DEPARTAMENTO DE FÍSICA TEÓRICA

SYMMETRY AND HOLONOMY IN M THEORY

ÓSCAR VARELA RIZO

UNIVERSITAT DE VALENCIA
Servei de Publicacions
2007

Aquesta Tesi Doctoral va ser presentada a Valencia el dia 7 de Juliol de 2006 davant un tribunal format per:

- D. Igor Bandos
- D. Eric Bergshoeff
- D. Jerome Gauntlett
- D. Dmitri Sorokin
- D. José Manuel Izquierdo Rodríguez

Va ser dirigida per:
D. José A. de Azcárraga Feliu

Depòsit legal:
I.S.B.N.:978-84-370-6793-3
Edita: Universitat de València
Servei de Publicacions
C/ Artes Gráficas, 13 bajo
46010 València
Spain
Telèfon: 963864115

SYMMETRY AND HOLONOMY IN M THEORY

OSCAR VARELA RIZO

Departament de Física Teòrica and IFIC
Universitat de València
46100 Burjassot (València), Spain

oscar.varela@ific.uv.es

TESIS DOCTORAL

Abril 2006

VNIVERSITAT DE VALÈNCIA
Departament de Física Teòrica
Institut de Física Corpuscular

D. José A. de Azcárraga Feliu, Catedrático de Física Teórica de la Universidad de Valencia,

certifica:

que la presente memoria, *Symmetry and holonomy in M Theory* ha sido realizada bajo su dirección en el Departamento de Física Teórica de la Universidad de Valencia por Óscar Varela Rizo y constituye su Tesis Doctoral.

Valencia, abril de 2006

José A. de Azcárraga Feliu

A Majo

Les principes de la Mecanique sont déjà si solidement établis,
qu'on auroit grand tort, si l'on vouloit encore douter de leur verité. [1] [2]

L. Euler, *Reflexions sur l'espace et le tems* (1748)

The necessity to depart from classical ideas when one wishes to account for the ultimate structure of matter may be seen, not only from experimentally established facts, but also from general philosophical grounds. [3]

P.A.M. Dirac, *The principles of quantum mechanics* (1930)

[1]The principles of Mechanics have already been so solidly established that it would be a great mistake to still question their truth.

[2]Los principios de la Mecánica han sido ya establecidos tan sólidamente que sería un gran error pretender aún dudar de su verdad.

[3]La necesidad de abandonar las ideas clásicas al tratar de dar cuenta de la estructura última de la materia puede verse no sólo a partir de hechos establecidos experimentalmente, sino también por motivos filosóficos generales.

Contents

Prefacio **ix**

Preface **xi**

Introducción **xiii**

La ruta hacia la Teoría M xiii

Contenidos de esta Tesis xxii

1 Introduction **1**

1.1 The road to M Theory 1

1.2 The contents of this Thesis 10

2 Eleven-dimensional supergravity **15**

2.1 The M Theory superalgebra 15

2.2 First order action of $D = 11$ supergravity 20

2.3 Supersymmetry and generalized holonomy 23

2.4 Equations of motion 26

2.5 The purely bosonic limit 28

2.6 Equations of motion and generalized curvature 30

3 Subtleties about generalized holonomy **35**

3.1 Killing spinors, holonomy and supersymmetry 36

3.2 Higher order integrability 40

3.3 Generalized holonomy of the M-branes 42

3.3.1 Generalized holonomy of the M5-brane 42

3.3.2 Generalized holonomy of the M2-brane 45

3.4 Higher order integrability and the squashed S^7 47

4 Generalized holonomy for BPS preons **53**
4.1 BPS preons 53
4.2 Moving G-frame 57
4.3 Generalized holonomy of preonic solutions 59
4.4 BPS preons in supergravity 62
4.4.1 BPS preons in Chern-Simons supergravity 62
4.4.2 Searching for preonic solutions of the free bosonic CJS equations 63
4.5 On possible preonic branes 65
4.5.1 Brane solutions and worldvolume actions 65
4.5.2 BPS preons in D'Auria-Fré supergravity 66

5 Interlude: Lie algebra expansions **69**
5.1 Three well-known ways to relate Lie (super)algebras . . . 70
5.2 The expansion method 72
5.2.1 The Lie algebras $\mathcal{G}(N)$ expanded from $\mathcal{G}$ 73
5.2.2 Structure of the expanded algebras $\mathcal{G}(N)$ 76
5.2.3 Limiting cases 77
5.3 The case in which $\mathcal{G}$ contains a subalgebra 77
5.3.1 The case in which $\mathcal{G}$ contains a symmetric coset . . 81
5.4 Rescaling with several different powers 84
5.5 Superalgebra expansions 89
5.6 The M Theory superalgebra as an expansion of $osp(1|32)$ 91

6 The underlying symmetry of $D = 11$ supergravity **95**
6.1 Free differential algebras, Lie algebras and cohomology . . 96
6.2 The $D = 11$ supergravity FDA 99
6.3 A family of extended superalgebras 102
6.4 The relation of $\tilde{\mathfrak{E}}(s)$ with $osp(1|32)$ 105
6.5 The composite nature of A_3 107
6.6 Dynamics with a composite A_3 111
6.7 Fields/extended superspace coordinates correspondence . 116

7 A 30/32-supersymmetric string in tensorial superspace **121**
7.1 Models in enlarged superspaces 122
7.2 A supersymmetric string in tensorial superspace 125
7.3 Equations of motion 128
7.4 Gauge symmetries 131
7.5 Hamiltonian analysis 132
7.6 Supertwistor form of the action 143
7.7 Hamiltonian analysis in the supertwistor formulation . . . 145
7.8 Hamiltonian analysis with 'unnormalized' supertwistors . 153
7.9 Supersymmetric p–branes in tensorial superspace 155

8 Conclusions and outlook **159**

A Second order integrability for the squashed S^7 **167**

B Expansion of $d\omega^{k_s,\alpha_s}$ **171**

C Symmetry breaking of the supertwistor string **175**

Bibliography **179**

Prefacio

A pesar de carecer actualmente de formulación dinámica, es posible obtener gran cantidad de información sobre la Teoría M, la teoría que se postula como unificadora de todas las interacciones, a partir de sus sectores perturbativo y de baja energía. En las regiones perturbativas adecuadas, la Teoría M adopta la apariencia de la Teoría de Cuerdas. Al considerar su límite de baja energía, surge la supergravedad en once dimensiones. Precisamente, esta Tesis Doctoral, basada en las referencias [1]–[8], discute algunos aspectos de la Teoría M desde el punto de vista de la supergravedad once-dimensional. En el capítulo 1 se esboza una visión de conjunto de la Tesis, y se argumenta la pertinencia del análisis de supergravedad para el estudio de cuestiones relativas a la Teoría M. Es en el capítulo 2 donde realmente comienza la discusión.

Esta Tesis ha sido realizada en su mayor parte en el Departamento de Física Teórica y el Instituto de Física Corpuscular de la Universidad de Valencia, con ayuda de una beca predoctoral de la Generalitat Valenciana. Reciban mi agradecimiento todas estas instituciones. Quisiera mostrar mi gratitud hacia mi director de tesis, José A. de Azcárraga, por haber aprendido de él tantas cosas, y no sólo sobre Física. Particular mención ha de hacerse también de Igor Bandos, con quien he disfrutado tantas conversaciones y de quien he recibido tanta ayuda. Estoy sumamente agradecido a ambos por su estímulo y apoyo. Quisiera expresar mi reconocimiento a José M. Izquierdo por conversaciones y colaboraciones y a Dmitri Sorokin por sus comentarios. Los agradecimientos han de hacerse extensivos a Moisés Picón por conversaciones y colaboraciones, a Miguel Nebot por conversaciones y a Luis J. Boya por una interesante discusión cuando esta Tesis estaba ya siendo finalizada.

Parte del contenido de esta Tesis fue llevado a cabo fuera de Valencia. Quisiera agradecer a Michael Duff su amable hospitalidad durante mi visita al Michigan Center for Theoretical Physics y a James Liu, Alex Batrachenko y Steve Wen por discusiones y colaboraciones. Asimismo, estoy muy agradecido a Kellogg Stelle y Jerome Gauntlett por su amable hospitalidad durante mi visita a Imperial College, y a J. Gauntlett, Daniel Waldram y Eoin O'Colgain por tan fructíferas discusiones y colaboraciones. Pasé un tiempo entrañable tanto en Ann Arbor como en Londres, y se lo debo a todos ellos.

Por último, quisiera agradecer el apoyo de mi familia, en especial el de mis padres Santiago y Antonia, y el amor, apoyo y afecto que recibo cada día de Majo Rodríguez.

O.V.

Valencia, abril de 2006

Preface

Despite its current lack of a dynamical formulation, a great deal of information about M Theory, the conjectured theory unifying all interactions, can be retrieved from its perturbative and low energy corners. In the suitable perturbative regions, M Theory adopts the ten-dimensional guise of String Theory. When its low energy limit is considered, eleven-dimensional supergravity arises. As a matter of fact, this PhD Thesis, based on references [1]–[8], is devoted to the discussion of some topics about M Theory from the eleven-dimensional supergravity point of view. A general overview is sketched in chapter 1, where the relevance of supergravity in order to study M-theoretical issues is discussed, and the contents of the Thesis outlined. It is, however, chapter 2 that really starts the discussion.

This Thesis has been mostly made at the Departamento de Física Teórica and the Instituto de Física Corpuscular of the Universidad de Valencia, with the help of a Valencian Government PhD fellowship. All these institutions are gratefully acknowledged. I am indebted to my supervisor, José A. de Azcárraga, from whom so much I have learned throughout all these years, and not only about Physics. Particular mention should also be made of Igor Bandos, with whom I have also enjoyed many discussions and from whom I have received so much help. I am extremely grateful to both of them for their encouragement and support. Discussions and collaborations with José M. Izquierdo are also gratefully acknowledged, as well as comments by Dmitri Sorokin. Kind acknowledgements are extended to Moisés Picón for discussions and collaborations, to Miguel Nebot for conversations and to Luis J. Boya for a useful discussion in the last stages of the writing up of this Thesis.

Some of the work contained in this Thesis was carried out outside Valencia. I would like to thank Michael Duff for kind hospitality during my visit to the Michigan Center for Theoretical Physics and James Liu, Alex Batrachenko and Steve Wen for discussions and collaborations. Likewise, I am grateful to Kellogg Stelle and Jerome Gauntlett for their kind hospitality during my visit to Imperial College, and to J. Gauntlett, Daniel Waldram and Eoin O'Colgain for fruitful discussions and collaborations. I had a wonderful time both in Ann Arbor and in London, and I owe it to all of them.

Last, I should like to thank the support from my family, especially from my parents Santiago and Antonia, and the love, support and affection I receive every day from Majo Rodríguez.

O.V.

Valencia, April 2006

Introducción

Pese al extraordinario avance experimentado en las últimas décadas por nuestro conocimiento de los procesos y leyes fundamentales que rigen el mundo físico, todavía quedan considerables cuestiones por resolver. Dos grandes descubrimientos, a saber, la Relatividad General y la Mecánica Cuántica, perfilaron el desarrollo de la Física del siglo XX. La primera, que supuso la culminación de la Física Clásica, es una generalización de la Relatividad Especial, la teoría que revisó las nociones galileanas y newtonianas de espacio y tiempo para ponerlas en pie de igualdad en un continuo espaciotiempo. La Relatividad General proporciona una descripción geométrica de la gravedad, además del marco para la formulación de los modelos cosmológicos actuales. La Mecánica Cuántica, por su parte, se aplica a los fenómenos físicos que ocurren (mayoritariamente) a escalas subatómicas, y es decisiva para la descripción del resto de interacciones fundamentales. La sustitución del carácter galileano original de la Mecánica Cuántica para hacerla compatible con la Relatividad Especial originó el desarrollo de la Teoría Cuántica de Campos. Fue entonces posible esgrimir argumentos de índole causal para insistir en una aplicación local, en vez de global, de ciertas simetrías. Las teorías de gauge, o de Yang-Mills, resultantes describen todas las interacciones fundamentales (las fuerzas electromagnéticas, débiles y fuertes) excepto la gravedad y son, junto con cierto contenido material prescrito, las piezas clave del Modelo Estándar de la física de partículas.

La ruta hacia la Teoría M

La búsqueda de una descripción unificada de diferentes fenómenos ha sido, históricamente, uno de los criterios que han guiado el progreso de la Física. Desde este punto de vista, parece natural buscar una teoría que combine las cuatro interacciones fundamentales dentro del mismo marco descriptivo. Un argumento de mayor envergadura, que trasciende lo meramente estético, lo proporciona el hecho de que las constantes de acoplamiento de las interacciones fundamentales, incluso la de la gravedad, parecen converger a muy altas energías: a la *escala de gran unificación* de alrededor de 10^{16} GeV. Sin embargo, incluso a bajas energías, las tres interacciones del Modelo Estándar se pueden describir mediante el mismo lenguaje de teoría gauge de Yang-Mills. La razón por la que la gravedad no encaja en este esquema es más grave de lo que podría pensarse a primera vista: la Relatividad General es una teoría clásica, de la que resulta imposible extraer resultados consistentes al aplicar los métodos habituales para dar cuenta de efectos cuánticos; en otras palabras, la Relatividad General es una teoría no-renormalizable. Con todo, la escala de energía a la que los efectos cuánticos serían significativos en gravedad, la llamada escala de Plank, siendo de 10^{19} GeV resulta relativamente cercana a la escala de gran unificación. Ello podría interpretarse como un indicio de la existencia de una teoría unificadora de todas las interacciones.

Tradicionalmente, la investigación en gravitación y en física de altas energías ha seguido caminos distintos, aunque algunos descubrimientos se han aplicado fructíferamente en ambos campos. Ese ha sido el caso de la supersimetría [9, 10, 11] (véase [12, 13] para artículos de repaso y [14] para una colección de reprints), una simetría entre bosones y fermiones basada en el concepto de superálgebra de Lie, estructura que contiene generadores de carácter tanto bosónico como fermiónico y en la que, por tanto, figuran tanto conmutadores como anticonmutadores. Poco después de su descubrimiento, se hizo notar que las teorías que poseían supersimetría local contenían automáticamente a la gravedad. A grandes rasgos, el argumento es el siguiente: el anticonmutador de dos generadores de supersimetría es una traslación; la aplicación *local* de supersimetría produce entonces una traslación *local*, que puede identificarse con un difeomorfismo o transformación local de coordenadas; invariancia bajo supersimetría local implica, por tanto, invariancia bajo difeomorfismos y, en definitiva, gravedad. Estas teorías de supersimetría local recibieron, pues, el nombre de supergravedades (véanse [15, 16, 17, 18]). La primera, y más simple, teoría de supergravedad en ser construida, fue su versión en cuatro dimensiones ($D = 4$) con una sola supercarga ($N = 1$) y fue llamada, en consecuencia, supergravedad $D = 4$, $N = 1$, o sencilla [19, 20] (véase el artículo de repaso [15]).

En los años setenta, la supergravedad se percibía como una firme candidata a convertirse en la teoría cuántica de la gravedad, al esperarse de las contribuciones fermiónicas a las expansiones perturbativas gravitatorias que cancelaran las amplitudes de dispersión divergentes. La supergravedad sencilla no logró materializar completamente estas expectativas puesto que, si bien se demostró su finitud incluso a segundo orden en teoría de perturbaciones [21], sus acoplamientos a materia dejaban de serlo ya a primer orden [22]. Las supergravedades extendidas (caracterizadas por $N > 1$ supercargas) se desarrollaron entonces, siendo posteriormente explorada la promoción de sus $N(N-1)/2$ campos de gauge abelianos a campos de gauge de $SO(N)$, lo que porporcionaba las llamadas *gauged supergravities*. De entre todas las supergravedades extendidas, la supergravedad $D = 4$, $N = 8$ [23, 24], máximamente extendida, poseía atractivas características: la gravedad, los campos de gauge y la materia formaban parte del mismo multiplete de supergravedad y, dada la ausencia de multipletes de materia con $N = 8$, la teoría (que se esperaba renormalizable) podría conseguir una verdadera unificación de la materia y todas las interacciones.

Una de las actividades que originó la investigación en supergravedad fue la construcción de teorías supergravitatorias en dimensiones espaciotemporales diversas (véase la colección de reprints [25]). En diez dimensiones podían existir tres teorías de supergravedad, una de ellas con supersimetría $N = 1$ (supergravedad de Tipo I) y dos versiones con $N = 2$: las supergravedades de Tipo IIA (no quiral) y la de Tipo IIB (quiral) (véase [25] y las referencias allí contenidas). Por el contrario, solo una teoría de supergravedad podía existir en once dimensiones, la máxima dimensión posible siempre que se pretendiera excluir campos de espín superior [26]. La supergravedad once-dimensional fue entonces construida por Cremmer, Julia y Scherk (CJS) en [27]. Las supergravedades máximas (aquéllas con un contenido máximo en supersimetría) en dimensiones inferiores, como la de Tipo IIA en $D = 10$ o la $N = 8$ en $D = 4$ resultaron ser reducciones dimensionales (es decir, compactificaciones toroidales) de la supergravedad en once dimensiones[4].

Estos descubrimientos provocaron el resurgimiento y actualización de las viejas ideas de (Nordström y) Kaluza-Klein, al explorarse las compactificaciones en variedades no triviales (véase [29] para un repaso de estas cuestiones) en las que las características de las teorías efectivas en cuatro dimensiones quedaban dictadas por las propiedades de la variedad de compactificación. Por ejemplo, el grupo de gauge y la supersimetría preservada en cuatro dimensiones quedaban determinados por el grupo de isometría y el grupo de holonomía [30], respectivamente, de la variedad de compactificación. Once era no sólo la máxima dimensión permitida por supersimetría, sino también la mínima dimensión que, tras

---

[4]De hecho, una de las mayores motivaciones originales para construir la supergravedad en $D = 11$ [27] era la de evitar las complicaciones técnicas que surgían al aplicar el procedimiento de Noether habitual a la construcción del lagrangiano de supergravedad $D = 4$, $N = 8$ [28]; así, el lagrangiano $N = 8$ completo se obtuvo mediante la reducción dimensional de su correspondiente en $D = 11$.

compactificación de las siete dimensiones extra, podía acomodar el grupo de gauge $SU(3) \times SU(2) \times U(1)$ del Modelo Estándar [31]. Además, existían compactificaciones 'espontáneas' [32], que permitían la aparición de variedades compactas de siete dimensiones de una forma natural. Sin embargo, era imposible obtener familias de fermiones quirales [33] a partir de la compactificación de supergravedad[5] en $D = 11$ (que es no-quiral). Estos hechos, junto con su carácter no-renormalizable, hicieron decaer el interés en las teorías de supergravedad como teorías de gravedad cuántica.

Por otro lado, surgían nuevos resultados relativos a la formulación del resto de interacciones. Mientras que las fuerzas electrodébiles eran descritas con éxito por la teoría de Yang-Mills $SU(2) \times U(1)$ espontáneamente rota, las interacciones fuertes carecían de una interpretación tan clara. Curiosamente, se propuso una descripción mediante una teoría de cuerdas, puesto que los resultados de las amplitudes de Veneziano y de las pendientes de Regge sugerían un origen de los hadrones como vibraciones de una cuerda fundamental [35]. Esta composición suponía la atrevida propuesta de sustituir las partículas puntuales por objetos extensos unidimensionales. Sin embargo, la exitosa aplicación de la teoría de Yang-Mills a la descripción de las interacciones fuertes mediante la Cromodinámica Cuántica (QCD), hizo mermar las simpatías hacía la descripción cuerdística, al tiempo que puso la formulación de las interacciones fuertes y electrodébiles en un satisfactorio pie de igualdad.

La Teoría de Cuerdas (véase [36, 37]) se recuperó de este revés tras la comprensión de que el campo de espín dos de su espectro se podía interpretar como el gravitón [38] (el cuanto del campo gravitatorio), siempre y cuando la escala de las cuerdas se moviera desde la escala de las interacciones fuertes hasta la escala de la gravedad cuántica. Además, las cuerdas proporcionaban una teoría renormalizable de la gravedad cuántica, al interaccionar en una región extensa del espaciotiempo y no sólo en un punto. La auténtica eclosión de la Teoría de Cuerdas vendría a mediados de los ochenta, al tener lugar la *primera revolución de las supercuerdas*.

Se sabía que la teoría clásica de supercuerdas (que incorporaba supersimetría) estaba bien definida en 3, 4, 6 y 10 dimensiones espaciotemporales [39], casos en los que existía un término de Wess-Zumino (véase [40]) capaz de dotar de simetría κ a la acción[6] (véase [36]) y, por tanto, de hacer corresponder correctamente los grados de libertad bosónicos y fermiónicos. Cuánticamente, se demostró que la teoría solo era consistente en diez dimensiones [43], al estar libre de anomalías sólo en ese caso. Es

[5]Esta cuestión fue revisada posteriormente, con el descubrimiento de que los espacios de compactificación con singularidades admitían fermiones quirales [34].

[6]En el caso de la partícula, la existencia de una simetría de gauge fermiónica fue probada en [41], en el caso con masa, y en [42] en el caso sin masa.

más, la cancelación de anomalías dejaba cinco posibles teorías de cuerdas diferentes, a saber, Tipo IIA, Tipo IIB, Tipo I, heterótica SO(32) y heterótica $E_8 \times E_8$ (véase [36] y las referencias allí contenidas). Por su parte, supergravedad quedaba incorporada en la Teoría de Cuerdas: según se desprendía del análisis de los modos sin masa (que describían la dinámica de baja energía) del espectro de las diferentes teorías de cuerdas, estos correspondían a los campos de los distintos multipletes de supergravedad, quizá acoplados a multipletes de super Yang-Mills. Para ser más precisos, se encontró que los límites de baja energía de las teorías de cuerdas de Tipos IIA y IIB coincidían, respectivamente, con las supergravedades del mismo nombre; y los de la de Tipo I y heteróticas, con la supergravedad de Tipo I acoplada al multiplete vectorial con $N = 1$ en diez dimensiones y grupo de gauge $SO(32)$ (para las cuerdas de Tipo I y la correspondiente heterótica) o $E_8 \times E_8$ (para la otra cuerda heterótica).

Aunque la signatura lorentziana había de imponerse, la consistencia de las cinco teorías perturbativas de cuerdas proporcionó por vez primera un argumento para un valor determinado de la dimensión del espaciotiempo. Se comenzó a aplicar un programa de tipo Kaluza-Klein a las compactificaciones de cuerdas, mediante el que se buscaban modelos realistas en los que el espaciotiempo diez-dimensional quedaba escindido en el espaciotiempo cuatrodimensional ordinario y una variedad euclídea y compacta de seis dimensiones. Es más, una elección adecuada de la variedad de compactificación permitía la obtención de modelos realistas cercanos al Modelo Estándar en cuatro dimensiones. La cuerda heterótica [44] $E_8 \times E_8$ se veía como particularmente adecuada para la búsqueda del Modelo Estándar, puesto que sus compactificaciones eran capaces de dar lugar a simetría de gauge[7] E_6 (un candidato para el grupo de gauge en teorías de gran unificación). Asimismo, escogiendo una variedad de Calabi-Yau para la compactificación [46], se satisfacía el requisito fenomenológico de conseguir familias de fermiones quirales en $D = 4$. Estos desarrollos contribuyeron al entusiasmo generalizado que hizo de la Teoría de Cuerdas la más firme candidata a proporcionar una descripción unificada de las interacciones fundamentales puesto que, propuesta como una teoría cuántica de la gravedad, también parecía incluir el Modelo

[7]Nótese que las compactificaciones de la cuerda heterótica diez-dimensional en variedades seis-dimensionales no contradicen el hecho mencionado anteriormente de que sólo las compactificaciones de once dimensiones en variedades de dimensión siete admiten el grupo de gauge del Modelo Estándar en la teoría cuatro-dimensional resultante. La simetría de gauge obtenida en las compactificaciones heteróticas es consecuencia de la presencia de campos de gauge de $E_8 \times E_8$ ya en la teoría diez-dimensional, y no de las isometrías de la variedad de compactificación. De hecho, las variedades de Calabi-Yau carecen de isometrías. Véase [45] para una obtención del grupo de gauge del Modelo Estándar a partir de compactificación en un contexto IIA.

Estándar como consecuencia de sus propias condiciones de autoconsistencia.

Pese a todos estos avances, aún quedaban muchos asuntos pendientes. En primer lugar, las (cinco diferentes) teorías de supercuerdas estaban definidas únicamente de forma perturbativa. Por otro lado, la existencia de cinco teorías perturbativas no era demasiado atractiva si la Teoría de Cuerdas había de unificar todas las interacciones. De hecho, entre las prescripciones habituales para el cálculo de amplitudes de dispersión se encuentran tanto la suma sobre todas las topologías posibles que puede adoptar la hoja de universo de la cuerda (la expansión del geno) como, en presencia de *backgrounds* no triviales, la expansión en *loops* de la constante de acoplamiento α' de la cuerda, para cada una de esas topologías. Aunque este enfoque perturbativo se puede explotar intensamente para obtener gran cantidad de información sobre la teoría que describe, parece asimismo evidente la existencia de estructura no-perturbativa fuera del alcance de esta prescripción.

Efectivamente, eso es así. Existen estados que son naturalmente no-perturbativos y similares, por el contrario, a las soluciones solitónicas presentes en otras teorías de campos. Así, algunas soluciones solitónicas especiales, llamadas estados de Bogomol'nyi-Prasad-Sommerfield (BPS)[8] pueden considerarse, a pesar de ser soluciones clásicas, como soluciones de la teoría cuántica no-perturbativa. Los multipletes BPS surgen en las representaciones con masa de las álgebras de supersimetría con $N \geq 1$ que saturan la cota de Bogomol'nyi, que relaciona los autovalores del momento (la masa) y de las cargas centrales. La característica fundamental de estos multipletes es la de venir caracterizados por menor número de estados que un multiplete con masa ordinario. Aunque tanto la masa como la carga pueden verse sometidas a renormalización en teoría de perturbaciones, la condición de BPS queda protegida de correcciones cuánticas. Heurísticamente, si no fuera así, a medida que los efectos cuánticos cobran importancia, un multiplete BPS podría convertirse en otro no BPS, no siendo de esperar tan drástica aparición de estados. En este sentido, los estados BPS, aun siendo soluciones clásicas, son estables ante correcciones cuánticas y pueden, por tanto, ser ascendidas a soluciones de la teoría no perturbativa completa. Esta es la razón por la que son tan útiles para sondear la estructura no-perturbativa de la teoría.

Así pues, la noción de estados BPS fue muy importante a la hora de explorar el régimen no-perturbativo de las cinco teorías de cuerdas. Estas investigaciones derivaron, a mediados de los años noventa, en importantes descubrimientos relativos a las relaciones entre ellas. Algunas relaciones perturbativas entre las compactificaciones de las distintas teorías de cuer-

[8]La terminología viene de [47, 48].

das, conocidas como T-dualidades (véase [49]), eran ya conocidas en aquél entonces: se sabía que las teorías de Tipo IIA y de Tipo IIB eran T-duales (equivalentes) tras compactificación en una circunferencia de radio R y α'/R, respectivamente, y viceversa. También eran T-duales las dos versiones de la cuerda heterótica. Las S-dualidades [50], por el contrario, eran de distinta naturaleza y consistían en una generalización de la dualidad eléctrica-magnética [51] de la electrodinámica clásica. Así, las S-dualidades relacionaban, por el contrario, el régimen de acoplamiento fuerte de una teoría de cuerdas (en el que las habituales prescripciones perturbativas dejan de tener validez) con el régimen de acoplamiento débil (que admitía un tratamiento en teoría de perturbaciones) de otra de las teorías de cuerdas. Las teorías de Tipo I y heterótica $SO(32)$ resultaron ser S-duales, y la de Tipo IIB, S-autodual. Las S y T-dualidades fueron unificadas bajo el marco de las U-dualidades [52], y la red de dualidades resultante logró establecer un panorama unificado de todas las teorías de cuerdas, que fueron entonces reinterpretadas como expansiones perturbativas de la misma teoría no-perturbativa subyacente, en torno a cinco vacíos diferentes. La importancia de este hecho motivó el que su descubrimiento fuera señalado como el del inicio de la *segunda revolución de las supercuerdas* (véase [53]).

El descubrimiento de la red de dualidades trajo consigo dos hechos importantes. El primero fue la comprensión de la necesidad de incluir en la teoría p-branas [54] (véase también [55]), objetos supersimétricos extensos que recorren un volumen de universo con p dimensiones espaciales [56]. Estas branas son soluciones clásicas a las ecuaciones de supergravedad y, al poseer cargas topológicas [57] (véase también [58]), asociadas a las cargas centrales del álgebra de supersimetría, siguen siendo soluciones en el régimen cuántico. Esa es la razón por la que son tan valiosas para sondear los sectores no-perturbativos de la teoría. La existencia de estos objetos extensos en Teoría de Cuerdas implica una *democracia de p-branas* [59], según la cual estas han de tratarse en pie de igualdad con las mismas cuerdas. El segundo hecho fue el descubrimiento de que las teorías de cuerdas de Tipo IIA y heterótica $E_8 \times E_8$ eran duales a una teoría no-perturbativa en once dimensiones [60, 61], llamada posteriormente Teoría M (véase [53] para un repaso y [62] para una colección de reprints). Las constantes de acoplamiento de esas teorías de cuerdas se tomaban como funciones crecientes de un radio de compactificación de modo que, en el régimen de acoplamiento fuerte, surgía una undécima dimensión. Otros hechos venían a respaldar esta dualidad; como ya se ha mencionado, se sabía que la supergravedad de Tipo IIA correspondía a la reducción dimensional de supergravedad en once dimensiones, y que la cuerda IIA derivaba de la membrana once-dimensional [63]. Todas estas razones permitieron conjeturar la supergravedad once-dimensional como el límite de baja energía de la Teoría M.

Once es, de hecho, la máxima dimensión del espaciotiempo que la supersimetría espaciotemporal local permite (como ya se ha mencionado), y que puede contener objetos supersimétricos extensos (véase [64]). A su vez, el supermultiplete correspondiente en once dimensiones, que determina el contenido en campos de la supergravedad en $D = 11$, es único y mucho más sencillo que sus análogos diez-dimensionales. Sólo contiene tres campos: la métrica (o gravitón) $g_{\mu\nu}$, su compañero supersimétrico ψ^{α}, y una tres-forma A_3. Desde este punto de vista, once dimensiones resultan ser más naturales que diez, aunque los argumentos de autoconsistencia proporcionados por la Teoría de Cuerdas eran irrefutables. A partir de mediados de los noventa, no obstante, se redefinió el lugar que la Teoría de Cuerdas debía ocupar, desde una teoría de objetos extensos unidimensionales en vibración a una teoría de objetos extensos en general. Más que teorías fundamentales por sí mismas, las teorías de cuerdas surgían como cinco sectores perturbativos y diez-dimensionales de una nueva y verdaderamente fundamental teoría en once dimensiones, la Teoría M.

Esta percepción de los sectores no-perturbativos de la teoría acarreó nueva actividad. Por ejemplo, las D-branas [65] (véase [66]) desbancaron a las compactificaciones heteróticas en la búsqueda del Modelo Estándar (véase [67]). Por otro lado, tuvieron lugar otros descubrimientos relativos a los sectores no-perturbativos de la teoría. Ese fue el caso de la correspondencia AdS/CFT [68] (véase [69] para un repaso), formulada originalmente en un contexto de Tipo IIB. Según la correspondencia, la Teoría de Cuerdas en un *background* diez-dimensional que contiene el espacio cinco-dimensional de Anti-de Sitter, AdS_5, es dual a una teoría superconforme de campos (CFT) en el borde conforme de AdS_5, a saber, el espacio de Minkowski cuatro-dimensional M_4. La formulación original de la correspondencia relacionaba la teoría de cuerdas de Tipo IIB en el *background* máximamente supersimétrico $AdS_5 \times S^5$, donde S^5 es la cinco-esfera ordinaria, con la teoría de super Yang-Mills con $N = 4$. Otras generalizaciones fueron propuestas con posterioridad; en particular, si se reemplaza S^5 por una variedad de Sasaki-Einstein, se obtiene mediante la correspondencia una teoría de campos superconforme con $N = 1$ [70].

El conocimiento actual de la Teoría M comprende las dualidades no-perturbativas que muestra, y el hecho de que su límite de baja energía es la supergravedad en once dimensiones. Algunas propiedades, como la estabilidad de los estados BPS, permiten explorar la Teoría M completa a partir de soluciones de supergravedad: las conocidas branas de la supergravedad en once dimensiones pueden ser descritas tanto por sus acciones en el volumen de universo [71, 72], como por soluciones de supergravedad [73, 74] (véanse los artículos de repaso [64, 75]) que, no obstante, siguen siendo soluciones de la Teoría M no-perturbativa completa. La soluciones de branas preservan la mitad de la cantidad máxima de supersimetría (véase [57]), y sus intersecciones preservan fracciones menores [76]. Llegar a conocer más soluciones supersimétricas de supergravedad, que preservaran diferentes fracciones ν de supersimetría, podría proporcionar una visión más profunda de la teoría.

Aunque se conocen soluciones de supergravedad en $D = 11$ que contienen campos tanto bosónicos como fermiónicos (véase [77]), la búsqueda ha estado restringida generalmente, por simplicidad, a configuraciones puramente bosónicas. Siendo configuraciones bosónicas, los campos fermiónicos pueden ponerse a cero en las ecuaciones de movimiento, y el requisito de que la solución sea supersimétrica se alcanza siempre y cuando la transformación de los fermiones bajo supersimetría también se anule. Las soluciones puramente geométricas, en las que la métrica es el único campo no nulo, se pueden clasificar mediante la holonomía riemanniana. Las soluciones bosónicas de supergravedad más generales pueden describirse de forma sugerente con una extensión de la noción de holonomía riemanniana mediante *holonomía generalizada* [78, 73], *G-estructuras* [79, 80], o *geometría espinorial* [81]. Se discutirán cuestiones relacionadas en esta Tesis.

Otra información muy valiosa sobre la Teoría M que se puede obtener también a partir de supergravedad, proviene del estudio de su álgebra de simetría [82, 59]. Las soluciones de branas de supergravedad en $D = 11$ suelen verse como los objetos fundamentales de la Teoría M, del mismo modo que las cuerdas eran los objetos básicos de la Teoría de Cuerdas. En particular, las cargas topológicas [57] de las branas M5 [58] y M2 se pueden incluir de forma natural en el álgebra de supersimetría, que pasa de esa forma a llamarse álgebra de la Teoría M. La falta de un principio de acción para la Teoría M se puede suplir en cierta medida mediante métodos de teoría de grupos y, de esta forma, el estudio de las representaciones de la superálgebra de la Teoría M sugiere que los estados que preservan 31 supersimetrías podrían tratarse como fundamentales, mientras que el resto estaría compuesto de ellos. Estos estados, introducidos en [83] y llamados *preones*, podrían ser considerados como los constituyentes fundamentales de la Teoría M. Como se probará en esta Tesis, estas nociones conducen de manera natural a la consideración de superespacios agrandados [8] y supertwistors (véase el artículo de repaso [84]); véase [85] para las ideas tempranas sobre superbranas y superespacios agrandados y [86] para un repaso en este contexto.

El estudio de las simetrías de supergravedad es una herramienta muy útil para investigar la estructura de la Teoría M. La simetría local (bosónica) obvia de supergravedad en $D = 11$ es el grupo de Lorentz $SO(1, 10)$ y, por tanto, las derivadas covariantes Lorentz de los campos de supergravedad aparecen de forma natural en el lagrangiano. Sin embargo, otra *derivada supercovariante* que toma valores sobre el álgebra de Lie de $SL(32, \mathbb{R})$ [87] aparece en la expresión de la variación bajo supersimetría local de su *campo de gauge*, el gravitino ψ^α. Se ha especulado [87], en consecuencia, con que la supergravedad en $D = 11$ posee una simetría oculta $SL(32, \mathbb{R})$, que podría cobrar importancia en Teoría M[9].

Estos comentarios son de aplicación a la formulación original de CJS de la supergravedad en $D = 11$ y, en particular, suponen un carácter fundamental de la tres-forma A_3 de supergravedad once-dimensional. Según se observó en [92], la falta de una formulación clara del grupo de simetría de supergravedad en $D = 11$ podría achacarse, precisamente, a la presencia de A_3. Siendo una tres-forma, A_3 no admite una interpretación como potencial de gauge de algún grupo de simetría. En consecuencia, se propuso [92] que A_3 fuera un compuesto de uno-formas de gauge potenciales de grupos adecuados, que podrían ser importantes [6, 7] en la formulación completa de la Teoría M. Esta formulación conduce de forma natural a álgebras de supersimetría *más grandes* que las álgebras de superPoincaré estándares. Otro marco natural en el que formular las simetrías de las teorías de supergravedad lo proporcionan las teorías de Chern-Simons (CS) [91, 93], en las que los lagrangianos se obtienen como formas de CS de grupos adecuados.

En resumidas cuentas, el análisis de su límite de baja energía, la supergravedad en $D = 11$, es capaz de proporcionar gran cantidad de información sobre la Teoría M. Las simetrías y estructura de la supergravedad merecen, pues, un mayor estudio. Es objeto de esta Tesis el contribuir modestamente al progreso hacia una formulación de estas simetrías y hacia la identificación de los constituyentes fundamentales de la Teoría M, desde el análisis de la supergravedad en $D = 11$.

Contenidos de esta Tesis

A continuación se resume el contenido de la Tesis.

El capítulo 1 contiene la versión inglesa de esta introducción. En el capítulo 2 se repasan los elementos de la supergravedad de CJS en $D = 11$ necesarios para el resto de la Tesis. Se presenta el álgebra de superPoincaré, y se extiende en la superálgebra de la Teoría M, que contiene los generadores asociados a las cargas centrales que se acoplan a las

[9] Otros grupos pueden tener también relevancia, como el grupo de Kac-Moody E_{11} de rango 11 [88], $OSp(1|64)$ [89, 90] o $OSp(1|32)$ (véase [91]).

M-branas básicas. La acción de la supergravedad en $D = 11$ se introduce a continuación, en un formalismo de primer orden que trata como campos dinámicos al vielbein, al gravitino y a la tres-forma del correspondiente multiplete de supergravedad. La conexión de espín está compuesta de ellos, y también se emplea una cuatro-forma auxiliar adicional que pasa a ser, sobre la superficie de las ecuaciones de movimiento, la curvatura de la tres-forma. Se discuten las diferentes simetrías de la acción, con particular énfasis en la supersimetría. La variación del gravitino bajo supersimetría local permite introducir una *conexión generalizada* que toma valores en el álgebra de Lie de $SL(32, \mathbb{R})$. Sin embargo, la simetría local (bosónica) de la supergravedad es (al menos cuando la tres-forma se ve como fundamental) sólo su subgrupo $SO(1, 10)$, de ahí el nombre de conexión generalizada. No obstante, la analogía se puede llevar más allá, pudiéndose definir una *curvatura generalizada* y su holonomía correspondiente como herramientas útiles para la discusión de soluciones supersimétricas. Se demuestra asimismo que, curiosamente, la curvatura generalizada codifica las ecuaciones de movimiento bosónicas de supergravedad en $D = 11$, no sólo en el límite puramente bosónico sino también en presencia de un gravitino no nulo [1].

En el capítulo 3 se estudian aspectos adicionales de la holonomía generalizada. Se suele decir que la holonomía de una conexión sobre un fibrado dado está generada por la curvatura. Una afirmación más precisa, sobre la que no suele hacerse hincapié, es que el álgebra de holonomía en un punto p está generada por la curvatura en p y en otros puntos que se puedan alcanzar desde el primero mediante transporte paralelo. El efecto de la curvatura en puntos vecinos de p se puede medir con las derivadas covariantes sucesivas de la curvatura *en* p. La definición del álgebra de holonomía involucra, por tanto, estas derivadas covariantes de la curvatura. Trasladado al problema del cómputo de supersimetrías de un vacío, esto significa que, en general, la integrabilidad de primer orden de la ecuación de espinores de Killing (que determina las supersimetrías preservadas por una solución de supergravedad) podría no ser suficiente para asegurar que la ecuación se satisfaga, siendo en este caso necesarias condiciones de integrabilidad superior para resolver la ecuación.

Haciendo uso de estas ideas, se examina en el capítulo 3 la holonomía generalizada de diversas soluciones supersimétricas de supergravedad. La holonomía generalizada de las branas usuales [94] se repasa demostrando que, en estos casos, las derivadas covariantes sucesivas de las curvaturas generalizadas correspondientes sólo cierran el álgebra obtenida de la curvatura. En este sentido, la integrabilidad de orden superior para las M branas no añade nueva información significativa al álgebra de holonomía. Las compactificaciones de Freund-Rubin, por el contrario, proporcionan ejemplos en los que las derivadas supercovariantes de la curvatura generalizada son fundamentales para determinar el álgebra de holonomía. De hecho, el álgebra de curvatura para la compactificación de Freund-Rubin sobre la *squashed* S^7 resulta ser el álgebra de Lie de G_2. Se argumenta que este no puede ser el resultado correcto, puesto que una holonomía G_2 no describe correctamente la supersimetría de las soluciones. Por el contrario, los generadores que proporciona la derivada supercovariante de curvatura generalizada realzan el álgebra de holonomía a $so(7)$ [95, 2], que es el resultado correcto por argumentos de supersimetría.

Nuestro estudio de la holonomía generalizada continúa en el capítulo 4 desde una perspectiva diferente, en el contexto de la hipótesis *preónica* [83]. Se argumenta que los estados BPS que preservan k supersimetrías de un total de 32 están compuestos de un número $\tilde{n} = 32 - k$ de preones, número que coincide con el de supersimetrías *rotas*, de modo que $k = 32$ indica un vacío máximamente supersimétrico. Los mismos preones están caracterizados por $k = 31$: son estados $\nu = 31/32$ y preservan todas las supersimetrías excepto una. Se demuestra que se puede introducir un conjunto de $\tilde{n}$ espinores bosónicos para describir estos estados. Por simplicidad, se supondrá que los estados preónicos son puramente bosónicos de modo que, siendo k-supersimétricos, también vienen caracterizados por k espinores de Killing. Se demuestra que los espinores preónicos y de Killing son ortogonales y, por tanto, proporcionan una descripción alternativa de las supersimetrías preservadas. De hecho, se puede explotar todavía más esta ortogonalidad [3]: el conjunto de espinores preónicos, por un lado, y el de espinores de Killing, por otro, pueden completarse, respectivamente, a dos bases del espacio de espinores, dual la una de la otra.

Cualesquiera de estas dos bases define un *G-frame solidario* [3], donde G es un grupo que puede elegirse convenientemente. La superálgebra de la Teoría M, cuyo análisis conduce a, y puede hacerse en términos de, la conjetura preónica, tiene un grupo máximo de automorfismos $GL(32, \mathbb{R})$ y, por tanto, resulta natural escoger $G = GL(32, \mathbb{R})$. No obstante, otros grupos son posibles, pudiéndose también adoptar las opciones más restrictivas $G = SL(32, \mathbb{R})$ (el grupo relevante en el enfoque de holonomía generalizada) o $G = Sp(32, \mathbb{R})$. El método del *G-frame* se aplica entonces a la caracterización del álgebra de holonomía de soluciones preónicas de supergravedad, a saber, soluciones de supergravedad hipotéticas que preservan 31 supersimetrías, asociadas a los estados preónicos BPS. No es posible dar una respuesta definitiva sobre su existencia en la supergravedad ordinaria de CJS. Sin embargo, se demuestra la existencia [3] de configuraciones preónicas en supergravedades de Chern-Simons. El capítulo 4 concluye con la introducción de una acción de brana que preserva 31 supersimetrías y que describe, por tanto, una solución preónica. Sin embargo, esta brana no está formulada en la supergravedad estándar de CJS, sino en el contexto del enfoque de D'Auria y Fré [92] de supergravedad (que supone una estructura compuesta de la tres-forma A_3 en términos de uno-formas de gauge adecuadas).

Este resultado nos da pie a explorar más profundamente la supergravedad à la D'Auria y Fré. Antes de ello, sin embargo, se hace una pausa para introducir en el capítulo 5 el *método de expansión* [4] para (super)álgebras de Lie, que será útil en el contexto de la supergravedad de D'Auria y Fré. El contenido matemático (en contraposición a físico) de este capítulo es significativamente superior al del resto de la Tesis. Se trata de un capítulo técnico en el que se dan los detalles del mecanismo de expansión y se describen las características de las álgebras obtenidas. En primer lugar, se repasan los métodos existentes (contracciones, deformaciones y extensiones) para obtener nuevas álgebras (y superálgebras) a partir de otras dadas. Después, se introduce el método de expansiones para álgebras de Lie $\mathcal{G}$ en general. Al igual que el método de contracción, se basa en una redefinición de las coordenadas grupales mediante un parámetro λ que provoca una expansión en serie de potencias infinita de las uno-formas de Maurer-Cartan (MC) del álgebra de Lie (dual), con coeficientes que son, a su vez, uno-formas. Las series pueden ser truncadas de forma consistente a ciertos órdenes, siempre y cuando los órdenes de corte satisfagan ciertas condiciones, y las uno-formas coeficientes corresponden, pues, a las formas de MC de nuevas álgebras *expandidas*. El método se aplica entonces a álgebras de Lie con una estructura particular de subespacios que hace, al final, inmediata su generalización a superálgebras de Lie. Como primera aplicación del método, la superálgebra de la Teoría M (con sus automorfismos de Lorentz incluidos) se obtiene [4] como la expansión $osp(1|32)(2,1,2)$ de $osp(1|32)$ (véase el capítulo 5 y [4] para la notación) .

El capítulo 6 regresa al principal asunto de esta Tesis, la supergravedad en $D = 11$. Se revisa la simetría subyacente a la supergravedad, en una formulación *à la* D'Auria y Fré [92]. En general, una teoría lagrangiana con álgebra de simetría local $\mathcal{G}$ viene descrita mediante uno-formas de gauge, asociadas a las formas de MC de $\mathcal{G}$, y por sus curvaturas. Sin embargo, contrariamente al caso de su análoga en $D = 4$ y $N = 1$, la supergravedad en $D = 11$ contiene, según ya se ha mencionado, además de las uno-formas correspondientes al vielbein e^a y al gravitino ψ^α, una tres-forma A_3 que, como tal, no puede asociarse a un generador de simetría. No obstante, se puede introducir dos nuevos campos uno-formas bosónicos B^{ab}, $B^{a_1 \cdots a_5}$ y otro fermiónico η^α para expresar A_3, junto con las anteriores uno-formas e^a, ψ^α, como un compuesto de estas uno-formas. Se puede asociar todas estas uno-formas a las formas de MC de una familia uniparamétrica de superálgebras de Lie. Estas formas de MC están definidas sobre las variedades grupales correspondientes, o superespacios agrandados. Se ha de hacer de ellas la interpretación de que describen la simetría gauge subyacente a la supergravedad en $D = 11$: la simetría está oculta al considerar A_3 como campo fundamental, pero resulta manifiesta cuando se trata a A_3 como un campo compuesto.

La naturaleza de este problema permite traer a colación las álgebras diferenciales libres (FDAs). Las FDAs [96, 92, 97, 18] son una generalización natural del (enfoque dual de) las álgebras de Lie, y contienen p-formas π_p de rango $p > 1$. Para un caso particular de FDAs (las mínimas), las diferenciales de las p-formas π_p de rango superior corresponden a cociclos no triviales de Chevalley-Eilenberg (CE) ω_{p+1} de cierta álgebra de Lie $\mathcal{G}$. Si existe otra álgebra $\tilde{\mathcal{G}}$ (*más grande*) cuyas formas de Maurer-Cartan permiten convertir en triviales los cociclos ω_{p+1}, no-triviales para $\mathcal{G}$, entonces las uno-formas de Maurer-Cartan de $\tilde{\mathcal{G}}$ permitirán expresar π_p como compuestas de ellas. El problema de la estructura compuesta de A_3 y la simetría subyacente a supergravedad encajan de forma natural en este lenguaje y, de hecho, se analiza [92, 6, 7] usando estos argumentos.

Las posibles consecuencias dinámicas de una A_3 compuesta se examinan también en el capítulo 6, mediante la sustitución de su expresión compuesta en la acción de primer orden de supergravedad. Se demuestra que las ecuaciones de movimiento de los nuevos campos implican las de A_3, aunque ahora esta última ha de considerarse compuesta de aquéllos, en vez de tratarse como fundamental. Puesto que estos campos poseen más grados de libertad que A_3, existen simetrías de gauge que convierten estos grados de libertad en puro gauge. El capítulo concluye resaltando el hecho de que algunas teorías se puedan formular mediante superespacios agrandados tales que su dimensión coincide con el número de campos presentes en la teoría: la estructura de gauge de la supergravedad en $D = 11$ es un ejemplo de esta correspondencia campos/coordenadas de superespacio extendido [85].

En el capítulo 7, vuelven a emplearse los superespacios agrandados. El superespacio que allí se considera es en realidad la variedad grupal asociada a la superálgebra de la Teoría M y sus generalizaciones con n coordenadas fermiónicas y $\frac{1}{2}n(n+1)$ bosónicas (estos superespacios se suelen llamar también 'maximales', 'máximamente agrandados' o 'tensoriales'). Se sabe que los modelos extensos en superespacios agrandados proporcionan modelos para objetos preónicos; ésa es, precisamente, nuestra motivación para el estudio de este sistema. En el capítulo 7, se propone un modelo de cuerda supersimétrica con tensión moviéndose en el espacio máximamente extendido, que puede interpretarse como una generalización de espín superior de la supercuerda de Green-Schwarz. El modelo no involucra matrices de Dirac y tampoco posee un término de Wess-Zumino (WZ). Al contrario, se formula mediante dos espinores bosónicos que tienen un papel análogo al de los espinores preónicos introducidos en el capítulo 4. La afirmación [8] de que (en $D = 11$) el modelo preserva 30 supersimetrías de 32 (o, en general, $n - 2$ de n) la prueba el hecho de poseer 30 simetrías κ, a pesar de carecer de un término de WZ. El número de grados de libertad bosónicos y fermiónicos del modelo se calcula recurriendo a un análisis hamiltoniano, que puede simplificarse con el uso de supertwistors ortosimplécticos. El capítulo concluye con una extensión de estas ideas a la construcción de modelos de super-p-branas en superespacios máximamente extendidos.

El capítulo 8 contiene nuestras conclusiones. Algunos detalles técnicos quedan relegados a los apéndices.

1

Introduction

Amazing as it is the advance experienced in the last decades by our understanding of the fundamental processes and laws that rule the physical world, many important questions remain unsolved. Two major developments, namely, General Relativity and Quantum Mechanics, contributed to shape the 20th century Physics. The former, culminating the framework of Classical Physics, is a generalization of Special Relativity, the theory that revised the Galilean and Newtonian notions of space and time and placed them on an equal footing in a continuum spacetime. General Relativity provides a geometrical description of gravity and the framework in which the current cosmological models are formulated. Quantum Mechanics, on the other hand, applies to physical phenomena occurring (mostly) at the subatomic level, and is crucial in the description of the rest of fundamental interactions. The replacement of the original Galilean character of Quantum Mechanics to make it consistent with Special Relativity led to the development of Quantum Field Theory. Causality arguments could then be invoked to insist on a local, rather than a global, realization of some of the symmetries. The resulting Yang-Mills, or gauge, theories describe all fundamental interactions (electromagnetic, weak and strong forces) but gravity and, together with a prescribed matter content, are the key components of the Standard Model of particle physics.

1.1 The road to M Theory

The search for a unified description of different phenomena has been historically a guiding principle for the progress of Physics. From this point of view, it seems natural to look for a theory that combines all four fundamental interactions within the same descriptive scheme. A more solid argument for the unification of fundamental interactions, going beyond aesthetical grounds, is provided by the fact that the coupling constants of the fundamental interactions, including gravity, seem to converge at very high energies: at the grand unification scale of about 10^{16} GeV. Even at low energies, though, the three interactions of the Standard Model admit a description with the same Yang-Mills gauge theory language. The reason why gravity does not fit in this scheme is more serious than it may seem at first sight: General Relativity is a classical theory, and no consistent results are obtained when the usual prescriptions to account for quantum effects are imposed; in other words, General Relativity is a non-renormalizable theory. And yet, the energy scale at which quantum gravity effects would be significant, the so-called Plank scale, is 10^{19} GeV, relatively close to the grand unification scale. This could be interpreted as a hint that a unifying theory describing all four fundamental interactions does indeed exist.

Research on gravitation and high energy physics has traditionally followed separate road maps, although some discoveries have been fruitfully applied to both. That has been the case of supersymmetry [9, 10, 11] (see [12, 13] for reviews and [14] for a collection of reprints), a symmetry between bosons and fermions based on the concept of Lie superalgebra, a structure containing generators of both bosonic and fermionic character and thus including both commutators and anticommutators. Soon after its discovery, it was noticed that theories in which supersymmetry was realized locally automatically contained gravity. Roughly, the argument goes as follows: the anticommutator of two supersymmetry generators is a translation; *locally* realized supersymmetry therefore produces a *local* translation, to be identified with a diffeomorphism or local coordinate transformation; invariance under local supersymmetry implies, thus, invariance under diffeomorphisms and hence gravity. Such theories of local supersymmetry were consequently called supergravities (see [15, 16, 17, 18]). The first and simplest supergravity theory to be constructed was its four-dimensional ($D = 4$) version with only one supercharge ($N = 1$), and it was consequently called $D = 4$, $N = 1$, or simple, supergravity [19, 20] (see [15] for a review).

In the seventies, supergravity was regarded as a promising candidate for a quantum theory of gravity, since fermionic contributions to the gravitational perturbative expansions were expected to cancel the divergent scattering amplitudes. Simple supergravity turned out not to completely fulfil this prospect since, although successfully proved finite even at two loops [21], its matter couplings failed to be so already at first order [22]. Extended supergravities (containing $N > 1$ supercharges) were then developed, the promotion of their $N(N-1)/2$ abelian gauge fields to $SO(N)$-gauge fields, yielding the so-called gauged supergravities, being subsequently explored. Among all extended supergravities, maximally extended $D = 4$, $N = 8$ supergravity [23, 24] had very attractive features: gravity, gauge fields and matter were all included in the same supergravity multiplet and, since no $N = 8$ matter multiplets existed, a truly unified theory (expected to be renormalizable) of matter and all interactions could be achieved.

Developments in supergravity research also included the construction of supergravity theories in diverse spacetime dimensions (see [25] for a collection of reprints), in particular, in ten and eleven dimensions. Three supergravity theories could exist in ten dimensions, a supergravity with $N = 1$ supersymmetry (Type I supergravity) and two versions with $N = 2$: Type IIA (non-chiral) and Type IIB (chiral) supergravities (see [25] and references therein). On the contrary, only one supergravity theory existed in eleven dimensions, which was proved to be the maximal dimension in which supergravity could exist if higher spin fields were to be excluded [26]. Eleven-dimensional supergravity was then constructed by Cremmer, Julia and Scherk (CJS) in [27]. Maximal lower dimensional supergravities (those with a maximum amount of supersymmetry), like Type IIA in $D = 10$ or $N = 8$ in $D = 4$, were shown to arise as dimensional reductions (*i.e.*, toroidal compactifications) of eleven-dimensional supergravity[1].

These developments encompassed a revival and update of the old (Nordström and) Kaluza-Klein ideas, and compactifications on non-trivial manifolds were explored (see [29]) where the features of the effective four-dimensional theories were dictated by the properties of the compactifying manifold. For instance, the gauge group and the preserved supersymmetry in four dimensions were related to the isometry group and the holonomy group [30] of the compactifying manifold, respectively. Eleven was not only the maximal dimension allowed by supersymmetry, but also the minimal dimension that, upon compactification of the extra seven dimensions, could accommodate the $SU(3) \times SU(2) \times U(1)$ gauge group of the Standard Model [31] as a subgroup of the isometry group. Moreover, it allowed for 'spontaneous' compactifications [32], in which compact seven-manifolds arose in a natural way. However, chiral fermion families could not be obtained [33] from compactification of (non-chiral) $D = 11$ supergravity[2]. These facts, together with their non-renormalizable character, damped the interest in supergravity theories as candidates for a quantum theory of gravity.

---

[1] As a matter of fact, one of the main original motivations to build up $D = 11$ supergravity [27] was to circumvent the technical difficulties arising in the application of the standard Noether procedure to the construction of the $D = 4$, $N = 8$ supergravity lagrangian [28]; the full $N = 8$ lagrangian was actually obtained [23] by dimensionally reducing its $D = 11$ counterpart.

[2] This issue was revised later on, with the discovery that compactification spaces with singularities allowed for chiral fermions [34].

On the other hand, new results were being obtained regarding a formulation of the other interactions. While the electroweak forces were successfully described by the spontaneously broken $SU(2) \times U(1)$ Yang-Mills theory, a description for the strong interactions was much less clear. Curiously enough, a description was proposed in terms of a theory of strings, since the results of the Veneziano amplitudes and Regge slopes suggested that hadrons could be described as vibrations of a fundamental string [35]. This setting assumed the bold proposal of substituting point particles for one-dimensional extended objects. However, the success of the application of Yang-Mills theory to the description of the strong interactions in terms of Quantum Chromodynamics (QCD), made the stringy description fall out of favour, while satisfactorily put the formulation of strong and electroweak forces on the same footing.

String Theory (see [36, 37]) recovered from this setback when the realization that the spin two field contained in its spectrum could be interpreted as the graviton [38] (the quantum of the gravitational field) provided that the string scale was moved up, from the scale of the strong interactions to that of quantum gravity. Moreover, strings provided a renormalizable quantum theory of gravity because their interaction was smeared over a region of spacetime, instead of taking place at a point. The real String Theory explosion would come in the mid eighties, when the *first superstring revolution* took place.

The classical theory of superstrings (incorporating supersymmetry) was known to be well defined in 3, 4, 6 and 10 spacetime dimensions [39], in which cases a Wess-Zumino term (see [40]) existed endowing the action with κ-symmetry[3] (see [36]), a local fermionic symmetry that allowed for the correct Bose-Fermi matching of degrees of freedom. At the quantum level, the theory was shown to be consistent only in ten dimensions, since only in that case it was anomaly free [43]. The anomaly cancelation left, moreover, five different possible string theories in ten dimensions, namely, Type IIA, Type IIB, Type I, SO(32) heterotic and $E_8 \times E_8$ heterotic (see [36] and references therein). Also, supergravity was incorporated into String Theory: an analysis of the massless modes (describing the low energy dynamics) in the spectrum of the different string theories showed that they consisted in the fields belonging to the different supergravity multiplets in ten dimensions, possibly coupled to super Yang-Mills multiplets. More precisely, the low energy limits of Type IIA and IIB string theories were found to be, respectively, the supergravities of the same name; and that of Type I and the heterotic strings, Type I supergravity coupled to the $N = 1$ vector multiplet in ten dimensions with gauge group

[3] In the particle case, the existence of a fermionic gauge symmetry was shown in [41] in the massive case and in [42] in the massless case.

$SO(32)$ (for Type I and the corresponding heterotic string) or $E_8 \times E_8$ (for the other heterotic string).

Although a Lorentzian signature had to be imposed, the consistency of the five perturbative string theories provided for the first time a theoretical argument for a particular value of the spacetime dimension. A Kaluza-Klein-type program was applied to string compactifications, that sought for realistic models in which ten-dimensional spacetime split into ordinary four-dimensional spacetime, and a compact six-dimensional Euclidean manifold. Interestingly enough, the compactifying manifold could be chosen so that realistic models close to the Standard Model were obtained in four dimensions. The heterotic [44] $E_8 \times E_8$ string was seen as particularly suitable for Standard Model building, since its compactifications managed to provide E_6 gauge symmetries[4] (a candidate gauge group in Grand Unification Theories). Moreover, choosing the compactification manifold to be Calabi-Yau [46], the phenomenological requirement of chiral fermion families provided by $N = 1$ supersymmetry in $D = 4$ was also fulfilled. All this added to the generalized enthusiasm that made String Theory the most promising candidate for the unified picture of the fundamental interactions since, proposed as a quantum theory of gravity, it also seemed to include the Standard Model as a result of its own selfconsistency conditions.

Despite all this headway, many issues were still unresolved. First of all, the (five different) theories of superstrings were only defined at the perturbative level. Moreover, the existence of five different perturbative theories was not too appealing if String Theory had to unify all interactions. In fact, the usual prescription for the computation of scattering amplitudes involves both the sum over all possible topologies the string worldsheet can display (the genus expansion) and, in the presence of nontrivial backgrounds, the loop expansion in the string coupling constant α' for each one of those topologies. Although this perturbative approach can be intensively exploited to obtain a great deal of information about the theory it describes, it was apparent that there was a lot of nonperturbative structure which could not be reached by this prescription.

That is indeed the case. There exist states that are naturally nonperturbative but similar, instead, to the solitonic solutions present in

---

[4] Notice that ten-dimensional heterotic compactifications on six-manifolds do not contradict the fact mentioned above that only compactifications from eleven dimensions on seven-manifolds allow for the Standard Model gauge group in the resulting four-dimensional theory. The resulting gauge symmetry in heterotic compactifications is a result of the presence of $E_8 \times E_8$ gauge fields already in the ten-dimensional theory, and not a consequence of the isometries of the compactifying manifold. Calabi-Yau manifolds have, indeed, no isometries at all. See [45] for a derivation of the Standard Model gauge group from compactification in a IIA context.

other field theories. There are some special solitonic solutions, the so-called Bogomol'nyi–Prasad–Sommerfield (BPS) states[5] which, despite being classical, can be considered as true solutions to the non-perturbative quantum theory. BPS multiplets arise in massive representations of the $N \geq 1$ extended supersymmetry algebras in which the Bogomol'nyi bound, relating the eigenvalues of the momentum (mass) and of the central charges, is saturated. Their basic feature is that they are characterized by containing significatively fewer states than a usual massive multiplet. Now, although both mass and charge may undergo renormalization in perturbation theory, the BPS condition is protected from quantum corrections. A heuristic argument to support this claim is that, otherwise, as quantum effects are being switched on a BPS multiplet could turn into a non-BPS one containing many more states than the former, and such drastic appearance of states is not expected to happen. It is in this sense that BPS states, albeit usually described by solutions of the classical equations of motion, are stable under quantum corrections and can be lifted, therefore, to solutions of the full, non-perturbative theory. This is why they are so useful to probe the non-perturbative structure of the theory.

Thus, the notion of BPS states turned out to be crucial in order to explore the non-perturbative regime of the five string theories. Such research derived, in the mid nineties, in important discoveries concerning the relations among them. Some perturbative relations, known as T-dualities (see [49]), among the compactifications of the different string theories were nevertheless already known at the time: Type IIA and Type IIB were known to be T-dual (equivalent) when compactified on a circle of radius R and α'/R, respectively, and viceversa. The two versions of the heterotic string were also T-dual. S-dualities [50], in contrast, were of a different nature and provided, instead, a generalization of the conjectured electric-magnetic duality of classical electrodynamics [51]. S-dualities related the strong coupling regime of a string theory (where the usual perturbative prescriptions break down) with the weak coupling regime (that could be treated in perturbation theory) of another string theory. Type I and $SO(32)$ heterotic were shown to be S-dual, and Type IIB to be S-selfdual. T and S-dualities were unified by U-duality [52], and the resulting web of dualities managed to give a unified picture of all string theories, which were then reinterpreted as perturbative expansions around five different vacua of the same underlying non-perturbative theory. The importance of this fact motivated that its discovery were marked as the beginning of the *second superstring revolution* (see [53]).

Two decisive developments came along with the discovery of the web

[5]The terminology comes from [47, 48]

of dualities. The first one was the realization that p-branes [54] (see also [55]), supersymmetric extended objects sweeping out a worldvolume of p spacelike dimensions, were to be included in the theory [56]. These branes are solutions of the classical supergravity equations and, since they carry topological charges [57] (see also [58]), associated to the central charges of the supersymmetry algebra, they keep on being solutions at the quantum level. That is why they are so valuable to probe the non-perturbative sectors of the theory. The existence of these extended objects in String Theory suggests a *p-brane democracy* [59], according to which they should be treated on the same footing as the strings themselves. The second development was the realization that Type IIA and $E_8 \times E_8$ heterotic string theories were dual to a non-perturbative theory in eleven dimensions [60, 61], subsequently dubbed M Theory (see [53] for a review and [62] for a collection of reprints). The coupling constants of those string theories were taken as growing functions of a compactification radius so that, in their strong coupling regime, an eleventh dimension arose. This duality was supported by other facts; as already mentioned, Type IIA supergravity was known to be the dimensional reduction of eleven-dimensional supergravity, and the IIA string could be derived [63] from the eleven-dimensional supermembrane. This added reasons to conjecture eleven-dimensional supergravity as the low energy limit of M Theory.

Eleven is, in fact, the maximum spacetime dimension that local spacetime supersymmetry permits (as already mentioned) and that can contain supersymmetric extended objects (see [64]). Also, the relevant supermultiplet in eleven dimensions that determines the field content of $D = 11$ supergravity is both unique and far simpler than its ten-dimensional counterparts. It only contains three fields: the metric (or graviton) $g_{\mu\nu}$, its supersymmetric partner, the gravitino ψ^{α}, and a three-form A_3. From this perspective, eleven dimensions are more natural than ten, although the selfconsistency arguments provided by String Theory were irrefutable. After the mid nineties, however, the place of String Theory was redefined, from a theory of vibrating one-dimensional extended objects, to a theory of extended objects in general. And, rather than being fundamental theories in themselves, string theories arose as five different ten-dimensional perturbative corners of a new and truly fundamental eleven-dimensional theory, M Theory.

These insights about the non-perturbative sectors of the theory allowed for new activity. For instance, D-branes [65] (see [66]) superseded heterotic string compactifications in Standard Model building (see [67]). On the other hand, other developments took place, exploring the non-perturbative sectors of the theory. That was the case of the AdS/CFT correspondence [68] (see [69] for a review), originally formulated in a Type

IIB context. According to it, String Theory on a ten-dimensional background containing five-dimensional Anti-de Sitter space, AdS_5, is dual to a superconformal field theory (CFT) in the conformal boundary of AdS_5, namely, four-dimensional Minkowski space M_4. The original formulation related IIB string theory on the maximally supersymmetric background $AdS_5 \times S^5$, where S^5 is the round five-sphere, to $N = 4$ super Yang-Mills theory. Subsequent generalizations were proposed; in particular, a dual $N = 1$ superconformal field theory is obtained if S^5 is replaced by a Sasaki-Einstein manifold [70].

The current knowledge of M Theory includes the non-perturbative dualities it displays and the fact that its low energy limit is eleven-dimensional supergravity. Properties such as the stability of BPS states allow to probe the full M Theory from supergravity solutions: the well-known eleven-dimensional branes can be described by their worldvolume actions [71, 72], or considered as supergravity solutions [73, 74] (see [64, 75] for reviews), that arguably remain solutions of the full non-perturbative M Theory. $D = 11$ supergravity thus provides a laboratory to explore basic features of M Theory. Brane solutions preserve one-half of the maximum amount of supersymmetry (see [57]), and their intersections preserve smaller fractions [76]. Getting to know more supersymmetric solutions of supergravity, preserving different fractions ν of supersymmetry, would still provide further insight into the theory.

Although supersymmetric solutions to $D = 11$ supergravity involving both bosonic and fermionic fields are known (see [77]), the search for supergravity solutions has been usually restricted, for simplicity, to purely bosonic configurations. Being bosonic, the fermion fields can be set to zero in the equations of motion and the requirement that the solution be supersymmetric is achieved provided the supersymmetry transformation of the fermions also vanishes. Supersymmetric purely geometrical solutions, in which the metric is the only non-vanishing field, can be classified by Riemannian holonomy. More general supergravity solutions can be suggestively discussed by an extension of Riemannian holonomy in terms of generalized holonomy [78, 73], G-structures [79, 80], or *spinorial geometry* [81]. We shall be discussing some related issues in this Thesis.

Another piece of valuable information about M Theory can be obtained, also at the supergravity level, by studying the symmetry algebra on which it is based [59, 82]. The brane solutions of $D = 11$ supergravity are often viewed as the fundamental objects of M Theory, in much the same way strings were the basic objects of String Theory. In particular the topological charges [57] of the M5 [58] and M2 branes can be naturally included in the supersymmetry algebra to give the so called M Theory algebra [59]. The lack of an action principle for M Theory can be partially overcome by group theoretical methods and, as a matter of fact, the study of the representations of the M Theory superalgebra suggests that states preserving 31 supersymmetries could be treated as fundamental, the rest being composed of them. These states, introduced in [83] and called *preons*, could be considered as fundamental constituents of M Theory. As it will be shown in this Thesis, these notions also lead naturally to consider enlarged superspaces [8] and supertwistors (see [84] for a review); see [85] for earlier ideas on superbranes and enlarged superspaces and [86] for a review in this context.

The study of the symmetries of supergravity is, arguably, a useful tool to obtain insights into the structure of M Theory. The obvious (bosonic) local symmetry group of $D = 11$ supergravity is the Lorentz group $SO(1, 10)$ and, hence, Lorentz covariant derivatives of the supergravity fields arise naturally in the lagrangian. However, another *supercovariant derivative* taking values on the Lie algebra of $SL(32, \mathbb{R})$ [87] arises in order to express the variation under local supersymmetry of its *gauge field*, the gravitino ψ^α. Indeed, the suggestion has been made [87] that $D = 11$ supergravity has a hidden $SL(32, \mathbb{R})$ symmetry, that might become relevant in M Theory[6]. However, no explicit formulation of $D = 11$ supergravity has been explicitly achieved so far exhibiting this symmetry.

This argument applies to the original, CJS formulation of $D = 11$ supergravity and, in particular, assumes a fundamental character for the three-form A_3 of $D = 11$ supergravity. The observation was made in [92] that the lack of a clear formulation for the gauge group of $D = 11$ supergravity could be put down, precisely, to the presence of A_3. Being a three-form, it did not admit an interpretation as a gauge potential of some symmetry group. In consequence, A_3 was proposed [92] to be composed of gauge one-form potentials of suitable groups which could play a role [6, 7] in the formulation of the fully-fledged M Theory. This formulation leads naturally to supersymmetry algebras *larger* than the standard superPoincaré algebras. Other natural setting to formulate the symmetries of supergravity theories is achieved by a Chern-Simons (CS) formulation [91, 93], in which the lagrangians are obtained as CS forms of suitable supergroups.

In summary, a great deal of information about M Theory can be obtained from the analysis of its low energy limit, $D = 11$ supergravity, the symmetries and structure of which are hence worth further study. This Thesis aims to make a modest progress towards a formulation of these symmetries and the identification of the fundamental constituents of M Theory from the analysis of $D = 11$ supergravity.

[6]Several groups may also play a role, as the rank 11 Kac-Moody group E_{11} [88], $OSp(1|64)$ [89, 90] or $OSp(1|32)$ (see [91]).

1.2 The contents of this Thesis

The plan of the Thesis is as follows.

In chapter 2, the elements of $D = 11$ CJS supergravity that will be needed in the rest of the Thesis are reviewed. The superPoincaré algebra is presented, and extended into the M Theory superalgebra, containing the central charge generators that couple to the basic M branes. The action of $D = 11$ supergravity is then introduced, in a first order formalism that treats the vielbein, gravitino and three-form of the corresponding supergravity multiplet, as dynamical fields. The spin connection is composed out of them, and an additional auxiliary four-form becomes related, on-shell, to the curvature of the three-form. The various symmetries of the action are discussed, with particular emphasis on supersymmetry. The variation of the gravitino under supersymmetry allows us to introduce a *generalized connection* taking values on the Lie algebra of $SL(32, \mathbb{R})$. The local (bosonic) symmetry of supergravity (at least when the three-form field is regarded as fundamental) is, however, only its subgroup $SO(1, 10)$, hence the name of generalized connection. Nevertheless, the analogy can be pushed forward, and a *generalized curvature* and its corresponding holonomy, consequently called *generalized holonomy*, can be introduced as a useful tool to discuss supersymmetric solutions. Interestingly enough, the generalized curvature is shown to encode the bosonic equations of motion of $D = 11$ supergravity, not only in the purely bosonic limit but also when the gravitino is not vanishing [1].

Further study about generalized holonomy is carried out in chapter 3. It is usually claimed that the holonomy of a connection on a given fiber bundle is generated by the curvature. A more precise statement, that is usually underemphasized, is that the Lie algebra of the holonomy group at a point p is generated by the curvature at p and at any other points that can be reached from the former by parallel transport. The effect of the curvature at neighbouring points of p can be measured by the successive covariant derivatives of the curvature *at* p. These covariant derivatives of the curvature are, thus, also involved in the definition of the Lie algebra of the holonomy group. Translated into the problem of counting the supersymmetries of a vacuum, this means that, in general, the first order integrability of the Killing spinor equation (which determines the supersymmetries preserved by a bosonic supergravity solution) might not be not enough to ensure that the equation is satisfied, and higher order integrability conditions could be needed to solve the equation.

Using these ideas, the generalized holonomy of several supersymmetric solutions of supergravity is revisited in chapter 3. The generalized holonomy of the usual brane solutions [94] is reviewed showing that, in these cases, successive covariant derivatives of the corresponding generalized curvatures only close the algebra obtained from the curvature. In this sense, higher order integrability for the M branes does not add significant new information to the Lie algebra of the holonomy group. Freund-Rubin compactifications, on the other hand, are given as examples in which the supercovariant derivatives of the generalized curvature are crucial to determine the Lie algebra of the generalized holonomy group. In fact, the curvature algebra for the Freund-Rubin compactification on the squashed S^7 turns out to be the Lie algebra of G_2. It is argued that this cannot be the right result, since a G_2 holonomy does not describe correctly the preserved supersymmetry of the solutions. On the contrary, the generators provided by the supercovariant derivative of the generalized curvature enhance the holonomy algebra to $so(7)$ [95, 2], which is also argued by supersymmetry to be the right result.

Our study of generalized holonomy continues in chapter 4 from a different point of view, in the context of the *preon* hypothesis [83]. BPS states preserving k supersymmetries out of 32 are argued to be composed of $\tilde{n} = 32 - k$ of preons, a number that coincides with that of *broken* supersymmetries so that $k = 32$ indicates a fully supersymmetric vacuum. Preons themselves are characterized by $k = 31$: they are $\nu = 31/32$ states and preserve all supersymmetries but one. It is shown that a set of $\tilde{n}$ bosonic spinors can be introduced in order to describe these states. For simplicity, it will be assumed that these states are purely bosonic so that, being k-supersymmetric, they are also characterized by k Killing spinors. The preonic and Killing spinors are shown to be orthogonal and, thus, provide an alternative description of the preserved supersymmetries. In fact, this orthogonality can be further exploited [3]: the set of preonic spinors, on the one hand, and the set of Killing spinors, on the other hand, can be completed, respectively, to two bases in the space of spinors, dual to each other.

Either one of these two bases define a *moving G-frame* [3], where G is a group that can be chosen for convenience. The M Theory superalgebra, the algebraic analysis of which leads to, and can be made in terms of, the preon conjecture, has a maximal automorphism group of $GL(32, \mathbb{R})$ and, thus, it is natural to choose $G = GL(32, \mathbb{R})$. Other groups are, however, possible and the more restrictive options $G = SL(32, \mathbb{R})$ (the relevant group in the generalized holonomy approach) or $G = Sp(32, \mathbb{R})$ may also be taken. The G-frame method is then applied to the characterization of the generalized holonomy of preonic supergravity solutions, namely, hypothetical supergravity solutions preserving 31 supersymmetries, associated to the BPS preon states. No definitive answer about their existence in ordinary CJS supergravity can be given from this analysis. However, preonic configurations are shown to exist [3] in the context of Chern-Simons supergravities. Chapter 4 then concludes with the introduction of a brane action preserving 31 supersymmetries and describing, hence, a preon state. This brane is not formulated on standard CJS supergravity, though, but in the context of D'Auria-Fré approach to supergravity [92] (that assumes a composite structure of the three-form A_3 in terms of suitable one-form gauge fields).

This result gives us reasons to further explore $D = 11$ supergravity *à la* D'Auria-Fré. Before doing so, however, a break is done to introduce in chapter 5 the *expansion method* [4] for Lie (super)algebras, since it will be useful in this context. The mathematical (*vs.* physical) content of chapter 5 is significatively higher than that of the rest of the Thesis. It is a technical chapter giving the details of how the expansion method works, and describing the features of the algebras obtained. First of all, a review is done of the existing methods (contractions, deformations and extensions) to obtain and derive new Lie algebras (and superalgebras) from given ones. Then the expansion method for Lie algebras $\mathcal{G}$ is introduced in general. Like the contraction method, it relies on a redefinition of the group coordinates by a parameter λ that makes the Maurer-Cartan (MC) one-forms of the (dual) Lie algebra expand in infinite power series of λ with one-form coefficients. The series can be consistently truncated at suitable orders, provided the cutting orders fulfil some conditions, and the retained one-form coefficients then correspond to the MC forms of the new, *expanded* algebras. The method is then applied to Lie algebras with a particular structure of subspaces that, in the end, makes straightforward its generalization to Lie superalgebras. As a first application of the method, the M Theory superalgebra (including its Lorentz automorphism part) is derived [4] as the expansion $osp(1|32)(2,1,2)$ of $osp(1|32)$ (see chapter 5 and [4] for the notation).

Chapter 6 returns to the main subject of this Thesis, $D = 11$ supergravity. The gauge symmetry underlying supergravity is revised, in a formulation *à la* D'Auria and Fré [92]. In general, a lagrangian theory with local symmetry algebra $\mathcal{G}$ is described in terms of gauge one-form fields, associated to the MC forms of $\mathcal{G}$, and its curvatures. However, as opposed to its $D = 4$, $N = 1$ counterpart, eleven-dimensional supergravity contains, as already mentioned, besides the vielbein e^a and gravitino ψ^α one-forms, a three-form A_3 that, as such, cannot be associated to a symmetry generator. However, two new bosonic one-form fields B^{ab}, $B^{a_1 \cdots a_5}$ and one fermionic η^α can be introduced to express A_3, together with the former one-forms e^a, ψ^α, as a composite of these one-forms. All these one-forms can be associated to Maurer-Cartan one-forms of a one parameter family of Lie superalgebras. These MC forms are defined on the corresponding group manifolds, or enlarged rigid superspaces. They are to be interpreted as describing the underlying gauge symmetry of $D = 11$ supergravity: the symmetry is hidden when A_3 is considered as a fundamental field, but becomes manifest when A_3 is treated as a composite field.

Free differential algebras (FDAs) are brought into the picture to deal with this problem. FDAs [96, 92, 97, 18] are a natural generalization of (the dual point of view of) Lie algebras, containing p-forms π_p of rank $p > 1$. For a particular case of FDAs (the minimal ones), the differentials of the higher rank p-forms π_p are nontrivial Chevalley-Eilenberg (CE) cocycles ω_{p+1} of a certain Lie algebra $\mathcal{G}$. If there exists another (*larger*) algebra $\tilde{\mathcal{G}}$ in terms of the Maurer-Cartan forms of which the non-trivial cocycles ω_{p+1} for $\mathcal{G}$ become trivial, then the MC forms of $\tilde{\mathcal{G}}$ will allow us to express π_p as composites of them. The problem of the composite structure of A_3 and the underlying symmetry of supergravity fits naturally in this language and, in fact, it is further studied [92, 6, 7] using these arguments.

The possible dynamical consequences of a composite A_3 are also analyzed in chapter 6, by substituting its composite expression into the supergravity first order action. The equations of motion of the new fields are shown to imply those of A_3, but now considering the later as composed of them, rather than as fundamental. Although the new fields carry more degrees of freedom than A_3 does, the formulation of supergravity due to D'Auria and Fré can be regarded as dynamically equivalent to the standard CJS formulation, since there exist gauge symmetries that make these extra degrees of freedom pure gauge. The chapter ends emphasizing how enlarged superspaces can be found for some theories such that their dimension coincides with the number of fields present in the theory: the gauge structure of $D = 11$ supergravity is an example of this extended superspace coordinates/fields correspondence [85].

In chapter 7, enlarged superspaces are again used. The relevant superspace there is actually the group manifold of the M Theory superalgebra and its generalizations with n fermionic and $\frac{1}{2}n(n+1)$ bosonic coordinates (also called 'maximal', 'maximally enlarged' or 'tensorial' superspace). Extended objects in enlarged superspaces are known to provide models for preonic objects; this is, in fact, our motivation for the study of this system. A model for a supersymmetric tensionful string moving in maximally enlarged superspace is proposed in chapter 7, which can be interpreted as a higher spin generalization of the Green-Schwarz superstring. The model neither involves Dirac matrices nor has a Wess-Zumino (WZ) term. Instead, it is formulated in terms of two bosonic spinors, a counterpart of the preonic spinors introduced in chapter 4. The claim [8] that (in $D=11$) the model preserves 30 supersymmetries out of 32 (or, in general, $n-2$ out of n) is supported by the fact that it possess 30 κ-symmetries, in spite of its lack of a WZ term. The number of bosonic and fermionic degrees of freedom of the model is worked out resorting to a hamiltonian analysis, which can be simplified with the use of orthosymplectic supertwistors. The chapter concludes with an extension of these ideas to the construction of super-p-branes models in maximally enlarged superspaces.

Chapter 8 contains our conclusions. Some technical details are relegated to the appendices.

2

Eleven-dimensional supergravity

The general framework in which the rest of the Thesis is developed is introduced in this chapter, devoted to the review of a number of topics about $D = 11$ Cremmer-Julia-Scherk (CJS) supergravity in order to fix the conventions and notation used in most of the subsequent chapters. After describing in section 2.1 the supersymmetry algebras and groups relevant for supergravity (with particular emphasis on the eleven-dimensional case), the action principle of the theory is introduced in section 2.2, in a first order formalism that turns out to be convenient for subsequent developments. The symmetries of the action are also discussed in that section, devoting the following section 2.3 to the supersymmetry of the theory and to the related notions of generalized connection, curvature and holonomy. The equations of motion are described both in general, in section 2.4, and in the purely bosonic limit, in section 2.5. Finally, in section 2.6 the equations of motion of eleven-dimensional supergravity are shown to be encoded in the generalized curvature even when the gravitino is non-vanishing [1].

2.1 The M Theory superalgebra

Eleven-dimensional CJS supergravity [27] is the locally supersymmetric field theory based on the (only) massless supermultiplet of the super-Poicaré group in eleven spacetime dimensions containing fields up to helicity two [26]. Not only the supergravity multiplet is unique in $D = 11$, but the theory does not allow modifications such as the presence of a cosmological constant [98]. It is thus worth starting the discussion about eleven-dimensional supergravity taking a look at the symmetry algebra on which it is based. The $D = 11$ superPoincaré (or standard supersymmetry) algebra[1] $\mathfrak{E} \rtimes so(1,10)$ is made up of the usual supertranslations algebra $\mathfrak{E}$ acted on semidirectly by the Lorentz algebra in eleven dimensions. The supertranslations algebra $\mathfrak{E}$ exponentiates into the supertranslations group Σ, the group manifold of which (also denoted Σ) corresponds to rigid superspace. In spacetime dimension D, the even part of the (N-extended) supertranslations algebra $\mathfrak{E}$ is generated by D bosonic (translation) generators P_a, $a = 1, \ldots, D$, and the odd part by N fermionic (supertranslation) generators, or supercharges, Q^i_α with n components, $i = 1, \ldots, N$, $\alpha = 1, \ldots, n$. The number n of components of each supercharge is that of the minimal spinor in spacetime dimension D, and the number N of supercharges has an upper bound depending on D (see below). In eleven dimensions, $D = 11$, there is only one supercharge, $N = 1$, that is a Majorana spinor with $n = 32$ components. Accordingly, we shall usually assume $N = 1$, in which case D and n are the bosonic and fermionic dimensions of superspace. When the dimensions are needed explicitly, to avoid confusion it will be written $\mathfrak{E}^{(D|n)}$ (for the superalgebra) and $\Sigma^{(D|n)}$ (for the supergroup, or superspace); in eleven dimensions, thus, the supertranslations algebra is $\mathfrak{E}^{(11|32)}$ and superspace is $\Sigma^{(11|32)}$. As a supergroup, the $\Sigma^{(11|32)}$ superspace can be regarded as a central extension[2] [85] by the generator P_a of the abelian fermionic translations group $\Sigma^{(0|32)}$ generated by Q_α. Further extensions and enlargements of the algebra are possible, as we shall shortly see.

