UNIVERSITAT POMPEU FABRA

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Expectations, Interest Rate and Limited Commitment

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A Giorgia e Alice

Contents

Chapter 1

Monetary Policy and the Transition to Rational Expectations

Under the assumption of bounded rationality, economic agents learn from their past mistaken predictions by combining new and old information to form new beliefs. The purpose of this chapter is to examine how the policy-maker, by affecting private agents' learning process, determines the speed at which the economy converges to the rational expectation equilibrium. I find that by reacting strongly to private agents' expected inflation, a central bank increases the speed of convergence.

I assess the relevance of the transition period when looking at a criterion for evaluating monetary policy decisions and suggest that a fast convergence is not always desirable.

1.1 Introduction

In the presence of structural changes, agents in the economy may need time in order to learn about the new environment: in the early stages of this process, previously held beliefs could lead to biased predictions. For this reason, there is now a substantial interest on whether rational expectations can be attained as the outcome of a learning process. In particular, the recent literature on monetary policy has emphasized that while rational expectations is an important and useful benchmark, a policy maker should consider the robustness of any equilibrium reached under a particular monetary policy to deviations from rational expectations. A common way to carry out this idea is to employ the techniques of Marcet and Sargent (1989a, 1989b) and Evans and Honkapohja (2001), assuming that agents in the model form expectations via econometric forecasts. In this environment, Evans and Honkapohja (2003a, 2003b) and Bullard and Mitra (2002) suggest that economic policies should be designed to be conducive to long-run convergence of private expectations to rational expectations (*E-Stability*)[1]. Failing to do so, gives rise to an equilibrium which is not robust to small expectational errors. In accordance with this literature, "good" policies are those that induce a determinate and learnable rational expectations equilibrium (see Bullard and Mitra, 2002).

A small but growing body of research is concerned with the properties of the convergence along the learning process[2]. The works of Giannitsarou (2003), Aoki and Nikolov (2003) and Orphanides and Williams (2003) analyze, in particular, the transition to the rational expectations equilibrium in the context of policy decisions, addressing the question of whether all policies that produce learnability and determinacy are equally good from a learning perspective.

This chapter takes up this point by adapting theoretical results of Benveniste, Metivier and Priouret (1990) and Marcet and Sargent (1995). I start by examining how the policy-maker, by affecting the private agents' learning process, can influence the transition to the rational expectations equilibrium (i.e. the speed of learning). I show that by reacting strongly to expected inflation, a central bank can shorten the length of the transition and increase the speed of convergence to the rational expectations equilibrium. Next, I focus on the notion of speed of learning to develop the analysis along three different directions. First, the concept is used to refine further the set of "good" policies: grounded on the set of policies that imply determinate and learnable rational expectations equilibria, I look at speed of learning as a criterion to characterize its elements. In this sense, I consider as an example, a policy described in Evans and Honkapohja (2003a), *EH policy,* and I show that even though this policy meets all of the objectives listed above (determinacy and learnability) and is optimal under discretion and rational expectations, it is not suitable from the perspective of the speed of learning. Second, I show that policies that drive the economy to the same rational expectations equilibrium may imply very different dynamics along the transition. I use this result in order to show how a policy maker who wants to reach in the long run the same rational expectations equilibrium determined as under the EH policy, can adjust the speed of learning of the private sector. Finally, I conduct a welfare analysis taking the speed of convergence to the rational expectation equilibrium into consideration. The interesting result is that a fast learning is not always desirable. While in the absence of an inflation bias, fast learning always increases

[1] An earlier paper by Howitt (1992) had already shown that under some interest rate rules the rational expectation equilibrium is not learnable.

[2] Papers on this topic include Evans and Honkapohja (1993), Marcet and Sargent (1995), Timmerman (1996), Sargent (1999), Marcet and Nicolini (2003).

the social welfare, in presence of such a bias, the relation between speed of convergence and welfare is not straightforward: if initial expected inflation is higher than the rational expectations equilibrium, the policy maker can substantially increase social welfare inducing a fast learning. If, instead, perceived inflation is initially lower, a slow transition might be preferable, since inflation would remain closer to the first best for a longer period of time.

The chapter is organized as follows. Section 1.2 presents the monetary policy problem, describing the learning dynamics under two different policy rules. The section ends showing that under the optimal policy described in Evans and Honkapohja (2003a), the transition to the rational expectations equilibrium is very slow. In section 1.3 I show how policies could be ordered according to their speed of convergence. In section 1.4 I study policies that allow the central bank to shorten (or extend) the transition without affecting the long-run equilibrium and in section 1.5 I analyze how these policies influence social welfare. Section 1.6 includes robustness checks and section 1.7 summarizes and concludes.

1.2 The framework

1.2.1 The baseline model

Much of the recent theoretical analysis on monetary policy has been conducted under the "New Phillips curve" paradigm reviewed in Clarida, Galí and Gertler (1999) and Woodford (2003). The baseline framework is a dynamic general equilibrium model with money and temporary nominal price rigidities. I consider the linearized reduced form of the economy with competitive monopolistic firms, staggered prices and private agents that maximize intertemporal utility. From the private agents' point of view there is an intertemporal *IS curve*[3]

$$x_t = E_t^* x_{t+1} - \varphi\left(i_t - E_t^* \pi_{t+1}\right) + g_t \tag{2.1}$$

and an aggregate supply (AS) modeled by an expectations-augmented *Phillips curve*[4]:

$$\pi_t = \alpha x_t + \beta E_t^* \pi_{t+1}, \tag{2.2}$$

[3]The IS relationship approximates the Euler equation characterizing optimal aggregate consumption choices and the parameter φ can be interpreted as the rate of intertemporal substitution.

[4]The AS relation approximates aggregate pricing emerging from monopolistically competitive firms' optimal behaviour in Calvo's model of staggered prices. Here I'm not considering cost-push shocks. Introducing cost-push shocks, would not change substantially the results on speed of convergence and the role of policy decisions along the transition. However, in section 6, the welfare analysis is handled also in presence of cost-push shocks.

where x_t is the output gap, measured as the log deviation of actual output (y_t) from potential output (z_t) (i.e., the level of output that would arise if wages and prices were perfectly competitive and flexible), π_t is actual inflation at time t, $E_t^*\pi_{t+1}$ is the level of inflation expected by private agents for period $t+1$, given the information at time t. Similarly $E_t^*x_{t+1}$ is the level of the output gap that private agents expect for period $t+1$, given the information at time t. I write E_t^* to indicate that expectations need not be rational (E_t without $*$ denotes RE); i_t is the short-term nominal interest rate and is taken to be the instrument for monetary policy; g_t is a demand shock, $g_t = \rho_g g_{t-1} + \varepsilon_{gt}$ with $\varepsilon_{gt} \sim N\left(0, \sigma_g^2\right)$ and i.i.d.

In order to complete the model, it is necessary to specify how the interest rate is settled and how agents form beliefs. I consider the nominal interest rate as the policy instrument and model it by means of a reaction function. Thus, a policy rule is just a functional relationship between a dependent variable (the interest rate) and some endogenous (expected inflation and output gap) and exogenous (shocks) variables. I consider three cases. I start with a simple expectations-based policy rule that helps me to introduce in a very simple and intuitive way the concept of speed of convergence. Then, I describe the optimal RE policy under discretion derived in Evans and Honkapohja (2003a)[5]. Finally, I introduce a set of expectations-based policy rules and I show how to characterize the elements of this set, using a measure of the speed of convergence.

Concerning beliefs, I start each analysis by considering the rational expectations hypothesis in order to focus and discuss subsequently the implications of bounded rationality.

1.2.2 A simple expectations-based reaction function

It has long been recognized that monetary policy needs a forward-looking dimension[6]. Let us assume that the central bank, in order to set the current interest rate, uses simple policy rules that feed back from expected values of future inflation and output

[5] I leave for future research a general study of the transition along the learning process for monetary policy problems under commitment.

[6] In a recent paper that analyzes monetary policy decisions in the US in the last two decades, A. Greenspan (2004), Chairman of the Board of Governors of the Federal Reserve System, writes "*In recognition of the lag in monetary policy's impact on economic activity, a preemptive response to potential for building inflationary pressures was made an important feature of policy. As a consequence, this approach elevated forecasting to an even more prominent place in policy deliberations*".

gap[7]

$$i_t = \gamma + \gamma_x E_t^* x_{t+1} + \gamma_\pi E_t^* \pi_{t+1}. \tag{2.3}$$

The class of *expectations-based reaction functions* that I first consider has $\gamma_x = \frac{1}{\varphi}$ in order to simplify the interaction between actual and expected variables. Under (2.3), in fact, the economy evolves according to the following system of equations:

$$Y_t = Q + F E_t^* Y_{t+1} + S g_t \tag{2.4}$$

where

$$Y_t = \begin{bmatrix} \pi_t \\ x_t \end{bmatrix},$$

$$Q = \begin{bmatrix} -\alpha\varphi\gamma \\ -\varphi\gamma \end{bmatrix}, \quad F = \begin{bmatrix} \beta + \alpha\varphi(1-\gamma_\pi) & 0 \\ \varphi(1-\gamma_\pi) & 0 \end{bmatrix}, \quad S = \begin{bmatrix} \alpha \\ 1 \end{bmatrix} \tag{2.5}$$

and neither the IS nor the AS are affected by expectations on output gap[8].

Under rational expectations (i.e. $E_t^* x_{t+1} = E_t x_{t+1}$ and $E_t^* \pi_{t+1} = E_t \pi_{t+1}$) it has been shown that the dynamic system defined by (2.4) has a unique non-explosive equilibrium (Bullard and Mitra, 2002). In particular, assuming for simplicity that $\rho_g = 0$, the equilibrium can be written as a linear function of a constant and the shock[9]

$$\pi_t = \overline{a}_\pi + \alpha g_t \quad \text{and} \quad x_t = \overline{a}_x + g_t, \tag{2.6}$$

while agents' forecasts are just constant

$$E_t \pi_{t+1} = \overline{a}_\pi \quad \text{and} \quad E_t x_{t+1} = \overline{a}_x. \tag{2.7}$$

Adaptive Learning

Let us assume now that private agents form expectations by learning from past experiences and update their forecasts through recursive least squares estimates[10].

We are firstly interested in studying whether the economy, in this case, might converge to the determinate equilibrium (2.6).

[7] One theme in the literature concerning rules of this type is that they tend to induce large regions of indeterminacy of the rational expectations equilibrium and are therefore undesirable (see for example, Bernanke and Woodford, 1997 and Bullard and Mitra, 2002). In the next sections I will focus on policies that, as a basic requirement, imply a determinate REE and I will consider learnability and speed of learning as additional constraint in this set of policies.

[8] For a more general class of expectations-based policy rules without restrictions on γ_x, I refer to section 3.

[9] Considering an i.i.d stochastic process instead of an AR(1) does not affect the results on speed of convergence. However, since the litterature usually consider AR(1) shocks, the welfare analysis in sections 5 and 6 is obtained assuming a persistent demand shock.

[10] See Marcet and Sargent (1989 a, b) or Evans and Honkapohja (2001) for a detailed analysis of least squares learning.

Since, under the *simple expectations-based reaction function* (2.3), neither the IS nor the AS relations depend on expected output gap, the system under learning can be described by focusing directly on beliefs regarding expected inflation[11].

I assume that agents do not know the effective value of $\overline{a}_\pi$ in equation (2.5), but estimate it using past information. In this case, private agents' expected inflation is given by:

$$E_t^* \pi_{t+1} = a_{\pi,t}, \tag{2.8}$$

where $a_{\pi,t}$ is a statistic inferred recursively from past data according to

$$a_{\pi,t} = a_{\pi,t-1} + t^{-1} \left(\pi_{t-1} - a_{\pi,t-1} \right). \tag{2.9}$$

Forecasts are updated by a term that depends on the last prediction error[12] weighted by the *gain sequence,* t^{-1}. It is well known that in this case the adaptive procedure is the result of a least squares regression of inflation on a constant, and perceived inflation is just equal to the sample mean of past inflations:

$$a_{\pi,t} = \frac{1}{t} \sum_{i=1}^{t} \pi_{i-1}. \tag{2.10}$$

An important aspect of least squares learning is that agents' beliefs may converge to RE, i.e., the estimated parameters $a_{\pi,t}$ may converge asymptotically to $\overline{a}_\pi$. The E-Stability principle (Evans and Honkapohja, 2001) provides conditions for asymptotic stability of the REE under least squares learning.

Before analyzing speed of convergence I describe briefly E-stability, since the building blocks of the two concepts are the same. The stability under learning (E-stability) of a particular equilibrium is addressed by studying the mapping from the estimated parameters (i.e., the perceived law of motion, PLM), to the true data generating process (i.e., the actual law of motion ALM).

When expectations in system (2.4) evolve according to expression (2.8), the inflation's ALM is

$$\pi_t = T\left(a_{\pi,t}\right) + \alpha g_t, \tag{2.11}$$

where

$$T\left(a_{\pi,t}\right) = -\alpha\varphi\gamma + \left[\beta + \alpha\varphi\left(1 - \gamma_\pi\right)\right] a_{\pi,t} \tag{2.12}$$

is the mapping from PLM to ALM of inflation.

[11] In the next section I show formally that this does not affect the results.

[12] This formula implies that private agents do not use today's inflation to formulate their forecasts. The assumption is made purely for convenience and it is often made in models of learning as it simplifies solving the model. The dynamics of the model are unlikely to change.

As shown in Marcet and Sargent (1999a,b) and Evans and Honkapohja (2001), it turns out that the dynamic system described by equations (2.9), (2.11) and (2.12) can be studied in terms of the associated *ordinary differential equation* (ODE)

$$\frac{da_\pi}{d\tau} = h(a_\pi) = T(a_\pi) - a_\pi, \tag{2.13}$$

where τ denotes "notional" or "artificial" time and $h(a_\pi)$ is the asymptotic mean prediction error (i.e. the mean distance between the ALM and the PLM):

$$h(a_\pi) = \lim_{t\to\infty} \left[T(a_{\pi,t}) - a_{\pi,t}\right]. \tag{2.14}$$

The REE is said to be E-stable if it is locally asymptotically stable under equation (2.14) and under some regularity conditions.

In our example E-stability conditions are readily obtained by computing the derivative of the ODE with respect to a_π and checking wether it is smaller than zero[13].

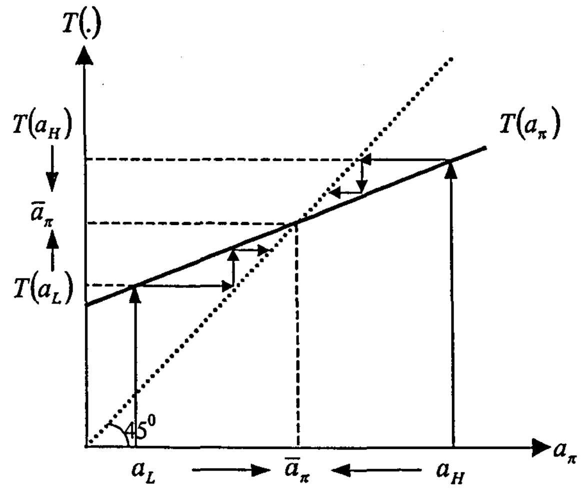
Speed of convergence to the REE

It turns out that policy decisions (i.e., choices concerning γ_π) are important not only to describe asymptotic properties of the equilibrium under learning, but also to determine the speed at which the distance between PLM and ALM shortens over time.

Figure 1 plots the mapping from PLM to ALM (2.12) and shows how private agents' estimates affect actual inflation along the transition to the REE.

Fig 1

Mapping from PLM to ALM



[13]This coincide with checking wether the derivative of the mapping from PLM to ALM is smaller than 1.

First of all note that if the slope of the mapping is smaller than 1, the determinate REE (2.6) is E-stable[14]. In other words, if the economy starts from a perceived level of inflation $a_L < \overline{a}_\pi$ or $a_H > \overline{a}_\pi$, the mean of the prediction error, $E\left[T\left(a_{\pi,t}\right) - a_{\pi,t}\right]$, decreases over time and asymptotically converges to zero.

Is there any difference between a policy that results in the slope of $T(.)$ equal to 0.01 and one with the slope equal to 0.99? The recent literature on monetary policy and learning (Evans and Honkapohja 2003a, 2003b, and Bullard and Mitra 2002), by focusing on asymptotic properties, does not provide an answer to this question. Since in both cases the REE is determinate and E-stable, both policies are "good"[15].

The concept of speed of learning can be used in order to refine the set of "good" policies.

In the literature, the problem of the speed of convergence of recursive least square learning algorithms has been analyzed mainly through numerical procedures and simulations. The few analytical results on the transition to the rational expectations equilibrium are obtained by using a theorem of Benveniste, Metiver and Priouret (1990) that relates the speed of convergence of the learning process to the eigenvalues of the associated ordinary differential equation (ODE) at the fixed point[16]. In the present case, the ODE to be analyzed is the one described in expression (2.14) and the associated eigenvalue coincides with the slope of the mapping from PLM to ALM (2.12).

The following propositions, adapting arguments from Marcet and Sargent (1995), show that by choosing the γ_π, the policy-maker not only determines the level of inflation in the long run, but also the speed at which the distance between perceived and actual inflation narrows over time.

Proposition 1 *Let us define*

$$S_1 = \left\{ \gamma_\pi : \gamma_\pi > \frac{\alpha\varphi + \beta - 1/2}{\alpha\varphi} \right\}$$

Under the simple expectations-based reaction function (2.3), if $\gamma_\pi \in S_1$, then

$$\sqrt{t}\left(a_{\pi,t} - \overline{a}_\pi\right) \xrightarrow{D} N\left(0, \sigma_a^2\right)$$

with

$$\sigma_a^2 = \frac{\alpha^2\sigma_g^2}{\left[1 - \beta - \alpha\varphi\left(1 - \gamma_\pi\right)\right]} \tag{2.15}$$

[14] For simplicity, let focus only on positive values of the slope. In this case a slope smaller than 1 is a necessary and sufficient condition for determinacy and E-stability.

[15] With "good" policy, I refer to the criterion used by Bullard and Mitra (2002) to evaluate policy rules, based on determinacy and E-stability of the REE.

[16] See Marcet and Sargent (1995) for an interpretation of the ODE.

Proof. See Appendix A. ■

If the conditions of Proposition 1 are satisfied, the estimates $a_{\pi,t}$ converge to the REE, $\overline{a}_\pi$, at root-t speed. Root-t is the speed at which, in classical econometrics, the mean of the distribution of the least square estimates converges to the true value of the parameters estimated. Note that the formula for the variance of the estimator a_π is modified with respect to the classical case where $\sigma_a^2 = \alpha^2\sigma_g^2$. Proposition 2 shows that for lower values of γ_π convergence is slower.

Proposition 2 *Under the simple expectations-based reaction function (2.3), if $\gamma_\pi \in S_1$, then the weaker the response to expected inflation (the smaller γ_π), the greater the asymptotic variance of the limiting distribution, σ_a^2.*

Proof. See Appendix B. ■

Looking at the formula for the asymptotic variance (2.15) it is easy to understand the role of policy decisions in determining the speed of convergence to the REE: for a weaker response to expected inflation, the convergence is slower in the sense that the asymptotic variance of the limiting distribution is greater.

