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Integro-differential equations: Regularity theory and Pohozaev identities

by

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Contents

Acknowledgements i

Contents iii

Summary 1

I Integro-differential equations 5

Introduction to Part I 7
Lévy processes . . . 7
Why studying integro-differential equations? . . . 8
Mathematical background . . . 10
Results of the thesis (Part I) . . . 13

1 Dirichlet problem for the fractional Laplacian: regularity up to the boundary 21
1.1 Introduction and results . . . 21
1.2 Optimal Hölder regularity for u . . . 26
1.3 Boundary regularity for u/δ^s . . . 32
1.4 Interior estimates for u/δ^s . . . 42
1.5 Appendix: Basic tools and barriers . . . 50

2 The Pohozaev identity for the fractional Laplacian 53
2.1 Introduction and results . . . 53
2.2 Star-shaped domains: Pohozaev identity and nonexistence . . . 61
2.3 Behavior of $(-\Delta)^{s/2}u$ near $\partial\Omega$ . . . 67
2.4 The operator $-\frac{d}{d\lambda}\big|_{\lambda=1^+}\int_{\mathbb{R}} w_\lambda w_{1/\lambda}$ . . . 75
2.5 Proof of the Pohozaev identity in non-star-shaped domains . . . 82
2.6 Appendix: Calculation of the constants c_1 and c_2 . . . 87

3 Nonexistence results for nonlinear nonlocal equations 93
3.1 Introduction and results . . . 93
3.2 Examples . . . 98
3.3 Sketch of the proof . . . 100
3.4 Proof of Proposition 3.1.4 . . . 102
3.5 Proof of Theorems 3.1.1 and 3.1.3 . . . 104
3.6 Proof of Proposition 3.1.2 . . . 106

4 Boundary regularity for fully nonlinear integro-differential equations 111

4.1 Introduction and results . . . 111
4.2 Properties of $\mathcal{L}_*$ and $\mathcal{L}_0$ . . . 120
4.3 Barriers . . . 126
4.4 Krylov's method . . . 129
4.5 Liouville type theorems . . . 135
4.6 Regularity by compactness . . . 139
4.7 Non translation invariant versions of the results . . . 152
4.8 Final comments and remarks . . . 155
4.9 Appendix . . . 155

II Regularity of stable solutions to elliptic equations 159

Introduction to Part II 161
Background and previous results . . . 161
Results of the thesis (Part II) . . . 162

5 Regularity of stable solutions in domains of double revolution 167
5.1 Introduction and results . . . 167
5.2 Proof of Proposition 5.1.6 . . . 173
5.3 Regularity of the extremal solution . . . 179
5.4 Weighted Sobolev inequality . . . 181

6 The extremal solution for the fractional Laplacian 185
6.1 Introduction and results . . . 185
6.2 Existence of the extremal solution . . . 191
6.3 An example case: the exponential nonlinearity . . . 194
6.4 Boundedness of the extremal solution in low dimensions . . . 196
6.5 Boundary estimates: the moving planes method . . . 202
6.6 H^s regularity of the extremal solution in convex domains . . . 205
6.7 L^p and C^β estimates for the linear Dirichlet problem . . . 207

7 Regularity for the fractional Gelfand problem up to dimension 7 213
7.1 Introduction and results . . . 213
7.2 Some preliminaries and remarks . . . 215
7.3 Proof of the main result . . . 218

III Isoperimetric inequalities with densities 223

Introduction to Part III 225
Background and results of the thesis (Part III) . . . 225

8 Sobolev and isoperimetric inequalities with monomial weights 229
8.1 Introduction and results . . . 229
8.2 Proof of the Isoperimetric inequality . . . 236
8.3 Weighted Sobolev inequality . . . 240

8.4 Best constant and extremal functions 245
8.5 Weighted Morrey inequality 248
8.6 Weighted Trudinger inequality and proof of Corollary 8.1.8 250

9 Sharp isoperimetric inequalities via the ABP method 253
9.1 Introduction and results 253
9.2 Examples of weights 266
9.3 Description of the proof 269
9.4 Proof of the classical Wulff inequality 271
9.5 Proof of Theorem 9.1.3: the case $w \equiv 0$ on $\partial\Sigma$ and $H = \|\cdot\|_2$ . . . 273
9.6 Proof of Theorem 9.1.3: the general case 276

Bibliography 283

Summary

We present here a brief overview of the contents of this thesis.

The main topic of the thesis is the study of Elliptic Partial Differential Equations. The thesis is divided into three Parts: (I) integro-differential equations; (II) stable solutions to reaction-diffusion problems; and (III) weighted isoperimetric and Sobolev inequalities.

Integro-differential equations arise naturally in the study of stochastic processes with jumps, and more precisely of Lévy processes. This type of processes, well studied in Probability, are of particular interest in Finance, Physics, or Ecology. Moreover, integro-differential equations appear naturally also in other contexts such as Image processing, Fluid Mechanics, and Geometry.

The most canonical example of elliptic integro-differential operator is the fractional Laplacian

$$(-\Delta)^s u(x) = c_{n,s}\mathrm{PV}\int_{\mathbb{R}^n} \frac{u(x)-u(x+y)}{|y|^{n+2s}}\,dy, \qquad s\in(0,1). \tag{1}$$

It is the infinitesimal generator of the radially symmetric and stable Lévy process of order $2s$.

In the first Part of this thesis we find and prove the Pohozaev identity for the fractional Laplacian. We also obtain boundary regularity results for the fractional Laplacian and for more general integro-differential operators, as explained next.

In the classical case of the Laplace operator, the Pohozaev identity applies to any solution of $-\Delta u = f(x,u)$ in Ω, $u=0$ on $\partial\Omega$. Its first immediate consequence is the nonexistence of solutions for critical and supercritical nonlinearities f. Still, Pohozaev-type identities have been used in many different contexts, and lead to monotonicity properties, concentration-compactness results, radial symmetry of solutions, uniqueness results, or partial regularity of stable solutions. Furthermore, they are also commonly used in nonlinear wave and heat equations, control theory, geometry, and harmonic maps.

Before our work, a Pohozaev identity for the fractional Laplacian was not known. It was not even known which form should it have, if any. In this thesis we find and establish such identity. Quite surprisingly, it involves a *local* boundary term, even though the operator is *nonlocal*.

For the Laplacian $-\Delta$, the Pohozaev identity follows easily from the divergence theorem or integration by parts formula in bounded domains. However, in the nonlocal framework these tools are not available. Our proof follows a different approach and requires fine regularity properties of solutions.

Namely, to prove the identity we need, among other things, the precise *boundary regularity* of solutions to the Dirichlet problem for the fractional Laplacian (1) in a bounded domain Ω. Solutions u to this problem were known to be comparable to d^s,

where $d(x) = \text{dist}(x, \partial\Omega)$, in the sense that $-Cd^s \leq u \leq Cd^s$ in $\overline{\Omega}$ for some constant C. However, to establish our Pohozaev identity we need a more precise boundary regularity result. Namely, we prove that the quotient u/d^s is Hölder continuous in $\overline{\Omega}$, i.e., that $u/d^s \in C^\gamma(\overline{\Omega})$ for some small $\gamma > 0$. In our Pohozaev identity, the quantity $\frac{u}{d^s}\big|_{\partial\Omega}$ plays the role that the normal derivative $\frac{\partial u}{\partial \nu}$ plays in second order PDEs.

Also in Part I, we establish boundary regularity results for fully nonlinear integro-differential equations. These equations arise in Stochastic Control Theory with jump processes and in zero-sum Stochastic Games. The interior regularity of their solutions has been recently studied by Caffarelli and Silvestre, among others. We show that solutions u to $Iu = g$ in Ω, $u = 0$ in $\mathbb{R}^n \setminus \Omega$, being I a fully nonlinear integro-differential operator of order $2s$, satisfy $u/d^s \in C^{s-\epsilon}(\overline{\Omega})$ for all $\epsilon > 0$. These boundary regularity results improve the best known ones even for linear equations.

Let us describe now our works on reaction-diffusion equations and weighted isoperimetric inequalities, which correspond to Parts II and III of the thesis.

Reaction-diffusion equations play a central role in PDE theory and its applications to other sciences. Our work on this field concerns the regularity of local minimizers to some elliptic equations, a classical problem in the Calculus of Variations. In fact, we treat a larger class than local minimizers: stable solutions.

More precisely, we study the regularity of stable solutions to reaction-diffusion equations of the form $-\Delta u = f(u)$ in $\Omega \subset \mathbb{R}^n$, $u = 0$ on $\partial\Omega$. It is a long standing open problem to prove that stable solutions to this equation are bounded, and thus regular, when $n \leq 9$. In dimensions $n \geq 10$ there are examples of singular stable solutions to the problem. Important examples of stable solutions are given by the extremal solutions of problems of the type $-\Delta u = \lambda f(u)$, where $\lambda > 0$.

The regularity of stable solutions is well understood for some particular nonlinearities f, essentially the exponential and power nonlinearities. In both cases a similar result holds: if $n \leq 9$ then all stable solutions are bounded for every domain Ω, while for $n \geq 10$ there are examples of unbounded stable solutions even in the unit ball.

For general nonlinearities f and general domains Ω, it is known that when $n \leq 4$ any stable solution is bounded. The problem is still open in dimensions $5 \leq n \leq 9$. A partial result in this direction is that all stable solutions are bounded in dimensions $n \leq 9$ when the domain Ω is a ball.

Here we study the regularity of stable solutions domains of double revolution (that is, symmetric with respect to the first m variables and with respect to the last $n-m$). Our main result is the boundedness of all stable solutions in dimensions $n \leq 7$ for all convex domains of double revolution. Except for the radial case, our result is the first partial answer valid for all nonlinearities f in dimensions $5 \leq n \leq 9$.

While studying this problem, we were led to some weighted Sobolev inequalities with *monomial weights* $w(x) = x_1^{A_1} \cdots x_n^{A_n}$ that were not treated in the literature. Below we explain our work on this subject, which is Part III of the thesis.

In Part II we also study the regularity of stable and extremal solutions to reaction problems with nonlocal diffusion, i.e., to problems of the form $(-\Delta)^s u = \lambda f(u)$ in Ω, $u = 0$ in $\mathbb{R}^n \setminus \Omega$, where $(-\Delta)^s$ is the fractional Laplacian. For the exponential nonlinearity $f(u) = e^u$, we obtain a sharp regularity result in domains which are convex in the x_i-direction and symmetric with respect to $\{x_i = 0\}$ for every $i = 1, ..., n$. This

result is new even in the unit ball. For more general nonlinearities f and in general domains Ω, we obtain L^∞ and H^s estimates which are sharp for s close to 1 but not for small values of $s \in (0,1)$.

In Part III we study the weighted Sobolev inequalities with monomial weights $w(x) = x_1^{A_1} \cdots x_n^{A_n}$ that arose in our work on stable solutions. These weights are not in the Muckenhoupt class and the inequalities had not been proved in the literature. We establish them for all weights with exponents $A_i \geq 0$, obtaining also the best constants and extremal functions.

The proof of such Sobolev inequalities is based on a new weighted isoperimetric inequality with monomial weights. We establish it by adapting a proof of the classical Euclidean isoperimetric inequality due to Cabré. Our proof uses a linear Neumann problem for the operator $x^{-A}\mathrm{div}(x^A \nabla \cdot)$ combined with the Alexandroff contact set method (or ABP method).

This type of isoperimetric inequalities with weights have attracted much attention recently. There are many results on existence, regularity, or boundedness of minimizers. However, the solution to the isoperimetric problem in $\mathbb{R}^n$ with a weight w is known only for very few weights, even in the case $n = 2$. Our result provides a class of weights (the monomial ones) for which we give the shape of the minimizers. It is quite surprising that, even if these weights are not radially symmetric, Euclidean balls (centered at the origin) solve the isoperimetric problem.

Also in Part III, we study more general weights. We obtain a family of new sharp isoperimetric inequalities with homogeneous weights in convex cones $\Sigma \subset \mathbb{R}^n$ (in the monomial case, Σ would correspond to $\{x_1 > 0, \cdots, x_n > 0\}$). We prove that Euclidean balls centered at the origin solve the isoperimetric problem in any open convex cone Σ of $\mathbb{R}^n$ (with vertex at the origin) for a certain class of *nonradial* homogeneous weights. More precisely, our result applies to all nonnegative continuous weights w which are positively homogeneous of degree $\alpha \geq 0$ and such that $w^{1/\alpha}$ is concave in the cone Σ.

Moreover, we also treat anisotropic perimeters, establishing similar inequalities for the same homogeneous weights as before. It is worth saying that, as a particular case of our results, we provide with totally new proofs of two classical results: the Wulff inequality for anisotropic perimeters, and the isoperimetric inequality in convex cones of Lions and Pacella.

The thesis is divided into three Parts. Each Part is divided into Chapters. Each Chapter corresponds to a paper or a preprint, as follows.

Part I:

- X. Ros-Oton, J. Serra. *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, J. Math. Pures Appl. 101 (2014), 275-302.

- X. Ros-Oton, J. Serra. *The Pohozaev identity for the fractional Laplacian*, Arch. Rat. Mech. Anal., to appear.

- X. Ros-Oton, J. Serra. *Nonexistence results for nonlocal equations with critical and supercritical nonlinearities*, Comm. Partial Differential Equations, to

appear.

- X. Ros-Oton, J. Serra. *Boundary regularity for fully nonlinear integro-differential equations*, Submitted. Available at arXiv (April 2014).

Part II:

- X. Cabré, X. Ros-Oton. *Regularity of stable solutions up to dimension 7 in domains of double revolution*, Comm. Partial Differential Equations 38 (2013), 135-154.

- X. Ros-Oton, J. Serra. *The extremal solution for the fractional Laplacian*, Calc. Var. Partial Differential Equations, to appear.

- X. Ros-Oton. *Regularity for the fractional Gelfand problem up to dimension 7*, J. Math. Anal. Appl., to appear.

Part III:

- X. Cabré, X. Ros-Oton. *Sobolev and isoperimetric inequalities with monomial weights*, J. Differential Equations 255 (2013), 4312-4336.

- X. Cabré, X. Ros-Oton, J. Serra. *Sharp isoperimetric inequalities via the ABP method*, Submitted. Available at arXiv (April 2013).

Part One

I

INTEGRO-DIFFERENTIAL EQUATIONS

Introduction to Part I

Partial Differential Equations are relations between the values of an unknown function and its derivatives of different orders. In order to check whether a PDE holds at a particular point, one needs to know only the values of the function in an arbitrarily small neighborhood, so that all derivatives can be computed. A *nonlocal* equation is a relation for which the opposite happens. In order to check whether a nonlocal equation holds at a point, information about the values of the function far from that point is needed. Most of the times, this is because the equation involves integral operators. A simple example of such operator is

$$Lu(x) = \mathrm{PV} \int_{\mathbb{R}^n} \big(u(x) - u(x+y)\big) K(y) dy \tag{2}$$

for some nonnegative symmetric kernel $K(y) = K(-y)$ satisfying

$$\int_{\mathbb{R}^n} \min\big(1, |y|^2\big) K(y) dy < +\infty.$$

In (2), PV denotes that the integral has to be understood in the principal value sense. When the singularity at the origin of the kernel K is not integrable, these operators are also called *integro-differential* operators. This is because, due to the singularity of K, the operator (2) differentiates (in some sense) the function u.

The most canonical example of an elliptic integro-differential operator is the fractional Laplacian

$$(-\Delta)^s u(x) = c_{n,s} \mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(x+y)}{|y|^{n+2s}}\, dy, \qquad s \in (0,1). \tag{3}$$

The Fourier symbol of this operator is $|\xi|^{2s}$ and, thus, one has that $(-\Delta)^t \circ (-\Delta)^s = (-\Delta)^{s+t}$ —this is why it is called fractional Laplacian.

Lévy processes

Integro-differential equations arise naturally in the study of stochastic processes with jumps, and more precisely in Lévy processes. A Lévy process is a stochastic process with independent and stationary increments. Informally speaking, it represents the random motion of a particle whose successive displacements are independent and statistically identical over different time intervals of the same length. These processes generalize the concept of Brownian motion, and may contain jump discontinuities.

By the Lévy-Khintchine Formula, the infinitesimal generator of any symmetric Lévy process is given by a linear integro-differential operator of the form

$$Lu(x) = -\sum_{i,j} a_{ij} \partial_{ij} u + \mathrm{PV} \int_{\mathbb{R}^n} \big(u(x) - u(x+y)\big) d\mu(y), \tag{4}$$

where $A = (a_{ij})$ is a nonnegative-definite matrix and μ is a measure satisfying

$$\int_{\mathbb{R}^n} \min\big(1, |y|^2\big) d\mu(y) < \infty.$$

For example, let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let us consider a Lévy process X_t, $t \geq 0$, starting at $x \in \Omega$. Let $u(x)$ be the expected first passage time, i.e., the expected time $\mathbb{E}[\tau]$, where $\tau = \inf\{t > 0 : X_t \notin \Omega\}$ is the first time at which the particle exits the domain. Then, u solves the following integro-differential equation

$$\begin{cases} Lu &= 1 \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \mathbb{R}^n \backslash \Omega, \end{cases}$$

where L is the infinitesimal generator of X_t —and thus, it is an operator of the form (4).

Recall that, when X_t is a Brownian motion, then L is the Laplace operator $-\Delta$. In the context of integro-differential equations, Lévy processes plays the same role that Brownian motion play in the theory of second order equations.

Most of the integro-differential equations appearing in this thesis have a probabilistic interpretation, the simplest example being the one given before.

Notice that an important difference and difficulty when studying integro-differential equations is that the "boundary data" is not given on the boundary, as in the classical case, but in the complement $\mathbb{R}^n \setminus \Omega$. This exhibits the fact that paths of the associated processes fail to be continuous.

A special class of Lévy processes are the so-called stable processes, well studied in probability. These processes satisfy a scaling property, and their infinitesimal generators are given by

$$Lu(x) = \text{PV} \int_{\mathbb{R}^n} \big(u(x) - u(x+y)\big) \frac{a\,(y/|y|)}{|y|^{n+2s}}\, dy. \tag{5}$$

Here, a is any nonnegative function (or, more generally, any finite measure) defined on S^{n-1}, usually called the spectral measure. In our works we will focus in the operators that satisfy the following uniform ellipticity condition

$$\lambda \leq a(\theta) \leq \Lambda \quad \text{on } S^{n-1},$$

where $0 < \lambda \leq \Lambda$ are constants. Note that, up to a multiplicative constant, the fractional Laplacian $(-\Delta)^s$ is the only radially symmetric stable process of order $2s$.

Why studying integro-differential equations?

To a great extent, the study of integro-differential equations is motivated by real world applications. Indeed, there are many situations in which a nonlocal equation gives a significantly better model than a PDE, as explained next.

In Mathematical Finance it is particularly important to study models involving jump processes, since the prices of assets are frequently modeled following a Lévy process. Note that jump processes are very natural in this situation, since asset prices can have sudden changes. These models have become increasingly popular for modeling

market fluctuations since the work of Merton [210] in 1976, both for risk management and option pricing purposes. For example, the obstacle problem for the fractional Laplacian can be used to model the pricing of American options [195, 235]; see also the nice introduction of [63] and also [271, 73]. Good references for financial modeling with jump processes are the books [100] and [269]; see also [231].

Just as an example, let us mention that in [229] Nolan examined the joint distribution of the German mark and the Japanese yen exchange rates, and observed that the distribution fits well in a Lévy stable model. Moreover, he estimated the value of the parameter $2s \approx 1.51$ and also the spectral measure a.

Integro-differential equations appear also in Ecology. Indeed, optimal search theory predicts that predators should adopt search strategies based on long jumps where prey is sparse and distributed unpredictably, Brownian motion being more efficient only for locating abundant prey; see [172, 244, 296]. Thus, reaction-diffusion problems with nonlocal diffusion such as

$$u_t + Lu = f(u) \quad \text{in} \quad \mathbb{R}^n \tag{6}$$

arise naturally when studying such population dynamics. Equation (6) appear also in physical models of plasmas and flames; see [205], [211], and references therein.

It is worth saying that in these problems the nonlocal diffusion (instead of a classical one) changes completely the behavior of the solutions. For example, consider problem (6) with $L = (-\Delta)^s$, $f(u) = u - u^2$, and with compactly supported initial data. Then, in both cases $s = 1$ and $s \in (0, 1)$, there is an invasion of the unstable state $u = 0$ by the stable one, $u = 1$. However, in the classical case ($s = 1$) the invasion front position is linear in time, while in case $s \in (0, 1)$ the front position will be exponential in time. This was heuristically predicted in [205] and [111], and rigorously proved in [48].

In Fluid Mechanics, many equations are nonlocal in nature. A clear example is the surface quasi-geostrophic equation, which is used in oceanography to model the temperature on the surface [99]. The regularity theory for this equation relies on very delicate regularity results for nonlocal equations in divergence form; see [76, 61, 77]. Another important example is the Benjamin-Ono equation

$$(-\Delta)^{1/2} u = -u + u^2,$$

which describes one-dimensional internal waves in deep water [5, 140]. Also, the half-Laplacian $(-\Delta)^{1/2}$ plays a very important role in the understanding of the gravity water waves equations in dimensions 2 and 3; see [152].

In Elasticity, there are also many models that involve nonlocal equations. An important example is the Peierls-Nabarro equation, arising in crystal dislocation models [289, 203, 116]. Also, other nonlocal models are used to take into account that in many materials the stress at a point depends on the strains in a region near that point [188, 123]. Long range forces have been also observed to propagate along fibers or laminae in composite materials [173], and nonlocal models are important also in composite analysis; see [119] and [216].

Other Physical models arising in macroscopic evolution of particle systems or in phase segregation lead to nonlocal diffusive models such as the fractional porous media equation; see [155, 262, 78]. Related evolution models with nonlocal effects are used in

superconductivity [90, 297]. Moreover, other continuum models for interacting particle systems involve nonlocal interaction potentials; see [81].

In Quantum Physics, the fractional Schrödinger equation arises when the Brownian quantum paths are replaced by the Lévy ones in the Feynman path integral [192, 193]. Similar nonlocal dispersive equations describe the dynamics and gravitational collapse of relativistic boson stars; see [121, 198, 170].

Other examples in which integro-differential equations are used are Image Processing (where nonlocal denoising algorithms are able to detect patterns and contours in a better way than the local PDE based models [300, 158, 181, 39]) and Geometry (where the conformally invariant operators, which encode information about the manifold, involve fractional powers of the Laplacian [160, 85]).

Finally, all Partial Differential Equations are a limit case (as $s \uparrow 1$) of integro-differential equations.

Mathematical background

Let us describe briefly the mathematical literature on integro-differential equations. As we will see, for many years these equations were studied by people in Probability, who treated mainly linear integro-differential equations. More recently, these equations have attracted much interest from people in Analysis and PDEs, with nonlinear equations being the focus of research.

Probability

The study of integro-differential equations started in the fifties with the works of Getoor, Blumenthal, and Kac, among others. Due to the relation with stochastic processes, they studied Dirichlet problems of the form

$$\begin{cases} Lu = g(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{7}$$

being L the infinitesimal generator of some stochastic process —in the simplest case, L would be the fractional Laplacian.

In 1959, the continuity up to the boundary of solutions was established, and also some spectral properties of such operators [153]. For the fractional Laplacian the asymptotic distribution of eigenvalues was obtained, as well as some comparison results between the Green's function in a domain and the fundamental solution in the entire space [22].

Later, sharp decay estimates for the heat kernel of the fractional Laplacian in the whole $\mathbb{R}^n$ were proved [23], and an explicit formula for the solution of

$$\begin{cases} (-\Delta)^s u = 1 & \text{in } B_1 \\ u = 0 & \text{in } \mathbb{R}^n \backslash B_1 \end{cases}$$

was found [178, 154]. Moreover, Green's function and the Poisson kernel for the fractional Laplacian in the unit ball B_1 were also explicitly computed by Getoor [24] and Riesz [245], respectively.

Potential theory for the fractional Laplacian in $\mathbb{R}^n$ enjoys an explicit formulation in terms of the Riesz potential, and thus it is similar to that of the Laplacian; see for example the classical book of Landkov [191]. However, the boundary potential theory for this operator presents more difficulties mainly due to its nonlocal character.

Fine boundary estimates for the Green's function and the heat kernel near the boundary have been established in the last twenty years. Namely, Green's function estimates were obtained by Kulczycki [190] and Chen-Song [92] in 1997 for $C^{1,1}$ domains, and in 2002 by Jakubowski for Lipschitz domains [176]. Later, Chen-Kim-Song [93] gave sharp explicit estimates for the heat kernel on $C^{1,1}$ domains, recently extended to Lipschitz and more general domains by Bogdan-Grzywny-Ryznar [26].

Related to this, Bogdan [25] in 1997 established the boundary Harnack principle for s-harmonic functions —solutions to $(-\Delta)^s u = 0$— in Lipschitz domains; see also [27] for an extension of this result to general bounded domains.

Dirichlet problems of the type (7) have also been considered for operators L which are infinitesimal generators of stable Lévy processes, i.e., for operators of the form (5). In this case, it is also possible to develop interior regularity results and boundary potential theory by using the associated fundamental solution; see for example [280, 28, 239, 29, 30, 281].

For more general integro-differential operators (2), regularity properties of solutions can not be proved by using the fundamental solution. Even if these are translation invariant operators, in general nothing can be said about their fundamental solution, and thus other methods are required.

To our knowledge, Bass-Levin [14] is the first work in this direction. It establishes interior Hölder regularity of solutions to $Lu = 0$, being L an operator with a kernel comparable to that of the fractional Laplacian. Their result applies also to non translation invariant equations, and more precisely to equations with "bounded measurable coefficients". After that, Song-Vondracek [276], Bass-Kassman [13], and Kassman-Mimica [180] extended the interior regularity results of [14] to more general classes of integro-differential operators. These works use probabilistic techniques. Their results are closely related to those obtained with analytical methods and described next.

Analysis and PDEs: nonlinear equations

In the last ten years the study of integro-differential equations has attracted much interest from people in Analysis and PDEs. The main motivation for this, as explained above, is that integro-differential equations appear in many models in different sciences.

In contrast with the probabilistic works above for linear equations, more recent results using analytical methods often concern *nonlinear* integro-differential equations.

In [57], Cabré and Solà-Morales studied layer solutions to a boundary reaction problem in $\mathbb{R}^{n+1}_+$,

$$\begin{cases} -\Delta v &= 0 & \text{in } \mathbb{R}^{n+1}_+ \\ \frac{\partial v}{\partial \nu} &= f(v) & \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases}$$

An important example is the Peierls-Nabarro equation, which corresponds to $f(v) = \sin(\pi v)$. As noticed in previous works of Amick and Toland [5, 289], this boundary reaction problem in all of $\mathbb{R}^{n+1}_+$ is equivalent to the integro-differential equation

$$(-\Delta)^{1/2} u = f(u) \quad \text{in } \mathbb{R}^n.$$

Indeed, given a function u in $\mathbb{R}^n$, one can compute its harmonic extension v in one more dimension, i.e., the solution to $\Delta v = 0$ in $\mathbb{R}^{n+1}_+$, $v = u$ on $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n$. Then, it turns out that the normal derivative $\partial_\nu v$ on $\mathbb{R}^n$ is exactly the half Laplacian $(-\Delta)^{1/2}u$.

On the other hand, motivated by applications to mathematical finance, Silvestre [271] studied the regularity of solutions to the obstacle problem for the fractional Laplacian $(-\Delta)^s$, $s \in (0,1)$. He obtained an almost-optimal regularity result for its solution, more precisely he proved the solution to be $C^{1+s-\epsilon}$ for all $\epsilon > 0$.

In case $s = 1/2$, thanks to the aforementioned extension method, the obstacle problem for the half-Laplacian in $\mathbb{R}^n$ is equivalent to the thin obstacle problem for the Laplacian in $\mathbb{R}^{n+1}$. For this latter problem, the optimal regularity of solutions and of free boundaries was well known; see [7, 8]. However, for fractional Laplacians with $s \neq 1/2$ no similar extension problem was available.

This situation changed when Caffarelli and Silvestre [68] introduced the extension problem for the fractional Laplacian $(-\Delta)^s$, $s \in (0,1)$. Thanks to this extension, in a joint work with Salsa [73] they established the optimal regularity of the solution and of the free boundary for the obstacle problem for the fractional Laplacian, for all $s \in (0,1)$.

These developments, and specially the extension problem for the fractional Laplacian, have led to a huge amount of new discoveries on nonlinear equations for fractional Laplacians. Just to mention some of them, we recall the important works on uniqueness of solutions for the equation $(-\Delta)^s u = f(u)$ in $\mathbb{R}^n$ [140, 141, 55, 56]; on the fractional Allen-Cahn equation [257, 97, 45, 46]; on nonlocal minimal surfaces [67, 74, 75, 136, 259]; on free boundary problems involving the fractional Laplacian [66, 103]; and many others [129, 287, 288, 127].

Of course, the extension problem is only available for $(-\Delta)^s$, and thus to obtain results for more general integro-differential operators, different methods are required. While variational methods usually do not need the use of the extension, other type of arguments seem to require its use. For example, it is still not known how to obtain optimal regularity results for the obstacle problem for other linear operators of order $2s$ different from the fractional Laplacian.

The regularity theory for nonlinear nonlocal equations is a very active field of research. For elliptic equations in divergence form, Kassmann obtained the nonlocal analog of the De Giorgi-Nash-Moser estimate [179] by adapting the Moser iteration method to this nonlocal framework. Later, motivated by their previous works on the surface quasi-geostrophic equation [76] and on the Navier-Stokes equation [293], Caffarelli-Chan-Vasseur established the regularity theory for nonlocal parabolic equations in divergence form [61].

On the other hand, the regularity for nonlinear equations in nondivergence form have been mostly developed by Caffarelli and Silvestre. In the foundational paper [69], they established the basis for the theory of fully nonlinear elliptic integro-differential equations of order $2s$. They obtained existence of viscosity solutions to

$$\begin{cases} \mathrm{I}u &= g \quad \text{in } \Omega \\ u &= h \quad \text{in } \mathbb{R}^n \backslash \Omega, \end{cases}$$

and $C^{1+\alpha}$ *interior* regularity of such solutions. Here, I denotes a fully nonlinear operator of order $2s$. Later, they established $C^{2s+\alpha}$ interior regularity for convex equations

[71], and developed a perturbative theory for non translation invariant equations [70]. More recently, this theory has been extended in many ways, for example to equations with lower order terms or to parabolic equations; see [86, 89, 64, 187, 261, 87, 88]. Other important works in this field are [165, 9, 180, 270, 72]. As explained later on in this Introduction, our main contribution to this field is a fine *boundary* regularity result for this type of fully nonlinear problems.

Let us come back to reaction-diffusion problems. They play a central role in PDE theory and its applications to other sciences. When the classical diffusion is replaced by a Lévy-type one, such reaction problems take the form

$$\begin{cases} Lu &= f(x,u) & \text{in } \Omega \\ u &= 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{8}$$

where L is an integro-differential operator. This type of nonlinear Dirichlet problems have attracted much attention in the last years. Many of the mathematical works in the literature deal with existence [131, 266, 265, 215, 137, 138], nonexistence [129, 11], symmetry [21, 95], regularity of solutions [10, 80], and other qualitative properties of solutions [130, 1].

For linear equations, the Lax-Milgram theorem and the Fredholm alternative lead to existence of solutions for very general integro-differential operators [131]. For semilinear equations, other variational methods (like the mountain pass lemma or linking theorems) lead also to existence results for subcritical nonlinearities [264, 265]. In case of critical nonlinearities like $f(u) = u^{\frac{n+2s}{n-2s}} + \lambda u$, a Brezis-Nirenberg type result has been obtained by Servadei and Valdinoci [266, 267].

A very important tool to obtain symmetry results for second order (local) equations $-\Delta u = f(u)$ is the moving planes method [263, 156]. This method was first adapted to nonlocal equations by Birkner, López-Mimbela, and Wakolbinger [21], who proved the radial symmetry of nonnegative solutions to $(-\Delta)^s u = f(u)$ in the unit ball $\Omega = B_1$. Later, the moving planes method has been used to solve Serrin's problem for the fractional Laplacian [105, 128], and also to show nonexistence of nonnegative solutions to supercritical and critical equations $(-\Delta)^s u = u^{\frac{n+2s}{n-2s}}$ in star-shaped domains [129]. This nonexistence result for the fractional Laplacian by Fall and Weth [129] uses the extension problem of Caffarelli-Silvestre and the fractional Kelvin transform to then apply the moving planes method.

Results of the thesis (Part I)

In the rest of this Introduction we explain our main results concerning integro-differential equations.

Chapters 1, 2, and 3 of this Part I of the thesis deal with linear and semilinear Dirichlet problems of the type (7) and (8). More precisely, in Chapter 1 we study fine boundary regularity properties of solutions to these problems in case $L = (-\Delta)^s$, $s \in (0,1)$. Then, in Chapter 2 we establish the Pohozaev identity for the fractional Laplacian. After that, we obtain in Chapter 3 nonexistence results for problem (8) for a wide class of nonlocal operators L.

Finally, the last chapter of this Part I, Chapter 4, is devoted to the study of the boundary regularity of solutions to fully nonlinear integro-differential equations.

Pohozaev-type identities

One of the main results of this thesis is the Pohozaev identity for the fractional Laplacian.

In the classical case of the Laplace operator, the Pohozaev identity applies to any solution to $-\Delta u = f(x,u)$ in Ω, $u = 0$ on $\partial\Omega$. This celebrated result due to S. Pohozaev [237] was originally used to prove nonexistence results for critical and supercritical nonlinearities f. For example, it gives the nonexistence of nonnegative solutions (with zero Dirichlet data) to the critical problem $-\Delta u = u^{\frac{n+2}{n-2}}$ in star-shaped domains —an equation appearing in some geometrical contexts such as the Yamabe problem.

Identities of Pohozhaev-type have been widely used in the analysis of elliptic PDEs [243, 237, 295, 118, 218, 236]. These identities are used to show monotonicity formulas, energy estimates for ground states in $\mathbb{R}^n$, unique continuation properties, radial symmetry of solutions, uniqueness results, or interior H^1 estimates for stable solutions. Moreover, they are also used in other contexts such as hyperbolic equations, harmonic maps, control theory, and geometry.

Before our work, a Pohozaev identity for the fractional Laplacian was not known. It was not even known which form should it have, if any. We find and establish here the Pohozaev identity for the fractional Laplacian, which reads as follows.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be any bounded $C^{1,1}$ domain, and let $d(x) = dist(x, \partial\Omega)$. Let u be any bounded solution of $(-\Delta)^s u = f(x,u)$ in Ω, $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$.*

Then u/d^s is Hölder continuous in $\overline{\Omega}$, and it holds the identity

$$\int_\Omega (x\cdot\nabla u)(-\Delta)^s u\,dx = \frac{2s-n}{2}\int_\Omega u(-\Delta)^s u\,dx - \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\left(\frac{u}{d^s}\right)^2 (x\cdot\nu)d\sigma, \quad (9)$$

where Γ is the Gamma function.

Note that the boundary term $u/d^s|_{\partial\Omega}$ has to be understood in the limit sense — note that one of the statements of the theorem is that u/d^s is continuous up to the boundary.

Let us mention some consequences of Theorem 1. First, when $f(x,u)$ does not depend on x, our identity can be written as

$$(2s-n)\int_\Omega uf(u)dx + 2n\int_\Omega F(u)dx = \Gamma(1+s)^2\int_{\partial\Omega}\left(\frac{u}{d^s}\right)^2 (x\cdot\nu)d\sigma,$$

where $F' = f$. Thus, when Ω is star-shaped, it immediately leads to the nonexistence of nontrivial solutions for supercritical nonlinearities, and also of nonnegative solutions for the critical power $f(u) = u^{\frac{n+2s}{n-2s}}$ —as explained above, this was previously showed in [129] for nonnegative solutions.

Second, (9) yields the following unique continuation property: if $f(x,u)$ is subcritical, then

$$\frac{u}{d^s}\Big|_{\partial\Omega} \equiv 0 \ \text{ on } \ \partial\Omega \quad \Longrightarrow \quad u \equiv 0 \ \text{ in } \ \Omega.$$

Note that since $f(u) = \lambda u$ is subcritical for all $\lambda > 0$, this unique continuation property holds for all eigenfunctions. Let us mention that this unique continuation property is

not known for critical or for supercritical nonlinearities —in this case it is only known for nonnegative solutions, thanks to the Hopf lemma.

Finally, from (9) we also deduce the following new integration-by-parts-type identity

$$\int_\Omega u_{x_i}(-\Delta)^s v\,dx = -\int_\Omega (-\Delta)^s u\, v_{x_i} dx + \Gamma(1+s)^2 \int_{\partial\Omega} \frac{u}{d^s}\,\frac{v}{d^s}\,\nu_i\,d\sigma,$$

where $u \equiv v \equiv 0$ in $\mathbb{R}^n \setminus \Omega$ and u and v have certain regularity properties (which are always satisfied by bounded solutions to linear or semilinear problems).

For the Laplacian $-\Delta$, the Pohozaev identity follows easily from integration by parts or the divergence theorem. However, in this nonlocal framework these tools are not available. The only known integration by parts formula for the fractional Laplacian was the one for the whole $\mathbb{R}^n$, which has no boundary terms. To our knowledge, our identities above are the first ones that involve an integro-differential operator and a boundary term (an integral over $\partial\Omega$). They are new even in dimension $n = 1$. In fact, the constant $\Gamma(1+s)^2$ in our identity seems to indicate that there is no trivial way to prove this identity without some work.

Is it important to observe that the quantity $\frac{u}{d^s}\big|_{\partial\Omega}$ plays the role that $\frac{\partial u}{\partial \nu}$ plays in second order equations.

Let us explain the main ideas appearing in the proof of the Pohozaev identity (9). We first assume the domain Ω to be star-shaped with respect to the origin. The result for general domains follows from the star-shaped case using an argument involving a partition of unity. When the domain is star-shaped, the idea of the proof is the following. First, one writes the left hand side of the identity as

$$\int_\Omega (x\cdot\nabla u)(-\Delta)^s u\;dx = \left.\frac{d}{d\lambda}\right|_{\lambda=1^+} \int_{\mathbb{R}^n} u_\lambda (-\Delta)^s u\;dx,$$

where $u_\lambda(x) = u(\lambda x)$, $\lambda > 1$. Then, integrating by parts and making the change of variables $y = \sqrt{\lambda}\,x$, we obtain

$$\int_{\mathbb{R}^n} u_\lambda (-\Delta)^s u\;dx = \lambda^{\frac{2s-n}{2}} \int_{\mathbb{R}^n} w_{\sqrt{\lambda}} w_{1/\sqrt{\lambda}}\;dy,$$

where

$$w(x) = (-\Delta)^{s/2} u(x).$$

Thus, differentiating with respect to λ at $\lambda = 1^+$, we find

$$\int_\Omega (x\cdot\nabla u)(-\Delta)^s u\;dx = \frac{2s-n}{2}\int_{\mathbb{R}^n} u(-\Delta)^s u\,dx + \frac{1}{2}\left.\frac{d}{d\lambda}\right|_{\lambda=1^+} I_\lambda,$$

where

$$I_\lambda = \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda} dy.$$

Therefore, the Pohozaev identity is equivalent to the following:

$$-\left.\frac{d}{d\lambda}\right|_{\lambda=1^+} \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda}\;dy = \Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{u}{d^s}\right)^2 (x\cdot\nu) d\sigma. \tag{10}$$

The quantity $\frac{d}{d\lambda}|_{\lambda=1^+} \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda}$ vanishes for any $C^1(\mathbb{R}^n)$ function w, as can be seen by differentiating under the integral sign. Instead, we prove that the function $w = (-\Delta)^{s/2}u$ has a singularity along $\partial\Omega$, and that (10) holds. The proof of (10) requires a fine analysis on the singularity of $(-\Delta)^{s/2}u$ near the boundary. More precisely, we show that, for x near $\partial\Omega$, one has

$$(-\Delta)^{s/2}u(x) = c_1\big\{\log d(x) + c_2\chi_\Omega(x)\big\} + h(x),$$

where h is a C^α function and c_1 and c_2 are constants depending only on s (which we compute explicitly). Of course, to find such fine behavior of $(-\Delta)^{s/2}u$ near $\partial\Omega$, a fine boundary regularity result for u is required. This is the object of other papers in the thesis and we describe them below.

In the proof of Theorem 1 we do not use the extension of Caffarelli-Silvestre [68] or other very particular properties of the fractional Laplacian, but only the scale invariance of the operator and some integration by parts in all of $\mathbb{R}^n$. Thanks to this, our methods can be used to show nonexistence of bounded solutions to some nonlinear problems involving quite general integro-differential operators. These nonexistence results follow from a general variational inequality in the spirit of the classical identity by Pucci and Serrin [240]. The proof of our variational inequality follows a similar approach to that in our proof of the Pohozaev identity. Here, instead of proving the equality (10), we show that its left hand side is nonnegative whenever the domain Ω is star-shaped. The operators under consideration are of the form

$$Lu(x) = -\sum a_{ij}\partial_{ij}u + \mathrm{PV}\int_{\mathbb{R}^n} (u(x) - u(x+y))K(y)dy,$$

where K is a symmetric kernel satisfying an appropriate monotonicity property. More precisely, we assume that either $a_{ij} \equiv 0$ and $K(y)|y|^{n+\sigma}$ is nondecreasing along rays from the origin for some $\sigma \in (0,2)$, or that (a_{ij}) is positive definite and $K(y)|y|^{n+2}$ is nondecreasing along rays from the origin. This is proved in Chapter 3, where we also give some concrete examples of operators to which our result applies. In addition, we establish an analogue result for quasilinear nonlocal equations.

Boundary regularity

To prove the Pohozaev identity for the fractional Laplacian we need, among other things, the precise boundary regularity of solutions to the Dirichlet problem

$$\begin{cases} (-\Delta)^s u &= g(x) & \text{in } \Omega \\ u &= 0 & \text{in } \mathbb{R}^n\backslash\Omega. \end{cases}$$

We prove in Chapter 1 that $u \in C^s(\mathbb{R}^n)$ and $u/d^s \in C^\gamma(\overline{\Omega})$ for some small $\gamma > 0$ whenever the right hand side g is bounded.

Moreover, to prove the Pohozaev identity we need also higher order interior Hölder estimates for the quotient u/d^s, which we prove by finding an equation satisfied by this quotient and using that it is $C^\gamma(\overline{\Omega})$; see Chapter 1 for more details.

To show the Hölder regularity of u/d^s, we adapt the method of Krylov for second order elliptic equations in nondivergence form with bounded measurable coefficients [189] to this nonlocal framework.

This method consists of using sub and supersolutions to prove an improvement of oscillation lemma for u/d^s, and then iterate it to deduce the Hölder regularity of this quantity. The method is quite general, and never uses the concrete structure of the operator: one only needs some barriers and an interior Harnack inequality. The main difficulty on applying this method to the fractional Laplacian is the nonlocal character of the operator, and more precisely of its Harnack inequality. Indeed, in contrast with the second order case, the Harnack inequality for the fractional Laplacian requires the function to be positive in *all of* $\mathbb{R}^n$, and not only in a larger ball. Thus, a careful control of the tails of the function is needed in order to adapt Krylov's method to nonlocal operators.

Our result on C^γ regularity of u/d^s is improved in Chapter 4. As explained below, we establish a $C^{s-\epsilon}(\overline{\Omega})$ estimate for u/d^s for all $\epsilon > 0$. The results of Chapter 4 apply not only to linear equations with the fractional Laplacian, but to fully nonlinear integro-differential equations. These equations arise in Stochastic Control with jump processes [231] and in zero-sum Stochastic Games, as described next.

Fully nonlinear equations appear when some random variable distributions depend on the choice of certain controls, and one looks for an optimal strategy to choose those controls in order to maximize the expected value of the random variable. This expected value, as a function of the starting point of the stochastic process, satisfies a fully nonlinear elliptic equation. If the stochastic processes involved are Brownian diffusions, the resulting PDEs are classical second order equations $F(D^2u) = 0$. Instead, if the stochastic processes are Lévy processes with jumps, then the equations are fully nonlinear integro-differential equations; see the books [231] and [167].

The most standard example is the Bellman equation. Consider a family of stochastic processes $\{X_t^\alpha\}$ indexed by a parameter $\alpha \in A$, whose corresponding infinitesimal generators are $\{L_\alpha\}$. We consider the following dynamic programming setting: the parameter α is a control that can be changed at any interval of time. We look for the optimal choice of the control that will maximize the expected value of a given payoff function h the first time that the process X_t exits a domain $\Omega \subset \mathbb{R}^n$. One can have also a running cost g, so that the quantity to maximize is the expected final payoff minus the expected total running cost. If we call this maximal possible expected value $u(x)$, in terms of the initial point $X_0 = x$, the function u will solve the following equation

$$\begin{cases} \sup_{\alpha \in A} L_\alpha u &= g \quad \text{in } \Omega \\ u &= h \quad \text{in } \mathbb{R}^n \backslash \Omega. \end{cases}$$

The equation has to be understood in the viscosity sense; see Chapter 4.

A more general fully nonlinear equation is the Isaacs equation. In this case, one has a stochastic zero-sum game with two players in which each player has one control. The resulting value function $u(x)$ satisfies the equation

$$\inf_\beta \sup_\alpha L_{\alpha\beta} u = g \quad \text{in} \quad \Omega.$$

When L_α (or $L_{\alpha\beta}$) are uniformly elliptic second order operators of the form $L_\alpha u = \sum_{i,j} a_{ij}^{(\alpha)} \partial_{ij} u$, then these equations can be written as

$$F(D^2u) = g \quad \text{in} \quad \Omega,$$

with F convex in the Bellman equation, and not necessarily convex in the Isaacs equation. Instead, if the operators L_α belong to some class $\mathcal{L}$ of integro-differential operators of the form (2) —or, more generally, of the form (4)—, then we have a fully nonlinear integro-differential equation.

The interior regularity for this type of equations is quite well understood —at least for kernels K which are comparable to $|y|^{-n-2s}$. In a series of three papers [69, 70, 71], Caffarelli and Silvestre obtained sharp interior regularity results for fully nonlinear integro-differential equations with kernels K in the class

$$\mathcal{L}_0 = \left\{ Lu(x) = PV \int_{\mathbb{R}^n} \big(u(x+y) - u(x)\big) K(y) dy \ : \ \frac{\lambda}{|y|^{n+2s}} \le K(y) \le \frac{\Lambda}{|y|^{n+2s}} \right\}$$

(note here the change of sign with respect to (2), just to be consistent with previous literature). They proved existence of viscosity solutions, established their $C^{1+\gamma}$ interior regularity [69], $C^{2s+\gamma}$ regularity in case of convex equations [71], and developed a perturbation theory for non translation invariant equations [70].

However, almost nothing was known on boundary regularity for fully nonlinear integro-differential equations. In Chapter 4 we develop such a theory.

As in the case of the fractional Laplacian, the correct notion of boundary regularity for equations of order $2s$ is the Hölder regularity of the quotient u/d^s. Recall that in such nonlocal equations the quantity $\frac{u}{d^s}\big|_{\partial\Omega}$ plays the role that $\frac{\partial u}{\partial \nu}$ plays in second order PDEs. This quantity appears not only in our Pohozaev identity, but also in free boundary problems [66], and in overdetermined problems for the fractional Laplacian [105, 128] that arise naturally in shape optimization problems.

Theorem 2, stated below, establishes boundary regularity for fully nonlinear integro-differential equations which are elliptic with respect to the class $\mathcal{L}_* \subset \mathcal{L}_0$ defined as follows:

$$\mathcal{L}_* = \left\{ Lu = \int_{\mathbb{R}^n} \big(u(x+y) - u(x)\big) K(y) dy; \quad K(y) = \frac{a\,(y/|y|)}{|y|^{n+2s}}, \quad \lambda \le a \le \Lambda \right\}.$$

Note that $\mathcal{L}_*$ consists of all infinitesimal generators of *stable* Lévy processes belonging to $\mathcal{L}_0$.

Theorem 2. *Let u be any solution of the following Bellman or Isaacs equation in a $C^{1,1}$ domain $\Omega \subset \mathbb{R}^n$,*

$$\begin{cases} \inf_\beta \sup_\alpha L_{\alpha\beta} u &= g \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \mathbb{R}^n \backslash \Omega, \end{cases}$$

with $L_{\alpha\beta} \in \mathcal{L}_*$ and with $g \in L^\infty(\Omega)$. Then, $u/d^s \in C^{s-\epsilon}(\overline{\Omega})$ for all $\epsilon > 0$, and

$$\|u/d^s\|_{C^{s-\epsilon}(\Omega)} \le C \left(\|u\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\Omega)} \right),$$

where C is a constant that depends only on n, s, and the ellipticity constants λ and Λ. Moreover, the constant C remains bounded as $s \uparrow 1$.

It is worth mentioning that this result applies to fully nonlinear operators, but it is new even for linear translation invariant equations $Lu = g$ with $L \in \mathcal{L}_*$.

We expect the Hölder exponent $s-\epsilon$ to be optimal (or almost optimal) for merely bounded right hand sides f. Moreover, we also expect the class $\mathcal{L}_*$ to be the largest scale invariant subclass of $\mathcal{L}_0$ for which this result is true.

For general elliptic equations with respect to $\mathcal{L}_0$, no fine boundary regularity results hold. In fact, as we show in Chapter 4, the class $\mathcal{L}_0$ is too large for all solutions to be comparable to d^s near the boundary. The same happens for the subclasses $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{L}_0$, which have more regular kernels and were considered in [69, 70, 71].

The proof of Theorem 2 relies on a C^γ boundary estimate for solutions to nonlocal equations with "bounded measurable coefficients", which is obtained via Krylov's method. Then, for solutions to fully nonlinear equations we push the small Hölder exponent $\gamma > 0$ up to the exponent $s-\epsilon$ in Theorem 2. To achieve this, new ideas are needed, and the procedure that we develop differs substantially from boundary regularity methods in second order equations. We use a new blow up and compactness method, combined with a new "boundary" Liouville-type theorem.

This compactness method has the advantage that allows to deal also with non translation invariant equations $\mathfrak{I}(u,x) = g(x)$ in Ω, with an exterior datum $u = h$ in $\mathbb{R}^n \setminus \Omega$; see Chapter 4 for more details.

Chapter One

THE DIRICHLET PROBLEM FOR THE FRACTIONAL LAPLACIAN: REGULARITY UP TO THE BOUNDARY

We study the regularity up to the boundary of solutions to the Dirichlet problem for the fractional Laplacian. We prove that if u is a solution of $(-\Delta)^s u = g$ in Ω, $u \equiv 0$ in $\mathbb{R}^n \backslash \Omega$, for some $s \in (0,1)$ and $g \in L^\infty(\Omega)$, then u is $C^s(\mathbb{R}^n)$ and $u/\delta^s|_\Omega$ is C^α up to the boundary $\partial\Omega$ for some $\alpha \in (0,1)$, where $\delta(x) = dist(x, \partial\Omega)$. For this, we develop a fractional analog of the Krylov boundary Harnack method.

Moreover, under further regularity assumptions on g we obtain higher order Hölder estimates for u and u/δ^s. Namely, the C^β norms of u and u/δ^s in the sets $\{x \in \Omega : \delta(x) \geq \rho\}$ are controlled by $C\rho^{s-\beta}$ and $C\rho^{\alpha-\beta}$, respectively.

These regularity results are crucial tools in our proof of the Pohozaev identity for the fractional Laplacian.

1.1 Introduction and results

Let $s \in (0,1)$ and $g \in L^\infty(\Omega)$, and consider the fractional elliptic problem

$$\begin{cases} (-\Delta)^s u &= g \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where

$$(-\Delta)^s u(x) = c_{n,s} PV \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy \tag{1.2}$$

and $c_{n,s}$ is a normalization constant.

Problem (1.1) is the Dirichlet problem for the fractional Laplacian. There are classical results in the literature dealing with the interior regularity of s-harmonic functions, or more generally for equations of the type (1.1). However, there are few results on regularity up to the boundary. This is the topic of study of the paper.

Our main result establishes the Hölder regularity up to the boundary $\partial\Omega$ of the function $u/\delta^s|_\Omega$, where

$$\delta(x) = dist(x, \partial\Omega).$$

For this, we develop an analog of the Krylov [189] boundary Harnack method for problem (1.1). As in Krylov's work, our proof applies also to operators with "bounded measurable coefficients". This will be treated in a future work [252]. In this paper we only consider the constant coefficient operator $(-\Delta)^s$, since in this case we can establish more precise regularity results. Most of them will be needed in our subsequent work [250], where we find and prove the Pohozaev identity for the fractional Laplacian, announced in [248]. For (1.1), in addition to the Hölder regularity up to the boundary for u/δ^s, we prove that any solution u is $C^s(\mathbb{R}^n)$. Moreover, when g is not only bounded but Hölder continuous, we obtain better interior Hölder estimates for u and u/δ^s.

The Dirichlet problem for the fractional Laplacian (1.1) has been studied from the point of view of probability, potential theory, and PDEs. The closest result to the one in our paper is that of Bogdan [25], establishing a boundary Harnack inequality for nonnegative s-harmonic functions. It will be described in more detail later on in the Introduction (in relation with Theorem 1.1.2). Related regularity results up to the boundary have been proved in [184] and [66]. In [184] it is proved that u/δ^s has a limit at every boundary point when u solves the homogeneous fractional heat equation. The same is proven in [66] for a free boundary problem for the fractional Laplacian.

Some other results dealing with various aspects concerning the Dirichlet problem are the following: estimates for the heat kernel (of the parabolic version of this problem) and for the Green function, e.g., [24, 93]; an explicit expression of the Poisson kernel for a ball [191]; and the explicit solution to problem (1.1) in a ball for $g \equiv 1$ [154]. In addition, the interior regularity theory for viscosity solutions to nonlocal equations with "bounded measurable coefficients" is developed in [69].

The first result of this paper gives the optimal Hölder regularity for a solution u of (1.1). The proof, which is given in Section 1.2, is based on two ingredients: a suitable upper barrier, and the interior regularity results for the fractional Laplacian. Given $g \in L^\infty(\Omega)$, we say that u is a solution of (1.1) when $u \in H^s(\mathbb{R}^n)$ is a weak solution (see Definition 1.2.1). When g is continuous, the notions of weak solution and of viscosity solution agree; see Remark 1.2.11.

We recall that a domain Ω satisfies the exterior ball condition if there exists a positive radius ρ_0 such that all the points on $\partial\Omega$ can be touched by some exterior ball of radius ρ_0.

Proposition 1.1.1. *Let Ω be a bounded Lipschitz domain satisfying the exterior ball condition, $g \in L^\infty(\Omega)$, and u be a solution of* (1.1)*. Then, $u \in C^s(\mathbb{R}^n)$ and*

$$\|u\|_{C^s(\mathbb{R}^n)} \le C\|g\|_{L^\infty(\Omega)},$$

where C is a constant depending only on Ω and s.

This C^s regularity is optimal, in the sense that a solution to problem (1.1) is not in general C^α for any $\alpha > s$. This can be seen by looking at the problem

$$\begin{cases} (-\Delta)^s u &= 1 \quad \text{in } B_r(x_0) \\ u &= 0 \quad \text{in } \mathbb{R}^n \backslash B_r(x_0), \end{cases} \tag{1.3}$$

for which its solution is explicit. For any $r > 0$ and $x_0 \in \mathbb{R}^n$, it is given by [154, 24]

$$u(x) = \frac{2^{-2s}\Gamma(n/2)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)}\left(r^2 - |x-x_0|^2\right)^s \qquad \text{in } B_r(x_0). \tag{1.4}$$

It is clear that this solution is C^s up to the boundary but it is not C^α for any $\alpha > s$.

Since solutions u of (1.1) are C^s up to the boundary, and not better, it is of importance to study the regularity of u/δ^s up to $\partial\Omega$. For instance, our recent proof [250, 248] of the Pohozaev identity for the fractional Laplacian uses in a crucial way that u/δ^s is Hölder continuous up to $\partial\Omega$. This is the main result of the present paper and it is stated next.

For local equations of second order with bounded measurable coefficients and in non-divergence form, the analog result is given by a theorem of N. Krylov [189], which states that u/δ is C^α up to the boundary for some $\alpha \in (0,1)$. This result is the key ingredient in the proof of the $C^{2,\alpha}$ boundary regularity of solutions to fully nonlinear elliptic equations $F(D^2u) = 0$ —see [182, 59].

For our nonlocal equation (1.1), the corresponding result is the following.

Theorem 1.1.2. *Let Ω be a bounded $C^{1,1}$ domain, $g \in L^\infty(\Omega)$, u be a solution of* (1.1)*, and $\delta(x) = dist(x, \partial\Omega)$. Then, $u/\delta^s|_\Omega$ can be continuously extended to $\overline{\Omega}$. Moreover, we have $u/\delta^s \in C^\alpha(\overline{\Omega})$ and*

$$\|u/\delta^s\|_{C^\alpha(\overline{\Omega})} \leq C\|g\|_{L^\infty(\Omega)}$$

for some $\alpha > 0$ satisfying $\alpha < \min\{s, 1-s\}$. The constants α and C depend only on Ω and s.

To prove this result we use the method of Krylov (see [182]). It consists of trapping the solution between two multiples of δ^s in order to control the oscillation of the quotient u/δ^s near the boundary. For this, we need to prove, among other things, that $(-\Delta)^s\delta_0^s$ is bounded in Ω, where $\delta_0(x) = dist(x, \mathbb{R}^n \setminus \Omega)$ is the distance function in Ω extended by zero outside. This will be guaranteed by the assumption that Ω is $C^{1,1}$.

To our knowledge, the only previous results dealing with the regularity up to the boundary for solutions to (1.1) or its parabolic version were the ones by K. Bogdan [25] and S. Kim and K. Lee [184]. The first one [25] is the boundary Harnack principle for nonnegative s-harmonic functions, which reads as follows: assume that u and v are two nonnegative functions in a Lipschitz domain Ω, which satisfy $(-\Delta)^s u \equiv 0$ and $(-\Delta)^s v \equiv 0$ in $\Omega \cap B_r(x_0)$ for some ball $B_r(x_0)$ centered at $x_0 \in \partial\Omega$. Assume also that $u \equiv v \equiv 0$ in $B_r(x_0) \setminus \Omega$. Then, the quotient u/v is $C^\alpha(\overline{B_{r/2}(x_0)})$ for some $\alpha \in (0,1)$. In [27] the same result is proven in open domains Ω, without any regularity assumption.

While the result in [27] assumes no regularity on the domain, we need to assume Ω to be $C^{1,1}$. This assumption is needed to compare the solutions with the function δ^s. As a counterpart, we allow nonzero right hand sides $g \in L^\infty(\Omega)$ and also changing-sign solutions. In $C^{1,1}$ domains, our results in Section 1.3 (which are local near any boundary point) extend Bogdan's result. For instance, assume that u and v satisfy $(-\Delta)^s u = g$ and $(-\Delta)^s v = h$ in Ω, $u \equiv v \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, and that h is positive in Ω. Then, by Theorem 1.1.2 we have that u/δ^s and v/δ^s are $C^\alpha(\overline{\Omega})$ functions. In addition, by the Hopf lemma for the fractional Laplacian we find that $v/\delta^s \geq c > 0$ in Ω. Hence, we obtain that the quotient u/v is C^α up to the boundary, as in Bogdan's result for s-harmonic functions.

A second result (for the parabolic problem) related to ours is contained in [184]. The authors show that any solution of $\partial_t u + (-\Delta)^s u = 0$ in Ω, $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$,

satisfies the following property: for any $t > 0$ the function u/δ^s is continuous up to the boundary $\partial\Omega$.

Our results were motivated by the study of nonlocal semilinear problems $(-\Delta)^s u = f(u)$ in Ω, $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, more specifically, by the Pohozaev identity that we establish in [250]. Its proof requires the precise regularity theory up to the boundary developed in the present paper (see Corollary 1.1.6 below). Other works treating the fractional Dirichlet semilinear problem, which deal mainly with existence of solutions and symmetry properties, are [258, 264, 129, 21].

In the semilinear case, $g = f(u)$ and therefore g automatically becomes more regular than just bounded. When g has better regularity, the next two results improve the preceding ones. The proofs of these results require the use of the following weighted Hölder norms, a slight modification of the ones in Gilbarg-Trudinger [157, Section 6.1].

Throughout the paper, and when no confusion is possible, we use the notation $C^\beta(U)$ with $\beta > 0$ to refer to the space $C^{k,\beta'}(U)$, where k is the is greatest integer such that $k < \beta$ and where $\beta' = \beta - k$. This notation is specially appropriate when we work with $(-\Delta)^s$ in order to avoid the splitting of different cases in the statements of regularity results. According to this, $[\,\cdot\,]_{C^\beta(U)}$ denotes the $C^{k,\beta'}(U)$ seminorm

$$[u]_{C^\beta(U)} = [u]_{C^{k,\beta'}(U)} = \sup_{x,y\in U,\ x\neq y} \frac{|D^k u(x) - D^k u(y)|}{|x-y|^{\beta'}}.$$

Moreover, given an open set $U \subset \mathbb{R}^n$ with $\partial U \neq \varnothing$, we will also denote

$$d_x = \operatorname{dist}(x, \partial U) \qquad \text{and} \qquad d_{x,y} = \min\{d_x, d_y\}.$$

Definition 1.1.3. *Let $\beta > 0$ and $\sigma \geq -\beta$. Let $\beta = k + \beta'$, with k integer and $\beta' \in (0,1]$. For $w \in C^\beta(U) = C^{k,\beta'}(U)$, define the seminorm*

$$[w]^{(\sigma)}_{\beta;U} = \sup_{x,y\in U} \left(d_{x,y}^{\beta+\sigma} \frac{|D^k w(x) - D^k w(y)|}{|x-y|^{\beta'}} \right).$$

For $\sigma > -1$, we also define the norm $\|\cdot\|^{(\sigma)}_{\beta;U}$ as follows: in case that $\sigma \geq 0$,

$$\|w\|^{(\sigma)}_{\beta;U} = \sum_{l=0}^{k} \sup_{x\in U} \left(d_x^{l+\sigma} |D^l w(x)| \right) + [w]^{(\sigma)}_{\beta;U},$$

while for $-1 < \sigma < 0$,

$$\|w\|^{(\sigma)}_{\beta;U} = \|w\|_{C^{-\sigma}(\overline{U})} + \sum_{l=1}^{k} \sup_{x\in U} \left(d_x^{l+\sigma} |D^l w(x)| \right) + [w]^{(\sigma)}_{\beta;U}.$$

Note that σ is the rescale order of the seminorm $[\,\cdot\,]^{(\sigma)}_{\beta;U}$, in the sense that $[w(\lambda\cdot)]^{(\sigma)}_{\beta;U/\lambda} = \lambda^\sigma [w]^{(\sigma)}_{\beta;U}$.

When g is Hölder continuous, the next result provides optimal estimates for higher order Hölder norms of u up to the boundary.

Proposition 1.1.4. *Let Ω be a bounded domain, and $\beta > 0$ be such that neither β nor $\beta + 2s$ is an integer. Let $g \in C^\beta(\Omega)$ be such that $\|g\|^{(s)}_{\beta;\Omega} < \infty$, and $u \in C^s(\mathbb{R}^n)$ be a solution of (1.1). Then, $u \in C^{\beta+2s}(\Omega)$ and*

$$\|u\|^{(-s)}_{\beta+2s;\Omega} \leq C\big(\|u\|_{C^s(\mathbb{R}^n)} + \|g\|^{(s)}_{\beta;\Omega}\big),$$

where C is a constant depending only on Ω, s, and β.

Next, the Hölder regularity up to the boundary of u/δ^s in Theorem 1.1.2 can be improved when g is Hölder continuous. This is stated in the following theorem, whose proof uses a nonlocal equation satisfied by the quotient u/δ^s in Ω —see (1.39)— and the fact that this quotient is $C^\alpha(\overline{\Omega})$.

Theorem 1.1.5. *Let Ω be a bounded $C^{1,1}$ domain, and let $\alpha \in (0,1)$ be given by Theorem 1.1.2. Let $g \in L^\infty(\Omega)$ be such that $\|g\|^{(s-\alpha)}_{\alpha;\Omega} < \infty$, and u be a solution of (1.1). Then, $u/\delta^s \in C^\alpha(\overline{\Omega}) \cap C^\gamma(\Omega)$ and*

$$\|u/\delta^s\|^{(-\alpha)}_{\gamma;\Omega} \leq C\big(\|g\|_{L^\infty(\Omega)} + \|g\|^{(s-\alpha)}_{\alpha;\Omega}\big),$$

where $\gamma = \min\{1, \alpha + 2s\}$ and C is a constant depending only on Ω and s.

Finally, we apply the previous results to the semilinear problem

$$\begin{cases} (-\Delta)^s u &= f(x,u) & \text{in } \Omega \\ u &= 0 & \text{on } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded $C^{1,1}$ domain and f is a Lipschitz nonlinearity.

In the following result, the meaning of "bounded solution" is that of "bounded weak solution" (see definition 1.2.1) or that of "viscosity solution". By Remark 1.2.11, these two notions coincide. Also, by $f \in C^{0,1}_{loc}(\overline{\Omega} \times \mathbb{R})$ we mean that f is Lipschitz in every compact subset of $\overline{\Omega} \times \mathbb{R}$.

Corollary 1.1.6. *Let Ω be a bounded and $C^{1,1}$ domain, $f \in C^{0,1}_{loc}(\overline{\Omega} \times \mathbb{R})$, u be a bounded solution of (1.5), and $\delta(x) = dist(x, \partial\Omega)$. Then,*

(a) $u \in C^s(\mathbb{R}^n)$ and, for every $\beta \in [s, 1+2s)$, u is of class $C^\beta(\Omega)$ and

$$[u]_{C^\beta(\{x \in \Omega \,:\, \delta(x) \geq \rho\})} \leq C\rho^{s-\beta} \qquad \text{for all } \ \rho \in (0,1).$$

(b) The function $u/\delta^s|_\Omega$ can be continuously extended to $\overline{\Omega}$. Moreover, there exists $\alpha \in (0,1)$ such that $u/\delta^s \in C^\alpha(\overline{\Omega})$. In addition, for all $\beta \in [\alpha, s+\alpha]$, it holds the estimate

$$[u/\delta^s]_{C^\beta(\{x \in \Omega \,:\, \delta(x) \geq \rho\})} \leq C\rho^{\alpha-\beta} \qquad \text{for all } \ \rho \in (0,1).$$

The constants α and C depend only on Ω, s, f, $\|u\|_{L^\infty(\mathbb{R}^n)}$, and β.

The paper is organized as follows. In Section 1.2 we prove Propositions 1.1.1 and 1.1.4. In Section 1.3 we prove Theorem 1.1.2 using the Krylov method. In Section 1.4 we prove Theorem 1.1.5 and Corollary 1.1.6. Finally, the Appendix deals with some basic tools and barriers which are used throughout the paper.

1.2 Optimal Hölder regularity for u

In this section we prove that, assuming Ω to be a bounded Lipschitz domain satisfying the exterior ball condition, every solution u of (1.1) belongs to $C^s(\mathbb{R}^n)$. For this, we first establish that u is C^β in Ω, for all $\beta \in (0, 2s)$, and sharp bounds for the corresponding seminorms near $\partial\Omega$. These bounds yield $u \in C^s(\mathbb{R}^n)$ as a corollary. First, we make precise the notion of weak solution to problem (1.1).

Definition 1.2.1. *We say that u is a weak solution of (1.1) if $u \in H^s(\mathbb{R}^n)$, $u \equiv 0$ (a.e.) in $\mathbb{R}^n \setminus \Omega$, and*

$$\int_{\mathbb{R}^n} (-\Delta)^{s/2} u (-\Delta)^{s/2} v \, dx = \int_\Omega g v \, dx$$

for all $v \in H^s(\mathbb{R}^n)$ such that $v \equiv 0$ in $\mathbb{R}^n \setminus \Omega$.

We recall first some well known interior regularity results for linear equations involving the operator $(-\Delta)^s$, defined by (1.2). The first one states that $w \in C^{\beta+2s}(\overline{B_{1/2}})$ whenever $w \in C^\beta(\mathbb{R}^n)$ and $(-\Delta)^s w \in C^\beta(\overline{B_1})$. Recall that, throughout this section and in all the paper, we denote by C^β, with $\beta > 0$, the space $C^{k,\beta'}$, where k is an integer, $\beta' \in (0, 1]$, and $\beta = k + \beta'$.

Proposition 1.2.2. *Assume that $w \in C^\infty(\mathbb{R}^n)$ solves $(-\Delta)^s w = h$ in B_1 and that neither β nor $\beta + 2s$ is an integer. Then,*

$$\|w\|_{C^{\beta+2s}(\overline{B_{1/2}})} \le C\big(\|w\|_{C^\beta(\mathbb{R}^n)} + \|h\|_{C^\beta(\overline{B_1})}\big),$$

where C is a constant depending only on n, s, and β.

Proof. Follow the proof of Proposition 2.1.8 in [270], where the same result is proved with B_1 and $B_{1/2}$ replaced by the whole $\mathbb{R}^n$. □

The second result states that $w \in C^\beta(\overline{B_{1/2}})$ for each $\beta \in (0, 2s)$ whenever $w \in L^\infty(\mathbb{R}^n)$ and $(-\Delta)^s w \in L^\infty(B_1)$.

Proposition 1.2.3. *Assume that $w \in C^\infty(\mathbb{R}^n)$ solves $(-\Delta)^s w = h$ in B_1. Then, for every $\beta \in (0, 2s)$,*

$$\|w\|_{C^\beta(\overline{B_{1/2}})} \le C\big(\|w\|_{L^\infty(\mathbb{R}^n)} + \|h\|_{L^\infty(B_1)}\big),$$

where C is a constant depending only on n, s, and β.

Proof. Follow the proof of Proposition 2.1.9 in [270], where the same result is proved in the whole $\mathbb{R}^n$. □

The third result is the analog of the first, with the difference that it does not need to assume $w \in C^\beta(\mathbb{R}^n)$, but only $w \in C^\beta(\overline{B_2})$ and $(1 + |x|)^{-n-2s} w(x) \in L^1(\mathbb{R}^n)$.

Corollary 1.2.4. *Assume that $w \in C^\infty(\mathbb{R}^n)$ is a solution of $(-\Delta)^s w = h$ in B_2, and that neither β nor $\beta + 2s$ is an integer. Then,*

$$\|w\|_{C^{\beta+2s}(\overline{B_{1/2}})} \leq C\left(\|(1+|x|)^{-n-2s} w(x)\|_{L^1(\mathbb{R}^n)} + \|w\|_{C^\beta(\overline{B_2})} + \|h\|_{C^\beta(\overline{B_2})} \right)$$

where the constant C depends only on n, s, and β.