As for the structure of the superPoincaré algebra, the supertranslations (anti)commutation relations defining $\mathfrak{E}$ are

$$\{Q_\alpha, Q_\beta\} = \Gamma^a_{\alpha\beta} P_a \ , \quad [P_a, Q_\alpha] = 0 \ , \quad [P_a, P_b] = 0 \ , \tag{2.1.1}$$

where ${\Gamma^a}_\alpha{}^\beta$ are 32×32 eleven-dimensional Dirac matrices defining the Clifford algebra $Cl(1,10)$,

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}\, I_{32} \ , \tag{2.1.2}$$

η_{ab} being the Minkowski metric and I_{32} the 32×32 identity matrix. The spinor indices are raised and lowered with the 32×32 skewsymmetric charge conjugation matrix $C_{\alpha\beta}$, which is understood in (2.1.1): $\Gamma^a_{\alpha\beta} \equiv {\Gamma^a}_\alpha{}^\gamma C_{\gamma\beta}$. The Lorentz group $SO(1,10)$ has generators J_{ab}, and its corresponding algebra $so(1,10)$ is defined by the commutation relations

$$[J_{ab}, J^{cd}] = -4 J_{[a}{}^{[c} \delta_{b]}{}^{d]} \ . \tag{2.1.3}$$

Its semidirect action on the supertranslations algebra $\mathfrak{E}$ is given by

$$[J_{ab}, Q_\alpha] = \tfrac{1}{4} \Gamma_{ab\,\alpha}{}^\beta Q_\beta \ , \quad [J_{ab}, P_c] = 2\eta_{c[a} P_{b]} \ , \tag{2.1.4}$$

[1] The symbol $\rtimes$ will be used throughout, either denoting semidirect *sum* (when used in a Lie *algebra* context) or semidirect *product* (when used in a Lie *group* context).

[2] See section 5.1 in chapter 5 for a brief review of Lie algebra extensions.

where Γ^{ab} is the antisymmetrized product of two Dirac matrices of the Clifford algebra $Cl(1,10)$; in general,

$$\Gamma^{a_1 \dots a_k} := \Gamma^{[a_1} \cdots \Gamma^{a_k]} \ , \tag{2.1.5}$$

where the brackets denote antisymmetrization with weight one. The set of (anti)commutators (2.1.1), (2.1.3), (2.1.4) defines the superPoincaré algebra $\mathfrak{E} \rtimes so(1,10)$.

The number of supersymmetries of a supergravity theory in any dimension must be at most 32, if interacting fields of spin higher than 2 are to be avoided[3]. This is the requirement that places an upper bound depending on the number of components n of the minimal spinor in dimension D. The 32-component supercharge Q_α of $D = 11$ supergravity makes its equations display maximal supersymmetry; the maximally supersymmetric supergravity in four dimensions has, instead, $N = 8$ 4-component supercharges Q^i_α, $i = 1, \dots, 8$, $\alpha = 1, \dots, 4$, so that it also displays 32 supersymmetries. N-extended supergravities in lower dimensions allow for supersymmetry algebras with a richer structure, since new generators commuting with the rest of the superPoincaré generators, and consequently called *central charges*, can be introduced on the right-hand-side of the anticommutator of two supercharges (the first equation in (2.1.1)). Eleven-dimensional supergravity has only one supercharge, $N = 1$, but extensions nevertheless do exist generalizing the anticommutator of two supercharges.

In fact, the anticommutator in (2.1.1) is symmetric in the spinor indices $(\alpha\beta)$ and takes values on the (even part $Cl(1,10)_+$ of the) Clifford algebra $Cl(1,10)$ generated by

$$\{I_{32}, \Gamma^{[1]}, \Gamma^{[2]}, \Gamma^{[3]}, \Gamma^{[4]}, \Gamma^{[5]}\} \ , \tag{2.1.6}$$

where the shorthand notation $\Gamma^{[k]}$ has been used to denote generically the antisymmetrized products of Dirac matrices, $\Gamma^{[k]} \equiv \Gamma^{a_1 \dots a_k}$. In eleven dimensions and Lorentzian signature, the matrices $\Gamma^{[1]}_{\alpha\beta} \equiv (\Gamma^{[1]}C)_{\alpha\beta}$, $\Gamma^{[2]}_{\alpha\beta} \equiv (\Gamma^{[2]}C)_{\alpha\beta}$ and $\Gamma^{[5]}_{\alpha\beta} \equiv (\Gamma^{[5]}C)_{\alpha\beta}$ are symmetric in $(\alpha\beta)$, whereas the rest in (2.1.6) are skewsymmetric. The standard supertranslations algebra $\mathfrak{E}$ can accordingly be extended by adding two more antisymmetric tensorial generators $Z_{ab} = Z_{[ab]}$, $Z_{a_1 \dots a_5} = Z_{[a_1 \dots a_5]}$ to the right-hand-side of the anticommutator of two supercharges [82]:

$$\{Q_\alpha, Q_\beta\} = \Gamma^a_{\alpha\beta} P_a + i \Gamma^{ab}_{\alpha\beta} Z_{ab} + \Gamma^{a_1 \dots a_5}_{\alpha\beta} Z_{a_1 \dots a_5} \ . \tag{2.1.7}$$

[3]See section 7.1 of chapter 7, and references therein, for some remarks about higher spin theories.

The generators Z_{ab}, $Z_{a_1 \dots a_5}$ commute among themselves and with the rest of supertranslation generators. They are, thus, central if the Lorentz group is ignored and, in fact, are also called central charges.

The extension (2.1.7) of the standard supertranslations algebra $\mathfrak{E} \equiv \mathfrak{E}^{(11|32)}$ by the bosonic generators Z_{ab}, $Z_{a_1 \dots a_5}$ gives the superalgebra $\mathfrak{E}^{(528|32)}$, with generators

$$P_a \,,\, Q_\alpha \,,\, Z_{a_1 a_2} \,,\, Z_{a_1 \dots a_5} \ , \tag{2.1.8}$$

and bosonic dimension $\binom{11}{1} + \binom{11}{2} + \binom{11}{5} = 11 + 55 + 462 = 528$. Being maximally extended (in the bosonic sector[4]), $\mathfrak{E}^{(528|32)}$ generalizes the superPoincaré algebra in eleven dimensions and is usually called the *M Theory superalgebra* or *M-algebra*[5] [59] (see [82, 99, 85]). Its associated group manifold $\Sigma^{(528|32)}$ corresponds to the maximally extended rigid superspace. The bosonic generators P_a, Z_{ab} and $Z_{a_1 \dots a_5}$ can be collected in a *generalized momentum* $P_{\alpha\beta} = P_{\beta\alpha}$ generator,

$$P_{\alpha\beta} = \Gamma^a_{\alpha\beta} P_a + i \Gamma^{a_1 a_2}_{\alpha\beta} Z_{a_1 a_2} + \Gamma^{a_1 \dots a_5}_{\alpha\beta} Z_{a_1 \dots a_5} \ , \tag{2.1.9}$$

in terms of which the (anti)commutation relations of the M Theory superalgebra $\mathfrak{E}^{(528|32)}$ can be written succinctly as

$$\{Q_\alpha, Q_\beta\} = P_{\alpha\beta} \ , \quad [Q_\alpha, P_{\beta\gamma}] = 0 \ . \tag{2.1.10}$$

In terms of the generalized momentum $P_{\alpha\beta}$, these (anti)commutation relations (2.1.10) exhibit a $GL(32,\mathbb{R})$ automorphism symmetry. When the decomposition (2.1.9) is used to write $P_{\alpha\beta}$ in terms of Dirac matrices, the $GL(32,\mathbb{R})$ automorphism symmetry is reduced down to the Lorentz group $SO(1,10)$. In some applications (see chapter 4), it is interesting to consider the maximal automorphism group of the M Theory superalgebra, $GL(32,\mathbb{R})$. For other developments, however, it is convenient to consider the reduced automorphism group $SO(1,10)$ for the M algebra since, after all, the supergravity equations only display a local Lorentz symmetry. The semidirect sum $\mathfrak{E}^{(528|32)} \rtimes so(1,10)$ becomes, then, the counterpart of the superPoincaré algebra $\mathfrak{E}^{(11|32)} \rtimes so(1,10)$ in the presence of additional tensorial central charges. In chapter 5, the M Theory superalgebra with $SO(1,10)$ automorphisms will be revisited in connection with the orthosymplectic superalgebra $osp(1|32)$ and shown to be an *expansion* of this group [4].

The M Theory superalgebra contains complete information about the non-perturbative BPS states of the hypothetical underlying M Theory:

[4] Further extensions are, however, possible in the fermionic sector: see chapter 6.

[5] See [99, 85, 92] for further generalizations of the M Theory superalgebra and for their structure.

the additional bosonic generators Z_{ab}, $Z_{a_1...a_5}$ of the M-algebra (2.1.10) are related to the topological charges [57] of the supermembrane and the super-M5-brane[6] [58] (see also [103]). These 'single brane' BPS states can be associated with $D = 11$ supergravity solutions [64, 75] or with fundamental M Theory objects described by their worldvolume actions [71, 72]. Although the M-algebra (2.1.10) leads naturally to a $D = 11$ Lorentz-covariant interpretation when the splitting (2.1.9) is used, it also allows both for a IIA and a IIB treatment. In the first case, this is allowed because the (relevant) Dirac matrices coincide in ten and eleven dimensions; in the IIB case, a counterpart of equation (2.1.9) [59, 104] can be written if the spinor indices α are split as $\alpha' i$, where $\alpha' = 1, \ldots, 16$ is a $D = 10$ Majorana-Weyl spinor index and $i = 1, 2$. As a result, the information about non-perturbative BPS states of the $D = 10$ superstring theories (including D-branes) can also be extracted from the algebra (2.1.10). This means that the M-algebra also encodes all the duality relations between different $D = 10$ and $D = 11$ superbranes. These facts add further reasons to call (2.1.10) the M Theory superalgebra [59].

To conclude this section, let us write, for future reference, the dual version of the algebras introduced above. It is usually convenient to resort to a dual point of view to deal with Lie algebras, especially to construct lagrangians invariant or quasi-invariant (*i.e.*, invariant up to a total derivative) under the symmetry transformations of the Lie algebra. This dual point of view will be particularly relevant in chapters 5 and 6. Let G be a Lie group with parameters g^i, $i = 1, \ldots, \mathrm{dim}G$, and $\mathcal{G}$ its Lie algebra, generated by the (left-, say) invariant vector fields $X_i(g)$ on the group manifold G, with commutation relations $[X_i, X_j] = c_{ij}^k X_k$. The coalgebra $\mathcal{G}^*$ is then spanned by the dual ($\omega^i(X_j) = \delta^i_j$), left-invariant Maurer-Cartan (MC) one-forms $\omega^i(g)$ on the group manifold G, subject to the Maurer-Cartan equations

$$d\omega^k = -\tfrac{1}{2} c_{ij}^k \omega^i \wedge \omega^j \ , \tag{2.1.11}$$

which contain the same information than the commutation relations in terms of generators X_i. In particular, the Jacobi identities $c_{i[j}^k c_{lm]}^i = 0$ arise from the requirement that the MC equations (2.1.11) be consistent with the nilpotency of the exterior differential, $dd \equiv 0$.

[6] This result was extended in [100] by showing that these generators also contain a contribution from the topological charges of the eleven-dimensional Kaluza-Klein monopole ($Z_{0\mu_1...\mu_4} \propto \epsilon_{0\mu_1...\mu_4\nu_1...\nu_6} \tilde{Z}^{\nu_1...\nu_6}$) and of the M9-brane ($Z_{0\mu} \propto \epsilon_{0\mu\nu_1...\nu_9} \tilde{Z}^{\nu_1...\nu_9}$) which is usually identified with the Hořava-Witten hyperplane [61] (for the Kaluza-Klein monopole and the M9 brane only bosonic actions are known [101, 102]).

Introducing the MC one-forms Π^a, π^α, σ^{ab} dual, respectively, to the generators P_a, Q_α, J_{ab}, the superPoincaré group can be described by the MC equations

$$
\begin{aligned}
&d\Pi^a = \Pi^b \wedge \sigma_b{}^a - i\pi^\alpha \wedge \pi^\beta \Gamma^a_{\alpha\beta} \ , \\
&d\pi^\alpha = \pi^\beta \wedge \sigma_\beta{}^\alpha \quad \left(\sigma_\alpha{}^\beta = \tfrac{1}{4}\sigma^{ab}\Gamma_{ab\,\alpha}{}^\beta\right) \ , \\
&d\sigma^{ab} = \sigma^{ac} \wedge \sigma_c{}^b \ , \qquad (2.1.12)
\end{aligned}
$$

offering a counterpart of the (anti)commutation relations (2.1.1), (2.1.3), (2.1.4). Finally, introducing the MC forms Π^{ab}, $\Pi^{a_1\ldots a_5}$ dual, respectively, to the generators Z_{ab}, $Z_{a_1\ldots a_5}$, the whole set

$$
\Pi^a \ , \ \pi^\alpha \ , \ \Pi^{a_1a_2} \ , \ \Pi^{a_1\ldots a_5} \ , \qquad (2.1.13)
$$

provides, setting aside the automorphisms part, the Maurer-Cartan one-forms of the M Theory superalgebra $\mathfrak{E}^{(528|32)}$, left-invariant on the corresponding group manifold (maximally extended rigid superspace) $\Sigma^{(528|32)}$. The one forms Π^a, Π^{ab}, $\Pi^{a_1\ldots a_5}$ can again be collected into the symmetric spin-tensor one-form

$$
\Pi^{\alpha\beta} = \frac{1}{32}\left(\Pi^a\Gamma_a^{\alpha\beta} - \frac{i}{2}\Pi^{a_1a_2}\Gamma_{a_1a_2}{}^{\alpha\beta} + \frac{1}{5!}\Pi^{a_1\ldots a_5}\Gamma_{a_1\ldots a_5}{}^{\alpha\beta}\right) \ , \qquad (2.1.14)
$$

dual to $P_{\alpha\beta}$, in terms of which the MC equations of the M Theory superalgebra, containing the same information as the (anti)commutation relations (2.1.10), can be written in compact form as

$$
d\Pi^{\alpha\beta} = -i\pi^\alpha \wedge \pi^\beta \ , \quad d\pi^\alpha = 0 \ . \qquad (2.1.15)
$$

2.2 First order action of $D = 11$ supergravity

The field content of N-extended supergravity in D dimensions is determined by the so-called supergravity multiplet, determined by the massless representation of the corresponding superPoincaré algebra containing fields up to helicity two. In particular, for the construction of the eleven-dimensional supergravity action, the central charges Z_{ab}, $Z_{a_1\ldots a_5}$ can be ignored. The fields involved in $D = 11$ supergravity [27] are, specifically, a Lorentzian metric (corresponding to the graviton) $g_{\mu\nu}$, a three-form A_3 and a Majorana Rarita-Schwinger field ψ^α (the gravitino). Actually, the presence of spinor fields makes it necessary to work in the vielbein approach, in which the metric is replaced by a vielbein field e^a_μ in tangent space, satisfying $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$, where η_{ab} is the Minkowski metric. Except in chapter 3, a *mostly minus* signature for the metric will be used throughout this Thesis. In the component approach, these fields are to be regarded as forms on eleven-dimensional spacetime[7] M^{11},

$$\begin{aligned} e^a(x) &= dx^\mu e_\mu^a(x) \ , \\ \psi^\alpha(x) &= dx^\mu \psi_\mu^\alpha(x) \equiv e^a \psi_a^\alpha(x) \ , \\ A_3(x) &= \tfrac{1}{3!}\ dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} A_{\mu_3\mu_2\mu_1}(x) \\ &\equiv \tfrac{1}{3!}\ e^{a_1} \wedge e^{a_2} \wedge e^{a_3} A_{a_3a_2a_1}(x) \ . \end{aligned} \tag{2.2.1}$$

Notice the 'superspace' reverse order convention for the components of the p-forms. The differential d will be taken to act from the right,

$$d\alpha_p = \tfrac{1}{p!} dx^{\mu_p} \wedge \ldots \wedge dx^{\mu_1} \wedge dx^\nu \partial_\nu \alpha_{\mu_1 \ldots \mu_p} \ . \tag{2.2.2}$$

As usual in supersymmetric theories, the number of bosonic and fermionic degrees of freedom match. In $D = 11$, e^a has $\frac{(D-2)(D-1)}{2} - 1 = 44$ on-shell degrees of freedom which, together with the $\binom{D-2}{3} = 84$ on-shell degrees of freedom provided by A_3, makes up 128 bosonic on shell degrees of freedom. That is the same number of on-shell degrees of freedom of fermionic character, associated to the gravitino ψ^α, namely, $\frac{1}{2} 2^{[D/2]}(D - 3) = 128$.

In addition to the forms (2.2.1), the first order action for $D = 11$ supergravity [92, 105],

$$S = \int_{M^{11}} \mathcal{L}_{11}[e^a, \psi^\alpha, A_3, \omega^{ab}, F_4] \ , \tag{2.2.3}$$

contains the Lorentz connection one-form ω^{ab} and the auxiliary four-form[8] F_4,

$$\begin{aligned} \omega^{ab}(x) &= dx^\mu \omega_\mu^{ab}(x) \ , \\ F_4(x) &= \tfrac{1}{4!}\ dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \wedge dx^{\mu_4} F_{\mu_4\mu_3\mu_2\mu_1}(x) \\ &\equiv \tfrac{1}{4!}\ e^{a_1} \wedge e^{a_2} \wedge e^{a_3} \wedge e^{a_4} F_{a_4a_3a_2a_1}(x) \ , \end{aligned} \tag{2.2.4}$$

that must be treated as independent fields in the variational problem, and acquire their usual, second order formalism values when considered on shell (see section 2.4). Notice that the on-shell counting of degrees of freedom coincides in the first and second order formalisms, since the auxiliary fields in the former become, on shell, functions of the fields defining the later.

[7] We shall be concerned with the spacetime component formulation of supergravity. For a review of the superspace formulation of supergravity, see *e.g.* [7].

[8] The first order formulation of [92] involved no four-form F_4 but a tensor zero-form $F_{a_1 \ldots a_4}$. The later can actually be replaced throughout by its contraction with four vielbeins to give an F_4 and, hence, both formulations are equivalent.

The action (2.2.3) is defined on eleven-dimensional spacetime M^{11} and the lagrangian that determines it reads [92, 105]

$$\begin{aligned}\mathcal{L}_{11} = &\tfrac{1}{4}R^{ab}\wedge e^{\wedge 9}_{ab} - D\psi^\alpha\wedge\psi^\beta\wedge\bar{\Gamma}^{(8)}_{\alpha\beta}\\ &+\tfrac{1}{4}\psi^\alpha\wedge\psi^\beta\wedge(T^a+\tfrac{i}{2}\,\psi\wedge\psi\,\Gamma^a)\wedge e_a\wedge\bar{\Gamma}^{(6)}_{\alpha\beta}\\ &+(dA_3-a_4)\wedge(*F_4+b_7)+\tfrac{1}{2}a_4\wedge b_7\\ &-\tfrac{1}{2}F_4\wedge *F_4-\tfrac{1}{3}A_3\wedge dA_3\wedge dA_3\ .\end{aligned}\tag{2.2.5}$$

Both the Riemann tensor and the torsion,

$$R^{ab} := d\omega^{ab} - \omega^{ac}\wedge\omega_c{}^b\ ,\tag{2.2.6}$$
$$T^a := De^a = de^a - e^b\wedge\omega_b{}^a\tag{2.2.7}$$

(where, in the last equation, D is the standard Lorentz covariant derivative) enter the first order lagrangian, the earlier in the Einstein-Hilbert term (the first one in the r.h.s. of (2.2.5)) characteristic of any gravitational lagrangian. Together with the curvature of A_3, these curvatures (2.2.6), (2.2.7) are the basic ingredients of the free differential algebra approach to $D = 11$ supergravity (see chapter 6).

The derivative acting on the gravitino in its kinetic term, the second of (2.2.5), is again the Lorentz covariant derivative,

$$D\psi^\alpha := d\psi^\alpha - \psi^\beta\wedge\omega_\beta{}^\alpha\ ,\tag{2.2.8}$$

now defined in terms of the spin connection,

$$\omega_\beta{}^\alpha = \tfrac{1}{4}\omega^{ab}(\Gamma_{ab})_\beta{}^\alpha\ ,\tag{2.2.9}$$

taking values on $so(1,10)$, the Lie algebra of the double cover of the eleven-dimensional Lorentz group, $Spin(1,10)$, generated by Γ^{ab}.

Following [105] (see also [1, 7]), we have introduced in the lagrangian (2.2.5) the notation

$$a_4 := \tfrac{1}{2}\psi^\alpha\wedge\psi^\beta\wedge\bar{\Gamma}^{(2)}_{\alpha\beta}\ ,\tag{2.2.10}$$
$$b_7 := \tfrac{i}{2}\psi^\alpha\wedge\psi^\beta\wedge\bar{\Gamma}^{(5)}_{\alpha\beta}\ ,\tag{2.2.11}$$

for the bifermionic 4- and 7-forms built up from the gravitino, as well as the compact notation

$$\bar{\Gamma}^{(k)}_{\alpha\beta} := \tfrac{1}{k!}e^{a_k}\wedge\ldots\wedge e^{a_1}\Gamma_{a_1\ldots a_k\alpha\beta}\ .\tag{2.2.12}$$

Finally, $*F_4$ is the Hodge dual of F_4,

$$*F_4 := -\tfrac{1}{4!}e^{\wedge 7}_{a_1\ldots a_4}F^{a_1\ldots a_4}\ ,\tag{2.2.13}$$

and the $(11-k)$-form

$$e^{\wedge(11-k)}_{a_1\ldots a_k} := \tfrac{1}{(11-k)!}\epsilon_{a_1\ldots a_k b_1\ldots b_{11-k}} e^{b_1} \wedge \ldots \wedge e^{b_{11-k}} \tag{2.2.14}$$

has been introduced for convenience[9].

As for the symmetries of the action, it should be noted that the usual general covariance of any gravitational action is implemented in this formalism by using differential forms to write the lagrangian (2.2.5). Local Lorentz symmetry is also straightforwardly implemented in the vielbein approach. The action is also invariant under abelian gauge symmetries of the three-form A_3, and locally supersymmetric, as we now discuss.

2.3 Supersymmetry and generalized holonomy

The action (2.2.3), (2.2.5) is locally supersymmetric, *i.e.* it is invariant under the following local supersymmetry transformations δ_ϵ parameterized by a fermionic $Spin(1,10)$-spinor parameter $\epsilon(x)$:

$$\delta_\epsilon e^a = -2i\psi^\alpha \Gamma^a_{\alpha\beta}\epsilon^\beta \ , \tag{2.3.1}$$

$$\delta_\epsilon \psi^\alpha = \mathcal{D}\epsilon^\alpha(x) \ , \tag{2.3.2}$$

$$\delta_\epsilon A_3 = \psi^\alpha \wedge \bar{\Gamma}^{(2)}_{\alpha\beta}\epsilon^\beta \ , \tag{2.3.3}$$

besides more complicated expressions for $\delta_\epsilon \omega^{ab}$ and $\delta_\epsilon F_{abcd}$, which can be found in [105] and that will not be needed below. Let us stress that, as shown in [105], the supersymmetry transformation rules of the physical fields are the same in the second and in the first order formalisms. The transformations (2.3.1)–(2.3.3) have the usual form expected from supersymmetry: the bosonic fields e^a and A_3 transform into the (only, in this case) fermionic field ψ^α which, in turn, transforms into e^a and A_3 (included, on shell, inside the supercovariant derivative $\mathcal{D}$: see below). Precisely, the transformation (2.3.2) in terms of the supersymmetric covariant derivative $\mathcal{D}$ is characteristic of locally realized supersymmetry, and allows for an interpretation of the gravitino as the *gauge field* of local supersymmetry.

The introduction of the *generalized covariant* (or *supersymmetric covariant*, or *supercovariant*) *derivative* $\mathcal{D}$ allows for a simple expression for the transformation rule (2.3.2) of the gravitino under supersymmetry. It can be written explicitly as

$$\begin{aligned}\delta_\epsilon \psi^\alpha = \mathcal{D}\epsilon^\alpha(x) &:= D\epsilon^\alpha(x) - \epsilon^\beta(x) t_\beta{}^\alpha(x) = \\ &= d\epsilon^\alpha(x) - \epsilon^\beta(x)\Omega_\beta{}^\alpha(x) \ ,\end{aligned} \tag{2.3.4}$$

[9] See [1] for the correspondence of this notation to that of [105].

in terms of the *generalized* (or *supersymmetric*) *connection* one-form

$$\Omega_\beta{}^\alpha = \tfrac{1}{4}\omega^{ab}\Gamma_{ab\beta}{}^\alpha + \tfrac{i}{144}e^a \left(\Gamma_{ab_1b_2b_3b_4\beta}{}^\alpha + 8\ \delta_{a[b_1}\Gamma_{b_2b_3b_4]\beta}{}^\alpha\right) F^{b_1b_2b_3b_4}\ , \tag{2.3.5}$$

that differs from the spin connection $\omega_\alpha{}^\beta = \frac{1}{4}\omega^{ab}\Gamma_{ab\beta}{}^\alpha$ (equation (2.2.9)) by the additional tensor one-form

$$t_\beta{}^\alpha = \tfrac{i}{144}e^a \left(\Gamma_{ab_1b_2b_3b_4\beta}{}^\alpha + 8\ \delta_{a[b_1}\Gamma_{b_2b_3b_4]\beta}{}^\alpha\right) F^{b_1b_2b_3b_4}, \tag{2.3.6}$$

depending on the auxiliary form F_4 (which, on-shell, reduces to the supercovariant field strength of A_3; see equation (2.4.4) below).

A connection one-form takes values on the Lie algebra $\mathcal{G}$ of the *structure group* G of a fiber bundle (see, *e.g.* [40]). The spin connection $\omega_\alpha{}^\beta$, for instance, takes values on the Lie algebra $so(1,10)$ of the structure group $Spin(1,10)$ of the spin bundle on eleven-dimensional spacetime M^{11}. It is, therefore, natural to ask what is the *generalized structure group*, on the Lie algebra of which the generalized connection $\Omega_\alpha{}^\beta$ takes values [78]. To this end, notice that when $F_4 = 0$ then $t_\alpha{}^\beta = 0$, and the generalized connection (2.3.5) reduces to the $so(1,10)$-valued spin connection (2.2.9). But, in general, $F_4 \neq 0$ and $t_\alpha{}^\beta$, as defined in (2.3.6), is non-vanishing. In this case, the presence of additional Dirac matrices makes the generalized connection to take values not on $so(1,10)$, but on the whole $Cl(1,10)_+$ generated by the antisymmetrized products of Dirac matrices in (2.1.6), namely, $\{I_{32}, \Gamma^{[1]}, \Gamma^{[2]}, \Gamma^{[3]}, \Gamma^{[4]}, \Gamma^{[5]}\}$. The dimension of the relevant even part $Cl(1,10)_+$ of the Clifford algebra is

$$\dim Cl(10,1)_+ = \binom{11}{0} + \binom{11}{1} + \binom{11}{2} + \binom{11}{3} + \binom{11}{4} + \binom{11}{5} = 1024\ . \tag{2.3.7}$$

The problem can still be analyzed in terms of Lie algebras, though. In fact, when $Cl(1,10)_+$ is endowed with the usual Lie bracket provided by matrix commutation, $[A,B] = AB - BA$, it coincides with $gl(32,\mathbb{R})$, the Lie algebra of the general linear group $GL(32,\mathbb{R})$, of dimension $\dim\ gl(32,\mathbb{R}) = 32^2 = 1024$. The Lie algebra $so(1,10)$, generated by $\Gamma^{[2]}$, on which the spin connection (2.2.9) takes values, is a subalgebra of $gl(32,\mathbb{R})$. Similarly, one may wonder what is the Lie subalgebra of $gl(32,\mathbb{R})$ on which the generalized connection $\Omega_\alpha{}^\beta$ takes values. An explicit computation reveals that the generators $\{\Gamma^{[2]}, \Gamma^{[3]}, \Gamma^{[5]}\}$ defining $\Omega_\alpha{}^\beta$ in equation (2.3.5) do not close into a Lie algebra by themselves, and that the presence of $\{\Gamma^{[1]}, \Gamma^{[4]}\}$ (not that of I_{32}) is also required to ensure closure under commutation [87]. In conclusion, the generalized connection $\Omega_\alpha{}^\beta$ takes values on the 1023–dimensional Lie subalgebra of $gl(32,\mathbb{R})$ spanned by its traceless generators,

$$\{\Gamma^{[1]},\Gamma^{[2]},\Gamma^{[3]},\Gamma^{[4]},\Gamma^{[5]}\}\ . \tag{2.3.8}$$

These are the generators of $sl(32,\mathbb{R})$, the Lie algebra of $SL(32,\mathbb{R})$ which is, therefore, to be interpreted as the *generalized structure group* of $D=11$ supergravity [87].

A remark about terminology is now in order. The local symmetry of $D=11$ supergravity is not $SL(32,\mathbb{R})$; as mentioned in the previous section, it is only $SO(1,10)$. In this sense, the generalized connection $\Omega_\alpha{}^\beta$ as defined by equation (2.3.5) may be said not to be a bona fide connection. However, it reduces to the spin connection when $F_4=0$ and, in a sense, generalizes it when F_4 is non-vanishing. Moreover, the role played by the Riemannian holonomy of the spin connection $\omega_\alpha{}^\beta$ in the classification of purely geometrical supersymmetric bosonic solutions of supergravity (for which the metric is the only non-vanishing field) can be taken over by its generalized counterpart $\Omega_\alpha{}^\beta$ when F_4 is turned on [78, 73, 87] (see chapter 3). This adds further reasons to call $\Omega_\alpha{}^\beta$ generalized connection.

Pushing this analogy further, the curvature two-form of the generalized connection $\Omega_\alpha{}^\beta$,

$$\begin{aligned}\mathcal{R}_\beta{}^\alpha &:= d\Omega_\beta{}^\alpha-\Omega_\beta{}^\gamma\wedge\Omega_\gamma{}^\alpha\\ &= \tfrac{1}{4}R^{ab}(\Gamma_{ab})_\alpha{}^\beta+Dt_\alpha{}^\beta-t_\alpha{}^\gamma\wedge t_\gamma{}^\beta\ ,\end{aligned} \tag{2.3.9}$$

can be introduced, and consequently referred to as the *generalized curvature*[10]. In general, the curvature two-form of a connection w takes values on a subalgebra $\mathcal{H}\equiv\mathrm{hol}(w)$ of the Lie algebra $\mathcal{G}$ of the structure group G. The corresponding group $H\equiv\mathrm{Hol}(w)$ is a subgroup of G and is called the *holonomy group* (of the connection w); its corresponding Lie algebra $\mathrm{hol}(w)$ will sometimes be called the holonomy algebra. Accordingly, the *generalized holonomy group* $\mathrm{Hol}(\Omega)$ [78] (see also [73, 87, 94, 107, 108, 109, 110, 95, 2, 3][11]) is the subgroup of $SL(32,\mathbb{R})$ on the Lie algebra of which the generalized curvature $\mathcal{R}_\alpha{}^\beta$ takes values. In general, however, the curvature *at a point* does not determine completely the Lie algebra of the holonomy group, but its successive covariant derivatives are needed to determine it (see, *e.g.* [112]). Generalized holonomy is no exception [2], as shown in chapter 3. See also section 3.1 for the

[10] A full expression for the generalized curvature $\mathcal{R}_\alpha{}^\beta$ corresponding to purely bosonic solutions of CJS supergravity can be found in [106, 80].

[11] See [111] for the role of generalized holonomy when vanishing F_4 is considered but higher order corrections to the supergravity equations of motion are taken into account.

role of generalized holonomy in the determination of the number of supersymmetries preserved by a bosonic solution of supergravity.

2.4 Equations of motion

Algebraic equations

Let us return to the analysis of the first order action of $D = 11$ supergravity, to obtain the equations of motion. The variations of the action (2.2.3), (2.2.5) with respect to the Lorentz connection ω^{ab} and the auxiliary four-form F_4 give algebraic constraints that can be used to define these auxiliary fields in terms of those of the supergravity multiplet (2.2.1). Indeed, from the variation of (2.2.3), (2.2.5) with respect to the Lorentz connection,

$$\frac{\delta S}{\delta \omega^{ab}} = \tfrac{1}{4} e^{\wedge 8}_{abc} \wedge (T^c + i\psi^\alpha \wedge \psi^\beta \, \Gamma^c_{\alpha\beta}) = 0 \ , \tag{2.4.1}$$

the torsion is seen to be given by

$$T^a = -i\psi^\alpha \wedge \psi^\beta \, \Gamma^a_{\alpha\beta} \ , \tag{2.4.2}$$

which, upon use of its definition (2.2.7), gives an algebraic equation for the Lorentz connection ω^{ab}, which allows us to solve for it in terms of the vielbein and the gravitino.

On the other hand, the variation of the action with respect to F_4,

$$\begin{aligned} \delta_F S = & \int_{M^{11}} (dA_3 - a_4 - F_4) \wedge *\delta F_4 = \\ & = -\tfrac{1}{4!} \int_{M^{11}} (dA_3 - a_4 - F_4) \wedge e^{\wedge 7}_{a_1 \ldots a_4} \, \delta F^{a_1 \ldots a_4} \ , \end{aligned} \tag{2.4.3}$$

produces an algebraic equation of motion, $\delta S / \delta F_4 = 0$, that makes of F_4 the 'supersymmetric' field strength of A_3,

$$dA_3 = a_4 + F_4 \ . \tag{2.4.4}$$

Making use of the expressions (2.4.2) and (2.4.4) for the torsion T^a and four-form F_4 into the first order lagrangian (2.2.5), the original second order CJS lagrangian [27] is recovered.

Dynamical equations

The variation of (2.2.3) with respect to the rest of the fields yields the dynamical equations of motion: the Einstein equations (arising from the variation with respect to e^a), the (generalization of the Maxwell) equation for A_3 and the Rarita-Schwinger equation for the gravitino ψ^α.

The explicit form of the Einstein equations,

$$M_{10\,a} := R^{bc} \wedge e^{\wedge 8}_{abc} + \ldots = 0 \;, \tag{2.4.5}$$

will not be needed in the remainder, so we will not be concerned with it[12]. The variation of the action with respect to the three-form A_3,

$$\delta_A S = \int_{M^{11}} \mathcal{G}_8 \wedge \delta A_3 \;, \qquad \frac{\delta S}{\delta A_3} := \mathcal{G}_8 \;, \tag{2.4.6}$$

gives the eight-form

$$\mathcal{G}_8 = d(*F_4 + b_7 - A_3 \wedge dA_3) \;, \tag{2.4.7}$$

and thus the equation of motion of A_3 is

$$\mathcal{G}_8 = d(*F_4 + b_7 - A_3 \wedge dA_3) = 0 \;. \tag{2.4.8}$$

Finally, the fermionic variation of the lagrangian (2.2.5) reads (*cf.* [105])

$$\begin{aligned} \delta_\psi \mathcal{L}_{11} \;=\; & -2\mathcal{D}\psi^\alpha \wedge \bar{\Gamma}^{(8)}_{\alpha\beta} \wedge \delta\psi^\beta + i(dA_3 - a_4 - F_4) \wedge \bar{\Gamma}^{(5)}_{\alpha\beta} \wedge \psi^\alpha \wedge \delta\psi^\beta \\ & + \left(i_a \bar{\Gamma}^{(8)}_{\alpha\beta} + \tfrac{1}{2} e_a \wedge \bar{\Gamma}^{(6)}_{\alpha\beta} \right) \wedge (T^a + i\psi^\alpha \wedge \psi^\beta\, \Gamma^a_{\alpha\beta}) \wedge \psi^\alpha \wedge \delta\psi^\beta \\ & - d\, \left[\psi^\alpha \wedge \bar{\Gamma}^{(8)}_{\alpha\beta} \wedge \delta\psi^\beta \right] \;, \end{aligned} \tag{2.4.9}$$

where i_a is defined by $i_a e^b = \delta^b_a$ so that, for $\alpha_p = \frac{1}{p!} e^{a_p} \wedge \ldots \wedge e^{a_1} \alpha_{a_1 \ldots a_p}$,

$$i_a \alpha_p = \tfrac{1}{(p-1)!} e^{a_p} \wedge \ldots \wedge e^{a_2} \alpha_{a a_2 \ldots a_p} \quad . \tag{2.4.10}$$

Imposing the algebraic constraints (2.4.2) and (2.4.4) and ignoring the total derivative term, equation (2.4.9) gives the gravitino equation of [27] written, as in [105], in differential form,

$$\Psi_{10\,\beta} := \mathcal{D}\psi^\alpha \wedge \bar{\Gamma}^{(8)}_{\alpha\beta} = 0 \;, \tag{2.4.11}$$

in terms of the supercovariant derivative

$$\mathcal{D}\psi^\alpha := d\psi^\alpha - \psi^\beta \wedge \Omega_\beta{}^\alpha \equiv D\psi^\alpha - \psi^\beta \wedge t_\beta{}^\alpha \;, \tag{2.4.12}$$

defined for the generalized connection (2.3.5).

---

[12] See [105] for the explicit expression of the Einstein equation in this formalism.

2.5 The purely bosonic limit

For many applications, it is interesting to consider the purely bosonic limit of $D = 11$ supergravity in which the gravitino vanishes, $\psi^\alpha = 0$. A torsion-free spacetime is then recovered (see equation (2.4.2)), described by the Einstein equations (2.4.5) which, in this limit, reduce to

$$E_{ab} := \mathrm{Ric}_{ab} - \tfrac{1}{3} F_{ac_1c_2c_3} F_b{}^{c_1c_2c_3} + \tfrac{1}{36}\eta_{ab} F_{c_1c_2c_3c_4} F^{c_1c_2c_3c_4} = 0 \ , \quad (2.5.1)$$

where Ric_{ab} is the Ricci tensor. The equation of motion (2.4.8) of A_3 reduces to

$$\mathcal{G}_8 = d * F_4 - F_4 \wedge F_4 = 0 \ . \quad (2.5.2)$$

The four-form F_4 that enters both (2.5.1) and (2.5.2) reduces, by virtue of the algebraic constraint (2.4.4), to the field strength of A_3, $F_4 = dA_3$; consequently, it is subject to the Bianchi identity

$$dF_4 \equiv 0 \ . \quad (2.5.3)$$

Interestingly enough, these bosonic equations are encoded in the generalized curvature $\mathcal{R}_\alpha{}^\beta$ of the generalized connection $\Omega_\alpha{}^\beta$, now still given by equations (2.3.9) and (2.3.5), respectively, but setting $\psi^\alpha = 0$ (and, consequently, $F_4 = dA_3$) in them. With these restrictions, $\mathcal{R}_\alpha{}^\beta$ obeys [80, 3]

$$\begin{aligned} \mathcal{N}_{a\,\beta}{}^\alpha := i_a \mathcal{R}_\alpha{}^\gamma \Gamma^a{}_\gamma{}^\beta = & -\tfrac{1}{4} e^b R_{b[c_1c_2c_3]} \Gamma^{c_1c_2c_3}{}_\alpha{}^\beta + \tfrac{1}{2} e^a E_{ab} \Gamma^b{}_\alpha{}^\beta \\ & + \tfrac{i}{36} e^a \, [*\mathcal{G}_8]_{b_1b_2b_3} (\Gamma_a{}^{b_1b_2b_3} + 6\delta_a^{[b_1}\Gamma^{b_2b_3]})_\alpha{}^\beta \\ & + \tfrac{i}{720} \, e^a \, [dF_4]_{b_1\ldots b_5} \, (\Gamma_a{}^{b_1\ldots b_5} + 10\delta_a^{[b_1}\Gamma^{b_2\ldots b_5]})_\alpha{}^\beta \ , \end{aligned} \quad (2.5.4)$$

where E_{ab}, $\mathcal{G}_8$ are the r.h.s's of the Einstein and the gauge field equations as defined in (2.5.1), (2.5.2) and i_a is defined in (2.4.10); in particular, $i_a \mathcal{R}_\alpha{}^\beta = e^b \mathcal{R}_{ab\alpha}{}^\beta$. The equality (2.5.4) implies that the set of the free bosonic equations for CJS supergravity, (2.5.1), (2.5.2), (2.5.3), is equivalent to the following simple equation for the generalized curvature (2.3.9), $e^b \mathcal{R}_{ab\alpha}{}^\gamma \Gamma^a{}_\gamma{}^\beta = 0$, or

$$i_a \mathcal{R}_\alpha{}^\gamma \, \Gamma^a{}_\gamma{}^\beta = 0 \ , \quad (2.5.5)$$

since the r.h.s. of equation (2.5.4) is zero on account of the equations of motion (2.5.1), (2.5.2) and the Bianchi identities for F_4 (equation (2.5.3)) and for the Riemann tensor, $R_{b[c_1c_2c_3]} = 0$.

Especially relevant are those solutions of the purely bosonic equations (2.5.1), (2.5.2), (2.5.3) of $D = 11$ supergravity preserving some supersymmetry (see section 3.1 of chapter 3 for the conditions that a purely bosonic supergravity solution must meet in order to preserve supersymmetry). In eleven dimensions, supergravity displays the maximum amount, 32, of supersymmetries permitted (see section 2.1) and hence, a supersymmetric bosonic solution of $D = 11$ supergravity preserves a number k of supersymmetries between 1 and 32. A supersymmetric solution can be referred to by the fraction of preserved supersymmetry as a $\nu = k/32$ solution.

The $\nu = 1/2$-supersymmetric solutions are usually regarded as the basic solutions of $D = 11$ supergravity[13]. These are the M-wave [77], the Kaluza-Klein monopole [113, 114, 100] and the elementary brane solutions, namely, the M2-brane [73] and the M5-brane [74] (the existence of an M9-brane has been conjectured in [102, 100]). See also [64, 75] and references therein. The M2-brane solution[14] solves the Einstein equation $E_{ab} = \mathcal{T}_{ab} - \frac{1}{9}\eta_{ab}\mathcal{T}_c{}^c$ with a singular energy-momentum tensor density source $\mathcal{T}_{ab} \propto \delta^3(x - \hat{x}(\xi))$ ($\hat{x}(\xi)$ being pulled-backed on the M2-brane worldvolume, parameterized by coordinates ξ). The gauge field equation also receives a singular source contribution J_8 in the r.h.s., $\mathcal{G}_8 = J_8$, similar to that of the electric current to the r.h.s. of Maxwell equations. In this sense, the M2-brane carries a supergravity counterpart of the electric charge in Maxwell electrodynamics (see [115] for a discussion). The other basic $\nu = 1/2$ brane solution of $D = 11$ supergravity, the M5-brane, is a counterpart of the Dirac monopole, *i.e.* of the magnetically charged particle. It is characterized by a modification of the Bianchi identities (equation (2.5.3)) with the analogue of a magnetic current in the r.h.s., $dF_4 = \mathcal{J}_5$.

Intersecting branes preserve less than one-half of the maximum supersymmetry, *i.e.*, they correspond to $\nu < 1/2$ supergravity solutions [76]. On the other hand, there also exist maximally supersymmetric solutions ($\nu = 1$) preserving, thus, all 32 supersymmetries. Four solutions exhaust the complete list of $\nu = 1$ solutions of $D = 11$ supergravity [109]: eleven-dimensional Minkowski space, the compactifications $AdS_4 \times S^7$, $AdS_7 \times S^4$ [32, 116] on round-spheres and the pp-wave of [117]. In spite of the fact that the supersymmetry algebra allowed, in principle, for all the fractions $\nu = k/32$, $k = 1, \ldots, 32$, to be preserved [118], no explicit solutions (other than those maximally supersymmetric) preserving more

[13] On the other hand, states with $\nu = 31/32$ (*BPS preons* [83]) can be argued, on purely algebraic grounds, to be fundamental in M Theory: see sec. 4.1 of chapter 4.

[14] See equations (3.3.10) and (3.3.1) of chapter 3 for the expressions of the metric and four-form corresponding, respectively, to the M2- and M5-brane solutions of $D = 11$ supergravity; that section, 3.3, also discusses the generalized holonomy of these brane solutions.

that $\nu = 1/2$ were known for some time. Solutions with extra supersymmetry were found indeed as pp-waves [119, 120, 121, 122] or Gödel universes [123, 124], preserving $k = 18, 20, 22, 24, 26$ and (in IIB supergravity) 28 supersymmetries.

Since the supergravity multiplet is the only one without higher spin fields in $D = 11$, no usual field-theoretical matter contribution to the r.h.s.'s of the equations of motion (2.5.1), (2.5.2), (2.5.3) may appear. Modifications to the equations might arise, however, not only due to the presence of the branes just mentioned, but also if higher order corrections to the curvature [125, 126, 127, 111] (a counterpart of the string α' corrections [128] in $D = 10$) are taken into account. These corrections should have an M Theoretical interpretation.

2.6 Equations of motion and generalized curvature

Let us now return to the general case of non-vanishing gravitino, $\psi^\alpha \neq 0$, and show that there exists a counterpart of equation (2.5.5) collecting the equations of motion of the bosonic fields in terms of the generalized curvature [1]. The gravitino equation of motion (2.4.11), $\Psi_{10\,\beta} = 0$, is expressed in terms of the supercovariant derivative $\mathcal{D}$ of ψ^α (equation (2.4.12)), defined in terms of the generalized connection ${\Omega_\alpha}^\beta$ (equations (2.3.5), (2.3.6)). As a result, the integrability/selfconsistency condition for equation (2.4.11) may be written in terms of the generalized curvature ${\mathcal{R}_\alpha}^\beta$ of equation (2.3.9). Using $\mathcal{D}\mathcal{D}\psi^\alpha = -\psi^\beta \wedge {\mathcal{R}_\beta}^\alpha$ and[15] ${t_{1[\beta}}^\gamma \wedge {\bar{\Gamma}^{(8)}}_{\alpha]\gamma} = 0$ which implies $\mathcal{D}\bar{\Gamma}^{(8)}_{\beta\alpha} = D\bar{\Gamma}^{(8)}_{\beta\alpha} = T^a \wedge i_a \bar{\Gamma}^{(8)}_{\beta\alpha}$, we obtain

$$\mathcal{D}\Psi_{10\,\alpha} = \mathcal{D}\psi^\beta \wedge (T^a + i\psi \wedge \psi\Gamma^a) \wedge i_a \bar{\Gamma}^{(8)}_{\beta\alpha} - \\ -\tfrac{i}{6}\psi^\beta \wedge \left[{\mathcal{R}_\beta}^\gamma \wedge e^{\wedge 8}_{abc}\Gamma^{abc}_{\gamma\alpha} + i\mathcal{D}\psi^\delta \wedge \psi^\gamma \wedge e^{\wedge 7}_{a_1\ldots a_4}\Gamma^{[a_1a_2a_3}_{\delta\alpha}\Gamma^{a_4]}_{\beta\gamma} \right] = 0\,. \tag{2.6.1}$$

The first term in the second part of equation (2.6.1) vanishes due to the algebraic constraint (2.4.2). Hence on the surface of constraints, the selfconsistency of the gravitino equation is guaranteed when [1]

$$\mathcal{M}_{10\,\alpha\beta} := {\mathcal{R}_\beta}^\gamma \wedge e^{\wedge 8}_{abc}\Gamma^{abc}_{\gamma\alpha} + i\mathcal{D}\psi^\delta \wedge \psi^\gamma \wedge e^{\wedge 7}_{a_1\ldots a_4}\Gamma^{[a_1a_2a_3}_{\delta\alpha}\Gamma^{a_4]}_{\beta\gamma} = 0\,. \tag{2.6.2}$$

As it will now be shown, *equation (2.6.2) collects all the equations of motion of the bosonic fields, (2.4.5), (2.4.8), and the corresponding Bianchi identities for the A_3 gauge field and for the Riemann curvature tensor* [1]. Equation (2.6.2) is, thus, the counterpart of equation (2.5.5) when the

[15] This follows from direct calculation: ${t_{1\,\alpha}}^\gamma \wedge \bar{\Gamma}^{(8)}_{\gamma\beta} = -\frac{i}{2}F_4 \wedge \bar{\Gamma}^{(5)}_{\alpha\beta} + \frac{1}{2} * F_4 \wedge \bar{\Gamma}^{(2)}_{\alpha\beta}$.

gravitino is non-vanishing. Let us stress that we distinguish between the algebraic equations or constraints (equations (2.4.2) and (2.4.4)) from the true dynamical equations ((2.4.5), (2.4.8)) and that our statement above refers to the dynamical equations; thus it is also true for the second order formalism.

To show this it is sufficient to use the second Noether theorem and/or the fact that the purely bosonic limit of (2.6.2) implies equation (2.5.5), which is equivalent to the set of all bosonic equations and Bianchi identities when $\psi^\alpha = 0$. According to the second Noether theorem, the local supersymmetry under (2.3.1)–(2.3.3) reflects (and is reflected by) the existence of an interdependence among the bosonic and fermionic equations of motion; such a relation is called a Noether identity. Furthermore, since the local supersymmetry variation of the gravitino (2.3.2) is given by the supercovariant derivative $\mathcal{D}\epsilon^\alpha$, the gravitino equation Ψ^α should enter the corresponding Noether identity through $\mathcal{D}\Psi^\alpha$. Thus, $\mathcal{D}\Psi^\alpha$ should be expressed in terms of the equations of motion for the bosonic fields, in our case including the algebraic equations for the auxiliary fields. Hence, due to the equations (2.6.1), (2.4.2), the l.h.s. of equation (2.6.2) vanishes when *all* the bosonic equations are taken into account.

Indeed, schematically, ignoring for simplicity the purely algebraic equations and neglecting the boundary contributions, the variation of the action (2.2.3), (2.2.5) (considered now in the second order formalism) reads

$$\delta S = \int_{M^{11}} (-2\Psi_{10\,\alpha} \wedge \delta\psi^\alpha + \mathcal{G}_8 \wedge \delta A_3 + M_{10\,a} \wedge \delta e^a) \ . \tag{2.6.3}$$

For the local supersymmetry transformations δ_ϵ, equations (2.3.1)–(2.3.3), one finds, integrating by parts

$$\begin{aligned} \delta_\epsilon S &= \int_{M^{11}} (-2\Psi_{10\,\alpha} \wedge \mathcal{D}\epsilon^\alpha + \mathcal{G}_8 \wedge \delta_\epsilon A_3 + M_{10\,a} \wedge \delta_\epsilon e^a) = \\ &= -\int_{M^{11}} (-2\mathcal{D}\Psi_{10\,\alpha} - \mathcal{G}_8 \wedge \psi^\beta \wedge \bar{\Gamma}^{(2)}_{\beta\alpha} + 2iM_{10\,a} \wedge \psi^\beta \Gamma^a_{\beta\alpha})\, \epsilon^\alpha = 0\,. \end{aligned} \tag{2.6.4}$$

Since $\delta_\epsilon S = 0$ is satisfied for an arbitrary fermionic function $\epsilon^\alpha(x)$, it follows that

$$\mathcal{D}\Psi_{10\,\alpha} = -\tfrac{1}{2}\psi^\beta \wedge \left(-2i\Gamma^a_{\beta\alpha} M_{10\,a} + \mathcal{G}_8 \wedge \bar{\Gamma}^{(2)}_{\beta\alpha}\right) \ . \tag{2.6.5}$$

By virtue of equations (2.6.1) and (2.6.5), and after the algebraic equations (2.4.2), (2.4.4) are taken into account,

$$\mathcal{M}_{10\ \alpha\beta} := \mathcal{R}_\beta{}^\gamma \wedge e^{\wedge 8}_{abc} \Gamma^{abc}_{\gamma\alpha} + i\mathcal{D}\psi^\delta \wedge \psi^\gamma \wedge e^{\wedge 7}_{a_1 \ldots a_4} \Gamma^{[a_1 a_2 a_3}_{\delta\alpha} \Gamma^{a_4]}_{\beta\gamma} =$$

$$= -3i \left(-2i\Gamma^a_{\beta\alpha} M_{10\,a} + \mathcal{G}_8 \wedge \bar{\Gamma}^{(2)}_{\beta\alpha} \right) . \qquad (2.6.6)$$

It then follows that the equation of motion for the bosonic fields (2.6.2), $\mathcal{M}_{10\;\alpha\beta} = 0$, is satisfied,

$$\mathcal{R}_\beta{}^\gamma \wedge e^{\wedge 8}_{abc} \Gamma^{abc}_{\gamma\alpha} = -i\mathcal{D}\psi^\delta \wedge \psi^\gamma \wedge e^{\wedge 7}_{a_1 \ldots a_4} \Gamma^{[a_1 a_2 a_3}_{\delta\alpha} \Gamma^{a_4]}_{\beta\gamma} , \qquad (2.6.7)$$

after the dynamical equations (2.4.5), (2.4.8) are used. Getting rid of the vielbein forms, the equation (2.6.2) (or (2.6.7)) can be written in terms of the components $\mathcal{R}_{ab\alpha}{}^\beta$, $(\mathcal{D}\psi)_{ab}{}^\alpha$ of the two-forms $\mathcal{R}_\alpha{}^\beta$, $\mathcal{D}\psi^\alpha$,

$$\mathcal{R}_\alpha{}^\beta = \tfrac{1}{2} e^b \wedge e^a \mathcal{R}_{ab\alpha}{}^\beta , \qquad (2.6.8)$$

$$\mathcal{D}\psi^\alpha = \tfrac{1}{2} e^b \wedge e^a (\mathcal{D}\psi)_{ab}{}^\alpha , \qquad (2.6.9)$$

as

$$\mathcal{R}_{bc\alpha}{}^\gamma \Gamma^{abc}{}_{\gamma\beta} = 4i((\mathcal{D}\psi)_{bc} \Gamma^{[abc})_\beta \, (\psi_d \Gamma^{d]})_\alpha . \qquad (2.6.10)$$

Equation (2.6.6) also shows what Lorentz-irreducible parts of the concise bosonic equations $\mathcal{M}_{10\;\alpha\beta} = 0$ coincide with the Einstein and with the 3-form gauge field equations. These are given, respectively, by

$$M_{10\,a} = -\tfrac{1}{192} \mathrm{tr}(\Gamma_a \mathcal{M}_{10}) , \qquad (2.6.11)$$

$$\mathcal{G}_8 \wedge e^a \wedge e^b = \tfrac{i}{96} \mathrm{tr}(\Gamma^{ab} \mathcal{M}_{10}) . \qquad (2.6.12)$$

All other Lorentz-irreducible parts in equation (2.6.2), $\mathcal{M}_{10\;\alpha\beta} = 0$, are satisfied either identically or due to the Bianchi identities that are the integrability conditions for the algebraic equations (2.4.2), (2.4.4) used in the derivation of (2.6.6).

In conclusion, we have proven that equation (2.6.2) collects all the dynamical bosonic equations of motion in the second order approach to supergravity. To see that it collects all the Bianchi identities as well, one may either perform a direct calculation or study its purely bosonic limit. The latter way is simpler and it also provides an alternative proof of the above statement as we now show.

For bosonic configurations, $\psi^\alpha = 0$, equation (2.6.2) reduces to

$$\mathcal{R}_\beta{}^\gamma \wedge e^{\wedge 8}_{abc} \Gamma^{abc}_{\gamma\alpha} = 0 . \qquad (2.6.13)$$

Decomposing $\mathcal{R}_\alpha{}^\beta$ on the vielbein basis as in (2.6.8), equation (2.6.13) implies

$$\mathcal{R}_{ab\;\beta}{}^\gamma \Gamma^{abc}_{\gamma\alpha} = 0 . \qquad (2.6.14)$$

Contracting (2.6.14) with $\Gamma^{\alpha\delta}_c$ one finds

$$\mathcal{R}_{ab\;\beta}{}^\gamma \Gamma^{ab}{}_\gamma{}^\delta = 0 . \qquad (2.6.15)$$

Then, contracting again with the Dirac matrix $\Gamma_d^{\alpha\delta}$ and using $\Gamma^{ab}\Gamma_d = \Gamma^{ab}{}_d + 2\Gamma^{[a}\delta_d{}^{b]}$ as well as equation (2.6.14), one recovers equation (2.5.4), $\mathcal{N}_{a\beta}{}^{\alpha} = 0$, namely,

$$i_a \mathcal{R}_\beta{}^\gamma \Gamma^{a\,\alpha}_\gamma \equiv e^b \mathcal{R}_{ab\beta}{}^\gamma \Gamma^{a\,\alpha}_\gamma = 0 \, . \tag{2.6.16}$$

Since (2.6.16) collects all the bosonic equations of $D = 11$ CJS supergravity as well as all the Bianchi identities in the purely bosonic limit [80, 3], $\psi^\alpha = 0$, the equivalence of equations (2.6.16) and (2.6.13) will imply that $\mathcal{M}_{10\ \alpha\beta} = 0$, equation (2.6.2), does the same for the case of non-vanishing fermions, $\psi^\alpha \neq 0$ [1].

The Bianchi identities $R_{a[bcd]} \equiv 0$ and $dF_4 \equiv 0$ appear as the irreducible parts $\mathrm{tr}(\Gamma_{c_1c_2c_3}\mathcal{N}_a)$ and $\mathrm{tr}(\Gamma_{c_1\ldots c_5}\mathcal{N}_a)$ of equation (2.5.4) [1]; more precisely, in the later case the relevant part in $\mathcal{N}_a$ is proportional to $[dF_4]_{b_1\ldots b_5}(\Gamma_a{}^{b_1\ldots b_5} + 10\delta_a{}^{[b_1}\Gamma^{b_2\ldots b_5]})$, but the two terms in the brackets are independent. Knowing this, one may also reproduce the terms that include the Bianchi identities in the concise equation (2.6.7) (equivalent to (2.6.10) or (2.6.6)) with a non-vanishing gravitino.

3

Subtleties about generalized holonomy

The generalized holonomy of some solutions of eleven-dimensional supergravity is reviewed in this chapter. It is done by paying particular attention to a feature of holonomy already mentioned in section 2.3, namely, that covariant derivatives of the curvature might be needed to define the Lie algebra of the holonomy group. In section 3.1, the supersymmetry transformations discussed in general in chapter 2 are particularized for purely bosonic solutions of supergravity. The Killing spinor equation that results as a consistency condition from the vanishing of the gravitino variation is presented and the usefulness of the integrability conditions of the equation exhibited. These (first order) integrability conditions are related to the generalized curvature. In section 3.2 it is argued that ordinary, first order integrability is in general not enough to characterize the holonomy, and that iterated commutators of the supercovariant derivatives may be needed to properly define the holonomy algebra.

To check for possible consequences of the higher order integrability conditions, the generalized holonomy of the usual M-branes is reviewed in section 3.3. It is found that, in these cases, successive commutators of the supercovariant derivatives only help to close the algebra obtained at first order (the curvature algebra) and that successive commutators do not add significant information. The situation is, however, different for other supergravity solutions: as section 3.4 shows, second order integrability conditions are necessary to compute the generalized holonomy of Freund-Rubin compactifications. Knowledge of the embedding of the generalized holonomy group in the generalized structure group is, moreover, needed to determine correctly the number of preserved supersymmetries. Some details are relegated to Appendix A.

This chapter follows closely reference [2], and uses the conventions therein. In particular, we temporarily resort to a mostly plus metric g_{MN}, $M, N, \ldots$, denoting eleven-dimensional spacetime indices ($\mu, \nu, \ldots$, and $a, b, \ldots$, will be reserved for lower dimensions) . With these conventions, the generalized connection (2.3.5) will be denoted Ω_M and its associated supercovariant derivative will act from the left and will be defined as

$$\mathcal{D}_M \equiv \partial_M + \tfrac{1}{4}\Omega_M = D_M - \tfrac{1}{288}({\Gamma_M}^{NPQR} - 8\delta_M^N \Gamma^{PQR})F_{NPQR}, \quad (3.0.1)$$

where D_M denotes the Lévi-Civita covariant derivative associated to the spin connection ω_M. The purely bosonic equations of motion (2.5.1) and (2.5.2) will read in this chapter:

$$\mathrm{Ric}_{MN} = \tfrac{1}{12}\left(F_{MPQR}{F_N}^{PQR} - \tfrac{1}{12}g_{MN}F^{PQRS}F_{PQRS}\right), \quad (3.0.2)$$

$$d * F_4 + \tfrac{1}{2}F_4 \wedge F_4 = 0. \quad (3.0.3)$$

Spinor indices will be omitted and derivatives will act from the left.

3.1 Killing spinors, holonomy and supersymmetry

For purely bosonic supergravity solutions, $\psi_M = 0$, the supersymmetry transformations simplify considerably. The bosonic fields, e^a and A_3, of such a solution are clearly invariant under supersymmetry,

$$\delta_\epsilon e^a = 0\ , \quad (3.1.1)$$

$$\delta_\epsilon A_3 = 0\ , \quad (3.1.2)$$

since their transformation rules, (2.3.1) and (2.3.3), respectively, are proportional to a vanishing gravitino. On the other hand, the invariance of the bosonic solution under supersymmetry implies, in particular, that the solution cannot change its bosonic character after the transformation, *i.e.*, that no gravitino is generated by the transformation. This amounts to requiring that the variation (2.3.2) of the gravitino under supersymmetry also vanishes. Namely, with the convention of (3.0.1),

$$\delta_\epsilon \psi_M \equiv \mathcal{D}_M \epsilon = 0\ . \quad (3.1.3)$$

It should be remarked that the expression (3.1.3) is not an identity, since the non-trivial character of the transformation of the gravitino, equation (2.3.2), will not allow for it to be identically satisfied for any spinor field ϵ. Equation (3.1.3) is, instead, a consistency requirement and only the spinors ϵ solving the equation will parameterize unbroken supersymmetries. The equation (3.1.3) is usually called *Killing spinor equation*, and its solutions, *Killing spinors*. The number k of preserved supersymmetries of a bosonic supergravity solution is, thus, given by the number of Killing spinors[1] ϵ_J, $J = 1, \ldots, k$.

[1] In lower dimensional supergravities, or in compactifications of $D = 11$ super-

In a fiber bundle, the notions of constancy with respect to the covariant derivative, invariance under parallel transport and invariance under the holonomy group come down to the same thing (see, *e.g.* [112]): in fact, the holonomy group is a measure of how vectors and tensors on the fiber transform under parallel transport around a closed loop at a point. Let us momentarily set $F_4 = 0$, so that, since we are dealing with bosonic supergravity solutions ($\psi_M = 0$), the only non-vanishing field is the metric; these configurations therefore correspond to purely geometrical solutions, to which the results of Riemannian geometry can be applied. In this case, the supercovariant derivative $\mathcal{D}_M$ (equation (3.0.1)) acting on spinors reduces to the covariant derivative associated to the Lévi-Civita-induced spin connection D_M (see (2.2.9)) taking values on the Lie algebra $so(1,10)$ of the tangent space group $SO(1,10)$ (the structure group). The Killing spinor equation (3.1.3) accordingly reduces to

$$D_M \epsilon = 0 \; . \tag{3.1.4}$$

Unbroken supersymmetries of purely geometrical supergravity solutions are, thus, parameterized by spinors parallel with respect to the spin connection (that is, satisfying (3.1.4)). Riemannian holonomy controls in this case the number of solutions to the equation (3.1.4) and, consequently, the number of preserved supersymmetries: solutions to (3.1.4) exist if, and only if, the spinor representation **32** of the structure group $SO(1,10)$, to which the spinor ϵ belongs, is not only reducible under the Riemannian holonomy group $\mathrm{Hol}(\omega)$, but also the identity representation arises in the decomposition of the **32** of $SO(1,10)$ under $\mathrm{Hol}(\omega)$. The number k of times that the identity shows up in this decomposition (*i.e.*, the number of *singlets* in this decomposition) corresponds to the number of invariant spinors ϵ_J, $J = 1, \ldots, k$, under the action of $\mathrm{Hol}(\omega)$. These are the spinors invariant under parallel transport and, thus, satisfying equation (3.1.4).

A heuristic argument can be given to support this result. A simpler equation for the parallel spinors is obtained if (3.1.4) is further differentiated,

$$[D_M, D_N]\epsilon = 0 \; . \tag{3.1.5}$$

From the computational point of view, this *(first order) integrability condition* of the spinor equation (3.1.4) is more convenient, because it is only

gravity, further spin 1/2 fermions might exist, their supersymmetry transformations being algebraic, instead of differential, in ϵ. In these cases, the invariance of purely bosonic solutions under supersymmetry requires that the variation of these fermions also vanishes, setting further algebraic constraints on the parameters ϵ_J if they are to parameterize preserved supersymmetries. We shall not encounter this situation in our discussion.

algebraic, whereas (3.1.4) is differential in ϵ. The commutator $[D_M, D_N]$ of two Lévi-Civita covariant derivatives is proportional to the Riemann tensor which, according to the Ambrose-Singer theorem [129] (see also [130]), determines the Lie algebra of the holonomy group. Obviously, equation (3.1.5) is only necessary for equation (3.1.4); however, for the relevant cases usually encountered in supergravity (including vanishing-flux compactifications), it is also sufficient[2]. The spinors solving equation (3.1.5) and, hence, invariant under the holonomy group, solve the equation (3.1.4) and correspond to preserved supersymmetries.

Notice that the existence of parallel spinors imply a holonomy reduction: the generic holonomy of a Riemannian manifold coincides with the structure group $SO(1,10)$. If parallel spinors exist, only when $\mathrm{Hol}(\omega) \subset SO(1,10)$ with strict inclusion, the spinor representation can be reducible under $\mathrm{Hol}(\omega)$. Riemannian holonomy groups have been classified by Berger [131] in the Euclidean case, such classification having been partially extended to the Lorentzian case by Bryant [132].

Let us now return to the case of non-vanishing four-form, $F_4 \neq 0$. This is the generic case in supergravity and, in fact, the presence of F_4 allows for supergravity solutions preserving exotic fractions of supersymmetry. As already discussed, the preserved supersymmetries of a bosonic solution when $F_4 \neq 0$ are now parameterized by the Killing spinors solving the Killing spinor equation (3.1.3). The relevant covariant derivative is not any longer the Lévi-Civita covariant derivative, but the supercovariant derivative (3.0.1) associated to the generalized connection Ω_M taking values on the Lie algebra of the generalized structure group $SL(32,\mathbb{R})$ [87] (see section 2.3). The presence of F_4 terms in the supercovariant derivative does not hamper, however, an analysis of the Killing spinor equation similar to that of its Riemannian counterpart. Again, the (first order) integrability condition of (3.1.3),

$$M_{MN}\epsilon \equiv [\mathcal{D}_M, \mathcal{D}_N]\epsilon = 0 \ , \tag{3.1.6}$$

is an algebraic, rather than a differential, equation for the Killing spinors. The commutator $M_{MN} = [\mathcal{D}_M, \mathcal{D}_N]$ of supercovariant derivatives now defines the generalized curvature $\mathcal{R}$ (in fact, M_{MN} contains the same information than equation (2.3.9)) taking values, again by the Ambrose-Singer theorem [129], in the Lie algebra $\mathrm{hol}(\Omega)$ of the generalized holonomy group. The proposal was then put forward in [78] (see also [73]) that the role of Riemannian holonomy in the determination of unbroken supersymmetries of a supergravity solution with non-vanishing F_4 was

[2]Were (3.1.5) not sufficient for (3.1.4), the holonomy argument would keep on being correct: further integrability conditions would then be needed to determine the holonomy (see next section).

taken over by generalized holonomy. In particular, in analogy with the purely geometrical, Riemannian case, the number of Killing spinors and, thus, the number of preserved supersymmetries of a purely bosonic solution of eleven-dimensional supergravity ought to be given by the number of singlets in the decomposition of the **32** representation of the generalized structure group ($SL(32,\mathbb{R})$ [87]) under the generalized holonomy group $\mathrm{Hol}(\Omega)$ [78, 73]. Notice that this argument does not apply to hypothetical preonic (31/32-supersymmetric) solutions [83], for which both the 31 unbroken supersymmetries and the only broken one are singlets. See [94, 107, 108, 109, 110, 2, 95, 3, 111] for further discussion about generalized holonomy.

Two remarks are in order. Firstly, both in the Riemannian and the generalized cases, the relevant structure and holonomy groups can be smaller: this is the case, *e.g.*, in compactification. In this case, the relevant representations of these groups are involved in the supersymmetry counting (see section 3.4 for an example). Secondly, spinors are assumed to be globally defined on the manifolds we are dealing with; namely, the manifolds M fulfilling the Einstein equations (3.0.2) (or (2.5.1) with the notation of chapter 2) are endowed with a spin structure[3] and, consequently, fulfil the topological restriction of having vanishing Stiefel-Whitney class (see [29] and references therein). The promotion of spinors from the (spinor representation) **32** of $SO(1,10)$ to the (fundamental representation) **32** of $SL(32,\mathbb{R})$ may encompass the loss of the information contained in the spin structure [134]. A different approach to deal with supersymmetric supergravity solutions, in which the spin structure is naturally incorporated, is that of G-structures [79, 80] (see also [135, 136, 137, 138, 139]). The later approach has proved to be very useful to build up explicit supergravity solutions (see [134, 140] for reviews, and [141, 142] for G-structures in the context of flux compactifications). See [81] for another recent approach to deal with the Killing spinor equation.

As in the Riemannian case, the presence of Killing spinors entails a generalized holonomy reduction: as shown in [107, 87], for a $D = 11$ supergravity solution to preserve k supersymmetries, the generalized holonomy group must be such that[4] $\mathrm{Hol}(\Omega) \subseteq SL(32-k,\mathbb{R}) \ltimes (\mathbb{R}^{32-k} \otimes \overset{k}{\ldots} \otimes \mathbb{R}^{32-k}) \equiv SL(32-k,\mathbb{R}) \ltimes \mathbb{R}^{k(32-k)}$ or, from the Lie algebra point of view,

$$\mathrm{hol}(\Omega) \subseteq sl(32-k,\mathbb{R}) \ltimes (\mathbb{R}^{32-k} \oplus \overset{k}{\ldots} \oplus \mathbb{R}^{32-k}) \,, \tag{3.1.7}$$

where $sl(32-k,\mathbb{R})$ acts on each of the k copies of $\mathbb{R}^{32-k}$ through the

[3] This could actually be a subtle issue: different spin structures on a manifold could yield different number of preserved supersymmetries [133].

[4] That is also the case in Type II $D = 10$ supergravities [108].

same, fundamental representation. The issue of classifying supersymmetric vacua may thus be mapped into one of classifying the generalized holonomy groups as subgroups of $SL(32, \mathbb{R})$. An investigation of basic supersymmetric configurations of M Theory was performed in [94] (see also [78, 87]), where a large variety of generalized holonomy groups were obtained. However, one of the striking results of the analysis of [94] was the fact that identical generalized holonomies may yield different amounts of supersymmetries. This shows that knowledge of the holonomy group is insufficient to fully classify the supergravity solution, and that knowledge of its embedding into the generalized structure group is also needed; in other words, knowledge of the decomposition of the 32-component spinor under $\mathrm{Hol}(\Omega)$ is also needed.

3.2 Higher order integrability

Being only algebraic in ϵ, the (first order) integrability condition (3.1.6) is more convenient than the Killing spinor equation itself, (3.1.3), in order to determine Killing spinors for a particular supergravity solution. It might happen, however, that the integrability condition (3.1.6) were only necessary, and not sufficient, for the Killing spinor equation (3.1.3). That is indeed the case for Freund-Rubin compactifications [32] of $D = 11$ supergravity, for which the preserved supersymmetry depends, in general, on the orientation chosen for the compactifying manifold (see [29]). Freund-Rubin compactification on the squashed seven-sphere (the coset space $SO(5) \times SU(2)/SU(2) \times SU(2)$) [30, 143], for instance, preserves $N = 1$ supersymmetry for one orientation (that can be referred to as left-squashing) while breaks it all for the other orientation (right-squashing). Accordingly, the Killing spinor equation (3.1.3) has solutions in the first case, but no solutions in the second one. And yet, both orientations share the same (first order) integrability condition (3.1.6) which is, therefore, not sufficient for (3.1.3). This issue can be resolved by going beyond first order integrability: successive covariant derivatives of equation (3.1.3) (*i.e.*, higher order integrability conditions) can give a set of additional algebraic equations for ϵ, sufficient for (3.1.3) [144].

This discussion can be put in a (generalized) holonomy context, by asking whether the Lie algebra generated by the curvature (expressed in the first order integrability condition (3.1.6)) agrees with the Lie algebra of the holonomy group. Actually, as shown in [94], in many cases the complete Lie algebra of $\mathrm{Hol}(\Omega)$ was not obtained from first order integrability (3.1.6), so that in particular the algebra had to be closed by hand. This issue is rather suggestive that the generalized curvature at a local point carries incomplete information of the generalized holonomy group, in apparent violation of the Ambrose-Singer theorem (but in agreement with the issue of left- versus right-squashing of S^7 mentioned above). However, the Ambrose-Singer theorem really indicates that $\mathrm{Hol}_p(\Omega)$ at a point p is spanned by elements of the generalized curvature (3.1.6) not just at point p, but at all points q connected to p by parallel transport (see *e.g.* [130, 112, 145]). Thus there is in fact no contradiction. Furthermore, this is rather suggestive that satisfying higher order integrability (representing motion from p to q, an information encoded in the successive covariant derivatives of the curvature) is in fact a necessary condition for identifying the proper generalized holonomy group [2].

In the remainder of this chapter, the interplay of higher order integrability and generalized holonomy will be explored, resorting to specific examples. We begin by revisiting the generalized holonomy of the M5 and M2-brane solutions of supergravity, and show that higher order integrability yields precisely the 'missing' generators that were needed to close the algebra. Other than this, however, the generalized holonomy groups for the M-branes identified in [94] are unchanged. Following this, we turn to the squashed S^7 [30, 143], where the situation is considerably different.

The importance of higher order integrability was of course previously recognized in [144] for the case of the squashed S^7. Here, we reinterpret the result of [144] in the language of generalized holonomy, and confirm the statement of [95] that while first order integrability yields the incorrect result $\mathrm{hol}^{(1)}(\Omega) = G_2 \subset so(7) \subset so(8)$, higher order integrability corrects this to $\mathrm{hol}(\Omega) = so_\pm(7) \subset so(8)$, where the two distinct possibilities $so(7)_-$ and $so(7)_+$ arise from left- and right-squashing, respectively, and correspond to the two different embeddings of $so(7)$ into $so(8)$. Since the spinor decomposes as either $\mathbf{8}_s \to \mathbf{7} + \mathbf{1}$ or $\mathbf{8}_s \to \mathbf{8}$ in the two cases, this explains the resulting $N = 1$ or $N = 0$ supersymmetry in four dimensions [95, 2] (see section 3.4).

Let us now introduce a convenient notation for the Lie algebra generators associated to the n-th order integrability conditions. For the supercovariant derivative (3.0.1) associated to the generalized connection Ω, first order integrability (3.1.6) of the Killing spinor equation (3.1.3) yields the generators

$$M_{MN} \equiv [\mathcal{D}_M(\Omega), \mathcal{D}_N(\Omega)] = \tfrac{1}{4}(\partial_M\Omega_N - \partial_N\Omega_M + \tfrac{1}{4}[\Omega_M, \Omega_N]) \equiv \tfrac{1}{4}\mathcal{R}_{MN}(\Omega), \tag{3.2.1}$$

where $\mathcal{R}_{MN}(\Omega)$ is the generalized curvature, *i.e.*, the curvature of Ω (see equation (2.3.9)). Higher order integrability expressions may be obtained by taking generalized covariant derivatives of (3.2.1). The corresponding generators will be taken to be

$$M_{MN_1N_2} \equiv [\mathcal{D}_M, M_{MN_1N_2}] \,, \tag{3.2.2}$$
$$M_{MN_1N_2N_3} \equiv [\mathcal{D}_N, M_{N_1N_2N_3}] \,, \tag{3.2.3}$$
$$M_{MN_1N_2N_3N_4} \equiv [\mathcal{D}_M, M_{N_1N_2N_3N_4}] \,, \tag{3.2.4}$$
$$\vdots$$

Higher order integrability conditions correspond to measuring the generalized curvature $\mathcal{R}_{MN}(\Omega)$ parallel transported away from the original base point p. In this sense, the information obtained from higher order integrability is precisely that required by the Ambrose-Singer theorem in making the connection between $\mathrm{Hol}_p(\Omega)$ and the curvature of the generalized connection.

3.3 Generalized holonomy of the M-branes

As examples of how higher order integrability may affect determination of the generalized holonomy group, we first revisit the case of the M5- and M2-brane solutions of supergravity. The generalized holonomy of these solutions, as well as several others, was originally investigated in [94]. For vacua with non-vanishing flux, including the brane solutions, it was seen that the Lie algebra generators obtained from first order integrability, (3.2.1), are insufficient for the closure of the algebra. In particular, additional generators must be obtained by further commutators. In [94], this was done by closing the algebra by hand. In the present context, however, additional commutators are readily available from the higher order integrability expressions, (3.2.2)–(3.2.4) [2].

3.3.1 Generalized holonomy of the M5-brane

The metric and four-form corresponding to the M5-brane solution of $D = 11$ supergravity are given by [74]

$$ds_{11}^2 = H_5^{-1/3}(dx^\mu)^2 + H_5^{2/3}(dy^i)^2 \,,$$
$$F_{ijkl} = \epsilon_{ijklm}\partial^m H_5 \,, \tag{3.3.1}$$

where x^μ, $\mu = 0, 1, \ldots, 5$, are coordinates corresponding to the world-volume directions, y^i, $i = 1, \ldots, 5$, are transverse space coordinates and $\epsilon_{ijklm} = \pm 1$ is the Lévi-Civita symbol, and $H_5(y^i)$ a function, in transverse space. Preservation of supersymmetry requires both the metric and four-form to be determined by the same function H_5 which is, in turn, demanded to be harmonic by the equations of motion (3.0.2), (3.0.3).

When acting on spinors, the generalized connection Ω_M defining the supercovariant derivative (3.0.1) for the solution (3.3.1) reads [94]

$$\Omega_\mu = \Omega_\mu^{\nu i} K_{\nu i} \ , \quad \Omega_i = -\tfrac{1}{3}\partial_i \ln H_5 \Gamma^{(M5)} + \tfrac{1}{2}\Omega_i^{jk} T_{jk} \ , \tag{3.3.2}$$

where

$$\Omega_\mu^{\nu i} = -\tfrac{2}{3} H_5^{-1/2} \delta_\mu^\nu \partial^i \ln H_5 \ , \quad \Omega_i^{jk} = \tfrac{8}{3}\delta_i^{[j}\partial^{k]} \ln H_5 \ , \tag{3.3.3}$$

and T_{ij}, $K_{\mu i}$ belong to the set

$$T_{ij} = \Gamma_{ij} P_5^+, \quad K_\mu = \Gamma_\mu P_5^+, \quad K_{\mu i} = \Gamma_{\mu i} P_5^+, \quad K_{\mu ij} = \Gamma_{\mu ij} P_5^+ \ , \tag{3.3.4}$$

of generators of a Lie algebra to be specified below (see equation (3.3.9)). In (3.3.2), $\Gamma^{(M5)} \equiv \frac{1}{5!}\epsilon_{ijklm}\Gamma^{ijklm}$ and, in (3.3.4), $P_5^+ \equiv \frac{1}{2}(1+\Gamma^{(M5)})$ is the M5-brane 1/2-supersymmetry projector. The generalized connection Ω_M of (3.3.2) includes the generator $\Gamma^{(M5)}$ in addition to T_{ij} and $K_{\mu i}$. However, the connection itself is not physical and, in fact, the terms containing $\Gamma^{(M5)}$ drop out from the expression of the generalized curvature (see below) and hence do not contribute to generalized holonomy.

The integrability conditions of the Killing spinor equation (3.1.3), posed with the supercovariant derivative associated to the generalized connection (3.3.2) of the M5-brane, can now be discussed. The first order integrability of the Killing spinor equation provides the set of generators (3.2.1) corresponding to the Lie algebra of the generalized curvature. For the M5-brane solution, these generators read [94]

$$\begin{aligned} M_{\mu\nu} &\equiv \tfrac{1}{4}\mathcal{R}_{\mu\nu} = 0, \\ M_{\mu i} &\equiv \tfrac{1}{4}\mathcal{R}_{\mu i} \\ &= H_5^{-1/2}\left[\tfrac{1}{6}(\partial_i\partial^j \ln H_5 - \tfrac{2}{3}\partial_i \ln H_5 \partial^j \ln H_5) + \tfrac{1}{18}\delta_i^j (\partial \ln H_5)^2\right] K_{\mu j}, \\ M_{ij} &\equiv \tfrac{1}{4}\mathcal{R}_{ij} \\ &= \left[\tfrac{2}{3}(\partial_l \partial_{[i} \ln H_5 - \tfrac{2}{3}\partial^l \ln H_5 \partial_{[i} \ln H_5)\delta_{j]}^k - \tfrac{2}{9}(\partial \ln H_5)^2 \delta_{[i}^k \delta_{j]}^l\right] T_{kl} \ . \end{aligned} \tag{3.3.5}$$

Only the generators T_{ij} and $K_{\mu i}$ show up in the expression for the Lie algebra (3.3.5) corresponding to the generalized curvature. As noticed in [94], the remaining generators K_μ and $K_{\mu ij}$ of (3.3.4) have to be obtained by closing the algebra defined by (3.3.5) 'by hand'. Alternatively, higher order integrability conditions, expressed as (3.2.2)–(3.2.4), can be used to obtain the remaining generators that ensure closure of the algebra [2].

In fact, the generators defining the second order integrability conditions, obtained for the M5-brane upon insertion of the corresponding generalized connection (3.3.2) into (3.2.2), take on the form [2]

$$M_{\mu\nu\lambda} = M_{\mu\nu\lambda}^{\rho i} K_{\rho i}, \quad M_{\mu\nu i} = \tfrac{1}{2} M_{\mu\nu i}^{jk} T_{jk}, \quad M_{\mu ij} = M_{\mu ij}^{\nu k} K_{\nu k} + \tfrac{1}{2} M_{\mu ij}^{\nu kl} K_{\nu kl},$$

$$M_{i\mu\nu}=0, \quad M_{i\mu j}=M^{\nu k}_{i\mu j}K_{\nu k}+\tfrac{1}{2}M^{\nu kl}_{i\mu j}K_{\nu kl}, \quad M_{ijk}=\tfrac{1}{2}M^{lm}_{ijk}T_{lm}, \quad (3.3.6)$$

where the component factors $M^{\cdot\cdot}_{AMN}$ are functions of H_5 and its derivatives. For example,

$$M^{\rho i}_{\mu\nu\lambda}=\tfrac{1}{36}H^{-3/2}[\partial^j \ln H_5\partial_j\partial^i \ln H_5-\tfrac{1}{3}\partial^i \ln H_5(\partial H_5)^2]\eta_{\mu[\nu}\delta^\rho_{\lambda]},$$
$$M^{jk}_{\mu\nu i}=\tfrac{4}{9}H^{-1}[\partial^{[j} \ln H_5\partial_i\partial^{k]} \ln H_5-\delta^{[j|}_i\partial^l \ln H_5\partial_l\partial^{|k]} \ln H_5]\eta_{\mu\nu}\ . \quad (3.3.7)$$

The other factors arising in (3.3.6) are similar and their explicit forms will not be needed. An additional generator $K_{\mu ij}$ arises at second order through the expressions $M_{\mu ij}\equiv[\mathcal{D}_\mu,\mathcal{R}_{ij}]$ and $M_{i\mu j}\equiv[\mathcal{D}_i,\mathcal{R}_{\mu j}]$ in (3.3.6). However, this does not still suffice to close the algebra. Pushing this procedure one step further into third order integrability (3.2.3), it is found that the generator K_μ arises through $M_{ki\mu j}\equiv[\mathcal{D}_k,[\mathcal{D}_i,\mathcal{R}_{\mu j}]]$. The complete set of generators (3.3.4) is then obtained and, actually, no new generator is found beyond third order [2].