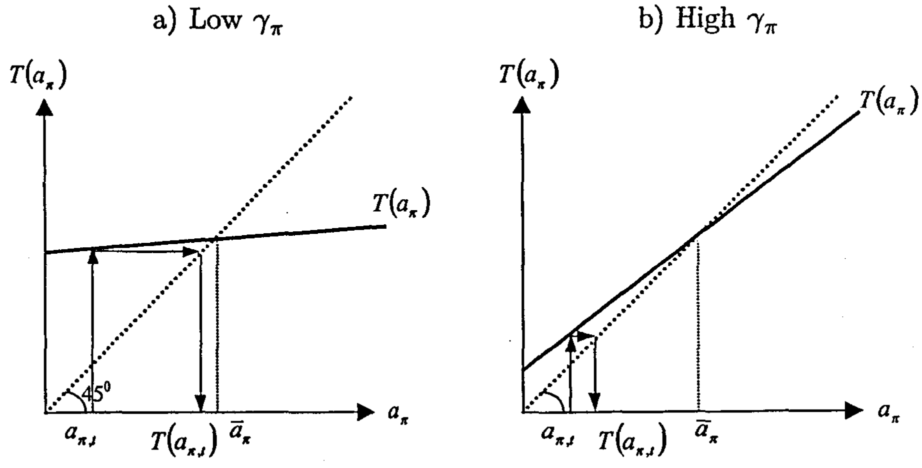
What happens when the slope of the mapping (2.12) is smaller than 1, but bigger than 0.5? Let us define $S_2 = \left\{\gamma_\pi : \frac{\alpha\varphi+\beta-1}{\alpha\varphi} < \gamma_\pi < \frac{\alpha\varphi+\beta-1/2}{\alpha\varphi}\right\}$. If $\gamma_\pi \in S_2$, the estimates $a_{\pi,t}$ converge to the REE $\overline{a}_\pi$, but at a speed different from root-t. In this case, as Marcet and Sargent (1995) suggests, the importance of initial conditions fails to die out at an exponential rate (as it is needed for *root-t convergence*) and agents' beliefs converge to rational expectations at a rate slower than root-t. In particular, also when $\gamma_\pi \in S_2$ it is possible to show by means of simulations that as the slope of the $T(.)$ mapping increases, the speed of convergence decreases[17].

Figure 2 shows examples for the two cases where $\gamma_\pi \in S_1$ and $\gamma_\pi \in S_2$. Since the least squares algorithm adjusts each parameter towards the truth when new information is received, the new belief $a_{\pi,t+1}$ will be an average of the previous beliefs $a_{\pi,t}$ and the actual value $T(a_{\pi,t})$ plus an error. When the reaction of the policy maker to expected inflation is *strong* ($\gamma_\pi \in S_1$), the derivative of $T(.)$ is smaller than (or equal to) 1/2 and $T(a_{\pi,t})$ is close to $\overline{a}_\pi$; when the reaction is *weak* ($\gamma_\pi \in S_2$), the derivative of $T(.)$ is larger than 1/2 and $T(a_{\pi,t})$ is close to $a_{\pi,t}$ instead of being close to $\overline{a}_\pi$, so the average can stay far from the REE for a long time.

[17]See section 4 for simulations that relate speed of convergence and the slope of the $T()$ mapping.

Fig. 2

Mapping from PLM to ALM and the speed of convergence



It is worth noting that even though the transition is quite different in the two cases analyzed here, the learning equilibrium could end up converging to the same REE and, according to policy-maker preferences, the speed of convergence could become a relevant variable in the policy decision problem.

1.2.3 Optimal monetary policy under discretion

The reason why I started with the *simple expectations-based reaction function* (2.3) was that it simplified the analysis of the dynamics under learning. I now consider the optimal monetary policy problem without commitment (discretionary policies), where any promises made in the past by the policy-maker do not constrain current decisions. In deriving the optimal discretionary policy, I follow Evans and Honkapohja (2003a), assuming that the policy-maker cannot manipulate private agent's beliefs. This assumption implies that the optimality conditions derived under learning are equivalent to the ones obtained under RE.

The policy problem consists in choosing the time path for the instrument i_t to engineer a contingent plan for the target variables π_t and $(x_t - \overline{x})$ that maximizes the objective function

$$\underset{x_t,\pi_t}{Max} - E_0 \sum_{t=0}^{\infty} \beta^t L(\pi_t, x_t)$$

where

$$L(\pi_t, x_t) = \frac{1}{2}\left[\pi_t^2 + \lambda (x_t - \overline{x})^2\right]$$

subject to the constraints (2.1) and (2.2) and $E_t^*\pi_{t+1}$, $E_t^*x_{t+1}$ given.

The solution of this problem[18], as derived in Evans and Honkapohja (2003a), yields to a reaction function that relates the policy instrument i_t to the current state of the economy and private agents' expectations:

$$i_t = \gamma^\star + \gamma_x^\star E_t^* x_{t+1} + \gamma_\pi^\star E_t^* \pi_{t+1} + \gamma_g^\star g_t \tag{2.16}$$

where $\gamma^\star = -\frac{\lambda}{(\lambda+\alpha^2)\varphi}\overline{x}$, $\gamma_x^\star = \gamma_g^\star = \frac{1}{\varphi}$ and $\gamma_\pi^\star = 1 + \frac{\alpha\beta}{(\lambda+\alpha^2)\varphi}$.

Since interest rate rule (2.16) states that the policy maker should react to expected inflation and expected output gap, it is sometimes called the *optimal expectations-based reaction function (*Evans and Honkapohja, 2003a). However, to stress the fact that this policy is optimal under rational expectations but is not necessarily optimal under learning, it would be worth to call it the *RE-optimal expectations-based reaction function*; in order to avoid notational flutter, I call it *Evans and Honkapohja policy,* or *EH*-policy.

Under rational expectations (i.e. $E_t^*x_{t+1} = E_t x_{t+1}$ and $E_t^*\pi_{t+1} = E_t\pi_{t+1}$) the equilibrium is:

$$\pi_t = E_t\pi_{t+1} = \overline{a}_\pi \quad \text{and} \quad x_t = E_t x_{t+1} = \overline{a}_x, \tag{2.17}$$

Assuming that private agents do not know $\overline{a}_\pi$ and $\overline{a}_x$ but estimate them recursively, the expected inflation and output gap evolve as described in section 2.2, while the mapping from PLM to ALM is now given by

$$T\left(a_{\pi,t}, a_{x,t}\right) = \left(\Phi^* + \Gamma^* a_{\pi,t}, \Phi^*\alpha^{-1} - \left(\beta - \Gamma^*\right)\alpha^{-1}a_{\pi,t}\right). \tag{2.18}$$

where

$$\Gamma^* = \frac{\lambda\beta}{(\lambda+\alpha^2)}, \Phi^* = \frac{\lambda\alpha}{(\lambda+\alpha^2)}\overline{x}$$

Again, since the right-hand side of (2.18) does not depend on $a_{x,t}$, properties of the equilibrium under learning can be described simply by focusing on the mapping from perceived inflation to actual inflation.

As the *EH*-policy results in a unique and E-stable REE, Evans and Honkapohja (2003a) concludes that the policy derived as the optimal solution of the problem under discretion and rational expectations is also "good" under learning.

However, if we simulate the model under the *EH-policy,* and Clarida, Galì and Gertler (2000) calibration[19], it turns out that the distance between the actual inflation

[18] I consider λ as an exogenous policy parameter, as is often done in the literature. An alternative approach is to obtain λ as the result of the general equilibrium problem. In this case λ would depend on representative consumer preferences and firms' price setting rules.

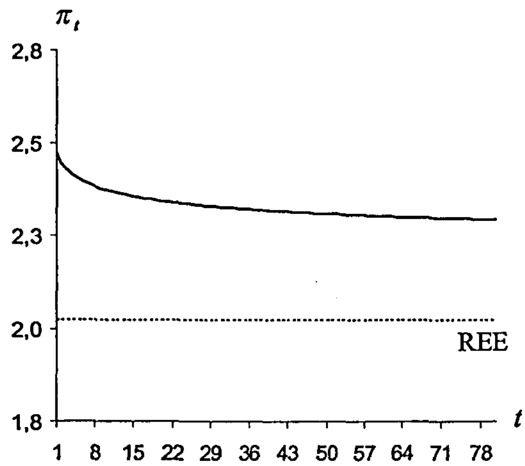
[19] Clarida, Galí and Gertler (CGG, 2000) derive from regressions on US data, $\varphi = 4$, $\alpha = 0.075$, $\beta = 0.99$; Woodford (W, 1999) finds $\varphi = (0.157)^{-1}$, $\alpha = 0.024$, $\beta = 0.99$. Both the Clarida, Galí

and the REE would be significatively different from zero for many periods[20].

Fig 3

Deviation of actual inflation from the REE[1]

($\pi^{RE} = 2\%$; initial expected inflation $= 2.5\%$; $\lambda = 0.1$)



[1]annualized data

Figure 3 shows the evolution of perceived inflation under learning. Assuming that the policy-maker follows a flexible inflation targeting policy rule with $\lambda = 0.1$, the output gap target[21] is $\overline{x} = 0.004$ and using CGG calibration, the REE for inflation is around 0.5 per cent (notice that since inflation here is measured as quarterly changes in the log of prices, the annualized inflation in the REE is around 2 per cent). I consider an initial expected inflation 0.5 percentage point higher (in annualized terms) than the REE. After 20 years ($t = 80$) perceived inflation is still 0.3 percentage points higher than the REE[22].

Applying a similar argument to that used in Propositions 1 and 2 it is possible to state the following proposition about the speed of convergence and the role of the relative weight to output gap, λ , in the loss function.

and Gertler (2000) and Woodford (1999) calibrations are for quarterly data. However the first work uses annualized data for inflation and interest rate, while the second one uses quarterly interst rates and measures inflation as quarterly changes in the log of prices. I uses Woodford convention and therefore my CGG calibration divides by 4 the α and multiply by 4 the φ reported by CGG (see also Honkapohja and Mitra, 2004).

[20]In general, the weight that is attributed to the initial belief plays an important role. In equation (2.9), an initial t very small would imply a much higher weight to the present than to the past. In other words, the bigger the t, the higher the weight given to previous belives. In all the simulations, I give very low initial weight to past data ($t_0 = 2$).

[21]I choose this value for the output gap target to match the annualized inflation rate of 2 per cent.

[22]Similar results obtain under Woodford (2003) calibration.

Proposition 3 *Under the EH policy, the speed of convergence of the learning process depends negatively on the weight that the policy-maker gives to output gap relative to inflation. In particular, the greater the weight to output gap, the slower the learning process.*

Proof. See Appendix C. ■

Proposition 3, by looking at the slope of the mapping from perceived to actual inflation, relates the speed of convergence of the learning equilibrium to the importance of output gap in the objective function.

Fig 4

Slope of the mapping from PLM to ALM

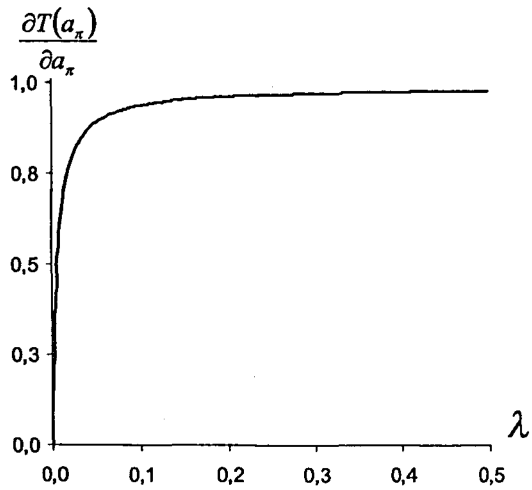


Figure 4 shows how the slope of the mapping from the PLM to ALM of inflation changes as the relative weight that the policy-maker gives to the output gap increases[23]. It is sufficient that the policy maker cares very little about the output gap in order that the slope of the mapping is already close to 1 (for example when $\lambda = 0.1$, the slope is 0.94). In particular, unless λ is smaller than 0.006, *root-t convergence* is never reached! Notice that also in the case where λ is obtained as the result of a general equilibrium problem (Woodford, 2003, suggests in this case a value for λ close to 0.05;) the slope is close to 0.9.

[23] I use the Clarida, Gali and Gertler calibration for US. Similar results obtain with the Woodford calibration.

The fact that the learning speed could be very slow (or very fast) depending on policy decisions[24], suggests that when they consider the monetary policy problem under learning, policy-makers should take into account the transition to the REE. E-stability and determinace are not sufficient to characterize policies in a context of adaptive learning. The *EH*-policy, which is optimal under rational expectations, may not be optimal under learning if policy makers take into account the transition.

1.3 Speed of convergence and policy design

Let us consider a third and more generic set of *expectations-based reaction functions*

$$i_t = \gamma + \gamma_x E_t^* x_{t+1} + \gamma_\pi E_t^* \pi_{t+1} + \gamma_g g_t \tag{3.1}$$

and show how to characterize the elements of this set using a measure of the speed of convergence.

Under a generic *expectations-based reaction function*, the economy evolves according to the following expression:

$$Y_t = Q + F \times E_t^* Y_{t+1} + S g_t, \tag{3.2}$$

where

$$Q = \begin{bmatrix} -\alpha\varphi\gamma \\ -\varphi\gamma \end{bmatrix}, \quad S = \begin{bmatrix} \alpha\left(1-\varphi\gamma_g\right) \\ \left(1-\varphi\gamma_g\right) \end{bmatrix} \tag{3.3}$$

$$F = \begin{bmatrix} (\beta + \alpha\varphi(1-\gamma_\pi)) & \alpha(1-\varphi\gamma_x) \\ \varphi(1-\gamma_\pi) & (1-\varphi\gamma_x) \end{bmatrix}. \tag{3.4}$$

and the REE is of the form

$$Y_t = \overline{A} + S g_t, \tag{3.5}$$

If private agents do not know $\overline{A}$ but estimate it recursively, expected inflation and output gap evolve in a more complex way then described in section 2. As both the IS and the AS relations also depend on the expected output gap[25], the learning process cannot be described only by focusing on beliefs regarding expected inflation (see Appendix D for a complete description of the learning mechanism in this case).

Expectations are given by:

$$E_t^* Y_{t+1} = A_t, \tag{3.6}$$

where elements in A_t are estimated similarly to (2.9).

[24] This result could be applied to the problem of "optimal delegation", justifing a conservative central bank when fast convergence is required.

[25] Under *generic expectations-based reaction functions* (3.1) the elements in the second column of the F matrix are not necessarily zero.

Lemma 4 is a slight generalization of a result obtained in Bullard and Mitra (2002) and describes the necessary and sufficient conditions under which the REE (3.5) is E-stable.

Lemma 4 *Under a generic expectations-based reaction function (3.1), the necessary and sufficient condition for a rational expectations equilibrium to be E-stable is*

$$\gamma_\pi > \max\left[1 - \frac{1-\beta}{\alpha}\gamma_x, 1 - \frac{1-\beta}{\alpha\varphi} - \frac{\gamma_x}{\alpha}\right]$$

Proof. See Appendix D. ∎

Figure 5 shows, with CGG calibration, all the combinations (γ_π, γ_x) under which the REE is E-stable.

Fig 5

E-stable region under the expectations-based policy rule

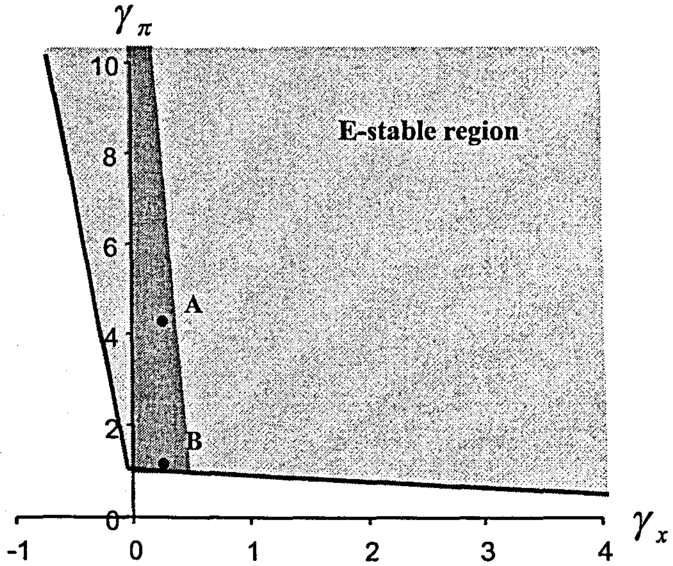


Figure 5 shows a well known result (see Bullard and Mitra 2002) that under policy rules like (3.1) the set of policies under which the REE is determinate (the darker area) is much smaller than the one where REE is E-stable (the lighter area).

Note that, since the *EH policy* (2.16) is an element of the set of *generic expectations-based policy rules* (3.1), points A and B represent the combination γ_π^*, γ_x^* in the two extreme cases where a policy-maker does not care about the output gap, $\lambda = 0$ (point A), and where he gives a relative small weight also to the output gap, $\lambda = 0.1$ (point B).

Figure 5 shows that in both cases the REE is determinate and E-Stable[26]. However, for $\lambda = 0.1$ this is already very close to the bounds of the E-stability region; in this case, if the policy-maker chooses the *EH policy rule,* but improperly calibrates the model it can easily end up outside the E-stable region, enforcing a non-stationary policy.

Finally, the fact that the origin is not in the stable region is consistent with the non-convergence result of Evans and Honkapohja (2003a): policies that react only to shocks, ignoring expectations, are unstable under learning.

1.3.1 The transition to the REE

In the previous sections, policy-makers settled the coefficients of matrix F, by means of the reaction function. This means that the evolution of estimated coefficients in private agents' forecasts (i.e., the speed at which private agents learn) strictly depends on policy decisions.

Proposition 5 provides conditions for root-t convergence.

Proposition 5 *Under expectations-based reaction functions (3.1), if*

$$\gamma_\pi > \max\left[1 + \frac{1-2\beta}{2\alpha\varphi} - \frac{1-2\beta}{\alpha}\gamma_x, 1 + \frac{\beta}{\alpha\varphi} - \frac{\gamma_x}{\alpha}\right], \tag{3.8}$$

then

$$\sqrt{t}\,(A_t - A) \xrightarrow{D} N\,(0, \Omega)$$

where the matrix Ω *satisfies*

$$\left[\frac{I}{2}(F-I)\right]\Omega + \Omega\left[\frac{I}{2}(F-I)\right]' + SS'\sigma_g^2 = 0 \tag{3.9}$$

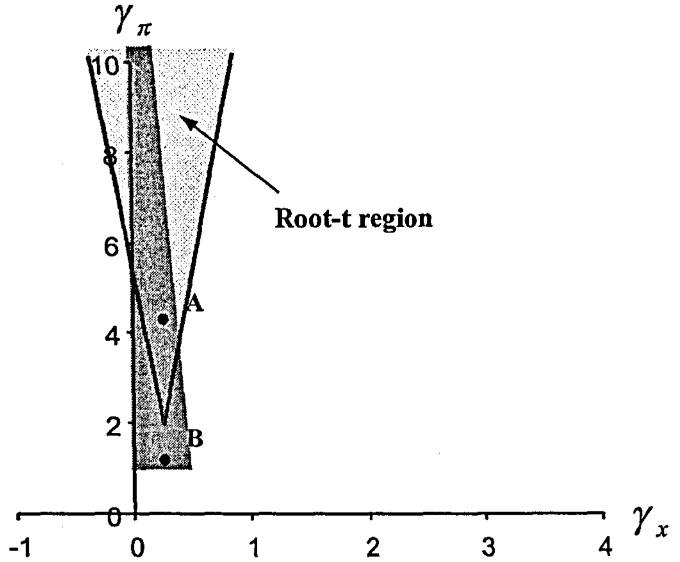
Proof. See Appendix E. ∎

Under the *generic expectations-based reaction function* (3.1), if the REE is E-stable but conditions in Proposition 5 are not satisfied, then not all the eigenvalues of the matrix F have real part smaller than one half. In this case, as suggested in section 2, the learning equilibrium converges to the REE at a slower rate than root-t. Figure 6 shows all combinations of γ_π and γ_x for which there is root-t convergence.

[26] It is possible, moreover, to show that for any positive and finite value of λ, i.e., for all *flexible inflation targeting* policies under the *EH policy* (2.16) the rational expectation equilibrium is E-Stable (Evans and Honkapohja, 2002).