Proof. Let $\eta \in C^\infty(\mathbb{R}^n)$ be such that $\eta \equiv 0$ outside B_2 and $\eta \equiv 1$ in $B_{3/2}$. Then $\tilde{w} := w\eta \in C^\infty(\mathbb{R}^n)$ and $(-\Delta)^s \tilde{w} = \tilde{h} := h - (-\Delta)^s\big(w(1-\eta)\big)$. Note that for $x \in B_{3/2}$ we have

$$(-\Delta)^s \left(w(1-\eta)\right)(x) = c_{n,s} \int_{\mathbb{R}^n \setminus B_{3/2}} \frac{-\big(w(1-\eta)\big)(y)}{|x-y|^{n+2s}} dy.$$

From this expression we obtain that

$$\|(-\Delta)^s \left(w(1-\eta)\right)\|_{L^\infty(B_1)} \leq C \|(1+|y|)^{-n-2s} w(y)\|_{L^1(\mathbb{R}^n)}$$

and for all $\gamma \in (0, \beta]$,

$$\begin{aligned} [(-\Delta)^s \left(w(1-\eta)\right)]_{C^\gamma(\overline{B_1})} &\leq C \|(1+|y|)^{-n-2s-\gamma} w(y)\|_{L^1(\mathbb{R}^n)} \\ &\leq C \|(1+|y|)^{-n-2s} w(y)\|_{L^1(\mathbb{R}^n)} \end{aligned}$$

for some constant C that depends only on n, s, β, and η. Therefore

$$\|\tilde{h}\|_{C^\beta(\overline{B_1})} \leq C\big(\|h\|_{C^\beta(\overline{B_2})} + \|(1+|x|)^{-n-2s} w(x)\|_{L^1(\mathbb{R}^n)} \big),$$

while we also clearly have

$$\|\tilde{w}\|_{C^\beta(\overline{\mathbb{R}^n})} \leq C \|w\|_{C^\beta(\overline{B_2})}.$$

The constants C depend only on n, s, β and η. Now, we finish the proof by applying Proposition 1.2.2 with w replaced by $\tilde{w}$. □

Finally, the fourth result is the analog of the second one, but instead of assuming $w \in L^\infty(\mathbb{R}^n)$, it only assumes $w \in L^\infty(B_2)$ and $(1+|x|)^{-n-2s} w(x) \in L^1(\mathbb{R}^n)$.

Corollary 1.2.5. *Assume that $w \in C^\infty(\mathbb{R}^n)$ is a solution of $(-\Delta)^s w = h$ in B_2. Then, for every $\beta \in (0, 2s)$,*

$$\|w\|_{C^\beta(\overline{B_{1/2}})} \leq C\left(\|(1+|x|)^{-n-2s} w(x)\|_{L^1(\mathbb{R}^n)} + \|w\|_{L^\infty(B_2)} + \|h\|_{L^\infty(B_2)} \right)$$

where the constant C depends only on n, s, and β.

Proof. Analog to the proof of Corollary 1.2.4. □

As a consequence of the previous results we next prove that every solution u of (1.1) is $C^s(\mathbb{R}^n)$. First let us find an explicit upper barrier for $|u|$ to prove that $|u| \leq C\delta^s$ in Ω. This is the first step to obtain the C^s regularity.

To construct this we will need the following result, which is proved in the Appendix.

Lemma 1.2.6 (Supersolution)**.** *There exist $C_1 > 0$ and a radial continuous function $\varphi_1 \in H^s_{loc}(\mathbb{R}^n)$ satisfying*

$$\begin{cases} (-\Delta)^s \varphi_1 \geq 1 & \text{in } B_4 \setminus B_1 \\ \varphi_1 \equiv 0 & \text{in } B_1 \\ 0 \leq \varphi_1 \leq C_1(|x|-1)^s & \text{in } B_4 \setminus B_1 \\ 1 \leq \varphi_1 \leq C_1 & \text{in } \mathbb{R}^n \setminus B_4\,. \end{cases} \tag{1.6}$$

The upper barrier for $|u|$ will be constructed by scaling and translating the supersolution from Lemma 1.2.6. The conclusion of this barrier argument is the following.

Lemma 1.2.7. *Let Ω be a bounded domain satisfying the exterior ball condition and let $g \in L^\infty(\Omega)$. Let u be the solution of* (1.1)*. Then,*

$$|u(x)| \leq C\|g\|_{L^\infty(\Omega)} \delta^s(x) \quad \text{for all } x \in \Omega\,,$$

where C is a constant depending only on Ω and s.

In the proof of Lemma 1.2.7 it will be useful the following

Claim 1.2.8. *Let Ω be a bounded domain and let $g \in L^\infty(\Omega)$. Let u be the solution of* (1.1)*. Then,*

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C(diam\,\Omega)^{2s}\|g\|_{L^\infty(\Omega)}$$

where C is a constant depending only on n and s.

Proof. The domain Ω is contained in a large ball of radius $diam\,\Omega$. Then, by scaling the explicit (super)solution for the ball given by (1.4) we obtain the desired bound. □

We next give the

Proof of Lemma 1.2.7. Since Ω satisfies the exterior ball condition, there exists $\rho_0 > 0$ such that every point of $\partial\Omega$ can be touched from outside by a ball of radius ρ_0. Then, by scaling and translating the supersolution φ_1 from Lemma 1.2.6, for each of this exterior tangent balls B_{ρ_0} we find an upper barrier in $B_{2\rho_0} \setminus B_{\rho_0}$ vanishing in $\overline{B_{\rho_0}}$. This yields the bound $u \leq C\delta^s$ in a ρ_0-neighborhood of $\partial\Omega$. By using Claim 1.2.8 we have the same bound in all of $\overline{\Omega}$. Repeating the same argument with $-u$ we find $|u| \leq C\delta^s$, as wanted. □

The following lemma gives interior estimates for u and yields, as a corollary, that every bounded weak solution u of (1.1) in a $C^{1,1}$ domain is $C^s(\mathbb{R}^n)$.

Lemma 1.2.9. *Let Ω be a bounded domain satisfying the exterior ball condition, $g \in L^\infty(\Omega)$, and u be the solution of* (1.1)*. Then, $u \in C^\beta(\Omega)$ for all $\beta \in (0, 2s)$ and for all $x_0 \in \Omega$ we have the following seminorm estimate in $B_R(x_0) = B_{\delta(x_0)/2}(x_0)$:*

$$[u]_{C^\beta(\overline{B_R(x_0)})} \leq CR^{s-\beta}\|g\|_{L^\infty(\Omega)}, \tag{1.7}$$

where C is a constant depending only on Ω, s, and β.

Proof. Recall that if u solves (1.1) in the weak sense and η_ϵ is the standard mollifier then $(-\Delta)^s(u * \eta_\epsilon) = g * \eta_\epsilon$ in B_R for ϵ small enough. Hence, we can regularize u, obtain the estimates, and then pass to the limit. In this way we may assume that u is smooth.

Note that $B_R(x_0) \subset B_{2R}(x_0) \subset \Omega$. Let $\tilde{u}(y) = u(x_0 + Ry)$. We have that

$$(-\Delta)^s \tilde{u}(y) = R^{2s} g(x_0 + Ry) \quad \text{in } B_1\,. \tag{1.8}$$

Furthermore, using that $|u| \le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)}\big)\delta^s$ in Ω —by Lemma 1.2.7— we obtain

$$\|\tilde{u}\|_{L^\infty(B_1)} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)}\big)R^s \tag{1.9}$$

and, observing that $|\tilde{u}(y)| \le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)}\big)R^s(1 + |y|^s)$ in all of $\mathbb{R}^n$,

$$\|(1 + |y|)^{-n-2s}\tilde{u}(y)\|_{L^1(\mathbb{R}^n)} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)}\big)R^s, \tag{1.10}$$

with C depending only on Ω and s.

Next we use Corollary 1.2.5, which taking into account (1.8), (1.9), and (1.10), yields

$$\|\tilde{u}\|_{C^\beta\left(\overline{B_{1/4}}\right)} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)}\big)R^s$$

for all $\beta \in (0, 2s)$, where $C = C(\Omega, s, \beta)$.

Finally, we observe that

$$[u]_{C^\beta\left(\overline{B_{R/4}(x_0)}\right)} = R^{-\beta}[\tilde{u}]_{C^\beta\left(\overline{B_{1/4}}\right)}.$$

Hence, by an standard covering argument, we find the estimate (1.7) for the C^β seminorm of u in $\overline{B_R(x_0)}$. □

We now prove the C^s regularity of u.

Proof of Proposition 1.1.1. By Lemma 1.2.9, taking $\beta = s$ we obtain

$$\frac{|u(x) - u(y)|}{|x - y|^s} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)}\big) \tag{1.11}$$

for all x, y such that $y \in B_R(x)$ with $R = \delta(x)/2$. We want to show that (1.11) holds, perhaps with a bigger constant $C = C(\Omega, s)$, for all $x, y \in \overline{\Omega}$, and hence for all $x, y \in \mathbb{R}^n$ (since $u \equiv 0$ outside Ω).

Indeed, observe that after a Lipschitz change of coordinates, the bound (1.11) remains the same except for the value of the constant C. Hence, we can flatten the boundary near $x_0 \in \partial\Omega$ to assume that $\Omega \cap B_{\rho_0}(x_0) = \{x_n > 0\} \cap B_1(0)$. Now, (1.11) holds for all x, y satisfying $|x - y| \le \gamma x_n$ for some $\gamma = \gamma(\Omega) \in (0, 1)$ depending on the Lipschitz map.

Next, let $z = (z', z_n)$ and $w = (w', w_n)$ be two points in $\{x_n > 0\} \cap B_{1/4}(0)$, and $r = |z - w|$. Let us define $\bar{z} = (z', z_n + r)$, $\bar{z} = (z', z_n + r)$ and $z_k = (1 - \gamma^k)z + \gamma^k \bar{z}$ and $w_k = \gamma^k w + (1 - \gamma^k)\bar{w}$, $k \ge 0$. Then, using that bound (1.11) holds whenever $|x - y| \le \gamma x_n$, we have

$$|u(z_{k+1}) - u(z_k)| \le C|z_{k+1} - z_k|^s = C|\gamma^k(z - \bar{z})(\gamma - 1)|^s \le C\gamma^k|z - \bar{z}|.$$

Moreover, since $x_n > r$ in all the segment joining $\bar{z}$ and $\bar{w}$, splitting this segment into a bounded number of segments of length less than γr, we obtain

$$|u(\bar{z}) - u(\bar{w})| \le C|\bar{z} - \bar{w}|^s \le Cr^s.$$

Therefore,

$$\begin{aligned} |u(z) - u(w)| &\le \sum_{k\ge 0} |u(z_{k+1}) - u(z_k)| + |u(\bar{z}) - u(\bar{w})| + \sum_{k\ge 0} |u(w_{k+1}) - u(w_k)| \\ &\le \left(C\sum_{k\ge 0} \big(\gamma^k r\big)^s + Cr^s \right) \big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)} \big) \\ &\le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)} \big) |z - w|^s, \end{aligned}$$

as wanted. □

The following lemma is similar to Proposition 1.2.2 but it involves the weighted norms introduced above. It will be used to prove Proposition 1.1.4 and Theorem 1.1.5.

Lemma 1.2.10. *Let s and α belong to $(0,1)$, and $\beta > 0$. Let U be an open set with nonempty boundary. Assume that neither β nor $\beta + 2s$ is an integer, and $\alpha < 2s$. Then,*

$$\|w\|^{(-\alpha)}_{\beta+2s;U} \le C\left(\|w\|_{C^\alpha(\mathbb{R}^n)} + \|(-\Delta)^s w\|^{(2s-\alpha)}_{\beta;U} \right) \tag{1.12}$$

for all w with finite right hand side. The constant C depends only on n, s, α, and β.

Proof. Step 1. We first control the $C^{\beta+2s}$ norm of w in balls $B_R(x_0)$ with $R = d_{x_0}/2$.

Let $x_0 \in U$ and $R = d_{x_0}/2$. Define $\tilde{w}(y) = w(x_0 + Ry) - w(x_0)$ and note that

$$\|\tilde{w}\|_{C^\alpha(B_1)} \le R^\alpha [w]_{C^\alpha(\mathbb{R}^n)}$$

and

$$\|(1 + |y|)^{-n-2s} \tilde{w}(y)\|_{L^1(\mathbb{R}^n)} \le C(n,s) R^\alpha [w]_{C^\alpha(\mathbb{R}^n)}.$$

This is because

$$|\tilde{w}(y)| = |w(x_0 + Ry) - w(x_0)| \le R^\alpha |y|^\alpha [w]_{C^\alpha(\mathbb{R}^n)}$$

and $\alpha < 2s$. Note also that

$$\|(-\Delta)^s \tilde{w}\|_{C^\beta(\overline{B_1})} = R^{2s+\beta} \|(-\Delta)^s w\|_{C^\beta(\overline{B_R(x_0)})} \le R^\alpha \|(-\Delta)^s w\|^{(2s-\alpha)}_{\beta;U}.$$

Therefore, using Corollary 1.2.4 we obtain that

$$\|\tilde{w}\|_{C^{\beta+2s}(\overline{B_{1/2}})} \le CR^\alpha \big([w]_{C^\alpha(\mathbb{R}^n)} + \|(-\Delta)^s w\|^{(2s-\alpha)}_{\beta;U} \big),$$

where the constant C depends only on n, s, α, and β. Scaling back we obtain

$$\begin{aligned} \sum_{l=1}^{k} R^{l-\alpha} \|D^l w\|_{L^\infty(B_{R/2}(x_0))} + R^{2s+\beta-\alpha} [w]_{C^{\beta+2s}(\overline{B_{R/2}(x_0)})} \le \\ \le C\big(\|w\|_{C^\alpha(\mathbb{R}^n)} + \|(-\Delta)^s w\|^{(2s-\alpha)}_{\alpha;U} \big), \end{aligned} \tag{1.13}$$

where k denotes the greatest integer less that $\beta + 2s$ and $C = C(n, s)$. This bound holds, with the same constant C, for each ball $B_R(x_0)$, $x_0 \in U$, where $R = d_{x_0}/2$.

Step 2. Next we claim that if (1.13) holds for each ball $B_{d_x/2}(x)$, $x \in U$, then (1.12) holds. It is clear that this already yields

$$\sum_{l=1}^{k} d_x^{k-\alpha} \sup_{x\in U} |D^k u(x)| \leq C\left(\|w\|_{C^\alpha(\mathbb{R}^n)} + \|(-\Delta)^s w\|_{\beta;U}^{(2s-\alpha)} \right) \tag{1.14}$$

where k is the greatest integer less than $\beta + 2s$.

To prove this claim we only have to control $[w]_{\beta+2s;U}^{(-\alpha)}$ —see Definition 1.1.3. Let $\gamma \in (0, 1)$ be such that $\beta + 2s = k + \gamma$. We next bound

$$\frac{|D^k w(x) - D^k w(y)|}{|x - y|^\gamma}$$

when $d_x \geq d_y$ and $|x - y| \geq d_x/2$. This will yield the bound for $[w]_{\beta+2s;U}^{(-\alpha)}$, because if $|x - y| < d_x/2$ then $y \in B_{d_x/2}(x)$, and that case is done in Step 1.

We proceed differently in the cases $k = 0$ and $k \geq 1$. If $k = 0$, then

$$d_x^{\beta+2s-\alpha} \frac{w(x) - w(y)}{|x - y|^{2s+\beta}} = \left(\frac{d_x}{|x - y|} \right)^{\beta+2s-\alpha} \frac{w(x) - w(y)}{|x - y|^\alpha} \leq C\|w\|_{C^\alpha(\mathbb{R}^n)}.$$

If $k \geq 1$, then

$$d_x^{\beta+2s-\alpha} \frac{|D^k w(x) - D^k w(y)|}{|x - y|^\gamma} \leq \left(\frac{d_x}{|x - y|} \right)^\gamma d_x^{\beta+2s-\alpha-\gamma} |D^k w(x) - D^k w(y)| \leq C\|w\|_{k;U}^{(-\alpha)},$$

where we have used that $\beta + 2s - \alpha - \gamma = k - \alpha$.

Finally, noting that for $x \in B_R(x_0)$ we have $R \leq d_{x_0} \leq 3R$, (1.12) follows from (1.13), (1.14) and the definition of $\|w\|_{\alpha+2s;U}^{(-\alpha)}$ in (1.1.3). □

Finally, to end this section, we prove Proposition 1.1.4.

Proof of Proposition 1.1.4. Set $\alpha = s$ in Lemma 1.2.10. □

Remark 1.2.11. When g is continuous, the notions of bounded weak solution and viscosity solution of (1.1) —and hence of (1.5)— coincide.

Indeed, let $u \in H^s(\mathbb{R}^n)$ be a weak solution of (1.1). Then, from Proposition 1.1.1 it follows that u is continuous up to the boundary. Let u_ε and g_ε be the standard regularizations of u and g by convolution with a mollifier. It is immediate to verify that, for ε small enough, we have $(-\Delta)^s u_\varepsilon = g_\varepsilon$ in every subdomain $U \subset\subset \Omega$ in the classical sense. Then, noting that $u_\varepsilon \to u$ and $g_\varepsilon \to g$ locally uniformly in Ω, and applying the stability property for viscosity solutions [69, Lemma 4.5], we find that u is a viscosity solution of (1.1).

Conversely, every viscosity solution of (1.1) is a weak solution. This follows from three facts: the existence of weak solution, that this solution is a viscosity solution as shown before, and the uniqueness of viscosity solutions [69, Theorem 5.2].

As a consequence of this, if g is continuous, any viscosity solution of (1.1) belongs to $H^s(\mathbb{R}^n)$ —since it is a weak solution. This fact, which is not obvious, can also

be proved without using the result on uniqueness of viscosity solutions. Indeed, it follows from Proposition 1.1.4 and Lemma 1.4.4, which yield a stronger fact: that $(-\Delta)^{s/2}u \in L^p(\mathbb{R}^n)$ for all $p < \infty$. Note that although we have proved Proposition 1.1.4 for weak solutions, its proof is also valid —with almost no changes— for viscosity solutions.

1.3 Boundary regularity for u/δ^s

In this section we study the precise behavior near the boundary of the solution u to problem (1.1), where $g \in L^\infty(\Omega)$. More precisely, we prove that the function $u/\delta^s|_\Omega$ has a $C^\alpha(\overline{\Omega})$ extension. This is stated in Theorem 1.1.2.

This result will be a consequence of the interior regularity results of Section 1.2 and an oscillation lemma near the boundary, which can be seen as the nonlocal analog of Krylov's boundary Harnack principle; see Theorem 4.28 in [182].

The following proposition and lemma will be used to establish Theorem 1.1.2. They are proved in the Appendix.

Proposition 1.3.1 (1-D solution in half space, [66])**.** *The function φ_0, defined by*

$$\varphi_0(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x^s & \text{if } x \ge 0\,, \end{cases} \tag{1.15}$$

satisfies $(-\Delta)^s \varphi_0 = 0$ in $\mathbb{R}_+$.

The lemma below gives a subsolution in $B_1 \setminus B_{1/4}$ whose support is $B_1 \subset \mathbb{R}^n$ and such that it is comparable to $(1-|x|)^s$ in B_1.

Lemma 1.3.2 (Subsolution)**.** *There exist $C_2 > 0$ and a radial function $\varphi_2 = \varphi_2(|x|)$ satisfying*

$$\begin{cases} (-\Delta)^s \varphi_2 \le 0 & \text{in } B_1 \setminus B_{1/4} \\ \varphi_2 = 1 & \text{in } B_{1/4} \\ \varphi_2(x) \ge C_2(1-|x|)^s & \text{in } B_1 \\ \varphi_2 = 0 & \text{in } \mathbb{R}^n \setminus B_1\,. \end{cases} \tag{1.16}$$

To prove Hölder regularity of $u/\delta^s|_\Omega$ up to the boundary, we will control the oscillation of this function in sets near $\partial\Omega$ whose diameter goes to zero. To do it, we will set up an iterative argument as it is done for second order equations.

Let us define the sets in which we want to control the oscillation and also auxiliary sets that are involved in the iteration.

Definition 1.3.3. *Let $\kappa > 0$ be a fixed small constant and let $\kappa' = 1/2 + 2\kappa$. We may take, for instance $\kappa = 1/16$, $\kappa' = 5/8$. Given a point x_0 in $\partial\Omega$ and $R > 0$ let us define*

$$D_R = D_R(x_0) = B_R(x_0) \cap \Omega$$

and

$$D^+_{\kappa' R} = D^+_{\kappa' R}(x_0) = B_{\kappa' R}(x_0) \cap \{x \in \Omega \,:\, -x \cdot \nu(x_0) \ge 2\kappa R\}\,,$$

where $\nu(x_0)$ is the unit outward normal at x_0; see Figure 1.1. By $C^{1,1}$ regularity of the domain, there exists $\rho_0 > 0$, depending on Ω, such that the following inclusions hold for each $x_0 \in \partial\Omega$ and $R \le \rho_0$:

$$B_{\kappa R}(y) \subset D_R(x_0) \quad \text{for all } y \in D^+_{\kappa' R}(x_0)\,, \tag{1.17}$$

and

$$B_{4\kappa R}(y^* - 4\kappa R\nu(y^*)) \subset D_R(x_0) \quad \text{and} \quad B_{\kappa R}(y^* - 4\kappa R\nu(y^*)) \subset D^+_{\kappa' R}(x_0) \tag{1.18}$$

for all $y \in D_{R/2}$, where $y^* \in \partial\Omega$ is the unique boundary point satisfying $|y - y^*| = dist(y, \partial\Omega)$. Note that, since $R \le \rho_0$, $y \in D_{R/2}$ is close enough to $\partial\Omega$ and hence the point $y^* - 4\kappa R\nu(y^*)$ lays on the line joining y and y^*; see Remark 1.3.4 below.

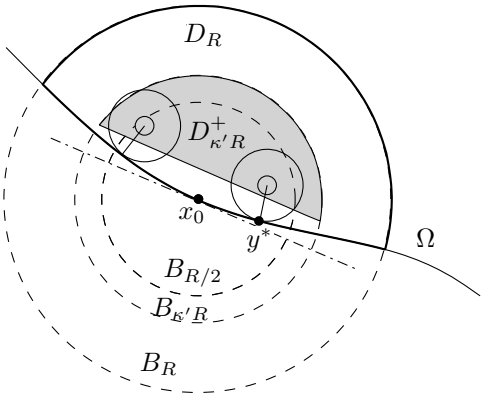

Figure 1.1: The sets D_R and $D^+_{\kappa' R}$

Remark 1.3.4. Throughout the paper, $\rho_0 > 0$ is a small constant depending only on Ω, which we assume to be a bounded $C^{1,1}$ domain. Namely, we assume that (1.17) and (1.18) hold whenever $R \le \rho_0$, for each $x_0 \in \partial\Omega$, and also that every point on $\partial\Omega$ can be touched from both inside and outside Ω by balls of radius ρ_0. In other words, given $x_0 \in \partial\Omega$, there are balls of radius ρ_0, $B_{\rho_0}(x_1) \subset \Omega$ and $B_{\rho_0}(x_2) \subset \mathbb{R}^n \setminus \Omega$, such that $\overline{B_{\rho_0}(x_1)} \cap \overline{B_{\rho_0}(x_2)} = \{x_0\}$. A useful observation is that all points y in the segment that joins x_1 and x_2 —through x_0— satisfy $\delta(y) = |y - x_0|$. Recall that $\delta = dist(\,\cdot\,, \partial\Omega)$.

In the rest of this section, by $|(-\Delta)^s u| \le K$ we mean that either $(-\Delta)^s u = g$ in the weak sense for some $g \in L^\infty$ satisfying $\|g\|_{L^\infty} \le K$ or that u satisfies $-K \le (-\Delta)^s u \le K$ in the viscosity sense.

The first (and main) step towards Theorem 1.1.2 is the following.

Proposition 1.3.5. *Let Ω be a bounded $C^{1,1}$ domain, and u be such that $|(-\Delta)^s u| \le K$ in Ω and $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, for some constant K. Given any $x_0 \in \partial\Omega$, let D_R be as in Definition 1.3.3.*

Then, there exist $\alpha \in (0,1)$ and C depending only on Ω and s —but not on x_0— such that

$$\sup_{D_R} u/\delta^s - \inf_{D_R} u/\delta^s \le CKR^\alpha \tag{1.19}$$

for all $R \le \rho_0$, where $\rho_0 > 0$ is a constant depending only on Ω.

To prove Proposition 1.3.5 we need three preliminary lemmas. We start with the first one, which might be seen as the fractional version of Lemma 4.31 in [182]. Recall that $\kappa' \in (1/2, 1)$ is a fixed constant throughout the section. It may be useful to regard the following lemma as a bound by below for $\inf_{D_{R/2}} u/\delta^s$, rather than an upper bound for $\inf_{D^+_{\kappa' R}} u/\delta^s$.

Lemma 1.3.6. *Let Ω be a bounded $C^{1,1}$ domain, and u be such that $u \geq 0$ in all of $\mathbb{R}^n$ and $|(-\Delta)^s u| \leq K$ in D_R, for some constant K. Then, there exists a positive constant C, depending only on Ω and s, such that*

$$\inf_{D^+_{\kappa' R}} u/\delta^s \leq C\big(\inf_{D_{R/2}} u/\delta^s + KR^s \big) \tag{1.20}$$

for all $R \leq \rho_0$, where $\rho_0 > 0$ is a constant depending only on Ω.

Proof. Step 1. We do first the case $K = 0$. Let $R \leq \rho_0$, and let us call $m = \inf_{D^+_{\kappa' R}} u/\delta^s \geq 0$. We have $u \geq m\delta^s \geq m(\kappa R)^s$ on $D^+_{\kappa' R}$. The second inequality is a consequence of (1.17).

We scale the subsolution φ_2 in Lemma 1.3.2 as follows, to use it as lower barrier:

$$\psi_R(x) := (\kappa R)^s \varphi_2\big(\tfrac{x}{4\kappa R}\big)\,.$$

By (1.16) we have

$$\begin{cases} (-\Delta)^s \psi_R \leq 0 & \text{in } B_{4\kappa R} \setminus B_{\kappa R} \\ \psi_R = (\kappa R)^s & \text{in } B_{\kappa R} \\ \psi_R \geq 4^{-s} C_2 (4\kappa R - |x|)^s & \text{in } B_{4\kappa R} \setminus B_{\kappa R} \\ \psi_R \equiv 0 & \text{in } \mathbb{R}^n \setminus B_{4\kappa R}\,. \end{cases}$$

Given $y \in D_{R/2}$, we have either $y \in D^+_{\kappa' R}$ or $\delta(y) < 4\kappa R$, by (1.18). If $y \in D^+_{\kappa' R}$ it follows from the definition of m that $m \leq u(y)/\delta(y)^s$. If $\delta(y) < 4\kappa R$, let y^* be the closest point to y on $\partial\Omega$ and $\tilde{y} = y^* + 4\kappa\nu(y^*)$. Again by (1.18), we have $B_{4\kappa R}(\tilde{y}) \subset D_R$ and $B_{\kappa R}(\tilde{y}) \subset D^+_{\kappa' R}$. But recall that $u \geq m(\kappa R)^s$ in $D^+_{\kappa' R}$, $(-\Delta)^s u = 0$ in Ω, and $u \geq 0$ in $\mathbb{R}^n$. Hence, $u(x) \geq m\psi_R(x - \tilde{y})$ in all $\mathbb{R}^n$ and in particular $u/\delta^s \geq 4^{-s} C_2 m$ on the segment joining y^* and $\tilde{y}$, that contains y. Therefore,

$$\inf_{D^+_{\kappa' R}} u/\delta^s \leq C \inf_{D_{R/2}} u/\delta^s\,. \tag{1.21}$$

Step 2. If $K > 0$ we consider $\tilde{u}$ to be the solution of

$$\begin{cases} (-\Delta)^s \tilde{u} = 0 & \text{in } D_R \\ \tilde{u} = u & \text{in } \mathbb{R}^n \setminus D_R. \end{cases}$$

By Step 1, (1.21) holds with u replaced by $\tilde{u}$.

On the other hand, $w = \tilde{u} - u$ satisfies $|(-\Delta)^s w| \leq K$ and $w \equiv 0$ outside D_R. Recall that points of $\partial\Omega$ can be touched by exterior balls of radius less than ρ_0. Hence, using the rescaled supersolution $KR^{2s}\varphi_1(x/R)$ from Lemma 1.2.6 as upper barrier and we readily prove, as in the proof of Lemma 1.2.7, that

$$|w| \leq C_1 K R^s \delta^s \quad \text{in } D_R\,.$$

Thus, (1.20) follows. □

The second lemma towards Proposition 1.3.5, which might be seen as the fractional version of Lemma 4.35 in [182], is the following.

Lemma 1.3.7. *Let Ω be a bounded $C^{1,1}$ domain, and u be such that $u \geq 0$ in all of $\mathbb{R}^n$ and $|(-\Delta)^s u| \leq K$ in D_R, for some constant K. Then, there exists a positive constant C, depending on Ω and s, such that*

$$\sup_{D^+_{\kappa' R}} u/\delta^s \leq C\Big(\inf_{D^+_{\kappa' R}} u/\delta^s + KR^s \Big) \tag{1.22}$$

for all $R \leq \rho_0$, where $\rho_0 > 0$ is a constant depending only on Ω.

Proof. Step 1. Consider first the case $K = 0$. In this case (1.22) follows from the Harnack inequality for the fractional Laplacian [191] —note that we assume $u \geq 0$ in all $\mathbb{R}^n$. Indeed, by (1.17), for each $y \in D^+_{\kappa' R}$ we have $B_{\kappa R}(y) \subset D_R$ and hence $(-\Delta)^s u = 0$ in $B_{\kappa R}(y)$. Then we may cover $D^+_{\kappa' R}$ by a finite number of balls $B_{\kappa R/2}(y_i)$, using the same (scaled) covering for all $R \leq \rho_0$, to obtain

$$\sup_{B_{\kappa R/2}(y_i)} u \leq C \inf_{B_{\kappa R/2}(y_i)} u.$$

Then, (1.22) follows since $(\kappa R/2)^s \leq \delta^s \leq (3\kappa R/2)^s$ in $B_{\kappa R/2}(y_i)$ by (1.17).

Step 2. When $K > 0$, we prove (1.22) by using a similar argument as in Step 2 in the proof of Proposition 1.3.6. □

Before proving Lemma 1.3.9 we give an extension lemma —see [125, Theorem 1, Section 3.1] where the case $\alpha = 1$ is proven in full detail.

Lemma 1.3.8. *Let $\alpha \in (0, 1]$ and $V \subset \mathbb{R}^n$ a bounded domain. There exists a (nonlinear) map $E : C^{0,\alpha}(\overline{V}) \to C^{0,\alpha}(\mathbb{R}^n)$ satisfying*

$$E(w) \equiv w \quad \text{in } \overline{V}, \quad [E(w)]_{C^{0,\alpha}(\mathbb{R}^n)} \leq [w]_{C^{0,\alpha}(\overline{V})}, \quad \text{and} \quad \|E(w)\|_{L^\infty(\mathbb{R}^n)} \leq \|w\|_{L^\infty(V)}$$

for all $w \in C^{0,\alpha}(\overline{V})$.

Proof. It is immediate to check that

$$E(w)(x) = \min\left\{ \min_{z \in \overline{V}} \left\{ w(z) + [w]_{C^\alpha(\overline{V})} |z - x|^\alpha \right\}, \|w\|_{L^\infty(V)} \right\}$$

satisfies the conditions since, for all x, y, z in $\mathbb{R}^n$,

$$|z - x|^\alpha \leq |z - y|^\alpha + |y - x|^\alpha .$$

□

We can now give the third lemma towards Proposition 1.3.5. This lemma, which is related to Proposition 1.3.1, is crucial. It states that $\delta^s|_\Omega$, extended by zero outside Ω, is an approximate solution in a neighborhood of $\partial\Omega$ inside Ω.

Lemma 1.3.9. *Let Ω be a bounded $C^{1,1}$ domain, and $\delta_0 = \delta\chi_\Omega$ be the distance function in Ω extended by zero outside Ω. Let $\alpha = \min\{s, 1-s\}$, and ρ_0 be given by Remark 1.3.4. Then,*

$$(-\Delta)^s \delta_0^s \qquad \text{belongs to } \; C^\alpha(\overline{\Omega_{\rho_0}}),$$

where $\Omega_{\rho_0} = \Omega \cap \{\delta < \rho_0\}$. In particular,

$$|(-\Delta)^s \delta_0^s| \le C_\Omega \quad \text{in } \Omega_{\rho_0},$$

where C_Ω is a constant depending only on Ω and s.

Proof. Fix a point x_0 on $\partial\Omega$ and denote, for $\rho > 0$, $B_\rho = B_\rho(x_0)$. Instead of proving that

$$(-\Delta)^s \delta_0^s = c_{n,s} PV \int_{\mathbb{R}^n} \frac{\delta_0(x)^s - \delta_0(y)^s}{|x-y|^{n+2s}} dy$$

is $C^\alpha(\overline{\Omega \cap B_{\rho_0}})$ —as a function of x—, we may equivalently prove that

$$PV \int_{B_{2\rho_0}} \frac{\delta_0(x)^s - \delta_0(y)^s}{|x-y|^{n+2s}} dy \qquad \text{belongs to} \qquad C^\alpha(\overline{\Omega \cap B_{\rho_0}}). \tag{1.23}$$

This is because the difference

$$\frac{1}{c_{n,s}}(-\Delta)^s \delta_0^s - PV \int_{B_{2\rho_0}} \frac{\delta_0(x)^s - \delta_0(y)^s}{|x-y|^{n+2s}} dy = \int_{\mathbb{R}^n \setminus B_{2\rho_0}} \frac{\delta_0(x)^s - \delta_0(y)^s}{|x-y|^{n+2s}} dy$$

belongs to $C^s(\overline{B_{\rho_0}})$, since δ_0^s is $C^s(\mathbb{R}^n)$ and $|x|^{-n-2s}$ is integrable and smooth outside a neighborhood of 0.

To see (1.23), we flatten the boundary. Namely, consider a $C^{1,1}$ change of variables $X = \Psi(x)$, where $\Psi : B_{3\rho_0} \to V \subset \mathbb{R}^n$ is a $C^{1,1}$ diffeomorphism, satisfying that $\partial\Omega$ is mapped onto $\{X_n = 0\}$, $\Omega \cap B_{3\rho_0}$ is mapped into $\mathbb{R}^n_+$, and $\delta_0(x) = (X_n)_+$. Such diffeomorphism exists because we assume Ω to be $C^{1,1}$. Let us respectively call V_1 and V_2 the images of B_{ρ_0} and $B_{2\rho_0}$ under Ψ. Let us denote the points of $V \times V$ by (X, Y). We consider the functions x and y, defined in V, by $x = \Psi^{-1}(X)$ and $y = \Psi^{-1}(Y)$. With these notations, we have

$$x - y = -D\Psi^{-1}(X)(X-Y) + \mathcal{O}\left(|X-Y|^2\right),$$

and therefore

$$|x-y|^2 = (X-Y)^T A(X)(X-Y) + \mathcal{O}\left(|X-Y|^3\right), \tag{1.24}$$

where

$$A(X) = \left(D\Psi^{-1}(X)\right)^T D\Psi^{-1}(X)$$

is a symmetric matrix, uniformly positive definite in $\overline{V_2}$. Hence,

$$PV \int_{B_{2\rho_0}} \frac{\delta_0(x)^s - \delta_0(y)^s}{|x-y|^{n+2s}} dy = PV \int_{V_2} \frac{(X_n)_+^s - (Y_n)_+^s}{|(X-Y)^T A(X)(X-Y)|^{\frac{n+2s}{2}}} g(X,Y) dY,$$

where we have denoted

$$g(X,Y)=\left(\frac{(X-Y)^TA(X)(X-Y)}{|x-y|^2}\right)^{\frac{n+2s}{2}}J(Y)$$

and $J=|\det D\Psi^{-1}|$. Note that we have $g\in C^{0,1}(\overline{V_2\times V_2})$, since Ψ is $C^{1,1}$ and we have (1.24).

Now we are reduced to proving that

$$\psi_1(X):=PV\int_{V_2}\frac{(X_n)_+^s-(Y_n)_+^s}{|(X-Y)^TA(X)(X-Y)|^{\frac{n+2s}{2}}}\,g(X,Y)dY, \tag{1.25}$$

belongs to $C^\alpha(\overline{V_1^+})$ (as a function of X), where $V_1^+=V_1\cap\{X_n>0\}$.

To prove this, we extend the Lipschitz function $g\in C^{0,1}(\overline{V_2\times V_2})$ to all $\mathbb{R}^n$. Namely, consider the function $g^*=E(g)\in C^{0,1}(\mathbb{R}^n\times\mathbb{R}^n)$ provided by Proposition 1.3.8, which satisfies

$$g^*\equiv g \text{ in } \overline{V_2\times V_2}\quad\text{and}\quad \|g^*\|_{C^{0,1}(\mathbb{R}^n\times\mathbb{R}^n)}\le\|g\|_{C^{0,1}(\overline{V_2\times V_2})}.$$

By the same argument as above, using that $V_1\subset\subset V_2$, we have that $\psi_1\in C^\alpha(\overline{V_1^+})$ if and only if so is the function

$$\psi(X)=PV\int_{\mathbb{R}^n}\frac{(X_n)_+^s-(Y_n)_+^s}{|(X-Y)^TA(X)(X-Y)|^{\frac{n+2s}{2}}}\,g^*(X,Y)dY.$$

Furthermore, from g^* define $\tilde g\in C^{0,1}(\overline{V_2}\times\mathbb{R}^n)$ by $\tilde g(X,Z)=g^*(X,X+MZ)\det M$, where $M=M(X)=D\Psi(X)$. Then, using the change of variables $Y=X+MZ$ we deduce

$$\psi(X)=PV\int_{\mathbb{R}^n}\frac{(X_n)_+^s-\big(e_n\cdot(X+MZ)\big)_+^s}{|Z|^{n+2s}}\,\tilde g(X,Z)dZ.$$

Next, we prove that $\psi\in C^\alpha(\mathbb{R}^n)$, which concludes the proof. Indeed, taking into account that the function $(X_n)_+^s$ is s-harmonic in $\mathbb{R}^n_+$ —by Proposition 1.3.1— we obtain

$$PV\int_{\mathbb{R}^n}\frac{(e'\cdot X')_+^s-(e'\cdot(X'+Z))_+^s}{|Z|^{n+2s}}dZ=0$$

for every $e'\in\mathbb{R}^n$ and for every X' such that $e'\cdot X'>0$. Thus, letting $e'=e_n^TM$ and $X'=M^{-1}X$ we deduce

$$PV\int_{\mathbb{R}^n}\frac{(X_n)_+^s-\big(e_n\cdot(X+MZ)\big)_+^s}{|Z|^{n+2s}}dZ=0$$

for every X such that $(e_n^TM)\cdot(M^{-1}X)>0$, that is, for every $X\in\mathbb{R}^n_+$.

Therefore, it holds

$$\psi(X)=\int_{\mathbb{R}^n}\frac{\phi(X,0)-\phi(X,Z)}{|Z|^{n+2s}}\big(\tilde g(X,Z)-\tilde g(X,0)\big)dZ,$$

where

$$\phi(X,Z)=(e_n\cdot(X+MZ))_+^s$$

satisfies $[\phi]_{C^s(\overline{V_2}\times\mathbb{R}^n)} \le C$, and $\|\tilde g\|_{C^{0,1}(\overline{V_2}\times\mathbb{R}^n)} \le C$.

Let us finally prove that ψ belongs to $C^\alpha(\overline{V_1^+})$. To do it, let X and $\bar X$ be in $\overline{V_1^+}$. Then, we have

$$\psi(X) - \psi(\bar X) = \int_{\mathbb{R}^n} \frac{\Theta(X,\bar X, Z)}{|Z|^{n+2s}} dZ,$$

where

$$\begin{aligned}\Theta(X,\bar X, Z) &= \big(\phi(X,0) - \phi(X,Z)\big)\big(\tilde g(X,Z) - \tilde g(X,0)\big)\\ &\quad - \big(\phi(\bar X,0) - \phi(\bar X,Z)\big)\big(\tilde g(\bar X,Z) - \tilde g(\bar X,0)\big)\\ &= \big(\phi(X,0) - \phi(X,Z) - \phi(\bar X,0) + \phi(\bar X,Z)\big)\big(\tilde g(X,Z) - \tilde g(X,0)\big)\\ &\quad - \big(\phi(\bar X,0) - \phi(\bar X,Z)\big)\big(\tilde g(X,Z) - \tilde g(X,0) - \tilde g(\bar X,Z) + \tilde g(\bar X,0)\big).\end{aligned} \tag{1.26}$$

Now, on the one hand, it holds

$$|\Theta(X,\bar X,Z)| \le C|Z|^{1+s}, \tag{1.27}$$

since $[\phi]_{C^s(\overline{V_2}\times\mathbb{R}^n)} \le C$ and $\|\tilde g\|_{C^{0,1}(\overline{V_2}\times\mathbb{R}^n)} \le C$.

On the other hand, it also holds

$$|\Theta(X,\bar X,Z)| \le C|X-\bar X|^s \min\{|Z|,|Z|^s\}. \tag{1.28}$$

Indeed, we only need to observe that

$$\begin{aligned}\big|\tilde g(X,Z) - \tilde g(X,0) - \tilde g(\bar X,Z) + \tilde g(\bar X,0)\big| &\le C\min\big\{\min\{|Z|,1\}, |X-\bar X|\big\}\\ &\le C\min\{|Z|^{1-s},1\}|X-\bar X|^s.\end{aligned}$$

Thus, letting $r = |X-\bar X|$ and using (1.27) and (1.28), we obtain

$$\begin{aligned}|\psi(X) - \psi(\bar X)| &\le \int_{\mathbb{R}^n} \frac{|\Theta(X,\bar X,Z)|}{|Z|^{n+2s}} dZ\\ &\le \int_{B_r} \frac{C|Z|^{1+s}}{|Z|^{n+2s}} dZ + \int_{\mathbb{R}^n\setminus B_r} \frac{Cr^s\min\{|Z|,|Z|^s\}}{|Z|^{n+2s}} dZ\\ &\le Cr^{1-s} + C\max\{r^{1-s}, r^s\},\end{aligned}$$

as desired. □

Next we prove Proposition 1.3.5.

Proof of Proposition 1.3.5. By considering u/K instead of u we may assume that $K = 1$, that is, that $|(-\Delta)^s u| \le 1$ in Ω. Then, by Claim 1.2.8 we have $\|u\|_{L^\infty(\mathbb{R}^n)} \le C$ for some constant C depending only on Ω and s.

Let $\rho_0 > 0$ be given by Remark 1.3.4. Fix $x_0 \in \partial\Omega$. We will prove that there exist constants $C_0 > 0$, $\rho_1 \in (0,\rho_0)$, and $\alpha \in (0,1)$, depending only on Ω and s, and monotone sequences (m_k) and (M_k) such that, for all $k \ge 0$,

$$M_k - m_k = 4^{-\alpha k}, \quad -1 \le m_k \le m_{k+1} < M_{k+1} \le M_k \le 1, \tag{1.29}$$

and

$$m_k \le C_0^{-1} u/\delta^s \le M_k \quad \text{in } D_{R_k} = D_{R_k}(x_0)\,, \quad \text{where } R_k = \rho_1 4^{-k}. \tag{1.30}$$

Note that (1.30) is equivalent to the following inequality in B_{R_k} instead of D_{R_k} — recall that $D_{R_k} = B_{R_k} \cap \Omega$.

$$m_k \delta_0^s \le C_0^{-1} u \le M_k \delta_0^s \quad \text{in } B_{R_k} = B_{R_k}(x_0)\,, \quad \text{where } R_k = \rho_1 4^{-k}\,. \tag{1.31}$$

If there exist such sequences, then (1.19) holds for all $R \le \rho_1$ with $C = 4^\alpha C_0/\rho_1^\alpha$. Then, by increasing the constant C if necessary, (1.19) holds also for every $R \le \rho_0$.

Next we construct $\{M_k\}$ and $\{m_k\}$ by induction.

By Lemma 1.2.7, we find that there exist m_0 and M_0 such that (1.29) and (1.30) hold for $k = 0$ provided we pick C_0 large enough depending on Ω and s.

Assume that we have sequences up to m_k and M_k. We want to prove that there exist m_{k+1} and M_{k+1} which fulfill the requirements. Let

$$u_k = C_0^{-1} u - m_k \delta_0^s\,. \tag{1.32}$$

We will consider the positive part u_k^+ of u_k in order to have a nonnegative function in all of $\mathbb{R}^n$ to which we can apply Lemmas 1.3.6 and 1.3.7. Let $u_k = u_k^+ - u_k^-$. Observe that, by induction hypothesis,

$$u_k^+ = u_k \quad \text{and} \quad u_k^- = 0 \quad \text{in } B_{R_k}\,. \tag{1.33}$$

Moreover, $C_0^{-1} u \ge m_j \delta_0^s$ in B_{R_j} for each $j \le k$. Therefore, by (1.32) we have

$$u_k \ge (m_j - m_k)\delta_0^s \ge (m_j - M_j + M_k - m_k)\delta_0^s \ge (-4^{-\alpha j} + 4^{-\alpha k})\delta_0^s \quad \text{in } B_{R_j}.$$

But clearly $0 \le \delta_0^s \le R_j^s = \rho_1^s 4^{-js}$ in B_{R_j}, and therefore using $R_j = \rho_1 4^{-j}$

$$u_k \ge -\rho_1^{-\alpha} R_j^s (R_j^\alpha - R_k^\alpha) \quad \text{in} \quad B_{R_j} \text{ for each } \ j \le k\,.$$

Thus, since for every $x \in B_{R_0} \setminus B_{R_k}$ there is $j < k$ such that

$$|x - x_0| < R_j = \rho_1 4^{-j} \le 4|x - x_0|,$$

we find

$$u_k(x) \ge -\rho_1^{-\alpha} R_k^{\alpha+s} \left|\frac{4(x-x_0)}{R_k}\right|^s \left(\left|\frac{4(x-x_0)}{R_k}\right|^\alpha - 1\right) \quad \text{outside } B_{R_k}\,. \tag{1.34}$$

By (1.34) and (1.33), at $x \in B_{R_k/2}(x_0)$ we have

$$\begin{aligned}
0 \le -(-\Delta)^s u_k^-(x) &= c_{n,s} \int_{x+y \notin B_{R_k}} \frac{u_k^-(x+y)}{|y|^{n+2s}}\, dy \\
&\le c_{n,s}\, \rho_1^{-\alpha} \int_{|y| \ge R_k/2} R_k^{\alpha+s} \left|\frac{8y}{R_k}\right|^s \left(\left|\frac{8y}{R_k}\right|^\alpha - 1\right) |y|^{-n-2s}\, dy \\
&= C \rho_1^{-\alpha} R_k^{\alpha-s} \int_{|z| \ge 1/2} \frac{|8z|^s (|8z|^\alpha - 1)}{|z|^{n+2s}}\, dz \\
&\le \varepsilon_0 \rho_1^{-\alpha} R_k^{\alpha-s},
\end{aligned}$$

where $\varepsilon_0 = \varepsilon_0(\alpha) \downarrow 0$ as $\alpha \downarrow 0$ since $|8z|^\alpha \to 1$.

Therefore, writing $u_k^+ = C_0^{-1}u - m_k\delta_0^s + u_k^-$ and using Lemma 1.3.9, we have

$$\begin{aligned}|(-\Delta)^s u_k^+| &\le C_0^{-1}|(-\Delta)^s u| + m_k|(-\Delta)^s\delta_0^s| + |(-\Delta)^s(u_k^-)| \\ &\le (C_0^{-1} + C_\Omega) + \varepsilon_0\rho_1^{-\alpha}R_k^{\alpha-s} \\ &\le \big(C_1\rho_1^{s-\alpha} + \varepsilon_0\rho_1^{-\alpha}\big)R_k^{\alpha-s} \qquad \text{in } D_{R_k/2}.\end{aligned}$$

In the last inequality we have just used $R_k \le \rho_1$ and $\alpha \le s$.

Now we can apply Lemmas 1.3.6 and 1.3.7 with u in its statements replaced by u_k^+, recalling that

$$u_k^+ = u_k = C_0^{-1}u - m_k\delta^s \quad \text{in } D_{R_k}$$

to obtain

$$\begin{aligned}\sup_{D^+_{\kappa' R_k/2}} (C_0^{-1}u/\delta^s - m_k) &\le C\left(\inf\nolimits_{D^+_{\kappa' R_k/2}}(C_0^{-1}u/\delta^s - m_k) + \big(C_1\rho_1^{s-\alpha} + \varepsilon_0\rho_1^{-\alpha}\big)R_k^\alpha\right) \\ &\le C\left(\inf\nolimits_{D_{R_k/4}}(C_0^{-1}u/\delta^s - m_k) + \big(C_1\rho_1^{s-\alpha} + \varepsilon_0\rho_1^{-\alpha}\big)R_k^\alpha\right) \qquad (1.35)\end{aligned}$$

Next we can repeat all the argument "upside down", that is, with the functions $u^k = M_k\delta^s - u$ instead of u_k. In this way we obtain, instead of (1.35), the following:

$$\sup_{D^+_{\kappa' R_k/2}} (M_k - C_0^{-1}u/\delta^s) \le C\left(\inf_{D_{R_k/4}} (M^k - C_0^{-1}u/\delta^s) + \big(C_1\rho_1^{s-\alpha} + \varepsilon_0\rho_1^{-\alpha}\big)R_k^\alpha\right). \quad (1.36)$$

Adding (1.35) and (1.36) we obtain

$$\begin{aligned}M_k - m_k &\le C\left(\inf_{D_{R_k/4}} (C_0^{-1}u/\delta^s - m_k) + \inf_{D_{R_k/4}} (M_k - C_0^{-1}u/\delta^s) + \big(C_1\rho_1^{s-\alpha} + \varepsilon_0\rho_1^{-\alpha}\big)R_k^\alpha\right) \\ &= C\left(\inf_{D_{R_{k+1}}} C_0^{-1}u/\delta^s - \sup_{D_{R_{k+1}}} C_0^{-1}u/\delta^s + M_k - m_k + \big(C_1\rho_1^{s-\alpha} + \varepsilon_0\rho_1^{-\alpha}\big)R_k^\alpha\right), \qquad (1.37)\end{aligned}$$

and thus, using that $M_k - m_k = 4^{-\alpha k}$ and $R_k = \rho_1 4^{-k}$,

$$\sup_{D_{R_{k+1}}} C_0^{-1}u/\delta^s - \inf_{D_{R_{k+1}}} C_0^{-1}u/\delta^s \le \big(\tfrac{C-1}{C} + C_1\rho_1^s + \varepsilon_0\big)4^{-\alpha k}\,.$$

Now we choose α and ρ_1 small enough so that

$$\frac{C-1}{C} + C_1\rho_1^s + \varepsilon_0(\alpha) \le 4^{-\alpha}.$$

This is possible since $\varepsilon_0(\alpha) \downarrow 0$ as $\alpha \downarrow 0$ and the constants C and C_1 do not depend on α nor ρ_1 —they depend only on Ω and s. Then, we find

$$\sup_{D_{R_{k+1}}} C_0^{-1}u/\delta^s - \inf_{D_{R_{k+1}}} C_0^{-1}u/\delta^s \le 4^{-\alpha(k+1)},$$

and thus we are able to choose m_{k+1} and M_{k+1} satisfying (1.29) and (1.30). □

Finally, we give the:

Proof of Theorem 1.1.2. Define $v = u/\delta^s|_\Omega$ and $K = \|g\|_{L^\infty(\Omega)}$. As in the proof of Proposition 1.3.5, by considering u/K instead of u we may assume that $|(-\Delta)^s u| \le 1$ in Ω and that $\|u\|_{L^\infty(\Omega)} \le C$ for some constant C depending only on Ω and s.

First we claim that there exist constants C, $M > 0$, $\widetilde{\alpha} \in (0,1)$ and $\beta \in (0,1)$, depending only on Ω and s, such that

(i) $\|v\|_{L^\infty(\Omega)} \le C$.

(ii) For all $x \in \Omega$, it holds the seminorm bound

$$[v]_{C^\beta(\overline{B_{R/2}(x)})} \le C\left(1 + R^{-M}\right),$$

where $R = dist(x, \mathbb{R}^n \setminus \Omega)$.

(iii) For each $x_0 \in \partial\Omega$ and for all $\rho > 0$ it holds

$$\sup_{B_\rho(x_0)\cap\Omega} v - \inf_{B_\rho(x_0)\cap\Omega} v \le C\rho^{\widetilde{\alpha}}.$$

Indeed, it follows from Lemma 1.2.7 that $\|v\|_{L^\infty(\Omega)} \le C$ for some C depending only on Ω and s. Hence, (i) is satisfied.

Moreover, if $\beta \in (0, 2s)$, it follows from Lemma 1.2.9 that for every $x \in \Omega$,

$$[u]_{C^\beta(B_{R/2}(x))} \le CR^{-\beta}, \qquad \beta \in (0, 2s),$$

where $R = \delta(x)$. But since Ω is $C^{1,1}$, then provided $\delta(x) < \rho_0$ we will have

$$\|\delta^{-s}\|_{L^\infty(B_{R/2}(x))} \le CR^{-s} \quad \text{and} \quad [\delta^{-s}]_{C^{0,1}(B_{R/2}(x))} \le CR^{-s-1}$$

and hence, by interpolation,

$$[\delta^{-s}]_{C^\beta(B_{R/2}(x))} \le CR^{-s-\beta}$$

for each $\beta \in (0,1)$. Thus, since $v = u\delta^{-s}$, we find

$$[v]_{C^\beta(B_{R/2}(x))} \le C\left(1 + R^{-s-\beta}\right)$$

for all $x \in \Omega$ and $\beta < \min\{1, 2s\}$. Therefore hypothesis (ii) is satisfied. The constants C depend only on Ω and s.

In addition, using Proposition 1.3.5 and that $\|v\|_{L^\infty(\Omega)} \le C$, we deduce that hypothesis (iii) is satisfied.

Now, we claim that (i)-(ii)-(iii) lead to

$$[v]_{C^\alpha(\overline{\Omega})} \le C,$$

for some $\alpha \in (0,1)$ depending only on Ω and s.

Indeed, let $x, y \in \Omega$, $R = dist(x, \mathbb{R}^n \setminus \Omega) \ge dist(y, \mathbb{R}^n \setminus \Omega)$, and $r = |x - y|$. Let us see that $|v(x) - v(y)| \le Cr^\alpha$ for some $\alpha > 0$.

If $r \geq 1$ then it follows from (i). Assume $r < 1$, and let $p \geq 1$ to be chosen later. Then, we have the following dichotomy:

Case 1. Assume $r \geq R^p/2$. Let $x_0, y_0 \in \partial\Omega$ be such that $|x - x_0| = dist(x, \mathbb{R}^n \setminus \Omega)$ and $|y - y_0| = dist(y, \mathbb{R}^n \setminus \Omega)$. Then, using (iii) and the definition of R we deduce

$$|v(x) - v(y)| \leq |v(x) - v(x_0)| + |v(x_0) - v(y_0)| + |v(y_0) - v(y)| \leq CR^{\widetilde{\alpha}} \leq Cr^{\widetilde{\alpha}/p}.$$

Case 2. Assume $r \leq R^p/2$. Hence, since $p \geq 1$, we have $y \in B_{R/2}(x)$. Then, using (ii) we obtain

$$|v(x) - v(y)| \leq C(1 + R^{-M})r^\beta \leq C\left(1 + r^{-M/p}\right) r^\beta \leq Cr^{\beta - M/p}.$$

To finish the proof we only need to choose $p > M/\beta$ and take $\alpha = \min\{\widetilde{\alpha}/p, \beta - M/p\}$. □

1.4 Interior estimates for u/δ^s

The main goal of this section is to prove the C^γ bounds in Ω for the function u/δ^s in Theorem 1.1.5.

To prove this result we find an equation for the function $v = u/\delta^s|_\Omega$, that is derived below. This equation is nonlocal, and thus, we need to give values to v in $\mathbb{R}^n \setminus \Omega$, although we want an equation only in Ω. It might seem natural to consider u/δ^s, which vanishes outside Ω since $u \equiv 0$ there, as an extension of $u/\delta^s|_\Omega$. However, such extension is discontinuous through $\partial\Omega$, and it would lead to some difficulties.

Instead, we consider a $C^\alpha(\mathbb{R}^n)$ extension of the function $u/\delta^s|_\Omega$, which is $C^\alpha(\overline{\Omega})$ by Theorem 1.1.2. Namely, throughout this section, let v be the $C^\alpha(\mathbb{R}^n)$ extension of $u/\delta^s|_\Omega$ given by Lemma 1.3.8.

Let $\delta_0 = \delta\chi_\Omega$, and note that $u = v\delta_0^s$ in $\mathbb{R}^n$. Then, using (1.1) we have

$$g(x) = (-\Delta)^s(v\delta_0^s) = v(-\Delta)^s\delta_0^s + \delta_0^s(-\Delta)^s v - I_s(v, \delta_0^s)$$

in $\Omega_{\rho_0} = \{x \in \Omega : \delta(x) < \rho_0\}$, where

$$I_s(w_1, w_2)(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{\big(w_1(x) - w_1(y)\big)\big(w_2(x) - w_2(y)\big)}{|x - y|^{n+2s}}\, dy \tag{1.38}$$

and ρ_0 is a small constant depending on the domain; see Remark 1.3.4. Here, we have used that $(-\Delta)^s(w_1w_2) = w_1(-\Delta)^s w_2 + w_2(-\Delta)^s w_1 - I_s(w_1, w_2)$, which follows easily from (1.2). This equation is satisfied pointwise in Ω_{ρ_0}, since g is C^α in Ω. We have to consider Ω_{ρ_0} instead of Ω because the distance function is $C^{1,1}$ there and thus we can compute $(-\Delta)^s\delta_0^s$. In all Ω the distance function δ is only Lipschitz and hence $(-\Delta)^s\delta_0^s$ is singular for $s \geq \frac{1}{2}$.

Thus, the following is the equation for v:

$$(-\Delta)^s v = \frac{1}{\delta_0^s}\Big(g(x) - v(-\Delta)^s\delta_0^s + I_s(v, \delta_0^s)\Big) \quad \text{in } \Omega_{\rho_0}\,. \tag{1.39}$$

From this equation we will obtain the interior estimates for v. More precisely, we will obtain a priori bounds for the interior Hölder norms of v, treating $\delta_0^{-s} I_s(v, \delta_0^s)$ as a

lower order term. For this, we consider the weighted Hölder norms given by Definition 1.1.3.

Recall that, in all the paper, we denote C^β the space $C^{k,\beta'}$, where $\beta = k + \beta'$ with k integer and $\beta' \in (0, 1]$.

In Theorem 1.1.2 we have proved that $u/\delta^s|_\Omega$ is $C^\alpha(\overline{\Omega})$ for some $\alpha \in (0, 1)$, with an estimate. From this C^α estimate and from the equation for v (1.39), we will find next the estimate for $\|u/\delta^s\|^{(-\alpha)}_{\gamma;\Omega}$ stated in Theorem 1.1.5.

The proof of this result relies on some preliminary results below.

Next lemma is used to control the lower order term $\delta_0^{-s} I_s(v, \delta_0^s)$ in the equation (1.39) for v.

Lemma 1.4.1. *Let Ω be a bounded $C^{1,1}$ domain, and $U \subset \Omega_{\rho_0}$ be an open set. Let s and α belong to $(0, 1)$ and satisfy $\alpha + s \le 1$ and $\alpha < s$. Then,*

$$\|I_s(w, \delta_0^s)\|^{(s-\alpha)}_{\alpha;U} \le C\left([w]_{C^\alpha(\mathbb{R}^n)} + [w]^{(-\alpha)}_{\alpha+s;U} \right), \tag{1.40}$$

for all w with finite right hand side. The constant C depends only on Ω, s, and α.

To prove Lemma 1.4.1 we need the next

Lemma 1.4.2. *Let $U \subset \mathbb{R}^n$ be a bounded open set. Let $\alpha_1, \alpha_2, \in (0, 1)$ and $\beta \in (0, 1]$ satisfy $\alpha_i < \beta$ for $i = 1, 2$, $\alpha_1 + \alpha_2 < 2s$, and $s < \beta < 2s$. Assume that $w_1, w_2 \in C^\beta(U)$. Then,*

$$\|I_s(w_1, w_2)\|^{(2s-\alpha_1-\alpha_2)}_{2\beta-2s;U} \le C \left([w_1]_{C^{\alpha_1}(\mathbb{R}^n)} + [w_1]^{(-\alpha_1)}_{\beta;U} \right) \left([w_2]_{C^{\alpha_2}(\mathbb{R}^n)} + [w_2]^{(-\alpha_2)}_{\beta;U} \right), \tag{1.41}$$

for all functions w_1, w_2 with finite right hand side. The constant C depends only on α_1, α_2, n, β, and s.

Proof. Let $x_0 \in U$ and $R = d_{x_0}/2$, and denote $B_\rho = B_\rho(x_0)$. Let

$$K = \left([w_1]_{C^{\alpha_1}(\mathbb{R}^n)} + [w_1]^{(-\alpha_1)}_{\beta;U} \right) \left([w_2]_{C^{\alpha_2}(\mathbb{R}^n)} + [w_2]^{(-\alpha_2)}_{\beta;U} \right).$$

First we bound $|I_s(w_1, w_2)(x_0)|$.

$$\begin{aligned}
|I_s(w_1, w_2)(x_0)| &\le C \int_{\mathbb{R}^n} \frac{\big|w_1(x_0) - w_1(y)\big|\big|w_2(x_0) - w_2(y)\big|}{|x_0 - y|^{n+2s}}\, dy \\
&\le C \int_{B_R(0)} \frac{R^{\alpha_1+\alpha_2-2\beta} [w_1]^{(-\alpha_1)}_{\beta;U} [w_2]^{(-\alpha_2)}_{\beta;U} |z|^{2\beta}}{|z|^{n+2s}}\, dz + \\
&\qquad + C \int_{\mathbb{R}^n \setminus B_R(0)} \frac{[w_1]_{C^{\alpha_1}(\mathbb{R}^n)} [w_2]_{C^{\alpha_2}(\mathbb{R}^n)} |z|^{\alpha_1+\alpha_2}}{|z|^{n+2s}}\, dz \\
&\le C R^{\alpha_1+\alpha_2-2s} K\,.
\end{aligned}$$

Let $x_1, x_2 \in B_{R/2}(x_0) \subset B_{2R}(x_0)$. Next, we bound $|I_s(w_1, w_2)(x_1) - I_s(w_1, w_2)(x_2)|$. Let η be a smooth cutoff function such that $\eta \equiv 1$ on $B_1(0)$ and $\eta \equiv 0$ outside $B_{3/2}(0)$. Define

$$\eta^R(x) = \eta\left(\frac{x - x_0}{R} \right) \quad \text{and} \quad \bar{w}_i = \big(w_i - w_i(x_0)\big)\eta^R\,, \quad i = 1, 2\,.$$

Note that we have

$$\|\bar{w}_i\|_{L^\infty(\mathbb{R}^n)} = \|\bar{w}_i\|_{L^\infty(B_{3R/2})} \le \left(\frac{3R}{2}\right)^{\alpha_i} [w_i]_{C^{\alpha_i}(\mathbb{R}^n)}$$

and

$$\begin{aligned}[\bar{w}_i]_{C^\beta(\mathbb{R}^n)} &\le C\left([w_i]_{C^\beta(\overline{B_{3R/2}})}\|\eta\|_{L^\infty(B_{3R/2})} + \|w_i - w_i(0)\|_{L^\infty(B_{3R/2})}[w_i]_{C^\beta(\overline{B_{3R/2}})}\right)\\ &\le CR^{\alpha_i-\beta}\left([w_i]_{C^{\alpha_i}(\mathbb{R}^n)} + [w_i]^{(-\alpha_i)}_{\beta;U}\right).\end{aligned}$$

Let

$$\varphi_i = w_i - w_i(x_0) - \bar{w}_i$$

and observe that φ_i vanishes in B_R. Hence, $\varphi_i(x_1) = \varphi_i(x_2) = 0$, $i = 1, 2$. Next, let us write

$$I_s(w_1, w_2)(x_1) - I_s(w_1, w_2)(x_2) = c_{n,s}\left(J_{11} + J_{12} + J_{21} + J_{22}\right),$$

where

$$\begin{aligned}J_{11} = \int_{\mathbb{R}^n} \frac{\big(\bar{w}_1(x_1) - \bar{w}_1(y)\big)\big(\bar{w}_2(x_1) - \bar{w}_2(y)\big)}{|x_1 - y|^{n+2s}}\,dy\\ - \int_{\mathbb{R}^n} \frac{\big(\bar{w}_1(x_2) - \bar{w}_1(y)\big)\big(\bar{w}_2(x_2) - \bar{w}_2(y)\big)}{|x_2 - y|^{n+2s}}\,dy,\end{aligned}$$

$$J_{12} = \int_{\mathbb{R}^n\setminus B_R} \frac{-\big(\bar{w}_1(x_1) - \bar{w}_1(y)\big)\varphi_2(y)}{|x_1 - y|^{n+2s}} + \frac{\big(\bar{w}_1(x_2) - \bar{w}_1(y)\big)\varphi_2(y)}{|x_2 - y|^{n+2s}}\,dy,$$

$$J_{21} = \int_{\mathbb{R}^n\setminus B_R} \frac{-\big(\bar{w}_2(x_1) - \bar{w}_2(y)\big)\varphi_1(y)}{|x_1 - y|^{n+2s}} + \frac{\big(\bar{w}_2(x_2) - \bar{w}_2(y)\big)\varphi_1(y)}{|x_2 - y|^{n+2s}}\,dy,$$

and

$$J_{22} = \int_{\mathbb{R}^n\setminus B_R} \frac{\varphi_1(y)\varphi_2(y)}{|x_1 - y|^{n+2s}} - \frac{\varphi_1(y)\varphi_2(y)}{|x_2 - y|^{n+2s}}\,dy.$$

We now bound separately each of these terms.

Bound of J_{11}. We write $J_{11} = J^1_{11} + J^2_{11}$ where

$$J^1_{11} = \int_{\mathbb{R}^n} \frac{\big(\bar{w}_1(x_1) - \bar{w}_1(x_1+z) - \bar{w}_1(x_2) + \bar{w}_1(x_2+z)\big)\big(\bar{w}_2(x_1) - \bar{w}_2(x_1+z)\big)}{|z|^{n+2s}}\,dz,$$

$$J^2_{11} = \int_{\mathbb{R}^n} \frac{\big(\bar{w}_1(x_2) - \bar{w}_1(x_2+z)\big)\big(\bar{w}_2(x_1) - \bar{w}_2(x_1+z) - \bar{w}_2(x_2) + \bar{w}_2(x_2+z)\big)}{|z|^{n+2s}}\,dz.$$

To bound $|J^1_{11}|$ we proceed as follows

$$\begin{aligned}|J^1_{11}| &\le \int_{B_r(0)} \frac{R^{\alpha_1-\beta}[w_1]^{(-\alpha_1)}_{\beta;U}|z|^\beta R^{\alpha_2-\beta}[w_2]^{(-\alpha_2)}_{\beta;U}|z|^\beta}{|z|^{n+2s}}\,dz +\\ &\qquad + \int_{\mathbb{R}^n\setminus B_r(0)} \frac{R^{\alpha_1-\beta}[w_1]^{(-\alpha_1)}_{\beta;U} r^\beta R^{\alpha_2-\beta}[w_2]^{(-\alpha_2)}_{\beta;U}|z|^\beta}{|z|^{n+2s}}\,dz\\ &\le CR^{\alpha_1+\alpha_2-2\beta} r^{2\beta-2s} K.\end{aligned}$$

Similarly, $|J^2_{11}| \le CR^{\alpha_1+\alpha_2-2\beta} r^{2\beta-2s} K$.

Bound of J_{12} and J_{21}. We write $J_{12} = J^1_{12} + J^2_{12}$ where

$$J^1_{12} = \int_{\mathbb{R}^n \setminus B_R} -\varphi_2(y) \frac{\bar{w}_1(x_1) - \bar{w}_1(x_2)}{|x_1 - y|^{n+2s}}\, dy$$

and

$$J^2_{12} = \int_{\mathbb{R}^n \setminus B_R} -\varphi_2(y)\big(\bar{w}_1(x_2) - \bar{w}_1(y)\big) \left\{ \frac{1}{|x_1 - y|^{n+2s}} - \frac{1}{|x_2 - y|^{n+2s}} \right\} dy\,.$$

To bound $|J^1_{12}|$ we recall that $\varphi_2(x_1) = 0$ and proceed as follows

$$\begin{aligned} |J^1_{12}| &\le C \int_{\mathbb{R}^n \setminus B_R} |x_1 - y|^{\alpha_2} [\varphi_2]_{C^{0,\alpha_2}(\mathbb{R}^n)} \frac{R^{\alpha_1-\beta} [w_1]^{(-\alpha_1)}_{\beta;U} r^\beta}{|x_1 - y|^{n+2s}}\, dy \\ &\le CR^{\alpha_1+\alpha_2-\beta-2s} r^\beta K \le CR^{\alpha_1+\alpha_2-2\beta} r^{2\beta-2s} K. \end{aligned}$$

We have used that $[\varphi_2]_{C^{\alpha_2}(\mathbb{R}^n)} = [w - \bar{w}]_{C^{\alpha_2}(\mathbb{R}^n)} \le 2[w]_{C^{\alpha_2}(\mathbb{R}^n)}$, $r \le R$, and $\beta < 2s$.

To bound $|J^2_{12}|$, let $\Phi(z) = |z|^{-n-2s}$. Note that, for each $\gamma \in (0, 1]$, we have

$$|\Phi(z_1 - z) - \Phi(z_2 - z)| \le C|z_1 - z_2|^\gamma |z|^{-n-2s-\gamma} \tag{1.42}$$

for all z_1, z_2 in $B_{R/2}(0)$ and $z \in \mathbb{R}^n \setminus B_R(0)$. Then, using that $\varphi_2(x_2) = 0$,

$$\begin{aligned} |J^2_{12}| &\le C \int_{\mathbb{R}^n \setminus B_R} |x_2 - y|^{\alpha_1+\alpha_2} [\varphi_2]_{C^{\alpha_2}(\mathbb{R}^n)} [\varphi_2]_{C^{\alpha_2}(\mathbb{R}^n)} \frac{|x_1 - x_2|^{2\beta-2s}}{|x_2 - y|^{n+2\beta}}\, dy \\ &\le CR^{\alpha_1+\alpha_2-2\beta} r^{2\beta-2s} K\,. \end{aligned}$$

This proves that $|J_{12}| \le CR^{\alpha_1+\alpha_2-2\beta} r^{2\beta-2s} K$. Changing the roles of α_1 and α_2 we obtain the same bound for $|J_{21}|$.

Bound of J_{22}. Using again $\varphi_i(x_i) = 0$, $i = 1, 2$, we write

$$J_{22} = \int_{\mathbb{R}^n \setminus B_R} \big(\varphi_1(x_1) - \varphi_1(y)\big)\big(\varphi_2(x_1) - \varphi_2(y)\big) \left(\frac{1}{|x_1 - y|^{n+2s}} - \frac{1}{|x_2 - y|^{n+2s}} \right) dy\,.$$

Hence, using again (1.42),

$$\begin{aligned} |J_{22}| &\le C \int_{\mathbb{R}^n \setminus B_R} |x_1 - y|^{\alpha_1+\alpha_2} [\varphi_2]_{C^{0,\alpha_2}(\mathbb{R}^n)} [\varphi_2]_{C^{0,\alpha_2}(\mathbb{R}^n)} \frac{|x_1 - x_2|^{2\beta-2s}}{|x_1 - y|^{n+2\beta}}\, dy \\ &\le CR^{\alpha_1+\alpha_2-2\beta} r^{2\beta-2s} K\,. \end{aligned}$$

Summarizing, we have proven that for all x_0 such that $d_x = 2R$ and for all $x_1, x_2 \in B_{R/2}(x_0)$ it holds

$$|I_s(\delta^s_0, w)(x_0)| \le CR^{\alpha_1-\alpha_2-2s} K$$

and

$$\frac{|I_s(\delta^s_0, w)(x_1) - I_s(\delta^s_0, w)(x_2)|}{|x_1 - x_2|^{2\beta-2s}} \le CR^{\alpha_1+\alpha_2-2\beta} \big([w]^{(-\alpha)}_{\alpha+s;U} + [w]_{C^\alpha(\mathbb{R}^n)} \big)\,.$$

This yields (1.41), as shown in Step 2 in the proof of Lemma 1.2.10. □

Next we prove Lemma 1.4.1.