The generators (3.3.4) thus generate the Lie algebra hol_{M5} of the generalized holonomy group of the M5-brane [94]. The $\binom{5}{2}=10$ generators T_{ij} correspond to $so(5)$, whereas the remaining $6+6\cdot 5+6\cdot\binom{5}{2}=96$ generators K_μ, $K_{\mu i}$, $K_{\mu ij}$ in (3.3.4) span the abelian Lie algebra $\mathbb{R}^{96}$, on which $so(5)$ acts semidirectly, *i.e.*, through a 96-dimensional representation. According to the general rule (3.1.7), as a supergravity solution preserving $k=16$ supersymmetries, the M5-brane (3.3.1) must have its generalized holonomy in $sl(32-k,\mathbb{R})\ltimes(\mathbb{R}^{32-k}\oplus\overset{k}{\cdots}\oplus\mathbb{R}^{32-k})$, namely,

$$\mathrm{hol}_{M5}\ \subseteq\ sl(16,\mathbb{R})\ltimes(\mathbb{R}^{16}\oplus\overset{16}{\cdots}\oplus\mathbb{R}^{16})\ . \quad (3.3.8)$$

The 96-dimensional representation of $so(5)\subset sl(16,\mathbb{R})$ on $\mathbb{R}^{96}$ must be, therefore, reducible into at most $k=16$ copies of the same (reducible or irreducible) representation of dimension $32-k=16$; thus, in this case, $\mathbb{R}^{96}=(\mathbb{R}^{16}\oplus\overset{6}{\cdots}\oplus\mathbb{R}^{16})\subset(\mathbb{R}^{16}\oplus\overset{16}{\cdots}\oplus\mathbb{R}^{16})$, where each of the six copies of $\mathbb{R}^{16}$ carries the same 16-dimensional representation of $so(5)$. This representation turns out to be further reducible into four 4-dimensional (spinor) representations $\mathbf{4}$ of $so(5)$. Introducing the convenient notation $\mathbb{R}^{4(\mathbf{4})}$ to denote this splitting of $\mathbb{R}^{16}$, the generalized holonomy algebra of the M5-brane solution of $D=11$ supergravity (3.3.1) is then [94]

$$\mathrm{hol}_{M5}=so(5)\ltimes(\mathbb{R}^{4(\mathbf{4})}\oplus\overset{6}{\cdots}\oplus\mathbb{R}^{4(\mathbf{4})})\ . \quad (3.3.9)$$

For the M5-brane case, higher order integrability conditions just provide the generators missing at first order, that can nevertheless be obtained by closing the generalized curvature algebra (defined by first order integrability) 'by hand'. In particular, higher order integrability conditions do not change the generalized holonomy (3.3.9) of the M5-brane, which remains the same as in [94].

3.3.2 Generalized holonomy of the M2-brane

The analysis of the M2-brane is similar to that of the M5-brane. The supergravity solution corresponding to the M2-brane is given by [73]

$$\begin{aligned} ds_{11}^2 &= H_2^{-2/3}(dx^\mu)^2 + H_2^{1/3}(dy^i)^2, \\ F_{\mu\nu\rho i} &= \epsilon_{\mu\nu\rho}\partial_i H_2^{-1}, \end{aligned} \tag{3.3.10}$$

where x^μ, $\mu = 0, 1, 2$, are coordinates corresponding to the worldvolume directions, y^i, $i = 1, \ldots, 8$, are transverse space coordinates and $\epsilon_{\mu\nu\rho} = \pm 1$. $H_2(y^i)$ is a harmonic function in transverse space.

Denoting by $P_2^+ = \frac{1}{2}(1 + \Gamma^{(M2)})$ the 1/2-supersymmetry projector of the M2-brane, where $\Gamma^{(M2)} \equiv \frac{1}{3!}\epsilon_{\mu\nu\rho}\Gamma^{\mu\nu\rho}$, the following generators

$$T_{ij} = \Gamma_{ij}P_2^+ \ , \quad K_{\mu i} = \Gamma_{\mu i}P_2^+, \quad K_{\mu ijk} = \Gamma_{\mu ijk}P_2^+, \tag{3.3.11}$$

can be introduced in order to express the generalized connection of the M2-brane solution (3.3.10) [94]:

$$\Omega_\mu = \Omega_\mu^{\nu i} K_{\nu i}, \quad \Omega_i = \tfrac{2}{3}\partial_i \ln H_2 \Gamma^{(M2)} + \tfrac{1}{2}\Omega_i^{jk} T_{jk} \ . \tag{3.3.12}$$

Here, the components of Ω_M are

$$\Omega_\mu^{\nu i} = -\tfrac{4}{3}H_2^{-1/2}\delta_\mu^\nu \partial^i \ln H_2 \ , \quad \Omega_i^{jk} = \tfrac{4}{3}\delta_i^{[j}\partial^{k]} \ln H_2. \tag{3.3.13}$$

The generators of the generalized curvature algebra corresponding to the M2-brane solution are again obtained through first order integrability of the Killing spinor equation (3.1.3), written for the supercovariant derivative associated to the generalized connection (3.3.12). These generators are [94]

$$\begin{aligned} M_{\mu\nu} &\equiv \tfrac{1}{4}\mathcal{R}_{\mu\nu} = 0, \\ M_{\mu i} &\equiv \tfrac{1}{4}\mathcal{R}_{\mu i} \\ &= \tfrac{1}{18}H_2^{-1/2}\left[6(\partial_i\partial^j \ln H_2 + 2\partial_i \ln H_2 \partial^j \ln H_2) - (\partial \ln H_2)^2\delta_i^j\right] K_{\mu j}, \\ M_{ij} &\equiv \tfrac{1}{4}\mathcal{R}_{ij} \\ &= \left[-\tfrac{1}{3}(\partial_l\partial_{[i} \ln H_2 - \tfrac{1}{3}\partial^l \ln H_2 \partial_{[i} \ln H_2)\delta_{j]}^k - \tfrac{1}{18}(\partial \ln H_2)^2\delta_{[i}^k\delta_{j]}^l\right] T_{kl}, \end{aligned} \tag{3.3.14}$$

and include terms proportional only to the generators T_{ij} and $K_{\mu i}$ of (3.3.11). As in the M5-brane case, the closure of the algebra spanned by the generators (3.3.14) can be achieved either 'by hand' [94] or by working out higher order integrability conditions [2]. The generators (3.2.2), corresponding to the second order integrability conditions, when considered for the M2-brane solution have the general form [2]

$$
\begin{aligned}
&M_{\mu\nu\lambda} = M^{\rho i}_{\mu\nu\lambda} K_{\rho i} \ , \quad M_{\mu\nu i} = \tfrac{1}{2} M^{jk}_{\mu\nu i} T_{jk} + M^{\nu k}_{\mu\nu i} K_{\nu k} \ , \\
&M_{\mu ij} = M^{\nu k}_{\mu ij} K_{\nu k} + \tfrac{1}{6} M^{\nu klm}_{\mu ij} K_{\nu klm} \ , \\
&M_{i\mu j} = M^{\nu k}_{i\mu j} K_{\nu k} + \tfrac{1}{6} M^{\nu klm}_{i\mu j} K_{\nu klm} + \tfrac{1}{2} M^{kl}_{i\mu j} T_{kl} \ , \\
&M_{i\mu\nu} = 0 \ , \quad M_{ijk} = \tfrac{1}{2} M^{lm}_{ijk} T_{lm} .
\end{aligned} \tag{3.3.15}
$$

where the explicit form of their components along the generators (3.3.11) will not be needed. The generators $M_{\mu ij} \equiv [\mathcal{D}_\mu, \mathcal{R}_{ij}]$ and $M_{i\mu j} \equiv [\mathcal{D}_i, \mathcal{R}_{\mu j}]$ in (3.3.15) give rise to the additional generator $K_{\mu ijk}$ of (3.3.11) which, together with T_{ij} and $K_{\mu i}$ generate the Lie algebra hol_{M2} of the generalized holonomy group of the M2-brane solution of $D = 11$ supergravity [94].

Since the M2-brane preserves $k = 16$ supersymmetries, hol_{M2} must be contained, by virtue of equation (3.1.7), in $sl(16,\mathbb{R}) \ltimes (\mathbb{R}^{16} \oplus \overset{16}{\cdots} \oplus \mathbb{R}^{16})$. In fact, T_{ij} in (3.3.11) generate $so(8) \subset sl(16,\mathbb{R})$ while $K_{\mu i}$, $K_{\mu ijk}$ are the generators of the abelian Lie algebra $\mathbb{R}^{192} = (\mathbb{R}^{16} \oplus \overset{12}{\cdots} \oplus \mathbb{R}^{16}) \subset (\mathbb{R}^{16} \oplus \overset{16}{\cdots} \oplus \mathbb{R}^{16})$. The representation of $so(8)$ on each $\mathbb{R}^{16}$ is reducible into two 8-dimensional (spinor) representations $\mathbf{8}_s$, making $\mathbb{R}^{16}$ split as $\mathbb{R}^{2(\mathbf{8}_s)}$ and yielding a generalized holonomy for the M2-brane [94]

$$
\mathrm{hol}_{M2} = so(8) \ltimes (\mathbb{R}^{2(\mathbf{8}_s)} \oplus \overset{12}{\cdots} \oplus \mathbb{R}^{2(\mathbf{8}_s)}) \ . \tag{3.3.16}
$$

Second order integrability is, thus, sufficient to guarantee the closure of the Lie algebra of the generalized holonomy group of the M2-brane.

Note that the generalized connection Ω_M contains complete information about the generalized holonomy of the spacetime, as the complete set of integrability conditions (3.2.1)–(3.2.4) may be obtained through commutators and derivatives of Ω_M. In this sense, the algebra of the holonomy group can never be larger than the algebra obtained through the generators in Ω_M itself. However it can certainly be smaller. This is apparent for the M5-brane, where the $\Gamma^{(M5)}$ generator is absent in the generalized curvature $\mathcal{R}_{MN}(\Omega)$ and its derivatives and also for the M2-brane, where $\Gamma^{(M2)}$ is absent. For these examples, and in fact for all vacua considered in [94, 110], the generators appearing in Ω_M and those appearing in $\mathcal{R}_{MN}(\Omega)$ are nearly identical. As a result, the generalized holonomy group may be correctly identified at first order in integrability, and the higher order conditions only serve to complete the set of generators needed for closure of the algebra.

A different situation may arise, however, if for some reason (such as accidental symmetries) a greatly reduced set of generators appear in

$\mathcal{R}_{MN}(\Omega)$. In such cases, examination of first order integrability may result in the misidentification of the actual generalized holonomy group. What happens here is that the algebra of the curvature $\mathcal{R}_{MN}(\Omega)$ at a single point p forms a subalgebra of the Lie algebra of the holonomy group. It is then necessary to explore the curvature at all points q connected by parallel transport to p in order to determine the actual holonomy. We demonstrate below that this incompleteness of first order integrability does arise in the case of generalized holonomy.

3.4 Higher order integrability and the squashed S^7

For an example of the need to resort to higher order integrability to characterize the generalized holonomy group $\mathrm{Hol}(\Omega)$, we turn to Freund-Rubin compactifications of eleven-dimensional supergravity. With vanishing gravitino, the Freund-Rubin ansatz [32] for the 4-form field strength F_4,

$$F_{\mu\nu\rho\sigma} = 3m\epsilon_{\mu\nu\rho\sigma}, \quad \mu = 0, 1, 2, 3, \tag{3.4.1}$$

with m constant and all other components vanishing, leads to spontaneous compactifications of the product form $AdS_4 \times X^7$. Here X^7 is a compact, Einstein, Euclidean 7-manifold. Decomposing the eleven-dimensional Dirac matrices Γ_M as

$$\Gamma_M = (\gamma_\mu \otimes 1, \gamma_5 \otimes \Gamma_m), \qquad \mu = 0, 1, 2, 3, \quad m = 1, \ldots, 7, \tag{3.4.2}$$

where γ_μ and Γ_m are four- and seven-dimensional Dirac matrices, respectively, and assuming the usual direct-product ansatz $\epsilon(x^\mu) \otimes \eta(y^m)$ for eleven-dimensional spinors, the Killing spinor equation (3.1.3) splits as

$$\mathcal{D}_\mu \epsilon = \left(\partial_\mu \; + \tfrac{1}{4}\omega_\mu{}^{\alpha\beta}\gamma_{\alpha\beta} + m\gamma_\mu\gamma_5\right)\epsilon = 0, \tag{3.4.3}$$

$$\mathcal{D}_m \eta = \left(\partial_m \; + \tfrac{1}{4}\omega_m{}^{ab}\Gamma_{ab} - \tfrac{i}{2}m\Gamma_m\right)\eta = 0. \tag{3.4.4}$$

Since AdS_4 admits the maximum number of Killing spinors (four in this case), the number N of supersymmetries preserved in the compactification coincides with the number of Killing spinors of the internal manifold X^7, that is, with the number of solutions to the Killing spinor equation (3.4.4). Therefore we only need to concern ourselves with the Killing spinors on X^7.

An orientation reversal of X^7 or, alternatively, a sign reversal of F_4, provides another solution to the equations of motion (3.0.2), (3.0.3) and, hence, another acceptable Freund-Rubin vacuum [143, 29]. For definiteness, we shall call *left*-orientation the solution corresponding to the choice of sign of F_4 in (3.4.1), that leads to the Killing spinor equation (3.4.4), and *right*-orientation the solution corresponding to the opposite choice of sign of F_4:

$$\text{(right)} \qquad F_{\mu\nu\rho\sigma} = -3m\epsilon_{\mu\nu\rho\sigma}, \quad \mu = 0,1,2,3, \tag{3.4.5}$$

leading to the Killing spinor equation

$$\text{(right)} \qquad \mathcal{D}_m\eta = \left(\partial_m \ + \tfrac{1}{4}\omega_m{}^{ab}\Gamma_{ab} + \tfrac{i}{2}m\Gamma_m\right)\eta = 0. \tag{3.4.6}$$

From either (3.4.4) or (3.4.6), we see that the generalized connection defining $\mathcal{D}_m$ takes values in the algebra spanned by $\{\Gamma_{ab}, \Gamma_a\}$ and therefore the generalized structure group is $SO(8)$. Notice, however, that both Killing spinor equations (3.4.4) and (3.4.6) share the same first order integrability condition [30, 29]

$$M_{mn}\eta \equiv [\mathcal{D}_m, \mathcal{D}_n]\eta = \tfrac{1}{4}\mathcal{R}_{mn}\eta \equiv \tfrac{1}{4}\mathcal{C}_{mn}\eta = \tfrac{1}{4}C_{mn}{}^{ab}\Gamma_{ab}\eta = 0, \tag{3.4.7}$$

where $C_{mn}{}^{ab}$ is the Weyl tensor of X^7 (thus demonstrating that, in this case the generalized curvature tensor is simply the Weyl tensor). Thus first order integrability is unable to distinguish between left and right orientations on the sphere. Then it might be possible that spinors η solving the integrability condition (3.4.7) will only satisfy the Killing spinor equation for one orientation, that is, satisfy (3.4.4) but not (3.4.6) (or the other way around). In fact, the skew-whiffing theorem [143, 29] for Freund-Rubin compactifications proves that this will, in general, be the case: it states that at most one orientation can give $N > 0$, with the exception of the round S^7, for which both orientations give maximal supersymmetry, $N = 8$. Since the preserved supersymmetry N is given by the number of singlets in the decomposition of the $\mathbf{8}_s$ of $SO(8)$ (the generalized structure group) under the generalized holonomy group $\mathrm{Hol}(\Omega)$, then, in general, each orientation must have either a different generalized holonomy, or the same generalized holonomy but a different decomposition of the $\mathbf{8}_s$.

To illustrate this feature, consider compactifications on the squashed S^7 [143, 30]. This choice for X^7 has the topology of the sphere, but the metric is distorted away from that of the round S^7; it is instead the coset space $SO(5) \times SU(2)/SU(2) \times SU(2)$ endowed with its Einstein metric [143, 30]. The compactification on the left-squashed S^7 preserves $N = 1$ supersymmetry whereas that on the right-squashed S^7 has $N = 0$; put another way, the integrability condition (3.4.7) has one non-trivial solution, corresponding in turn to a solution to the Killing spinor equation (3.4.4) (making the left-squashed S^7 preserve $N = 1$), but not to a solution to (3.4.6), which in fact has no solutions (yielding $N = 0$ for the right-squashed S^7). On the other hand, an analysis of the Weyl tensor of the squashed S^7 shows that there are only 14 linear combinations $\mathcal{C}_{mn}$ of gamma matrices in (3.4.7), corresponding to the generators of G_2 [30, 29]. Though appealing, G_2 cannot be, however, the generalized holonomy since the $\mathbf{8}_s$ of $SO(8)$ would decompose as $\mathbf{8}_s \to \mathbf{8} \to \mathbf{7} + \mathbf{1}$ under $SO(8) \supset SO(7) \supset G_2$ regardless of the orientation, giving $N = 1$ for both left- and right-squashed solutions. We thus conclude that in this case the first order integrability condition (3.4.7) is insufficient to determine the generalized holonomy.

The resolution to this puzzle is naturally given by higher order integrability. In the case of the squashed S^7, it turns out that the second order integrability condition (3.2.2) is sufficient. For a general Freund-Rubin internal space X^7 this condition reads[5] [144]

$$M_{lmn}\eta \equiv \tfrac{1}{4}[\mathcal{D}_l, \mathcal{C}_{mn}]\eta = \tfrac{1}{4}\left(D_l C_{mn}{}^{ab}\Gamma_{ab} \mp 2imC_{mnl}{}^{a}\Gamma_a\right)\eta = 0 \ , \quad (3.4.8)$$

the $-$ sign corresponding to the left solution, and the $+$ to the right. For the squashed S^7, we find that only 21 of the M_{lmn} are linearly independent combinations of the Dirac matrices [2]. The details are provided in Appendix A. Following the notation of [30, 29], we split the index m as $m = (0, i, \hat{\imath})$, with $i = 1, 2, 3$, $\hat{\imath} = 4, 5, 6 = \hat{1}, \hat{2}, \hat{3}$; then, with a suitable normalization, the linearly independent generators in (3.4.8) may be chosen to be [2]

$$\begin{aligned}
&\mathcal{C}_{0i} = \Gamma_{0i} + \tfrac{1}{2}\epsilon_{ikl}\Gamma^{\hat{k}\hat{l}} \ , \quad \mathcal{C}_{ij} = \Gamma_{ij} + \Gamma_{\hat{\imath}\hat{\jmath}} \ , \\
&\mathcal{C}_{i\hat{\jmath}} = -\Gamma_{i\hat{\jmath}} - \tfrac{1}{2}\Gamma_{j\hat{\imath}} + \tfrac{1}{2}\delta_{ij}\delta^{kl}\Gamma_{k\hat{l}} - \tfrac{1}{2}\epsilon_{ijk}\Gamma^{0\hat{k}} \ , && (3.4.9) \\
&M_{ij} = \Gamma_{\hat{\imath}\hat{\jmath}} \mp \tfrac{2}{3}\sqrt{5}im\epsilon_{ijk}\Gamma^{\hat{k}} \ , \quad M_i = \Gamma_{0\hat{\imath}} \mp \tfrac{2}{3}\sqrt{5}im\Gamma_i \ , \\
&M = \delta^{kl}\Gamma_{k\hat{l}} \pm 2\sqrt{5}im\Gamma_0, && (3.4.10)
\end{aligned}$$

the $-$ sign in front of m corresponding to the left solution and the $+$ to the right. Notice that there are 8 linearly independent generators in $\mathcal{C}_{i\hat{\jmath}}$ of (3.4.9), since $\delta^{kl}\mathcal{C}_{k\hat{l}} \equiv \mathcal{C}_{1\hat{1}} + \mathcal{C}_{2\hat{2}} + \mathcal{C}_{3\hat{3}} = 0$. The $3+3+8 = 14$ generators $\mathcal{C}_{0i}$, $\mathcal{C}_{ij}$, $\mathcal{C}_{i\hat{\jmath}}$ span G_2 [30, 29], and are the same as those obtained from the first integrability condition (3.4.7), while the $3+3+1 = 7$ additional generators M_{ij}, M_i, M of (3.4.10) were not contained in (3.4.7). Taken together, they generate the 21 dimensional algebra $so(7)$, regardless of the orientation, provided [2]

$$m^2 = \tfrac{9}{20}, \quad (3.4.11)$$

[5] In (3.4.7), $\mathcal{D}_l$ (the generalized covariant derivative in equation (3.4.4)) should not be confused with D_l (the Lévi-Civita covariant derivative).

in agreement with the Einstein equation for the squashed S^7 [29].

The embedding of $so(7)$ into $so(8)$ is, however, different for each orientation. We use $so(7)_-$ to denote the embedding corresponding to the left solution and $so(7)_+$ the right. While the spinor η transforms as an $\mathbf{8}_s$ of the generalized structure group $SO(8)$, the decomposition of the $\mathbf{8}_s$ is different under left- and right-squashing. With our Dirac conventions, it turns out that $\mathbf{8}_s \to \mathbf{7} + \mathbf{1}$ under $so(8) \supset so(7)_-$, giving $N = 1$ for the left-squashed S_7, while $\mathbf{8}_s \to \mathbf{8}$ under $so(8) \supset so(7)_+$, giving $N = 0$ for the right-squashed S^7.

Since $so(7)$ is the subalgebra of $so(8)$ that yields the correct branching rules of the $\mathbf{8}_s$ of $SO(8)$, we conclude that second order integrability is sufficient in this case to identify all generators of the Lie algebra of the holonomy $\mathrm{Hol}(\Omega_m)$ of the connection Ω_m defining the supercovariant derivative $\mathcal{D}_m$ in (3.4.4) . Hence the generalized holonomy algebra of the Freund-Rubin compactification on the squashed S^7 is given precisely by $\mathrm{hol}(\Omega_m) = so(7)$ [95, 2][6]. In this case, it is the embedding of $so(7)$ in $so(8)$ (with corresponding spinor decomposition $\mathbf{8}_s \to \mathbf{7} + \mathbf{1}$ or $\mathbf{8}_s \to \mathbf{8}$) that determines the number of preserved supersymmetries. This indicates that, for generalized holonomy, knowledge of the holonomy group *and* the embedding are both necessary in order to understand the number of preserved supersymmetries. While this was already observed in [94, 87] for non-compact groups, here we see that this is also true when the generalized holonomy group is compact.

The analysis of the squashed S^7, along with that of the brane solutions of the previous section, highlights several features of generalized holonomy. For the squashed S^7, the Lie algebra of the generalized holonomy group is in fact larger than that generated locally by the Weyl curvature at a point p. In this case, the algebra arising from lowest order integrability is already closed, but is only a subalgebra of the correct holonomy algebra. It is then mandatory to examine the second order integrability expression (3.4.8) in order to identify the generalized holonomy group. On the other hand, for the M2 and M5-branes, lowest order integrability, while lacking a complete set of generators, nevertheless closes on the correct holonomy algebra, and no really new information is gained at higher

[6]For a d-dimensional manifold X_d, the *cone* $C(X_d)$ *over* X_d is the $(d{+}1)$-dimensional manifold defined to have topology $\mathbb{R}^+ \times X_d$ and metric $g(C(X_d)) = dr^2 + r^2 g(X_d)$, where $g(X_d)$ is the metric on X_d and r parameterizes $\mathbb{R}^+$. In a supergravity context, *Killing spinors* on the Freund-Rubin compactifying manifold X_d correspond to *parallel spinors with respect to the Lévi-Civita connection* on $C(X_d)$ [146, 70]. Thus, in this case, the *generalized holonomy* of X_d corresponds to the *Riemannian holonomy* of $C(X_d)$. A compactifying 7-manifold X_7 preserves $N = 1$ supersymmetry for one orientation if its corresponding 8-dimensional cone $C(X_7)$ has Spin(7) holonomy [70], in agreement with this result for the generalized holonomy of the squashed S^7 [95, 2].

order. Of course, in all cases, complete information is contained in the generalized connection Ω_M itself. However, examination of Ω_M directly can be misleading, as it may contain gauge degrees of freedom, which are unphysical. This is most clearly seen in the case of the round S^7, where $\Omega_m = \omega_m^{ab}\Gamma_{ab} - 2im\Gamma_m$ is certainly non-vanishing, while the generalized curvature $\mathcal{R}_{mn}$, given by the Weyl tensor, is trivial, $\mathcal{R}_{mn} = 0$.

For generalized holonomy to be truly useful, it ought to go beyond simply a classification scheme, and must yield methods for constructing new supersymmetric solutions. In much the same way that the rich structure of Riemannian holonomy teaches us a great deal about the geometry of Killing spinors on Riemannian manifolds, the formal analysis of generalized holonomy via connections on Clifford bundles may lead to a similar expansion of knowledge of supergravity structures and manifolds with fluxes. Such an analysis is well beyond the scope of this Thesis and, instead, we now continue with the study of generalized holonomy to characterize supersymmetric solutions of supergravity, from a different point of view.

4

Generalized holonomy for BPS preons

The observation [83] that BPS states that break $\tilde{n} = 32 - k$ supersymmetries can be treated as composites of those preserving all but one supersymmetries, suggests that the $k = 31$-supersymmetric states might be considered as fundamental constituents of M Theory. These $\nu = 31/32$ BPS states were accordingly named *BPS preons* in [83]. In this chapter we apply the ideas previously developed about generalized holonomy to the study of hypothetical preonic solutions of eleven-dimensional supergravity. In section 4.1, the notion of preonic states is reviewed. States composed of $\tilde{n}$ preons are shown to be characterized by $\tilde{n}$ bosonic spinors that parameterize the broken supersymmetries. In section 4.2, these spinors are shown to be orthogonal to the Killing spinors characterizing the unbroken supersymmetries. A moving G-frame (where the group G can chosen to be $G = GL(32, \mathbb{R})$, $SL(32, \mathbb{R})$ or $Sp(32, \mathbb{R})$) defined by both preonic and Killing spinors can be consequently used to describe the corresponding states. We then apply, in section 4.3, this moving G-frame method to the study of the generalized holonomies of hypothetical preonic solutions of supergravity. Although no definite answer to the question of the existence of preonic solutions for the standard $D = 11$ supergravity is given here, we do show, in section 4.4, that $\nu = 31/32$ supersymmetric preonic configurations exist in Chern-Simons (CS) supergravity *i.e.*, that CS supergravity does have preonic solutions. To conclude this chapter, we propose in section 4.5 a worldvolume action for BPS preons in the background of the D'Auria-Fré formulation of $D = 11$ supergravity [92]. The notation and conventions are restored to those of chapter 2. This chapter follows closely reference [3].

4.1 BPS preons

Group theoretical methods usually help with the lack of a dynamical description of M Theory; in particular, the representation theory of the M-algebra $\mathfrak{E}^{(528|32)}$ (see section 2.1 of chapter 2) can shed some light into the structure of M Theory. Bogomoln'yi-Prasad-Sommerfield (BPS) states saturate the Bogomoln'yi bound associated to the M Theory superalgebra (2.1.10) and are, therefore, protected from corrections as argued in the Introduction (chapter 1). They are, thus, intrinsically non-perturbative and are expected to be fundamental states of the fully-fledged M Theory.

A BPS state $|BPS\,,\,k\rangle$ described by a supergravity solution preserving k supersymmetries is characterized by k spinors ϵ_J^α, $J = 1, \ldots, k \leq 32$ parameterizing the supersymmetry transformations (2.3.1)–(2.3.3) of the spacetime fields. In particular, it will be assumed that the state $|BPS\,,\,k\rangle$ corresponds to a purely bosonic supergravity solution, so that the spinors ϵ_J^α are Killing and satisfy the Killing spinor equation (3.1.3). These spinors parameterize the unbroken supersymmetries that leave invariant the supersymmetric state; that is, at the level of generators acting on $|BPS\,,\,k\rangle$,

$$\epsilon_J{}^\alpha Q_\alpha |BPS\,,\,k\rangle = 0\,, \quad J = 1, \ldots, k\,, \quad k \leq 32\,. \tag{4.1.1}$$

Here, Q_α are the supersymmetry generators, that we shall take to be in the maximally extended supersymmetry algebra, namely, the M Theory superalgebra $\mathfrak{E}^{(528|32)}$, whose (anti)commutation relations are given in (2.1.10): $\{Q_\alpha, Q_\beta\} = P_{\alpha\beta}$, $[Q_\alpha, P_{\beta\gamma}] = 0$, $\alpha, \beta, \gamma = 1, 2, \ldots, 32$, so that $P_{\alpha\beta} = P_{\beta\alpha}$. The generalized momentum $P_{\alpha\beta}$ can be decomposed in the basis of $D = 11$ $Spin(1, 10)$ (32×32) Dirac matrices as in (2.1.9), namely, $P_{\alpha\beta} = P_a \Gamma^a_{\alpha\beta} + iZ_{ab}\Gamma^{ab}_{\alpha\beta} + Z_{a_1\ldots a_5}\Gamma^{a_1\ldots a_5}_{\alpha\beta}$, containing the standard $D = 11$ momentum P_a and the tensorial 'central' charge generators Z_{ab}, $Z_{a_1\ldots a_5}$. As discussed in section 2.1, these central charges are associated to the basic M Theory branes.

In a formal, quantum-mechanical discussion, a $\nu = k/32$-supersymmetric BPS state $|BPS, k\rangle$ can also be defined as an eigenstate of the generalized momentum operator $P_{\alpha\beta}$,

$$P_{\alpha\beta}|BPS, k\rangle = p^{(k)}_{\alpha\beta}|BPS, k\rangle \tag{4.1.2}$$

with eigenvalue $p^{(k)}_{\alpha\beta}$ such that $\det p^{(k)}_{\alpha\beta} = 0$, as justified below. The vanishing determinant condition implies that the matrix $p^{(k)}_{\alpha\beta}$ has rank less than the maximal possible rank 32. More precisely, a $\nu = k/32$-BPS state $|BPS, k\rangle$ is such that

$$\text{rank } p^{(k)}_{\alpha\beta} \equiv \tilde{n} = 32 - k\,, \quad 1 \leq k < 32\,. \tag{4.1.3}$$

Recall from the discussion of section 2.1 that the maximal automorphism group of the M-algebra $\mathfrak{E}^{(528|32)}$ is $GL(32, \mathbb{R})$. Then the matrix $p^{(k)}_{\alpha\beta}$ can be diagonalized by a $GL(32, \mathbb{R})$ transformation ${g_\alpha}^{(\gamma)}$,

$$p^{(k)}_{\alpha\beta} = {g_\alpha}^{(\gamma)} p_{(\gamma)(\delta)} {g_\beta}^{(\delta)} \ . \tag{4.1.4}$$

In (4.1.4), $p_{(\gamma)(\delta)}$ is a diagonal matrix that can be put in the canonical form

$$p_{(\gamma)(\delta)} = \mathrm{diag}(\underbrace{1, \ldots, 1, -1, \ldots, -1}_{\tilde{n}=32-k}, \underbrace{0, \ldots, 0}_{k}) \ , \tag{4.1.5}$$

where the number of non-vanishing elements, all $+1$ or -1, is equal to $\tilde{n} = \mathrm{rank}(p^{(k)}_{\alpha\beta})$. However, the usual assumptions for the supersymmetric quantum mechanics describing BPS states do not allow for negative eigenvalues of $P_{\alpha\beta} = \{Q_\alpha, Q_\beta\}$ ($p_{11} = -1$, *e.g.*, would imply $(Q_1)^2|BPS, k\rangle = -|BPS, k\rangle$, contradicting unitarity). Thus, only positive eigenvalues are allowed and

$$p_{(\gamma)(\delta)} = \mathrm{diag}(\underbrace{1, \ldots, 1}_{\tilde{n}=32-k}, \underbrace{0, \ldots, 0}_{k}) \ . \tag{4.1.6}$$

Substituting (4.1.6) into (4.1.4), one arrives at

$$p^{(k)}_{\alpha\beta} = {g_\alpha}^{(\gamma)} \ \mathrm{diag}(\underbrace{1, \ldots, 1}_{\tilde{n}=32-k}, \underbrace{0, \ldots, 0}_{k})_{(\gamma)(\delta)} \ \ {g_\beta}^{(\delta)} \ , \tag{4.1.7}$$

or, equivalently, introducing the $\tilde{n}$ vectors $\lambda^1_\alpha, \ldots, \lambda^{\tilde{n}}_\alpha$ of $GL(32, \mathbb{R})$, defined by ${g_\alpha}^1 = {\lambda_\alpha}^1$, …, ${g_\alpha}^{\tilde{n}} = {\lambda_\alpha}^{\tilde{n}}$,

$$\begin{aligned} P_{\alpha\beta}|BPS, k\rangle =& \sum_{r=1}^{\tilde{n}=32-k} {\lambda_\alpha}^r {\lambda_\beta}^r |BPS, k\rangle \\ \equiv& \big({\lambda_\alpha}^1 {\lambda_\beta}^1 + \ldots + {\lambda_\alpha}^{\tilde{n}} {\lambda_\beta}^{\tilde{n}}\big) \, |BPS, k\rangle \ . \end{aligned} \tag{4.1.8}$$

Taking suitable linear combinations of the supertranslations, namely, $Q^{(0)}_\alpha = (g^{-1})^\beta{}_\alpha Q_\beta$, the algebra diagonalizes on BPS states,

$$\begin{aligned} &\{Q^{(0)}_r, Q^{(0)}_s\}|BPS, k\rangle = \delta_{rs}|BPS, k\rangle \ , \\ &\{Q^{(0)}_r, Q^{(0)}_J\}|BPS, k\rangle = \{Q^{(0)}_J, Q^{(0)}_K\}|BPS, k\rangle = 0 \ , \end{aligned} \tag{4.1.9}$$

where $r, s = 1, \ldots, \tilde{n}$, $J, K = 1, \ldots, k$, so that the set of 32 supercharges $Q^{(0)}_\alpha = (Q^{(0)}_r, Q^{(0)}_J)$ acting on the BPS state $|BPS, k\rangle$ splits into k generators $Q^{(0)}_J$ of supersymmetry that preserve the BPS state (and correspond to the generators of (4.1.1), $Q_J^{(0)}|k\rangle = 0$), and $\tilde{n} = 32 - k$ generators $Q_r^{(0)}$ corresponding the set of broken supersymmetries.

Equation (4.1.8) suggests that all BPS states can be considered as composites of states with rank $p^{(k)}_{\alpha\beta} = 1$ [83], that is, preserving $k = 31$ supersymmetries. The hypothetical objects carrying these "elementary values" of $p^{(31)}_{\alpha\beta}$ are called *BPS preons* [83]. For a BPS preon state, the index r in equation (4.1.8) assumes only one value and can therefore be suppressed. In summary, a BPS preon [83] state $|BPS\,,\,31\rangle \equiv |\lambda\rangle$ preserves 31 supersymmetries (hence the notation $|BPS\,,\,31\rangle$) and is characterized by the following choice of central charges matrix

$$p_{\alpha\beta} = g^{\gamma}{}_{\alpha}\; p^{(0)}_{\gamma\delta}\; g^{\delta}{}_{\beta} = \lambda_\alpha \lambda_\beta \; , \tag{4.1.10}$$

in terms of a single bosonic spinor[1] (hence the notation $|\lambda\rangle$) such that

$$P_{\alpha\beta}|\lambda\rangle = \lambda_\alpha \lambda_\beta |\lambda\rangle \; . \tag{4.1.11}$$

Equation (4.1.8) may be looked at as a manifestation of the *composite structure* of the $\nu = k/32$ BPS state $|BPS, k\rangle$,

$$|BPS, k\rangle = |\lambda^1\rangle \otimes \ldots \otimes |\lambda^{\tilde{n}}\rangle \; , \tag{4.1.12}$$

where $|\lambda^1\rangle$, $\ldots$, $|\lambda^{\tilde{n}}\rangle$, $\tilde{n} = 32 - k$, are BPS elementary, preonic states characterized by the spinors $\lambda_\alpha{}^1$, $\ldots$, $\lambda_\alpha{}^{\tilde{n}}$, respectively.

From this point of view, all the single-brane solutions of 11-dimensional supergravity, which preserve 16 out of 32 supersymmetries (see section 2.5 of chapter 2), correspond to composites of 16 BPS preons. By the same token, intersecting branes, preserving *less* than 16 supersymmetries ($\nu < 1/2$) correspond to composites of *more* than 16 preons, and solutions with extra supersymmetry ($\nu > 1/2$) can be considered as composites of less than 16 BPS preons. Initially, it seemed that solutions preserving all supersymmetries but one, *i.e.* describing the excitations of a BPS preon, could not exist, and indeed they were not found by means of the standard brane ansatzes used to solve the usual 11-dimensional supergravity [27] equations. A more general study in the context of standard $D = 11$ supergravity has shown that the existence of such solutions is not ruled out [87, 78].

The possible existence of brane solutions with extra supersymmetries should not be excluded, although these solutions would describe quite unusual branes. The reason why the 'standard' brane solutions (like

[1] By construction, λ_α is a $GL(32,\mathbb{R})$ vector. However, we keep the 'spinor' name for it bearing in mind the possibility of a spacetime treatment, although this is not straightforward and would require additional study.

M-waves, M2 and M5-branes in $D = 11$) always break 1/2 of the supersymmetry is that their κ-symmetry projector (the bosonic part of which is identical to the projector defining the preserved supersymmetries [147, 58]) has the form $(1 - \bar{\Gamma})$ with $tr\bar{\Gamma} = 0$, $\bar{\Gamma}^2 = I$. However, worldvolume actions for branes with a different form for the κ-symmetry projector are known [148, 149, 150, 8, 84] although in an enlarged superspace (see [85]): see chapter 7 for an explicit example. A question arises, whether such actions may be written in usual spacetime or superspace.

However, and independently of whether BPS preons can be associated with solutions of standard supergravity or there is, instead, a *BPS preon conspiracy* preventing their existence in *standard* D=11 spacetime or superspace, preons do provide an algebraic classification of the M Theory BPS states [83]. In this perspective such a BPS preon conspiracy, if it exists, would perhaps indicate the necessity of a wider geometric framework for a suitable description of M Theory, such as extended superspaces and supertwistors. If, on the contrary, solitonic solutions with the properties of BPS preons were actually found, extended superspaces would still provide a useful tool for a description of M Theory[2]. One is led to expect that the additional tensorial coordinates of these superspaces carry a counterpart of the information which, in the framework of standard $D = 10, 11$ supergravity, is encoded in the antisymmetric tensor gauge fields entering the supergravity multiplets (*cf.* [85]). This point of view may be also supported by the observation that in the standard topological charge treatment of the tensorial generators of the M–algebra [57], these topological charges are associated just with these gauge fields.

4.2 Moving G-frame

When a BPS state $|k\rangle$ is realized as a solitonic solution of supergravity, it is characterized by k Killing spinors $\epsilon_J{}^{\beta}(x)$ or by the $\tilde{n} = 32 - k$ bosonic spinors $\lambda_{\alpha}{}^{r}(x)$ associated with the $\tilde{n}$ BPS preonic components of the state $|BPS, k\rangle$. The Killing spinors and the preonic spinors are orthogonal. Indeed, using the (anti)commutation relations (2.1.10) of the M-algebra, if the preserved supersymmetries correspond to the generators $\epsilon_J{}^{\alpha} Q_{\alpha}$, $J = 1, \ldots, k$, equation (4.1.1), then

$$\sum_{r=1}^{\tilde{n}=32-k} \epsilon_{(J}{}^{\alpha} \lambda_{\alpha}{}^{r}\, \epsilon_{K)}{}^{\beta} \lambda_{\beta}{}^{r} = 0 \; , \tag{4.2.1}$$

[2] There are also related reasons to consider more general superspaces, as the ensuing *fields/extended superspace coordinates correspondence* [85, 86] associated with extended superspaces: see section 6.7 of chapter 6 and further references therein.

which implies the orthogonality of Killing and preonic spinors [3],

$$\epsilon_J{}^\alpha \lambda_\alpha{}^r = 0 \; , \quad J = 1, ..., k \; , \quad r = 1, ..., \tilde{n} \; , \tag{4.2.2}$$

explaining the relation $\tilde{n} = 32 - k$ between the number of preons $\tilde{n} = \mathrm{rank}(p^{(k)}_{\alpha\beta})$ and the number of preserved supersymmetries k.

Then, BPS preonic ($\lambda_\alpha{}^r$) and Killing ($\epsilon_J{}^\alpha$) spinors provide an alternative (dual) characterization of a ν-supersymmetric solution; either one can be used and, for solutions with extra supersymmetries ($\nu > 1/2$) [119, 120, 121, 122, 123, 124], the characterization provided by BPS preons is a more economic one. Moreover, the use of both BPS preonic spinors and Killing spinors allows us to develop a *moving G-frame* method [3], which we now introduce, and that may be useful in the search for new supersymmetric solutions of supergravity.

The set of Killing and preonic spinors can be completed to obtain bases in the spaces of spinors with upper and with lower indices by introducing $\tilde{n} = 32 - k$ spinors $w_r{}^\alpha$ and k spinors $u_\alpha{}^L$ satisfying

$$w_s{}^\alpha \lambda_\alpha{}^r = \delta_s^r \; , \quad w_s{}^\alpha u_\alpha{}^J = 0 \; , \quad \epsilon_J{}^\alpha u_\alpha{}^K = \delta_J{}^K \; . \tag{4.2.3}$$

Either of these two dual bases defines a *generalized moving G-frame* described by the nondegenerate matrices

$$g_\alpha{}^{(\beta)} = \left(\lambda_\alpha{}^s \, , u_\alpha{}^J\right) \; , \qquad g^{-1}{}_{(\beta)}{}^\alpha = \begin{pmatrix} w_s{}^\alpha \\ \epsilon_J{}^\alpha \end{pmatrix} \; , \tag{4.2.4}$$

where $(\alpha) = (s, J) = (1, \ldots, 32 - k; J = 1, \ldots, k)$. Indeed, $g^{-1}{}_{(\beta)}{}^\gamma g_\gamma{}^{(\alpha)} = \delta_{(\beta)}{}^{(\alpha)}$ is equivalent to Eqs. (4.2.3) and (4.2.2), while

$$\delta_\alpha{}^\beta = g_\alpha{}^{(\gamma)} g^{-1}{}_{(\gamma)}{}^\beta \equiv \lambda_\alpha{}^r w_r{}^\beta + u_\alpha{}^J \epsilon_J{}^\beta \tag{4.2.5}$$

provides the unity I_{32} decomposition or completeness relation in terms of these dual bases.

One may consider the dual basis $g^{-1}{}_{(\beta)}{}^\alpha$ to be constructed from the bosonic spinors in $g_\alpha{}^{(\beta)}$ by solving equation (4.2.5) or $g^{-1} g = I_{32}$ (Eqs. (4.2.3) and (4.2.2)). Alternatively, one may think of $w_r{}^\alpha$ and $u_\alpha{}^J$ as being constructed from $\epsilon_J{}^\alpha$ and $\lambda_\alpha{}^r$ through a solution of the same constraints. In this sense [3] *the generalized moving G-frame (4.2.4) is constructed from k Killing spinors $\epsilon_J{}^\alpha$ characterizing the supersymmetries preserved by a BPS state (realized as a solution of the supergravity equations) and from the $\tilde{n} = 32 - k$ bosonic spinors $\lambda_\alpha{}^r$ characterizing the BPS preons from which the BPS state is composed.* Although many of the considerations below are general, we shall be mainly interested here in the cases $G = SL(32, \mathbb{R})$ and $G = Sp(32, \mathbb{R})$.

In $D = 11$, the charge conjugation matrix $C^{\alpha\beta} = -C^{\beta\alpha}$ allows us to express explicitly the dual basis g^{-1} in terms of the original one g or *vice versa*. In particular, in the preonic $k = 31$ case one finds that, since $\lambda_\alpha C^{\alpha\beta}\lambda_\beta \equiv 0$, then $\lambda^\alpha = C^{\alpha\beta}\lambda_\beta$ has to be expressed as $\lambda^\alpha = \lambda^I \epsilon_I{}^\alpha$, for some coefficients λ^I, $I = 1, \ldots, 31$. In general (as *e.g.*, in CJS supergravity with nonvanishing F_4), the charge conjugation matrix is not 'covariantly constant', $\mathcal{D}C^{\alpha\beta} = -2\Omega^{[\alpha\beta]} \neq 0$, where $\Omega_\alpha{}^\beta$ is the $D = 11$ supergravity generalized connection (2.3.5) (see section 2.3 of chapter 2). This relates the coefficients $\lambda^I = \lambda^\alpha u_\alpha{}^I$ to the antisymmetric (non-symplectic) part of the generalized connection, $\Omega^{[\alpha\beta]} = C^{[\alpha\gamma}\Omega_\gamma{}^{\beta]}$ by[3] $d\lambda^I - A\lambda^I = 2\lambda_\alpha \Omega^{[\alpha\beta]} u_\beta{}^I$. In $OSp(1|32)$-related models, $\Omega^{[\alpha\beta]} = 0$ and $A = 0$, hence λ^I is constant and we may set $\lambda^I = \delta^I_{31}$ using the *global* transformations of $GL(31, \mathbb{R})$, which is a rigid symmetry of the system of Killing spinors. This allows us to identify λ^α itself with one of the Killing spinors

$$G = Sp(32, \mathbb{R}) :$$
$$\epsilon_I{}^\alpha = (\epsilon_i{}^\alpha, \lambda^\alpha) , \quad \lambda^\alpha := C^{\alpha\beta}\lambda_\beta \quad i = 1, \ldots, 30 . \tag{4.2.6}$$

Without specifying a solution of the constraints (4.2.5) (or $g^{-1}g = I_{32}$), the moving frame possesses a $G = GL(32, \mathbb{R})$ symmetry. One may impose as additional constraints $\det(g) = 1$ or $\det(g^{-1}) = 1$ reducing G to $SL(32, \mathbb{R})$,

$$G = SL(32, \mathbb{R}) \ : \quad \det(g_\beta^{(\alpha)}) = 1 = \det(g^{-1}{}_{(\alpha)}{}^\beta) . \tag{4.2.7}$$

For instance, in the preonic case $k = 31$ this would imply

$$w^\alpha = \tfrac{1}{(31)!}\varepsilon^{\alpha\beta_1\ldots\beta_{31}} u_{\beta_1}{}^1 \ldots u_{\beta_{31}}{}^{31} . \tag{4.2.8}$$

Such a frame is most convenient to study the bosonic solutions of CJS supergravity, since the corresponding generalized holonomy must be a subgroup of $SL(32, \mathbb{R})$ (see section 2.3 of chapter 2 and references therein).

4.3 Generalized holonomy of preonic solutions

The Killing equation (3.1.3) for a $\nu = k/32$ supersymmetric solution,

$$\mathcal{D}\epsilon_J{}^\alpha = d\epsilon_J{}^\alpha - \epsilon_J{}^\beta \Omega_\beta{}^\alpha = 0 , \qquad J = 1, \ldots, k , \tag{4.3.1}$$

implies the following equations for the other components of the moving G-frame

$$\mathcal{D}\lambda_\alpha{}^r := d\lambda_\alpha{}^r + \Omega_\alpha{}^\beta \, \lambda_\beta{}^r = \lambda_\alpha{}^s \, A_s{}^r , \tag{4.3.2}$$

[3] To see this, one calculates $d\lambda^I = \mathcal{D}\lambda^I = (\mathcal{D}C^{\alpha\beta})\lambda_\beta u_\alpha{}^I + C^{\alpha\beta}(\mathcal{D}\lambda_\beta)u_\alpha{}^I + C^{\alpha\beta}\lambda_\beta \mathcal{D}u_\alpha{}^I$ and use equation (4.3.9), (4.3.10) to find $d\lambda^I = A\lambda^I + 2\lambda_\alpha\Omega^{[\alpha\beta]}u_\beta{}^I$.

$$\mathcal{D}u_\alpha{}^J := du_\alpha{}^J + \Omega_\alpha{}^\beta u_\beta{}^J = \lambda_\alpha{}^r \, B_r^J \; , \qquad (4.3.3)$$

$$\mathcal{D}w_r{}^\alpha := dw_r{}^\alpha - w_r{}^\beta \Omega_\beta{}^\alpha = -A_r{}^s w_s{}^\alpha - B_r^J \epsilon_J{}^\alpha \; , \qquad (4.3.4)$$

where $\alpha, \beta = 1, \ldots, 32,\ J = 1, \ldots, k,\ r, s = 1, \ldots, (32-k)$, and $A_s{}^r$ and $B_r{}^I$ are $(32-k) \times (32-k)$ and $(32-k) \times k$ arbitrary one-form matrices. To obtain the equations (4.3.2), (4.3.3), (4.3.4) one can take firstly the derivative $\mathcal{D}$ of the orthogonality relations (4.2.2), (4.2.3). After using equation (4.3.1), this results in

$$\epsilon_I{}^\alpha \mathcal{D}\lambda_\alpha{}^r = 0 \; , \quad \epsilon_I{}^\alpha \mathcal{D}u_\alpha{}^J = 0 \; , \qquad (4.3.5)$$

$$w_s{}^\alpha \mathcal{D}\lambda_\alpha{}^r = -\mathcal{D}w_s{}^\alpha \, \lambda_\alpha{}^r \; , \quad w_s{}^\alpha \mathcal{D}u_\alpha{}^J = -\mathcal{D}w_s{}^\alpha \, u_\alpha{}^J \; . \qquad (4.3.6)$$

Then, for instance, to derive (4.3.2), one uses the unity decomposition (4.2.5) to express $\mathcal{D}\lambda_\alpha{}^r$ through the contractions $w_s{}^\alpha \mathcal{D}\lambda_\alpha{}^r$ and $\epsilon_I{}^\alpha \mathcal{D}\lambda_\alpha{}^r$: $\mathcal{D}\lambda_\alpha{}^r \equiv \lambda_\alpha{}^s \, w_s{}^\beta \mathcal{D}\lambda_\beta{}^r + u_\alpha{}^I \, \epsilon_I{}^\beta \mathcal{D}\lambda_\beta{}^r$. The second term vanishes due to (4.3.5), while the first one is not restricted by the consequences of the Killing spinor equations and may be written as in equation (4.3.2) in terms of an arbitrary form $A_s{}^r \equiv w_s{}^\alpha \mathcal{D}\lambda_\alpha{}^r$.

Notice that, using the unity decomposition (4.2.5), one may also solve formally equations (4.3.1), (4.3.2), (4.3.3), (4.3.4) with respect to the generalized connection $\Omega_\alpha{}^\beta$ of equation (2.3.5),

$$\Omega_\alpha{}^\beta = A_r{}^s \, \lambda_\alpha{}^r w_s{}^\beta + B_r{}^J \lambda_\alpha{}^r \epsilon_J{}^\beta - (dgg^{-1})_\alpha{}^\beta \; , \qquad (4.3.7)$$

where $g_\alpha^{(\beta)}$ and $g^{-1}{}_{(\beta)}{}^\alpha$ are defined in equation (4.2.4) and, hence,

$$(dgg^{-1})_\alpha{}^\beta = d\lambda_\alpha{}^r \, w_r{}^\beta + du_\alpha{}^I \, \epsilon_I{}^\beta \; . \qquad (4.3.8)$$

For a BPS $\nu = 31/32$, preonic configuration, equations (4.3.2), (4.3.3), (4.3.4) read

$$\mathcal{D}\lambda_\alpha := d\lambda_\alpha + \Omega_\alpha{}^\beta \lambda_\beta = A\lambda_\alpha \; , \qquad (4.3.9)$$

$$\mathcal{D}u_\alpha{}^I := du_\alpha{}^I + \Omega_\alpha{}^\beta u_\beta{}^I = B^I \lambda_\alpha \; , \qquad (4.3.10)$$

$$\mathcal{D}w^\alpha := dw^\alpha - w^\beta \Omega_\beta{}^\alpha = -A \, w^\alpha - B^I \epsilon_I{}^\alpha \qquad (4.3.11)$$

and contain $1 + 31 = 32$ arbitrary one-forms A and B^I.

For $G = SL(32, \mathbb{R})$ one may choose $\det(g) = 1$, equation (4.2.7), which implies $\mathrm{tr}(dgg^{-1}) := (dgg^{-1})_\alpha{}^\alpha = 0$. Then the $sl(32, \mathbb{R})$-valued generalized connection $\Omega_\alpha{}^\beta$ ($\Omega_\alpha{}^\alpha = 0$) allowing for a $\nu = k/32$ supersymmetric configuration is determined by equation (4.3.7) with $A_r{}^r = 0$,

$$G = SL(32, \mathbb{R}) \; : \qquad A_r{}^r = 0 \; . \qquad (4.3.12)$$

In particular, the $sl(32, \mathbb{R})$-valued generalized connection allowing for a BPS preonic, $\nu = 31/32$, configuration, should have the form [3]

$$G = SL(32, \mathbb{R}) \; , \nu = 31/32 \; : \; \Omega_\alpha{}^\beta = B^I \; \lambda_\alpha \epsilon_I{}^\beta - (dgg^{-1})_\alpha{}^\beta \qquad (4.3.13)$$

in terms of 31 arbitrary one-forms B^I, $I = 1, \ldots, 31$.

Assuming a definite form for the generalized connection $\Omega_\alpha{}^\beta$, one finds that Eqs. (4.3.7) become differential equations for k Killing spinors $\epsilon_J{}^\alpha$ *and* $n = 32 - k$ *BPS preonic spinors* $\lambda_\alpha{}^r$ once $(dgg^{-1}) = d\lambda_\alpha{}^r w_r{}^\beta - u_\alpha{}^I d\epsilon_I{}^\beta$ (equation (4.3.8)) is taken into account. On the other hand, one might reverse the argument and ask for the structure of a theory allowing for $\nu = k/32$ supersymmetric solutions. This question is especially interesting for the case of BPS preonic and $\nu = 30/32$ solutions as, for the moment, such solutions are unknown in the standard $D = 11$ CJS and $D = 10$ Type II supergravities.

The simplest application of the moving G-frame construction is to find an explicit form for the general solution of the integrability conditions,

$$\epsilon_J{}^\beta \mathcal{R}_\beta{}^\alpha = 0 \; , \tag{4.3.14}$$

which are necessary for the Killing spinor equation (4.3.1). In (4.3.14), $\mathcal{R}_\beta{}^\alpha$ is the generalized curvature (2.3.9) corresponding to the $D = 11$ supergravity generalized connection $\Omega_\alpha{}^\beta$ of (2.3.5). To make things simpler, we shall consider that the solutions we are dealing with are such that their generalized holonomy is fully determined by $\mathcal{R}_\beta{}^\alpha$ and, like in the M2 and M5-brane cases (see section 3.3 of chapter 3), further supercovariant derivatives of $\mathcal{R}_\beta{}^\alpha$ do not provide additional essential information.

Since the Killing spinor equation (4.3.1) implies Eqs. (4.3.2), (4.3.3), one may solve instead the selfconsistency conditions for these equations,

$$\mathcal{D}\mathcal{D}\lambda_\alpha{}^r = \mathcal{R}_\alpha{}^\beta \lambda_\beta{}^r = \lambda_\alpha{}^s (dA - A \wedge A)_s{}^r \tag{4.3.15}$$

$$\mathcal{D}\mathcal{D}u_\alpha{}^I = \mathcal{R}_\alpha{}^\beta u_\beta{}^I = \lambda_\alpha{}^r (dB^I_r + B^I_s \wedge A_r{}^s) \; . \tag{4.3.16}$$

Using the unity decomposition (4.2.5), which implies $\mathcal{R}_\alpha{}^\beta = \mathcal{R}_\alpha{}^\gamma \lambda_\gamma{}^r \, w_r{}^\beta + \mathcal{R}_\alpha{}^\gamma u_\gamma{}^I \, \epsilon_I{}^\beta$, one finds the following expression for the generalized curvature

$$\mathcal{R}_\alpha{}^\beta = G_r{}^s \, \lambda_\alpha{}^r w_s{}^\beta + \nabla B^I_r \lambda_\alpha{}^r \epsilon_I{}^\beta \; , \tag{4.3.17}$$

where

$$G_r{}^s := (dA - A \wedge A)_r{}^s \; , \tag{4.3.18}$$

$$\nabla B^I_r := dB^I_r - A_r{}^s \wedge B^I_s \; , \tag{4.3.19}$$

For $k = 31$, corresponding to the case of a BPS preon, equation (4.3.17) simplifies to [3]

$$\mathcal{R}_\alpha{}^\beta = dA \, \lambda_\alpha w^\beta + (dB^I + B^I \wedge A) \, \lambda_\alpha \epsilon_I{}^\beta \; . \tag{4.3.20}$$

Equations (4.3.17) and (4.3.20) imply $\mathcal{R}_\alpha{}^\beta = \lambda_\alpha{}^r(\cdots)_r{}^\beta$ and, thus, due to the orthogonality condition (4.2.2) they solve equation (4.3.14), $\epsilon_I{}^\beta \mathcal{R}_\beta{}^\alpha = 0$.

The conditions $G \subset SL(32,\mathbb{R})$ and hence, also for the generalized holonomy group, $\mathrm{Hol}(\Omega) \subset SL(32,\mathbb{R})$, $\mathcal{R}_\alpha{}^\alpha = 0$ (which is always the case for bosonic solutions of 'free' CJS [87, 107] and Type II supergravities [108]), imply $A_r{}^r = 0$ in equation (4.3.17) [see equation (4.3.12)], while for $k = 31$ equation (4.3.20) simplifies to [3]

$$\mathrm{Hol}(\Omega) \subset SL(32,\mathbb{R}) \ , \ k = 31 \ : \quad \mathcal{R}_\alpha{}^\beta = dB^I \lambda_\alpha \epsilon_I{}^\beta \ . \qquad (4.3.21)$$

Finally, for $G \subset Sp(32,\mathbb{R})$ $\Omega^{[\alpha\beta]} = 0$, then $\mathrm{Hol}(\Omega) \subset Sp(32,\mathbb{R})$, $\mathcal{R}^{\alpha\beta} := C^{\alpha\gamma}\mathcal{R}_\gamma{}^\beta = \mathcal{R}^{(\alpha\beta)}$, and equation (4.3.21) reduces to [3]

$$\mathrm{Hol}(\Omega) \subset Sp(32,\mathbb{R}) \ , \ k = 31 \ : \quad \mathcal{R}_\alpha{}^\beta = dB\, \lambda_\alpha \lambda^\beta \ , \qquad (4.3.22)$$

where only one arbitrary one-form B appears [to obtain (4.3.22) one has to keep in mind that $\epsilon_I{}^\alpha = (\epsilon_i{}^\alpha, C^{\alpha\beta}\lambda_\beta)$, $I = (i, 31)$, equation (4.2.6)]. *Eqs. (4.3.21), (4.3.22) solve equation (4.3.14) for preons when* $G = SL(32,\mathbb{R})$ *and* $G = Sp(32,\mathbb{R})$*, respectively.*

Equation (4.3.17) with $A_r{}^r = 0$ (Eq. (4.3.12), and, hence, $(dA - A \wedge A)_r{}^r = 0$) provides an explicit expression for the result of equation (3.3.8), namely, for the fact that a k-supersymmetric solution of either $D = 11$ or $D = 10$ Type II supergravities must have its generalized holonomy group contained in $\mathrm{Hol}(\Omega) \subset SL(32-k,\mathbb{R}) \ltimes \mathbb{R}^{k(32-k)}$. For a BPS preon $k = 31$, and $\mathrm{Hol}(\Omega) \subset \mathbb{R}^{31}$ as expressed by equation (4.3.21). However, our explicit expressions for the $[sl(32-k,\mathbb{R}) \ltimes (\mathbb{R}^{(32-k)} \oplus \overset{k}{\dots} \oplus \mathbb{R}^{(32-k)}]$-valued generalized curvatures $\mathcal{R}_\alpha{}^\beta$, Eqs. (4.3.17), (4.3.21), given in terms of the Killing spinors $\epsilon_I{}^\beta$ and bosonic spinors $\lambda_\alpha{}^r$ characterizing the BPS preon contents of a $\nu = k/32$ BPS state, may be useful in searching for new supersymmetric solutions, including preonic $\nu = 31/32$ ones. Some steps in this direction are taken in the next section.

4.4 BPS preons in supergravity

4.4.1 BPS preons in Chern-Simons supergravity

The first observation is that the generalized curvature allowing for a BPS preonic ($k = 31$ supersymmetric) configuration for the case of $\mathrm{Hol}(\Omega) \subset SL(32,\mathbb{R})$ holonomy, equation (4.3.21), is nilpotent

$$\mathcal{R}_\alpha{}^\gamma \wedge \mathcal{R}_\gamma{}^\beta = 0 \ , \quad \text{for} \quad \mathrm{Hol}(\Omega) \subset SL(32,\mathbb{R}) \ , \ k = 31 \ . \qquad (4.4.1)$$

As a result it solves [3] the purely bosonic equations of a Chern-Simons supergravity (see [91]),

$$\mathcal{R}_\alpha{}^{\gamma_1} \wedge \mathcal{R}_{\gamma_1}{}^{\gamma_2} \wedge \mathcal{R}_{\gamma_2}{}^{\gamma_3} \wedge \mathcal{R}_{\gamma_3}{}^{\gamma_4} \wedge \mathcal{R}_{\gamma_4}{}^\beta = 0 \ . \qquad (4.4.2)$$

The same is true for $\mathrm{Hol}(\Omega) \subset Sp(32,\mathbb{R}) \subset SL(32,\mathbb{R})$, where $\mathcal{R}$ is given by equation (4.3.22). Thus, there exist BPS preonic solutions in CS supergravity theories, including $OSp(1|32)$-type ones.

Note that equation (4.4.1) follows in general for a preonic configuration only. In fact, it implies that the generalized holonomy algebra is abelian, in agreement with the fact noted above that 31-supersymmetric solutions have their generalized holonomy groups $\mathrm{Hol}(\Omega)$ in $\mathbb{R}^{31}$. For configurations preserving $k \leq 30$ of the 32 supersymmetries, the bosonic equations of a CS supergravity, Eqs. (4.4.2) reduce to (see (4.3.18), (4.3.19))

$$\begin{aligned} &G_s{}^{s_2} \wedge G_{s_2}{}^{s_3} \wedge G_{s_3}{}^{s_4} \wedge G_{s_4}{}^{s_5} \wedge G_{s_5}{}^{r} = 0 \ , \\ &G_s{}^{s_2} \wedge G_{s_2}{}^{s_3} \wedge G_{s_3}{}^{s_4} \wedge G_{s_4}{}^{r} \wedge \nabla B_r{}^I = 0 \ , \end{aligned} \tag{4.4.3}$$

which are not satisfied identically for $G_r{}^r = 0$. Eqs. (4.4.3) are satisfied *e.g.*, by configurations with $G_s{}^r = 0$, for which the generalized holonomy group is reduced down to $\mathrm{Hol}(\Omega) \subset \mathbb{R}^{k(32-k)}$, $\mathcal{R}_\beta{}^\alpha = \nabla B_r^I \lambda_\beta{}^r \epsilon_I{}^\alpha$. Thus, *only* the preonic, $\nu = 31/32$, configurations *always* solve the Chern-Simons supergravity equations (4.4.2).

4.4.2 Searching for preonic solutions of the free bosonic CJS equations

We now go back to the question of whether BPS $\nu = 31/32$ (preonic) solutions exist for the standard CJS supergravity [27]. This problem can be addressed step by step, beginning by studying the existence of preonic solutions of the 'free' bosonic CJS equations. To this aim it is useful to observe [80, 3, 1] that these equations may be collected in a compact expression for the generalized curvature, $i_a \mathcal{R}_\alpha{}^\gamma \Gamma^a{}_\gamma{}^\beta = 0$ (equation (2.5.5) of chapter 2). The generalized curvature of a BPS preonic configuration satisfies equation (4.3.21), and thus it solves the 'free' bosonic CJS supergravity equations (2.5.5) if [3]

$$i_a dB^I \, \epsilon_I{}^\alpha \Gamma^a{}_\alpha{}^\beta = 0 \, . \tag{4.4.4}$$

Actually, equation (4.3.21) substituted in (2.5.5) gives

$$\lambda_\alpha \, i_a dB^I \, \epsilon_I{}^\gamma \Gamma^a{}_\gamma{}^\beta = 0 \, . \tag{4.4.5}$$

However, since $\lambda_\alpha \neq 0$, this is equivalent to (4.4.4).

Equation (4.4.4) contains a summed $I = 1, \ldots, 31$ index and, as a result, it is not easy to handle. It would be much easier to deal with the expression $\Gamma^a{}_\alpha{}^\gamma i_a \mathcal{R}_\gamma{}^\beta$ which, with equation (4.3.21) is equal to $\Gamma^a{}_\alpha{}^\gamma \lambda_\gamma i_a dB^J \epsilon_J{}^\beta$. Indeed, $(\Gamma^a \lambda)_\alpha i_a dB^J \epsilon_J{}^\beta = 0$, for instance, would imply $(\Gamma^a\lambda)_\alpha i_a dB^J = 0$ which may be shown to have only trivial solutions. However, $\Gamma^a{}_\alpha{}^\gamma i_a \mathcal{R}_\gamma{}^\beta \neq 0$ in general *for a solution of the 'free' bosonic CJS equations* (equation (2.5.5)),

$$\Gamma^a{}_\alpha{}^\gamma i_a \mathcal{R}_\gamma{}^\beta = -\tfrac{i}{12}\left(D\hat{F}_\alpha{}^\beta + \mathcal{O}(F\,F)\right) , \tag{4.4.6}$$

where $D = e^a D_a$ is the Lorentz covariant derivative (not to be confused with $\mathcal{D}$ defined in Eqs. (4.3.1), (2.3.5)),

$$\hat{F}_\alpha{}^\beta = F_{a_1a_2a_3a_4}(\Gamma^{a_1a_2a_3a_4})_\alpha{}^\beta , \tag{4.4.7}$$

and $\mathcal{O}(F\,F)$ denotes the terms of second order in $F_{c_1c_2c_3c_4}$,

$$\begin{aligned}\mathcal{O}(F\,F)=&\tfrac{1}{(3!)^2\,4!}\, e^a \left(\Gamma_a{}^{b_1b_2b_3} + 2\delta_a^{[b_1}\Gamma^{b_2b_3]}\right)\epsilon_{b_1b_2b_3c_1\ldots c_4d_1\ldots d_4}F^{c_1\ldots c_4}F^{d_1\ldots d_4}\\ &+\tfrac{2i}{3}e^a\left(\Gamma_a{}^{b_1b_2b_3b_4} + 3\delta_a^{[b_1}\Gamma^{b_2b_3b_4]}\right)F_{cdb_1b_2}F^{cd}{}_{b_3b_4}\\ &+\tfrac{8i}{9}e^a\Gamma^{b_1b_2b_3b_4b_5}F_{acb_1b_2}F^c{}_{b_3b_4b_5} .\end{aligned} \tag{4.4.8}$$

Equation (4.3.21) then implies that for a hypothetical preonic solution of the 'free' bosonic CJS equations, the gauge field strength F_{abcd} should be nonvanishing (otherwise $dB^J = 0$ and $\mathcal{R}_\alpha{}^\beta = 0$, see above) and satisfy

$$\Gamma^a{}_\alpha{}^\gamma\lambda_\gamma\, i_a dB^J\, \epsilon_J{}^\beta = -\tfrac{i}{12}\left(D\hat{F}_\alpha{}^\beta + \mathcal{O}(F\,F)\right) . \tag{4.4.9}$$

Using (4.2.3), Eqs. (4.4.9) split into a set of restrictions for F_{abcd},

$$\left(D\hat{F} + \mathcal{O}(F\,F)\right)_\alpha{}^\beta\lambda_\beta = 0 , \tag{4.4.10}$$

and equations for dB^I,

$$\Gamma^a{}_\alpha{}^\gamma\lambda_\gamma\, i_a dB^I = -\tfrac{i}{12}\left(D\hat{F} + \mathcal{O}(F\,F)\right)_\alpha{}^\beta\, u_\beta{}^I. \tag{4.4.11}$$

Eq. (4.4.9) or, equivalently, Eqs. (4.4.10),(4.4.11) are the equations to be satisfied by a CJS preonic configuration [3]. Note that if a non-trivial solution of the above equations with some $F_{abcd} \neq 0$ and some $dB^I \neq 0$ is found, one would have then to check in particular that such a solution satisfies $ddB^I = 0$ and $D_{[e}F_{abcd]} = 0$.

On the other hand, if the general solution of the above equation turned out to be trivial, $dB^I = 0$, this would imply $\mathcal{R}_\alpha{}^\beta = 0$ and, thus, a trivial generalized holonomy group, $\mathrm{Hol}(\Omega) = 1$. However, this is the necessary condition for fully supersymmetric, $k = 32$, solutions [109]. Hence a general trivial solution for Eqs. (4.4.10), (4.4.11) would indicate that a solution preserving 31 supersymmetries possesses all 32 ones (thus corresponding to a fully supersymmetric vacuum) and, hence, that there are no preonic, $\nu = 31/32$ solutions of the *free bosonic* CJS supergravity equations (2.5.1), (2.5.3), (2.5.2) and (2.4.2) and (2.4.4). If this happened to be the case, one would have to study the existence of preonic solutions for the CJS supergravity equations with non-trivial right hand sides. These could be produced by corrections of higher-order in curvature [125, 126, 61] and by the presence of sources (from some possibly exotic p-branes).

4.5 On possible preonic branes

4.5.1 Brane solutions and worldvolume actions

As far as supersymmetric p-brane solutions of supergravity equations are concerned, the usual situation is that to $\nu = 1/2$ supersymmetric solutions ($\nu = 16/32$ in the $D = 11$ and $D = 10$ Type II cases) there also exist worldvolume actions in the corresponding ($D = 11$ or $D = 10$ Type II) superspaces possessing 16 κ-symmetries, exactly the number of supersymmetries preserved by the supergravity solitonic solutions. The *κ-symmetry–preserved supersymmetry* correspondence was further discussed and extended for the case of $\nu < 1/2$ multi-brane solutions in [147, 58].

In this perspective one may expect that if preonic $\nu = 31/32$ supersymmetric solutions of the CJS equations with a source do exist, a worldvolume action possessing 31 κ-symmetries should also exist in a curved $D = 11$ superspace. For the time being, no such actions are known in the *standard* $D = 11$ superspace, but they do exist in a superspace enlarged with additional tensorial 'central' charge coordinates (see chapter 7 and [8, 148, 149, 150]). One might expect that the role of these additional tensorial coordinates could be taken over by the tensorial fields of supergravity. But this would imply that the corresponding action does not exist in the flat standard $D = 11$ superspace as it would require a contribution from the above additional field degrees of freedom (replacing the tensorial coordinate ones as in [85]). This lack of a clear flat *standard* superspace limit hampers the way towards a hypothetical worldvolume action for a BPS preon in the usual curved $D = 11$ superspace.

Nevertheless, a shortcut in the search for such an action may be provided by the observation [151] that the superfield description of the dynamical supergravity–superbrane interacting system, described by the sum of the *superfield* action for supergravity (still unknown for $D = 10, 11$) and the super-p-brane action, is gauge equivalent to the much simpler dynamical system described by the sum of the spacetime, component action for supergravity and the action *for the purely bosonic limit* of the super-p-brane. This bosonic p-brane action carries the memory of being the bosonic limit of a super-p-brane by still possessing 1/2 of the spacetime local supersymmetries [152]; this preservation of local supersymmetry reflects the κ-symmetry of the original super-p-brane action.

Thus the κ-symmetric worldvolume actions for super-p-branes have a clear spacetime counterpart: the purely bosonic actions in spacetime possessing a part of local spacetime supersymmetry of a 'free' supergravity theory. This fact, although explicitly discussed for the standard, $\nu = 1/2$ superbranes in [151], is general since it follows from symmetry considerations only and thus it applies to any superbrane, including a hypothetical preonic one. The number of supersymmetries possessed by this bosonic brane action coincides with the number of κ-symmetries of the parent super-p-brane action. Moreover, these supersymmetries are extracted by a projector which may be identified with the bosonic limit of the κ-symmetry projector for the superbrane. With this guideline in mind one may simplify, in a first stage, the search for a worldvolume action for a BPS preon in standard supergravity (or in a model minimally extending the standard supergravity) by discussing the bosonic limit that such a hypothetical action should have.

4.5.2 BPS preons in D'Auria-Fré supergravity

Consider [3] a symmetric spin-tensor one-form $e^{\alpha\beta} = e^{\beta\alpha} = dx^\mu e_\mu^{\alpha\beta}(x)$, transforming under local supersymmetry as

$$\delta_\varepsilon e^{\alpha\beta} = -2i\psi^{(\alpha}\,\varepsilon^{\beta)}\ , \qquad (4.5.1)$$

where ψ^α is a fermionic one-form,

$$\psi^\alpha = dx^\mu \psi_\mu^\alpha(x)\ , \qquad (4.5.2)$$

which we may identify with the gravitino. Let us consider for simplicity the worldline action (*cf.* [148])

$$\begin{aligned} S &= \int_{W^1} \lambda_\alpha(\tau)\,\lambda_\beta(\tau)\hat{e}^{\alpha\beta} \\ &= \int_{W^1} d\tau \lambda_\alpha(\tau)\,\lambda_\beta(\tau)\, e_\mu^{\alpha\beta}(\hat{x}(\tau))\,\partial_\tau \hat{x}^\mu(\tau)\ , \end{aligned} \qquad (4.5.3)$$

where τ parameterizes the worldline W^1 in $D = 11$ spacetime, $\hat{e}^{\alpha\beta} := d\tau\partial_\tau\hat{x}^\mu(\tau)\, e_\mu^{\alpha\beta}(\hat{x}(\tau))$ and $\lambda_\alpha(\tau)$ is an auxiliary spinor field on the worldline W^1. The extended ($p \geq 1$) object counterpart of this worldline action is the following action for tensionless p-branes (cf. [149, 150])

$$S_{p+1} = \int_{W^{p+1}} \lambda_\alpha\lambda_\beta\hat{\rho} \wedge \hat{e}^{\alpha\beta}$$

$$= \int_{W^{p+1}} d^{p+1}\xi\, \rho^k \lambda_\alpha \lambda_\beta \hat{e}_\mu^{\alpha\beta} \partial_k \hat{x}^\mu \ , \qquad (4.5.4)$$

where $\hat{\rho}(\xi)$ is a p-form auxiliary field, and $\rho^k(\xi)$ is the worldvolume vector density (see [153, 154]) related to $\hat{\rho}(\xi)$ by $\hat{\rho}(\xi) = (1/p!)d\xi^{j_p} \wedge \ldots \wedge d\xi^{j_1} \rho_{j_1 \ldots j_p}(\xi) = (1/p!)d\xi^{j_p} \wedge \ldots \wedge d\xi^{j_1} \epsilon_{j_1 \ldots j_p k} \rho^k(\xi)$.

The action (4.5.3) possesses all but one of the local spacetime supersymmetries[4], equation (4.5.1), 31 for $\alpha, \beta = 1, \ldots, 32$ corresponding to $D = 11$. Indeed, performing a supersymmetric variation δ_ε of (4.5.3) assuming $\delta_\varepsilon \lambda_\alpha(\tau) = 0$, one finds

$$\delta_\varepsilon S \ = \ -2i \int_{W^1} \hat{\psi}^\alpha \lambda_\alpha(\tau)\, \hat{\varepsilon}^\beta \lambda_\beta(\tau) \ . \qquad (4.5.5)$$

Thus, one sees that $\delta_\varepsilon S = 0$ for the supersymmetry parameters on W^1 that obey (*cf.* (4.2.2))

$$\hat{\varepsilon}^\beta \lambda_\beta(\tau) = 0 \quad (\hat{\varepsilon}^\beta := \varepsilon^\beta(\hat{x}(\tau))) \ . \qquad (4.5.6)$$

Equation (4.5.6) possesses 31 solutions, which may be expressed through worldvolume spinors $\hat{\epsilon}_I{}^\alpha(\tau)$ (the worldline counterparts of the Killing spinors) orthogonal to $\lambda_\alpha(\tau)$, $\hat{\epsilon}_I{}^\alpha(\tau)\lambda_\alpha(\tau) = 0$, as

$$\hat{\varepsilon}^\beta = \varepsilon^I(\tau) \hat{\epsilon}_I{}^\beta \quad , \quad I = 1, \ldots, 31 \quad , \qquad (4.5.7)$$

for some arbitrary $\varepsilon^I(\tau)$. The same is true for the tensionless p-branes described by the action (4.5.4).

Thus, the actions (4.5.3), (4.5.4) possess 31 of the 32 local spacetime supersymmetries (4.5.1) and, in the light of the discussion of the previous subsection, can be considered as the spacetime counterparts of a superspace BPS-preonic action (hypothetical in the standard superspace but known [148, 155, 149, 150] in flat maximally enlarged or tensorial superspaces).

The question that remains to be settled is the meaning of the symmetric spin-tensor one-form $e^{\alpha\beta}$ with the local supersymmetry transformation rule (4.5.1) in $D = 11$ supergravity. The contraction of $e^{\alpha\beta}$ with the Dirac matrix Γ^a,

$$e^a = e^{\alpha\beta} \Gamma^a_{\alpha\beta} \quad , \qquad (4.5.8)$$

may be identified with the $D = 11$ vielbein. Decomposing $e^{\alpha\beta}$ in the basis of the $D = 11$ $Spin(1, 10)$ gamma-matrices,

$$e^{\alpha\beta} \ = \ e^{\beta\alpha} = \tfrac{1}{32} \left(e^a \Gamma_a^{\alpha\beta} - \tfrac{i}{2!} B^{ab} \Gamma_{ab}^{\alpha\beta} + \tfrac{1}{5!} B^{a_1 \ldots a_5} \Gamma^{\alpha\beta}_{a_1 \ldots a_5} \right) \ , \qquad (4.5.9)$$

[4] Notice that when a brane action is considered in a supergravity *background*, the local spacetime supersymmetry is not a gauge symmetry of that action but rather a transformation of the background; it becomes a gauge symmetry only when a supergravity action is added to the brane one so that supergravity is dynamical.

one finds [3] that $e^{\alpha\beta}$ also contains the antisymmetric tensor one-forms $B^{ab}(x) = dx^\mu B_\mu^{ab}(x)$ and $B^{a_1 \cdots a_5}(x) = dx^\mu B_\mu^{a_1 \cdots a_5}(x)$. Such fields, whose supersymmetry transformation properties follow from (4.5.9) and (4.5.1), also appear among the additional fields introduced in [92] in order to investigate the hidden gauge symmetry of $D = 11$ supergravity, which will be discussed in chapter 6. In this case, however, the degrees of freedom of the B fields in (4.5.3) will not be reduced by the gauge symmetry to be discussed in chapter 6. Thus, the action (4.5.3), preserving 31 out of 32 supersymmetries, could be treated as a worldline action for a BPS preon in the presence of supergravity with additional fields *à la* D'Auria and Fré [92].

The formulation of $D = 11$ supergravity due to D'Auria and Fré [92] is, actually, closely related to enlarged superspaces so, in this sense, it is not surprising that preonic branes would exist in such a context (given that preonic actions are known in enlarged superspaces, see [84] for a review). It is, then, worthwhile both to take a closer look at the D'Auria-Fré approach to supergravity and to further study actions for supersymmetric extended objects in enlarged superspaces. We shall, thus, turn our attention to these issues in chapters 6 and 7 respectively. In particular, the symmetry algebras underlying the construction of supergravity *à la* D'Auria and Fré will be reviewed in chapter 6. These algebras were known to be fermionic central extensions of the M Theory superalgebra, but their expected relation to the orthosymplectic superalgebra $osp(1|32)$ was quite unclear. In chapter 6 this relation will be discussed in terms of *Lie algebra expansions*, a new method of building up new algebras from given ones. It seems appropriate, then, to stop our physical discussion momentarily and open up a purely mathematical parenthesis to introduce the expansion method in the next chapter.

5

Interlude: Lie algebra expansions

Setting aside the problem of finding whether an algebra is a subalgebra of another one there are, essentially, three different ways of relating and/or obtaining new algebras from given ones: contractions, deformations and extensions. In this chapter we explore a fourth way to obtain new algebras of increasingly higher dimensions from a given one $\mathcal{G}$. The idea, originally considered in [156] in a less general context and developed in general in [4] (see also [5]) consists in looking at the algebra $\mathcal{G}$ as described by the Maurer-Cartan (MC) forms[1] on the manifold of its associated group G and, after rescaling some of the group parameters by a factor λ, in expanding the MC forms as a series in λ. The resulting *expansion* method is different from the three above albeit, when the algebra dimension does no change in the process, it may lead to a simple Inönü-Wigner (IW) or IW-generalized contraction (see section 5.1), but not always. Furthermore, the algebras to which it leads have in general a higher dimension than the original one (hence the *expansion* name), in which case they cannot be related to it by any contraction or deformation process.

A description of the expansion method is given in this rather technical chapter. Our main concern will be its application to Lie superalgebras, so we proceed step by step towards that goal. First of all, the brief review in section 5.1 of the three already known methods to obtain new Lie algebras from given ones will be useful in order to discuss the properties and structure of the algebras encountered in the rest of the chapter (and other parts of this Thesis). Section 5.2 introduces the expansion method for Lie algebras $\mathcal{G}$. When further assumptions are made about the structure of the original Lie algebras, the results provided by the expansion method are more interesting. That is why the existence of a subalgebra in $\mathcal{G}$ is assumed in section 5.3, and the further existence of a symmetric coset is

[1] See section 2.1 of chapter 2 for the dual formulation of Lie algebras in terms of MC one-forms.

assumed in subsection 5.3.1. Section 5.4 generalizes in a convenient way the case in which $\mathcal{G}$ contains a subalgebra, by assuming that there is a certain subspace splitting of $\mathcal{G}$. All these cases are actually combined in section 5.5 to discuss the expansions of Lie superalgebras. The chapter concludes with an explicit example: the derivation of the M Theory algebra from an expansion of $osp(1|32)$. Appendix B contains some technical details. This chapter follows closely references [4] and [5].

5.1 Three well-known ways to relate Lie (super)algebras

Contractions

The *first* one is the *contraction* procedure [157, 158, 159]. In its İnönü and Wigner (IW) simple form [158], the contraction $\mathcal{G}_c$ of a Lie algebra $\mathcal{G}$ is performed with respect to a subalgebra $\mathcal{L}_0$ by rescaling the basis generators of the coset $\mathcal{G}/\mathcal{L}_0$ by means of a parameter, and then by taking a singular limit for this parameter. The generators in $\mathcal{G}/\mathcal{L}_0$ become abelian in the contracted algebra $\mathcal{G}_c$, and the subalgebra $\mathcal{L}_0 \subset \mathcal{G}_c$ acts on them. As a result, $\mathcal{G}_c$ has a semidirect structure, and the abelian generators determine an ideal of $\mathcal{G}_c$; obviously, $\mathcal{G}_c$ has the same dimension as $\mathcal{G}$. The contraction process has well known physical applications as *e.g.*, in understanding the non-relativistic limit from a group theoretical point of view, or to explain the appearance of dimensionful generators when the original algebra $\mathcal{G}$ is semisimple (and hence with dimensionless generators). This is achieved by using a dimensionful contraction parameter, as in the derivation of the Poincaré group from the de Sitter groups (there, the parameter is the radius R of the universe, and the limit is $R \to \infty$). There have been many discussions and variations of the IW contraction procedure (see [160, 161, 162, 163, 164, 165] to name a few), but all of them have in common that $\mathcal{G}$ and $\mathcal{G}_c$ have, necessarily, the same dimension as vector spaces.

This procedure can be extended to generalized IW contractions in the sense of Weimar-Woods (W-W) [165]. These are defined when $\mathcal{G}$ can be split in a sum of vector subspaces

$$\mathcal{G} = V_0 \oplus V_1 \oplus \cdots \oplus V_n = \bigoplus_{s=0}^{n} V_s, \tag{5.1.1}$$

(V_0 being the vector space of the subalgebra $\mathcal{L}_0$), such that the following conditions are satisfied:

$$c_{i_p j_q}^{k_s} = 0 \text{ if } s > p+q \qquad i.e. \qquad [V_p, V_q] \subset \bigoplus_s V_s,\ s \le p+q\ , \tag{5.1.2}$$

where i_p labels the generators of $\mathcal{G}$ in V_p, and c_{ij}^k are structure constants of $\mathcal{G}$. Then the W-W [165] contracted algebra is obtained by rescaling the group parameters as $g^{i_p} \mapsto \lambda^p g^{i_p}$, $p = 0, \ldots, n$, and then by taking a singular limit for λ. The contracted Lie algebra obtained this way, $\mathcal{G}_c$, has the same dimension as $\mathcal{G}$. The case $n = 1$ corresponds to the simple IW contraction.

