tag{2.71}$$

Notice that this additional condition together with 2.69 implies that under the subjective probability measure, $\mathcal{P}$, the discounted returns (under subjective SDF) on bonds and stocks must be the same. I am afraid (almost sure) that in this case we will get unusually high volatility of interest rates. One way to break this is to consider also subjective beliefs for bond prices which will be the next natural extension.

1. Approximate the consumption policy using a

$$CD_t^i = \Psi_c^i = \Psi_c(S_{t-1}^i, PD_t, WD_t, \frac{B_{t-1^i}}{D_t}, \beta_t^i) = B^i(\beta_t^i)\left(WD_t + (PD_t+1)S_{t-1}^i + \frac{B_{t-1}^i}{D_t}\right) \tag{2.72}$$

where

$$B_t^i = B(\beta_t^i) = 1 - \chi_1^i \beta_t^i \tag{2.73}$$

where χ_1^i are unknown parameters (N parameters) that will be estimated via PEA to be discussed below. The consumption policy function is linear in wealth and the propensity to consume depends negatively on expectations.

2. Approximate the FOC with respect to bond holdings. The latter is given by 2.81 and holds for each agent i. If we would approximate the same conditional expectation for each agent the system would be over-identified and we would not be able to solve for B_t^i or prices which we need to simulate the model. Instead, following Marcet and Singleton, 1999, we rewrite the FOC for some agents in the following way (for the case of $N = 3$ agents)

$$\begin{aligned} q_t e^{B_t^1} &= \delta \mathbb{E}_t^{\mathcal{P}_1}\left(\left(\frac{CD_{t+1}^1}{CD_t^1}\Delta D_{t+1}\right)^{-\gamma} e^{B_t^1}\right) = \Psi_b^1 \\ q_t e^{B_t^2} &= \delta \mathbb{E}_t^{\mathcal{P}_2}\left(\left(\frac{CD_{t+1}^2}{CD_t^2}\Delta D_{t+1}\right)^{-\gamma} e^{B_t^2}\right) = \Psi_b^2 \\ q_t &= \delta \mathbb{E}_t^{\mathcal{P}_3}\left(\left(\frac{CD_{t+1}^3}{CD_t^3}\Delta D_{t+1}\right)^{-\gamma}\right) = \Psi_b^3 \end{aligned} \tag{2.74}$$

where

$$\begin{aligned} \Psi_b^i &= B_b^i(\beta_t^i)\left(WD_t + (PD_t + 1)S_{t-1}^i + \frac{B_{t-1}^i}{D_t}\right), \\ B_b^i &= B_b(\beta_t^i) = 1 - \chi_2^i \beta_t^i. \end{aligned} \tag{2.75}$$

Dividing the first equation in 2.74 by the second and second by the third we obtain

$$\begin{aligned} B_t^1 &= log\left(\frac{\Psi_b^1}{\Psi_b^2}\right) \\ B_t^2 &= log\left(\frac{\Psi_b^2}{\Psi_b^3}\right) \\ B_t^3 &= -(B_t^1 + B_t^2) \end{aligned} \tag{2.76}$$

where the last equation follows from the marketing clearing for bonds and voila we hae bond holdings. Bond prices, q_t, are obtained from

$$q_t = \delta \mathbb{E}_t^{\mathcal{P}_3}\left(\left(\frac{CD_{t+1}^3}{CD_t^3}\Delta D_{t+1}\right)^{-\gamma}\right) = \Psi_b^3. \tag{2.77}$$

Now we are back to the main algorithm.

3. Obtain the stock holdings policy function by plugging the consumption policy in the budget constraint:

$$S_t^i = \frac{(1 - B_t^i)\left(WD_t^i + (PD_t + 1)S_{t-1}^i + \frac{B_{t-1}^i}{D_t}\right) - q_t BD_t^i}{PD_t}. \tag{2.78}$$

4. Determine market-clearing prices by adding individual demands, equating them to the aggregate supply, imposing 0 net supply in the bond market and solving for prices. In this case,

$$\frac{P_t}{D_t} = \frac{\sum_{i=1}^{M} \mu_i (1 - B_t^i)(S_{t-1}^i + \frac{W_t^i}{D_t} + \frac{B_{t-1}^i}{D_t})}{S^s - \mu_i \sum_{i=1}^{M} (1 - B_t^i) S_{t-1}^i}. \tag{2.79}$$

The only unknown at this point are the parameters from equation 2.73 $\chi_j^i, j = 1, 2; i = 1 : N$ which we gather in the vector χ. To obtain this parameter vector we make use of PEA on the first order conditions of the agent which we rewrite as

$$(CD_t^i)^{-\gamma}\delta^{-1} = \mathbb{E}_t^{\mathcal{P}}\left(\left(\frac{D_{t+1}}{D_t}\right)^{1-\gamma}\frac{(PD_{t+1} + 1)}{PD_t}(CD_{t+1}^i)^{-\gamma}\right), \tag{2.80}$$

$$q_t = \delta\mathbb{E}_t^{\mathcal{P}}\left(\left(\frac{CD_{t+1}^i}{CD_t^i}\Delta D_{t+1}\right)^{-\gamma}\right). \tag{2.81}$$

The PEA algorithm involves the following steps:

1. Draw a series of the exogenous processes for a large T.

2. For a given $\chi \in 2X\mathbb{R}^N$, recursively compute the series of the endogenous variables.

3. Minimize the Euler Equation square residuals

$$G(\chi) = \underset{\chi}{\operatorname{argmin}} \sum_i \sum_t \left[\left(\left(\frac{D_{t+1}^{\mathcal{P}}}{D_t}\right)^{1-\gamma}\frac{(PD_{t+1}^{\mathcal{P}} + 1)}{PD_t}(CD(\chi)_{t+1}^{i,\mathcal{P}})^{-\gamma}\right) - \frac{(CD(\chi)_t^i)^{-\gamma}}{\delta}\right]^2 + \left[q_t(\chi) - \delta\left[\frac{CD(\chi)_{t+1}^{i,\mathcal{P}}}{CD_t^i(\chi)}\Delta D_{t+1}\right]^{-\gamma}\right]^2 \tag{2.82}$$

Note the interior of the expectation must be computed according to investor's beliefs. Since investors know the process for dividends and wage-dividends, the only problematic objects are PD_{t+1} and CD_{t+1}. Using agents subjective price model

$$\beta_{t+1}^{i,\mathcal{P}} = \beta_t^i \nu_{t+1} \Rightarrow \left(\frac{P_{t+1}}{P_t}\right)^{\mathcal{P}} = \beta_t^i \nu_{t+1} \varepsilon_{t+1}^p \Rightarrow \left(\frac{P_{t+1}}{D_{t+1}}\right)^{\mathcal{P}} = \left(\frac{P_{t+1}}{P_t}\right)^{\mathcal{P}} \frac{D_t}{D_{t+1}} \frac{P_t}{D_t}$$

In turn, expected consumption reads

$$CD(\chi)_{t+1}^{i,\mathcal{P}} = (1 - \chi\beta_{t+1}^{i,\mathcal{P}}) \left[WD_{t+1}^i + \left(\left(\frac{P_{t+1}}{D_{t+1}}\right)^{\mathcal{P}} + 1 \right) S_t^i + \frac{B_t^i}{D_{t+1}} \right]$$

Notice that given consumption policy function 2.72 the price of the bond will depend on the subjective expectations of marginal consumption at $t + 1$ which will depend on B_t^i.

4. Find a fixed point $\chi = G(\chi)$. For that, update χ following

$$\chi^{j+1} = \chi^j + d(G(\chi^j) - \chi^j) \tag{2.83}$$

where j iteration number and d the dampening parameter.

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Chapter 3

The Fiscal Channel of Quantitative Easing

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Abstract

This paper is a theoretical examination of the role of fiscal distortions in shaping the effects of Quantitative Easing (QE). The presence of deadweight losses from taxation breaks Wallace's neutrality since QE influences the level and volatility of such losses. Under some conditions, QE can stimulate demand by removing tax distortions, but it increases the risk premium. This differs from the standard view that QE stimulates demand precisely by lowering risk premiums due to the relaxation of financial frictions. A Central Bank must strike the right balance between the efficiency gain of more QE against the additional risks it entails. By exploiting the risk premium from capital ownership, QE emerges as an alternative to costly taxation, suggesting an efficiency-risk trade-off for public finances.

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3.1. Introduction

This paper is a theoretical examination of the role of fiscal distortions in shaping the effects of Quantitative Easing (QE). The presence of tax deadweight losses breaks Wallace, 1981's irrelevance, making asset purchases capable of influencing aggregate demand and risk premiums but in a different way than commonly thought. Our goal is to uncover this fiscal channel and study its consequences for the conduct of QE.

Fiscal policy is paramount to determining the Central Bank's ability to affect the economy via asset purchases (Wallace, 1981, Leeper and Leith, 2016, Benigno and Nisticò, 2020). This is because asset purchases originate gains and losses that enter the consolidated budget constraint of the State and must be balanced out by movements in other fiscal or monetary items. How fiscal variables react determines then the autonomy degree of monetary policy and the ultimate effects of asset purchases.

Following Wallace, 1981, a common assumption in the literature is that fiscal policy offsets QE gains/losses by using lump-sum taxes. Thus, the differences in interest earnings implied by QE are paid out in the form of taxes to the asset-holders that participate in the QE program. In this way, the allocation of resources is unchanged; the same agents get the same flow of resources, although under a different cover -a fiscal transfer instead of an asset payoff. As a result, the change in the public portfolio does not affect the competitive equilibrium. In other words, full fiscal support (in the sense of Del Negro and Sims, 2015) implemented via lump-sum taxes makes QE irrelevant.

Some papers deviate from Wallace's fiscal policy by assuming that fiscal deficits do not react to QE flows.[1] Without fiscal adjustment, there is a reallocation of resources and risk between the private and public sectors that impacts the equilibrium. For instance, in the event of a loss that is transferred to a Treasury with a given path of deficits, public debt must buffer it out, representing a wealth transfer to the private sector and a potential need for an alternative monetary policy path to stabilize the debt, both factors inducing inflation.[2] The role of sticky taxes in shaping QE outcomes has been studied in detail by Benigno and Nisticò, 2020 and also in Hollmayr and Kühl, 2019; and has been used to study issues such as debt sustainability (Elenev et al., 2021) or QE's exit strategies (Airaudo, 2022).[3]

[1] This is the case, for instance, of some active fiscal policy, in the sense of **leeper1991equilibria**. However, active fiscal policy does not break neutrality by itself, as there exist fiscal rules that qualify as active but still fully react to QE flows, as shown by Benigno and Nisticò, 2020.