Proof of Lemma 1.4.1. The distance function δ_0 is $C^{1,1}$ in $\overline{\Omega_{\rho_0}}$ and since $U \subset \Omega_{\rho_0}$ we have $d_x \leq \delta_0(x)$ for all $x \in U$. Hence, it follows that

$$[\delta_0^s]_{C^s(\mathbb{R}^n)} + [\delta_0^s]_{\beta;U}^{(-s)} \leq C(\Omega, \beta)$$

for all $\beta \in [s, 2]$.

Then, applying Lemma 1.4.2 with $w_1 = w$, $w_2 = \delta_0^s$, $\alpha_1 = \alpha$, $\alpha_2 = s$, and $\beta = s+\alpha$, we obtain

$$\|I_s(w, \delta_0^s)\|_{2\alpha;U}^{(s-\alpha)} \leq C\bigg([w]_{C^\alpha(\mathbb{R}^n)} + [w]_{\alpha+s;U}^{(-\alpha)} \bigg),$$

and hence (1.40) follows. □

Using Lemma 1.4.1 we can now prove Theorem 1.1.5 and Corollary 1.1.6.

Proof of Theorem 1.1.5. Let $U \subset\subset \Omega_{\rho_0}$. We prove first that there exist $\alpha \in (0,1)$ and C, depending only on s and Ω —and not on U—, such that

$$\|u/\delta^s\|_{\alpha+2s;U}^{(-\alpha)} \leq C\left(\|g\|_{L^\infty(\Omega)} + \|g\|_{\alpha;\Omega}^{(s-\alpha)} \right).$$

Then, letting $U \uparrow \Omega_{\rho_0}$ we will find that this estimate holds in Ω_{ρ_0} with the same constant.

To prove this, note that by Theorem 1.1.2 we have

$$\|u/\delta^s\|_{C^\alpha(\overline{\Omega})} \leq C\big(s, \Omega\big) \|g\|_{L^\infty(\Omega)}.$$

Recall that v denotes the $C^\alpha(\mathbb{R}^n)$ extension of $u/\delta^s|_\Omega$ given by Lemma 1.3.8, which satisfies $\|v\|_{C^\alpha(\mathbb{R}^n)} = \|u/\delta^s\|_{C^\alpha(\overline{\Omega})}$. Since $u \in C^{\alpha+2s}(\Omega)$ and $\delta \in C^{1,1}(\Omega_{\rho_0})$, it is clear that $\|v\|_{\alpha+2s;U}^{(-\alpha)} < \infty$ —it is here where we use that we are in a subdomain U and not in Ω_{ρ_0}. Next we obtain an a priori bound for this seminorm in U. To do it, we use the equation (1.39) for v:

$$(-\Delta)^s v = \frac{1}{\delta^s}\bigg(g(x) - v(-\Delta)^s \delta_0^s + I(\delta_0^s, v) \bigg) \quad \text{in } \Omega_{\rho_0} = \{x \in \Omega \,:\, \delta(x) < \rho_0\}.$$

Now we will se that this equation and Lemma 1.2.10 lead to an a priori bound for $\|v\|_{\alpha+2s;U}^{(-\alpha)}$. To apply Lemma 1.2.10, we need to bound $\|(-\Delta)^s v\|_{\alpha;U}^{(2s-\alpha)}$. Let us examine the three terms on the right hand side of the equation.

First term. Using that

$$d_x = dist(x, \partial U) < dist(x, \partial\Omega) = \delta(x)$$

for all $x \in U$ we obtain that, for all $\alpha \leq s$,

$$\|\delta^{-s} g\|_{\alpha;U}^{(2s-\alpha)} \leq C\big(s, \Omega\big) \|g\|_{\alpha;\Omega}^{(s-\alpha)}.$$

Second term. We know from Lemma 1.3.9 that, for $\alpha \leq \min\{s, 1-s\}$,

$$\|(-\Delta)^s \delta_0^s\|_{C^\alpha(\overline{\Omega_{\rho_0}})} \leq C\big(s, \Omega\big).$$

Hence,

$$\begin{aligned}\|\delta^{-s}v(-\Delta)^s\delta_0^s\|_{\alpha;U}^{(2s-\alpha)} &\le diam(\Omega)^s\|\delta^{-s}v(-\Delta)^s\delta_0^s\|_{\alpha;U}^{(s-\alpha)} \le C\big(s,\Omega\big)\|v\|_{C^\alpha(\mathbb{R}^n)}\\ &\le C\big(s,\Omega\big)\|g\|_{L^\infty(\Omega)}\,.\end{aligned}$$

Third term. From Lemma 1.4.1 we know that

$$\|I(v,\delta_0^s)\|_{\alpha;U}^{(s-\alpha)} \le C(n,s,\alpha)\bigg(\|v\|_{C^\alpha(\mathbb{R}^n)} + [v]_{\alpha+s;U}^{(-\alpha)}\bigg)\,,$$

and hence

$$\begin{aligned}\|\delta^{-s}I(v,\delta_0^s)\|_{\alpha;U}^{(2s-\alpha)} &\le C(n,s,\Omega,\alpha)\bigg(\|v\|_{C^\alpha(\mathbb{R}^n)} + [v]_{\alpha+s;U}^{(-\alpha)}\bigg)\\ &\le C(n,s,\Omega,\alpha,\varepsilon_0)\|v\|_{C^\alpha(\mathbb{R}^n)} + \varepsilon_0\|v\|_{\alpha+2s;U}^{(-\alpha)}\end{aligned}$$

for each $\epsilon_0 > 0$. The last inequality is by standard interpolation.

Now, using Lemma 1.2.10 we deduce

$$\begin{aligned}\|v\|_{\alpha+2s;U}^{(-\alpha)} &\le C\left(\|v\|_{C^\alpha(\mathbb{R}^n)} + \|(-\Delta)^s v\|_{\alpha;U}^{(2s-\alpha)}\right)\\ &\le C\left(\|v\|_{C^\alpha(\mathbb{R}^n)} + \|\delta^{-s}g\|_{\alpha;U}^{(2s-\alpha)} + \|\delta^{-s}v(-\Delta)^s\delta_0^s\|_{\alpha;U}^{(2s-\alpha)} + \|I(v,\delta_0^s)\|_{\alpha;U}^{(s-\alpha)}\right)\\ &\le C(s,\Omega,\alpha,\varepsilon_0)\left(\|g\|_{L^\infty(\Omega)} + \|g\|_{\alpha;\Omega}^{(s-\alpha)}\right) + C\varepsilon_0\|v\|_{\alpha+2s;U}^{(-\alpha)},\end{aligned}$$

and choosing ε_0 small enough we obtain

$$\|v\|_{\alpha+2s;U}^{(-\alpha)} \le C\left(\|g\|_{L^\infty(\Omega)} + \|g\|_{\alpha;\Omega}^{(s-\alpha)}\right).$$

Furthermore, letting $U \uparrow \Omega_{\rho_0}$ we obtain that the same estimate holds with U replaced by Ω_{ρ_0}.

Finally, in $\Omega \setminus \Omega_{\rho_0}$ we have that u is $C^{\alpha+2s}$ and δ^s is uniformly positive and $C^{0,1}$. Thus, we have $u/\delta^s \in C^\gamma(\Omega \setminus \Omega_{\rho_0})$, where $\gamma = \min\{1, \alpha+2s\}$, and the theorem follows. □

Next we give the

Proof of Corollary 1.1.6. (a) It follows from Proposition 1.1.1 that $u \in C^s(\mathbb{R}^n)$. The interior estimate follow by applying repeatedly Proposition 1.1.4.

(b) It follows from Theorem 1.1.2 that $u/\delta^s|_\Omega \in C^\alpha(\overline{\Omega})$. The interior estimate follows from Theorem 1.1.5. □

The following two lemmas are closely related to Lemma 1.4.2 and are needed in [250] and in Remark 1.2.11 of this paper.

Lemma 1.4.3. *Let U be an open domain and α and β be such that $\alpha \le s < \beta$ and $\beta - s$ is not an integer. Let k be an integer such that $\beta = k + \beta'$ with $\beta' \in (0,1]$. Then,*

$$[(-\Delta)^{s/2}w]_{\beta-s;U}^{(s-\alpha)} \le C\big(\|w\|_{C^\alpha(\mathbb{R}^n)} + \|w\|_{\beta;U}^{(-\alpha)}\big)\,, \tag{1.43}$$

for all w with finite right hand side. The constant C depends only on n, s, α, and β.

Proof. Let $x_0 \in U$ and $R = d_{x_0}/2$, and denote $B_\rho = B_\rho(x_0)$. Let η be a smooth cutoff function such that $\eta \equiv 1$ on $B_1(0)$ and $\eta \equiv 0$ outside $B_{3/2}(0)$. Define

$$\eta^R(x) = \eta\left(\frac{x - x_0}{R}\right) \quad \text{and} \quad \bar{w} = \big(w - w(x_0)\big)\eta^R .$$

Note that we have

$$\|\bar{w}\|_{L^\infty(\mathbb{R}^n)} = \|\bar{w}\|_{L^\infty(\overline{B_{3R/2}})} \le \left(\frac{3R}{2}\right)^\alpha [w]_{C^\alpha(\mathbb{R}^n)} .$$

In addition, for each $1 \le l \le k$

$$\begin{aligned}\|D^l\bar{w}\|_{L^\infty(\mathbb{R}^n)} &\le C\sum_{m=0}^{l} \|D^m(w - w(x_0))D^{l-m}\eta^R\|_{L^\infty(\overline{B_{3R/2}})} \\ &\le CR^{-l+\alpha}\left([w]_{C^\alpha(\mathbb{R}^n)} + \sum_{m=1}^{l}[w]_{m,U}^{(-\alpha)}\right).\end{aligned}$$

Hence, by interpolation, for each $0 \le l \le k-1$

$$\|D^l\bar{w}\|_{C^{l+\beta'}(\mathbb{R}^n)} \le CR^{-l-\beta'+\alpha}\left([w]_{C^\alpha(\mathbb{R}^n)} + \sum_{m=1}^{l}[w]_{m,U}^{(-\alpha)}\right),$$

and therefore

$$[D^k\bar{w}]_{C^{\beta'}(\mathbb{R}^n)} \le CR^{-\beta+\alpha}\|w\|_{\beta;U}^{(-\alpha)} . \tag{1.44}$$

Let $\varphi = w - w(x_0) - \bar{w}$ and observe that φ vanishes in B_R and, hence, $\varphi(x_1) = \varphi(x_2) = 0$.

Next we proceed differently if $\beta' > s$ or if $\beta' < s$. This is because $C^{\beta-s}$ equals either $C^{k,\beta'-s}$ or $C^{k-1,1+\beta'-s}$.

Case 1. Assume $\beta' > s$. Let $x_1, x_2 \in B_{R/2}(x_0) \subset B_{2R}(x_0)$. We want to bound $|D^k(-\Delta)^{s/2}w(x_1) - D^k(-\Delta)^{s/2}w(x_2)|$, where D^k denotes any k-th derivative with respect to a fixed multiindex. We have

$$(-\Delta)^{s/2}w = (-\Delta)^{s/2}\bar{w} + (-\Delta)^{s/2}\varphi \quad \text{in } B_{R/2} .$$

Then,

$$D^k(-\Delta)^{s/2}w(x_1) - D^k(-\Delta)^{s/2}w(x_2) = c_{n,\frac{s}{2}}(J_1 + J_2) ,$$

where

$$J_1 = \int_{\mathbb{R}^n}\left\{\frac{D^k\bar{w}(x_1) - D^k\bar{w}(y)}{|x_1 - y|^{n+s}} - \frac{D^k\bar{w}(x_2) - D^k\bar{w}(y)}{|x_2 - y|^{n+s}}\right\}dy$$

and

$$J_2 = D^k\int_{\mathbb{R}^n\setminus B_R}\frac{-\varphi(y)}{|x_1 - y|^{n+s}}\,dy - D^k\int_{\mathbb{R}^n\setminus B_R}\frac{-\varphi(y)}{|x_2 - y|^{n+s}}\,dy .$$

To bound $|J_1|$ we proceed as follows. Let $r = |x_1 - x_2|$. Then, using (1.44),

$$\begin{aligned}|J_1| &= \left| \int_{\mathbb{R}^n} \frac{D^k\bar{w}(x_1) - D^k\bar{w}(x_1+z) - D^k\bar{w}(x_2) + D^k\bar{w}(x_2+z)}{|z|^{n+s}}\, dz \right| \\ &\leq \int_{B_r} \frac{R^{\alpha-\beta}\|w\|^{(-\alpha)}_{\beta;U}|z|^{\beta'}}{|z|^{n+s}}\, dz + \int_{\mathbb{R}^n\setminus B_r} \frac{R^{\alpha-\beta}\|w\|^{(-\alpha)}_{\beta;U} r^{\beta'}}{|z|^{n+s}}\, dz \\ &\leq CR^{\alpha-\beta} r^{\beta'-s}\|w\|^{(-\alpha)}_{\beta;U}\,.\end{aligned}$$

Let us bound now $|J_2|$. Writing $\Phi(z) = |z|^{-n-s}$ and using that $\varphi(x_0) = 0$,

$$\begin{aligned}|J_2| &= \left| \int_{\mathbb{R}^n\setminus B_R} \varphi(y)\big(D^k\Phi(x_1-y) - D^k\Phi(x_2-y)\big)\, dy \right| \\ &\leq C\int_{\mathbb{R}^n\setminus B_R} |x_0-y|^\alpha [w]_{C^\alpha(\mathbb{R}^n)} \frac{|x_1-x_2|^{\beta'-s}}{|x_0-y|^{n+\beta}}\, dy \\ &\leq CR^{\alpha-\beta} r^{\beta'-s}[w]_{C^\alpha(\mathbb{R}^n)},\end{aligned}$$

where we have used that

$$|D^k\Phi(z_1-z) - D^k\Phi(z_2-z)| \leq C|z_1-z_2|^{\beta'-s}|z|^{-n-\beta}$$

for all z_1, z_2 in $B_{R/2}(0)$ and $z \in \mathbb{R}^n \setminus B_R$.

Hence, we have proved that

$$[(-\Delta)^{s/2}w]_{C^{\beta-s}(\overline{B_R(x_0)})} \leq CR^{\alpha-\beta}\|w\|^{(-\alpha)}_{\beta;U}.$$

Case 2. Assume $\beta' < s$. Let $x_1, x_2 \in B_{R/2}(x_0) \subset B_{2R}(x_0)$. We want to bound $|D^{k-1}(-\Delta)^{s/2}w(x_1) - D^{k-1}(-\Delta)^{s/2}w(x_2)|$. We proceed as above but we now use

$$\begin{aligned}&|D^{k-1}\bar{w}(x_1) - D^{k-1}\bar{w}(x_1+y) - D^{k-1}\bar{w}(x_2) + D^{k-1}\bar{w}(x_2+y)| \leq \\ &\qquad \leq \big|D^k\bar{w}(x_1) - D^k\bar{w}(x_2)\big|\,|y| + |y|^{1+\beta'}\|\bar{w}\|_{C^\beta(\mathbb{R}^n)} \\ &\qquad \leq \big(|x_1-x_2|^{\beta'}|y| + |y|^{1+\beta'}\big)R^{\alpha-\beta}\|w\|^{(-\alpha)}_{\beta;U}\end{aligned}$$

in B_r, and

$$\begin{aligned}&|D^{k-1}\bar{w}(x_1) - D^{k-1}\bar{w}(x_1+y) - D^{k-1}\bar{w}(x_2) + D^{k-1}\bar{w}(x_2+y)| \leq \\ &\qquad \leq \big|D^k\bar{w}(x_1) - D^k\bar{w}(x_1+y)\big|\,|x_1-x_2| + |x_1-x_2|^{1+\beta'}\|\bar{w}\|_{C^\beta(\mathbb{R}^n)} \\ &\qquad \leq \big(|y|^{\beta'}|x_1-x_2| + |x_1-x_2|^{1+\beta'}\big)R^{\alpha-\beta}\|w\|^{(-\alpha)}_{\beta;U}\end{aligned}$$

in $\mathbb{R}^n\setminus B_r$. Then, as in Case 1 we obtain $[(-\Delta)^{s/2}w]_{C^{\beta-s}(\overline{B_R(x_0)})} \leq CR^{\alpha-\beta}\|w\|^{(-\alpha)}_{\beta;U}$.

This yields (1.43), as in Step 2 of Lemma 1.2.10. □

Next lemma is a variation of the previous one and gives a pointwise bound for $(-\Delta)^{s/2}w$. It is used in Remark 1.2.11.

Lemma 1.4.4. *Let $U \subset \mathbb{R}^n$ be an open set, and let $\beta > s$. Then, for all $x \in U$*

$$|(-\Delta)^{s/2}w(x)| \leq C(\|w\|_{C^s(\mathbb{R}^n)} + \|w\|^{(-s)}_{\beta;U})\Big(1 + |\log dist(x,\partial U)|\Big),$$

whenever w has finite right hand side. The constant C depends only on n, s, and β.

Proof. We may assume $\beta < 1$. Let $x_0 \in U$ and $R = d_{x_0}/2$, and define $\bar{w}$ and φ as in the proof of the previous lemma. Then,

$$(-\Delta)^{s/2} w(x_0) = (-\Delta)^{s/2} \bar{w}(x_0) + (-\Delta)^{s/2} \varphi(x_0) = c_{n,\frac{s}{2}} (J_1 + J_2),$$

where

$$J_1 = \int_{\mathbb{R}^n} \frac{\bar{w}(x_0) - \bar{w}(x_0 + z)}{|z|^{n+s}}\, dz \quad \text{and} \quad J_2 = \int_{\mathbb{R}^n \setminus B_R} \frac{-\varphi(x_0 + z)}{|z|^{n+s}}\, dz.$$

With similar arguments as in the previous proof we readily obtain $|J_1| \le C(1 + |\log R|)\|w\|^{(-s)}_{\beta;U}$ and $|J_2| \le C(1 + |\log R|)\|w\|_{C^s(\mathbb{R}^n)}$. □

1.5 Appendix: Basic tools and barriers

In this appendix we prove Proposition 1.3.1 and Lemmas 1.3.2 and 1.2.6. Proposition 1.3.1 is well-known (see [66]), but for the sake of completeness we sketch here a proof that uses the Caffarelli-Silvestre extension problem [68].

Proof of Proposition 1.3.1. Let (x, y) and (r, θ) be Cartesian and polar coordinates of the plane. The coordinate $\theta \in (-\pi, \pi)$ is taken so that $\{\theta = 0\}$ on $\{y = 0,\ x > 0\}$. Use that the function $r^s \cos(\theta/2)^{2s}$ is a solution in the half-plane $\{y > 0\}$ to the extension problem [68],

$$div(y^{1-2s} \nabla u) = 0 \quad \text{in } \{y > 0\},$$

and that its trace on $y = 0$ is φ_0. □

The fractional Kelvin transform has been studied thoroughly in [31].

Proposition 1.5.1 (Fractional Kelvin transform)**.** *Let u be a smooth bounded function in $\mathbb{R}^n \setminus \{0\}$. Let $x \mapsto x^* = x/|x|^2$ be the inversion with respect to the unit sphere. Define $u^*(x) = |x|^{2s-n} u(x^*)$. Then,*

$$(-\Delta)^s u^*(x) = |x|^{-2s-n} (-\Delta)^s u(x^*)\,, \tag{1.45}$$

for all $x \neq 0$.

Proof. Let $x_0 \in \mathbb{R}^n \setminus \{0\}$. By subtracting a constant to u^* and using $(-\Delta)^s |x|^{2s-n} = 0$ for $x \neq 0$, we may assume $u^*(x_0) = u(x_0^*) = 0$. Recall that

$$|x - y| = \frac{|x^* - y^*|}{|x^*||y^*|}\,.$$

Thus, using the change of variables $z = y^* = y/|y|^2$,

$$\begin{aligned}
(-\Delta)^s u^*(x_0) &= c_{n,s}\ \mathrm{PV} \int_{\mathbb{R}^n} \frac{-u^*(y)}{|x_0 - y|^{n+2s}}\, dy \\
&= c_{n,s}\ \mathrm{PV} \int_{\mathbb{R}^n} \frac{-|y|^{2s-n} u(y^*)}{|x_0^* - y^*|^{n+2s}} |x_0^*|^{n+2s} |y^*|^{n+2s}\, dy \\
&= c_{n,s} |x_0|^{-n-2s}\ \mathrm{PV} \int_{\mathbb{R}^n} \frac{-|z|^{n-2s} u(z)}{|x_0^* - z|^{n+2s}} |z|^{n+2s}\, |z|^{-2n} dz \\
&= c_{n,s} |x_0|^{-n-2s}\ \mathrm{PV} \int_{\mathbb{R}^n} \frac{-u(z)}{|x_0^* - z|^{n+2s}} dz \\
&= |x_0|^{-n-2s} (-\Delta)^s u(x_0^*)\,.
\end{aligned}$$

□

Now, using Proposition 1.5.1 we prove Lemma 1.2.6.

Proof of Lemma 1.2.6. Let us denote by ψ (instead of u) the explicit solution (1.4) to problem (1.3) in B_1, which satisfies

$$\begin{cases} (-\Delta)^s \psi = 1 & \text{in } B_1 \\ \psi \equiv 0 & \text{in } \mathbb{R}^n \setminus B_1 \\ 0 < \psi < C(1-|x|)^s & \text{in } B_1 \,. \end{cases} \tag{1.46}$$

From ψ, the supersolution φ_1 in the exterior of the ball is readily built using the fractional Kelvin transform. Indeed, let ξ be a radial smooth function satisfying $\xi \equiv 1$ in $\mathbb{R}^n \setminus B_5$ and $\xi \equiv 0$ in B_4, and define φ_1 by

$$\varphi_1(x) = C|x|^{2s-n}\psi(1-|x|^{-1}) + \xi(x)\,. \tag{1.47}$$

Observe that $(-\Delta)^s \xi \geq -C_2$ in B_4, for some $C_2 > 0$. Hence, if we take $C \geq 4^{2s+n}(1 + C_2)$, using (1.45), we have

$$(-\Delta)^s \varphi_1(x) \geq C|x|^{-2s-n} + (-\Delta)^s \xi(x) \geq 1 \quad \text{in } B_4\,.$$

Now it is immediate to verify that φ_1 satisfies (1.6) for some $c_1 > 0$.

To see that $\varphi_1 \in H^s_{loc}(\mathbb{R}^n)$ we observe that from (1.47) it follows

$$|\nabla \varphi_1(x)| \leq C(|x|-1)^{s-1} \quad \text{in } \mathbb{R}^n \setminus B_1$$

and hence, using Lemma 1.4.4, we have $(-\Delta)^{s/2}\varphi_1 \in L^p_{loc}(\mathbb{R}^n)$ for all $p < \infty$. □

Next we prove Lemma 1.3.2.

Proof of Lemma 1.3.2. We define

$$\psi_1(x) = (1-|x|^2)^s \chi_{B_1}(x)\,.$$

Since (1.4) is the solution of problem (1.3), we have $(-\Delta)^s \psi_1$ is bounded in B_1. Hence, for $C > 0$ large enough the function $\psi = \psi_1 + C\chi_{\overline{B_{1/4}}}$ satisfies $(-\Delta)^s \psi \leq 0$ in $B_1 \setminus \overline{B_{1/4}}$ and it can be used as a viscosity subsolution. Note that ψ is upper semicontinuous, as required to viscosity subsolutions, and it satisfies pointwise (if C is large enough)

$$\begin{cases} \psi \equiv 0 & \text{in } \mathbb{R}^n \setminus B_1 \\ (-\Delta)^s \psi \leq 0 & \text{in } B_1 \setminus \overline{B_{1/4}} \\ \psi = 1 & \text{in } \overline{B_{1/4}} \\ \psi(x) \geq c(1-|x|)^s & \text{in } B_1. \end{cases}$$

If we want a subsolution which is continuous and $H^s(\mathbb{R}^n)$ we may construct it as follows. We consider the viscosity solution (which is also a weak solution by Remark 1.2.11) of

$$\begin{cases} (-\Delta)^s \varphi_2 = 0 & \text{in } B_1 \setminus B_{1/4} \\ \varphi_2 \equiv 0 & \text{in } \mathbb{R}^n \setminus B_1 \\ \varphi_2 = 1 & \text{in } \overline{B_{1/4}}. \end{cases}$$

Using ψ as a lower barrier, it is now easy to prove that φ_2 satisfies (1.16) for some constant $c_2 > 0$. □

Chapter Two

The Pohozaev identity for the fractional Laplacian

In this paper we prove the Pohozaev identity for the semilinear Dirichlet problem $(-\Delta)^s u = f(u)$ in Ω, $u \equiv 0$ in $\mathbb{R}^n \backslash \Omega$. Here, $s \in (0,1)$, $(-\Delta)^s$ is the fractional Laplacian in $\mathbb{R}^n$, and Ω is a bounded $C^{1,1}$ domain.

To establish the identity we use, among other things, that if u is a bounded solution then $u/\delta^s|_\Omega$ is C^α up to the boundary $\partial\Omega$, where $\delta(x) = dist(x, \partial\Omega)$. In the fractional Pohozaev identity, the function $u/\delta^s|_{\partial\Omega}$ plays the role that $\partial u/\partial\nu$ plays in the classical one. Surprisingly, from a nonlocal problem we obtain an identity with a boundary term (an integral over $\partial\Omega$) which is completely local.

As an application of our identity, we deduce the nonexistence of nontrivial solutions in star-shaped domains for supercritical nonlinearities.

2.1 Introduction and results

Let $s \in (0,1)$ and consider the fractional elliptic problem

$$\begin{cases} (-\Delta)^s u &= f(u) & \text{in } \Omega \\ u &= 0 & \text{in } \mathbb{R}^n \backslash \Omega \end{cases} \tag{2.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where

$$(-\Delta)^s u(x) = c_{n,s} PV \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy \tag{2.2}$$

is the fractional Laplacian. Here, $c_{n,s}$ is a normalization constant given by (2.50).

When $s = 1$, a celebrated result of S. I. Pohozaev states that any solution of (2.1) satisfies an identity, which is known as the Pohozaev identity [237]. This classical result has many consequences, the most immediate one being the nonexistence of nontrivial bounded solutions to (2.1) for supercritical nonlinearities f.

The aim of this paper is to give the fractional version of this identity, that is, to prove the Pohozaev identity for problem (2.1) with $s \in (0,1)$. This is the main result of the paper, and it reads as follows. Here, since the solution u is bounded, the notions of weak and viscosity solutions agree (see Remark 2.1.5).

Theorem 2.1.1. *Let Ω be a bounded and $C^{1,1}$ domain, f be a locally Lipschitz function, u be a bounded solution of (2.1), and*

$$\delta(x) = dist(x, \partial\Omega).$$

Then,

$$u/\delta^s|_\Omega \in C^\alpha(\overline{\Omega}) \qquad \text{for some } \ \alpha \in (0,1),$$

meaning that $u/\delta^s|_\Omega$ has a continuous extension to $\overline{\Omega}$ which is $C^\alpha(\overline{\Omega})$, and the following identity holds

$$(2s-n)\int_\Omega uf(u)dx + 2n\int_\Omega F(u)dx = \Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x\cdot\nu)d\sigma,$$

where $F(t) = \int_0^t f$, ν is the unit outward normal to $\partial\Omega$ at x, and Γ is the Gamma function.

Note that in the fractional case the function $u/\delta^s|_{\partial\Omega}$ plays the role that $\partial u/\partial\nu$ plays in the classical Pohozaev identity. Moreover, if one sets $s = 1$ in the above identity one recovers the classical one, since $u/\delta|_{\partial\Omega} = \partial u/\partial\nu$ and $\Gamma(2) = 1$.

It is quite surprising that from a nonlocal problem (2.1) we obtain a completely local boundary term in the Pohozaev identity. That is, although the function u has to be defined in all $\mathbb{R}^n$ in order to compute its fractional Laplacian at a given point, knowing u only in a neighborhood of the boundary we can already compute $\int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x\cdot\nu)d\sigma$.

Recall that problem (2.1) has an equivalent formulation given by the Caffarelli-Silvestre [68] associated extension problem —a local PDE in $\mathbb{R}^{n+1}_+$. For such extension, some Pohozaev type identities are proved in [33, 46, 58]. However, these identities contain boundary terms on the cylinder $\partial\Omega \times \mathbb{R}^+$ or in a half-sphere $\partial B_R^+ \cap \mathbb{R}^{n+1}_+$, which have no clear interpretation in terms of the original problem in $\mathbb{R}^n$. The proofs of these identities are similar to the one of the classical Pohozaev identity and use PDE tools (differential calculus identities and integration by parts).

Sometimes it may be useful to write the Pohozaev identity as

$$2s[u]^2_{H^s(\mathbb{R}^n)} - 2n\mathcal{E}[u] = \Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x\cdot\nu)d\sigma,$$

where $\mathcal{E}$ is the energy functional

$$\mathcal{E}[u] = \frac{1}{2}[u]^2_{H^s(\mathbb{R}^n)} - \int_\Omega F(u)dx, \tag{2.3}$$

$F' = f$, and

$$[u]_{H^s(\mathbb{R}^n)} = \||\xi|^s \mathcal{F}[u]\|_{L^2(\mathbb{R}^n)} = \frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy. \tag{2.4}$$

We have used that if u and v are $H^s(\mathbb{R}^n)$ functions and $u \equiv v \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, then

$$\int_\Omega v(-\Delta)^s u\,dx = \int_{\mathbb{R}^n} (-\Delta)^{s/2} u (-\Delta)^{s/2} v\,dx, \tag{2.5}$$

which yields

$$\int_\Omega uf(u)dx = \int_{\mathbb{R}^n} |(-\Delta)^{s/2}u|^2 dx = [u]_{H^s(\mathbb{R}^n)}.$$

As a consequence of our Pohozaev identity we obtain nonexistence results for problem (2.1) with supercritical nonlinearities f in star-shaped domains Ω. In Section 2.2 we will give, however, a short proof of this result using our method to establish the Pohozaev identity. This shorter proof will not require the full strength of the identity.

Corollary 2.1.2. *Let Ω be a bounded, $C^{1,1}$, and star-shaped domain, and let f be a locally Lipschitz function. If*

$$\frac{n-2s}{2n}uf(u) \geq \int_0^u f(t)dt \qquad \text{for all } \ u \in \mathbb{R}, \tag{2.6}$$

then problem (2.1) *admits no positive bounded solution. Moreover, if the inequality in* (2.6) *is strict, then* (2.1) *admits no nontrivial bounded solution.*

For the pure power nonlinearity, the result reads as follows.

Corollary 2.1.3. *Let Ω be a bounded, $C^{1,1}$, and star-shaped domain. If $p \geq \frac{n+2s}{n-2s}$, then problem*

$$\begin{cases} (-\Delta)^s u &= |u|^{p-1}u & \text{in } \Omega \\ u &= 0 & \text{in } \mathbb{R}^n \backslash \Omega \end{cases} \tag{2.7}$$

admits no positive bounded solution. Moreover, if $p > \frac{n+2s}{n-2s}$ then (2.7) *admits no nontrivial bounded solution.*

The nonexistence of changing-sign solutions to problem (2.7) for the critical power $p = \frac{n+2s}{n-2s}$ remains open.

Recently, M. M. Fall and T. Weth [129] have also proved a nonexistence result for problem (2.1) with the method of moving spheres. In their result no regularity of the domain is required, but they need to assume the solutions to be positive. Our nonexistence result is the first one allowing changing-sign solutions. In addition, their condition on f for the nonexistence —(2.16) in our Remark 2.1.14— is more restrictive than ours, i.e., (2.6) and, when $f = f(x,u)$, condition (2.15).

The existence of weak solutions $u \in H^s(\mathbb{R}^n)$ to problem (2.1) for subcritical f has been recently proved by R. Servadei and E. Valdinoci [268].

The Pohozaev identity will be a consequence of the following two results. The first one establishes $C^s(\mathbb{R}^n)$ regularity for u, $C^\alpha(\overline{\Omega})$ regularity for $u/\delta^s|_\Omega$, and higher order interior Hölder estimates for u and u/δ^s. It is proved in our paper [249].

Throughout the article, and when no confusion is possible, we will use the notation $C^\beta(U)$ with $\beta > 0$ to refer to the space $C^{k,\beta'}(U)$, where k is the is greatest integer such that $k < \beta$, and $\beta' = \beta - k$. This notation is specially appropriate when we work with $(-\Delta)^s$ in order to avoid the splitting of different cases in the statements of regularity results. According to this, $[\cdot]_{C^\beta(U)}$ denotes the $C^{k,\beta'}(U)$ seminorm

$$[u]_{C^\beta(U)} = [u]_{C^{k,\beta'}(U)} = \sup_{x,y\in U,\ x\neq y} \frac{|D^k u(x) - D^k u(y)|}{|x-y|^{\beta'}}.$$

Here, by $f \in C^{0,1}_{loc}(\overline{\Omega} \times \mathbb{R})$ we mean that f is Lipschitz in every compact subset of $\overline{\Omega} \times \mathbb{R}$.

Theorem 2.1.4 ([249]). *Let Ω be a bounded and $C^{1,1}$ domain, $f \in C^{0,1}_{loc}(\overline{\Omega} \times \mathbb{R})$, u be a bounded solution of*

$$\begin{cases} (-\Delta)^s u &= f(x,u) & \text{in } \Omega \\ u &= 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{2.8}$$

and $\delta(x) = dist(x, \partial\Omega)$. Then,

(a) $u \in C^s(\mathbb{R}^n)$ and, for every $\beta \in [s, 1+2s)$, u is of class $C^\beta(\Omega)$ and

$$[u]_{C^\beta(\{x\in\Omega\,:\,\delta(x)\geq\rho\})} \leq C\rho^{s-\beta} \qquad \text{for all } \ \rho \in (0,1).$$

(b) The function $u/\delta^s|_\Omega$ can be continuously extended to $\overline{\Omega}$. Moreover, u/δ^s belongs to $C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$ depending only on Ω, s, f, $\|u\|_{L^\infty(\mathbb{R}^n)}$. In addition, for all $\beta \in [\alpha, s+\alpha]$, it holds the estimate

$$[u/\delta^s]_{C^\beta(\{x\in\Omega\,:\,\delta(x)\geq\rho\})} \leq C\rho^{\alpha-\beta} \qquad \text{for all } \ \rho \in (0,1).$$

The constant C depends only on Ω, s, f, $\|u\|_{L^\infty(\mathbb{R}^n)}$, and β.

Remark 2.1.5. For bounded solutions of (2.8), the notions of energy and viscosity solutions coincide (see more details in Remark 2.9 in [249]). Recall that u is an energy (or weak) solution of problem (2.8) if $u \in H^s(\mathbb{R}^n)$, $u \equiv 0$ in $\mathbb{R}^n \backslash \Omega$, and

$$\int_{\mathbb{R}^n} (-\Delta)^{s/2} u (-\Delta)^{s/2} v \, dx = \int_\Omega f(x,u) v \, dx$$

for all $v \in H^s(\mathbb{R}^n)$ such that $v \equiv 0$ in $\mathbb{R}^n \setminus \Omega$.

By Theorem 2.1.4 (a), any bounded weak solution is continuous up to the boundary and solve equation (2.8) in the classical sense, i.e., in the pointwise sense of (2.2). Therefore, it follows from the definition of viscosity solution (see [69]) that bounded weak solutions are also viscosity solutions.

Reciprocally, by uniqueness of viscosity solutions [69] and existence of weak solution for the linear problem $(-\Delta)^s v = f(x, u(x))$, any viscosity solution u belongs to $H^s(\mathbb{R}^n)$ and it is also a weak solution. See [249] for more details.

The second result towards Theorem 2.1.1 is the new Pohozaev identity for the fractional Laplacian. The hypotheses of the following proposition are satisfied for any bounded solution u of (2.8) whenever $f \in C^{0,1}_{loc}(\overline{\Omega} \times \mathbb{R})$, by our results in [249] (see Theorem 2.1.4 above).

Proposition 2.1.6. *Let Ω be a bounded and $C^{1,1}$ domain. Assume that u is a $H^s(\mathbb{R}^n)$ function which vanishes in $\mathbb{R}^n \setminus \Omega$, and satisfies*

(a) $u \in C^s(\mathbb{R}^n)$ and, for every $\beta \in [s, 1+2s)$, u is of class $C^\beta(\Omega)$ and

$$[u]_{C^\beta(\{x\in\Omega\,:\,\delta(x)\geq\rho\})} \leq C\rho^{s-\beta} \qquad \text{for all } \ \rho \in (0,1).$$

(b) The function $u/\delta^s|_\Omega$ can be continuously extended to $\overline{\Omega}$. Moreover, there exists $\alpha \in (0,1)$ such that $u/\delta^s \in C^\alpha(\overline{\Omega})$. In addition, for all $\beta \in [\alpha, s+\alpha]$, it holds the estimate

$$[u/\delta^s]_{C^\beta(\{x\in\Omega\,:\,\delta(x)\geq\rho\})} \leq C\rho^{\alpha-\beta} \qquad \text{for all } \ \rho \in (0,1).$$

(c) $(-\Delta)^s u$ is pointwise bounded in Ω.

Then, the following identity holds

$$\int_\Omega (x\cdot\nabla u)(-\Delta)^s u \, dx = \frac{2s-n}{2}\int_\Omega u(-\Delta)^s u \, dx - \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\Big(\frac{u}{\delta^s}\Big)^2 (x\cdot\nu)d\sigma,$$

where ν is the unit outward normal to $\partial\Omega$ at x, and Γ is the Gamma function.

Remark 2.1.7. Note that hypothesis (a) ensures that $(-\Delta)^s u$ is defined pointwise in Ω. Note also that hypotheses (a) and (c) ensure that the integrals appearing in the above identity are finite.

Remark 2.1.8. By Propositions 1.1 and 1.4 in [249], hypothesis (c) guarantees that $u \in C^s(\mathbb{R}^n)$ and $u/\delta^s \in C^\alpha(\overline{\Omega})$, but not the interior estimates in (a) and (b). However, under the stronger assumption $(-\Delta)^s u \in C^\alpha(\overline{\Omega})$ the whole hypothesis (b) is satisfied; see Theorem 1.5 in [249].

As a consequence of Proposition 2.1.6, we will obtain the Pohozaev identity (Theorem 2.1.1) and also a new integration by parts formula related to the fractional Laplacian. This integration by parts formula follows from using Proposition 2.1.6 with two different origins.

Theorem 2.1.9. *Let Ω be a bounded and $C^{1,1}$ domain, and u and v be functions satisfying the hypotheses in Proposition 2.1.6. Then, the following identity holds*

$$\int_\Omega (-\Delta)^s u \; v_{x_i}\, dx = -\int_\Omega u_{x_i}(-\Delta)^s v \, dx + \Gamma(1+s)^2 \int_{\partial\Omega} \frac{u}{\delta^s}\frac{v}{\delta^s}\,\nu_i \, d\sigma$$

for $i = 1, ..., n$, where ν is the unit outward normal to $\partial\Omega$ at x, and Γ is the Gamma function.

To prove Proposition 2.1.6 we first assume the domain Ω to be star-shaped with respect to the origin. The result for general domains will follow from the star-shaped case, as seen in Section 2.5. When the domain is star-shaped, the idea of the proof is the following. First, one writes the left hand side of the identity as

$$\int_\Omega (x\cdot\nabla u)(-\Delta)^s u \, dx = \frac{d}{d\lambda}\Big|_{\lambda=1^+}\int_\Omega u_\lambda(-\Delta)^s u \, dx,$$

where

$$u_\lambda(x) = u(\lambda x).$$

Note that $u_\lambda \equiv 0$ in $\mathbb{R}^n\backslash\Omega$, since Ω is star-shaped and we take $\lambda > 1$ in the above derivative. As a consequence, we may use (2.5) with $v = u_\lambda$ and make the change of variables $y = \sqrt{\lambda}x$, to obtain

$$\int_\Omega u_\lambda(-\Delta)^s u \, dx = \int_{\mathbb{R}^n}(-\Delta)^{s/2}u_\lambda(-\Delta)^{s/2}u \, dx = \lambda^{\frac{2s-n}{2}}\int_{\mathbb{R}^n} w_{\sqrt{\lambda}} w_{1/\sqrt{\lambda}} \, dy,$$

where

$$w(x) = (-\Delta)^{s/2} u(x).$$

Thus,

$$\begin{aligned}\int_\Omega (x\cdot\nabla u)(-\Delta)^s u\, dx &= \frac{d}{d\lambda}\Big|_{\lambda=1^+} \left\{ \lambda^{\frac{2s-n}{2}} \int_{\mathbb{R}^n} w_{\sqrt{\lambda}} w_{1/\sqrt{\lambda}}\, dy \right\} \\ &= \frac{2s-n}{2}\int_{\mathbb{R}^n} w^2 dx + \frac{d}{d\lambda}\Big|_{\lambda=1^+} I_{\sqrt{\lambda}} \\ &= \frac{2s-n}{2}\int_{\mathbb{R}^n} u(-\Delta)^s u\, dx + \frac{1}{2}\,\frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda,\end{aligned} \tag{2.9}$$

where

$$I_\lambda = \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda} dy.$$

Therefore, Proposition 2.1.6 is equivalent to the following equality

$$-\frac{d}{d\lambda}\Big|_{\lambda=1^+} \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda}\, dy = \Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x\cdot\nu) d\sigma. \tag{2.10}$$

The quantity $\frac{d}{d\lambda}|_{\lambda=1^+} \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda}$ vanishes for any $C^1(\mathbb{R}^n)$ function w, as can be seen by differentiating under the integral sign. Instead, we will prove that the function $w = (-\Delta)^{s/2} u$ has a singularity along $\partial\Omega$, and that (2.10) holds.

Next we give an easy argument to give a direct proof of the nonexistence result for supercritical nonlinearities without using neither equality (2.10) nor the behavior of $(-\Delta)^{s/2} u$; the detailed proof is given in Section 2.2.

Indeed, in contrast with the delicate equality (2.10), the inequality

$$\frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda \le 0 \tag{2.11}$$

follows easily from Cauchy-Schwarz. Namely,

$$I_\lambda \le \|w_\lambda\|_{L^2(\mathbb{R}^n)} \|w_{1/\lambda}\|_{L^2(\mathbb{R}^n)} = \|w\|^2_{L^2(\mathbb{R}^n)} = I_1,$$

and hence (2.11) follows.

With this simple argument, (2.9) leads to

$$-\int_\Omega (x\cdot\nabla u)(-\Delta)^s u\, dx \ge \frac{n-2s}{2}\int_\Omega u(-\Delta)^s u\, dx,$$

which is exactly the inequality used to prove the nonexistence result of Corollary 2.1.2 for supercritical nonlinearities. Here, one also uses that, when u is a solution of (2.1), then

$$\int_\Omega (x\cdot\nabla u)(-\Delta)^s u\, dx = \int_\Omega (x\cdot\nabla u) f(u) dx = \int_\Omega x\cdot\nabla F(u) dx = -n\int_\Omega F(u) dx.$$

This argument can be also used to obtain nonexistence results (under some decay assumptions) for weak solutions of (2.1) in the whole $\mathbb{R}^n$; see Remark 2.2.2.

The identity (2.10) is the difficult part of the proof of Proposition 2.1.6. To prove it, it will be crucial to know the precise behavior of $(-\Delta)^{s/2} u$ near $\partial\Omega$ —from both inside and outside Ω. This is given by the following result.

Proposition 2.1.10. *Let Ω be a bounded and $C^{1,1}$ domain, and u be a function such that $u \equiv 0$ in $\mathbb{R}^n \backslash \Omega$ and that u satisfies (b) in Proposition 2.1.6. Then, there exists a $C^\alpha(\mathbb{R}^n)$ extension v of $u/\delta^s|_\Omega$ such that*

$$(-\Delta)^{s/2} u(x) = c_1 \left\{ \log^- \delta(x) + c_2 \chi_\Omega(x) \right\} v(x) + h(x) \quad \text{in} \quad \mathbb{R}^n, \tag{2.12}$$

where h is a $C^\alpha(\mathbb{R}^n)$ function, $\log^- t = \min\{\log t, 0\}$,

$$c_1 = \frac{\Gamma(1+s)\sin\left(\frac{\pi s}{2}\right)}{\pi}, \qquad \text{and} \qquad c_2 = \frac{\pi}{\tan\left(\frac{\pi s}{2}\right)}. \tag{2.13}$$

Moreover, if u also satisfies (a) in Proposition 2.1.6, then for all $\beta \in (0, 1+s)$

$$[(-\Delta)^{s/2} u]_{C^\beta(\{x \in \mathbb{R}^n : \delta(x) \geq \rho\})} \leq C \rho^{-\beta} \qquad \text{for all} \quad \rho \in (0,1), \tag{2.14}$$

for some constant C which does not depend on ρ.

The values (2.13) of the constants c_1 and c_2 in (2.12) arise in the expression for the $s/2$ fractional Laplacian, $(-\Delta)^{s/2}$, of the 1D function $(x_n^+)^s$, and they are computed in the Appendix.

Writing the first integral in (2.10) using spherical coordinates, equality (2.10) reduces to a computation in dimension 1, stated in the following proposition. This result will be used with the function φ in its statement being essentially the restriction of $(-\Delta)^{s/2} u$ to any ray through the origin. The constant γ will be chosen to be any value in $(0, s)$.

Proposition 2.1.11. *Let A and B be real numbers, and*

$$\varphi(t) = A \log^- |t-1| + B \chi_{[0,1]}(t) + h(t),$$

where $\log^- t = \min\{\log t, 0\}$ and h is a function satisfying, for some constants α and γ in $(0,1)$, and $C_0 > 0$, the following conditions:

(i) $\|h\|_{C^\alpha([0,\infty))} \leq C_0$.

(ii) For all $\beta \in [\gamma, 1+\gamma]$

$$\|h\|_{C^\beta((0,1-\rho)\cup(1+\rho,2))} \leq C_0 \rho^{-\beta} \qquad \text{for all} \quad \rho \in (0,1).$$

(iii) $|h'(t)| \leq C_0 t^{-2-\gamma}$ *and* $|h''(t)| \leq C_0 t^{-3-\gamma}$ *for all* $t > 2$.

Then,

$$-\left.\frac{d}{d\lambda}\right|_{\lambda=1^+} \int_0^\infty \varphi(\lambda t)\, \varphi\left(\frac{t}{\lambda}\right) dt = A^2 \pi^2 + B^2.$$

Moreover, the limit defining this derivative is uniform among functions φ satisfying (i)-(ii)-(iii) with given constants C_0, α, and γ.

From this proposition one obtains that the constant in the right hand side of (2.10), $\Gamma(1+s)^2$, is given by $c_1^2(\pi^2+c_2^2)$. The constant c_2 comes from an involved expression and it is nontrivial to compute (see Proposition 2.3.2 in Section 5 and the Appendix). It was a surprise to us that its final value is so simple and, at the same time, that the Pohozaev constant $c_1^2(\pi^2+c_2^2)$ also simplifies and becomes $\Gamma(1+s)^2$.

Instead of computing explicitly the constants c_1 and c_2, an alternative way to obtain the constant in the Pohozaev identity consists of using an explicit nonlinearity and solution to problem (2.1) in a ball. The one which is known [154, 24] is the solution to problem

$$\begin{cases} (-\Delta)^s u &= 1 \quad \text{in } B_r(x_0) \\ u &= 0 \quad \text{in } \mathbb{R}^n \backslash B_r(x_0). \end{cases}$$

It is given by

$$u(x) = \frac{2^{-2s}\Gamma(n/2)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)}\left(r^2-|x-x_0|^2\right)^s \qquad \text{in } \ B_r(x_0).$$

From this, it is straightforward to find the constant $\Gamma(1+s)^2$ in the Pohozaev identity; see Remark 2.6.4 in the Appendix.

Using Theorem 2.1.4 and Proposition 2.1.6, we can also deduce a Pohozaev identity for problem (2.8), that is, allowing the nonlinearity f to depend also on x. In this case, the Pohozaev identity reads as follows.

Proposition 2.1.12. *Let Ω be a bounded and $C^{1,1}$ domain, $f \in C^{0,1}_{loc}(\overline{\Omega}\times\mathbb{R})$, u be a bounded solution of (2.8), and $\delta(x) = dist(x,\partial\Omega)$. Then*

$$u/\delta^s|_\Omega \in C^\alpha(\overline{\Omega}) \qquad \text{for some } \ \alpha\in(0,1),$$

and the following identity holds

$$(2s-n)\int_\Omega uf(x,u)dx + 2n\int_\Omega F(x,u)dx =$$

$$= \Gamma(1+s)^2\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2(x\cdot\nu)d\sigma - 2\int_\Omega x\cdot F_x(x,u)dx,$$

where $F(x,t)=\int_0^t f(x,\tau)d\tau$, ν is the unit outward normal to $\partial\Omega$ at x, and Γ is the Gamma function.

From this, we deduce nonexistence results for problem (2.8) with supercritical nonlinearities f depending also on x. This has been done also in [129] for positive solutions. Our result allows changing sign solutions as well as a slightly larger class of nonlinearities (see Remark 2.1.14).

Corollary 2.1.13. *Let Ω be a bounded, $C^{1,1}$, and star-shaped domain, $f\in C^{0,1}_{loc}(\overline{\Omega}\times\mathbb{R})$, and $F(x,t)=\int_0^t f(x,\tau)d\tau$. If*

$$\frac{n-2s}{2}uf(x,t) \geq nF(x,t) + x\cdot F_x(x,t) \qquad \text{for all } \ x\in\Omega \ \text{ and } \ t\in\mathbb{R}, \tag{2.15}$$

then problem (2.8) admits no positive bounded solution. Moreover, if the inequality in (2.15) is strict, then (2.8) admits no nontrivial bounded solution.

Remark 2.1.14. For locally Lipschitz nonlinearities f, condition (2.15) is more general than the one required in [129] for their nonexistence result. Namely, [129] assumes that for each $x \in \Omega$ and $t \in \mathbb{R}$, the map

$$\lambda \mapsto \lambda^{-\frac{n+2s}{n-2s}} f(\lambda^{-\frac{2}{n-2s}} x, \lambda t) \qquad \text{is nondecreasing for } \lambda \in (0,1]. \tag{2.16}$$

Such nonlinearities automatically satisfy (2.15).

However, in [129] they do not need to assume any regularity on f with respect to x.

The paper is organized as follows. In Section 2.2, using Propositions 2.1.10 and 2.1.11 (to be established later), we prove Proposition 2.1.6 (the Pohozaev identity) for strictly star-shaped domains with respect to the origin. We also establish the nonexistence results for supercritical nonlinearities, and this does not require any result from the rest of the paper. In Section 2.3 we establish Proposition 2.1.10, while in Section 2.4 we prove Proposition 2.1.11. Section 2.5 establishes Proposition 2.1.6 for non-star-shaped domains and all its consequences, which include Theorems 2.1.1 and 2.1.9 and the nonexistence results. Finally, in the Appendix we compute the constants c_1 and c_2 appearing in Proposition 2.1.10.

2.2 Star-shaped domains: Pohozaev identity and nonexistence

In this section we prove Proposition 2.1.6 for strictly star-shaped domains. We say that Ω is strictly star-shaped if, for some $z_0 \in \mathbb{R}^n$,

$$(x - z_0) \cdot \nu > 0 \qquad \text{for all } \ x \in \partial\Omega. \tag{2.17}$$

The result for general $C^{1,1}$ domains will be a consequence of this strictly star-shaped case and will be proved in Section 2.5.

The proof in this section uses two of our results: Proposition 2.1.10 on the behavior of $(-\Delta)^{s/2} u$ near $\partial\Omega$ and the one dimensional computation of Proposition 2.1.11.

The idea of the proof for the fractional Pohozaev identity is to use the integration by parts formula (2.5) with $v = u_\lambda$, where

$$u_\lambda(x) = u(\lambda x), \quad \lambda > 1,$$

and then differentiate the obtained identity (which depends on λ) with respect to λ and evaluate at $\lambda = 1$. However, this apparently simple formal procedure requires a quite involved analysis when it is put into practice. The hypothesis that Ω is star-shaped is crucially used in order that u_λ, $\lambda > 1$, vanishes outside Ω so that (2.5) holds.

Proof of Proposition 2.1.6 for strictly star-shaped domains. Let us assume first that Ω is strictly star-shaped with respect to the origin, that is, $z_0 = 0$.

Let us prove that

$$\int_\Omega (x \cdot \nabla u)(-\Delta)^s u \, dx = \left. \frac{d}{d\lambda} \right|_{\lambda = 1^+} \int_\Omega u_\lambda (-\Delta)^s u \, dx, \tag{2.18}$$

where $\frac{d}{d\lambda}\big|_{\lambda=1^+}$ is the derivative from the right side at $\lambda = 1$. Indeed, let $g = (-\Delta)^s u$. By assumption (a) g is defined pointwise in Ω, and by assumption (c) $g \in L^\infty(\Omega)$. Then, making the change of variables $y = \lambda x$ and using that $\operatorname{supp} u_\lambda = \frac{1}{\lambda}\Omega \subset \Omega$ since $\lambda > 1$, we obtain

$$\begin{aligned}\frac{d}{d\lambda}\bigg|_{\lambda=1^+} \int_\Omega u_\lambda(x) g(x)dx &= \lim_{\lambda\downarrow 1} \int_\Omega \frac{u(\lambda x) - u(x)}{\lambda - 1} g(x)dx \\ &= \lim_{\lambda\downarrow 1} \lambda^{-n} \int_{\lambda\Omega} \frac{u(y) - u(y/\lambda)}{\lambda - 1} g(y/\lambda)dy \\ &= \lim_{\lambda\downarrow 1} \int_\Omega \frac{u(y) - u(y/\lambda)}{\lambda - 1} g(y/\lambda)dy + \lim_{\lambda\downarrow 1} \int_{(\lambda\Omega)\setminus\Omega} \frac{-u(y/\lambda)}{\lambda - 1} g(y/\lambda)dy.\end{aligned}$$

By the dominated convergence theorem,

$$\lim_{\lambda\downarrow 1} \int_\Omega \frac{u(y) - u(y/\lambda)}{\lambda - 1} g(y/\lambda)\, dy = \int_\Omega (y \cdot \nabla u) g(y)\, dy,$$

since $g \in L^\infty(\Omega)$, $|\nabla u(\xi)| \le C\delta(\xi)^{s-1} \le C\lambda^{1-s}\delta(y)^{s-1}$ for all ξ in the segment joining y and y/λ, and δ^{s-1} is integrable. The gradient bound $|\nabla u(\xi)| \le C\delta(\xi)^{s-1}$ follows from assumption (a) used with $\beta = 1$. Hence, to prove (2.18) it remains only to show that

$$\lim_{\lambda\downarrow 1} \int_{(\lambda\Omega)\setminus\Omega} \frac{-u(y/\lambda)}{\lambda - 1} g(y/\lambda)dy = 0.$$

Indeed, $|(\lambda\Omega)\setminus\Omega| \le C(\lambda - 1)$ and —by (a)— $u \in C^s(\mathbb{R}^n)$ and $u \equiv 0$ outside Ω. Hence, $\|u\|_{L^\infty((\lambda\Omega)\setminus\Omega)} \to 0$ as $\lambda \downarrow 1$ and (2.18) follows.

Now, using the integration by parts formula (2.5) with $v = u_\lambda$,

$$\begin{aligned}\int_\Omega u_\lambda(-\Delta)^s u\, dx &= \int_{\mathbb{R}^n} u_\lambda(-\Delta)^s u\, dx \\ &= \int_{\mathbb{R}^n} (-\Delta)^{s/2} u_\lambda (-\Delta)^{s/2} u\, dx \\ &= \lambda^s \int_{\mathbb{R}^n} \left((-\Delta)^{s/2} u\right)(\lambda x)(-\Delta)^{s/2} u(x)dx \\ &= \lambda^s \int_{\mathbb{R}^n} w_\lambda w\ dx,\end{aligned}$$

where

$$w(x) = (-\Delta)^{s/2} u(x) \qquad \textit{and} \qquad w_\lambda(x) = w(\lambda x).$$

With the change of variables $y = \sqrt{\lambda}x$ this integral becomes

$$\lambda^s \int_{\mathbb{R}^n} w_\lambda w\, dx = \lambda^{\frac{2s-n}{2}} \int_{\mathbb{R}^n} w_{\sqrt{\lambda}} w_{1/\sqrt{\lambda}}\, dy,$$

and thus

$$\int_\Omega u_\lambda(-\Delta)^s u\, dx = \lambda^{\frac{2s-n}{2}} \int_{\mathbb{R}^n} w_{\sqrt{\lambda}} w_{1/\sqrt{\lambda}}\, dy.$$

Furthermore, this leads to

$$\begin{aligned}\int_\Omega (\nabla u\cdot x)(-\Delta)^s u\,dx &= \frac{d}{d\lambda}\Big|_{\lambda=1^+}\left\{\lambda^{\frac{2s-n}{2}}\int_{\mathbb{R}^n} w_{\sqrt{\lambda}}w_{1/\sqrt{\lambda}}\,dy\right\}\\ &= \frac{2s-n}{2}\int_{\mathbb{R}^n}|(-\Delta)^{s/2}u|^2\,dx+\frac{d}{d\lambda}\Big|_{\lambda=1^+}\int_{\mathbb{R}^n} w_{\sqrt{\lambda}}w_{1/\sqrt{\lambda}}\,dy\\ &= \frac{2s-n}{2}\int_\Omega u(-\Delta)^s u\,dx+\frac{1}{2}\,\frac{d}{d\lambda}\Big|_{\lambda=1^+}\int_{\mathbb{R}^n} w_\lambda w_{1/\lambda}\,dy. \quad (2.19)\end{aligned}$$

Hence, it remains to prove that

$$-\frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda = \Gamma(1+s)^2\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2(x\cdot\nu)\,d\sigma, \quad (2.20)$$

where we have denoted

$$I_\lambda = \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda}\,dy. \quad (2.21)$$

Now, for each $\theta\in S^{n-1}$ there exists a unique $r_\theta>0$ such that $r_\theta\theta\in\partial\Omega$. Write the integral (2.21) in spherical coordinates and use the change of variables $t=r/r_\theta$:

$$\begin{aligned}\frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda &= \frac{d}{d\lambda}\Big|_{\lambda=1^+}\int_{S^{n-1}} d\theta\int_0^\infty r^{n-1}w(\lambda r\theta)w\left(\frac{r}{\lambda}\theta\right)dr\\ &= \frac{d}{d\lambda}\Big|_{\lambda=1^+}\int_{S^{n-1}} r_\theta d\theta\int_0^\infty (r_\theta t)^{n-1}w(\lambda r_\theta t\theta)w\left(\frac{r_\theta t}{\lambda}\theta\right)dt\\ &= \frac{d}{d\lambda}\Big|_{\lambda=1^+}\int_{\partial\Omega}(x\cdot\nu)d\sigma(x)\int_0^\infty t^{n-1}w(\lambda tx)w\left(\frac{tx}{\lambda}\right)dt,\end{aligned}$$

where we have used that

$$r_\theta^{n-1}\,d\theta = \left(\frac{x}{|x|}\cdot\nu\right)d\sigma = \frac{1}{r_\theta}(x\cdot\nu)\,d\sigma$$

with the change of variables $S^{n-1}\to\partial\Omega$ that maps every point in S^{n-1} to its radial projection on $\partial\Omega$, which is unique because of the strictly star-shapedness of Ω.

Fix $x_0\in\partial\Omega$ and define

$$\varphi(t) = t^{\frac{n-1}{2}}w\,(tx_0) = t^{\frac{n-1}{2}}(-\Delta)^{s/2}u(tx_0).$$

By Proposition 2.1.10,

$$\varphi(t) = c_1\{\log^-\delta(tx_0)+c_2\chi_{[0,1]}\}v(tx_0)+h_0(t)$$

in $[0,\infty)$, where v is a $C^\alpha(\mathbb{R}^n)$ extension of $u/\delta^s|_\Omega$ and h_0 is a $C^\alpha([0,\infty))$ function. Next we will modify this expression in order to apply Proposition 2.1.11.

Using that Ω is $C^{1,1}$ and strictly star-shaped, it is not difficult to see that $\frac{|r-r_\theta|}{\delta(r\theta)}$ is a Lipschitz function of r in $[0,\infty)$ and bounded below by a positive constant (independently of x_0). Similarly, $\frac{|t-1|}{\delta(tx_0)}$ and $\frac{\min\{|t-1|,1\}}{\min\{\delta(tx_0),1\}}$ are positive and Lipschitz functions of t in $[0,\infty)$. Therefore,

$$\log^-|t-1|-\log^-\delta(tx_0)$$

is Lipschitz in $[0,\infty)$ as a function of t.

Hence, for $t \in [0,\infty)$,

$$\varphi(t) = c_1\{\log^- |t-1| + c_2\chi_{[0,1]}\}v(tx_0) + h_1(t),$$

where h_1 is a C^α function in the same interval.

Moreover, note that the difference

$$v(tx_0) - v(x_0)$$

is C^α and vanishes at $t = 1$. Thus,

$$\varphi(t) = c_1\{\log^- |t-1| + c_2\chi_{[0,1]}(t)\}v(x_0) + h(t)$$

holds in all $[0,\infty)$, where h is C^α in $[0,\infty)$ if we slightly decrease α in order to kill the logarithmic singularity. This is condition (i) of Proposition 2.1.11.

From the expression

$$h(t) = t^{\frac{n-1}{2}}(-\Delta)^{s/2}u\,(tx_0) - c_1\{\log^- |t-1| + c_2\chi_{[0,1]}(t)\}v(x_0)$$

and from (2.14) in Proposition 2.1.10, we obtain that h satisfies condition (ii) of Proposition 2.1.11 with $\gamma = s/2$.

Moreover, condition (iii) of Proposition 2.1.11 is also satisfied. Indeed, for $x \in \mathbb{R}^n\backslash(2\Omega)$ we have

$$(-\Delta)^{s/2}u(x) = c_{n,\frac{s}{2}}\int_\Omega \frac{-u(y)}{|x-y|^{n+s}}dy$$

and hence

$$|\partial_i(-\Delta)^{s/2}u(x)| \le C|x|^{-n-s-1} \quad \text{and} \quad |\partial_{ij}(-\Delta)^{s/2}u(x)| \le C|x|^{-n-s-2}.$$

This yields $|\varphi'(t)| \le Ct^{\frac{n-1}{2}-n-s-1} \le Ct^{-2-\gamma}$ and $|\varphi''(t)| \le Ct^{\frac{n-1}{2}-n-s-2} \le Ct^{-3-\gamma}$ for $t > 2$.

Therefore we can apply Proposition 2.1.11 to obtain

$$\frac{d}{d\lambda}\bigg|_{\lambda=1^+}\int_0^\infty \varphi(\lambda t)\varphi\left(\frac{t}{\lambda}\right)dt = (v(x_0))^2\,c_1^2(\pi^2+c_2^2),$$

and thus

$$\frac{d}{d\lambda}\bigg|_{\lambda=1^+}\int_0^\infty t^{n-1}w(\lambda t x_0)w\left(\frac{tx_0}{\lambda}\right)dt = (v(x_0))^2\,c_1^2(\pi^2+c_2^2)$$

for each $x_0 \in \partial\Omega$.

Furthermore, by uniform convergence on x_0 of the limit defining this derivative (see Proposition 2.4.2 in Section 2.4), this leads to

$$\frac{d}{d\lambda}\bigg|_{\lambda=1^+} I_\lambda = c_1^2(\pi^2+c_2^2)\int_{\partial\Omega}(x_0\cdot\nu)\left(\frac{u}{\delta^s}(x_0)\right)^2 dx_0.$$

Here we have used that, for $x_0 \in \partial\Omega$, $v(x_0)$ is uniquely defined by continuity as

$$\left(\frac{u}{\delta^s}\right)(x_0) = \lim_{x\to x_0,\ x\in\Omega}\frac{u(x)}{\delta^s(x)}.$$

Hence, it only remains to prove that

$$c_1^2(\pi^2 + c_2^2) = \Gamma(1+s)^2.$$

But

$$c_1 = \frac{\Gamma(1+s)\sin\left(\frac{\pi s}{2}\right)}{\pi} \qquad \textit{and} \qquad c_2 = \frac{\pi}{\tan\left(\frac{\pi s}{2}\right)},$$

and therefore

$$\begin{aligned} c_1^2(\pi^2 + c_2^2) &= \frac{\Gamma(1+s)^2 \sin^2\left(\frac{\pi s}{2}\right)}{\pi^2}\left(\pi^2 + \frac{\pi^2}{\tan^2\left(\frac{\pi s}{2}\right)}\right) \\ &= \Gamma(1+s)^2 \sin^2\left(\frac{\pi s}{2}\right)\left(1 + \frac{\cos^2\left(\frac{\pi s}{2}\right)}{\sin^2\left(\frac{\pi s}{2}\right)}\right) \\ &= \Gamma(1+s)^2. \end{aligned}$$

Assume now that Ω is strictly star-shaped with respect to a point $z_0 \neq 0$. Then, Ω is strictly star-shaped with respect to all points z in a neighborhood of z_0. Then, making a translation and using the formula for strictly star-shaped domains with respect to the origin, we deduce

$$\begin{aligned} \int_\Omega \{(x-z)\cdot\nabla u\}(-\Delta)^s u\,dx = \frac{2s-n}{2}\int_\Omega u(-\Delta)^s u\,dx + \\ - \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2 (x-z)\cdot\nu\,d\sigma \end{aligned} \tag{2.22}$$

for each z in a neighborhood of z_0. This yields

$$\int_\Omega u_{x_i}(-\Delta)^s u\,dx = -\frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2 \nu_i\,d\sigma \tag{2.23}$$

for $i = 1, ..., n$. Thus, by adding to (2.22) a linear combination of (2.23), we obtain

$$\int_\Omega (x\cdot\nabla u)(-\Delta)^s u\,dx = \frac{2s-n}{2}\int_\Omega u(-\Delta)^s u\,dx - \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2 x\cdot\nu\,d\sigma.$$

□

Next we prove the nonexistence results of Corollaries 2.1.2, 2.1.3, and 2.1.13 for supercritical nonlinearities in star-shaped domains. Recall that star-shaped means $x\cdot\nu \geq 0$ for all $x \in \partial\Omega$. Although these corollaries follow immediately from Proposition 2.1.12 —as we will see in Section 2.5—, we give here a short proof of their second part, i.e., nonexistence when the inequality (2.6) or (2.15) is strict. That is, we establish the nonexistence of nontrivial solutions for supercritical nonlinearities (not including the critical case).

Our proof follows the method above towards the Pohozaev identity but does not require the full strength of the identity. In addition, in terms of regularity results for the equation, the proof only needs an easy gradient estimate for solutions u. Namely,

$$|\nabla u| \leq C\delta^{s-1} \quad \text{in} \quad \Omega,$$

which follows from part (a) of Theorem 2.1.4, proved in [249].

Proof of Corollaries 2.1.2, 2.1.3, and 2.1.13 for supercritical nonlinearities. We only have to prove Corollary 2.1.13, since Corollaries 2.1.2 and 2.1.3 follow immediately from it by setting $f(x,u)=f(u)$ and $f(x,u)=|u|^{p-1}u$ respectively.

Let us prove that if Ω is star-shaped and u is a bounded solution of (2.8), then

$$\frac{2s-n}{2}\int_\Omega uf(x,u)dx + n\int_\Omega F(x,u)dx - \int_\Omega x\cdot F_x(x,u)dx \geq 0. \tag{2.24}$$

For this, we follow the beginning of the proof of Proposition 2.1.6 (given above) to obtain (2.19), i.e., until the identity

$$\int_\Omega (\nabla u\cdot x)(-\Delta)^s u\,dx = \frac{2s-n}{2}\int_\Omega u(-\Delta)^s u\,dx + \frac{1}{2}\,\frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda,$$

where

$$I_\lambda = \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda}\,dx, \qquad w(x)=(-\Delta)^{s/2}u(x), \qquad \text{and} \qquad w_\lambda(x)=w(\lambda x).$$

This step of the proof only need the star-shapedness of Ω (and not the strictly star-shapedness) and the regularity result $|\nabla u|\leq C\delta^{s-1}$ in Ω, which follows from Theorem 2.1.4, proved in [249].

Now, since $(-\Delta)^s u = f(x,u)$ in Ω and

$$(\nabla u\cdot x)(-\Delta)^s u = x\cdot\nabla F(x,u) - x\cdot F_x(x,u),$$

by integrating by parts we deduce

$$-n\int_\Omega F(x,u)dx - \int_\Omega x\cdot F_x(x,u)dx = \frac{2s-n}{2}\int_\Omega uf(x,u)dx + \frac{1}{2}\,\frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda.$$

Therefore, we only need to show that

$$\frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda \leq 0. \tag{2.25}$$

But applying Hölder's inequality, for each $\lambda>1$ we have

$$I_\lambda \leq \|w_\lambda\|_{L^2(\mathbb{R}^n)}\|w_{1/\lambda}\|_{L^2(\mathbb{R}^n)} = \|w\|^2_{L^2(\mathbb{R}^n)} = I_1,$$

and (2.25) follows. □

Remark 2.2.1. For this nonexistence result the regularity of the domain Ω is only used for the estimate $|\nabla u|\leq C\delta^{s-1}$. This estimate only requires Ω to be Lipschitz and satisfy an exterior ball condition; see [249]. In particular, our nonexistence result for supercritical nonlinearities applies to any convex domain, such as a square for instance.

Remark 2.2.2. When $\Omega=\mathbb{R}^n$ or when Ω is a star-shaped domain with respect to infinity, there are two recent nonexistence results for subcritical nonlinearities. They use the method of moving spheres to prove nonexistence of bounded positive solutions in these domains. The first result is due to A. de Pablo and U. Sánchez [236], and

they obtain nonexistence of bounded positive solutions to $(-\Delta)^s u = u^p$ in all of $\mathbb{R}^n$, whenever $s > 1/2$ and $1 < p < \frac{n+2s}{n-2s}$. The second result, by M. Fall and T. Weth [129], gives nonexistence of bounded positive solutions of (2.8) in star-shaped domains with respect to infinity for subcritical nonlinearities.

Our method in the previous proof can also be used to prove nonexistence results for problem (2.7) in star-shaped domains with respect to infinity or in the whole $\mathbb{R}^n$. However, to ensure that the integrals appearing in the proof are well defined, one must assume some decay on u and ∇u. For instance, in the supercritical case $p > \frac{n+2s}{n-2s}$ we obtain that the only solution to $(-\Delta)^s u = u^p$ in all of $\mathbb{R}^n$ decaying as

$$|u| + |x \cdot \nabla u| \leq \frac{C}{1 + |x|^\beta},$$

with $\beta > \frac{n}{p+1}$, is $u \equiv 0$.

In the case of the whole $\mathbb{R}^n$, there is an alternative proof of the nonexistence of solutions which decay fast enough at infinity. It consists of using a Pohozaev identity in all of $\mathbb{R}^n$, that is easily deduced from the pointwise equality

$$(-\Delta)^s(x \cdot \nabla u) = 2s(-\Delta)^s u + x \cdot \nabla(-\Delta)^s u.$$

The classification of solutions in the whole $\mathbb{R}^n$ for the critical exponent $p = \frac{n+2s}{n-2s}$ was obtained by W. Chen, C. Li, and B. Ou in [94]. They are of the form

$$u(x) = c\left(\frac{\mu}{\mu^2 + |x - x_0|^2}\right)^{\frac{n-2s}{2}},$$

where μ is any positive parameter and c is a constant depending on n and s.