Deformations

The *deformation* of algebras, and Lie algebras in particular [166, 167, 168, 169] (see also [170, 171]), allows us to obtain algebras *close*, but not isomorphic, to a given one. This leads to the important notion of rigidity [166, 167, 169] (or physical stability): an algebra is called *rigid* when any attempt to deform it leads to an equivalent (isomorphic) one. From a physical point of view, the deformation process is essentially the inverse to the contraction one (see [170] and the second ref. in [165]), and the dimensions of the original and deformed Lie algebras are again the same. For instance, the Poincaré algebra is not rigid, but the de Sitter algebras, being semisimple, have trivial second cohomology group by the Whitehead lemma and, as a result, they are rigid. One may also consider the Poincaré algebra as a deformation of the Galilei algebra, so that this deformation may be read as a group theoretical prediction of relativity. Thus, the mathematical deformation may be physically considered as a tool for developing a physical theory from another pre-existing one.

Deformations are performed by modifying the r.h.s. of the original commutators by adding new terms that depend on a parameter t in the form

$$[X, Y]_t = [X, Y]_0 + \sum_{i=1}^{\infty} \omega_i(X, Y) t^i \ , \quad X, Y \in \mathcal{G} \ , \quad \omega_i(X, Y) \in \mathcal{G} \ . \quad (5.1.3)$$

Checking the Jacobi identities up to $O(t^2)$, it is seen that the expression satisfied by ω_1 characterizes it as a two-cocycle so that the second Lie algebra cohomology group $H^2(\mathcal{G}, \mathcal{G})$ of $\mathcal{G}$ with coefficients in the Lie algebra $\mathcal{G}$ itself is the group of infinitesimal deformations of $\mathcal{G}$. Thus $H^2(\mathcal{G}, \mathcal{G}) = 0$ is a sufficient condition for rigidity [166, 167, 168, 169].

Extensions

In contrast with the previous procedures, the initial data of the extension problem include *two* algebras $\mathcal{G}$ and $\mathcal{A}$. A Lie algebra $\tilde{\mathcal{G}}$ is an extension of the Lie algebra $\mathcal{G}$ by the Lie algebra $\mathcal{A}$ if $\mathcal{A}$ is an ideal of $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}/\mathcal{A} = \mathcal{G}$. As a result, $\dim \tilde{\mathcal{G}} = \dim \mathcal{G} + \dim \mathcal{A}$, so this process is also 'dimension preserving'.

Given $\mathcal{G}$ and $\mathcal{A}$, in order to obtain an extension $\tilde{\mathcal{G}}$ of $\mathcal{G}$ by $\mathcal{A}$ it is necessary to specify first an action ρ of $\mathcal{G}$ on $\mathcal{A}$ *i.e.*, a Lie algebra homomorphism $\rho : \mathcal{G} \longrightarrow \mathrm{End}\,\mathcal{A}$. The different possible extensions $\tilde{\mathcal{G}}$ for $(\mathcal{G}, \mathcal{A}, \rho)$ and the possible obstructions to the extension process are, once again, governed by cohomology (see [40] and references therein). To be more explicit, let $\mathcal{A}$ be abelian. The extensions are governed by $H^2_\rho(\mathcal{G}, \mathcal{A})$. Some special cases are: 1) trivial action $\rho = 0$, $H^2_0(\mathcal{G}, \mathcal{A}) \neq 0$. These are central extensions, in which $\mathcal{A}$ belongs to the centre of $\tilde{\mathcal{G}}$; they are determined by non-trivial $\mathcal{A}$-valued two-cocycles on $\mathcal{G}$, and non-equivalent extensions correspond to non-equivalent cocycles; 2) non-trivial action $\rho \neq 0$, $H^2_\rho(\mathcal{G}, \mathcal{A}) = 0$ (semidirect extension of $\mathcal{G}$ by $\mathcal{A}$); and 3) $\rho = 0$, $H^2(\mathcal{G}, \mathcal{A}) = 0$ (direct sum of $\mathcal{G}$ and $\mathcal{A}$, $\tilde{\mathcal{G}} = \mathcal{G} \oplus \mathcal{A}$, or trivial extension).

Well-known examples of extensions in Physics include the centrally extended Galilei algebra, which is relevant in quantum mechanics, or the M Theory superalgebra that, without the Lorentz automorphisms part, is the maximal central extension of the abelian $D = 11$ supertranslations algebra (see section 2.1 of chapter 2 and [82, 59, 85]).

5.2 The expansion method

Let G be a Lie group, of local coordinates g^i, $i = 1, \ldots, r = \dim G$. Let $\mathcal{G}$ be its Lie algebra[2] of basis $\{X_i\}$, which may be realized by left-invariant generators $X_i(g)$ on the group manifold. Let $\mathcal{G}^*$ be the coalgebra, and let $\{\omega^i(g)\}$, $i = 1, \ldots, r = \dim G$ be the basis determined by the (dual, left-invariant) Maurer-Cartan (MC) one-forms on G. Then, when $[X_i, X_j] = c_{ij}^k X_k$, the MC equations read

$$d\omega^k(g) = -\frac{1}{2} c_{ij}^k \omega^i(g) \wedge \omega^j(g) \ , \quad i, j, k = 1, \ldots, r \quad . \tag{5.2.1}$$

We wish to show in this section how we may obtain new algebras by means of a redefinition $g^l \to \lambda g^l$ of some of the group parameters and by looking at the power series expansion in λ of the resulting one-forms $\omega^i(g, \lambda)$. Let θ be the left-invariant canonical form on G,

$$\theta(g) = g^{-1} dg = e^{-g^i X_i}\, d e^{g^i X_i} \equiv \omega^i X_i \ . \tag{5.2.2}$$

Since

$$e^{-A}\, de^{A} = dA + \frac{1}{2}[dA, A] + \frac{1}{3!}[[dA, A], A] + \frac{1}{4!}[[[dA, A], A], A] + \ldots$$

[2] Calligraphic $\mathcal{G}$, $\mathcal{L}$, $\mathcal{W}$ will denote both the Lie algebras and their underlying vector spaces; V, W etc. will be used for vector spaces that are not necessarily Lie algebras.

$$= dA + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} [\overset{n}{\ldots} [dA, A], \ldots, A], A] \ , \tag{5.2.3}$$

one obtains, for $A \equiv g^k X_k$, $dA = (dg^j)X_j$, the expansion of $\theta(g)$ and of the MC forms $\omega^i(g)$ as polynomials in the group coordinates g^i :

$$\theta(g) = \Big[\delta^i_j + \frac{1}{2!} c^i_{jk} g^k + \frac{1}{3!} c^{h_1}_{jk_1} c^i_{h_1 k_2} g^{k_1} g^{k_2} + \frac{1}{4!} c^{h_1}_{jk_1} c^{h_2}_{h_1 k_2} c^i_{h_2 k_3} g^{k_1} g^{k_2} g^{k_3} + \ldots \Big] dg^j X_i \ , \tag{5.2.4}$$

$$\omega^i(g) = \Big[\delta^i_j + \frac{1}{2!} c^i_{jk} g^k + \sum_{n=2}^{\infty} \frac{1}{(n+1)!} c^{h_1}_{jk_1} c^{h_2}_{h_1 k_2} \ldots c^{h_{n-1}}_{h_{n-1} k_{n-1}} c^i_{h_{n-1} k_n} g^{k_1} g^{k_2} \ldots g^{k_{n-1}} g^{k_n} \Big] dg^j \ . \tag{5.2.5}$$

Looking at (5.2.5), it is evident that the redefinition

$$g^l \to \lambda g^l \tag{5.2.6}$$

of *some* coordinates g^l will produce an expansion of the MC one-forms $\omega^i(g, \lambda)$ as a sum of one-forms $\omega^{i,\alpha}(g)$ on G multiplied by the corresponding powers λ^α of λ.

5.2.1 The Lie algebras $\mathcal{G}(N)$ expanded from $\mathcal{G}$

Consider, as a first example, the splitting of $\mathcal{G}^*$ into the sum of two (arbitrary) vector subspaces,

$$\mathcal{G}^* = V_0^* \oplus V_1^* \ , \tag{5.2.7}$$

V_0^*, V_1^* being generated by the MC forms $\omega^{i_0}(g)$, $\omega^{i_1}(g)$ of $\mathcal{G}^*$ with indices corresponding, respectively, to the unmodified and modified parameters,

$$g^{i_0} \to g^{i_0} \ , \ g^{i_1} \to \lambda g^{i_1} \quad , \quad i_0 \ (i_1) = 1, \ldots, \dim V_0 \ (\dim V_1) \ . \tag{5.2.8}$$

In general, the series of $\omega^{i_0}(g, \lambda) \in V_0^*$, $\omega^{i_1}(g, \lambda) \in V_1^*$, will involve all powers of λ,

$$\omega^{i_p}(g, \lambda) = \sum_{\alpha=0}^{\infty} \lambda^\alpha \omega^{i_p,\alpha}(g) = \omega^{i_p,0}(g) + \lambda \omega^{i_p,1}(g) + \lambda^2 \omega^{i_p,2}(g) + \ldots \ , \tag{5.2.9}$$

for $p = 0, 1$ and $\omega^{i_p}(g, 1) = \omega^{i_p}(g)$. We will see in the following sections what restrictions on $\mathcal{G}$ make zero certain coefficient one-forms $\omega^{i_p,\alpha}$.

With the above notation, the MC equations (5.2.1) for $\mathcal{G}$ can be rewritten as

$$d\omega^{k_s} = -\frac{1}{2} c_{i_p j_q}^{k_s} \, \omega^{i_p} \wedge \omega^{j_q} \quad (p, q, s = 0, 1) \tag{5.2.10}$$

or, explicitly

$$d\omega^{k_0} = -\frac{1}{2} c_{i_0 j_0}^{k_0} \omega^{i_0} \wedge \omega^{j_0} - c_{i_0 j_1}^{k_0} \omega^{i_0} \wedge \omega^{j_1} - \frac{1}{2} c_{i_1 j_1}^{k_0} \omega^{i_1} \wedge \omega^{j_1} \ , \tag{5.2.11}$$

$$d\omega^{k_1} = -\frac{1}{2} c_{i_0 j_0}^{k_1} \omega^{i_0} \wedge \omega^{j_0} - c_{i_0 j_1}^{k_1} \omega^{i_0} \wedge \omega^{j_1} - \frac{1}{2} c_{i_1 j_1}^{k_1} \omega^{i_1} \wedge \omega^{j_1} \ . \tag{5.2.12}$$

Inserting now the expansions (5.2.9) into the MC equations (5.2.10) and using (B.1) in appendix B, the MC equations are expanded in powers of λ:

$$\sum_{\alpha=0}^{\infty} \lambda^\alpha d\omega^{k_s,\alpha} = \sum_{\alpha=0}^{\infty} \lambda^\alpha \left[-\frac{1}{2} c_{i_p j_q}^{k_s} \sum_{\beta=0}^{\alpha} \omega^{i_p,\beta} \wedge \omega^{j_q,\alpha-\beta} \right] \ . \tag{5.2.13}$$

The equality of the two λ-polynomials in (5.2.13) requires the equality of the coefficients of equal power λ^α. This implies that the coefficient one-forms $\omega^{i_p,\alpha}$ in the expansions (5.2.9) satisfy the identities:

$$d\omega^{k_s,\alpha} = -\frac{1}{2} c_{i_p j_q}^{k_s} \sum_{\beta=0}^{\alpha} \omega^{i_p,\beta} \wedge \omega^{j_q,\alpha-\beta} \quad (p, q, s = 0, 1) \quad . \tag{5.2.14}$$

We can rewrite (5.2.14) in the form

$$d\omega^{k_s,\alpha} = -\frac{1}{2} C_{i_p,\beta \; j_q,\gamma}^{k_s,\alpha} \, \omega^{i_p,\beta} \wedge \omega^{j_q,\gamma} \ , \quad C_{i_p,\beta \; j_q,\gamma}^{k_s,\alpha} = \begin{cases} 0 \, , & \text{if } \beta + \gamma \neq \alpha \\ c_{i_p j_q}^{k_s} \, , & \text{if } \beta + \gamma = \alpha \end{cases} \ . \tag{5.2.15}$$

We now ask ourselves whether we can use the expansion coefficients $\omega^{k_0,\alpha}$, $\omega^{k_1,\beta}$ up to given orders $N_0 \geq 0$, $N_1 \geq 0$, $\alpha = 0, 1, \ldots, N_0$, $\beta = 0, 1, \ldots, N_1$, so that equation (5.2.15) (or (5.2.14)) determines the MC equations of a new Lie algebra. The answer is affirmative. More precisely, *the vector space generated by*

$$\{\omega^{i_0,0}, \omega^{i_0,1}, \ldots, \omega^{i_0,N}, \omega^{i_1,0}, \omega^{i_1,1}, \ldots, \omega^{i_1,N}\} \ , \tag{5.2.16}$$

together with the MC equations (5.2.15) for the structure constants

$$C^{k_s,\alpha}_{i_p,\beta\, j_q,\gamma} = \begin{cases} 0, & \text{if } \beta+\gamma \neq \alpha \\ c^{k_s}_{i_p j_q}, & \text{if } \beta+\gamma = \alpha \end{cases} \qquad (\alpha,\beta,\gamma = 1,\ldots,N\ ;\ p,q,s=0,1)\ , \tag{5.2.17}$$

determines a Lie algebra $\mathcal{G}(N)$ *for each expansion order* $N \geq 0$ *of dimension* $\dim \mathcal{G}(N) = (N+1)\dim \mathcal{G}$ [4].
To see why, consider the one-forms

$$\{\omega^{i_0,\alpha_0}\,;\,\omega^{i_1,\alpha_1}\} = \{\omega^{i_0,0},\omega^{i_0,1},\ldots,\omega^{i_0,N_0};\omega^{i_1,0},\omega^{i_1,1},\ldots,\omega^{i_1,N_1}\} \tag{5.2.18}$$

where we have not assumed *a priori* the same range for the expansions of the one-forms of V_0^* and V_1^*. To see whether the vector space $V^*(N_0,N_1)$ of basis (5.2.18) determines a Lie algebra $\mathcal{G}(N_0,N_1)$, it is sufficient to check that a) the exterior algebra generated by (5.2.18) is closed[3] under the exterior derivative d and that b) the Jacobi identities for $\mathcal{G}$ are satisfied.

To have closure under d we need that the r.h.s. of equations (5.2.15) does not contain one-forms that are not already present in (5.2.18). Consider the forms ω^{i_s,β_s}, $s=0,1$, that contribute to $d\omega^{k_s,\alpha_s}$ up to order $\alpha = N_s$. Looking at equations (5.2.14) it follows trivially that

$$N_0 = N_1 \qquad (=N)\ . \tag{5.2.19}$$

To check the Jacobi identities for $\mathcal{G}(N)$, it is sufficient to see that $dd\omega^{k_s,\alpha} \equiv 0$ in (5.2.15) is consistent with the definition of $C^{k_s,\alpha}_{i_p,\beta\, j_q,\gamma}$. Equation (5.2.15) gives

$$0 = C^{k_s,\alpha}_{i_p,\beta\, j_q,\gamma} C^{i_p,\beta}_{l_t,\rho\, m_u,\sigma}\omega^{j_q,\gamma}\wedge\omega^{l_t,\rho}\wedge\omega^{m_u,\sigma} \qquad (\alpha,\beta,\gamma,\rho,\sigma = 1,\ldots,N)\ , \tag{5.2.20}$$

which implies

$$C^{k_s,\alpha}_{i_p,\beta\,[j_q,\gamma} C^{i_p,\beta}_{l_t,\rho\, m_u,\sigma]} = 0\ . \tag{5.2.21}$$

Now, on account of definition (5.2.17), the terms in the l.h.s. above are either zero (when $\alpha \neq \gamma+\rho+\sigma$) or give zero due to the Jacobi identities for $\mathcal{G}$, $c^{k_s}_{i_p[j_q}c^{i_p}_{l_t m_u]} = 0$. Thus, the $C^{k_s,\alpha}_{i_p,\beta\, j_q,\gamma}$ satisfy the Jacobi identities (5.2.21) and define the Lie algebra $\mathcal{G}(N,N) \equiv \mathcal{G}(N)$ [4].

[3] An algebra of forms closed under d defines in general a free differential algebra (FDA): see chapter 6 and references therein.

Explicitly, the resulting algebras for the first few orders are [4]:

$N = 0$, $\mathcal{G}(0)$:

$$d\omega^{k_s,0} = -\frac{1}{2} c_{i_p j_q}^{k_s} \omega^{i_p,0} \wedge \omega^{j_q,0} \quad (p, q, s = 0, 1) \ , \tag{5.2.22}$$

i.e., $\mathcal{G}(0)$ reproduces the original algebra $\mathcal{G}$.

$N = 1$, $\mathcal{G}(1)$:

$$d\omega^{k_s,0} = -\frac{1}{2} c_{i_p j_q}^{k_s} \omega^{i_p,0} \wedge \omega^{j_q,0} \ , \tag{5.2.23}$$

$$d\omega^{k_s,1} = -c_{i_p j_q}^{k_s} \omega^{i_p,0} \wedge \omega^{j_q,1} \quad (p, q, s = 0, 1) \ . \tag{5.2.24}$$

$N = 2$, $\mathcal{G}(2)$:

$$d\omega^{k_s,0} = -\frac{1}{2} c_{i_p j_q}^{k_s} \omega^{i_p,0} \wedge \omega^{j_q,0} \ , \tag{5.2.25}$$

$$d\omega^{k_s,1} = -c_{i_p j_q}^{k_s} \omega^{i_p,0} \wedge \omega^{j_q,1} \ , \tag{5.2.26}$$

$$d\omega^{k_s,2} = -c_{i_p j_q}^{k_s} \omega^{i_p,0} \wedge \omega^{j_q,2} - \frac{1}{2} c_{i_p j_q}^{k_s} \omega^{i_p,1} \wedge \omega^{j_q,1} \ (p, q, s = 0, 1) \ . \tag{5.2.27}$$

In sight of the above results, the following remark is in order. Since $\omega^{i_p,0}(g) \neq \omega^{i_p}(g)$, one might wonder how the MC equations for $\mathcal{G}(0) = \mathcal{G}$ can be satisfied by $\omega^{i_p,0}(g)$. The $\dim \mathcal{G}$ MC forms $\omega^{i_p}(g)$ are left-invariant forms on the group manifold G of $\mathcal{G}$. The $(N+1)\dim \mathcal{G}$ $\omega^{i_p,\alpha}(g)$ ($\alpha = 0, 1, \ldots, N$) determined by the expansions (5.2.9) are also one-forms on G, but they are no longer left-invariant under G-translations. They cannot be, since there are only $\dim G = r$ linearly independent MC forms on G. Nevertheless, equations (5.2.15) determine the MC relations that will be satisfied by the MC forms on the manifold of the *higher dimensional* group $G(N)$ associated with $\mathcal{G}(N)$. These MC forms on $G(N)$ will depend on the $(N+1)\dim \mathcal{G}(N)$ coordinates of $G(N)$ associated with the generators (forms) $X_{i_p,\alpha}$ ($\omega^{i_p,\alpha}$) that determine $\mathcal{G}(N)$ ($\mathcal{G}^*(N)$).

5.2.2 Structure of the expanded algebras $\mathcal{G}(N)$

Let $V_{p,\alpha}$ be, at each order $\alpha = 0, 1, \ldots, N$, the vector space spanned by the generators $X_{i_p,\alpha}$, $p = 0, 1$; clearly, $V_{p,\alpha} \approx V_p$. Let

$$W_\alpha = V_{0,\alpha} \oplus V_{1,\alpha} \quad , \quad \mathcal{G}(N) = \bigoplus_{\alpha=0}^{N} W_\alpha \quad . \tag{5.2.28}$$

We first notice that $\mathcal{G}(N-1)$ is a vector subspace of $\mathcal{G}(N)$, but not a subalgebra for $N \geq 2$. Indeed, for $N \geq 2$ there always exist $\alpha, \beta \leq N-1$

such that $\alpha+\beta = N$. Denoting by $C^{(N)\,k_s,\gamma}_{i_p,\alpha\; j_q,\beta}$ and $C^{(N-1)\,k_s,\gamma}_{i_p,\alpha\; j_q,\beta}$ the structure constants of $\mathcal{G}(N)$ and $\mathcal{G}(N-1)$ respectively, one sees that, for $\alpha+\beta = N$, $C^{(N-1)\,k_s,\alpha+\beta}_{i_p,\alpha\; j_q,\beta} = 0$ in $\mathcal{G}(N-1)$ (since $\alpha+\beta > N-1$) while, in general, $C^{(N)\,k_s,\alpha+\beta}_{i_p,\alpha\; j_q,\beta} \neq 0$ in $\mathcal{G}(N)$. In other words, $\mathcal{G}(N-1)$ is not a subalgebra of $\mathcal{G}(N)$ because the structure constants for the elements of the various subspaces $V_{p,\alpha}$ depend on N and they are different, in general, for $\mathcal{G}(N-1)$ and $\mathcal{G}(N)$. Likewise, $\mathcal{G}(M)$ for $1 \leq M < N$ is not a subalgebra of $\mathcal{G}(N)$.

We now show that the Lie algebras $\mathcal{G}(N)$ have a Lie algebra extension structure for $N \geq 1$. More precisely, *the Lie algebra $\mathcal{G}(0)$ is a subalgebra of $\mathcal{G}(N)$, for all $N \geq 0$. For $N \geq 1$, W_N is an abelian ideal $\mathcal{W}_N \subset \mathcal{G}(N)$ and $\mathcal{G}(N)/\mathcal{W}_N = \mathcal{G}(N-1)$ i.e., $\mathcal{G}(N)$ is an extension of $\mathcal{G}(N-1)$ by $\mathcal{W}_N$ which is not semidirect for $N \geq 2$* [4]. To prove this result, notice that $\mathcal{G}(0) \subset \mathcal{G}(N)$ is a subalgebra by construction, since $C^{(N)\,k_s,\alpha}_{i_p,0\; j_q,0} = 0$, $\alpha = 1,\ldots,N$, by equation (5.2.15). For the second part, notice that, since $\alpha + N > N$ for $\alpha \neq 0$, $[W_\alpha, \mathcal{W}_N] = 0$; in particular, $\mathcal{W}_N$ is an abelian subalgebra. Furthermore $[\mathcal{W}_0, \mathcal{W}_N] \subset \mathcal{W}_N$, so that $\mathcal{W}_N$ is an ideal of $\mathcal{G}(N)$. Now, the vector space $\mathcal{G}(N)/W_N$ is isomorphic to $\mathcal{G}(N-1)$. $\mathcal{G}(N-1)$ is a Lie algebra the MC equations of which are (5.2.15), and $\mathcal{G}(N)/\mathcal{W}_N \approx \mathcal{G}(N-1)$. Since $\mathcal{G}(N-1)$ is not a subalgebra of $\mathcal{G}(N)$ for $N \geq 2$, the extension is not semidirect.

5.2.3 Limiting cases

Let us discuss the limiting cases $V_0 = 0, V_1 = V$ and $V_0 = V, V_1 = 0$ When $V_1 = V$, all the group parameters are modified by (5.2.8). In this case $\mathcal{G}(0)$ is the trivial $\mathcal{G}(0) = 0$ subalgebra of $\mathcal{G}(N)$. The first order $N = 1$, $\omega^{i_1,1} = dg^{i_1}$, corresponds to an abelian algebra with the same dimension as $\mathcal{G}$ (in fact, $\mathcal{G}(1)$ is the IW contraction of $\mathcal{G}$ with respect to the trivial $V_0 = 0$ subalgebra). For $N \geq 2$ we will have extensions with the structure in section 5.2.2.

For the other limiting case, $V_1 = 0$, there is obviously no expansion and we have $\mathcal{G}(0) = \mathcal{G}$.

5.3 The case in which $\mathcal{G}$ contains a subalgebra

Let $\mathcal{G} = V_0 \oplus V_1$ as before, where now V_0 is a subalgebra $\mathcal{L}_0$ of $\mathcal{G}$. Then,

$$c^{k_1}_{i_0 j_0} = 0 \qquad (i_p = 1, \ldots, \dim V_p\,,\; p = 0, 1)\;, \tag{5.3.1}$$

and the basis one-forms ω^{i_0} are associated with the (sub)group parameters g^{i_0} unmodified under the rescaling (5.2.8). The MC equations for $\mathcal{G}$

become

$$d\omega^{k_0} = -\frac{1}{2}c^{k_0}_{i_0j_0}\omega^{i_0}\wedge\omega^{j_0} - c^{k_0}_{i_0j_1}\omega^{i_0}\wedge\omega^{j_1} - \frac{1}{2}c^{k_0}_{i_1j_1}\omega^{i_1}\wedge\omega^{j_1}\ , \quad (5.3.2)$$

$$d\omega^{k_1} = -c^{k_1}_{i_0j_1}\omega^{i_0}\wedge\omega^{j_1} - \frac{1}{2}c^{k_1}_{i_1j_1}\omega^{i_1}\wedge\omega^{j_1}\ . \quad (5.3.3)$$

Using (5.3.1) in equation (5.2.5), one finds that the expansions of $\omega^{i_0}(g,\lambda)$ ($\omega^{i_1}(g,\lambda)$) start with the power λ^0 (λ^1):

$$\omega^{i_0}(g,\lambda) = \sum_{\alpha=0}^{\infty}\lambda^\alpha\omega^{i_0,\alpha}(g) = \omega^{i_0,0}(g) + \lambda\omega^{i_0,1}(g) + \lambda^2\omega^{i_0,2}(g) + \ldots \quad (5.3.4)$$

$$\omega^{i_1}(g,\lambda) = \sum_{\alpha=1}^{\infty}\lambda^\alpha\omega^{i_1,\alpha}(g) = \lambda\omega^{i_1,1}(g) + \lambda^2\omega^{i_1,2}(g) + \lambda^3\omega^{i_1,3}(g) + \ldots\ . \quad (5.3.5)$$

Inserting them into the MC equations (5.3.2) and (5.3.3) and using equation (B.1) of appendix B when the double sums begin with $(0,0)$, $(0,1)$ and $(1,1)$, we get

$$\begin{aligned}\sum_{\alpha=0}^{\infty}\lambda^\alpha d\omega^{k_0,\alpha} &= -\frac{1}{2}c^{k_0}_{i_0j_0}\omega^{i_0,0}\wedge\omega^{j_0,0}\\ &+\lambda\left[-c^{k_0}_{i_0j_0}\omega^{i_0,0}\wedge\omega^{j_0,1} - c^{k_0}_{i_0j_1}\omega^{i_0,0}\wedge\omega^{j_1,1}\right]\\ &+\sum_{\alpha=2}^{\infty}\lambda^\alpha\left[-\frac{1}{2}c^{k_0}_{i_0j_0}\sum_{\beta=0}^{\alpha}\omega^{i_0,\beta}\wedge\omega^{j_0,\alpha-\beta}\right.\\ &\left.\qquad -c^{k_0}_{i_0j_1}\sum_{\beta=0}^{\alpha-1}\omega^{i_0,\beta}\wedge\omega^{j_1,\alpha-\beta} - \frac{1}{2}c^{k_0}_{i_1j_1}\sum_{\beta=1}^{\alpha-1}\omega^{i_1,\beta}\wedge\omega^{j_1,\alpha-\beta}\right]\ ,\end{aligned} \quad (5.3.6)$$

$$\begin{aligned}\sum_{\alpha=1}^{\infty}\lambda^\alpha d\omega^{k_1,\alpha} &= -\lambda c^{k_1}_{i_0j_1}\omega^{i_0,0}\wedge\omega^{j_1,1}\\ &+\sum_{\alpha=2}^{\infty}\lambda^\alpha\left[-c^{k_1}_{i_0j_1}\sum_{\beta=0}^{\alpha-1}\omega^{i_0,\beta}\wedge\omega^{j_1,\alpha-\beta} - \frac{1}{2}c^{k_1}_{i_1j_1}\sum_{\beta=1}^{\alpha-1}\omega^{i_1,\beta}\wedge\omega^{j_1,\alpha-\beta}\right]\ .\end{aligned} \quad (5.3.7)$$

Again, the equality of the coefficients of equal power λ^α in (5.3.6), (5.3.7) leads to the equalities:

$\alpha = 0$:

$$d\omega^{k_0,0} = -\frac{1}{2} c^{k_0}_{i_0 j_0} \omega^{i_0,0} \wedge \omega^{j_0,0} \quad ; \tag{5.3.8}$$

$\alpha = 1$:

$$d\omega^{k_0,1} = -c^{k_0}_{i_0 j_0} \omega^{i_0,0} \wedge \omega^{j_0,1} - c^{k_0}_{i_0 j_1} \omega^{i_0,0} \wedge \omega^{j_1,1} \quad , \tag{5.3.9}$$

$$d\omega^{k_1,1} = -c^{k_1}_{i_0 j_1} \omega^{i_0,0} \wedge \omega^{j_1,1} \quad ; \tag{5.3.10}$$

$\alpha \geq 2$:

$$d\omega^{k_0,\alpha} = -\frac{1}{2} c^{k_0}_{i_0 j_0} \sum_{\beta=0}^{\alpha} \omega^{i_0,\beta} \wedge \omega^{j_0,\alpha-\beta} - c^{k_0}_{i_0 j_1} \sum_{\beta=0}^{\alpha-1} \omega^{i_0,\beta} \wedge \omega^{j_1,\alpha-\beta}$$
$$-\frac{1}{2} c^{k_0}_{i_1 j_1} \sum_{\beta=1}^{\alpha-1} \omega^{i_1,\beta} \wedge \omega^{j_1,\alpha-\beta} \, , \tag{5.3.11}$$

$$d\omega^{k_1,\alpha} = -c^{k_1}_{i_0 j_1} \sum_{\beta=0}^{\alpha-1} \omega^{i_0,\beta} \wedge \omega^{j_1,\alpha-\beta} - \frac{1}{2} c^{k_1}_{i_1 j_1} \sum_{\beta=1}^{\alpha-1} \omega^{i_1,\beta} \wedge \omega^{j_1,\alpha-\beta} \, . \tag{5.3.12}$$

To allow for a different range in the orders α of each $\omega^{i_p,\alpha}$, we now denote the coefficient one-forms in (5.3.4) ((5.3.5)) ω^{i_0,α_0} (ω^{i_1,α_1}), $\alpha_0 = 0, 1, \ldots, N_0$ ($\alpha_1 = 1, 2, \ldots, N_1$). With this notation, the above relations take the generic form

$$d\omega^{k_s,\alpha_s} = -\frac{1}{2} C^{k_s,\alpha_s}_{i_p,\beta_p \; j_q,\gamma_q} \; \omega^{i_p,\beta_p} \wedge \omega^{j_q,\gamma_q} \quad , \tag{5.3.13}$$

where

$$C^{k_s,\alpha_s}_{i_p,\alpha_p \; j_q,\alpha_q} = \begin{cases} 0, & \text{if } \beta_p + \gamma_q \neq \alpha_s \\ c^{k_s}_{i_p j_q}, & \text{if } \beta_p + \gamma_q = \alpha_s \end{cases} \qquad \begin{array}{l} p,q,s = 0,1 \\ i_{p,q,s} = 1,2,\ldots,\dim V_{p,q,s} \\ \alpha_0,\beta_0,\gamma_0 = 0,1,\ldots,N_0 \\ \alpha_1,\beta_1,\gamma_1 = 1,2,\ldots,N_1 \, . \end{array} \tag{5.3.14}$$

As in the preceding case, we now ask ourselves whether the expansion coefficients ω^{k_0,α_0}, ω^{k_1,α_1} up to a given order N_0, N_1 determine the MC equations (5.3.13) of a new Lie algebra $\mathcal{G}(N_0, N_1)$. It is obvious from (5.3.8) that the zeroth order of the expansion in λ corresponds to $N_0 =$

$0 = N_1$ (omitting all ω^{i_1,α_1} and thus allowing N_1 to be zero), and that $\mathcal{G}(0,0) = \mathcal{L}_0$. It is seen directly that the terms up to first order give two possibilities: $\mathcal{G}(0,1)$ (equations (5.3.8), (5.3.10) for $\omega^{k_0,0}, \omega^{k_1,1}$) and $\mathcal{G}(1,1)$ (equations (5.3.8), (5.3.9), (5.3.10) for $\omega^{k_0,0}, \omega^{k_1,1}, \omega^{k_0,1}$). Thus, we see that now (and due to (5.3.1)) one does not need to retain *all* ω^{i_p,α_p} up to a given order to obtain a Lie algebra. To look at the general $N_0 \geq 0, N_1 \geq 1$ case, consider the vector space $V^*(N_0, N_1)$, generated by

$$\{\omega^{i_0,\alpha_0}\,;\,\omega^{i_1,\alpha_1}\} = \{\omega^{i_0,0}, \omega^{i_0,1}, \omega^{i_0,2}, \ldots, \omega^{i_0,N_0}; \omega^{i_1,1}, \omega^{i_1,2}, \ldots, \omega^{i_1,N_1}\}\ . \tag{5.3.15}$$

To see that it determines a Lie algebra $\mathcal{G}(N_0, N_1)$ of dimension

$$\dim \mathcal{G}(N_0, N_1) = (N_0 + 1) \dim V_0 + N_1 \dim V_1\ , \tag{5.3.16}$$

we first notice that the Jacobi identities in $\mathcal{G}(N_0, N_1)$ will follow from those in $\mathcal{G}$. To find the conditions that N_0 and N_1 must satisfy to have closure under d, we look at the orders β_p of the forms ω^{i_p,β_p} that appear in the expression (5.3.13) of $d\omega^{k_s,\alpha_s}$ up to a given order $\alpha_s \geq s$. Looking at equations (5.3.8) to (5.3.12) we find the following table:

$\alpha_s \geq s$	ω^{i_0,β_0}	ω^{i_1,β_1}
$d\omega^{k_0,\alpha_0}$	$\beta_0 \leq \alpha_0$	$\beta_1 \leq \alpha_0$
$d\omega^{k_1,\alpha_1}$	$\beta_0 \leq \alpha_1 - 1$	$\beta_1 \leq \alpha_1$

Table 5.1. Orders β_p of the forms ω^{i_p,β_p} that contribute to $d\omega^{k_s,\alpha_s}$

Since there must be enough one-forms in (5.3.15) for the MC equations (5.3.13) to be satisfied, the $N_0 + 1$ and N_1 one-forms ω^{i_0,α_0} ($\alpha_0 = 0, 1, \ldots, N_0$) and ω^{i_1,α_1} ($\alpha_1 = 1, 2, \ldots, N_1$) in (5.3.15) should include, at least, those appearing in their differentials. Thus, the previous table 5.1 implies the reverse inequalities

$\alpha_s \geq s$	ω^{i_0,β_0}	ω^{i_1,β_1}
$d\omega^{k_0,\alpha_0}$	$N_0 \geq N_0$	$N_1 \geq N_0$
$d\omega^{k_1,\alpha_1}$	$N_0 \geq N_1 - 1$	$N_1 \geq N_1$

Table 5.2. Conditions on the number N_0 (N_1) of one-forms $\omega^{i_0,\alpha_0}(\omega^{i_1,\alpha_1})$

Hence, in this case there are two ways of cutting the expansions (5.3.4), (5.3.5), namely for

$$N_1 = N_0\ , \tag{5.3.17}$$

$$\text{or} \qquad N_1 = N_0 + 1\ . \tag{5.3.18}$$

Besides (5.2.19) there is now an additional type of solutions, equation (5.3.18). For the $N_0 = 0, N_1 = 1$ values equation (5.3.16) yields $\dim\mathcal{G}(0,1) = \dim\mathcal{G}$. Then, $\alpha_0 = 0$ and $\alpha_1 = 1$ only, the label α_p may be dropped and the structure constants (5.3.14) for $\mathcal{G}(0,1)$ read

$$C^{k_s}_{i_p\, j_q} = \begin{cases} 0, & \text{if } p+q \neq s \qquad p=0,1 \\ c^{k_s}_{i_p j_q}, & \text{if } p+q = s \qquad i_{p,q,s} = 1,2,\ldots,\dim V_{p,q,s} \quad , \end{cases} \tag{5.3.19}$$

which shows that V_1 is an abelian ideal of $\mathcal{G}(0,1)$. Hence, $\mathcal{G}(0,1)$ is just the (simple) IW contraction of $\mathcal{G}$ with respect to the subalgebra $\mathcal{L}_0$, as it may be seen by taking the $\lambda \to 0$ limit in (5.3.6)-(5.3.7), which reduce to equations (5.3.8) and (5.3.10).

To summarize, *let* $\mathcal{G} = V_0 \oplus V_1$, *where* V_0 *is a subalgebra* $\mathcal{L}_0$ *and let the coordinates* g^{i_p} *of* G *be rescaled by* $g^{i_0} \to g^{i_0}, g^{i_1} \to \lambda g^{i_1}$ *(equation (5.2.8)). Then, the coefficient one-forms* $\{\omega^{i_0,\alpha_0},\, \omega^{i_1,\alpha_1}\}$ *of the expansions (5.3.4), (5.3.5) of the Maurer-Cartan forms of* $\mathcal{G}^*$ *determine Lie algebras* $\mathcal{G}(N_0,N_1)$ *when* $N_1 = N_0$ *or* $N_1 = N_0+1$ *of dimension* $\dim\mathcal{G}(N_0,N_1) = (N_0+1)\dim V_0 + N_1 \dim V_1$ *and with structure constants (5.3.14),*

$$C^{k_s,\alpha_s}_{i_p,\beta_p\; j_q,\gamma_q} = \begin{cases} 0, & \text{if } \beta_p+\gamma_q \neq \alpha_s \\ c^{k_s}_{i_p j_q}, & \text{if } \beta_p+\gamma_q = \alpha_s \end{cases} \qquad \begin{array}{l} p,q,s = 0,1 \\ i_{p,q,s} = 1,2,\ldots,\dim V_{p,q,s} \\ \alpha_0,\beta_0,\gamma_0 = 0,1,\ldots,N_0 \\ \alpha_1,\beta_1,\gamma_1 = 1,2,\ldots,N_1\ . \end{array}$$

In particular, $\mathcal{G}(0,0) = \mathcal{L}_0$ *and* $\mathcal{G}(0,1)$ *(equation (5.3.18) for* $N_0 = 0$*) is the simple IW contraction of* $\mathcal{G}$ *with respect to the subalgebra* $\mathcal{L}_0$ [4].

5.3.1 The case in which $\mathcal{G}$ contains a symmetric coset

Let us now particularize to the case in which $\mathcal{G}/\mathcal{L}_0 = V_1$ is a symmetric coset *i.e.*,

$$[V_0,\, V_0] \subset V_0\ , \quad [V_0,\, V_1] \subset V_1\ , \quad [V_1,\, V_1] \subset V_0\ , \tag{5.3.20}$$

($[V_p,\, V_q] \subset V_{p+q}$, $(p+q)\mathrm{mod}\,2$). This applies, for instance, to all superalgebras where V_0 is the bosonic subspace and V_1 the fermionic one. Then, if $c^{k_s}_{i_p j_q}$ ($p,q,s = 0,1$; $i_p = 1,\ldots \dim V_p$) are the structure constants of $\mathcal{G}$, $c^{k_s}_{i_p j_q} = 0$ if $s \neq (p+q)\mathrm{mod}\,2$, the MC equations reduce to

$$d\omega^{k_0} = -\frac{1}{2}c^{k_0}_{i_0 j_0}\omega^{i_0}\wedge\omega^{j_0} - \frac{1}{2}c^{k_0}_{i_1 j_1}\omega^{k_1}\wedge\omega^{j_1}\ , \tag{5.3.21}$$

$$d\omega^{k_1} = -c^{k_1}_{i_0 j_1}\omega^{i_0}\wedge\omega^{j_1}\ . \tag{5.3.22}$$

In this case, the rescaling (5.2.8) leads to an even (odd) power series in λ for the MC forms $\omega^{i_0}(g,\lambda)$ ($\omega^{i_1}(g,\lambda)$):

$$\begin{aligned} \omega^{i_0}(g,\lambda) &= \omega^{i_0,0}(g) + \lambda^2\omega^{i_0,2}(g) + \lambda^4\omega^{i_0,4}(g) + \dots \\ \omega^{i_1}(g,\lambda) &= \lambda\omega^{i_1,1}(g) + \lambda^3\omega^{i_1,3}(g) + \lambda^5\omega^{i_1,5}(g) + \dots\ , \end{aligned} \tag{5.3.23}$$

namely, $\omega^{i_{\overline{\alpha}}}(g,\lambda) = \sum_{\alpha=0}^{\infty} \lambda^\alpha \omega^{i_{\overline{\alpha}},\alpha}(g)\, ;\ \overline{\alpha} = \alpha \,(\mathrm{mod}\, 2)$.

Indeed, under (5.2.8) $dg^{i_0} \to dg^{i_0}$, $dg^{i_1} \to \lambda\, dg^{i_1}$, which contributes with λ^0 (λ) to $\omega^{i_0}(g,\lambda)$ ($\omega^{i_1}(g,\lambda)$); $c^{i_p}_{j_q k_s}$ vanish trivially unless $p = (q+s)\,\mathrm{mod}\, 2$. Then, under (5.2.8), the $g^{k_s} dg^{j_q}$ terms in (5.2.5) with one g^{k_s} rescale as

$$\begin{aligned} p=0 \quad &: \quad c^{i_0}_{j_0 k_0}\, g^{k_0} dg^{j_0} \to c^{i_0}_{j_0 k_0}\, g^{k_0} dg^{j_0}\ ,\ c^{i_0}_{j_1 k_1}\, g^{k_1} dg^{j_1} \to \lambda^2 c^{i_0}_{j_1 k_1}\, g^{k_1} dg^{j_1}\ ; \\ p=1 \quad &: \quad c^{i_1}_{j_0 k_1}\, g^{k_1} dg^{j_0} \to \lambda\, c^{i_1}_{j_0 k_1}\, g^{k_1} dg^{j_0}\ , \end{aligned} \tag{5.3.24}$$

so that the powers λ^0 and λ^2 (λ) contribute to $\omega^{i_0}(g,\lambda)$ ($\omega^{i_1}(g,\lambda)$). For the terms in (5.2.5) involving the products of n g^{k_s}'s,

$$c^{h_{t_1}}_{j_q k_{s_1}} c^{h_{t_2}}_{h_{t_1} k_{s_2}} \dots c^{h_{t_{n-1}}}_{h_{t_{n-2}} k_{t_{n-1}}} c^{i_p}_{h_{t_{n-1}} k_{s_n}} g^{k_{s_1}} g^{k_{s_2}} \dots g^{k_{s_{n-1}}} g^{k_{s_n}} dg^{j_q}\ , \tag{5.3.25}$$

the fact that $V_1 = \mathcal{G}/\mathcal{L}_0$ is a symmetric space requires that $p = q + s_1 + s_2 \ldots + s_n\,(\mathrm{mod}2)$. Thus, after the rescaling (5.2.8), only even (odd) powers of λ, from λ^0 (λ) up to the closest (lower or equal to) $n+1$ even (odd) power λ^{n+1}, contribute to $\omega^{i_0}(g,\lambda)$ ($\omega^{i_1}(g,\lambda)$).

Structure of $\mathcal{G}(N_0,N_1)$ in the symmetric coset case

Inserting the power series above into the MC equations (5.3.21) and (5.3.22), we arrive at the equalities:

$$\begin{aligned} d\omega^{k_0,2\sigma} = &-\frac{1}{2} c^{k_0}_{i_0 j_0} \sum_{\rho=0}^{\sigma} \omega^{i_0,2\rho} \wedge \omega^{j_0,2(\sigma-\rho)} \\ &-\frac{1}{2} c^{k_0}_{i_1 j_1} \sum_{\rho=1}^{\sigma} \omega^{i_1,2\rho-1} \wedge \omega^{j_1,2(\sigma-\rho)+1}\ , \end{aligned} \tag{5.3.26}$$

$$d\omega^{k_1,2\sigma+1} = -c^{k_1}_{i_0 j_1} \sum_{\rho=0}^{\sigma} \omega^{i_0,2\rho} \wedge \omega^{j_1,2(\sigma-\rho)+1}\ , \tag{5.3.27}$$

where the expansion orders α are either $\alpha = 2\sigma$ or $\alpha = 2\sigma + 1$. From them it follows that the vector spaces generated by

$$\{\omega^{i_0,0}, \omega^{i_0,2}, \omega^{i_0,4}, \dots, \omega^{i_0,N_0}; \omega^{i_1,1}, \omega^{i_1,3}, \dots, \omega^{i_1,N_1}\}\ , \tag{5.3.28}$$

where $N_0 \geq 0$ (and even) and $N_1 \geq 1$ (and odd), will determine a Lie algebra when

$$N_1 = N_0 - 1 \quad , \tag{5.3.29}$$
$$\text{or} \quad N_1 = N_0 + 1 \quad . \tag{5.3.30}$$

Notice that we have a new type of solutions (5.3.29) with respect to the preceding case (equations (5.3.17), (5.3.18)), and that the previous solution $N_0 = N_1$ is not allowed now since N_0 (N_1) is necessarily even (odd). Then, for the symmetric case, the algebras $\mathcal{G}(N_0, N_1)$ may also be denoted $\mathcal{G}(N)$, where $N = \max\{N_0, N_1\}$, and are obtained at each order by adding alternatively copies of V_0 and V_1. Its structure constants are given by

$$C^{k_{\overline{\alpha}},\alpha}_{i_{\overline{\beta}},\beta\; j_{\overline{\gamma}},\gamma} = \begin{cases} 0, & \text{if } \beta+\gamma \neq \alpha \\ c^{k_{\overline{\alpha}}}_{i_{\overline{\beta}} j_{\overline{\gamma}}}, & \text{if } \beta+\gamma = \alpha \;;\; \overline{\alpha}, \overline{\beta}, \overline{\gamma} = \alpha, \beta, \gamma \,(\text{mod} 2)\;. \end{cases} \tag{5.3.31}$$

Let us write explicitly the MC equations for the first algebras obtained. If we allow for $N_1 = 0$, we get the trivial case

$\mathcal{G}(0,0) = \mathcal{G}(0)$:

$$d\omega^{k_0,0} = -\frac{1}{2} c^{k_0}_{i_0 j_0} \omega^{i_0,0} \wedge \omega^{j_0,0} \tag{5.3.32}$$

i.e., $\mathcal{G}(0,0)$ is the subalgebra $\mathcal{L}_0$ of the original algebra $\mathcal{G}$.

$\mathcal{G}(0,1) = \mathcal{G}(1)$:

$$d\omega^{k_0,0} = -\frac{1}{2} c^{k_0}_{i_0 j_0} \omega^{i_0,0} \wedge \omega^{j_0,0} \;, \tag{5.3.33}$$
$$d\omega^{k_1,1} = -c^{k_1}_{i_0 j_1} \omega^{i_0,0} \wedge \omega^{j_1,1} \;, \tag{5.3.34}$$

so that $\mathcal{G}(0,1)$ is again the IW contraction of $\mathcal{G}$ with respect to $\mathcal{L}_0$.

$\mathcal{G}(2,1) = \mathcal{G}(2)$:

$$d\omega^{k_0,0} = -\frac{1}{2} c^{k_0}_{i_0 j_0} \omega^{i_0,0} \wedge \omega^{j_0,0} \;, \tag{5.3.35}$$
$$d\omega^{k_1,1} = -c^{k_1}_{i_0 j_1} \omega^{i_0,0} \wedge \omega^{j_1,1} \;, \tag{5.3.36}$$
$$d\omega^{k_0,2} = -c^{k_0}_{i_0 j_0} \omega^{i_0,0} \wedge \omega^{j_0,2} - \frac{1}{2} c^{k_0}_{i_1 j_1} \omega^{i_1,1} \wedge \omega^{j_1,1} \;. \tag{5.3.37}$$

The structure of the Lie algebras $\mathcal{G}(N)$ can be summarized as follows. *The Lie algebra $\mathcal{G}(0) = \mathcal{L}_0$ is a subalgebra of $\mathcal{G}(N)$ for all $N \geq 0$. W_α in (5.2.28) reduces here to*

$$W_\alpha = \begin{cases} V_{0,\alpha} \;, & \text{if } \alpha \, \text{even} \\ V_{1,\alpha} \;, & \text{if } \alpha \text{ odd} \quad . \end{cases} \tag{5.3.38}$$

For $N \geq 1$, W_N is an abelian ideal $\mathcal{W}_N$ of $\mathcal{G}(N)$ and $\mathcal{G}(N)/\mathcal{W}_N = \mathcal{G}(N-1)$, i.e., $\mathcal{G}(N)$ is an extension of $\mathcal{G}(N-1)$ by $\mathcal{W}_N$. Further, for N even and $\mathcal{L}_0$ abelian, the extension $\mathcal{G}(N)$ of $\mathcal{G}(N-1)$ by $\mathcal{W}_N$ is central [4].
The proof of the first part of the claim proceeds as in section 5.2.2. For the second part, notice that, for $N \geq 1$, the only thing that prevents the abelian ideal $\mathcal{W}_N$ from being central is its failure to commute with $\mathcal{W}_0 \approx \mathcal{L}_0$, since $[W_\alpha, \mathcal{W}_N] = 0$ for $\alpha = 1, 2, \ldots, N$. But for N even, $C_{i_0,0\, j_0,N}^{k_0,N} = c_{i_0 j_0}^{k_0}$, which vanish for $\mathcal{L}_0$ abelian. Thus $\mathcal{W}_N$ becomes a central ideal, and $\mathcal{G}(N)$ a central extension of $\mathcal{G}(N-1)$ by $\mathcal{W}_N$.

5.4 Rescaling with several different powers

Let us extend now the above results to the case where the group parameters are multiplied by arbitrary integer powers of λ. Let $\mathcal{G}$ be split into a sum of $n+1$ vector subspaces,

$$\mathcal{G} = V_0 \oplus V_1 \oplus \cdots \oplus V_n = \bigoplus_0^n V_p \ , \tag{5.4.1}$$

and let the rescaling

$$\begin{aligned} & g^{i_0} \to g^{i_0} \ , \quad g^{i_1} \to \lambda g^{i_1} \ , \ \ldots \ , \quad g^{i_n} \to \lambda^n g^{i_n} \\ & (g^{i_p} \to \lambda^p g^{i_p}, \ p = 0, \ldots, n) \end{aligned} \tag{5.4.2}$$

of the group coordinates g^{i_p} be subordinated to the splitting (5.4.1) in an obvious way. We found in the previous section ($p = 0, 1$) that, when the rescaling (5.2.8) was performed, having V_0 as a subalgebra $\mathcal{L}_0$ proved to be convenient (though not necessary) since it led to more types of solutions ((5.3.17)-(5.3.18),*cf.* (5.2.19)). Furthermore, the first order algebra $\mathcal{G}(0,1)$ for that case was found to be the simple IW contraction of $\mathcal{G}$ with respect to $\mathcal{L}_0$. By the same reason, we will consider here conditions on $\mathcal{G}$ that will lead to a richer new algebras structure, including the generalized IW contraction of $\mathcal{G}$ in the sense [165] of Weimar-Woods (W-W). In terms of the structure constants of $\mathcal{G}$ we will then require that they fulfil the condition (5.1.2), namely,

$$c_{i_p j_q}^{k_s} = 0 \quad \text{if } s > p + q \tag{5.4.3}$$

i.e., that the Lie bracket of elements in V_p, V_q is in $\oplus_s V_s$ for $s \leq p + q$. This condition leads, through (5.2.5), to a power series expansion of the one-forms ω^{i_p} in V_p^* that, for each $p = 0, 1, \ldots, n$, starts precisely with the power λ^p,

$$\omega^{i_0}(g,\lambda) = \sum_{\alpha=0}^{\infty} \lambda^\alpha \omega^{i_0,\alpha}(g) = \omega^{i_0,0}(g) + \lambda\omega^{i_0,1}(g) + \lambda^2\omega^{i_0,2}(g) + \dots \ , \tag{5.4.4}$$

$$\omega^{i_1}(g,\lambda) = \sum_{\alpha=1}^{\infty} \lambda^\alpha \omega^{i_1,\alpha}(g) = \lambda\omega^{i_1,1}(g) + \lambda^2\omega^{i_1,2}(g) + \lambda^3\omega^{i_1,3}(g) + \dots \ , \tag{5.4.5}$$

$$\vdots$$

$$\omega^{i_n}(g,\lambda) = \sum_{\alpha=n}^{\infty} \lambda^\alpha \omega^{i_n,\alpha}(g) = \lambda^n\omega^{i_n,n}(g) + \lambda^{n+1}\omega^{i_n,n+1}(g) + \dots \ . \tag{5.4.6}$$

We may extend all the sums so that they begin at $\alpha = 0$ by setting $\omega^{i_p,\alpha} \equiv 0$ when $\alpha < p$. Then, inserting the expansions of $\omega^{i_p,\alpha}$ in the MC equations and using (B.1) we get (5.2.14) for $p, q, s = 0, 1, \dots, n$. If we now introduce the notation ω^{i_p,α_p} with different ranges for the expansion orders, $\alpha_p = p, p+1, \dots N_p$ for each p, we see that the MC equations take the form

$$d\omega^{k_s,\alpha_s} = -\frac{1}{2} C^{k_s,\alpha_s}_{i_p,\beta_p\, j_q,\gamma_q}\ \omega^{i_p,\beta_p} \wedge \omega^{j_q,\gamma_q} \quad , \tag{5.4.7}$$

where

$$C^{k_s,\alpha_s}_{i_p,\beta_p\, j_q,\gamma_q} = \begin{cases} 0, & \text{if } \beta_p + \gamma_q \neq \alpha_s \\ c^{k_s}_{i_p j_q}, & \text{if } \beta_p + \gamma_q = \alpha_s \end{cases} \qquad \begin{array}{l} p, q, s = 0, 1, \dots, n \\ i_{p,q,s} = 1, 2, \dots, \dim V_{p,q,s} \\ \alpha_p, \beta_p, \gamma_p = p, p+1, \dots, N_p \end{array} \tag{5.4.8}$$

and the $c^{k_s}_{i_p j_q}$ satisfy (5.4.3). To find now the ω^{i_p,β_p}'s that enter $d\omega^{k_s,\alpha_s}$, $s = 0, 1, \dots, n$, we need an explicit expression for it. This is found in appendix B, equations (B.7)-(B.10). From them we read that $d\omega^{k_s,\alpha_s}$, $s = 0, 1, \dots, n$, is expressed in terms of products of the forms ω^{i_p,β_p} in the following table:

$\alpha_s \geq s$	ω^{i_0,β_0}	ω^{i_1,β_1}	ω^{i_2,β_2}	$\cdots$	ω^{i_n,β_n}
$d\omega^{k_0,\alpha_0}$	$\beta_0 \leq \alpha_0$	$\beta_1 \leq \alpha_0$	$\beta_2 \leq \alpha_0$	$\cdots$	$\beta_n \leq \alpha_0$
$d\omega^{k_1,\alpha_1}$	$\beta_0 \leq \alpha_1 - 1$	$\beta_1 \leq \alpha_1$	$\beta_2 \leq \alpha_1$	$\cdots$	$\beta_n \leq \alpha_1$
$d\omega^{k_2,\alpha_2}$	$\beta_0 \leq \alpha_2 - 2$	$\beta_1 \leq \alpha_2 - 1$	$\beta_2 \leq \alpha_2$	$\cdots$	$\beta_n \leq \alpha_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$d\omega^{k_n,\alpha_n}$	$\beta_0 \leq \alpha_n - n$	$\beta_1 \leq \alpha_n - n + 1$	$\beta_2 \leq \alpha_n - n + 2$	$\cdots$	$\beta_n \leq \alpha_n$

Table 5.3. Types and orders of the forms ω^{i_p,β_p} needed to express $d\omega^{k_s,\alpha_s}$

Now let $V^*(N_0, \dots, N_n)$ be the vector space generated by

$$\{\omega^{i_0,\alpha_0}\ ;\omega^{i_1,\alpha_1}\ ;\dots;\omega^{i_n,\alpha_n}\} = \\ = \{\omega^{i_0,0}, \omega^{i_0,1}, \overset{N_0+1}{\dots}, \omega^{i_0,N_0};\ \omega^{i_1,1}, \overset{N_1}{\dots}, \omega^{i_1,N_1};\ \dots;\ \omega^{i_n,n}, \overset{N_n-n+1}{\dots}, \omega^{i_n,N_n}\}\ . \tag{5.4.9}$$

These one-forms determine a Lie algebra $\mathcal{G}(N_0, N_1, \dots, N_n)$, of dimension

$$\dim \mathcal{G}(N_0, \dots, N_n) = \sum_{p=0}^{n} (N_p - p + 1) \dim V_p\ . \tag{5.4.10}$$

More precisely, *let $\mathcal{G} = V_0 \oplus V_1 \oplus \cdots \oplus V_n$ be a splitting of $\mathcal{G}$ into $n+1$ subspaces and let $\mathcal{G}$ fulfil the Weimar-Woods contraction condition (5.4.3) subordinated to this splitting, $c_{i_p j_q}^{k_s} = 0$ if $s > p + q$. The one-form coefficients ω^{i_p,α_p} of (5.4.9) resulting from the expansion of the Maurer-Cartan forms ω^{i_p} in which $g^{i_p} \to \lambda^p g^{i_p}$, $p = 0, \dots, n$ (equation (5.4.2)), determine Lie algebras $\mathcal{G}(N_0, N_1, \dots, N_n)$ of dimension (5.4.10) and structure constants*

$$C_{i_p,\beta_p\, j_q,\gamma_q}^{k_s,\alpha_s} = \begin{cases} 0, & \text{if } \beta_p + \gamma_q \neq \alpha_s \\ c_{i_p j_q}^{k_s}, & \text{if } \beta_p + \gamma_q = \alpha_s \end{cases} \qquad \begin{array}{l} p, q, s = 0, 1, \dots, n \\ i_{p,q,s} = 1, 2, \dots, \dim V_{p,q,s} \\ \alpha_p, \beta_p, \gamma_p = p, p+1, \dots, N_p\ , \end{array}$$

(equation(5.4.8)) if $N_q = N_{q+1}$ or $N_q = N_{q+1} - 1$ $(q = 0, 1, \dots, n-1)$ in $(N_0, N_1, \dots, N_n)$. In particular, the $N_p = p$ solution determines the algebra $\mathcal{G}(0, 1, \dots, n)$, which is the generalized İnönü-Wigner contraction of $\mathcal{G}$ [4].

Let us prove this statement. To enforce the closure under d of the exterior algebra generated by the one-forms in (5.4.9) and to find the conditions that the various N_p must meet, we require, as in section 5.3, that all the forms ω^{i_p,β_p} present in $d\omega^{k_s,\alpha_s}$ are already in (5.4.9). Looking at equations (B.7)-(B.10) and at table 5.3 above, we find the restrictions

$\alpha_s \geq s$	ω^{i_0,β_0}	ω^{i_1,β_1}	ω^{i_2,β_2}	$\cdots$	ω^{i_n,β_n}
$d\omega^{k_0,\alpha_0}$	$N_0 \geq N_0$	$N_1 \geq N_0$	$N_2 \geq N_0$	$\cdots$	$N_n \geq N_0$
$d\omega^{k_1,\alpha_1}$	$N_0 \geq N_1 - 1$	$N_1 \geq N_1$	$N_2 \geq N_1$	$\cdots$	$N_n \geq N_1$
$d\omega^{k_2,\alpha_2}$	$N_0 \geq N_2 - 2$	$N_1 \geq N_2 - 1$	$N_2 \geq N_2$	$\cdots$	$N_n \geq N_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$d\omega^{k_n,\alpha_n}$	$N_0 \geq N_n - n$	$N_1 \geq N_n - n + 1$	$N_2 \geq N_n - n + 2$	$\cdots$	$N_n \geq N_n$

Table 5.4. Closure conditions on the number N_p of one-forms ω^{i_p,α_p}

It then follows that there are 2^n types of solutions[4] characterized by $(N_0, N_1, \dots, N_n)$, $N_p \geq p$, $p = 0, 1, \dots, n$, where

[4] This number may be found, *e.g.* for $n = 3$, by writing symbolically the solution

$$N_{q+1} = N_q \quad \text{or} \quad N_{q+1} = N_q + 1 \quad (q = 0, 1, \ldots, n-1) \ . \tag{5.4.11}$$

The Jacobi identities for $\mathcal{G}(N_0, \ldots, N_n)$,

$$C^{k_s,\alpha_s}_{i_p,\beta_p\ [j_q,\gamma_q} C^{i_p,\beta_p}_{l_t,\rho_t\ m_u,\sigma_u]} = 0 =$$
$$C^{k_s,\alpha_s}_{i_p,\beta_p\ j_q,\gamma_q} C^{i_p,\beta_p}_{l_t,\rho_t\ m_u,\sigma_u} + C^{k_s,\alpha_s}_{i_p,\beta_p\ m_u,\sigma_u} C^{i_p,\beta_p}_{j_q,\gamma_q\ l_t,\rho_t} + C^{k_s,\alpha_s}_{i_p,\beta_p\ l_t,\rho_t} C^{i_p,\beta_p}_{m_u,\sigma_u\ j_q,\gamma_q} \ , \tag{5.4.12}$$

are again satisfied through the ones for $\mathcal{G}$. This is a consequence of the fact that, for $\mathcal{G}$, the exterior derivative of the λ-expansion of the MC equations is the λ-expansion of their exterior derivative, but it may also be seen directly.

Indeed, we only need to check that (5.4.12) reduces to the Jacobi identities for $\mathcal{G}$ when the order in the upper index is the sum of those in the lower ones since the C's are zero otherwise. First we see that, when $\alpha_s = \gamma_q + \rho_t + \sigma_u$, all three terms in the r.h.s. of (5.4.12) give non-zero contributions. This is so because the range of β_p is only limited by $\beta_p \leq \alpha_s$, which holds when $\beta_p = \rho_t + \sigma_u$, $\beta_p = \gamma_q + \rho_t$ and $\beta_p = \sigma_u + \gamma_q$. Secondly, and since $\beta_p \geq p$, we also need that the terms in the i_p sum that are suppressed in (5.4.12) when $p > \beta_p$ be also absent in the Jacobi identities for $\mathcal{G}$ so that (5.4.12) does reduce to the Jacobi identities for $\mathcal{G}$. Consider *e.g.*, the first term in the r.h.s. of (5.4.12). If $p > \beta_p$, then $p > \rho_t + \sigma_u$ and hence $p > t + u$. Thus, by the W-W condition (5.4.3), this term will not contribute to the Jacobi identities for $\mathcal{G}$ and no sum over the subspace V_p index i_p will be lost as a result. The argument also applies to the other two terms for their corresponding β_p's.

A particular solution to (5.4.11) is obtained by setting $N_p = p$, $p = 0, 1, \ldots, n$, which defines $\mathcal{G}(0, 1, \ldots, n)$, with $\dim \mathcal{G}(0, 1, \ldots, n) = \dim \mathcal{G} = r$ (from (5.4.10)). Since in this case α_p takes only one value ($\alpha_p = N_p = p$) for each $p = 0, 1, \ldots, n$, we may drop this label. Then, the structure constants (5.4.8) for $\mathcal{G}(0, 1, \ldots, n)$ read

$$C^{k_s}_{i_p\, j_q} = \begin{cases} 0, & \text{if } p + q \neq s \qquad p = 0, 1, \ldots, n \\ c^{k_s}_{i_p j_q}, & \text{if } p + q = s \qquad i_{p,q,s} = 1, 2, \ldots, \dim V_{p,q,s} \ , \end{cases} \tag{5.4.13}$$

types in (5.4.11) as [0,0,0,0] for $N_0 = N_1 = N_2 = N_3$; [0,0,0,1] for $N_0 = N_1 = N_2, N_3 = N_2 + 1$; [0,0,1,0] for $N_0 = N_1, N_2 = N_1 + 1 = N_3$; [0,0,1,1] for $N_0 = N_1, N_2 = N_1 + 1, N_3 = N_2 + 1$; [0,1,0,0] for $N_0, N_1 = N_0 + 1 = N_2 = N_3$; [0,1,0,1] for $N_0, N_1 = N_0 + 1 = N_2, N_3 = N_2 + 1$; [0,1,1,0] for $N_0, N_1 = N_0 + 1, N_2 = N_1 + 1 = N_3$ and [0,1,1,1] for $N_0, N_1 = N_0 + 1, N_2 = N_1 + 1, N_3 = N_2 + 1$. This notation numbers the solutions in base 2; since [0,1,1,1] corresponds to $2^3 - 1$ we see, adding [0,0,0,0], that there are 2^3 ways of cutting the expansions that determine Lie algebras $\mathcal{G}(N_0, N_1, N_2, N_3)$, and 2^n in the general $\mathcal{G}(N_0, N_1, \ldots, N_n)$ case.

which shows that $\mathcal{G}(0, 1, \ldots, n)$ is the generalized IW contraction of $\mathcal{G}$, in the sense of [165], subordinated to the splitting (5.4.1). Of course, when $n = 1$ $(p = 0, 1)$, $V = V_0 \oplus V_1$, $\mathcal{L}_0$ is a subalgebra and equations (5.4.11) ((5.4.13)) reduce to (5.3.17) or (5.3.18) ((5.3.19)), what concludes the proof of the statement.

For instance, for the case $\mathcal{G} = V_0 \oplus V_1 \oplus V_2$ there are four types of algebras[5] $\mathcal{G}(N_0, N_1, N_2)$

$$N_0 = N_1 = N_2 \tag{5.4.14}$$
$$N_0 = N_1 = N_2 - 1 \;, \tag{5.4.15}$$
$$N_0 = N_1 - 1 = N_2 - 1 \;, \tag{5.4.16}$$
$$N_0 = N_1 - 1 = N_2 - 2 \;. \tag{5.4.17}$$

Since in the above theorem $\alpha_p \geq p$ for all $p = 0, \ldots, n$ was assumed, all types of one-forms ω^{i_p,α_p} with indices i_p in all subspaces V_p were present in the basis of $\mathcal{G}(N_0, N_1, \ldots, N_n)$. However, one may consider keeping terms in the expansion up to a certain order l, $l < n$ in which case due to (5.4.6), the forms ω^{i_p,α_p} with $p > l$ will not appear. Those with $p \leq l$ will determine the vector space $V^*(N_0, N_1, \ldots, N_l)$ where N_l is the highest order l and hence α_l takes only the value $N_l = l = \alpha_l$. This vector space, of dimension

$$\dim V^*(N_0, \ldots, N_l) = \sum_{p=0}^{l} (N_p - p + 1) \dim V_p \quad , \tag{5.4.18}$$

determines a Lie algebra $\mathcal{G}(N_0, N_1, \ldots, N_l)$, as claims the following statement.

Let $\mathcal{G} = \oplus_0^n V_p$, satisfy the Weimar-Woods conditions (5.4.3). Then, up to a certain order $N_l = l < n$, the one-forms

$$\begin{aligned}
&\{\omega^{i_0,\alpha_0}\,;\omega^{i_1,\alpha_1}\,;\ldots;\omega^{i_l,\alpha_l}\} = \\
&\quad = \{\omega^{i_0,0}, \omega^{i_0,1}, \overset{N_0+1}{\ldots}, \omega^{i_0,N_0}\,;\, \omega^{i_1,1}, \overset{N_1}{\ldots}, \omega^{i_1,N_1}\,;\,\ldots;\omega^{i_l,N_l}\} \;,
\end{aligned} \tag{5.4.19}$$

where $N_l = l = \alpha_l$, determine a Lie algebra $\mathcal{G}(N_0, N_1, \ldots N_l)$ of dimension (5.4.18) and structure constants given by

$$C^{k_s,\alpha_s}_{i_p,\beta_p\; j_q,\gamma_q} = \begin{cases} 0, & \text{if } \beta_p + \gamma_q \neq \alpha_s \\ c^{k_s}_{i_p j_q}, & \text{if } \beta_p + \gamma_q = \alpha_s \end{cases} \quad \begin{array}{l} p,q,s = 0,1,\ldots,l \\ i_{p,q,s} = 1,2,\ldots,\dim V_{p,q,s} \\ \alpha_p, \beta_p, \gamma_p = p, p+1, \ldots, N_p \;\leq l, \end{array} \tag{5.4.20}$$

[5] With the notation of footnote 4, these correspond, respectively, to [0,0,0], [0,0,1], [0,1,0] and [0,1,1].

if $N_q = N_{q+1}$ *or* $N_q = N_{q+1} - 1$, *(*$q = 0, 1, \ldots, l-1$*)* [4].

To see that this is indeed the case, notice that the restriction $\alpha_p \leq N_l = l < n$ on the order α_p of the one-forms ω^{i_p,α_p} implies, due to (5.4.6), that V_l is monodimensional and that $\omega^{i_l,l}$ is the last form entering (5.4.19). Then, looking at the closure conditions in table 5.4, we can restrict ourselves to the box delimited by ω^{i_p,β_p}, $d\omega^{k_s,\alpha_s}$ with $p, s \leq l$. This box will give spaces $V^*(N_0, N_1, \ldots, N_l)$, where $N_q = N_{q+1}$ or $N_q = N_{q+1} - 1$ ($q = 0, 1, \ldots, l-1$), and these spaces will determine Lie algebras if the Jacobi identities for (5.4.20)

$$C^{k_s,\alpha_s}_{i_p,\beta_p\;[j_q,\gamma_q}C^{i_p,\beta_p}_{l_t,\rho_t\;m_u,\sigma_u]} = 0\;,\quad i_{p,q,s} = 1, 2, \ldots, \mathrm{dim} V_{p,q,s} \qquad (5.4.21)$$

i.e., if $c^{k_s}_{i_p[j_q}c^{i_p}_{l_t m_u]} = 0\,,\; s, q, t, u \leq l$, is satisfied when $\alpha_s = \gamma_q + \rho_t + \sigma_u$ above. Note that this is not the Jacobi identities for $\mathcal{G}$ since i_p now runs over the basis of $\oplus_0^l V_p \subset \mathcal{G}$ only since $p \leq l$, and we are thus removing the values corresponding to the basis of $\oplus_{l+1}^n V_p$. However, if $p > l$ it is also *e.g.* $p > \beta_p = \rho_t + \sigma_u \geq t + u$ in which case $c^{i_p}_{l_t\, m_u} = 0$ by (5.4.3), what concludes the proof.

Notice that, since the structure constants (5.4.20) are obtained from those of $\mathcal{G}$ by restricting the i_p indices to be in the subspaces V_p, $p \leq l$, $\mathcal{G}(N_0, N_1, \ldots, N_l)$ is *not* a subalgebra of $\mathcal{G}(N_0, N_1, \ldots, N_n)$.

5.5 Superalgebra expansions

The above general procedure of generating Lie algebras from a given one does not rely on the antisymmetry of the structure constants of the original Lie algebra. Hence, with the appropriate changes to account for the grading, the method is applicable when $\mathcal{G}$ is a Lie superalgebra, a case which we consider explicitly in this section.

Let G be a supergroup and $\mathcal{G}$ its superalgebra. It is natural to consider a splitting of $\mathcal{G}$ into the sum of three subspaces $\mathcal{G} = V_0 \oplus V_1 \oplus V_2$, V_1 being the fermionic part of $\mathcal{G}$ and $V_0 \oplus V_2$ the bosonic part, so that the notation reflects the $\mathbb{Z}_2$-grading of $\mathcal{G}$. The even space is always a subalgebra of $\mathcal{G}$ but it may be convenient to consider it further split into the sum $V_0 \oplus V_2$ to allow for the case in which a subspace (V_0) of the bosonic space is itself a subalgebra $\mathcal{L}_0$.

Notice that, since V_0 is a Lie algebra $\mathcal{L}_0$, the $\mathbb{Z}_2$-graduation of $\mathcal{G}$ implies that the splitting $\mathcal{G} = V_0 \oplus V_1 \oplus V_2$ satisfies the W-W contraction conditions (5.4.3). Indeed, let $c^{k_s}_{i_p j_q}$ ($i_{p,q,s} = 1, \ldots, \dim V_{p,q,s}$, $p, q, s = 0, 1, 2$) be the structure constants of $\mathcal{G}$. The $\mathbb{Z}_2$-graduation of $\mathcal{G}$ obviously implies

$$c^{k_1}_{i_0 j_0} = c^{k_2}_{i_0 j_1} = 0\;, \qquad (5.5.1)$$

$$c_{i_0 j_1}^{k_0} = c_{i_1 j_1}^{k_1} = c_{i_0 j_2}^{k_1} = c_{i_2 j_1}^{k_0} = c_{i_2 j_1}^{k_2} = c_{i_2 j_2}^{k_1} = 0 \ . \tag{5.5.2}$$

The first set of restrictions (5.5.1), together with the assumed subalgebra condition for V_0 (which, in addition, requires $c_{i_0 j_0}^{k_2} = 0$), are indeed the W-W conditions (5.4.3) for $\mathcal{G}$; note that these conditions alone allow for $c_{i_1 j_1}^{k_0} \neq 0$, and $c_{i_1 j_1}^{k_2} \neq 0$ (and for $c_{i_1 j_1}^{k_1} \neq 0$, although here $c_{i_1 j_1}^{k_1} = 0$ due to the $\mathbb{Z}_2$-grading).

To apply now the above general procedure one must rescale the group parameters. The rescaling (5.4.2) for $V = V_0 \oplus V_1 \oplus V_2$ takes the form

$$g^{i_0} \to g^{i_0} \ , \ g^{i_1} \to \lambda g^{i_1} \ , \ g^{i_2} \to \lambda^2 g^{i_2} \ . \tag{5.5.3}$$

The present $\mathbb{Z}_2$-graded case fits into the preceding general discussion for $n = 2$, but with additional restrictions besides the W-W ones that follow from the $\mathbb{Z}_2$-grading. This situation is described by the following statement.

Let $\mathcal{G} = V_0 \oplus V_1 \oplus V_2$ be a Lie superalgebra, V_1 its odd part, and $V_0 \oplus V_2$ the even one. Let further V_0 be a subalgebra $\mathcal{L}_0$. As a result, $\mathcal{G}$ satisfies the W-W conditions (5.4.3) and, further, V_1 is a symmetric coset. Then, the coefficients of the expansion of the Maurer-Cartan forms of $\mathcal{G}$ rescaled by (5.5.3) determine Lie superalgebras $\mathcal{G}(N_0, N_1, N_2)$, $N_p \geq p$, $p = 0, 1, 2$, of dimension

$$\dim \mathcal{G}(N_0, N_1, N_2) = \left[\frac{N_0 + 2}{2}\right] \dim V_0 + \left[\frac{N_1 + 1}{2}\right] \dim V_1 + \left[\frac{N_2}{2}\right] \dim V_2 \ , \tag{5.5.4}$$

and structure constants

$$C_{i_p,\beta_p\, j_q,\gamma_q}^{k_s,\alpha_s} = \begin{cases} 0, & \text{if } \beta_p + \gamma_q \neq \alpha_s \qquad p, q, s = 0, 1, 2 \\ c_{i_p j_q}^{k_s}, & \text{if } \beta_p + \gamma_q = \alpha_s \qquad i_{p,q,s} = 1, 2, \ldots, \dim V_{p,q,s} \ , \end{cases} \tag{5.5.5}$$

and $\alpha_p, \beta_p, \gamma_p = p, p+2, \ldots, N_p-2, N_p$, where [ ] denotes integer part and the N_0, N_2 (even) and N_1 (odd) integers satisfy one of the three conditions below

$$N_0 = N_1 + 1 = N_2 \tag{5.5.6}$$
$$N_0 = N_1 - 1 = N_2 \ , \tag{5.5.7}$$
$$N_0 = N_1 - 1 = N_2 - 2 \ . \tag{5.5.8}$$

In particular, the superalgebra $\mathcal{G}(0, 1, 2)$ (equation (5.5.8) for $N_0 = 0$) is the generalized İnönü-Wigner contraction of $\mathcal{G}$ [4].

Indeed, since V_1 is a symmetric coset the rescaling (5.5.3) leads to an even (odd) power series in λ for the one-forms $\omega^{i_0}(g,\lambda)$ and $\omega^{i_2}(g,\lambda)$ ($\omega^{i_1}(g,\lambda)$), as in Sec. 5.3.1 (equations (5.3.23)). Thus, the conditions N_0, N_2 even, N_1 odd, have to be added to those that follow from the closure inequalities in table 5.4. This gives the conditions

$$N_0 + 1 \geq N_1 \geq N_0 - 1 \tag{5.5.9}$$
$$N_1 + 1 \geq N_2 \geq N_1 - 1 \tag{5.5.10}$$
$$N_0 + 2 \geq N_2 \geq N_0 \; , \tag{5.5.11}$$

from which equations (5.5.6)–(5.5.8) follow.

5.6 The M Theory superalgebra as an expansion of $osp(1|32)$

Let us work out an explicit example to illustrate the expansion method. The M Theory superalgebra (see section 2.1 of chapter 2 and references therein) is sometimes regarded (see *e.g.* [59]) as an IW contraction of the superalgebra $osp(1|32)$. That is indeed the case if the 55 Lorentz generators are excluded, otherwise there are not enough generators in $osp(1|32)$ to give the M-algebra by the dimension-preserving method of contraction (see section 5.1). In other words, the M Theory superalgebra, when its Lorentz automorphism generators are included, $\mathfrak{E}^{(528|32)} \rtimes so(1,10)$, cannot be obtained as a contraction of $osp(1|32)$. Let us show that, in contrast, the former is an expansion of the later [4].

The orthosymplectic superalgebra is defined by the 528 bosonic MC forms $\rho^{\alpha\beta} = \rho^{\beta\alpha}$ of the symplectic algebra $sp(32)$ and by the 32 fermionic MC forms ν^α satisfying the MC equations

$$\begin{aligned} d\rho^{\alpha\beta} &= -i\rho^{\alpha}{}_{\gamma} \wedge \rho^{\gamma\beta} - i\nu^\alpha \wedge \nu^\beta \\ d\nu^\alpha &= -i\rho^{\alpha}{}_{\beta} \wedge \nu^\beta \qquad (\alpha,\beta = 1,\ldots,32) \; . \end{aligned} \tag{5.6.1}$$

The spinor indices are raised and lowered by the 32×32 symplectic form $C_{\alpha\beta}$, which can be interpreted, as in section 2.1, as the $D = 11$ charge conjugation matrix. It is useful to use Dirac matrices to decompose $\rho^{\alpha\beta}$ as

$$\rho^{\alpha\beta} = \tfrac{1}{32}\left(\rho^a\Gamma_a - \tfrac{i}{2}\rho^{ab}\Gamma_{ab} + \tfrac{1}{5!}\rho^{a_1\ldots a_5}\Gamma_{a_1\ldots a_5}\right)^{\alpha\beta} \; . \tag{5.6.2}$$

In terms of the one-forms ρ^a, ρ^{ab}, $\rho^{a_1\ldots a_5}$ entering the decomposition (5.6.2) of $\rho^{\alpha\beta}$ the MC equations (5.6.1) of $osp(1|32)$ can be rewritten as

$$d\rho_a = -\tfrac{1}{16}\rho_n \wedge \rho^b{}_a + \tfrac{1}{32(5!)^2}\epsilon^{b_1\ldots b_{10}}{}_a \rho_{b_1\ldots b_5} \wedge \rho_{b_6\ldots b_{10}} - \nu^\alpha(\Gamma_a)_{\alpha\beta} \wedge \nu^\beta \; ,$$

$$
\begin{aligned}
d\rho_{ab} &= -\tfrac{1}{16} e_a \wedge e_b - \tfrac{1}{16}\rho_{ac} \wedge \rho^c{}_b - \tfrac{1}{16(4!)}\rho_{ac_1\ldots c_4} \wedge \rho^{c_4\ldots c_1}{}_b \\
&\quad -\nu^\alpha (\Gamma_{ab})_{\alpha\beta} \wedge \nu^\beta \ , \\
d\rho_{a_1\ldots a_5} &= \tfrac{1}{16(5!)}\epsilon^{bc_1\ldots c_5}{}_{a_1\ldots a_5}\rho_b \wedge \rho_{c_1\ldots c_5} + \tfrac{5}{16}\rho^b{}_{[a_1\ldots a_4} \wedge \rho_{a_5]b} \\
&\quad + \tfrac{1}{4(4!)^2}\epsilon^{b_1\ldots b_6}{}_{a_1\ldots a_5}\rho_{b_1b_2b_3c_1c_2} \wedge \rho^{c_2c_1}{}_{b_4b_5b_6} - \nu^\alpha(\Gamma_{a_1\ldots a_5})_{\alpha\beta} \wedge \nu^\beta \ , \\
d\nu^\alpha &= \tfrac{1}{32}\left(\rho^a\Gamma_a - \tfrac{i}{2}\rho^{ab}\Gamma_{ab} + \tfrac{1}{5!}\rho^{a_1\ldots a_5}\Gamma_{a_1\ldots a_5}\right)^\alpha{}_\beta \wedge \nu^\beta \ . \qquad (5.6.3)
\end{aligned}
$$

This form of the MC equations of $osp(1|32)$ suggests a splitting of the underlying vector space into three subspaces $osp(1|32) = V_0 \oplus V_1 \oplus V_2$, where V_0 is the space generated by the 55 MC forms $\rho^{ab} = \rho^{\alpha\beta}(\gamma^{ab})_{\alpha\beta}$ of the Lorentz subalgebra of $osp(1|32)$, V_1 the fermionic subspace generated by ν^α, and V_2 the space generated by the remaining 11+462 bosonic generators $\rho^a = \rho^{\alpha\beta}(\Gamma^a)_{\alpha\beta}$, $\rho^{a_1\cdots a_5} = \rho^{\alpha\beta}(\Gamma^{a_1\cdots a_5})_{\alpha\beta}$. Moreover, this splitting fulfils the general conditions discussed for superalgebras in section 5.5. It then follows that, after the redefinition (5.5.3) of the group parameters of $osp(1|32)$, the expansions of the forms in V_0 contain even powers of λ starting from λ^0, that those of the forms in V_1 include only odd powers in λ starting from λ^1, and that those of V_2 contain even orders starting with λ^2, *i.e.*,

$$
V_0 \ : \quad \rho^{ab} = \sum_{n=0}^{\infty} \lambda^{2n}\rho^{ab,2n} = \rho^{ab,0} + \lambda^2\rho^{ab,2} + \cdots \ ; \qquad (5.6.4)
$$

$$
V_1 \ : \quad \nu^\alpha = \sum_{n=0}^{\infty} \lambda^{2n+1}\nu^{\alpha,2n+1} = \lambda\nu^{\alpha,1} + \lambda^3\nu^{\alpha,3} + \cdots \ ; \qquad (5.6.5)
$$

$$
V_2 \ : \quad \begin{cases} \rho^a = \sum_{n=1}^{\infty} \lambda^{2n}\rho^{a,2n} = \lambda^2\rho^{a,2} + \cdots \ , \\ \rho^{a_1\ldots a_5} = \sum_{n=1}^{\infty} \lambda^{2n}\rho^{a_1\ldots a_5,2n} = \lambda^2\rho^{a_1\ldots a_5,2} + \cdots \ . \end{cases} \qquad (5.6.6)
$$

The restriction (5.5.6) allow to cut the series (5.6.4)–(5.6.6) at orders $N_0 = 2$, $N_1 = 1$, $N_2 = 2$, respectively, to obtain the MC equations of the expansion $osp(1|32)(2,1,2)$:

$$
\begin{aligned}
d\rho^{ab,0} &= -\tfrac{1}{16}\rho^{ac,0} \wedge \rho_c{}^{b,0} \ , \\
d\rho^{a\,,2} &= -\tfrac{1}{16}\rho^{b,2} \wedge \rho_b{}^{a,0} - i\nu^{\alpha,1} \wedge \nu^{\beta,1}\Gamma^a_{\alpha\beta} \quad , \\
d\rho^{ab,2} &= -\tfrac{1}{16}\left(\rho^{ac,0} \wedge \rho_c{}^{b,2} + \rho^{ac,2} \wedge \rho_c{}^{b,0}\right) - \nu^{\alpha,1} \wedge \nu^{\beta,1}\Gamma^{ab}_{\alpha\beta} \quad , \\
d\rho^{a_1\ldots a_5,2} &= \tfrac{5}{16}\rho^{b[a_1\ldots a_4|\,,2} \wedge \rho_b{}^{|a_5],0} - i\nu^{\alpha,1} \wedge \nu^{\beta,1}\Gamma^{a_1\ldots a_5}_{\alpha\beta} \quad , \\
d\nu^{\alpha,1} &= -\tfrac{1}{64}\nu^{\beta,1} \wedge \rho^{ab,0}\Gamma_{ab\beta}{}^{\alpha} \quad , \qquad (5.6.7)
\end{aligned}
$$

Now, setting $\rho^{ab,0} \equiv -16\sigma^{ab}$ and identifying $\rho^{a,2} \equiv \Pi^a$, $\rho^{ab,2} \equiv \Pi^{ab}$, $\rho^{a_1\cdots a_5,2} \equiv \Pi^{a_1\cdots a_5}$ and $\nu^{\alpha,1} \equiv \pi^\alpha$, the set of equations (5.6.7) coincides with the MC equations of the M Theory superalgebra containing the Lorentz group $\mathfrak{E}^{(528|32)} \rtimes so(1,10)$ (equations (2.1.15) when the Lorentz part is restored) [4]. As a check, notice that the dimensional counting is correct since, by equation (5.5.4),

$$\begin{aligned} \dim osp(1|32)(2,1,2) =& 2 \cdot 55 + 32 + 473 = 583 + 32 = \\ =& \dim \left(\mathfrak{E}^{(528|32)} \rtimes so(1,10) \right) . \end{aligned} \tag{5.6.8}$$

In conclusion, from the supergroup point of view [4],

$$\Sigma^{(528|32)} \rtimes SO(1,10) \approx OSp(1|32)(2,1,2) . \tag{5.6.9}$$

This concludes this mathematical parenthesis, and we now return to $D = 11$ supergravity. In chapter 4 we were able to write down a worldvolume action for a *preonic* brane in a D'Auria and Fré supergravity background. This formulation of supergravity is closely related to the notions of enlarged supersymmetry algebras and superspaces. In chapter 7, a worldvolume action for a string describing the excitations of two preons will be formulated, in fact, in an enlarged superspace. In the next chapter, D'Auria-Fré supergravity will be revisited, and the expansion method for Lie algebras will be find useful to describe the origin of the underlying symmetry algebras.