[2] Alternatively, the loss can be kept within the Central Bank, perhaps as a deferred asset. This can be simply viewed as another form of public debt, backed by seignorage. If the debt is big enough, it might need an alternative monetary policy to be stabilized, breaking neutrality. Given this equivalence, in the paper, we assume full fiscal support; see Benigno and Nisticò, 2020 for deviations from this.

[3] Elenev et al., 2021 is the only that considers distortionary taxes. As a result, there is a fiscal channel of QE since it affects debt, taxes and then the allocation of labour. However, they don't really explore this channel,

Nonetheless, what if the Government fully support the Central Bank but finds it easier to cut spending rather than to raise taxes in the event of a loss? Or to finance a targeted public program instead of transferring the resources back to the investors? What if, despite wanting to pass the gains or losses to the asset-holders that participated in the program, lump-sum taxes are not available? If public spending is adjusted and the public goods financed or defunded are not perfect substitutes for the private consumption of the asset-holders, QE ends up causing a reallocation of resources between the public sector and the investors. Besides, adjustments in costly taxes can influence tax deadweight losses, factor allocations, etc. Both adjustments imply the non-neutrality of asset purchases. They are the focus of this paper.

To study these cases, we use a two-period endowment stochastic economy, with a Lucas tree, a society of identical investors and an institutional framework characterized by: i) an independent Central Bank that is fully supported by the Government, in the sense of Del Negro and Sims, 2015; ii) a Government that uses costly taxes to finance valuable public goods. We abstract from inflation and nominal variables, such that all the adjustments are in terms of goods. This, along with the risk-free nature of the public debt, implies a passive fiscal policy.[4] The Central Bank can only choose its balance sheet, that is, it chooses the holdings of the private risky asset financed by issuing risk-free public liabilities.[5] The independence of the Central Bank is reflected in the timing; it is a Stackelberg leader that decides its balance sheet policy in the first period. Given that policy, investors save and consume, and later, in the second period, the Government chooses taxes and spending in a discretionary way, after observing the gains or losses originated by QE.

The setup differs from the literature in one key dimension. Instead of imposing an exogenous public spending and forcing tax adjustments or, alternatively, conjecturing some ad-hoc fiscal rules, we endogenously derive the fiscal reaction functions by asking the Government to achieve the best possible equilibrium. Thus, fiscal policy functions depend on the structure of the economy in a precise way. To make this problem interesting, we add two important ingredients. First, tax deadweight losses. Following Bohn, 1992, we use a reduced-form formulation whereby some resources are lost in taxation, alluding to various possible reasons such as collection costs, allocation distortions, resources devoted to tax evasion activities and the likes. Second, public goods are valuable. This introduces a trade-off between costly taxation and desirable spending that might resemble the one actually faced by governments.

which operates together with several financial frictions and changes between active-passive regimes. We do the opposite, focusing exclusively on this channel.

[4]Our mix of passive fiscal policy and full fiscal support is one of the examples of neutrality used by Benigno and Nisticò, 2020. We show how non-neutrality can emerge even within this setup, which, for instance, closely follows the institutional arrangement in the UK.

[5]This particular definition of QE facilitates some of the derivations. However, it is not crucial at all for our results, which holds as long as the Central Bank finance the program by issuing liabilities that are less riskier than the assets it acquires.

A new non-neutrality result emerges in this economy, related to the effects triggered by fiscal adjustment to QE.[6] Consider a QE program originating some gains. A rational government facing costly taxes and certain demand of public goods would rationally choose to react to the gains by lowering taxes and increasing spending, with an intensity depending on the severity of tax costs. Lower taxes reduce the tax deadweight losses; this efficiency gain is distributed between private and public spending, increasing ex-post welfare in the last period.

What are the consequences for period-1 variables? Forward-looking investors adjust their expectations on future consumption, becoming more optimistic, which leads to an increase in first-period consumption due to a consumption smoothing motive. On the downside, the efficiency gain increases the exposure of future consumption to output fluctuations; this generates a precautionary savings motive. Depending on the relative strength of each effect, QE can stimulate or depress the private goods demand and then deflate or inflate asset prices. Finally, the risk premium is also affected since QE increases the covariance between private consumption and the asset payoff. Thus, if QE deflates asset prices, the higher covariance would deflate risky prices further, widening the risk premium. Altogether, QE moves the economy to a new equilibrium with a higher mean-variance consumption profile, with effects on aggregate demand, asset prices and risk premiums.

This channel implies a boost (dampening) of aggregate demand, asset price deflation (inflation) and risk premium widening (narrowing) simultaneously. This differs strongly from standard views of QE, whereby the reduction of long-run interest rates (and risk premiums) cohabits with higher aggregate demand. In fact, popular models conjecture that QE reduces risk premiums due to market segmentation or other financial frictions, boosting then aggregate demand (e.g., Gertler and Karadi, 2011, Vayanos and Vila, 2021). On the contrary, in our model, aggregate demand is stimulated by removing tax distortions from the economy, but at the cost of increasing the risk, which eventually affects the risk premiums. In this sense, the fiscal channel is a complementary transmission channel that might counteract some of the effects of the financial channels, perhaps helping to understand the lasting uncertainty about QE's effects. Additionally, the relevant dimension of QE is the stock rather than the flow, as in Harrison, 2017, due to its relation with the level of tax deadweight losses.