2.3 Behavior of $(-\Delta)^{s/2}u$ near $\partial\Omega$

The aim of this section is to prove Proposition 2.1.10. We will split this proof into two propositions. The first one is the following, and compares the behavior of $(-\Delta)^{s/2}u$ near $\partial\Omega$ with the one of $(-\Delta)^{s/2}\delta_0^s$, where $\delta_0(x) = \operatorname{dist}(x, \partial\Omega)\chi_\Omega(x)$.

Proposition 2.3.1. *Let Ω be a bounded and $C^{1,1}$ domain, u be a function satisfying (b) in Proposition 2.1.6. Then, there exists a $C^\alpha(\mathbb{R}^n)$ extension v of $u/\delta^s|_\Omega$ such that*

$$(-\Delta)^{s/2}u(x) = (-\Delta)^{s/2}\delta_0^s(x)v(x) + h(x) \quad \text{in} \quad \mathbb{R}^n,$$

where $h \in C^\alpha(\mathbb{R}^n)$.

Once we know that the behavior of $(-\Delta)^{s/2}u$ is comparable to the one of $(-\Delta)^{s/2}\delta_0^s$, Proposition 2.1.10 reduces to the following result, which gives the behavior of $(-\Delta)^s\delta_0^s$ near $\partial\Omega$.

Proposition 2.3.2. *Let Ω be a bounded and $C^{1,1}$ domain, $\delta(x) = dist(x, \partial\Omega)$, and $\delta_0 = \delta\chi_\Omega$. Then,*

$$(-\Delta)^{s/2}\delta_0^s(x) = c_1\left\{\log^- \delta(x) + c_2\chi_\Omega(x)\right\} + h(x) \quad \text{in} \quad \mathbb{R}^n,$$

where c_1 and c_2 are constants, h is a $C^\alpha(\mathbb{R}^n)$ function, and $\log^- t = \min\{\log t, 0\}$. *The constants c_1 and c_2 are given by*

$$c_1 = c_{1,\frac{s}{2}} \qquad \text{and} \qquad c_2 = \int_0^\infty \left\{ \frac{1-z^s}{|1-z|^{1+s}} + \frac{1+z^s}{|1+z|^{1+s}} \right\} dz,$$

where $c_{n,s}$ is the constant appearing in the singular integral expression (2.2) for $(-\Delta)^s$ in dimension n.

The fact that the constants c_1 and c_2 given by Proposition 2.3.2 coincide with the ones from Proposition 2.1.10 is proved in the Appendix.

In the proof of Proposition 2.3.1 we need to compute $(-\Delta)^{s/2}$ of the product $u = \delta_0^s v$. For it, we will use the following elementary identity, which can be derived from (2.2):

$$(-\Delta)^s(w_1 w_2) = w_1(-\Delta)^s w_2 + w_2(-\Delta)^s w_1 - I_s(w_1, w_2),$$

where

$$I_s(w_1, w_2)(x) = c_{n,s}\mathrm{PV} \int_{\mathbb{R}^n} \frac{\big(w_1(x) - w_1(y)\big)\big(w_2(x) - w_2(y)\big)}{|x-y|^{n+2s}}\, dy. \tag{2.26}$$

Next lemma will lead to a Hölder bound for $I_s(\delta_0^s, v)$.

Lemma 2.3.3. *Let Ω be a bounded domain and $\delta_0 = dist(x, \mathbb{R}^n \setminus \Omega)$. Then, for each $\alpha \in (0,1)$ the following a priori bound holds*

$$\|I_{s/2}(\delta_0^s, w)\|_{C^{\alpha/2}(\mathbb{R}^n)} \le C[w]_{C^\alpha(\mathbb{R}^n)}, \tag{2.27}$$

where the constant C depends only on n, s, and α.

Proof. Let $x_1, x_2 \in \mathbb{R}^n$. Then,

$$|I_{s/2}(\delta_0^s, w)(x_1) - I_{s/2}(\delta_0^s, w)(x_2)| \le c_{n,\frac{s}{2}}(J_1 + J_2),$$

where

$$J_1 = \int_{\mathbb{R}^n} \frac{\big|w(x_1) - w(x_1+z) - w(x_2) + w(x_2+z)\big|\big|\delta_0^s(x_1) - \delta_0^s(x_1+z)\big|}{|z|^{n+s}}\, dz$$

and

$$J_2 = \int_{\mathbb{R}^n} \frac{\big|w(x_2) - w(x_2+z)\big|\big|\delta_0^s(x_1) - \delta_0^s(x_1+z) - \delta_0^s(x_2) + \delta_0^s(x_2+z)\big|}{|z|^{n+s}}\, dz\,.$$

Let $r = |x_1 - x_2|$. Using that $\|\delta_0^s\|_{C^s(\mathbb{R}^n)} \le 1$ and $supp\, \delta_0^s = \overline{\Omega}$,

$$\begin{aligned} J_1 &\le \int_{\mathbb{R}^n} \frac{\big|w(x_1) - w(x_1+z) - w(x_2) + w(x_2+z)\big| \min\{|z|^s, (diam\,\Omega)^s\}}{|z|^{n+s}}\, dz \\ &\le C \int_{\mathbb{R}^n} \frac{[w]_{C^\alpha(\mathbb{R}^n)} r^{\alpha/2} |z|^{\alpha/2} \min\{|z|^s, 1\}}{|z|^{n+s}}\, dz \\ &\le C r^{\alpha/2} [w]_{C^\alpha(\mathbb{R}^n)}\,. \end{aligned}$$

Analogously,

$$J_2 \le Cr^{\alpha/2}[w]_{C^\alpha(\mathbb{R}^n)}\,.$$

The bound for $\|I_{s/2}(\delta_0^s, w)\|_{L^\infty(\mathbb{R}^n)}$ is obtained with a similar argument, and hence (2.27) follows. □

Before stating the next result, we need to introduce the following weighted Hölder norms; see Definition 1.3 in [249].

Definition 2.3.4. *Let $\beta > 0$ and $\sigma \ge -\beta$. Let $\beta = k + \beta'$, with k integer and $\beta' \in (0,1]$. For $w \in C^\beta(\Omega) = C^{k,\beta'}(\Omega)$, define the seminorm*

$$[w]^{(\sigma)}_{\beta;\Omega} = \sup_{x,y\in\Omega}\left(\min\{\delta(x),\delta(y)\}^{\beta+\sigma}\frac{|D^k w(x) - D^k w(y)|}{|x-y|^{\beta'}}\right).$$

For $\sigma > -1$, we also define the norm $\|\cdot\|^{(\sigma)}_{\beta;\Omega}$ as follows: in case that $\sigma \ge 0$,

$$\|w\|^{(\sigma)}_{\beta;\Omega} = \sum_{l=0}^{k}\sup_{x\in\Omega}\left(\delta(x)^{l+\sigma}|D^l w(x)|\right) + [w]^{(\sigma)}_{\beta;\Omega}\,,$$

while for $-1 < \sigma < 0$,

$$\|w\|^{(\sigma)}_{\beta;\Omega} = \|w\|_{C^{-\sigma}(\overline{\Omega})} + \sum_{l=1}^{k}\sup_{x\in\Omega}\left(\delta(x)^{l+\sigma}|D^l w(x)|\right) + [w]^{(-\alpha)}_{\beta;\Omega}.$$

The following lemma, proved in [249], will be used in the proof of Proposition 2.3.1 below —with w replaced by v— and also at the end of this section in the proof of Proposition 2.1.10 —with w replaced by u.

Lemma 2.3.5 ([254, Lemma 4.3])**.** *Let Ω be a bounded domain and α and β be such that $\alpha \le s < \beta$ and $\beta - s$ is not an integer. Let k be an integer such that $\beta = k + \beta'$ with $\beta' \in (0,1]$. Then,*

$$[(-\Delta)^{s/2}w]^{(s-\alpha)}_{\beta-s;\Omega} \le C\big(\|w\|_{C^\alpha(\mathbb{R}^n)} + \|w\|^{(-\alpha)}_{\beta;\Omega}\big) \tag{2.28}$$

for all w with finite right hand side. The constant C depends only on n, s, α, and β.

Before proving Proposition 2.3.1, we give an extension lemma —see [125, Theorem 1, Section 3.1] where the case $\alpha = 1$ is proven in full detail.

Lemma 2.3.6. *Let $\alpha \in (0,1]$ and $V \subset \mathbb{R}^n$ a bounded domain. There exists a (nonlinear) map $E : C^{0,\alpha}(\overline{V}) \to C^{0,\alpha}(\mathbb{R}^n)$ satisfying*

$$E(w) \equiv w \quad \text{in } \overline{V}, \quad [E(w)]_{C^{0,\alpha}(\mathbb{R}^n)} \le [w]_{C^{0,\alpha}(\overline{V})}, \quad \text{and} \quad \|E(w)\|_{L^\infty(\mathbb{R}^n)} \le \|w\|_{L^\infty(V)}$$

for all $w \in C^{0,\alpha}(\overline{V})$.

Proof. It is immediate to check that

$$E(w)(x) = \min\left\{ \min_{z\in\overline{V}} \left\{ w(z) + [w]_{C^\alpha(\overline{V})}|z-x|^\alpha \right\}, \|w\|_{L^\infty(V)} \right\}$$

satisfies the conditions since, for all x, y, z in $\mathbb{R}^n$,

$$|z-x|^\alpha \le |z-y|^\alpha + |y-x|^\alpha .$$

□

Now we can give the

Proof of Proposition 2.3.1. Since $u/\delta^s|_\Omega$ is $C^\alpha(\overline{\Omega})$ —by hypothesis (b)— then by Lemma 2.3.6 there exists a $C^\alpha(\mathbb{R}^n)$ extension v of $u/\delta^s|_\Omega$.

Then, we have that

$$(-\Delta)^{s/2}u(x) = v(x)(-\Delta)^{s/2}\delta_0^s(x) + \delta_0(x)^s(-\Delta)^{s/2}v(x) - I_{s/2}(v,\delta_0^s),$$

where

$$I_{s/2}(v,\delta_0^s) = c_{n,\frac{s}{2}} \int_{\mathbb{R}^n} \frac{\big(v(x)-v(y)\big)\big(\delta_0^s(x)-\delta_0^s(y)\big)}{|x-y|^{n+s}}\, dy\,,$$

as defined in (2.26). This equality is valid in all of $\mathbb{R}^n$ because $\delta_0^s \equiv 0$ in $\mathbb{R}^n\backslash\Omega$ and $v \in C^{\alpha+s}$ in Ω —by hypothesis (b). Thus, we only have to see that $\delta_0^s(-\Delta)^{s/2}v$ and $I_{s/2}(v,\delta_0^s)$ are $C^\alpha(\mathbb{R}^n)$ functions.

For the first one we combine assumption (b) with $\beta = s+\alpha < 1$ and Lemma 2.3.5. We obtain

$$\|(-\Delta)^{s/2}v\|_{\alpha;\Omega}^{(s-\alpha)} \le C, \tag{2.29}$$

and this yields $\delta_0^s(-\Delta)^{s/2}v \in C^\alpha(\mathbb{R}^n)$. Indeed, let $w = (-\Delta)^{s/2}v$. Then, for all $x, y \in \Omega$ such that $y \in B_R(x)$, with $R = \delta(x)/2$, we have

$$\frac{|\delta^s(x)w(x) - \delta^s(y)w(y)|}{|x-y|^\alpha} \le \delta(x)^s \frac{|w(x)-w(y)|}{|x-y|^\alpha} + |w(x)|\frac{|\delta^s(x)-\delta^s(y)|}{|x-y|^\alpha}.$$

Now, since

$$|\delta^s(x)-\delta^s(y)| \le CR^{s-\alpha}|x-y|^\alpha \le C\min\{\delta(x),\delta(y)\}^{s-\alpha}|x-y|^\alpha,$$

using (2.29) and recalling Definition 2.3.4 we obtain

$$\frac{|\delta^s(x)w(x) - \delta^s(y)w(y)|}{|x-y|^\alpha} \le C \quad \text{whenever } y \in B_R(x)\,,\ R = \delta(x)/2.$$

This bound can be extended to all $x, y \in \Omega$, since the domain is regular, by using a dyadic chain of balls; see for instance the proof of Proposition 1.1 in [249].

The second bound, that is,

$$\|I_{s/2}(v,\delta_0^s)\|_{C^\alpha(\mathbb{R}^n)} \le C,$$

follows from assumption (b) and Lemma 2.3.3 (taking a smaller α if necessary). □

To prove Proposition 2.3.2 we need some preliminaries.
Fixed $\rho_0 > 0$, define $\phi \in C^s(\mathbb{R})$ by

$$\phi(x) = x^s \chi_{(0,\rho_0)}(x) + \rho_0^s \chi_{(\rho_0,+\infty)}(x). \tag{2.30}$$

This function ϕ is a truncation of the s-harmonic function x_+^s. We need to introduce ϕ because the growth at infinity of x_+^s prevents us from computing its $(-\Delta)^{s/2}$.

Lemma 2.3.7. *Let $\rho_0 > 0$, and let $\phi : \mathbb{R} \to \mathbb{R}$ be given by* (2.30)*. Then, we have*

$$(-\Delta)^{s/2}\phi(x) = c_1\{\log|x| + c_2\chi_{(0,\infty)}(x)\} + h(x)$$

for $x \in (-\rho_0/2, \rho_0/2)$, where $h \in C^s([-\rho_0/2, \rho_0/2])$. The constants c_1 and c_2 are given by

$$c_1 = c_{1,\frac{s}{2}} \qquad \text{and} \qquad c_2 = \int_0^\infty \left\{ \frac{1-z^s}{|1-z|^{1+s}} + \frac{1+z^s}{|1+z|^{1+s}} \right\} dz,$$

where $c_{n,s}$ is the constant appearing in the singular integral expression (2.2) *for $(-\Delta)^s$ in dimension n.*

Proof. If $x < \rho_0$,

$$(-\Delta)^{s/2}\phi(x) = c_{1,\frac{s}{2}} \left(\int_{-\infty}^{\rho_0} \frac{x_+^s - y_+^s}{|x-y|^{1+s}}\, dy + \int_{\rho_0}^\infty \frac{x_+^s - \rho_0^s}{|x-y|^{1+s}}\, dy \right).$$

We need to study the first integral:

$$J(x) = \int_{-\infty}^{\rho_0} \frac{x_+^s - y_+^s}{|x-y|^{1+s}}\, dy = \begin{cases} J_1(x) = \displaystyle\int_{-\infty}^{\rho_0/x} \frac{1-z_+^s}{|1-z|^{1+s}}\, dz & \text{if } x > 0 \\ \\ J_2(x) = \displaystyle\int_{-\infty}^{\rho_0/|x|} \frac{-z_+^s}{|1+z|^{1+s}}\, dz & \text{if } x < 0\,, \end{cases} \tag{2.31}$$

since

$$(-\Delta)^{s/2}\phi(x) - c_1 J(x) = c_1 \int_{\rho_0}^\infty \frac{x_+^s - \rho_0^s}{|x-y|^{1+s}}\, dy \tag{2.32}$$

belongs to $C^s([-\rho_0/2, \rho_0/2])$ as a function of x.

Using L'Hôpital's rule we find that

$$\lim_{x\downarrow 0} \frac{J_1(x)}{\log|x|} = \lim_{x\uparrow 0} \frac{J_2(x)}{\log|x|} = 1.$$

Moreover,

$$\begin{aligned} \lim_{x\downarrow 0} x^{1-s}\left(J_1'(x) - \frac{1}{x} \right) &= \lim_{x\downarrow 0} x^{1-s}\left(-\frac{\rho_0}{x^2} \frac{1-(\rho_0/x)^s}{((\rho_0/x)-1)^{1+s}} - \frac{1}{x} \right) \\ &= \rho_0^{-s} \lim_{y\downarrow 0} y^{1-s}\left(\frac{1-y^s}{y(1-y)^{1+s}} - \frac{(1-y)^{1+s}}{y(1-y)^{1+s}} \right) \\ &= \rho_0^{-s} \lim_{y\downarrow 0} \frac{1-y^s-(1-y)^{1+s}}{y^s} = -\rho_0^{-s} \end{aligned}$$

and

$$\begin{aligned}\lim_{x\uparrow 0}(-x)^{1-s}\left(J_2'(x)-\frac{1}{x}\right) &= \lim_{x\uparrow 0}(-x)^{1-s}\left(\frac{\rho_0}{x^2}\frac{-(-\rho_0/x)^s}{(1+(-\rho_0/x))^{1+s}}-\frac{1}{-x}\right)\\ &= \rho_0^{-s}\lim_{y\downarrow 0} y^{1-s}\left(\frac{-1}{y(1+y)^{1+s}}+\frac{(1+y)^{1+s}}{y(1+y)^{1+s}}\right)\\ &= \rho_0^{-s}\lim_{y\downarrow 0}\frac{(1+y)^{1+s}-1}{y^s}=0\,.\end{aligned}$$

Therefore,

$$(J_1(x)-\log|x|)'\le C|x|^{s-1}\quad \text{in } (0,\rho_0/2]$$

and

$$(J_2(x)-\log|x|)'\le C|x|^{s-1}\quad \text{in } [-\rho_0/2,0),$$

and these gradient bounds yield

$$(J_1-\log|\cdot|)\in C^s([0,\rho_0/2])\quad\text{and}\quad (J_2-\log|\cdot|)\in C^s([-\rho_0/2,0]).$$

However, these two Hölder functions do not have the same value at 0. Indeed,

$$\begin{aligned}\lim_{x\downarrow 0}\big\{(J_1(x)-\log|x|)-(J_2(-x)-\log|-x|)\big\} &= \lim_{x\downarrow 0}\{J_1(x)-J_2(-x)\}\\ &= \int_{-\infty}^{\infty}\left\{\frac{1-z_+^s}{|1-z|^{1+s}}+\frac{z_+^s}{|1+z|^{1+s}}\right\}dz\\ &= \int_0^{\infty}\left\{\frac{1-z^s}{|1-z|^{1+s}}+\frac{1+z^s}{|1+z|^{1+s}}\right\}dz=c_2.\end{aligned}$$

Hence, the function $J(x)-\log|x|-c_2\chi_{(0,\infty)}(x)$, where J is defined by (2.31), is $C^s([-\rho_0/2,\rho_0/2])$. Recalling (2.32), we obtain the result. □

Next lemma will be used to prove Proposition 2.3.2. Before stating it, we need the following

Remark 2.3.8. From now on in this section, $\rho_0>0$ is a small constant depending only on Ω, which we assume to be a bounded $C^{1,1}$ domain. Namely, we assume that that every point on $\partial\Omega$ can be touched from both inside and outside Ω by balls of radius ρ_0. In other words, given $x_0\in\partial\Omega$, there are balls of radius ρ_0, $B_{\rho_0}(x_1)\subset\Omega$ and $B_{\rho_0}(x_2)\subset\mathbb{R}^n\setminus\Omega$, such that $\overline{B_{\rho_0}(x_1)}\cap\overline{B_{\rho_0}(x_2)}=\{x_0\}$. A useful observation is that all points y in the segment that joins x_1 and x_2 —through x_0— satisfy $\delta(y)=|y-x_0|$.

Lemma 2.3.9. *Let Ω be a bounded $C^{1,1}$ domain, $\delta(x)=dist(x,\partial\Omega)$, $\delta_0=\delta\chi_\Omega$, and ρ_0 be given by Remark 2.3.8. Fix $x_0\in\partial\Omega$, and define*

$$\phi_{x_0}(x)=\phi\left(-\nu(x_0)\cdot(x-x_0)\right)$$

and

$$S_{x_0}=\{x_0+t\nu(x_0),\ t\in(-\rho_0/2,\rho_0/2)\},\tag{2.33}$$

where ϕ is given by (2.30) *and $\nu(x_0)$ is the unit outward normal to $\partial\Omega$ at x_0. Define also $w_{x_0}=\delta_0^s-\phi_{x_0}$.*

Then, for all $x \in S_{x_0}$,

$$|(-\Delta)^{s/2}w_{x_0}(x) - (-\Delta)^{s/2}w_{x_0}(x_0)| \le C|x - x_0|^{s/2},$$

where C depends only on Ω and ρ_0 (and not on x_0).

Proof. We denote $w = w_{x_0}$. Note that, along S_{x_0}, the distance to $\partial\Omega$ agrees with the distance to the tangent plane to $\partial\Omega$ at x_0; see Remark 2.3.8. That is, denoting $\delta_\pm = (\chi_\Omega - \chi_{\mathbb{R}^n\setminus\Omega})\delta$ and $d(x) = -\nu(x_0)\cdot(x - x_0)$, we have $\delta_\pm(x) = d(x)$ for all $x \in S_{x_0}$. Moreover, the gradients of these two functions also coincide on S_{x_0}, i.e., $\nabla\delta_\pm(x) = -\nu(x_0) = \nabla d(x)$ for all $x \in S_{x_0}$.

Therefore, for all $x \in S_{x_0}$ and $y \in B_{\rho_0/2}(0)$, we have

$$|\delta_\pm(x + y) - d(x + y)| \le C|y|^2$$

for some C depending only on ρ_0. Thus, for all $x \in S_{x_0}$ and $y \in B_{\rho_0/2}(0)$,

$$|w(x + y)| = |(\delta_\pm(x + y))_+^s - (d(x + y))_+^s| \le C|y|^{2s}, \tag{2.34}$$

where C is a constant depending on Ω and s.

On the other hand, since $w \in C^s(\mathbb{R}^n)$, then

$$|w(x + y) - w(x_0 + y)| \le C|x - x_0|^s. \tag{2.35}$$

Finally, let $r < \rho_0/2$ to be chosen later. For each $x \in S_{x_0}$ we have

$$\begin{aligned}
|(-\Delta)^{s/2}w(x) - (-\Delta)^{s/2}w(x_0)| &\le C\int_{\mathbb{R}^n} \frac{|w(x + y) - w(x_0 + y)|}{|y|^{n+s}}\,dy \\
&\le C\int_{B_r} \frac{|w(x + y) - w(x_0 + y)|}{|y|^{n+s}}\,dy + C\int_{\mathbb{R}^n\setminus B_r} \frac{|w(x + y) - w(x_0 + y)|}{|y|^{n+s}}\,dy \\
&\le C\int_{B_r} \frac{|y|^{2s}}{|y|^{n+s}}\,dy + C\int_{\mathbb{R}^n\setminus B_r} \frac{|x - x_0|^s}{|y|^{n+s}}\,dy \\
&= C(r^s + |x - x_0|^s r^{-s}),
\end{aligned}$$

where we have used (2.34) and (2.35). Taking $r = |x - x_0|^{1/2}$ the lemma is proved. □

The following is the last ingredient needed to prove Proposition 2.3.2.

Claim 2.3.10. *Let Ω be a bounded $C^{1,1}$ domain, and ρ_0 be given by Remark 2.3.8. Let w be a function satisfying, for some $K > 0$,*

(i) w is locally Lipschitz in $\{x \in \mathbb{R}^n : 0 < \delta(x) < \rho_0\}$ and

$$|\nabla w(x)| \le K\delta(x)^{-M} \quad \text{in} \quad \{x \in \mathbb{R}^n : 0 < \delta(x) < \rho_0\}$$

for some $M > 0$.

(ii) There exists $\alpha > 0$ such that

$$|w(x) - w(x^*)| \le K\delta(x)^\alpha \quad \text{in} \quad \{x \in \mathbb{R}^n : 0 < \delta(x) < \rho_0\},$$

where x^* is the unique point on $\partial\Omega$ satisfying $\delta(x) = |x - x^*|$.

(iii) For the same α, it holds

$$\|w\|_{C^\alpha(\{\delta\geq\rho_0\})}\leq K.$$

Then, there exists $\gamma>0$, depending only on α and M, such that

$$\|w\|_{C^\gamma(\mathbb{R}^n)}\leq CK, \tag{2.36}$$

where C depends only on Ω.

Proof. First note that from (ii) and (iii) we deduce that $\|w\|_{L^\infty(\mathbb{R}^n)}\leq CK$. Let $\rho_1\leq\rho_0$ be a small positive constant to be chosen later. Let $x,y\in\{\delta\leq\rho_0\}$, and $r=|x-y|$.

If $r\geq\rho_1$, then

$$\frac{|w(x)-w(y)|}{|x-y|^\gamma}\leq\frac{2\|w\|_{L^\infty(\mathbb{R}^n)}}{\rho_1^\gamma}\leq CK.$$

If $r<\rho_1$, consider

$$x'=x^*+\rho_0 r^\beta\nu(x^*)\ \text{ and }\ y'=y^*+\rho_0 r^\beta\nu(y^*),$$

where $\beta\in(0,1)$ is to be determined later. Choose ρ_1 small enough so that the segment joining x' and y' contained in the set $\{\delta>\rho_0 r^\beta/2\}$. Then, by (i),

$$|w(x')-w(y')|\leq CK(\rho_0 r^\beta/2)^{-M}|x'-y'|\leq Cr^{1-\beta M}. \tag{2.37}$$

Thus, using (ii) and (2.37),

$$\begin{aligned}|w(x)-w(y)|&\leq|w(x)-w(x^*)|+|w(x^*)-w(x')|+\\&\quad+|w(y)-w(y^*)|+|w(y^*)-w(y')|+|w(x')-w(y')|\\&\leq K\delta(x)^\alpha+K\delta(y)^\alpha+2K(\rho_0 r^\beta)^\alpha+CKr^{1-\beta M}.\end{aligned}$$

Taking $\beta<1/M$ and $\gamma=\min\{\alpha\beta,1-\beta M\}$, we find

$$|w(x)-w(y)|\leq CKr^\gamma=CK|x-y|^\gamma.$$

This proves

$$[w]_{C^\gamma(\{\delta\leq\rho_0\})}\leq CK.$$

To obtain the bound (2.36) we combine the previous seminorm estimate with (iii). □

Finally, we give the proof of Proposition 2.3.2.

Proof of Proposition 2.3.2. Let

$$h(x)=(-\Delta)^{s/2}\delta_0^s(x)-c_1\left\{\log^-\delta(x)+c_2\chi_\Omega(x)\right\}.$$

We want to prove that $h\in C^\alpha(\mathbb{R}^n)$ by using Claim 2.3.10.

On one hand, by Lemma 2.3.7, for all $x_0\in\partial\Omega$ and for all $x\in S_{x_0}$, where S_{x_0} is defined by (2.33), we have

$$h(x)=(-\Delta)^{s/2}\delta_0^s(x)-(-\Delta)^{s/2}\phi_{x_0}(x)+\tilde{h}\big(\nu(x_0)\cdot(x-x_0)\big),$$

where $\tilde{h}$ is the $C^s([-\rho_0/2,\rho_0/2])$ function from Lemma 2.3.7. Hence, using Lemma 2.3.9, we find

$$|h(x)-h(x_0)|\le C|x-x_0|^{s/2}\quad \text{for all } x\in S_{x_0}$$

for some constant independent of x_0.

Recall that for all $x\in S_{x_0}$ we have $x^*=x_0$, where x^* is the unique point on $\partial\Omega$ satisfying $\delta(x)=|x-x^*|$. Hence,

$$|h(x)-h(x^*)|\le C|x-x^*|^{s/2}\quad \text{for all } x\in\{\delta<\rho_0/2\}\,. \tag{2.38}$$

Moreover,

$$\|h\|_{C^\alpha(\{\delta\ge\rho_0/2\})}\le C \tag{2.39}$$

for all $\alpha\in(0,1-s)$, where C is a constant depending only on α, Ω and ρ_0. This last bound is found using that $\|\delta_0^s\|_{C^{0,1}(\{\delta\ge\rho_0/2\})}\le C$, which yields

$$\|(-\Delta)^{s/2}\delta_0^s\|_{C^\alpha(\{\delta\ge\rho_0\})}\le C$$

for $\alpha<1-s$.

On the other hand, we claim now that if $x\notin\partial\Omega$ and $\delta(x)<\rho_0/2$, then

$$|\nabla h(x)|\le|\nabla(-\Delta)^{s/2}\delta_0^s(x)|+c_1|\delta(x)|^{-1}\le C|\delta(x)|^{-n-s}. \tag{2.40}$$

Indeed, observe that $\delta_0^s\equiv 0$ in $\mathbb{R}^n\backslash\Omega$, $|\nabla\delta_0^s|\le C\delta_0^{s-1}$ in Ω, and $|D^2\delta_0^s|\le C\delta_0^{s-2}$ in Ω_{ρ_0}. Then, $r=\delta(x)/2$,

$$\begin{aligned}|(-\Delta)^{s/2}\nabla\delta_0^s(x)|&\le C\int_{\mathbb{R}^n}\frac{|\nabla\delta_0^s(x)-\nabla\delta_0^s(x+y)|}{|y|^{n+s}}\,dy\\&\le C\int_{B_r}\frac{Cr^{s-2}|y|\,dy}{|y|^{n+s}}+C\int_{\mathbb{R}^n\backslash B_r}\left(\frac{|\nabla\delta_0^s(x)|}{|y|^{n+s}}+\frac{|\nabla\delta_0^s(x+y)|}{r^{n+s}}\right)dy\\&\le\frac{C}{r}+\frac{C}{r}+\frac{C}{r^{n+s}}\int_{\mathbb{R}^n}\delta_0^{s-1}\le\frac{C}{r^{n+s}},\end{aligned}$$

as claimed.

To conclude the proof, we use bounds (2.38), (2.39), and (2.40) and Claim 2.3.10. □

To end this section, we give the

Proof of Proposition 2.1.10. The first part follows from Propositions 2.3.1 and 2.3.2. The second part follows from Lemma 2.3.5 with $\alpha=s$ and $\beta\in(s,1+2s)$. □

2.4 The operator $-\frac{d}{d\lambda}\big|_{\lambda=1^+}\int_{\mathbb{R}} w_\lambda w_{1/\lambda}$

The aim of this section is to prove Proposition 2.1.11. In other words, we want to evaluate the operator

$$\mathfrak{I}(w)=-\frac{d}{d\lambda}\bigg|_{\lambda=1^+}\int_0^\infty w(\lambda t)\,w\left(\frac{t}{\lambda}\right)dt \tag{2.41}$$

on

$$w(t) = A \log^- |t-1| + B\chi_{[0,1]}(t) + h(t),$$

where $\log^- t = \min\{\log t, 0\}$, A and B are real numbers, and h is a function satisfying, for some constants $\alpha \in (0,1)$, $\gamma \in (0,1)$, and C_0, the following conditions:

(i) $\|h\|_{C^\alpha((0,\infty))} \leq C_0$.

(ii) For all $\beta \in [\gamma, 1+\gamma]$,

$$\|h\|_{C^\beta((0,1-\rho)\cup(1+\rho,2))} \leq C_0 \rho^{-\beta} \qquad \text{for all } \ \rho \in (0,1).$$

(iii) $|h'(t)| \leq Ct^{-2-\gamma}$ and $|h''(t)| \leq Ct^{-3-\gamma}$ for all $t > 2$.

We will split the proof of Proposition 2.1.11 into three parts. The first part is the following, and evaluates the operator $\mathfrak{I}$ on the function

$$w_0(t) = A \log^- |t-1| + B\chi_{[0,1]}(t). \tag{2.42}$$

Lemma 2.4.1. *Let w_0 and $\mathfrak{I}$ be given by* (2.42) *and* (2.41)*, respectively. Then,*

$$\mathfrak{I}(w_0) = A^2\pi^2 + B^2.$$

The second result towards Proposition 2.1.11 is the following.

Lemma 2.4.2. *Let h be a function satisfying (i), (ii), and (iii) above, and $\mathfrak{I}$ be given by* (2.41)*. Then,*

$$\mathfrak{I}(h) = 0.$$

Moreover, there exist constants C and $\nu > 1$, depending only on the constants α, γ, and C_0 appearing in (i)-(ii)-(iii), such that

$$\left| \int_0^\infty \left\{ h(\lambda t)\, h\left(\frac{t}{\lambda}\right) - h(t)^2 \right\} dt \right| \leq C|\lambda - 1|^\nu$$

for each $\lambda \in (1, 3/2)$.

Finally, the third one states that $\mathfrak{I}(w_0 + h) = \mathfrak{I}(w_0)$ whenever $\mathfrak{I}(h) = 0$.

Lemma 2.4.3. *Let w_1 and w_2 be $L^2(\mathbb{R})$ functions. Assume that the derivative at $\lambda = 1^+$ in the expression $\mathfrak{I}(w_1)$ exists, and that*

$$\mathfrak{I}(w_2) = 0.$$

Then,

$$\mathfrak{I}(w_1 + w_2) = \mathfrak{I}(w_1).$$

Let us now give the proofs of Lemmas 2.4.1, 2.4.2, and 2.4.3. We start proving Lemma 2.4.3. For it, is useful to introduce the bilinear form

$$(w_1, w_2) = -\frac{1}{2} \frac{d}{d\lambda}\bigg|_{\lambda=1^+} \int_0^\infty \left\{ w_1(\lambda t)\, w_2\left(\frac{t}{\lambda}\right) + w_1\left(\frac{t}{\lambda}\right) w_2(\lambda t) \right\} dt,$$

and more generally, the bilinear forms

$$(w_1, w_2)_\lambda = -\frac{1}{2(\lambda-1)}\int_0^\infty \left\{ w_1(\lambda t)\, w_2\left(\frac{t}{\lambda}\right) + w_1\left(\frac{t}{\lambda}\right) w_2(\lambda t) - 2w_1(t)w_2(t) \right\} dt, \tag{2.43}$$

for $\lambda > 1$.

It is clear that $\lim_{\lambda\downarrow 1}(w_1, w_2)_\lambda = (w_1, w_2)$ whenever the limit exists, and that $(w, w) = \mathfrak{I}(w)$. The following lemma shows that these bilinear forms are positive definite and, thus, they satisfy the Cauchy-Schwarz inequality.

Lemma 2.4.4. *The following properties hold.*

(a) $(w_1, w_2)_\lambda$ *is a bilinear map.*

(b) $(w, w)_\lambda \geq 0$ *for all* $w \in L^2(\mathbb{R}_+)$.

(c) $|(w_1, w_2)_\lambda|^2 \leq (w_1, w_1)_\lambda (w_2, w_2)_\lambda$.

Proof. Part (a) is immediate. Part (b) follows from the Hölder inequality

$$\|w_\lambda w_{1/\lambda}\|_{L^1} \leq \|w_\lambda\|_{L^2}\|w_{1/\lambda}\|_{L^2} = \|w\|_{L^2}^2,$$

where $w_\lambda(t) = w(\lambda t)$. Part (c) is a consequence of (a) and (b). □

Now, Lemma 2.4.3 is an immediate consequence of this Cauchy-Schwarz inequality.

Proof of Lemma 2.4.3. By Lemma 2.4.4 (iii) we have

$$0 \leq |(w_1, w_2)_\lambda| \leq \sqrt{(w_1, w_1)_\lambda}\sqrt{(w_2, w_2)_\lambda} \longrightarrow 0.$$

Thus, $(w_1, w_2) = \lim_{\lambda\downarrow 1}(w_1, w_2)_\lambda = 0$ and

$$\mathfrak{I}(w_1 + w_2) = \mathfrak{I}(w_1) + \mathfrak{I}(w_2) + 2(w_1, w_2) = \mathfrak{I}(w_1).$$

□

Next we prove that $\mathfrak{I}(h) = 0$. For this, we will need a preliminary lemma.

Lemma 2.4.5. *Let* h *be a function satisfying (i), (ii), and (iii) in Propostion 2.1.11,* $\lambda \in (1, 3/2)$*, and* $\tau \in (0, 1)$ *be such that* $\tau/2 > \lambda - 1$*. Let* α*,* γ*, and* C_0 *be the constants appearing in (i)-(ii)-(iii). Then,*

$$\left| h(\lambda t) h\left(\frac{t}{\lambda}\right) - h(t)^2 \right| \leq \begin{cases} C \max\left\{ |t-\lambda|^\alpha, |t-1/\lambda|^\alpha \right\} & t \in (1-\tau, 1+\tau) \\ C(\lambda-1)^{1+\gamma}|t-1|^{-1-\gamma} & t \in (0, 1-\tau) \cup (1+\tau, 2) \\ C(\lambda-1)^2 t^{-1-\gamma} & t \in (2, \infty), \end{cases}$$

where the constant C *depends only on* C_0.

Proof. Let $t \in (1-\tau, 1+\tau)$. Let us denote $\widetilde{h} = h - h(1)$. Then,

$$h(\lambda t)\, h\left(\frac{t}{\lambda}\right) - h(t)^2 = \widetilde{h}(\lambda t)\, \widetilde{h}\left(\frac{t}{\lambda}\right) - \widetilde{h}(t)^2 + h(1)\left(\widetilde{h}(\lambda t) + \widetilde{h}\left(\frac{t}{\lambda}\right) - 2\widetilde{h}(t)\right).$$

Therefore, using that $|\widetilde{h}(t)| \le C_0|t-1|^\alpha$ and $\|h\|_{L^\infty(\mathbb{R})} \le C_0$, we obtain

$$\begin{aligned}\left|h(\lambda t)\, h\left(\frac{t}{\lambda}\right) - h(t)^2\right| &\le C\,|\lambda t - 1|^\alpha \left|\frac{t}{\lambda} - 1\right|^\alpha + C|t-1|^{2\alpha} + C|\lambda t - 1|^\alpha + \\ &\quad + C\left|\frac{t}{\lambda} - 1\right|^\alpha + C|t-1|^\alpha \\ &\le C \max\left\{|t-\lambda|^\alpha, \left|t - \frac{1}{\lambda}\right|^\alpha\right\}.\end{aligned}$$

Let now $t \in (0, 1-\tau) \cup (1+\tau, 2)$ and recall that $\lambda \in (1, 1+\tau/2)$. Define, for $\mu \in [1, \lambda]$,

$$\psi(\mu) = h(\mu t)\, h\left(\frac{t}{\mu}\right) - h(t)^2.$$

By the mean value theorem, $\psi(\lambda) = \psi(1) + \psi'(\mu)(\lambda - 1)$ for some $\mu \in (1, \lambda)$. Moreover, observing that $\psi(1) = \psi'(1) = 0$, we deduce

$$|\psi(\lambda)| \le (\lambda - 1)|\psi'(\mu) - \psi'(1)|.$$

Next we claim that

$$|\psi'(\mu) - \psi'(1)| \le C|\mu - 1|^\gamma |t-1|^{-1-\gamma}. \tag{2.44}$$

This yields the desired bound for $t \in (0, 1-\tau) \cup (1+\tau, 2)$.

To prove this claim, note that

$$\psi'(\mu) = t h'(\mu t)\, h\left(\frac{t}{\mu}\right) - \frac{t}{\mu^2} h(\mu t)\, h'\left(\frac{t}{\mu}\right).$$

Thus, using the bounds from (ii) with β replaced by γ, 1, and $1+\gamma$,

$$\begin{aligned}|\psi'(\mu) - \psi'(1)| &\le t|h'(\mu t) - h'(t)| \left|h\left(\frac{t}{\mu}\right)\right| + t\left|h\left(\frac{t}{\mu}\right) - h(t)\right| |h'(t)| + \\ &\quad + t\left|h'\left(\frac{t}{\mu}\right) - h'(t)\right| \frac{|h(\mu t)|}{\mu^2} + t\left|\frac{h(\mu t)}{\mu^2} - h(t)\right| |h'(t)| \\ &\le C|\mu t - t|^\gamma m^{-1-\gamma} + C\left|\frac{t}{\mu} - t\right|^\gamma m^{-\gamma}|t-1|^{-1} + \frac{C}{\mu^2}\left|\frac{t}{\mu} - t\right|^\gamma m^{-1-\gamma} + \\ &\quad + \frac{C}{\mu^2}|\mu t - t|^\gamma m^{-\gamma}|t-1|^{-1} + C(\mu - 1)|t-1|^{-1} \\ &\le C(\mu - 1)^\gamma m^{-1-\gamma},\end{aligned}$$

where $m = \min\{|\mu t - 1|, |t-1|, |t/\mu - 1|\}$.

Furthermore, since $\mu - 1 < |t-1|/2$, we have $m \ge \frac{1}{4}|t-1|$, and hence (2.44) follows.

Finally, if $t \in (2, \infty)$, with a similar argument but using the bound (iii) instead of (ii), we obtain

$$|\psi(\lambda)| \le C(\lambda - 1)^2 t^{-1-\gamma},$$

and we are done. □

Let us now give the

Proof of Lemma 2.4.2. Let us call

$$I_\lambda = \int_0^\infty \left\{ h\left(\lambda t\right) h\left(\frac{t}{\lambda}\right) - h(t)^2 \right\} dx.$$

For each $\lambda \in (1, 3/2)$, take $\tau \in (0,1)$ such that $\lambda - 1 < \tau/2$ to be chosen later. Then, by Lemma 2.4.5,

$$\begin{aligned} |I_\lambda| &\leq C(\lambda-1)^{1+\gamma}\int_0^{1-\tau} |t-1|^{-1-\gamma}dt + C\int_{1-\tau}^{1} |t-\lambda|^\alpha\, dt + \\ &\quad + C\int_1^{1+\tau} \left|t - \frac{1}{\lambda}\right|^\alpha dt + C(\lambda-1)^{1+\gamma}\int_{1+\tau}^{2} |t-1|^{-1-\gamma}dt + \\ &\quad + C(\lambda-1)^2\int_2^\infty t^{-1-s}dt \\ &\leq C(\lambda-1)^{1+\gamma}\tau^{-\gamma} + C\left(\tau+\lambda-1\right)^{\alpha+1} + C(\lambda-1)^{1+\gamma}\tau^{-\gamma} + \\ &\quad + C\left(\tau + 1 - \frac{1}{\lambda}\right)^{\alpha+1} + C(\lambda-1)^2. \end{aligned}$$

Choose now

$$\tau = (\lambda-1)^\theta,$$

with $\theta < 1$ to be chosen later. Then,

$$\tau + \lambda - 1 \leq 2\tau \qquad \text{and} \qquad \tau + 1 - \frac{1}{\lambda} \leq 2\tau,$$

and hence

$$|I_\lambda| \leq C(\lambda-1)^{(\alpha+1)\theta} + C(\lambda-1)^{1+\gamma-\theta\gamma} + C(\lambda-1)^2.$$

Finally, choose θ such that $(\alpha+1)\theta > 1$ and $1+\gamma-\theta\gamma > 1$, that is, satisfying

$$\frac{1}{1+\alpha} < \theta < 1.$$

Then, for $\nu = \min\{(\alpha+1)\theta, 1+\gamma-\gamma\theta\} > 1$, it holds

$$\left| \int_0^\infty \left\{ h\left(\lambda t\right) h\left(\frac{t}{\lambda}\right) - h(t)^2 \right\} dt \right| \leq C|\lambda-1|^\nu,$$

as desired. □

Next we prove Lemma 2.4.1.

Proof of Lemma 2.4.1. Let

$$w_1(t) = \log^- |t-1| \qquad \text{and} \qquad w_2(t) = \chi_{[0,1]}(t).$$

We will compute first $\mathfrak{I}(w_1)$.

Define

$$\Psi(t) = \int_0^t \frac{\log|r-1|}{r} dr.$$

It is straightforward to check that, if $\lambda > 1$, the function

$$\begin{aligned}\vartheta_\lambda(t) &= \left(t - \frac{1}{\lambda}\right)\log|\lambda t - 1|\log\left|\frac{t}{\lambda} - 1\right| + (\lambda - t)\log\left|\frac{t}{\lambda} - 1\right| \\ &\quad -\frac{\lambda^2-1}{\lambda}\log(\lambda^2-1)\log\left|\frac{t}{\lambda} - 1\right| - \frac{\lambda^2-1}{\lambda}\Psi\left(\frac{\lambda(\lambda-t)}{\lambda^2-1}\right) \\ &\quad +2t - \frac{\lambda t - 1}{\lambda}\log|\lambda t - 1|\end{aligned}$$

is a primitive of $\log|\lambda t - 1|\log\left|\frac{t}{\lambda} - 1\right|$. Denoting $I_\lambda = \int_0^\infty w_1(\lambda t)\, w_1\left(\frac{t}{\lambda}\right) dt$, we have

$$\begin{aligned}I_\lambda - I_1 &= \int_0^{\frac{2}{\lambda}} \log|\lambda t - 1|\log\left|\frac{t}{\lambda} - 1\right| dt - \int_0^2 \log^2|t-1| dt \\ &= \vartheta_\lambda\left(\frac{2}{\lambda}\right) - \vartheta_\lambda(0) - 4 \\ &= \left(\frac{\lambda^2-1}{\lambda}\right)\left\{\Psi\left(\frac{\lambda^2}{\lambda^2-1}\right) - \Psi\left(\frac{\lambda^2-2}{\lambda^2-1}\right)\right\} + \left(\lambda - \frac{2}{\lambda}\right)\log\left(\frac{2}{\lambda^2} - 1\right) + \\ &\quad + \left(\lambda - \frac{1}{\lambda}\right)\log(\lambda^2-1)\log\left(\frac{2}{\lambda^2} - 1\right) - \frac{4(\lambda-1)}{\lambda},\end{aligned}$$

where we have used that

$$I_1 = \int_0^2 \log^2|t-1| dt = 2\int_0^1 \log^2 t' dt' = 2\int_0^\infty r^2 e^{-r} dr = 2\Gamma(3) = 4.$$

Therefore, dividing by $\lambda - 1$ and letting $\lambda \downarrow 1$,

$$\begin{aligned}\left.\frac{d}{d\lambda}\right|_{\lambda=1^+} I_\lambda &= 2\lim_{\lambda\downarrow 1}\int_{\frac{\lambda^2-2}{\lambda^2-1}}^{\frac{\lambda^2}{\lambda^2-1}} \frac{\log|t-1|}{t} dt + \\ &\quad + \lim_{\lambda\downarrow 1}\left\{2\log(\lambda^2-1)\log\left(\frac{2}{\lambda^2} - 1\right) - \frac{\log\left(\frac{2}{\lambda^2} - 1\right)}{\lambda - 1} - \frac{4}{\lambda}\right\}.\end{aligned}$$

The first term equals to

$$\lim_{M\to+\infty}\int_{-M}^{M} \frac{2\log|t-1|}{t} dt,$$

while the second, using that $\log(1+x) \sim x$ for $x \sim 0$, equals to

$$\lim_{\lambda\downarrow 1}\left\{2\log(\lambda^2-1)\left(\frac{2}{\lambda^2} - 2\right) - \frac{\frac{2}{\lambda^2} - 2}{\lambda - 1} - \frac{4}{\lambda}\right\} = 0 + 4 - 4 = 0.$$

Hence,

$$\begin{aligned}\frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda &= \lim_{M\to+\infty}\int_{-M}^{M}\frac{2\log|t-1|}{t}dt = \lim_{M\to+\infty}\int_{-M}^{M}\frac{2\log|t|}{t+1}dt\\ &= \lim_{M\to+\infty}\left\{\int_{-M}^{0}\frac{2\log(-t)}{t+1}dt+\int_{0}^{M}\frac{2\log t}{t+1}dt\right\}\\ &= \lim_{M\to+\infty}\left\{\int_{0}^{M}\frac{2\log t}{1-t}dt+\int_{0}^{M}\frac{2\log t}{t+1}dt\right\} = \int_0^{+\infty}\frac{4\log t}{1-t^2}dt\\ &= \int_0^1\frac{4\log t}{1-t^2}dt+\int_1^{+\infty}\frac{-4\log\frac{1}{t}}{\frac{1}{t^2}-1}\frac{dt}{t^2} = 2\int_0^1\frac{4\log t}{1-t^2}dt.\end{aligned}$$

Furthermore, using that $\frac{1}{1-t^2}=\sum_{n\geq 0}t^{2n}$ and that

$$\int_0^1 t^n\log t\;dt = -\int_0^1\frac{t^{n+1}}{n+1}\frac{1}{t}dt = -\frac{1}{(n+1)^2},$$

we obtain

$$\int_0^1\frac{\log t}{1-t^2}dt = -\sum_{n\geq 0}\frac{1}{(2n+1)^2} = -\frac{\pi^2}{8},$$

and thus

$$\mathfrak{I}(w_1) = -\frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda = \pi^2.$$

Let us evaluate now $\mathfrak{I}(w_2)=\mathfrak{I}(\chi_{[0,1]})$. We have

$$\int_0^{+\infty}\chi_{[0,1]}(\lambda t)\,\chi_{[0,1]}\left(\frac{t}{\lambda}\right)dt = \int_0^{\frac{1}{\lambda}}dt = \frac{1}{\lambda}.$$

Therefore, differentiating with respect to λ we obtain $\mathfrak{I}(w_2)=1$.

Let us finally prove that $(w_1,w_2)=0$, i.e., that

$$\frac{d}{d\lambda}\Big|_{\lambda=1^+}\left\{\int_0^\lambda\log|1-\lambda t|dt+\int_0^{\frac{1}{\lambda}}\log\left|1-\frac{t}{\lambda}\right|dt\right\}=0. \tag{2.45}$$

We have

$$\begin{aligned}\int_0^\lambda\log|1-\lambda t|dt &= \frac{1}{\lambda}\big[(\lambda t-1)\log|1-\lambda t|-\lambda t\big]_0^\lambda\\ &= \left(\lambda-\frac{1}{\lambda}\right)\log(\lambda^2-1)-\lambda,\end{aligned}$$

and similarly,

$$\int_0^{\frac{1}{\lambda}}\log\left|1-\frac{t}{\lambda}\right|dt = \left(\frac{1}{\lambda}-\lambda\right)\log\left(1-\frac{1}{\lambda^2}\right)-\frac{1}{\lambda}.$$

Thus,

$$\left|\int_0^\lambda\log|1-\lambda t|dt+\int_0^{\frac{1}{\lambda}}\log\left|1-\frac{t}{\lambda}\right|dt-2\int_0^1\log|1-t|dt\right| =$$

$$= \left| \frac{2(\lambda^2-1)}{\lambda} \log \lambda - \frac{(\lambda-1)^2}{\lambda} \right| \leq 4(\lambda-1)^2.$$

Therefore (2.45) holds, and the proposition is proved. □

Finally, to end this section, we give the:

Proof of Proposition 2.1.11. Let us write $\varphi = w_0 + h$, where w_0 is given by (2.42). Then, for each $\lambda > 1$ we have

$$(\varphi, \varphi)_\lambda = (w_0, w_0)_\lambda + 2(w_0, h)_\lambda + (h, h)_\lambda,$$

where $(\cdot, \cdot)_\lambda$ is defined by (2.43). Using Lemma 2.4.4 (c) and Lemma 2.4.2, we deduce

$$\left|(\varphi, \varphi)_\lambda - A^2\pi^2 - B^2\right| \leq \left|(w_0, w_0)_\lambda - A^2\pi^2 - B^2\right| + C|\lambda - 1|^\nu.$$

The constants C and ν depend only on α, γ, and C_0, and by Lemma 2.4.1 the right hand side goes to 0 as $\lambda \downarrow 1$, since $(w_0, w_0)_\lambda \to \mathfrak{I}(w_0)$ as $\lambda \downarrow 1$. □

2.5 Proof of the Pohozaev identity in non-star-shaped domains

In this section we prove Proposition 2.1.6 for general $C^{1,1}$ domains. The key idea is that every $C^{1,1}$ domain is locally star-shaped, in the sense that its intersection with any small ball is star-shaped with respect to some point. To exploit this, we use a partition of unity to split the function u into a set of functions u_1, ..., u_m, each one with support in a small ball. However, note that the Pohozaev identity is quadratic in u, and hence we must introduce a bilinear version of this identity, namely

$$\begin{aligned} \int_\Omega (x \cdot \nabla u_1)(-\Delta)^s u_2 \, dx + \int_\Omega (x \cdot \nabla u_2)(-\Delta)^s u_1 \, dx = \frac{2s-n}{2} \int_\Omega u_1 (-\Delta)^s u_2 \, dx + \\ + \frac{2s-n}{2} \int_\Omega u_2 (-\Delta)^s u_1 \, dx - \Gamma(1+s)^2 \int_{\partial\Omega} \frac{u_1}{\delta^s} \frac{u_2}{\delta^s} (x \cdot \nu) \, d\sigma. \end{aligned} \tag{2.46}$$

The following lemma states that this bilinear identity holds whenever the two functions u_1 and u_2 have disjoint compact supports. In this case, the last term in the previous identity equals 0, and since $(-\Delta)^s u_i$ is evaluated only outside the support of u_i, we only need to require $\nabla u_i \in L^1(\mathbb{R}^n)$.

Lemma 2.5.1. *Let u_1 and u_2 be $W^{1,1}(\mathbb{R}^n)$ functions with disjoint compact supports K_1 and K_2. Then,*

$$\begin{aligned} \int_{K_1} (x \cdot \nabla u_1)(-\Delta)^s u_2 \, dx + \int_{K_2} (x \cdot \nabla u_2)(-\Delta)^s u_1 \, dx = \\ = \frac{2s-n}{2} \int_{K_1} u_1 (-\Delta)^s u_2 \, dx + \frac{2s-n}{2} \int_{K_2} u_2 (-\Delta)^s u_1 \, dx. \end{aligned}$$

Proof. We claim that

$$(-\Delta)^s(x\cdot\nabla u_i) = x\cdot\nabla(-\Delta)^s u_i + 2s(-\Delta)^s u_i \qquad \text{in } \mathbb{R}^n\backslash K_i. \tag{2.47}$$

Indeed, using $u_i \equiv 0$ in $\mathbb{R}^n\setminus K_i$ and the definition of $(-\Delta)^s$ in (2.2), for each $x\in\mathbb{R}^n\setminus K_i$ we have

$$\begin{aligned}
(-\Delta)^s(x\cdot\nabla u_i)(x) &= c_{n,s}\int_{K_i}\frac{-y\cdot\nabla u_i(y)}{|x-y|^{n+2s}}dy\\
&= c_{n,s}\int_{K_i}\frac{(x-y)\cdot\nabla u_i(y)}{|x-y|^{n+2s}}dy + c_{n,s}\int_{K_i}\frac{-x\cdot\nabla u_i(y)}{|x-y|^{n+2s}}dy\\
&= c_{n,s}\int_{K_i} div_y\left(\frac{x-y}{|x-y|^{n+2s}}\right)u_i(y)dy + x\cdot(-\Delta)^s\nabla u_i(x)\\
&= c_{n,s}\int_{K_i}\frac{-2s}{|x-y|^{n+2s}}u_i(y)dy + x\cdot\nabla(-\Delta)^s u_i(x)\\
&= 2s(-\Delta)^s u_i(x) + x\cdot\nabla(-\Delta)^s u_i(x),
\end{aligned}$$

as claimed.

We also note that for all functions w_1 and w_2 in $L^1(\mathbb{R}^n)$ with disjoint compact supports W_1 and W_2, it holds the integration by parts formula

$$\int_{W_1} w_1(-\Delta)^s w_2 = \int_{W_1}\int_{W_2}\frac{-w_1(x)w_2(y)}{|x-y|^{n+2s}}dy\,dx = \int_{W_2} w_2(-\Delta)^s w_1. \tag{2.48}$$

Using that $(-\Delta)^s u_2$ is smooth in K_1 and integrating by parts,

$$\int_{K_1}(x\cdot\nabla u_1)(-\Delta)^s u_2 = -n\int_{K_1} u_1(-\Delta)^s u_2 - \int_{K_1} u_1 x\cdot\nabla(-\Delta)^s u_2.$$

Next we apply the previous claim and also the integration by parts formula (2.48) to $w_1 = u_1$ and $w_2 = x\cdot\nabla u_2$. We obtain

$$\begin{aligned}
\int_{K_1} u_1 x\cdot\nabla(-\Delta)^s u_2 &= \int_{K_1} u_1(-\Delta)^s(x\cdot\nabla u_2) - 2s\int_{K_1} u_1(-\Delta)^s u_2\\
&= \int_{K_2}(-\Delta)^s u_1(x\cdot\nabla u_2) - 2s\int_{K_1} u_1(-\Delta)^s u_2.
\end{aligned}$$

Hence,

$$\int_{K_1}(x\cdot\nabla u_1)(-\Delta)^s u_2 = -\int_{K_2}(-\Delta)^s u_1(x\cdot\nabla u_2) + (2s-n)\int_{K_1} u_1(-\Delta)^s u_2.$$

Finally, again by the integration by parts formula (2.48) we find

$$\int_{K_1} u_1(-\Delta)^s u_2 = \frac{1}{2}\int_{K_1} u_1(-\Delta)^s u_2 + \frac{1}{2}\int_{K_2} u_2(-\Delta)^s u_1,$$

and the lemma follows. □

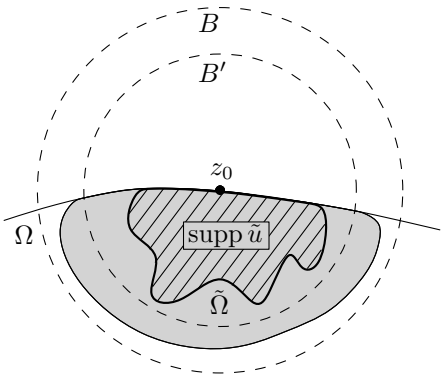

Figure 2.1:

The second lemma states that the bilinear identity (2.46) holds whenever the two functions u_1 and u_2 have compact supports in a ball B such that $\Omega \cap B$ is star-shaped with respect to some point z_0 in $\Omega \cap B$.

Lemma 2.5.2. *Let Ω be a bounded $C^{1,1}$ domain, and let B be a ball in $\mathbb{R}^n$. Assume that there exists $z_0 \in \Omega \cap B$ such that*

$$(x - z_0) \cdot \nu(x) > 0 \qquad \text{for all } x \in \partial\Omega \cap \overline{B}.$$

Let u be a function satisfying the hypothesis of Proposition 2.1.6, and let $u_1 = u\eta_1$ and $u_2 = u\eta_2$, where $\eta_i \in C_c^\infty(B)$, $i = 1, 2$. Then, the following identity holds

$$\int_B (x \cdot \nabla u_1)(-\Delta)^s u_2 \, dx + \int_B (x \cdot \nabla u_2)(-\Delta)^s u_1 \, dx = \frac{2s-n}{2} \int_B u_1 (-\Delta)^s u_2 \, dx +$$

$$+ \frac{2s-n}{2} \int_B u_2 (-\Delta)^s u_1 \, dx - \Gamma(1+s)^2 \int_{\partial\Omega \cap B} \frac{u_1}{\delta^s} \frac{u_2}{\delta^s} (x \cdot \nu) \, d\sigma.$$

Proof. We will show that given $\eta \in C_c^\infty(B)$ and letting $\tilde{u} = u\eta$ it holds

$$\int_B (x \cdot \nabla \tilde{u})(-\Delta)^s \tilde{u} \, dx = \frac{2s-n}{2} \int_B \tilde{u} (-\Delta)^s \tilde{u} \, dx - \Gamma(1+s)^2 \int_{\partial\Omega \cap B} \left(\frac{\tilde{u}}{\delta^s} \right)^2 (x \cdot \nu) d\sigma. \tag{2.49}$$

From this, the lemma follows by applying (2.49) with $\tilde{u}$ replaced by $(\eta_1 + \eta_2)u$ and by $(\eta_1 - \eta_2)u$, and subtracting both identities.

We next prove (2.49). For it, we will apply the result for strictly star-shaped domains, already proven in Section 2.2. Observe that there is a $C^{1,1}$ domain $\tilde{\Omega}$ satisfying

$$\{\tilde{u} > 0\} \subset \tilde{\Omega} \subset \Omega \cap B \quad \text{and} \quad (x - z_0) \cdot \nu(x) > 0 \quad \text{for all } x \in \partial\tilde{\Omega}.$$

This is because, by the assumptions, $\Omega \cap B$ is a Lipschitz polar graph about the point $z_0 \in \Omega \cap B$ and *supp* $\tilde{u} \subset B' \subset\subset B$ for some smaller ball B'; see Figure 2.1. Hence, there is room enough to round the corner that $\Omega \cap B$ has on $\partial\Omega \cap \partial B$.

Hence, it only remains to prove that $\tilde{u}$ satisfies the hypotheses of Proposition 2.1.6. Indeed, since u satisfies (a) and η is $C_c^\infty(B')$ then $\tilde{u}$ satisfies

$$[\tilde{u}]_{C^\beta(\{x \in \tilde{\Omega} \,:\, \tilde{\delta}(x) > \rho\})} \leq C\rho^{s-\beta}$$

for all $\beta \in [s, 1+2s)$, where $\tilde{\delta}(x) = dist(x, \partial\tilde{\Omega})$.

On the other hand, since u satisfies (b) and we have $\eta\delta^s/\tilde{\delta}^s$ is Lipschitz in $supp\,\tilde{u}$ —because $dist(x, \partial\tilde{\Omega} \setminus \partial\Omega) \geq c > 0$ for all $x \in supp\,\tilde{u}$—, then we find

$$\left[\tilde{u}/\tilde{\delta}^s\right]_{C^\beta(\{x\in\tilde{\Omega}\,:\,\tilde{\delta}(x)>\rho\})} \leq C\rho^{\alpha-\beta}$$

for all $\beta \in [\alpha, s+\alpha]$.

Let us see now that $\tilde{u}$ satisfies (c), i.e., that $(-\Delta)^s\tilde{u}$ is bounded. For it, we use

$$(-\Delta)^s(u\eta) = \eta(-\Delta)^s u + u(-\Delta)^s\eta - I_s(u,\eta)$$

where I_s is given by (2.26), i.e.,

$$I_s(u,\eta)(x) = c_{n,s}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}}\,dy\,.$$

The first term is bounded since $(-\Delta)^s u$ so is by hypothesis. The second term is bounded since $\eta \in C_c^\infty(\mathbb{R}^n)$. The third term is bounded because $u \in C^s(\mathbb{R}^n)$ and $\eta \in Lip(\mathbb{R}^n)$.

Therefore, $\tilde{u}$ satisfies the hypotheses of Proposition 2.1.6 with Ω replaced by $\tilde{\Omega}$, and (2.49) follows taking into account that for all $x_0 \in \partial\tilde{\Omega} \cap supp\,\tilde{u} = \partial\Omega \cap supp\,\tilde{u}$ we have

$$\lim_{x\to x_0,\,x\in\tilde{\Omega}} \frac{\tilde{u}(x)}{\tilde{\delta}^s(x)} = \lim_{x\to x_0,\,x\in\Omega} \frac{\tilde{u}(x)}{\delta^s(x)}.$$

□

We now give the

Proof of Proposition 2.1.6. Let $B_1, ..., B_m$ be balls of radius $r > 0$ covering $\overline{\Omega}$. By regularity of the domain, if r is small enough, for each i, j such that $\overline{B_i} \cap \overline{B_j} \neq \varnothing$ there exists a ball B containing $B_i \cup B_j$ and a point $z_0 \in \Omega \cap B$ such that

$$(x - z_0)\cdot\nu(x) > 0 \qquad \text{for all } x \in \partial\Omega \cap B.$$

Let $\{\psi_k\}_{k=1,...,m}$ be a partition of the unity subordinated to $B_1, ..., B_m$, that is, a set of smooth functions $\psi_1, ..., \psi_m$ such that $\psi_1 + \cdots + \psi_m = 1$ in Ω and that ψ_k has compact support in B_k for each $k = 1, ..., m$. Define $u_k = u\psi_k$.

Now, for each $i, j \in \{1, ..., m\}$, if $\overline{B_i} \cap \overline{B_j} = \varnothing$ we use Lemma 2.5.1, while if $\overline{B_i} \cap \overline{B_j} \neq \varnothing$ we use Lemma 2.5.2. We obtain

$$\begin{aligned}\int_\Omega (x\cdot\nabla u_i)(-\Delta)^s u_j\,dx + \int_\Omega (x\cdot\nabla u_j)(-\Delta)^s u_i\,dx = \frac{2s-n}{2}\int_\Omega u_i(-\Delta)^s u_j\,dx + \\ + \frac{2s-n}{2}\int_\Omega u_j(-\Delta)^s u_i\,dx - \Gamma(1+s)^2\int_{\partial\Omega}\frac{u_i}{\delta^s}\frac{u_j}{\delta^s}(x\cdot\nu)\,d\sigma\end{aligned}$$

for each $1 \leq i \leq m$ and $1 \leq j \leq m$. Therefore, adding these identities for $i = 1, ..., m$ and for $j = 1, ..., m$ and taking into account that $u_1 + \cdots + u_m = u$, we find

$$\int_\Omega (x\cdot\nabla u)(-\Delta)^s u\,dx = \frac{2s-n}{2}\int_\Omega u(-\Delta)^s u\,dx - \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2(x\cdot\nu)\,d\sigma,$$

and the proposition is proved. □

To end this section we prove Theorem 2.1.1, Proposition 2.1.12, Theorem 2.1.9, and Corollaries 2.1.2, 2.1.3, and 2.1.13.

Proof of Proposition 2.1.12 and Theorem 2.1.1. By Theorem 2.1.4, any solution u to problem (2.8) satisfies the hypothesis of Proposition 2.1.6. Hence, using this proposition and that $(-\Delta)^s u = f(x,u)$, we obtain

$$\int_\Omega (\nabla u \cdot x) f(x,u) dx = \frac{2s-n}{2} \int_\Omega u f(x,u) dx + \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x \cdot \nu) d\sigma.$$

On the other hand, note that $(\nabla u \cdot x) f(x,u) = \nabla\left(F(x,u)\right) \cdot x - x \cdot F_x(x,u)$. Then, integrating by parts,

$$\int_\Omega (\nabla u \cdot x) f(x,u) dx = -n \int_\Omega F(x,u) dx - \int_\Omega x \cdot F_x(x,u) dx.$$

If f does not depend on x, then the last term do not appear, as in Theorem 2.1.1. □

Proof of Theorem 2.1.9. As shown in the final part of the proof of Proposition 2.1.6 for strictly star-shaped domains given in Section 2.2, the freedom for choosing the origin in the identity from this proposition leads to

$$\int_\Omega w_{x_i} (-\Delta)^s w \, dx = \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{w}{\delta^s}\right)^2 \nu_i \, d\sigma$$

for each $i = 1, ..., n$. Then, the theorem follows by using this identity with $w = u + v$ and with $w = u - v$ and subtracting both identities. □

Proof of Corollaries 2.1.2, 2.1.3, and 2.1.13. We only have to prove Corollary 2.1.13, since Corollaries 2.1.2 and 2.1.3 follow immediately from it by setting $f(x,u) = f(u)$ and $f(x,u) = |u|^{p-1}u$ respectively.

By hypothesis (2.15), we have

$$\frac{n-2s}{2} \int_\Omega u f(x,u) dx \geq n \int_\Omega F(x,u) dx + \int_\Omega x \cdot F_x(x,u) dx.$$

This, combined with Proposition 2.1.12 gives

$$\int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x \cdot \nu) d\sigma \leq 0.$$

If Ω is star-shaped and inequality in (2.15) is strict, we obtain a contradiction. On the other hand, if inequality in (2.15) is not strict but u is a positive solution of (2.8), then by the Hopf Lemma for the fractional Laplacian (see, for instance, [66] or Lemma 3.2 in [249]) the function u/δ^s is strictly positive in $\overline{\Omega}$, and we also obtain a contradiction. □

2.6 Appendix: Calculation of the constants c_1 and c_2

In Proposition 2.3.2 we have obtained the following expressions for the constants c_1 and c_2:

$$c_1 = c_{1,\frac{s}{2}}, \qquad \textit{and} \qquad c_2 = \int_0^\infty \left\{ \frac{1-x^s}{|1-x|^{1+s}} + \frac{1+x^s}{|1+x|^{1+s}} \right\} dx,$$

where $c_{n,s}$ is the constant appearing in the singular integral expression for $(-\Delta)^s$ in dimension n.

Here we prove that the values of these constants coincide with the ones given in Proposition 2.1.10. We start by calculating c_1.

Proposition 2.6.1. *Let $c_{n,s}$ be the normalizing constant of $(-\Delta)^s$ in dimension n. Then,*

$$c_{1,\frac{s}{2}} = \frac{\Gamma(1+s)\sin\left(\frac{\pi s}{2}\right)}{\pi}.$$

Proof. Recall that

$$c_{n,s} = \frac{s 2^{2s} \Gamma\left(\frac{n+2s}{2}\right)}{\pi^{n/2}\Gamma(1-s)}. \tag{2.50}$$

Thus,

$$c_{1,\frac{s}{2}} = \frac{s 2^{s-1} \Gamma\left(\frac{1+s}{2}\right)}{\sqrt{\pi}\Gamma\left(1-\frac{s}{2}\right)}.$$

Now, using the properties of the Gamma function (see for example [6])

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z) \qquad \text{and} \qquad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

we obtain

$$c_{1,\frac{s}{2}} = \frac{s2^{s-1}}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}\right)} = \frac{s2^{s-1}}{\sqrt{\pi}} \cdot \frac{2^{1-s}\sqrt{\pi}\Gamma(s)}{\pi/\sin\left(\frac{\pi s}{2}\right)} = \frac{s\Gamma(s)\sin\left(\frac{\pi s}{2}\right)}{\pi}.$$

The result follows by using that $z\Gamma(z) = \Gamma(1+z)$. □

Let us now compute the constant c_2.

Proposition 2.6.2. *Let $0 < s < 1$. Then,*

$$\int_0^\infty \left\{ \frac{1-x^s}{|1-x|^{1+s}} + \frac{1+x^s}{|1+x|^{1+s}} \right\} dx = \frac{\pi}{\tan\left(\frac{\pi s}{2}\right)}.$$

For it, we will need some properties of the hypergeometric function ${}_2F_1$, which we prove in the next lemma. Recall that this function is defined as

$${}_2F_1(a,b;c;z) = \sum_{n\geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \qquad \text{for } \ |z|<1,$$

where $(a)_n = a(a+1)\cdots(a+n-1)$, and by analytic continuation in the whole complex plane.

Lemma 2.6.3. *Let ${}_2F_1(a,b;c;z)$ be the ordinary hypergeometric function, and $s \in \mathbb{R}$. Then,*

(i) For all $z \in \mathbb{C}$,

$$\frac{d}{dz}\left\{\frac{z^{s+1}}{s+1}\,{}_2F_1(1+s,1+s;2+s;z)\right\} = \frac{z^s}{(1-z)^{1+s}}.$$

(ii) If $s \in (0,1)$, then

$$\lim_{x\to 1}\left\{\frac{1}{s+1}\,{}_2F_1(1+s,1+s;2+s;x) - \frac{1}{s(1-x)^s}\right\} = -\frac{\pi}{\sin(\pi s)}.$$

(iii) If $s \in (0,1)$, then

$$\lim_{x\to+\infty}\left\{\frac{(-x)^{s+1}}{s+1}\,{}_2F_1(1+s,1+s;2+s;x) - \frac{x^{s+1}}{s+1}\,{}_2F_1(1+s,1+s;2+s;-x)\right\} = i\pi,$$

where the limit is taken on the real line.

Proof. (i) Let us prove the equality for $|z|<1$. In this case,

$$\frac{d}{dz}\left\{\frac{z^{s+1}}{s+1}\,{}_2F_1(1+s,1+s;2+s;z)\right\} = \frac{d}{dz}\sum_{n\geq 0}\frac{(1+s)_n^2}{(2+s)_n}\frac{z^{n+1+s}}{n!(s+1)} =$$
$$= \sum_{n\geq 0}\frac{(1+s)_n}{n!}z^{n+s} = z^s\sum_{n\geq 0}\binom{-1-s}{n}(-z)^n = z^s(1-z)^{-1-s},$$

where we have used that $(2+s)_n = \frac{n+1+s}{1+s}(1+s)_n$ and that $\frac{(a)_n}{n!} = (-1)^n\binom{-a}{n}$. Thus, by analytic continuation the identity holds in $\mathbb{C}$.