6

The underlying symmetry of $D = 11$ supergravity

The problem of the hidden or underlying geometry of $D = 11$ supergravity was raised already in the pioneering paper by Cremmer-Julia-Scherk (CJS) [27] (see also [172]), where the possible relevance of $OSp(1|32)$ was suggested. It was specially considered by D'Auria and Fré [92], where the search for the local supergroup of $D = 11$ supergravity was formulated as a search for a composite structure of its three-form A_3. Indeed, while the graviton and gravitino are given by one-form fields $e^a = dx^\mu e^a_\mu(x)$, $\psi^\alpha = dx^\mu \psi^\alpha_\mu(x)$ and can be considered, together with the spin connection $\omega^{ab} = dx^\mu \omega^{ab}_\mu(x)$, the gauge fields for the standard superPoincaré group [173], the $A_{\mu_1\mu_2\mu_3}(x)$ abelian gauge field is not associated with a symmetry generator and it rather corresponds to a three-form A_3. However, one may ask whether it is possible to introduce a set of additional one-form fields such that they, together with e^a and ψ^α, can be used to express A_3 in terms of products of one-forms. If so, the 'old' and 'new' one-form fields may be considered as gauge fields of a larger supergroup, and all the CJS supergravity fields can then be treated as gauge fields, with A_3 expressed in terms of them. This is what is meant here by the underlying gauge group structure of $D = 11$ supergravity: it is hidden when the standard $D = 11$ supergravity multiplet is considered, and manifest when A_3 becomes a composite of the one-form gauge fields associated with the extended group. The solution to this problem is equivalent to the trivialization of a standard $D = 11$ supersymmetry algebra four-cocycle (related to dA_3) on an enlarged superalgebra.

The notion of free differential algebras (FDAs) is a natural extension of that of Lie algebras, particularly suitable to account for the p-form fields present in supergravity theories. The notion of FDA and their construction as a process governed by cohomology will be reviewed in section 6.1. All this is put in its due context in section 6.2, where the

FDA of $D = 11$ supergravity is presented, and dA_3 seen to be related to a non-trivial supersymmetry algebra cocycle. We then apply these ideas to discuss the trivialization of FDAs (the process of obtaining *Lie* algebras from FDAs) and its implications for the physics they may describe. To that end, a family of extensions $\tilde{\mathfrak{E}}(s)$ of the supersymmetry superalgebra is described in section 6.3 and its relation to $osp(1|32)$ discussed in section 6.4. In section 6.5, the cocycle associated to dA_3 is shown to be trivialized by any member of the family, except for $s \neq 0$ [6, 7], extending previous results [92]. Section 6.6 analyzes the possible dynamical consequences of a composite structure of A_3 and shows the presence of additional gauge symmetries in the action for a composite A_3. Section 6.7 concludes this chapter with some remarks about a conjectured fields/enlarged superspace coordinates correspondence. The main results of this chapter can be found in references [6, 7].

6.1 Free differential algebras, Lie algebras and cohomology

The presence of forms of orders higher than one in the supergravity lagrangians makes it especially convenient to resort to free differential algebras in order to discuss the geometry associated to those theories. In fact, the discussion of this section about the relation of free differential algebras and Lie algebras can be straightforwardly extended to account for their superalgebra counterparts, the structures of interest in supergravity theories.

A free differential algebra (FDA) [96, 92, 18, 97] (termed Cartan integrable system in [92]) is an exterior algebra with constant coefficients, generated by a set of forms (not necessarily of the same rank) closed under the action of the exterior differential d. The dual formulation of a Lie algebra $\mathcal{G}$, in terms of Maurer-Cartan (MC) one-forms[1] π^i left-invariant on the corresponding group manifold, provides the simplest example of an FDA. As an FDA, $\mathcal{G}$ is to be regarded as generated by a collection of one-forms π^i, $i = 1, \dots, \mathrm{dim}\mathcal{G}$, and two-forms $d\pi^i$, related through the MC equations of $\mathcal{G}$ (equation (5.2.1)) and closed under d due to the Jacobi identity.

A more interesting application of FDAs is the description of the local symmetry of a theory through the *gauging* of Lie algebras. The gauge FDA associated to the Lie algebra $\mathcal{G}$ is obtained by replacing the MC one-forms π^i of $\mathcal{G}$ by their *gauge field* or *soft* (see [18]) *one-form* counterparts A^i, and by introducing *two-form curvatures* satisfying a generalization of

[1] In chapter 5, the MC one-forms of the Lie algebra $\mathcal{G}$ were denoted as ω^i. Here, the notation π^i is preferred to reserve ω for non-trivial Chevalley-Eilenberg cocycles.

the MC equations of $\mathcal{G}$ (see equation (5.2.1)),

$$F^k = dA^k + \frac{1}{2} c^k_{ij} A^i \wedge A^j . \tag{6.1.1}$$

The curvatures then satisfy the consistency conditions expressed by the Bianchi identities

$$dF^k = c^k_{ij} F^i \wedge A^j \ . \tag{6.1.2}$$

The structure equations (6.1.1) and the Bianchi identities (6.1.2) then define the gauge FDA associated with the Lie algebra $\mathcal{G}$. Dynamically, the relevance of the FDA constructed this way from the Lie algebra $\mathcal{G}$ is reflected by the fact that the lagrangian of a theory with local symmetry $\mathcal{G}$ is built up from the gauge potentials A^i and their curvatures F^i.

An FDA is called *minimal* [96] when the differential of any p-form in the FDA is expressible *only* in terms of sums of wedge products of q-forms in the FDA, with $q \leq p$. The FDA is *contractible* if it is generated by pairs of forms π_p, π_{p+1} such that $d\pi_p = \pi_{p+1}$, $d\pi_{p+1} = 0$. According to *Sullivan's first theorem* [96], which is the counterpart for FDAs of the Lévi-Mal'čev theorem (see [40]) for Lie algebras, the most general FDA is the semidirect sum of a contractible with a minimal one. For instance, regarded as an FDA, a Lie algebra $\mathcal{G}$ is minimal, whereas the FDA (6.1.1), (6.1.2) is contractible. If the 'flat limit' of the contractible algebra (6.1.1), (6.1.2) is considered, in which all the curvatures are set to zero, $F^i = 0$, the gauge potentials A_i turn out to satisfy the same equations than π^i (*i.e.*, (6.1.1) reduces to (5.2.1)), and the minimal algebra, which in this case coincides with the Lie algebra $\mathcal{G}$, is recovered.

The minimal FDA does not need to be a Lie algebra, though. Indeed, it is the typical case in supergravity theories that their lagrangian contains not only one-forms and their curvature two-forms, but also[2] p-forms A^i_p, $p > 1$, and their curvature $(p+1)$-forms F^i_{p+1}. This is precisely the case of $D = 11$ supergravity, the lagrangian (2.2.5) of which involves not only the one-forms e^a and ψ^α and their curvatures, but also the three-form A_3 and, in the first order approach, an auxiliary four-form F_4 which is related, on-shell, to its curvature. For notational analogy with the gauging of Lie algebras, it is convenient to introduce the *rigid* p-form counterparts π^i_p of the *soft* [18] p-forms A^i_p, such that, in the 'flat limit' in which all the curvatures are set to zero, $F^i_{p+1} = 0$, the forms A^i_p satisfy the same structure equations than π^i_p. The FDA thus reduces to the minimal FDA, which is nevertheless not a Lie algebra since it contains forms π^i_p of rank higher than one. And yet, from a physical point of view, despite

[2] Here, the superindex i is again used to label the forms, and a subindex showing their rank is added.

the unclear relation in this case of the minimal FDA with a Lie algebra, a lagrangian built up from A_p^i and F_{p+1}^i possesses a local symmetry: that is the case in supergravity theories, which are the field theories of local supersymmetry.

Consider a Lie algebra $\mathcal{G}$ defined through the MC, left invariant, one-forms π_1^i on the group manifold G of $\mathcal{G}$ satisfying the MC equations (5.2.1). *Sullivan's second theorem* [96] determines the structure of minimal FDAs, built by using the MC forms π_1^i, through an iterative process [96, 18] (see also [174]). First, the MC one-forms π_1^i of any minimal FDA close into the original Lie algebra $\mathcal{G}$. The structure equations for additional p-forms π_p^i of the minimal FDA can be written as

$$d\pi_p^i = \omega_{p+1}^i(\pi_1^j) , \tag{6.1.3}$$

where ω_{p+1}^i are nontrivial Chevalley-Eilenberg (CE) [175, 40] $(p+1)$-cocycles on the Lie algebra $\mathcal{G}$. In other words, for each i, ω_{p+1}^i is a closed $(p+1)$-form built up as a sum of exterior products (*i.e.*, as an *exterior polynomial*) of the MC one-forms π_1^i (and, thus, invariant under $\mathcal{G}$) which is not the differential of a p-form invariant under $\mathcal{G}$, namely, π_p^i is not an (exterior) polynomial in π_1^i. The process can be iterated by adding new q-forms $\pi_q'^i$ such that their differentials are non-trivial $(q+1)$-cocycles depending on (π_1^i, π_p^j), then on $(\pi_1^i, \pi_p^j, \pi_q'^i)$, and so on.

In general, the non-trivial character of the cocycles ω_{p+1}^i defining a minimal FDA can be interpreted by saying that there are not enough MC one-forms π_1^i in the Lie algebra $\mathcal{G}$ to write the p-forms π_p^i in equation (6.1.3) in terms of them. But it may happen that the introduction of an algebra $\tilde{\mathcal{G}}$ *larger* than $\mathcal{G}$ allows for the $\mathcal{G}$-cocycles to be written in terms of the new MC one-forms of $\tilde{\mathcal{G}}$, so that they are (left-)invariant under the corresponding group $\tilde{G}$. Bearing this in mind, the question of whether there exists a Lie algebra describing the same local symmetry than a given FDA can be put in precise mathematical terms, at least for minimal FDAs: if there exists an extension[3] $\tilde{\mathcal{G}}$ of the Lie algebra $\mathcal{G}$ for which the cocycles ω_{p+1}^i become trivial, *i.e.*, such that for each i, the p-forms π_p^i (related to ω_{p+1}^i through equation (6.1.3)) can be expressed as (exterior) polynomials in the MC one-forms of $\tilde{\mathcal{G}}$, then the FDA can be 'trivialized', by writing its forms in terms of the MC forms of the Lie algebra $\tilde{\mathcal{G}}$. In a supergravity context, it is in this sense that $\tilde{\mathcal{G}}$ can be said to be the *underlying gauge symmetry* of the theory under consideration.

Notice that the trivialization problem might have either no solution at all (see [97] for an example) or more than one solution: there might

[3] See section 5.1 of the previous chapter and references therein.

exist more than one enlarged algebra $\tilde{\mathcal{G}}$ that trivializes the cocycles defining the minimal FDA. The later is the case for $D = 11$ supergravity, for which two superalgebras were already found in [92] to account for its underlying symmetry. It will be shown in section 6.5 that, in fact, not only the two algebras of [92] solve the problem, but that there exists a whole one-parameter family of Lie superalgebras describing the underlying symmetry of $D = 11$ supergravity. First, we shall introduce the FDA corresponding to $D = 11$ supergravity.

6.2 The $D = 11$ supergravity FDA

Let us first consider, momentarily, the case of four-dimensional simple supergravity, where the only fields involved are the graviton and gravitino (in the $N = 1$, $D = 4$ supergravity multiplet) and the Lorentz connection. These can actually be considered as the gauge fields of simple $D = 4$ supergravity [173] and can be described by a gauge (super)FDA constructed as discussed in section 6.1. Indeed, replacing the MC one-forms Π^a, π^α, σ^{ab} of the superPoincaré algebra by the gauge field one-forms e^a, ψ^α, ω^{ab}, respectively, and introducing their corresponding curvatures, $\mathbf{R}^a$, $\mathbf{R}^\alpha$, $\mathbf{R}^{ab}$, the superPoincaré MC equations (2.1.12) can be promoted to the structure equations (see equation (6.1.1))

$$\begin{aligned}
&\mathbf{R}^a := de^a - e^b \wedge \omega_b{}^a + i\psi^\alpha \wedge \psi^\beta \Gamma^a_{\alpha\beta} = T^a + i\psi^\alpha \wedge \psi^\beta \Gamma^a_{\alpha\beta} \, , \\
&\mathbf{R}^\alpha := d\psi^\alpha - \psi^\beta \wedge \omega_\beta{}^\alpha \quad \left(\omega_\alpha{}^\beta = \tfrac{1}{4}\omega^{ab}\Gamma_{ab\,\alpha}{}^\beta\right) \, , \\
&\mathbf{R}^{ab} := d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b \, ,
\end{aligned} \tag{6.2.1}$$

where $T^a := De^a = de^a - e^b \wedge \omega_b{}^a$ is the torsion (see equation (2.2.7)). The equations (6.2.1), together with their selfconsistency or integrability conditions (see (6.1.2))

$$\begin{aligned}
&D\mathbf{R}^a = -e^b \wedge \mathbf{R}_b{}^a + 2i\psi^\alpha \wedge \mathbf{R}^\beta \Gamma^a_{\alpha\beta} \, , \\
&D\mathbf{R}^\alpha = -\tfrac{1}{4}\psi^\beta \wedge \mathbf{R}^{ab}\Gamma_{ab\beta}{}^\alpha \, , \\
&D\mathbf{R}^{ab} = 0 \, ,
\end{aligned} \tag{6.2.2}$$

where D is the Lorentz covariant derivative, form the gauge FDA of the superPoincaré group. When all the curvatures are set to zero, $\mathbf{R}^a = 0$, $\mathbf{R}^\alpha = 0$, $\mathbf{R}^{ab} = 0$, the Bianchi identities (6.2.2) are trivially satisfied and, as discussed in the previous section, the structure equations (6.2.1) of the FDA reduce to the MC equations (2.1.12) of the superPoincaré algebra. The minimal FDA is, in this case, a Lie superalgebra (superPoincaré), the local symmetry of simple supergravity.

This FDA description is, however, incomplete for $D = 11$ supergravity, due to the presence of the three-form field A_3. When A_3 is taken into account, the FDA defined by equations (6.2.1) must be completed by the definition of the four-form field strength [92]

$$\mathbf{R}_4 \;\; := \;\; dA_3 + \tfrac{1}{4}\psi^\alpha \wedge \psi^\beta \wedge e^a \wedge e^b \Gamma_{ab\alpha\beta} \; , \qquad (6.2.3)$$

supplemented by its corresponding Bianchi identity[4],

$$d\mathbf{R}_4 \;\; = \;\; -\psi^\alpha \wedge \mathbf{R}^\beta \wedge \bar{\Gamma}^{(2)}_{\alpha\beta} - \tfrac{1}{2}\psi^\alpha \wedge \psi^\beta \wedge e^b \wedge \mathbf{R}^a \Gamma_{ab\alpha\beta} \; . \qquad (6.2.4)$$

Equations (6.2.1), (6.2.3) are the structure equations for the $D = 11$ *supergravity FDA*, their corresponding Bianchi identities being (6.2.2), (6.2.4). The definition (6.2.3) of the curvature $\mathbf{R}_4$ of A_3 is obviously inspired in the algebraic constraint (2.4.4) that relates F_4 to A_3. Indeed, resorting to the superspace formulation of supergravity and setting $\mathbf{R}^a = 0$ and $\mathbf{R}_4 = F_4 := \frac{1}{4!} e^{a_4} \wedge \ldots \wedge e^{a_1} F_{a_1 \ldots a_4}$, the on-shell $D = 11$ superspace supergravity constraints [176, 177] are recovered (see also [7]).

In contrast with the $D = 4$ case, the above FDA for vanishing curvatures cannot be associated with the MC equations of a *Lie* superalgebra due to the presence of the *three*-form A_3. In fact, according to the general discussion in section 6.1, for vanishing curvatures the bi-fermionic four-form

$$a_4 = -\tfrac{1}{4}\psi^\alpha \wedge \psi^\beta \wedge e^a \wedge e^b \Gamma_{ab\alpha\beta} \qquad (6.2.5)$$

entering the definition (6.2.3) of the curvature $\mathbf{R}_4$ of A_3 (see also equation (2.2.10)) becomes a CE four-cocycle on the supertranslations algebra $\mathfrak{E} \equiv \mathfrak{E}^{(11|32)}$ given by

$$\omega_4(x^a, \theta^\alpha) = -\tfrac{1}{4}\pi^\alpha \wedge \pi^\beta \wedge \Pi^a \wedge \Pi^b \Gamma_{ab\alpha\beta} = d\omega_3(x^a, \theta^\alpha) \; , \qquad (6.2.6)$$

where $\Pi^a = dx^a - i d\theta^\alpha \Gamma^a_{\alpha\beta} \theta^\beta$ and $\pi^\alpha = d\theta^\alpha$. In equation (6.2.6), the dependence of the forms ω_3 and ω_4 on the coordinates $Z = (x^a, \theta^\alpha)$ of rigid superspace $\Sigma \equiv \Sigma^{(11|32)}$, the group manifold of the $D = 11$ supertranslations group, has been written explicitly. The Lorentz group, being simple and not adding anything to the cohomology, can be neglected in this discussion.

As discussed in general, the $\mathfrak{E}$-invariant and closed four-cocycle ω_4 is, furthermore, non-trivial in the CE cohomology, since the three-form $\omega_3 = \omega_3(x^a, \theta^\alpha)$ in (6.2.6) cannot be expressed in terms of the invariant MC forms Π^a, π^α of $\mathfrak{E}$. Now, we may ask whether there exists an extension $\tilde{\mathfrak{E}}$ of the standard $D = 11$ supersymmetry algebra $\mathfrak{E}$, with MC

[4] See section 2.1 for the notation.

forms defined on its associated enlarged superspace $\tilde{\Sigma}$, on which the CE four-cocycle ω_4 becomes trivial. In this way, the problem of writing the original A_3 field in terms of one-form fields becomes purely geometrical: it is equivalent to looking, in the spirit of the fields/extended superspace variables correspondence of [85] (see section 6.7), for an *enlarged* supergroup manifold $\tilde{\Sigma}$ on which one can find a new three-form $\tilde{\omega}_3$ (corresponding to A_3) that can be expressed in terms of sums of exterior products of $\tilde{\mathfrak{E}}$ MC forms on $\tilde{\Sigma}$ (that will correspond to one-form gauge fields), hence depending on the coordinates $\tilde{Z}$ of $\tilde{\Sigma}$. That such a $\tilde{\Sigma}$-invariant form $\tilde{\omega}_3(\tilde{Z})$ should exist is also not surprising if we recall that the CE $(p+2)$-cocycles on $\mathfrak{E}$ that characterize [178] the Wess-Zumino terms of the super-p-brane actions and their associated FDAs, can also be trivialized on larger superalgebras $\tilde{\mathfrak{E}}$ [179, 85] (see also [180]) associated to extended superspaces $\tilde{\Sigma}$, and that the pull-back of $\tilde{\omega}_3(\tilde{Z})$ to the supermembrane worldvolume defines an invariant Wess-Zumino term.

To summarize, the minimal FDA of $D = 11$ supergravity is obtained by enlarging the supertranslations algebra $\mathfrak{E}$ (containing the MC one-forms Π^a, π^α, corresponding to the gauge fields e^a, ψ^α, respectively) with the three-form ω_3 (corresponding to A_3) such that its differential is the CE four-cocycle ω_4 on $\mathfrak{E}$ (equation (6.2.6)). Notice, however, that further enlargements are possible, within the FDA construction scheme of section 6.1. Actually, the closed seven-form

$$\omega_7 = -\omega_3 \wedge \omega_4 + \tfrac{i}{2\cdot 5!}\pi^\alpha \wedge \pi^\beta \wedge \Pi^{a_5} \wedge \ldots \wedge \Pi^{a_1}\, \Gamma_{a_1 \ldots a_5\, \alpha\beta} \tag{6.2.7}$$

is a non-trivial cocycle[5] *on the FDA generated by* $(\Pi^a, \pi^\alpha, \omega_3)$ ([18], vol. II, p. 866). The seven-cocycle ω_7 is nothing but the 'flat limit', $\mathbf{R}_7 = 0$, of dA_6, where the six-form A_6 is the dual six-form of A_3, defined by $F_7 = *F_4$, where $F_4 = dA_3$ and $F_7 = dA_6 + A_3 \wedge dA_3$. The structure equation of A_6,

$$\mathbf{R}_7 := dA_6 + A_3 \wedge dA_3 - \tfrac{i}{2}\psi^\alpha \wedge \psi^\beta \wedge \bar{\Gamma}^{(5)}_{\alpha\beta} \tag{6.2.8}$$

(see (2.2.12) for the notation), together with its corresponding Bianchi identity,

$$\begin{aligned} d\mathbf{R}_7 = & \left(\mathbf{R}_4 + \tfrac{1}{2}\psi \wedge \psi \wedge \bar{\Gamma}^{(2)}\right) \wedge \left(\mathbf{R}_4 + \tfrac{1}{2}\psi \wedge \psi \wedge \bar{\Gamma}^{(2)}\right) \\ & + i\psi^\alpha \wedge \mathbf{R}^\beta \wedge \bar{\Gamma}^{(5)}_{\alpha\beta} - \tfrac{i}{2\cdot 4!}\psi^\alpha \wedge \psi^\beta \wedge e^{c_4} \wedge \ldots \wedge e^{c_1} \wedge \mathbf{R}^a \Gamma_{ac_1 \ldots c_4 \alpha\beta} \\ & - \tfrac{1}{4}\psi^\alpha \wedge \psi^\beta \wedge \psi^\gamma \wedge \psi^\delta \wedge \bar{\Gamma}^{(2)}_{\alpha\beta} \wedge \bar{\Gamma}^{(2)}_{\gamma\delta} \equiv 0 \; , \end{aligned} \tag{6.2.9}$$

[5] Notice that ω_7 is not a CE seven-cocycle *on the standard, $D = 11$ supertranslations algebra* $\mathfrak{E} \equiv \mathfrak{E}^{(11|32)}$, since it explicitly involves ω_3.

may thus be added to the FDA (6.2.1), (6.2.3), (6.2.2), (6.2.4). We shall, however, ignore this enlargement of the algebra and work with the FDA generated by $(\Pi^a, \pi^\alpha, \omega_3)$ whose gauging, described (neglecting the Lorentz part) by the structure equations (6.2.1) and (6.2.3) and their Bianchi identities (6.2.2), (6.2.4), involves only e^a, ψ^α, A_3 and their curvatures $\mathbf{R}^a$, $\mathbf{R}^\alpha$, $\mathbf{R}_4$. Nevertheless, it would be an interesting question for further study to determine whether the Lie algebras $\tilde{\mathfrak{E}}(s)$ introduced below to trivialize the four-cocycle ω_4 (of equation (6.2.6)) also allow for the trivialization of ω_7 (of equation (6.2.7)). Namely, whether there exists a six-form $\tilde{\omega}_6$ (corresponding to the 'flat limit' of A_6) constructed as an exterior polynomial of the MC one-forms of $\tilde{\mathfrak{E}}(s)$ and such that $\omega_7 = d\tilde{\omega}_6$. This would correspond to the problem of finding the *underlying gauge symmetry* of the duality-symmetric formulation of $D = 11$ supergravity (see [181] for the action).

6.3 A family of extended superalgebras

As stated in [92], the problem is whether the $D = 11$ supergravity FDA (6.2.1), (6.2.3), may be completed with a number of additional *one*-forms and their curvatures in such a way that the three-form A_3 obeying (6.2.3) is constructed from one-forms, becoming composite rather than fundamental or 'elementary'. This problem, when attacked in the flat limit achieved by setting all the curvatures to zero, is equivalent to trivializing the Σ four-cocycle ω_4 (equation (6.2.6)) on the algebra $\tilde{\mathfrak{E}}$ of an *enlarged* superspace group $\tilde{\Sigma}$. A one-parameter family of Lie superalgebra extensions $\tilde{\mathfrak{E}}(s)$ (with the notation of [6]) of the M Theory superalgebra was first proposed by D'Auria and Fré in [92] as an ansatz to solve the problem. All the superalgebras in the family contain a set of 528 bosonic and $32 + 32 = 64$ fermionic generators,

$$P_a \, , \, Q_\alpha \, , \, Z_{a_1 a_2} \, , \, Z_{a_1 \ldots a_5} \, , \, Q'_\alpha \, , \tag{6.3.1}$$

including the M Theory superalgebra ones (see section 2.1 of chapter 2) plus a central fermionic generator Q'_α, and are defined through the (anti)commutation relations

$$\begin{aligned}
&\{Q_\alpha, Q_\beta\} = \Gamma^a_{\alpha\beta} P_a + i\Gamma^{a_1 a_2}_{\alpha\beta} Z_{a_1 a_2} + \Gamma^{a_1 \ldots a_5}_{\alpha\beta} Z_{a_1 \ldots a_5} \, , \\
&[P_a, Q_\alpha] = \delta \, \Gamma_{a \, \alpha}{}^\beta Q'_\beta \, , \\
&[Z_{a_1 a_2}, Q_\alpha] = i\gamma_1 \Gamma_{a_1 a_2 \, \alpha}{}^\beta Q'_\beta \, , \\
&[Z_{a_1 \ldots a_5}, Q_\alpha] = \gamma_2 \Gamma_{a_1 \ldots a_5 \, \alpha}{}^\beta Q'_\beta \, , \\
&[Q'_\alpha, \text{all}\} = 0 \, ,
\end{aligned} \tag{6.3.2}$$

that display their structure as central extensions of the M Theory superalgebra by the fermionic generator Q'_α. With the notation of section 2.1 of chapter 2, the family of enlarged superalgebras $\tilde{\mathfrak{E}}(s)$ could be denoted as $\mathfrak{E}^{(528|64)}(s)$ or, in order to emphasize the presence of two independent fermionic generators, as $\mathfrak{E}^{(528|32+32)}(s)$. The corresponding group manifolds can accordingly be denoted $\tilde{\Sigma}(s)$ or $\Sigma^{(528|32+32)}(s)$. The notation $\tilde{\mathfrak{E}}(s)$, $\tilde{\Sigma}(s)$ will be preferred, although $\mathfrak{E}^{(528|32+32)}(s)$, $\Sigma^{(528|32+32)}(s)$ will sometimes be used to avoid confusion.

In (6.3.2), δ, γ_1, γ_2 are real parameters only restricted by the requirement that (6.3.2) be indeed a superalgebra, *i.e.*, that the Jacobi identities are satisfied. This translates into a relation for the parameters [92]:

$$\delta + 10\gamma_1 - 6!\gamma_2 = 0 \ . \tag{6.3.3}$$

One parameter (γ_1 if nonvanishing, δ otherwise) can be removed by rescaling the new fermionic generator Q'_α and it is thus inessential. Hence equations (6.3.2) describe, effectively, a one-parameter family of Lie superalgebras that may be denoted $\tilde{\mathfrak{E}}(s)$ by using a parameter s given by

$$s := \frac{\delta}{2\gamma_1} - 1 \ , \ \gamma_1 \neq 0 \qquad \Rightarrow \qquad \begin{cases} \delta \ = 2\gamma_1(s+1) \, , \\ \gamma_2 \ = 2\gamma_1(\frac{s}{6!} + \frac{1}{5!}) \ . \end{cases} \tag{6.3.4}$$

This notation also accounts for the case $\gamma_1 = 0$, by considering the limit $\gamma_1 \to 0$, $s \to \infty$ and $\gamma_1 s \to \delta/2 \neq 0$, and the corresponding algebra can be denoted $\tilde{\mathfrak{E}}(\infty)$. In terms of s, the algebra (6.3.2) reads:

$$\begin{aligned}
&\{Q_\alpha, Q_\beta\} = \Gamma^a_{\alpha\beta} P_a + i\Gamma^{a_1 a_2}_{\alpha\beta} Z_{a_1 a_2} + \Gamma^{a_1 \ldots a_5}_{\alpha\beta} Z_{a_1 \ldots a_5} \ , \\
&[P_a, Q_\alpha] = 2\gamma_1(s+1) \ \Gamma_{a\ \alpha}{}^{\beta} Q'_\beta \ , \\
&[Z_{a_1 a_2}, Q_\alpha] = i\gamma_1 \Gamma_{a_1 a_2\ \alpha}{}^{\beta} Q'_\beta \ , \\
&[Z_{a_1 \ldots a_5}, Q_\alpha] = 2\gamma_1(\frac{s}{6!} + \frac{1}{5!}) \Gamma_{a_1 \ldots a_5\ \alpha}{}^{\beta} Q'_\beta \ , \\
&[Q'_\alpha, \text{all}\} = 0 \ .
\end{aligned} \tag{6.3.5}$$

Introducing the MC one-forms

$$\Pi^a \ , \ \pi^\alpha \ , \ \Pi^{a_1 a_2} \ , \ \Pi^{a_1 \ldots a_5} \ , \ \pi'^\alpha \ , \tag{6.3.6}$$

dual to the generators (6.3.1), and left-invariant on the corresponding group manifolds $\tilde{\Sigma}(s) \equiv \Sigma^{(528|32+32)}(s)$, the family of superalgebras $\tilde{\mathfrak{E}}(s)$ can be equivalently described by the MC equations

$$\begin{aligned}
&d\Pi^a = -i\pi^\alpha \wedge \pi^\beta \Gamma^a_{\alpha\beta} \ , \\
&d\pi^\alpha = 0 \ ,
\end{aligned}$$

$$
\begin{aligned}
d\Pi^{a_1a_2} &= -\pi^\alpha \wedge \pi^\beta\, \Gamma^{a_1a_2}_{\alpha\beta}\ , \\
d\Pi^{a_1\dots a_5} &= -i\pi^\alpha \wedge \pi^\beta\, \Gamma^{a_1\dots a_5}_{\alpha\beta}\ , \\
d\pi'^\alpha &= \pi^\beta \wedge \left(-i\,\delta\,\Pi^a\Gamma_{a\,\beta}{}^\alpha - \gamma_1\,\Pi^{ab}\Gamma_{ab\,\beta}{}^\alpha - i\,\gamma_2\,\Pi^{a_1\dots a_5}\Gamma_{a_1\dots a_5\beta}{}^\alpha\right)\ .
\end{aligned}
\tag{6.3.7}
$$

In the dual, MC formulation of the family of superalgebras $\tilde{\mathfrak{E}}(s)$, the parameters δ, γ_1, γ_2 are only involved in the last equation of (6.3.7), the MC equation for the extra fermionic MC one-form π'^α, and the relation (6.3.3) among the parameters is obtained from the integrability condition $dd\pi'^\alpha = 0$. In terms of the parameter s defined in (6.3.4), the last equation in (6.3.7) reads

$$
d\pi'^\alpha = -2\gamma_1\pi^\beta \wedge \left(i(s+1)\Pi^a\Gamma_a + \tfrac{1}{2}\Pi^{ab}\Gamma_{ab} + i\left(\tfrac{s}{6!}+\tfrac{s}{5!}\right)\Pi^{a_1\dots a_5}\Gamma_{a_1\dots a_5}\right)_\beta{}^\alpha . \tag{6.3.8}
$$

Finally, introducing the 'soft' one-form fields,

$$
e^a\ ,\ \psi^\alpha\ ,\ B^{a_1a_2}\ ,\ B^{a_1\dots a_5}\ ,\ \eta^\alpha\ , \tag{6.3.9}
$$

corresponding to the MC one-forms (6.3.6), and their corresponding curvatures,

$$
\mathbf{R}^a\ ,\ \mathbf{R}^\alpha\ ,\ \mathcal{B}^{a_1a_2}\ ,\ \mathcal{B}^{a_1\dots a_5}\ ,\ \mathcal{B}^\alpha\ , \tag{6.3.10}
$$

the family of gauge FDAs corresponding to $\tilde{\mathfrak{E}}(s)$ is described by the equations (6.2.1), (6.2.3) together with the corresponding equations for the new one-forms and their curvatures, namely, by

$$
\begin{aligned}
&\mathbf{R}^a := de^a - e^b \wedge \omega_b{}^a + i\psi^\alpha \wedge \psi^\beta\Gamma^a_{\alpha\beta} = T^a + i\psi^\alpha \wedge \psi^\beta\Gamma^a_{\alpha\beta}\ , \\
&\mathbf{R}^\alpha := d\psi^\alpha - \psi^\beta \wedge \omega_\beta{}^\alpha \quad \left(\omega_\alpha{}^\beta = \tfrac{1}{4}\omega^{ab}\Gamma_{ab\,\alpha}{}^\beta\right)\ , \\
&\mathbf{R}^{ab} := d\omega^{ab} - \omega^{ac}\wedge\omega_c{}^b\ , \\
&\mathbf{R}_4 := dA_3 + \tfrac{1}{4}\psi^\alpha \wedge \psi^\beta \wedge e^a \wedge e^b\Gamma_{ab\alpha\beta}\ , \\
&\mathcal{B}_2^{ab} = DB^{ab} + \psi^\alpha \wedge \psi^\beta\,\Gamma^{ab}_{\alpha\beta}\ , \\
&\mathcal{B}_2^{a_1\dots a_5} = DB^{a_1\dots a_5} + i\psi^\alpha \wedge \psi^\beta\,\Gamma^{a_1\dots a_5}_{\alpha\beta}\ , \\
&\mathcal{B}_2^\alpha = D\eta^\alpha + \psi^\beta \wedge \left(i\,\delta\,e^a\Gamma_{a\,\beta}{}^\alpha + \gamma_1\,B^{ab}\Gamma_{ab\,\beta}{}^\alpha + i\,\gamma_2\,B^{a_1\dots a_5}\Gamma_{a_1\dots a_5\ \beta}{}^\alpha\right)\ ,
\end{aligned}
\tag{6.3.11}
$$

and their corresponding Bianchi identities.

6.4 The relation of $\tilde{\mathfrak{E}}(s)$ with $osp(1|32)$

For $s \neq 0$, the superalgebras $\tilde{\mathfrak{E}}(s)$ are non-trivial deformations (see section 5.1) of $\tilde{\mathfrak{E}}(0)$. Actually, the superalgebra $\tilde{\mathfrak{E}}(0)$ is singled out within the family $\tilde{\mathfrak{E}}(s)$ for having an enhanced automorphism group. Introducing, as in (2.1.9), the generalized momentum $P_{\alpha\beta} = \Gamma^a_{\alpha\beta} P_a + i\Gamma^{a_1 a_2}_{\alpha\beta} Z_{a_1 a_2} + \Gamma^{a_1 \ldots a_5}_{\alpha\beta} Z_{a_1 \ldots a_5}$, the $D = 11$ decomposition

$$\delta_{(\alpha}{}^{\gamma}\delta_{\beta)}{}^{\delta} = \tfrac{1}{32}\left(\Gamma^a_{\alpha\beta}\Gamma^{\gamma\delta}_a - \tfrac{1}{2}\Gamma^{a_1a_2}{}_{\alpha\beta}\Gamma_{a_1a_2}{}^{\gamma\delta} + \tfrac{1}{5!}\Gamma^{a_1\ldots a_5}{}_{\alpha\beta}\Gamma_{a_1\ldots a_5}{}^{\gamma\delta}\right) \tag{6.4.1}$$

allows us to write the superalgebra (6.3.5) for $s = 0$ as

$$\begin{aligned} &\{Q_\alpha, Q_\beta\} = P_{\alpha\beta} \ , \\ &[P_{\alpha\beta}, Q_\gamma] = 64\ \gamma_1\ C_{\gamma(\alpha}Q'_{\beta)} \ , \\ &[Q'_\alpha, \text{all}\} = 0 \ . \end{aligned} \tag{6.4.2}$$

Similarly, it is possible to collect the MC one-forms Π^a, $\Pi^{a_1a_2}$, $\Pi^{a_1\cdots a_5}$ in a symmetric spin-tensor one-form (2.1.14), $\Pi^{\alpha\beta} = \frac{1}{32}(\Pi^a\Gamma_a - \frac{i}{2}\Pi^{a_1a_2}\Gamma_{a_1a_2} + \frac{1}{5!}\Pi^{a_1\ldots a_5}\Gamma_{a_1\ldots a_5})^{\alpha\beta}$ that allows us to write, for $s = 0$, the MC equations (6.3.7) of $\tilde{\mathfrak{E}}(0)$ in compact form as

$$\begin{aligned} &d\Pi^{\alpha\beta} = -i\pi^\alpha \wedge \pi^\beta \ , \\ &d\pi^\alpha = 0 \ , \\ &d\pi'^\alpha = -64i\gamma_1\, \pi^\beta \wedge \Pi_\beta{}^\alpha \ . \end{aligned} \tag{6.4.3}$$

The explicit appearance in equation (6.4.2) of the $Sp(32)$-invariant eleven-dimensional 32×32 charge conjugation matrix $C_{\alpha\beta}$ or, alternatively, its concealed appearance in the contraction of spinor indices in (6.4.3), exhibits $Sp(32)$ as the automorphism symmetry of $\tilde{\mathfrak{E}}(0)$. In contrast, the rest of superalgebras $\tilde{\mathfrak{E}}(s)$, $s \neq 0$, have a reduced automorphism symmetry $SO(1,10)$, since they involve explicitly the $SO(1,10)$ Dirac matrices.

Hence, the generalizations of the superPoincaré group $\Sigma \rtimes SO(1,10)$ for the $s \neq 0$ and $s = 0$ cases are, respectively, the semidirect products $\tilde{\Sigma}(s) \rtimes SO(1,10)$ and $\tilde{\Sigma}(0) \rtimes Sp(32)$. Precisely for $s = 0$, both $\tilde{\Sigma}(0) \rtimes SO(1,10)$ and $\tilde{\Sigma}(0) \rtimes Sp(32)$ can be obtained from $OSp(1|32)$ by the expansion method of chapter 5; they are given, respectively, by the expansions $Osp(1|32)(2,3,2)$ and $Osp(1|32)(2,3)$ [6] as it will now be shown.

The derivation of $\tilde{\mathfrak{E}}(0) \rtimes so(1,10)$ as an expansion of $osp(1|32)$ fits into the general discussion of section 5.5 of the expansion method for superalgebras. In fact, it follows the same steps that led to the M Theory superalgebra in section 5.6, the only difference being the way the cutting orders of the series expansions of the MC one-forms of $osp(1|32)$ are chosen. Consider the 528 $sp(32)$ bosonic $\rho^{\alpha\beta}$ and 32 fermionic ν^α MC forms of $osp(1|32)$, satisfying the MC equations (5.6.1). Again, it is useful to decompose $\rho^{\alpha\beta}$ in terms of Dirac matrices as in (5.6.2), $\rho^{\alpha\beta} = \frac{1}{32}\left(\rho^a\Gamma_a - \frac{i}{2}\rho^{ab}\Gamma_{ab} + \frac{1}{5!}\rho^{a_1\ldots a_5}\Gamma_{a_1\ldots a_5}\right)^{\alpha\beta}$; this decomposition is adapted to the splitting $osp(1|32) = V_0 \oplus V_1 \oplus V_2$, where V_0 is generated by ρ^{ab}, V_1 by ν^α and V_2 by ρ^a and $\rho^{a_1\ldots a_5}$. In terms of ρ^{ab}, ν^α, ρ^a and $\rho^{a_1\ldots a_5}$, the superalgebra $osp(1|32)$ takes the form (5.6.3).

After the redefinition (5.5.3) of the group parameters of $osp(1|32)$, the MC forms expand as in (5.6.4)–(5.6.6), namely,

$$
\begin{aligned}
&\rho^{ab} = \rho^{ab,0} + \lambda^2\rho^{ab,2} + \lambda^4\rho^{ab,4} + \cdots\ ,\\
&\rho^a = \lambda^2\rho^{a,2} + \lambda^4\rho^{a,4} + \cdots\ ,\\
&\rho^{a_1\ldots a_5} = \lambda^2\rho^{a_1\ldots a_5,2} + \lambda^4\rho^{a_1\ldots a_5,4} + \cdots\ ,\\
&\nu^\alpha = \lambda\nu^{\alpha,1} + \lambda^3\nu^{\alpha,3} + \cdots\ .
\end{aligned}
\tag{6.4.4}
$$

Choosing the cutting orders $N_0 = 2$, $N_1 = 3$, $N_2 = 2$, as allowed by the restriction (5.5.7), the MC equations of the expanded algebra $osp(1|32)(2,3,2)$ are obtained:

$$
\begin{aligned}
&d\rho^{ab,0} = -\tfrac{1}{16}\rho^{ac,0}\wedge\rho_c{}^{b,0}\ ,\\
&d\rho^{a\,,2} = -\tfrac{1}{16}\rho^{b,2}\wedge\rho_b{}^{a,0} - i\nu^{\alpha,1}\wedge\nu^{\beta,1}\Gamma^a_{\alpha\beta}\ ,\\
&d\rho^{ab,2} = -\tfrac{1}{16}\left(\rho^{ac,0}\wedge\rho_c{}^{b,2} + \rho^{ac,2}\wedge\rho_c{}^{b,0}\right) - \nu^{\alpha,1}\wedge\nu^{\beta,1}\Gamma^{ab}_{\alpha\beta}\ ,\\
&d\rho^{a_1\ldots a_5,2} = \tfrac{5}{16}\rho^{b[a_1\ldots a_4|\,,2}\wedge\rho_b{}^{|a_5],0} - i\nu^{\alpha,1}\wedge\nu^{\beta,1}\Gamma^{a_1\ldots a_5}_{\alpha\beta}\ ,\\
&d\nu^{\alpha,1} = -\tfrac{1}{64}\nu^{\beta,1}\wedge\rho^{ab,0}\Gamma_{ab\beta}{}^\alpha\ ,\\
&d\nu^{\alpha,3} = -\tfrac{1}{64}\nu^{\beta,3}\wedge\rho^{ab,0}\Gamma_{ab\beta}{}^\alpha\\
&\qquad -\tfrac{1}{32}\nu^{\beta,1}\wedge\left(i\rho^{a,2}\Gamma_a + \tfrac{1}{2}\rho^{ab,2}\Gamma_{ab} + \tfrac{i}{5!}\rho^{a_1\ldots a_5,2}\Gamma_{a_1\ldots a_5}\right)_\beta{}^\alpha\ .
\end{aligned}
\tag{6.4.5}
$$

Now, setting $\rho^{ab,0} \equiv -16\omega^{ab}$ and identifying $\rho^{a,2} \equiv \Pi^a$, $\rho^{ab,2} \equiv \Pi^{ab}$, $\rho^{a_1\cdots a_5,2} \equiv \Pi^{a_1\cdots a_5}$, $\nu^{\alpha,1} \equiv \pi^\alpha$ and $\nu^{\alpha,3} \equiv \pi'^\alpha/64\gamma_1$ (notice that $\gamma_1 \neq 0$ just defines the scale of Q'_α), the set of equations (6.4.5) coincides with the MC equations of $\tilde{\mathfrak{E}}(0) \rtimes so(1,10)$ (obtained by restoring the Lorentz part in equations (6.4.3)). As a check, notice that the dimensional counting is correct since, by equation (5.5.4),

$$
\begin{aligned}
\dim osp(1|32)(2,3,2) &= 2\cdot 55 + 2\cdot 32 + 473 = 583 + 64 =\\
&= \dim\left(\tilde{\mathfrak{E}}(0)\rtimes so(1,10)\right)\ .
\end{aligned}
\tag{6.4.6}
$$

In conclusion [6],

$$
\tilde{\Sigma}(0) \rtimes SO(1,10) \approx OSp(1|32)(2,3,2)\ . \tag{6.4.7}
$$

The algebra $\tilde{\mathfrak{E}}(0) \rtimes sp(32)$ with its enhanced automorphism symmetry $Sp(32)$ can also be obtained as an expansion of $osp(1|32)$. Indeed, consider instead the splitting $osp(1|32) = V_0 \oplus V_1$ where V_0 is generated by all the bosonic generators $\rho^{\alpha\beta}$ and V_1 by the fermionic ones, ν^α. This splitting makes V_1 a symmetric coset and, indeed, makes the algebra have the structure discussed in section 5.3.1 of chapter 5. Cutting the corresponding series at orders $N_0 = 2$ and $N_1 = 3$, in agreement with condition (5.3.30), the MC equations corresponding to the expansion $osp(1|32)(2,3)$ are obtained:

$$\begin{aligned}
d\rho^{\alpha\beta,0} &= -i\rho^{\alpha\gamma,0} \wedge \rho_\gamma{}^{\beta,0} \ , \\
d\rho^{\alpha\beta,2} &= -i\left(\rho^{\alpha\gamma,0} \wedge \rho_\gamma{}^{\beta,2} + \rho^{\alpha\gamma,2} \wedge \rho_\gamma{}^{\beta,0}\right) - i\nu^{\alpha,1} \wedge \nu^{\beta,1} \ , \\
d\nu^{\alpha,1} &= -i\nu^{\beta,1} \wedge \rho_\beta{}^{\alpha,0} \ , \\
d\nu^{\alpha,3} &= -i\nu^{\beta,3} \wedge \rho_\beta{}^{\alpha,0} - i\nu^{\beta,1} \wedge \rho_\beta{}^{\alpha,2} \ . \qquad (6.4.8)
\end{aligned}$$

Identifying $\rho^{\alpha\beta,0}$ in (6.4.8) with an $sp(32)$ connection, equations (6.4.8) are those of $\tilde{\mathfrak{E}}(0) \rtimes sp(32)$ (given by (6.4.3) when $sp(32)$-automorphisms are included) with $\rho^{\alpha\beta,2} \equiv \Pi^{\alpha\beta}$, $\nu^{\alpha,1} \equiv \pi^\alpha$ and $\nu^{\alpha,3} \equiv \pi'^\alpha/64\gamma_1$. Again, by equations (5.5.4), the dimensions agree,

$$\dim osp(1|32)(2,3) = 2 \cdot 528 + 64 = \dim\left(\tilde{\mathfrak{E}}(0) \rtimes sp(32)\right) \ , \qquad (6.4.9)$$

and [6]

$$\tilde{\Sigma}(0) \rtimes Sp(32) \approx OSp(1|32)(2,3) \ . \qquad (6.4.10)$$

6.5 The composite nature of A_3

We will now show how the set of one-forms (6.3.9) of the gauge FDA (6.3.11) associated to $\tilde{\mathfrak{E}}(s)$ allows for a composite structure of A_3,

$$A_3 = A_3(e^a, \psi^\alpha \ ; \ B^{ab}, B^{abcde}, \eta^\alpha) \ . \qquad (6.5.1)$$

According to the discussion of section 6.2, based on the general arguments of section 6.1, the problem is equivalent to the trivialization of the super-Poincaré algebra CE four-cocycle ω_4 of equation (6.2.6) in an extended superalgebra. Thus, we are looking for a three-form $\tilde{\omega}_3$ built up as an exterior polynomial of the one-forms (6.3.6) of the family of extensions $\tilde{\mathfrak{E}}(s)$ that fulfils equation (6.2.6), namely,

$$d\tilde{\omega}_3 = \omega_4 \equiv -\tfrac{1}{4}\pi^\alpha \wedge \pi^\beta \wedge \Pi^a \wedge \Pi^b \Gamma_{ab\alpha\beta} \ . \qquad (6.5.2)$$

In the process, it will be made apparent which of the superalgebras in the family $\tilde{\mathfrak{E}}(s)$ allow for a trivialization of ω_4.

The most general expression for $\tilde{\omega}_3$ as an exterior polynomial of the one-forms (6.3.6) of $\tilde{\mathfrak{E}}(s)$ is

$$
\begin{aligned}
4\,\tilde{\omega}_3 = & \lambda \Pi^{ab} \wedge \Pi_a \wedge \Pi_b \; - \alpha_1 \Pi_{ab} \wedge \Pi^b{}_c \wedge \Pi^{ca} \\
& - \alpha_2 \Pi_{b_1 a_1 \ldots a_4} \wedge \Pi^{b_1}{}_{b_2} \wedge \Pi^{b_2 a_1 \ldots a_4} \\
& - \alpha_3 \epsilon_{a_1 \ldots a_5 b_1 \ldots b_5 c} \Pi^{a_1 \ldots a_5} \wedge \Pi^{b_1 \ldots b_5} \wedge \Pi^c \\
& - \alpha_4 \epsilon_{a_1 \ldots a_6 b_1 \ldots b_5} \Pi^{a_1 a_2 a_3}{}_{c_1 c_2} \wedge \Pi^{a_4 a_5 a_6 c_1 c_2} \wedge \Pi^{b_1 \ldots b_5} \\
& - 2i \pi^\beta \wedge \pi'^\alpha \wedge \left(\beta_1 \, \Pi^a \Gamma_{a \alpha\beta} - i\beta_2 \, \Pi^{ab} \Gamma_{ab\, \alpha\beta} + \beta_3 \, \Pi^{a_1 \ldots a_5} \Gamma_{a_1 \ldots a_5\, \alpha\beta} \right) \,,
\end{aligned}
\qquad (6.5.3)
$$

where $\alpha_1, \ldots, \alpha_4$, $\beta_1, \ldots, \beta_3$ [92] *and* λ [6, 7] are constants to be determined by the requirement that $\tilde{\omega}_3$ obeys equation (6.5.2). The numerical factors in the right hand side of (6.5.3) have been introduced to make the definition of the coefficients coincide with that in [92] while keeping our notation for the FDA. The only essential difference with [92] is the inclusion of the arbitrary coefficient λ in the first term; as we show below this leads to a one-parametric family of solutions that includes the two D'Auria-Fré ones.

Using the MC equations (6.3.7) for $\tilde{\mathfrak{E}}(s)$, the application in (6.5.3) of the differential d leads to [92]

$$
\begin{aligned}
4\, d\tilde{\omega}_3 = & -(\lambda - 2\delta\beta_1) \; \pi^\alpha \wedge \pi^\beta \wedge \Pi^a \wedge \Pi^b \Gamma_{ab\alpha\beta} \\
& + 2(\beta_1 + 10\beta_2 - 6!\beta_3) \; \pi^\alpha \wedge \pi^\beta \wedge \pi^\gamma \wedge \pi'^\delta \Gamma^a{}_{\alpha\beta} \Gamma_{a\gamma\delta} \\
& + 2i(\lambda - 2\gamma_1\beta_1 - 2\delta\beta_2) \; \pi^\alpha \wedge \pi^\beta \wedge \Pi^{ab} \wedge \Pi_b \Gamma_{a\alpha\beta} \\
& + (3\alpha_1 + 8\gamma_1\beta_2) \; \pi^\alpha \wedge \pi^\beta \wedge \Pi^a{}_c \wedge \Pi^{cb} \Gamma_{ab\alpha\beta} \\
& + 2i(\alpha_2 - 10\gamma_1\beta_3 - 10\gamma_2\beta_2) \; \pi^\alpha \wedge \pi^\beta \wedge \Pi^{a_1}{}_c \wedge \Pi^{c a_2 \cdots a_5} \Gamma_{a_1 \cdots a_5 \alpha\beta} \\
& + \tfrac{2i}{5!}(5!\alpha_3 - \delta\beta_3 - \gamma_2\beta_1) \; \epsilon_{a_1 \cdots a_5 b_1 \cdots b_5 c} \pi^\alpha \wedge \pi^\beta \wedge \Pi^{b_1 \cdots b_5} \wedge \Pi^c \Gamma_{a_1 \cdots a_5 \alpha\beta} \\
& - (\alpha_2 - 5!\, 10\gamma_2\beta_3) \; \pi^\alpha \wedge \pi^\beta \wedge \Pi^{a_1}{}_{b_1 \cdots b_4} \wedge \Pi^{a_2 b_1 \cdots b_4} \Gamma_{a_1 a_2 \alpha\beta} \\
& + i(\alpha_3 - 2\gamma_2\beta_3) \; \epsilon_{a_1 \cdots a_5 b_1 \cdots b_5 c} \pi^\alpha \wedge \pi^\beta \wedge \Pi^{a_1 \cdots a_5} \wedge \Pi^{b_1 \cdots b_5} \Gamma^c{}_{\alpha\beta} \\
& + \tfrac{i}{3}(9\alpha_4 + 10\gamma_2\beta_3) \; \epsilon_{a_1 \cdots a_6 b_1 \cdots b_5} \pi^\alpha \wedge \pi^\beta \wedge \Pi^{a_1 a_2 a_3}{}_{c_1 c_2} \wedge \Pi^{a_4 a_5 a_6 c_1 c_2} \Gamma^{b_1 \cdots b_5}{}_{\alpha\beta} .
\end{aligned}
\qquad (6.5.4)
$$

Finally, comparing the expressions for $d\tilde{\omega}_3$ in (6.5.4) and (6.5.2), and equating the coefficients of the different, independent four-form terms, the following non-homogeneous linear system of nine equations and eight unknowns λ, $\alpha_1, \ldots, \alpha_4$, $\beta_1, \ldots, \beta_3$, dependent on the parameters δ, γ_1,

γ_2 is found[6] [92, 6, 7]

$$
\begin{aligned}
&\lambda - 2\delta\beta_1 = 1 \ , \\
&\beta_1 + 10\beta_2 - 6!\beta_3 = 0 \ , \\
&\lambda - 2\gamma_1\beta_1 - 2\delta\beta_2 = 0 \ , \\
&3\alpha_1 + 8\gamma_1\beta_2 = 0 \ , \\
&\alpha_2 - 10\gamma_1\beta_3 - 10\gamma_2\beta_2 = 0 \ , \\
&5!\alpha_3 - \delta\beta_3 - \gamma_2\beta_1 = 0 \ , \\
&\alpha_2 - 5!\,10\gamma_2\beta_3 = 0 \ , \\
&\alpha_3 - 2\gamma_2\beta_3 = 0 \ , \\
&9\alpha_4 + 10\gamma_2\beta_3 = 0 \ .
\end{aligned} \tag{6.5.5}
$$

The existence of solutions to this system depends on the values of the parameters δ, γ_1, γ_2 that define it. Actually, the system (6.5.5) depends effectively on one parameter s (defined in (6.3.4)) only, for the same reason as the family of superalgebras $\tilde{\mathfrak{E}}(s)$ does: one parameter among δ, γ_1, γ_2 can be eliminated by means of the relation (6.3.3), and another one by a redefinition of the extra fermionic generator Q'_α in (6.3.5) (or π'^α in (6.3.7)). The system (6.5.5) turns out to be incompatible for $s = 0$ (namely, for $\delta = 2\gamma_1$, $\gamma_2 = 2\gamma_1/5!$) but has, otherwise, a unique solution for each $s \neq 0$ given by [6, 7]

$$
\begin{aligned}
&\lambda = \tfrac{1}{5}\,\tfrac{s^2+2s+6}{s^2} \ , \ \beta_1 = -\tfrac{1}{10\gamma_1}\,\tfrac{2s-3}{s^2} \ , \ \beta_2 = \tfrac{1}{20\gamma_1}\,\tfrac{s+3}{s^2} \ , \ \beta_3 = \tfrac{3}{10\cdot 6!\gamma_1}\,\tfrac{s+6}{s^2} \ , \\
&\alpha_1 = -\tfrac{1}{15}\,\tfrac{2s+6}{s^2} \ , \ \alpha_2 = \tfrac{1}{6!}\,\tfrac{(s+6)^2}{s^2} \ , \ \alpha_3 = \tfrac{1}{5\cdot 6!5!}\,\tfrac{(s+6)^2}{s^2} \ , \ \alpha_4 = -\tfrac{1}{9\cdot 6!5!}\,\tfrac{(s+6)^2}{s^2} \ .
\end{aligned} \tag{6.5.6}
$$

The two particular solutions in [92] are recovered by adjusting s (*i.e.*, δ, γ_1 in equation (6.3.4)) so that $\lambda = 1$ in equation (6.5.6). This is achieved for $\delta = 5\gamma_1$ (δ non vanishing but otherwise arbitrary), or for $\delta = 0$ (with γ_1 non vanishing but otherwise arbitrary). Thus, the two D'Auria and Fré decompositions of A_3 are characterized by $s = 3/2$,

$$
\begin{aligned}
&\tilde{\mathfrak{E}}(3/2) \ : \ \delta = 5\gamma_1 \neq 0 \ , \ \gamma_2 = \tfrac{\gamma_1}{2\cdot 4!} \ , \\
&\qquad \lambda = 1 \ , \quad \beta_1 = 0 \ , \quad \beta_2 = \tfrac{1}{10\gamma_1} \ , \quad \beta_3 = \tfrac{1}{6!\,\gamma_1} \ , \\
&\qquad \alpha_1 = -\tfrac{4}{15} \ , \quad \alpha_2 = \tfrac{25}{6!} \ , \quad \alpha_3 = \tfrac{1}{6!4!} \ , \quad \alpha_4 = -\tfrac{1}{54\,(4!)^2} \ ,
\end{aligned} \tag{6.5.7}
$$

and by $s = -1$,

$$
\tilde{\mathfrak{E}}(-1) \ : \ \delta = 0 \ , \ \gamma_1 \neq 0 \ , \ \gamma_2 = \tfrac{\gamma_1}{3\cdot 4!} \ ,
$$

[6] The factor 5! in the equation $5!\alpha_3 - \delta\beta_3 - \gamma_2\beta_1 = 0$, and the factor 9 in the equation $9\alpha_4 + 10\gamma_2\beta_3 = 0$ were both missing in footnote 6 in [6], and the later factor was missing in equation (4.39) of [7].

$$\lambda = 1 \;, \quad \beta_1 = \frac{1}{2\gamma_1} \;, \quad \beta_2 = \frac{1}{10\gamma_1} \;, \quad \beta_3 = \frac{1}{4{\cdot}5!\,\gamma_1} \;,$$
$$\alpha_1 = -\frac{4}{15} \;, \quad \alpha_2 = \frac{25}{6!} \;, \quad \alpha_3 = \frac{1}{6!\,4!} \;, \quad \alpha_4 = -\frac{1}{54\,(4!)^2} \;. \qquad (6.5.8)$$

Here it has been shown that not only these two superalgebras solve the problem, but that *all the superalgebras in the family* $\tilde{\mathfrak{E}}(s)$*, except* $\tilde{\mathfrak{E}}(0)$*, allow for a trivialization of the four-cocycle* ω_4 *in (6.5.2)* [6, 7]. The three-form $\tilde{\omega}_3$ that trivializes it in $\tilde{\mathfrak{E}}(s)$, $s \neq 0$, is given by the expression (6.5.3) with the coefficients (6.5.6). The composite expression (6.5.1) of the three-form A_3 in terms of the soft one-form counterparts (6.3.9) of the MC one-forms of $\tilde{\mathfrak{E}}(s)$, $s \neq 0$, is thus given explicitly by

$$\begin{aligned} 4A_3 = {} & \lambda B^{ab} \wedge e_a \wedge e_b \; - \alpha_1 B_{ab} \wedge B^b{}_c \wedge B^{ca} \\ & -\alpha_2 B_{b_1 a_1 \ldots a_4} \wedge B^{b_1}{}_{b_2} \wedge B^{b_2 a_1 \ldots a_4} \\ & -\alpha_3 \epsilon_{a_1 \ldots a_5 b_1 \ldots b_5 c} B^{a_1 \ldots a_5} \wedge B^{b_1 \ldots b_5} \wedge e^c \\ & -\alpha_4 \epsilon_{a_1 \ldots a_6 b_1 \ldots b_5} B^{a_1 a_2 a_3}{}_{c_1 c_2} \wedge B^{a_4 a_5 a_6 c_1 c_2} \wedge B^{b_1 \ldots b_5} \\ & -2i\psi^\beta \wedge \eta^\alpha \wedge \Big(\beta_1\, e^a \Gamma_{a\alpha\beta} - i\beta_2\, B^{ab} \Gamma_{ab\,\alpha\beta} + \beta_3\, B^{a_1 \ldots a_5} \Gamma_{a_1 \ldots a_5\,\alpha\beta}\Big) \;, \end{aligned} \qquad (6.5.9)$$

where the coefficients are given by (6.5.6).

It is worth stressing that allowing the coefficient λ not to be fixed from the onset, produces more possibilities for the trivializing algebras than those in [92] (equations (6.5.7) and (6.5.8)). A particularly interesting superalgebra within the family $\tilde{\mathfrak{E}}(s)$ is achieved for $s = -6$. The trivialization of the four-cocycle ω_4 (6.5.2) associated to dA_3 on $\tilde{\mathfrak{E}}(-6)$ is obtained for the coefficients (6.5.6) with $s = -6$, namely,

$$\begin{aligned} \tilde{\mathfrak{E}}(-6) \;:\; & \delta = -10\gamma_1 \neq 0 \;, \; \gamma_2 = 0 \;, \\ & \lambda = \frac{1}{6} \;, \quad \beta_1 = \frac{1}{4!\gamma_1} \;, \quad \beta_2 = -\frac{1}{2{\cdot}5!\gamma_1} \;, \quad \beta_3 = 0 \;, \\ & \alpha_1 = \frac{1}{90} \;, \quad \alpha_2 = 0 \;, \quad \alpha_3 = 0 \;, \quad \alpha_4 = 0 \;. \end{aligned} \qquad (6.5.10)$$

Interestingly enough, the vanishing coefficients α_2, α_3, α_4, β_3 are those in front of each term involving the one-form $\Pi^{a_1 \ldots a_5}$ in the expression of the trivializing three-form $\tilde{\omega}_3$ (6.5.3). In consequence, the expression (6.5.9) for A_3 as a composite of the gauge one-form fields of $\tilde{\mathfrak{E}}(-6)$ becomes especially simple,

$$\begin{aligned} A_3 = {} & \frac{1}{4!} B^{ab} \wedge e_a \wedge e_b \; - \frac{1}{3{\cdot}5!} B_{ab} \wedge B^b{}_c \wedge B^{ca} \\ & - \frac{i}{4{\cdot}5!\,\gamma_1} \psi^\beta \wedge \eta^\alpha \wedge \Big(10\, e^a \Gamma_{a\alpha\beta} + i\, B^{ab} \Gamma_{ab\,\alpha\beta}\Big) \;, \end{aligned} \qquad (6.5.11)$$

since it does not involve the gauge one-form field $B^{a_1 \ldots a_5}$. This is, nevertheless, not surprising since, by the definition (6.3.4), the choice $s = -6$

fixes the parameter γ_2 in the algebra (6.3.2) to $\gamma_2 = 0$, rendering central the generator $Z_{a_1 \dots a_5}$ associated to the gauge field $B^{a_1 \dots a_5}$.

For $s = -6$ the central generator $Z_{a_1 \dots a_5}$ plays, in short, no role in the trivialization of ω_4. One can, therefore, get rid of it and consider the smaller, $(66 + 64)$-dimensional superalgebra $\tilde{\mathfrak{E}}_{min}$ [6, 7], the extension of which by the central charge $Z_{a_1 \dots a_5}$ gives the superalgebra $\tilde{\mathfrak{E}}(-6)$. The 'minimal' superalgebra $\tilde{\mathfrak{E}}_{min}$ is given explicitly by setting $\gamma_2 = 0$ (and, hence, $\delta = -10\gamma_1$) in (6.3.2) or, equivalently, by setting $s = -6$ in (6.3.5) and by removing the generator $Z_{a_1 \dots a_5}$:

$$
\begin{aligned}
&\{Q_\alpha, Q_\beta\} = \Gamma^a_{\alpha\beta} P_a + i\Gamma^{a_1 a_2}_{\alpha\beta} Z_{a_1 a_2} \ , \\
&[P_a, Q_\alpha] = -10\gamma_1 \ \Gamma_{a\ \alpha}{}^{\beta} Q'_\beta \ , \\
&[Z_{a_1 a_2}, Q_\alpha] = i\gamma_1 \Gamma_{a_1 a_2\ \alpha}{}^{\beta} Q'_\beta \ , \\
&[Q'_\alpha, \text{all}\} = 0 \ .
\end{aligned}
\qquad (6.5.12)
$$

It is worth noting that $\tilde{\mathfrak{E}}_{min}$ does not belong to the family $\tilde{\mathfrak{E}}(s)$ and yet it provides a composite structure of A_3 in terms of the soft field counterparts of its MC one-forms. The expression of A_3 in terms of the gauge field one-forms associated to $\tilde{\mathfrak{E}}_{min}$ is precisely (6.5.11), and it coincides with the composite expression achieved for $\tilde{\mathfrak{E}}(-6)$. In summary, the most economic extension of the standard supertranslations algebra that allows for a composite structure of A_3 is $\tilde{\mathfrak{E}}_{min}$, corresponding to the most economical extended supergroup manifold $\tilde{\Sigma}_{min} = \Sigma^{(66|32+32)}$ on which ω_4 corresponding to dA_3 becomes trivial [6, 7].

6.6 Dynamics with a composite A_3

In section 2.4 of chapter 2, the equations of motion of ordinary $D = 11$ CJS supergravity were derived (in a first order formalism) from its action S, given by equations (2.2.3), (2.2.5). It is now our aim to check for possible dynamical consequences of a composite structure of A_3 (equation (6.5.9)) in terms of the gauge one-forms (6.3.9) associated to any of the superalgebras $\tilde{\mathfrak{E}}(s)$, $s \neq 0$. As noticed in [92], to perform the analysis in general, the expression (6.5.9) with the coefficients (6.5.6) for a composed A_3 in terms of gauge one-forms of $\tilde{\mathfrak{E}}(s)$, $s \neq 0$, would have to be introduced in the first order action (2.2.3), (2.2.5) of supergravity. We shall only consider here the case in which A_3 is given by the expression (6.5.11) in terms of the soft one-forms of the minimal algebra $\tilde{\mathfrak{E}}_{min}$ (equation (6.5.12)). The conclusions, however, are general and can be translated to the general case in which A_3 is given by equation (6.5.9).

Consider, thus, the expression (6.5.11) for A_3 in terms of the gauge one-forms of the superalgebra $\tilde{\mathfrak{E}}_{min}$. Ignoring the variation of the vielbein

e^a and gravitino ψ^α fields, that would contribute, respectively, to the Einstein (2.4.5) and the Rarita-Schwinger (2.4.11) equations with on-shell-vanishing terms, the variation of A_3 as given by (6.5.11) when the fields $B^{a_1 a_2}$ and η^α are varied reads

$$\begin{aligned}\delta A_3 = &\left(\tfrac{1}{4!} e_a \wedge e_b - \tfrac{1}{5!} B_a{}^c \wedge B_{cb} + \tfrac{1}{5!\,4} \psi^\beta \wedge \eta^\alpha \Gamma_{ab\,\beta\alpha}\right) \wedge \delta B^{ab} \\ &+ \tfrac{i}{5!\,4} \psi^\beta \wedge (10\, e^a \Gamma_{a\alpha\beta} + i B^{ab} \Gamma_{ab\,\alpha\beta}) \wedge \delta\eta^\alpha \,. \qquad (6.6.1)\end{aligned}$$

Once the composite A_3 given in (6.5.11) has been introduced into the supergravity action S (equations (2.2.3), (2.2.5)), the variation of S with respect to the field B^{ab} can be worked out, taking into account that B^{ab} enters the action S only through A_3:

$$\begin{aligned}\frac{\delta S}{\delta B_{ab}} &= \frac{\delta S}{\delta A_3} \wedge \frac{\delta A_3}{\delta B_{ab}} \\ &= \tfrac{1}{4!} \mathcal{G}_8 \wedge \left(e^a \wedge e^b - \tfrac{1}{5} B^{ac} \wedge B_c{}^b + \tfrac{1}{20} \psi \wedge \eta\, \Gamma_{ab}\right) \,. \qquad (6.6.2)\end{aligned}$$

In this expression, the eight-form $\mathcal{G}_8$ is the variation of the action S with respect to A_3, $\delta S/\delta A_3 = \mathcal{G}_8$, and its explicit expression is given in equation (2.4.7), namely, $\mathcal{G}_8 = d(*F_4 + b_7 - A_3 \wedge dA_3)$.

In the component approach we are dealing with, the action S is defined on eleven-dimensional spacetime M^{11}. Consequently, all the forms involved in the action, including B^{ab} and η^α take arguments on M^{11} and can, therefore, be expressed in terms of the vielbein basis e^a; in particular, $B^{ab} = e^c B_c{}^{ab}$, $\eta^\alpha = e^c \eta_c{}^\alpha$. Thus, introducing the matrix

$$\mathcal{K}_{cd}{}^{ab} = \delta_{[c}{}^a \delta_{d]}^b + \frac{1}{5} B_{[c}{}^{ae} B_{d]}{}^b{}_e + \frac{1}{20} \psi_{[c}{}^\beta\, \eta_{d]}{}^\alpha\, \Gamma^{ab}_{\alpha\beta} \,, \qquad (6.6.3)$$

the variation (6.6.2) of the action with respect to B^{ab} can be written as

$$\frac{\delta S}{\delta B_{ab}} = \frac{1}{4!} \mathcal{G}_8 \wedge e^c \wedge e^d\, \mathcal{K}_{cd}{}^{ab} \,. \qquad (6.6.4)$$

Now, as it can be seen *e.g.* at the linearized level, in which the fields B^{ab} are weak, the matrix $\mathcal{K}_{cd}{}^{ab}$ can be supposed to be invertible and the requirement that the action be invariant under variations of B^{ab} leads to

$$\det(\mathcal{K}_{ab}{}^{cd}) \neq 0 \;:\; \frac{\delta S}{\delta B^{ab}} = 0 \quad \Rightarrow \quad \mathcal{G}_8 \wedge e^c \wedge e^d = 0 \,. \qquad (6.6.5)$$

The last equation then implies the standard equations of motion for A_3, equation (2.4.8), but now for a composite, rather than fundamental A_3. Thus one may state, at least within the $\det(\mathcal{K}_{ab}{}^{cd}) \neq 0$ assumption, that the variation with respect to the B^{ab} field produces the same equations as the variation with respect to the CJS three-form A_3,

$$\det(\mathcal{K}_{ab}{}^{cd}) \neq 0 \ : \ \frac{\delta S}{\delta B^{ab}} = 0 \quad \Rightarrow \quad \mathcal{G}_8 := \frac{\delta S}{\delta A_3} = 0 \ . \tag{6.6.6}$$

Notice, however, that the B^{ab} field carries more degrees of freedom than A_3 does. In fact, the three index tensor $B_c{}^{ab} = -B_c{}^{ba}$ has reducible symmetry properties (product of two Young tableaux),

$$B_{c\,ab} \ \sim \ \square \otimes \begin{smallmatrix}\square\\ \square\end{smallmatrix} = \begin{smallmatrix}\square & \square\\ \square & \end{smallmatrix} \oplus \begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix} \tag{6.6.7}$$

whereas the components A_{abc} of $A_3 = \frac{1}{3!} e^c \wedge e^b \wedge e^a A_{abc}$ are completely antisymmetric, $A_{abc} = A_{[abc]}$,

$$A_{abc} \ \sim \ \begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix} \ . \tag{6.6.8}$$

Then, since a variation of the action with respect to B^{ab} produces (for $\det(\mathcal{K}_{[ab]}{}^{[cd]}) \neq 0$) the same equations as the variation with respect to A_3, one concludes that the action for a composite A_3 must possess local symmetries that make the *extra* degrees of freedom in B^{ab} (*i.e*, $\begin{smallmatrix}\square & \square\\ \square & \end{smallmatrix}$ but *not* $\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}$) pure gauge. Similarly, one may expect to have an extra local fermionic symmetry under which the new fermionic fields η_a^α in $\eta^\alpha = e^a \eta_a{}^\alpha$ are also pure gauge.

This is indeed the case [7]. Actually, the fact that the above $\delta B^{ab} = e^c \delta B_c{}^{ab}$ variation produces the same result as the variation with respect to $\delta A_{abc} = \delta A_{[abc]}$ plays the role of Noether identities for all these 'extra' gauge symmetries. Let us show, for instance, that the supergravity action with A_3 with the simple composite structure of equation (6.5.11) does possess extra fermionic gauge symmetries with a spinorial one-form parameter. Indeed, the equations of motion for η^α,

$$\frac{\delta S}{\delta \eta^\alpha} = 0 \quad \Rightarrow \quad \mathcal{G}_8 \wedge \psi^\beta \wedge \Big(10\, e^a \Gamma_{a\alpha\beta} + i\, B^{ab} \Gamma_{ab\,\alpha\beta}\Big) = 0 \ , \tag{6.6.9}$$

are satisfied identically on the B^{ab} equations of motion ($\mathcal{G}_8 = 0$ for $\det(\mathcal{K}_{[ab]}{}^{[cd]}) \neq 0$, equations (6.6.5)). This is a Noether identity that indicates the presence of a local fermionic symmetry with spinorial one-form parameter χ^α, $\chi^\alpha = e^a \chi_a{}^\alpha$, such that

$$\begin{aligned} &\delta_\chi \eta^\alpha = \chi^\alpha \ , \\ &\delta_\chi B^{ab} = \frac{i}{16} \mathcal{K}^{-1\,[ab]\,[cd]}\, \psi_c{}^\alpha (10\Gamma_d + i B_d{}^{ef} \Gamma_{ef})_{\alpha\beta}\, \chi^\beta \ . \end{aligned} \tag{6.6.10}$$

We can see that the transformations (6.6.10), leave invariant the composite three-form A_3 (6.5.11) considered as a form on spacetime. In the same way, having in mind that the contribution of *any* variation of the fundamental fields in δA_3 on M^{11} is always given by an *antisymmetric* third-rank tensor contribution, one concludes that *any* contribution to δA_3 from an arbitrary variation of the $\yng(2,1)$ irreducible part of $\delta B_c{}^{ab}$ (which carries also an antisymmetric contribution) can always be compensated by a contribution of a proper transformation of its completely antisymmetric part $\delta B_{[cba]}$, $\yng(1,1,1)$.

When the more general form for A_3, (equations (6.5.9), (6.5.6)) is considered, the same reasoning shows that any transformations of the new form $B^{a_1\ldots a_5}$ can be compensated by some properly chosen B^{ab} transformations. The key point is that the coefficient λ in (6.5.6) never vanishes. Hence (omitting δe^a and $\delta\psi^\alpha$),

$$\begin{aligned}\delta A_3 &= -\tfrac{\lambda}{4} e^c\wedge e^d\wedge \mathcal{K}_{cd}{}^{ab}\delta B_{ab} \;+\mathcal{S}_{2a_1\ldots a_5}\wedge\delta B^{a_1\ldots a_5}+\mathcal{S}_2^\alpha\wedge\delta\eta_\alpha\\ &= -\tfrac{\lambda}{4} e^a\wedge e^b\wedge e^c\,\delta B_{[c\,ab]}+\mathcal{O}(B\wedge B)+\mathcal{O}(\psi\wedge\eta)\;, \qquad (6.6.11)\end{aligned}$$

$$\mathcal{K}_{cd}{}^{ab} = \delta_{[c}{}^a\delta_{d]}{}^b+\mathcal{O}(B\wedge B)+\mathcal{O}(\psi\wedge\eta)\;, \qquad (6.6.12)$$

$$\lambda=\frac{(20\gamma_1^2+\delta^2)}{5(2\gamma_1-\delta)^2}\equiv\frac{1}{5}\;\frac{s^2+2s+6}{s^2}\;\;\neq 0$$

and the variation of the completely antisymmetric part $B_{[abc]}$ of $B^{ab}=e^cB_c{}^{ab}$ always reproduces (for an invertible $\mathcal{K}$ (6.6.12)) the same equation $\mathcal{G}_8=0$ as it would an independent, fundamental three-form A_3 [7].

One might also wonder whether the equations of motion of the first order action with a composite A_3 produce any relations for the curvatures $\mathcal{B}_2^{ab}$, $\mathcal{B}_2^{a_1\ldots a_5}$ and $\mathcal{B}_2^\alpha$ of the new fields B^{ab}, $B^{a_1\ldots a_5}$ and η^α, in the same way that they fix the curvatures $\mathbf{R}^a$ and $\mathbf{R}_4$ of e^a and A_3 to be $\mathbf{R}^a=0$ and $\mathbf{R}_4=F_4$, where F_4 is the auxiliary four-form of the first order supergravity action. An expression for the curvature $\mathbf{R}_4$ of A_3 in terms of the curvatures $\mathcal{B}_2^{ab}$, $\mathcal{B}_2^{a_1\ldots a_5}$, $\mathcal{B}_2^\alpha$ and $\mathbf{R}^\alpha$ may be obtained by substituting the composite expression (6.5.9) for A_3 in the expression (6.2.3) for $\mathbf{R}_4$ [7],

$$\begin{aligned}\mathbf{R}_4 =\; &\tfrac{\lambda}{4}\mathcal{B}_2^{ab}\wedge e_a\wedge e_b\;-\tfrac{3\alpha_1}{4}\mathcal{B}_{2ab}\wedge B^b{}_c\wedge B^{ca}\\ &-\tfrac{\alpha_2}{2}\mathcal{B}_{2\,a_1\ldots a_5}\wedge B^{a_1}{}_b\wedge B^{ba_2\ldots a_5}+\tfrac{\alpha_2}{4}B_{a_1\ldots a_5}\wedge\mathcal{B}_2^{a_1}{}_b\wedge B^{ba_2\ldots a_5}\\ &-\tfrac{\alpha_3}{2}\epsilon_{a_1\ldots a_5b_1\ldots b_5c}e^c\wedge B^{a_1\ldots a_5}\wedge\mathcal{B}_2^{b_1\ldots b_5}\\ &-\tfrac{\alpha_4}{4}\epsilon_{a_1\ldots a_6b_1\ldots b_5}B^{a_1a_2a_3}{}_{c_1c_2}\wedge B^{a_4a_5a_6c_1c_2}\wedge\mathcal{B}_2^{b_1\ldots b_5}\\ &-\tfrac{\alpha_4}{2}\epsilon_{a_1\ldots a_6b_1\ldots b_5}B^{a_4a_5a_6c_1c_2}\wedge B^{b_1\ldots b_5}\wedge\mathcal{B}_2^{a_1a_2a_3}{}_{c_1c_2}\end{aligned}$$

$$
\begin{aligned}
&-\tfrac{i}{2}\psi^\beta \wedge \eta_1^\alpha \wedge \Big(-i\beta_2\, \mathcal{B}_2^{ab}\Gamma_{ab\;\alpha\beta} + \beta_3\, \mathcal{B}_2^{abcde}\Gamma_{abcde\;\alpha\beta}\Big) \\
&+\tfrac{i}{2}\psi^\beta \wedge \Big(\beta_1\, e^a\Gamma_{a\alpha\beta} - i\beta_2\, B^{ab}\Gamma_{ab\;\alpha\beta} + \beta_3\, B^{abcde}\Gamma_{abcde\;\alpha\beta}\Big) \wedge \mathcal{B}_2^\alpha \\
&+\tfrac{i}{2}\eta^\alpha \wedge \Big(\beta_1\, e^a\Gamma_{a\alpha\beta} - i\beta_2\, B^{ab}\Gamma_{ab\;\alpha\beta} + \beta_3\, B^{abcde}\Gamma_{abcde\;\alpha\beta}\Big) \wedge \mathbf{R}^\beta\;,
\end{aligned}
\tag{6.6.13}
$$

where $\mathbf{R}^a = 0$ has been assumed by consistency with the equations of motion for e^a, and the coefficients are given by (6.5.6).

If the condition $\mathbf{R}_4 = F_4$, where $F_4 = \frac{1}{4!}e^{a_4} \wedge \ldots \wedge e^{a_1} F_{a_1\ldots a_4}$, is now imposed, equation (6.6.13) sets the value of $F_{a_1\ldots a_4}$ in terms of the curvatures $\mathcal{B}_2^{ab}$, $\mathcal{B}_2^{a_1\ldots a_5}$, $\mathcal{B}_2^\alpha$ and $\mathbf{R}^\alpha$. This reflects the existence of the extra gauge symmetries that makes the theory with a composite A_3 carry the same number of degrees of freedom than the standard theory with a fundamental A_3, as discussed previously in this section at the level of the equations of motion. Indeed, equation (6.6.13) with $\mathbf{R}_4 = F_4$ is the only relation imposed on the new field strengths $\mathcal{B}_2^{ab}$, $\mathcal{B}_2^{a_1\ldots a_5}$, $\mathcal{B}_2^\alpha$ by the first-order $D = 11$ supergravity action (2.2.3), (2.2.5) with a composite A_3. This makes the detailed properties of the curvatures $\mathcal{B}_2^{ab}$, $\mathcal{B}_2^{a_1\ldots a_5}$, $\mathcal{B}_2^\alpha$ of the additional gauge fields inessential: their only relevant properties are that the field strength F_4 is constructed out of them in agreement with equation (6.6.13), and that such a composite field strength obeys the equation of motion (2.4.8), $\mathcal{G}_8 = 0$.

In summary, on the one hand, the underlying gauge group structure implied by the new one-form fields allows us to treat $D = 11$ supergravity as a gauge theory of the supergroup $\tilde{\Sigma}(s) \rtimes SO(1,10)$, $s \neq 0$, that replaces superPoincaré. On the other hand, the supergravity action (2.2.3), (2.2.5) with a composite A_3 also possesses 'extra' gauge symmetries (*i.e.*, not in $\tilde{\Sigma}(s) \rtimes SO(1,10)$, $s \neq 0$) that make the *additional* degrees of freedom in the 'new' fields B^{ab}, $B^{a_1\ldots a_5}$, η^α pure gauge (*i.e.* B^{ab}, $B^{a_1\ldots a_5}$, η^α carry in all the same number of physical degrees of freedom as the fundamental A_3 field). One might conjecture that the superfluous degrees of freedom in the 'new' one-form fields, which are pure gauge in the pure supergravity action, could become 'alive' when supergravity is coupled to some M Theory objects. These could not be the usual M-branes as they couple to the standard fields and, hence, all the gauge symmetries preserving the composite A_3 would remain preserved. Thus one might think of some coupling of supergravity through some new action containing explicitly the new one-form fields. A guide in the search for such an action would be the preservation of the gauge symmetries of the underlying $\tilde{\Sigma}(s) \rtimes SO(1,10)$, $s \neq 0$, gauge supergroup.

6.7 Fields/extended superspace coordinates correspondence

In the previous section the additional one-forms $B^{a_1 a_2}$, $B^{a_1 \dots a_5}$ and η^α were introduced as forms on conventional $D = 11$ spacetime. In contrast, in section 6.5, the trivialization of the four-cocycle ω_4 associated to dA_3 was carried out assuming that all those forms, together with e^a and ψ^α, were independent. This was explicitly used in the derivation of the linear system of equations (6.5.5) for the coefficients of the trivializing three-form $\tilde{\omega}_3$ from the expression (6.5.4) of $d\tilde{\omega}_3$. From this point of view, for each value of the parameter s, the natural space on which the MC one-forms Π^a, $\Pi^{a_1 a_2}$, $\Pi^{a_1 \dots a_5}$, π^α and π'^α of the superalgebra $\tilde{\mathfrak{E}}(s)$ are defined, is the corresponding group manifold $\tilde{\Sigma}(s)$ of $\tilde{\mathfrak{E}}(s)$, the (rigid) enlarged superspace manifold.