If swapping risk-free for riskier assets impacts the competitive equilibrium simply due to the presence of costly taxation and valuable public goods, a natural question is: How much QE should be done? We answer the question by asking the Central Bank to choose the quantity Q of risky asset purchases to maximize the expected social welfare taking into account the optimal reactions of all the other agents to its policy. It turns out

[6] Note this is different from an active tax rule that does not react to QE flows and forces monetary policy to act. Here fiscal policy acts and this action triggers the effects in the model. In this sense, this is a fiscal transmission channel of QE, rather than a monetary policy channel induced by fiscal inaction.

that under linear tax costs, the efficiency gain always dominates the higher risk. More QE is always better: less distortions without too much additional risk ($Q^* = 1$). On the contrary, introducing quadratic costs delivers a $Q* < 1$. The key factor is the marginal productivity of tax cuts. With linear costs, it was constant; every additional unit of QE generates the same efficiency gain such that it always dominates the additional risk brought up by a larger QE program. With quadratic costs, however, the efficiency gains are decreasing with QE such that, at some point, QE begin to deteriorate the mean-variance consumption equilibrium.

All in all, QE emerges as an alternative way of collecting resources for the State: exploiting the risk premium from capital ownership rather than imposing costly taxes on households. In this vein, it shares with Farhi, 2010 the exploration of capital ownership as an optimal alternative to taxation. In our economy, ownership is more efficient but at the price of increasing risk-taking. Thus, the paper suggests an efficiency-risk trade-off for public finances.

Practically, fifteen years of widespread QE employment (or more than twenty for Bank of Japan) has opened up the prospect of a more permanent or conventional use of QE. As *normalization* of monetary policy takes hold, there is a reasonable challenge regarding which of the unconventional measures may end up in the conventional monetary toolkit of the future. This paper provides a rationale and goal for a more conventional use of QE. Looking ahead, we show how QE can be used to finance valuable public goods, complementing the view described in Reis, 2017. This has sometimes been called the "Fiscal QE" (Selgin, 2020). Recent targeted QE programs, such as Bank of England's and ECB's Green Corporate Bond Programs or the ECB's Transmission Protection Instrument, can be seen as examples of this use.

The rest of the paper is structured as follows. Section 2 sets out the model and solves the fiscal and savings-consumption problems. Section 3 deals with the Central Bank problem.Section 4 concludes pointing out some promising extensions.

3.2. QE and Fiscal Policy

In this section, we describe a two-period model economy and use it to study QE. The model abstracts from all sorts of frictions considered in the QE literature, and focuses on fiscal elements that allude to tax collection costs and the utility of public goods. QE consists of purchasing risky private assets by issuing risk-free public assets.[7] The problem is set in two stages and solved backwards. In the last stage, the consolidated government observes QE gains or losses and decides how to adjust taxes and spending to satisfy the

[7] We assume QE buys risky private assets instead of long bonds, but the results go through with long-term public bonds as well. We key is to buy an asset with a higher mean-variance profile.

budget constraint. In the second stage, investors solve a consumption-savings problem before knowing QE gains or losses but anticipating how the government would eventually react to them. We analize the cases that make QE (non-)neutral and uncover a fiscal channel related to the efficiency gain of QE due to the removal of tax deadweight losses.

3.2.1. The model

The economy is populated by a continuum of measure 1 of identical investors. They last for 2 periods, indexed by t = 0, 1, 2. There is a single perishable good in the economy that also acts as the numeraire of the economy. There exist two assets: a single risky asset, call it "stock" S, in fixed supply in the form of a contract that delivers D_t goods each period and is marketable at an uncertain price P; a safe public bond B, that is issued at discount 1/R and delivers 1 unit of goods with certainty. When the time starts, each investor is endowed with one unit of the stock ($S^i_{-1} = 1$). Payments D_t are exogenous and stochastic, following a Normal distribution with mean μ and standard deviation σ. This is the only source of risk in the economy.

Financial markets are competitive but incomplete, as output D can materialize in a continuum of outcomes, but there are only two assets available. The goods market behaves also competitively. Investors possess full information about the economy's structure and are rational.

We consider a State that participates in the economy by determining monetary and fiscal variables. In particular, it is in charge of public spending G_t; costly taxes T_t with an associated tax cost function $H : T \rightarrow \mathbb{R}$ with $0 < H'(T) < 1$ such that the government has to collect 1+H(T) units of goods from the private sector to be able to spend 1 unit with $H(T) = \alpha T$ for $\alpha > 0$;[8] risk-free government debt B; and purchases of risky assets QE. Assume that the economy starts without debt. Thus, the State budget constraints read as:

$$G_0 + QP = T_0 + \frac{B}{R} \tag{3.1}$$

$$G_1 + B = T_1 + D_1 QE \tag{3.2}$$

These constraints can be collapsed into this intertemporal constraint

$$\underbrace{Q\left(P - \frac{D_1}{R}\right)}_{\text{QE losses}} = \underbrace{T_0 + \frac{T_1}{R} - G_0 - \frac{G_1}{R}}_{\text{Primary Surplus}} \tag{3.3}$$