Fig 6

Root-t convergence under the expectations-based policy rule



By comparing Figure 5 and Figure 6, it is clear that the set of combinations (γ_x, γ_π) resulting in root-t convergence is much smaller than the one under which E-stability holds. Moreover, the region where both root-t convergence and determinateness hold is smaller than the one of E-stability and determinateness. As derived in section 2, point B in Figure 6 shows that when the policy-maker gives weight $\lambda = 0.1$, in the *EH policy,* the economy converges at a speed slower than *root-t.*

In the previous sections, in order to characterize how policies determine the speed of converge to REE, I focused only on one policy parameter at a time (γ_π in section 2.2 and λ in section 2.3). Here, on the contrary, since the speed of convergence is determined by the eigenvalues of F and this matrix depends on both γ_π and γ_x, it is necessary to focus on two policy parameters at a time. For this reason I define the speed of convergence isoquants that map elements of the set of *expectations-based reaction functions* into a speed of convergence measure[27].

Definition 6 *A speed of convergence isoquant is a curve in R^2 along which all points (i.e., combinations (γ_π, γ_x) of an expectations-based reaction function (3.1)) result in the same real part of the largest eigenvalue z_1 of the matrix F.*

[27] In the definition I relate speed of convergence to the eigenvalues of the matrix F. In general, as shown in previous sections, the speed of convergence is related to the eigenvalues of the derivatives of the mapping from PLM to ALM, $T(A)$. In this case, the derivative is equal to F.

For simplicity I restrict the analysis to the set

$$\Gamma = \{\gamma_\pi, \gamma_x : \gamma_\pi > 0, \gamma_x > 0 \text{ and } 0 \leq z_1 < 1\}.$$

The following definition and proposition describe the main properties of the speed of convergence isoquants:

Definition 7 *The speed of convergence, represented by the speed of convergence isoquants, is monotonically increasing in the reaction to expected inflation (γ_π) if, given the reaction to the expected output gap (γ_x), the real part of the largest eigenvalue z_1 of the matrix F is decreasing in γ_π.*

A similar definition for monotonicity with respect to the expected output gap could be settled.

Proposition 8 *The speed of convergence relation, represented by the speed of convergence isoquants and defined over Γ is: (i) monotonically increasing in γ_π, (ii) not monotonic with respect to γ_x.*

Proof. See Appendix F. ∎

Proposition 8 states that, for a given reaction to output gap expectations, the policy-maker, by increasing the reaction to expected inflation increases monotonically the speed at which private agents learn. On the contrary, for a given reaction to expected inflation, by increasing the reaction to the expected output gap, private agents could learn both faster or slower, depending on the value of γ_π.

Figure 7 shows the speed of convergence isoquants: the lower the isoquant, the slower the convergence. In fact, the larger the real part of z_1, the lower the isoquant and, from Marcet and Sargent (1995), the larger the real part of z_1, the slower the convergence.

Fig 7

The speed of learning isoquants

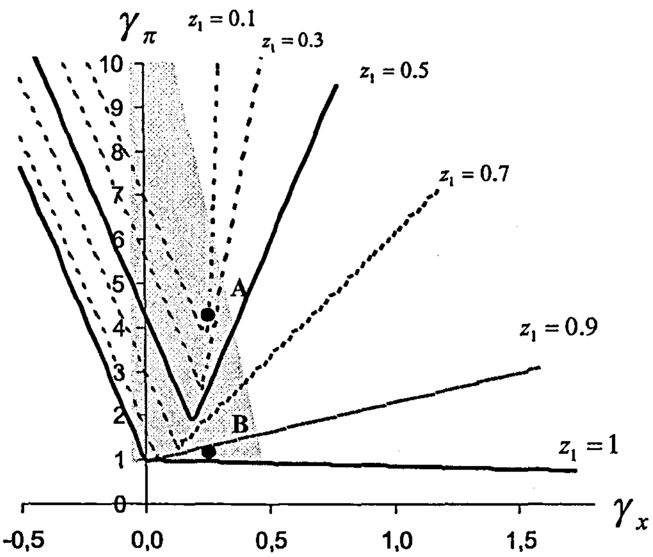


Figure 7 illustrates a practical way of using speed of learning in order to characterize monetary policies. For example, a combination of γ_π and γ_x just above the isoquant $z_1 = 1$ (point B) determines an E-stable REE, but would imply very slow convergence. Combinations of γ_π and γ_x placed above the isoquant $z_1 = 0.1$ imply a very fast learning process. The combinations of γ_π and γ_x that stay above the isoquant $z_1 = 0.5$ imply a learning process that converges to the REE at a *root-t* speed.

Let us now see how to make active use of the speed of convergence in the study of optimal policies under discretion.

1.4 Discretionary policy and learning

In section 2 we have seen that, in order to identify *EH policy* (2.16) as the optimal policy under discretion and learning, the crucial assumption is that "*the policy-maker does not make active use of learning behavior on the part of agents*" (Evans and Honkapohja, 2003a). Under rational expectations, the problem of optimal "discretionary policy" implies, by definition, that policy-makers cannot affect private agents' expectations. However, under the hypothesis of bounded rational private agents a rational policy-maker with full information should take transition into account. In fact, if private agents' expectations are the result of estimations that depend on past values of the policy instrument, the policy-maker's decisions will affect future estimates and, consequently, the private agents' learning process.

Therefore, *EH policy* (2.16) is not necessarily optimal under learning but could be defined as *asymptotically-optimal.* However, if private agents' PLM is well specified, not only the EH-policy is asymptotically optimal under discrertion, but we know that there is a continuum of *expectations-based policy rules* that result in the same REE.

Lemma 9 *Under rational expectations, in the set of expectations-based reaction functions (3.1) there are infinitely many elements, i.e. combinations of* $\gamma, \gamma_x, \gamma_\pi, \gamma_g$ *that result in the optimal REE for* $\{\pi_t, x_t\}$ *defined in (2.17).*

Proof. See Appendix G. ■

Since all the policies that in the long run result in the same optimal allocation, could determine different transitions to the REE, a device for discriminating between them is required[28]. The speed of convergence isoquants derived in the previous section could be a useful starting point.

Let us consider a set of *asymptotically-optimal expectations-based reaction functions* that allow to completely offset demand shocks, as under the *EH policy.*

Proposition 10 *The maximum speed of convergence of the learning process that could be reached under the restricted set of asymptotically-optimal expectations-based reaction functions,*

$$i_t = \gamma' + \gamma'_x E_t x_{t+1} + \gamma'_\pi E_t \pi_{t+1} + \gamma'_g g_t, \tag{4.1}$$

with $\gamma'_g = \gamma^*_g = \frac{1}{\varphi}$, $\gamma' = \gamma^* = -\frac{\lambda}{(\lambda+\alpha^2)\varphi}\overline{x}$ *and* $\gamma'_\pi = \frac{(1+\alpha\varphi)(\lambda+\alpha^2)-\lambda\beta}{(\lambda+\alpha^2)\varphi\alpha} - \frac{(1-\beta)}{\alpha}\gamma'_x$, *depends negatively on the weight that the policy-maker gives to output gap relative to inflation.*

Proof. See Appendix H. ■

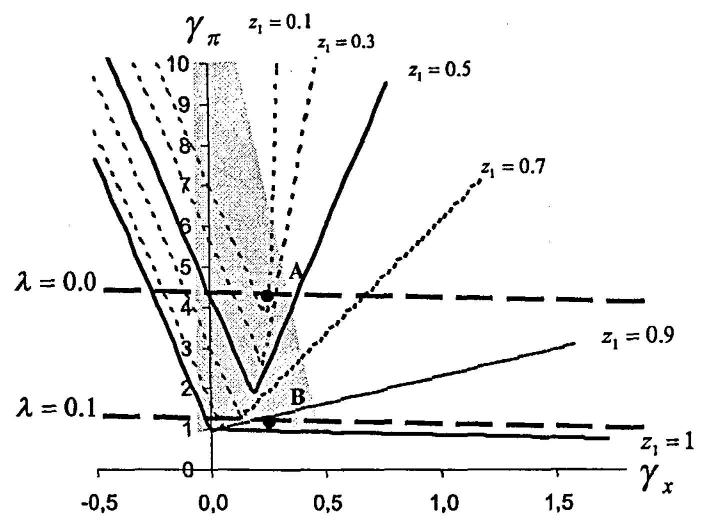
Proposition 10 states that under the set of reaction functions (4.1), the economy converges asymptotically to the optimal REE under discretion, but for a given λ the policy-maker can bring about a different speed of convergence. Note, instead, that under EH policy (2.16), each λ was associated with a given speed of convergence. In particular, under *asymptotically optimal expectations-based reaction functions* (4.1), the larger the relative weight on output gap, λ, the larger will be the real part of the biggest eigenvalue of the F matrix and the slower the fastest speed of convergence that a policy-maker can reach. Figure 8 shows, in the same picture, the speed of learning

[28] *"There is no single policy rule that is uniquely consistent with the optimal equilibrium. Many rules may be consistent with the same equilibrium, even though they are not equivalent insofar as they imply a commitment to different sorts of out-of equilibrium behaviour" (Svensson and Woodford, 1999).*

isoquants and, for given λ, combinations of γ_x and γ_x under which the economy will converge asymptotically to the optimal REE under discretion.

Fig 8

Asymptotically-optimal expectations-based reaction functions



The line $\lambda = 0$ shows that if the policy-maker does not care about the output gap, by imposing $\gamma_\pi = \gamma'_\pi$, he can choose combinations of γ_π and γ_x that imply a fast convergence: in the region where the REE is determinate (dark area), the line $\lambda = 0$, in fact, intersects all the isoquants lower or equal to $z_1 = 0.5$. If, instead, the relative weight to output gap is equal to 0.1, the policy-maker could choose only combinations of γ_π and γ_x such that the speed of convergence is slower than root-t: the line $\lambda = 0.1$ does not intersect any isoquant with $z_1 \leq 0.5$.

Points A and B in Figure 8 also show another important result that will be analyzed further in the next section: for a given value of λ there are infinitely many expectations-based policies that determine asymptotically the same REE, but induce a faster (or slower) speed of convergence than the one determined by EH policy (2.16).

1.4.1 The mapping from PLM to ALM

In order to show how the central bank can make active use of private agents' learning behavior in the monetary policy problem under discretion, I now consider more in detail the mapping from perceived to actual variables.

In section 2.3 I have shown that the analysis of the transition to the REE, under EH policy stands on the mapping from perceived inflation to actual inflation

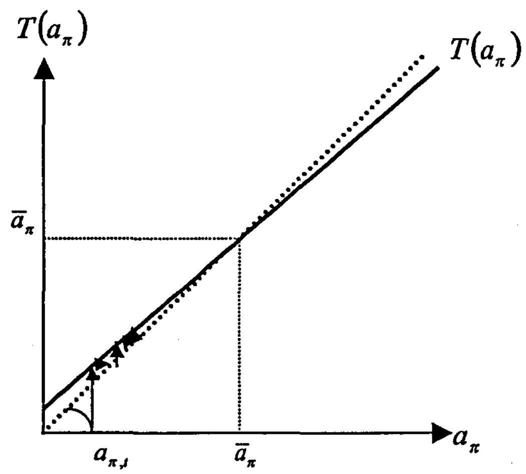
$$T\left(a_{\pi,t}\right) = \Phi^* + \Gamma^* a_{\pi,t} \tag{4.2}$$

and the necessary and sufficient condition for E-stability reduces to $\Gamma^* < 1$.

To give an example, since I consider λ to be an exogenous policy parameter, let us assume that the policy-maker gives a positive weight $\lambda = 0.1$ (note that with this weight the policy-maker cares 10 times more about inflation than about output gap). In this case the mapping $T\left(a_{\pi,t}\right)$ has a slope equal to 0.94 under CGG parametrization. Figure 9 shows the mapping from PLM to ALM. Even if initial perceived inflation is not too far from the REE, since the slope of the $T(.)$ mapping is close to 1, the transition from the learning to the RE equilibrium is very slow.

Fig 9

The mapping from PLM to ALM under the EH policy
($\lambda = 0.1$)



1.4.2 Adjusting the learning speed

The question now is whether a policy-maker who wants to reach in the long run the same REE determined by the EH policy (2.16) can speed up or slow down the private agents' learning process. To answer to this question I consider a subset of the asymptotically optimal policies that allow to offset not only demand shocks, but also expected output gap movements, as under the EH policy[29].

[29] At the beginning of this section I have analyzed a more generic set of asymptotically-optimal policies (4.1) that allowed to choose among different speeds of convergence. However, under that policy, the analysis of the learning dynamics involved a mapping from PLM to ALM with both

Definition 11 *The Adjusted Learning Speed-Γ' (ALS-Γ') policy rule, is an expectations-based reaction function*

$$i_t^{ALS}(\Gamma') = \gamma^{ALS} + \gamma_x^{ALS} E_t x_{t+1} + \gamma_\pi^{ALS} E_t \pi_{t+1} + \gamma_g^{ALS} g_t \quad (4.3)$$

with coefficients $\gamma^{ALS} = -\frac{\Phi^*(1-\Gamma')}{(1-\Gamma^*)\alpha\varphi}$, $\gamma_x^{ALS} = \gamma_g^{ALS} = \frac{1}{\varphi}$, $\gamma_\pi^{ALS} = \left(1 + \frac{\beta-\Gamma'}{\alpha\varphi}\right)$, *where* $1 < \Gamma' < 0$ *is the slope of the new mapping from perceived inflation to actual inflation obtained under the ALS-Γ' policy:*

$$T'(a_{\pi,t}) = \frac{(1-\Gamma')}{(1-\Gamma^*)}\Phi^* + \Gamma' a_{\pi,t}. \quad (4.4)$$

Note that, under least square learning, the ALS-Γ' policy leads to a mapping from PLM to ALM

$$T'(a_{\pi,t}, a_{x,t}) = \left(\frac{(1-\Gamma')}{(1-\Gamma^*)}\Phi^* + \Gamma' a_\pi, \frac{\Phi^*(1-\Gamma')}{(1-\Gamma^*)\alpha} - \frac{(\beta-\Gamma')}{\alpha} a_{\pi,t}\right) \quad (4.5)$$

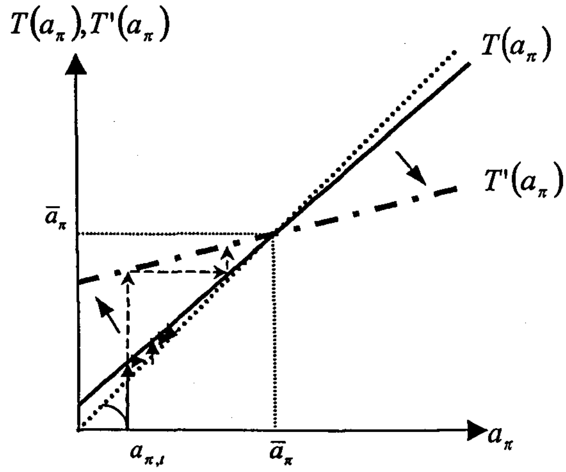
that does not depend on the perceived output gap. Therefore, in order to study convergence to the REE, as under EH-policy, the analysis can focus on the mapping from perceived to actual inflation.

Figure 10 shows the new mapping $T'(a_{\pi,t})$ under the *ALS-Γ'policy.*

Fig 10

The mapping from PLM to ALM under the ALS-Γ' policy

$(\Gamma' < \Gamma^*)$



perceived inflation and output gap. Here, instead, I consider a policy that allows a choice between diffent speeds of convergence just by looking at a mapping from perceived to actual inflation, as under the *EH Policy.*

In particular, it can be observed that $T'(a_{\pi,t})$ has the same fixed point, $\overline{a}_{\pi}$, as under the *EH policy*, but the intercept and the slope are different. The policy-maker, in order to speed up (slow down) the transition to the REE can follow an expectations-based reaction function that induces a rotation of the mapping from PLM to ALM around the fixed-point (i.e., the REE), with a slope Γ' lower (higher) than under the EH policy.

The following proposition formalizes this result.

Proposition 12 *Under rational expectations, the ALS-Γ' policy results in the same REE for $\{\pi_t, x_t\}$ derived under the EH policy. Under least squares learning, the ALS-Γ' policy results asymptotically in the same REE for $\{\pi_t, x_t\}$ derived under the EH policy.*

Proof. See Appendix I. ■

Taking parameters α, φ, β as given, under the EH policy, the speed of convergence relies entirely on λ: by choosing a λ the policy-maker is also choosing the slope of the $T(.)$ mapping (in the previous example, with $\lambda = 0.1$, the slope was equal to 0. 94) and, therefore, he determines the speed of convergence. However, under *ALS-Γ' policy,* the policy-maker could choose separately the relative weight on output gap and the speed at which agents learn without affecting the REE.

Lemma 13 *Under ALS-Γ' policy (4.3) the speed of convergence does not depend on the relative weight on output gap.*

Proof. See Appendix L. ■

Comparing now *EH* and *ALS-Γ'* policies, we have that

Lemma 14 *The response of interest rate to a rise in expected inflation is higher under the ALS-Γ' than under the EH policy if $\Gamma' < \Gamma^*$, is lower if $\Gamma' > \Gamma^*$.*

Proof. See Appendix M. ■

The following proposition and its corollary formally compare the transition under ALS-Γ' and under the EH policies.

Proposition 15 *Assume that private agents form expectations through recursive least squares learning and that initial perceived inflation is the same under both ALS-Γ' and EH policies but different from the REE. Now, if the reaction to expected inflation is stronger under ALS-Γ' than under EH policy, i.e. $\gamma_{\pi}^{ALS} > \gamma_{\pi}^{*}$, then perceived and actual inflation will be closer to the REE under ALS-Γ' than under the EH policy, along the transition. The opposite is true when $\gamma_{\pi}^{ALS} < \gamma_{\pi}^{*}$.*

Proof. See Appendix N. ■

Corollary 16 *Consider two ALS policies* $i^{ALS}(\Gamma_1')$ *and* $i^{ALS}(\Gamma_2')$ *with* $0 < \Gamma_1' < \Gamma_2' < 1$ *and* $a_{\pi,0}\left(i^{ALS}(\Gamma_1')\right) = a_{\pi,0}\left(i^{ALS}(\Gamma_2')\right) \neq \overline{a}_{\pi}$. *Along the transition, perceived and actual inflation will be closer to the REE under ALS-*$\Gamma'(\Gamma_1')$ *than under ALS-*$\Gamma'(\Gamma_2')$ *policy.*

The intuition is the following: if the policy-maker reacts strongly to a change in expected inflation, the difference between private agents' expectations and actual inflation will be greater and the prediction error will be initially bigger; if private agents make larger errors they will adapt their estimates faster and both expected and actual inflation will move closer to the REE. In other words, the stronger the policy-maker's response to a change in private agents' expectations, the faster private agents learn and the shorter the transition to the REE[30].

The fact that under the ALS policy for every $0 < t < \infty$ the distance from the REE could be smaller (greater) than under the EH policy brings to the following question: how long does it take under the two policies to get ε-close to the REE, i.e., starting from the same distance from the REE, $|a_{\pi,0} - \overline{a}_{\pi}| > \varepsilon$, how many periods are needed under the two policies in order to get $|\pi_t - \overline{a}_{\pi}| < \varepsilon$?

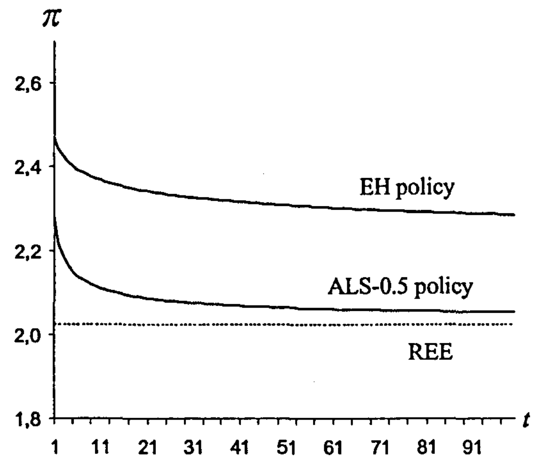
Assuming that the policy-maker follows a flexible inflation targeting policy rule with $\lambda = 0.1$, the output gap target is $\overline{x} = 0.004$ and using CGG calibration, figure 11 compares the results of a simulation under the EH policy and under an ALS-Γ' policy with $\Gamma' = 0.5$ (i.e., root-t convergence is imposed). Given that the REE for annual inflation is around 2 per cent, I consider an initial expected annualized inflation 0.5 percentage point higher than the REE.