(ii) Recall the Euler transformation (see for example [6])

$${}_2F_1(a,b;c;x) = (1-x)^{c-a-b}\,{}_2F_1(c-a,c-b;c;x), \tag{2.51}$$

and the value at $x=1$

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad \text{whenever} \qquad a+b<c. \tag{2.52}$$

Hence,

$$\frac{1}{s+1}\,{}_2F_1(1+s,1+s;2+s;x) - \frac{1}{s(1-x)^s} = \frac{\frac{1}{s+1}\,{}_2F_1(1,1;2+s;x) - \frac{1}{s}}{(1-x)^s},$$

and we can use l'Hôpital's rule,

$$\begin{aligned}
\lim_{x\to 1}\frac{\frac{1}{s+1}\,{}_2F_1(1,1;2+s;x)-\frac{1}{s}}{(1-x)^s} &= \lim_{x\to 1}\frac{\frac{1}{s+1}\frac{d}{dx}\,{}_2F_1(1,1;2+s;x)}{-s(1-x)^{s-1}}\\
&= -\lim_{x\to 1}\frac{(1-x)^{1-s}}{s(s+1)(s+2)}\,{}_2F_1(2,2;3+s;x)\\
&= -\lim_{x\to 1}\frac{1}{s(s+1)(s+2)}\,{}_2F_1(1+s,1+s;3+s;x)\\
&= -\frac{1}{s(s+1)(s+2)}\,{}_2F_1(1+s,1+s;3+s;1)\\
&= -\frac{1}{s(s+1)(s+2)}\frac{\Gamma(3+s)\Gamma(1-s)}{\Gamma(2)\Gamma(2)}\\
&= -\Gamma(s)\Gamma(1-s)\\
&= -\frac{\pi}{\sin(\pi s)}.
\end{aligned}$$

We have used that

$$\frac{d}{dx}\,{}_2F_1(1,1;2+s;x)=\frac{1}{s+2}\,{}_2F_1(2,2;3+s;x),$$

the Euler transformation (2.51), and the properties of the Γ function

$$x\Gamma(x)=\Gamma(x+1),\qquad \Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin(\pi x)}.$$

(iii) In [20] it is proved that

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,{}_2F_1(a,b;a+b;x)=\log\frac{1}{1-x}+R+o(1)\qquad \text{for}\quad x\sim 1, \tag{2.53}$$

where

$$R=-\psi(a)-\psi(b)-\gamma,$$

ψ is the digamma function, and γ is the Euler-Mascheroni constant. Using the Pfaff transformation [6]

$${}_2F_1(a,b;c;x)=(1-x)^{-a}\,{}_2F_1\left(a,c-b;c;\frac{x}{x-1}\right)$$

and (2.53), we obtain

$$\begin{aligned}
\frac{(1-x)^{1+s}}{1+s}\,{}_2F_1(1+s,1+s;2+s;x) &= \frac{1}{1+s}\,{}_2F_1\left(1+s,1;2+s;\frac{x}{x-1}\right)\\
&= \log\frac{1}{1-x}+R+o(1)\quad \text{for}\quad x\sim\infty.
\end{aligned}$$

Thus, it also holds

$$\frac{(-x)^{1+s}}{1+s}\,{}_2F_1(1+s,1+s;2+s;x)=\log\frac{1}{1-x}+R+o(1)\quad \text{for}\quad x\sim\infty,$$

and therefore the limit to be computed is now

$$\lim_{x\to+\infty}\left\{\left(\log\frac{1}{1-x}+R\right)-\left(\log\frac{1}{1+x}+R\right)\right\}=i\pi.$$

□

Next we give the:

Proof of Proposition 2.6.2. Let us compute separately the integrals

$$I_1=\int_0^1\left\{\frac{1-x^s}{|1-x|^{1+s}}+\frac{1+x^s}{|1+x|^{1+s}}\right\}dx$$

and

$$I_2=\int_1^\infty\left\{\frac{1-x^s}{|1-x|^{1+s}}+\frac{1+x^s}{|1+x|^{1+s}}\right\}dx.$$

By Lemma 2.6.3 (i), we have that

$$\int\left\{\frac{1-x^s}{(1-x)^{1+s}}+\frac{1+x^s}{(1+x)^{1+s}}\right\}dx=\frac{1}{s}(1-x)^{-s}-\frac{x^{s+1}}{s+1}\,{}_2F_1(1+s,1+s;2+s;x)$$
$$-\frac{1}{s}(1+x)^{-s}+\frac{x^{s+1}}{s+1}\,{}_2F_1(1+s,1+s;2+s;-x).$$

Hence, using 2.6.3 (ii),

$$I_1=\frac{\pi}{\sin(\pi s)}-\frac{1}{s2^s}+\frac{1}{s+1}\,{}_2F_1(1+s,1+s;2+s;-1).$$

Let us evaluate now I_2. As before, by Lemma 2.6.3 (i),

$$\int\left\{\frac{1-x^s}{(x-1)^{1+s}}+\frac{1+x^s}{(x+1)^{1+s}}\right\}dx=\frac{1}{s}(x-1)^{-s}+(-1)^s\frac{x^{s+1}}{s+1}\,{}_2F_1(1+s,1+s;2+s;x)$$
$$-\frac{1}{s}(1+x)^{-s}+\frac{x^{s+1}}{s+1}\,{}_2F_1(1+s,1+s;2+s;-x).$$

Hence, using 2.6.3 (ii) and (iii),

$$\begin{aligned}I_2 &= -i\pi+(-1)^s\frac{\pi}{\sin(\pi s)}+\frac{1}{s2^s}-\frac{1}{s+1}\,{}_2F_1(1+s,1+s;2+s;-1)\\ &= -i\pi+\cos(\pi s)\frac{\pi}{\sin(\pi s)}+i\sin(\pi s)\frac{\pi}{\sin(\pi s)}+\\ &\qquad+\frac{1}{s2^s}-\frac{1}{s+1}\,{}_2F_1(1+s,1+s;2+s;-1)\\ &= \frac{\pi}{\tan(\pi s)}+\frac{1}{s2^s}-\frac{1}{s+1}\,{}_2F_1(1+s,1+s;2+s;-1).\end{aligned}$$

Finally, adding up the expressions for I_1 and I_2, we obtain

$$\begin{aligned}\int_0^\infty\left\{\frac{1-x^s}{|1-x|^{1+s}}+\frac{1+x^s}{|1+x|^{1+s}}\right\}dx &= \frac{\pi}{\sin(\pi s)}+\frac{\pi}{\tan(\pi s)}=\pi\cdot\frac{1+\cos(\pi s)}{\sin(\pi s)}\\ &= \pi\cdot\frac{2\cos^2\left(\frac{\pi s}{2}\right)}{2\sin\left(\frac{\pi s}{2}\right)\cos\left(\frac{\pi s}{2}\right)}=\frac{\pi}{\tan\left(\frac{\pi s}{2}\right)},\end{aligned}$$

as desired. □

Remark 2.6.4. It follows from Proposition 2.1.11 that the constant appearing in (2.10) (and thus in the Pohozaev identity), $\Gamma(1+s)^2$, is given by

$$c_3 = c_1^2(\pi^2 + c_2^2).$$

We have obtained the value of c_3 by computing explicitly c_1 and c_2. However, an alternative way to obtain c_3 is to exhibit an explicit solution of (2.1) for some nonlinearity f and apply the Pohozaev identity to this solution. For example, when $\Omega = B_1(0)$, the solution of

$$\begin{cases} (-\Delta)^s u &= 1 \quad \text{in } B_1(0) \\ u &= 0 \quad \text{in } \mathbb{R}^n \backslash B_1(0) \end{cases}$$

can be computed explicitly [154, 24]:

$$u(x) = \frac{2^{-2s}\Gamma(n/2)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)}\left(1-|x|^2\right)^s. \tag{2.54}$$

Thus, from the identity

$$(2s-n)\int_{B_1(0)} u\,dx + 2n\int_{B_1(0)} u\,dx = c_3\int_{\partial B_1(0)}\left(\frac{u}{\delta^s}\right)^2 (x\cdot\nu)d\sigma \tag{2.55}$$

we can obtain the constant c_3, as follows.

On the one hand,

$$\begin{aligned}
\int_{B_1(0)} u\,dx &= \frac{2^{-2s}\Gamma(n/2)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)}\int_{B_1(0)}\left(1-|x|^2\right)^s dx \\
&= \frac{2^{-2s}\Gamma(n/2)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)}|S^{n-1}|\int_0^1 r^{n-1}(1-r^2)^s dr \\
&= \frac{2^{-2s}\Gamma(n/2)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)}|S^{n-1}|\frac{1}{2}\int_0^1 r^{n/2-1}(1-r)^s dr \\
&= \frac{2^{-2s}\Gamma(n/2)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)}|S^{n-1}|\frac{1}{2}\frac{\Gamma(n/2)\Gamma(1+s)}{\Gamma(n/2+1+s)},
\end{aligned}$$

where we have used the definition of the Beta function

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$$

and the identity

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

On the other hand,

$$\int_{\partial B_1(0)}\left(\frac{u}{\delta^s}\right)^2 (x\cdot\nu)d\sigma = \left(\frac{2^{-2s}\Gamma(n/2)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)}\right)^2 |S^{n-1}|2^{2s}.$$

Thus, (2.55) is equivalent to

$$(n+2s)\frac{2^{-2s}\Gamma(n/2)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)}\frac{1}{2}\frac{\Gamma(n/2)\Gamma(1+s)}{\Gamma(n/2+1+s)} = c_3\left(\frac{2^{-2s}\Gamma(n/2)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)}\right)^2 2^{2s}.$$

Hence, after some simplifications,

$$c_3 = \frac{\Gamma(1+s)^2}{\Gamma(n/2+1+s)}\frac{n+2s}{2}\Gamma\left(\frac{n+2s}{2}\right),$$

and using that

$$z\Gamma(z) = \Gamma(1+z)$$

one finally obtains

$$c_3 = \Gamma(1+s)^2,$$

as before.

Chapter Three

NONEXISTENCE RESULTS FOR NONLOCAL EQUATIONS WITH CRITICAL AND SUPERCRITICAL NONLINEARITIES

We prove nonexistence of nontrivial bounded solutions to some nonlinear problems involving nonlocal operators of the form

$$Lu(x) = -\sum a_{ij}\partial_{ij}u + PV\int_{\mathbb{R}^n}(u(x)-u(x+y))K(y)dy.$$

These operators are infinitesimal generators of symmetric Lévy processes. Our results apply to even kernels K satisfying that $K(y)|y|^{n+\sigma}$ is nondecreasing along rays from the origin, for some $\sigma \in (0,2)$ in case $a_{ij} \equiv 0$ and for $\sigma = 2$ in case that (a_{ij}) is a positive definite symmetric matrix.

Our nonexistence results concern Dirichlet problems for L in star-shaped domains with critical and supercritical nonlinearities (where the criticality condition is in relation to n and σ).

We also establish nonexistence of bounded solutions to semilinear equations involving other nonlocal operators such as the higher order fractional Laplacian $(-\Delta)^s$ (here $s > 1$) or the fractional p-Laplacian. All these nonexistence results follow from a general variational inequality in the spirit of a classical identity by Pucci and Serrin.

3.1 Introduction and results

The aim of this paper is to prove nonexistence results for the following type of nonlinear problems

$$\begin{cases} Lu &= f(x,u) & \text{in } \Omega \\ u &= 0 & \text{in } \mathbb{R}^n\backslash\Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, f is a critical or supercritical nonlinearity (as defined later), and L is an integro-differential elliptic operator. Our main results concern operators of the form

$$Lu(x) = PV\int_{\mathbb{R}^n}\big(u(x)-u(x+y)\big)K(y)dy \tag{3.2}$$

and

$$Lu(x) = -\sum_{i,j} a_{ij}\partial_{ij}u + PV \int_{\mathbb{R}^n} \big(u(x) - u(x+y)\big)K(y)dy, \tag{3.3}$$

where (a_{ij}) is a positive definite matrix (independent of $x \in \Omega$) and K is a nonnegative kernel satisfying

$$K(y) = K(-y) \quad \text{and} \quad \int_{\mathbb{R}^n} \frac{|y|^2}{1+|y|^2} K(y)dy < \infty. \tag{3.4}$$

These operators are infinitesimal generators of symmetric Lévy processes.

We will state two different nonexistence results, one corresponding to (3.2) and the other to (3.3).

On the one hand, we consider operators (3.2) that may not have a definite order but only satisfy, for some $\sigma \in (0,2)$,

$$K(y)|y|^{n+\sigma} \quad \text{is nondecreasing along rays from the origin.} \tag{3.5}$$

Heuristically, (3.5) means that even if the order is not defined, σ acts as an upper bound for the order of the operator —see Section 3.2 for some examples. For these operators we prove, under some additional technical assumptions on the kernel, nonexistence of nontrivial bounded solutions to (3.1) in star-shaped domains for supercritical nonlinearities. When $f(x,u) = |u|^{q-1}u$, the critical power for this class of operators is $q = \frac{n+\sigma}{n-\sigma}$.

On the other hand, we establish the analogous result for second order integro-differential elliptic operators (3.3) with kernels K satisfying (3.5) with $\sigma = 2$. In this case, the critical power is $q = \frac{n+2}{n-2}$.

Moreover, we can use the same ideas to prove an abstract variational inequality that applies to more general problems. For instance, we can obtain nonexistence results for semilinear equations involving the higher order fractional Laplacian $(-\Delta)^s$ (i.e., with $s > 1$) or the fractional p-Laplacian.

When L is the Laplacian $-\Delta$, the nonexistence of nontrivial solutions to (3.1) for critical and supercritical nonlinearities in star-shaped domains follows from the celebrated Pohozaev identity [237]. For positive solutions, this result can also be proved with the moving spheres method [279, 242]. For more general elliptic operators (such as the p-Laplacian, the bilaplacian Δ^2, or k-hessian operators), the nonexistence of regular solutions usually follows from Pohozaev-type or Pucci-Serrin identities [240].

When L is the fractional Laplacian $(-\Delta)^s$ with $s \in (0,1)$, which corresponds to $K(y) = c_{n,s}|y|^{-n-2s}$ in (3.2), this nonexistence result for problem (3.1) was first obtained by Fall-Weth for positive solutions [129] (by using the moving spheres method). In $C^{1,1}$ domains, the nonexistence of nontrivial solutions (not necessarily positive) can be deduced from the Pohozaev identity for the fractional Laplacian, recently established by the authors in [250, 248].

Both the local operator $-\Delta$ and the nonlocal operator $(-\Delta)^s$ satisfy a property of invariance under scaling. More precisely, denoting $w_\lambda(x) = w(\lambda x)$, these operators satisfy $Lw_\lambda(x) = \lambda^\sigma Lw(\lambda x)$, with $\sigma = 2$ in case $L = -\Delta$ and $\sigma = 2s$ in case $L = (-\Delta)^s$. These scaling exponents are strongly related to the critical powers $q = \frac{n+2}{n-2}$ and $q = \frac{n+2s}{n-2s}$ obtained for power nonlinearities $f(x,u) = |u|^{q-1}u$ in (3.1).

Here, we prove a nonexistence result for problem (3.1) with operators L that may not satisfy a scale invariance condition but satisfy (3.5) instead. Our arguments are in the same philosophy as Pucci-Serrin [240], where they proved a general variational identity that applies to many second order problems. Here, we prove a variational inequality that applies to the previous integro-differential problems.

Before stating our results recall that, given $\sigma > 0$ and $\Omega \subset \mathbb{R}^n$, the nonlinearity $f \in C^{0,1}_{loc}(\overline{\Omega} \times \mathbb{R})$ is said to be *supercritical* if

$$\frac{n-\sigma}{2}\, t\, f(x,t) > nF(x,t) + x \cdot F_x(x,t) \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad t \neq 0, \tag{3.6}$$

where $F(x,t) = \int_0^t f(x,\tau)d\tau$. When $f(x,u) = |u|^{q-1}u$, this corresponds to $q > \frac{n+\sigma}{n-\sigma}$.

As explained later on in this Introduction, by bounded solution of (3.1) we mean a critical point $u \in L^\infty(\Omega)$ of the associated energy functional.

Our first nonexistence result reads as follows. Note that it applies not only to positive solutions but also to changing-sign ones.

In the first two parts of the theorem, we assume the solution u to be $W^{1,r}$ for some $r > 1$. This is a natural assumption that is satisfied when L is a pure fractional Laplacian and also for those operators L with kernels K satisfying an additional assumption on its "order", as stated in part (c).

Theorem 3.1.1. *Let K be a nonnegative kernel satisfying* (3.4), (3.5) *for some $\sigma \in (0,2)$, and*

$$K \text{ is } C^1(\mathbb{R}^n \setminus \{0\}) \text{ and } |\nabla K(y)| \le C\,\frac{K(y)}{|y|} \quad \text{for all } y \neq 0 \tag{3.7}$$

for some constant C. Let L be given by (3.2). *Let $\Omega \subset \mathbb{R}^n$ be any bounded star-shaped domain, and $f \in C^{0,1}_{loc}(\overline{\Omega} \times \mathbb{R})$ be a supercritical nonlinearity, i.e., satisfying* (3.6). *Let u be any bounded solution of* (3.1). *The following statements hold:*

(a) If $u \in W^{1,r}(\Omega)$ for some $r > 1$, then $u \equiv 0$.

(b) Assume that $K(y)|y|^{n+\sigma}$ is not constant along some ray from the origin, and that the nonstrict inequality

$$\frac{n-\sigma}{2}\, t\, f(x,t) \ge nF(x,t) + x \cdot F_x(x,t) \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad t \in \mathbb{R} \tag{3.8}$$

holds instead of (3.6). *If $u \in W^{1,r}(\Omega)$ for some $r > 1$, then $u \equiv 0$.*

(c) Assume that in addition Ω is convex, that the kernel K satisfies

$$K(y)|y|^{n+\epsilon} \quad \text{is nonincreasing along rays from the origin} \tag{3.9}$$

for some $\epsilon \in (0,\sigma)$, and that

$$\max_{\partial B_r} K(y) \le C \min_{\partial B_r} K(y) \quad \text{for all } r \in (0,1) \tag{3.10}$$

for some constant C. Then, $u \in W^{1,r}(\Omega)$ for some $r > 1$, and therefore statements (a) and (b) hold without the assumption $u \in W^{1,r}(\Omega)$.

Note that in part (c) we have the additional assumption that the domain Ω is convex. This is used to prove the $W^{1,r}$ regularity of bounded solutions to (3.1) (and it is not needed for example when the operator is the fractional Laplacian, see Remark 3.6.7). Note also that condition (3.5) means in some sense that L has order at most σ, while (3.9) means that L is at least of order ϵ for some small $\epsilon > 0$.

Some examples to which our result applies are sums of fractional Laplacians of different orders, anisotropic operators (i.e., with nonradial kernels), and also operators whose kernels have a singularity different of a power at the origin. More examples are given in Section 3.2.

Note that for $f(x,u) = |u|^{q-1}u$, part (a) gives nonexistence for supercritical powers $q > \frac{n+\sigma}{n-\sigma}$, while part (b) establishes nonexistence also for the critical power $q = \frac{n+\sigma}{n-\sigma}$. The nonexistence of nontrivial solutions for the critical power in case that $K(y)|y|^{n+\sigma}$ is constant along all rays from the origin remains an open problem. Even for the fractional Laplacian $(-\Delta)^s$, this has been only established for positive solutions, and it is not known for changing-sign solutions.

The existence of nontrivial solutions in (3.1) for subcritical nonlinearities was obtained by Servadei and Valdinoci [268] by using the mountain pass theorem. Their result applies to nonlocal operators of the form (3.2) with symmetric kernels K satisfying $K(y) \geq \lambda |y|^{-n-\sigma}$.

As stated in Theorem 3.1.1, the additional hypotheses of part (c) lead to the $W^{1,r}(\Omega)$ regularity of bounded solutions for some $r > 1$. This is a consequence of the following proposition.

Proposition 3.1.2. *Let $\Omega \subset \mathbb{R}^n$ be any bounded and convex domain. Let L be an operator satisfying the hypotheses of Theorem 3.1.1 (c), i.e., satisfying* (3.2), (3.4), (3.5), (3.7), (3.9), *and* (3.10). *Let $f \in C^{0,1}_{loc}(\overline{\Omega} \times \mathbb{R})$, and let u be any bounded solution of* (3.1). *Then,*

$$\|u\|_{C^{\epsilon/2}(\mathbb{R}^n)} \leq C \qquad \text{and} \qquad |\nabla u(x)| \leq C\delta(x)^{\frac{\epsilon}{2}-1} \quad \text{in} \quad \Omega, \tag{3.11}$$

where $\delta(x) = dist(x, \partial\Omega)$ and C is a constant that depends only on Ω, ϵ, σ, f, and $\|u\|_{L^\infty(\Omega)}$.

Note that (3.11) and the fact that Ω is convex imply $u \in W^{1,r}(\Omega)$ for all $1 < r < \frac{1}{1-\epsilon/2}$. In (3.11) the exponents $\epsilon/2$ are optimal, as seen when $L = (-\Delta)^{\epsilon/2}$ (see [254]).

Our second nonexistence result, stated next, deals with operators of the form (3.3). Here, the additional assumptions on Ω and K leading to the $W^{1,r}$ regularity of solutions are not needed thanks to the presence of the second order constant coefficients regularizing term.

Theorem 3.1.3. *Let L be an operator of the form* (3.3), *where (a_{ij}) is a positive definite symmetric matrix and K is a nonnegative kernel satisfying* (3.4). *Assume in addition that* (3.7) *holds, and that*

$$K(y)|y|^{n+2} \text{ is nondecreasing along rays from the origin.} \tag{3.12}$$

Let $\Omega \subset \mathbb{R}^n$ be any bounded star-shaped domain, $f \in C^{0,1}_{loc}(\overline{\Omega} \times \mathbb{R})$, and u be any bounded solution of (3.1). *If* (3.8) *holds with $\sigma = 2$, then $u \equiv 0$.*

Note that for $f(x,u) = |u|^{q-1}u$ we obtain nonexistence for critical and supercritical powers $q \geq \frac{n+2}{n-2}$.

The proofs of Theorems 3.1.1 and 3.1.3 follow some ideas introduced in our proof of the Pohozaev identity for the fractional Laplacian [250]. The key ingredient in all these proofs is the scaling properties both of the bilinear form associated to L and of the potential energy associated to f. These two terms appear in the variational formulation of (3.1), as explained next.

Recall that solutions to problem (3.1), with L given by (3.2) or (3.3), are critical points of the functional

$$\mathcal{E}(u) = \frac{1}{2}(u,u) - \int_\Omega F(x,u) \tag{3.13}$$

among all functions u satisfying $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. Here, $F(x,u) = \int_0^u f(x,t)dt$, and $(\cdot,\cdot)$ is the bilinear form associated to L. More precisely, in case that L is given by (3.2), we have

$$(u,v) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(u(x) - u(x+y)\big)\big(v(x) - v(x+y)\big)K(y)dx\,dy, \tag{3.14}$$

while in case that L is given by (3.3), we have

$$(u,v) = \int_\Omega A(\nabla u, \nabla v)dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(u(x)-u(x+y)\big)\big(v(x)-v(x+y)\big)K(y)dx\,dy, \tag{3.15}$$

where $A(p,q) = p^T Aq$ and $A = (a_{ij})$ is the matrix in (3.3).

Both Theorems 3.1.1 and 3.1.3 are particular cases of the more general result that we state next. This result establishes nonexistence of bounded solutions $u \in W^{1,r}(\Omega)$, $r > 1$, to problems of the form (3.1) with variational operators L satisfying a scaling inequality.

Proposition 3.1.4. *Let E be a Banach space contained in $L^1_{loc}(\mathbb{R}^n)$, and $\|\cdot\|$ be a seminorm in E. Assume that for some $\alpha > 0$ the seminorm $\|\cdot\|$ satisfies*

$$w_\lambda \in E \quad \text{and} \quad \|w_\lambda\| \leq \lambda^{-\alpha}\|w\| \ \text{ for every } w \in E \text{ and } \lambda > 1, \tag{3.16}$$

where $w_\lambda(x) = w(\lambda x)$.

Let $\Omega \subset \mathbb{R}^n$ be any bounded star-shaped domain with respect to the origin, $p > 1$, and $f \in C^{0,1}_{loc}(\overline{\Omega} \times \mathbb{R})$. Consider the energy functional

$$\mathcal{E}(u) = \frac{1}{p}\|u\|^p - \int_\Omega F(x,u), \tag{3.17}$$

where $F(x,u) = \int_0^u f(x,t)dt$, and let u be a critical point of $\mathcal{E}$ among all functions $u \in E$ satisfying $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$.

Assume that f is supercritical, in the sense that

$$\alpha t\, f(x,t) > nF(x,t) + x \cdot F_x(x,t) \quad \text{for all } \ x \in \Omega \ \text{ and } \ t \neq 0. \tag{3.18}$$

If $u \in L^\infty(\Omega) \cap W^{1,r}(\Omega)$ for some $r > 1$, then $u \equiv 0$.

Some examples to which this result applies are second order variational operators such as the Laplacian or the p-Laplacian, the nonlocal operators in Theorems 3.1.1 or 3.1.3, or the higher order fractional Laplacian $(-\Delta)^s$ (here $s > 1$). See Section 3.2 for more examples.

Remark 3.1.5. Proposition 3.1.4 establishes nonexistence of nontrivial bounded solutions belonging to $W^{1,r}(\Omega)$, $r > 1$. In general, removing the $W^{1,r}$ assumption may be done in two different situations:

First, it may happen that the space $E_\Omega = \{u \in E : u \equiv 0 \text{ in } \mathbb{R}^n \setminus \Omega\}$ is embedded in $W^{1,r}(\Omega)$, $r > 1$. This happens for instance when considering the natural functional spaces associated to the Laplacian, the p-Laplacian with $p > 1$, the higher order fractional Laplacian $(-\Delta)^s$ (with $s \geq 1$), and of the nonlocal operators considered in Theorem 3.1.3.

Second, even if the space E_Ω is not embedded in $W^{1,r}$, it is often the case that by some regularity estimates one can prove that critical points of (3.17) belong to $W^{1,r}$, $r > 1$. This occurs when the operator if the fractional Laplacian, and also in Theorem 3.1.1 (c), thanks to Proposition 3.1.2.

As said before, for local operators of order 2, the nonexistence of regular solutions usually follows from Pohozaev-type or Pucci-Serrin identities [240]. Our proofs are in the spirit of these identities. However, for nonlocal operators this type of identity is only known for the fractional Laplacian $(-\Delta)^s$ with $s \in (0, 1)$ [250], and requires a precise knowledge of the boundary behavior of solutions to (3.1) [254] (that are not available for most L). To overcome this, instead of proving an identity we prove an inequality which is sufficient to prove nonexistence. This approach allows us to require much less regularity on the solution u and, thus, to include a wide class of operators in our results.

The paper is organized as follows. In Section 3.2 we give a list of examples of operators to which our results apply. In Section 3.3 we present the main ideas appearing in the proofs of our results. In Section 3.4 we prove Proposition 3.1.4. In Section 3.5 we prove Theorems 3.1.1 and 3.1.3. Finally, in Section 3.6 we prove Proposition 3.1.2.

3.2 Examples

In this Section we give a list of examples to which our results apply.

(i) First, note that if $K_1, ..., K_m$ are kernels satisfying the hypotheses of Theorem 3.1.1, and $a_1, ..., a_m$ are nonnegative numbers, then $K = a_1K_1 + \cdots + a_mK_m$ also satisfies the hypotheses. In particular, our nonexistence result applies to operators of the form

$$L = a_1(-\Delta)^{\alpha_1} + \cdots + a_m(-\Delta)^{\alpha_m},$$

with $a_i \geq 0$ and $\alpha_i \in (0, 1)$. The critical exponent is $q = \frac{n+2\max\alpha_i}{n-2\max\alpha_i}$.

(ii) Theorem 3.1.1 may be applied to anisotropic operators L of the form (3.2) with nonradial kernels such as

$$K(y) = H(y)^{-n-\sigma},$$

where H is any homogeneous function of degree 1 whose restriction to S^{n-1} is positive and C^1. These operators are infinitesimal generators of σ-stable symmetric Lévy processes. The critical exponent is $q = \frac{n+\sigma}{n-\sigma}$.

(iii) Theorem 3.1.1 applies also to operators with kernels that do not have a power-like singularity at the origin. For example, the one given by the kernel

$$K(y) = \frac{c}{|y|^{n+\sigma} \log\left(2 + \frac{1}{|y|}\right)}, \qquad \sigma \in (0,2),$$

whose singularity at $y = 0$ is comparable to $|y|^{-n-\sigma}\big|\log|y|\big|^{-1}$. In this example we also have that the critical exponent is $q = \frac{n+\sigma}{n-\sigma}$.

Other examples of operators that may not have a definite order are given by infinite sums of fractional Laplacians, such as $L = \sum_{k\geq 1} \frac{1}{k^2}(-\Delta)^{s-\frac{1}{k}}$.

(iv) Theorem 3.1.3 applies to operators such as $L = -\Delta + (-\Delta)^s$, with $s \in (0,1)$, and also anisotropic operators whose nonlocal part is given by nonradial kernels, as in example (ii). For all these operators, the critical power is $q = \frac{n+2}{n-2}$.

(v) One may take in (3.17) the $W^{s,p}(\mathbb{R}^n)$ seminorm

$$\|u\|^p = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy.$$

This leads to nonexistence results for the s-fractional p-Laplacian operator, considered for example in [60, 143]. The critical power for this operator is $q = \frac{n+ps}{n-ps}$.

(vi) Our results can also be used to obtain a generalization of Theorem 8 in [240], where Pucci and Serrin proved nonexistence results for the bilaplacian Δ^2 and the polylaplacian $(-\Delta)^K$, with K positive integer. More precisely, Proposition 3.1.4 can be applied to the $H^s(\mathbb{R}^n)$ seminorm to obtain nonexistence of bounded solutions u to (3.1) with $L = (-\Delta)^s$, $s > 1$. Note that the hypotheses $u \in W^{1,r}(\Omega)$ is always satisfied, since the fractional Sobolev embeddings yield that any function $u \in H^s(\mathbb{R}^n)$ that vanishes outside Ω belongs to $W^{1,r}(\Omega)$ for $r = 2$ (see Remark 3.1.5).

As an example, when $n > 2s$ and $f(u) = \lambda u + |u|^{q-1}u$, one obtains nonexistence of bounded solutions for $\lambda < 0$ and $q \geq \frac{n+2s}{n-2s}$ and also for $\lambda \leq 0$ and $q > \frac{n+2s}{n-2s}$, as in [240].

(vii) Proposition 3.1.4 can be applied to the usual $W^{1,p}(\Omega)$ norm to obtain nonexistence of bounded weak solutions to (3.1) with $L = -\Delta_p$, the p-Laplacian. These nonexistence results were obtained by Otani in [232] via a Pohozaev-type inequality.

More generally, we may consider nonlinear anisotropic operators that come from setting

$$\|u\|^p = \int_\Omega H(\nabla u)^p |x|^\gamma dx$$

in (3.17), where H is any norm in $\mathbb{R}^n$. In this case, the critical power is $q = \frac{n+\gamma+p}{n+\gamma-p}$. For $\gamma = 0$, some problems involving this class of operators were studied in [17, 132, 112]. For $\gamma \neq 0$, nonexistence results for these type of problems were studied in [2].

(viii) From Proposition 3.1.4 one may obtain also nonexistence results for k-Hessian operators $S_k(D^2u)$ with $2k < n$. Recall that $S_k(D^2u)$ are defined in terms of the elementary symmetric polynomials acting on the eigenvalues of D^2u, and that these are variational operators. In the two extreme cases $k = 1$ and $k = n$, we have $S_1(D^2u) = \Delta u$ and $S_n(D^2u) = \det D^2u$.

Tso studied this problem in [292], and obtained nonexistence of solutions $u \in C^4(\Omega) \cap C^1(\overline{\Omega})$ in smooth star-shaped domains via a Pohozaev identity. Our results give only nonexistence for supercritical powers $q > \frac{(n+2)k}{n-2k}$, and not for the critical one. As a counterpart, we only need to assume the solution u to be $L^\infty(\Omega) \cap W^{1,r}(\Omega)$.

3.3 Sketch of the proof

In this section we sketch the proof of the nonexistence of critical points to functionals of the form

$$\mathcal{E}(u) = \frac{1}{2}(u,u) + \int_\Omega F(u), \tag{3.19}$$

where $(\cdot,\cdot)$ is a bilinear form satisfying, for some $\alpha > 0$,

$$u_\lambda \in E \quad \text{and} \quad \|u_\lambda\| := (u_\lambda, u_\lambda)^{1/2} \leq \lambda^{-\alpha}(u,u)^{1/2} \ \text{ for all } \ \lambda \geq 1, \tag{3.20}$$

where $u_\lambda(x) = u(\lambda x)$. Of course, this is a particular case of Proposition 3.1.4, in which $p = 2$, E is a Hilbert space, and f does not depend on x. Note that in this case condition (3.16) reads as (3.20). In case of Theorems 3.1.1 and 3.1.3, the bilinear form is given by (3.14) and (3.15), respectively.

The proof goes as follows. Since u is a critical point of (3.19), then we have that

$$(u,\varphi) = \int_\Omega f(u)\varphi\,dx \quad \text{for all } \varphi \in E \text{ satisfying } \varphi \equiv 0 \text{ in } \mathbb{R}^n \setminus \Omega.$$

Next we use $\varphi = u_\lambda$, with $\lambda > 1$, as a test function. Note that, by (3.20), we have $u_\lambda \in E$, and since Ω is star-shaped, then $u_\lambda \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. Hence u_λ is indeed an admissible test function. We obtain

$$(u,u_\lambda) = \int_\Omega f(u)u_\lambda\,dx \ \text{ for all } \ \lambda \geq 1. \tag{3.21}$$

Now, we differentiate with respect to λ in both sides of (3.21). On the one hand, since $u \in L^\infty(\Omega) \cap W^{1,r}(\Omega)$, one can show —see Lemma 3.4.2— that

$$\frac{d}{d\lambda}\bigg|_{\lambda=1^+} \int_\Omega f(u)u_\lambda\,dx = \int_\Omega (x\cdot\nabla u)f(u)\,dx = \int_\Omega x\cdot\nabla F(u)dx = -n\int_\Omega F(u)dx.$$

On the other hand,

$$\left.\frac{d}{d\lambda}\right|_{\lambda=1^+}(u,u_\lambda)=\left.\frac{d}{d\lambda}\right|_{\lambda=1^+}\left\{\lambda^{-\alpha}I_\lambda\right\}=-\alpha(u,u)+\left.\frac{d}{d\lambda}\right|_{\lambda=1^+}I_\lambda,$$

where

$$I_\lambda=\lambda^\alpha(u,u_\lambda). \tag{3.22}$$

We now claim that

$$\left.\frac{d}{d\lambda}\right|_{\lambda=1^+}I_\lambda\le 0. \tag{3.23}$$

Indeed, using (3.20) and the Cauchy-Schwarz inequality, we deduce

$$I_\lambda\le\lambda^\alpha\|u\|\|u_\lambda\|\le\|u\|^2=I_1,$$

and thus (3.23) follows. Therefore, we find

$$-n\int_\Omega F(u)dx=-\alpha\,(u,u)+\left.\frac{d}{d\lambda}\right|_{\lambda=1^+}I_\lambda\le-\alpha\,(u,u),$$

and since $(u,u)=\int_\Omega uf(u)dx$,

$$\int_\Omega uf(u)dx\le\frac{n}{\alpha}\int_\Omega F(u)dx.$$

From this, the nonexistence of nontrivial solutions for supercritical nonlinearities follows immediately.

In case of Theorem 3.1.1 (b) and Theorem 3.1.3, with a little more effort we will be able to prove that (3.23) holds with strict inequality, and this will yield the nonexistence result for critical nonlinearities.

When the previous bilinear form is invariant under scaling, in the sense that (3.20) holds with an equality instead of an inequality, then one has $I_\lambda=(u_{\sqrt{\lambda}},u_{1/\sqrt{\lambda}})$. In the case $L=(-\Delta)^s$, it is proven in [250] that

$$\left.\frac{d}{d\lambda}\right|_{\lambda=1^+}I_\lambda=\Gamma(1+s)\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2(x\cdot\nu)dS,$$

where $\delta(x)=\mathrm{dist}(x,\partial\Omega)$. This gives the boundary term in the Pohozaev identity for the fractional Laplacian.

Remark 3.3.1. This method can also be used to prove nonexistence results in star-shaped domains with respect to infinity or in the whole space $\Omega=\mathbb{R}^n$. However, one need to assume some decay on u and its gradient ∇u, which seems a quite restrictive hypothesis. More precisely, when $f(u)=|u|^{q-1}u$ and the operator is the fractional Laplacian $(-\Delta)^s$, this proof yields nonexistence of bounded solutions (decaying at infinity) for subcritical nonlinearities $q<\frac{n+2s}{n-2s}$ in star-shaped domains with respect to infinity, and for noncritical nonlinearities $q\neq\frac{n+2s}{n-2s}$ in the whole $\mathbb{R}^n$. The classification of entire solutions in $\mathbb{R}^n$ for the critical power $q=\frac{n+2s}{n-2s}$ was obtained in [94].

3.4 Proof of Proposition 3.1.4

In this section we prove Proposition 3.1.4. For it, we will need the following lemma, which can be viewed as a Hölder-type inequality in normed spaces. For example, for $\|u\| = \left(\int_{\mathbb{R}^n} |u|^p\right)^{1/p}$, we recover the usual Hölder inequality (assuming that the Minkowski inequality holds).

Lemma 3.4.1. *Let E be a normed space, and $\|\cdot\|$ a seminorm in E. Let $p > 1$, and define $\Phi = \frac{1}{p}\|\cdot\|^p$. Assume that Φ is Gateaux differentiable at $u \in E$, and let $D\Phi(u)$ be the Gateaux differential of Φ at u. Then, for all v in E,*

$$\langle D\Phi(u), v\rangle \leq p\,\Phi(u)^{1/p'}\,\Phi(v)^{1/p},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, equality holds whenever $v = u$.

Proof. Since $\Phi^{1/p}$ is a seminorm, then by the triangle inequality we find that

$$\Phi(u+\varepsilon v) \leq \left\{\Phi(u)^{1/p} + \varepsilon\,\Phi(v)^{1/p}\right\}^p$$

for all u and v in E and for all $\varepsilon \in \mathbb{R}$. Hence, since these two quantities coincide for $\varepsilon = 0$, we deduce

$$\langle D\Phi(u), v\rangle = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \Phi(u+\varepsilon v) \leq \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \left\{\Phi(u)^{1/p} + \varepsilon\,\Phi(v)^{1/p}\right\}^p = p\,\Phi(u)^{1/p'}\Phi(v)^{1/p},$$

and the lemma follows. □

Before giving the proof of Proposition 3.1.4, we also need the following lemma.

Lemma 3.4.2. *Let $\Omega \subset \mathbb{R}^n$ be any bounded domain, and let $u \in W^{1,r}(\Omega)$, $r > 1$. Then,*

$$\frac{u_\lambda - u}{\lambda - 1} \rightharpoonup x \cdot \nabla u \quad \text{weakly in } L^1(\Omega),$$

where $u_\lambda(x) = u(\lambda x)$.

Proof. Similarly to [124, Theorem 5.8.3], it can be proved that

$$\int_\Omega \left|\frac{u_\lambda - u}{\lambda - 1}\right|^r dx \leq C \int_\Omega |\nabla u|^r dx.$$

Thus, since $1 < r \leq \infty$, then $L^r \cong (L^{r'})'$ and hence there exists a sequence $\lambda_k \to 1$, and a function $v \in L^r(\Omega)$, such that

$$\frac{u_{\lambda_k} - u}{\lambda_k - 1} \rightharpoonup v \quad \text{weakly in } L^r(\Omega).$$

On the other hand note that, for each $\phi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega u\,(x\cdot\nabla\phi)\,dx = -\int_\Omega (x\cdot\nabla u)\,\phi\,dx - n\int_\Omega u\phi\,dx.$$

Moreover, it is immediate to see that, for λ sufficiently close to 1,

$$\int_\Omega u\,\frac{\phi_\lambda-\phi}{\lambda-1}\,dx=-\lambda^{-n-1}\int_\Omega \frac{u_{1/\lambda}-u}{1/\lambda-1}\,\phi\,dx+\frac{\lambda^{-n}-1}{\lambda-1}\int_\Omega u\phi\,dx.$$

Therefore,

$$\begin{aligned}\int_\Omega u\,(x\cdot\nabla\phi)\,dx&=\lim_{k\to\infty}\int_\Omega u\,\frac{\phi_{1/\lambda_k}-\phi}{1/\lambda_k-1}\,dx\\&=\lim_{k\to\infty}-\int_\Omega\frac{u_{\lambda_k}-u}{\lambda_k-1}\,\phi\,dx-n\int_\Omega u\phi\,dx\\&=-\int_\Omega v\phi\,dx-n\int_\Omega u\phi\,dx.\end{aligned}$$

Thus, it follows that $v=x\cdot\nabla u$.

Now, note that this argument yields also that for each sequence $\mu_k\to 1$ there exists a subsequence $\lambda_k\to 1$ such that

$$\frac{u_{\lambda_k}-u}{\lambda_k-1}\rightharpoonup x\cdot\nabla u\quad\text{weakly in } L^r(\Omega).$$

Since this can be done for any sequence μ_k, then this implies that

$$\frac{u_{\lambda}-u}{\lambda-1}\rightharpoonup x\cdot\nabla u\quad\text{weakly in } L^r(\Omega).$$

Finally, since $L^r(\Omega)\subset L^1(\Omega)$, the lemma follows. □

We can now give the:

Proof of Proposition 3.1.4. Define $\Phi=\frac{1}{p}\|\cdot\|^p$. Since u is a critical point of (3.17), then

$$\langle D\Phi(u),\varphi\rangle=\int_\Omega f(x,u)\varphi\,dx \tag{3.24}$$

for all $\varphi\in E$ satisfying $\varphi\equiv 0$ in $\mathbb{R}^n\setminus\Omega$. Since Ω is star-shaped, we may choose $\varphi=u_\lambda$, with $\lambda\geq 1$, as a test function in (3.24). We find

$$\langle D\Phi(u),u_\lambda\rangle=\int_\Omega f(x,u)u_\lambda dx\quad\text{for all }\lambda\geq 1. \tag{3.25}$$

We compute now the derivative with respect to λ at $\lambda=1^+$ in both sides of (3.25). On the one hand, using Lemma 3.4.2 we find that

$$\begin{aligned}\frac{d}{d\lambda}\Big|_{\lambda=1^+}\int_\Omega u_\lambda f(x,u)dx&=\int_\Omega(x\cdot\nabla u)f(x,u)\,dx\\&=\int_\Omega\big\{x\cdot\nabla\big(F(x,u)\big)-x\cdot F_x(x,u)\big\}dx\\&=-\int_\Omega\big\{nF(x,u)+x\cdot F_x(x,u)\big\}dx.\end{aligned}\tag{3.26}$$

Note that here we have used also that $F(x,u) \in W^{1,1}(\Omega)$, which follows from $u \in L^\infty(\Omega)$, $(x\cdot\nabla u)f(x,u) \in L^r(\Omega)$, and $x\cdot F_x(x,u) \in L^\infty$.

On the other hand, let

$$I_\lambda = \lambda^\alpha \langle D\Phi(u), u_\lambda\rangle. \tag{3.27}$$

Then,

$$\begin{aligned}\frac{d}{d\lambda}\Big|_{\lambda=1^+}\langle D\Phi(u),u_\lambda\rangle &= -\alpha\,\langle D\Phi(u),u\rangle + \frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda \\ &= -\alpha\int_\Omega uf(x,u)dx + \frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda,\end{aligned} \tag{3.28}$$

where we have used that $\langle D\Phi(u),u\rangle = \int_\Omega uf(x,u)dx$, which follows from (3.25).

Now, using Lemma 3.4.1 and the scaling condition (3.16), we find

$$\begin{aligned}I_\lambda = \lambda^\alpha\langle D\Phi(u),u_\lambda\rangle &\le p\,\lambda^\alpha \Phi(u)^{1/p'}\Phi(u_\lambda)^{1/p} = \lambda^\alpha\|u\|^{p/p'}\|u_\lambda\| \\ &\le \|u\|^{p/p'+1} = \|u\|^p = p\,\Phi(u) = \langle D\Phi(u),u\rangle = I_1,\end{aligned}$$

where $1/p + 1/p' = 1$. Therefore,

$$\frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda \le 0.$$

Thus, it follows from (3.25), (3.26), and (3.28) that

$$-\int_\Omega \big\{nF(x,u) + x\cdot F_x(x,u)\big\}dx \le -\alpha\int_\Omega uf(x,u)dx,$$

which contradicts (3.18) unless $u \equiv 0$. □

3.5 Proof of Theorems 3.1.1 and 3.1.3

This section is devoted to give the

Proof of Theorem 3.1.1. Recall that u is a weak solution of (3.1) if and only if

$$(u,\varphi) = \int_\Omega f(x,u)\varphi\,dx \tag{3.29}$$

for all φ satisfying $(\varphi,\varphi) < \infty$ and $\varphi \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, where $(\cdot,\cdot)$ is given by (3.14). Note that (3.16) is equivalent to (3.5). Thus, part (a) follows from Proposition 3.1.4, where $\alpha = \frac{n-\sigma}{2}$.

Moreover, it follows from the proof of Proposition 3.1.4 that

$$-\int_\Omega \big\{nF(x,u) + x\cdot F_x(x,u)\big\}dx = \frac{\sigma-n}{2}\int_\Omega uf(x,u)dx + \frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda, \tag{3.30}$$

where

$$I_\lambda = \lambda^{\frac{n-\sigma}{2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big(u(x)-u(x+y)\big)\big(u_\lambda(x)-u_\lambda(x+y)\big)K(y)dx\,dy.$$

Thus, to prove part (b), it suffices to show that

$$\frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda < 0. \tag{3.31}$$

Following the proof of Proposition 3.1.4, by the Cauchy-Schwarz inequality we find

$$\begin{aligned} I_\lambda &\le \lambda^{\frac{n-\sigma}{2}} \|u\| \, \|u_\lambda\| \\ &= \sqrt{I_1} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \big(u(x) - u(x+z)\big)^2 \lambda^{-n-\sigma} K(z/\lambda) dx\, dz \right)^{1/2} \\ &= \frac{I_1}{2} + \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \big(u(x) - u(x+z)\big)^2 \lambda^{-n-\sigma} K(z/\lambda) dx\, dz \\ &\le I_1. \end{aligned}$$

Denote now $K(y) = g(y)/|y|^{n+\sigma}$. Then,

$$\begin{aligned} I_1 - I_\lambda &\ge \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \big(u(x) - u(x+y)\big)^2 \left\{ K(y) - \lambda^{-n-\sigma} K(y/\lambda) \right\} dx\, dy \\ &= \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\big(u(x) - u(x+y)\big)^2}{|y|^{n+\sigma}} \big\{ g(y) - g(y/\lambda) \big\} dx\, dy, \end{aligned}$$

and therefore, by the Fatou lemma

$$-\frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda \ge \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\big(u(x) - u(x+y)\big)^2}{|y|^{n+\sigma}} \, y \cdot \nabla g(y) dx\, dy.$$

Now, recall that $g \in C^1(\mathbb{R}^n \setminus \{0\})$ is nondecreasing along all rays from the origin and nonconstant along some of them. Then, we have that $y \cdot \nabla g(y) \ge 0$ for all y, with strict inequality in a small ball B. This yields that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\big(u(x) - u(x+y)\big)^2}{|y|^{n+\sigma}} \, y \cdot \nabla g(y) dx\, dy > 0$$

unless $u \equiv 0$. Indeed, if $u(x) - u(x+y) = 0$ for all $x \in \mathbb{R}^n$ and $y \in B$ then u is constant in a neighborhood of x, and thus u is constant in all of $\mathbb{R}^n$.

Therefore, using (3.30) we find that if u is a nontrivial bounded solution then

$$\frac{n-\sigma}{2} \int_\Omega u f(x,u) dx < \int_\Omega \big\{ nF(x,u) + x \cdot F_x(x,u) \big\} dx,$$

which is a contradiction with (3.8).

Finally, part (c) follows from (a), (b), and Proposition 3.1.2. □

To end this section, we give the

Proof of Theorem 3.1.3. As explained in the Introduction, weak solutions to problem (3.1) with L given by (3.3) are critical points to (3.17) with $p = 2$ and with

$$\|u\|^2 = \int_\Omega A(\nabla u, \nabla u) dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \big(u(x) - u(x+y)\big)^2 K(y) dx dy,$$

where $A(p,q) = p^T Aq$ and $A = (a_{ij})$ is the matrix in (3.3). It is immediate to see that this norm satisfies (3.16) with $\alpha = \frac{n-2}{2}$ whenever (3.12) holds. Moreover, since A is positive definite by assumption, then $\|u\|_{W^{1,2}(\Omega)} \leq c\|u\|^2$, and hence $u \in W^{1,r}(\Omega)$ with $r = 2$.

Then, it follows from the proof of Proposition 3.1.4 that

$$\frac{n-2}{2}\int_\Omega uf(x,u)dx = \int_\Omega \{nF(x,u) + x\cdot F_x(x,u)\}\,dx + \frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda,$$

where

$$\begin{aligned} I_\lambda = {} & \lambda^{\frac{n-2}{2}}\int_\Omega A(\nabla u, \nabla u_\lambda)dx + \\ & + \lambda^{\frac{n-2}{2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big(u(x)-u(x+y)\big)\big(u_\lambda(x)-u_\lambda(x+y)\big)K(y)dxdy. \end{aligned} \tag{3.32}$$

Now, as in the proof of Theorem 3.1.1, we find

$$I_1 - I_\lambda \geq \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\big(u(x)-u(x+y)\big)^2}{|y|^{n+2}}\big\{g(y)-g(y/\lambda)\big\}dy,$$

where $g(y) = K(y)|y|^{n+2}$. Thus, differentiating with respect to λ, we find that

$$\frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda \geq \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\big(u(x)-u(x+y)\big)^2}{|y|^{n+2}}\,y\cdot\nabla g(y)dy.$$

Moreover, since $\int_{\mathbb{R}^n}\frac{|y|^2}{1+|y|^2}K(y)dy < \infty$ and g is radially nondecreasing, then it follows that $\lim_{t\to 0} g(t\tau) = 0$ for almost all $\tau \in S^{n-1}$. Thus, if K is not identically zero then $y\cdot\nabla g(y)$ is positive in a small ball B, and hence

$$\frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda > 0$$

unless $u \equiv 0$, which yields the desired result. □

3.6 Proof of Proposition 3.1.2

In this section we prove Proposition 3.1.2. To prove it, we follow the arguments used in [254], where we studied the regularity up to the boundary for the Dirichlet problem for the fractional Laplacian. The main ingredients in the proof of this result are the interior estimates of Silvestre [270] and the supersolution given by the next lemma.

Lemma 3.6.1. *Let L be an operator of the form (3.2), with K symmetric, positive, and satisfying (3.9). Let $\psi(x) = (x_n)_+^{\epsilon/2}$. Then,*

$$L\psi \geq 0 \quad \text{in} \quad \mathbb{R}^n_+,$$

where $\mathbb{R}^n_+ = \{x_n > 0\}$.

Proof. Assume first $n = 1$. Let $x \in \mathbb{R}_+$. Since K is symmetric, we have

$$L\psi(x) = \frac{1}{2}\int_{-\infty}^{+\infty} \big(2\psi(x) - \psi(x+y) - \psi(x-y)\big)K(y)dy.$$

Then, it is immediate to see that there exists $\rho > 0$ such that

$$2\psi(x) - \psi(x+y) - \psi(x-y) > 0 \quad \text{for } |y| < \rho$$

and

$$2\psi(x) - \psi(x+y) - \psi(x-y) < 0 \quad \text{for } |y| > \rho.$$

Thus, using that $K(y)|y|^{1+\epsilon}$ is nonincreasing in $(0, +\infty)$, and that $(-\Delta)^{\epsilon/2}\psi = 0$ in $\mathbb{R}_+$, we find

$$\begin{aligned}
L\psi(x) &= \frac{1}{2}\int_{|y|<\rho} \big(2\psi(x) - \psi(x+y) - \psi(x-y)\big)K(y)dy \\
&\qquad + \frac{1}{2}\int_{|y|>\rho} \big(2\psi(x) - \psi(x+y) - \psi(x-y)\big)K(y)dy \\
&\geq \frac{1}{2}\int_{|y|<\rho} \big(2\psi(x) - \psi(x+y) - \psi(x-y)\big)\frac{K(\rho)|\rho|^{1+\epsilon}}{|y|^{1+\epsilon}}dy \\
&\qquad + \frac{1}{2}\int_{|y|>\rho} \big(2\psi(x) - \psi(x+y) - \psi(x-y)\big)\frac{K(\rho)|\rho|^{1+\epsilon}}{|y|^{1+\epsilon}}dy \\
&= K(\rho)|\rho|^{1+\epsilon}\frac{1}{2}\int_{-\infty}^{+\infty} \frac{2\psi(x) - \psi(x+y) - \psi(x-y)}{|y|^{1+\epsilon}}dy \\
&= K(\rho)|\rho|^{1+\epsilon}(-\Delta)^{\epsilon/2}\psi(x) = 0.
\end{aligned}$$

Thus, the lemma is proved for $n = 1$.

Assume now $n > 1$, and let $x \in \mathbb{R}^n_+$. Then,

$$\begin{aligned}
L\psi(x) &= \frac{1}{2}\int_{\mathbb{R}^n} \big(2\psi(x) - \psi(x+y) - \psi(x-y)\big)K(y)dy \\
&= \frac{1}{4}\int_{S^{n-1}} \left(\int_{-\infty}^{+\infty} \big(\psi(x) - \psi(x+t\tau) - \psi(x-t\tau)\big)t^{n-1}K(t\tau)dt\right)d\tau.
\end{aligned} \tag{3.33}$$

Now, for each $\tau \in S^{n-1}$, the kernel $K_1(t) := t^{n-1}K(t\tau)$ satisfies $K_1(t)t^{1+\epsilon}$ is nonincreasing in $(0, +\infty)$, and in addition

$$\psi(x + \tau t) = (x_n + \tau_n t)_+^{\epsilon/2} = \tau_n^{\epsilon/2}(x_n/\tau_n + t)_+^{\epsilon/2}.$$

Thus, by using the result in dimension $n = 1$, we find

$$\int_{-\infty}^{+\infty} \big(\psi(x) - \psi(x+t\tau) - \psi(x-t\tau)\big)t^{n-1}K(t\tau)dt \geq 0. \tag{3.34}$$

Therefore, we deduce from (3.33) and (3.34) that $L\psi(x) \geq 0$ for all $x \in \mathbb{R}^n_+$, and the lemma is proved. □

The following result is the analog of Lemma 2.7 in [254].

Lemma 3.6.2. *Under the hypotheses of Proposition 3.1.2, it holds*

$$|u(x)| \leq C\delta(x)^{\epsilon/2} \quad \text{for all } x \in \Omega,$$

where C is a constant depending only on Ω, ϵ, and $\|u\|_{L^\infty(\Omega)}$.

Proof. By Lemma 3.6.1, we have that $\psi(x) = (x_n)_+^{\epsilon/2}$ satisfies $L\psi \geq 0$ in $\mathbb{R}^n_+$. Thus, we can truncate this $1D$ supersolution in order to obtain a strict supersolution ϕ satisfying $\phi \equiv \psi$ in $\{x_n < 1\}$, $\phi \equiv 1$ in $\{x_n > 1\}$, and $L\phi \geq c_0$ in $\{0 < x_n < 1\}$.

We can now use $C\phi$ as a supersolution at each point of the boundary $\partial\Omega$ to deduce $|u| \leq C\delta^{\epsilon/2}$ in Ω; see Lemma 2.7 in [254] for more details. □

We next prove the following result, which is the analog of Proposition 2.3 in [254].

Proposition 3.6.3. *Under the hypotheses of Proposition 3.1.2, assume that $w \in L^\infty(\mathbb{R}^n)$ solves $Lw = g$ in B_1, with $g \in L^\infty$. Then, there exists $\alpha > 0$ such that*

$$\|w\|_{C^\alpha(B_{1/2})} \leq C\left(\|g\|_{L^\infty(B_1)} + \|w\|_{L^\infty(\mathbb{R}^n)}\right), \tag{3.35}$$

where C depends only on n, ϵ, σ, and the constant in (3.10).

Proof. With slight modifications, the results in [270] yield the desired result.

Indeed, given $\delta > 0$ conditions (3.5), (3.9), and (3.10) yield

$$\kappa Lb(x) + 2\int_{\mathbb{R}^n \setminus B_{1/4}} \big(|8y|^\eta - 1\big)K(y)dy < \frac{1}{2}\inf_{A \subset B_2,\ |A|>\delta} \int_A K(y)dy \tag{3.36}$$

for some κ and η depending only on n, ϵ, σ, and the constant in (3.10). Moreover, since our hypotheses are invariant under scaling, then (3.36) holds at every scale. Note that (3.36) is exactly hypothesis (2.1) in [270].

Then, as mentioned by Silvestre in [270, Remark 4.3], Lemma 4.1 in [270] holds also with (4.1) therein replaced by $Lw \leq \nu_0$ in B_1, with ν_0 depending on κ. Therefore, the Hölder regularity of w with the desired estimate (3.35) follows from [270, Theorem 5.1].

Note that it is important to have σ strictly less than 2, since otherwise condition (3.36) does not hold. □

The following is the analog of Proposition 2.2 in [254].

Proposition 3.6.4. *Under the same hypotheses of Proposition 3.1.2, assume that $w \in C^\beta(\mathbb{R}^n)$ solves $Lw = g$ in B_1, with $g \in C^\beta$, $\beta \in (0,1)$. Then, there exists $\alpha > 0$ such that*

$$\|w\|_{C^{\beta+\alpha}(B_{1/2})} \leq C\left(\|g\|_{C^\beta(B_1)} + \|w\|_{C^\beta(\mathbb{R}^n)}\right) \quad \text{if} \quad \beta + \alpha < 1,$$

$$\|w\|_{C^{0,1}(B_{1/2})} \leq C\left(\|g\|_{C^\beta(B_1)} + \|w\|_{C^\beta(\mathbb{R}^n)}\right) \quad \text{if} \quad \beta + \alpha > 1,$$

where C and α depend only on n, ϵ, σ, and the constants in (3.10) *and* (3.7).

Proof. It follows from the previous Proposition applied to the incremental quotients $(w(x+h)-w(x))/|h|^\beta$ and from Lemma 5.6 in [59]. □

As a consequence of the last two propositions, we find the following corollaries. The first one is the analog of Corollary 2.5 in [254].

Corollary 3.6.5. *Under the same hypotheses of Proposition 3.1.2, assume that $w \in L^\infty(\mathbb{R}^n)$ solves $Lw = g$ in B_1, with $g \in L^\infty$. Then, there exists $\alpha > 0$ such that*

$$\|w\|_{C^\alpha(B_{1/2})} \leq C\left(\|g\|_{L^\infty(B_1)} + \|w\|_{L^\infty(B_2)} + \|(1+|y|)^{-n-\epsilon}w(y)\|_{L^1(\mathbb{R}^n)}\right),$$

where C depends only on n, ϵ, σ, and the constants in (3.7) *and* (3.10).

Proof. Using (3.7), the proof is exactly the same as the one in [254, Corollary 2.5]. □

The second one is the analog of Corollary 2.4 in [254].

Corollary 3.6.6. *Under the same hypotheses of Proposition 3.1.2, assume that $w \in C^\beta(\mathbb{R}^n)$ solves $Lw = g$ in B_1, with $g \in C^\beta$, $\beta \in (0,1)$. Then, there exists $\alpha > 0$ such that*

$$\|w\|_{C^{\beta+\alpha}(B_{1/2})} \leq C\left(\|g\|_{C^\beta(B_1)} + \|w\|_{C^\beta(\overline{B_2})} + \|(1+|y|)^{-n-\epsilon}w(y)\|_{L^1(\mathbb{R}^n)}\right)$$

if $\beta + \alpha < 1$, while

$$\|w\|_{C^{0,1}(B_{1/2})} \leq C\left(\|g\|_{C^\beta(B_1)} + \|w\|_{C^\beta(\overline{B_2})} + \|(1+|y|)^{-n-\epsilon}w(y)\|_{L^1(\mathbb{R}^n)}\right)$$

if $\beta + \alpha > 1$. The constant C depends only on n, ϵ, σ and the constants in (3.7) *and* (3.10).

Proof. Using (3.7), the proof is the same as the one in [254, Corollary 2.4]. □

We can finally give the

Proof of Proposition 3.1.2. Let now $x \in \Omega$, and $2R = \operatorname{dist}(x, \partial\Omega)$. Then, one may rescale problem (3.1)-(3.2) in $B_R = B_R(x)$, to find that $w(y) := u(x+Ry)$ satisfies $\|w\|_{L^\infty(B_2)} \leq CR^{\epsilon/2}$, $|w(y)| \leq CR^{\epsilon/2}(1+|y|^{\epsilon/2})$ in $\mathbb{R}^n$, and $\|L_R w\|_{L^\infty(B_1)} \leq CR^\epsilon$, where

$$L_R w(y) = \int_{\mathbb{R}^n} \big(w(y) - w(y+z)\big) K_R(y)dy$$

and $K_R(y) = K(Ry)R^{n+\epsilon}$.

Moreover, it is immediate to check that (3.7) yields

$$|\nabla K_R(y)| \leq C\frac{K_R(y)}{|y|},$$

with the same constant C for each $R \in (0,1)$. The other hypotheses of Proposition (3.1.2) are clearly satisfied by the kernels K_R for each $R \in (0,1)$.

Hence, one may apply Corollaries 3.6.5 and 3.6.6 (repeatedly) to obtain

$$|\nabla w(0)| \leq CR^{\epsilon/2}.$$

From this, we deduce that $|\nabla u(x)| \leq CR^{\frac{\epsilon}{2}-1}$, and since this can be done for any $x \in \Omega$, we find

$$|\nabla u(x)| \leq C\delta(x)^{\frac{\epsilon}{2}-1} \quad \text{in } \Omega,$$

as desired. The $C^{\epsilon/2}(\mathbb{R}^n)$ regularity of u follows immediately from this gradient bound. □

Remark 3.6.7. The convexity of the domain has been only used in the construction of the supersolution. To establish Proposition 3.1.2 in general $C^{1,1}$ domains, one only needs to construct a supersolution which is not $1D$ but it is radially symmetric and with support in $\mathbb{R}^n \setminus B_1$, as in [254, Lemma 2.6], where it is done for the fractional Laplacian.

Chapter Four

BOUNDARY REGULARITY FOR FULLY NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

We study fine boundary regularity properties of solutions to fully nonlinear elliptic integro-differential equations of order $2s$, with $s \in (0,1)$.

We consider the class of nonlocal operators $\mathcal{L}_* \subset \mathcal{L}_0$, which consists of all the infinitesimal generators of stable Lévy processes belonging to the class $\mathcal{L}_0$ of Caffarelli-Silvestre. For fully nonlinear operators I elliptic with respect to $\mathcal{L}_*$, we prove that solutions to $Iu = f$ in Ω, $u = 0$ in $\mathbb{R}^n \setminus \Omega$, satisfy $u/d^s \in C^{s-\epsilon}(\overline{\Omega})$ for all $\epsilon > 0$, where d is the distance to $\partial\Omega$ and $f \in L^\infty$.

We expect the Hölder exponent $s - \epsilon$ to be optimal (or almost optimal) for right hand sides $f \in L^\infty$. Moreover, we also expect the class $\mathcal{L}_*$ to be the largest scale invariant subclass of $\mathcal{L}_0$ for which this result is true. In this direction, we show that the class $\mathcal{L}_0$ is too large for all solutions to behave like d^s.

The constants in all the estimates in this paper remain bounded as the order of the equation approaches 2.

4.1 Introduction and results

This paper is concerned with boundary regularity for fully nonlinear elliptic integro-differential equations.

Since the foundational paper of Caffarelli and Silvestre [69], ellipticity for a nonlinear integro-differential operator is defined relatively to a given set $\mathcal{L}$ of linear translation invariant elliptic operators. This set $\mathcal{L}$ is called the ellipticity class.

The reference ellipticity class from [69] is the class $\mathcal{L}_0 = \mathcal{L}_0(s)$, containing all operators L of the form

$$Lu(x) = \int_{\mathbb{R}^n} \left(\frac{u(x+y) + u(x-y)}{2} - u(x) \right) K(y)\, dy \tag{4.1}$$

with even kernels $K(y)$ bounded between two positive multiples of $(1-s)|y|^{-n-2s}$, which is the kernel of the fractional Laplacian $(-\Delta)^s$.

In the three papers [69, 70, 71], Caffarelli and Silvestre studied the interior regularity for solutions u to

$$\begin{cases} \mathrm{I}u &= f \quad \text{in } \Omega \\ u &= g \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{4.2}$$

being I a translation invariant fully nonlinear integro-differential operator of order $2s$ (see the definition later on in this Introduction). They proved existence of viscosity solutions, established $C^{1+\alpha}$ interior regularity of solutions [69], $C^{2s+\alpha}$ regularity in case of convex equations [71], and developed a perturbative theory for non translation invariant equations [70]. Thus, the interior regularity for these equations is well understood.

However, very few is known about the boundary regularity for fully nonlinear problems of fractional order.

When I is the fractional Laplacian $-(-\Delta)^s$, the boundary regularity of solutions u to (4.2) is now well understood. The first result in this direction was obtained by Bogdan, who established the boundary Harnack principle for s-harmonic functions [25] —i.e., for solutions to $(-\Delta)^s u = 0$. More recently, we proved in [249] that if $f \in L^\infty$, $g \equiv 0$, and Ω is $C^{1,1}$ then $u \in C^s(\mathbb{R}^n)$ and $u/d^s \in C^\alpha(\overline{\Omega})$ for some small $\alpha > 0$, where d is the distance to the boundary $\partial\Omega$. Moreover, the limit of $u(x)/d^s(x)$ as $x \to \partial\Omega$ is typically nonzero (in fact it is positive if $f < 0$), and thus the C^s regularity of u is optimal. After this, Grubb [163] showed that when $f \in C^\beta$ with $\beta > 0$ (resp. $f \in L^\infty$), $g \equiv 0$, and Ω is smooth, then $u/d^s \in C^{\beta+s-\epsilon}(\overline{\Omega})$ (resp. $u/d^s \in C^{s-\epsilon}(\overline{\Omega})$) for all $\epsilon > 0$. In particular, $f \in C^\infty$ leads to $u/d^s \in C^\infty(\overline{\Omega})$. Thus, the correct notion of boundary regularity for equations of order $2s$ is the Hölder regularity of the quotient u/d^s.

Besides these works for the fractional Laplacian, no other result on fine boundary regularity for more general operators was known —not even for linear equations.

Here, we obtain boundary regularity for fully nonlinear integro-differential problems of the form (4.2) which are elliptic with respect to a class $\mathcal{L}_* \subset \mathcal{L}_0$ defined as follows. $\mathcal{L}_*$ consists of all linear operators of the form

$$Lu(x) = (1-s) \int_{\mathbb{R}^n} \left(\frac{u(x+y) + u(x-y)}{2} - u(x) \right) \frac{a(y/|y|)}{|y|^{n+2s}} \, dy, \tag{4.3}$$

with

$$a \in L^\infty(S^{n-1}) \quad \text{satisfying} \quad \lambda \le a \le \Lambda, \tag{4.4}$$

where $0 < \lambda \le \Lambda$ are called ellipticity constants. The class $\mathcal{L}_*$ consists of all infinitesimal generators of *stable* Lévy processes belonging to $\mathcal{L}_0$. Our main result establishes that when $f \in L^\infty$, $g \equiv 0$, and Ω is $C^{1,1}$, viscosity solutions u satisfy

$$u/d^s \in C^{s-\epsilon}(\overline{\Omega}) \text{ for all } \epsilon > 0. \tag{4.5}$$

We also obtain boundary regularity for problem (4.2) with exterior data $g \in C^2$, and also for non translation invariant operators $\mathfrak{I}(u, x)$. These results apply to fully nonlinear equations, but they are new even for linear translation invariant equations $Lu = f$ with L as in (4.3).

We believe the Hölder exponent $s - \epsilon$ in (4.5) to be optimal (or almost optimal) for merely bounded right hand sides f. Moreover, we expect the class $\mathcal{L}_*$ to be the largest scale invariant subclass of $\mathcal{L}_0$ for which this result is true.

For general elliptic equations with respect to $\mathcal{L}_0$, no fine boundary regularity results like (4.5) hold. In fact, the class $\mathcal{L}_0$ is too large for all solutions to be comparable to d^s near the boundary. Indeed, we show in Section 2 that there are powers $0 < \beta_1 < s < \beta_2$ for which the functions $(x_n)_+^{\beta_1}$ and $(x_n)_+^{\beta_2}$ satisfy

$$M^+_{\mathcal{L}_0}(x_n)_+^{\beta_1} = 0 \quad \text{and} \quad M^-_{\mathcal{L}_0}(x_n)_+^{\beta_2} = 0 \quad \text{in } \{x_n > 0\},$$

where $M^+_{\mathcal{L}_0}$ and $M^-_{\mathcal{L}_0}$ are the extremal operators for the class $\mathcal{L}_0$; see their definition in Section 2. Hence, since $(-\Delta)^s(x_n)_+^s = 0$ in $\{x_n > 0\}$, we have at least three functions which solve fully nonlinear elliptic equations with respect to $\mathcal{L}_0$ but which are not even comparable near the boundary $\{x_n = 0\}$. As we show in Section 2, the same happens for the subclasses $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{L}_0$, which have more regular kernels and were considered in [69, 70, 71].

4.1.1 The class $\mathcal{L}_*$

The class $\mathcal{L}_*$ consists of all infinitesimal generators of stable Lévy processes belonging to $\mathcal{L}_0$. This type of Lévy processes are well studied in probability, as explained next. In that context, the function $a \in L^\infty(S^{n-1})$ is called the spectral measure.

Stable processes are by several reasons a natural extension of Gaussian processes. For instance, the Generalized Central Limit Theorem states that the distribution of a sum of independent identically distributed random variables with heavy tails converges to a stable distribution; see [255], [196], or [12] for a precise statement of this result. Thus, stable processes are often used to model sums of many random independent perturbations with heavy-tailed distributions —i.e., when large outcomes are not unlikely. In particular, they arise frequently in financial mathematics, internet traffic statistics, or signal processing; see for instance [236, 213, 214, 228, 229, 230, 3, 186, 233, 168] and the books [227, 255].

Linear equations $Lu = f$ with L in the class $\mathcal{L}_*$ have been already studied, specially by Sztonyk and Bogdan; see for instance [280, 28, 239, 29, 30, 281]. Although there were some results on the boundedness of u/d^s, the Hölder regularity for the quotient u/d^s was not known. In this paper we establish it for linear and for fully nonlinear equations.

Notice that all second order linear uniformly elliptic operators are recovered as limits of operators in $\mathcal{L}_* = \mathcal{L}_*(s)$ as $s \to 1$. In particular, all second order fully nonlinear equations $F(D^2u, x) = f(x)$ are recovered as limits of the fully nonlinear integro-differential equations that we consider. Furthermore, when $s < 1$ the class of translation invariant linear operators $\mathcal{L}_*(s)$ is much richer than the one of second order uniformly elliptic operators. Indeed, while any operator in the latter class is determined by a positive definite $n \times n$ matrix, a function $a : S^{n-1} \to \mathbb{R}^+$ is needed to determine an operator in $\mathcal{L}_*(s)$.

A key feature of the class $\mathcal{L}_*$ for boundary regularity issues is that

$$L(x_n)_+^s = 0 \quad \text{in } \{x_n > 0\} \quad \text{for all } L \in \mathcal{L}_*.$$

This is essential first to construct barriers which are comparable to d^s, and later to prove finer boundary regularity.