The one-forms Π^a and π^α are the usual MC one-forms of the supertranslations algebra $\mathfrak{E} \equiv \mathfrak{E}^{(11|32)}$, defined on the supertranslations group manifold, that is, rigid superspace $\Sigma \equiv \Sigma^{(11|32)}$. In eleven spacetime dimensions, a set of 11 bosonic coordinates x^a and 32 fermionic coordinates θ^α can be introduced to parameterize the standard rigid superspace,

$$\Sigma \equiv \Sigma^{(11|32)} \ : \quad Z^M = (x^a, \theta^\alpha) \ . \tag{6.7.1}$$

The MC equations of the supertranslations algebra (obtained from the MC equations (2.1.12) of the superPoincaré algebra disregarding the Lorentz part) can be solved, accordingly, in terms of superspace coordinates as

$$\begin{aligned} \Pi^a &= dx^a - id\theta^\alpha \Gamma^a_{\alpha\beta} \theta^\beta \ , \\ \pi^\alpha &= d\theta^\alpha \ . \end{aligned} \tag{6.7.2}$$

On standard superspace Σ, any (left-invariant) differential form can be expressed in the basis provided by the MC one-forms Π^a, π^α (with constant coefficients). However, the assumption that the one-forms Π^{ab}, $\Pi^{a_1 \dots a_5}$ (or their 'soft' counterparts B^{ab}, $B^{a_1 \dots a_5}$) are independent is equivalent to the assumption that the expressions

$$\begin{aligned} d\Pi^{ab} &= -d\theta^\alpha \wedge d\theta^\beta \Gamma^{ab}_{\alpha\beta} \ , \\ d\Pi^{a_1 \dots a_5} &= -id\theta^\alpha \wedge d\theta^\beta \Gamma^{a_1 \dots a_5}_{\alpha\beta} \end{aligned} \tag{6.7.3}$$

(see equation (6.3.7)) cannot be solved in terms of the left-invariant MC one-forms Π^a, π^α on standard superspace Σ. Although the forms Π^{ab}, $\Pi^{a_1 \dots a_5}$ are actually de Rham trivial (exact) and can indeed be solved in terms of the coordinates $Z^M = (x^a, \theta^\alpha)$ of Σ, the resulting expressions $\Pi^{ab} = -d\theta^\alpha \Gamma^{ab}_{\alpha\beta} \theta^\beta$, $\Pi^{a_1 \dots a_5} = -id\theta^\alpha \Gamma^{a_1 \dots a_5}_{\alpha\beta} \theta^\beta$ fail to be left invariant on

Σ. In contrast, the introduction of new parameters y^{ab}, $y^{a_1\dots a_5}$ does allow for a solution for Π^{ab}, $\Pi^{a_1\dots a_5}$ in terms of them,

$$\begin{aligned} \Pi^{ab} &= dy^{ab} - d\theta^\alpha \Gamma^{ab}_{\alpha\beta}\theta^\beta \ , \\ \Pi^{a_1\dots a_5} &= dy^{a_1\dots a_5} - id\theta^\alpha \Gamma^{a_1\dots a_5}_{\alpha\beta}\theta^\beta \ , \end{aligned} \tag{6.7.4}$$

such that, under suitable (and straightforward) transformation rules for the new parameters, the forms Π^{ab} and $\Pi^{a_1\dots a_5}$ become left invariant MC one-forms of an enlarged algebra. The corresponding group manifold $\Sigma^{(528|32)}$ is parameterized by the 11 bosonic x^a and 32 fermionic θ^α coordinates of standard superspace, together with the additional $\binom{11}{2}+\binom{11}{5}=517$ bosonic coordinates y^{ab}, $y^{a_1\dots a_5}$; $\Sigma^{(528|32)}$ is, precisely, the group manifold associated to the M Theory superalgebra[7] $\mathfrak{E}^{(528|32)}$:

$$\Sigma^{(528|32)} \ : \quad (x^a, y^{ab}, y^{a_1\dots a_5}, \theta^\alpha) \ . \tag{6.7.5}$$

When the curvatures are not zero, and in particular $\mathcal{B}_2^{ab} \neq 0$, $\mathcal{B}_2^{a_1\dots a_5} \neq 0$ *i.e.*, the invariant one-forms $\Pi^{a_1a_2}$, $\Pi^{a_1\dots a_5}$ become 'soft', rendering $\Sigma^{(528|32)}$ non-flat and no longer a group manifold.

Likewise, if the additional fermionic one-form π'^α is considered, 32 new coordinates θ'^α must be introduced to solve for π'^α in

$$d\pi'^\alpha = -id\theta^\beta \wedge \left(\delta\, \Pi^a\Gamma_a - i\gamma_1\Pi^{ab}\Gamma_{ab} + \gamma_2\Pi^{a_1\dots a_5}\Gamma_{a_1\dots a_5}\right)_\beta{}^\alpha \tag{6.7.6}$$

(see the last equation of (6.3.7)), where Π^a, Π^{ab} and $\Pi^{a_1\dots a_5}$ are given by (6.7.2), (6.7.4). In terms of the new coordinates, π'^α reads

$$\begin{aligned} \pi'^\alpha = d\theta'^\alpha &+ i\theta^\beta \left(\delta\, \Pi^a\Gamma_a - i\gamma_1\Pi^{ab}\Gamma_{ab} + \gamma_2\Pi^{a_1\dots a_5}\Gamma_{a_1\dots a_5}\right)_\beta{}^\alpha \\ &-\tfrac{2}{3}\delta\, d\theta\Gamma^a\theta\,(\Gamma_a\theta)^\alpha + \tfrac{2}{3}\gamma_1 d\theta\Gamma^{ab}\theta\,(\Gamma_{ab}\theta)^\alpha - \tfrac{2}{3}\gamma_2 d\theta\Gamma^{a_1\dots a_5}\theta\,(\Gamma_{a_1\dots a_5}\theta)^\alpha \ . \end{aligned} \tag{6.7.7}$$

All these coordinates thus define the enlarged superspaces $\tilde{\Sigma}(s)$ parameterized by the coordinates

$$\tilde{\Sigma}(s) \equiv \Sigma^{(528|32+32)}(s) \ : \mathcal{Z}^{\mathcal{N}} := \left(x^a \ , \ y^{ab} \ , \ y^{a_1\dots a_5} \ ; \ \theta^\alpha \ , \ \theta'^\alpha\right), \tag{6.7.8}$$

[7] The $\Sigma^{(528|32)}$ extended superspace group may be found in our spirit by searching for a trivialization of the $\mathbb{R}^{528}$-valued two-cocycle $d\mathcal{E}^{\alpha\beta} = -id\theta^\alpha \wedge d\theta^\beta$, which leads to the one-form $\mathcal{E}^{\alpha\beta} = dX^{\alpha\beta} - id\theta^{(\alpha}\theta^{\beta)}$. This introduces in a natural way the 528 bosonic coordinates $X^{\alpha\beta}$ including the coordinates x^a, y^{ab}, $y^{a_1\dots a_5}$ in (6.7.5) (see equation (7.1.1) of next chapter). The transformation law $\delta_\epsilon X^{\alpha\beta} = i\theta^{(\alpha}\epsilon^{\beta)}$ makes $\mathcal{E}^{\alpha\beta}$ invariant, and hence leads to a central extension structure for the extended superspace group $\Sigma^{(528|32)}$. Thus, the (maximally extended in the bosonic sector) superspace $\Sigma^{(528|32)}$ transformations make of $\mathcal{E}^{\alpha\beta}$ a MC form that trivializes, on the extended superalgebra $\mathfrak{E}^{(528|32)}$, the non-trivial CE two-cocycle on the original odd abelian algebra $\Sigma^{(0|32)}$.

which correspond to the group manifolds of the extended superalgebras $\tilde{\mathfrak{E}}(s) \equiv \mathfrak{E}^{(528|32+32)}(s)$. Again, when the curvatures are not zero the invariant MC one-forms become 'soft', and $\tilde{\Sigma}(s)$ non-flat and no longer a group manifold.

The gauging of the superalgebra $\tilde{\mathfrak{E}}(s)$ (6.3.7) leads to the associated FDA (6.3.11), including as many one-form gauge fields (6.3.9) as group parameters (6.7.8) correspond to the enlarged superspace $\tilde{\Sigma}(s)$. This points out to the existence of a *fields/extended superspace coordinates correspondence* [85, 86], according to which all the (here, spacetime) fields entering the physical action are in one-to-one correspondence with the coordinates (that is, group parameters of their corresponding group manifolds) of suitably enlarged superspaces. This one-to-one correspondence is further supported by the fact that enlarged superalgebras also arise in the description [179, 85] of the strictly invariant Wess-Zumino (WZ) terms of the scalar p-branes. These invariant WZ terms trivialize their characterizing Chevalley-Eilenberg (CE) $(p+2)$-cocycles [178] on the standard supersymmetry algebras $\mathfrak{E}^{(D|n)}$, including that of the $D = 11$ supermembrane, since its WZ term is given by the pull-back to $\mathcal{W}$ of the three-form potential of the dA_3 superspace four-cocycle. See [86] for a review.

Enlarged superspaces can also be used in the case of D-branes [182, 85, 180]. Moreover, whereas the coordinates corresponding to the enlarged superspaces enter trivially (through a total derivative) the scalar p-brane actions, that is not the case for D-branes or the M5-brane. The Born-Infeld fields of D-branes and the antisymmetric tensor field of the M5-brane are usually defined as 'fundamental' gauge fields *i.e.*, they are given, respectively, by one-forms $A_1(\xi)$ and a two-form $A_2(\xi)$ directly defined on the worldvolume $\mathcal{W}$. It was shown in [85] (see also [182]) that both $A_1(\xi)$ and $A_2(\xi)$ can be expressed through pull-backs to $\mathcal{W}$ of forms defined on superspaces Σ' suitably enlarged by additional bosonic and fermionic coordinates, in agreement with the worldvolume fields/extended superspace coordinates correspondence for superbranes [85] (see also [183]). The extra degrees of freedom that are introduced by considering $A_1(\xi)$ and $A_2(\xi)$ to be the pull-backs to $\mathcal{W}$ of forms given on Σ', and that produce the composite structure of the Born-Infeld fields to be used in the superbrane actions, are removed by the appearance of extra gauge symmetries [85, 183], as it is here the case for the composite A_3 field of D=11 supergravity. Of course, these two problems are not identical: for instance, in the case of D=11 supergravity with a composite A_3, the suitably enlarged *flat* superspace $\tilde{\Sigma}(s) = \Sigma^{(528|32+32)}(s)$ solves, for $s \neq 0$, the associated problem of trivializing the CE cocycle, but a dynamical A_3 field requires 'softening' the $\tilde{\mathfrak{E}}(s) = \mathfrak{E}^{(528|32+32)}(s)$, $s \neq 0$, MC equations by introducing nonvanishing curvatures; in contrast, the Born-Infeld worldvolume fields $A_1(\xi)$ and the tensor field $A_2(\xi)$ are already dynamical in the flat superspace situation considered in [85]. Nevertheless, in both these seemingly different situations the fields/extended superspace coordinates correspondence leads us to the convenience of enlarging standard superspace $\Sigma \equiv \Sigma^{(11|32)}$. In this way, all the fields in the theory under consideration (be them on spacetime or on the worldvolume) correspond to coordinates of a suitably enlarged superspace.

Superalgebras $\tilde{\mathfrak{E}}$ enlarged with additional (bosonic and, possibly, fermionic) generators have been shown here to arise naturally when the underlying gauge structure of $D = 11$ supergravity is studied. The corresponding group manifolds are, thus, superspaces $\tilde{\Sigma}$ enlarged with additional (bosonic and, possibly, fermionic) coordinates, that generalize ordinary superspace $\Sigma \equiv \Sigma^{(11|32)}$. The role in supergravity of these enlarged superspaces merits further investigation. Just like ordinary supersymmetric objects are formulated as dynamical systems in standard superspace Σ, it is, thus, natural to pose actions describing the dynamics of objects moving in the backgrounds provided by enlarged superspaces $\tilde{\Sigma}$ and look for an interpretation for these actions. As an example, the next chapter studies the dynamics of a supersymmetric string moving in an enlarged superspace (corresponding, in fact, to the group manifold associated to the M Theory superalgebra). Such string can be conveniently described in terms of supertwistors and, interestingly enough, the model is argued to describe the excitations of a system composed of two BPS preons.

7

A 30/32-supersymmetric string in tensorial superspace

In the early period of superstring theory, when it was found that all $D = 10$ supergravities appear as low energy limits of superstring models, a question arose: what is the origin of maximally extended $D = 11$ supergravity? Its relation with the supermembrane [71] was established by studying the supermembrane action in a supergravity background; however, the quantization of the supermembrane was fraught with difficulties. An indication was found [184] that the quantum state spectrum of the supermembrane is continuous, a problem now sorted out by treating [185] the supermembrane as an object composed of D0-branes in the framework of the Matrix model approach [186]. Another aspect of the same problem was that the membrane was shown to develop string-like instabilities [184]. The Green-Schwarz superstring is free from these problems, but it is a $D = 10$ theory. Thus, it was tempting to search for possible new $D = 11$ superstring models hoping that their low energy limit would be eleven-dimensional supergravity. Such a search requires going beyond the standard superspace framework: in moving from $D = 10$ to $D = 11$ one has to add also extra bosonic degrees of freedom, thus arriving to an *enlarged* $D = 11$ superspace rather than to the standard one.

In section 7.1, models in enlarged superspaces are argued to provide higher spin generalizations of their standard superspace counterparts. A supersymmetric string action in maximal, or tensorial superspace (the supergroup manifold corresponding to the M Theory superalgebra, and its generalizations containing n fermionic and $\frac{1}{2}n(n+1)$ bosonic coordinates) is subsequently introduced in section 7.2. The model does not use $D = 11$ gamma-matrices, but instead includes two auxiliary bosonic spinor variables[1], λ^+_α and λ^-_α. As a consequence, the resulting supersymmetric string action possesses 30 local fermionic κ-symmetries, although

[1] Actually, the model possesses $Sp(32)$ symmetry besides the $SO(1,10)$ one, so that

it does not include a Wess-Zumino term as that of [191]. In the general formulation in terms of n fermionic coordinates, the number of preserved κ-symmetries is $n-2$, so that, for any n, the models provide an extended object action for the excitations of a state composed of two BPS preons.

Sections 7.3 and 7.4 describe, respectively, the equations of motion and gauge symmetries of the model, including the $(n-2)$ κ-symmetries and their 'superpartners', the $(n-1)(n-2)/2$ bosonic gauge b-symmetries. The gauge symmetries are studied in section 7.5 in the Hamiltonian approach. We also discuss there the number of degrees of freedom of our model. In Sec. 7.6 we show that its action may be formulated in terms of a pair of constrained $OSp(2n|1)$ supertwistors (see [148]) which, by definition, are invariant under both κ- and b-symmetries. The hamiltonian analysis then simplifies considerably, as shown in section 7.7. Section 7.8 contains the hamiltonian analysis in terms of unconstrained supertwistors. The generalization of the model to the super-p-brane case is given in Sec. 7.9. Some details about the supertwistor formulation of the model are included in Appendix C. This chapter follows reference [8].

7.1 Models in enlarged superspaces

A first example of a supersymmetric string action in an enlarged $D=11$ superspace was found in [191]. The model, possessing 32 supersymmetries and 16 κ-symmetries, was constructed in the enlarged superspace $\Sigma^{(528|32)}$. This contains 32 fermionic coordinates θ^α and 528 bosonic coordinates $x^\mu, y^{\mu\nu}, y^{\mu_1\ldots\mu_5}$ ($y^{\mu\nu}=-y^{\nu\mu}\equiv y^{[\mu\nu]}$, $y^{\mu_1\ldots\mu_5}=y^{[\mu_1\ldots\mu_5]}$) which may be collected in a symmetric spin-tensor $X^{\alpha\beta}=X^{\beta\alpha}$,

$$X^{\alpha\beta}=\tfrac{1}{32}\left(x^\mu\Gamma_\mu^{\alpha\beta}-\tfrac{i}{2!}y^{\mu\nu}\Gamma_{\mu\nu}^{\alpha\beta}+\tfrac{1}{5!}y^{\mu_1\ldots\mu_5}\Gamma_{\mu_1\ldots\mu_5}^{\alpha\beta}\right)\ , \qquad (7.1.1)$$

so that the coordinates of $\Sigma^{(528|32)}$ are (see equation (6.7.5)):

$$\mathcal{Z}^{\mathcal{M}}=(X^{\alpha\beta},\theta^\alpha)\ ,\quad X^{\alpha\beta}=X^{\beta\alpha}\ ,\quad \alpha,\beta=1,2,\ldots,32\ . \qquad (7.1.2)$$

Recall that the $\Sigma^{(528|32)}$ superspace has a special interest because it is the supergroup manifold associated with the maximal $D=11$ supersymmetry algebra $\mathfrak{E}^{(528|32)}$, the M Theory superalgebra, defined in chapter 2 by the (anti)commutators (2.1.10) or the MC equations (2.1.15). In fact, the coordinates $X^{\alpha\beta}$ parameterizing the superspace $\Sigma^{(528|32)}$ can be seen as canonically conjugate to the generalized momentum $P_{\alpha\beta}$ entering the M–algebra and defined in equation (2.1.9). Also, the MC one-forms Π^a, π^α,

---

$\lambda_\alpha^\pm$ may be considered as symplectic vectors (called 's-vectors' in [187, 188]) rather than Lorentz spinors. See footnote 1 of chapter 4. See [189] for a spacetime treatment of a $\mathbb{CP}^3$ sigma model *i.e.*, of a string theory in twistor space, and its relation to Yang-Mills amplitudes. See [190] and references therein for very recent work in this subject.

Π^{ab} and $\Pi^{a_1\ldots a_5}$ can be solved by the coordinates $X^{\alpha\beta}$ (see equations (6.7.2), (6.7.4)).

The model of [191] may also be restricted to the superspaces[2] $\Sigma^{(66|32)}$ ($(x^\mu, y^{\mu\nu}, \theta^\alpha)$, with 66 bosonic coordinates) and $\Sigma^{(462|32)}$ ($(x^\mu, y^{\mu_1\ldots\mu_5}, \theta^\alpha)$, with 462 bosonic coordinates). For the sake of definiteness, we shall call here *maximal*, or *tensorial superspaces*, those with bosonic coordinates of symmetric 'spin-tensorial' type, like $\Sigma^{(528|32)}$ and its counterparts $\Sigma^{(\frac{n(n+1)}{2}|n)}$,

$$\Sigma^{(\frac{n(n+1)}{2}|n)} : \mathcal{Z}^\Sigma = (X^{\alpha\beta}, \theta^\alpha),\ X^{\alpha\beta} = X^{\beta\alpha},\ \alpha, \beta = 1, 2, \ldots n, \quad (7.1.3)$$

where $n = 2^l$ for suitable l, to allow for an interpretation of n as the number fermionic coordinates. This name distinguishes the $\Sigma^{(\frac{n(n+1)}{2}|n)}$ superspaces from other, not maximally extended (in the bosonic sector) superspaces like $\Sigma^{(66|32)}$ and $\Sigma^{(462|32)}$ whose bosonic coordinates may be described by a spin-tensor $X^{\alpha\beta}$ only if it satisfies some conditions.

The main problem of the approach in [191] is how to treat the large number of additional bosonic degrees of freedom[3] in the coset(s) $\Sigma^{(528|32)}/\Sigma^{(11|32)}$ (or $\Sigma^{(462|32)}/\Sigma^{(11|32)}$, $\Sigma^{(66|32)}/\Sigma^{(11|32)}$), where, as usual, $\Sigma \equiv \Sigma^{(11|32)}$ is the standard $D = 11$ superspace (see (6.7.1)). Actually, the same problem arises in any approach dealing with enlarged superspaces [85, 193, 148, 155, 83, 187, 188, 194, 195, 150, 149, 8]. Thus, one has to find a mechanism that either suppresses the additional (with respect to the usual spacetime/superspace $\Sigma^{(D|n)}$) degrees of freedom or provides a physical interpretation for them. In this respect $\Sigma^{(\frac{n(n+1)}{2}|n)}$, despite having a maximal bosonic part, has some advantages with respect to non-maximally extended superspaces. Indeed, the bosonic sector of the tensorial superspace (7.1.3),

$$\Sigma^{(\frac{n(n+1)}{2}|0)} \ :\ X^{\alpha\beta} = X^{\beta\alpha}\ , \quad \alpha, \beta = 1, 2, \ldots, n\ , \quad (7.1.4)$$

was proposed for $n = 4$ [196] as a basis for the construction of $D = 4$ higher-spin theories [197, 187, 188]. Moreover, it was shown in [155] that the quantization of a simple superparticle model [148] in $\Sigma^{(\frac{n(n+1)}{2}|n)}$ for $n = 2, 4, 8, 16$ results in a wavefunction describing a tower of massless

[2] All these superspaces $\Sigma^{(528|32)}$, $\Sigma^{(462|32)}$ and $\Sigma^{(66|32)}$, considered as supergroup manifolds, may be seen as central extensions of an abelian 32-dimensional fermionic group by tensorial (equation (7.1.1)) bosonic groups [85]. See footnote 7 of chapter 6.

[3] See [192] for a later related search based on an attempt to replace the κ-symmetry requirement by a dynamically generated projection constraint on the spinor coordinate functions. This approach also suffers from the problem of additional bosonic degrees of freedom.

fields of all possible spins (helicities). Such an infinite tower of higher spin fields allows for a non-trivial interaction in AdS spacetimes [198, 197][4].

To give an idea of the relation between higher spin theories and maximally extended superspaces, let us consider the free bosonic massless higher-spin equations proposed in [187] (for $n = 4$). These can be collected as the following set of equations for a scalar function b on $\Sigma^{(\frac{n(n+1)}{2}|0)}$

$$\partial_{\alpha[\beta}\partial_{\gamma]\delta}b(X) = 0 \ , \tag{7.1.5}$$

where $\partial_{\alpha\beta} = \partial/\partial X^{\alpha\beta}$. Equation (7.1.5) states that $\partial_{\alpha\beta}\partial_{\gamma\delta}$ is fully symmetric on a non-trivial solution. In the generalized momentum representation equation (7.1.5) reads

$$k_{\alpha[\beta}k_{\gamma]\delta}b(k) = 0 \ . \tag{7.1.6}$$

This implies that $b(k)$ has support on the $\frac{n(n+1)}{2} - \frac{n(n-1)}{2} = n$-dimensional surface in momentum space $\Sigma^{(\frac{n(n+1)}{2}|0)}$ (actually, in $\Sigma^{(\frac{n(n+1)}{2}|0)}\backslash\{0\}$) on which the rank of the matrix $k_{\gamma\delta}$ is equal to unity [188]. This is the surface defined by $k_{\alpha\beta} = \lambda_\alpha\lambda_\beta$ (or $-\lambda_\alpha\lambda_\beta$) characterized by the n components of λ_α. In a '$GL(n,\mathbb{R})$-preferred' frame (an analogue of the standard frame for lightlike ordinary momentum), $\lambda_\alpha = (1, 0, \ldots, 0)$ and the surface is the $GL(n,\mathbb{R})$-orbit of the point $k_{\alpha\beta} = \delta_{\alpha 1}\delta_{\beta 1}$. Thus, equation (7.1.6) may also be written as

$$(k_{\alpha\beta} - \lambda_\alpha\lambda_\beta)b = 0 \ , \tag{7.1.7}$$

which is equivalent to writing equation (7.1.5) in the form

$$(i\partial_{\alpha\beta} - \lambda_\alpha\lambda_\beta)b = 0 \ . \tag{7.1.8}$$

Equations (7.1.7) and (7.1.8) may be considered [8] as the generalized momentum ($k_{\alpha\beta}$) and coordinate ($X^{\alpha\beta}$) representations of the definition (4.1.10) of a BPS preon [83] (see section 4.1 of chapter 4). The solutions of equations (7.1.7), (7.1.8) are the momentum and coordinate 'wavefunctions' corresponding to a BPS preon state $|\lambda\rangle$, $b(X) = \langle X|\lambda\rangle$, $b(k) = \langle k|\lambda\rangle$. These equations also appear as a result of the quantization [155] of the superparticle model in [148][5].

---

[4] A relation between the generalized $n = 4$ superparticle wavefunctions [155] and Vasiliev's 'unfolded' equations for higher spin fields was noted in [187]. This was elaborated in detail in [199], where the quantization of an AdS superspace generalization of the $n = 4$ model of [148] was also carried out (see also [200] for a related study of higher spin theories in the maximal generalized AdS_4 superspace).

[5] In [187, 201] equation (7.1.8) was written as $(\partial_{\alpha\beta} - \frac{\partial}{\partial\mu^\alpha}\frac{\partial}{\partial\mu^\beta})b(X,\mu) = 0$ which is an equivalent 'momentum' representation obtained by a Fourier transformation with respect to λ_α, see [199].

Thus, in contrast with other extended superspaces, the models in the tensorial superspaces $\Sigma^{(\frac{n(n+1)}{2}|n)}$ can be regarded as higher spin generalizations of the models in standard superspace[6] $\Sigma^{(D|n)}$. For a recent review of higher spin theory see [203].

7.2 A supersymmetric string in tensorial superspace

A superstring in $\Sigma^{(\frac{n(n+1)}{2}|n)}$ is described by worldsheet functions $X^{\alpha\beta}(\xi)$, $\theta^\alpha(\xi)$, where $\xi = (\tau, \sigma)$ are the worldsheet W^2 coordinates. We propose the following action [8]:

$$S = \frac{1}{\alpha'} \int_{W^2} [e^{++} \wedge \Pi^{\alpha\beta}\, \lambda^-_\alpha \lambda^-_\beta - e^{--} \wedge \Pi^{\alpha\beta}\, \lambda^+_\alpha \lambda^+_\beta - e^{++} \wedge e^{--}] \ , \quad (7.2.1)$$

where

$$\begin{aligned} &\Pi^{\alpha\beta}(\xi) = dX^{\alpha\beta}(\xi) - id\theta^{(\alpha}\,\theta^{\beta)}(\xi) = d\tau \Pi^{\alpha\beta}_\tau + d\sigma \Pi^{\alpha\beta}_\sigma \ ; \\ &\alpha, \beta = 1, \ldots, n \ , \quad m = 0, 1 \ , \quad \xi^m = (\tau, \sigma) \ , \end{aligned} \quad (7.2.2)$$

with dimensions $[1/\alpha'] = \mathrm{ML}^{-1}$, $[\Pi^{\alpha\beta}] = \mathrm{L}$, $[e^{\pm\pm}] = \mathrm{L}$ $(c = 1)$. The two auxiliary worldvolume fields, the *bosonic* spinors $\lambda^-_\alpha(\xi), \lambda^+_\alpha(\xi)$, are *dimensionless* and constrained by

$$C^{\alpha\beta} \lambda^+_\alpha \lambda^-_\beta = 1 \ ; \quad (7.2.3)$$

$e^{\pm\pm}(\xi) = d\xi^m e^{\pm\pm}_m(\xi) = d\tau e^{\pm\pm}_\tau(\xi) + d\sigma e^{\pm\pm}_\sigma(\xi)$ are two auxiliary worldvolume one-forms. The one-forms $e^{\pm\pm}$ are assumed to be linearly independent and, hence, define an auxiliary worldsheet zweibein

$$e^a = (e^0, e^1) = d\xi^m e^a_m(\xi) = (\tfrac{1}{2}(e^{++} + e^{--}), \tfrac{1}{2}(e^{++} - e^{--})). \quad (7.2.4)$$

The $C^{\alpha\beta}$ in (7.2.3) is an invertible constant antisymmetric matrix

$$C^{\alpha\beta} = -C^{\beta\alpha} \ , \qquad dC^{\alpha\beta} = 0 \ , \quad (7.2.5)$$

which can be used to rise and lower the spinor indices (as the charge conjugation matrix in Minkowski spacetimes). The invertibility of the matrix $C^{\alpha\beta}$ requires n to be even; this is not really a limitation since, after all, we are interested in $n = 2^l$ to allow for a spinor treatment of the α, β indices.

We shall refer to this $n = 32$ ($\Sigma^{(528|32)}$) model as a $D = 11$ superstring, which implies the decomposition of equation (2.1.9) for the generalized momentum. Nevertheless, the $n = 32$ case also admits a $D = 10$, Type

---

[6] Formulations of higher spin theories are currently known up to spacetime dimension $D = 10$ [202].

IIA treatment, which uses the same $C^{\alpha\beta}$ of the $D = 11$ case, and in which the decomposition (2.1.9) is replaced by its $D = 10$, IIA counterpart obtained from (2.1.9) by separating the eleventh value of the vector index.

The action (7.2.1) is invariant under the supersymmetry transformations

$$\delta_\epsilon X^{\alpha\beta} = i\theta^{(\alpha}\epsilon^{\beta)} \ , \quad \delta_\epsilon \theta^\alpha = \epsilon^\alpha \ , \quad \delta_\epsilon \lambda^\pm_\alpha = 0 \ , \quad \delta_\epsilon e^{\pm\pm} = 0 \ , \qquad (7.2.6)$$

as well as under rigid $Sp(n,\mathbb{R})$ 'rotations' acting on the α, β indices. Note also that, although formally the action (7.2.1) possesses a manifest $GL(n,\mathbb{R})$ invariance, the constraint (7.2.3) breaks it down to $Sp(n,\mathbb{R}) \subset GL(n,\mathbb{R})$. Under the action of $Sp(n,\mathbb{R})$, the Grassmann coordinate functions $\theta^\alpha(\xi)$ and the auxiliary fields $\lambda^\pm_\alpha(\xi)$ are transformed as symplectic vectors and $X^{\alpha\beta}(\xi)$ as a symmetric symplectic tensor. Nevertheless, we keep for them the 'spinor' and 'spin-tensor' terminology having in mind their transformation properties under the subgroup $Spin(t, D-t) \subset Sp(n,\mathbb{R})$, which would appear in a 'standard' $(t, D-t)$ spacetime treatment.

The above $\Sigma^{(\frac{n(n+1)}{2}|n)}$ superstring model may also be described by an action written in terms of *dimensionful* unconstrained spinors $\Lambda^\pm_\alpha(\xi)$, $[\Lambda^\pm_\alpha] = (\mathrm{ML}^{-1})^{1/2}$ [8],

$$S = \int_{W^2} [e^{++} \wedge \Pi^{\alpha\beta}\, \Lambda^-_\alpha \Lambda^-_\beta - e^{--} \wedge \Pi^{\alpha\beta}\, \Lambda^+_\alpha \Lambda^+_\beta$$
$$-\alpha' e^{++} \wedge e^{--} (C^{\alpha\beta}\Lambda^+_\alpha \Lambda^-_\beta)^2] \ . \qquad (7.2.7)$$

Indeed, one can see that the action (7.2.7) possesses two independent scaling gauge symmetries defined by the transformation rules

$$e^{++}(\xi) \to e^{2\alpha(\xi)} e^{++}(\xi) \ , \quad \Lambda^-_\alpha(\xi) \to e^{-\alpha(\xi)} \Lambda^-_\alpha(\xi) \qquad (7.2.8)$$

and

$$e^{--}(\xi) \to e^{2\beta(\xi)} e^{--}(\xi) \ , \quad \Lambda^+_\alpha(\xi) \to e^{-\beta(\xi)} \Lambda^+_\alpha(\xi) \ . \qquad (7.2.9)$$

This allows one to obtain $C^{\alpha\beta}\Lambda^+_\alpha\Lambda^-_\beta = 1/\alpha'$ as a gauge fixing condition. Then the gauge fixed version of the action (7.2.7) coincides with (7.2.1) up to the trivial redefinition $\Lambda^\pm_\alpha = (\alpha')^{-1/2}\lambda^\pm_\alpha$. The gauge $C^{\alpha\beta}\Lambda^+_\alpha\Lambda^-_\beta = 1/\alpha'$ (equivalent to equation (7.2.3)) is preserved by a one-parametric combination of (7.2.8) and (7.2.9) with $\alpha = -\beta$, which is exactly the $SO(1,1)$ gauge symmetry (worldvolume Lorentz symmetry) of the action (7.2.1),

$$e^{\pm\pm}(\xi) \to e^{\pm 2\alpha(\xi)} e^{\pm\pm}(\xi) \ , \ \ \lambda^\pm_\alpha(\xi) \to e^{\pm\alpha(\xi)} \lambda^\pm_\alpha(\xi) \, . \qquad (7.2.10)$$

The tension parameter $T = 1/\alpha'$ enters in the last ('cosmological') term of the action (7.2.7) only. Setting in it $\alpha' = 0$ one finds that the model is non-trivial only for $e^{++} \propto e^{--}$ and $\Lambda^+ \propto \Lambda^-$ in which case one arrives at the tensionless super-p-brane action (with $p = 1$) of reference [149], $S = \int d^2\xi \rho^{++m} \Pi_m^{\alpha\beta} \Lambda_\alpha^- \Lambda_\beta^-$. As we are not interested in this case, we set $\alpha' = 1$ below since the α' factors can be restored by dimensional considerations.

The most interesting feature of the model (7.2.1), (7.2.7) is that, being formulated in the tensorial $\Sigma^{(\frac{n(n+1)}{2}|n)}$ superspace with n fermionic coordinates, it possesses $(n-2)$ κ-symmetries [8]; we will prove this in section 7.4. For a supersymmetric extended object in standard superspace, the κ-symmetry of its worldvolume action determines the number k of supersymmetries which are preserved by the ground state, which is a $\nu = \frac{k}{n}$ BPS state made out of $\tilde{n} = n - k$ preons if at least one supersymmetry, $k \geq 1$, is preserved (see section 4.1 of chapter 4). In the present case, we may expect that the ground state of our model should preserve $(n-2)$ out of n supersymmetries, *i.e.* that it is a $\nu = \frac{n-2}{n}$ BPS state ($\tilde{n} = 2$, $\frac{30}{32}$ BPS state for the $D = 11$ tensorial superspace $\Sigma^{(528|32)}$).

For $n = 2$, $X^{\alpha\beta}$ provides a representation of the 3-dimensional Minkowski space coordinates, $X^{\alpha\beta} \propto \Gamma_\mu^{\alpha\beta} x^\mu$ ($\alpha, \beta = 1, 2$; $\mu = 0, 1, 2$). Thus the $n = 2$ model (7.2.1) describes a string in the $D = 3$ standard $\Sigma^{(3|2)}$ superspace. However, in the light of the above discussion, it does not possess any κ-symmetry and, hence, its ground state is not a BPS state since it does not preserve any supersymmetry. The situation becomes different starting with the $n = 4$ model (7.2.1), which possesses two κ-symmetries, the same number as the Green-Schwarz superstring in the standard $D = 4$ superspace. For $D \geq 6$, $n \geq 8$ the number of κ-symmetries of our model exceeds $n/2$ and thus the model describes the excitations of BPS states with extra supersymmetries, a $\frac{30}{32}$ BPS state in the $D = 11$ $\Sigma^{(528|32)}$ superspace.

The number of *bosonic* degrees of freedom of our model is $4n - 6$ [8] (see section 7.5). It is not as large as it might look at first sight due to the 'momentum space dimensional reduction mechanism' [155] which occurs due to the presence of auxiliary spinor variables entering the generalized Cartan-Penrose relation (see equation (7.5.8) below) generated by our model. However, it is larger than that of the ($D = 3, 4, 6, 10$) Green-Schwarz superstring (which has D [$2n = 4(D-2)$] bosonic [fermionic] configuration space real degrees of freedom, which reduce to $D-2$ [$2(D-2)$] after taking into account reparameterization invariance (κ-symmetry), thus resulting in $2(D-2)$ bosonic and $2(D-2)$ fermionic phase space degrees of freedom). Thus, the relation of models in tensorial superspaces to higher spin theories mentioned in section 7.1, allows us to consider our model as a higher spin generalization of the Green-Schwarz superstring, containing additional information about the non-perturbative states of the String/M Theory.

The number of *fermionic* degrees of freedom of our model is 2 for any n, less than that of the $D=4,6,10$ $(N=2)$ Green-Schwarz superstring.

7.3 Equations of motion

Consider the variation of the action (7.2.1). Allowing for integration by parts one finds

$$\begin{aligned}\delta S = & \int_{W^2} d(e^{--}\lambda^+_\alpha\lambda^+_\beta - e^{++}\lambda^-_\alpha\lambda^-_\beta)\, i_\delta\Pi^{\alpha\beta} \\ & -2i\int_{W^2} e^{++}\wedge d\theta^\alpha\lambda^-_\alpha\;\delta\theta^\beta\lambda^-_\beta + 2i\int_{W^2} e^{--}\wedge d\theta^\alpha\lambda^+_\alpha\;\delta\theta^\beta\lambda^+_\beta \\ & +\int_{W^2}(\Pi^{\alpha\beta}\,\lambda^+_\alpha\lambda^+_\beta - e^{++})\wedge\delta e^{--} - \int_{W^2}(\Pi^{\alpha\beta}\lambda^-_\alpha\lambda^-_\beta - e^{--})\wedge\delta e^{++} \\ & +\delta_\lambda S\,, \end{aligned} \tag{7.3.1}$$

where $i_\delta\Pi^{\alpha\beta}\equiv\delta X^{\alpha\beta} - i\delta\theta^{(\alpha}\theta^{\beta)}$ and the last term

$$\delta_\lambda S = +\int_{W^2} 2e^{++}\wedge\Pi^{\alpha\beta}\lambda^-_\beta\delta\lambda^-_\alpha - \int_{W^2} 2e^{--}\wedge\Pi^{\alpha\beta}\lambda^+_\beta\delta\lambda^+_\alpha\;, \tag{7.3.2}$$

collects the variations of the bosonic spinors $\lambda^\pm_\alpha(\xi)$.

The equations of motion for the bosonic coordinate functions, $\delta S/\delta X^{\alpha\beta}$ $(=\delta S/i_\delta\Pi^{\alpha\beta})=0$, turn out to restrict the auxiliary spinors and auxiliary one-forms,

$$d(e^{--}\lambda^+_\alpha\lambda^+_\beta - e^{++}\lambda^-_\alpha\lambda^-_\beta) = 0\,. \tag{7.3.3}$$

The equations for the fermionic coordinate functions, $\delta S/\delta\theta^\alpha=0$, read

$$e^{++}\wedge d\theta^\alpha\lambda^-_\alpha\lambda^-_\beta - e^{--}\wedge d\theta^\alpha\lambda^+_\alpha\lambda^+_\beta = 0\;, \tag{7.3.4}$$

which, due to the linear independence of the spinors λ^+_α and λ^-_α, imply

$$e^{++}\wedge d\theta^\alpha\lambda^-_\alpha = 0\;, \qquad e^{--}\wedge d\theta^\alpha\lambda^+_\alpha = 0\;. \tag{7.3.5}$$

The equations for the one-forms $e^{\pm\pm}(\xi)$ express them through the worldsheet covariant bosonic form (7.2.2) of the $\Sigma^{(\frac{n(n+1)}{2}|n)}$ superspace and the spinors $\lambda^\pm_\alpha(\xi)$,

$$e^{++} = \Pi^{\alpha\beta}\,\lambda^+_\alpha\lambda^+_\beta\;, \tag{7.3.6}$$

$$e^{--} = \Pi^{\alpha\beta}\lambda^-_\alpha\lambda^-_\beta\;. \tag{7.3.7}$$

This reflects the auxiliary nature of $e^{\pm\pm}$ and implies that equations (7.3.3) and (7.3.5) actually restrict $\Pi^{\alpha\beta}$ and $d\theta^\alpha$,

$$d(\Pi^{\gamma\delta}\lambda_\gamma^-\lambda_\delta^- \; \lambda_\alpha^+\lambda_\beta^+ - \Pi^{\gamma\delta}\lambda_\gamma^+\lambda_\delta^+ \; \lambda_\alpha^-\lambda_\beta^-) = 0\,, \tag{7.3.8}$$

$$\Pi^{\gamma\delta}\lambda_\gamma^+\lambda_\delta^+ \wedge d\theta^\alpha\lambda_\alpha^- = 0\;, \tag{7.3.9}$$

$$\Pi^{\gamma\delta}\lambda_\gamma^-\lambda_\delta^- \wedge d\theta^\alpha\lambda_\alpha^+ = 0\;. \tag{7.3.10}$$

The necessity of the constraints (7.2.3) on the bosonic spinor variables can be seen to stem from the equations (7.3.6), (7.3.7). Indeed, were the constraints (7.2.3) ignored, the variation of the action (7.3.2) with respect to unconstrained $\lambda_\alpha^\pm$ would yield $e^{++}\wedge\Pi^{\alpha\beta}\,\lambda_\beta^- = 0$ and $e^{--}\wedge\Pi^{\alpha\beta}\,\lambda_\beta^+ = 0$. By (7.3.6) (or (7.3.7)) this would imply, in particular, $e^{++}\wedge e^{--} = 0$, contradicting the original assumption of independence of the one-forms e^{++} and e^{--} and, actually, reducing the present model to a $p=1$ version of the tensionless p-brane model [149].

As $\lambda_\alpha^\pm$ are restricted by the constraint (7.2.3), this constraint has to be taken into account in the variational problem. Instead of applying the Lagrange multiplier technique, one may restrict the variations to those that preserve (7.2.3), *i.e.* such that

$$C^{\alpha\beta}\delta\lambda_\alpha^+\lambda_\beta^- + C^{\alpha\beta}\lambda_\alpha^+\delta\lambda_\beta^- = 0\;. \tag{7.3.11}$$

One can solve (7.3.11) by introducing a set of $n-2$ auxiliary spinors u_α^I 'orthogonal' to the $\lambda^\pm$ (*cf.* [154, 204]),

$$C^{\alpha\beta}u_\alpha^I\lambda_\beta^\pm = 0\;, \qquad I = 1,\ldots,n-2 \quad , \tag{7.3.12}$$

and normalized by

$$C^{\alpha\beta}u_\alpha^I u_\beta^J = C^{IJ}\;, \qquad C^{IJ} = -C^{JI}\;, \tag{7.3.13}$$

where C^{IJ} is an antisymmetric constant invertible $(n-2)\times(n-2)$ matrix.

The n spinors

$$\{\lambda_\alpha^+\;,\;\lambda_\alpha^-\;,u_\alpha^I\}\;,\; I = 1,\ldots,n-2\;, \tag{7.3.14}$$

provide a basis that can be used to decompose an arbitrary spinor worldvolume function (*cf.* [205]), and in particular the variations $\delta\lambda^+$, $\delta\lambda^-$. Then one finds that the only consequence of equation (7.3.11) is that the sum of the coefficient for λ^+ in the decomposition of $\delta\lambda^+$ and that of λ^- in the decomposition of $\delta\lambda^-$ vanishes . In other words, the general solution of equation (7.3.11) reads

$$\delta\lambda_\alpha^+ = \omega(\delta)\lambda_\alpha^+ + \Omega^{++}(\delta)\lambda_\alpha^- + \Omega_I^+(\delta)u_\alpha^I\;, \tag{7.3.15}$$

$$\delta\lambda^-_\alpha = -\omega(\delta)\lambda^-_\alpha + \Omega^{--}(\delta)\lambda^+_\alpha + \Omega^-_I(\delta)u^I_\alpha \, , \tag{7.3.16}$$

where $\Omega^\pm_I(\delta)$, $\Omega^{\pm\pm}(\delta)$ and $\omega(\delta)$ are arbitrary variational parameters. Substituting equations (7.3.15), (7.3.16) into (7.3.2), one finds

$$\begin{aligned}\delta_\lambda S = &- \int_{W^2} (2e^{++} \wedge \Pi^{\alpha\beta}\lambda^-_\beta\lambda^-_\alpha + 2e^{--} \wedge \Pi^{\alpha\beta}\lambda^+_\beta\lambda^+_\alpha)\omega(\delta) \\ &+ \int_{W^2} 2e^{++} \wedge \Pi^{\alpha\beta}\lambda^-_\beta\lambda^+_\alpha\Omega^{--}(\delta) \\ &+ \int_{W^2} 2e^{--} \wedge \Pi^{\alpha\beta}\lambda^+_\beta\lambda^-_\alpha\Omega^{++}(\delta) \\ &+ \int_{W^2} 2e^{++} \wedge \Pi^{\alpha\beta}\lambda^-_\beta u^I_\alpha\Omega^-_I(\delta) \\ &- \int_{W^2} 2e^{--} \wedge \Pi^{\alpha\beta}\lambda^+_\beta u^I_\alpha\Omega^+_I(\delta) \, . \end{aligned} \tag{7.3.17}$$

Now we can write the complete set of equations of motion which include, in addition to equations (7.3.3), (7.3.5), (7.3.6), (7.3.7), the set of equations for $\lambda^\pm_\alpha$, which follows from $\delta S/\omega(\delta) = 0$, $\delta S/\Omega^{++}(\delta) = 0$, $\delta S/\Omega^+_I(\delta) = 0$, $\delta S/\Omega^{--}(\delta) = 0$, and $\delta S/\Omega^-_I(\delta) = 0$, namely

$$e^{++} \wedge \Pi^{\alpha\beta}\lambda^-_\beta\lambda^-_\alpha + e^{--} \wedge \Pi^{\alpha\beta}\lambda^+_\beta\lambda^+_\alpha = 0 \, , \tag{7.3.18}$$

$$e^{++} \wedge \Pi^{\alpha\beta}\lambda^-_\beta\lambda^+_\alpha = 0 \, , \tag{7.3.19}$$

$$e^{--} \wedge \Pi^{\alpha\beta}\lambda^+_\beta\lambda^-_\alpha = 0 \, , \tag{7.3.20}$$

$$e^{++} \wedge \Pi^{\alpha\beta}\lambda^-_\beta u^I_\alpha = 0 \, , \tag{7.3.21}$$

$$e^{--} \wedge \Pi^{\alpha\beta}\lambda^+_\beta u^I_\alpha = 0 \, . \tag{7.3.22}$$

Due to the linear independence of both one-forms $e^{++} = d\xi^m e^{++}_m(\xi)$ and $e^{--} = d\xi^m e^{--}_m(\xi)$, equations (7.3.19), (7.3.20) imply

$$\Pi^{\alpha\beta}\lambda^-_\beta\lambda^+_\alpha = 0 \, . \tag{7.3.23}$$

Decomposing the bosonic invariant one form $\Pi^{\alpha\beta} = d\xi^m\Pi^{\alpha\beta}_m$ in the basis provided by $e^{\pm\pm}$,

$$\Pi^{\alpha\beta} = e^{++}\Pi^{\alpha\beta}_{++} + e^{--}\Pi^{\alpha\beta}_{--} \, , \tag{7.3.24}$$

$$\Pi^{\alpha\beta}_{\pm\pm} = \nabla_{\pm\pm}X^{\alpha\beta} - i\nabla_{\pm\pm}\theta^{(\alpha}\,\theta^{\beta)} \, , \tag{7.3.25}$$

where $\nabla_{\pm\pm}$ is defined by

$$d \equiv e^{\pm\pm}\nabla_{\pm\pm} = e^{++}\nabla_{++} + e^{--}\nabla_{--} \, , \tag{7.3.26}$$

one finds that equations (7.3.21) and (7.3.22) restrict only the derivatives (∇_{++}, ∇_{--}) of the bosonic coordinate function $X^{\alpha\beta}(\xi)$, respectively,

$$\Pi^{\alpha\beta}_{--}\lambda^-_\beta u^I_\alpha \equiv (\nabla_{--}X^{\alpha\beta} - i\nabla_{--}\theta^{(\alpha}\,\theta^{\beta)})\,\lambda^-_\beta u^I_\alpha = 0\ , \tag{7.3.27}$$

$$\Pi^{\alpha\beta}_{++}\lambda^+_\beta u^I_\alpha \equiv (\nabla_{++}X^{\alpha\beta} - i\nabla_{++}\theta^{(\alpha}\,\theta^{\beta)})\,\lambda^+_\beta u^I_\alpha = 0\ . \tag{7.3.28}$$

In the same manner, equations (7.3.5) can be written as

$$\nabla_{--}\theta^\alpha\,\lambda^-_\alpha = 0\ , \qquad \nabla_{++}\theta^\alpha\,\lambda^+_\alpha = 0\ . \tag{7.3.29}$$

The analysis of the above set of equations in the tensorial superspace, the search for solutions and their reinterpretation in standard D-dimensional spacetime, possibly along the fields/extended superspace coordinates correspondence of [85] (see section 6.7 of chapter 6) , or of the 'two-time physics' approach of [206], lies beyond the scope of this Thesis.

7.4 Gauge symmetries

The expression (7.3.1), with (7.3.17), for the general variation of the supersymmetric string action (7.2.1) shows that the model possesses n supersymmetries and $(n-2)$ κ-symmetries of the form [8]

$$\delta_\kappa\theta^\alpha(\xi) = C^{\alpha\beta}u^I_\beta(\xi)\kappa_I(\xi)\ , \tag{7.4.1}$$

$$\delta_\kappa X^{\alpha\beta}(\xi) = i\delta_\kappa\theta^{(\alpha}(\xi)\theta^{\beta)}(\xi)\ , \tag{7.4.2}$$

$$\delta_\kappa\lambda^\pm_\alpha(\xi) = 0\ , \quad \delta_\kappa e^{\pm\pm}_m(\xi) = 0\ , \tag{7.4.3}$$

with $(n-2)$ fermionic gauge parameters $\kappa_I(\xi)$ (30 for $\Sigma^{(528|32)}$). In the framework of the second Noether theorem this κ-symmetry is reflected by the fact that only 2 of the n fermionic equations (7.3.4) are independent. We stress that the $(n-2)$ $GL(n,\mathbb{R})$ vector fields u^I_α defined by (7.3.12) are auxiliary. They allow us to write explicitly the general solution of the equations

$$\delta_\kappa\theta^\alpha(\xi)\lambda^\pm_\alpha(\xi) = 0\ , \tag{7.4.4}$$

which define implicitly the κ-symmetry transformation (7.4.1). Note that the dynamical system is κ-symmetric despite it does not contain a Wess-Zumino term. This property seems to be specific of models defined on tensorial superspaces.

Our model also possesses $\frac{1}{2}(n-1)(n-2)$ b-symmetries, which are the bosonic 'superpartners' of the fermionic κ-symmetries, defined by

$$\delta_b X^{\alpha\beta} = b_{IJ}(\xi)u^{\alpha I}u^{\beta J}\ , \delta_b\theta^\alpha = 0\ , \quad \delta_b\lambda^\pm_\alpha = 0\ , \quad \delta_b e^{\pm\pm} = 0\ , \tag{7.4.5}$$

where $b_{IJ}(\xi)$ is symmetric and $I, J = 1, \ldots, n-2$. They are reflected by the $(n-1)(n-2)/2$ Noether identities stating that the contractions of the bosonic equations (7.3.3) with the $u^{\alpha I} u^{\beta J}$ bilinears of the $(n-2)$ auxiliary bosonic spinors $u^{\alpha I} (= C^{\alpha\beta} u^I_\beta)$ vanish [7].

The remaining gauge symmetries of the action (7.2.1) are the $SO(1,1)$ worldsheet Lorentz invariance

$$\delta X^{\alpha\beta} = 0, \quad \delta\theta^\alpha = 0, \quad \delta\lambda^\pm_\alpha = \pm\omega(\delta)\lambda^\pm_\alpha, \quad \delta e^{\pm\pm} = \pm 2\omega(\delta) e^{\pm\pm}, \quad (7.4.6)$$

which is reflected by the fact that equation (7.3.18) is satisfied identically when equations (7.3.6), (7.3.7) are taken into account, and the symmetry under worldvolume general coordinate transformations.

As customary in string models, the general coordinate invariance and the $SO(1,1)$ gauge symmetry allows one to fix locally the conformal gauge where $e_m{}^a(\xi) = e^{\phi(\xi)} \delta^a_m$ or, equivalently

$$e^{++} = e^{\phi(\xi)}(d\tau + d\sigma)\ , \qquad e^{--} = e^{\phi(\xi)}(d\tau - d\sigma)\ , \qquad (7.4.7)$$

$$\Leftrightarrow \quad e^{++}_\sigma = e^{++}_\tau = e^{\phi(\xi)}\ , \quad e^{--}_\sigma = -e^{--}_\tau = -e^{\phi(\xi)}\ . \qquad (7.4.8)$$

This indicates that it makes sense to consider the fields $e^{\pm\pm}_\sigma(\tau,\sigma)$ as nonsingular ($\frac{1}{e^{\pm\pm}_\sigma} = \pm e^{-\phi(\xi)}$ in the conformal gauge), a fact used in the Hamiltonian analysis below.

According to the correspondence [147, 58] between the κ-symmetry of the worldvolume action and the supersymmetry preserved by a BPS state (*e.g.* by a solitonic solution of the supergravity equations of motion), the action (7.2.1) defines a dynamical model for the excitations of a BPS state preserving *all but two* supersymmetries. Such a BPS state can be treated as a composite of two BPS preons ($\tilde{n} = 32 - 30$). This will be proved after the Hamiltonian analysis of next section.

7.5 Hamiltonian analysis

The gauge symmetry structure of the model has already been shown in the Lagrangian framework. However, our dynamical system possesses additional, second class, constraints [207], one of which is condition (7.2.3). The Hamiltonian analysis of our $\Sigma^{(\frac{n(n+1)}{2}|n)}$ superstring model [8], that

[7] In the massless $\Sigma^{(\frac{n(n+1)}{2}|n)}$ superparticle and tensionless super-p-brane models the b-symmetry [148, 155, 149] is $n(n-1)/2$ parametric. This comes from the fact that such models contain a single bosonic spinor λ_α and the non-trivial b-symmetry variation is the general solution of the spinorial equation $\delta_b X^{\alpha\beta}\lambda_\alpha = 0$. In our tensionful superstring model with two bosonic spinors $\lambda^\pm_\alpha(\xi)$, the $(n-1)(n-2)/2$ parametric b-symmetry transformations (equation (7.4.5)) are the solutions of two equations $\delta_b X^{\alpha\beta}\lambda^+_\alpha = 0$ and $\delta_b X^{\alpha\beta}\lambda^-_\alpha = 0$.

we perform in this section, will allow us to find the number of field theoretical degrees of freedom of our model and to establish its relation with the notion of BPS preons [83] (see section 4.1 of chapter 4).

The Lagrangian density $\mathcal{L}$ for the action (7.2.1),

$$S = \int_{W^2} d\tau d\sigma \, \mathcal{L} \; , \tag{7.5.1}$$

is given by

$$\begin{aligned}\mathcal{L} = & (e_\tau^{++}\Pi_\sigma^{\alpha\beta} - e_\sigma^{++}\Pi_\tau^{\alpha\beta})\lambda_\alpha^- \lambda_\beta^- - (e_\tau^{--}\Pi_\sigma^{\alpha\beta} - e_\sigma^{--}\Pi_\tau^{\alpha\beta})\lambda_\alpha^+ \lambda_\beta^+ \\ & -(e_\tau^{++}e_\sigma^{--} - e_\sigma^{++}e_\tau^{--}) \; ,\end{aligned} \tag{7.5.2}$$

where

$$\Pi_\tau^{\alpha\beta} = \partial_\tau X^{\alpha\beta} - i\partial_\tau \theta^{(\alpha}\theta^{\beta)} \; , \quad \Pi_\sigma^{\alpha\beta} = \partial_\sigma X^{\alpha\beta} - i\partial_\sigma \theta^{(\alpha}\theta^{\beta)} \; , \tag{7.5.3}$$

are the worldsheet components of the one-form (7.2.2).

The momenta $P_{\mathcal{M}}$ canonically conjugate to the configuration space variables

$$\mathcal{Z}^{\mathcal{M}} \equiv \mathcal{Z}^{\mathcal{M}}(\tau,\sigma) := \Big(X^{\alpha\beta} \, , \, \theta^\alpha , \lambda_\alpha^\pm \, , \, e_\tau^{\pm\pm} \, , \, e_\sigma^{\pm\pm} \Big) \tag{7.5.4}$$

are defined as usual:

$$P_{\mathcal{M}} = (P_{\alpha\beta} \, , \, \pi_\alpha \, , \, P_\pm^{\alpha(\lambda)}, \, P_{\pm\pm}^\tau \, , \, P_{\pm\pm}^\sigma) = \frac{\partial \mathcal{L}}{\partial(\partial_\tau \mathcal{Z}^{\mathcal{M}})} \; . \tag{7.5.5}$$

The canonical equal τ graded Poisson brackets,

$$[\mathcal{Z}^{\mathcal{N}}(\sigma) \, , \, P_{\mathcal{M}}(\sigma')\}_P = -(-1)^{\mathcal{N}\mathcal{M}} [P_{\mathcal{M}}(\sigma') \, , \, \mathcal{Z}^{\mathcal{N}}(\sigma)\}_P \; , \tag{7.5.6}$$

are defined by

$$[\mathcal{Z}^{\mathcal{N}}(\sigma') \, , \, P_{\mathcal{M}}(\sigma)\}_P \;\; := \;\; (-1)^{\mathcal{N}} \delta_{\mathcal{M}}^{\mathcal{N}} \delta(\sigma - \sigma') \; , \tag{7.5.7}$$

where $(-1)^{\mathcal{N}} \equiv (-1)^{\deg(\mathcal{N})}$ and the degree $\deg(\mathcal{N}) \equiv \deg(\mathcal{Z}^{\mathcal{N}})$ is 0 for the bosonic fields, $\mathcal{Z}^{\mathcal{N}} = X^{\alpha\beta}, \lambda_\alpha^\pm, e_m^{\pm\pm}$ (or for the 'bosonic indices' $\mathcal{N} = (\alpha\beta), (\alpha\pm), (\pm\pm), m$), and 1 for the fermionic fields $\mathcal{Z}^{\mathcal{N}} = \theta^\alpha$ (or for the 'fermionic indices' $\mathcal{N} = \alpha$ and $\mathcal{N} = \pm$).

Since the action (7.2.1) is of first order type, it is not surprising that the expression of every momentum results in a primary [207] constraint. Explicitly,

$$\mathcal{P}_{\alpha\beta} = P_{\alpha\beta} + e_\sigma^{++}\lambda_\alpha^- \lambda_\beta^- - e_\sigma^{--}\lambda_\alpha^+ \lambda_\beta^+ \approx 0 \; , \tag{7.5.8}$$

$$\mathcal{D}_\alpha = \pi_\alpha + i\theta^\beta P_{\alpha\beta} \approx 0 \; , \tag{7.5.9}$$

$$P_{\pm}^{\alpha(\lambda)} \approx 0 \ , \qquad (7.5.10)$$
$$P_{\pm\pm}^{\sigma} \approx 0 \ , \qquad (7.5.11)$$
$$P_{\pm\pm}^{\tau} \approx 0 \ , \qquad (7.5.12)$$

where only $\mathcal{D}_\alpha$ is fermionic. Condition (7.2.3),

$$\mathcal{N} := C^{\alpha\beta}\lambda_\alpha^+\lambda_\beta^- - 1 \approx 0 \ , \qquad (7.5.13)$$

imposed on the bosonic spinors from the beginning, is also a primary constraint and has to be treated on the same footing as equations (7.5.8)-(7.5.12).

The *canonical* Hamiltonian density $\mathcal{H}_0$,

$$\mathcal{H}_0 = \partial_\tau \mathcal{Z}^{\mathcal{M}} P_{\mathcal{M}} - \mathcal{L} \quad , \qquad (7.5.14)$$

calculated on the primary constraints (7.5.8)–(7.5.12) hypersurface reads

$$\mathcal{H}_0 = e_\tau^{--}\Pi_\sigma^{\alpha\beta}\lambda_\alpha^+\lambda_\beta^+ - e_\tau^{++}\Pi_\sigma^{\alpha\beta}\lambda_\alpha^-\lambda_\beta^- + (e_\tau^{++}e_\sigma^{--} - e_\sigma^{++}e_\tau^{--}) \ . \qquad (7.5.15)$$

The evolution of any functional $f(\mathcal{Z}^{\mathcal{M}}, P_{\mathcal{N}})$ is then defined by

$$\partial_\tau f = [f \ , \int d\sigma \mathcal{H}']_P \ , \qquad (7.5.16)$$

involving the total Hamiltonian, $\int d\sigma\mathcal{H}'$, where the Hamiltonian density $\mathcal{H}'$ is the sum of $\mathcal{H}_0$ in equation (7.5.15) and the terms given by integrals of the primary constraints (7.5.8)-(7.5.12) multiplied by arbitrary functions (Lagrange multipliers) [207]. Then one has to check that the primary constraints are preserved under the evolution, $\partial_\tau \mathcal{P}_{\alpha\beta} \approx 0$, etc. At this stage additional, secondary constraints may be obtained. This is the case for our system.

Indeed, since the constraints (7.5.12) have zero Poisson brackets with any other primary constraint, their time evolution is just determined by the canonical Hamiltonian $\mathcal{H}_0$, $\partial_\tau \mathcal{P}^\tau_{\pm\pm} = [\mathcal{P}^\tau_{\pm\pm}, \int d\sigma\mathcal{H}_0]_P$. Then $\partial_\tau \mathcal{P}^\tau_{\pm\pm} \approx 0$ can be seen to produce a pair of secondary constraints,

$$\begin{aligned} \Phi_{\pm\pm} \ &:= \ \Pi_\sigma^{\alpha\beta}\lambda_\alpha^\mp\lambda_\beta^\mp - e_\sigma^{\mp\mp} \\ &= \ (\partial_\sigma X^{\alpha\beta} - i\partial_\sigma\theta^{(\alpha}\theta^{\beta)})\lambda_\alpha^\mp\lambda_\beta^\mp - e_\sigma^{\mp\mp} \approx 0 \ . \end{aligned} \qquad (7.5.17)$$

Slightly more complicated calculations with the total $\mathcal{H}'$ show that we also have the secondary constraint

$$\Phi^{(0)} := \Pi_\sigma^{\alpha\beta}\lambda_\alpha^+\lambda_\beta^- = (\partial_\sigma X^{\alpha\beta} - i\partial_\sigma\theta^{(\alpha}\theta^{\beta)})\lambda_\alpha^+\lambda_\beta^- \approx 0 \qquad (7.5.18)$$

(details about its derivation can be found below equation (7.5.32)). The appearance of this secondary constraint may be understood as well by comparing with the results of the Lagrangian approach: it is just the σ component of the differential form equation (7.3.23).

The secondary constraints (7.5.17) imply that the canonical Hamiltonian $\mathcal{H}_0$, equation (7.5.14), vanishes on the surface of constraints (7.5.17),

$$\mathcal{H}_0 \approx 0\,, \tag{7.5.19}$$

a characteristic property of theories with general coordinate invariance. Hence the total Hamiltonian reduces to a linear combination of the constraints (7.5.8)–(7.5.12), (7.5.17), (7.5.18),

$$\begin{aligned}\mathcal{H} = & -e_\tau^{++}\Phi_{++} + e_\tau^{--}\Phi_{--} + l^{(0)}\Phi^{(0)} + L^{\alpha\beta}\mathcal{P}_{\alpha\beta} + \xi^\alpha \mathcal{D}_\alpha \\ & + l_\alpha^{\pm} P_{\pm}^{\alpha(\lambda)} + L^{\pm\pm} P_{\pm\pm}^{\sigma} + h^{\pm\pm} P_{\pm\pm}^{\tau} + L^{(n)}\mathcal{N}\end{aligned} \tag{7.5.20}$$

where $l^{(0)}$, $L^{\alpha\beta}$, ξ^α, $l_\alpha^{\pm}$, $L^{\pm\pm}$, $h^{\pm\pm}$, $L^{(n)}$ *and* $\pm e_\tau^{\pm\pm}$ are Lagrangian multipliers whose form should be fixed from the preservation of all the primary and secondary constraints under τ-evolution.

Note that the constraints (7.5.12) are trivially first class, since their Poisson brackets with all the other constraints, including (7.5.17) and (7.5.18), vanish. This allows us to state that $e_\tau^{\pm\pm}(\xi)$ are not dynamical fields but rather Lagrange multipliers (as the time component of electromagnetic potential A_0 in electrodynamics). Nevertheless, the appearance of these Lagrange multipliers from the τ components of the zweibein $e_m^{\pm\pm}$ puts a 'topological' restriction on a possible gauge fixing; in particular the gauge $e_\tau^{\pm\pm} = 0$ is not allowed. Indeed, the nondegeneracy of the zweibein, assumed from the beginning, reads

$$\det\left(e_m^a(\xi)\right) \equiv \tfrac{1}{2}(e_\tau^{--}e_\sigma^{++} - e_\tau^{++}e_\sigma^{--}) \neq 0\,. \tag{7.5.21}$$

Just due to this restriction, studying the τ-preservation of the primary constraints, one finds the secondary constraint (7.5.18).

If by checking the (primary and secondary) constraints preservation under τ-evolution one finds that some lagrangian multipliers remain unfixed, then they correspond to *first class constraints* [207] which generate gauge symmetries of the system through the Poisson brackets. In other words, since the canonical Hamiltonian vanishes in the weak sense, the total Hamiltonian is a linear combination of all first class constraints [207]. If some of the equations resulting from the τ-evolution of the constraints (or their linear combinations) do not restrict the Lagrangian multiplier, but imply the vanishing of a combination of the canonical variables, they correspond to new secondary constraints, which have to be added with new Lagrange multipliers to obtain a new total Hamiltonian. In this case the check that all the constraints are preserved under τ–evolution has to be repeated.

This does not happen for our dynamical system: a further check of the constraints τ–preservation does not result in the appearance of new constraints. Indeed, it leads to the following set of equations for the Lagrange multipliers

$$\partial_\sigma(e_\tau^{--}\lambda_\alpha^+\lambda_\beta^+ - e_\tau^{++}\lambda_\alpha^-\lambda_\beta^- + l^{(0)}\lambda^+_{(\alpha}\lambda^-_{\beta)}) \\ -2e_\sigma^{--}\lambda^+_{(\alpha}l^+_{\beta)} + 2e_\sigma^{++}\lambda^-_{(\alpha}l^-_{\beta)} + L^{++}\lambda_\alpha^-\lambda_\beta^- - L^{--}\lambda_\alpha^+\lambda_\beta^+ \approx 0 \,, \quad (7.5.22)$$

$$\lambda_\alpha^-\,[2ie_\tau^{++}(\partial_\sigma\theta\lambda^-) - il^{(0)}(\partial_\sigma\theta\lambda^+) + 2ie_\sigma^{++}(\xi\lambda^-)] \\ -\lambda_\alpha^+[2ie_\tau^{--}(\partial_\sigma\theta\lambda^+) + il^{(0)}(\partial_\sigma\theta\lambda^-) + 2ie_\sigma^{--}(\xi\lambda^+)] \approx 0 \,, \quad (7.5.23)$$

$$-2e_\tau^{--}\Pi_\sigma^{\alpha\beta}\lambda_\beta^+ - l^{(0)}\Pi_\sigma^{\alpha\beta}\lambda_\beta^- + 2e_\sigma^{--}L^{\alpha\beta}\lambda_\beta^+ - L^{(n)}C^{\alpha\beta}\lambda_\beta^- \approx 0 \quad (7.5.24)$$

$$2e_\tau^{++}\Pi_\sigma^{\alpha\beta}\lambda_\beta^- - l^{(0)}\Pi_\sigma^{\alpha\beta}\lambda_\beta^+ - 2e_\sigma^{++}L^{\alpha\beta}\lambda_\beta^- - L^{(n)}C^{\alpha\beta}\lambda_\beta^+ \approx 0 \,, \quad (7.5.25)$$

$$e_\tau^{--} - L^{\alpha\beta}\lambda_\alpha^-\lambda_\beta^- \approx 0 \,, \quad (7.5.26)$$

$$e_\tau^{++} - L^{\alpha\beta}\lambda_\alpha^+\lambda_\beta^+ \approx 0 \,, \quad (7.5.27)$$

$$l_\alpha^+C^{\alpha\beta}\lambda_\beta^- - l_\alpha^-C^{\alpha\beta}\lambda_\beta^+ \approx 0 \,, \quad (7.5.28)$$

$$\partial_\sigma L^{\alpha\beta}\lambda_\alpha^-\lambda_\beta^- + 2i(\xi\lambda^-)(\partial_\sigma\theta\lambda^-) + 2l^-\Pi_\sigma\lambda^- - L^{--} \approx 0 \,, \quad (7.5.29)$$

$$\partial_\sigma L^{\alpha\beta}\lambda_\alpha^+\lambda_\beta^+ + 2i(\xi\lambda^+)(\partial_\sigma\theta\lambda^+) + 2l^+\Pi_\sigma\lambda^+ - L^{++} \approx 0 \,, \quad (7.5.30)$$

$$\partial_\sigma L^{\alpha\beta}\lambda_\alpha^+\lambda_\beta^- + i(\xi\lambda^+)(\partial_\sigma\theta\lambda^-) - i(\xi\lambda^-)(\partial_\sigma\theta\lambda^+) + \\ +l^+\Pi_\sigma\lambda^- + l^-\Pi_\sigma\lambda^+ \approx 0 \,, \quad (7.5.31)$$

where the weak equality sign is used to stress that one may use the constraints in solving the above system of equations. For brevity, in equations (7.5.22)–(7.5.31) and below we often omit spinor indices in the contractions

$$(\partial_\sigma\theta\lambda^\pm) \equiv \partial_\sigma\theta^\beta\,\lambda_\beta^\pm \,, \quad (\xi\lambda^\pm) \equiv \xi^\beta\,\lambda_\beta^\pm \,, \\ l^\pm\Pi_\sigma\lambda^\pm \equiv l_\alpha^\pm\Pi_\sigma^{\alpha\beta}\lambda_\beta^\pm \,, \quad l^\pm L\lambda^\pm \equiv l_\alpha^\pm L_\sigma^{\alpha\beta}\lambda_\beta^\pm \,. \quad (7.5.32)$$

Note that equations (7.5.22)–(7.5.28) come from the requirement of τ–preservation of the primary constraints, while that for the secondary constraints leads to equations (7.5.29)–(7.5.31). Thus the above statement about the appearance of the secondary constraint (7.5.18) can be checked by studying equations (7.5.22)–(7.5.28) with $l^{(0)} = 0$. In this case the contraction of equation (7.5.24) with $(-\lambda^-_\alpha)$ and of equation (7.5.25) with λ^+_α results, respectively, in the equations $e^{--}_\tau \lambda^+ \Pi_\sigma \lambda^- - e^{--}_\sigma \lambda^+ L \lambda^- \approx 0$ and $e^{++}_\tau \lambda^+ \Pi_\sigma \lambda^- - e^{++}_\sigma \lambda^+ L \lambda^- \approx 0$. Due to the nondegeneracy of the zweibein, equation (7.5.21), the solution to these two equations is trivial, *i.e.* it implies $\lambda^+ L \lambda^- \approx 0$ and $\lambda^+ \Pi_\sigma \lambda^- \approx 0$, the last of which is just the secondary constraint (7.5.18).

To solve this system of equations for the Lagrange multipliers and thus to describe explicitly the first class constraints, we can use the auxiliary spinor fields $u^I_\alpha(\xi)$ defined as in (7.3.12), (7.3.13). The general solution of equations (7.5.22)–(7.5.31) obtained in such a framework reads

$$
\begin{aligned}
L^{\alpha\beta} &= b_{IJ} u^{\alpha I} u^{\beta J} \\
&\quad + \frac{e^{++}_\tau}{e^{++}_\sigma} \Big[e^{++}_\sigma \lambda^{-\alpha} \lambda^{-\beta} + 2 \Big(\lambda^-_\gamma \Pi^{\gamma(\alpha}_\sigma \lambda^{+\beta)} - (\lambda^- \Pi_\sigma \lambda^+) \lambda^{-(\alpha} \lambda^{+\beta)} \\
&\qquad\qquad + (\lambda^- \Pi_\sigma \lambda^-) \lambda^{+(\alpha} \lambda^{+\beta)} \Big) \Big] \\
&\quad + \frac{e^{--}_\tau}{e^{--}_\sigma} \Big[e^{--}_\sigma \lambda^{+\alpha} \lambda^{+\beta} - 2 \Big(\lambda^+_\gamma \Pi^{\gamma(\alpha}_\sigma \lambda^{-\beta)} - (\lambda^+ \Pi_\sigma \lambda^+) \lambda^{-(\alpha} \lambda^{-\beta)} \\
&\qquad\qquad + (\lambda^+ \Pi_\sigma \lambda^-) \lambda^{+(\alpha} \lambda^{-\beta)} \Big) \Big] ,
\end{aligned}
\tag{7.5.33}
$$

$$
\xi^\alpha = \kappa_I \, u^{\alpha I} + \frac{e^{++}_\tau}{e^{++}_\sigma} (\partial_\sigma \theta \lambda^-) \lambda^{+\alpha} - \frac{e^{--}_\tau}{e^{--}_\sigma} (\partial_\sigma \theta \lambda^+) \lambda^{-\alpha} , \tag{7.5.34}
$$

$$
\begin{aligned}
l^+_\alpha &= \omega^{(0)} \lambda^+_\alpha + \frac{e^{--}_\tau}{e^{--}_\sigma} \Big(\partial_\sigma \lambda^+_\alpha - \Omega^{(0)}_\sigma \lambda^+_\alpha \Big) \\
&\quad + \frac{e^{--}_\tau}{2 e^{--}_\sigma e^{--}_\sigma} \Big[- e^{--}_\sigma \Omega^{++}_\sigma - e^{++}_\sigma \Omega^{--}_\sigma + i \partial_\sigma \theta \lambda^+ \partial_\sigma \theta \lambda^- \\
&\qquad\qquad - \Pi^{\alpha\beta}_\sigma (\partial_\sigma \lambda^+_\alpha \lambda^-_\beta - \lambda^+_\alpha \partial_\sigma \lambda^-_\beta) \Big] \lambda^-_\alpha \\
&\quad + \frac{e^{++}_\tau}{2 e^{++}_\sigma e^{--}_\sigma} \Big[e^{--}_\sigma \Omega^{++}_\sigma + e^{++}_\sigma \Omega^{--}_\sigma + i \partial_\sigma \theta \lambda^+ \partial_\sigma \theta \lambda^- \\
&\qquad\qquad + \Pi^{\alpha\beta}_\sigma (\partial_\sigma \lambda^+_\alpha \lambda^-_\beta - \lambda^+_\alpha \partial_\sigma \lambda^-_\beta) \Big] \lambda^-_\alpha ,
\end{aligned}
\tag{7.5.35}
$$

$$
\begin{aligned}
l^-_\alpha &= -\omega^{(0)} \lambda^-_\alpha + \frac{e^{++}_\tau}{e^{++}_\sigma} \Big(\partial_\sigma \lambda^-_\alpha + \Omega^{(0)}_\sigma \lambda^-_\alpha \Big) + \\
&\quad + \frac{e^{--}_\tau}{2 e^{++}_\sigma e^{--}_\sigma} \Big[- e^{--}_\sigma \Omega^{++}_\sigma - e^{++}_\sigma \Omega^{--}_\sigma + i \partial_\sigma \theta \lambda^+ \partial_\sigma \theta \lambda^- - \\
&\qquad\qquad - \Pi^{\alpha\beta}_\sigma (\partial_\sigma \lambda^+_\alpha \lambda^-_\beta - \lambda^+_\alpha \partial_\sigma \lambda^-_\beta) \Big] \lambda^+_\alpha +
\end{aligned}
$$

$$+\frac{e_\tau^{++}}{2e_\sigma^{++}e_\sigma^{++}}\Big[e_\sigma^{--}\Omega_\sigma^{++} + e_\sigma^{++}\Omega_\sigma^{--} + i\partial_\sigma\theta\lambda^+\partial_\sigma\theta\lambda^- + \\ +\Pi_\sigma^{\alpha\beta}(\partial_\sigma\lambda_\alpha^+\lambda_\beta^- - \lambda_\alpha^+\partial_\sigma\lambda_\beta^-)\Big]\lambda_\alpha^+ \;, \quad (7.5.36)$$

$$L^{\pm\pm} = \partial_\sigma e_\tau^{\pm\pm} + 2e_\tau^{\pm\pm}\Omega_\sigma^{(0)} \pm 2e_\sigma^{\pm\pm}\omega^{(0)} \;, \qquad (7.5.37)$$

$$L^{(n)} = -4\det(e_m^a) \equiv -2(e_\tau^{--}e_\sigma^{++} - e_\tau^{++}e_\sigma^{--}) \;, \qquad (7.5.38)$$

$$l^{(0)} = 0 \;, \qquad (7.5.39)$$

where, $\Omega_\sigma^{\pm\pm}$ and $\Omega_\sigma^{(0)}$ (*cf.* equations (7.3.15)) are given by

$$\Omega_\sigma^{++} := \partial_\sigma\lambda^+ C\lambda^+ \;, \qquad \Omega_\sigma^{--} := \partial_\sigma\lambda^- C\lambda^- \;, \qquad (7.5.40)$$

$$\Omega_\sigma^{(0)} := \frac{1}{2}(\partial_\sigma\lambda^+ C\lambda^- - \lambda^+ C\partial_\sigma\lambda^-) \;. \qquad (7.5.41)$$

In this solution the parameters

$$\text{bosonic}: \quad b^{IJ} = b^{JI} \;, \quad \omega^{(0)} \;, \quad e_\tau^{\pm\pm} \;, \quad h^{\pm\pm} \;, \qquad (7.5.42)$$

$$\text{fermionic}: \quad \kappa_I \;, \qquad (7.5.43)$$

are indefinite. They correspond to the first class constraints

$$\mathcal{P}^{IJ} := \mathcal{P}_{\alpha\beta}u^{\alpha I}u^{\beta J} \approx 0 \;, \qquad (7.5.44)$$

$$\mathcal{D}^I := \mathcal{D}_\alpha u^{\alpha I} \approx 0 \;, \qquad (7.5.45)$$

$$G^{(0)} := \lambda_\alpha^+ P_+^{\alpha(\lambda)} - \lambda_\alpha^- P_-^{\alpha(\lambda)} + 2e_\sigma^{++}P_{++}^\sigma - 2e_\sigma^{--}P_{--}^\sigma \approx 0 \;, \qquad (7.5.46)$$

$$\tilde{\Phi}_{++} = \Phi_{++} + \partial_\sigma P_{++}^\sigma - 2\Omega_\sigma^{(0)}P_{++}^\sigma - 2e_\sigma^{--}\mathcal{N} \\ -\frac{1}{e_\sigma^{++}}(\partial_\sigma\lambda_\alpha^- + \Omega_\sigma^{(0)}\lambda_\alpha^-)P_-^{\alpha(\lambda)} - \frac{1}{e_\sigma^{++}}(\partial_\sigma\theta\lambda^-)(\lambda^{+\alpha}\mathcal{D}_\alpha) - \\ -\Big[\lambda^{-\alpha}\lambda^{-\beta} + \frac{2}{e_\sigma^{++}}\Big(\lambda_\gamma^-\Pi_\sigma^{\gamma\alpha}\lambda^{+\beta} - (\lambda^-\Pi_\sigma\lambda^+)\lambda^{-\alpha}\lambda^{+\beta} \\ +(\lambda^-\Pi_\sigma\lambda^-)\lambda^{+\alpha}\lambda^{+\beta}\Big)\Big]\mathcal{P}_{\alpha\beta} \\ -\frac{1}{2e_\sigma^{++}}\Big[e_\sigma^{--}\Omega_\sigma^{++} + e_\sigma^{++}\Omega_\sigma^{--} + i\partial_\sigma\theta\lambda^+\partial_\sigma\theta\lambda^- \\ +\Pi_\sigma^{\alpha\beta}(\partial_\sigma\lambda_\alpha^+\lambda_\beta^- - \lambda_\alpha^+\partial_\sigma\lambda_\beta^-)\Big]\left[\frac{\lambda_\alpha^- P_+^{\alpha(\lambda)}}{e_\sigma^{--}} + \frac{\lambda_\alpha^+ P_-^{\alpha(\lambda)}}{e_\sigma^{++}}\right] \quad (7.5.47)$$

$$\tilde{\Phi}_{--} := \Phi_{--} - \partial_\sigma P_{--}^\sigma + 2\Omega_\sigma^{(0)}P_{--}^\sigma - 2e_\sigma^{++}\mathcal{N}$$

$$-\frac{1}{e_\sigma^{--}}(\partial_\sigma\theta\lambda^+)(\lambda^{-\alpha}\mathcal{D}_\alpha)+\frac{1}{e_\sigma^{--}}(\partial_\sigma\lambda_\alpha^+ +\Omega_\sigma^{(0)}\lambda_\alpha^+)P_+^{\alpha(\lambda)}+$$
$$+\Big[\lambda^{+\alpha}\lambda^{+\beta}-\frac{2}{e_\sigma^{--}}\big(\lambda_\gamma^+\Pi_\sigma^{\gamma\alpha}\lambda^{-\beta}-(\lambda^+\Pi_\sigma\lambda^+)\lambda^{-\alpha}\lambda^{-\beta}+$$
$$+(\lambda^+\Pi_\sigma\lambda^-)\lambda^{+\alpha}\lambda^{-\beta}\big)\Big]\mathcal{P}_{\alpha\beta}$$
$$+\frac{1}{2e_\sigma^{--}}\Big[-e_\sigma^{--}\Omega_\sigma^{++}-e_\sigma^{++}\Omega_\sigma^{--}+i\partial_\sigma\theta\lambda^+\partial_\sigma\theta\lambda^-$$
$$-\Pi_\sigma^{\alpha\beta}(\partial_\sigma\lambda_\alpha^+\lambda_\beta^- -\lambda_\alpha^+\partial_\sigma\lambda_\beta^-)\Big]\left[\frac{\lambda_\alpha^-P_+^{\alpha(\lambda)}}{e_\sigma^{--}}+\frac{\lambda_\alpha^+P_-^{\alpha(\lambda)}}{e_\sigma^{++}}\right] \quad (7.5.48)$$

and

$$P_{\pm\pm}^\tau \approx 0\,. \quad (7.5.49)$$

In equations (7.5.47), (7.5.48) the relation

$$\delta_\alpha{}^\beta \approx \lambda_\alpha^+\lambda^{-\beta}-\lambda_\alpha^-\lambda^{+\beta}-u_\alpha^I u^{J\beta}C_{IJ}\,, \quad (7.5.50)$$
$$\lambda^{\pm\beta}:=C^{\beta\alpha}\lambda_\alpha^\pm\,,\quad u^{I\beta}:=C^{\beta\alpha}u_\alpha^I\,, \quad (7.5.51)$$

has been used to remove the auxiliary variables u_α^I in all places where it is possible. Note that (7.5.50) is a consequence of the constraint (7.5.13) and of the definition of the u_α^I spinors, equations (7.3.12), (7.3.13) (see further discussion on the use of u variables below). Thus we are allowed to use them in the solution of the equation for the Lagrange multipliers and, then, in the definition of the first class constraints, as the product of any two constraints is a first class one since its Poisson brackets with any other constraint vanishes weakly.

Using the Poisson brackets (7.5.7), the first class constraints generate gauge symmetries. In our dynamical system the fermionic first class constraints (7.5.45) are the generators of the $(n-2)$–parametric κ–symmetry (7.4.1)–(7.4.3). The $\mathcal{P}^{IJ}$ in equation (7.5.44) are the $\frac{1}{2}(n-1)(n-2)$ generators of the b-symmetry (7.4.5). The constraint $G^{(0)}$ (7.5.46) generates the $SO(1,1)$ gauge symmetry (7.2.10). Finally, the constraints $\tilde{\Phi}_{\pm\pm}$, equations (7.5.47), (7.5.48), generate worldvolume reparameterizations. They provide a counterpart of the Virasoro constraints characteristic of the Green–Schwarz superstring action. Thus, as it could be expected, our $\Sigma^{(\frac{n(n+1)}{2}|n)}$ superstring is a two-dimensional conformal field theory. As it was noted above, the presence of the first class constraints (7.5.49) indicates the pure gauge nature of the fields $e_\tau^{\pm\pm}(\xi)$; the freedom of the gauge fixing is, nevertheless, restricted by the 'topological' conditions (7.5.21).

Note that the κ–symmetry and b–symmetry generators, in (7.5.45) and (7.5.44), are the u_α^I and $u_\alpha^I u_\beta^J$ components of equation (7.5.9) and equation (7.5.8), respectively, while all other first class constraints can be defined without any reference to auxiliary variables. The use of the auxiliary spinors $u^I_\alpha(\xi)$ to define the first class constraints requires some discussion. Any spinor can be decomposed in the basis (7.3.14), but the use of u^I_α to define constraints requires, to be rigorous, to consider them as (auxiliary) dynamical variables, to introduce momenta, and to take into account any additional constraints for them, including equations (7.3.13) and the vanishing of the momenta conjugate to u^I_α (*cf.* [208]).