[30] Orphanides and Williams (2003), independently obtained the result that policies that take account of private learning should call for aggressive responses to inflation in order to improve performances in stabilizing fluctuations in the economy. Their analysis is mainly focused on numerical simulations, while here I use the concept of speed of learning in order to justify analytically the result. In this sense this paper could be considered a theoretical justification for Orphanides and Williams work.

Fig 11

Deviation of actual inflation from the REE

($\pi^{RE} = 2\%$; initial expected inflation $= 2.5\%$; $\lambda = 0.1$)



Under the ALS-Γ' policy, after 1 quarter the initial error is already halved, after 1 year is below $0,2$ percentage points and after 5 years the distance from the REE is smaller than $0,1$ percentage point. On the contrary, under EH policy, after 1 quarter inflation is still 0.5 percentage point higher than the REE and after 20 years is still $0,3$ percentage points higher.

Table 2 compares the transition to the REE for different ALS-Γ' policies.

Tab 2

Transition under the ALS-Γ′ policy[31]

(Quarters needed in order to have $(\pi_t - \overline{\pi}_{REE})$ smaller than)

		0.4	0.3	0.2	0.1
γ_π^{ALS}	Γ'				
3.6	0.2	1	1	1	2
2.9	0.4	1	1	2	4
2.6	0.5	1	1	2	8
2.3	0.6	1	2	4	22
1.6	0.8	2	7	50	> 1000
1.3	0.9	6	98	> 1000	> 1000
1.2[1]	0.94	24	> 1000	> 1000	> 1000
1.03	0.98	> 1000	> 1000	> 1000	> 1000

Let us consider, for example, the ALS-Γ′ with $\gamma_\pi^{ALS} = 2.6$. Given that in equilibrium inflation is 2 per cent and assuming an initial expected inflation equal to 2.5 percent, inflation can be reduced by more than 0.3 percentage point, in half of the time needed under the ALS-Γ′ with $\gamma_\pi^{ALS} = 2.3$, approximately 1/25 of the time needed under the ALS-Γ′ with $\gamma_\pi^{ALS} = 1.6$ and more than 1/1000 of the time needed under the EH-policy.

This section looked at the role of policy decisions in determining the speed of convergence under learning, focusing on the mapping from perceived inflation to actual inflation. Before asking how the policy-maker can make use of his role to increase social welfare, the following lemma concerns the behavior of the output gap along the transition.

Lemma 17 *Under EH and ALS policies, when initial perceived inflation is higher (lower) than the REE, the output gap converges to the REE from below (above).*

Proof. See Appendix N. ■

Now it is possible to return to the question addressed at the beginning of the chapter: is the *EH policy* still optimal under learning? Are policies that speed up the learning process always better than policies that involve a slow transition to the REE?

[31] When $\lambda = 0.1$, the ALS policy with $\gamma_\pi^{ALS} = 1.2$ coincides with EH-policy.

1.5 Welfare analysis

In January 1999, with the start of stage 3 of the Economic and Monetary Union, monetary competencies were transferred from each country of the European Union to the European Central Bank. Before that date people were accustomed to take into account the monetary policy of their own country when making economic decisions. After the start of stage 3, they faced a new policy-maker (and a new monetary policy) and inflation and output gap equilibria determined under the new policy regime were, in some cases, different from the ones implied by the previous policies. Let us consider, for example, countries like Italy or Spain, whose rates of inflation are historically higher than in other member states, and assume that in those two countries expected inflation at the start of the EMU was higher than the REE determined by the new monetary regime. Under the assumption that private agents need time to learn the new equilibrium, it is clear that the dynamics of the learning equilibrium along the transition to the REE play an important role in the analysis of monetary policy decisions based on welfare measures. Questions like the ones raised at the end of the previous section show up spontaneously.

To answer to those questions I consider separately the two cases where initial expected inflation is higher than the REE and where it is lower. The reason why I proceed in this way is twofold. First, under adaptive learning, when the policy-maker chooses the policy, he already knows private agents' expectations and he could infer wether the initial bias in agents' prediction is positive or negative. Second, the welfare implications differs in the two cases. In the literature it is well known that under the loss function described in section 2.3 the first best plan would be, for all t, to have inflation and output gap at their target levels, i.e., $\pi_t^{FB} = 0$ and $x_t^{FB} = \overline{x}$. As many works have shown, under no commitment, the first best solution is not feasible if $\overline{x} \neq 0$. The optimal (time-consistent) policy in this case leads to a REE with inflation higher than the first best and output gap lower:

$$\pi_t^{REE} = \frac{\lambda\alpha}{(\lambda+\alpha^2)-\lambda\beta}\overline{x} > \pi_t^{FB} \quad \text{for all } t$$

$$x_t^{REE} = \frac{\lambda(1-\beta)}{(\lambda+\alpha^2)-\lambda\beta}\overline{x} < x_t^{FB} \quad \text{for all } t$$

Under learning, however, inflation and output gap could remain far from the REE for a long time. Therefore, if initial perceived inflation is higher than the REE, as in previous section, actual inflation will be higher and output gap lower than the REE along the transition. In this case, a policy-maker who bases decisions on the loss function described in section 2.3 would prefer policies that make inflation fall and output gap rise quickly to the REE. On the contrary, if initial perceived inflation is lower than the REE, the policy-maker would prefer policies that make inflation climbing and output gap landing slowly to the REE. Since *EH policy* is not taking into account the transition, I claim that there are *ALS*-Γ' *policies* that will make our economy better off.

In order to verify this claim, let us start by assuming that the *EH policy* (which is optimal under RE) is also optimal when private agents form expectations through adaptive learning. The aim is to compute the welfare cost of alternative monetary policies, i.e., ALS-Γ', that asymptotically result in the same REE as the *EH policy*, but along the transition result in different learning equilibria.

The social loss associated with *EH policy* is defined as:

$$L_0^{EH} = E_0 \sum_{t=0}^{\infty} \beta^t L\left(\pi_t\left(i^{EH}\right), x_t\left(i^{EH}\right)\right),$$

where $L\left(\pi_t\left(i^{EH}\right), x_t\left(i^{EH}\right)\right)$ is the period t loss function defined above and $\pi_t\left(i^{EH}\right), x_t\left(i^{EH}\right)$ denote the contingent plans for inflation and output gap under *EH policy*. Similarly, the social loss associated with *ALS*-Γ' *policies* is defined as

$$L_0^{ALS}\left(\Gamma'\right) = E_0 \sum_{t=0}^{\infty} \beta^t L\left(\pi_t\left(i^{ALS}\left(\Gamma'\right)\right), x_t\left(i^{ALS}\left(\Gamma'\right)\right)\right).$$

I measure the welfare cost (or gain) of adopting policy *ALS*-Γ' instead of the reference *EH policy* as the percentage increase (decrease) in the social loss of moving from *EH* to *ALS*-Γ'*policy*:

$$\omega\left(L_0^{ALS}\left(\Gamma'\right)\right) = \left(\frac{L_0^{ALS}\left(\Gamma'\right) - L_0^{EH}}{L_0^{EH}}\right) * 100.$$

Note that for values of $\omega\left(L_0^{ALS}\left(\Gamma'\right)\right) < 0$ there is a welfare gain in adopting ALS-Γ' policy instead of EH, while for $\omega\left(L_0^{ALS}\left(\Gamma'\right)\right) > 0$, there is a welfare loss.

I run simulations[32] for 10000 periods, assuming that the policy-maker follows a flexible inflation targeting policy rule with $\lambda = 0.1$, the output gap target is $\overline{x} = 0.004$ and using CGG calibration. The annualized inflation in the REE is around 2 per cent and the initial expected inflation is 0.5 percentage point higher (in annualized terms) than the REE. I compute social losses under the EH and ALS-Γ' policies for different values of γ_π^{ALS} (i.e., different Γ').

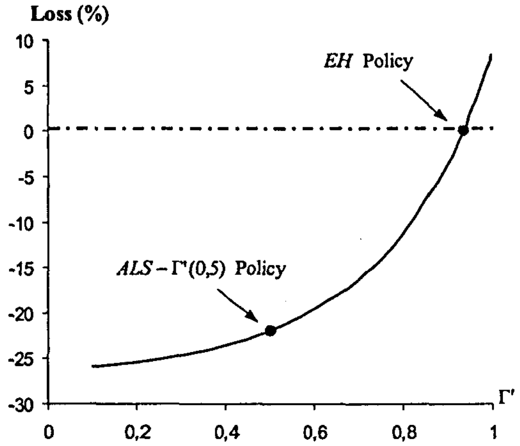
Figure 12 shows that ALS-Γ' policies with $\Gamma' < \Gamma^*$, by inducing a fast convergence, reduce the social loss up to 25 per cent relative to EH policy. Policies with $\Gamma' > \Gamma^*$, on the contrary, increase the social loss by up to 10 per cent. In particular, a central

[32] In the following simulations I consider as AR(1) stochastic process for the demand shock, with $\rho_g = 0.95$ and $\varepsilon_{gt} \sim N(0, 0.005)$

bank that follows an ALS-Γ' policy with $\gamma_\pi^{ALS} = 2.6$ can, by increasing the speed of convergence to root-t, lower the value of the loss function by 20 per cent relative to the EH policy.

Fig 12

Percentage loss in total welfare ($\pi_0 > \pi^{RE}$)



In order to analyze how the percentage increase (decrease) in the social loss evolves along the transition, simulations are also run for $T < 10000$ periods. Table 3 shows the results, pointing out that most of the gain from using an ALS-Γ' policy with fast transition is concentrated in the first 40 quarters.

Tab 3

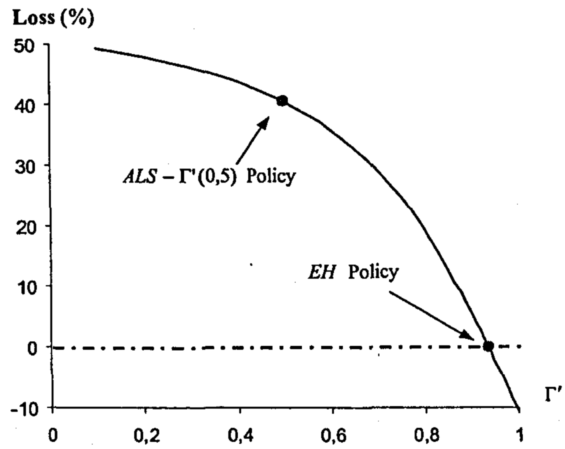
Percentage loss in total welfare after Tquarters ($\pi_0 > \pi^{RE}$)

γ_π^{ALS}	Γ'	T=10	T=20	T=40	T=100	T=10000
4.0	0.2	−11	−18	−22	−24.5	−25.5
3.3	0.4	−10	−16	−19	−22.5	−23.5
2.6	0.5	−9	−14	−17	−21	−22
2.3	0.6	−8	−12	−15	−18	−19.5
1.3	0.9	−1,5	−2	−3	−3,5	−4
1.03	0.98	3	4	5	6	7

Figure 13 and Table 4 show that under the assumption of an initial expected inflation 0.5 percentage point lower than the REE, by inducing a slower convergence, the policy-maker could reduce the welfare loss.

Fig 13

Percentage loss in total welfare ($\pi_0 < \pi^{RE}$)



A central bank that follows an ALS-Γ' policy with $\gamma_\pi^{ALS} = 1.03$ can, by increasing the slope of the mapping from perceived inflation to actual inflation to $\Gamma' = 0.98$, slow down the transition and lower the value of the loss function by approximately 10 per cent relative to the EH policy. On the contrary, a policy-maker who speeds up the transition to root-t convergence, following an ALS-Γ' policy with $\gamma_\pi^{ALS} = 2.6$, would increase the value of the loss function by approximately 40 per cent relative to the EH policy! Again, Table 4 shows that most of the loss from using an *ALS*-Γ' *policy* with fast transition is concentrated in the first 20 quarters, while advantages from inducing a slower convergence are distributed along the transition.

Tab 4

Percentage loss in total welfare after T quarters ($\pi_0 < \pi^{RE}$)

γ_π^{ALS}	Γ'	T=10	T=20	T=40	T=100	T=10000
4.0	0.2	56	53	51	49	48
3.3	0.4	43	43	44	44	43
2.6	0.5	36	38	39	40	40
2.3	0.6	28	30	33	35	36
1.3	0.9	3	4	5	5,5	6
1.03	0.98	−3	−5	−6	−8	−9

Before concluding I wish to emphasize some aspects concerning the robustness of welfare results.

1.6 Robustness

In the previous section I studied the speed of convergence and welfare by running simulations with $\lambda = 0.1$ and $\overline{x} = 0.004$. Changing these parameters would not change the finding that *EH policy* is not optimal under learning when the central bank makes active use of learning and that, when initial perceived inflation is higher than the REE, the central bank could increase welfare by inducing a faster transition. However, in the extreme case where $\overline{x} = 0$, if initial expected inflation is 0.5 percentage point lower (in annualized terms) than the REE, the finding that a slower convergence to the REE increases welfare does not hold anymore. In fact, when $\overline{x} = 0$, in our model, the optimal policy under discretion results in a REE with inflation and output gap equal to the first best, and a faster transition will always be better (table 5).

Tab 5

Percentage loss in total welfare, when $\overline{x} = 0$
$\left(\omega\left(L_0^{ALS}(\Gamma')\right);\ \mathbf{T} = 10000\right)$

γ_π^{ALS}	Γ'	$\pi_0 > \pi^{RE}$	$\pi_0 < \pi^{RE}$
4.0	0.2	−57	−57
3.3	0.4	−59	−59
2.6	0.5	−58	−58
2.3	0.6	−56	−56
1.3	0.9	−20	−20
1.03	0.98	69	69

Table 6 shows what happens if we change the weight that the policy-maker gives to output gap relative to inflation. In the table are reported the results of a simulation with $\overline{x} = 0.004$ and $\lambda = 0.05$ (i.e. the value obtained as the result of a general equilibrium problem in Woodford, 2003). In this case, the effects on welfare are only slightly different from the ones obtained in the previous section.

Tab 6

Percentage loss in total welfare, when $\lambda = 0.05$
$\left(\omega\left(L_0^{ALS}(\Gamma')\right);\ \overline{x} = 0.004,\ \mathbf{T} = 10000\right)$

γ_π^{ALS}	Γ'	$\pi_0 > \pi^{RE}$	$\pi_0 < \pi^{RE}$
4.0	0.2	−21	34.5
3.3	0.4	−19	31
2.6	0.5	−18	28
2.3	0.6	−15	24
1.3	0.9	1	−1
1.03	0.98	14	−15.5

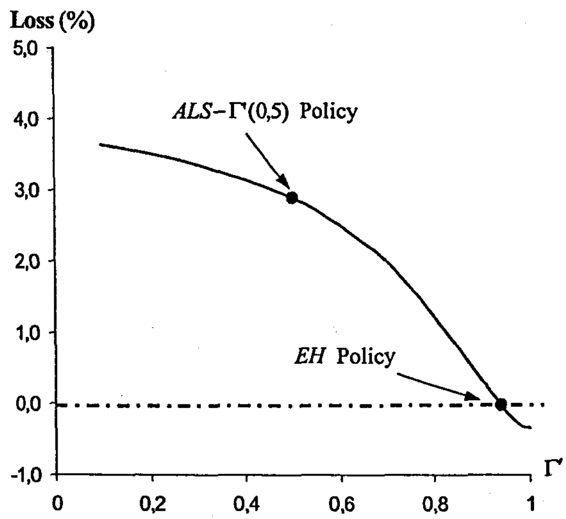
1.6.1 Initial Expected Inflation Symmetrically Distributed around the REE

Under the assumption that initial expected inflation is random and distributed symmetrically[33] around the REE figure 14 shows that if the policy maker does not take into account the initial bias in agents' prediction, *ALS*-Γ' *policies* that induce a slower convergence than under EH policy would be slightly preferable than policies that determine a fast convergence.

Fig 14

Percentage loss in total welfare

(π_0 symmetrically distributed around π^{RE})



1.6.2 An Economy with Cost-push Shocks

The new-Keynesian model analyzed in this chapter is derived assuming that only one shock affects the economy. Under this assumption the policy-maker neutralizes real effects of the shock whether it follows the *EH policy* or an *ALS*-Γ' *policy*, i.e., $\gamma_g^* = \gamma_g^{ALS} = \frac{1}{\varphi}$. However, when an additional shock hits the economy (for example, a "cost-push shock", u_t) the policy-maker cannot, in general, neutralize both shocks at the same time. In this case, since the two policies along the transition to the REE would react differently to u_t, welfare analysis could be affected. Simulations show that the introduction of a cost-push shock affects the results only in the amount of the welfare gain (or loss).

[33] Here I assume an uniform distribution between −0.5 and +0.5 percentage point around the REE.

Tab 7

Percentage loss in total welfare with cost-push shocks

$\left(\omega\left(L_0^{ALS}(\Gamma')\right);\ \mathrm{T}=10000\right)$

γ_π^{ALS}	Γ'	$\pi_0 > \pi^{RE}$	$\pi_0 < \pi^{RE}$
4.0	0.2	−23	36
3.3	0.4	−22	32
2.6	0.5	−20	29
2.3	0.6	−19	26
1.3	0.9	−4	4.4
1.03	0.98	8.3	−8.1

Table 7 shows that adding an AR(1) shock u_t in the aggregate supply equation[34], when initial private agents' perceived inflation is 0.5 percentage point higher (in annualized terms) than the REE, a central bank that follows an ALS-Γ' policy with $\gamma_\pi^{ALS} = 2.6$ can lower the value of the loss function by approximately 20 per cent relative to the EH policy (22 per cent without cost-push shocks); when initial private agents' perceived inflation is 0.5 percentage point lower than the REE, an ALS-Γ' policy with $\gamma_\pi^{ALS} = 1.01$ can lower the value of the loss function by approximately 8 per cent (10 per cent without cost-push shocks).

The results obtained in this section show that optimal policies derived under RE are not optimal under learning. Using results for the speed of convergence could help to increase social welfare by taking into account the transition from learning equilibrium to the REE. Solving for the true optimal policy under discretion and learning would involve taking into account that the policy-maker could make active use of private agents' learning behavior. However, since the optimal monetary policy has to be derived by substituting the private agents' PLM into the objective function, it would be time-dependent. Further analysis in this direction is required and will be left for future research.

1.7 Conclusions

In this chapter of the thesis I have shown that considering learning in a model of monetary policy design is particularly important in order to describe not only the asymptotic properties of rational expectations equilibrium to which the economy could

[34] I assume $\lambda = 0.1$, $\overline{x} = 0.004$, $u_t = \rho_u u_{t-1} + \varepsilon_{u,t}$ with $\rho_u = 0.35$ and $\varepsilon_{u,t} \sim N(0, 0.005)$

converge, but even to describe the dynamics that characterize the transition to this equilibrium.

The central message is that policy-makers should not only look at monetary policies that determine a stable equilibrium under learning, but also take into account how policy decisions affect the speed at which learning converges to rational expectations. In particular, under certain policies, the REE is E-stable, but the period needed to converge to this equilibrium could be incredibly long. Reacting strongly to expected inflation, a central bank would shorten the transition and increase the speed of convergence from the learning equilibrium to the REE.

A policy-maker who considers his role in determining the dynamics of the private agents' learning process could choose a policy rule that induces agents to learn at a given speed, affecting the welfare of society. In particular, if the policy-maker knows that after a regime change private agents' perceived inflation would be higher than the REE, by choosing a policy that reacts strongly to expected inflation he would determine a fast convergence and could increase social welfare. If, instead, perceived inflation is initially lower than the REE, a slow transition is preferred when the output gap target is greater than zero.