4.1.2 Equations with "bounded measurable coefficients"

The first result of in this paper, and on which all the other results rely, is Proposition 4.1.1 below.

Here, and throughout the article, we use the definition of viscosity solutions and inequalities of [69]. Moreover, for $r > 0$ we denote

$$B_r^+ = B_r \cap \{x_n > 0\} \quad \text{and} \quad B_r^- = B_r \cap \{x_n < 0\},$$

and the constants λ and Λ in (4.4) are called ellipticity constants.

The extremal operators associated to the class $\mathcal{L}_*$ are denoted by $M^+_{\mathcal{L}_*}$ and $M^-_{\mathcal{L}_*}$,

$$M^+_{\mathcal{L}_*} u = \sup_{L \in \mathcal{L}_*} Lu \quad \text{and} \quad M^-_{\mathcal{L}_*} u = \inf_{L \in \mathcal{L}_*} Lu.$$

Note that, since $\mathcal{L}_* \subset \mathcal{L}_0$, then $M^-_{\mathcal{L}_0} \le M^-_{\mathcal{L}_*} \le M^+_{\mathcal{L}_*} \le M^+_{\mathcal{L}_0}$.

Proposition 4.1.1. *Let $s_0 \in (0,1)$ and $s \in [s_0, 1)$. Assume that $u \in C(B_1) \cap L^\infty(\mathbb{R}^n)$ is a viscosity solution of*

$$\begin{cases} M^+_{\mathcal{L}_*} u \ge -C_0 & \text{in } B_1^+ \\ M^-_{\mathcal{L}_*} u \le C_0 & \text{in } B_1^+ \\ u = 0 & \text{in } B_1^-, \end{cases} \tag{4.6}$$

for some nonnegative constant C_0. Then, u/x_n^s is $C^\alpha(\overline{B^+_{1/2}})$ for some $\alpha > 0$, with the estimate

$$\|u/x_n^s\|_{C^\alpha(B^+_{1/2})} \le C\left(C_0 + \|u\|_{L^\infty(\mathbb{R}^n)}\right). \tag{4.7}$$

The constants α and C depend only on n, s_0, and the ellipticity constants.

It is important to remark that the constants in our estimate remain bounded as $s \to 1$. This means that from Proposition 4.1.1 we can recover the classical boundary Harnack inequality of Krylov [189].

The estimate of Proposition 4.1.1 is only a first step towards our results. It is obtained via a nonlocal version of the method of Krylov [189] for second order equations with bounded measurable coefficients; see also Section 9.2 in [59]. This method has been adapted to nonlocal equations by the authors in [249], where we proved estimate (4.7) for the fractional Laplacian $(-\Delta)^s$ in $C^{1,1}$ domains.

As explained before, our main result is the $C^{s-\epsilon}$ regularity of u/d^s in $C^{1,1}$ domains for solutions u to fully nonlinear integro-differential equations (see the next subsection). Thus, for solutions to the nonlinear equations we push the small Hölder exponent $\alpha > 0$ in (4.7) up to the exponent $s-\epsilon$ in (4.5). To achieve this, new ideas are needed, and the procedure that we develop differs substantially from that in second order equations. We use a new compactness method and the "boundary" Liouville-type Theorem 4.1.5, stated later on in the Introduction. This Liouville theorem relies on Proposition 4.1.1.

4.1.3 Main result

Before stating our main result, let us recall the definition and motivations of fully nonlinear integro-differential operators.

As defined in [69], a fully nonlinear operator I is said to be elliptic with respect to a subclass $\mathcal{L} \subseteq \mathcal{L}_0$ when

$$M^-_{\mathcal{L}}(u-v)(x) \leq \mathrm{I}u(x) - \mathrm{I}v(x) \leq M^+_{\mathcal{L}}(u-v)(x)$$

for all test functions u, v which are C^2 in a neighborhood of x and having finite integral against $\omega_s(x) = (1-s)(1+|x|^{-n-2s})$. Moreover, if

$$\mathrm{I}\left(u(x_0+\cdot)\right)(x) = (\mathrm{I}u)(x_0+x),$$

then we say that I is translation invariant.

Fully nonlinear elliptic integro-differential equations naturally arise in stochastic control and games. In typical examples, a single player or two players control some parameters (e.g. the volatilities of the assets in a portfolio) affecting the joint distribution of the random increments of n variables $X(t) \in \mathbb{R}^n$. The game ends when $X(t)$ exits for the first time a certain domain Ω (as when having automated orders to sell assets when their prices cross certain limits).

The *value* or *expected payoff* of these games $u(x)$ depends on the starting point $X(0) = x$ (initial prices of all assets in the portfolio). A remarkable fact is that value $u(x)$ solves an equation of the type $\mathrm{I}u = 0$, where

$$\mathrm{I}u(x) = \sup_{\alpha}\big(L_\alpha u + c_\alpha\big) \quad \text{or} \quad \mathrm{I}u(x) = \inf_{\beta}\sup_{\alpha}\big(L_{\alpha\beta}u + c_{\alpha\beta}\big). \tag{4.8}$$

The first equation, known as the Bellman equation, arises in control problems (a single player), while the second one, known as the Isaacs equations, arises in zero-sum games (two players). The linear operators L_α and $L_{\alpha\beta}$ are infinitesimal generators of Lévy processes, standing for all the possible choices of the distribution of time increments of $X(t)$. The constants c_α and $c_{\alpha\beta}$ are costs associated to the choice of the operators L_α and $L_{\alpha\beta}$. More involved equations with zeroth order terms and right hand sides have also meanings in this context as interest rates or running costs. See [60, 273, 231, 100, 69], and references therein for more information on these equations.

When all L_α and $L_{\alpha\beta}$ belong to $\mathcal{L}_*$, then (4.8) are fully nonlinear translation invariant operators elliptic with respect to $\mathcal{L}_*$, as defined above.

A fractional Monge-Ampère operator has been recently introduced by Caffarelli-Charro [62]. It is a fully nonlinear integro-differential operator which, by the main result in [62], is elliptic with respect to $\mathcal{L}_*$ whenever the right hand side is uniformly positive.

The interior regularity for fully nonlinear integro-differential elliptic equations was mainly established by Caffarelli and Silvestre in the well-known paper [69]. More precisely, for some small $\alpha > 0$, they obtain $C^{1+\alpha}$ interior regularity for fully nonlinear elliptic equations with respect to the class $\mathcal{L}_1$ made of kernels in $\mathcal{L}_0$ which are C^1 away from the origin. For $s > \frac{1}{2}$, the same result in the class $\mathcal{L}_0$ has been recently proved by Kriventsov [187]. These estimates are uniform as the order of the equations approaches two, so they can be viewed as a natural extension of the interior regularity for fully nonlinear equations of second order. There were previous interior estimates by Bass and Levin [14] and by Silvestre [270] which are not uniform as the order of the equation approaches 2. An interesting aspect of [270] is that its proof is short and

uses only elementary analysis tools, taking advantage of the nonlocal character of the equations. This is why same ideas have been used in other different contexts [72, 272].

For convex equations elliptic with respect to $\mathcal{L}_2$ (i.e., with kernels in $\mathcal{L}_0$ which are C^2 away from the origin), Caffarelli and Silvestre obtained $C^{2s+\alpha}$ interior regularity [71]. This is the nonlocal extension of the Evans-Krylov theorem. Other important references concerning interior regularity for nonlocal equations in nondivergence form are [241, 180, 86, 9, 165].

To give local boundary regularity results for $C^{1,1}$ domains it is useful the following:

Definition 4.1.2. *We say that Γ is $C^{1,1}$ surface with radius $\rho_0 > 0$ splitting B_1 into Ω^+ and Ω^- if the following happens.*

- *The two disjoint domains Ω^+ and Ω^- partition B_1, i.e., $\overline{B_1} = \overline{\Omega^+} \cup \overline{\Omega^-}$.*
- *The boundary $\Gamma := \partial\Omega^+ \setminus \partial B_1 = \partial\Omega^- \setminus \partial B_1$ is $C^{1,1}$ surface with $0 \in \Gamma$.*
- *All points on $\Gamma \cap \overline{B_{3/4}}$ can be touched by two balls of radii ρ_0, one contained in Ω^+ and the other contained in Ω^-.*

Our main result reads as follows.

Theorem 4.1.3. *Let Γ be a $C^{1,1}$ surface with radius ρ_0 splitting B_1 into Ω^+ and Ω^-; see Definition 4.1.2. Let $d(x) = dist\,(x, \Gamma)$.*

Let $s_0 \in (0,1)$ and $s \in [s_0, 1)$. Assume that I is a fully nonlinear and translation invariant operator, elliptic with respect to $\mathcal{L}_*(s)$, with $I0 = 0$. Let $f \in C\big(\overline{\Omega^+}\big)$, and $u \in L^\infty(\mathbb{R}^n) \cap C\big(\overline{\Omega^+}\big)$ be a viscosity solution of

$$\begin{cases} Iu &= f \quad \text{in } \Omega^+ \\ u &= 0 \quad \text{in } \Omega^-. \end{cases}$$

Then, u/d^s belongs to $C^{s-\epsilon}\big(\overline{\Omega^+ \cap B_{1/2}}\big)$ for all $\epsilon > 0$ with the estimate

$$\big\|u/d^s\big\|_{C^{s-\epsilon}(\Omega^+ \cap B_{1/2})} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\Omega^+)}\big),$$

where the constant C depends only on ρ_0, s_0, ϵ, ellipticity constants, and dimension.

Remark 4.1.4. As in the case of the fractional Laplacian, under the hypotheses of Theorem 4.1.3 we have that $u \in C^s\big(\overline{\Omega^+ \cap B_{1/2}}\big)$, with the estimate $\|u\|_{C^s(\Omega^+ \cap B_{1/2})} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\Omega^+)}\big)$. Indeed, one only needs to combine the interior estimates in [69, 187, 261] (stated in Theorem 4.2.6) with the supersolution in Lemma 4.3.3, exactly as we did in [249, Proposition 1.1] for $(-\Delta)^s$.

It is important to notice that our result is not only an a priori estimate for classical solutions but also applies to viscosity solutions. For local equations of second order $F(D^2u, Du, x) = f(x)$, the boundary regularity for viscosity solutions to fully nonlinear equations has been recently obtained by Silvestre-Sirakov [274]. The methods that we introduce here to prove Theorem 4.1.3 can be used also to give a new proof of the results for such second order fully nonlinear equations; see Section 4.8 for more details.

Besides its own interest, the boundary regularity of solutions to integro-differential equations plays an important role in different contexts. For example, it is needed in

overdetermined problems arising in shape optimization [105, 128] and also in Pohozaev-type or integration by parts identities [250]. Moreover, boundary regularity issues appear naturally in free boundary problems [66, 271].

Theorem 4.1.3 is, to our knowledge, the first boundary regularity result for fully nonlinear integro-differential equations. It was only known that solutions u to these equations are C^α up to the boundary for some small $\alpha > 0$ (a result for u but not for the quotient u/d^s). For solutions u to elliptic equations with respect to $\mathcal{L}_*$, our result gives a quite accurate description of the boundary behavior. Namely, u/d^s is $C^{s-\epsilon}$ for all $\epsilon > 0$, where d is the distance to the boundary.

This result is close to being optimal, at least when the right-hand sides f are just bounded. Indeed, let us compare it with the best known boundary regularity results for the fractional Laplacian $(-\Delta)^s$, due to Gerd Grubb [163]. These results use powerful machinery from Hörmander's theory. One of the main results in [163] applies to solutions u of the linear problem

$$\begin{cases} (-\Delta)^s u &= f \quad \text{in } U \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus U \end{cases} \tag{4.9}$$

in a C^∞ domain U. It states that if f is C^β for some $\beta \in [0, +\infty]$, then u/d^s is also $C^{\beta+s-\epsilon}$ up to the boundary for all $\epsilon > 0$. These estimates in Hölder spaces are actually particular cases of sharp estimates in Hörmander's μ-spaces. Needless to say, these remarkable results almost close the theory of boundary regularity for bounded solutions of (4.9), and they are a major improvement of the previously available results by the authors [249]. However, these techniques are only available for linear operators that satisfy the so called μ-transmission property. Such operators are mainly powers of second order linear elliptic operators. We find thus interesting to have reached, when f is just L^∞, the same boundary regularity for fully nonlinear equations.

4.1.4 A Liouville theorem and other ingredients of the proof

Theorem 4.1.3 follows by combining an estimate *on* the boundary, (4.10) below, with the known interior regularity estimates in [69, 187]. The estimate on the boundary reads as follows. If u satisfies the hypotheses of Theorem 4.1.3, then for all $z \in \Gamma \cap \overline{B_{1/2}}$ there exists $Q(z) \in \mathbb{R}$ for which

$$\left| u(x) - Q(z)\big((x-z)\cdot\nu(z)\big)_+^s \right| \le C|x-z|^{2s-\epsilon} \quad \text{for all } x \in B_1. \tag{4.10}$$

Here, $\nu(z)$ is the unit normal vector to Γ at z pointing towards Ω^+.

Our proof of (4.10) differs substantially from boundary regularity methods in second order equations. A main reason for this is not only the nonlocal character of the estimates, but also that tangential and normal derivatives of the solution behave differently on the boundary; recall that the solution is C^s but cannot be Lipschitz up to the boundary.

The estimate on the boundary (4.10) relies heavily on two ingredients, as explained next.

The first ingredient is the following Liouville-type theorem for solutions in a half space.

Theorem 4.1.5. *Let $u \in C(\mathbb{R}^n)$ be a viscosity solution of*

$$\begin{cases} Iu &= 0 \quad in \quad \{x_n > 0\} \\ u &= 0 \quad in \quad \{x_n < 0\}, \end{cases}$$

where I is a fully nonlinear and translation invariant operator, elliptic with respect to $\mathcal{L}_*$ and with $I0 = 0$. Assume that for some positive $\beta < 2s$, u satisfies the growth control at infinity

$$\|u\|_{L^\infty(B_R)} \leq CR^\beta \quad \text{for all } R \geq 1. \tag{4.11}$$

Then,

$$u(x) = K(x_n)_+^s$$

for some constant $K \in \mathbb{R}$.

To prove Theorem 4.1.5, we apply Proposition 4.1.1 to incremental quotients of u in the first $(n-1)$-variables. After this, rescaling the obtained estimates and using (4.11), we find that such incremental quotients are zero, and thus that u is a 1D solution. Then, we use that for 1D functions all operators $L \in \mathcal{L}_*$ coincide up to a multiplicative constant with the fractional Laplacian $(-\Delta)^s$; see Lemma 4.2.1. Therefore, we only need to prove a Liouville theorem for solutions to $(-\Delta)^s w = 0$ in $\mathbb{R}_+$, $w = 0$ in $\mathbb{R}_-$ satisfying a growth control at infinity, which is done in Lemma 4.5.2.

The second ingredient towards (4.10) is the following compactness argument. With u as in Theorem 4.1.3, we suppose by contradiction that (4.10) does not hold, and we blow up the fully nonlinear equation at a boundary point (after subtracting appropriate terms to the solution). We then show that the blow up sequence converges to an entire solution in $\{x \cdot \nu > 0\}$ for some unit vector ν. For this, we need to develop a boundary version of a method introduced by the second author in [261]. The method was conceived there to prove interior regularity for integro-differential equations with rough kernels. Finally, the contradiction is reached by applying the Liouville-type theorem stated above to the entire solution in $\{x \cdot \nu > 0\}$.

These are the main ideas used to prove (4.10). A byproduct of this blow-up method is that the same proof yields results for non translation invariant equations; see Theorem 4.1.6 below.

Finally, Theorem 4.1.3 follows by combining (4.10) with the interior regularity estimates in [69, 187].

4.1.5 Non translation invariant equations

An interesting feature of the blow up and compactness argument used in this paper is that it allows to deal also with equations depending continuously on the x variable. For example, consider

$$\mathfrak{I}(u, x) = f(x) \quad \text{in } \Omega^+,$$

where $\mathfrak{I}$ is an operator of the form

$$\mathfrak{I}(u, x) = \inf_\beta \sup_\alpha \left(\int_{\mathbb{R}^n} \big\{ u(x+y) + u(x-y) - 2u(x) \big\} K_{\alpha\beta}(x, y)\, dy + c_{\alpha\beta}(x) \right). \tag{4.12}$$

The kernels $K_{\alpha\beta}$ are of the form

$$K_{\alpha\beta}(x,y) = (1-s)\frac{a_{\alpha\beta}(x,y/|y|)}{|y|^{n+2s}}, \tag{4.13}$$

and satisfy, for all α and β,

$$0 < \frac{\lambda}{|y|^{n+2s}} \le K_{\alpha\beta}(x,y) \le \frac{\Lambda}{|y|^{n+2s}} \quad \text{for all } x \in \Omega^+ \text{ and } y \in \mathbb{R}^n, \tag{4.14}$$

$$\inf_{\beta}\sup_{\alpha} c_{\alpha\beta}(x) = 0 \quad \text{for all } x \in \Omega^+, \qquad \|c_{\alpha\beta}\|_{L^\infty} \le \Lambda \tag{4.15}$$

and

$$\big|a_{\alpha\beta}(x_1,\theta) - a_{\alpha\beta}(x_2,\theta)\big| \le \mu\big(|x_1-x_2|\big) \tag{4.16}$$

for all $x_1, x_2 \in \Omega^+$ and $\theta \in S^{n-1}$, where μ is some modulus of continuity.

As proved in [70], the operator $\mathfrak{I}$ defined above satisfies the ellipticity condition

$$M^-_{\mathcal{L}_*}(u-v)(x) \le \mathfrak{I}(u,x) - \mathfrak{I}(v,x) \le M^+_{\mathcal{L}_*}(u-v)(x).$$

The assumption (4.15) guarantees that $\mathfrak{I}(0,x) = 0$.

The following is our result for non translation invariant equations. In this result, we also consider a nonzero Dirichlet condition $g(x)$.

Theorem 4.1.6. *Let Γ be a $C^{1,1}$ hypersurface with radius $\rho_0 > 0$ splitting B_1 into Ω^+ and Ω^-; see Definition 4.1.2.*

Let $s_0 \in (0,1)$ and $s \in [s_0, 1)$. Assume that $\mathfrak{I}$ is an operator of the form (4.12)-(4.16). Let $f \in C\big(\overline{\Omega^+}\big)$, $g \in C^2(B_1)$, and $u \in L^\infty(\mathbb{R}^n) \cap C\big(\overline{\Omega^+}\big)$ be a viscosity solution of

$$\begin{cases} \mathfrak{I}(u,x) &= f(x) \quad \text{in } \Omega^+ \\ u &= g(x) \quad \text{in } \Omega^-. \end{cases}$$

Then, given $\epsilon > 0$, for all $z \in \Gamma \cap \overline{B_{1/2}}$ there exists $Q(z) \in \mathbb{R}$ with $|Q(z)| \le CC_0$ for which

$$\Big|u(x) - g(x) - Q(z)\big((x-z)\cdot\nu(z)\big)^s_+\Big| \le CC_0|x-z|^{2s-\epsilon} \quad \text{for all } x \in B_1,$$

where

$$C_0 = \|f\|_{L^\infty(\Omega^+)} + \|g\|_{C^2(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}$$

and $\nu(z)$ is the unit normal vector to Γ at z pointing towards Ω^+. The constant C depends only on n, ρ_0, s_0, ϵ, μ, and ellipticity constants.

In case $g \equiv 0$, the proof of Theorem 4.1.6 is almost the same as that of Theorem 4.1.3. On the other hand, the full Theorem 4.1.6 follows from the case $g \equiv 0$ by applying it to the function $\tilde{u} = u - g$.

In Theorem 4.1.6, the C^2 norm of g may be replaced by the $C^{2s+\epsilon}$ norm for any $\epsilon > 0$. This easily follows from the proof of the result.

Remark 4.1.7. When the kernels $K_{\alpha\beta}$ belong to $\mathcal{L}_1$, interior regularity estimates for the operators $\mathfrak{I}$ are proved in [70]. For operators $\mathfrak{I}$ elliptic with respect to $\mathcal{L}_0$, these interior estimates can be proved by using the methods of the second author [261]. Once proved these interior estimates, it follows from Theorem 4.1.6 that $(u-g)/d^s \in C^{s-\epsilon}(\overline{\Omega^+ \cap B_{1/2}})$, as in Theorem 4.1.3.

The paper is organized as follows. In Section 4.2 we give some important results on $\mathcal{L}_*$ and $\mathcal{L}_0$. In Section 4.3 we construct some sub and supersolutions that will be used later. In Section 4.4 we prove Proposition 4.1.1. In Section 4.5 we show Theorem 4.1.5. Then, in Section 4.6 we prove our main result, Theorem 4.1.3. Finally, in Section 4.7 we prove results for non-translation-invariant equations.

4.2 Properties of $\mathcal{L}_*$ and $\mathcal{L}_0$

This section has two main purposes: to show that the class $\mathcal{L}_* \subset \mathcal{L}_0$ is the appropriate one to obtain fine boundary regularity results, and to give some important results on $\mathcal{L}_*$ and $\mathcal{L}_0$.

4.2.1 The class $\mathcal{L}_*$

For $s \in (0,1)$, we define the ellipticity class $\mathcal{L}_* = \mathcal{L}_*(s)$ as the set of all linear operators L of the form (4.3)-(4.4).

Throughout the paper, the extremal operators (as defined in [69]) for the class $\mathcal{L}_*$ are denoted by M^+ and M^-, that is,

$$M^+u(x) = M^+_{\mathcal{L}_*}u(x) = \sup_{L\in\mathcal{L}_*} Lu(x) \quad \text{and} \quad M^-u(x) = M^-_{\mathcal{L}_*}u(x) = \inf_{L\in\mathcal{L}_*} Lu(x). \tag{4.17}$$

The following useful formula writes an operator $L \in \mathcal{L}_*$ as a weighted integral of one dimensional fractional Laplacians in all directions.

$$\begin{aligned} Lu &= (1-s)\int_{S^{n-1}} d\theta\, \frac{1}{2}\int_{-\infty}^{\infty} dr \left(\frac{u(x+r\theta)+u(x-r\theta)}{2} - u(x)\right)\frac{a(\theta)}{|r|^{n+2s}}\, r^{n-1} \\ &= -\frac{1-s}{2c_{1,s}}\int_{S^{n-1}} d\theta\, a(\theta)\,(-\partial_{\theta\theta})^s u(x), \end{aligned} \tag{4.18}$$

where

$$-(-\partial_{\theta\theta})^s u(x) = c_{1,s}\int_{-\infty}^{\infty}\left(\frac{u(x+\theta r)+u(x-\theta r)}{2} - u(x)\right)\frac{dr}{|r|^{1+2s}}$$

is the one-dimensional fractional Laplacian in the direction θ, whose Fourier symbol is $-|\theta\cdot\xi|^{2s}$.

The following is an immediate consequence of the formula (4.18).

Lemma 4.2.1. *Let u be a function depending only on variable x_n, i.e. $u(x) = w(x_n)$, where $w : \mathbb{R} \to \mathbb{R}$. Then,*

$$Lu(x) = -\frac{1-s}{2c_{1,s}}\left(\int_{S^{n-1}} |\theta_n|^{2s} a(\theta)\, d\theta\right)(-\Delta)^s_{\mathbb{R}} w(x_n),$$

where $(-\Delta)^s_{\mathbb{R}}$ denotes the fractional Laplacian in dimension one.

Proof. Using (4.18) we find

$$\begin{aligned} Lu(x) &= \frac{1-s}{2c_{1,s}} \int_{S^{n-1}} -(-\Delta)^s_{\mathbb{R}}\big(w(x_n+\theta_n\cdot)\big)\, a(\theta)\, d\theta \\ &= \frac{1-s}{2c_{1,s}} \int_{S^{n-1}} -|\theta_n|^{2s}(-\Delta)^s_{\mathbb{R}}\big(w(x_n+\cdot)\big)\, a(\theta)\, d\theta, \end{aligned}$$

as wanted. □

Another consequence of (4.18) is that M^+ and M^- admit the following "closed formulae":

$$M^+u(x) = \frac{1-s}{2c_{1,s}} \int_{S^{n-1}} \Big\{\Lambda\big(-(-\partial_{\theta\theta})^s w(x)\big)^+ - \lambda\big(-(-\partial_{\theta\theta})^s w(x)\big)^-\Big\}\, d\theta$$

and

$$M^-u(x) = \frac{1-s}{2c_{1,s}} \int_{S^{n-1}} \Big\{\lambda\big(-(-\partial_{\theta\theta})^s w(x)\big)^+ - \Lambda\big(-(-\partial_{\theta\theta})^s w(x)\big)^-\Big\}\, d\theta.$$

In all the paper, given $\nu \in S^{n-1}$ and $\beta \in (0,2s)$ we denote by $\varphi^\beta : \mathbb{R} \to \mathbb{R}$ and $\varphi^\beta_\nu : \mathbb{R}^n \to \mathbb{R}$ the functions

$$\varphi^\beta(x) := (x_+)^\beta \qquad \text{and} \qquad \varphi^\beta_\nu(x) := (x\cdot\nu)^\beta_+. \tag{4.19}$$

A very important property of $\mathcal{L}_*$ is the following.

Lemma 4.2.2. *For any unit vector $\nu \in S^{n-1}$, the function φ^s_ν satisfies $M^+\varphi^s_\nu = M^-\varphi^s_\nu = 0$ in $\{x\cdot\nu > 0\}$ and $\varphi^s_\nu = 0$ in $\{x\cdot\nu < 0\}$.*

Proof. We use Lemma 4.2.1 and the well-known fact that the function $\varphi^s(x) = (x_+)^s$ is satisfies $(-\Delta)^s_{\mathbb{R}}\varphi^s = 0$ in $\{x > 0\}$; see for instance [249, Proposition 3.1]. □

Next we give a useful property of M^+ and M^-.

Lemma 4.2.3. *Let $\beta \in (0,2s)$, and let M^+ and M^- be defined by (4.17). For any unit vector $\nu \in S^{n-1}$, the function φ^β_ν satisfies $M^+\varphi^\beta_\nu(x) = \overline{c}(s,\beta)(x\cdot\nu)^{\beta-2s}$ and $M^-\varphi^\beta_\nu(x) = \underline{c}(s,\beta)(x\cdot\nu)^{\beta-2s}$ in $\{x\cdot\nu > 0\}$, and $\varphi^\beta_\nu = 0$ in $\{x\cdot\nu < 0\}$. Here, $\overline{c}$ and $\underline{c}$ are constants depending only on s, β, n, and ellipticity constants.*

Moreover, $\overline{c}$ and $\underline{c}$ satisfy $\overline{c} \ge \underline{c}$, and they are continuous as functions of the variables (s,β) in $\{0 < s \le 1,\ 0 < \beta < 2s\}$. In addition, we have

$$\overline{c}(s,\beta) > \underline{c}(s,\beta) > 0 \quad \text{for all } \beta \in (s,2s). \tag{4.20}$$

and

$$\lim_{\beta\nearrow 2s} \underline{c}(s,\beta) = \begin{cases} +\infty & \text{for all } s \in (0,1) \\ C > 0 & \text{for } s = 1. \end{cases} \tag{4.21}$$

Proof. Given $L \in \mathcal{L}_*$, by Lemma 4.2.1 we have

$$L\varphi_\nu^\beta(x) = -\frac{1-s}{2c_{1,s}} \left(\int_{S^{n-1}} |\theta_n|^{2s} a(\theta)\, d\theta \right) (-\Delta)^s_{\mathbb{R}} \varphi^\beta(x \cdot \nu).$$

Hence, using the scaling properties of the fractional Laplacian and of the function φ^β we obtain that, for $x \cdot \nu > 0$,

$$M^+\varphi_\nu^\beta(x) = C\,(x \cdot \nu)^{\beta-2s} \max\left\{-\Lambda(-\Delta)^s_{\mathbb{R}}\varphi^\beta(1), -\lambda(-\Delta)^s_{\mathbb{R}}\varphi^\beta(1)\right\}$$

and

$$M^-\varphi_\nu^\beta(x) = C\,(x \cdot \nu)^{\beta-2s} \min\left\{-\Lambda(-\Delta)^s_{\mathbb{R}}\varphi^\beta(1), -\lambda(-\Delta)^s_{\mathbb{R}}\varphi^\beta(1)\right\},$$

where $C = (1-s)/(2c_{1,s}) > 0$.

Therefore, to prove that the two functions $\overline{c}$ and $\underline{c}$ are continuous in the variables (s, β) in $\{0 < s \leq 1,\, 0 < \beta < 2s\}$, and that (4.20)-(4.21) holds, it is enough to prove the same for

$$(s, \beta) \longmapsto -(-\Delta)^s_{\mathbb{R}}\varphi^\beta(1).$$

We first prove continuity in β. If β and β' belong to $(0, 2s)$, then as $\beta' \to \beta$, we have $\varphi^{\beta'} \to \varphi^\beta$ in $C^2([1/2, 3/2])$ and

$$\int_{\mathbb{R}} \left|\varphi^{\beta'} - \varphi^\beta\right|(x)\,(1+|x|)^{-1-2s}\, dx \to 0.$$

As a consequence, $(-\Delta)^s_{\mathbb{R}}\varphi^{\beta'}(1) \to (-\Delta)^s_{\mathbb{R}}\varphi^\beta(1)$. It is easy to see that if s and s' belong to $(0, 1]$, and $\beta < 2s$, then $(-\Delta)^{s'}_{\mathbb{R}}\varphi^\beta(1) \to (-\Delta)^s_{\mathbb{R}}\varphi^\beta(1)$ as $s' \to s$.

Moreover, note that whenever $\beta > s$, the function φ^β is touched by below by the function $\varphi^s - C$ at some point $x_0 > 0$ for some constant $C > 0$. Hence, we have $(-\Delta)^s_{\mathbb{R}}\varphi^\beta(x_0) > (-\Delta)^s_{\mathbb{R}}\varphi^s(x_0) = 0$. This yields (4.20).

Finally, (4.21) follows from an easy computation using the definition of $(-\Delta)^s_{\mathbb{R}}$, and thus the proof is finished. □

4.2.2 The class $\mathcal{L}_0$

As defined in [69], for $s \in (0, 1)$ the ellipticity class $\mathcal{L}_0 = \mathcal{L}_0(s)$ consists of all operators L of the form

$$Lu(x) = (1-s) \int_{\mathbb{R}^n} \left(\frac{u(x+y) + u(x-y)}{2} - u(x) \right) \frac{b(y)}{|y|^{n+2s}}\, dy.$$

where

$$b \in L^\infty(\mathbb{R}^n) \quad \text{satisfies} \quad \lambda \leq b \leq \Lambda.$$

It is clear that

$$\mathcal{L}_* \subsetneq \mathcal{L}_0.$$

The extremal operators for the class $\mathcal{L}_0$ are denoted here by $M^+_{\mathcal{L}_0}$ and $M^-_{\mathcal{L}_0}$. Since $\mathcal{L}_* \subset \mathcal{L}_0$, we have

$$M^-_{\mathcal{L}_0} \leq M^- \leq M^+ \leq M^+_{\mathcal{L}_0}.$$

Hence, all elliptic equations with respect to $\mathcal{L}_*$ are elliptic with respect to $\mathcal{L}_0$ and all the definitions and results in [69] apply to the elliptic equations considered in this paper.

As in [69, 70] we consider the weighted L^1 spaces $L^1(\mathbb{R}^n, \omega_s)$, where

$$\omega_s(x) = (1-s)(1+|x|)^{-n-2s}. \tag{4.22}$$

The utility of this weighted space is that, if $L \in \mathcal{L}_0(s)$, then $Lu(x)$ can be evaluated classically and is continuous in $B_{\epsilon/2}$ provided $u \in C^2(B_\epsilon) \cap L^1(\mathbb{R}^n, \omega_s)$. One can then consider viscosity solutions to elliptic equations with respect to $\mathcal{L}_0(s)$ which are not bounded but belong to $L^1(\mathbb{R}^n, \omega_s)$. The weighted norm appears in stability results; see [70].

As said in the Introduction, the definitions we follow of viscosity solutions and viscosity inequalities are the ones in [69].

Next we state the interior Harnack inequality and the C^α estimate from [69].

Theorem 4.2.4 ([69])**.** *Let $s_0 \in (0,1)$ and $s \in [s_0, 1]$. Let $u \geq 0$ in $\mathbb{R}^n$ satisfy in the viscosity sense $M^-_{\mathcal{L}_0} u \leq C_0$ and $M^+_{\mathcal{L}_0} u \geq -C_0$ in B_R. Then,*

$$u(x) \leq C\big(u(0) + C_0 R^{2s}\big) \quad \text{for every } x \in B_{R/2},$$

for some constant C depending only on n, s_0, and ellipticity constants.

Theorem 4.2.5 ([69])**.** *Let $s_0 \in (0,1)$ and $s \in [s_0, 1]$. Let $u \in C(\overline{B_1}) \cap L^1(\mathbb{R}^n, \omega_s)$ satisfy in the viscosity sense $M^-_{\mathcal{L}_0} u \leq C_0$ and $M^+_{\mathcal{L}_0} u \geq -C_0$ in B_1. Then, $u \in C^\alpha\big(\overline{B_{1/2}}\big)$ with the estimate*

$$\|u\|_{C^\alpha(B_{1/2})} \leq C\big(C_0 + \|u\|_{L^\infty(B_1)} + \|u\|_{L^1(\mathbb{R}^n, \omega_s)}\big),$$

where α and C depend only on n, s, and ellipticity constants.

The following result is a consequence of the results in [187] in the case $s \in (1/2, 1)$. In the case $s \leq 1/2$ it follows as a particular case of the results for parabolic equations in [261].

Theorem 4.2.6 ([187], [261])**.** *Let $s_0 \in (0,1)$ and $s \in [s_0, 1]$. Let $f \in C(\overline{B_1})$ and $u \in C(\overline{B_1}) \cap L^\infty(\mathbb{R}^n)$ be a viscosity solution of $Iu = f(x)$ in B_1, where I is translation invariant and elliptic with respect to $\mathcal{L}_0(s)$, with $I0 = 0$. Then, $u \in C^s\big(\overline{B_{1/2}}\big)$ with the estimate*

$$\|u\|_{C^s(B_{1/2})} \leq C\big(\|f\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}\big),$$

where C depends only on n, s_0, and ellipticity constants.

In fact, [187, 261] establish not only a C^s estimate, but also a C^β one, for all $\beta < \min\{2s, 1+\alpha\}$. However, in this paper we only need the C^s estimate.

4.2.3 No fine boundary regularity for $\mathcal{L}_0$

The aim of this subsection is to show that the class $\mathcal{L}_0$ is too large for all solutions to behave comparably near the boundary. Moreover, we give necessary conditions on a subclass $\mathcal{L} \subset \mathcal{L}_0$ to have comparability of all solutions near the boundary. These necessary conditions lead us to the class $\mathcal{L}_*$.

In the next result we show that, for any scale invariant class $\mathcal{L} \subseteq \mathcal{L}_0$ that contains the fractional Laplacian $(-\Delta)^s$, and any unit vector ν, there exist powers $0 \le \beta_1 \le s \le \beta_2$ such that $M^+_{\mathcal{L}} \varphi^{\beta_1}_\nu = 0$ and $M^-_{\mathcal{L}} \varphi^{\beta_2}_\nu = 0$ in $\{x \cdot \nu > 0\}$. Before stating this result, we give the following

Definition 4.2.7. *We say that a class of operators $\mathcal{L}$ is* scale invariant *of order $2s$ if for each operator L in $\mathcal{L}$, and for all $R > 0$, the rescaled operator L_R, defined by*

$$(L_R u)(R\,\cdot) = R^{-2s} L\big(u(R\,\cdot)\big),$$

also belongs to $\mathcal{L}$.

The proposition reads as follows.

Proposition 4.2.8. *Assume that $\mathcal{L} \subset \mathcal{L}_0(s)$ is scale invariant of order $2s$. Then,*

(a) For every $\nu \in S^{n-1}$ and $\beta \in (0, 2s)$ the function φ^β_ν defined in (4.19) satisfies

$$\begin{aligned} M^+_{\mathcal{L}} \varphi^\beta_\nu(x) &= \overline{C}(\beta, \nu)(x \cdot \nu)^{\beta - 2s} \quad \text{in} \quad \{x \cdot \nu > 0\}, \\ M^-_{\mathcal{L}} \varphi^\beta_\nu(x) &= \underline{C}(\beta, \nu)(x \cdot \nu)^{\beta - 2s} \quad \text{in} \quad \{x \cdot \nu > 0\}. \end{aligned} \tag{4.23}$$

Here, $\overline{C}$ and $\underline{C}$ are constants depending only on s, β, ν, n, and ellipticity constants.

(b) The functions $\overline{C}$ and $\underline{C}$ are continuous in β and, for each unit vector ν, there are $\beta_1 \le \beta_2$ in $(0, 2s)$ such that

$$\overline{C}(\beta_1, \nu) = 0 \quad \text{and} \quad \underline{C}(\beta_2, \nu) = 0. \tag{4.24}$$

Moreover, for all $\beta \in (0, 2s)$,

$$\overline{C}(\beta, \nu) - \overline{C}(\beta_1, \nu) \text{ has the same sign as } \beta - \beta_1 \tag{4.25}$$

and

$$\underline{C}(\beta, \nu) - \underline{C}(\beta_2, \nu) \text{ has the same sign as } \beta - \beta_2. \tag{4.26}$$

(c) If in addition the fractional Laplacian $-(-\Delta)^s$ belongs to $\mathcal{L}$, then we have $\beta_1 \le s \le \beta_2$.

Proof. The scale invariance of $\mathcal{L}$ is equivalent to a scaling property of the extremal operators $M^+_{\mathcal{L}}$ and $M^-_{\mathcal{L}}$. Namely, for all $R > 0$, we have

$$M^\pm_{\mathcal{L}}\big(u(R\,\cdot)\big) = R^{2s}(M^\pm_{\mathcal{L}} u)(R\,\cdot).$$

(a) By this scaling property it is immediate to prove that given $\beta \in (0, 2s)$ and $\nu \in S^{n-1}$, the function φ_ν^β satisfies (4.23), where

$$\overline{C}(\beta, \nu) := M_{\mathcal{L}}^+ \varphi_\nu^\beta(\nu) \qquad \text{and} \qquad \underline{C}(\beta, \nu) := M_{\mathcal{L}}^- \varphi_\nu^\beta(\nu).$$

Of course, $\overline{C}$ and $\underline{C}$ depend also on s and the ellipticity constants, but these are fixed constants in this proof.

(b) Note that, as $\beta' \to \beta \in [0, 2s)$, we have $\varphi_\nu^{\beta'} \to \varphi_\nu^\beta$ in $C^2(\overline{B_{1/2}(\nu)})$ and in $L^1(\mathbb{R}^n, \omega_s)$. As a consequence, $\underline{C}$ and $\overline{C}$ are continuous in β in the interval $[0, 2s)$. Since $\varphi_\nu^\beta \to \chi_{\{x\cdot\nu>0\}}$ as $\beta \to 0$, we have that

$$\underline{C}(\nu, 0) \le \overline{C}(\nu, 0) < 0.$$

On the other hand, it is easy to see that

$$M_{\mathcal{L}_0}^- \varphi_\nu^\beta(\nu) \longrightarrow +\infty \quad \text{as } \beta \nearrow 2s.$$

Hence, using that $M_{\mathcal{L}_0}^- \le M_{\mathcal{L}}^-$, we obtain

$$0 < \underline{C}(\nu, \beta) \le \overline{C}(\nu, \beta) \quad \text{for } \beta \text{ close to } 2s.$$

Therefore, by continuity, there are β_1 and β_2 in $(0, 2s)$ such that

$$\overline{C}(\beta_1, \nu) = 0 \quad \text{and} \quad \underline{C}(\beta_2, \nu) = 0.$$

To prove (4.25), we observe that if $\beta > \beta_1$ the function φ_ν^β is be touched by below by $\varphi_\nu^{\beta_1} - C$ at some $x_0 \in \{x \cdot \nu > 0\}$ for some $C > 0$. It follows that

$$M_{\mathcal{L}}^+ \varphi_\nu^\beta(x_0) - M_{\mathcal{L}}^+ \varphi_\nu^{\beta_1}(x_0) \ge M_{\mathcal{L}_0}^- \big(\varphi_\nu^\beta - \varphi_\nu^{\beta_1}\big)(x_0) > 0.$$

Since the sign of $M_{\mathcal{L}}^+ \varphi_\nu^\beta$ is constant in $\{x \cdot \nu > 0\}$ it follows that $\overline{C}(\nu, \beta) > 0$ when $\beta > \beta_1$. Similarly one proves that $\overline{C}(\nu, \beta) < 0$ when $\beta < \beta_1$, and hence (4.26).

(c) It is an immediate consequence of the results in parts (a) and (b) and the fact that $-(-\Delta)^s \varphi_\nu^s = 0$ in $\{x \cdot \nu > 0\}$. □

Clearly, to hope for some good description of the boundary behavior of solutions to all elliptic equations with respect to a scale invariant class $\mathcal{L}$, it must be $\beta_1 = \beta_2$ for every direction ν. Typical classes $\mathcal{L}$ contain the fractional Laplacian $-(-\Delta)^s$. Thus, for them, we must have $\beta_1 = \beta_2 = s$ for all $\nu \in S^{n-1}$. If this happens, then

$$L\varphi_\nu^s = 0 \quad \text{in } \{x \cdot \nu > 0\} \quad \text{for all } L \in \mathcal{L}, \text{ and for all } \nu \in S^{n-1}, \tag{4.27}$$

since $M_{\mathcal{L}}^- \le L \le M_{\mathcal{L}}^+$ for all $L \in \mathcal{L}$.

As a consequence, we find the following.

Corollary 4.2.9. *Let β_1, β_2 be given by* (4.24) *in Proposition 4.2.8. Then, for the classes $\mathcal{L}_0$, $\mathcal{L}_1$, and $\mathcal{L}_2$ we have $\beta_1 < s < \beta_2$.*

Proof. Let us show that for $\mathcal{L} = \mathcal{L}_0$ the condition (4.27) is not satisfied. Indeed, we may easily cook up $L \in \mathcal{L}_0$ so that $L\varphi^s_{e_n}(x', 1) \neq 0$ for $x' \in \mathbb{R}^{n-1}$. Namely, if we take

$$b(y) = \Big(\lambda + (\Lambda - \lambda)\chi_{B_{1/2}}(y)\Big),$$

then at points $x = (x', 1)$ we have

$$0 > L\varphi^s_{e_n}(x) = (1-s)\int_{\mathbb{R}^n} \left(\frac{u(x+y) + u(x-y)}{2} - u(x)\right) \frac{b(y)}{|y|^{n+2s}}\, dy,$$

since $\varphi^s_{e_n}$ is concave in $B_{1/2}(x', 1)$ and $(-\Delta)^s \varphi^s_{e_n} = 0$ in $\{x_n > 0\}$.

By taking an smoothed version of $b(y)$, we obtain that both $\mathcal{L}_1$ and $\mathcal{L}_2$ fail to satisfy (4.27). □

By the results in Subsection 2.1, we have that the class $\mathcal{L}_*$ satisfies the necessary condition (4.27). Although we do not have a rigorous mathematical proof, we believe that $\mathcal{L}_*$ is actually the largest scale invariant subclass of $\mathcal{L}_0$ satisfying (4.27).

4.3 Barriers

In this section we construct supersolutions and subsolutions that are needed in our analysis. From now on, all the results are for the class $\mathcal{L}_*$ (and not for $\mathcal{L}_0$).

First we give two preliminary lemmas.

Lemma 4.3.1. *Let $s_0 \in (0, 1)$ and $s \in [s_0, 1)$. Let*

$$\varphi^{(1)}(x) = \big(dist(x, B_1)\big)^s \quad \text{and} \quad \varphi^{(2)}(x) = \big(dist(x, \mathbb{R}^n \setminus B_1)\big)^s.$$

Then,

$$0 \le M^-\varphi^{(1)}(x) \le M^+\varphi^{(1)}(x) \le C\left\{1 + (1-s)\big|\log(|x|-1)\big|\right\} \quad \text{in } B_2 \setminus B_1. \tag{4.28}$$

and

$$0 \ge M^+\varphi^{(2)}(x) \ge M^-\varphi^{(2)}(x) \ge -C\left\{1 + (1-s)\big|\log(1-|x|)\big|\right\} \text{ in } B_1 \setminus B_{1/2}. \tag{4.29}$$

The constant C depends only on s_0, n, and ellipticity constants.

Note that the above bounds are much better than $\big||x|-1\big|^{-s}$, which would be the expected bound given by homogeneity. This is since $\varphi^{(1)}$ and $\varphi^{(2)}$ are in some sense close to the 1D solution $(x_+)^s$.

Proof of Lemma 4.3.1. Let $L \in \mathcal{L}_*$. For points $x \in \mathbb{R}^n$ we use the notation $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$. To prove (4.28) let us estimate $L\varphi^{(1)}(x_\rho)$ where $x_\rho = (0, 1+\rho)$ for $\rho \in (0, 1)$ and for a generic $L \in \mathcal{L}_*$. To do it, we subtract the function $\psi(x) = (x_n - 1)^s_+$, which satisfies $L\psi(x_\rho) = 0$. Note that

$$\big(\varphi^{(1)} - \psi\big)(x_\rho) = 0 \quad \text{for all } \rho > 0$$

and that, for $|y| < 1$,

$$\big|dist\,(x_\rho + y, B_1) - (1 + \rho + y_n)_+\big| \le C|y'|^2.$$

This is because the level sets of the two previous functions are tangent on $\{y' = 0\}$.

Thus,

$$0 \le \big(\varphi_1^{(1)} - \psi\big)(x_\rho + y) \le \begin{cases} C\rho^{s-1}|y'|^2 & \text{for } y = (y', y_n) \in B_{\rho/2} \\ C|y'|^{2s} & \text{for } y = (y', y_n) \in B_1 \setminus B_{\rho/2} \\ C|y|^s & \text{for } y \in \mathbb{R}^n \setminus B_1. \end{cases}$$

The bound in $B_{\rho/2}$ follows from the inequality $a^s - b^s \le (a-b)b^{s-1}$ for $a > b > 0$.

Therefore, we have

$$\begin{aligned} 0 \le L\varphi^{(1)}(x_\rho) &= L\big(\varphi^{(1)} - \psi\big)(x_\rho) \\ &= (1-s)\int \frac{\big(\varphi_1^{(1)} - \psi\big)(x_\rho + y) + \big(\varphi_1^{(1)} - \psi\big)(x_\rho - y)}{2} \frac{a(y/|y|)}{|y|^{n+2s}}\, dy \\ &\le C(1-s)\Lambda \left(\int_{B_{\rho/2}} \frac{\rho^{s-1}|y'|^2 dy}{|y|^{n+2s}} + \int_{B_1 \setminus B_{\rho/2}} \frac{|y'|^{2s} dy}{|y|^{n+2s}} + \int_{\mathbb{R}^n \setminus B_1} \frac{|y|^s dy}{|y|^{n+2s}} \right) \\ &\le C\big(1 + (1-s)|\log \rho|\big). \end{aligned}$$

This establishes (4.28). The proof of (4.29) is similar. □

In the next result, instead, the bounds are those given by the homogeneity. In addition, the constant in the bounds has the right sign to construct (together with the previous lemma) appropriate barriers.

Lemma 4.3.2. *Let $s_0 \in (0,1)$ and $s \in [s_0, 1)$. Let*

$$\varphi^{(3)}(x) = \big(dist(x, B_1)\big)^{3s/2} \quad \text{and} \quad \varphi^{(4)}(x) = \big(dist(x, \mathbb{R}^n \setminus B_1)\big)^{3s/2}.$$

Then,

$$M^-\varphi^{(3)}(x) \ge c(|x| - 1)^{-s/2} \quad \text{for all } x \in B_2 \setminus B_1. \tag{4.30}$$

and

$$M^-\varphi^{(4)}(x) \ge c(1 - |x|)^{-s/2} - C \quad \text{for all } x \in B_1 \setminus B_{1/2}. \tag{4.31}$$

The constants $c > 0$ and C depend only on n, s_0, and ellipticity constants.

Proof. Let $L \in \mathcal{L}_*$. For points $x \in \mathbb{R}^n$ we use the notation $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$. To prove (4.31) let us estimate $L\varphi^{(4)}(x_\rho)$ where $x_\rho = (0, 1+\rho)$ for $\rho \in (0,1)$ and for a generic $L \in \mathcal{L}_*$. To do it we subtract the function $\psi(x) = (1 - x_n)_+^{3s/2}$, which by Lemma 4.2.3 satisfies $L\psi(x_\rho) = c\rho^{-s/2}$ for some $c > 0$. We note that

$$\big(\varphi^{(4)} - \psi\big)(x_\rho) = 0$$

and, similarly as in the proof of Lemma 4.3.1,

$$0 \ge \big(\varphi^{(4)} - \psi\big)(x_\rho + y) \ge \begin{cases} -C\rho^{3s/2-1}|y'|^2 & \text{for } y = (y', y_n) \in B_{\rho/2} \\ -C|y'|^{3s} & \text{for } y = (y', y_n) \in B_1 \setminus B_{\rho/2} \\ -C|y|^{3s/2} & \text{for } y \in \mathbb{R}^n \setminus B_1. \end{cases}$$

Hence,

$$\begin{aligned} L\varphi^{(4)}(x_\rho) - c\rho^{-s/2} &= L\big(\varphi^{(4)} - \psi\big)(x_\rho) \\ &\geq -C(1-s)\Lambda\left(\int_{B_{\rho/2}} \frac{\rho^{3s/2-1}|y'|^2 dy}{|y|^{n+2s}} + \int_{B_1\setminus B_{\rho/2}} \frac{|y'|^{3s} dy}{|y|^{n+2s}} + \int_{\mathbb{R}^n\setminus B_1} \frac{|y|^{s/2} dy}{|y|^{n+2s}}\right) \\ &\geq -C. \end{aligned}$$

This establishes (4.31). To prove (4.30), we now define $\psi(x) = (x_n - 1)_+^{3s/2}$, and we use Lemma 4.2.3 and the fact that $\varphi^{(3)} - \psi$ is nonnegative in all of $\mathbb{R}^n$ and vanishes on the positive x_n axis. □

We can now construct the sub and supersolutions that will be used in the next section.

Lemma 4.3.3. *Let $s_0 \in (0,1)$ and $s \in [s_0, 1)$. There are positive constants ϵ and C, and a radial, bounded, continuous function φ_1 which is $C^{1,1}$ in $B_{1+\epsilon} \setminus \overline{B}_1$ and satisfies*

$$\begin{cases} M^+\varphi_1(x) \leq -1 & \text{in } B_{1+\epsilon} \setminus \overline{B_1} \\ \varphi_1(x) = 0 & \text{in } B_1 \\ \varphi_1(x) \leq C\big(|x| - 1\big)^s & \text{in } \mathbb{R}^n \setminus B_1 \\ \varphi_1(x) \geq 1 & \text{in } \mathbb{R}^n \setminus B_{1+\epsilon} \end{cases}$$

The constants ϵ, c and C depend only on n, s_0, and ellipticity constants.

Proof. Let

$$\psi = \begin{cases} 2\varphi^{(1)} - \varphi^{(3)} & \text{in } B_2 \\ 1 & \text{in } \mathbb{R}^n \setminus B_2. \end{cases}$$

By Lemmas 4.3.1 and 4.3.2, for $|x| > 1$ it is

$$M^+\psi \leq C\left\{1 + (1-s)\Big|\log(|x|-1)\Big|\right\} - c(|x|-1)^{-s/2} + C.$$

Hence, we may take $\epsilon > 0$ small enough so that $M^+\psi \leq -1$ in $B_{1+\epsilon} \setminus \overline{B_1}$. We then set $\varphi_1 = C\psi$ with $C \geq 1$ large enough so that $\varphi_1 \geq 1$ outside $B_{1+\epsilon}$. □

Lemma 4.3.4. *Let $s_0 \in (0,1)$ and $s \in [s_0, 1)$. There is $c > 0$, and a radial, bounded, continuous function φ_2 that satisfies*

$$\begin{cases} M^-\varphi_2(x) \geq c & \text{in } B_1 \setminus B_{1/2} \\ \varphi_2(x) = 0 & \text{in } \mathbb{R}^n \setminus B_1 \\ \varphi_2(x) \geq c\big(1 - |x|\big)^s & \text{in } B_1 \\ \varphi_2(x) \leq 1 & \text{in } \overline{B_{1/2}}. \end{cases}$$

The constants ϵ, c and C depend only on n, s_0, and ellipticity constants.

Proof. We first construct a subsolution ψ in the annulus $B_1 \setminus \overline{B_{1-\epsilon}}$, for some small $\epsilon > 0$. Then, using it, we will construct the desired subsolution in $B_1 \setminus B_{1/2}$. Let

$$\psi = \varphi^{(2)} + \varphi^{(4)}.$$

By Lemmas 4.3.1 and 4.3.2, for $1/2 < |x| < 1$ it is

$$M^-\psi \geq -C\left\{1 + (1-s)\big|\log(1-|x|)\big|\right\} + c(1-|x|)^{-s/2} - C.$$

Hence, we can take $\epsilon > 0$ small enough so that $M^-\psi \geq 1$ in $B_1 \setminus \overline{B_{1-\epsilon}}$.

Let us now construct a subsolution in $B_1 \setminus \overline{B_{1/2}}$ from ψ, which is a subsolution only in $B_1 \setminus \overline{B_{1-\epsilon}}$. We consider

$$\Psi(x) = \max_{0 \leq k \leq N} C^k \psi(2^{k/N} x),$$

where N is a large integer and $C > 1$. Notice that, for C large enough, the set $\{x \in B_1 \ : \ \Psi(x) = \psi(x)\}$ is an annulus contained in $B_1 \setminus \overline{B_{1-\epsilon}}$.

Consider, for $k \geq 0$,

$$A_k = \left\{x \in B_1 \ : \ \Psi(x) = C^k \psi(2^{k/N} x)\right\}.$$

Since $A_0 \subset B_1 \setminus \overline{B_{1-\epsilon}}$, then Ψ satisfies $M^-\Psi \geq 1$ in A_0.

Observe that $A_k = 2^{-k/N} A_0$, since $C^{-1}\Psi(2^{1/n}x) = \Psi(x)$ in the annulus $\{1/2 < |x| < 2^{-1/n}\}$. Hence, for $x \in A_k$ we have $2^{k/N}x \in A_0 \subset B_1 \setminus \overline{B_{1-\epsilon}}$ and

$$M^-\Psi(x) > M^-\big(C^k \psi(2^{k/N}\,\cdot\,)\big)(x) = C^k 2^{2sk/N} M^-\psi(2^{k/N}x) > 1.$$

We then set $\varphi_2 = c\Psi$ with $c > 0$ small enough so that $\varphi_2(x) \leq 1$ in $\overline{B_{1/2}}$. □

Remark 4.3.5. Notice that the subsolution φ_2 constructed above is $C^{1,1}$ by below in $B_1 \setminus \overline{B}_{1/2}$, in the sense that it can be touched by below by paraboloids. This is important when considering non translation invariant equations for which a comparison principle for viscosity solutions is not available.

4.4 Krylov's method

The goal of this section is to prove Proposition 4.1.1. Its proof combines the interior Hölder regularity results of Caffarelli and Silvestre [69] and the next key Lemma.

Lemma 4.4.1. *Let $s_0 \in (0,1)$, $s \in [s_0, 1)$, and $u \in C\big(\overline{B_1^+}\big)$ be a viscosity solution of (4.6). Then, there exist $\alpha \in (0,1)$ and C depending only on n, s_0, and ellipticity constants, such that*

$$\sup_{B_r^+} u/x_n^s - \inf_{B_r^+} u/x_n^s \leq C r^\alpha \left(C_0 + \|u\|_{L^\infty(\mathbb{R}^n)}\right) \tag{4.32}$$

for all $r \leq 3/4$.

To prove Lemma 4.4.1 we need two preliminary lemmas.

We start with the first, which is a nonlocal version of Lemma 4.31 in [182]. Throughout this section we denote

$$D_r^* := B_{9r/10} \cap \{x_n > 1/10\}.$$

Lemma 4.4.2. *Let $s_0 \in (0,1)$ and $s \in [s_0, 1)$. Assume that u satisfies $u \geq 0$ in all of $\mathbb{R}^n$ and*

$$M^- u \leq C_0 \quad \text{in } B_r^+,$$

for some $C_0 > 0$. Then,

$$\inf_{D_r^*} u/x_n^s \leq C \left(\inf_{B_{r/2}^+} u/x_n^s + C_0 r^s \right) \tag{4.33}$$

for all $r \leq 1$, where C is a constant depending only on s_0, ellipticity constants, and dimension.

Proof. Step 1. Assume $C_0 = 0$. Let us call

$$m = \inf_{D_r^*} u/x_n^s \geq 0.$$

We have

$$u \geq m x_n^s \geq m(r/10)^s \quad \text{in } D_r^*. \tag{4.34}$$

Let us scale and translate the subsolution φ_2 in Lemma 4.3.4 as follows to use it as lower barrier:

$$\psi_r(x) := (r/10)^s \, \varphi_2\big(\tfrac{10(x-x_0)}{2r}\big) \, . \tag{4.35}$$

We then have, for some $c > 0$,

$$\begin{cases} M^- \psi_r \geq 0 & \text{in } B_{2r/10}(x_0) \setminus B_{r/10}(x_0) \\ \psi_r = 0 & \text{in } \mathbb{R}^n \setminus B_{2r/10}(x_0) \\ \psi_r \geq c\big(\frac{2r}{10} - |x|\big)^s & \text{in } B_{2/10}(x_0) \\ \psi_r \leq (r/10)^s & \text{in } B_{r/10}(x_0). \end{cases}$$

It is immediate to verify that $B_{r/2}^+$ is covered by balls of radius $2r/10$ such that the concentric ball of radius $r/10$ is contained in D_r^*, that is,

$$B_{r/2}^+ \subset \bigcup \big\{ B_{2r/10}(x_0) \, : \, B_{r/10}(x_0) \subset D_r^* \big\} \, .$$

Now, if we choose some ball $B_{r/10}(x_0) \subset D_r^*$ and define ψ_r by (4.35), then by (4.34) we have $u \geq m\psi_r$ in $B_{r/10}(x_0)$. On the other hand $u \geq m\psi_r$ outside $B_{2r/10}(x_0)$, since ψ_r vanishes there and $u \geq 0$ in all of $\mathbb{R}^n$ by assumption. Finally, $M^+ \psi_r \leq 0$, and since $C_0 = 0$, $M^- u \geq 0$ in the annulus $B_{2r/10}(x_0) \setminus B_{r/10}(x_0)$.

Therefore, it follows from the comparison principle that $u \geq m\psi_r$ in $B_{2r/10}(x_0)$. Since these balls of radius $2r/10$ cover $B_{r/2}^+$ and $\psi_r \geq c\big(\frac{2r}{10} - |x|\big)^s$ in $B_{2/10}(x_0)$, we obtain

$$u \geq c m x_n^s \qquad \text{in } B_{r/2}^+,$$

which yields (4.33).

Step 2. If $C_0 > 0$ we argue as follows. First, let

$$\phi(x) = \min\{1,\ 2(x_n)_+^s - (x_n)_+^{3s/2}\}.$$

By Lemma 4.2.3, we have that $M^+\phi \le -c$ in $\{0 < x_n < \epsilon\}$ for some $\epsilon > 0$ and some $c > 0$. By scaling ϕ and reducing c, we may assume $\epsilon = 1$.

We then consider

$$\tilde{u}(x) = u(x) + \frac{C_0}{c} r^{2s} \phi(x/r).$$

The function $\tilde{u}$ satisfies in $\{0 < x_n < r\}$

$$M^-\tilde{u} - M^-u \le M^+\left(\frac{C_0}{c} r^{2s}\phi(x/r)\right) \le -C_0$$

and hence

$$M^-\tilde{u} \le 0.$$

Using that $u(x) \le \tilde{u}(x) \le u(x) + CC_0 r^s (x_n)_+^s$ and applying Step 1 to $\tilde{u}$, we obtain (4.33). □

The second lemma towards Proposition 4.4.1 is a nonlocal version of Lemma 4.35 in [182]. It is an immediate consequence of the Harnack inequality of Caffarelli and Silvestre [69].

Lemma 4.4.3. *Let $s_0 \in (0,1)$, $s \in [s_0, 1)$, $r \le 1$, and u satisfy $u \ge 0$ in all of $\mathbb{R}^n$ and*

$$M^+u \ge -C_0 \quad \text{and} \quad M^-u \le C_0 \quad \text{in} \quad B_r^+.$$

Then,

$$\sup_{D_r^*} u/x_n^s \le C\left(\inf_{D_r^*} u/x_n^s + C_0 r^s\right),$$

for some constant C depending only on n, s_0, and ellipticity constants.

Proof. The lemma is a consequence of Theorem 4.2.4. Indeed, covering the set D_r^* with balls contained in B_r^+ and with radii comparable to r —using the same (scaled) covering for all r—, Theorem 4.2.4 yields

$$\sup_{D_r^*} u \le C\left(\inf_{D_r^*} u + C_0 r^{2s}\right).$$

Then, the lemma follows by noting that x_n^s is comparable to r^s in D_r^*. □

Next we prove Lemma 4.4.1.

Proof of Lemma 4.4.1. First, dividing u by a constant, we may assume that $C_0 + \|u\|_{L^\infty(\mathbb{R}^n)} \le 1$.

We will prove that there exist constants $C_1 > 0$ and $\alpha \in (0, s)$, depending only on n, s_0, and ellipticity constants, and monotone sequences $(m_k)_{k\ge1}$ and $(\overline{m}_k)_{k\ge1}$ satisfying the following. For all $k \ge 1$,

$$\overline{m}_k - m_k = 4^{-\alpha k}, \quad -1 \le m_k \le m_{k+1} < \overline{m}_{k+1} \le \overline{m}_k \le 1, \tag{4.36}$$

and

$$m_k \le C_1^{-1} u/x_n^s \le \overline{m}_k \quad \text{in } B_{r_k}^+, \quad \text{where } r_k = 4^{-k}. \tag{4.37}$$

Note that since $u = 0$ in B_1^- then we have that (4.37) is equivalent to the following inequality in B_{r_k} instead of $B_{r_k}^+$

$$m_k (x_n)_+^s \le C_1^{-1} u \le \overline{m}_k (x_n)_+^s \quad \text{in } B_{r_k}, \quad \text{where } r_k = 4^{-k}. \tag{4.38}$$

Clearly, if such sequences exist, then (4.32) holds for all $r \le 1/4$ with $C = 4^\alpha C_1$. Moreover, for $1/4 < r \le 3/4$ the result follows from (4.39) below. Hence, we only need to construct $\{m_k\}$ and $\{\overline{m}_k\}$.

Next we construct these sequences by induction.

Using the supersolution φ_1 in Lemma 4.3.3 we find that

$$-\frac{C_1}{2}(x_n)_+^s \le u \le \frac{C_1}{2}(x_n)_+^s \quad \text{in } B_{3/4}^+ \tag{4.39}$$

whenever C_1 is large enough. Thus, we may take $m_1 = -1/2$ and $\overline{m}_1 = 1/2$.

Assume now that we have sequences up to m_k and $\overline{m}_k$. We want to prove that there exist m_{k+1} and $\overline{m}_{k+1}$ which fulfill the requirements. Let

$$u_k = C_1^{-1} u - m_k (x_n)_+^s.$$

We will consider the positive part u_k^+ of u_k in order to have a nonnegative function in all of $\mathbb{R}^n$ to which we can apply Lemmas 4.4.2 and 4.4.3. Let $u_k = u_k^+ - u_k^-$. Observe that, by induction hypothesis,

$$u_k^+ = u_k \quad \text{and} \quad u_k^- = 0 \quad \text{in } B_{r_k}.$$

Moreover, $C_1^{-1} u \ge m_j (x_n)_+^s$ in B_{r_j} for each $j \le k$. Therefore, we have

$$u_k \ge (m_j - m_k)(x_n)_+^s \ge (m_j - \overline{m}_j + \overline{m}_k - m_k)(x_n)_+^s = (-4^{-\alpha j} + 4^{-\alpha k})(x_n)_+^s \quad \text{in } B_{r_j}.$$

But clearly $0 \le (x_n)_+^s \le r_j^s$ in B_{r_j}, and therefore using $r_j = 4^{-j}$

$$u_k \ge -r_j^s (r_j^\alpha - r_k^\alpha) \quad \text{in } B_{r_j} \text{ for each } j \le k.$$

Thus, since for every $x \in B_1 \setminus B_{r_k}$ there is $j < k$ such that

$$|x| < r_j = 4^{-j} \le 4|x|,$$

we find

$$u_k(x) \ge -r_k^{\alpha+s} \left|\frac{4x}{r_k}\right|^s \left(\left|\frac{4x}{r_k}\right|^\alpha - 1\right) \quad \text{outside } B_{r_k}. \tag{4.40}$$

Now let $L \in \mathcal{L}_*$. Using (4.40) and that $u_k^- \equiv 0$ in B_{r_k}, then for all $x \in B_{r_k/2}$ we have

$$\begin{aligned}
0 \le L u_k^-(x) &= (1-s) \int_{x+y \notin B_{r_k}} u_k^-(x+y) \frac{a(y/|y|)}{|y|^{n+2s}}\, dy \\
&\le (1-s) \int_{|y| \ge r_k/2} r_k^{\alpha+s} \left|\frac{8y}{r_k}\right|^s \left(\left|\frac{8y}{r_k}\right|^\alpha - 1\right) \frac{\Lambda}{|y|^{n+2s}}\, dy \\
&= (1-s)\Lambda r_k^{\alpha-s} \int_{|z| \ge 1/2} \frac{|8z|^s (|8z|^\alpha - 1)}{|z|^{n+2s}}\, dz \\
&\le \varepsilon_0 r_k^{\alpha-s},
\end{aligned}$$

where $\varepsilon_0 = \varepsilon_0(\alpha) \downarrow 0$ as $\alpha \downarrow 0$ since $|8z|^\alpha \to 1$. Since this can be done for all $L \in \mathcal{L}_*$, u_k^- vanishes in B_{r_k} and satisfies pointwise

$$0 \le M^- u_k^- \le M^+ u_m^- \le \varepsilon_0 r_k^{\alpha-s} \quad \text{in } B^+_{r_k/2}.$$

Therefore, recalling that

$$u_k^+ = C_1^{-1} u - m_k (x_n)_+^s + u_k^-,$$

and using that $M^+(x_n)_+^s = M^-(x_n)_+^s = 0$ in $\{x_n > 0\}$, we obtain

$$\begin{aligned} M^- u_k^+ &\le C_1^{-1} M^- u + M^+(u_k^-) \\ &\le C_1^{-1} + \varepsilon_0 r_k^{\alpha-s} \qquad \text{in } B^+_{r_k/2}. \end{aligned}$$

Also clearly

$$M^+ u_k^+ \ge M^+ u_k \ge -C_1^{-1} \qquad \text{in } B^+_{r_k/2}.$$

Now we can apply Lemmas 4.4.2 and 4.4.3 with u in its statements replaced by u_k^+. Recalling that

$$u_k^+ = u_k = C_1^{-1} u - m_k x_n^s \quad \text{in } B^+_{r_k},$$

we obtain

$$\begin{aligned} \sup_{D^*_{r_k/2}} (C_1^{-1} u/x_n^s - m_k) &\le C \left(\inf_{D^*_{r_k/2}} (C_1^{-1} u/x_n^s - m_k) + C_1^{-1} r_k^s + \varepsilon_0 r_k^\alpha \right) \\ &\le C \left(\inf_{B^+_{r_k/4}} (C_1^{-1} u/x_n^s - m_k) + C_1^{-1} r_k^s + \varepsilon_0 r_k^\alpha \right). \end{aligned} \tag{4.41}$$

On the other hand, we can repeat the same reasoning "upside down", that is, considering the functions $\overline{u}_k = \overline{m}_k (x_n)_+^s - u$ instead of u_k. In this way we obtain, instead of (4.41), the following

$$\sup_{D^*_{r_k/2}} (\overline{m}_k - C_1^{-1} u/x_n^s) \le C \left(\inf_{B^+_{r_k/4}} (\overline{m}^k - C_1^{-1} u/x_n^s) + C_1^{-1} r_k^s + \varepsilon_0 r_k^\alpha \right). \tag{4.42}$$

Adding (4.41) and (4.42) we obtain

$$\begin{aligned} \overline{m}_k - m_k &\le C \left(\inf_{B^+_{r_k/4}} (C_1^{-1} u/x_n^s - m_k) + \inf_{B^+_{r_k/4}} (\overline{m}_k - C_1^{-1} u/x_n^s) + C_1^{-1} r_k^s + \varepsilon_0 r_k^\alpha \right) \\ &= C \left(\inf_{B^+_{r_{k+1}}} C_1^{-1} u/x_n^s - \sup_{B^+_{r_{k+1}}} C_1^{-1} u/x_n^s + \overline{m}_k - m_k + C_1^{-1} r_k^s + \varepsilon_0 r_k^\alpha \right). \end{aligned}$$

Thus, using that $\overline{m}_k - m_k = 4^{-\alpha k}$, $\alpha < s$, and $r_k = 4^{-k} \le 1$, we obtain

$$\sup_{B^+_{r_{k+1}}} C_1^{-1} u/x_n^s - \inf_{B^+_{r_{k+1}}} C_1^{-1} u/x_n^s \le \left(\tfrac{C-1}{C} + C_1^{-1} + \varepsilon_0 \right) 4^{-\alpha k}.$$

Now we choose α small and C_1 large enough so that

$$\frac{C-1}{C} + C_1^{-1} + \varepsilon_0(\alpha) \le 4^{-\alpha}.$$

This is possible since $\varepsilon_0(\alpha) \downarrow 0$ as $\alpha \downarrow 0$ and the constant C depends only on n, s_0, and ellipticity constants. Then, we find

$$\sup_{B^+_{r_{k+1}}} C_1^{-1} u/x_n^s - \inf_{B^+_{r_{k+1}}} C_1^{-1} u/x_n^s \leq 4^{-\alpha(k+1)},$$

and thus we are able to choose m_{k+1} and $\overline{m}_{k+1}$ satisfying (4.36) and (4.37). □

To end this section, we give the

Proof of Proposition 4.1.1. Let $x \in B^+_{1/2}$ and let x_0 be its nearest point on $\{x_n = 0\}$. Let

$$d = dist\,(x, x_0) = x_n = dist\,(x, B_1^-).$$

By Theorem 4.2.5 (rescaled), we have

$$\|u\|_{C^\alpha\left(B_{d/2}(x)\right)} \leq C d^{-\alpha} \left(\|u\|_{L^\infty(\mathbb{R}^n)} + C_0\right).$$

Hence, since $\|(x_n)^{-s}\|_{C^\alpha\left(B_{d/2}(x)\right)} \leq C d^{-s}$, then for $r \leq d/2$

$$osc_{B_r(x)} u/x_n^s \leq C r^\alpha d^{-s-\alpha} \left(\|u\|_{L^\infty(\mathbb{R}^n)} + C_0\right). \tag{4.43}$$

On the other hand, by Lemma 4.4.1, for all $r \geq d/2$ we have

$$osc_{B_r(x)\cap B^+_{3/4}} u/x_n^s \leq C r^\alpha \left(\|u\|_{L^\infty(\mathbb{R}^n)} + C_0\right). \tag{4.44}$$

In both previous estimates $\alpha \in (0,1)$ depends only on n, s_0, and ellipticity constants. Let us call

$$M = \left(\|u\|_{L^\infty(\mathbb{R}^n)} + C_0\right).$$

Then, given $\theta > 1$ we have the following alternatives

(i) If $r \leq d^\theta/2$ then, by (4.43),

$$osc_{B_r(x)} u/x_n^s \leq C r^\alpha d^{-s-\alpha} M \leq C r^{\alpha-(s+\alpha)/\theta} M.$$

(ii) If $d^\theta/2 < r \leq d/2$ then, by (4.44),

$$osc_{B_r(x)} u/x_n^s \leq osc_{B_{d/2}(x)} u/x_n^s \leq C d^\alpha M \leq C r^{\alpha/\theta} M.$$

(iii) If $d/2 < r$, then by (4.44)

$$osc_{B_r(x)\cap B^+_{3/4}} u/x_n^s \leq C r^\alpha M.$$

Choosing $\theta > \frac{s+\alpha}{\alpha}$ (so that the exponent in (i) is positive), we obtain

$$osc_{B_r(x)\cap B^+_{3/4}} u/x_n^s \leq C r^{\alpha'} M \quad \text{whenever } x \in B^+_{1/2} \quad \text{and} \quad r > 0, \tag{4.45}$$

for some $\alpha' \in (0, \alpha)$. This means that $\|u/x_n^s\|_{C^{\alpha'}(B^+_{1/2})} \leq CM$, as desired. □

4.5 Liouville type theorems

The goal of this section is to prove Theorem 4.1.5.