[8] This is a reduced form for distortionary taxes that simplifies some computations. Bohn, 1992 shown its equivalence with labor income taxes. It can also be broadly related to the Okun's "leaky bucket", whereby resources spent by the government are less than the ones collected, due to all sort of potential inefficiencies, distorted decisions, etc. The reduce-form remains silent about the sources of such inefficiencies.

that points out that the present value of the primary surplus must offset QE losses. Note this model implies there is fiscal support in the sense of Del Negro and Sims, 2015 and fiscal policy is passive since it adjusts surpluses given the actions of the Central Bank. In the literature, typically (G_0, G_1) are exogenous, for instance $(0, 0)$, such that passive government must adjust taxes. On the contrary, we allow the government to choose G, giving rise to infinitely many possible combinations of taxes and spending adjustments satisfying the intertemporal budget constraint.

Private investors solve a standard savings-consumption problem. We assume welfare depends on current consumption and a convex combination of utility derived from future consumption and public spending, with y denoting the weight attached to the utility derived from consumption. Utility is given by a CARA function $u(x) = \frac{-1}{\gamma} exp(-\gamma x)$, with γ being the parameter of absolute risk aversion.

Competitive Equilibrium. Given $S^i_{-1} = 1$, a Competitive Equilibrium is a vector of non-negative asset prices $\{P, 1/R\}$ and allocations $\{C^i_0, C^i_1, S^i, B^i\}$ indexed by an economic policy made of a fiscal policy $\{G_0, G_1, T_0, T_1\}$ and a balance sheet policy $\{B, QE\}$ that satisfies:

1. Investor's Euler Equations for stocks and bonds.
2. Investor's budget constraints.
3. The State's intertemporal budget constraint.
4. Assets market clearing conditions

$$\int_0^1 S^i di + QE = 1; \quad \int_0^1 B^i di = B \tag{3.4}$$

There are 12 endogenous variables and 7 optimality conditions. It follows that economic policy needs to target 5 variables out of $\{G_0, G_1, T_0, T_1, B, QE\}$. Without loss of generality, assume there is no public spending and tax collection in period 0. QE is defined as the vector $\{B/R, QE\} = \{QP, Q\}$. The remaining two fiscal variables, T_1 and G_1, will be set optimally by the government to absorb the flows originated by QE.

3.2.2. The fiscal problem

QE originates a flow of funds in the government's consolidated budget constraint. Let $X = Q(D_1 - PR)$ be such flow. Standard practice is to assume $G_1 = 0$ such that the period-1 budget constraint would impose $T_1 = -X$. Instead, we allow the government to select how to react to QE rationally once all the outcomes have been observed. The rational response, call it T^* and G^*, is given by

$$T^*, G^* = arg \max_{\{T,G\}} yu(C_1) + (1-y)u(G) \tag{3.5}$$

subject to

$$G + B = T + QD_1 \tag{3.6}$$

$$C_1 + T + H(T) = SD_1 + B \tag{3.7}$$

Given $S = 1 - Q$, $B = QPR$, P and R. Thus, fiscal policy exhibits no commitment; it is selected to maximize ex-post social welfare subject to period-1 restrictions and taken as given period-0 equilibrium outcomes.

The FOC determining optimal taxes is equal to:

$$y(1+\alpha)exp\{-\gamma C_1\} = (1-y)exp\{-\gamma G\} \tag{3.8}$$

Manipulating this expression a bit using investor's and gov's budget constraints:

$$ln[y(1+\alpha)] - ln(1-y) = -\gamma(G - C_1) = -\gamma(X + T - D_1 + X + T(1+\alpha)) = -\gamma(2X + (2+\alpha)T - D_1) \tag{3.9}$$

Solving for T:

$$T^* = \frac{1}{2+\alpha}D_1 - \frac{2}{2+\alpha}X - \underbrace{\frac{ln[y(1+\alpha)] - ln[(1-y)]}{\gamma(2+\alpha)}}_{\equiv a} \tag{3.10}$$

In words, taxes should increase with output D_1, decline with QE gains X (as QE gains are an alternative way of financing G), decline with distortions α (the more costly taxation is, the less it should be used) and increase with the social weight on public goods $(1-y)$. Using the government's budget constraint:

$$G^* = X + T^* = \frac{1}{2+\alpha}D_1 + \frac{\alpha}{2+\alpha}X - a \tag{3.11}$$

It is optimal for the government to raise spending to offset part of the QE gains, but less than one-to-one (as $\alpha > 0$). If taxes are costless ($\alpha = 0$), $T^*/X = -1$ and $G^*/X = 0$ as in Wallace, 1981 and the literature following him. If public spending is useless ($y = 1$), the government would not spend anything before QE, and then all the adjustments would go through taxes as well.[9]

[9] Imagine that for some reason, $G_1 = \bar{G}$ before QE and $y = 1$. In this case, QE losses would be optimally absorbed by cutting G, and QE gains by lowering T.