1.8 Appendix: Proofs of Propositions and Lemmas

A. Proof of Proposition 1

Given the recursive stochastic algorithm

$$a_{\pi,t} = a_{\pi,t-1} + t^{-1}\left(-\alpha\varphi\gamma + \left[\beta + \alpha\varphi\left(1-\gamma_\pi\right)\right] a_{\pi,t-1} + \alpha g_{t-1} - a_{\pi,t-1}\right)$$

let

$$h\left(a_\pi\right) = \left[-\alpha\varphi\gamma + \left[\beta + \alpha\varphi\left(1-\gamma_\pi\right)\right] a_\pi - a_\pi\right]$$

and let a_π be such that $h\left(a_\pi\right) = 0$. By the theorem of Benveniste et. al.(Theorem 3, page 110), if the derivative of $h\left(a_\pi\right)$ is smaller than $-1/2$, then

$$\sqrt{t}\left(a_{\pi,t} - \overline{a}_\pi\right) \xrightarrow{D} N\left(0, \sigma_a^2\right)$$

where σ_a^2 satisfies

$$\left[h'\left(\overline{a}_\pi\right)\right]\sigma_a^2 + E\left[-\alpha\varphi\gamma + \left[\beta + \alpha\varphi\left(1-\gamma_\pi\right)\right]\overline{a}_\pi - \overline{a}_\pi + \alpha g_t\right]^2 = 0$$

Note that the derivative of $E\left[-\alpha\varphi\gamma + \left[\beta + \alpha\varphi\left(1-\gamma_\pi\right)\right] a_\pi - a_\pi\right]$ being smaller than $-1/2$ coincides with $\left[\beta + \alpha\varphi\left(1-\gamma_\pi\right)\right]$ being smaller than $1/2$, i.e., γ_π being larger than $1 - \frac{1/2-\beta}{\alpha\varphi}$

B. Proof of Proposition 2

The formula for the asymptotic variance of the limiting distribution is

$$\sigma_a^2 = \frac{\alpha^2}{[1-\beta-\alpha\varphi(1-\gamma_\pi)]}\sigma_g^2$$

and the derivative,

$$\frac{\partial\sigma_a^2}{\partial\gamma_\pi} = -\frac{\alpha\varphi}{[1-\beta-\alpha\varphi(1-\gamma_\pi)]^2}\alpha^2\sigma_g^2 < 0$$

C. Proof of Proposition 3

The argument is similar to the one used in the proof of Propositions 1 and 2. In order to have root-t convergence,

$$\lambda < \frac{\alpha^2}{2\beta-1}$$

For values of $\lambda > \frac{\alpha^2}{2\beta-1}$ there is no root-t convergence and convergence will be slower.

D. Proof of Lemma 4

In the context of the present model, expected inflation and output gap are

$$E_t Y_{t+1} = \begin{pmatrix} a_{\pi,t} \\ a_{x,t} \end{pmatrix} = A_t$$

where $a_{\pi,t}$ and $a_{x,t}$ are estimated recursively

$$a_{\pi,t} = a_{\pi,t-1} + t^{-1}(\pi_{t-1} - a_{\pi,t-1})$$

$$a_{x,t} = a_{x,t-1} + t^{-1}(x_{t-1} - a_{x,t-1})$$

The ALM of inflation and output gap is

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = Q + FA_t$$

Thus the mapping from PLM to ALM takes the form

$$T(A_t') = Q + FA_t$$

Consider the stability under learning (E-stability) of the rational expectation solution $\overline{A}$ as the situation where the estimated parameters A_t converge to $\overline{A}$ over time.

From Evans and Honkapohja (2001), the E-stability is determined by the following differential equation

$$\frac{d}{d\tau}(A') = T(A') - A'$$

For this framework E-stability conditions are readily obtained by computing the derivative of $T(A') - A'$ and imposing that the determinant of the matrix with the derivatives of the previous differential equation with respect to A is greater than zero and the trace of the matrix with the derivative is greater than zero. In particular, the eigenvalues of F, z_1 and z_2, must have real parts less than one (let us define the biggest eigenvalue of the F matrix as z_1).

Then, let us distinguish between the two cases:

1. The "real" case.

   In this case two conditions must be satisfied in order to have convergence to the REE:

   (a) For reality

   $$(\alpha\varphi(1-\gamma_\pi) + \beta + (1-\varphi\gamma_x))^2 - 4\beta(1-\varphi\gamma_x) > 0$$

   (b) $\boxed{z_1 < 1}$ implies

   $$\gamma_\pi > 1 - \frac{(1-\beta)}{\alpha}\gamma_x$$

   Since by hypothesis $z_1 \geq z_2$, if $z_1 < 1$ then also $z_2 < 1$.

2. The "complex" case.

   In this case two conditions must be satisfied in order to have convergence to the REE:

   (a) For the solution to be imaginary,

   $$(\alpha\varphi(1-\gamma_\pi) + \beta + (1-\varphi\gamma_x))^2 - 4\beta(1-\varphi\gamma_x) < 0$$

   (b) $\boxed{\text{Real part of } z_1 < 1}$ implies

   $$\frac{(\alpha\varphi(1-\gamma_\pi) + \beta + (1-\varphi\gamma_x))}{2} < 1$$

   That is

   $$\gamma_\pi > 1 - \frac{1-\beta}{\alpha\varphi} - \frac{\gamma_x}{\alpha}$$

   Since by hypothesis $z_1 \geq z_2$, if $z_1 < 1$ then also $z_2 < 1$.

From case 1 and case 2, we obtain the necessary and sufficient condition for E-stability,

$$\gamma_\pi > \max\left[1 - \frac{1-\beta}{\alpha}\gamma_x, 1 - \frac{1-\beta}{\alpha\varphi} - \frac{\gamma_x}{\alpha}\right]$$

E. Proof of Proposition 5

Consider again the mapping from PLM to ACL under the least square learning hypothesis:

$$T\left(A_t'\right) = Q + FA_t$$

From Marcet and Sargent (1992) it follows that in order to have root-t convergence the eigenvalues of F must have the real part smaller than $\frac{1}{2}$.

Then, let us distinguish between the two cases:

1. The "real" case.

   In this case two conditions must be satisfied in order to have convergence to the REE:

   (a) For reality

   $$(\alpha\varphi(1-\gamma_\pi) + \beta + (1-\varphi\gamma_x))^2 - 4\beta(1-\varphi\gamma_x) > 0$$

   (b) $\boxed{z_1 < 0.5}$ implies

   $$\gamma_\pi > 1 + \frac{1-2\beta}{2\alpha\varphi} - \frac{1-2\beta}{\alpha}\gamma_x$$

   Note that if $z_1$is smaller than $\frac{1}{2}$ then even z_2 is smaller than $\frac{1}{2}$.

2. The "complex" case.

   In this case two conditions to be satisfied in order to have root-t convergence:

   (a) For the solution to be imaginary,

   $$(\alpha\varphi(1-\gamma_\pi) + \beta + (1-\varphi\gamma_x))^2 - 4\beta(1-\varphi\gamma_x) < 0$$

   (b) $\boxed{\text{Real part of } z_1 < \frac{1}{2}}$ implies

   $$\frac{(\alpha\varphi(1-\gamma_\pi) + \beta + (1-\varphi\gamma_x))}{2} < \frac{1}{2}$$

   That is

   $$\gamma_\pi > 1 + \frac{\beta}{\alpha\varphi} - \frac{\gamma_x}{\alpha}$$

   Note that if $z_1$is smaller than $\frac{1}{2}$ then even z_2 is smaller than $\frac{1}{2}$.

From case 1 and case 2, we obtain the necessary and sufficient condition for root-t convergence,

$$\gamma_\pi > \max\left[1+\frac{1-2\beta}{2\alpha\varphi}-\frac{1-2\beta}{\alpha}\gamma_x, 1+\frac{\beta}{\alpha\varphi}-\frac{\gamma_x}{\alpha}\right]$$

F. PROOF OF PROPOSITION 8

Consider the set $\Gamma = \{\gamma_\pi, \gamma_x : \gamma_\pi > 0, \gamma_x > 0 \text{ and } 0 \leq z_1 < 1\}$.

Monotonically increasing with respect to γ_π: for every $h = (\gamma_\pi^1, \gamma_x^1) \in \Gamma$ and $w = (\gamma_\pi^2, \gamma_x^1) \in \Gamma$, with $\gamma_\pi^2 \geq \gamma_\pi^1$, w implies a value for the real part of z_1 smaller or equal to the one with h.

Proof. z_1 is the biggest eigenvalue of F:

$$z_1 = \frac{(\alpha\varphi(1-\gamma_\pi)+\beta+(1-\varphi\gamma_x))}{2} + \\ +\frac{\sqrt{(\alpha\varphi(1-\gamma_\pi)+\beta+(1-\varphi\gamma_x))^2-4\beta(1-\varphi\gamma_x)}}{2}$$

Consider a $h = (\gamma_\pi^1, \gamma_x^1) \in \Gamma$ such that $z_1 = z_1^1$ is real. In this case

$$(\alpha\varphi(1-\gamma_\pi)+\beta+(1-\varphi\gamma_x))^2-4\beta(1-\varphi\gamma_x) > 0$$

For every $\varepsilon \geq 0$ there is a $w = (\gamma_\pi^2, \gamma_x^2) = (\gamma_\pi^1+\varepsilon, \gamma_x^1) \in \Gamma$ with $\gamma_\pi^2 \geq \gamma_\pi^1$.

For the combination $(\gamma_\pi, \gamma_x) = w$, the biggest eigenvalue of F, z_1^2 is equal to

$$z_1^2 = \frac{(\alpha\varphi(1-(\gamma_\pi^1+\varepsilon))+\beta+(1-\varphi(\gamma_x^1)))}{2} + \\ +\frac{\sqrt{(\alpha\varphi(1-(\gamma_\pi^1+\varepsilon))+\beta+(1-\varphi(\gamma_x^1)))^2-4\beta(\gamma_x^1)}}{2}$$

There could be two cases:

1. w is such that z_1^2 is real. In this case

   $$(\alpha\varphi(1-(\gamma_\pi^1+\varepsilon))+\beta+(1-\varphi\gamma_x^1))^2-4\beta\gamma_x^1 > 0$$

   Now, it is obvious that $z_1^2 - z_1^1 < 0$ and monotonicity with respect to γ_π is satisfied.

2. w is such that z_1^2 is complex. In this case z_1^1 should be compared with the real part of z_1^2

   Since

   $$\frac{(\alpha\varphi(1-(\gamma_\pi^1+\varepsilon))+\beta+(1-\varphi\gamma_x^1))}{2}-\frac{(\alpha\varphi(1-\gamma_\pi)+\beta+(1-\varphi\gamma_x))}{2} < 0$$

   monotonicity with respect to γ_π is satisfied.

Consider an $h = (\gamma_\pi^1, \gamma_x^1)$ such that z_1^1 is complex. In this case only the real part of z_1^1 is of interest.

Take a $w = (\gamma_\pi^1 + \varepsilon, \gamma_x^1)$, in this case $\|w - h\| = \left[(\gamma_\pi^1 + \varepsilon - \gamma_\pi^1)^2\right]^{\frac{1}{2}} = \varepsilon$. In the point $(\gamma_\pi, \gamma_x) = w$, the biggest eigenvalue of F, z_1^2 is equal to

$$
\begin{aligned}
z_1^2 &= \frac{(\alpha\varphi(1 - (\gamma_\pi^1 + \varepsilon)) + \beta + (1 - \varphi\gamma_x^1))}{2} + \\
&+ \frac{\sqrt{(\alpha\varphi(1 - (\gamma_\pi^1 + \varepsilon)) + \beta + (1 - \varphi\gamma_x^1))^2 - 4\beta\gamma_x^1}}{2}
\end{aligned}
$$

Note now, that if z_1^1 is complex, z_1^2 cannot be real: if z_1^1 is complex $4\beta\gamma_x^1 > (\alpha\varphi(1 - \gamma_\pi^1) + \beta + (1 - \varphi\gamma_x^1))^2$.

Now, since

$$
(\alpha\varphi(1 - \gamma_\pi^1) + \beta + (1 - \varphi\gamma_x^1))^2 > (\alpha\varphi(1 - (\gamma_\pi^1 + \varepsilon)) + \beta + (1 - \varphi\gamma_x^1))^2
$$

then $4\beta\gamma_x^1 > (\alpha\varphi(1 - (\gamma_\pi^1 + \varepsilon)) + \beta + (1 - \varphi\gamma_x^1))^2$, i.e., z_1^2 is complex. In this case it is obvious that monotonicity with respect to γ_π is satisfied. ■

No Monotonicity with respect to γ_x: Consider an $h = (\gamma_\pi^1, \gamma_x^1) \in \Gamma$ and a $w = (\gamma_\pi^1, \gamma_x^2) = (\gamma_\pi^1, \gamma_x^1 + \varepsilon) \in \Gamma$ such that z_1^1 and z_1^2 are complex. In this case it is easy to see (using a similar argument to the previous proof) that $z_1^1 \leq z_1^2$; take now $h = (\gamma_\pi^1, \gamma_x^1) \in \Gamma$ and a $w = (\gamma_\pi^1, \gamma_x^2) = (\gamma_\pi^1, \gamma_x^1 + \varepsilon) \in \Gamma$ such that z_1^1 and z_1^2 are real and it is easy to see that $z_1^2 \leq z_1^1$.

G. Proof of Lemma 9

Substituting the value of the conditional expectations into (2.21), the optimal policy rule could be written as:

$$
i_t = \gamma^R + \gamma_g^R g_t
$$

$$
\begin{aligned}
\gamma^R &= \frac{\lambda\alpha}{(\lambda + \alpha^2) - \lambda\beta}\overline{x} \\
\gamma_g^R &= \frac{1}{\varphi}
\end{aligned}
$$

This expression says that the policy-maker should offset demand shocks (g_t) by adjusting the nominal interest rate in order to neutralize any shock to the IS curve. Since this optimal policy rule involves only the fundamentals of the economy (demand and supply shocks), it could be defined as the *optimal fundamentals-based reaction function* under rational expectations (Evans and Honkapohja (2003a))[35].

[35] Many autors (see for example Woodford (1999)) have shown that this interest rate rule leads to indeterminacy, i.e., a multiplicity of rational expectations equilibria.

Now, consider a generic *expectations-based* policy rule of the form:

$$i_t = \gamma + \gamma_x E_t x_{t+1} + \gamma_\pi E_t \pi_{t+1} + \gamma_g g_t$$

Assuming rational expectations, expected values could be substituted in the previous expression to obtain the following policy rule:

$$i_t = (\gamma + \gamma_x a_x + \gamma_\pi a_\pi) + \gamma_g g_t$$

By comparing this equation with the *optimal fundamentals-based policy rule,* a system of two equations on four unknowns $(\gamma, \gamma_x, \gamma_\pi, \gamma_g)$ is obtained:

$$\begin{aligned} \gamma^R &= (\gamma + \gamma_x a_x + \gamma_\pi a_\pi) \\ \gamma_g^R &= \gamma_g \end{aligned}$$

Obviously, this system has multiple solutions.

H. Proof of proposition 10

By considering the values of the coefficients of the reaction function γ_g^*, γ^*,γ^R and the rational expectations values a_x, a_π as given, the combinations of γ_x and γ_π are obtained imposing asymptotically the same equilibrium derived under the *optimal expectations-based reaction function* (2.21):

$$\gamma_\pi = \frac{(\lambda + \alpha^2)(1 + \alpha\varphi) - \lambda\beta}{\alpha(\lambda + \alpha^2)\varphi} - \frac{(1-\beta)}{\alpha}\gamma_x$$

Consider the isoquants of Figure 8:

$$\gamma_\pi = 1 - \frac{(1-z_1)(\beta - z_1)}{z_1 \alpha\varphi} + \frac{(\beta - z_1)}{z_1\alpha}\gamma_x \qquad \text{for } \gamma_x < \widehat{\gamma}_x$$

$$\gamma_\pi = 1 + \frac{\beta + 1 - 2z_1}{\alpha\varphi} - \frac{1}{\alpha}\gamma_x \qquad \text{for } \gamma_x \geq \widehat{\gamma}_x$$

with a kink on

$$\left(\widehat{\gamma}_x = \frac{-z_1^2 + \beta}{\varphi\beta}, \widehat{\gamma}_\pi = 1 + \frac{(\beta - z_1)^2}{\alpha\varphi\beta}\right)$$

Restricting the analysis to the set $\Gamma_\beta = \{\gamma_\pi, \gamma_x : 0 < z_1 < \beta, \gamma_\pi > 0, \gamma_x > 0\}$, now the maximum speed of convergence problem defined for $0 < z_1 < \beta$:

$$\begin{aligned} &\min_{\gamma_\pi, \gamma_x} Z(\gamma_\pi, \gamma_x) \\ s.t. \quad \gamma_\pi &= \frac{(\lambda + \alpha^2)(1 + \alpha\varphi) - \lambda\beta}{\alpha(\lambda + \alpha^2)\varphi} - \frac{(1-\beta)}{\alpha}\gamma_x \end{aligned}$$

has a solution (use proposition 3.D.1 in Mas-Colell et al., 1995), and there is also an indirect speed of convergence function $v(\lambda)$ that is strictly decreasing on λ (use proposition 3.D.3 in Mas-Colell et al., 1995). The maximum speed of convergence that could be induced by a combination (γ_π, γ_x) for a given λ will always coincide with the kink. Note that

$$\frac{\partial \widehat{\gamma}_x}{\partial z_1} = \frac{-2z_1}{\varphi\beta} < 0$$

$$\frac{\partial \widehat{\gamma}_\pi}{\partial z_1} = \frac{-2(\beta - z_1)}{\alpha\varphi\beta} < 0 \quad \text{for} \quad z_1 < \beta$$

Now, since the higher the level curve, the faster the convergence, it must be shown that as λ increases, the line

$$\gamma_\pi = \frac{(\lambda + \alpha^2)(1 + \alpha\varphi) - \lambda\beta}{\alpha(\lambda + \alpha^2)\varphi} - \frac{(1-\beta)}{\alpha}\gamma_x$$

moves downward and the fastest speed of convergence that is feasible is lower, or in other words the smallest z_1 that can be reached is larger.

I. Proof of Proposition 12

Under the EH policy, the economy evolves according to the following dynamic system:

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} \Phi^* \\ \frac{\Phi^*}{\alpha} \end{bmatrix} + \begin{bmatrix} \Gamma^* & 0 \\ -\frac{(\beta - \Gamma^*)}{\alpha} & 0 \end{bmatrix} \begin{bmatrix} E_t\pi_{t+1} \\ E_t x_{t+1} \end{bmatrix}$$

Under the ALS-Γ' policy, the economy evolves according to the following dynamic system:

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} \frac{\Phi^*(1-\Gamma')}{(1-\Gamma^*)} \\ \frac{\Phi^*(1-\Gamma')}{(1-\Gamma^*)\alpha} \end{bmatrix} + \begin{bmatrix} \Gamma' & 0 \\ -\frac{(\beta - \Gamma')}{\alpha} & 0 \end{bmatrix} \begin{bmatrix} E_t\pi_{t+1} \\ E_t x_{t+1} \end{bmatrix}$$

The REE under both policies is

$$\pi_t = \frac{\Phi^*}{(1-\Gamma^*)} \text{ and } x_t = \frac{\Phi^*(1-\beta)}{(1-\Gamma^*)\alpha}$$

and under learning, when both Γ' and Γ^* are smaller than one the REE is E-stable.

L. Proof of Lemma 13

Since under ALS-Γ' policy,

$$\Gamma' = \left(1 - \gamma_\pi^{ALS}\right)\alpha\varphi + \beta,$$

given the parameters α, φ, β, each value of the policy reaction parameter γ_π^{ALS} has a corresponding slope of the $T(.)$ mapping, Γ', independently from λ.