First, as a consequence of Proposition 4.1.1 we obtain the following Liouville-type result involving here the extremal operators (in contrast with Theorem 4.1.5). Note also that the growth condition CR^β in this lemma holds for $\beta < s + \alpha$ (with α small), whereas we have $\beta < 2s$ in the Liouville Theorem 4.1.5.

Proposition 4.5.1. *Let $s_0 \in (0,1)$ and $s \in [s_0, 1)$. Let $\alpha > 0$ be the exponent given by Proposition 4.1.1. Assume that $u \in C(\mathbb{R}^n)$ is a viscosity solution of*

$$\begin{aligned} M^+u \geq 0 \quad \text{and} \quad M^-u \leq 0 \quad &\text{in} \quad \{x_n > 0\}, \\ u = 0 \quad &\text{in} \quad \{x_n < 0\}. \end{aligned}$$

Assume that, for some positive $\beta < s + \alpha$, u satisfies the growth control at infinity

$$\|u\|_{L^\infty(B_R)} \leq CR^\beta \quad \text{for all } R \geq 1. \tag{4.46}$$

Then,

$$u(x) = K(x_n)_+^s$$

for some constant $K \in \mathbb{R}$.

Proof. Given $\rho \geq 1$, let $v_\rho(x) = \rho^{-\beta}u(\rho x)$. Note that for all $\rho \geq 1$ the function v_ρ satisfies the same growth control (4.46) as u. Indeed,

$$\|v_\rho\|_{L^\infty(B_R)} = \rho^{-\beta}\|u\|_{L^\infty(B_{\rho R})} \leq \rho^{-\beta}C(\rho R)^\beta = CR^\beta.$$

In particular $\|v_\rho\|_{L^\infty(B_1)} \leq C$ and $\|v_\rho\|_{L^1(\mathbb{R}^n,\omega_s)} \leq C$, with C independent of ρ. Hence, the function $\tilde{v}_\rho = v_\rho\chi_{B_1}$ satisfies $M^+\tilde{v}_\rho \geq -C$ and $M^-\tilde{v}_\rho \leq C$ in $B_{1/2} \cap \{x_n > 0\}$, and $\tilde{v}_\rho = 0$ in $\{x_n < 0\}$. Also, $\|\tilde{v}_\rho\|_{L^\infty(B_{1/2})} \leq C$. Therefore, by Proposition 4.1.1 we obtain that

$$\left\|v_\rho/x_n^s\right\|_{C^\alpha(B^+_{1/4})} = \left\|\tilde{v}_\rho/x_n^s\right\|_{C^\alpha(B^+_{1/4})} \leq C.$$

Scaling this estimate back to u we obtain

$$\left[u/x_n^s\right]_{C^\alpha(B^+_{\rho/4})} = \rho^{-\alpha}\left[u(\rho x)/(\rho x_n)^s\right]_{C^\alpha(B^+_{1/4})} = \rho^{\beta-s-\alpha}\left[v_\rho/(x_n)^s\right]_{C^\alpha(B^+_{1/4})} \leq C\rho^{\beta-s-\alpha}.$$

Using that $\beta < s + \alpha$ and letting $\rho \to \infty$ we obtain

$$\left[u/x_n^s\right]_{C^\alpha(\mathbb{R}^n \cap \{x_n > 0\})} = 0,$$

which means $u = K\big(x_n\big)_+^s$. □

The previous proposition will be applied to tangential derivatives of a solution to $Iu = 0$ as in the situation of Theorem 4.1.5. It gives that u is in fact a function of x_n alone. To proceed, we need the following crucial lemma It is a Liouville-type result for the fractional Laplacian in dimension 1, and classifies all functions which are s-harmonic in $\mathbb{R}_+$, vanish in $\mathbb{R}_-$, and grow at infinity less than $|x|^\beta$ for some $\beta < 2s$.

Lemma 4.5.2. *Let u satisfy $(-\Delta)^s u = 0$ in $\mathbb{R}_+$ and $u = 0$ in $\mathbb{R}_-$. Assume that, for some $\beta \in (0, 2s)$, u satisfies the growth control $\|u\|_{L^\infty(0,R)} \le CR^\beta$ for all $R \ge 1$. Then $u(x) = K(x_+)^s$.*

To establish the lemma, we will need the following result. It classifies all homogeneous solutions (with no growth condition) that vanish in a half line of the extension problem of Caffarelli and Silvestre [68] in dimension $1 + 1$.

Lemma 4.5.3. *Let $s \in (0, 1)$. Let (x, y) denote a point in $\mathbb{R}^2$, and $r > 0$, $\theta \in (-\pi, \pi)$ be polar coordinates defined by the relations $x = r\cos\theta$, $y = r\sin\theta$. Assume that $\nu > -s$, and $q_\nu = r^{s+\nu}\Theta_\nu(\theta)$ is even with respect y (or equivalently with respect to θ) and solves*

$$\begin{cases} div\,(|y|^{1-2s}\nabla q_\nu) = 0 & \text{in } \{y \neq 0\} \\ \lim_{y\to 0} |y|^{1-2s}\partial_y q_\nu = 0 & \text{on } \{y = 0\} \cap \{x > 0\} \\ q_\nu = 0 & \text{on } \{y = 0\} \cap \{x < 0\}. \end{cases} \tag{4.47}$$

Then,

(a) ν belongs to $\mathbb{N} \cup \{0\}$ and

$$\Theta_\nu(\theta) = K\,|\sin\theta|^s\, P_\nu^s\big(\cos\theta\big),$$

where P_ν^μ is the associated Legendre function of first kind. Equivalently,

$$\Theta_\nu(\theta) = C\,\left|\cos\left(\frac{\theta}{2}\right)\right|^{2s} {}_2F_1\left(-\nu, \nu+1; 1-s; \frac{1-\cos\theta}{2}\right),$$

where ${}_2F_1$ is the hypergeometric function.

(b) The functions $\big\{\Theta_\nu\big\}_{\nu\in\mathbb{N}\cup\{0\}}$ are a complete orthogonal system in the subspace of even functions of the weighted space $L^2\big((-\pi,\pi), |\sin\theta|^{1-2s}\big)$.

Proof. We differ the proof to the Appendix. □

We can now give the

Proof of Lemma 4.5.2. Let

$$P_s(x, y) = \frac{p_{1,s}}{y}\frac{1}{\big(1 + (x/y)^2\big)^{\frac{1+2s}{2}}}$$

be the Poisson kernel for the extension problem of Caffarelli and Silvestre; see [68, 55].

Given the growth control $u(x) \le C|x|^\beta$ at infinity and $\beta < 2s$, the convolution

$$v(\,\cdot\,, y) = u * P_s(\,\cdot\,, y)$$

is well defined and is a solution of the extension problem

$$\begin{cases} div(y^{1-2s}\nabla v) = 0 & \text{in } \{y > 0\} \\ v(x, 0) = u(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Since $(-\Delta)^s u = 0$ in $\{x > 0\}$ and $u = 0$ in $\{x < 0\}$, the function v satisfies

$$\lim_{y \searrow 0} y^{1-2s} \partial_y v(x, y) = 0 \quad \text{for } x > 0 \quad \text{and} \quad v(x, 0) = 0 \quad \text{for } x < 0.$$

Hence, v solves (4.47).

Let Θ_ν, $\nu \in \mathbb{N} \cup \{0\}$, be as in Lemma 4.5.3. Recall that $r^{s+\nu}\Theta_\nu(\theta)$ also solve (4.47). By standard separation of variables, in every ball $B_R^+(0)$ of $\mathbb{R}^2$ the function v can be written as a series

$$v(x, y) = v(r\cos\theta, r\sin\theta) = \sum_{\nu=0}^{\infty} a_\nu r^{s+\nu} \Theta_\nu(\theta). \tag{4.48}$$

To obtain this expansion we use that, by Lemma 4.5.3 (b), the functions $\{\Theta_\nu\}_{\nu \in \mathbb{N} \cup \{0\}}$ are a complete orthogonal system in the subspace of even functions in the weighted space $L^2\big((-\pi, \pi), |\sin\theta|^{1-2s}\big)$, and hence are complete in $L^2\big((0, \pi), (\sin\theta)^{1-2s}\big)$.

Moreover, by uniqueness, the coefficients a_ν are independent of R and hence the series (4.48) provides a representation formula for $v(x, y)$ in the whole $\{y > 0\}$.

Now, we claim that the growth control $\|u\|_{L^\infty(-R,R)} \le CR^\beta$ with $\beta \in (0, 2s)$ is transferred to v (perhaps with a bigger constant C), that is,

$$\|v\|_{L^\infty(B_R^+)} \le CR^\beta.$$

To see this, consider the rescaled function $u_R(x) = R^{-\beta}u(Rx)$, which satisfy the same growth control of u. Then,

$$v_R = R^{-\beta} v(R\,\cdot) = u_R * P_s.$$

Since the growth control for u_R is independent of R we find a bound for $\|v_R\|_{L^\infty(B_1^+)}$ that is independent of R, and this means that v is controlled by CR^β in B_R^+, as claimed.

Next, since we may assume that $\int_0^\pi |\Theta_\nu(\theta)|^2 |\sin\theta|^a \, d\theta = 1$ for all $\nu \ge 0$, Parseval's identity yields

$$\int_{\partial^+ B_R} \big|v(x, y)\big|^2 y^a \, d\sigma = \sum_{\nu=0}^{\infty} |a_\nu|^2 R^{2s+2\nu+1+a},$$

where $\partial^+ B_R = \partial B_R \cap \{y > 0\}$. But by the growth control, we have

$$\int_{\partial^+ B_R} \big|v(x, y)\big|^2 y^a \, d\sigma \le CR^{2\beta} \int_{\partial^+ B_R} y^a \, d\sigma = CR^{2\beta+1+a}.$$

Finally, since $2\beta < 4s < 2s + 2$, this implies $a_\nu = 0$ for all $\nu \ge 1$, and hence $u(x) = K(x_+)^s$, as desired. □

The following basic Hölder estimate up to the boundary follows from [70, Section 3]. It is also a consequence of Lemma 4.6.4, which we prove in Section 4.6.

Lemma 4.5.4 ([70])**.** *Let $s_0 \in (0, 1)$ and $s \in [s_0, 1]$. Let u be a solution of $M^+u \ge 0$ and $M^-u \le 0$ in B_1^+, $u = 0$ in B_1^- and assume that $u \in L^1(\mathbb{R}^n, \omega_s)$. Then, for some $\alpha > 0$ it is $u \in C^\alpha\big(\overline{B}_{1/2}\big)$ and*

$$\|u\|_{C^\alpha(B_{1/2})} \le C\big(\|u\|_{L^\infty(B_1)} + \|u\|_{L^1(\mathbb{R}^n, \omega_s)}\big).$$

The constants α and C depend only on n, s_0, and ellipticity constants.

To end this section, we finally prove Theorem 4.1.5.

Proof of Theorem 4.1.5. Note that, since $\beta < 2s$, the growth control (4.11) yields $u \in L^1(\mathbb{R}^n, \omega_s)$.

Given $\rho \geq 1$, let $v_\rho = \rho^{-\beta} u(\rho\,\cdot)$. As in the proof of Proposition 4.5.1, v_ρ satisfies the same growth control as u, namely, $\|v_\rho\|_{L^\infty(B_R)} \leq CR^\beta$. Hence,

$$\|v_\rho\|_{L^\infty(B_1)} \leq C \quad \text{and} \quad \|v_\rho\|_{L^1(\mathbb{R}^n,\omega_s)} \leq C.$$

Moreover, since u satisfies $\mathrm{I}u = 0$ in $\{x_n > 0\}$ and $\mathrm{I}0 = 0$ we have that $M^+u \geq 0$ and $M^-u \leq 0$ in $\{x_n > 0\}$. This implies that $M^+v_\rho \geq 0$ and $M^-v_\rho \leq 0$ in B_1^+. Then it follows from Lemma 4.5.4 that

$$\|v_\rho\|_{C^\alpha(B_{1/2})} \leq C.$$

Scaling the previous estimate back to u and setting $\rho = R$, we obtain

$$[u]_{C^\alpha(B_R)} \leq CR^{\beta-\alpha}.$$

Next, given $\tau \in S^{n-1}$ with $\tau_n = 0$ and given $h > 0$, we consider the "tangential" incremental quotients $v^{(1)}(x) = \frac{u(x+h\tau)-u(x)}{h^\alpha}$. We have shown that

$$\|v^{(1)}\|_{L^\infty(B_R)} \leq CR^{\beta-\alpha}.$$

Moreover, since I is translation invariant, $v^{(1)}$ satisfies $M^+v^{(1)} \geq 0$ and $M^-v^{(1)} \leq 0$ in $\{x_n > 0\}$. Hence, we can apply again the previous scaling argument to $v^{(1)}$ and obtain

$$[v^{(1)}]_{C^\alpha(B_R)} \leq CR^{\beta-2\alpha} \quad \text{for all } R \geq 1.$$

Thus, we have a new growth control for $v^{(2)}(x) = \frac{u(x+h\tau)-u(x)}{h^{2\alpha}}$. We can keep iterating in this way until we obtain (after a finite number N of iterations)

$$\left\| \frac{u(x+h\tau)-u(x)}{h} \right\|_{L^\infty(B_R)} \leq CR^{\beta-1}. \tag{4.49}$$

Now, $v^{(N)} = \frac{u(x+h\tau)-u(x)}{h}$ satisfies $M^+v^{(N)} \geq 0$, $M^-v^{(N)} \leq 0$ in $\{x_n > 0\}$ and $v^{(N)} = 0$ in $\{x_n < 0\}$. Moreover, $v^{(N)}$ satisfies the growth control (4.49) with exponent $\beta - 1 < 2s - 1 < s$. Hence, using Proposition 4.5.1 we conclude that $v^{(N)} \equiv 0$. Therefore, $u(x + h\tau) = u(x)$ for all $h > 0$ and for all unit vector τ with $\tau_n = 0$. This means that u depends only on the variable x_n. That is, $u(x) = w(x_n)$ for some function $w : \mathbb{R} \longrightarrow \mathbb{R}$.

Now, if $\tilde{u}$ is a test function of the form $\tilde{u}(x) = \tilde{w}(x_n)$, Lemma 4.2.1 yields

$$\begin{aligned} M^+\tilde{u}(x) &= \sup_{L\in\mathcal{L}_*} L\tilde{u} \\ &= \sup_{\lambda\leq a\leq\Lambda} \frac{1-s}{2c_{1,s}} \left(\int_{S^{n-1}} |\theta_n|^{2s} a(\theta)\, d\theta \right) (-\Delta)^s_{\mathbb{R}} \tilde{w}(x_n) \\ &= C\left\{ \Lambda\big(-(-\Delta)^s_{\mathbb{R}}\tilde{w}(x_n)\big)^+ - \lambda\big(-(-\Delta)^s_{\mathbb{R}}\tilde{w}(x_n)\big)^- \right\}. \end{aligned} \tag{4.50}$$

Similarly,

$$M^-\tilde{u}(x) = C\left\{\lambda\big(-(-\Delta)^s\tilde{w}(x_n)\big)^+ - \Lambda\big(-(-\Delta)^s\tilde{w}(x_n)\big)^-\right\}. \tag{4.51}$$

Finally, recall that u solves $Iu = 0$ in $\mathbb{R}^n_+$, and $I0 = 0$. In particular we have $M^+u \geq 0$ and $M^-u \leq 0$ in $\mathbb{R}^n_+$ in the viscosity sense. Note that, since $u(x) = w(x_n)$, then we may test the viscosity inequalities using only test functions of the type $\tilde{u}(x) = \tilde{w}(x_n)$. Hence, using (4.50) and (4.51) we deduce that w is a viscosity solution of $(-\Delta)^s w = 0$ in $\mathbb{R}_+$ and $w = 0$ in $\mathbb{R}_-$. Clearly, w satisfies the growth control $\|w\|_{L^\infty(0,R)} \leq CR^\beta$. Therefore we deduce, using Lemma 4.5.2, that $u(x) = w(x_n) = K(x_n^+)^s$. □

4.6 Regularity by compactness

In this section we prove the main result of the paper: the boundary regularity in $C^{1,1}$ domains for fully nonlinear elliptic equations with respect to the class $\mathcal{L}_*$, given by Theorem 4.1.3.

As explained in the Introduction, the following result is the main ingredient in the proof of Theorem 4.1.3.

Proposition 4.6.1. *Let $s_0 \in (0,1)$, $\delta \in (0, s_0/4)$, $\rho_0 > 0$, and $\beta = 2s_0 - \delta$ be given constants.*

Let Γ be a $C^{1,1}$ hypersurface with radius ρ_0 splitting B_1 into Ω^+ and Ω^-; see Definition 4.1.2.

Let $s \in [s_0, \max\{1, s_0 + \delta\}]$ and $f \in C(\overline{\Omega^+})$. Assume that $u \in C(\overline{B_1}) \cap L^\infty(\mathbb{R}^n)$ is a solution of $Iu = f$ in Ω^+ and $u = 0$ in Ω^-, where I is a fully nonlinear translation invariant operator elliptic with respect to $\mathcal{L}_*(s)$.

Then, for all $z \in \Gamma \cap \overline{B_{1/2}}$ there is a constant $Q(z)$ with $|Q(z)| \leq CC_0$ for which

$$\Big|u(x) - Q(z)\big((x-z)\cdot\nu(z)\big)_+^s\Big| \leq CC_0|x-z|^\beta \quad \text{for all } x \in B_1,$$

where $\nu(z)$ is the unit normal vector to Γ at z pointing towards Ω^+ and

$$C_0 = \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\Omega^+)}.$$

The constant C depends only on n, ρ_0, s_0, δ, and ellipticity constants.

The proof of Proposition 4.6.1 is by contradiction, using a blow up and compactness argument. In order to fix ideas, we prove first the following reduced version of the statement.

> *Let $u \in C(\overline{B_1}) \cap L^\infty(\mathbb{R}^n)$ be a viscosity solution of $Iu = 0$ in B_1^+ and $u = 0$ in B_1^-. Then, given $\beta \in (s, 2s)$, there are $Q \in \mathbb{R}$ and $C > 0$ such that*
>
> $$\big|u(x) - Q(x_n)_+^s\big| \leq C|x|^\beta \quad \text{for all } x \in B_1. \tag{4.52}$$
>
> *The constant C is independent of x, but it could depend on everything else, also on u.*

We next prove (4.52) by contradiction. If (4.52) were false then it would be (by the contraposition of Lemma 4.6.2 below)

$$\sup_{r>0} r^{-\beta} \big\| u - Q_*(r)(x_n)_+^s \big\|_{L^\infty(B_r)} = +\infty,$$

where

$$Q_*(r) := arg\, min_{Q\in\mathbb{R}} \int_{B_r} \big(u(x) - Q(x_n)_+^s\big)^2\, dx = \frac{\int_{B_r} u(x)\,(x_n)_+^s\, dx}{\int_{B_r} (x_n)_+^{2s}\, dx}. \tag{4.53}$$

Then, a useful trick is to define the monotone in r quantity

$$\theta(r) = \sup_{r'>r} (r')^{-\beta} \max\Big\{ \big\| u - Q_*(r')(x_n)_+^s \big\|_{L^\infty(B_{r'})}, \, (r')^s \big| Q_*(2r') - Q_*(r') \big| \Big\},$$

which satisfies $\theta(r) \nearrow \infty$ as $r \searrow 0$. Then, there is a sequence $r_m \searrow 0$ such that

$$(r_m)^{-\beta} \max\Big\{ \big\| u - Q_*(r_m)(x_n)_+^s \big\|_{L^\infty(B_{r_m})}, \, (r_m)^s \big| Q_*(2r_m) - Q_*(r_m) \big| \Big\} \geq \frac{\theta(r_m)}{2}. \tag{4.54}$$

We then consider the blow up sequence

$$v_m(x) = \frac{u(r_m x) - (r_m)^s Q_*(r_m)(x_n)_+^s}{(r_m)^\beta \theta(r_m)}.$$

Note that (4.54) is equivalent to

$$\max\left\{ \|v_m\|_{L^\infty(B_1)}, \, \left| \frac{\int_{B_2} v_m(x)\,(x_n)_+^s\, dx}{\int_{B_2} (x_n)_+^{2s}\, dx} - \frac{\int_{B_1} v_m(x)\,(x_n)_+^s\, dx}{\int_{B_1} (x_n)_+^{2s}\, dx} \right| \right\} \geq 1/2. \tag{4.55}$$

Also, by definition of $Q_*(r_m)$, we have

$$\int_{B_1} v_m(x)(x_n)_+^s\, dx = 0, \tag{4.56}$$

which is the optimality condition of "least squares".

In addition, by definition of θ, we have

$$\frac{(r')^{s-\beta} |Q_*(2r') - Q_*(r')|}{\theta(r)} \leq 1 \quad \text{for all } r' \geq r.$$

Thus, for $R = 2^N$ we have

$$\begin{aligned} \frac{r^{s-\beta} |Q_*(rR) - Q_*(r)|}{\theta(r)} &\leq \sum_{j=0}^{N-1} 2^{j(\beta-s)} \frac{(2^j r)^{s-\beta} |Q_*(2^{j+1} r) - Q_*(2^j r)|}{\theta(r)} \\ &\leq \sum_{j=0}^{N-1} 2^{j(\beta-s)} \leq C 2^{N(\beta-s)} = C R^{\beta-s}. \end{aligned}$$

Moreover, v_m satisfy the growth control

$$\begin{aligned}\|v_m\|_{L^\infty(B_R)} &= \frac{1}{\theta(r_m)(r_m)^\beta}\big\|u - Q_*(r_m)(x_n)_+^s\big\|_{L^\infty(B_{r_m R})} \\ &\le \frac{R^\beta}{\theta(r_m)(r_m R)^\beta}\big\|u - Q_*(r_m R)(x_n)_+^s\big\|_{L^\infty(B_{r_m R})} + \\ &\qquad + \frac{1}{\theta(r_m)(r_m)^\beta}|Q_*(r_m R) - Q_*(r_m)|\,(r_m R)^s \qquad (4.57)\\ &\le \frac{R^\beta\theta(r_m R)}{\theta(r_m)} + CR^\beta \\ &\le CR^\beta,\end{aligned}$$

for all $R \ge 1$, where we have used the definition θ and its monotonicity.

In addition, since $M^+(x_n)_+^s = M^-(x_n)_+^s = 0$ in $\{x_n > 0\}$, and $\mathrm{I}u = 0$ in B_1^+, we obtain that

$$\tilde{\mathrm{I}}_m v_m = 0 \qquad \text{in } B^+_{1/r_m},$$

for some $\tilde{\mathrm{I}}_m$ translation invariant and elliptic with respect to $\mathcal{L}_*$. It follows, using the basic C^α estimate up to the boundary of Lemma 4.5.4 that (up to taking a subsequence)

$$v_m \longrightarrow v \quad \text{ locally uniformly in } \mathbb{R}^n.$$

Moreover, since all the v_m's satisfy the growth control (4.70), and $\beta < 2s$, by the dominated convergence theorem we obtain that

$$\int_{\mathbb{R}^n} \big|v_m - v\big|(x)\,\omega_s(x)\,dx \to 0.$$

Also, by Theorem 42 in [70] a subsequence of $\tilde{\mathrm{I}}_m$ converges weakly to some translation invariant operator $\tilde{\mathrm{I}}$ elliptic with respect to $\mathcal{L}_*$. Hence, the stability result in [70] yields

$$\tilde{\mathrm{I}}v = 0 \quad \text{in } \{x_n > 0\} \quad \text{ and } \quad v = 0 \quad \text{in } \{x_n < 0\}.$$

Furthermore, passing to the limit the growth control (4.70) we obtain $\|v\|_{L^\infty(B_R)} \le R^\beta$ for all $R \ge 1$. Thus, the Liouville type Theorem 4.1.5 implies

$$v(x) = K(x_n)_+^s.$$

Passing (4.56) to the limit (using uniform convergence) we find

$$\int_{B_1} v(x)(x_n)_+^s\,dx = 0.$$

But passing (4.55) to the limit, we obtain a contradiction. □

To prove Proposition 4.6.1 we will need a more involved version of this argument, but the main idea is essentially contained in the previous reduced version. Before proving Proposition 4.6.1, let us give some preliminary results.

The following lemma is for general continuous functions u, not necessarily solutions to some equation.

Lemma 4.6.2. *Let $\beta > s$ and $\nu \in S^{n-1}$ be some unit vector. Let $u \in C(B_1)$ and define*

$$\phi_r(x) := Q_*(r)\,(x\cdot\nu)_+^s, \tag{4.58}$$

where

$$Q_*(r) := arg\,min_{Q\in\mathbb{R}} \int_{B_r} \big(u(x) - Q(x\cdot\nu)_+^s\big)^2\,dx = \frac{\int_{B_r} u(x)\,(x\cdot\nu)_+^s\,dx}{\int_{B_r}(x\cdot\nu)_+^{2s}\,dx}.$$

Assume that for all $r \in (0,1)$ we have

$$\big\|u - \phi_r\big\|_{L^\infty(B_r)} \le C_0 r^\beta. \tag{4.59}$$

Then, there is $Q \in \mathbb{R}$ satisfying $|Q| \le C\big(C_0 + \|u\|_{L^\infty(B_1)}\big)$ such that

$$\big\|u - Q(x\cdot\nu)_+^s\big\|_{L^\infty(B_r)} \le CC_0 r^\beta$$

for some constant C depending only on β and s.

Proof. We may assume $\|u\|_{L^\infty(B_1)} = 1$. By (4.59), for all $x' \in B_r$ we have

$$\big|\phi_{2r}(x') - \phi_r(x')\big| \le \big|u(x') - \phi_{2r}(x')\big| + \big|u(x') - \phi_r(x')\big| \le CC_0 r^\beta.$$

But this happening for every $x' \in B_r$ yields, recalling (4.58),

$$\big|Q_*(2r) - Q_*(r)\big| \le CC_0 r^{\beta-s}.$$

In addition, since $\|u\|_{L^\infty(B_1)} = 1$, we clearly have that

$$|Q_*(1)| \le C. \tag{4.60}$$

Since $\beta > s$, this implies the existence of the limit

$$Q := \lim_{r\searrow 0} Q_*(r).$$

Moreover, using again $\beta - s > 0$,

$$\big|Q - Q_*(r)\big| \le \sum_{m=0}^{\infty}\big|Q_*(2^{-m}r) - Q_*(2^{-m-1}r)\big| \le \sum_{m=0}^{\infty} CC_0 2^{-m(\beta-s)} r^{\beta-s} \le CC_0 r^{\beta-s}.$$

In particular, using (4.60) we obtain

$$|Q| \le C(C_0 + 1). \tag{4.61}$$

We have thus proven that for all $r \in (0,1)$

$$\begin{aligned}\big\|u - Q(x\cdot\nu)_+^s\|_{L^\infty(B_r)} &\le \|u - Q_*(r)(x\cdot\nu)_+^s\|_{L^\infty(B_r)} + \\ &\quad + \|Q_*(r)(x\cdot\nu)_+^s - Q(x\cdot\nu)_+^s\|_{L^\infty(B_r)} \\ &\le C_0 r^\beta + |Q_*(r) - Q| r^s \le C(C_0+1) r^\beta.\end{aligned}$$

□

The following lemma will be used in the proof of Theorem 4.1.3 to obtain compactness for sequences of elliptic operators of variable order. Its proof is almost the same as the proof of Lemma 3.1 of [261].

Lemma 4.6.3. *Let $s_0 \in (0,1)$, $s_m \in [s_0, 1]$, and I_m such that*

- *I_m is a fully nonlinear translation invariant operator elliptic with respect to $\mathcal{L}_*(s_m)$.*
- *$I_m 0 = 0$.*

Then, a subsequence of $s_m \to s \in [s_0, 1]$ and a subsequence of I_m converges weakly (with the weight ω_{s_0}) to some fully nonlinear translation invariant operator I elliptic with respect to $\mathcal{L}_*(s)$.

Proof. We may assume by taking a subsequence that $s_m \to s \in [s_0, 1]$. Consider the class $\mathcal{L} = \bigcup_{s\in[s_0,1]} \mathcal{L}_*(s)$. This class satisfies Assuptions 23 and 24 of [70]. Also, each I_m is elliptic with respect to $\mathcal{L}$. Hence using Theorem 42 in [70] there is a subsequence of I_m converging weakly (with the weight ω_{s_0}) to a translation invariant operator I, also elliptic with respect to $\mathcal{L}$. Let us see next that I is in fact elliptic with respect to $\mathcal{L}_*(s) \subset \mathcal{L}$. Indeed, for test functions u and v that are quadratic polynomials in a neighborhood of x and that belong to $L^1(\mathbb{R}^n, \omega_{s_0})$, the inequalities

$$M^-_{s_m} v(x) \le \mathrm{I}_m(u+v)(x) - \mathrm{I}_m u(x) \le M^+_{s_m} v(x)$$

pass to the limit to obtain

$$M^-_{s} v(x) \le \mathrm{I}(u+v)(x) - \mathrm{I} u(x) \le M^+_{s} v(x).$$

□

The following lemma will be used to obtain a C^γ estimate up to the boundary for solutions to fully nonlinear integro-differential equations. This estimate will be useful in the proof of Proposition 4.6.1. It is essentially a consequence of the proof of Theorem 3.3 in [70]. Note that, in contrast with Proposition 4.6.1, in this lemma the assumption of regularity of the domain is only "from the exterior". Namely, we only assume that the exterior ball condition is satisfied.

Lemma 4.6.4. *Assume that B_1 is divided into two disjoint subdomains Ω_1 and Ω_2 such that $\overline{B_1} = \overline{\Omega}_1 \cup \overline{\Omega}_2$. Assume that $\Gamma := \partial\Omega_1 \setminus \partial B_1 = \partial\Omega_2 \setminus \partial B_1$ is a $C^{0,1}$ surface and that $0 \in \Gamma$. Moreover assume that, for some $\rho_0 > 0$, all the points on $\Gamma \cap \overline{B_{3/4}}$ can be touched by a ball of radius $\rho_0 \in (0, 1/4)$ contained in Ω_2.*

Let $s_0 \in (0,1)$ and $s \in [s_0, 1]$. Let $\alpha \in (0,1)$, $g \in C^\alpha\big(\overline{\Omega_2}\big)$, and $u \in C(\overline{B_1}) \cap L^1(\mathbb{R}^n, \omega_s)$ satisfy in the viscosity sense

$$M^+ u \ge -C_0 \ \text{ and } \ M^- u \le C_0 \ \text{ in } \ \Omega_1, \quad u = g \ \text{ in } \ \Omega_2.$$

Then, there is $\gamma \in (0, \alpha)$ such that $u \in C^\gamma\big(\overline{B_{1/2}}\big)$ with the estimate

$$\|u\|_{C^\gamma(B_{1/2})} \le C\big(\|u\|_{L^\infty(B_1)} + \|g\|_{C^\alpha(\Omega_2)} + \|u\|_{L^1(\mathbb{R}^n,\omega_s)} + C_0\big).$$

The constants C and γ depend only on n, s_0, α, ρ_0, and ellipticity constants.

Proof. Let $\tilde{u} = u\chi_{B_1}$. Then $\tilde{u}$ satisfies $M^+\tilde{u} \geq -C_0'$ and $M^-\tilde{u} \leq C_0'$ in $\Omega_1 \cap B_{3/4}$ and $\tilde{u} = g$ in Ω_2, where $C_0' \leq C\big(C_0 + \|u\|_{L^1(\mathbb{R}^n,\omega_s)}\big)$. Here, the constant C depends only on n, s_0, and ellipticity constants.

The proof consists of two steps.

First step. We next prove that there are $\delta > 0$ and C such that for all $z \in \Gamma \cap \overline{B_{1/2}}$ it is

$$\|\tilde{u} - g(z)\|_{L^\infty(B_r(z))} \leq Cr^\delta \quad \text{for all } r \in (0,1), \tag{4.62}$$

where δ and C depend only on n, s_0, C_0', $\|u\|_{L^\infty(B_1)}$, $\|g\|_{C^\alpha(\Omega_2)}$, and ellipticity constants.

Let $z \in \Gamma \cap \overline{B_{1/2}}$. By assumption, for all $R \in (0, \rho_0)$ there $y_R \in \Omega_2$ such that a ball $B_R(y_R) \subset \Omega_2$ touches Γ at z, i.e., $|z - y_R| = R$.

Let φ_1 and $\epsilon > 0$ be the supersolution and the constant in Lemma 4.3.3. Take

$$\psi(x) = g(y_R) + \|g\|_{C^\alpha(\Omega_2)}\big((1+\epsilon)R\big)^\alpha + \big(C_0' + \|u\|_{L^\infty(B_1)}\big)\varphi_1\left(\frac{x - y_R}{R}\right).$$

Note that ψ is above $\tilde{u}$ in $\Omega_2 \cap B_{(1+\epsilon)R}$. On the other hand, from the properties of φ_1, it is $M^+\psi \leq -\big(C_0' + \|u\|_{L^\infty(B_1)}\big)R^{-2s} \leq -C_0'$ in the annulus $B_{(1+\epsilon)R}(y_R) \setminus B_R(y_R)$, while $\psi \geq \|u\|_{L^\infty(B_1)} \geq \tilde{u}$ outside $B_{(1+\epsilon)R}(y_R)$. It follows that $\tilde{u} \leq \psi$ and thus we have

$$\tilde{u}(x) - g(z) \leq C\big(R^\alpha + (r/R)^s\big) \quad \text{for all } x \in B_r(z) \quad \text{and for all } r \in (0, \epsilon R) \text{ and } R \in (0, \rho_0).$$

Here, C denotes a constant depending only on n, s_0, C_0', $\|u\|_{L^\infty(B_1)}$, $\|g\|_{C^\alpha(\Omega_2)}$, and ellipticity constants. Taking $R = r^{1/2}$ and repeating the argument up-side down we obtain

$$|\tilde{u}(x) - g(z)| \leq C\big(r^{\alpha/2} + r^{s/2}\big) \leq Cr^\delta \quad \text{for all } x \in B_r(z) \text{ and } r \in (0, \epsilon^{1/2})$$

for $\delta = \frac{1}{2}\min\{\alpha, s_0\}$. Taking a larger constant C, (4.62) follows.

Second step. We now show that (4.62) and the interior estimates in Theorem 4.2.5 imply $\|u\|_{C^\gamma(B_{1/2})} \leq C$, where C depends only on the same quantities as above.

Indeed, given $x_0 \in \Omega_1 \cap B_{1/2}$, let $z \in \Gamma$ and $r > 0$ be such that

$$d = dist\,(x_0, \Gamma) = dist\,(x_0, z).$$

Let us consider

$$v(x) = \tilde{u}\left(x_0 + \frac{d}{2}x\right) - g(z).$$

We clearly have

$$\|v\|_{L^\infty(B_1)} \leq C \quad \text{and} \quad \|v\|_{L^1(\mathbb{R}^n,\omega_s)} \leq C.$$

On the other hand, v satisfies

$$M^+v(x) = (d/2)^{2s}M^+\tilde{u}(x_0 + rx) \leq C_0' \quad \text{in } B_1$$

and

$$M^-v(x) = (d/2)^{2s}M^-\tilde{u}(x_0 + rx) \geq -C_0' \quad \text{in } B_1.$$

Therefore, Theorem 4.2.5 yields

$$\|v\|_{C^\alpha(B_{1/2})} \leq C$$

or equivalently

$$[u]_{C^\alpha(B_{d/4}(x_0))} \le Cd^{-\alpha}. \tag{4.63}$$

Combining (4.62) and (4.63), using a similar argument as in the proof of Proposition 4.1.1, we obtain

$$\|u\|_{C^\gamma(\Omega_1 \cap B_{1/2})} \le C,$$

as desired. □

We can now give the

Proof of Proposition 4.6.1. The proof is by contradiction. Assume that there are sequences Γ_k, Ω_k^+, Ω_k^-, s_k, f_k, u_k, and I_k that satisfy the assumptions of the proposition. That is, for all $k \ge 1$:

- Γ_k is a $C^{1,1}$ hyper surface with radius ρ_0 splitting B_1 into Ω_k^+ and Ω_k^-.
- $s_k \in [s_0, \max\{1, s_0 + \delta\}]$.
- I_k is translation invariant and elliptic with respect to $\mathcal{L}_*(s_k)$.
- $\|u_k\|_{L^\infty(\mathbb{R}^n)} + \|f_k\|_{L^\infty(\Omega_k^+)} = 1$ (by scaling we may assume $C_0 = 1$).
- u_k is a solution of $\mathrm{I}_k u_k = f_k$ in Ω_k^+ and $u_k = 0$ in Ω_k^-.

Suppose for a contradiction that the conclusion of the proposition does not hold. That is, for all $C > 0$, there are k and $z \in \Gamma_k \cap \overline{B_{1/2}}$ for which no constant $Q \in \mathbb{R}$ satisfies

$$\left| u_k(x) - Q\big((x-z)\cdot \nu_k(z)\big)_+^{s_k} \right| \le C|x-z|^\beta \quad \text{for all } x \in B_1. \tag{4.64}$$

Above, $\nu_k(z)$ denotes the unit normal vector to Γ_k at z, pointing towards Ω_k^+.

In particular, noting that $s_k \in [s_0, s_0 + \delta]$ and $\beta \ge s_0 + 2\delta$ by assumption, and using Lemma 4.6.2, we obtain

$$\sup_k \sup_{z \in \Gamma_k \cap B_{1/2}} \sup_{r>0} r^{-\beta} \left\| u_k - \phi_{k,z,r} \right\|_{L^\infty(B_r(z))} = \infty, \tag{4.65}$$

where

$$\phi_{k,z,r}(x) = Q_{k,z}(r) \left((x-z)\cdot \nu_k(z)\right)_+^{s_k} \tag{4.66}$$

and

$$\begin{aligned} Q_{k,z}(r) &:= arg\,min_{Q\in\mathbb{R}} \int_{B_r(z)} \left| u_k(x) - Q\big((x-z)\cdot\nu_k(z)\big)_+^{s_k} \right|^2 dx \\ &= \frac{\int_{B_r(z)} u_k(x)\big((x-z)\cdot\nu_k(z)\big)_+^{s_k} dx}{\int_{B_r(z)} \big((x-z)\cdot\nu_k(z)\big)_+^{2s_k} dx}. \end{aligned}$$

Next define the monotone in r quantity

$$\theta(r) := \sup_k \sup_{z\in\Gamma_k\cap B_{1/2}} \sup_{r'>r} (r')^{-\beta} \max\Big\{ \left\| u_k - \phi_{k,z,r'} \right\|_{L^\infty(B_{r'}(x_0))}, \\ (r')^s \left| Q_{k,z}(2r') - Q_{k,z}(r') \right| \Big\}.$$

We have $\theta(r) < \infty$ for $r > 0$ and $\theta(r) \nearrow \infty$ as $r \searrow 0$. Clearly, there are sequences $r_m \searrow 0$, k_m, and $z_m \to z \in \overline{B}_{1/2}$, for which

$$(r_m)^{-\beta} \max\Big\{ \|u_{k_m} - \phi_{k_m,z_m,r_m}\|_{L^\infty(B_{r_m}(x_m))} , \\ (r_m)^s \left|Q_{k_m,z_m}(2r_m) - Q_{k_m,z_m}(r_m)\right| \Big\} \geq \theta(r_m)/2. \tag{4.67}$$

From now on in this proof we denote $\phi_m = \phi_{k_m,z_m,r_m}$, $\nu_m = \nu_{k_m}(z_m)$, and $s_m = s_{k_m}$.

In this situation we consider

$$v_m(x) = \frac{u_{k_m}(z_m + r_m x) - \phi_m(z_m + r_m x)}{(r_m)^\beta \theta(r_m)}.$$

Note that, for all $m \geq 1$,

$$\int_{B_1} v_m(x) \big(x \cdot \nu_m\big)_+^{s_m} \, dx = 0. \tag{4.68}$$

This is the optimality condition for least squares.

Note also that (4.67) is equivalent to

$$\max\left\{ \|v_m\|_{L^\infty(B_1)} , \left| \frac{\int_{B_2} v_m(x) \, (x \cdot \nu_m)_+^{s_m} \, dx}{\int_{B_2} (x \cdot \nu_m)_+^{2s_m} \, dx} - \frac{\int_{B_1} v_m(x) \, (x \cdot \nu_m)_+^{s_m} \, dx}{\int_{B_1} (x \cdot \nu_m)_+^{2s_m} \, dx} \right| \right\} \geq 1/2, \tag{4.69}$$

which holds for all $m \geq 1$.

In addition, by definition of θ, for all k and z we have

$$\frac{(r')^{s-\beta} |Q_{k,z}(2r') - Q_{k,z}(r')|}{\theta(r)} \leq 1 \quad \text{for all } r' \geq r > 0.$$

Thus, for $R = 2^N$ we have

$$\begin{aligned} \frac{r^{s_k-\beta} |Q_{k,z}(rR) - Q_{k,z}(r)|}{\theta(r)} &\leq \sum_{j=0}^{N-1} 2^{j(\beta - s_k)} \frac{(2^j r)^{s_k-\beta} |Q_{k,z}(2^{j+1} r) - Q_{k,z}(2^j r)|}{\theta(r)} \\ &\leq \sum_{j=0}^{N-1} 2^{j(\beta - s_k)} \leq C 2^{N(\beta - s_k)} = C R^{\beta - s_k}, \end{aligned}$$

where we have used $\beta - s_k \geq \delta$.

Moreover, we have

$$\begin{aligned} \|v_m\|_{L^\infty(B_R)} &= \frac{1}{\theta(r_m)(r_m)^\beta} \big\| u_{k_m} - Q_{k_m,z_m}(r_m) \big((x - z_m) \cdot \nu_m\big)_+^{s_m} \big\|_{L^\infty(B_{r_m R})} \\ &\leq \frac{R^\beta}{\theta(r_m)(r_m R)^\beta} \big\| u_{k_m} - Q_{k_m,z_m}(r_m R) \big((x - z_m) \cdot \nu_m\big)_+^{s_m} \big\|_{L^\infty(B_{r_m R})} + \\ &\qquad + \frac{1}{\theta(r_m)(r_m)^\beta} |Q_{k_m,z_m}(r_m R) - Q_{k_m,z_m}(r_m)| \, (r_m R)^{s_m} \\ &\leq \frac{R^\beta \theta(r_m R)}{\theta(r_m)} + C R^\beta, \end{aligned}$$

and hence v_m satisfy the growth control

$$\|v_m\|_{L^\infty(B_R)} \le CR^\beta \quad \text{for all } R \ge 1. \tag{4.70}$$

We have used the definition $\theta(r)$ and its monotonicity.

Now, without loss of generality (taking a subsequence), we assume that

$$\nu_m \longrightarrow \nu \in S^{n-1}.$$

Then, the rest of the proof consists mainly in showing the following Claim.

Claim. *A subsequence of v_m converges locally uniformly in $\mathbb{R}^n$ to some function v which satisfies $\tilde{I}v = 0$ in $\{x \cdot \nu > 0\}$ and $v = 0$ in $\{x \cdot \nu < 0\}$, for some $\tilde{I}$ translation invariant and elliptic with respect to $\mathcal{L}_*$.*

Once we know this, a contradiction is immediately reached using the Liouville type Theorem 4.1.5, as seen at the end of the proof.

To prove the Claim, given $R \ge 1$ and m such that $r_m R < 1/2$ define

$$\Omega^+_{R,m} = \big\{x \in B_R \ : \ (z_m + r_m x) \in \Omega^+_{k_m} \quad \text{and} \quad x \cdot \nu_m(z_m) > 0\big\}.$$

Notice that for all R and k, the origin 0 belongs to the boundary of $\Omega^+_{R,m}$.

We will use that v_m satisfies an elliptic equation in $\Omega^+_{R,m}$. Namely,

$$\tilde{\mathrm{I}}_m v_m(x) = \frac{(r_m)^{2s_m}}{(r_m)^\beta \theta(r_m)} f_{k_m}(z_m + r_m x) \quad \text{ in } \Omega^+_{R,m}. \tag{4.71}$$

where $\tilde{\mathrm{I}}_m$ is defined by

$$\tilde{\mathrm{I}}_m \left(\frac{w(z_m + r\,\cdot) - \phi_m(z_m + r\,\cdot)}{(r_m)^\beta \theta(r_m)} \right)(x) = \frac{(r_m)^{2s_m}}{(r_m)^\beta \theta(r_m)} \big(\mathrm{I}_{k_m} w\big)(z_m + rx),$$

for all test function w. Equivalently, for all test function v,

$$\begin{aligned} \tilde{\mathrm{I}}_m v(x) &\overset{(*)}{=} \frac{(r_m)^{2s_m}}{(r_m)^\beta \theta(r_m)} \mathrm{I}_{k_m} \left((r_m)^\beta \theta(r_m)\ v\left(\frac{\cdot - z_m}{r} \right) + \phi_m(\,\cdot\,) \right)(z_m + r_m x) \\ &\overset{(**)}{=} \frac{(r_m)^{2s_m}}{(r_m)^\beta \theta(r_m)} \mathrm{I}_{k_m} \left((r_m)^\beta \theta(r_m)\ v\left(\frac{\cdot - z_m}{r_m} \right) \right)(z_m + r_m x), \end{aligned}$$

the last identity being valid only in $\{x \cdot \nu_m > 0\}$ since $M^+\phi_m = M^-\phi_m = 0$ in $\{(x - z) \cdot \nu_m > 0\}$.

Note that the right hand side of (4.71) converges uniformly to 0 as $r_m \searrow 0$, since $\beta = 2s_0 - \delta < 2s_m$ and $\theta(r_m) \nearrow \infty$.

Using that I_{k_m} is translation invariant and elliptic with respect to $\mathcal{L}_*(s_m)$ and that $\mathrm{I}_{k_m} 0 = 0$ we readily show that $\tilde{\mathrm{I}}_m$ is also elliptic with respect to $\mathcal{L}_*(s_m)$ (i.e., with the same ellipticity constants Λ and λ, which are always fixed). Also, since the domains $\Omega^+_{R,m}$ are always contained in $\{(x - z_m) \cdot \nu_m > 0\}$ we may define $\tilde{\mathrm{I}}_m$ by $(**)$, and hence it is a translation invariant operator.

In order to prove the convergence of a subsequence of v_m we first obtain, for every fixed $R \ge 1$, a uniform in m bound for $\|v_m\|_{C^\delta(B_R)}$, for some small $\delta > 0$. Then the local

uniform convergence of a subsequence of v_m follows from the Arzelà-Ascoli theorem. Let us fix $R \geq 1$ and consider that m is always large enough so that $r_m R < 1/4$.

Let Σ_m^- be the half space which is "tangent" to $\Omega_{k_m}^-$ at z_m, namely,

$$\Sigma_m^- := \big\{(x - z_m) \cdot \nu(z_m) < 0\big\}.$$

The first step is showing that, for all m and for all $r < 1/4$,

$$\big\|u_{k_m} - \phi_m\big\|_{L^\infty\left(B_r(z_m) \cap (\Omega_{k_m}^- \cup \Sigma_m^-)\right)} \leq C r^{2s_m} \leq C r^{2s_0} \tag{4.72}$$

for some constant C depending only on s_0, ρ_0, ellipticity constants, and dimension.

Indeed, we may rescale and slide the supersolution φ_1 from Lemma 4.3.3 and use the fact that all points of $\Gamma_{k_m} \cap B_{3/4}$ can be touched by balls of radius ρ_0 contained in $\Omega_{k_m}^-$. We obtain that

$$|u_{k_m}| \leq C\big(dist\,(x, \Omega_{k_m}^-)\big)^{s_m},$$

with C depending only on n, s_0, ρ_0, and ellipticity constants. On the other hand, by definition of ϕ_m we have

$$|\phi_m| \leq C\big(dist\,(x, \Sigma_m^-)\big)^{s_m}.$$

But by assumption, points on $\Gamma_k \cap B_{3/4}$ can be also touched by balls of radius ρ_0 from the $\Omega_{k_m}^+$ side, and hence we have a quadratic control (depending only on ρ_0) on on how Γ_{k_m} separates from the hyperplane $\partial\Sigma_m^-$. As a consequence, in $B_r(z_m) \cap (\Omega_{k_m}^- \cup \Sigma_m^-)$ we have

$$C\big(dist\,(x, \Omega_{k_m}^-)\big)^{s_m} \leq C r^{2s_m} \quad \text{and} \quad C\big(dist\,(x, \Sigma_m^-)\big)^{s_m} \leq C r^{2s_m}.$$

Hence, (4.72) holds.

We use now Lemma 4.6.4 to obtain that, for some small $\gamma \in (0, s_0)$,

$$\|u_{k_m}\|_{C^\gamma(B_{1/8}(z_m))} \leq C \quad \text{for all } m.$$

On the other hand, clearly

$$\|\phi_m\|_{C^\gamma(B_{1/8}(z_m))} \leq C \quad \text{for all } m.$$

Hence,

$$\big\|u_{k_m} - \phi_m\big\|_{C^\gamma\left(B_r(z_m) \cap (\Omega_{k_m}^- \cup \Sigma_m^-)\right)} \leq C. \tag{4.73}$$

Next, interpolating (4.72) and (4.73) we obtain, for some positive $\delta < \gamma$ small enough (depending on γ, s_0, and δ),

$$\big\|u_{k_m} - \phi_m\big\|_{C^\delta\left(B_r(z_m) \cap (\Omega_{k_m}^- \cup \Sigma_m^-)\right)} \leq C r^{2s_0 - \delta} = C r^\beta. \tag{4.74}$$

Therefore, scaling (4.74) we find that

$$\big\|v_m\big\|_{C^\delta\left(B_R \setminus \Omega_{R,m}^+\right)} \leq C \quad \text{for all } m \text{ with } r_m R < 1/4. \tag{4.75}$$

Next we observe that the boundary points on $\partial\Omega_{R,m}^+ \cap B_{3R/4}$ can be touched by balls of radius $(\rho_0/r_m) \geq \rho_0$ contained in $B_R \setminus \Omega_{R,m}^+$. We then apply Lemma 4.6.4

(rescaled) to v_m. Indeed, we have that v_m solves (4.71) and satisfies (4.75). Thus, we obtain, for some $\delta' \in (0, \delta)$,

$$\|v_m\|_{C^{\delta'}(B_{R/2})} \leq C(R), \quad \text{for all } m \text{ with } r_m R < 1/4, \tag{4.76}$$

where we write $C(R)$ to emphasize the dependence on R of the constant, which also depends on s_0, ρ_0, ellipticity constants, and dimension, but not on m.

As said above, the Arzelà-Ascoli theorem and the previous uniform (in m) $C^{\delta'}$ estimate (4.76) yield the local uniform convergence in $\mathbb{R}^n$ of a subsequence of v_m to some function v.

Next, since all the v_m's satisfy the growth control (4.70), and $2s_0 > \beta$, by the dominated convergence theorem we have $v_m \longrightarrow v$ in $L^1(\mathbb{R}^n, \omega_{s_0})$.

In addition, by Lemma 4.6.3 there is a subsequence of s_m converging to some $s \in [s_0, \min\{1, s_0 + \delta\}]$ and a subsequence of $\tilde{\mathrm{I}}_m$ which converges weakly to some translation invariant operator $\tilde{\mathrm{I}}$, which is elliptic with respect to $\mathcal{L}_*(s)$. Hence, it follows from the stability result in [70, Lemma 5] that $\tilde{\mathrm{I}} v = 0$ in all of $\mathbb{R}^n$. Thus, the Claim is proved.

Finally, passing to the limit the growth control (4.70) on v_m we find $\|v\|_{L^\infty(B_R)} \leq R^\beta$ for all $R \geq 1$. Hence, by Theorem 4.1.5, it must be

$$v(x) = K\big(x \cdot \nu(z)\big)_+^s.$$

Passing (4.68) to the limit, we find

$$\int_{B_1} v(x)\big(x \cdot \nu(z)\big)_+^s \, dx = 0.$$

But passing (4.69) to the limit, we reach the contradiction. □

Before giving the proof of Theorem 4.1.3, we prove the following.

Lemma 4.6.5. *Let Γ be a $C^{1,1}$ surface of radius $\rho_0 > 0$ splitting B_1 into Ω^+ and Ω^-; see Definition 4.1.2. Let $d(x) = dist\,(x, \Omega^-)$. Let $x_0 \in B_{1/2}$ and $z \in \Gamma$ be such that*

$$dist\,(x_0, \Gamma) = dist\,(x_0, z) =: 2r.$$

Then,

$$\Big\| \big((x - z) \cdot \nu(z)\big)_+^s - d^s(x) \Big\|_{L^\infty(B_r(x_0))} \leq C r^{2s}, \tag{4.77}$$

$$\Big[d^s - \big((x - z) \cdot \nu(z)\big)_+^s \Big]_{C^{s-\epsilon}(B_r(x_0))} \leq C r^s, \tag{4.78}$$

and

$$\big[d^{-s}\big]_{C^{s-\epsilon}(B_r(x_0))} \leq C r^{-2s+\epsilon}. \tag{4.79}$$

The constant C depends only on ρ_0.

Proof. Let us denote

$$\bar{d}(x) = \big((x-z)\cdot\nu(z)\big)_+.$$

First, since Γ is $C^{1,1}$ with curvature radius bounded below by ρ_0, we have that $|\bar{d}-d| \le Cr^2$ in $B_r(x_0)$, and thus (4.77) follows.

To prove (4.78) we use on the one hand that

$$\big\|\nabla d - \nabla\bar{d}\big\|_{L^\infty(B_r(x_0))} \le Cr, \tag{4.80}$$

which also follows from the fact that Γ is $C^{1,1}$. On the other hand, using the inequality $|a^{s-1}-b^{s-1}| \le |a-b|\max\{a^{s-2},b^{s-2}\}$ for $a,b>0$, we find

$$\big\|d^{s-1}-\bar{d}^{s-1}\big\|_{L^\infty(B_r(x_0))} \le Cr^2 \max\left\{\big\|d^{s-2}\big\|_{L^\infty(B_r(x_0))}\,,\,\big\|\bar{d}^{s-2}\big\|_{L^\infty(B_r(x_0))}\right\} \le Cr^s. \tag{4.81}$$

Thus, using (4.80) and (4.81), we deduce

$$\big[d^s-\bar{d}^s\big]_{C^{0,1}(B_r(x_0))} = \big\|d^{s-1}\nabla d - \bar{d}^{s-1}\nabla\bar{d}\,\big\|_{L^\infty(B_r(x_0))} \le Cr^s.$$

Therefore, (4.78) follows.

Finally, interpolating the inequalities

$$\big[d^{-s}\big]_{C^{0,1}(B_r(x_0))} = \|d^{-s-1}\nabla d\|_{L^\infty(B_r(x_0))} \le Cr^{-s-1} \quad\text{and}\quad \|d^{-s}\|_{L^\infty(B_r(x_0))} \le Cr^{-s},$$

(4.79) follows. □

We can finally give the

Proof of Theorem 4.1.3. As usual, we may assume that

$$\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\Omega^+)} \le 1.$$

First, note that by Proposition 4.6.1 we have that, for all $z \in \Gamma \cap \overline{B_{1/2}}$, there is $Q = Q(z)$ such that

$$|Q(z)| \le C \quad\text{and}\quad \|u - Q\left((x-z)\cdot\nu(z)\right)_+^s\|_{L^\infty(B_R(z))} \le CR^{2s-\epsilon} \tag{4.82}$$

for all $R>0$, where C depends only on n, s_0, ρ_0, ϵ, and ellipticity constants.

Indeed, let $\delta = \min\{\epsilon/2, s_0/4\}$ and take a partition $s_0 < s_1 < \cdots < s_N = 1$ of $[s_0,1]$ satisfying $|s_{j+1}-s_j| \le \delta$. Then, using Proposition 4.6.1 with s_0 replaced by s_j, (4.82) holds for all $s \in [s_j, s_{j+1}]$ with a constant C_j depending only on n, s_j, ρ_0, and ellipticity constants. Taking $C = \max_j C_j$, (4.82) holds for all $s \in [s_0,1]$.

Now, to prove the $C^{s-\epsilon}$ estimate up to the boundary for u/d^s we must combine the C^s interior estimate for u in Theorem 4.2.6 with (4.82). To do it, we will use a similar argument for "glueing estimates" as in the proof of Proposition 4.1.1. However, here we need to be more precise in the argument because we want to obtain the best possible Hölder exponent.

Let x_0 be a point in $\Omega^+ \cap B_{1/4}$, and let $z \in \Gamma$ be such that

$$2r := dist\,(x_0,\Gamma) = dist\,(x_0,z) < \rho_0.$$

Note that $B_r(x_0) \subset B_{2r}(x_0) \subset \Omega^+$ and that $z \in \Gamma \cap B_{1/2}$ (since $0 \in \Gamma$).

We claim now that there is $Q = Q(x_0)$ such that $|Q(x_0)| \leq C$,

$$\|u - Qd^s\|_{L^\infty(B_r(x_0))} \leq Cr^{2s-\epsilon}, \tag{4.83}$$

and

$$[u - Qd^s]_{C^{s-\epsilon}(B_r(x_0))} \leq Cr^s, \tag{4.84}$$

where the constant C depends only on n, s_0, ϵ, ρ_0, and ellipticity constants.

Indeed, (4.83) follows immediately combining (4.82) and (4.77).

To prove (4.84), let

$$v_r(x) = r^{-s}u(z + rx) - Q\,(x \cdot \nu(z))_+^s.$$

Then, (4.82) implies

$$\|v_r\|_{L^\infty(B_4)} \leq Cr^{s-\epsilon}$$

and

$$\|v_r\|_{L^1(\mathbb{R}^n, \omega_s)} \leq Cr^{s-\epsilon}.$$

Moreover, v_r solves the equation

$$\tilde{\mathrm{I}} v_r = r^s f(z + rx) \quad \text{in } B_2(\tilde{x}_0),$$

where $\tilde{x}_0 = (x_0 - z)/r$ satisfies $|\tilde{x}_0 - z| = 2$ and $\tilde{\mathrm{I}}$ is translation invariant and elliptic with respect to $\mathcal{L}_*$. Hence, using the interior estimate in Theorem 4.2.6 we obtain $[v_r]_{C^{s-\epsilon}(B_1(\tilde{x}_0))} \leq Cr^{s-\epsilon}$. This yields that

$$r^{s-\epsilon}\left[u - Q\left((x - z) \cdot \nu(z)\right)_+^s\right]_{C^{s-\epsilon}(B_r(x_0))} = r^s[v]_{C^{s-\epsilon}(B_1(\tilde{x}_0))} \leq Cr^s r^{s-\epsilon}.$$

Therefore, using (4.78), (4.84) follows.

Let us finally show that (4.83)-(4.84) yield the desired result. Indeed, note that, for all x_1 and x_2 in $B_r(x_0)$,

$$\frac{u}{d^s}(x_1) - \frac{u}{d^s}(x_2) = \frac{\big(u - Qd^s\big)(x_1) - \big(u - Qd^s\big)(x_2)}{d^s(x_1)} + \big(u - Qd^s\big)(x_2)\big(d^{-s}(x_1) - d^{-s}(x_2)\big).$$

By (4.84), and using that d is comparable to r in $B_r(x_0)$, we have

$$\frac{\big|\big(u - Qd^s\big)(x_1) - \big(u - Qd^s\big)(x_2)\big|}{d^s(x_1)} \leq C|x_1 - x_2|^{s-\epsilon}.$$

Also, by (4.83) and (4.79),

$$\big|u - Qd^s\big|(x_2)\big|d^{-s}(x_1) - d^{-s}(x_2)\big| \leq C|x_1 - x_2|^{s-\epsilon}.$$

Therefore,

$$[u/d^s]_{C^{s-\epsilon}(B_r(x_0))} \leq C.$$

From this, we obtain the desired estimate for $\|u/d^s\|_{C^{s-\epsilon}(\Omega^+ \cap B_{1/2})}$ by summing a geometric series, as in the proof of Proposition 1.1 in [249]. □

4.7 Non translation invariant versions of the results

Proposition 4.7.1. *Let $s_0 \in (0,1)$, $\delta \in (0, s_0/4)$, $\rho_0 > 0$, and $\beta = 2s_0 - \delta$ be given constants.*

Let Γ be a $C^{1,1}$ hypersurface with radius $\rho_0 > 0$ splitting B_1 into Ω^+ and Ω^-; see Definition 4.1.2.

Let $s \in [s_0, \max\{1, s_0 + \delta\}]$, and $f \in C(\overline{\Omega^+})$. Assume that $u \in C(\overline{B_1}) \cap L^\infty(\mathbb{R}^n)$ is a viscosity solution of $\mathfrak{I}(u,x) = f(x)$ in Ω^+ and $u = 0$ in Ω^-, where $\mathfrak{I}$ is an operator of the form (4.12)-(4.16).

Then, for all $z \in \Gamma \cap \overline{B_{1/2}}$ there exists $Q(z) \in \mathbb{R}$ with $|Q(z)| \le C$ for which

$$\left| u(x) - Q(z)\big((x-z)\cdot\nu(z)\big)_+^s \right| \le C|x-z|^\beta \quad \text{for all } x \in B_1,$$

where $\nu(z)$ is the unit normal vector to Γ at x pointing towards Ω^+. The constant C depends only on n, ρ_0, s_0, δ, $\|u\|_{L^\infty(\mathbb{R}^n)}$, $\|f\|_{L^\infty(\Omega^+)}$, the modulus of continuity μ, and ellipticity constants.

Proof. It is a variation of the Proof of Proposition 4.6.1. Hence, it is again by contradiction. Assume that there are sequences Γ_k, Ω_k^+, Ω_k^-, s_k, $\mathfrak{I}_k$, f_k, and u_k that satisfy the assumptions of the proposition. That is, for all $k \ge 1$:

- Γ_k is a $C^{1,1}$ hyper surface with radius ρ_0 splitting B_1 into Ω_k^+ and Ω_k^-.
- $s_k \in [s_0, \max\{1, s_0 + \delta\}]$.
- $\mathfrak{I}_k$ is elliptic with respect to $\mathcal{L}_*(s_k)$ and satisfies (4.12)-(4.16) (with $\mathfrak{I}$ and s replaced by $\mathfrak{I}_k$ and s_k, respectively).
- $\|u_k\|_{L^\infty(\mathbb{R}^n)} + \|f_k\|_{L^\infty(\Omega_k^+)} = 1$.
- u_k is a solution of $\mathfrak{I}_k(u_k, x) = f_k(x)$ in Ω_k^+ and $u_k = 0$ in Ω_k^-.

But suppose that the conclusion of the proposition does not hold. That is, for all $C > 0$, there are k and $z \in \Gamma_k \cap \overline{B_{1/2}}$ for which no constant $Q \in \mathbb{R}$ satisfies

$$\left| u_k(x) - Q\big((x-z)\cdot\nu_k(z)\big)_+^{s_k} \right| \le C|x-z|^\beta \quad \text{for all } x \in B_1. \tag{4.85}$$

Above, $\nu_k(z)$ denotes the unit normal vector to Γ_k at z, pointing towards Ω_k^+.

As in the proof of Proposition 4.6.1, using Lemma 4.6.2, we have that

$$\sup_k \sup_{z \in \Gamma_k \cap B_{1/2}} \sup_{r>0} r^{-\beta} \left\| u_k - \phi_{k,z,r} \right\|_{L^\infty(B_r(z))} = \infty. \tag{4.86}$$

where $\phi_{k,z,r}$ is given by (4.66).

We next define $\theta(r)$ and the sequences $r_m \searrow 0$, k_m, ϕ_m, ν_m, and $z_m \to z \in \overline{B}_{1/2}$ as in the proof of Proposition 4.6.1.

Again, we also define

$$v_m(x) = \frac{u_{k_m}(z_m + r_m x) - \phi_m(z_m + r_m x)}{(r_m)^\beta \theta(r_m)},$$

which satisfies (4.68), (4.69), and the growth control (4.70).

Note that, up to a subsequence, we may assume that $\nu_m \to \nu \in S^{n-1}$.

The rest of the proof consists in showing

Claim. *A subsequence of v_m converges locally uniformly in $\mathbb{R}^n$ to some function v which satisfies $\tilde{I}v = 0$ in $\{x \cdot \nu > 0\}$ and $v = 0$ in $\{x \cdot \nu < 0\}$, for some $\tilde{I}$ translation invariant and elliptic with respect to $\mathcal{L}_*$.*

Once we know this, a contradiction is immediately reached using the Liouville type Theorem 4.1.5, as seen at the end of the proof.

To prove the Claim, given $R \geq 1$ and m such that $r_m R < 1/2$ define

$$\Omega^+_{R,m} = \big\{x \in B_R \ : \ (z_m + r_m x) \in \Omega^+_{k_m} \quad \text{and} \quad x \cdot \nu_m(z_m) > 0\big\}.$$

Notice that for all R and k, the origin 0 belongs to the boundary of $\Omega^+_{R,m}$.

We will use that v_m satisfies an elliptic equation in $\Omega^+_{R,m}$. Namely,

$$\tilde{\mathfrak{I}}_m(v_m, x) = \frac{(r_m)^{2s_{k_m}}}{(r_m)^\beta \theta(r_m)} f(z_m + r_m x) \quad \text{in } \Omega^+_{R,m}. \tag{4.87}$$

where $\tilde{\mathfrak{I}}_m$ is defined by

$$\tilde{\mathfrak{I}}_m \left(\frac{w(z_m + r\,\cdot) - \phi_m(z_m + r\,\cdot)}{(r_m)^\beta \theta(r_m)}\,,\, x \right) = \frac{(r_m)^{2s_{k_m}}}{(r_m)^\beta \theta(r_m)} \mathfrak{I}_{k_m}(w\,,\, z_m + rx),$$

for all test function w. Equivalently, for all test function v,

$$\begin{aligned}
\tilde{\mathfrak{I}}_m(v, x) &\overset{(*)}{=} \frac{(r_m)^{2s_{k_m}}}{(r_m)^\beta \theta(r_m)} \mathfrak{I}_{k_m} \left((r_m)^\beta \theta(r_m) v \left(\frac{\cdot - z_m}{r_m} \right) + \phi_m(\,\cdot\,)\,,\, z_m + r_m x \right) \\
&\overset{(**)}{=} \frac{(r_m)^{2s_{k_m}}}{(r_m)^\beta \theta(r_m)} \mathfrak{I}_{k_m} \left((r_m)^\beta \theta(r_m) v \left(\frac{\cdot - z_m}{r_m} \right) \right) (z_m + r_m x) \\
&\overset{(***)}{=} \inf_\beta \sup_\alpha \left(\int_{\mathbb{R}^n} \big\{ v(x+y) + v(x-y) - 2v(x) \big\} K^{(m)}_{\alpha\beta}(z_m + r_m x, y)\, dy + \right. \\
&\qquad \left. + \frac{(r_m)^{2s_{k_m}} c^{(m)}_{\alpha\beta}(z_m + rx)}{(r_m)^\beta \theta(r_m)} \right).
\end{aligned}$$

The last two identities hold only in $\{x \cdot \nu_m > 0\}$ since $M^+\phi_m = M^-\phi_m = 0$ in $\{(x - z) \cdot \nu_m > 0\}$.

Note that the right hand side of (4.87) converges uniformly to 0 as $r_m \searrow 0$ since $\beta = 2s_0 - \delta < 2s_{k_m}$ and $\theta(r_m) \nearrow \infty$.

Using that $\mathfrak{I}_{k_m}$ is elliptic with respect to $\mathcal{L}_*(s_{k_m})$ and that $\mathfrak{I}_{k_m}(0, x) = 0$, we readily show that $\tilde{\mathfrak{I}}_m$ is also elliptic with respect to $\mathcal{L}_*(s_{k_m})$.

Note that, since $\mathfrak{I}_m$ is elliptic with respect to $\mathcal{L}_*(s_{k_m})$, and $\|f_{k_m}\|_{L^\infty} \leq 1$, then

$$M^+_{s_{k_m}} u_{k_m} \geq -1 \quad \text{and} \quad M^-_{s_{k_m}} u_{k_m} \leq 1 \quad \text{in } \Omega^+,$$

and the same inequalities hold for v_m. Hence, by the same argument as in the proof of Proposition 4.6.1, we find that

$$\big\|v_m\big\|_{C^{\delta'}(B_{R/2})} \leq C(R), \quad \text{for all } m \text{ with } r_m R < 1/4,$$

where $C(R)$ depends only on R, n, s_0, ρ_0, and ellipticity constants, but not on m.

Then, the Arzelà-Ascoli theorem yields the local uniform convergence in $\mathbb{R}^n$ of a subsequence of v_m to some function v. Thus, the Claim is proved.

Next, since all the v_m's satisfy the growth control (4.70), and $2s_0 > \beta$, by the dominated convergence theorem $v_m \to v$ in $L^1(\mathbb{R}^n, \omega_{s_0})$.

Let now $\tilde{\mathrm{I}}_m$ be the sequence of translation invariant operators defined by

$$\tilde{\mathrm{I}}_m w = \inf_\beta \sup_\alpha \left(\int_{\mathbb{R}^n} \big\{ w(x+y) + w(x-y) - 2w(x) \big\} K^{(m)}_{\alpha\beta}(z_m, y)\, dy \right).$$

Note that, for all test functions w,

$$\tilde{\mathfrak{I}}_m(w,x) - \tilde{\mathrm{I}}_m(w) \longrightarrow 0 \quad \text{uniformly in compact sets of } \{(x-z)\cdot\nu > 0\}. \tag{4.88}$$

Indeed, by (4.16),

$$\left| K^{(m)}_{\alpha\beta}(z_m + r_m x, y) - K^{(m)}_{\alpha\beta}(z_m, y) \right| \le \big(1 - s_{k_m}\big) \frac{\mu(Cr_m)}{|y|^{n+2s_{k_m}}} \longrightarrow 0$$

and

$$\left| \frac{(r_m)^{2s_{k_m}} c^{(m)}_{\alpha\beta}(z_m + rx)}{(r_m)^\beta \theta(r_m)} \right| \le \Lambda (r_m)^{2s_{k_m} - \beta} \longrightarrow 0,$$

where μ is the modulus of continuity of the kernels $K_{\alpha\beta}(x,y)$ with respect to x.

On the other hand, by Lemma 4.6.3 there is a subsequence of s_{k_m} converging to some $s \in [s_0, \min\{1, 2s_0 - \delta\}]$ and a subsequence of $\tilde{\mathrm{I}}_m$ which converges weakly to some translation invariant operator $\tilde{\mathrm{I}}$, which is elliptic with respect to $\mathcal{L}_*(s)$. Hence, by (4.88), it follows that $\tilde{\mathfrak{I}}_m \to \tilde{\mathrm{I}}$ weakly in compact subsets of $\{x\cdot\nu > 0\}$. Therefore, using the stability result in [70, Lemma 5], $\tilde{\mathrm{I}} v = 0$ in $\{x\cdot\nu > 0\}$.