3.2.3. The investor's problem

The representative investor solves the following consumption-savings problem at period-0 (superindex i has been eliminated to save notation):

$$\max_{\{C_0, C_1, S, B\}} u(C_0) + \delta \mathbb{E}_0 \{ y u(C_1) + (1 - y) u(G) \} \tag{3.12}$$

subject to

$$C_0 + PS + \frac{B}{R} = (P + D_0) S_{-1} \tag{3.13}$$

$$C_1 + T + H(T) = D_1 S + B \tag{3.14}$$

Optimality conditions boil down to two Euler Equation for bonds and stocks.

$$\frac{1}{R} = \delta y \mathbb{E}_0 \left(\frac{exp\{-\gamma C_1\}}{exp(-\gamma C_0\}} \right) \tag{3.15}$$

$$P = \delta y \mathbb{E}_0 \left(\frac{exp\{-\gamma C_1\}}{exp(-\gamma C_0\}} D_1 \right) \tag{3.16}$$

In equilibrium, individual and aggregate consumption coincide such that $\{C_0, C_1\} = \{D_0, C_1^*\}$. Given the rational reaction of the government in period-1,

$$C_1^* = D_1 - G^* - \alpha T^* = \frac{1}{2+\alpha} D_1 + \frac{\alpha}{2+\alpha} X + a(1+\alpha) \tag{3.17}$$

Equilibrium consumption grows with output, and QE gains. Notice there is a constant gap between private and public consumption determined by $C_1^* - G^* = (ln[y(1+\alpha)] - ln[(1-y)])\gamma^{-1}$. Intuitively, the gap increases with both the private consumption weight and the cost of taxation. Apart from that, consumption and public spending react symmetrically to output and QE gains.

Using this expression in the Euler Equations and operating, it can be shown that equilibrium asset prices are given by

$$\frac{1}{R^*} = \delta y exp\left\{ -\gamma \left(a(1+\alpha) + \frac{\mu}{2+\alpha} + \frac{\gamma \sigma^2}{2(2+\alpha)^2} (\alpha^2 Q^2 - 1) - D_0 \right) \right\} \tag{3.18}$$

and

$$P^* = \frac{\mu - \gamma \left(\frac{1+\alpha Q}{2+\alpha} \right) \sigma^2}{R^*} \tag{3.19}$$

From these expressions, we can evaluate the effect of QE on demand, asset prices and the risk premium. To begin with,

$$\frac{1/R^*}{Q} = \delta yexp\Big\{-\gamma\Big(a(1+\alpha)+\frac{\mu}{2+\alpha}+\frac{\gamma\sigma^2}{2(2+\alpha)^2}(\alpha^2Q^2-1)-D_0\Big)\Big\}\Big(-\frac{\gamma\sigma^2\alpha^2 2Q}{2(2+\alpha)^2}\Big) < 0 \tag{3.20}$$

The effect of QE on bond prices is unambiguously negative. To understand the reason behind it, rewrite the bond price as

$$\begin{aligned}\frac{1}{R} =&\delta y\mathbb{E}_0\left(\frac{exp\{-\gamma C_1^*\}}{exp(-\gamma C_0^*\}}\right) = \delta yexp(\gamma D_0)\mathbb{E}_0\left(exp\{-\gamma(C_1^*)\}\right)\\ =&\delta yexp(\gamma D_0)exp\{-\gamma\mu_c + \frac{\gamma^2\sigma_c^2}{2}\}\end{aligned} \tag{3.21}$$

Hence, the effect of QE on bond prices is transmitted through its effects on consumption mean and variance. How is this effect? Using the equilibrium period-1 consumption (3.17), it turns out

$$\mathbb{E}(C_1^*) \equiv \mu_c = \frac{\mu}{2+\alpha} + \frac{\alpha\gamma\sigma^2 Q(1+\alpha Q)}{(2+\alpha)^2} + a(1+\alpha) \tag{3.22}$$

$$\mathbb{V}ar(C_1^*) \equiv \sigma_c^2 = \left(\frac{1+\alpha Q}{2+\alpha}\right)^2\sigma^2 \tag{3.23}$$

QE pushes the mean-variance consumption equilibrium up. A higher mean pushes bond prices down, as higher future consumption leads to reduce savings today (consumption smoothing). On the contrary, a higher variance pushes bond prices up due to a boost in precautionary savings. Which effect does dominate? It turns out the distance between the mean and the risk-weighted variance that determines the QE effect on bond prices increases with Q:

$$\frac{(\mu_c - \gamma\sigma_c^2/2)}{Q} = \frac{\alpha^2\gamma\sigma^2}{(2+\alpha)^2}Q \tag{3.24}$$

Therefore, a larger QE increases the expected future consumption more than its risk, with the net effect of reducing savings and asset prices in period 0.

Behind the increase in mean consumption, there is a reduction in the tax deadweight loss αT. This is easy to check since $C_1^* = D_1 - G^* - \alpha T^*$, and the dividend mean and the gap between consumption and public spending are both unaffected by Q. A lower tax deadweight loss implies investors and government enjoy a larger amount of goods, becoming both more exposed to goods' volatility.

To understand the effect of QE on stock prices, equation (3.19) can be rewritten as

$$P = \frac{1}{R^*}\mu - \mathbb{C}ov(exp\{-\gamma(C_1 - C_O)\}, D_1) \tag{3.25}$$

with

$$\mathbb{C}ov(exp\{-\gamma(C_1 - C_O)\}, D_1) = \frac{\gamma\left(\frac{1+\alpha Q}{2+\alpha}\right)\sigma^2}{R^*} \tag{3.26}$$

Thus, the effect of QE on stock prices can be decomposed into two terms. First, stock and bond prices comove positively such that QE deflates stock prices for the same reasons it reduces bond prices. Additionally, the stock price decreases more than the bond price with Q because QE increases the covariance between consumption and the asset payoff. The asymmetric asset price deflation leads to a widening of the risk premium.