M. Proof of Lemma 14

If

$$1 > \Gamma^* > \Gamma'$$

then

$$\gamma'_\pi = \left(1 + \frac{(\beta - \Gamma')}{\alpha\varphi}\right) > \left(1 + \frac{\beta - \Gamma^*}{\alpha\varphi}\right) = \gamma^*_\pi$$

Similarly if

$$1 > \Gamma' > \Gamma^*$$

N. Proof of Proposition 15

Define $a_{\pi,t}\left(i^{ALS}(\Gamma')\right)$ and $a_{\pi,t}\left(i^{EH}\right)$ the perceived inflation under ALS-Γ' and EH policies, $\pi_t\left(i^{ALS}(\Gamma')\right)$ and $\pi_t\left(i^{EH}\right)$ actual inflation under ALS-Γ' and EH policies. Assume that the economy starts from a point where the learning equilibrium and the REE do not coincide,

$$a_{\pi,0}\left(i^{ALS}(\Gamma')\right) = a_{\pi,0}\left(i^{EH}\right) \neq \overline{a}_\pi$$

I have to show that if $\gamma_\pi^{ALS} > \gamma^*_\pi$, then for every $0 < t < \infty$

$$\left|a_{\pi,t}\left(i^{ALS}(\Gamma')\right) - \overline{a}_\pi\right| < \left|a_{\pi,t}\left(i^{EH}\right) - \overline{a}_\pi\right|$$

and

$$\left|\pi_t\left(i^{ALS}(\Gamma')\right) - \overline{a}_\pi\right| < \left|\pi_t\left(i^{EH}\right) - \overline{a}_\pi\right|$$

while, if $\gamma_\pi^{ALS} < \gamma^*_\pi$, then for every $0 < t < \infty$

$$\left|a_{\pi,t}\left(i^{ALS}(\Gamma')\right) - \overline{a}_\pi\right| > \left|a_{\pi,t}\left(i^{EH}\right) - \overline{a}_\pi\right|$$

and

$$\left|\pi_t\left(i^{ALS}(\Gamma')\right) - \overline{a}_\pi\right| > \left|\pi_t\left(i^{EH}\right) - \overline{a}_\pi\right|$$

I will prove the proposition for $\gamma_\pi^{ALS} > \gamma^*_\pi$. A similar procedure could be used for $\gamma_\pi^{ALS} < \gamma^*_\pi$.

Let $\gamma_\pi^{ALS} > \gamma^*_\pi$, then

$$\Gamma' < \Gamma^*$$

Now, for $t = 0$, since

$$\Gamma^*\left(a_{\pi,0} - \frac{\Phi^*}{(1-\Gamma^*)}\right) > \Gamma'\left(a_{\pi,0} - \frac{\Phi^*}{(1-\Gamma^*)}\right)$$

then

$$\left|\pi_0\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_\pi\right|<\left|\pi_0\left(i^{EH}\right)-\overline{a}_\pi\right|$$

For $t=1$, since

$$a_{\pi,1}\left(i^{EH}\right)-\overline{a}_\pi=\pi_0\left(i^{EH}\right)-\overline{a}_\pi$$
$$a_{\pi,t}\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_\pi=\pi_0\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_\pi$$

then

$$\left|a_{\pi,t}\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_\pi\right|<\left|a_{\pi,1}\left(i^{EH}\right)-\overline{a}_\pi\right|$$

Moreover, since

$$\pi_1\left(i^{EH}\right)-\overline{a}_\pi=\Gamma^{*2}\left(a_{\pi,0}-\frac{\Phi^*}{(1-\Gamma^*)}\right)$$

$$\pi_1\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_\pi=\Gamma'^2\left(a_{\pi,0}-\frac{\Phi^*}{(1-\Gamma^*)}\right)$$

and since $\Gamma^{*2}>\Gamma'^2$, then

$$\left|\pi_1\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_\pi\right|<\left|\pi_1\left(i^{EH}\right)-\overline{a}_\pi\right|$$

Similarly for $t>1$.

N. Proof of Lemma 17

Given that

$$x^{REE}=\frac{\Phi^*(1-\beta)}{(1-\Gamma^*)\,\alpha}$$

it must be shown that if $a_{\pi,0}\left(i^{ALS}\left(\Gamma'\right)\right)=a_{\pi,0}\left(i^{EH}\right)>\overline{a}_\pi$, then for every $0\leq t<\infty$,

$$x_t\left(i^{ALS}\left(\Gamma'\right)\right),x_t\left(i^{EH}\right)<x^{REE}$$

and for all $0\leq t',t<\infty$, with $t'>t$

$$x_t\left(i^{ALS}\left(\Gamma'\right)\right)<x_{t'}\left(i^{ALS}\left(\Gamma'\right)\right)<x^{REE}\ \text{ and }\ x_t\left(i^{EH}\right)<x_{t'}\left(i^{EH}\right)<x^{REE}$$

If $a_{\pi,0}\left(i^{ALS}\left(\Gamma'\right)\right)=a_{\pi,0}\left(i^{EH}\right)<\overline{a}_\pi$, then for every $0\leq t<\infty$,

$$x_t\left(i^{ALS}\left(\Gamma'\right)\right),x_t\left(i^{EH}\right)>x^{REE}$$

and for all $0\leq t',t<\infty$, with $t'>t$

$$x_t\left(i^{ALS}\left(\Gamma'\right)\right)>x_{t'}\left(i^{ALS}\left(\Gamma'\right)\right)>x^{REE}\ \text{ and }\ x_t\left(i^{EH}\right)>x_{t'}\left(i^{EH}\right)>x^{REE}$$

$$x_t\left(i^{EH}\right) = \frac{\Phi^*}{\alpha} - \frac{(\beta - \Gamma^*)}{\alpha} a_{\pi,t}\left(i^{EH}\right)$$

$$x_t\left(i^{ALS}(\Gamma')\right) = \frac{\Phi^*(1-\Gamma')}{(1-\Gamma^*)\alpha} - \frac{(\beta-\Gamma')}{\alpha} a_{\pi,t}\left(i^{ALS}(\Gamma')\right)$$

Let $a_{\pi,0} < \overline{a}$. Since

$$\frac{(\Gamma'-\beta)}{\alpha}, \frac{(\Gamma^*-\beta)}{\alpha} < 0$$

and

$$x_0\left(i^{EH}\right) - x^{REE} = \frac{(\Gamma^*-\beta)}{\alpha}\left(a_{\pi,0} - \frac{\Phi^*}{(1-\Gamma^*)}\right)$$

$$x_0\left(i^{ALS}(\Gamma')\right) - x^{REE} = \frac{(\Gamma'-\beta)}{\alpha}\left(a_{\pi,0} - \frac{\Phi^*}{(1-\Gamma^*)}\right)$$

then $x_0\left(i^{EH}\right), x_0\left(i^{ALS}(\Gamma')\right) < 0$.

For $t = 1$, we have

$$x_1\left(i^{EH}\right) - x^{REE} = \Gamma^* \frac{(\Gamma^*-\beta)}{\alpha}\left(a_{\pi,0} - \frac{\Phi^*}{(1-\Gamma^*)}\right)$$

$$x_1\left(i^{ALS}(\Gamma')\right) - \overline{a}_\pi = \Gamma' \frac{(\Gamma'-\beta)}{\alpha}\left(a_{\pi,0} - \frac{\Phi^*}{(1-\Gamma^*)}\right)$$

and again it is obvious that $x_1\left(i^{EH}\right), x_1\left(i^{ALS}(\Gamma')\right) < 0$ and

$$x_1\left(i^{ALS}(\Gamma')\right) > x_0\left(i^{ALS}(\Gamma')\right) \text{ and } x_1\left(i^{EH}\right) > x_0\left(i^{EH}\right)$$

similarly for t=2,3,... and for the case $a_{\pi,0}\left(i^{ALS}(\Gamma')\right) = a_{\pi,0}\left(i^{EH}\right) > \overline{a}_\pi$.

Chapter 2

Quantitative Implication of Limited Commitment and Temporary Exclusion

A burgeoning literature studies models of risk-sharing where the planner is able to enforce contracts under the threat of permanent exclusion. Permanent exclusion has been often criticized for not being a credible threat. We study the problem when exclusion can only be enforced temporarily. We show how to adapt recursive methods to compute the optimal allocation and compare it with the one enforceable under the threat of permanent exclusion. We study when the exclusion period is large enough and we use this to comment on various issues of risk default and international borrowing and lending[1].

2.1 Introduction

The literature on risk-sharing has provided a framework to think about the role of assets in diversifying risks. The effects on asset prices, consumption and investment, both within a country and across countries, have been thoroughly studied under complete markets. In this case, with full information, the optimal consumption allocation depends on the current value of the aggregate state only; this outcome reflects the opportunities to insure risks that markets provide.

One stylized fact on consumption data, however, is that, conditional on per capita consumption, individual consumption is positively correlated with current and lagged individual income. One way to model this phenomena is to assume that financial

[1]This chapter is part of a project on which I'm working with Albert Marcet.

markets are incomplete, so that agents cannot eliminate all idiosyncratic risk. The drawback of this approach is that the results it implies depend on the arbitrary set of securities that are made available to agents. A recent strand of literature considers risk-sharing agreements under limited participation, introducing an element of default in the equilibrium. The main assumption is that agents are able to default on the risk-sharing agreement if they find a better outside alternative. An enforceable contract should deliver at all times at least the same utility of the outside option. Several studies assume that the outside option is defined by some autarkic solution; references include, among others, Kehoe and Levine (1993), Kocherlakota (1996), Marcet and Marimon (1998), Alvarez and Jermann (2000, 2001) and Kehoe and Perri (2002). All these works consider *permanent* exclusion as the harshest punishment for default and they study optimal contracts under the constraint that agents will never be worse off with respect to permanent autarky. The introduction of these constraints provides agents with incentive to commit to their agreements at the cost of reducing risk sharing.

Kehoe and Levine (1993) and Kocherlakota (1996) study efficient allocations in economies where participation constraints ensure that agents would at no time be better off by reverting permanently to autarchy. Alvarez and Jermann (2000) show that by modelling participation constraints as portfolio constraints, the efficient allocations can be decentralized as a competitive equilibrium with solvency constraints. Kehoe and Perri (2002) go a step further by assuming that the constraints that private agents face are explicitly chosen as part of the equilibrium.

The common factor of all these works is that the optimal solution is enforced by the threat to leave the defaulting agent in autarky forever. In most real life situations this is hardly a credible threat. In most situations where default may take place, agents would have an incentive to renegotiate after having been in autarky, because this punishment eliminates all future mutual gains from intertemporal exchange. In general if a relationship is broken (be it a credit contract, a commercial relationship, a political alliance, an employment agreement or, of course, a sentimental relationship) permanent exclusion is not a credible threat, but in most of these cases some time has to elapse before a new similar relationship can be established.

In this chapter we relax the assumption of permanent autarky as the harshest credible punishment for default on a risk-sharing agreement[2]. We assume that a defaulting agent will be excluded for T periods. After this period of time the agent can start anew with a (restricted) optimal contract that, in turn, will take into account the possibility of default. In solving the model, we assume that the planner that will reoptimize after the T periods, will have the same preferences as his predecessor

[2]While we were working at this project, we realized that Scholl (2004) has independently developed a similar framework in the context of international business cyle theory.

(equivalently, this is the same planner, who could not enforce autarky any longer)[3]. There are many real world situations that can be used to motivate this structure: a typical example of temporary autarky, is the case described by the Chapter 7 of the US Bankruptcy Code in the States. Borrowers can default on their loans by filing for bankruptcy; post-bankruptcy the household has serious difficulty in getting new loans for a period of about ten years, but once the ten year period is over, federal law mandates that the record of the filing be deleted from the household's credit history. Other examples come from international borrowing and lending agreements: it is not credible that a country would be permanently excluded from international borrowing and lending after default, given that governments (both of the defaulting and the lending countries) change after a few years, so permanent exclusion is simply not an option. In fact, all countries in history that have defaulted have been, eventually, brought back in the international capital markets.

We present preliminary results. First, we characterize analytically the constrained efficient allocation. A recursive formulation is needed to compute equilibria. The most general ways of formulating optimal dynamic contracts recursively are the Lagrangian approach of Marcet and Marimon (M&M, 1998) and the promised utility approach of Abreu, Pierce and Stachetti (APS, 1990). Temporary punishment is not an immediate extension of these setups, because the outside option now depends on the solution of the optimal contract, since the optimal contract will be enforced again after T periods. We show how to adapt these methods to formulate a recursive solutions by solving an appropriate fixed point problem that uses M&M or APS as an accessory in an inner loop iteration. We develop an algorithm that implements the recursive solution. We show, perhaps surprisingly, that the computational costs of finding a numerical solution to the temporary punishment model are, in a certain dimension, the same as for the case of permanent autarky. Therefore, maintaining this assumption is not justified by appealing to numerical simplicity.

We also show that in some cases the risk-sharing agreement may give rise to an empty feasible set. This may occur when the fixed point that defines an equilibrium is stated in terms of arbitrary functions or, even in cases where a feasible set is known to be non-empty, along the iterations to find the solution of the problem. We overcome the problem by modifying slightly the original model, introducing a penalty function in the objective of the planner.

Finally, we decentralize the allocation by a system of borrowing constraint and prices a la Alvarez & Jermann in order to discuss the properties of the pricing kernel and obtain predictions about asset prices. We document quantitative implications

[3]Studying in detail the game theorethic justification for this assumption is beyond the scope of this paper and is left for future research.

and compare them with the ones obtained under permanent reversion to autarchy. We show how to use our framework to asses a few issues of interest, such as the length of the punishment needed in order to get close to the equilibrium under the threat of permanent exclusion; the behavior of prices and interest rate under the two punishment strategies. The first issue is important in determining the effects of different periods of exclusion: are the 10 years of Chapter 7 very costly in terms of efficiency?, do they imply a very different interest rate with respect to the case of permanent exclusion?, how long should the IMF exclude a defaulting country before it is again a candidate to receive loans?. The second issue is important to determine whether temporary punishment can help or hurt in explaining some empirical puzzles.

The chapter is organized in the following way. Section 2.2 describes the environment. Section 2.3 discusses the equilibrium allocation and the computational method. Section 2.4 shows the quantitative results and section 2.5 concludes.

2.2 The Environment

There are two *ex ante* identical households denoted $i = 1, 2$ with preferences over consumption streams ordered by $E_0 \sum_{t=0}^{\infty} \delta^t u_i\left(c_t^i\right)$, where $u(c)$ is an increasing, strictly concave, and twice continuously differentiable function, and $\delta \in (0,1)$ is a discount factor.

Each household receives a stochastic endowment stream $\left\{\omega_t^i\right\}_{t=0}^{\infty}$ for $i = 1, 2$ where ω_t^1, ω_t^2 is a Markov process with ω_0^1, ω_0^2 given and

$$\log \omega_t^i = (1 - \rho_\omega)\overline{\omega} + \rho_\omega \log \omega_{t-1}^i + \varepsilon_t^i$$

with $\varepsilon_t \sim N(0, \Omega)$ and *i.i.d.*; all variables dated t are contingent on realizations of the shocks up to t.

A planner allocates goods to households in ways designed to get them to share resources voluntarily given that they have an outside option. The planner objective function consists in maximizing

$$E_0 \sum_{t=0}^{\infty} \delta^t \left[\alpha u_1\left(c_t^1\right) + (1 - \alpha) u_2\left(c_t^2\right)\right] \tag{1}$$

subject to the stochastic endowment law of motion and the feasibility constraint:

$$c_t^1 + c_t^2 \leq \omega_t^1 + \omega_t^2 \tag{2}$$

The planner is committed to honor his promises, while private agents are free to walk away at any time and choose autarky. In order to ensure that agents share their resources, efficient contracts would issue rewards that depend on the public observable outcomes.

A fairly common assumption in the literature on limited enforcement is that if the participants walk away from the arrangement they would be punished with *permanent exclusion.* In our work we focus on a more general case where the household that chooses to default, will have to spend only finitely many periods in autarky. We refer to this period as "crossing the desert". We also assume that once the punishment is lifted the defaulting agent regains the status that he had at time 0. This assumption could be loosely justified as corresponding to a situation where the defaulting agent, after having crossed the desert, finds some planner and some other agent to provide, from then on, the same contract that he has faced at $t = 0$, contingent on the state variables being the same as the ones after crossing the desert.

2.2.1 Temporary Exclusion

Let $c_{t'}^{i,t}$ for $t' \geq t$ denote the consumption (contingent on the realization) of agent i in period t' if the agent has defaulted in t. In order to enforce agent i's participation, the planner should make sure that, at any time, the continuation utility delivered by his allocation is at least as high as the one implied by staying in autarchy for T periods and after that to a utility to be determined by the contract,

$$E_t \sum_{j=0}^{\infty} \delta^j u_i \left(c_{t+j}^i\right) \geq E_t \sum_{j=0}^{T-1} \delta^j u_i \left(\omega_{t+j}^i\right) + E_t \sum_{j=T}^{\infty} \delta^j u_i \left(c_{t+j}^{i,t}\right). \tag{3}$$

The first sum in the right side refers to the punishment period, when the agent consumes only his endowment, the second sum allows for the consumption T-periods after default to be determined by the contract.

Some assumptions on the nature of the contract that the defaulting agent obtains after the exclusion period expires are needed. We assume that if agent i defaults at time t, the allocations after T periods do not depend on shocks that occurred before $t + T$. In addition, we assume that the allocations after crossing the desert are the same (given the shocks) regardless of when default occurred. Let $s_t = (\omega_t^1, \omega_t^2)$ be the vector of states with support $\mathcal{S}$ and $s^t = (s_t, s_{t-1}, ...s_0)$, then, we assume that there are functions $c_j^{i,D} : \mathcal{S}^j \rightarrow R_k$ such that

$$c_{t+T+j}^{i,t}(s^{t+T+j}) = c_j^{i,D}(s_{t+T}, s_{t+T+1}, ..., s_{t+T+j}) \qquad j = 0, 1, ...$$

Restricting consumption after default in this way is reasonable in most situations having to do with limited participation. It captures the idea that after the punishment is over, the allocation does not depend on the fact that the default has occurred, either because of bankruptcy law (as in Chapter 7) or because another planner is going to offer a similar contract to the agent (as in banking, when competitive banks offer contracts that yield the same to the same agents) or because of lack of commitment on the part of the planner which causes the planner to deny the promises it made after T periods.

Given $c_j^{i,D}$, we can define the expected utility after "crossing the desert" as a function $F^{i,D} : S \to R$ such that

$$F^{i,D}(s_{t+T}) = E_{t+T} \sum_{j=0}^{\infty} \delta^j u_i \left(c_j^{i,D} \left(s^{t+T+j} \right) \right) \tag{4}$$

In words, equation (4) says that the expected value of the continuation utility after crossing the desert of an agent that decided to default at time t depends only on time $t+T$ state variables. Therefore, the only consumption paths that are feasible are those that satisfy

$$E_t \sum_{j=0}^{\infty} \delta^j u_i \left(c_{t+j}^i \right) \geq E_t \sum_{j=0}^{T-1} \delta^j u_i \left(\omega_{t+j}^i \right) + \delta^T E_t F^{i,D}(s_{t+T}) \tag{5}$$

for all periods and realizations.

Thus, given any set of functions $F = (F^1, F^2)$ such that the set of consumption allocations satisfying (5) is non-empty, and denoting

$$V_i^{Aut,T}(\omega_t; F) \equiv E_t \sum_{j=0}^{T-1} \delta^j u_i \left(\omega_{t+j}^i \right) + \delta^T E_t F^i(\omega_{t+T}) \tag{6}$$

we have to solve the

F-Planner Problem

$$\max_{\{c_t\}} \; E_0 \sum_{t=0}^{\infty} \delta^t \left[\alpha u_1 \left(c_t^1 \right) + (1-\alpha) u_2 \left(c_t^2 \right) \right] \tag{7}$$

$$\text{s.t.} \quad E_t \sum_{j=0}^{\infty} \delta^j u_i \left(c_{t+j}^i \right) \geq V_i^{Aut,T}(\omega_t; F) \qquad \forall t, i \tag{8}$$

$$c_t^1 + c_t^2 \leq \omega_t^1 + \omega_t^2 \tag{9}$$

$$\text{given } \left(\omega_0^1, \omega_0^2 \right) \tag{10}$$

The contract that enforces the optimal allocation obtained from solving the F-planner problem is a sequence of functions $c_t^i = h_t\left(s^t; F\right)$ for $t \geq 0$ that assigns a history-dependent consumption stream c_t^i to the households.