An alternative is to consider these auxiliary spinors as defined by (7.3.12), (7.3.13) and by the gauge symmetries of these constraints, *i.e.* to treat them as some implicit functions of $\lambda^\pm_\alpha$ (*cf.* [209]). Such a description can be obtained rigorously by the successive gauge fixings of all the additional gauge symmetries that act only on u^I_α and by introducing Dirac brackets accounting for all the second class constraints for the u^I_α variables. Nevertheless, with some precautions, the above simpler alternative can be used from the beginning. In this case, one has to keep in mind, in particular, that the u^I_α's do not commute with $P^{\alpha(\lambda)}_\pm$. Indeed, as conditions (7.3.12) have to be treated in a strong sense, one has to assume $[P^{\alpha(\lambda)}_\pm(\sigma), u^I_\beta(\sigma')]_P \approx \pm\lambda^\pm_\beta C^{\alpha\gamma} u^I_\gamma \delta(\sigma-\sigma')$. However, one notices that this does not change the result of the analysis of the number of first and second class constraints among equations (7.5.8)–(7.5.13), (7.5.17), (7.5.18), which do not involve $u^I_\gamma(\xi)$. The reason is that one only uses $u^I_\gamma(\xi)$ as multipliers needed to extract the first and second class constraints from the mixed ones (7.5.8), (7.5.9). Thus, the Poisson brackets of the projected constraints $\mathcal{P}_{\alpha\beta}u^{\alpha I}u^{\beta J}$, $u^{\alpha I}\mathcal{D}_\alpha$ with other constraints (*e.g.*, $[\mathcal{P}_{\alpha\beta}u^{\alpha I}u^{\beta J}, \ldots]_P$) and the projected Poisson brackets of the original constraints $\mathcal{P}_{\alpha\beta}$, $\mathcal{D}_\alpha$ with the same ones (*e.g.*, $u^{\alpha I}u^{\beta J}[\mathcal{P}_{\alpha\beta}, \ldots]_P$) are equivalent in the sense that a non-zero difference ($[\mathcal{P}_{\alpha\beta}u^{\alpha I}u^{\beta J}, \ldots]_P - u^{\alpha I}u^{\beta J}[\mathcal{P}_{\alpha\beta}, \ldots]_P$) will be proportional to $\mathcal{P}_{\alpha\beta}$ or $\mathcal{D}_\alpha$ and, hence, will vanish weakly. This observation allows us to use the basis (7.3.14) to solve the equations (7.5.22)–(7.5.28), that is to say, to decompose the constraints (7.5.8)–(7.5.13), (7.5.17), (7.5.18) into first and second class ones, without introducing momenta for the $u^I_\gamma(\xi)$ and without studying the constraints restricting these variables.

The remaining constraints are second class. In particular, these are the $\lambda^\pm$ components of the fermionic constraints (7.5.9),

$$\mathcal{D}^\pm = \mathcal{D}_\alpha \lambda^{\pm\alpha} = \pi_\alpha \lambda^{\pm\alpha} + ie^{\pm\pm}_\sigma \theta^\beta \lambda^\mp_\beta \approx 0 \qquad (7.5.52)$$

with Poisson brackets

$$\{\mathcal{D}^+(\sigma), \mathcal{D}^+(\sigma')\}_P \approx +2ie^{++}_\sigma \delta(\sigma-\sigma')\ ,$$
$$\{\mathcal{D}^+(\sigma), \mathcal{D}^+(\sigma')\}_P \approx -2ie^{--}_\sigma \delta(\sigma-\sigma')\ ,$$

$$\{\mathcal{D}^+(\sigma), \mathcal{D}^+(\sigma')\}_P \approx 0 \qquad (7.5.53)$$

(recall that, having in mind the possibility of fixing the conformal gauge (7.4.7), we assume nondegeneracy of $e_\sigma^{\pm\pm}(\sigma)$, *i.e.* that the expression $1/e_\sigma^{\pm\pm}(\sigma)$ is well defined). The selection of the basic second class constraints and the simplification of their Poisson bracket algebra is a technically involved problem.

In the next section we show that the dynamical degrees of freedom of our superstring in $\Sigma^{(\frac{n(n+1)}{2}|n)}$, may be presented in a more economic way in terms of constrained $OSp(2n|1)$ supertwistors. The Hamiltonian mechanics also simplifies in this symplectic supertwistor formulation. In particular, all the first class constraints can be extracted without using the auxiliary fields u_α^I. The reason is that the supertwistor variables are invariant under both κ– and b–symmetry. Thus, moving to the twistor form of our action means rewriting it in terms of trivially κ– and b–invariant quantities, effectively removing all variables that transform non-trivially under these gauge symmetries. Since the description of κ– and b–symmetries is the one requiring the introduction of the $u_\alpha^I(\xi)$ fields, it is natural that these are not needed in the supertwistor Hamiltonian approach.

This consideration already allows us to calculate the number of the (field theoretical worldsheet) degrees of freedom of our superstring model [8]. The dynamical system described by the action (7.2.1) possesses $\frac{1}{2}(n-1)(n-2)+5$ bosonic first class constraints (equations (7.5.44), (7.5.46), (7.5.47), (7.5.48) and (7.5.49)) out of a total number of $\frac{1}{2}n(n+1)+2n+8$ constraints (equations (7.5.8), (7.5.10), (7.5.11), (7.5.12), (7.5.13), (7.5.17) and (7.5.18)). This leaves $4n+2$ bosonic second class constraints. Since the phase space dimension corresponding to the worldvolume bosonic fields $\mathcal{Z}^{\mathcal{M}}(\tau,\sigma) = (X^{\alpha\beta}, \lambda_\alpha^\pm, e_\sigma^{\pm\pm}, e_\tau^{\pm\pm})$ is $2(\frac{n(n+1)}{2}+2n+4)$, the action (7.2.1) turns out to have $(4n-6)$ bosonic degrees of freedom.

Likewise, the $(n-2)$ fermionic first class constraints (7.5.45) and the 2 fermionic second class constraints, equations (7.5.52), reduce the original $2n$ phase space fermionic degrees of freedom of the action (7.2.1) down to 2.

Thus our supersymmetric string model in $\Sigma^{(\frac{n(n+1)}{2}|n)}$ superspace carries $(4n-6)$ bosonic and 2 fermionic worldvolume field theoretical degrees of freedom. Treating the number n as the number of components of an irreducible spinor representation of the D-dimensional Lorentz group $SO(1,D-1)$, one finds [8]

D	n	$\#_{bosonic\ d.o.f.}$ $= 4n-6$	$\#_{fermionic\ d.o.f.}$ $= 2$	BPS states
3	2	2	2	NO
4	4	10	2	1/2
6	8	26	2	6/8
10	16	58	2	14/16
11	32	122	2	30/32

Table 7.1. Bosonic and fermionic degrees of freedom of the model in various dimensions

Thus, the number of bosonic degrees of freedom of our $\Sigma^{(\frac{n(n+1)}{2}|n)}$ superstring model exceeds that of the Green–Schwarz superstring (where it exists, $4n-6 > 2(D-2)$), while the number of fermionic dimensions, 2, is smaller than that of the Green–Schwarz superstring for $D = 6, 10$. Note that here the #(bosons) = #(fermions) rule is not valid. The additional bosonic degrees of freedom might be treated as higher spin degrees of freedom and/or as corresponding to the additional 'brane' central charges in the maximal supersymmetry algebra (2.1.10). The smaller number of physical fermionic degrees of freedom just reflects the presence of extra κ–symmetries ($(n-2) > n/2$ for $n > 4$) in our $\Sigma^{(528|32)}$ superstring model. Our $\Sigma^{(\frac{n(n+1)}{2}|n)}$ superstring model describes, as argued, the excitations of a BPS state preserving $k = (n-2)$ supersymmetries (a $\frac{30}{32}$ BPS state for the $D = 11$ superstring in $\Sigma^{(528|32)}$).

The search for solitonic solutions of the usual $D = 11$ and $D = 10$ Type II supergravities preserving exotic fractions of supersymmetry is a subject of recent interest. If successful, it would be interesting to study how the additional bosonic degrees of freedom of our model are mapped into the moduli of these solutions, presumably related to the gauge fields of the supergravity multiplet (*cf.* [85]). Nevertheless, if it were shown that such solutions do not appear in the standard $D = 11$ supergravity, this could indicate that M Theory does require an extension of the usual superspace for its adequate description.

To conclude this section we comment on the BPS preon interpretation of our model. In agreement with [83], it can be argued to describe a composite of $\tilde{n} = n - k = 2$ BPS preons. To support this conclusion one can have a look at the constraint (7.5.8). As we have shown, it is a mixture of first and second class constraints. However, performing a 'conversion' of the second class constraints [210] to obtain first class constraints (in a way similar to the one carried out for a point–like model in [155]), one arrives at the first class constraint

$$\mathcal{P}_{\alpha\beta} = P_{\alpha\beta} + e_\sigma^{++}\tilde{\lambda}_\alpha^-\tilde{\lambda}_\beta^- - e_\sigma^{--}\tilde{\lambda}_\alpha^+\tilde{\lambda}_\beta^+ \approx 0 \,, \qquad (7.5.54)$$

where the $\tilde{\lambda}^{\pm}_{\alpha}$ are related to $\lambda^{\pm}_{\alpha}$. In the quantum theory this first class constraint can be imposed on quantum states giving rise to a relation similar to equation (4.1.8) with $\tilde{n} = n - k = 2$.

7.6 Supertwistor form of the action

Further analysis of the Hamiltonian mechanics of our $\Sigma^{(\frac{n(n+1)}{2}|n)}$ superstring model would become quite involved. Instead, we present in this section a more economic description of the system.

The action (7.2.1) can be rewritten ($\alpha' = 1$) in the form [8]

$$\begin{aligned} S = \int_{W^2} [& e^{++} \wedge (d\mu^{-\alpha}\lambda^{-}_{\alpha} - \mu^{-\alpha} d\lambda^{-}_{\alpha} - i d\eta^{-}\eta^{-}) \\ & - e^{--} \wedge (d\mu^{+\alpha}\lambda^{+}_{\alpha} - \mu^{+\alpha} d\lambda^{+}_{\alpha} - i d\eta^{+}\eta^{+}) \\ & - e^{++} \wedge e^{--}] \,, \end{aligned} \tag{7.6.1}$$

where the bosonic $\mu^{\pm\alpha}$ and the fermionic $\eta^{\pm}$ are defined by

$$\mu^{\pm\alpha} = X^{\alpha\beta}\lambda^{\pm}_{\beta} - \frac{i}{2}\theta^{\alpha}\theta^{\beta}\lambda^{\pm}_{\beta} \,, \quad \eta^{\pm} = \theta^{\beta}\lambda^{\pm}_{\beta} \,. \tag{7.6.2}$$

Equations (7.6.2) are reminiscent of the Ferber generalization [211] of the Penrose correspondence relation [212] (see also [83, 148]). The two sets of $2n+1$ variables belonging to the same real one-dimensional (Majorana–Weyl spinor) representation of the worldsheet Lorentz group $SO(1,1)$,

$$(\mu^{+\alpha}, \lambda^{+}_{\alpha}, \eta^{+}) := Y^{+\Sigma} \,\,, \,\, (\mu^{-\alpha}, \lambda^{-}_{\alpha}, \eta^{-}) := Y^{-\Sigma} \,, \tag{7.6.3}$$

can be treated as the components of two $OSp(2n|1)$ supertwistors, $Y^{+\Sigma}$ and $Y^{-\Sigma}$. However, equations (7.6.2) considered together imply the following constraint:

$$\lambda^{+}_{\alpha}\mu^{-\alpha} - \lambda^{-}_{\alpha}\mu^{+\alpha} - i\eta^{-}\eta^{+} = 0 \,. \tag{7.6.4}$$

One has to consider as well the 'kinematic' constraint (7.2.3), which breaks $GL(n,\mathbb{R})$ down to $Sp(n,\mathbb{R})$. In terms of the two supertwistors $Y^{\pm\Sigma}$ the action (7.2.1) describing our tensionful string model and the constraints (7.2.3), (7.6.4) can be written as follows[8] [8]

$$\begin{aligned} S = \int_{W^2} [& e^{++} \wedge dY^{-\Sigma}\,\Omega_{\Sigma\Pi} Y^{-\Pi} \\ & - e^{--} \wedge dY^{+\Sigma}\,\Omega_{\Sigma\Pi} Y^{+\Pi} \; - \; e^{++} \wedge e^{--}] \,; \end{aligned} \tag{7.6.5}$$

[8]See [213] for a recent construction of massive particle actions in terms of only one supertwistor.

$$Y^{+\Sigma}\, C_{\Sigma\Pi} Y^{-\Pi} = 1 \,, \tag{7.6.6}$$
$$Y^{+\Sigma}\, \Omega_{\Sigma\Pi} Y^{-\Pi} = 0 \,, \tag{7.6.7}$$

where the nondegenerate matrix $\Omega_{\Sigma\Pi} = -(-1)^{\deg(\pm\Sigma)\deg(\pm\Pi)}\Omega_{\Pi\Sigma}$ is the orthosymplectic metric,

$$\Omega_{\Sigma\Pi} = \begin{pmatrix} 0 & \delta_\alpha{}^\beta & 0 \\ -\delta_\beta{}^\alpha & 0 & 0 \\ 0 & 0 & -i \end{pmatrix} \quad , \tag{7.6.8}$$

preserved by $OSp(2n|1)$. The degenerate matrix $C_{\Sigma\Pi}$ in equation (7.6.6) has the form

$$C_{\Sigma\Pi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C^{\alpha\beta} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.6.9}$$

with $C^{\alpha\beta}$ defined in (7.2.5).

One can also find the orthosymplectic twistor form for the action (7.6.1) with unconstrained spinors. It reads [8]

$$\begin{aligned} S = \int_{W^2} [& e^{++} \wedge (d\mathcal{M}^{-\alpha}\Lambda^-_\alpha - \mathcal{M}^{-\alpha} d\Lambda^-_\alpha - id\chi^-\chi^-) \\ & -e^{--} \wedge (d\mathcal{M}^{+\alpha}\Lambda^+_\alpha - \mathcal{M}^{+\alpha} d\Lambda^+_\alpha - id\chi^+\chi^+) \\ & -e^{++} \wedge e^{--}(C^{\alpha\beta}\Lambda^+_\alpha\Lambda^-_\beta)^2] \,, \end{aligned} \tag{7.6.10}$$

where

$$\mathcal{M}^{\pm\alpha} = X^{\alpha\beta}\Lambda^\pm_\beta - \frac{i}{2}\theta^\alpha\theta^\beta\Lambda^\pm_\beta \,, \quad \chi^\pm = \theta^\beta\Lambda^\pm_\beta \,. \tag{7.6.11}$$

Equation (7.6.11) differs from (7.6.2) only by replacement of the constrained dimensionless $\lambda^\pm$ by the unconstrained dimensionful $\Lambda^\pm$. But, as a result, the $OSp(2n|1)$ supertwistors

$$\Upsilon^{\pm\Sigma} := (\mathcal{M}^{\pm\alpha}, \Lambda^\pm_\alpha, \chi^\pm) \,, \tag{7.6.12}$$

are restricted by only one condition similar to (7.6.7),

$$\Upsilon^{+\Sigma}\, \Omega_{\Sigma\Pi} \Upsilon^{-\Pi} = 0 \,. \tag{7.6.13}$$

The action in terms of $\Upsilon^{\pm\Sigma}$ includes the degenerate matrix $C_{\Sigma\Pi}$, and reads [8]

$$\begin{aligned} S = \int_{W^2} [& e^{++} \wedge d\Upsilon^{-\Sigma}\, \Omega_{\Sigma\Pi} \Upsilon^{-\Pi} - e^{--} \wedge d\Upsilon^{+\Sigma}\, \Omega_{\Sigma\Pi} \Upsilon^{+\Pi} \\ & - e^{++} \wedge e^{--}\, (\Upsilon^{+\Sigma}\, C_{\Sigma\Pi} \Upsilon^{-\Pi})^2] \,. \end{aligned} \tag{7.6.14}$$

The orthosymplectic supertwistors $\Upsilon^{\pm\Sigma}$ are both in the fundamental representation of the $OSp(2n|1)$ supergroup. The constraints (7.6.7) (or (7.6.13)) are also $OSp(2n|1)$ invariant. However, condition (7.6.6) (or the last term in the action (7.6.14)) breaks the $OSp(2n|1)$ invariance down to the semidirect product $\Sigma^{(\frac{n(n+1)}{2}|n)} \rtimes Sp(n,\mathbb{R})$, generalizing superPoincaré, of $Sp(n,\mathbb{R}) \subset Sp(2n,\mathbb{R})$ and the maximal superspace group $\Sigma^{(\frac{n(n+1)}{2}|n)}$ (see appendix C). In contrast, both the point–like model in [148] and the tensionless superbrane model of [149] possess full $OSp(2n|1)$ symmetry. This is in agreement with treating $OSp(2n|1)$ as a generalized superconformal group, as the standard conformal and superconformal symmetry is broken in any model with mass, tension or another dimensionful parameter.

7.7 Hamiltonian analysis in the supertwistor formulation

The Hamiltonian analysis simplifies in the supertwistor formulation (7.6.5) of the action (7.2.1) [8]. This is due to the fact that moving from (7.2.1) to (7.6.5) reduces the number of fields involved in the model.

The Lagrangian of the action (7.6.5) reads

$$\begin{aligned}\mathcal{L} = &(e_\tau^{++}\partial_\sigma Y^{-\Sigma} - e_\sigma^{++}\partial_\tau Y^{-\Sigma})\Omega_{\Sigma\Pi}Y^{-\Pi} \\ &-(e_\tau^{--}\partial_\sigma Y^{+\Sigma} - e_\sigma^{--}\partial_\tau Y^{+\Sigma})\Omega_{\Sigma\Pi}Y^{+\Pi} \\ &-(e_\tau^{++}e_\sigma^{--} - e_\sigma^{++}e_\tau^{--}) \, ,\end{aligned} \tag{7.7.1}$$

and involves the $2(2n+1+2) = 4n+6$ configuration space worldvolume fields

$$\widetilde{\mathcal{Z}}^{\tilde{\mathcal{M}}} \equiv \widetilde{\mathcal{Z}}^{\tilde{\mathcal{M}}}(\tau,\sigma) = \left(Y^{\pm\Sigma} \, , \, e_\tau^{\pm\pm} \, , \, e_\sigma^{\pm\pm}\right) \, . \tag{7.7.2}$$

The calculation of their canonical momenta

$$\widetilde{P}_{\tilde{\mathcal{M}}} = (P_{\pm\Sigma} \, , \, P^\tau_{\pm\pm} \, , \, P^\sigma_{\pm\pm}) = \frac{\partial\mathcal{L}}{\partial(\partial_\tau \widetilde{\mathcal{Z}}^{\tilde{\mathcal{M}}})} \tag{7.7.3}$$

provides the following set of primary constraints:

$$\mathcal{P}_{\pm\Sigma} = P_{\pm\Sigma} \mp e_\sigma^{\mp\mp}\Omega_{\Sigma\Pi}Y^{\pm\Pi} \approx 0 \, , \tag{7.7.4}$$

$$P^\sigma_{\pm\pm} \approx 0 \, , \tag{7.7.5}$$

$$P^\tau_{\pm\pm} \approx 0 \, . \tag{7.7.6}$$

Conditions (7.6.7), (7.6.6) should also be taken into account after all the Poisson brackets are calculated and, hence, are also primary constraints,

$$\mathcal{U} := Y^{+\Sigma}\,\Omega_{\Sigma\Pi}Y^{-\Pi} \approx 0 \, , \tag{7.7.7}$$

$$\mathcal{N} := Y^{+\Sigma} C_{\Sigma\Pi} Y^{-\Pi} - 1 \approx 0 \,. \tag{7.7.8}$$

The *canonical* Hamiltonian density $\mathcal{H}_0$ corresponding to the action (7.6.5), reads

$$\begin{aligned} \mathcal{H}_0 = & [-e_\tau^{++}\partial_\sigma Y^{-\Sigma}\Omega_{\Sigma\Pi} Y^{-\Pi} + e_\tau^{--}\partial_\sigma Y^{+\Sigma}\Omega_{\Sigma\Pi} Y^{+\Pi} \\ & +(e_\tau^{++}e_\sigma^{--} - e_\sigma^{++}e_\tau^{--})] \,. \end{aligned} \tag{7.7.9}$$

The preservation of the primary constraints under τ–evolution (see section 7.5) leads to the secondary constraints

$$\Phi_{++} = \partial_\sigma Y^{-\Sigma}\Omega_{\Sigma\Pi} Y^{-\Pi} - e_\sigma^{--} \approx 0 \,, \tag{7.7.10}$$
$$\Phi_{--} = \partial_\sigma Y^{+\Sigma}\Omega_{\Sigma\Pi} Y^{+\Pi} - e_\sigma^{++} \approx 0 \,, \tag{7.7.11}$$
$$\Phi^{(0)} = \partial_\sigma Y^{+\Sigma}\Omega_{\Sigma\Pi} Y^{-\Pi} - Y^{+\Sigma}\Omega_{\Sigma\Pi}\partial_\sigma Y^{-\Pi} \approx 0 \,. \tag{7.7.12}$$

Again (see section 7.5) the canonical Hamiltonian vanishes on the surface of constraints (7.7.10), (7.7.11), and thus the τ–evolution is defined by the Hamiltonian density (*cf.* (7.5.20))

$$\begin{aligned} \mathcal{H}' = & -e_\tau^{++}\Phi_{++} + e_\tau^{--}\Phi_{--} + l^{(0)}\Phi^{(0)} + L^{\pm\Sigma}\mathcal{P}_{\pm\Sigma} + \\ & +L^{(0)}\mathcal{U} + L^{(n)}\mathcal{N} + L^{\pm\pm}P^\sigma_{\pm\pm} + h^{\pm\pm}P^\tau_{\pm\pm} \end{aligned} \tag{7.7.13}$$

and the canonical Poisson brackets

$$[P_{\pm\Lambda}(\sigma)\,,\, Y^{\pm\Sigma}(\sigma')\}_P = -\delta_\Lambda^\Sigma\, \delta(\sigma-\sigma'), \tag{7.7.14}$$
$$[e_\sigma^{\pm\pm}(\sigma)\,,\, P^\sigma_{\pm\pm}(\sigma')]_P = \delta(\sigma-\sigma') \,, \tag{7.7.15}$$
$$[e_\tau^{\pm\pm}(\sigma)\,,\, P^\tau_{\pm\pm}(\sigma')]_P = \delta(\sigma-\sigma') \,, \tag{7.7.16}$$

Then the τ–preservation requirement of the primary and secondary constraints results in the following system of equations for the Lagrange multipliers

$$\begin{aligned} L^{+\Sigma} \approx & \frac{e_\tau^{--}}{e_\sigma^{--}}\partial_\sigma Y^{+\Sigma} + \frac{\partial_\sigma e_\tau^{--} - L^{--}}{2e_\sigma^{--}} Y^{+\Sigma} + \frac{l^{(0)}}{e_\sigma^{--}}\partial_\sigma Y^{-\Sigma} \\ & + \frac{\partial_\sigma l^{(0)} - L^{(0)}}{2e_\sigma^{--}} Y^{-\Sigma} - \frac{L^{(n)}}{2e_\sigma^{--}} Y^{-\Pi}(C\Omega)_\Pi{}^\Sigma \,, \end{aligned} \tag{7.7.17}$$

$$\begin{aligned} L^{-\Sigma} \approx & \frac{e_\tau^{++}}{e_\sigma^{++}}\partial_\sigma Y^{-\Sigma} + \frac{\partial_\sigma e_\tau^{++} - L^{++}}{2e_\sigma^{++}} Y^{-\Sigma} - \frac{l^{(0)}}{e_\sigma^{++}}\partial_\sigma Y^{+\Sigma} \\ & - \frac{\partial_\sigma l^{(0)} + L^{(0)}}{2e_\sigma^{++}} Y^{+\Sigma} - \frac{L^{(n)}}{2e_\sigma^{++}} Y^{+\Pi}(C\Omega)_\Pi{}^\Sigma \,, \end{aligned} \tag{7.7.18}$$

$$L^{+\Sigma}\Omega_{\Sigma\Pi}Y^{-\Pi} \approx L^{-\Sigma}\Omega_{\Sigma\Pi}Y^{+\Pi} \ , \tag{7.7.19}$$
$$L^{+\Sigma}C_{\Sigma\Pi}Y^{-\Pi} \approx L^{-\Sigma}C_{\Sigma\Pi}Y^{+\Pi} \ , \tag{7.7.20}$$
$$L^{-\Sigma}\Omega_{\Sigma\Pi}Y^{-\Pi} \approx e_\tau^{--} \ , \tag{7.7.21}$$
$$L^{+\Sigma}\Omega_{\Sigma\Pi}Y^{+\Pi} \approx e_\tau^{++} \ , \tag{7.7.22}$$

and

$$L^{--} \approx \partial_\sigma L^{-\Sigma}\Omega_{\Sigma\Pi}Y^{-\Pi} - L^{-\Sigma}\Omega_{\Sigma\Pi}\partial_\sigma Y^{-\Pi} \ , \tag{7.7.23}$$
$$L^{++} \approx \partial_\sigma L^{+\Sigma}\Omega_{\Sigma\Pi}Y^{+\Pi} - L^{+\Sigma}\Omega_{\Sigma\Pi}\partial_\sigma Y^{+\Pi} \ , \tag{7.7.24}$$
$$\sum_{\pm}\left(\partial_\sigma L^{\pm\Sigma}\Omega_{\Sigma\Pi}Y^{\mp\Sigma} - L^{\pm\Sigma}\Omega_{\Sigma\Pi}\partial_\sigma Y^{\mp\Sigma}\right) \approx 0 \ . \tag{7.7.25}$$

where $(C\Omega)_\Pi{}^\Sigma := C_{\Pi\Lambda}\Omega^{\Lambda\Sigma}$ and $\Omega^{\Sigma\Pi} = -\Omega_{\Sigma\Pi}$ is the inverse of the orthosymplectic metric (7.6.8),

$$\Omega_{\Sigma\Lambda}\Omega^{\Lambda\Pi} = \delta_\Sigma^\Pi \ , \quad \Omega^{\Sigma\Pi} = \begin{pmatrix} 0 & -\delta_\beta{}^\alpha & 0 \\ \delta_\alpha{}^\beta & 0 & 0 \\ 0 & 0 & i \end{pmatrix} \ . \tag{7.7.26}$$

Equations (7.7.17)–(7.7.22) come from the preservation of the primary constraints, while equations (7.7.23)–(7.7.25) from the preservation of the secondary constraints. Again, as in section 7.5, one can follow the appearance of the secondary constraint (7.7.12) by considering equations (7.7.17)–(7.7.22) with $l^{(0)} = 0$. Denoting

$$A_\sigma^{(0)} = \frac{1}{2}\left(\partial_\sigma Y^{+\Sigma}C_{\Sigma\Pi}Y^{-\Pi} - Y^{+\Sigma}C_{\Sigma\Pi}\partial_\sigma Y^{-\Pi}\right) \ , \tag{7.7.27}$$
$$A_\sigma^{++} = \partial_\sigma Y^{+\Sigma}C_{\Sigma\Pi}Y^{+\Pi} \ , \tag{7.7.28}$$
$$A_\sigma^{--} = \partial_\sigma Y^{-\Sigma}C_{\Sigma\Pi}Y^{-\Pi} \ , \tag{7.7.29}$$
$$B^{(0)} = \mathcal{S} - \frac{\partial_\sigma Y^+\Omega\partial_\sigma Y^-}{2e_\sigma^{++}e_\sigma^{--}} \ , \tag{7.7.30}$$
$$\mathcal{S} = \frac{1}{2}\left(\frac{A_\sigma^{++}}{e_\sigma^{++}} + \frac{A_\sigma^{--}}{e_\sigma^{--}}\right) \ , \tag{7.7.31}$$

one can write the general solution of equations (7.7.17)–(7.7.22) in the form

$$L^{+\Sigma} \approx \omega^{(0)}Y^{+\Sigma} + \\ +\frac{e_\tau^{--}}{e_\sigma^{--}}\left(\partial_\sigma Y^{+\Sigma} - A_\sigma^{(0)}Y^{+\Sigma} - e_\sigma^{++}B^{(0)}Y^{-\Sigma} + e_\sigma^{++}(Y^-C\Omega)^\Sigma\right)$$

$$+e_\tau^{++}\left(B^{(0)}Y^{-\Sigma} - (Y^- C\Omega)^{\Sigma}\right) , \tag{7.7.32}$$

$$\begin{aligned} L^{-\Sigma} \approx & -\omega^{(0)}Y^{-\Sigma} \\ & +\frac{e_\tau^{++}}{e_\sigma^{++}}\left(\partial_\sigma Y^{-\Sigma} + A_\sigma^{(0)}Y^{-\Sigma} + e_\sigma^{--}B^{(0)}Y^{+\Sigma} - e_\sigma^{--}(Y^+C\Omega)^{\Sigma}\right) \\ & -e_\tau^{--}\left(B^{(0)}Y^{+\Sigma} - (Y^+C\Omega)^{\Sigma}\right) , \end{aligned} \tag{7.7.33}$$

$$L^{(0)} = 2(e_\tau^{--}e_\sigma^{++} - e_\tau^{++}e_\sigma^{--})\, B^{(0)} , \tag{7.7.34}$$

$$L^{\pm\pm} = \partial_\sigma e_\tau^{\pm\pm} \mp 2e_\tau^{\pm\pm}A_\sigma^{(0)} \pm 2e_\sigma^{\pm\pm}\omega^{(0)} , \tag{7.7.35}$$

$$L^{(n)} = -4\det(e_m^a) \equiv -2(e_\tau^{--}e_\sigma^{++} - e_\tau^{++}e_\sigma^{--}) , \tag{7.7.36}$$

$$l^{(0)} = 0 . \tag{7.7.37}$$

Note that equations (7.7.36), (7.7.37) have the same form as (7.5.38), (7.5.39) for the Lagrange multipliers in the original formulation, and equations (7.7.35) are similar to equations (7.5.37).

The above solution contains the indefinite worldsheet field parameters $h^{\pm\pm}(\xi)$, $\omega^{(0)}(\xi)$ and $e_\tau^{\pm\pm}(\xi)$ corresponding to the five first class constraints which generate the gauge symmetries of the symplectic twistor formulation of our $\Sigma^{(\frac{n(n+1)}{2}|n)}$ superstring model. They are

$$P^\tau_{\pm\pm} \approx 0 \tag{7.7.38}$$

and

$$G^{(0)} := Y^{+\Sigma}\mathcal{P}_{+\Sigma} - Y^{-\Sigma}\mathcal{P}_{-\Sigma} + 2e_\sigma^{++}P^\sigma_{++} - 2e_\sigma^{--}P^\sigma_{--} \approx 0 , \tag{7.7.39}$$

$$\begin{aligned} \tilde{\Phi}_{++} := & \Phi_{++} + \partial_\sigma P^\sigma_{++} + 2A_\sigma^{(0)}P^\sigma_{++} + 2e_\sigma^{--}B^{(0)}\mathcal{U} \\ & -2e_\sigma^{--}\mathcal{N} + \mathcal{F}_{++}^{\pm\Sigma}\mathcal{P}_{\pm\Sigma} , \end{aligned} \tag{7.7.40}$$

$$\begin{aligned} \tilde{\Phi}_{--} := & \Phi_{--} - \partial_\sigma P^\sigma_{--} + 2A_\sigma^{(0)}P^\sigma_{--} + 2e_\sigma^{++}B^{(0)}\mathcal{U} \\ & -2e_\sigma^{++}\mathcal{N} + \mathcal{F}_{--}^{\pm\Sigma}\mathcal{P}_{\pm\Sigma} , \end{aligned} \tag{7.7.41}$$

where

$$\mathcal{F}_{++}^{+\Sigma} = -B^{(0)}Y^{-\Sigma} + (Y^-C\Omega)^{\Sigma} , \tag{7.7.42}$$

$$\mathcal{F}_{++}^{-\Sigma} = -\frac{1}{e_\sigma^{++}}[\partial_\sigma Y^{-\Sigma} + A_\sigma^{(0)}Y^{-\Sigma} + B^{(0)}e_\sigma^{--}Y^{+\Sigma} - e_\sigma^{--}(Y^+C\Omega)^{\Sigma}] , \tag{7.7.43}$$

$$\mathcal{F}_{--}^{+\Sigma} = \frac{1}{e_\sigma^{--}}[\partial_\sigma Y^{+\Sigma} - A_\sigma^{(0)}Y^{+\Sigma} - B^{(0)}e_\sigma^{++}Y^{-\Sigma} + e_\sigma^{++}(Y^-C\Omega)^{\Sigma}], \tag{7.7.44}$$

$$\mathcal{F}_{--}^{-\Sigma} = -B^{(0)}Y^{+\Sigma} + (Y^+C\Omega)^{\Sigma} . \tag{7.7.45}$$

Using Poisson brackets, the constraint (7.7.39) generates the $SO(1,1)$ worldsheet Lorentz gauge symmetry, (7.7.40) and (7.7.41) are the reparameterization (Virasoro) generators, and the symmetry generated by equations (7.7.38) indicates the pure gauge nature of the $e_\tau^{\pm\pm}(\xi)$ fields (again, subject to the nondegeneracy condition (7.5.21) that restricts the gauge choice freedom for them).

Note that both the b–symmetry and the κ–symmetry generators, equations (7.5.44) and (7.5.45), are not present in the symplectic supertwistor formulation. Actually, the number of variables in this formulation minus the constraint among them, equation (7.6.7), is $(4n+6)-1$ and equal to the number of variables in the previous formulation $(\frac{n(n+1)}{2}+n+2n+4)$, minus the number of b– and κ–symmetry generators $(\frac{(n-1)(n-2)}{2}+(n-2))$. This indicates that the transition to the supertwistor form of the action corresponds to an implicit gauge fixing of these symmetries and the removal of the additional variables, since the remaining supertwistor ones are invariant under both b– and κ–symmetry[9].

Other constraints are second class. Indeed, *e.g.* the algebra of the constraints $\mathcal{P}_{\pm\Sigma}$, equation (7.7.4),

$$[\mathcal{P}_{+\Sigma}(\sigma)\,,\,\mathcal{P}_{+\Lambda}(\sigma')\}_P = 2e_\sigma^{--}\Omega_{\Lambda\Sigma}\delta(\sigma-\sigma')\,, \tag{7.7.46}$$

$$[\mathcal{P}_{-\Sigma}(\sigma)\,,\,\mathcal{P}_{-\Lambda}(\sigma')\}_P = -2e_\sigma^{++}\Omega_{\Lambda\Sigma}\delta(\sigma-\sigma')\,, \tag{7.7.47}$$

$$[\mathcal{P}_{+\Sigma}(\sigma)\,,\,\mathcal{P}_{-\Lambda}(\sigma')\}_P = 0\,, \tag{7.7.48}$$

shows their second class nature. As such, one can introduce the graded Dirac (or starred [207]) brackets that allows one to put them strongly equal to zero. For any arbitrary two (bosonic or fermionic) functionals f and g of the canonical variables (7.7.2), (7.7.3) they are defined by

$$\begin{aligned}[f(\sigma_1),g(\sigma_2)\}_D &= [f(\sigma_1),g(\sigma_2)\}_P \\ &-\frac{1}{2}\int d\sigma\left(\frac{1}{e_\sigma^{--}(\sigma)}[f(\sigma_1),\mathcal{P}_{+\Sigma}(\sigma)\}_P\Omega^{\Pi\Sigma}[\mathcal{P}_{+\Pi}(\sigma),g(\sigma_2)\}_P\right. \\ &\left.-\frac{1}{e_\sigma^{++}(\sigma)}[f(\sigma_1),\mathcal{P}_{-\Sigma}(\sigma)\}_P\Omega^{\Pi\Sigma}[\mathcal{P}_{-\Pi}(\sigma),g(\sigma_2)\}_P\right)\,.\end{aligned} \tag{7.7.49}$$

Using these and reducing further the number of phase space degrees of freedom by setting $P_{\pm\Sigma}=0$ strongly, the supertwistor becomes a self-conjugate variable,

$$[Y^{\pm\Sigma}(\sigma),Y^{\pm\Pi}(\sigma')\}_D = \mp\frac{1}{2e_\sigma^{\mp\mp}}\,\Omega^{\Sigma\Pi}\,\delta(\sigma-\sigma')\,. \tag{7.7.50}$$

[9]This invariance was known for the massless superparticle and the tensionless superstring cases, see *e.g.* [148, 149, 155, 214].

For the 'components' of the supertwistor, equation (7.7.50) implies

$$[\lambda^{\pm}_{\alpha}(\sigma), \mu^{\pm\beta}(\sigma')]_D = \mp \frac{1}{2e^{\mp\mp}_{\sigma}} \delta^{\beta}_{\alpha} \delta(\sigma - \sigma') \; , \tag{7.7.51}$$

$$\{\eta^{\pm}(\sigma), \eta^{\pm}(\sigma')\}_D = \mp \frac{i}{2e^{\mp\mp}_{\sigma}} \; \delta(\sigma - \sigma') \; . \tag{7.7.52}$$

The Dirac brackets for $e^{\pm\pm}_{\sigma}$, $e^{\pm\pm}_{\tau}$ and $P^{\tau}_{\pm\pm}$ coincide with the Poisson brackets, while for $P^{\sigma}_{\pm\pm}$ one finds

$$[P^{\sigma}_{++}(\sigma), ...]_D = [P^{\sigma}_{++}(\sigma), ...]_P - \frac{1}{2e^{++}_{\sigma}} Y^{-\Sigma}(\sigma)[\mathcal{P}_{-\Sigma}(\sigma), ...\}_P , \tag{7.7.53}$$

$$[P^{\sigma}_{--}(\sigma), ...]_D = [P^{\sigma}_{--}(\sigma), ...]_P - \frac{1}{2e^{--}_{\sigma}} Y^{+\Sigma}(\sigma)[\mathcal{P}_{+\Sigma}(\sigma), ...\}_P . \tag{7.7.54}$$

However, $P^{\sigma}_{\pm\pm}(\sigma)$ still commute among themselves,

$$[P^{\sigma}_{\pm\pm}(\sigma), P^{\sigma}_{\pm\pm}(\sigma')]_D = 0 = [P^{\sigma}_{++}(\sigma), P^{\sigma}_{--}(\sigma')]_D \; . \tag{7.7.55}$$

When the constraints (7.7.4) are taken as strong equations, the first class constraints (7.7.39)–(7.7.41) simplify to

$$G^{(0)} := 2e^{++}_{\sigma} P^{\sigma}_{++} - 2e^{--}_{\sigma} P^{\sigma}_{--} \approx 0 \, , \tag{7.7.56}$$

$$\tilde{\Phi}_{++} := \Phi_{++} + \partial_{\sigma} P^{\sigma}_{++} + 2A^{(0)}_{\sigma} P^{\sigma}_{++} + 2e^{--}_{\sigma} B^{(0)} \mathcal{U} - 2e^{--}_{\sigma} \mathcal{N} \approx 0, \tag{7.7.57}$$

$$\tilde{\Phi}_{--} := \Phi_{--} - \partial_{\sigma} P^{\sigma}_{--} + 2A^{(0)}_{\sigma} P^{\sigma}_{--} + 2e^{++}_{\sigma} B^{(0)} \mathcal{U} - 2e^{++}_{\sigma} \mathcal{N} \approx 0, \tag{7.7.58}$$

and the remaining second class constraints can be taken in the form

$$K^{(0)} := e^{++}_{\sigma} P^{\sigma}_{++} + e^{--}_{\sigma} P^{\sigma}_{--} \approx 0 \, , \tag{7.7.59}$$

$$\mathcal{N} = Y^{+\Sigma} C_{\Sigma\Pi} Y^{-\Pi} - 1 \approx 0 \; , \tag{7.7.60}$$

$$\mathcal{U} = Y^{+\Sigma} \Omega_{\Sigma\Pi} Y^{-\Pi} \approx 0 \; , \tag{7.7.61}$$

$$\Phi^{(0)} = \partial_{\sigma} Y^{+\Sigma} \Omega_{\Sigma\Pi} Y^{-\Pi} - Y^{+\Sigma} \Omega_{\Sigma\Pi} \partial_{\sigma} Y^{-\Pi} \approx 0 \; . \tag{7.7.62}$$

One has to take into account that, under the Dirac brackets, $P^{\sigma}_{\pm\pm}$ and $Y^{\mp\Sigma}$ do not commute,

$$[P^{\sigma}_{++}(\sigma), Y^{-\Sigma}(\sigma')]_D = \frac{1}{2e^{++}_{\sigma}} Y^{-\Sigma}(\sigma) \delta(\sigma - \sigma') \; , \tag{7.7.63}$$

$$[P^{\sigma}_{--}(\sigma), Y^{+\Sigma}(\sigma')]_D = \frac{1}{2e^{--}_{\sigma}} Y^{+\Sigma}(\sigma) \delta(\sigma - \sigma') \; . \tag{7.7.64}$$

Then one checks that, under Dirac brackets, $G^{(0)}$ generates the $SO(1,1)$ transformations of the supertwistors,

$$[G^{(0)}(\sigma), Y^{\pm\Sigma}(\sigma')]_D = \mp Y^{\pm\Sigma}(\sigma) \delta(\sigma - \sigma') \; . \tag{7.7.65}$$

On the other hand, one finds that the second class constraint $\mathcal{U}$ interchanges the $Y^{+\Sigma}$ and $Y^{-\Sigma}$ supertwistors,

$$[\mathcal{U}(\sigma)\,, Y^{+\Sigma}(\sigma')]_D = \frac{1}{2e_\sigma^{--}} Y^{-\Sigma}(\sigma)\delta(\sigma-\sigma')\ ,$$
$$[\mathcal{U}(\sigma), Y^{-\Sigma}(\sigma')]_D = \frac{1}{2e_\sigma^{++}} Y^{+\Sigma}(\sigma)\delta(\sigma-\sigma')\ . \qquad (7.7.66)$$

It is interesting to note that in the original supertwistor formulation of the $D=4$, $N=1$ superparticle [211] there exists a counterpart of the $\mathcal{U}$ constraint; however, there it is the first class constraint generating the internal $U(1)$ symmetry[10].

The Dirac brackets of the above second class constraints (7.7.59)–(7.7.62) are:

$$\begin{aligned}
&[\Phi^{(0)}(\sigma),\mathcal{U}(\sigma')]_D = \\
= \;& -\frac{1}{2}\left(\frac{\partial_\sigma Y^{+\Sigma}(\sigma)\Omega_{\Sigma\Pi}Y^{+\Pi}(\sigma)}{e_\sigma^{++}(\sigma)} + \frac{\partial_\sigma Y^{-\Sigma}(\sigma)\Omega_{\Sigma\Pi}Y^{-\Pi}(\sigma)}{e_\sigma^{--}(\sigma)}\right)\delta(\sigma-\sigma') \\
= \;& -\frac{1}{2}\left(\frac{\Phi_{++}(\sigma)}{e_\sigma^{++}(\sigma)} + \frac{\Phi_{--}(\sigma)}{e_\sigma^{--}(\sigma)} + 2\right)\delta(\sigma-\sigma') \\
\approx \;& -\delta(\sigma-\sigma')\ ,
\end{aligned} \qquad (7.7.67)$$

$$\begin{aligned}
&[\Phi^{(0)}(\sigma),\mathcal{N}(\sigma')]_D = \\
= \;& -\frac{1}{2}\left(\frac{\partial_\sigma Y^{+\Sigma}(\sigma)C_{\Sigma\Pi}Y^{+\Pi}(\sigma)}{e_\sigma^{++}(\sigma)} + \frac{\partial_\sigma Y^{-\Sigma}(\sigma)C_{\Sigma\Pi}Y^{-\Pi}(\sigma)}{e_\sigma^{--}(\sigma)}\right)\delta(\sigma-\sigma') \\
= \;& -\frac{1}{2}\left(\frac{A_\sigma^{++}(\sigma)}{e_\sigma^{++}(\sigma)} + \frac{A_\sigma^{--}(\sigma)}{e_\sigma^{--}(\sigma)}\right)\delta(\sigma-\sigma') \equiv -\mathcal{S}(\sigma)\delta(\sigma-\sigma')\ ,
\end{aligned} \qquad (7.7.68)$$

$$\begin{aligned}
[K^{(0)}(\sigma),\mathcal{U}(\sigma')]_D &= Y^{+\Sigma}(\sigma)\Omega_{\Sigma\Pi}Y^{-\Pi}(\sigma)\ \delta(\sigma-\sigma') \\
&= \mathcal{U}\ \delta(\sigma-\sigma') \approx 0\ ,
\end{aligned} \qquad (7.7.69)$$

$$\begin{aligned}
[K^{(0)}(\sigma),\mathcal{N}(\sigma')]_D &= Y^{+\Sigma}(\sigma)C_{\Sigma\Pi}Y^{-\Pi}(\sigma)\ \delta(\sigma-\sigma') \\
&= (\mathcal{N}+1)\ \delta(\sigma-\sigma') \approx \delta(\sigma-\sigma')\ ,
\end{aligned} \qquad (7.7.70)$$

$$\begin{aligned}
&[K^{(0)}(\sigma), \Phi^{(0)}(\sigma')]_D = \\
&= \tfrac{1}{2}\left(\partial_\sigma Y^{+\Sigma}(\sigma)\Omega_{\Sigma\Pi}Y^{-\Pi}(\sigma) - Y^{+\Sigma}(\sigma)\Omega_{\Sigma\Pi}\partial_\sigma Y^{-\Pi}(\sigma)\right)\delta(\sigma-\sigma') \\
&= \tfrac{1}{2}\Phi^{(0)}\ \delta(\sigma-\sigma') \approx 0\ ,
\end{aligned} \qquad (7.7.71)$$

[10]See [215] for a detailed study of the Hamiltonian mechanics in the twistor–like formulation of the $D=4$ superparticle, where the possibility of constraint class transmutation was noted.

where, in (7.7.68), $\mathcal{S}(\sigma) \equiv \frac{1}{2}\left(\frac{A_\sigma^{++}(\sigma)}{e_\sigma^{++}(\sigma)} + \frac{A_\sigma^{--}(\sigma)}{e_\sigma^{--}(\sigma)}\right)$ (equation (7.7.31)) and $\delta_{\sigma\,\sigma'} \equiv \delta(\sigma - \sigma')$.

These Dirac brackets (7.7.67)-(7.7.71) can be summarized schematically in the following table

$[\ldots\downarrow, \ldots\rightarrow\}_D \approx$	$(\Phi^{(0)}(\sigma')+$ $+\mathcal{S}K^{(0)}(\sigma'))$	$\mathcal{U}(\sigma')$	$K^{(0)}(\sigma')$	$\mathcal{N}(\sigma')$
$(\Phi^{(0)} + \mathcal{S}K^{(0)})(\sigma)$	0	$-\delta_{\sigma\,\sigma'}$	0	0
$\mathcal{U}(\sigma)$	$\delta_{\sigma\,\sigma'}$	0	0	0
$K^{(0)}(\sigma)$	0	0	0	$\delta_{\sigma\,\sigma'}$
$\mathcal{N}(\sigma)$	0	0	$-\delta_{\sigma\,\sigma'}$	0

Table 7.2. Schematic Dirac brackets of the second class constraints in the supertwistor formalism

This table indicates that the $K^{(0)}$ constraint is canonically conjugate to $\mathcal{N}$ while the second class constraint $\Phi^{(0)} + \mathcal{S}K^{(0)}$ is conjugate to $\mathcal{U}$. One may pass to the (doubly starred) Dirac brackets with respect to the above mentioned four second class constraints. However, the new Dirac brackets for the supertwistor variables would have a very complicated form, so that it looks more practical either to apply the formalism using (singly starred) Dirac brackets (equation (7.7.49)) and simple first and second class constraints, equations (7.7.56)–(7.7.58) and (7.7.59)–(7.7.62), or to search for a conversion [210] of the remaining second class constraints into first class ones. Note that a phenomenon similar to conversion occurs when one moves from (7.6.5) to the dynamical system with unnormalized twistors described by the action (7.6.14). We discuss on this in more detail in the next section.

As the simplest application of the above Hamiltonian analysis let us calculate the number of field theoretical degrees of freedom of the dynamical system (7.6.5). In this supertwistor formulation one finds from equations (7.7.2) and (7.6.3) $(4n + 4)$ bosonic and 2 fermionic configuration space variables, which corresponds to a phase space with $2(4n + 4)$ and 4 fermionic 'dimensions'. The system has 5 bosonic first class constraints, equations (7.7.38)-(7.7.41), out of a total number of $4n + 9$ bosonic constraints (the bosonic components of (7.7.4) and (7.7.5), (7.7.6), (7.7.10)–(7.7.12)). Thus, in agreement with section 7.5, one finds that the $\Sigma^{(\frac{n(n+1)}{2}|n)}$ supersymmetric string described by the action (7.6.5) possesses $4n - 6$ bosonic degrees of freedom. Likewise, the 2 fermionic constraints of the action (the fermionic components of (7.7.4)) reduce to 2 the fermionic degrees of freedom [8].

7.8 Hamiltonian analysis with 'unnormalized' supertwistors

As shown in Section 7.6, the action (7.6.5) may be considered as a gauge fixed form of the action (7.6.14) written in terms of supertwistors (7.6.12) restricted by only one Lagrangian constraint (7.6.13). The second constraint (7.6.6), the 'normalization' condition that distinguishes among the $Y^{\pm\Sigma}$ and $\Upsilon^{\pm\Sigma}$ supertwistors, may be obtained by gauge fixing the direct product of the two scaling gauge symmetries (7.2.8) and (7.2.9) down to the $SO(1,1)$ worldsheet Lorentz symmetry (7.2.10) of the action (7.6.5). As a result, one may expect that the Hamiltonian structure of the model (7.6.14) will differ from the one of the model (7.6.5) by the absence of one second class constraint (7.7.60) and the presence of one additional first class constraint replacing (7.7.59).

This is indeed the case [8]. An analysis similar to the one carried out in Section 7.7 allows one to find the following set of primary

$$\mathcal{P}_{\pm\Sigma} = P_{\pm\Sigma} \mp e_\sigma^{\mp\mp} \Omega_{\Sigma\Pi} \Upsilon^{\pm\Pi} \approx 0 \, , \tag{7.8.1}$$
$$P^\sigma_{\pm\pm} \approx 0 \, , \tag{7.8.2}$$
$$P^\tau_{\pm\pm} \approx 0 \, , \tag{7.8.3}$$
$$\mathcal{U} \, := \Upsilon^{+\Sigma} \Omega_{\Sigma\Pi} \Upsilon^{-\Pi} \equiv \Upsilon^{+} \Omega \Upsilon^{-} \approx 0 \, , \tag{7.8.4}$$

and secondary constraints

$$\Phi_{++} = \partial_\sigma \Upsilon^- \Omega \Upsilon^- - e_\sigma^{--} (\Upsilon^+ C \Upsilon^-)^2 \approx 0 \, , \tag{7.8.5}$$
$$\Phi_{--} = \partial_\sigma \Upsilon^+ \Omega \Upsilon^+ - e_\sigma^{++} (\Upsilon^+ C \Upsilon^-)^2 \approx 0 \, , \tag{7.8.6}$$
$$\Phi^{(0)} = \partial_\sigma \Upsilon^+ \Omega \Upsilon^- - \Upsilon^+ \Omega \partial_\sigma \Upsilon^- \approx 0 \, , \tag{7.8.7}$$

that restrict the phase space variables

$$\widetilde{\mathcal{Z}}^{\tilde{\mathcal{M}}} \equiv \widetilde{\mathcal{Z}}^{\tilde{\mathcal{M}}}(\tau,\sigma) = \left(\Upsilon^{\pm\Sigma} \, , \, e_\tau^{\pm\pm} \, , \, e_\sigma^{\pm\pm} \right) \, , \tag{7.8.8}$$

$$\widetilde{P}_{\tilde{\mathcal{M}}} = (P_{\pm\Sigma} \, , \, P^\tau_{\pm\pm} \, , \, P^\sigma_{\pm\pm}) = \frac{\partial \mathcal{L}}{\partial(\partial_\tau \widetilde{\mathcal{Z}}^{\tilde{\mathcal{M}}})} \, . \tag{7.8.9}$$

The set (7.8.1)–(7.8.7) contains 6 first class constraints (versus five first class constraints (7.7.38)–(7.7.41) in the system (7.6.5)), namely

$$P^\tau_{\pm\pm} \approx 0 \, , \tag{7.8.10}$$
$$2e_\sigma^{++} P^\sigma_{++} - \Upsilon^{-\Sigma} \mathcal{P}_{-\Sigma} \approx 0 \, , \tag{7.8.11}$$
$$2e_\sigma^{--} P^\sigma_{--} - \Upsilon^{+\Sigma} \mathcal{P}_{+\Sigma} \approx 0 \, , \tag{7.8.12}$$

$$\begin{aligned}
&\tilde{\Phi}_{++} = \Phi_{++} \\
&+\frac{2e_\sigma^{--}\mathcal{B}^{(0)}}{(\Upsilon^+ C\Upsilon^-)^2}\mathcal{U} - \frac{\mathcal{B}^{(0)}}{(\Upsilon^+ C\Upsilon^-)^2}\Upsilon^{-\Sigma}\mathcal{P}_{+\Sigma} - \partial_\sigma P^\sigma_{++} + (\Upsilon^+ C\Upsilon^-)\Upsilon^- C\Omega\mathcal{P}_+ \\
&-\frac{1}{e_\sigma^{++}}\Big[\partial_\sigma\Upsilon^{-\Sigma}\mathcal{P}_{-\Sigma} + \frac{e_\sigma^{--}\mathcal{B}^{(0)}}{(\Upsilon^+ C\Upsilon^-)^2}\Upsilon^{+\Sigma}\mathcal{P}_{-\Sigma} - e_\sigma^{--}(\Upsilon^+ C\Upsilon^-)\Upsilon^+ C\Omega\mathcal{P}_-\Big] \\
&\approx 0\,,
\end{aligned} \tag{7.8.13}$$

$$\begin{aligned}
&\tilde{\Phi}_{--} = \Phi_{--} \\
&+\frac{2e_\sigma^{++}\mathcal{B}^{(0)}}{(\Upsilon^+ C\Upsilon^-)^2}\mathcal{U} - \frac{\mathcal{B}^{(0)}}{(\Upsilon^+ C\Upsilon^-)^2}\Upsilon^{+\Sigma}\mathcal{P}_{-\Sigma} + \partial_\sigma P^\sigma_{--} + (\Upsilon^+ C\Upsilon^-)\Upsilon^+ C\Omega\mathcal{P}_- \\
&+\frac{1}{e_\sigma^{--}}\Big[\partial_\sigma\Upsilon^{+\Sigma}\mathcal{P}_{+\Sigma} - \frac{e_\sigma^{++}\mathcal{B}^{(0)}}{(\Upsilon^+ C\Upsilon^-)^2}\Upsilon^{-\Sigma}\mathcal{P}_{+\Sigma} + e_\sigma^{++}(\Upsilon^+ C\Upsilon^-)(\Upsilon^- C\Omega\mathcal{P}_+)\Big] \\
&\approx 0\,,
\end{aligned} \tag{7.8.14}$$

where (*cf.* (7.7.30))

$$\begin{aligned}
&\mathcal{B}^{(0)} = \\
&= \frac{1}{2}\left(\frac{\partial_\sigma\Upsilon^+ C\Upsilon^+\,(\Upsilon^+ C\Upsilon^-)}{e_\sigma^{++}} + \frac{\partial_\sigma\Upsilon^- C\Upsilon^-\,(\Upsilon^+ C\Upsilon^-)}{e_\sigma^{--}} + \frac{\partial_\sigma\Upsilon^+\Omega\partial_\sigma\Upsilon^-}{e_\sigma^{++}e_\sigma^{--}}\right)\,.
\end{aligned} \tag{7.8.15}$$

Using Dirac brackets to account for the second class constraints (7.8.1), where (*cf.* (7.7.50))

$$[\Upsilon^{\pm\Sigma}(\sigma), \Upsilon^{\pm\Pi}(\sigma')\}_D = \mp\frac{1}{2e_\sigma^{\mp\mp}}\,\Omega^{\Sigma\Pi}\,\delta(\sigma-\sigma')\,, \tag{7.8.16}$$

the first class constraints simplify to

$$P^\tau_{\pm\pm} \approx 0\,, \tag{7.8.17}$$

$$P^\sigma_{++} \approx 0\,, \tag{7.8.18}$$

$$P^\sigma_{--} \approx 0\,, \tag{7.8.19}$$

$$\tilde{\Phi}_{++} = \Phi_{++} + \frac{2e_\sigma^{--}\mathcal{B}^{(0)}}{(\Upsilon^+ C\Upsilon^-)^2}\mathcal{U} \approx 0\,, \tag{7.8.20}$$

$$\tilde{\Phi}_{--} = \Phi_{--} + \frac{2e_\sigma^{++}\mathcal{B}^{(0)}}{(\Upsilon^+ C\Upsilon^-)^2}\mathcal{U} \approx 0\,, \tag{7.8.21}$$

which corresponds to the set of constraints (7.7.56)–(7.7.58) of the description in terms of ‘normalized’ supertwistors with the addition of the constraint (7.7.59), which is now ‘converted’ into a first class one due to the disappearance of the normalization constraint (7.7.60).

The remaining two bosonic constraints, equations (7.8.4) and (7.8.7), are second class. Their Dirac bracket

$$[\mathcal{U}(\sigma)\,,\,\Phi^{(0)}(\sigma')]_D = (\Upsilon^+ C\Upsilon^-)^2\delta(\sigma-\sigma') + \left(\frac{\Phi_{++}}{2e_\sigma^{++}} + \frac{\Phi_{--}}{2e_\sigma^{--}}\right)\delta(\sigma-\sigma')$$
$$\approx (\Upsilon^+ C\Upsilon^-)^2\delta(\sigma-\sigma') \qquad (7.8.22)$$

is nonvanishing due to the linear independence of the $\Upsilon^{+\Sigma}$ and $\Upsilon^{-\Sigma}$ supertwistors (7.6.12) (coming from the linear independence of their Λ^+_α and Λ^-_α components, $\Lambda^+_\alpha C^{\alpha\beta}\Lambda^-_\alpha \neq 0$). For a further simplification of the Hamiltonian formalism it might be convenient to make a conversion of this pair of second class constraints into first class by adding a pair of canonically conjugate variables, $q(\xi)$ and $P^{(q)}(\xi)$, ($[q(\sigma)\,,\,P^{(q)}(\sigma')]_P = \delta(\sigma-\sigma')$) to our phase space.

The above Hamiltonian formalism and its further development can be applied to quantize the $\Sigma^{(\frac{n(n+1)}{2}|n)}$ superstring model. This should produce a quantum higher spin generalization of the Green–Schwarz superstring for $n = 4, 8, 16$ and, for $n = 32$, an exactly solvable quantum description of a conformal field theory carrying, somehow, information about the non-perturbative brane BPS states of M Theory.

7.9 Supersymmetric p–branes in tensorial superspace

The model may be generalized to describe higher-dimensional extended objects (supersymmetric p–branes) in $\Sigma^{(\frac{n(n+1)}{2}|n)}$. The expression of the supersymmetric p–brane action in terms of dimensionful unconstrained bosonic spinors reads (cf. (7.2.7)) [8]

$$S_p = \int_{W^{p+1}} e_a^{\wedge p} \wedge \Pi^{\alpha\beta}(\Lambda^r_\alpha \rho^a_{rs}\Lambda^s_\beta)$$
$$-(-\alpha')^p \int_{W^{p+1}} e^{\wedge(p+1)} \det(C^{\alpha\beta}\Lambda^r_\alpha\Lambda^s_\beta)\,, \qquad (7.9.1)$$

where $a = 0, 1, \ldots, p\,,\ r = 1, \ldots, \tilde{n}(p)\,,\ \alpha = 1, \ldots, n\,,$

$$e_a^{\wedge p} \equiv \tfrac{1}{p!}\epsilon_{ab_1\ldots b_p} e^{b_1} \wedge \ldots \wedge e^{b_p}\,, \qquad (7.9.2)$$

(see equation (2.2.14)) and $e^{\wedge(p+1)}$ is the W^{p+1} volume element

$$e^{\wedge(p+1)} \equiv \tfrac{1}{(p+1)!}\epsilon_{b_1\ldots b_{p+1}} e^{b_1} \wedge \ldots \wedge e^{b_{p+1}}\ . \qquad (7.9.3)$$

In equation (7.9.1), the $(p+1)$ $e^a = d\xi^m e^a_m(\xi)$ are auxiliary worldvolume vielbein fields, $\xi^m = (\tau, \sigma^1, \ldots, \sigma^p)$ are the worldvolume W^{p+1} local coordinates and $\Lambda^r_\alpha(\xi)$ is a set of $\tilde{n} = \tilde{n}(p)$ unconstrained auxiliary real bosonic fields with a 'spacetime' spinorial (actually, a $Sp(n)$–vector) index $\alpha = 1, ..., n$. The number $\tilde{n}(p)$ of real spinor fields $\Lambda^r_\alpha(\xi)$ as well as the meaning of the symmetric real matrices ρ^a_{rs} depend on the worldvolume dimension $d = p + 1$. For $d = 2, 3, 4$ (mod 8), where a Majorana spinor representation exists, the ρ^a_{rs} are $Spin(1, p)$ Dirac matrices multiplied by the charge conjugation matrix or sigma matrices, provided they are symmetric. If not, it is always possible to find a real symmetric matrix by doubling the index r, $r' = ri$ $(i = 1, 2)$, as in the case of $d = 6$ symplectic Majorana spinors. For dimensions with only Dirac spinors (like $d = 5$) $\Lambda^r_\alpha \rho^a_{rs} \Lambda^s_\beta$ should be understood as $\bar{\Lambda}_\alpha \rho^a \Lambda_\beta + \bar{\Lambda}_\beta \rho^a \Lambda_\alpha$, etc.

For simplicity we present equation (7.9.1) and other formulae of this section for 'Majorana dimensions' d with symmetric $C\rho$–matrices; the generalization to the other cases is straightforward, although one should be careful determining the value of $\tilde{n}(p)$ for a given $d = p + 1$. For $p = 1$, where the irreducible Majorana–Weyl spinor is one-dimensional ($Spin(1, 1)$ is abelian), one needs Λ^r_α to be in a reducible Majorana representation in the worldsheet spinor index r, i.e. $\Lambda^r_\alpha = (\Lambda^+_\alpha, \Lambda^-_\alpha)$; otherwise the second term in (7.9.1) would be zero and the action would become that of a tensionless $\Sigma^{(\frac{n(n+1)}{2}|n)}$ supersymmetric string. Then, the action (7.9.1) reduces to (7.2.7) using (7.2.4).

The fermionic variation δ_f of the action (7.9.1), $\delta_f S_p$, comes only from the variation of $\Pi^{\alpha\beta}$. Let us simplify it by taking $\delta_f X^{\alpha\beta} = i\delta_f \theta^{(\alpha} \theta^{\beta)}$ (*cf.* below equation (7.3.1)), so that $i_{\delta_f} \Pi^{\alpha\beta} = 0$ and $\delta_f \Pi^{\alpha\beta} = -2id\theta^{(\alpha} \delta\theta^{\beta)}$. As $\Pi^{\alpha\beta}$ enters the action (7.9.1) only through its contraction with $\Lambda^r_\alpha \rho^a_{rs} \Lambda^s_\beta$ we find

$$\delta_f S_p = -2i \int_{W^{p+1}} e^{\wedge p}_a \wedge d\theta^\alpha \Lambda^r_\alpha \, \rho^a_{rs} \, \Lambda^s_\beta \delta\theta^\beta \ . \tag{7.9.4}$$

Thus only $\tilde{n}(p)$ fermionic variations $\delta\theta^\beta \Lambda^s_\beta$ out of the n variations $\delta\theta^\beta$ are effectively involved in $\delta_f S_p$.

This reflects the presence of $(n - \tilde{n}(p))$ κ–symmetries in the dynamical system described by the supersymmetric p–brane action (7.9.1). They are defined by

$$\delta_\kappa X^{\alpha\beta} = i\delta_\kappa \theta^{(\alpha} \theta^{\beta)} \ , \quad \delta_\kappa e^a = 0 \ , \tag{7.9.5}$$

and by the following condition on $\delta_\kappa \theta^\alpha$,

$$\delta_\kappa \theta^\alpha \Lambda^r_\alpha = 0 \ , \qquad r = 1, \ldots, \tilde{n}(p) \ . \tag{7.9.6}$$

This can be solved, using the auxiliary spinor fields $u^{\alpha J}$ [where now $J = 1, \ldots, (n - \tilde{n}(p))$] orthogonal to Λ^r_α, as

$$\delta_\kappa \theta^\alpha \ = \ \kappa_J(\xi) \, u^{\alpha J}(\xi) \ , \qquad u^{\alpha J}(\xi) \, \Lambda^r_\alpha(\xi) = 0 \ ,$$

$$J = 1, \ldots, (n - \tilde{n}(p)) \, , \quad r = 1, \ldots, \tilde{n}(p) \, . \tag{7.9.7}$$

The κ–symmetry (7.9.5), (7.9.7) implies the preservation of all but $\tilde{n}(p)$ supersymmetries by the corresponding $\nu = \frac{n-\tilde{n}(p)}{n}$ BPS state.

For instance, for $p = 2$, $n = 32$, it is $\tilde{n} = 2$. The action (7.9.1) then describes excitations of a membrane BPS state preserving all but 2 supersymmetries, a $\frac{30}{32}$ BPS state. For $p = 5$ and $\tilde{n} = 8$ the action (7.9.1) with $n = 32$ describes a $\frac{24}{32}$ supersymmetric 5–brane model in $\Sigma^{(528|32)}$. Both the supermembrane (M2–brane) and the super–5–brane (M5–brane) are known in the standard $D = 11$ superspace, where they correspond to $\frac{16}{32}$ BPS states. The speculation could be made that the 'usual' M2 and M5 superbranes are related to the generalized $\Sigma^{(528|32)}$ supersymmetric 2–brane and 5–brane described by the action (7.9.4) for $p = 2$ and 5. For instance, they might be related with some particular solutions to the equations of motion of the corresponding $\frac{30}{32}$ and $\frac{24}{32}$ $\Sigma^{(528|32)}$ models preserving 16 supersymmetries and/or with the result of a dimensional reduction of them. For the $p = 5$ case a question of a special interest would be the role of the M5 selfdual worldvolume gauge field in the $\Sigma^{(528|32)}$ superspace description (see [85] for a related discussion). For $p = 3$ and $\tilde{n} = 4$ we have a $\frac{28}{32}$ BPS state, a BPS 3–brane. Neither the Green–Schwarz superstring nor the super–3–brane exist in the standard $D = 11$ superspace, but a super–D3–brane does exist in the $D = 10$ Type IIB superspace, as the superstring does. The possible relation of these preonic branes with the usual Type II branes would require further study.

8

Conclusions and outlook

A number of topics about eleven-dimensional supergravity, the low energy limit of the hypothetical M Theory, have been covered in this Thesis. The role of (generalized) holonomy in the description of supersymmetric solutions of supergravity has been discussed and applied, in particular, to the search of possible preonic solutions. The notion of BPS preons leads naturally to that of enlarged superspaces and supersymmetry algebras, the role of which in supergravity has also been explored. In particular, enlarged superspaces have been shown to allow for the construction of supersymmetric objects with a manifest content of BPS preons, and enlarged superalgebras have appeared in the discussion of the underlying symmetry of $D = 11$ supergravity.

After an introductory chapter 1, the conventions and notation used throughout the Thesis (with the exception of chapter 3), were set in chapter 2. The later contains a general discussion of topics about eleven-dimensional supergravity relevant for the remainder of the Thesis. The M Theory superalgebra is introduced, and the Lagrangian, equations of motion and symmetries of $D = 11$ supergravity discussed. Especial attention has been payed to supersymmetry, in relation to which the notions of generalized connection, curvature and holonomy have been reviewed. The interplay between the generalized curvature and the equations of motion has also been discussed. In particular, we have shown [1] that all the bosonic equations of D=11 supergravity can be collected in a single equation, (2.6.2), written in terms of the generalized curvature (2.3.9) which takes values in the algebra of the generalized holonomy group. The concise form (2.6.2) of all the bosonic equations is obtained by factoring out the fermionic one–form ψ^β in the selfconsistency (or integrability) conditions $\mathcal{D}\Psi_{10\,\beta} = 0$ [Eqs. (2.6.1)], for the gravitino equations $\Psi_{10\,\alpha} = 0$, Eqs. (2.4.11). In this sense, one can say that in (the second order formalism of) $D = 11$ CJS supergravity all the equations of motion and Bianchi identities are encoded in the fermionic gravitino equation

$\Psi_{10\,\beta} := \mathcal{D}\psi^\alpha \wedge \bar{\Gamma}^{(8)}_{\alpha\beta} = 0$ (equation (2.4.11)). Actually this should be expected for a supergravity theory including only one fermionic field, the gravitino, and whose supersymmetry algebra closes on shell. As we have discussed, the basis for such an expectation is provided by the second Noether theorem.

Generalized holonomy has been further explored in chapter 3, where especial emphasis has been made in the role of supercovariant derivatives of the curvature to characterize the holonomy algebra. After a general review of how higher order integrability conditions might be necessary to properly determine the Killing spinors characterizing purely bosonic supersymmetric solutions of supergravity, the generalized holonomy of some well known solutions has been revised. Supercovariant derivatives of the generalized curvatures corresponding to the M2 and M5 brane solutions of supergravity only turn out to help to close the generalized holonomy algebra obtained at first order. The situation is different for other solutions, such as Freund-Rubin compactifications. An example has been provided by the compactification on the squashed S^7. Left squashing preserves $N = 1$ supersymmetry, while its right counterpart breaks all supersymmetries. This situation cannot be described if the generalized holonomy is G_2, as obtained in first order. Second order integrability, namely, the supercovariant derivative of the generalized curvature yields a holonomy algebra of $so(7)$ [2], which gives the correct decompositions of the Killing spinor for both left and right compactifications.

In chapter 4, the role of the BPS preon notion in the analysis of supersymmetric solutions of $D = 11$ supergravity is studied. This notion suggests the moving G–frame method [3], which is proposed as a useful tool in the search for supersymmetric solutions of $D = 11$ and $D = 10$ supergravity. We used this method here to make a step towards answering whether the standard CJS supergravity possesses a solution preserving 31 supersymmetries, a solution that would correspond to a BPS preon state. Although this question has not been settled for the CJS supergravity case, we have shown in our framework that preonic, $\nu = 31/32$ solutions do exist [3] in a Chern–Simons type $D = 11$ supergravity.

Although the main search for preonic solutions concerns the 'free' bosonic CJS supergravity equations, one should not exclude other possibilities, both inside and outside the CJS standard supergravity framework. When, *e.g.*, super–p–brane sources are included, the Einstein equation, and possibly the gauge field equations and even the Bianchi identities, acquire r.h.s.'s and the situation would have to be reconsidered. Another source of modification of the CJS supergravity equations might be due to 'radiative' corrections of higher order in curvature. Such modified equations might also allow for preonic solutions not present in the unmodified ones. If it were found that only the inclusion of these higher–order curvature terms allows for preonic BPS solutions, this would indicate that BPS preons cannot be seen in a classical low energy approximation of M Theory and, hence, that they are intrinsically quantum objects.

The special role of BPS preons in the algebraic classification of all the M Theory BPS states allows us to conjecture that they are elementary ('quark-like') necessary ingredients of any model providing a more complete description of M Theory. In such a framework, if the standard supergravity did not contain $\nu = 31/32$ solutions, neither in its 'free' form, nor in the presence of a super–p–brane source, this might just indicate the need for a wider framework for an effective description of M Theory. Such an approach could include Chern–Simons supergravities [91] and/or the use of larger, extended superspaces (see [85, 4] and refs. therein), in particular with additional tensorial coordinates (also relevant in the description of massless higher–spin theories [155, 187, 203]). In this perspective our observation that the BPS preonic configurations do solve the bosonic equations of Chern–Simons supergravity models looks interesting. It might be also worthwhile to look at the role of vectors and higher order tensors that may be constructed from the preonic spinors ${\lambda_\alpha}^r$, in analogy with the use of the Killing vectors $K^a_{IJ} = \epsilon_I \Gamma^a \epsilon_J$ and higher order bilinears $\epsilon_I \Gamma^{a_1 \cdots a_s} \epsilon_J$ made in references [79, 80, 135, 136, 137, 138, 139, 103].

Chapter 5 contains the technical details of the expansion method [4, 5], a procedure of obtaining new (super)algebras $\mathcal{G}(N_0, \ldots, N_p)$ from a given one $\mathcal{G}$. It is based in the power expansion of the Maurer-Cartan equations that results from rescaling some group parameters. These expansions are in principle infinite, but some truncations are consistent and define the Maurer-Cartan equations of new (super)algebras, the structure constants of which are obtained from those of the original (super)algebra $\mathcal{G}$. We have considered the different possible $\mathcal{G}(N_0, \ldots, N_p)$ algebras subordinated to various splittings of $\mathcal{G}$ and discussed their structure. We have seen that in some cases (when the splitting of $\mathcal{G}$ satisfies the Weimar-Woods conditions) the resulting algebras include the simple or generalized İnönü-Wigner contractions of $\mathcal{G}$, but that this is not always the case. In general, the new 'expanded' algebras have higher dimension than the original one. Since $\mathcal{G}$ is the only ingredient of the expansion method, it is clear that the extension procedure (which involves *two* algebras) is richer when one is looking for new (super)algebras; the expansion method is more constrained. Nevertheless, we have used it to obtain the M Theory superalgebra, including its Lorentz part, from the orthosymplectic superalgebra $osp(1|32)$ as the expansion $osp(1|32)(2,1,2)$ [4].

The expansion method is also applied, already in chapter 6, to discuss the relation of the gauge structure of $D = 11$ supergravity with $osp(1|32)$.

In this chapter, the consequences of a possible composite structure, *à la* D'Auria-Fré, of the three–form field A_3 of the standard CJS $D = 11$ supergravity are studied. In particular, we have shown that A_3 may be expressed in terms of the one-form gauge fields B^{ab}, $B^{a_1 \cdots a_5}$, η^α, e^a, ψ^α associated to a *family* of superalgebras $\tilde{\mathfrak{E}}(s) \equiv \tilde{\mathfrak{E}}^{(528|32+32)}(s)$, $s \neq 0$, corresponding to the supergroups $\tilde{\Sigma}(s) \equiv \tilde{\Sigma}^{(528|32+32)}(s)$ [6, 7]. Two values of the parameter s recover the two earlier D'Auria–Fré [92] decompositions of A_3, while one value of s, $s = -6$ leads to a simple expression for A_3 that does not involve $B^{a_1 \cdots a_5}$. Indeed, the generator $Z_{a_1 \ldots a_5}$ associated to $B^{a_1 \cdots a_5}$ is central in $\tilde{\mathfrak{E}}(-6)$, so that the smaller supergroup Σ_{min} obtained by removing $Z_{a_1 \ldots a_5}$ from $\tilde{\Sigma}(-6)$ can be regarded as the minimal underlying gauge supergroup of supergravity [6, 7]. The supergroups $\tilde{\Sigma}(s) \rtimes SO(1,10)$ with $s \neq 0$ are non-trivial (non-isomorphic) deformations of the $\tilde{\Sigma}(0) \rtimes SO(1,10) \subset \tilde{\Sigma}(0) \rtimes Sp(32)$ supergroup, which is itself the expansion [6, 4] $OSp(1|32)(2,3,2)$ of $OSp(1|32)$. For any $s \neq 0$, $\tilde{\Sigma}(s) \rtimes SO(1,10)$ may be looked at as a hidden gauge symmetry of the $D = 11$ CJS supergravity generalizing the D=11 superPoincaré group $\Sigma^{(11|32)} \rtimes SO(1,10)$.

To study the possible dynamical consequences of the composite structure of A_3 we have followed the original proposal [92] of substituting the composite A_3 for the fundamental A_3 in the first order CJS supergravity action [92, 105] of chapter 2. It has been seen that such an action possesses the right number of 'extra' gauge symmetries to make the number of degrees of freedom the same as in the standard CJS supergravity [7]. These are symmetries under the transformations of the new one–form fields that leave the composite A_3 field invariant; their presence is related to the fact that the new gauge fields enter the supergravity action only inside the A_3 field. In other words, the extra degrees of freedom carried by the new fields B^{ab}, $B^{a_1 \cdots a_5}$, η^α are pure gauge. One may conjecture that these extra degrees of freedom might be important in M Theory and that, correspondingly, the extra gauge symmetries that remove them would be broken by including in the supergravity action some exotic 'matter' terms that couple to the 'new' additional one–form gauge fields. In constructing such an 'M–theoretical matter' action, the preservation of the $\tilde{\Sigma}(s) \rtimes SO(1,10)$ gauge symmetry would provide a guiding principle.

We have stressed the equivalence between the problem of searching for a composite structure of the A_3 field and, hence, for a hidden gauge symmetry of $D = 11$ supergravity, and that of trivializing a four–cocycle of the standard $D = 11$ supersymmetry algebra $\mathfrak{E} \equiv \mathfrak{E}^{(11|32)}$ cohomology on the enlarged superalgebras $\tilde{\mathfrak{E}}(s)$, $s \neq 0$. The generators of $\tilde{\mathfrak{E}}(s)$ are in one–to–one correspondence with the one–form fields e^a, ψ^α, B^{ab}, $B^{a_1 \ldots a_5}$, η^α. For zero curvatures these fields satisfy the same equations as the $\tilde{\Sigma}(s)$–

invariant Maurer-Cartan forms of $\tilde{\mathfrak{E}}(s)$ which, before pulling them back to a bosonic eleven–dimensional spacetime surface, are expressed through the coordinates $(x^a, \theta^\alpha, y^{ab},\ y^{a_1 \ldots a_5}, \theta'^\alpha)$ of the $\tilde{\Sigma}(s)$ superspace. This is the content of the fields/extended superspace coordinates correspondence principle, that conjectures that for the relevant supergravity theories there always exists an enlarged superspace whose coordinates are in one-to-one correspondence with the fields of the theory [85, 86]. $D = 11$ supergravity itself can be conjectured to be embedded in a larger superspace (see [7]).

Several interesting questions arise concerning the composite nature of A_3. The first one was already sketched in chapter 7 and refers to the trivialization of the FDA seven-cocycle ω_7 related to the dual formulation of $D = 11$ supergravity. It would be interesting to check if ω_7 is already trivial on the family of superalgebras $\tilde{\mathfrak{E}}(s)$, or further extensions are needed instead to trivialize it. Another issue that would be worth studying is the trivialization of the FDAs corresponding to lower dimensional supergravities. It would be interesting, in particular, to study whether the FDA corresponding to IIA and IIB supergravities can be trivialized by some superalgebra and, in case it were possible, to study its relation with the family $\tilde{\mathfrak{E}}(s)$ trivializing the $D = 11$ supergravity FDA. Another question that would be interesting to analyze would be the implications of the composite nature of A_3 in the problem of the cosmological constant of $D = 11$ supergravity, argued in [98] to be forbidden on cohomological grounds.

In chapter 7, we have presented a supersymmetric string model in the 'maximal' or 'tensorial' superspace $\Sigma^{(\frac{n(n+1)}{2}|n)}$ with additional tensorial central charge coordinates (for $n > 2$) [8]. The model possesses n rigid supersymmetries and $n-2$ local fermionic κ–symmetries. This implies that it provides the worldsheet action for the excitations of a BPS state preserving $(n-2)$ supersymmetries. In particular, for $n = 32$ our model describes a supersymmetric string with 30 κ–symmetries in $\Sigma^{(528|32)}$, which corresponds to a BPS state preserving 30 out of 32 supersymmetries. This model can be treated as a composite of two BPS preons [83] and is the second (after the $D = 11$ Curtright model [191]) tensionful extended object model in $\Sigma^{(528|32)}$. In contrast with the Curtright model [191], our supersymmetric string action in the enlarged $D = 11$ superspace $\Sigma^{(528|32)}$ does not involve any gamma–matrices, but instead makes use of two constrained bosonic spinor variables, λ^+_α and λ^-_α, corresponding to the two BPS preons from which the superstring BPS state is composed. As a result, our model preserves the $Sp(32)$ subgroup of the $GL(32, \mathbb{R})$ automorphism symmetry of the $D = 11$ M–algebra. Our $\Sigma^{(\frac{n(n+1)}{2}|n)}$ supersymmetric string model can be treated as a higher spin generalization of the classical Green–Schwarz superstring. At the same time, the additional bosonic tensorial coordinate fields of the $n = 32$ case might contain information about topological charges corresponding to the higher branes of the superstring/M Theory [71].

The $\Sigma^{(\frac{n(n+1)}{2}|n)}$ model may also be formulated in terms of a pair of constrained worldvolume $OSp(2n|1)$ supertwistors. The transition to the supertwistor formulation is similar to that for the massless superparticle and the tensionless $\Sigma^{(\frac{n(n+1)}{2}|n)}$ supersymmetric p–branes [148, 149]. In our case, however, the supertwistors are restricted by a constraint that breaks the generalized superconformal $OSp(64|1)$ symmetry down to a generalization of the super–Poincaré group, $\Sigma^{(528|32)} \rtimes Sp(32)$. Such a breaking is characteristic of tensionful models. We note that this constrained $OSp(2n|1)$ supertwistor framework might also be useful for massive higher spin theories.

We have developed the Hamiltonian formalism, both in the original and in the symplectic supertwistor representation, and found that, while the Hamiltonian analysis in the original formulation requires the use of the additional auxiliary spinor variables u^I_α $(I = 1, ..., (n-2))$ orthogonal to $\lambda^\pm_\alpha$, the symplectic supertwistor Hamiltonian mechanics can be discussed in terms of the original phase space variables. Moreover, under Dirac brackets, supertwistors become selfconjugate variables and the symplectic structure of the phase space simplifies considerably. A natural application of the Hamiltonian approach developed here would be the BRST quantization of the $\Sigma^{(\frac{n(n+1)}{2}|n)}$ superstring model, which might provide a 'higher spin' counterpart of the usual string field theory.

We have also presented a generalization of our $\Sigma^{(\frac{n(n+1)}{2}|n)}$ supersymmetric string model for supersymmetric p–branes in $\Sigma^{(\frac{n(n+1)}{2}|n)}$. They correspond to BPS states preserving all but $\tilde{n}(p)$ (see below (7.9.1)) supersymmetries, composites of $\tilde{n}(p)$ BPS preons ($\tilde{n}(2) = 2\,,\ \tilde{n}(3) = 4\,,\ \tilde{n}(5) = 8$). In particular, the $\Sigma^{(528|32)}$ supersymmetric membrane ($p = 2$) also corresponds to $\frac{30}{32}$ a BPS state.

In this Thesis, preonic solutions have been shown to exist in enlarged superspaces or in the context of Chern-Simons supergravities. It would be very interesting to determine whether preonic solutions also occur as solutions of standard CJS $D = 11$ supergravity. The definitive answer would be provided by a complete classification of supergravity solutions, that would also shed light into the structure of M Theory itself. As future work, we aim to make some steps towards a complete classification of CJS supergravity solutions. The study of the interplay between the approaches to classify general supergravity solutions, is expected to give us new insights towards that classification. In particular, the presence of Killing spinors implies the existence of a G-structure [79, 80] on the tangent bundle, that is, a sub-bundle of the frame bundle with structure group G. Its existence can be seen from the fact that a set of covariantly constant tensors exist on the tangent bundle, built as bilinears of the Killing spinors. The differential and algebraic conditions that these tensors satisfy turn out to constrain the geometry (the metric) and the fluxes.

The generalized holonomy approach [78, 87], on the contrary, deals with the supergravity connection as a Clifford algebra valued connection, as discussed in chapter 2, without focussing on the tangent bundle or the spinor bundle of the background. The difficulty in relating both approaches could be put down to that fact. By dealing with an $sl(32, \mathbb{R})$-valued connection, the Killing spinors are not any longer regarded as spinors in the tangent bundle (transforming in suitable representations of the tangent bundle structure group, $SO(1, 10)$), but are instead promoted to vectors of $SL(32, \mathbb{R})$. It is not obvious that this step does not entail any loss of the information contained in the spin bundle [134] so, if this were indeed the case, supplementary conditions should be derived to account for the fact that the relevant $SL(32, \mathbb{R})$-vectors are also tangent bundle spinors.