All the previous derivations can be summarized in the following two results:

Result 1: QE non-neutrality with tax deadweight losses. If $\{C_0, C_1, B, S, P, 1/R\}$ is an equilibrium for the policy $\{G_0, G_1, T_0, T_1, B/R, QE\} = \{0, G, 0, T, QP, Q\}$, then $\{C_0, C_1, \hat{B}, \hat{S}, P, 1/R\}$ is an equilibrium for the policy $\{0, \hat{G}, 0, \hat{T}, \hat{Q}P, \hat{Q}\}$ only if $\alpha = 0$ (*Neutrality with lump-sum taxes*). For $\alpha > 0$, $\{0, \hat{G}, 0, \hat{T}, \hat{Q}P, \hat{Q}\}$ implies a different equilibrium $\{\hat{C}_0, \hat{C}_1, \hat{B}, \hat{S}, \hat{P}, 1/\hat{R}\}$ (*Non-neutrality with tax collection costs*).

Result 2: The fiscal channel of QE. A program of asset purchases Q financed by issuing risk-free public debt $B/R = QP$ in an economy where collecting taxes is costly ($\alpha > 0$) and public goods are of some utility ($\gamma < 1$) has the following consequences:

- Efficiency gain. QE reduces the tax deadweight loss, increasing expected future consumption. This increases consumption at time 0 due to a consumption smoothing motive.
- Higher risk. The gain in efficiency increases the consumption exposure to output fluctuations; this generates a precautionary savings motive at time 0.
- Private demand stimulus. The efficiency gain dominates the higher risk, and its distance increases with Q. The increase in consumption dominates the precautionary savings motive.
- Asset price deflation. The precautionary savings motive is dominated.
- Risk premium widening. QE increases the covariance between consumption and the asset payoff.

3.3. The Central Bank problem

The previous section showed that swapping risk-free for riskier assets impacts the competitive equilibrium in the context of costly taxation. A natural question is: with that knowledge, how much QE should be done? We answer the question by posing an optimal policy problem for the agency determining QE. It turns out that linear tax collection costs lead to a corner solution. We then explore the question with quadratic collection costs, which poses a true trade-off for QE.

3.3.1. Linear tax collection costs

Assume $H(T) = \alpha T$, as in the previous section. In this context, a public agency (typically, a Central Bank) would choose the optimal QE, call it Q^* such that

$$Q^* = arg \max_{Q} \mathbb{E}_0\{yu(C_1) + (1-y)v(G_1)\} \tag{3.27}$$

given the equilibrium policy functions C_1^*, G^*, T^* and the equilibrium pricing functions P^*, R^*. In words, the Central Bank chooses Q to maximize the expected welfare in the last period taking into account the optimal reactions of all the other agents to its policy.

From the previous section, we know that both C_1^* and G^* are normally distributed as they depend only on D_1. The mean and variance of equilibrium consumption are given by () and (). Likewise, for public spending they read as

$$\mathbb{E}(G^*) \equiv \mu_g = \frac{\mu}{2+\alpha} + \frac{\alpha\gamma\sigma^2 Q(1+\alpha Q)}{(2+\alpha)^2} - a = \mu_c - a(2+\alpha) \tag{3.28}$$

$$\mathbb{V}ar(G^*) \equiv \sigma_g^2 = \left(\frac{1+\alpha Q}{2+\alpha}\right)^2 \sigma^2 = \sigma_c^2 \tag{3.29}$$

Note that maximizing

$$-\frac{y}{\gamma} exp\left\{-\gamma\mu_c + \frac{\gamma^2\sigma_c^2}{2}\right\} - \frac{1-y}{\gamma} exp\left\{-\gamma\mu_g + \frac{\gamma^2\sigma_g^2}{2}\right\} \tag{3.30}$$

is equivalent to maximize

$$\begin{aligned} &-\frac{y}{\gamma}\left(-\gamma\mu_c + \frac{\gamma^2\sigma_c^2}{2}\right) - \frac{(1-y)}{\gamma}\left(-\gamma\mu_g + \frac{\gamma^2\sigma_g^2}{2}\right) \\ &=\mu_c - \frac{\gamma\sigma_c^2}{2} + (1-y)a(2+\alpha) \end{aligned} \tag{3.31}$$

where the second line uses the equivalences between the mean and variances of C and G. Altogether, the Central Bank's problem boils down to choosing the Q that maximizes the distance between the consumption mean and variance, accounting for the level of risk aversion

$$\begin{aligned} Q^* =& arg \max_Q \mu_c - \frac{\gamma \sigma_c^2}{2} \\ =& arg \max_Q \frac{\gamma \sigma^2}{2(2+\alpha)^2}(\alpha^2 Q^2 - 1) \end{aligned} \tag{3.32}$$

where the last equality uses the expressions for μ_c and σ_c^2. Then, it is optimal for the Central Bank to do as much QE as possible; in our economy, $Q^* = 1$ (since the total stocks in the economy are normalized to 1). The reason was apparent already in the previous section: the more the QE, the more the efficiency gain dominates the increase in risk-taking; the more the QE, the larger the risk premium so that the more the taxes can be reduced and the more efficient the economy gets. QE represents a more efficient way of funding public goods, trading a larger government's balance sheet with a lower tax economy.