Finally, we assume that the consumption profile obtained after crossing the desert is the same as the optimal profile that would be chosen at $t = 0$. This is the usual case considered under time consistency, where it is known that the planner is going to re-optimize (in this case after the punishment period of length T). As a result, the planner sets the variables taking into account the fact that in the future there would be a planner willing to offer the same $t = 0$ contractual conditions to agents who have crossed the desert.

Now, for each agent we can compute the value function

$$V^i(s_0; F; \alpha) \equiv E_0 \sum_{j=0}^{\infty} \delta^j u_i\left(c_t^{i*}\right) \tag{11}$$

for all $s_t \in S$ where "$*$" denotes the optimal solution of the planner's problem. Notice that α, in principle, may depend on time; here, however, we assume for simplicity that it is constant[4]. In this way we can maintain implicit the dependence of the optimal solution and the value function on α, since we assume that the future planners will use the same α. Since, by assumption, the planner that will re-optimize after T-periods of autarchy, has the same preferences of his predecessor, we have to look for a solution $W = (W^1, W^2)$ where

$$V^i(\cdot, W) = W^i(\cdot) \quad \text{for } i = 1, 2 \tag{12}$$

Definition An optimal contract with temporary autarky and reversion to initial value is defined as the solution to the planner problem described above when $F = W$ satisfying (12).

2.2.2 Recursive Formulation

In order to reduce the dimension of the argument of the h_t function we apply a recursive formulation of the history-dependent contracts. The key is to realize that if F is fixed, the function $V_i^{Aut,T}(\cdot; F)$ is a given function, known before the optimal solution is found, so that the Lagrangian approach of Marcet and Marimon (1998) can be used to formulate recursively the solution to the F-planner's problem.

Let $\gamma_{i,t}$ be the Lagrange multiplier associated to the time-t participation constraint of agent i. The planner's Lagrangean would be

$$\begin{aligned} L \;=\; & E_0 \sum_{t=0}^{\infty} \delta^t \{\alpha u_1\left(c_t^1\right) + (1-\alpha) u_2\left(c_t^2\right) + \\ & \sum_{i=1}^{2} \gamma_{i,t} E_t \left[\sum_{j=0}^{\infty} \delta^j u_i\left(c_{t+j}^i\right) - V_i^{Aut,T}\left(\omega_t; F\right) \right] \}. \end{aligned}$$

After defining the co-state variable as in Marcet and Marimon (1992),

$$\mu_{i,t} = \mu_{i,t-1} + \gamma_{i,t} \quad \text{for } i = 1, 2 \tag{13}$$

[4] We leave for further reasearch the case where α is endogenous and changes across periods.

with $\mu_{1,-1} = \alpha$, $\mu_{2,-1} = 1 - \alpha$, the Lagrangian could be written as:

$$\begin{aligned} L &= E_0 \sum_{t=0}^{\infty} \delta^t \{ (\mu_{1,t-1} + \gamma_{1,t}) u_1 (c_t^1) + (\mu_{2,t-1} + \gamma_{2,t}) u_2 (c_t^2) - \\ & - \sum_{i=1}^{2} (\mu_{i,t} - \mu_{i,t-1}) V_i^{Aut,T} (\omega_t; F) \} \end{aligned} \tag{14}$$

The solution of the F-planner problem satisfies the recursive saddle point functional equation (SPFE):

$$\begin{aligned} W(\omega, \mu) &= \inf_{\gamma_i \geq 0} \sup_{c^i} \{ (\mu_1 + \gamma_1) u_1 (c^1) + (\mu_2 + \gamma_2) u_2 (c^2) - \\ & - \sum_{i=1}^{2} \gamma_i V_i^{Aut,T} (\omega^i; F) + \delta E [W(\omega', \mu') \mid \omega] \} \\ \text{s.t.} \quad \mu_i' &= \mu_i + \gamma_i \quad \text{for } i = 1, 2 \end{aligned} \tag{15}$$

and

$$\begin{bmatrix} c \\ \gamma \end{bmatrix} = f(\omega, \mu; F) \tag{16}$$

for a time-invariant policy function f, where $\mu_{1,-1} = \alpha$, $\mu_{2,-1} = 1 - \alpha$ and μ_t evolves according to (13). Solving the model reduces to finding the policy function f. Once this function is approximated we can generate consumption sequences and find the agent value functions $V(\cdot; F)$. In section 3 we describe the computational method used to find f and V.

Once we know how to approximate the f function (and, consequently, V), the temporary punishment equilibrium can be computed by iterating on F until we find the fixed point satisfying (12).

Described in this form, finding the temporary punishment equilibrium may appear very complicated, because we need an inner iteration (to solve for $f(\cdot; F)$ given F) nested in an outer iteration (to iterate on F and find W). The good news is that we can subsume both iterations in one. We start with a candidate f, test if the FOC and (12) are satisfied, and iterate until both the FOC are satisfied and the resulting agent-value function satisfies the fixed point requirement (12). More formally, let $\mathcal{W}$ be the function that maps a candidate decision function f into the agent-value function V :

$$\mathcal{W}_i(f)(s) \equiv E \left(\sum_{t=0}^{\infty} \delta^t u_i(c_t^i(f)) \middle| s \right) \tag{17}$$

where $c_t^i(f)$ is the consumption series generated by f. The policy function in the temporary exclusion algorithm satisfies[5]

$$f^{TE}(\cdot;\mathcal{W}(f^{TE})) \tag{18}$$

Before characterizing the optimal allocation under the threat of temporary exclusion and describing the computational algorithm used to find the numerical solution, we would like to remark two points.

First of all, notice that in order to recover the case of limited enforcement under the threat of permanent exclusion it is sufficient to consider the model with temporary exclusion and T sufficiently large. In fact as $T \to \infty$, participation constraints reduce to

$$E_t \sum_{j=0}^{\infty} \delta^j u_i\left(c_{t+j}^i\right) \geq E_t \sum_{j=0}^{\infty} \delta^j u_i\left(\omega_{i,t+j}\right) \tag{19}$$

where the right hand side is the utility of the outside option when the agent consumes only his endowment. Notice that under the threat of permanent exclusion we don't have to solve the outer fixed point problem (i.e., we don't need to iterate on F and find W). In this case, Marcet and Marimon (1992) have shown that the optimal allocations satisfy

$$\frac{u_1'\left(c_t^1\right)}{u_2'\left(c_t^2\right)} = \frac{\mu_{2,t}}{\mu_{1,t}}. \tag{20}$$

Finally, notice that full enforcement is equivalent to assume that the planner has the ability to punish any deviation from the optimal plan with some action that leaves the defaulting agent with arbitrarily low utility. In this case, the optimal allocations satisfy the sharing rule

$$\frac{u_1'\left(c_t^1\right)}{u_2'\left(c_t^2\right)} = \frac{1-\alpha}{\alpha} \tag{21}$$

Equation (21) tells us that consumption of both agents depends on the current value of the aggregate state only.

2.3 Characterizing the optimal allocation

From the recursive formulation given by the SPFE (15) we can easily obtain the first order conditions for the optimal consumption paths of the agents, which, combined, will define the following optimal sharing rule:

$$\frac{u_1'\left(c_t^1\right)}{u_2'\left(c_t^2\right)} = \frac{\mu_{2,t}}{\mu_{1,t}}. \tag{22}$$

[5] We are actually working in showing that in order to find the temporary punishment equilibrium it is sufficient to find an algorithm that at the n-th iteration obtains an approximate function f_n using $F_n^D = \mathcal{W}(f_{n-1})$ in the FOC.

Equation (22) shows that in an optimal allocation the planner should choose efficiently the weights $\mu_{2,t}$ and $\mu_{1,t}$, making sure that agents are induced not to default by increasing consumption not only in the period in which they are tempted to default but also in the future. That is, "individual paths of consumption depend on individual histories not just on the initial wealth distribution and the aggregate consumption path, as in the Arrow-Debrau competitive allocations" (Marcet and Marimon, 1998). Comparing equations (20) and (22) we can see the similarities between permanent and temporary exclusion: given the lagrange multipliers, the two optimality conditions are the same. However, as the participation constraints differ in the two models, the lagrange multipliers and the optimal allocations should differ.

Notice that when the participation constraints are never binding (for example, if the endowment of each agent can be sufficiently small with a positive probability) equation (22) reduces to (21) and the efficient allocation is the same as under full enforcement.

We analyze now in detail the computational algorithm.

2.3.1 Computational Algorithm

For convenience, we normalize the multipliers by defining $\widetilde{\gamma}_t^i \equiv \frac{\gamma_t^i}{\mu_{1,t-1}}$ and $\lambda_t \equiv \frac{\mu_{2,t}}{\mu_{1,t}}$[6]. This allows us to keep track of only the relative weight λ_t instead of the two absolute weights $\mu_{1,t}$ and $\mu_{2,t}$. The transition law for λ can be written as follows,

$$\lambda_t \equiv \frac{\lambda_{t-1} + \widetilde{\gamma}_t^2}{1 + \widetilde{\gamma}_t^1} \tag{23}$$

With the normalized multipliers we can summarize the sharing rule (22) by

$$\frac{u_1'\left(c_t^1\right)}{u_2'\left(c_t^2\right)} = \frac{\lambda_{t-1} + \widetilde{\gamma}_t^2}{1 + \widetilde{\gamma}_t^1}. \tag{24}$$

To approximate the non linear functions that enter in the participation constraints we employ the parametrized expectation approach (PEA) described in den Haan and Marcet (1994).

We specify a second-degree polynomial in the state and co-state variables to approximate the left-hand side of the participation constraint,

$$E_t \sum_{j=1}^{\infty} \delta^{j-1} u\left(c_{t+j}^i\right) = \psi\left(a; S_{i,t}\right) \tag{25}$$

[6] Since we assume that co-state variables at time $t = -1$ are equal to zero, the initial welfare weights' ratio is $\lambda_{-1} \equiv \frac{1-\alpha}{\alpha}$.

where $S_{1,t} = (1, \log\omega_t^1, \log\omega_t^2, \log\lambda_{t-1})$, $S_{2,t} = (1, \log\omega_t^2, \log\omega_t^1, \log 1/\lambda_{t-1})$ is the vector of the state and co-state variables and a is a vector of parameters that we compute with the iterative algorithm described below[7].

To approximate the expected utility in the T-periods of autarchy we use a second degree polynomial on ω_t^i,

$$E_t \sum_{j=1}^{T-1} \delta^{j-1} u\left(\omega_{t+j}^i\right) = \varphi\left(b; \omega_t^i\right). \tag{26}$$

In order to compute b in (26), we run one non-linear least square regression of $Aut_{i,t}^T$ on ω_t^i, where

$$Aut_{i,t}^T = \sum_{j=1}^{T-1} \delta^{j-1} u\left(\omega_{t+j}^i\right) \quad \text{for } t = 0, ...N.$$

and we define with the vector $\widehat{b}$ the result of this regression[8].

In order to approximate $F^{i,D}(\omega_{t+T})$ we use the same function as in equation (25) with $S_{1,t}^D = (1, \log\omega_t^1, \log\omega_t^2, \log\frac{1-\alpha}{\alpha})$ and $S_{2,t}^D = (1, \log\omega_t^2, \log\omega_t^1, \log\frac{\alpha}{1-\alpha})$ instead of $S_{i,t}$:

$$F^{i,D}(\omega_{t+T}) = \psi\left(a; S_{i,t+T}^D\right)$$

Finally to obtain $E_t F^{i,D}(\omega_{t+T})$ we use a second-degree polynomial on ω_t^i

$$E_t F^{i,D}(\omega_{t+T}) = E_t\left[\psi\left(a; S_{i,t+T}^D\right)\right] = \varphi\left(d; \omega_t^i\right) \tag{27}$$

where d is computed by the iterative algorithm.

The first three steps in the algorithm compute the endogenous variables of the model for a given set of parametrized expectations; steps 4 and 5 are used to compute the coefficients a and b. Thus, for given values of the coefficients we simulate the system and find the polynomials with the highest predictive power (see Marcet and Singleton, 1999). We look for second-degree polynomials that generate simulations such that these polynomials are, precisely, the ones with the highest predictive power. Formally the algorithm is expressed as follows:

Step 1: Because of the Kuhn-Tucker conditions, we have to consider several cases in order to calculate for the endogenous variables. Given a realization of the shocks $(\varepsilon_{1,t}, \varepsilon_{2,t})$ and the endowments $(\omega_{1,t}, \omega_{2,t})$, first of all we tentatively try the solution where the participation constraints are not binding. In this case, the normalized

[7] As the problem is symmetric for the two agents (in fact the variables in the function that approximate the left-hand side of the agent 1's participation constraint are symmetric to the ones for agent 2), we can use the same parameters a for both participation constraints.

[8] Notice that we can run the regression before starting the iterative procedure, since the expected utility in the T-periods of autarchy depends only on exogenous variables.

lagrange multipliers are equal to zero and $\lambda_t = \lambda_{t-1}$. Consumption is determined from the feasibility constraint (2) and the optimality condition (24).

Step 2: Check wether this allocation satisfies the participation constraints[9]. Given the parameters $\widehat{a}^n$ and $\widehat{d}^n$, we define the variable

$$pc_{i,t} = u\left(c_t^i\right) - u\left(\omega_t^i\right) + \delta\left[\psi\left(\widehat{a}^n; S_t^i\right) + \varphi\left(\widehat{b}; \omega_t^i\right) + \delta^{T-1}\varphi\left(\widehat{d}^n; \omega_t^i\right)\right], \quad (28)$$

the following cases can occur[10]:

- $pc_{1,t} \geq 0$ and $pc_{2,t} \geq 0$
- $pc_{1,t} < 0$ and $pc_{2,t} \geq 0$
- $pc_{2,t} < 0$ and $pc_{1,t} \geq 0$

In the first case go to step 3; otherwise consumption of agent i for whom the participation constraint is binding is computed by imposing $pc_{i,t} = 0$ in (28). The lagrange multiplier associated to the PC of the defaulting agent is obtained from (24), while λ_t and consumption of the other agent is obtained from the definition (23) and from the feasibility constraint (2).

Step 3: Repeat Steps 1-2 for $t = 1, ...N$.

Step 4: Now we are ready to compute the new value for the parameters a.

Compute the discounted sum of future utilities

$$\mathcal{V}_t^i = \sum_{j=0}^{K} \delta^j u\left(c_{t+j}^i\right) \quad \text{for } t = 0, ...N \text{ and } K \text{ big enough}$$

In particular, compute first $\mathcal{V}_N^i = \psi\left(a; S_N^i\right)$, then obtain backward the remaining values. In order to find $\widehat{a}^{n+1}$, we run a non-linear least square regression of $\mathcal{V}_t^i$ on $S_{i,t}$

$$\mathcal{V}_t^i = \psi\left(\widehat{a}^{n+1}; S_{i,t}\right) + \varepsilon_{\mathcal{V}_i,t}$$

where $\varepsilon_{\mathcal{V}_i,t} \sim N\left(0, \sigma^2\right)$ and i.i.d.

Step 5: In order to compute $\widehat{d}^{n+1}$, generate the time series $\left\{\psi\left(\widehat{a}^{n+1}; S_{i,t+T}^D\right)\right\}_{t=1}^N$ for $i = 1, 2$ and run a non-linear regression

$$\psi\left(\widehat{a}^{n+1}; S_{i,t+T}^D\right) = \varphi\left(d^{n+1}; \omega_t^i\right) + \varepsilon_{\psi,t}.$$

where $\varepsilon_{\psi,t} \sim N\left(0, \sigma^2\right)$ and i.i.d.

Repeat **Steps 1-5** in order to find a new value of the parameters a and d. The iteration ends when $\widehat{a}^{n+1} \approx \widehat{a}^n$ and $\widehat{d}^{n+1} \approx \widehat{d}^n$, that is when ψ and φ converge to the correct approximating polynomials.

[9] We have to go down through all possible cases, until we find one where all the conditions are satisfied. By proceeding in this manner, all the inequalities implied by the Kuhn-Tucker conditions are satisfied automatically (see Marcet and Singleton, 1999).

[10] Along the iteration to find the equilibrium, also the case where $pc_{1,t} \leq 0$ and $pc_{2,t} \leq 0$ may occur. In section 3.2 we show how to modify the computational algorithm to take care of this event.

2.3.2 Feasibility along the Iteration

With the treat of permanent exclusion, Kocherlakota (1996) shows that, under fairly general assumptions, for any state there is a feasible continuation of consumption plans that makes at least one of the agents better off than under autarky. However during the iteration process to find the optimum, any algorithm would replace the left hand side in the participation constraints by a function W_i that approximates the discounted sum of future consumption in the contract. In particular, if we use PEA as in Marcet and Marimon (1992) then we approximate the participation constraints as follows

$$u(c_{i,t}) + \delta\psi\left(a^n; S_{i,t}\right) \geq v_i^a(\omega_t) \qquad \text{for all } i, t$$

As during the iteration process $\psi\left(a^n; S_{i,t}\right)$ is not exactly equal to the discounted sum, Kocherlakota's result may not apply. It can indeed happen that there are no values of (c_1, c_2) that satisfy the equation. In this case the algorithm breaks down because it is asked to compute feasible consumption when a feasible consumption does not exist for the approximate $\psi(.)$. This is common to any algorithm solving this problem: in PEA the simulation step can not be performed in some periods, and in MWR algorithms the Euler equation can not be evaluated at some point in the state space so that the residual can not be computed at that state point. This problem appears to be even more of serious in the temporary exclusion model, as the function $\psi(.)$ enters also in the right hand side.

We propose the following solution. Introduce a third agent that can give units of consumption to the economy whenever there are no feasible consumption paths that satisfy the participation constraint. Such agent will be asked to provide additional consumption only when it is absolutely necessary; the aim is to gradually make this contribution less generous. Formally, we change the technology constraints to be as follows,

$$\begin{aligned} \sum_{j=1}^{2} c_{j,t} &\leq \sum_{j=1}^{2} \omega_{j,t} + \theta_t \\ \theta_t &\geq 0 \end{aligned} \tag{29}$$

We could think about many different alternatives of changing the objective function of the planner, here we just consider the simple case where we solve the following planner's problem.

Modified-Planner's Problem

$$\max_{\{c_t\}} \; \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\left(\sum_{j=1}^{2} \alpha_j \, u(c_{j,t}) \right) - \rho\theta_t^2 \right] \tag{30}$$

for a "large" $\rho > 0$. Since the set of participation constraints does not change, the modified problem will merely[11] consists of maximizing (30) subject to the original participation constraints and (29).

The contribution of the third agent plays a similar role as the penalty function in the computation of static constrained maximization problems. In this way we alter the objective function so that it is painful to make active use of the third agent. We label this agent as the "penalty agent". He will provide additional consumption to insure feasibility, but at the same time reducing utility of the planner. As in models with penalty functions, using a high enough ρ, would insure that the fixed point is a solution to the original problem.

Notice that for the modified problem there is always a feasible solution. The algorithms will not break down for the modified problem in a period or realization with unfeasible consumption. Setting a positive θ whenever it is necessary the algorithm would continue working. With PEA the simulation step can be performed for as many periods as we wish, and if some MWR algorithm is used the Euler equation can be evaluated at any point in the state space. For large ρ's the policy function will be selected setting $\theta > 0$ not very often, because of the term $-\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \rho \theta_t^2$ in the planner's objective function.

The new procedure to find the optimal allocation described in the previous section should then be slightly modified. The Kuhn-Tucker condition of the modified problem is now given by the following expression,

$$u'(c_{i,t}) = \rho \theta_t \quad \text{if } \theta_t > 0 \tag{31}$$

Then if there is a feasible consumption for the original constraints, we proceed as before. If there are no feasible consumptions, we add a new step in the algorithm and compute endogenous variables using the Kuhn-Tucker condition of the modified problem.

2.4 Quantitative Results

We now describe some quantitative results obtained for different lengths of the exclusion period, T, and compare them with those obtained under the threat of permanent exclusion.