An interesting question that would shed light into the relation of both approaches is what subgroups H of the generalized structure group can actually arise as generalized holonomy groups of supergravity solutions. A refined definition of holonomy taking into account covariant derivatives of the curvature (higher order commutators of the covariant derivatives) [2] could be relevant with this regard. It could also be worth studying the effects of the effect successive covariant derivatives of the curvature when no fluxes are considered but higher order gravitational corrections are taken into account [111]. Alternatively, the study of the relevant G-structures of solutions including higher order corrections could allow us both to generate new examples, and help us to understand the origin of this higher order gravitational terms in the fully-fledged M Theory.

The classification of supergravity solutions is not only expected to provide insights into the structure of M Theory, but also to provide new backgrounds for Standard Model building and for the AdS/CFT correspondence. In the later case, the G-structure approach has provided solutions containing AdS factors [134, 136, 137] suitable to test the correspondence (see [134, 79] and references therein for a review). We expect to be able to make progress also in this direction.

Appendix A

Second order integrability for the squashed S^7

In this Appendix we present the details of the derivation of the linearly independent generators (3.4.9) and (3.4.10) of the generalized holonomy algebra hol(Ω_m) = $so(7)$ of the squashed S^7, associated to the second order integrability condition (3.4.8). For convenience, we rewrite (3.4.8) with a modified normalization

$$M_{abc} = 5\left(\sqrt{5}D_a C_{bcde}\Gamma^{de} \; - m' C_{bcad}\Gamma^d\right) \; , \tag{A.1}$$

where we have defined

$$m' = 2\sqrt{5}im, \tag{A.2}$$

and have chosen the $-$ sign in front of m' for definiteness.

To obtain M_{abc}, we have computed both the Weyl tensor C_{bcad} (given in [29]) and its (Lévi-Civita) covariant derivative $D_a C_{bcde}$. We obtain, for the non-vanishing generators (A.1) [2]:

$$M_{00j} = 4\Gamma_{0\hat{j}} - \epsilon_{jkl}\Gamma^{k\hat{l}} - 2m'\Gamma_j \; , \tag{A.3}$$

$$M_{00\hat{j}} = 4\Gamma_{0j} + \epsilon_{jkl}\Gamma^{kl} + 2m'\Gamma_{\hat{j}} \; , \tag{A.4}$$

$$M_{0ij} = 2\epsilon_{ijk}\Gamma^{0\hat{k}} + \Gamma_{i\hat{j}} - \Gamma_{j\hat{i}} \; , \tag{A.5}$$

$$M_{0i\hat{j}} = -\epsilon_{ijk}\Gamma^{0k} + \Gamma_{ij} - 3\Gamma_{\hat{i}\hat{j}} + m'\epsilon_{ijk}\Gamma^{\hat{k}} \; , \tag{A.6}$$

$$M_{0\hat{i}\hat{j}} = -3\Gamma_{i\hat{j}} + 3\Gamma_{j\hat{i}} - 2m'\epsilon_{ijk}\Gamma^k \; , \tag{A.7}$$

$$M_{h0j} = \epsilon_{hjk}\Gamma^{0\hat{k}} + 2\Gamma_{h\hat{j}} + \delta_{hj}\delta^{kl}\Gamma_{k\hat{l}} + \Gamma_{j\hat{h}} + 2m'\delta_{hj}\Gamma_0 \; , \tag{A.8}$$

$$M_{h0\hat{j}} = -\epsilon_{hjk}\Gamma^{0k} + \Gamma_{hj} + 3\Gamma_{\hat{h}\hat{j}} - m'\epsilon_{hjk}\Gamma^{\hat{k}} \; , \tag{A.9}$$

$$M_{hij} = \delta_{hi}\Gamma_{0\hat{j}} - \delta_{hj}\Gamma_{0\hat{i}} + 4\epsilon_{ij}{}^{k}\Gamma_{h\hat{k}} - \epsilon_{hij}\delta^{kl}\Gamma_{k\hat{l}} - \epsilon_{ij}{}^{k}\Gamma_{k\hat{h}} \\ +2m'(\delta_{hj}\Gamma_i - \delta_{hi}\Gamma_j) \ , \tag{A.10}$$

$$M_{hi\hat{j}} = (2\epsilon_{jkl}\delta_{hi} - \tfrac{1}{2}\epsilon_{hkl}\delta_{ij} - \tfrac{1}{2}\epsilon_{ikl}\delta_{hj})\Gamma^{kl} - 3\epsilon_{hi}{}^{k}\Gamma_{\hat{k}\hat{j}} + 2\delta_{hi}\Gamma_{0j} \\ +\delta_{ij}\Gamma_{0h} + \delta_{hj}\Gamma_{0i} + m'(2\delta_{hi}\Gamma_{\hat{j}} - \delta_{ij}\Gamma_{\hat{h}} + \delta_{hj}\Gamma_{\hat{i}}) \ , \tag{A.11}$$

$$M_{h\hat{i}\hat{j}} = 3\delta_{hi}\Gamma_{0\hat{j}} - 3\delta_{hj}\Gamma_{0\hat{i}} + 3\epsilon_{hi}{}^{k}\Gamma_{k\hat{j}} - 3\epsilon_{hj}{}^{k}\Gamma_{k\hat{i}} \\ +2m'(\epsilon_{hij}\Gamma_0 + \delta_{hj}\Gamma_i - \delta_{hi}\Gamma_j) \ , \tag{A.12}$$

$$M_{\hat{h}0j} = -6\Gamma_{\hat{h}\hat{j}} + 2m'\epsilon_{hjk}\Gamma^{\hat{k}} \ , \tag{A.13}$$

$$M_{\hat{h}0\hat{j}} = 3\Gamma_{j\hat{h}} - 3\delta_{hj}\delta^{kl}\Gamma_{k\hat{l}} - m'(2\delta_{hj}\Gamma_0 + \epsilon_{hjk}\Gamma^{k}) \ , \tag{A.14}$$

$$M_{\hat{h}ij} = 6\epsilon_{ij}{}^{k}\Gamma_{\hat{k}\hat{h}} + 2m'(\delta_{hj}\Gamma_{\hat{i}} - \delta_{hi}\Gamma_{\hat{j}}) \ , \tag{A.15}$$

$$M_{\hat{h}i\hat{j}} = 3\delta_{hj}\Gamma_{0\hat{i}} - 3\delta_{ij}\Gamma_{0\hat{h}} - 3\delta_{hi}\epsilon_{jkl}\Gamma^{k\hat{l}} - 3\epsilon_{ij}{}^{l}\Gamma_{h\hat{l}} \\ +m'(-\epsilon_{hij}\Gamma_0 - 2\delta_{hj}\Gamma_i + \delta_{ij}\Gamma_h - \delta_{hi}\Gamma_j) \ , \tag{A.16}$$

$$M_{\hat{h}\hat{i}\hat{j}} = 6\delta_{hj}\Gamma_{0i} - 6\delta_{hi}\Gamma_{0j} - 6\epsilon_{ij}{}^{k}\Gamma_{kh} + 4m'(\delta_{hj}\Gamma_{\hat{i}} - \delta_{hi}\Gamma_{\hat{j}}) \ . \tag{A.17}$$

Not all the generators included in (A.3)–(A.17) are linearly independent, however. After all, they are built up from Dirac matrices $\{\Gamma_{ab}, \Gamma_a\}$, that is, from generators of SO(8), so at most 28 can be linearly independent.

In fact, only 21 linearly independent generators are contained in (A.3)–(A.17), as we will now show. Some redundant generators are straightforward to detect, since the Bianchi identities for the Weyl tensor, $D_{[a}C_{bc]de} = 0$ and $C_{[bca]d} = 0$ place the restrictions

$$M_{[abc]} = 0 \ . \tag{A.18}$$

Further manipulations show that only the generators (A.11) and (A.16) are relevant, the rest being linear combinations of them. The generators (A.4), (A.6), (A.9), (A.13), (A.15) and (A.17) are obtained from (A.11):

$$M_{00\hat{j}} = \tfrac{1}{5}\delta^{kl}(M_{kl\hat{j}} + M_{kj\hat{l}} + M_{jk\hat{l}}) \ , \tag{A.19}$$

$$M_{0i\hat{j}} = \tfrac{1}{5}\epsilon_{[i|}{}^{kl}(4M_{k|j]\hat{l}} - M_{|j]k\hat{l}}) - \tfrac{1}{5}\epsilon_{ij}{}^{k}\delta^{lm}M_{lm\hat{k}} \ , \tag{A.20}$$

$$M_{i0\hat{j}} = -\tfrac{1}{5}\epsilon_{[i|}{}^{kl}(M_{k|j]\hat{l}} - 4M_{|j]k\hat{l}}) - \tfrac{1}{5}\epsilon_{ij}{}^{k}\delta^{lm}M_{lm\hat{k}} \ , \tag{A.21}$$

$$M_{\hat{j}0i} = -\epsilon_{[i|}{}^{kl}(M_{k|j]\hat{l}} - M_{|j]k\hat{l}}) \ , \tag{A.22}$$

$$M_{\hat{h}ij} = M_{ji\hat{h}} - M_{ij\hat{h}} \ , \tag{A.23}$$

$$M_{\hat{h}\hat{i}\hat{j}} = \tfrac{1}{5}(M_{hi\hat{j}} - M_{hj\hat{i}} + M_{ih\hat{j}} - M_{jh\hat{i}}) - \tfrac{4}{5}\delta^{kl}(\delta_{hi}M_{kl\hat{j}} - \delta_{hj}M_{kl\hat{i}}) \,, \tag{A.24}$$

while (A.3), (A.5), (A.7), (A.8), (A.10), (A.12) and (A.14) are linear combinations of (A.16):

$$M_{00j} = \tfrac{1}{3}\delta^{kl}(M_{\hat{k}j\hat{l}} - M_{\hat{j}k\hat{l}}) \,, \tag{A.25}$$

$$M_{0hj} = -\tfrac{1}{3}\epsilon_h{}^{kl}(M_{\hat{k}l\hat{j}} + 3M_{\hat{j}k\hat{l}}) + \tfrac{1}{3}\epsilon_j{}^{kl}\left(M_{\hat{k}l\hat{h}} + 3M_{\hat{h}k\hat{l}}\right) \,, \tag{A.26}$$

$$M_{0\hat{h}\hat{j}} = \epsilon_h{}^{kl}(M_{\hat{k}l\hat{j}} + 2M_{\hat{j}k\hat{l}}) - \epsilon_j{}^{kl}(M_{\hat{k}l\hat{h}} + 2M_{\hat{h}k\hat{l}}) \,, \tag{A.27}$$

$$M_{h0j} = -\tfrac{1}{6}\epsilon_h{}^{kl}(2M_{\hat{k}l\hat{j}} + 5M_{\hat{j}k\hat{l}}) + \tfrac{1}{6}\epsilon_j{}^{kl}M_{\hat{h}k\hat{l}} \,, \tag{A.28}$$

$$M_{hij} = \tfrac{1}{2}\delta^{kl}\left(\delta_{hi}(M_{\hat{k}l\hat{j}} - 2M_{\hat{j}k\hat{l}}) - \delta_{hj}(M_{\hat{k}l\hat{i}} - 2M_{\hat{i}k\hat{l}})\right) + M_{\hat{i}j\hat{h}} - M_{\hat{j}i\hat{h}} + \tfrac{7}{3}(M_{\hat{h}i\hat{j}} - M_{\hat{h}j\hat{i}}) - \tfrac{2}{3}\epsilon_h{}^{kl}\epsilon_{ij}{}^{m}(M_{\hat{k}l\hat{m}} + 4M_{\hat{m}k\hat{l}}) \,, \tag{A.29}$$

$$M_{h\hat{i}\hat{j}} = M_{\hat{i}h\hat{j}} - M_{\hat{j}h\hat{i}} \,, \tag{A.30}$$

$$M_{\hat{h}0\hat{j}} = \epsilon_h{}^{kl}(M_{\hat{k}l\hat{j}} + M_{\hat{j}k\hat{l}}) - \epsilon_j{}^{kl}M_{\hat{h}k\hat{l}}, \tag{A.31}$$

Moreover, both (A.11) and (A.16) contain redundant generators. The following combinations obtained from (A.11):

$$\mathcal{C}_{0i} = \tfrac{1}{6}\delta^{kl}M_{ik\hat{l}} \,, \tag{A.32}$$

$$\mathcal{C}_{ij} = -\tfrac{1}{30}\epsilon_{[i|}{}^{kl}(M_{k|j]\hat{l}} - 9M_{|j]k\hat{l}}) - \tfrac{1}{30}\epsilon_{ij}{}^{k}\delta^{lm}M_{lm\hat{k}} \,, \tag{A.33}$$

$$M_{ij} = \tfrac{1}{6}M_{\hat{j}0i} = -\tfrac{1}{6}\epsilon_{[i|}{}^{kl}(M_{k|j]\hat{l}} - M_{|j]k\hat{l}}) \tag{A.34}$$

(the expressions of which in terms of Dirac matrices are the first two equations in (3.4.9) and the first equation in (3.4.10), respectively) are linearly independent. Thus (A.11) [and so (A.4), (A.6), (A.9), (A.13), (A.15) and (A.17)] can be uniquely written in terms of them:

$$M_{hi\hat{j}} = 2\delta_{hi}\mathcal{C}_{0j} + \delta_{ij}\mathcal{C}_{0h} + \delta_{hj}\mathcal{C}_{0i} - 3\delta_{hi}\epsilon_j{}^{kl}M_{kl} - 3\epsilon_{hi}{}^{k}M_{kj} + (2\epsilon_j{}^{kl}\delta_{hi} - \tfrac{1}{2}\epsilon_h{}^{kl}\delta_{ij} - \tfrac{1}{2}\epsilon_i{}^{kl}\delta_{hj})\mathcal{C}_{kl} \,. \tag{A.35}$$

Similarly, the following combinations contained in (A.16):

$$\mathcal{C}_{i\hat{j}} = \tfrac{1}{3}\epsilon_i{}^{kl}M_{\hat{j}k\hat{l}} - \tfrac{1}{6}\epsilon_j{}^{kl}(M_{\hat{k}l\hat{i}} + M_{\hat{i}k\hat{l}}) \,, \tag{A.36}$$

$$M_i = \tfrac{1}{12}\delta^{kl}(M_{\hat{k}l\hat{i}} - 2M_{\hat{i}k\hat{l}}) \,, \tag{A.37}$$

$$M = -\tfrac{1}{6}\epsilon^{hij}M_{\hat{h}i\hat{j}} \tag{A.38}$$

(which can be written in terms of Dirac matrices as in the final equation of (3.4.9) and the last two equations of (3.4.10), respectively) are linearly independent. Hence (A.16) [and so (A.3), (A.5), (A.7), (A.8), (A.10), (A.12) and (A.14)] can be uniquely written in terms of them:

$$\begin{aligned} M_{\hat{h}i\hat{j}} = & 6\delta_{hi}\epsilon_j{}^{kl}\mathcal{C}_{k\hat{l}} - 2\epsilon_{ij}{}^k(\mathcal{C}_{k\hat{h}} - 2\mathcal{C}_{h\hat{k}}) \\ & +6\delta_{hj}M_i + 3\delta_{ih}M_j - 3\delta_{ij}M_h - \epsilon_{hij}M \ . \end{aligned} \tag{A.39}$$

In summary, the linearly independent generators associated to the second order integrability condition (3.4.8) are the 21 linearly independent generators (3.4.9) and (3.4.10), namely $\{\mathcal{C}_{0i}, \mathcal{C}_{ij}, \mathcal{C}_{i\hat{j}}, M_{ij}, M_i, M\}$ (notice that $\mathcal{C}_{i\hat{j}}$ contains 8 generators, since it is traceless), which close into an algebra whenever m^2 takes the value required by the equations of motion, $m^2 = \frac{9}{20}$. Since the only condition for the generators to close the algebra is placed on m^2, they will close regardless of the orientation (*i.e.*, of the sign of m). In fact, they generate the 21-dimensional algebra $so(7)$, for both orientations [2].

Note that, by further choosing linear combinations of (3.4.9), the 14 generators $\{\mathcal{C}_{0i}, \mathcal{C}_{ij}, \mathcal{C}_{i\hat{j}}\}$ of G_2 may be re-expressed in symmetric form

$$\begin{aligned} &\Gamma_{1\hat{1}} - \Gamma_{2\hat{2}}, && \Gamma_{1\hat{1}} - \Gamma_{3\hat{3}}, && \\ &\Gamma_{0\hat{i}} + \Gamma_{j\hat{k}}, && \Gamma_{0\hat{i}} + \Gamma_{\hat{j}k}, && (i,j,k = 123, 231, 312) \\ &\Gamma_{0i} + \Gamma_{\hat{j}\hat{k}}, && \Gamma_{0i} - \Gamma_{jk}, && (i,j,k = 123, 231, 312). \end{aligned} \tag{A.40}$$

The 7 additional generators $\{M_{ij}, M_i, M\}$ of (3.4.10) extending (A.40) to $so(7)$ may also be simplified in appropriate linear combinations. One possible set of generators is given by [2]:

$$\begin{aligned} &\Gamma_{1\hat{1}} \pm i\Gamma_0, && \\ &\Gamma_{0\hat{i}} \mp i\Gamma_i, && \\ &\Gamma_{\hat{j}\hat{k}} \mp i\Gamma_{\hat{i}}, && (i,j,k = 123, 231, 312). \end{aligned} \tag{A.41}$$

Appendix B

Expansion of $d\omega^{k_s,\alpha_s}$

This appendix contains the details of the derivation of the results summarized in table 5.3, about what one-form coefficients ω^{i_p,β_p} are needed to express $d\omega^{k_s,\alpha_s}$ when the original algebra $\mathcal{G}$ is split as in (5.4.1), with the structure constants satisfying (5.4.3).

Inserting (5.4.4)-(5.4.6) into (5.2.10) where now $p, q, s = 0, 1, \ldots, n$, and using

$$\left(\sum_{\alpha=p}^{\infty} \lambda^\alpha \omega^{i_p,\alpha}\right) \wedge \left(\sum_{\alpha=q}^{\infty} \lambda^\alpha \omega^{j_q,\alpha}\right) = \sum_{\alpha=p+q}^{\infty} \lambda^\alpha \sum_{\beta=p}^{\alpha-q} \omega^{i_p,\beta} \wedge \omega^{j_q,\alpha-\beta} \, , \tag{B.1}$$

we obtain the expansion of the MC equations for $\mathcal{G}$,

$$\sum_{\alpha=s}^{\infty} \lambda^\alpha d\omega^{k_s,\alpha} = \sum_{\alpha=s}^{\infty} \lambda^\alpha \left[-\frac{1}{2} c_{i_p j_q}^{k_s} \sum_{\beta=0}^{\alpha} \omega^{i_p,\beta} \wedge \omega^{j_q,\alpha-\beta} \right] \, , \tag{B.2}$$

since the W-W conditions (5.4.3) will give zero in the r.h.s. unless $\alpha = p+q \geq s$, in agreement with the l.h.s. equation (B.2) can be made explicit for $p, q, s = 0, 1, \ldots, n$ as follows [4]:

$$\begin{aligned}
\sum_{\alpha=s}^{\infty} \lambda^\alpha d\omega^{k_s,\alpha} = -\frac{1}{2} \Bigg[& c_{i_0 j_0}^{k_s} \sum_{\alpha=0}^{\infty} \lambda^\alpha \sum_{\beta=0}^{\alpha} \omega^{i_0,\beta} \wedge \omega^{j_0,\alpha-\beta} + \\
& +2 c_{i_0 j_1}^{k_s} \sum_{\alpha=1}^{\infty} \lambda^\alpha \sum_{\beta=0}^{\alpha-1} \omega^{i_0,\beta} \wedge \omega^{j_1,\alpha-\beta} + \ldots + \\
& +2 c_{i_0 j_n}^{k_s} \sum_{\alpha=n}^{\infty} \lambda^\alpha \sum_{\beta=0}^{\alpha-n} \omega^{i_0,\beta} \wedge \omega^{j_n,\alpha-\beta} + \\
& + c_{i_1 j_1}^{k_s} \sum_{\alpha=2}^{\infty} \lambda^\alpha \sum_{\beta=1}^{\alpha-1} \omega^{i_1,\beta} \wedge \omega^{j_1,\alpha-\beta} + \ldots +
\end{aligned}$$

$$+2c^{k_s}_{i_1 j_n}\sum_{\alpha=1+n}^{\infty}\lambda^\alpha\sum_{\beta=1}^{\alpha-n}\omega^{i_1,\beta}\wedge\omega^{j_n,\alpha-\beta}+\ldots+$$

$$+c^{k_s}_{i_{n-1}j_{n-1}}\sum_{\alpha=2n-2}^{\infty}\lambda^\alpha\sum_{\beta=n-1}^{\alpha-n+1}\omega^{i_{n-1},\beta}\wedge\omega^{j_{n-1},\alpha-\beta}+$$

$$+2c^{k_s}_{i_{n-1}j_n}\sum_{\alpha=2n-1}^{\infty}\lambda^\alpha\sum_{\beta=n-1}^{\alpha-n}\omega^{i_{n-1},\beta}\wedge\omega^{j_n,\alpha-\beta}+$$

$$+c^{k_s}_{i_n j_n}\sum_{\alpha=2n}^{\infty}\lambda^\alpha\sum_{\beta=n}^{\alpha-n}\omega^{i_n,\beta}\wedge\omega^{j_n,\alpha-\beta}\Bigg] \quad . \tag{B.3}$$

Rearranging powers we get

$$\sum_{\alpha=s}^{\infty}\lambda^\alpha d\omega^{k_s,\alpha}=-\frac{1}{2}\Bigg[c^{k_s}_{i_0j_0}\omega^{i_0,0}\wedge\omega^{j_0,0}+$$
$$+\lambda\Big(c^{k_s}_{i_0j_0}\sum_{\beta=0}^{1}\omega^{i_0,\beta}\wedge\omega^{j_0,1-\beta}+2c^{k_s}_{i_0j_1}\omega^{i_0,0}\wedge\omega^{j_1,1}\Big)+$$
$$+\lambda^2\Big(c^{k_s}_{i_0j_0}\sum_{\beta=0}^{2}\omega^{i_0,\beta}\wedge\omega^{j_0,2-\beta}+$$
$$+2c^{k_s}_{i_0j_1}\sum_{\beta=0}^{1}\omega^{i_0,\beta}\wedge\omega^{j_1,2-\beta}+2c^{k_s}_{i_0j_2}\omega^{i_0,0}\wedge\omega^{j_2,2}+$$
$$+c^{k_s}_{i_1j_1}\omega^{i_1,1}\wedge\omega^{j_1,1}\Big)+\ldots\Bigg]\,. \tag{B.4}$$

Equation (B.4) now gives

$$\sum_{\alpha=s}^{\infty}\lambda^\alpha d\omega^{k_s,\alpha} = -\frac{1}{2}c^{k_s}_{i_0j_0}\omega^{i_0,0}\wedge\omega^{j_0,0}-$$
$$-\sum_{\alpha=1}^{n-1}\lambda^\alpha\Bigg[\frac{1}{2}\sum_{p=0}^{[\frac{\alpha}{2}]}c^{k_s}_{i_pj_p}\sum_{\beta=p}^{\alpha-p}\omega^{i_p,\beta}\wedge\omega^{j_p,\alpha-\beta}+$$
$$+\sum_{p=0}^{[\frac{\alpha-1}{2}]}\sum_{q=p+1}^{\alpha-p}c^{k_s}_{i_pj_q}\sum_{\beta=p}^{\alpha-q}\omega^{i_p,\beta}\wedge\omega^{j_q,\alpha-\beta}\Bigg]-$$
$$-\sum_{\alpha=n}^{2n-1}\lambda^\alpha\Bigg[\frac{1}{2}\sum_{p=0}^{[\frac{\alpha}{2}]}c^{k_s}_{i_pj_p}\sum_{\beta=p}^{\alpha-p}\omega^{i_p,\beta}\wedge\omega^{j_p,\alpha-\beta}+$$

$$+\sum_{p=0}^{[\frac{\alpha-1}{2}]}\sum_{q=p+1}^{\min\{\alpha-p\,,n\}} c^{k_s}_{i_p j_q}\sum_{\beta=p}^{\alpha-q}\omega^{i_p,\beta}\wedge\omega^{j_q,\alpha-\beta}\Bigg]-$$

$$-\sum_{\alpha=2n}^{\infty}\lambda^{\alpha}\left[\frac{1}{2}\sum_{p=0}^{n}c^{k_s}_{i_p j_p}\sum_{\beta=p}^{\alpha-p}\omega^{i_p,\beta}\wedge\omega^{j_p,\alpha-\beta}+\right.$$

$$\left.+\sum_{p=0}^{n-1}\sum_{q=p+1}^{n}c^{k_s}_{i_p j_q}\sum_{\beta=p}^{\alpha-q}\omega^{i_p,\beta}\wedge\omega^{j_q,\alpha-\beta}\right], \quad \text{(B.5)}$$

that is

$$\sum_{\alpha=s}^{\infty}\lambda^{\alpha}d\omega^{k_s,\alpha}=-\frac{1}{2}c^{k_s}_{i_0 j_0}\omega^{i_0,0}\wedge\omega^{j_0,0}-$$

$$-\sum_{\alpha=1}^{\infty}\lambda^{\alpha}\left[\frac{1}{2}\sum_{p=0}^{\min\left\{[\frac{\alpha}{2}]\,,n\right\}}c^{k_s}_{i_p j_p}\sum_{\beta=p}^{\alpha-p}\omega^{i_p,\beta}\wedge\omega^{j_p,\alpha-\beta}+\right.$$

$$\left.+\sum_{p=0}^{\min\left\{[\frac{\alpha-1}{2}]\,,n-1\right\}}\sum_{q=p+1}^{\min\{\alpha-p\,,n\}}c^{k_s}_{i_p j_q}\sum_{\beta=p}^{\alpha-q}\omega^{i_p,\beta}\wedge\omega^{j_q,\alpha-\beta}\right], \quad \text{(B.6)}$$

from which we obtain, upon explicit imposition of the contraction condition (5.4.3) on the structure constants c's [4]:

$\alpha = s = 0$:

$$d\omega^{k_0,0}=-\frac{1}{2}c^{k_0}_{i_0 j_0}\omega^{i_0,0}\wedge\omega^{j_0,0}\ ; \quad \text{(B.7)}$$

$\alpha = s \geq 1$, s odd:

$$d\omega^{k_s,s}=-\sum_{p=0}^{\frac{s-1}{2}}c^{k_s}_{i_p j_{s-p}}\omega^{i_p,p}\wedge\omega^{j_{s-p},s-p}\ ; \quad \text{(B.8)}$$

$\alpha = s \geq 1$, s even:

$$d\omega^{k_s,s}=-\frac{1}{2}c^{k_s}_{i_{\frac{s}{2}} j_{\frac{s}{2}}}\omega^{i_{\frac{s}{2}},\frac{s}{2}}\wedge\omega^{j_{\frac{s}{2}},\frac{s}{2}}-\sum_{p=0}^{\frac{s-2}{2}}c^{k_s}_{i_p j_{s-p}}\omega^{i_p,p}\wedge\omega^{j_{s-p},s-p}\ ; \quad \text{(B.9)}$$

$\alpha > s \geq 0$:

$$d\omega^{k_s,\alpha} = -\frac{1}{2} \sum_{p=\left[\frac{s+1}{2}\right]}^{\min\left\{\left[\frac{\alpha}{2}\right],n\right\}} c^{k_s}_{i_p j_p} \sum_{\beta=p}^{\alpha-p} \omega^{i_p,\beta} \wedge \omega^{j_p,\alpha-\beta} -$$

$$- \sum_{p=0}^{\min\left\{\left[\frac{\alpha-1}{2}\right],n-1\right\}} \sum_{q=\max\{s-p,p+1\}}^{\min\{\alpha-p,n\}} c^{k_s}_{i_p j_q} \sum_{\beta=p}^{\alpha-q} \omega^{i_p,\beta} \wedge \omega^{j_q,\alpha-\beta} \,. \tag{B.10}$$

Appendix C

Symmetry breaking of the supertwistor string

This appendix contains the details of the breaking of the $OSp(2n|1)$ symmetry down to the supergroup $\Sigma^{(\frac{n(n+1)}{2}|n)} \rtimes Sp(n)$, generalizing super-Poincaré, in the supertwistor formulation (section 7.6) of the supersymmetric string model in tensorial superspace of chapter 7.

The supergroup $OSp(2n|1)$ is characterized by the $(2n+1)\times(2n+1)$ supermatrices $\mathcal{G}_\Sigma{}^\Pi$ that preserve the graded-antisymmetric matrix $\Omega_{\Sigma\Pi} = -(-1)^{\deg(\Sigma)\deg(\Pi)}\Omega_{\Pi\Sigma}$, 'orthosymplectic metric',

$$\mathcal{G}_\Sigma{}^{\Sigma'}\Omega_{\Sigma'\Pi'}\mathcal{G}_\Pi{}^{\Pi'}(-1)^{\deg(\Pi)(\deg(\Pi')+1)} = \Omega_{\Sigma\Pi} \ , \tag{C.1}$$

the canonical form of which is given by equation (7.6.8). The grading is defined by

$$(-1)^{\deg(\Sigma)} = \begin{cases} 1 & \text{for } \Sigma = 1, \dots, 2n \\ -1 & \text{for } \Sigma = 2n+1 \end{cases} \tag{C.2}$$

and coincides with $\deg(\pm\Sigma)$ for $Y^{\pm\Sigma}$ (see below equation (7.5.7)). The fundamental representation of $OSp(2n|1)$ acts on supertwistors

$$Y^\Sigma = (\mu^\alpha, \lambda_\alpha, \eta) \ , \tag{C.3}$$

with even $\mu^\alpha, \lambda_\alpha$ and odd η. Near the unity,

$$\mathcal{G}_\Sigma{}^\Pi \sim \delta_\Sigma{}^\Pi + \Xi_\Sigma{}^\Pi \ , \tag{C.4}$$

where $\Xi_\Sigma{}^\Pi$ is an element of the $osp(2n|1)$ superalgebra. It has the form

$$\Xi_\Sigma{}^\Pi = \begin{pmatrix} G_\alpha{}^\beta & K_{\alpha\beta} & \zeta_\alpha \\ A^{\alpha\beta} & -G_\beta{}^\alpha & \epsilon^\alpha \\ i\epsilon^\beta & -i\zeta_\beta & 0 \end{pmatrix} \ , \tag{C.5}$$

where the even $n \times n$ matrix $G_\alpha{}^\beta$ is arbitrary and the even $n \times n$ $K_{\alpha\beta} = K_{\beta\alpha}$ and $A^{\alpha\beta} = A^{\beta\alpha}$ matrices are symmetric. They define a $gl(n)$ and two $sp(n)$ subalgebras of $osp(2n|1)$,

$$G_\alpha{}^\beta \in gl(n) \ , \quad A^{\alpha\beta} \in sp(n) \ , \quad K_{\alpha\beta} \in sp(n) \ . \tag{C.6}$$

Exploiting the analogy with the matrix representation of the standard 4-dimensional conformal algebra $su(2,2|N)$ and the 4-dimensional super-Poincaré algebra, one can look at the $gl(n)$ boxes G as a generalization of the $spin(1, D-1)$ and dilatation algebras $(L_\alpha{}^\beta + \delta_\alpha{}^\beta D)$, at the elements $A^{\alpha\beta} \in sp(n)$ as a generalization of the translation one, and at $K_{\alpha\beta} \in sp(n)$ as a generalization of the special conformal transformations. Equation (C.5) also contains two fermionic parameters, ϵ^α and ζ_α, which can be identified as those of the of 'usual' and special conformal supersymmetries. A specific check is provided by the $n = 2$ case, where $SL(2,\mathbb{R}) = Spin(1,2)$, the symmetric spin-tensor provides an equivalent representation for a $SO(1,2)$ vector, and the superconformal group is $OSp(2|1)$.

If we now demand in addition that the degenerate matrix $C_{\Sigma\Pi}$ (equation (7.6.9)) is preserved,

$$\mathcal{G}_\Sigma{}^{\Sigma'} C_{\Sigma'\Pi'} \mathcal{G}_\Pi{}^{\Pi'} (-1)^{\deg(\Pi)(\deg(\Pi')+1)} = C_{\Sigma\Pi} \ , \tag{C.7}$$

we see that this is satisfied by the $osp(2n|1)$ elements of the form

$$\Xi_\Sigma{}^\Pi = \begin{pmatrix} S_\alpha{}^\beta & 0 & 0 \\ A^{\alpha\beta} & -S_\beta{}^\alpha & \epsilon^\alpha \\ i\epsilon^\beta & 0 & 0 \end{pmatrix} \equiv \Xi_\Sigma{}^\Pi(S, A, \epsilon) \quad , \tag{C.8}$$

where $S_\alpha{}^\beta \in sp(n)$,

$$S^{\alpha\beta} \equiv C^{\alpha\gamma} S_\gamma{}^\beta = S^{\beta\alpha} \ , \tag{C.9}$$

i.e. by those of (C.5) with $K_{\alpha\beta} = 0$, $\zeta_\alpha = 0$ and $G_\alpha{}^\beta = S_\alpha{}^\beta \in sp(n)$. Thus the condition (C.7) not only reduces $GL(n)$ symmetry down to $Sp(n)$, but also breaks the generalized special conformal transformations and the superconformal supersymmetry.

The right action of $\mathcal{G}_\Sigma{}^\Pi(S, A, \epsilon)$ (Eqs. (C.4), (C.8)) on the supertwistor (C.3), $\delta Y^\Sigma = Y^\Pi \Xi_\Pi{}^\Sigma$, defines the generalized super-Poincaré transformation of the supertwistor components,

$$\begin{aligned} &\delta\mu^\alpha = \mu^\beta S_\beta{}^\alpha + \lambda_\beta A^{\beta\alpha} + i\epsilon^\alpha \eta \ , \\ &\delta\lambda_\alpha = -S_\alpha{}^\beta \lambda_\beta \ , \qquad \delta\eta = \epsilon^\alpha \lambda_\alpha \ . \end{aligned} \tag{C.10}$$

These can be reproduced from the following transformations of the coordinates of $\Sigma^{(\frac{n(n+1)}{2}|n)}$,

$$\delta X^{\alpha\beta} = A^{\alpha\beta} + i\theta^{(\alpha}\epsilon^{\beta)} + 2X^{(\alpha|\gamma} S_\gamma{}^{|\beta)} \ , \quad \delta\theta^\alpha = \epsilon^\alpha + \theta^\beta A_\beta{}^\alpha \ , \tag{C.11}$$

using the generalization [148] of the Penrose correspondence relation [211, 212] given in equation (7.6.2),

$$\mu^\alpha = X^{\alpha\beta}\lambda_\beta - \frac{i}{2}\theta^\alpha\theta^\beta\lambda_\beta \ , \quad \eta = \theta^\alpha\lambda_\alpha \ . \tag{C.12}$$

The transformations (C.11) of the $\Sigma^{(\frac{n(n+1)}{2}|n)}$ variables are a straightforward generalization of the super-Poincaré transformations of the standard superspace coordinates. This justifies calling the resulting supergroup $\Sigma^{(\frac{n(n+1)}{2}|n)} \rtimes Sp(n)$ a generalization of the super-Poincaré group.

Going back to $osp(2n|1)$, let us note that the generalized special superconformal transformations $(K_{\alpha\beta}, \zeta_\alpha)$ act on the supertwistor components by

$$\delta\mu^\alpha = 0 \ , \quad \delta\lambda_\alpha = \mu^\beta K_{\beta\alpha} - i\eta\zeta_\alpha \ , \quad \delta\eta = \mu^\beta\zeta_\beta \ . \tag{C.13}$$

Using equation (C.12) one may find from (C.13) the generalized special superconformal transformations of the $\Sigma^{(\frac{n(n+1)}{2}|n)}$ coordinates

$$\begin{aligned} \delta X^{\alpha\beta} &= i\theta^{(\alpha}X^{\beta)\gamma}\zeta_\gamma - (XKX)^{\alpha\beta} \ , \\ \delta\theta^\alpha &= X^{\alpha\beta}\zeta_\beta - \frac{i}{2}(\theta\zeta)\,\theta^\alpha - (\theta K X)^\alpha \ . \end{aligned} \tag{C.14}$$

Note that (C.11) follows as well from a nonlinear realization of the generalized super-Poincaré group $\Sigma^{(\frac{n(n+1)}{2}|n)} \rtimes Sp(n)$ on the $\Sigma^{(\frac{n(n+1)}{2}|n)}$ coset, *i.e.* from the left action of $\mathcal{G}_\Sigma{}^\Pi(S,A,\epsilon) \sim \delta_\Sigma{}^\Pi + \Sigma_\Sigma{}^\Pi(S,A,\epsilon)$ (C.8) on $\mathcal{K}_\Sigma{}^\Pi(X,\theta) \sim \delta_\Sigma{}^\Pi + K_\Sigma{}^\Pi(X,\theta)$ with

$$K_\Sigma{}^\Pi(X,\theta) = \begin{pmatrix} 0 & 0 & 0 \\ X^{\alpha\beta} & 0 & \theta^\alpha \\ i\theta^\beta & 0 & 0 \end{pmatrix} \quad . \tag{C.15}$$

Indeed, the infinitesimal form of

$$\mathcal{G}_\Sigma{}^\Pi(S,A,\epsilon)\mathcal{K}_\Sigma{}^\Pi(X,\theta) = \mathcal{K}_\Sigma{}^\Pi(X',\theta')\mathcal{G}_\Sigma{}^\Pi(A,0,0) \tag{C.16}$$

reads

$$\begin{aligned} K(\delta X, \delta\theta) = {} & \Xi(0,A,\epsilon) + \Xi(0,A,\epsilon)K(X,\theta) \\ & + [\Xi(S,0,0)\,,\, K(X,\theta)] \end{aligned} \tag{C.17}$$

and reproduces the generalized super-Poincaré transformations (C.11) [8].

Bibliography

[1] I.A. Bandos, J.A. de Azcárraga, M. Picón and O. Varela, *Generalized curvature and the equations of $D = 11$ supergravity*, Phys. Lett. **B615**, 127–133 (2005) [arXiv:hep-th/0501007].

[2] A. Batrachenko, J.T. Liu, O. Varela and W.Y. Wen, *Higher order integrability in generalized holonomy*, arXiv:hep-th/0412154.

[3] I.A. Bandos, J.A. de Azcárraga, J.M. Izquierdo, M. Picón and O. Varela, *BPS preons, generalized holonomies and $D = 11$ supergravities*, Phys. Rev. **D69**, 105010 (2004) [arXiv:hep-th/0312266].

[4] J.A. de Azcárraga, J.M. Izquierdo, M. Picón and O. Varela, *Generating Lie and gauge free differential (super)algebras by expanding Maurer-Cartan forms and Chern-Simons supergravity*, Nucl. Phys. **B662**, 185–219 (2003) [arXiv:hep-th/0212347].

[5] J.A. de Azcárraga, J.M. Izquierdo, M. Picón and O. Varela, *Extensions, expansions, Lie algebra cohomology and enlarged superspaces*, Class. Quantum Grav. **21**, S1375–S1384 (2004) [arXiv:hep-th/0401033].

[6] I.A. Bandos, J.A. de Azcárraga, J.M. Izquierdo, M. Picón and O. Varela, *On the underlying gauge group structure of $D = 11$ supergravity*, Phys. Lett. **B596**, 145–155 (2004) [arXiv:hep-th/0406020].

[7] I.A. Bandos, J.A. de Azcárraga, M. Picón and O. Varela, *On the formulation of $D = 11$ supergravity and the composite nature of its three-form gauge field*, Ann. Phys. **317**, 238–279 (2005) [arXiv:hep-th/0409100].

[8] I.A. Bandos, J.A. de Azcárraga, M. Picón and O. Varela, *Supersymmetric string model with 30 κ-syymetries in an extended $D = 11$*

superspace and 30/32 BPS states, Phys. Rev. **D69**, 085007 (2004) [arXiv:hep-th/0307106].

[9] Yu.A. Gol'fand and E.P. Likhtman, *Extensions of the algebra of Poicaré group generators and violation of P-invariance*, JETP Lett. **13**, 323–326 (1972).

[10] D.V. Volkov and V.P. Akulov, *Possible universal neutrino interaction*, JETP Lett. **16**, 438–440 (1972).

[11] J. Wess and B. Zumino, *Supergauge transformations in four dimensions*, Nucl. Phys. **B70**, 39–50 (1974).

[12] P. Fayet and S. Ferrara, *Supersymmetry*, Phys. Rept. **32**, 249–334 (1977).

[13] M.F. Sohnius, *Introducing supersymmetry*, Phys. Rept. **128**, 39–204 (1985).

[14] S. Ferrara (ed.), *Supersymmetry* (vols. I,II), World Sci. (1987).

[15] P. van Nieuwenhuizen, *Supergravity*, Phys. Rept. **68**, 189–398 (1981).

[16] A. Van Proeyen, *Structure of supergravity theories*, Proceedings of the XI Fall Workshop on Geometry and Physics, Oviedo, (2002) [arXiv:hep-th/0301005].

[17] T. Ortín, *Gravity and strings*, Cambridge University Press (2004).

[18] L. Castellani, R. D'Auria and P. Fré, *Supergravity and superstrings: a geometric perspective*, vols. I, II, III, World Scientific (1991).

[19] D.Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, *Progress towards a theory of supergravity*, Phys. Rev. **D13**, 3214–3218 (1976).

[20] S. Deser, B. Zumino, *Consistent supergravity*, Phys. Lett. **B62**, 335–337 (1976).

[21] M.T. Grisaru, *Two-loop renormalizability of supergravity*, Phys. Lett. **B66**, 75–76 (1977).

[22] P. van Nieuwenhuizen and J.A.M. Vermaseren, *Finiteness of $SO(3)$ and $SO(4)$ supergravity and nonrenormalizability of the scalar multiplet in explicit one-loop examples*, Phys. Rev. **D16**, 298–303 (1977).

[23] E. Cremmer and B. Julia, *The $N = 8$ supergravity theory. I. The lagrangian*, Phys. Lett. **B80**, 48–51 (1978); *The $SO(8)$ supergravity*, Nucl. Phys. **B159**, 141–212 (1979).

[24] B. de Wit and H. Nicolai, *$N = 8$ supergravity with local $SO(8) \times SU(8)$ invariance*, Phys. Lett. **B108**, 285–290 (1982); *$N = 8$ supergravity*, Nucl. Phys. **B208**, 323–364 (1982).

[25] A. Salam and E. Sezgin, *Supergravities in diverse dimensions*, Elsevier and World Scientific (1989).

[26] W. Nahm, *Supersymmetries and their representations*, Nucl. Phys. **B135**, 149–166 (1978).

[27] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in eleven dimensions*, Phys. Lett. **B76**, 409–412 (1978).

[28] B. de Wit and D. Freedman, *On $SO(8)$ extended supergravity*, Nucl. Phys. **B130**, 105–113 (1977).

[29] M.J. Duff, B.E.W. Nilsson and C.N. Pope, *Kaluza-Klein supergravity*, Phys. Rept. **130**, 1–142 (1986).

[30] M.A. Awada, M.J. Duff and C.N. Pope, *$N = 8$ supergravity breaks down to $N = 1$*, Phys. Rev. Lett. **50**, 294 (1983).

[31] E. Witten, *Search for a realistic Kaluza-Klein theory*, Nucl. Phys. **B186**, 412 (1981).

[32] P.G.O. Freund and M.A. Rubin, *Dynamics of dimensional reduction*, Phys. Lett. **B97**, 233 (1980).

[33] E. Witten, *Fermion quantum numbers in Kaluza-Klein theory*, in *Shelter Island II*, R. Jackiw, N. Khuri, S. Weinberg and E. Witten eds., M.I.T. Press (1985).

[34] B. Acharya and E. Witten, *Chiral fermions from manifolds of G_2 holonomy*, arXiv:hep-th/0109152.

[35] G. Veneziano, *Construction of a crossing-symmetric, Regge behaved amplitude for linearly rising trajectories*, Nuovo Cim. **A57**, 190–197 (1968).

[36] M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory*, vols. I, II, Cambridge University Press (1987).

[37] J. Polchinski, *String Theory*, vols. I, II, Cambridge University Press (1998).

[38] J. Scherk and J.H. Schwarz, *Dual models for non-hadrons*, Nucl. Phys. **B81**, 118–144 (1974).

[39] M.B. Green and J.H. Schwarz, *Covariant description of superstrings*, Phys. Lett. **B136**, 367 (1984).

[40] J.A. de Azcárraga and J.M. Izquierdo, *Lie groups, Lie algebras, cohomology and some applications in physics*, Cambridge University Press (1995).

[41] J.A. de Azcárraga and J. Lukierski, *Supersymmetric particles with internal symmetries and central charges*, Phys. Lett. **B113**, 170 (1982); *Supersymmetric particles in $N = 2$ superespace: phase space variables and hamiltonian dynamics*, Phys. Rev. **D28**, 1337 (1983).

[42] W. Siegel, *Hidden local supersymmetry in the supersymmetric particle action*, Phys. Lett. **B128**, 397-399 (1983); *Spacetime-supersymmetric quantum mechanics*, Class. Quantum Grav. **2**, L95–L97 (1985).

[43] M.B. Green and J.H. Schwarz, *Anomaly cancellation in supersymmetric $D = 10$ gauge theory and superstring theory*, Phys. Lett. **B149**, 117–122 (1984).

[44] D.J. Gross, J.A. Harvey, E.J. Martinec and R. Rohm, *Heterotic string*, Phys. Rev. Lett. **54**, 502–505 (1985).

[45] D.V. Volkov, D.P. Sorokin and V.I. Tkach, *Mechanisms of spontaneous compactification of $N = 2$, $d = 10$ supergravitation*, JETP Lett. **38**, 481–485 (1983).

[46] P. Candelas, G.T. Horowitz, A. Strominger and E. Witten, *Vacuum configurations for superstrings*, Nucl. Phys. **B258**, 46–74 (1985).

[47] E.B. Bogomol'nyi, *The stability of classical solutions*, Sov. J. Nucl. Phys. **24** 449 (1976).

[48] M.K. Prasad and C.M. Sommerfield, *An exact classical solution for the 't Hooft monopole and the Julia-Zee dyon*, Phys. Rev. Lett. **35**, 760–762 (1975).

[49] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, Phys. Rept. **244**, 77–202 (1994) [arXiv:hep-th/9401139].

[50] A. Font, L. Ibáñez, D. Lüst and F. Quevedo, *Strong–Weak coupling duality and nonperturbative effects in string theory*, Phys. Lett. **B249**, 35–43 (1990).

[51] C. Montonen, D.I. Olive, *Magnetic monopoles as gauge particles?*, Phys. Lett. **B72**, 117 (1977).

[52] C. Hull and P. Townsend, *Unity of superstring dualities*, Nucl. Phys. **B450**, 69 (1995) [arXiv:hep-th/9410167].

[53] J.H. Schwarz, *The second superstring revolution*, in Proc. of the *Second Int. Sakharov conference on physics, Moscow, 1996*, I.M. Dremin and A.M. Semikhatov eds., World Sci. 1997, pp. 562-569 [arXiv:hep-th/9607067]; *Lectures on superstring and M theory dualities*, Nucl. Phys. Proc. Suppl. **55B**, 1-32 (1997) [arXiv:hep-th/9607201].

M.J. Duff, *M Theory (the theory formerly known as Strings)*, Int. J. Mod. Phys. **A11**, 5623–5642 (1996) [arXiv:hep-th/9608117].

[54] A. Achúcarro, J.M. Evans, P.K. Townsend and D.L. Wiltshire, *Super p-branes*, Phys. Lett. **B198**, 441 (1987).

[55] J.M. Evans, *Super p-brane Wess-Zumino terms*, Class. Quantum Grav. **5**, L87–L90 (1988).

[56] P.K. Townsend, *The eleven-dimensional supermembrane revisited*, Phys. Lett. **B350**, 184–188 (1995) [arXiv:hep-th/9501068].

[57] J.A. de Azcárraga, J.P. Gauntlett, J.M. Izquierdo and P.K. Townsend, *Topological extensions of the supersymmetry algebra for extended objects*, Phys. Rev. Lett. **63**, 2443 (1989).

[58] D.P. Sorokin and P.K. Townsend, *M Theory superalgebra from the M five-brane*, Phys. Lett. **B412**, 265 (1997) [arXiv:hep-th/9708003].

[59] P.K. Townsend, *p-brane democracy*, in *Particles, strings and cosmology*, J. Bagger *et al.* eds., World Sci., 271–285 (1996) [arXiv:hep-th/9507048]; *M-theory from its superalgebra*, in *Strings, branes and dualities*, NATO ASI Ser. C, Math. and Phys. Sci., **520**, 141–177 (1999) [arXiv:hep-th/9712004], and refs. therein.

[60] E. Witten, *String theory dynamics in various dimensions*, Nucl. Phys. **B443**, 85–126 (1995) [arXiv:hep-th/9503124].

[61] P. Hořava and E. Witten, *Eleven-Dimensional Supergravity on a Manifold with Boundary*, Nucl. Phys. **B475**, 94 (1996) [arXiv:hep-th/9603142].

[62] M.J. Duff, *The world in eleven dimensions: supergravity, supermembranes and M-theory*, Institute of Physics Publishing (1999).

[63] M.J. Duff, P.S. Howe, T. Inami and K.S. Stelle, *Superstrings in D=10 from supermembranes in D=11*, Phys. Lett. **B191**, 70 (1987).

[64] M.J. Duff, R.R. Khuri and J.X. Lu, *String solitons*, Phys. Rept. **259**, 213 (1995) [arXiv:hep-th/9412184].

[65] J. Polchinski, *Dirichlet branes and Ramond-Ramond charges*, Phys. Rev. Lett. **75**, 4724–4727 (1995) [arXiv:hep-th/9510017].

[66] C.V. Johnson, *D-Branes*, Cambridge University Press (2003).

[67] E. Kiritsis, *D-branes in standard model building, gravity and cosmology*, Fortsch. Phys. **52**, 200–263 (2004) and Phys. Rept. **421**, 105–190 (2005) [arXiv:hep-th/0310001].

[68] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2**, 231–252 (1998) and Int. J. Theor. Phys. **38**, 1113–1133 (1999) [arXiv:hep-th/9711200].

[69] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri, Y. Oz, *Large N field theories, string theory and gravity*, Phys. Rept. **323**, 183–386 (2000).

[70] B.S. Acharya, J.M. Figueroa-O'Farrill, C.M. Hull and B.J. Spence, *Branes at conical singularities and holography*, Adv. Theor. Math. Phys. **2**, 1249–1286 (1999) [arXiv:hep-th/9808014].

[71] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Supermembranes and eleven-dimensional supergravity*, Phys. Lett. **B189**, 75–78 (1987); *Properties of the eleven dimensional super membrane theory*, Ann. Phys. (NY) **185**, 330 (1988);

[72] I. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D.P. Sorokin and M. Tonin, *Covariant action for the super-five-brane of M-theory*, Phys. Rev. Lett. **78**, 4332 (1997) [arXiv:hep-th/9701149].

M. Aganagic, J. Park, C. Popescu and J.H. Schwarz, *World-volume action of the M-theory five-brane*, Nucl. Phys. **B 496**, 191 (1997) [arXiv:hep-th/9701166].

[73] M.J. Duff and K.S. Stelle, *Multi–membrane solutions of $D = 11$ Supergravity*, Phys. Lett. **B253**, 113–118 (1991).

[74] R. Güven, *Black p-brane solutions of $D = 11$ supergravity theory*, Phys. Lett. **B276**, 49–55 (1992).

[75] K.S. Stelle, *BPS branes in supergravity*, in *High energy physics and cosmology 1997: Proceedings of the ICTP Summer school in high energy physics and cosmology* (E. Gava et al eds.), 29–127 (1997) [arXiv:hep-th/9803116].

[76] G. Papadopoulos and P.K. Townsend, *Intersecting M-branes*, Phys. Lett. **B380**, 273 (1996) [arXiv:hep-th/9603087].

A.A. Tseytlin, *Harmonic superpositions of M-branes*, Nucl. Phys. **B475**, 149 (1996) [arXiv:hep-th/9604035].

J.P. Gauntlett, D.A. Kastor and J.H. Traschen, *Overlapping branes in M-Theory*, Nucl. Phys. **B478**, 544 (1996) [arXiv:hep-th/9604179].

E. Bergshoeff, M. de Roo, E. Eyras, B. Janssen and J.P. van der Schaar, *Multiple intersections of D-branes and M-branes*, Nucl. Phys. **B494**, 119 (1997) [arXiv:hep-th/9612095].

[77] C.M. Hull, *Exact pp wave solutions of 11–dimensional supergravity*, Phys. Lett. **B139**, 39–41 (1984).

[78] M.J. Duff and J.T. Liu, *Hidden spacetime symmetries and generalized holonomy in M-theory*, Nucl. Phys. **B674**, 217–230 (2003) [arXiv:hep-th/0303140].

[79] J.P. Gauntlett, D. Martelli, S. Pakis and D. Waldram, *G-structures and wrapped NS5-branes*, Comm. Math. Phys. **247**, 421–445 (2004) [arXiv:hep-th/0205050].

[80] J.P. Gauntlett and S. Pakis, *The geometry of* $D = 11$ *Killing spinors*, JHEP **0304**, 039 (2003) [arXiv:hep-th/0212008].

[81] J. Gillard, U. Gran and G. Papadopoulos, *The spinorial geometry of supersymmetric backgrounds*, Class. Quant. Grav. **22** (2005) 1033 [arXiv:hep-th/0410155].

[82] J.W. van Holten and A. Van Proeyen, *N=1 supersymmetry algebras in d=2,3,4 mod 8*, J. Phys. **A15**, 3763–3783 (1982).

[83] I.A. Bandos, J.A. de Azcárraga, J.M. Izquierdo and J. Lukierski, *BPS states in M theory and twistorial constituents*, Phys. Rev. Lett. **86**, 4451 (2001) [arXiv:hep-th/0101113].

[84] I.A. Bandos, *BPS preons in supergravity and higher spin theories. An overview from the hill of twistor approach*, in *Fundamental interactions and twistor-like methods*, J. Lukierski and D. Sorokin eds., AIP Conf. Proc. **767**, 141–171 (2005) [arXiv:hep-th/0501115].

[85] C. Chryssomalakos, J.A. de Azcárraga, J.M. Izquierdo and J.C. Pérez Bueno, *The geometry of branes and extended superspaces*, Nucl. Phys. **B567**, 293–330 (2000) [arXiv:hep-th/9904137].

J.A. de Azcárraga and J.M. Izquierdo, *Superalgebra cohomology, the geometry of extended superspaces and superbranes*, in *New developments in fundamental interactions theories. Procs. of 37th Karpacz Winter School of Theoretical Physics*, AIP Conf. Proc. **589**, 3–17 (2001) [arXiv:hep-th/0105125].

[86] J.A. de Azcárraga, *Superbranes,* $D = 11$ *CJS supergravity and enlarged superspace coordinates/fields correspondence*, in *Fundamental interactions and twistor-like methods*, J. Lukierski and D. Sorokin eds., AIP Conf. Proc. **767**, 243–267 (2005) [arXiv:hep-th/0501198].

[87] C. Hull, *Holonomy and symmetry in M-theory*, arXiv:hep-th/0305039.

[88] P. West, E_{11} *and M theory*, Class. Quantum Grav. **18**, 4443-4460 (2001) [arXiv:hep-th/0104081].

[89] I. Bars, C. Deliduman and D. Minic, *Lifting M-theory to two-time physics*, Phys. Lett. **B457**, 275-284 (1999) [arXiv:hep-th/9904063].

[90] P. West, *Hidden superconformal symmetry in M theory*, JHEP **0008**, 007 (2000) [arXiv:hep-th/0005270].

[91] R. Troncoso and J. Zanelli, *New gauge supergravity in seven and eleven dimensions*, Phys. Rev. **D58**, 101703 (1998) [arXiv:hep-th/9710180];

J. Zanelli, *Chern-Simons gravity: from* $2+1$ *to* $2n+1$ *dimensions*, Braz. J. Phys. **30**,251–267 (2000) [arXiv:hep-th/0010049]; *(Super)-gravities beyond 4 dimensions*, arXiv:hep-th/0206199.

P. Hořava, *M-theory as a holographic field theory*, Phys. Rev. **D59**, 046004 (1999) [arXiv:hep-th/9712130];

H. Nastase, *Towards a Chern-Simons M theory of* $OSp(1|32) \times OSp(1|32)$, arXiv:hep-th/0306269;

M. Bañados, *The linear spectrum of OSp(32|1) Chern-Simons supergravity in eleven dimensions*, Phys. Rev. Lett. **88**, 031301 (2002) [arXiv:hep-th/0107214].

[92] R. D'Auria and P. Fré, *Geometric supergravity in D = 11 and its hidden supergroup*, Nucl. Phys. **B201**, 101–140 (1982) [*Erratum-ibid.* **B206**, 496 (1982)].

[93] M. Hassaine, R. Troncoso and J. Zanelli, *Poincaré invariant gravity with local supersymmetry as a gauge theory for the M-algebra*, Phys. Lett. **B596**, 132–137 (2004) [arXiv:hep-th/0306258].

[94] A. Batrachenko, M.J. Duff, J.T. Liu and W.Y. Wen, *Generalized holonomy of M-theory vacua*, Nucl. Phys. **B726**, 275–293 (2005) [arXiv:hep-th/0312165].

[95] M.J. Duff, *M-theory on manifolds of G_2 holonomy: The first twenty years*, arXiv:hep-th/0201062.

[96] D. Sullivan, *Infinitesimal computations in topology*, Inst. des Haut. Étud. Sci., Pub. Math. **47**, 269–331 (1977).

[97] P. van Nieuwenhuizen, *Free graded differential superalgebras*, in *Group theoretical methods in physics*, M. Serdaroğlu and E. İnönü Eds., Lect. Notes in Phys. **180**, 228–247 (1983).

[98] K. Bautier, S. Deser, M. Henneaux and D. Seminara, *No cosmological $D = 11$ supergravity*, Phys. Lett. **B406**, 49-53 (1997) [arXiv:hep-th/9704131].

[99] E. Sezgin, *The M-algebra*, Phys. Lett. **392**, 323–331 (1997) [arXiv:hep-th/9609086].

[100] C.M. Hull, *Gravitational Duality, Branes and Charges*, Nucl. Phys. **B509**, 216 (1998) [arXiv:hep-th/9705162].

[101] E. Bergshoeff, E. Eyras and Y. Lozano, *The massive Kaluza-Klein monopole*, Phys. Lett. **B430**, 77 (1998) [arXiv:hep-th/9802199];

E. Eyras, Y. Lozano, *The Kaluza-Klein monopole in a massive IIA background*, Nucl. Phys. **B546**, 197–218 (1999) [arXiv:hep-th/9812188].

[102] E. Bergshoeff and J.P. van der Schaar, *On M-9-branes*, Class. Quantum Grav. **16**, 23 (1999) [arXiv:hep-th/9806069].

[103] E.J. Hackett-Jones, D.C. Page and D.J. Smith, *Topological charges for branes in M-theory*, JHEP **0310**, 005 (2003) [arXiv:hep-th/0306267].

[104] I. Bars, *Duality covariant type IIB supersymmetry and nonperturbative consequences*, Phys. Rev. **D56**, 7954 (1997) [arXiv:hep-th/9706185].

[105] B. Julia and S. Silva, *On first order formulations of supergravities*, JHEP **0001**, 026 (2000) [arXiv:hep-th/9911035].

[106] B. Biran, F. Englert, B. de Wit and H. Nicolai, *Gauged $N = 8$ supergravity and its breaking from spontaneous compactification*, Phys. Lett. **B124**, 45 (1983) [E.:*ibid.* **B 128**, 461 (1983)].

[107] G. Papadopoulos and D. Tsimpis, *The holonomy of the supercovariant connection and Killing spinors*, JHEP **0307**, 018 (2003) [arXiv:hep-th/0306117].

[108] G. Papadopoulos and D. Tsimpis, *The holonomy of IIB supercovariant connection*, Class. Quantum Grav. **20**, L253 (2003) [arXiv:hep-th/0307127].

[109] J. Figueroa O'Farrill and G. Papadopoulos, *Maximally supersymmetric solutions of ten and eleven-dimensional supergravities*, JHEP **0303**, 048 (2003) [arXiv:hep-th/0211089].

[110] A. Batrachenko, W.Y. Wen, *Generalized holonomy of supergravities with 8 real supercharges*, Nucl. Phys. **B690**, 331–340 (2004) [arXiv:hep-th/0402141].

[111] H. Lu, C.N. Pope, K.S. Stelle, *Generalized holonomy for higher order corrections to supersymmetric backgrounds in String and M-Theory*, Nucl. Phys. **B741**, 17–23 (2006) [arXiv:hep-th/0509057].

[112] A.L. Besse, *Einstein manifolds*, Springer-Verlag (1987).

[113] D.J. Gross and M.J. Perry, *Magnetic Monopoles In Kaluza-Klein Theories*, Nucl. Phys. **B226**, 29 (1983).

[114] R.D. Sorkin, *Kaluza-Klein Monopole*, Phys. Rev. Lett. **51**, 87 (1983).

[115] M.J. Duff and J.X. Lu, *A Duality between strings and five-branes*, Class. Quantum Grav. **9**, 1 (1992).

[116] K. Pilch, P. van Nieuwenhuizen and P. K. Townsend, *Compactification of $D = 11$ supergravity on S(4) (or $11 = 7 + 4$, too)*, Nucl. Phys. **B242**, 377 (1984).

[117] J. Kowalski-Glikman, *Vacuum states in supersymmetric Kaluza-Klein theory*, Phys. Lett. **B134**, 194 (1984).

P.T. Chrusciel and J. Kowalski-Glikman, *The isometry group and killing spinors for the p p wave space-time in $D = 11$ supergravity*, Phys. Lett. **B149**, 107 (1984).

[118] J.P. Gauntlett and C.M. Hull, *BPS states with extra supersymmetry*, JHEP 0001 **004** (2000) [arXiv:hep-th/9909098].

[119] M. Cvetič, H. Lü and C.N. Pope, *M-theory pp-waves, Penrose limits and supernumerary supersymmetries*, Nucl. Phys. **B644**, 65 (2002) [arXiv:hep-th/0203229].

[120] J.P. Gauntlett and C.M. Hull, *PP-waves in 11-dimensions with extra supersymmetry*, JHEP **0206**, 013 (2002) [arXiv:hep-th/0203255].

[121] I. Bena and R. Roiban, *A supergravity pp-wave solutions with 28 and 24 supercharges*, Phys. Rev. **D67**, 125014 (2003) [arXiv:hep-th/0206195].

[122] J. Michelson, *(Twisted) toroidal compactification of pp-waves*, Phys. Rev. D **66**, 066002 (2002) [arXiv:hep-th/0203140]; *A pp-Wave With 26 Supercharges*, Class. Quantum Grav. **19**, 5935 (2002) [arXiv:hep-th/0206204].

[123] J.P. Gauntlett, J.B. Gutowski, C.M. Hull, S. Pakis and H.S. Reall, *All supersymmetric solutions of minimal supergravity in five dimensions*, Class. Quant. Grav. **20** (2003) 4587 [arXiv:hep-th/0209114].

[124] T. Harmark and T. Takayanagi, *Supersymmetric Gödel universes in string theory*, Nucl. Phys. **B662**, 3 (2003) [arXiv:hep-th/0301206].

[125] K. Peeters, P. Vanhove and A. Westerberg, *Supersymmetric higher-derivative actions in ten and eleven dimensions, the associated superalgebras and their formulation in superspace*, Class. Quantum Grav. **18**, 843 (2001) [arXiv:hep-th/0010167]; *Supersymmetric R^4 actions and quantum corrections to superspace torsion constraints*, in *Kiev 2000, Noncommutative structures in mathematics and physics* [arXiv:hep-th/0010182].

[126] P.S. Howe and D. Tsimpis, *On higher-order corrections in M theory*, JHEP **0309**, 038 (2003) [arXiv:hep-th/0305129].

[127] H. Lu, C.N. Pope, K.S. Stelle, P.K. Townsend, *Supersymmetric deformations of G_2 manifolds from higher order corrections to string and M Theory*, JHEP **0410**, 019 (2004) [arXiv:hep-th/0312002]; *String and M Theory deformations of manifolds with special holonomy*, JHEP **0507**, 075 (2005) [arXiv:hep-th/0410176].

[128] B. Zwiebach, *Curvature squared terms and string theories*, Phys. Lett. **B156**, 315 (1985).

[129] W. Ambrose and I.M. Singer, *A theorem on holonomy*, Trans. Amer. Math. Soc. **75**, 428 (1953).

[130] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vols. I, II, John Wiley & Sons (1963).

[131] M. Berger, *Sur les groupes d'holonomie homogene des varietes a connexion affine et des varietes riemanniennes*, Bull. Soc. Math. France **83**, 225 (1955).

[132] R.L. Bryant, *Pseudo-Riemannian metrics with parallel spinor fields and non-vanishing Ricci tensor*, Sémin. Congr., **4**, 53–94, Soc. Math. France (2000) [arXiv:math.DG/0004073].

[133] J.M. Figueroa-O'Farrill and S. Gadhia, *Supersymmetry and spin structures*, Class. Quantum Grav. **22**, L121–L126 [arXiv:hep-th/0506229].

[134] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Supersymmetric AdS backgrounds in string and M Theory*, published in *Strasbourg 2003, AdS/CFT correspondence*, 217–252 [arXiv:hep-th/0411194].

[135] J.P. Gauntlett, J.B. Gutowski and S. Pakis, *The geometry of $D = 11$ null Killing spinors*, JHEP **0312**, 049 (2003) [arXiv:hep-th/0311112].

[136] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Supersymmetric AdS_5 solutions of M Theory*, Class. Quantum Grav. **21**, 4335–4366 (2004) [arXiv:hep-th/0402153].

[137] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Supersymmetric AdS_5 solutions of IIB supergravity*, arXiv:hep-th/0510125.

[138] O.A.P. Mac Conamhna, *Refining G-structure classifications*, Phys. Rev. **D70** (2004) 105024 [arXiv:hep-th/0408203].

[139] C.N. Gowdigere, D. Nemeschansky and N.P. Warner, *Supersymmetric solutions with fluxes from algebraic spinors*, Adv. Theor. Math. Phys. **7**, 787–806 (2004) [arXiv:hep-th/0306097].

[140] J.P. Gauntlett, *Classifying supergravity solutions*, Fortsch. Phys. **53**, 468–479 (2005) [arXiv:hep-th/0501229].

[141] M. Graña, *Flux compactifications in string theory: A comprehensive review*, Phys. Rept. **423**, 91–158 (2006) [arXiv:hep-th/0509003].

[142] K. Behrndt, M. Cvetič and T. Liu, *Classification of supersymmetric flux vacua in M Theory*, arXiv:hep-th/0512032.

[143] M.J. Duff, B.E.W. Nilsson and C.N. Pope, *Spontaneous supersymmetry breaking by the squashed seven-sphere*, Phys. Rev. Lett. **50**, 2043 (1983).

[144] P. van Nieuwenhuizen and N. Warner, *Integrability conditions for Killing spinors*, Commun. Math. Phys. **93**, 277 (1984).

[145] D.D. Joyce, *Compact Manifolds with Special Holonomy*, Oxford University Press, Oxford, (2000).

[146] C. Bär, *Real Killing spinors and holonomy*, Comm. Math. Phys. **154**, 509–521 (1993).

[147] E. Bergshoeff, R. Kallosh, T. Ortín and G. Papadopoulos, *Kappa-Symmetry, Supersymmetry and Intersecting Branes*, Nucl. Phys. **B502**, 149–169 (1997) [arXiv:hep-th/9705040].

[148] I. Bandos and J. Lukierski, *Tensorial central charges and new superparticle models with fundamental spinor coordinates*, Mod. Phys. Lett. **14**, 1257 (1999) [arXiv:hep-th/9811022].

[149] I.A. Bandos, *BPS preons and tensionless super–p–branes in generalized superspace*, Phys. Lett. **B558**, 197 (2003) [arXiv:hep-th/0208110].

[150] A.A. Zheltukhin and D.V. Uvarov, *An Inverse Penrose Limit and Supersymmetry Enhancement in the Presence of Tensor Central Charges*, JHEP **0208**, 008 (2002) [arXiv:hep-th/0206214].

[151] I.A. Bandos, J.A. de Azcárraga, J.M. Izquierdo and J. Lukierski, $D = 4$ *supergravity dynamically coupled to a massless superparticle in a superfield Lagrangian approach*, Phys. Rev. **D67**, 065003 (2003)[arXiv:hep-th/0207139]; *On dynamical supergravity interacting with super–p–brane sources*, Invited talk at the 3rd Sakharov International Conference, Moscow, June 24-29, 2002, arXiv:hep-th/0211065;

I.A. Bandos and J.M. Isidro, $D = 4$ *supergravity dynamically coupled to superstring in a superfield Lagrangian approach*, Phys. Rev. **D69**, 085009 (2004) [arXiv:hep-th/0308102].

[152] I.A. Bandos, J.A. de Azcárraga and J.M. Izquierdo, *Supergravity interacting with bosonic p–branes and local supersymmetry*, Phys. Rev. **D65**, 105010 (2002) [arXiv:hep-th/0112207].

[153] I. Bandos, *Superparticle in Lorentz harmonic superspace*, Sov. J. Nucl. Phys. **51**, 906–914 (1990).

[154] I.A. Bandos and A.A. Zheltukhin, *Null Super-p-branes in 4-dimensional Space-time*, Fortschr. Phys. **41**, 619–676 (1993).

[155] I.A. Bandos, J. Lukierski and D. Sorokin, *Superparticle models with tensorial central charges*, Phys. Rev. **D61**, 045002 (2000) [arXiv:hep-th/9904109].

[156] M. Hatsuda and M. Sakaguchi, *Wess-Zumino term for the AdS superstring and generalized İnönü-Wigner contraction*, Prog. Theor. Phys. **109**, 853-869 (2003) [arXiv:hep-th/0106114].

[157] I.E. Segal, *A class of operator algebras which are determined by groups*, Duke Math. J. **18**, 221–265 (1951).

[158] E. İnönü and E.P. Wigner, *On the contraction of groups and their representations*, Proc. Nat. Acad. Sci. U.S.A. **39**, 510–524 (1953); E. İnönü, *contractions of Lie groups and their representations*, in *Group theoretical concepts in elementary particle physics*, F.Gürsey ed., Gordon and Breach, 391–402 (1964).

[159] E.J. Saletan, *Contractions of Lie groups*, J. Math. Phys. **2**, 1–21 (1961).

[160] D. Arnal and J.C. Cortet, *Contractions and group representations*, J. Math. Phys. **20**, 556–563 (1979).

[161] E. Celeghini and M. Tarlini, *Contractions of group representations*, Nuov. Cim. **61B**, 265–277, 172–180 (1981), *ibid.* **68B**, 133-141 (1982).

[162] E.A. Lord, *Geometrical interpretation of İnönü-Wigner contractions*, Int. J. Theor. Phys. **24**, 723–730 (1985).

[163] M. de Montigny and J. Patera, *Discrete and continuous graded contractions of Lie algebras and superalgebras*, J. Phys. **A24**, 525-547 (1991); R.V. Moody and J. Patera, *Discrete and continuous graded contractions of representations of Lie algebras*, J. Phys. **A24**, 2227–2257 (1991).

[164] F. Herranz, M. De Montigny, M.A. del Olmo and M. Santander, *Cayley-Klein algebras as graded contractions of* $so(N+1)$, J. Phys. **A27**, 2515–2526 (1994) [arXiv:hep-th/9312126].

[165] E. Weimar-Woods,*Contractions of Lie algebras: generalized İnönü-Wigner contractions versus graded contractions*, J. Math. Phys. **36**,

4519–4548 (1995); *Contractions, generalized İnönü and Wigner contractions and deformations of finite-dimensional Lie algebras*, Rev. Math. Phys. **12**, 1505–1529 (2000).

[166] M. Gerstenhaber, *On the deformations of rings and algebras*, Ann. Math. **79**, 59–103 (1964) .

[167] A. Nijenhuis and R.W. Richardson Jr., *Cohomology and deformations in graded Lie algebras*, Bull. A, Math. Soc. **72**, 1–29 (1966).

[168] A. Nijenhuis and R.W. Richardson Jr., *Deformations of Lie algebra structures*, J. Math. Mech. **171**, 89–105 (1967).

[169] R.W. Richardson, *On the rigidity of semi-direct products of Lie algebras*, Pac. J. Math. **22**, 339–344 (1967).

[170] M. Lévy-Nahas, J. Math. Phys., *Deformation and contraction of Lie algebras*, J. Math. Phys. **8**, 1211–1222 (1967).

[171] R. Hermann, *Analytic continuation of group representations III*, Commun. Math. Phys. **3**, 75-97 (1996); *Vector bundles in mathematical physics* (vol. II), W.A. Benjamin (N.Y.), 107 (1970).

[172] R.E. Kallosh, *Geometry of eleven-dimensional supergravity*, Phys. Lett. **B143**, 373–387 (1984).

I. Bars and S.W. MacDowell, *Gravity with extra gauge symmetry*, Phys. Lett. **B129**, 182–191 (1983).

I. Bars and A. Higuchi, *First order formulation and geometrical interpretation of $D = 11$ supergravity*, Phys. Lett. **145B**, 329–332 (1984).

[173] S.W. MacDowell and F. Mansouri, *Unified geometric theory of gravity and supergravity*, Phys. Rev. Lett. **38**, 739–742 (1977).

F. Mansouri, *Superunified theories based on the geometry of local (super-)gauge invariance*, Phys. Rev. **D16**, 2456–2467 (1977).

[174] P. Fré, *M Theory FDA, twisted tori and Chevalley cohomology*, arXiv:hep-th/0510068.

[175] C. Chevalley and S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Am. Math. Soc. **63**, 85–124 (1948).

[176] E. Cremmer and S. Ferrara, *Formulation of eleven-dimensional supergravity in superspace*, Phys. Lett. **B91**, 61 (1980).

[177] L. Brink and P.S. Howe, *Eleven-dimensional supergravity on the mass-shell in superspace*, Phys. Lett. **B91**, 384 (1980).

[178] J.A. de Azcárraga and P.K. Townsend, *Superspace geometry and the formulation of supersymmetric extended objects*, Phys. Rev. Lett. **62**, 2579–2582 (1989)

[179] E. Bergshoeff and E. Sezgin, *Super p-brane theories and new space-time superalgebras*, Phys. Lett. **B354**, 256–263 (1995) [arXiv:hep-th/9504140].

[180] L. Anguelova and P.A. Grassi, *Super D-branes from BRST symmetry*, JHEP **0311** (2003) 010 [arXiv:hep-th/0307260].

[181] I.A. Bandos, N. Berkovits, D.P. Sorokin, *Duality-symmetric eleven-dimensional supergravity and its coupling to M-branes*, Nucl. Phys. **B522**, 214–233 (1998) [arXiv:hep-th/9711055].

[182] M. Sakaguchi, *Type-II superstrings and new spacetime superalgebras*, Phys. Rev. **D59**, 046007 (1999) [arXiv:hep-th/9809113].

[183] J.A. de Azcárraga and J.M. Izquierdo and C. Miquel-Espanya, *Spacetime scale-invariant super-p-brane actions on enlarged superspaces and the geometry of κ-symmetry*, Nucl. Phys. **B706**, 181–203 (2005) [arXiv:hep-th/0407238].

[184] B. de Wit, J. Hoppe and H. Nicolai, *On the quantum mechanics of supermembranes*, Nucl. Phys. **B305**, 545 (1988).

[185] H. Nicolai and J. Plefka, *Supersymmetric effective action of matrix theory*, Phys. Lett. **B477**, 309 (2000) [arXiv:hep-th/0001106].

A. Dasgupta, H. Nicolai and J. Plefka, *Vertex operators for the supermembrane*, JHEP **0005**, 007 (2000) [arXiv:hep-th/0003280]; *An introduction to the quantum supermembrane*, Grav. Cosmol. **8**, 1 (2002) [arXiv:hep-th/0201182], and refs. therein.

[186] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M–theory as a matrix model: a conjecture*, Phys. Rev. **D55**, 5112 (1997) [arXiv:hep-th/9610043].

[187] M.A. Vasiliev, *Conformal higher spin symmetries of 4d massless supermultiplets and $osp(L, 2M)$ invariant equations in generalized (super)space*, Phys. Rev. **D66**, 066006 (2002) [arXiv:hep-th/0106149].

[188] M.A. Vasiliev, *Relativity, causality, locality, quantization and duality in the $Sp(2M)$ invariant generalized space-Time*, in *Multiple facets of quantization and supersymmetry* (*Marinov's memorial volume*, M.Olshanetsky and A.Vainshtein eds.), 826 [arXiv:hep-th/0111119].

[189] E. Witten, *Perturbative gauge theory and a string theory in twistor space*, Commun. Math. Phys. **252**, 189–258 (2004) [arXiv:hep-th/0312171].

[190] I.A. Bandos, J.A. de Azcárraga and C. Miquel-Espanya, *Superspace formulations of the (super)twistor string*, arXiv:hep-th/0604037.

[191] T. Curtright, *Are there any superstrings in eleven dimensions?*, Phys. Rev. Lett. **60**, 393 (1988).

[192] A.A. Deriglazov and A.V. Galajinsky, *A covariant action for the eleven dimensional superstring*, Mod. Phys. Lett. **A12**, 2993 (1997) [arXiv:hep-th/9711196].

N. Berkovits, *A Problem with the superstring action of Deriglazov and Galajinsky*, Phys. Rev. **D59**, 048901 (1999) [arXiv:hep-th/9712056].

[193] I. Rudychev and E. Sezgin, *Superparticles in $D = 11$*, Phys. Lett. **B415**, 363 (1997); **B424**, 411 (1998) [arXiv:hep-th/9704057].

[194] R. Manvelyan and R. Mkrtchyan, *Towards $SO(2,10)$-invariant M-Theory: multilagrangian fields* Mod. Phys. Lett. **A15**, 747 (2000) [arXiv:hep-th/9907011]; *Free fields equations for space–time algebras with tensorial momentum*, Mod. Phys. Lett. **A17**, 1393 (2002) [arXiv:hep-th/0112233].

R. Mkrtchyan, *Little groups and statistics of branes*, arXiv:hep-th/0209040; *On an (interacting) field theories with tensorial momentum*, arXiv:hep-th/0209175; *Brain content of branes' states*, Phys. Lett. **B558**, 205–212 (2003) [arXiv:hep-th/0212174].

[195] A.A. Zheltukhin and U. Lindström, *Strings in a space with tensor central charge coordinates*, Nucl. Phys. Proc. Suppl. **102**, 126 (2001) [arXiv:hep-th/0103101]; *Hamiltonian of tensionless strings with tensor central charge coordinates*, JHEP **0201**, 034 (2002) [arXiv:hep-th/0112206].

[196] C. Fronsdal, *Massless particles, orthosymplectic symmetry and another type of Kaluza–Klein theory*, in *Essays on Supersymmetry* (C.

Fronsdal ed.), Mathematical Physics Studies **8**, D. Reidel Pub. Co., 163 (1986).

[197] M.A. Vasiliev, *Progress in higher spin gauge theories*, in Moscow 2000, *Quantization, gauge theory and strings*, **vol I** 452 (2001) [arXiv:hep-th/0104246], and refs. therein.

[198] M.A. Vasiliev, *Consistent equations for interacting masless fields of all spins in the first order in curvatures*, Ann. Phys. **190**, 59 (1989).

[199] M. Plyushchay, D. Sorokin and M. Tsulaia, *Higher spins from tensorial charges and $OSp(N|2n)$ symmetry*, JHEP **0304**, 013 (2003) [arXiv:hep-th/0301067].

[200] V.E. Didenko and M.A. Vasiliev, *Free field dynamics in the generalized AdS (super)space*, J. Math. Phys. **45** 197–215 [arXiv:hep-th/0301054].

M.A. Vasiliev, *Higher-spin theories and $Sp(2M)$ invariant space–time*, arXiv:hep-th/0301235.

[201] O.A. Gelfond and M.A. Vasiliev, *Higher rank conformal fields in the $Sp(2M)$ symmetric generalized space-time*, Theor. Math. Phys. **145**, 1400–1424 (2005) [arXiv:hep-th/0304020].

[202] I.Bandos, X. Bekaert, J.A. de Azcárraga, D. Sorokin, M. Tsulaia, *Dynamics of higher spin fields and tensorial space*, JHEP 0505 **031** (2005) [arXiv:hep-th/0501113].

[203] D. Sorokin, *Introduction to the classical theory of higher spins*, AIP Conf. Proc. **767**, 172–202 (2005) [arXiv:hep-th/0405069].

[204] I.A. Bandos and A.A. Zheltukhin, $N = 1$ *Super-p-branes in twistor–like Lorentz harmonic formulation*, Class. Quantum Grav. **12**, 609 (1995) [arXiv:hep-th/9405113].

[205] E.R. Nissimov and S.J. Pacheva, *Quantization of the $N = 1$, $N = 2$ superparticle with irreducible contraints*, Phys. Lett. **B189**, 57 (1986).

[206] I. Bars and C. Deliduman, *High Spin Gauge Fields and Two-Time Physics*, Phys. Rev. **D64**, 045004 (2001) [arXiv:hep-th/0103042], and refs. therein.

[207] P.A.M. Dirac, *Lectures on quantum mechanics*, Academic Press, NY (1967).

[208] I.A. Bandos and A.A. Zheltukhin, *Spinor Cartan N hedron, Lorentz harmonic formulations of superstrings and kappa symmetry*, JETP Lett. **54**, 421 (1991); *Green-Schwarz superstrings in spinor moving frame formalism*, Phys. Lett. **B288**, 77 (1992).

[209] P.A. Grassi, G. Policastro and M. Porrati, *Covariant quantization of the Brink–Schwarz superparticle*, Nucl. Phys. **B606**, 380 (2001) [arXiv:hep-th/0009239].

[210] I.A. Batalin and E.S. Fradkin, *Operatorial quantization of dynamical systems subject to second class constraints*, Nucl. Phys. **B279**, 514 (1987).

L.D. Faddeev and S.L. Shatashvili, *Realization of the Schwinger term in the Gauss law and the possibility of correct quantization of a theory with anomalies*, Phys. Lett. **B167**, 225 (1986).

E.S. Egorian and R.P. Manvelyan, *BRST quantization of Hamiltonian systems with second class constraints*, *Preprint* YERPHI-1056-19-88 [unpublished]; Theor. Math. Phys. **94** 173–181 (1993).

I.A. Batalin, E.S. Fradkin and T.E. Fradkina, *Another version for operatorial quantization of dynamical systems with irreducible constraints*, Nucl. Phys. **B314**, 158 (1989).

[211] A. Ferber, *Supertwistor and conformal supersymmetry*, Nucl. Phys. **B132**, 55 (1978).

T. Shirafuji, *Lagrangian mechanics of massless particles with spin*, Prog. Theor. Phys. **70**, 18 (1983).

[212] R. Penrose and M.A.H. MacCallum, *Twistor theory: an approach to the quantization of fields and space-time*, Phys. Rept. **6**, 241 (1972).

R. Penrose, *The twistor program*, Rep. Math. Phys. **12**, 65 (1977).

[213] I. Bars and M. Picón, *Twistor transform in d dimensions and a unifying role for twistors*, Phys. Rev. **D73**, 064033 (2006) [arXiv:hep-th/hep-th/0512348]; *Single twistor description of massless, massive, AdS, and other interacting particles*, Phys. Rev. **D73**, 064002 (2006) [arXiv:hep-th/0512091].

[214] E. Sokatchev, *Light cone harmonic superspace and its applications*, Phys. Lett. **B169**, 209 (1986); *Harmonic superparticle*, Class. Quantum Grav. **4**, 237 (1987).

[215] A.I. Gumenchuk and D.P. Sorokin, *Relativistic superparticle dynamics and twistor correspondence*, Sov. J. Nucl. Phys. **51**, 350 (1990).