3.3.2. Quadratic tax collection costs

Consider $H(T) = \alpha T^2$, capturing the possibility that tax collection is increasingly costly, perhaps due to the complexity of managing a higher volume of resources. The introduction of quadratic costs makes the closed-form solution infeasible but enrich the properties of the model. The key factor is the marginal productivity of tax cuts. With linear costs, it was constant; every additional unit of QE generates the same efficiency gain such that it always dominates the additional risk brought up by a larger QE program. On the contrary, with quadratic costs, the efficiency gains are decreasing with QE such that, at some point, QE begin to deteriorate the mean-variance consumption equilibrium, reverting the sign of the effects summarized in *Result 2*; more QE increases precautionary savings, driving asset prices up and spreads down.

Figure 3.1 plots many of the model's variables as a function of Q, illustrating *Result 2* and its reversion with quadratic deadweight losses. As shown analytically for the linear case, Q^* is the one that delivers the best mean-variance equilibrium for private future consumption, balancing the gains from lower taxes against the . For quadratic tax collection costs, that point is reached for $Q^* < 1$, given the decreasing marginal gains from lower taxes. Interestingly, for the quadratic case, this point coincides with a minimum for asset prices but not with a maximum for the expected consumption and risk premium (that peak at $Q > Q^*$). In simulations, we found that Q^* decreases with the asset's payoff risk and the weight of private consumption on the welfare function but increases with the level of risk aversion.

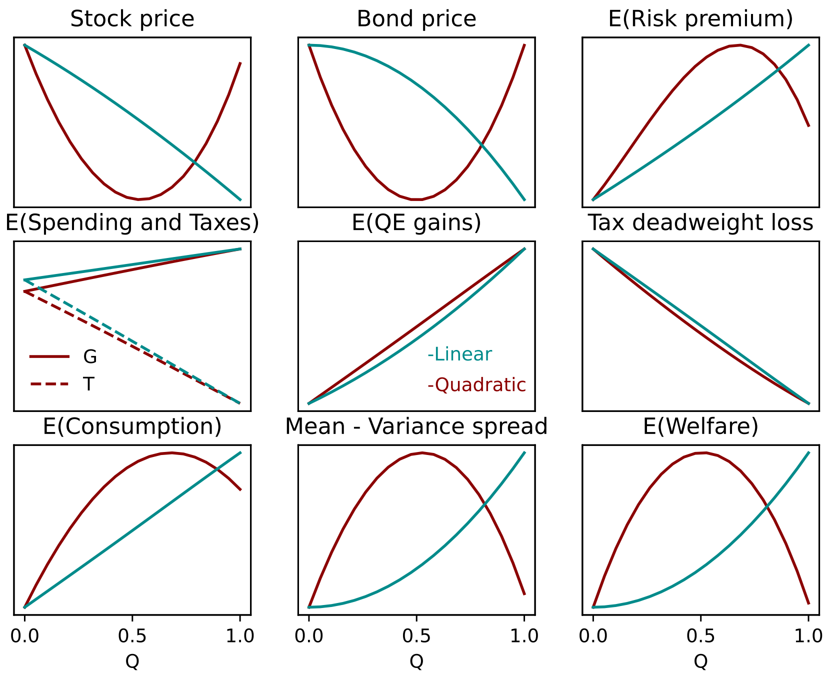

Figure 3.1. **Effects of Quantitative Easing with Linear (blue) and Quadratic (red) tax collection costs.**

3.4. Conclusions

This paper explores the effects of public risky asset purchases financed by risk-free public liabilities in the context of fiscal distortions. These distortions break Wallace's neutrality even in the context of monetary dominance since the profits from these purchases affect taxes and their associated deadweight loss. However, the effects of asset purchases are different from the ones emerging from models with financial frictions. Typically QE relaxes some financial constraints, lowering risk premiums and then, boosting aggregate demand. Differently, through the fiscal channel, QE reduces tax distortions but increases the consumption risk, potentially boosting demand but also risk premiums.

The fiscal channel poses an efficiency-risk trade-off for QE: more QE removes inefficiencies from the economy but increases private risks. Central Banks can exploit this trade-off to deliver the optimal mean-variance equilibrium. From a broader perspective, QE can be viewed as an alternative to taxation to fund public goods, based on capital ownership and the exploitation of risk premiums.

We presented these arguments in a stylized two-period model that isolates the fiscal channel as much as possible. In this enterprise, we abstract from conventional monetary policy and all the intricacies of monetary economics. The presence of nominal variables and rigidities would introduce additional adjustment mechanisms, such as inflation or output, that might potentially change some of the outcomes. Additionally, an exploration of the interaction between the fiscal channel and other transmission channels seems pertinent towards a holistic evaluation of the possibilities of QE. Moreover, the model misses dynamics, which would make the game between Government and Central Bank more complex. Finally, from a public finance standpoint, the model raises questions for optimal debt management under QE as well as the right mix of capital taxes and capital ownership to fund public goods. We are currently exploring some of these issues.

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