In the benchmark model, we set the initial weight $\alpha = 0.5$ and the relative risk aversion parameter equal to 1 (logarithmic utility). We choose the discount factor to match the average real risk-less interest rate of 2 per cent per year in a model where

[11] Since it is concave, the Modified-planner's problem is well defined.

reversion to autarchy lasts for 10 years[12] and we assume perfect negative correlation between the endowments of the two agents[13]. In order to simulate our model we also need to assume values for the parameters ρ_ω, $\overline{\omega}$ and Ω. We choose these to match the tightness of the participation constraints found in Alvarez and Jermann (2000) in the model with permanent exclusion[14].

Statistics for the benchmark model

δ	σ_{ε_i}	$cov(\varepsilon_1, \varepsilon_2)$	$\overline{\omega}$	ρ_ω	σ_{ω_i}	$cov(\omega_i, \omega_i)$	$corr(\omega_i, \omega_i)$
0.965	0,07	−0.004	2	0.88	0,27	−0.07	−1

2.4.1 Temptation to Default

We analyze quantitatively how, in presence of limited enforcement, the incentive to default modifies as the length of the punishment period shortens.

Table 1 shows the average "temptation to default", measured by the expected value of the normalized lagrange multiplier $\widetilde{\gamma}_t^i$.

Tab. 1 Average Temptation to Default and Outside Option

T	∞	100	48	40	36	28	24
$E\left[\widetilde{\gamma}_t^1 \mid \lambda_{t-1}= 1\right]$	0.11	0.11	0.12	0.13	0.15	0.20	0.20
$E\left[\widetilde{\gamma}_t^1 \mid \lambda_{t-1}= 0.66\right]$	0.01	0.01	0.01	0.03	0.04	0.10	0.10
$E\left[\widetilde{\gamma}_t^1 \mid \lambda_{t-1}= 1.5\right]$	0.68	0.66	0.61	0.62	0.66	0.75	0.75
$E\left[V_{i,t}^{Aut,T}\right]$	20.01	20.01	20.03	20.03	20.03	20.04	20.04

When both agents were equally lucky in the past ($\lambda_{t-1} = 1$; row 1 in the table), the temptation to default increases as the punishment period shortens. To give an intuition let's consider first the threat of permanent exclusion. In this case the incentive to default is small, as the outside option (the last row in the table) is the worst possible. Thus the compensation (see equation (24)) for not defaulting will be low. As the length

[12] If we calibrate our model to match the average real riskless interest rate of 2 per cent per year in a model with permanent exclusion (as in Alvarez and Jermann, 2000), it turns out that the average interest rate in the case of an exclusion that lasts 10 years would be negative. For this reason, one possible use of our model is in the evaluation of the effectiveness of calibrating the limited enforcement model on the permanent exclusion case.

[13] The fact that endowments are perfectly negatively correlated allows us to reduce the set of variables in the approximated function ψ on which we run regressions, $S_{1,t} = (1, \log\omega_t^1, \log\lambda_{t-1})$ and $S_{2,t} = (1, \log\omega_t^2, \log 1/\lambda_{t-1})$.

[14] Alvarez and Jermann (2000) found in a model with permanent esclusion that the participation constraints of an agents is binding 25 per cent of the time. Therefore we calibrate ρ_ω, $\overline{\omega}$ and Ω to have a similar tightness in a model where exclusion lasts 10 years.

of the punishment shortens, the expected utility of the outside option rises increasing the incentive to default.

When agent 1 has already a long history of good shocks (the normalized weight, λ_t, is lower than 1; second row in Tab 1) his temptation to leave is very low and remains almost unchanged for punishment periods bigger than $T = 40$. In this case, in fact, his share of aggregate consumption in the agreement is high (in equation (24), a low λ_{t-1} implies a high consumption) and an increase in the length of the punishment is not sufficient to reduce further the temptation to default. However, for $T < 40$, even if agent 1's consumption in the agreement is already high, the temptation to leave quickly increases, if we reduce T, as he knows that after a short period in autarchy he would enter a new contract.

Finally, when agent 1 has a long history of bad shocks (the co-state variable is bigger than 1; third row in Tab 1), his share of aggregate consumption in the agreement is relatively low and his temptation to default when he face a good shock is very high for all T. As T increases, for $T > 40$, the temptation slightly increases, while for low levels of T, the incentive to default increases as T decreases. Again, the fact that after a short period in autarchy the defaulting agent would face a new contract (with a new initial modified relative weight equal to 1) makes it more appealing to default.

2.4.2 Tightness of Participation Constraints

We analyze now how the inverse relationship between the incentive to default and the length of exclusion translates in terms of tightness of the participation constraints. In Figure 1 we report the expected combinations of $pc_{1,t}$ and $pc_{2,t}$ (see equation (28)) for different relative weights, $E\left[pc_t^i \mid \lambda_{t-1} = \overline{\lambda}\right]$. As T decreases, the region where $pc_{1,t}$ and $pc_{2,t}$ are both non-negative shrinks and the probability that the participation constraint of one agent is binding increases. The reason is that as the punishment period shortens, the incentive to default increases (see also table 1), since the defaulting agent knows that in T periods he will rejoin the agreement.

Fig. 1 Participation Constraints

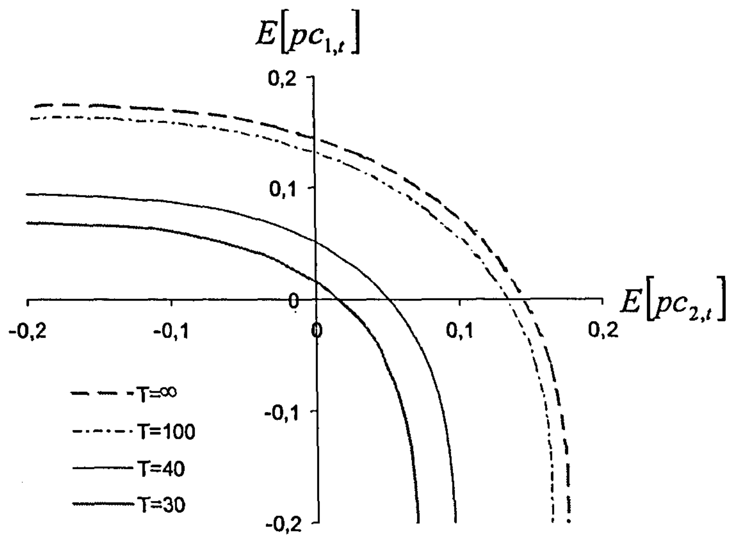


Table 2 reports the percentage of time that the participation constraint is binding for one agent. Clearly, the longer the period in autarchy the lower the percentage of periods when the participation constraint of is binding.

Tab. 2 Tightness of Participation Constraints

T	∞	200	100	60	44	40	36	20
PC agent i binding (%)	16.5	16.5	17.5	20.5	24.0	25.0	27.0	31.0

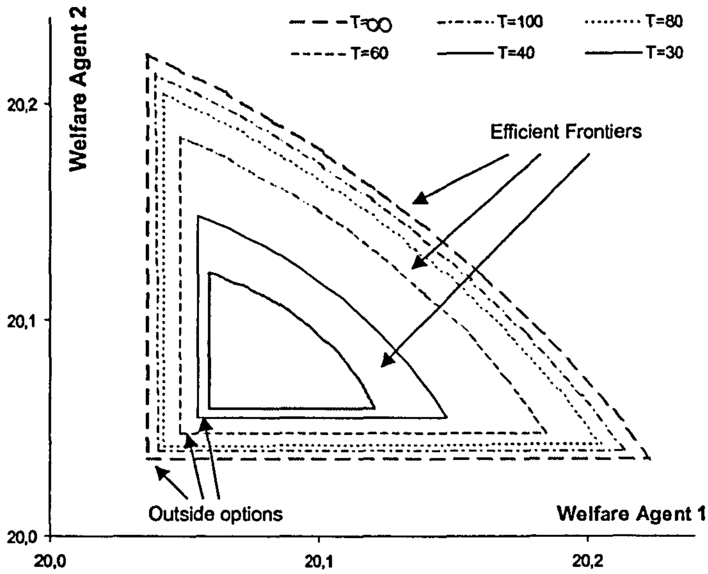
In the economy with a temporary exclusion of 40 years ($T = 100$), the probability that one agents is tempted to default is (almost) the same that we would have under permanent exclusion. Reducing the length of the punishment below 40 years, sensibly increases the percentage of periods when the participation constraints are binding.

2.4.3 Welfare analysis

We want now see whether the increase in the tightening of the participation constraints as the length of the punishment period shortens, brings about a change in total welfare.

Figure 2 draws the efficient constrained frontier for different T's: each frontier shows the combinations of the expected continuation values, as described in equation (11), for different relative weights; the left-corners, on the contrary, shows the expected utility of the outside option (see also the fourth row in table 1).

Fig. 2 The efficient frontier



As the length of the punishment period decreases, the outside option increases, and the frontier of efficient allocations shrinks. The decrease in total welfare is the cost to be paid in order to convince agents to commit to the risk-sharing agreement.

In order to quantify these costs, Table 3 reports welfare reduction with respect to an economy with full enforcement. We measure welfare loss in terms of consumption equivalence. This measure, denoted by ce^T, is computed as the fraction of consumption that an economy with full enforcement should give up, in order to have the same welfare as the economy with limited enforcement and temporary exclusion (of length-T). Formally the "consumption equivalent" is implicitly defined by:

$$E_t \sum_{j=0}^{\infty} \delta^j \sum_{i=1}^{2} \left[\alpha_i u \left(c_{t+j}^i \right) \right] = E_t \sum_{j=0}^{\infty} \delta^j \sum_{i=1}^{2} \left[\alpha_i u \left(c_{t+j}^i \left(1 - ce^T \right) \right) \right] \tag{32}$$

Tab. 3 Welfare Analysis

T	∞	100	80	60	40	36	24
Consumption equivalence	0.01	0.01	0.01	0.011	0.011	0.015	0.015

With permanent exclusion the decrease in welfare with respect to the full enforcement economy corresponds to a permanent reduction of quarterly consumption of around 1.0 per cent (4.0 per cent, annualized), while under the threat of a temporary exclusion of 10 years ($T = 40$), quarterly consumption fall by 1.1 per cent (4.4 per cent, annualized). An increase in the length of the punishment period from 10 years to ∞ would determine a very low increase in terms of welfare. This result, together with the one obtained in the previous section, suggests that tighter participation constraints under temporary punishment do not convey a comparable loss in terms of welfare. Notice, however, that in this model, assuming a long exclusion is not costly, as in equilibrium default never occurs, while in real word situations default takes place and lasting exclusions from an agreement could reveal very costly. Thus, in front of possibly high costs for increasing the length of the punishment period above 10 years, table 3 tells us that the benefits would be very low.

2.4.4 Consumption and the Business Cycle

In the previous section we have seen how the length of the punishment period affects welfare, here we study the quantitative properties of the consumption allocation for different T's.

Under full enforcement agents can full insure against all idiosyncratic risks. In this case the consumption ratio, $c_{1,t}/c_{2,t}$, would be constant. Under limited enforcement an agent is induced not to default by increasing his consumption not only in the period when he is tempted to default, but also in the future. The fact that as T decreases the temptation to default and the tightness of the participation constraints increase implies that also the variability of the consumption ratio increases (Figure 3).

Fig. 3 Consumptions

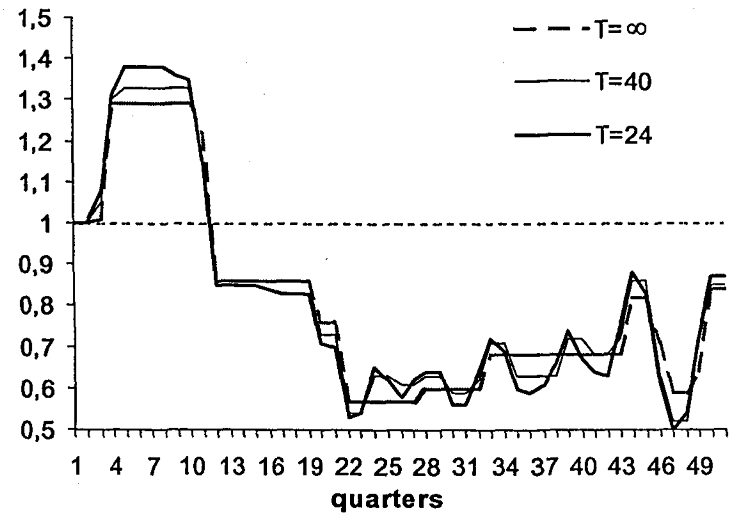


Table 4 reports the ratio between the standard deviation of consumption of one agent and the standard deviation of his endowment, as a measure of risk-sharing. In presence of no risk sharing the ratio would be equal to 1.

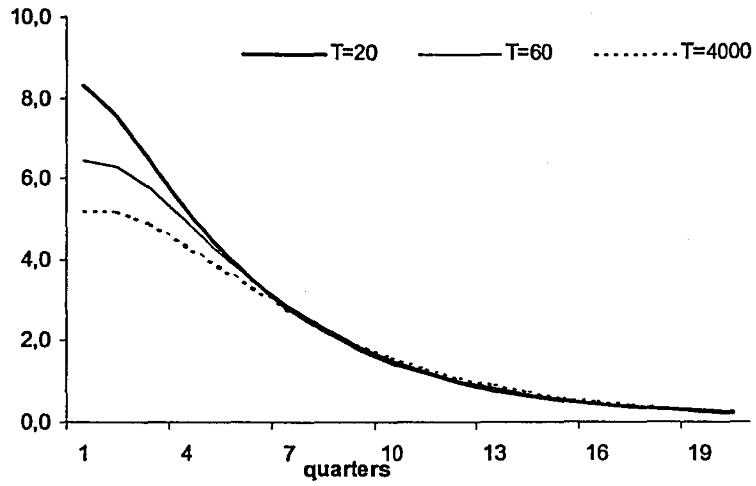
Tab. 4 Risk Sharing

T	∞	200	100	80	60	40	36	20
$\text{Std}(c_i)/\text{Std}(\omega_i)$	0.84	0.84	0.85	0.86	0.87	0.89	0.9	0.91

Risk sharing increases with the length of the exclusion period since, as T increases, the incentive to default and the tightness of the participation constraints decrease. Notice that risk-sharing is already very low under the threat of permanent exclusion and for $T \geqslant 100$ this would not change with T. Reducing below that level the length of the punishment would slightly reduce risk-sharing.

Finally, in figure 4 we plot the average effects of a 10% shock to endowments of agent 1 on consumption of agent 1. In the long run the impulse responses look the same under different Ts, but in the first 2 years the impact on consumption is higher, the smaller the T.

Fig. 4 Impulse response of agent i consumption to a 10 % increase in agent i endowment



A way to interpret this result is the following: as T decreases, the temptation to default when the agent faces a good shock increases. Therefore the compensation that the planner has to give him, at t, in order to stay in the agreement increases. Since also the continuation value rises, the higher increase in consumption persists for some periods.

2.4.5 Decentralizing the Efficient Allocations: Interest rate and Asset prices

In a limited commitment environment with permanent reversion to autarchy, Alvarez and Jermann (2000) show how to decentralize the planner solution into an equilibrium with endogenous solvency constraints. The main intuition behind this decentralization is that an agent would like to sell state-contingent claims on future consumption in those states in which he will be well endowed. But in those high endowment states he will also have an incentive to default; limiting the volume of debt that he is able to carry into those high endowment states will restrain him from doing so. As a consequence, his consumption and continuation value increase when he enters one high endowment state precisely because he has been prevented from selling enough claims to smooth his consumption.

With the definition of competitive equilibrium with solvency constraints we have a simple and intuitive representation of the prices of securities. In this context, one-period contingent claims (Arrow securities) are priced by the agent with the highest marginal rate of substitution, which is the agent that is not constrained with respect to his holding asset. The equilibrium Arrow price corresponds to the highest marginal rate of substitution of the two agents, so that for the one period (gross) return of a risk free asset R_t^f , the following must hold (see Alvarez and Jermann, 2000):

$$\left[R_t^f\right]^{-1} = E_t\left[\max_{j=1,2}\left(\frac{\beta u'\left(c_{j,t+1}\right)}{u'\left(c_{j,t}\right)}\right)\right]. \tag{31}$$

The idea is that the agent with the highest valuation of an asset is going to buy it. Buyers of state contingent securities are unconstrained, so the price of the asset is equal to their marginal rate of substitution.

The constraint of the agent with a low endowment will not bind. Unconstrained agents expect to have a declining consumption allocation, since the continuation values of the constrained agents have to increase. Low interest rates can reconcile buyers to accept a declining continuation value. A central result in Alvarez and Jermann (2000) is that interest rates are lower in economies with solvency constraints than in corresponding economies without such constraints.

Here we study how the interest rate behaves under different lengths of the punishment period. For lower T's, the economy displays less risk sharing, meaning that individual consumptions become more dissimilar and thus the maximum of the marginal rate of substitutions increases, leading to an increase in the pricing kernel. The first row in Tab. 7 compares the behavior of risk free annual (gross) interest rate for different Ts.

Tab. 7 Risk-free (Gross) interest rate and risk-premium

T	PO	∞	100	60	44	40	36	20
$E\left[R_t^f\right]$	1.14	1.05	1.047	1.035	1.024	1.02	1.012	1.00
$Std\left[R_t^f\right]$	0	0.01	0.01	0.01	0.02	0.03	0.04	0.04
$E\left[R_t^a - R_t^f\right]$	0.01	0.1	0.1	0.11	0.12	0.12	0.13	0.14

In economies with tighter participation constraints and less risk sharing, the interest rate is lower. In these economies, unconstrained agents' consumption will decline more than in economies with more risk sharing, as the consumption of the constrained agents increases more (Figure 4). Lower interest rates will make them accept the stronger decline in consumption. In particular, going from an economy with permanent exclusion to one with a 10 years exclusion, the risk free interest rate reduces by 3 percentage points. Notice that in an economy with a 25 years punishment ($T = 100$) the equilibrium interest rate is very close to the one under permanent exclusion.

In Table 7 we also consider the risk-premium for one-period assets that pay a random dividend $d_{t+1} = \omega_{1t+1}$. The (gross) return on these assets at time t is denoted by R_t^a. As the number of periods that the agent has to stay in autarchy decreases, the risk premium $E\left[R_t^a - R_t^f\right]$ increases, as a consequence of the increase in individual consumption volatility. Increasing the number of punishment periods from 10 to 25 years, reduces the risk premium of about 2 percentage points.

2.5 Conclusions

The objective of this chapter of the thesis was to derive the optimal solution and to document quantitative implications in a model of limited enforcement under the threat of temporary exclusion. First of all we have shown how to formulate the model recursively and we have developed an algorithm that implements the recursive solution involving computational costs similar to those faced under the threat of permanent exclusion. Maintaining the assumption of permanent exclusion cannot be justified by just appealing to numerical simplicity.

We have shown how to use our framework in a two-agent endowment economy to determine the effects of different lengths of exclusion. We found that for shorter periods of exclusion participation constraints are binding more often and consumption is more volatile. We have obtained that a punishment corresponding to staying 10 years (as in Chapter 7 legislation) in autarchy implies an incentive to default and a frequency of binding constraints considerably higher than under permanent exclusion and an amount of risk-sharing sensibly lower.

We have also analyzed the welfare implications of the reduction in risk-sharing. We found that a period of 10 years is not very costly if compared to permanent exclusion: the permanent decrease in consumption, with respect to the perfect risk-sharing economy, is around half percentage point higher in an economy with a 10 year punishment than in one with permanent exclusion. Finally, we have decentralized the allocation into an equilibrium with endogenous solvency constraints a la Alvarez & Jermann, in order to obtain predictions on the risk-free interest rate and the risk premium. The risk-free interest rate was 3 percentage point lower in an economy with an exclusion of 10 years than in that with permanent exclusion, while the risk-premium was 2 percentage points lower in this last case. Finally, we have found that in an economy with a 25 years punishment the equilibrium interest rate and the risk-premium were very close to those obtained under the threat of permanent exclusion.

Since in most real life situations the threat of permanent exclusion is hardly credible, we think that our approach improves on the standard limited enforcement economies. Further analysis is needed to determine whether temporary punishment can help or hurt in explaining empirical evidences.

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