Finally, passing to the limit the growth control (4.70) on v_m, we find $\|v\|_{L^\infty(B_R)} \le CR^\beta$ for all $R \ge 1$. Hence, by Theorem 4.1.5, it must be

$$v(x) = K\big(x\cdot\nu(z)\big)^s_+.$$

But passing (4.68) and (4.69) to the limit we find a contradiction. □

We next prove Theorem 4.1.6.

Proof of Theorem 4.1.6. In case that $g \equiv 0$, the result follows from Proposition 4.7.1 by using the same argument as is the proof of Theorem 4.1.3 (partition of $[s_0, 1]$ into intervals of length smaller than $\epsilon/2$).

When g is not zero, we consider $\bar{u} = u - g\chi_{B_1}$. Then $\bar{u}$ satisfies $\bar{u} \equiv 0$ in Ω^- and

$$\bar{\mathfrak{I}}(\bar{u}, x) = \bar{f}(x) \quad \text{in } \Omega^+ \cap B_{3/4},$$

where

$$\bar{\mathfrak{I}}(w,x) = \mathfrak{I}(w + g\chi_{B_1}, x) - \mathfrak{I}(g\chi_{B_1}, x)$$

and

$$\bar{f}(x) = \mathfrak{I}(g\chi_{B_1}, x) + f(x).$$

Then, applying the result for $g \equiv 0$ to the function $\tilde{u}$, the theorem follows. □

4.8 Final comments and remarks

Here we would like to make a few remarks and talk about some open problems and future research directions.

Higher regularity of u/d^s. In the proof of the Liouville-type Theorem 4.1.5, one starts with a solution satisfying $|u(x)| \leq C(1+|x|^\beta)$. Then, one proves that the tangential derivatives satisfy $|\partial_\tau u(x)| \leq C(1+|x|^{\beta-1})$. Hence, if $\beta - 1 < s$, Proposition 4.5.1 implies that $\partial_\tau u \equiv 0$, and thus u is 1D.

The fact that we only use $\beta < 1+s$ seems to indicate that the quotient u/d^s could belong to $C^{1-\epsilon}$, and not only to $C^{s-\epsilon}$. However, for functions with growth at infinity $2s \leq \beta < 1+s$, the integro-differential operators cannot be evaluated.

In fact, only having $\beta - 1 < s + \alpha$ would suffice to obtain $\partial_\tau u = c(x_n)_+^s$, and this seems enough to classify solutions in the half space. However, as before, such approach would require to give sense to the equation for functions that grow "too much" at infinity.

Therefore, the following question remains open. Is it possible to prove that u/d^s belongs to $C^{1+\alpha}$ when considering more regular kernels and right hand sides?

More general linear equations. In a future work we are planning to obtain $C^{s-\epsilon}$ regularity up to the boundary of u/d^s for linear equations involving general operators L of the form (4.3), where a is any measure (not supported in an hyperplane) which does not necessarily satisfy (4.4). We will also obtain higher order regularity of u/d^s for linear equations when $a \in C^k(S^{n-1})$, $f \in C^k(\overline{\Omega})$, and Ω is C^{k+2}.

Equations with lower order terms. We could have included lower order terms in the equations. Indeed, the compactness methods in Section 4.6 involve a blow up procedure. We have seen in Section 4.7 that non translation invariant equations with continuous dependence on x become translation invariant after blow up, and hence our methods still apply. Similarly, we could have considered equations with certain lower order terms, which disappear after blow up.

Second order fully nonlinear equations. As said in the introduction, with the methods developed in this paper one can prove the $C^{1,\alpha}$ and $C^{2,\alpha}$ boundary estimates for fully nonlinear equations $F(D^2u, Du, x) = f(x)$.

Obstacle and free boundary problems. The regularity theory for the obstacle problem (or other free boundary problems) is related to the boundary regularity of solutions to fully nonlinear elliptic equations. In this paper we have shown that $\mathcal{L}_*$ is the appropriate class to obtain fine regularity properties up to the boundary. We therefore wonder if one could obtain regularity results for free boundary problems involving operators in $\mathcal{L}_*$ similar to those for the fractional Laplacian [271].

4.9 Appendix

In this appendix we give the

Proof of Lemma 4.5.3. Let us show first the statement (a). Recall that $a = 1-2s$. We first note that the Caffarelli-Silvestre extension equation $\Delta u + \frac{a}{y}\partial_y u = 0$ is written

in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $r > 0$, $\theta \in (0, \pi)$ as

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{a}{r\sin\theta}\left(\sin\theta\, u_r + \cos\theta\,\frac{u_\theta}{r}\right) = 0.$$

Note the homogeneity of the equation in the variable r. If we seek for (bounded at 0) solutions of the form $u = r^{s+\nu}\Theta_\nu(\theta)$, then it must be $\nu > -s$ and

$$\Theta_\nu'' + a\, cotg\,\theta\,\Theta_\nu' + (s+\nu)(s+\nu+a)\Theta_\nu = 0.$$

If we want u to satisfy the boundary conditions

$$u(x,0) = 0 \quad \text{for } x < 0 \qquad \text{and} \qquad |y|^a \partial_y u(x,y) \to 0 \quad \text{as } y \to 0,$$

then Θ_ν must satisfy

$$\begin{cases} \Theta_\nu(\theta) = \Theta_\nu(0) + o\big((\sin\theta)^{2s}\big) \to 0 \quad \text{as } \theta \searrow 0 \\ \Theta_\nu(\pi) = 0. \end{cases} \tag{4.89}$$

We have used that, for $x > 0$

$$\lim_{y\searrow 0} y^a \partial_y u(x,y) = 0 \quad \Rightarrow \quad u(x,y) = u(x,0) + o(y^{2s}),$$

since $a = 1 - 2s$.

To solve this ODE, consider

$$\Theta_\nu(\theta) = (\sin\theta)^s h(\cos\theta).$$

After some computations and the change of variable $z = \cos\theta$ one obtains the following ODE for $h(z)$:

$$(1-z^2)h''(z) - 2zh'(z) + \left(\nu + \nu^2 - \frac{s^2}{1-z^2}\right)h(z) = 0.$$

This is the so called "associated Legendre differential equation". All solutions to this second order ODE solutions are given by

$$h(z) = C_1 P_\nu^s(z) + C_2 Q_\nu^s(z),$$

where P_ν^s and Q_ν^s are the "associated Legendre functions" of first and second kind, respectively.

Translating (4.89) to the function h, using that $\sin\theta \sim (1-\cos\theta)^{1/2}$ as $\theta \searrow 0$ and $\sin\theta \sim (1+\cos\theta)^{1/2}$ as $\theta \nearrow \pi$, we obtain

$$\begin{cases} (1-z)^{s/2}h(z) = c + o\big((1-z)^s\big) \quad \text{as } z \nearrow 1 \\ \lim_{z\searrow -1}(1+z)^{s/2}h(z) = 0. \end{cases} \tag{4.90}$$

Let us prove that P_ν^s fulfill all these requirements only for $\nu = 0, 1, 2, 3, \ldots$, while Q_ν^s have to be discarded. To have a good description of the singularities of $P_\nu^s(z)$ at $z = \pm 1$ we use its expression as an hypergeometric function

$$P_\nu^s(z) = \frac{1}{\Gamma(1-s)}\frac{(1+z)^{s/2}}{(1-z)^{s/2}}\ {}_2F_1\left(-\nu, \nu+1; 1-s; \frac{1-z}{2}\right).$$

Using this and the definition of ${}_2F_1$ as a power series we obtain

$$P^s_\nu(z) = \frac{1}{\Gamma(1-s)}\frac{2^{s/2}}{(1-z)^{s/2}}\left\{1 - \frac{\nu(\nu+1)}{1-s}\frac{1-z}{2} + o\big((1-z)^2\big)\right\} \quad \text{as } z \nearrow 1.$$

Hence, $(1-z)^{s/2}P^s_\nu(z) = c + O\big(1-z\big) = c + o\big((1-z)^s\big)$ as desired.

For the analysis as $z \searrow -1$ we need to use Euler's transformation

$${}_2F_1(a,b;c;x) = (1-x)^{c-b-a}\,{}_2F_1(c-a,c-b;c;x),$$

obtaining

$$P^s_\nu(z) = \frac{1}{\Gamma(1-s)}\frac{(1+z)^{s/2}}{2^{s/2}}\left(\frac{1+z}{2}\right)^{-s}\left\{\,{}_2F_1(1-s-\nu,-s-\nu;1-s;1) + o(1)\right\}$$

as $z \searrow -1$. It follows that the zero boundary condition is satisfied if and only if

$${}_2F_1(1-s-\nu,-s-\nu;1-s;1) = \frac{\Gamma(1-s)\Gamma(s)}{\Gamma(-\nu)\Gamma(1+\nu)} = 0.$$

This implies $\nu = 0, 1, 2, 3, \ldots$, so that $\Gamma(-\nu) = \infty$.

With a similar analysis one easily finds that the functions $Q^s_\nu(x)$ do not satisfy (4.90) for any $\nu \geq -s$.

The statement (b) of the Lemma follows from the Sturm-Liouville theory after observing that the ODE

$$\Theta''_\nu + a\,cotg\,\theta\,\Theta'_\nu - \lambda\Theta_\nu = 0$$

can be written as

$$\big(|\sin\theta|^a\,\Theta'_\nu\big)' = \lambda|\sin\theta|^a\Theta_\nu. \tag{4.91}$$

Indeed, we may regularize the problem by solving, for $\theta \in (-\pi,\pi)$, the ODE

$$\big((\sin^2\theta+\epsilon^2)^{a/2}\,f'_\epsilon\big)' = \lambda(\sin^2\theta+\epsilon^2)^{a/2}f_\epsilon \tag{4.92}$$

with the regularized boundary conditions

$$\begin{cases} f(-\theta) = f(\theta) \\ f(-\pi) = f(\pi) = 0. \end{cases}$$

For (4.92), we obtain a complete orthonormal system $\{f_{\epsilon,k}\}_{k\geq 0}$ in the subspace of even functions of weighted space $L^2\big((-\pi,\pi),(\sin^2\theta+\epsilon^2)^{a/2}\big)$. Then one proves that as $\epsilon \to 0$ the functions $f_{\epsilon,k}$ converges in $(0,\pi)$ to a solution of (4.91) satisfying the boundary conditions (4.89). Since the limit of a complete orthogonal system is a also complete orthogonal system and we have obtained all the solutions to the limiting equation, these have to be a complete system. □

II

Part Two

REGULARITY OF STABLE SOLUTIONS TO ELLIPTIC EQUATIONS

Introduction to Part II

This second part of the thesis is devoted to study the regularity of stable solutions to reaction-diffusion equations.

Reaction-diffusion equations play a central role in PDE theory and its applications to other sciences. They model many problems, running from Physics (fluids, combustion, etc.), Biology and Ecology (population evolution, illness propagation, etc.), to Financial Mathematics and Economy (Black-Scholes equation, price formation, etc.). They also play an important role in some geometric problems: the problem of prescribing a curvature on a manifold, conformal classification of varieties, and parabolic flows on manifolds.

Background and previous results

The regularity of minimizers to nonlinear elliptic equations is a classical problem in the Calculus of Variations appearing, for instance, in Hilbert's 19th problem. An important example in Geometry is the regularity of minimal hypersurfaces of $\mathbb{R}^n$ which are minimizers of the area functional. A deep result from the seventies states that these hypersurfaces are smooth if $n \leq 7$, while in $\mathbb{R}^8$ the Simons cone

$$S = \left\{ x \in \mathbb{R}^8 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2 \right\} \tag{4.93}$$

is a minimizing minimal hypersurface with a singularity at $x = 0$ [159]. The same phenomenon —the fact that regularity holds in low dimensions— happens for other nonlinear equations in bounded domains. For instance, let u be a solution of

$$\begin{cases} -\Delta u &= f(u) & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega. \end{cases} \tag{4.94}$$

It is still an open problem to show that local minimizers (and, more generally, stable solutions) of this equation are bounded when $n \leq 9$. In dimensions $n \geq 10$ there are examples of singular solutions to this problem which are local minimizers. Namely,

$$u(x) = \log \frac{1}{|x|^2} \text{ is a solution of (4.94) with } f(u) = 2(n-2)e^u \text{ and } \Omega = B_1,$$

which is stable if $n \geq 10$ and a local minimizer if $n \geq 11$ [44].

Of special importance is the following class of reaction-diffusion problems with interior reaction. Consider

$$\begin{cases} -\Delta u &= \lambda f(u) & \text{in } \Omega \subset \mathbb{R}^n \\ u &> 0 & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{4.95}$$

with $\lambda > 0$, posed in a bounded smooth domain Ω. We assume the nonlinearity f to satisfy

$$f \text{ is } C^2, \text{ nondecreasing, convex, } f(0) > 0, \text{ and } \lim_{t\to+\infty} \frac{f(t)}{t} = +\infty. \tag{4.96}$$

Typical examples are $-\Delta u = \lambda e^u$ (known as Gelfand problem, used to model combustion processes) or $-\Delta u = \lambda(1+u)^p$, with $p > 1$.

Under these conditions, it is well known that there exists an extremal value $\lambda^* \in (0, +\infty)$ of the parameter λ such that for each $0 < \lambda < \lambda^*$ there exists a positive minimal solution u_λ of (4.95), while for $\lambda > \lambda^*$ the problem has no solution, even in the weak sense. Here, minimal means the smallest positive solution. For $\lambda = \lambda^*$, there exists a weak solution, called the *extremal solution* of (4.95), which is given by

$$u^*(x) = \lim_{\lambda\uparrow\lambda^*} u_\lambda(x).$$

In 1997 H. Brezis and J. L. Vázquez [36] raised the question of studying the regularity of the extremal solution u^*, i.e., to decide whether u^* is or is not a classical solution depending on f and Ω. This is equivalent to determine whether u^* is bounded or unbounded. The importance of the problem stems in the fact that the existence of other non-minimal solutions for $\lambda < \lambda^*$ depends strongly on the regularity of the extremal solution [120].

The regularity of stable solutions was studied in the seventies and eighties for different nonlinearities f, essentially exponential or power nonlinearities. In both cases a similar result holds: if $n \le 9$ then any stable solution u is bounded for every domain Ω [177, 102, 212], while for $n \ge 10$ there are examples of unbounded stable solutions even in the unit ball —as the one given before.

At present, it is known that this result holds true for all nonlinearities f when the domain Ω is a ball [43], and also in general domains for a class of nonlinearities that satisfy a quite restrictive condition at infinity —the limit in (4.98) to exist.

The case of general f was studied first by Nedev in 2000 [225], who proved that the extremal solution of (4.95) is bounded for every nonlinearity f satisfying (4.96) and for every domain Ω if $n \le 3$. He also gave L^p estimates for u^* for $n \ge 4$, and proved that $u^* \in H^1(\Omega)$ in every dimension when the domain is strictly convex. Finally, the best known result so far states that all stable solutions are bounded in dimensions $n \le 4$, for any nonlinearity f and any domain Ω [42, 295].

The problem is still open in dimensions $5 \le n \le 9$. As mentioned before, a partial result in that direction is that all stable solutions are bounded in dimensions $n \le 9$ when the domain is a ball [43].

Results of the thesis (Part II)

In Chapter 5 we study the regularity of stable solutions u to (4.94) in the class of domains that we call of double revolution. These are those domains which are invariant under rotations of the first m variables and of the last $n-m$ variables, that is,

$$\Omega = \big\{(x^1, x^2) \in \mathbb{R}^m \times \mathbb{R}^{n-m} : (s = |x^1|, t = |x^2|) \in \Omega_2\big\},$$

where $\Omega_2 \subset \mathbb{R}^2$ is a bounded domain even (or symmetric) with respect to each coordinate. We prove the following.

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$ be any bounded and convex domain of double revolution. Let f be any nonlinearity satisfying* (4.96)*, and let u^* be the extremal solution of* (4.95)*. Then, u^* is bounded whenever $n \leq 7$.*

Except for the radial case, our result is the first partial answer valid for all nonlinearities in dimensions $5 \leq n \leq 9$.

The proofs of the results in [102, 225, 43, 256, 42] use heavily the stability of the extremal solution u^*. In fact, one first proves estimates for any regular stable solution u of (4.94), then one applies them to the minimal solutions u_λ, and finally by monotone convergence such estimates also hold for the extremal solution u^*.

Recall that a solution of (4.94) is said to be stable if the second variation of energy at u is nonnegative, i.e., if

$$Q_u(\xi) = \int_\Omega |\nabla \xi|^2 - f'(u)\xi^2 \geq 0$$

for all C^1 functions ξ vanishing on $\partial\Omega$. Obviously, every local minimizer of the energy functional

$$E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(u),$$

where $F' = f$, is a stable solution of (4.94).

For the exponential nonlinearity $f(u) = e^u$, the proof of Crandall-Rabinowitz [102] is based on the choice $\xi = e^{\alpha u} - 1$ in the stability condition, with $\alpha > 0$ chosen appropriately. Combining the stability condition with the equation, they find an L^p bound for e^u if $p < 5$. Since $-\Delta u = \lambda e^u$, then $u \in W^{2,p}$ and, by the Sobolev embeddings, $u \in L^\infty$ if $n < 10$. Nedev's result for $n \leq 3$ [225] uses $\xi = h(u)$ in the stability condition, with h chosen appropriately depending on f.

The proofs of the estimates in [43, 42], instead, use as a test function $\xi = |\nabla u|\eta$ (or $\xi = u_r\eta$ in the radial case), and then compute $Q_u(|\nabla u|\eta)$ in the stability property satisfied by u. The expression of Q_u in terms of η does not depend on f, and a clever choice of the test function η leads to L^∞ and L^p bounds depending on the dimension n (but not on f). This idea was inspired on the proof of Simons theorem on the nonexistence of singular minimal cones in $\mathbb{R}^n$ for $n \leq 7$; see the survey [44].

Our proof of Theorem 3 uses as test functions in the stability condition $\xi = u_s\eta_1$ and $\xi = u_t\eta_2$. Taking appropriate functions η_1 and η_2, this leads to inequalities of the form

$$\int_{\Omega_2} \left(s^{-\alpha}u_s^2 + t^{-\beta}u_t^2\right) dsdt \leq C,$$

where s and t are the two radial coordinates describing Ω. Here, the values of α and β depend on n and m. When $n \leq 7$, these values are large enough to deduce an L^∞ bound for u, as stated in Theorem 3. When $n \geq 8$, we obtain L^p bounds for the solution u via some new weighted Sobolev inequalities established in Chapter 5 (see also the Introduction to Part III).

Chapters 6 and 7 deal with the regularity of extremal solutions to semilinear problems involving now the fractional Laplacian $(-\Delta)^s$; see the Introduction to Part I

for the definition, motivation, and mathematical background on this type of nonlocal problems.

The regularity of the extremal solution was investigated for the *spectral* fractional Laplacian A^s in the unit ball $\Omega = B_1$ by Capella-Dávila-Dupaigne-Sire [80]. They proved the boundedness of all extremal solutions in dimensions $n \leq 6$ for all $s \in (0,1)$. Recall that the spectral fractional Laplacian A^s is defined via the Dirichlet eigenfunctions of the Laplacian $-\Delta$ in Ω. It can be also defined through an extension problem in the cylinder $\Omega \times \mathbb{R}_+$. Thus, this operator is different but related to the fractional Laplacian $(-\Delta)^s$ —recall the extension problem for $(-\Delta)^s$ explained in the Introduction to Part I. Also in this direction, Davila-Dupaigne-Montenegro [107] studied the extremal solution for a boundary reaction problem with mixed Dirichlet-Neumann condition. Thus, as before, this problem is related to the half-Laplacian.

Here, we study the extremal solution to

$$\begin{cases} (-\Delta)^s u &= \lambda f(u) & \text{in } \Omega \\ u &= 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{4.97}$$

Our results (Theorems 4 and 5 below) are the first ones on extremal solutions for the fractional Laplacian $(-\Delta)^s$.

In Chapter 6 we prove the following.

Theorem 4. *Let Ω be any bounded smooth domain in $\mathbb{R}^n$, $s \in (0,1)$, f be a function satisfying* (4.96)*. Let u^* be the extremal solution of* (4.97)*.*

(i) Assume that Ω is convex. Then, u^* belongs to $H^s(\mathbb{R}^n)$ for all $n \geq 1$ and all $s \in (0,1)$.

(ii) Assume that the following limit exists

$$\tau := \lim_{t \to +\infty} \frac{f(t) f''(t)}{f'(t)^2}. \tag{4.98}$$

Then, u^* is bounded whenever $n < 10s$.

The limit (4.98) exists for exponential and power type nonlinearities. Thus, their extremal solutions are bounded whenever $n < 10s$. It is important to notice that, in the limit $s \uparrow 1$, $n < 10$ is optimal.

Regarding part (i), as in the case $s = 1$, a priori one only knows that u^* and $f(u^*)$ are in L^1, but not $u^* \in H^s$. To prove the H^s regularity of the extremal solution we follow the ideas of Nedev for $s = 1$. This requires two main ingredients: the Pohozaev identity for the fractional Laplacian proved in Part I, and an L^∞ estimate near the boundary of convex domains. We establish this L^∞ boundary estimate via the moving planes method.

To prove part (ii) of the result, we argue similarly to the classical case $s = 1$, following the approach of Nedev [225] and Sanchón [256]. When trying to adapt their arguments to the fractional Laplacian, some identities that for $s = 1$ come from local integration by parts are no longer available for $s < 1$. We succeed to replace these identities by appropriate inequalities. They are sharp for $s \to 1$, but not for smaller values of s.

In this direction, although the condition $n < 10s$ in Theorem 4(ii) is optimal for s close to 1, it is not optimal for small values of $s \in (0,1)$. In fact, Theorem 4 does not give any L^∞ estimate for $s \leq 0.1$, while we expect extremal solutions to be bounded in dimensions $n \leq 7$ for all $s \in (0,1)$. The following result goes in this direction.

In Chapter 7 we prove, under some symmetry assumptions on the domain Ω, a sharp boundedness result for extremal solutions with the exponential nonlinearity $f(u) = e^u$. The result reads as follows.

Theorem 5. *Let Ω be a bounded smooth domain in $\mathbb{R}^n$ which is, for every $i = 1, ..., n$, convex in the x_i-direction and symmetric with respect to $\{x_i = 0\}$. Let $s \in (0,1)$, and let u^* be the extremal solution of problem (4.97) with $f(u) = e^u$.*

Then, u^* is bounded for all $s \in (0,1)$ whenever $n \leq 7$. Moreover, the same holds if $n = 8$ and $s \gtrsim 0'28206...$, or if $n = 9$ and $s \gtrsim 0'63237...$.

The result is new even in the unit ball.

The hypotheses of Theorem 5 on n and s —i.e., $n \leq 7$, or $n = 8$ and $s \gtrsim 0'28206...$, or $n = 9$ and $s \gtrsim 0'63237...$— are equivalent to the following inequality

$$\frac{\Gamma\left(\frac{n}{2}\right)\Gamma(1+s)}{\Gamma\left(\frac{n-2s}{2}\right)} > \frac{\Gamma^2\left(\frac{n+2s}{4}\right)}{\Gamma^2\left(\frac{n-2s}{4}\right)}, \tag{4.99}$$

where Γ is the Gamma function.

Condition (4.99) makes Theorem 5 sharp in the following sense. One can find a singular stable solution to $(-\Delta)^s u = \lambda e^u$ in B_1, with a certain nonzero exterior condition g in $\mathbb{R}^n \setminus B_1$, whenever (4.99) does not hold. Indeed, the function $u(x) = \log|x|^{-2s}$ solves

$$\begin{cases} -\Delta u &= \lambda_0 e^u & \text{in } B_1 \\ u &= g(x) & \text{in } \mathbb{R}^n \setminus B_1 \end{cases}$$

for $g(x) = \log|x|^{-2s}$ in $\mathbb{R}^n \setminus B_1$. See Chapter 7 for more details.

To prove Theorem 5, one may think on extending the classical proof of Crandall-Rabinowitz [102], i.e., using $\xi = e^{\alpha u} - 1$ as a test function in the stability condition. When doing this, one only obtains regularity in dimensions $n < 10s$. Thus, different methods are needed. Our proof goes as follows. We first assume by contradiction that u^* is singular, and we prove a lower bound for u^* near its singular point. More precisely, we show that for all $\epsilon > 0$ there exists $r > 0$ such that

$$u^*(x) \geq (1-\epsilon)\log\frac{1}{|x|^{2s}} \quad \text{in } B_r.$$

This is why we need to assume the domain Ω to be even and convex —in this case, the singular point is necessarily the origin. Then, in the stability condition we take an explicit function $\xi(x) \sim |x|^{-\beta}$, with β chosen appropriately. In case that (4.99) holds, this argument leads to a contradiction, and hence the extremal solution is bounded. Similar ideas were already used by Dávila-Dupaigne-Montenegro [107] when studying the extremal solution for the boundary reaction problem described before.

Chapter Five

REGULARITY OF STABLE SOLUTIONS IN DOMAINS OF DOUBLE REVOLUTION

We consider the class of semi-stable positive solutions to semilinear equations $-\Delta u = f(u)$ in a bounded domain $\Omega \subset \mathbb{R}^n$ of double revolution, that is, a domain invariant under rotations of the first m variables and of the last $n-m$ variables. We assume $2 \le m \le n-2$. When the domain is convex, we establish a priori L^p and H_0^1 bounds for each dimension n, with $p = \infty$ when $n \le 7$. These estimates lead to the boundedness of the extremal solution of $-\Delta u = \lambda f(u)$ in every convex domain of double revolution when $n \le 7$. The boundedness of extremal solutions is known when $n \le 3$ for any domain Ω, in dimension $n = 4$ when the domain is convex, and in dimensions $5 \le n \le 9$ in the radial case. Except for the radial case, our result is the first partial answer valid for all nonlinearities f in dimensions $5 \le n \le 9$.

5.1 Introduction and results

Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain, and consider the problem

$$\begin{cases} -\Delta u &= \lambda f(u) & \text{in } \Omega \\ u &> 0 & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where λ is a positive parameter and the nonlinearity $f : [0, \infty) \longrightarrow \mathbb{R}$ satisfies

$$f \text{ is } C^1, \text{ nondecreasing, } f(0) > 0, \text{ and } \lim_{\tau \to \infty} \frac{f(\tau)}{\tau} = \infty. \tag{5.2}$$

It is well known (see the excellent monograph [120] and references therein) that there exists an extremal parameter $\lambda^* \in (0, \infty)$ such that if $0 < \lambda < \lambda^*$ then problem (5.1) admits a minimal classical solution u_λ, while for $\lambda > \lambda^*$ it has no solution, even in the weak sense. Here, minimal means smallest. Moreover, the set $\{u_\lambda : 0 < \lambda < \lambda^*\}$ is increasing in λ, and its pointwise limit $u^* = \lim_{\lambda \to \lambda^*} u_\lambda$ is a weak solution of problem (5.1) with $\lambda = \lambda^*$. It is called the extremal solution of (5.1).

When $f(u) = e^u$, it is well known that $u \in L^\infty(\Omega)$ if $n \le 9$, while $u^*(x) = \log \frac{1}{|x|^2}$ if $n \ge 10$ and $\Omega = B_1$. An analogous result holds for $f(u) = (1+u)^p$, $p > 1$. In the nineties H. Brezis and J.L. Vázquez [36] raised the question of determining the regularity of u^*, depending on the dimension n, for general convex nonlinearities satisfying (5.2). The first general results were proved by G. Nedev [225, 226] —see [69] for the statement and proofs of the results of [226].

Theorem 5.1.1 ([225],[226]). *Let Ω be a smooth bounded domain, f be a function satisfying* (5.2) *which in addition is convex, and u^* be the extremal solution of* (5.1).

i) If $n \leq 3$, then $u^* \in L^\infty(\Omega)$.

ii) If $n \geq 4$, then $u^* \in L^p(\Omega)$ for every $p < \frac{n}{n-4}$.

iii) Assume either that $n \leq 5$ or that Ω is strictly convex. Then $u^* \in H_0^1(\Omega)$.

In 2006, the first author and A. Capella [43] studied the radial case. Their result establishes optimal L^∞ and L^p regularity results in every dimension for general f.

Theorem 5.1.2 ([43]). *Let $\Omega = B_1$ be the unit ball in $\mathbb{R}^n$, f be a function satisfying* (5.2), *and u^* be the extremal solution of* (5.1).

i) If $n \leq 9$, then $u^* \in L^\infty(\Omega)$.

ii) If $n \geq 10$, then $u^* \in L^p(\Omega)$ for every $p < p_n$, where

$$p_n = 2 + \frac{4}{\frac{n}{2+\sqrt{n-1}} - 2}. \tag{5.3}$$

iii) For every dimension n, $u^* \in H^3(\Omega)$.

The best known result was established in 2010 by the first author [42] and establishes the boundedness of u^* in convex domains in dimension $n = 4$. Related ideas recently allowed the first author and M. Sanchón [69] to improve Nedev's L^p estimates of Theorem 5.1.1 when $n \geq 5$:

Theorem 5.1.3 ([42],[69]). *Let $\Omega \subset \mathbb{R}^n$ be a convex, smooth and bounded domain, f be a function satisfying* (5.2), *and u^* be the extremal solution of* (5.1).

i) If $n \leq 4$, then $u^* \in L^\infty(\Omega)$.

ii) If $n \geq 5$, then $u^* \in L^p(\Omega)$ for every $p < \frac{2n}{n-4} = 2 + \frac{4}{\frac{n}{2}-2}$.

The boundedness of extremal solutions remains an open question in dimensions $5 \leq n \leq 9$, even in the case of convex domains and convex nonlinearities.

The aim of this paper is to study the regularity of the extremal solution u^* of (5.1) in a class of domains that we call of double revolution. The class contains domains much more general than balls, but is much simpler than general convex domains. In this class of domains our main result establishes the boundedness of the extremal solution u^* in dimensions $n \leq 7$, whenever Ω is convex. An interesting point of our work is that it has led us to a new Sobolev and isoperimetric inequality (Proposition 5.1.7 below) with a monomial weight or density. In a future paper [50], we treat a more general version of these Sobolev and isoperimetric inequalities with densities (see Remark 5.1.8 below) for which we can compute best constants, as well as extremal sets and functions. They are in the spirit of recent works on manifolds with a density; see F. Morgan's survey [218] for more information.

Let $n \geq 4$ and

$$\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k \text{ with } n = m + k,\ m \geq 2, \text{ and } k \geq 2. \tag{5.4}$$

For each $x \in \mathbb{R}^n$ we define the variables

$$\begin{cases} s &= \sqrt{x_1^2 + \cdots + x_m^2} \\ t &= \sqrt{x_{m+1}^2 + \cdots + x_n^2}. \end{cases}$$

We say that a domain $\Omega \subset \mathbb{R}^n$ is a *domain of double revolution* if it is invariant under rotations of the first m variables and also under rotations of the last k variables. Equivalently, Ω is of the form $\Omega = \{x \in \mathbb{R}^n : \ (s,t) \in \Omega_2\}$ where Ω_2 is a domain in $\mathbb{R}^2$ symmetric with respect to the two coordinate axes. In fact, $\Omega_2 = \{(y_1, y_2) \in \mathbb{R}^2 : \ x = (x_1 = y_1, x_2 = 0, ..., x_m = 0, x_{m+1} = y_2, x_{m+2} = 0, ..., x_n = 0) \in \Omega\}$ is the intersection of Ω with the (x_1, x_{m+1})-plane. Note that Ω_2 is smooth if and only if Ω is smooth. Let us call $\widetilde{\Omega}$ the intersection of Ω_2 with the positive quadrant of $\mathbb{R}^2$, i.e.,

$$\begin{aligned} \widetilde{\Omega} = \big\{(s,t) \in \mathbb{R}^2 : s > 0, t > 0, \text{ and} \\ (x_1 = s, x_2 = 0, ..., x_m = 0, x_{m+1} = t, x_{m+2} = 0, ..., x_n = 0) \in \Omega\big\}. \end{aligned} \tag{5.5}$$

Since $\{s = 0\}$ and $\{t = 0\}$ have zero measure in $\mathbb{R}^2$, we have that

$$\int_\Omega v \ dx = c_{m,k} \int_{\widetilde{\Omega}} v(s,t) s^{m-1} t^{k-1} ds dt$$

for every $L^1(\Omega)$ function $v = v(x)$ which depends only on the radial variables s and t. Here, $c_{m,k}$ is a positive constant depending only on m and k.

In the previous theorems, the regularity of u^* is proved using its semi-stability. More precisely, the minimal solutions u_λ of (5.1) turn out to be semi-stable solutions. A solution is semi-stable if the second variation of energy at the solution is nonnegative; see (5.9) below. We will prove that any semi-stable classical solution u of (5.1), and more generally of (5.8) below, depends only on s and t, and hence we can identify it with a function $u = u(s,t)$ defined in $(\mathbb{R}_+)^2 = (0,\infty)^2$ which satisfies the equation

$$u_{ss} + u_{tt} + \frac{m-1}{s} u_s + \frac{k-1}{t} u_t + f(u) = 0 \quad \text{for } (s,t) \in \widetilde{\Omega}. \tag{5.6}$$

Moreover, in the case of convex domains we will also have $u_s \leq 0$ and $u_t \leq 0$ (for $s > 0$, $t > 0$) and hence, $u(0) = \|u\|_{L^\infty}$ (see Remark 5.2.1).

The following is our main result. We prove that, in convex domains of double revolution, the extremal solution u^* is bounded when $n \leq 7$, and it belongs to H_0^1 and certain L^p spaces when $n \geq 8$. We also prove that in dimension $n = 4$ the convexity of the domain is not required for the boundedness of u^* (in [42], convexity of Ω was a requirement in general domains of $\mathbb{R}^4$).

Theorem 5.1.4. *Assume* (5.4). *Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain of double revolution, f be a function satisfying* (5.2), *and u^* be the extremal solution of* (5.1).

a) Assume either that $n = 4$ or that $n \le 7$ and Ω is convex. Then, $u^* \in L^\infty(\Omega)$.

b) If $n \ge 8$ and Ω is convex, then $u^* \in L^p(\Omega)$ for all $p < p_{m,k}$, where

$$p_{m,k} = 2 + \frac{4}{\frac{m}{2+\sqrt{m-1}} + \frac{k}{2+\sqrt{k-1}} - 2}. \tag{5.7}$$

c) Assume either that $n \le 6$ or that Ω is convex. Then, $u^* \in H_0^1(\Omega)$.

Remark 5.1.5. Let $q_{m,k} = \frac{m}{2+\sqrt{m-1}} + \frac{k}{2+\sqrt{k-1}}$. Since $q(x) := \frac{x}{2+\sqrt{x-1}}$ is a concave function in $[2,\infty)$, we have $q'(x) - q'(n-x) \ge 0$ in $[2, \frac{n}{2}]$, and thus $q(x)+q(n-x)$ is nondecreasing in $[2, \frac{n}{2}]$. Hence, $q_{2,n-2} \le q_{m,k} \le q_{\frac{n}{2},\frac{n}{2}}$, and therefore $p_{\frac{n}{2},\frac{n}{2}} \le p_{m,k} \le p_{2,n-2}$. Thus, asymptotically as $n \to \infty$,

$$2 + \frac{2\sqrt{2}}{\sqrt{n}} \simeq p_{\frac{n}{2},\frac{n}{2}} \le p_{m,k} \le p_{2,n-2} \simeq 2 + \frac{4}{\sqrt{n}}.$$

Instead, in a general convex domain, L^p estimates are only known for $p \simeq 2 + \frac{8}{n}$ (see Theorem 5.1.3 ii above), while in the radial case one has L^p estimates for $p \simeq 2 + \frac{4}{\sqrt{n}}$ (see Theorem 5.1.2 ii).

The proofs of the results in [225, 226, 43, 42, 69] use the semi-stability of the extremal solution u^*. In fact, one first proves estimates for any regular semi-stable solution u of

$$\begin{cases} -\Delta u &= f(u) & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{5.8}$$

then one applies these estimates to the minimal solutions u_λ (which are semi-stable), and finally by monotone convergence the estimates also hold for the extremal solution u^*.

Recall that a classical solution u of (5.8) is said to be *semi-stable* if the second variation of energy at u is nonnegative, i.e., if

$$Q_u(\xi) = \int_\Omega \left\{ |\nabla \xi|^2 - f'(u)\xi^2 \right\} dx \ge 0 \tag{5.9}$$

for all $\xi \in C_0^1(\overline{\Omega})$. For instance, every local minimizer of the energy is a semi-stable solution.

The proof of the estimates in [43, 42, 69] was inspired by the proof of Simons theorem on the nonexistence of singular minimal cones in $\mathbb{R}^n$ for $n \le 7$ (see [44] for more details). The key idea is to take $\xi = |\nabla u|\eta$ (or $\xi = u_r\eta$ in the radial case) and compute $Q_u(|\nabla u|\eta)$ in the semi-stability property satisfied by u. In this way the expression of Q_u in terms of η turns out not to depend on f and, thanks to this, a clever choice of the test function η leads to L^p and L^∞ bounds depending on the dimension n but valid for all nonlinearities f.

In this paper we will proceed in a similar way, proving first results for general positive semi-stable solutions of (5.8) and then applying them to u_λ to deduce estimates for u^*. We will take $\xi = u_s\eta$ and $\xi = u_t\eta$ separately instead of $\xi = |\nabla u|\eta$, and this will lead to bounds for

$$\int_\Omega u_s^2 s^{-2\alpha-2} dx \qquad \text{and} \qquad \int_\Omega u_t^2 t^{-2\beta-2} dx \tag{5.10}$$

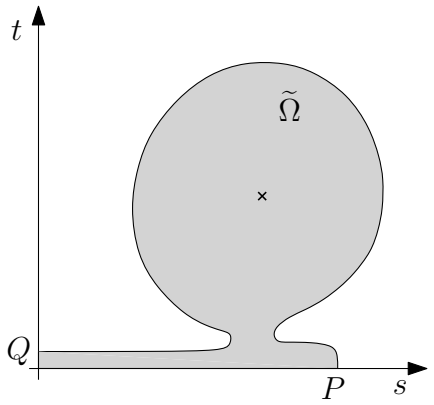

Figure 5.1: A non-convex domain for which the maximum of u^* will not be $u^*(0)$

for any $\alpha < \sqrt{m-1}$ and $\beta < \sqrt{k-1}$.

When the domain Ω is convex, we will have the additional information $\|u\|_{L^\infty} = u(0)$, $u_s \leq 0$, and $u_t \leq 0$, which combined with (5.10) will lead to L^∞ and L^p estimates for u^*.

Instead, when the domain Ω is not convex the maximum of u may not be achieved at the origin —see Figure 1 for an example in which $u(0)$ will be much smaller than $\|u\|_{L^\infty}$. Thus, in nonconvex domains we can not apply the same argument. However, if the maximum is away from $\{s = 0\}$ and $\{t = 0\}$ (as in Figure 1) then the problem is essentially two dimensional near the maximum, since $dx = c_{m,k} s^{m-1} t^{k-1} dsdt$ and both s and t will be positive and bounded below around the maximum. Thus, the two dimensional Sobolev inequality will hold near the maximum. We will still have to prove some boundary estimates, for instance estimates near the boundary points P and Q in Figure 1. But, by the same reason as before, near P the coordinate s is positive and bonded below. Thus, the problem near P will be essentially $1 + k$ dimensional, and we assume $k = n - m \leq n - 2$. This will allow us, if $1 + k \leq n - 1$ are small enough, to use Nedev's [225] $W^{2,p}$ estimates to obtain boundary estimates.

Our result for general positive semi-stable solutions of (5.8) reads as follows. It states global estimates controlled in terms of boundary estimates.

Proposition 5.1.6. *Assume* (5.4). *Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain of double revolution, f be any C^1 function, and u be a positive bounded semi-stable solution of* (5.8).

Let δ be any positive real number, and define

$$\Omega_\delta = \{x \in \Omega : dist(x, \partial\Omega) < \delta\}.$$

Then, for some constant C depending only on Ω, δ, n, and also p in part b) below, one has:

a) If $n \leq 7$ and Ω is convex, then $\|u\|_{L^\infty(\Omega)} \leq C \left(\|u\|_{L^\infty(\Omega_\delta)} + \|f(u)\|_{L^\infty(\Omega_\delta)}\right)$.

b) If $n \geq 8$ and Ω is convex, then $\|u\|_{L^p(\Omega)} \leq C \left(\|u\|_{L^\infty(\Omega_\delta)} + \|f(u)\|_{L^\infty(\Omega_\delta)}\right)$ for each $p < p_{m,k}$, where $p_{m,k}$ is given by (5.7).

c) For all $n \geq 4$, $\|u\|_{H_0^1(\Omega)} \leq C\|u\|_{H^1(\Omega_\delta)}$.

To prove part b) of Proposition 5.1.6 we will need a new weighted Sobolev inequality in $(\mathbb{R}_+)^2 = \{(\sigma,\tau) \in \mathbb{R}^2 : \sigma > 0, \tau > 0\}$. We will use this inequality in the (σ,τ)-plane defined after the change of variables

$$\sigma = s^{2+\alpha}, \quad \tau = t^{2+\beta},$$

where α and β are the exponents in (5.10). It states the following.

Proposition 5.1.7. *Let $a > -1$ and $b > -1$ be real numbers, being positive at least one of them, and let*

$$D = 2 + a + b.$$

Let u be a nonnegative Lipschitz function with compact support in $\mathbb{R}^2$ such that $u \in C^1(\{u > 0\})$,

$$u_\sigma \leq 0 \quad \textit{and} \quad u_\tau \leq 0 \quad \textit{in} \quad (\mathbb{R}_+)^2,$$

with strict inequalities whenever $u > 0$. Then, for each $1 \leq q < D$ there exists a constant C, depending only on a, b, and q, such that

$$\left(\int_{(\mathbb{R}_+)^2} \sigma^a \tau^b |u|^{q^*} d\sigma d\tau \right)^{1/q^*} \leq C \left(\int_{(\mathbb{R}_+)^2} \sigma^a \tau^b |\nabla u|^q d\sigma d\tau \right)^{1/q}, \tag{5.11}$$

where $q^* = \frac{Dq}{D-q}$.

Remark 5.1.8. When a and b are nonnegative integers, inequality (5.11) is a direct consequence of the classical Sobolev inequality in $\mathbb{R}^D$. Namely, define in $\mathbb{R}^D = \mathbb{R}^{a+1} \times \mathbb{R}^{b+1}$ the radial variables $\sigma = |(x_1, \ldots, x_{a+1})|$ and $\tau = |(x_{a+2}, \ldots, x_D)|$. Then, for functions u defined in $\mathbb{R}^D$ depending only on the variables σ and τ, write the integrals appearing in the classical Sobolev inequality in $\mathbb{R}^D$ in terms of σ and τ. Since $dx = c_{a,b}\sigma^a \tau^b d\sigma d\tau$, the obtained inequality is precisely the one given in Proposition 5.1.7.

Thus, the previous proposition extends the classical Sobolev inequality to the case of non-integer exponents a and b. In another article, [50], we prove inequality (5.11) with $(\mathbb{R}_+)^2$ replaced by $(\mathbb{R}_+)^d$ and with $\sigma^a \tau^b$ replaced by the monomial weight

$$x^A := x_1^{A_1} \cdots x_d^{A_d},$$

where $A_1, ..., A_d$ are nonnegative real numbers. We also prove a related isoperimetric inequality with best constant, a weighted Morrey's inequality, and we determine extremal sets and functions for some of these inequalities.

In section 4 we establish the weighted Sobolev inequality of Proposition 5.1.7 as a consequence of a new weighted isoperimetric inequality. Our proof is simple but does not give the best constant (in contrast with the more involved proof that we will give in [50] giving the best constant). When a and b belong to $(0, q-1)$ —i.e., $(0,1)$ when $q = 2$, as in our application) inequality (5.11) also follows from a result of P. Hajlasz [166] in a very general framework of weights or measures. His result does not give the best constant and, besides, its constant depends on the support of the function.

We will need to use the proposition for some exponents a and b in $(-1, 0)$ —this happens for instance when $m = 2$ or $m = 3$. In this case the assumption $u_\sigma \leq 0$, $u_\tau \leq 0$ is crucial for the inequality to hold with the optimal exponent q^*. Without

this assumption, a Sobolev inequality is still true but with a smaller exponent than q^* (this also follows from the results in [166]). For $a > q-1$ the weight is no longer in the Muckenhoupt class A_q and the results in [166] do not apply.

The paper is organized as follows. In section 2 we prove the estimates of Proposition 5.1.6. Section 3 deals with the regularity of the extremal solution of (5.1). Finally, in section 4 we prove the weighted Sobolev inequality of Proposition 5.1.7.

5.2 Proof of Proposition 5.1.6

We start with a remark on the symmetry and monotonicity properties of solutions to (5.8), as well as on the regularity of the functions u_s and u_t.

Remark 5.2.1. Note that when the domain is of double revolution, any bounded semi-stable solution u of (5.8) will depend only on the variables s and t. To prove this, define $v = x_i u_{x_j} - x_j u_{x_i}$, with $i \neq j$. Note that u will will depend only on s and t if and only if $v \equiv 0$ for each $i, j \in \{1, ..., m\}$ and for each $i, j \in \{m+1, ..., n\}$.

We first see that, for such indexes i and j, v is a solution of the linearized equation of (5.8):

$$\begin{aligned}
\Delta v &= \Delta(x_i u_{x_j} - x_j u_{x_i}) \\
&= x_i \Delta u_{x_j} + 2\nabla x_i \cdot \nabla u_{x_j} - x_j \Delta u_{x_i} - 2\nabla x_j \cdot \nabla u_{x_i} \\
&= x_i (\Delta u)_{x_j} - x_j (\Delta u)_{x_i} \\
&= -f'(u)\{x_i u_{x_j} - x_j u_{x_i}\} \\
&= -f'(u) v.
\end{aligned}$$

Note that v is a tangential derivative of u along $\partial\Omega$ since Ω is a domain of double revolution. Therefore, since $u = 0$ on $\partial\Omega$ then $v = 0$ on $\partial\Omega$. Thus, multiplying the equation by v and integrating by parts, we obtain

$$\int_\Omega \{|\nabla v|^2 - f'(u) v^2\} dx = 0.$$

But since u is semi-stable, the first Dirichlet eigenvalue $\lambda_1(\Delta + f'(u); \Omega) \geq 0$.

If $\lambda_1(\Delta + f'(u); \Omega) > 0$, the previous inequality leads to $v \equiv 0$.

If $\lambda_1(\Delta + f'(u); \Omega) = 0$, then we must have $v = K\phi_1$, where K is a constant and ϕ_1 is the first Dirichlet eigenfunction of $\Delta + f'(u)$, which we may take to be positive in Ω. But since v is the derivative of u along the vector field $\partial_t = x_i \partial_{x_j} - x_j \partial_{x_i}$, and its integral curves are closed, v can not have constant sign. Thus, $K = 0$, that is, $v \equiv 0$.

Hence, we have seen that any classical semi-stable solution u of (5.8) depends only on the variables s and t. Moreover, by the classical result of Gidas-Ni-Nirenberg [156], when Ω is even and convex with respect each coordinate and u is a positive solution, we have $u_{x_i} \leq 0$ when $x_i > 0$, for $i = 1, ..., n$. In particular, when Ω is a convex domain of double revolution, we have that $u_s < 0$ and $u_t < 0$ for $s > 0$, $t > 0$, $(s, t) \in \tilde{\Omega}$. In particular,

$$\|u\|_{L^\infty(\Omega)} = u(0).$$

On the other hand, by standard elliptic regularity for (5.8) and its linearization, every bounded solution u of (5.8) satisfies $u \in W^{3,p}(\Omega) \cap C^{2,\nu}(\overline{\Omega})$ for all $p < \infty$ and

$0 < \nu < 1$. In particular,

$$u_s \in H^2_{loc}(\Omega\backslash\{s=0\}) \qquad \text{and} \qquad u_t \in H^2_{loc}(\Omega\backslash\{t=0\}),$$

since $u_s = u_{x_1}\frac{x_1}{s} + \cdots + u_{x_m}\frac{x_m}{s}$ and $u_t = u_{x_{m+1}}\frac{x_{m+1}}{t} + \cdots + u_{x_n}\frac{x_n}{t}$. In addition, since $u = u(s,t)$ is the restriction to the first quadrant of the (x_1, x_{m+1})-plane of an even $C^{2,\nu}$ function of x_1 and x_{m+1}, we deduce that

$$u_s \in Lip(\overline{\Omega}),\ u_t \in Lip(\overline{\Omega}),\ u_s = 0 \text{ when } s = 0, \text{ and } u_t = 0 \text{ when } t = 0. \tag{5.12}$$

We note that u_s and u_t do not belong to $C^1(\overline{\Omega})$, neither to $H^2(\Omega)$. For instance, the solution of $-\Delta u = 1$ in $B_1 \subset \mathbb{R}^n$ is given by $u = \frac{1}{2n}(1 - s^2 - t^2)$ and, thus, $u_s = -\frac{1}{n}s$ is only Lipschitz in Ω.

Before proving Proposition 5.1.6, we will need two preliminary results. The first one, Lemma 5.2.2, was already used in [43, 42]. In this paper we use it taking the function c on its statement to be u_s and u_t. Note that $c = u_s \in H^2_{loc}(\Omega\backslash\{s=0\})$ but u_s is not H^2 in a neighborhood in Ω of $\{s = 0\}$.

Lemma 5.2.2. *Let u be a bounded semi-stable solution of (5.8), V be an open set with $V \subset \Omega$, and c be a $H^2_{loc}(V)$ function. Then,*

$$\int_\Omega c\{\Delta c + f'(u)c\}\eta^2 dx \le \int_\Omega c^2|\nabla\eta|^2 dx$$

for all $\eta \in C^1(V)$ with compact support in V.

Proof. It suffices to set $\xi = c\eta$ in the semi-stability condition (5.9) and then integrate by parts in V. □

We now apply Lemma 5.2.2 separately with $c = u_s$ and with $c = u_t$, and then we choose appropriately the test function η to get the following result. This estimate is the key ingredient in the proof of Proposition 5.1.6.

Lemma 5.2.3. *Assume (5.4). Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain of double revolution, f be any C^1 function, and u be a positive bounded semi-stable solution of (5.8). Let α and β be such that*

$$0 \le \alpha < \sqrt{m-1} \quad \text{and} \quad 0 \le \beta < \sqrt{k-1}.$$

Then, for each $\delta > 0$ there exists a constant C, which depends only on Ω, δ, n, α, and β, such that

$$\left(\int_\Omega \left\{u_s^2 s^{-2\alpha-2} + u_t^2 t^{-2\beta-2}\right\} dx\right)^{1/2} \le C\left(\|u\|_{L^\infty(\Omega_\delta)} + \|f(u)\|_{L^\infty(\Omega_\delta)}\right), \tag{5.13}$$

where

$$\Omega_\delta = \{x \in \Omega : dist(x, \partial\Omega) < \delta\}.$$

Proof. We will prove only the estimate for $u_s^2 s^{-2\alpha-2}$; the other term can be estimated similarly.

Differentiating (5.6) with respect to s, we obtain

$$\Delta u_s - (m-1)\frac{u_s}{s^2} + f'(u)u_s = 0 \qquad \text{in } \Omega\backslash\{s=0\}.$$

Hence, setting $c = u_s$ in Lemma 5.2.2 (recall that $c = u_s \in H^2_{loc}(\Omega\backslash\{s=0\})$ by Remark 5.2.1), we have that

$$(m-1)\int_\Omega u_s^2 \frac{\eta^2}{s^2} dx \le \int_\Omega u_s^2 |\nabla\eta|^2 dx \tag{5.14}$$

for all $\eta \in C^1(\Omega\backslash\{s=0\})$ with compact support in $\Omega\backslash\{s=0\}$.

We claim now that inequality (5.14) is valid for each $\eta \in C^1(\Omega)$ with compact support in Ω. Namely, take any such function η, and let ζ_δ be a smooth function satisfying $0 \le \zeta_\delta \le 1$, $\zeta_\delta \equiv 0$ in $\{s \le \delta\}$, $\zeta_\delta \equiv 1$ in $\{s \ge 2\delta\}$, and $|\nabla\zeta_\delta| \le C/\delta$. Applying (5.14) with η replaced by $\eta\zeta_\delta$ (which is C^1 and has compact support in $\Omega\backslash\{s=0\}$), we obtain

$$(m-1)\int_\Omega u_s^2 \frac{\eta^2\zeta_\delta^2}{s^2} dx \le \int_\Omega u_s^2 |\nabla(\eta\zeta_\delta)|^2 dx. \tag{5.15}$$

Now, we find

$$\begin{aligned}
\int_\Omega u_s^2 |\nabla(\eta\zeta_\delta)|^2 dx &= \int_\Omega u_s^2 \left\{ |\nabla\eta|^2\zeta_\delta^2 + \eta^2|\nabla\zeta_\delta|^2 + 2\eta\zeta_\delta\nabla\eta\nabla\zeta_\delta \right\} dx \\
&\le \int_\Omega u_s^2 |\nabla\eta|^2\zeta_\delta^2 dx + \frac{C}{\delta^2}\int_{\{\delta\le s\le 2\delta\}\cap\Omega} u_s^2 dx \\
&\le \int_\Omega u_s^2 |\nabla\eta|^2\zeta_\delta^2 dx + C\delta^{m-2}\|u_s\|^2_{L^\infty(\{\delta\le s\le 2\delta)},
\end{aligned}$$

where C denote different positive constants, and we have used that η and $|\nabla\eta|$ are bounded. Since u_s is continuous in $\overline{\Omega}$ and $u_s = 0$ on $\{s=0\}$ by (5.12), we have $\|u_s\|_{L^\infty(\{s\le 2\delta\})} \to 0$ as $\delta \to 0$. Recall also that $m - 2 \ge 0$. Therefore, letting $\delta \to 0$ in (5.15) we obtain (5.14), and our claim is proved.

Moreover, by approximation by $C^1(\Omega)$ functions with compact support in Ω, we see that (5.14) is valid also for each $\eta \in Lip(\Omega)$ with compact support in Ω.

Let us set $\eta = \eta_\epsilon$ in (5.14), where

$$\eta_\epsilon = \begin{cases} s^{-\alpha}\rho & \text{if } s > \epsilon \\ \epsilon^{-\alpha}\rho & \text{if } s \le \epsilon \end{cases} \qquad \text{and} \qquad \rho = \begin{cases} 0 & \text{in } \Omega_{\delta/3} \\ 1 & \text{in } \Omega\backslash\Omega_{\delta/2}, \end{cases}$$

and ρ is a smooth function. Note that $\eta_\epsilon \in \mathrm{Lip}(\Omega)$ and has compact support in Ω. Then, since $\alpha^2 < \frac{1}{2}(\alpha^2 + m - 1) < m - 1$,

$$|\nabla\eta_\epsilon|^2 \le \begin{cases} \frac{1}{2}(\alpha^2+m-1)s^{-2\alpha-2}\rho^2 & \text{in } (\Omega\backslash\Omega_{\delta/2})\cap\{s>\epsilon\} \\ \frac{1}{2}(\alpha^2+m-1)s^{-2\alpha-2}\rho^2 + Cs^{-2\alpha} & \text{in } \Omega_{\delta/2}\cap\{s>\epsilon\} \\ C\epsilon^{-2\alpha} & \text{in } \Omega\cap\{s\le\epsilon\}, \end{cases}$$

we deduce from (5.14)

$$\frac{m-1-\alpha^2}{2}\int_{\Omega\cap\{s>\epsilon\}} u_s^2 s^{-2\alpha-2}\rho^2 dx \le C\int_{\Omega_{\delta/2}\cap\{s>\epsilon\}} u_s^2 s^{-2\alpha}dx + C\epsilon^{-2\alpha}\int_{\Omega\cap\{s\le\epsilon\}} u_s^2 dx,$$

where C denote different constants depending only on the quantities appearing in the statement of the lemma. Note that we can bound the dependence of the constants in m and k by a constant depending on n, since for each n there is a finite number of possible m and k. Now, since $u_s \in L^\infty(\Omega)$, the last term is bounded by $C\|u_s\|^2_{L^\infty}\epsilon^{m-2\alpha}$. Making $\epsilon \to 0$ and using that

$$2\alpha < 2\sqrt{m-1} \le m, \tag{5.16}$$

we deduce

$$\int_\Omega u_s^2 s^{-2\alpha-2}\rho^2 dx \le C\int_{\Omega_{\delta/2}} u_s^2 s^{-2\alpha}dx.$$

Hence, since $\rho \equiv 1$ in $\Omega\backslash\Omega_{\delta/2}$,

$$\int_{\Omega\backslash\Omega_{\delta/2}} u_s^2 s^{-2\alpha-2}dx \le C\int_{\Omega_{\delta/2}} u_s^2 s^{-2\alpha}dx \le C\int_{\Omega_{\delta/2}} u_s^2 s^{-2\alpha-2}dx. \tag{5.17}$$

From this we deduce that, for another constant C,

$$\int_\Omega u_s^2 s^{-2\alpha-2}dx \le C\int_{\Omega_{\delta/2}} u_s^2 s^{-2\alpha-2}dx. \tag{5.18}$$

Let $0 < \nu < 1$ to be chosen later. On the one hand, using that $u_s \in \mathrm{Lip}(\overline{\Omega})$ and $u_s(0,t) = 0$ (by (5.12)), and that Ω is smooth, we deduce that $|u_s(s,t)| \le Cs^\nu \|u_s\|_{C^{0,\nu}(\overline{\Omega_{\delta/2}})}$ in $\Omega_{\delta/2}\cap\{s<\delta\}$. Moreover, since $-\Delta u = f(u)$ in Ω_δ and $u|_{\partial\Omega} = 0$, by $W^{2,p}$ estimates we have $\|u\|_{C^{1,\nu}(\overline{\Omega_{\delta/2}})} \le C\left(\|u\|_{L^\infty(\Omega_\delta)} + \|f(u)\|_{L^\infty(\Omega_\delta)}\right)$. It follows that

$$\|s^{-\nu}u_s\|_{L^\infty(\Omega_{\delta/2}\cap\{s<\delta\})} \le C\left(\|u\|_{L^\infty(\Omega_\delta)} + \|f(u)\|_{L^\infty(\Omega_\delta)}\right).$$

Thus, also in all $\Omega_{\delta/2}$ we have

$$\|s^{-\nu}u_s\|_{L^\infty(\Omega_{\delta/2})} \le C\left(\|u\|_{L^\infty(\Omega_\delta)} + \|f(u)\|_{L^\infty(\Omega_\delta)}\right). \tag{5.19}$$

On the other hand, recalling (5.16) and taking ν sufficiently close to 1 such that $m - 2\alpha - 2 + 2\nu > 0$, we will have

$$\int_{\Omega_{\delta/2}} u_s^2 s^{-2\alpha-2}dx \le \|s^{-\nu}u_s\|^2_{L^\infty(\Omega_{\delta/2})}\int_{\Omega_{\delta/2}} s^{-2\alpha-2+2\nu}dx \le C\|s^{-\nu}u_s\|^2_{L^\infty(\Omega_{\delta/2})}.$$

Hence, using also (5.18) and (5.19),

$$\int_\Omega u_s^2 s^{-2\alpha-2}dx \le C\left(\|u\|_{L^\infty(\Omega_\delta)} + \|f(u)\|_{L^\infty(\Omega_\delta)}\right)^2,$$

as claimed. □

Using Lemma 5.2.3 we can now establish Proposition 5.1.6.

Proof of Proposition 5.1.6. Using Lemma 5.2.3 and making the change of variables

$$\sigma = s^{2+\alpha}, \quad \tau = t^{2+\beta}$$

in the integral in (5.13), one has

$$\begin{cases} s^{m-1}ds &= c_\alpha \sigma^{\frac{m}{2+\alpha}-1} d\sigma \\ t^{k-1}dt &= c_\beta \tau^{\frac{k}{2+\beta}-1} d\tau, \end{cases}$$

and thus,

$$\int_{\widetilde{U}} \sigma^{\frac{m}{2+\alpha}-1} \tau^{\frac{k}{2+\beta}-1} (u_\sigma^2 + u_\tau^2) d\sigma d\tau \leq C \left(\|u\|_{L^\infty(\Omega_\delta)} + \|f(u)\|_{L^\infty(\Omega_\delta)} \right)^2. \tag{5.20}$$

Here, $\widetilde{U}$ denotes the image of the two dimensional domain $\widetilde{\Omega}$ in (5.5) after the transformation $(s,t) \mapsto (\sigma, \tau)$. The constant in (5.20) depends on α and β. However, later we will choose α and β depending only on m and k and hence the constants will be controlled by constants depending only on n (since for each n there are a finite number of integers m and k).

a) We assume Ω to be convex. Recall that in this case $\|u\|_{L^\infty} = u(0)$; see Remark 5.2.1.

From (5.20), setting $\rho = \sqrt{\sigma^2 + \tau^2}$ and taking into account that in $\{\tau < \sigma < 2\tau\}$ we have $\frac{\rho}{2} < \sigma < \rho$ and $\frac{\rho}{3} < \tau < \rho$, we obtain

$$\int_{\widetilde{U} \cap \{\tau < \sigma < 2\tau\}} \rho^{\frac{m}{2+\alpha} + \frac{k}{2+\beta} - 2} (u_\sigma^2 + u_\tau^2) d\sigma d\tau \leq C \left(\|u\|_{L^\infty(\Omega_\delta)} + \|f(u)\|_{L^\infty(\Omega_\delta)} \right)^2. \tag{5.21}$$

Now, for each angle θ we have

$$u(0) \leq \int_{l_\theta} |\nabla_{(\sigma,\tau)} u| d\rho,$$

where l_θ is the segment of angle θ in the (σ, τ)-plane from the origin to $\partial \widetilde{U}$. Integrating in $\arctan \frac{1}{2} < \theta < \arctan 1 = \frac{\pi}{4}$,

$$u(0) \leq C \int_{\arctan \frac{1}{2}}^{\frac{\pi}{4}} \int_{l_\theta} |\nabla_{(\sigma,\tau)} u| d\rho d\theta = C \int_{\widetilde{U} \cap \{\tau < \sigma < 2\tau\}} \frac{|\nabla_{(\sigma,\tau)} u|}{\rho} d\sigma d\tau. \tag{5.22}$$

Now, applying Schwarz's inequality and taking into account (5.21) and (5.22),

$$u(0) \leq C \left(\|u\|_{L^\infty(\Omega_\delta)} + \|f(u)\|_{L^\infty(\Omega_\delta)} \right) \left(\int_{\widetilde{U} \cap \{\tau < \sigma < 2\tau\}} \rho^{-\left(\frac{m}{2+\alpha} + \frac{k}{2+\beta}\right)} d\sigma d\tau \right)^{1/2}.$$

This integral is finite when

$$\frac{m}{2+\alpha} + \frac{k}{2+\beta} < 2.$$

Therefore, if

$$\frac{m}{2+\sqrt{m-1}}+\frac{k}{2+\sqrt{k-1}}<2 \tag{5.23}$$

then we can choose $\alpha<\sqrt{m-1}$ and $\beta<\sqrt{k-1}$ such that the integral is finite. Hence, since $\|u\|_{L^\infty(\Omega)}=u(0)$, if condition (5.23) is satisfied then

$$\|u\|_{L^\infty(\Omega)}\le C\left(\|u\|_{L^\infty(\Omega_\delta)}+\|f(u)\|_{L^\infty(\Omega_\delta)}\right).$$

Let

$$q_{m,k}=\frac{m}{2+\sqrt{m-1}}+\frac{k}{2+\sqrt{k-1}}.$$

If $n\le 7$ then by Remark 5.1.5 we have that $q_{m,k}\le q_{\frac{n}{2},\frac{n}{2}}\le q_{\frac{7}{2},\frac{7}{2}}<2$ (note that the function $q=q(x)$ in the remark is increasing in x). Instead, if $n\ge 8$ then $q_{m,k}\ge q_{2,n-2}\ge q_{2,6}>2$. Hence, (5.23) is satisfied if and only if $n\le 7$.

b) We assume that Ω is convex and that $n\ge 8$. Note that $q_{\frac{n}{2},\frac{n}{2}}=\frac{n}{2+\sqrt{\frac{n}{2}-1}}<\frac{n}{2}$, and thus

$$p_{m,k}>2+\frac{4}{\frac{n}{2}-2}=\frac{2n}{n-4}.$$

Hence, without loss of generality we may assume that

$$\frac{2n}{n-4}\le p<p_{m,k}$$

and we can choose nonnegative numbers α and β such that $\alpha^2<m-1$, $\beta^2<k-1$, and

$$p=2+\frac{4}{\frac{m}{2+\alpha}+\frac{k}{2+\beta}-2}. \tag{5.24}$$

This is because the expression (5.24) is increasing in α and β, and its value for $\alpha=\beta=0$ is $\frac{2n}{n-4}$. In addition, since $q_{m,k}\ge q_{2,n-2}\ge q_{2,6}>2$, we have that $\frac{m}{2+\alpha}+\frac{k}{2+\beta}-2>0$ and that one of the numbers $\frac{m}{2+\alpha}-1$ or $\frac{k}{2+\beta}-1$ is positive.

Hence, we can apply now Proposition 5.1.7 to $u=u(\sigma,\tau)$ with $a=\frac{m}{2+\alpha}-1$, $b=\frac{k}{2+\beta}-1$ and $q=2<D=\frac{m}{2+\alpha}+\frac{k}{2+\beta}$. We deduce that

$$\left(\int_{\widetilde{U}}\sigma^{\frac{m}{2+\alpha}-1}\tau^{\frac{k}{2+\beta}-1}|u|^p d\sigma d\tau\right)^{1/p}\le C\left(\int_{\widetilde{U}}\sigma^{\frac{m}{2+\alpha}-1}\tau^{\frac{k}{2+\beta}-1}|\nabla_{(\sigma,\tau)}u|^2 d\sigma d\tau\right)^{1/2}.$$

Here we have extended u by zero outside $\widetilde{U}$, obtaining a nonnegative Lipschitz function. By Remark 5.2.1 it satisfies $u_s<0$ and $u_t<0$ whenever $u>0$, $s>0$, and $t>0$ since Ω is convex, and therefore $u_\sigma<0$ and $u_\tau<0$ whenever $u>0$, $\sigma>0$, and $\tau>0$. Note also that $q^*=2^*=\frac{2D}{D-2}=2+\frac{4}{D-2}=p$. Thus, combining the last inequality with (5.20), we have

$$\left(\int_{\widetilde{U}}\sigma^{\frac{m}{2+\alpha}-1}\tau^{\frac{k}{2+\beta}-1}|u|^p d\sigma d\tau\right)^{1/p}\le C\left(\|u\|_{L^\infty(\Omega_\delta)}+\|f(u)\|_{L^\infty(\Omega_\delta)}\right).$$

Finally, since

$$\int_{\widetilde{U}} \sigma^{\frac{m}{2+\alpha}-1}\tau^{\frac{k}{2+\beta}-1}|u|^p d\sigma d\tau = c_{\alpha,\beta}\int_{\widetilde{\Omega}} s^{m-1}t^{k-1}|u|^p ds dt = c_{\alpha,\beta,m,k}\|u\|^p_{L^p(\Omega)},$$

we conclude

$$\|u\|_{L^p(\Omega)} \leq C\left(\|u\|_{L^\infty(\Omega_\delta)} + \|f(u)\|_{L^\infty(\Omega_\delta)}\right).$$

c) Here we do not assume Ω to be convex. We set $\alpha = 0$ in Lemma 5.2.3. Estimate (5.17) in its proof gives

$$\int_{\Omega\setminus\Omega_{\delta/2}} u_s^2 s^{-2} dx \leq C\int_{\Omega_{\delta/2}} u_s^2 dx,$$

and therefore, for a different constant C,

$$\int_{\Omega} u_s^2 dx \leq C\int_{\Omega_{\delta/2}} u_s^2 dx.$$

Since, for $1 \leq i \leq m$ and $m+1 \leq j \leq n$, $u_{x_i} = u_s\frac{x_i}{s}$ and $u_{x_j} = u_t\frac{x_j}{t}$, this leads to

$$\|u\|_{H^1_0(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)} \leq C\|u\|_{H^1(\Omega_\delta)},$$

as claimed. □

5.3 Regularity of the extremal solution

This section is devoted to give the proof of Theorem 5.1.4. The estimates for convex domains will follow easily from Proposition 5.1.6 and the boundary estimates in convex domains of de Figueiredo, Lions, and Nussbaum [110]. These boundary estimates (see also [42] for their proof) follow easily from the moving planes method [156].

Theorem 5.3.1 ([110],[156])**.** *Let Ω be a smooth, bounded, and convex domain, f be any Lipschitz function, and u be a bounded positive solution of* (5.8)*. Then, there exist constants $\delta > 0$ and C, both depending only on Ω, such that*

$$\|u\|_{L^\infty(\Omega_\delta)} \leq C\|u\|_{L^1(\Omega)},$$

where $\Omega_\delta = \{x \in \Omega : dist(x, \partial\Omega) < \delta\}$.

We can now give the proof of Theorem 5.1.4. The main part of the proof are the estimates for non-convex domains. They will be proved by interpolating the $W^{1,p}$ and $W^{2,p}$ estimates of Nedev [225] and our estimate of Lemma 5.2.3, and by applying the classical Sobolev inequality as explained in Remark 5.1.8.

Proof of Theorem 5.1.4. As we have pointed out, the estimates for convex domains are a consequence of Proposition 5.1.6 and Theorem 5.3.1. Namely, we can apply the estimates of Proposition 5.1.6 to the bounded and semi-stable minimal solutions u_λ of (5.1) for $\lambda < \lambda^*$, and then by monotone convergence the estimates hold for the extremal solution u^*. Note that $\|u_\lambda\|_{L^1(\Omega)} \leq \|u^*\|_{L^1(\Omega)} < \infty$ for all $\lambda < \lambda^*$.

To prove part c) for convex domains, we use part c) of Proposition 5.1.6 with δ replaced by $\delta/2$ and δ given by Theorem 5.3.1. We then control $\|u\|_{H^1(\Omega_{\delta/2})}$ by $\|u\|_{L^\infty(\Omega_\delta)} + \|f(u)\|_{L^\infty(\Omega_\delta)}$ using boundary estimates. Finally, we use Theorem 5.3.1. Next we prove the estimates in parts a) and c) for non-convex domains.

We start by proving part a) when Ω is not convex. We have that $n = 4$, i.e. $m = k = 2$. In [225] (see its Remark 1) it is proved that the extremal solution satisfies $u^* \in W^{1,p}(\Omega)$ for all $p < \frac{n}{n-3}$. Thus, since $n = 4$, for each $p < 4$ we have

$$\int_\Omega |u_s^*|^p dx \le C \qquad \text{and} \qquad \int_\Omega |u_t^*|^p dx \le C.$$

Assume that $\|u^*\|_{L^\infty(\Omega_\delta)} \le C$ for some $\delta > 0$ —which we will prove later. Then, by Lemma 5.2.3, for all $\gamma < 4$ we have

$$\int_\Omega s^{-\gamma}|u_s^*|^2 dx \le C \qquad \text{and} \qquad \int_\Omega t^{-\gamma}|u_t^*|^2 dx \le C.$$

Hence, for each $\lambda \in [0,1]$,

$$\int_\Omega (s^{-\lambda\gamma}|u_s^*|^{p-\lambda(p-2)} + t^{-\lambda\gamma}|u_t^*|^{p-\lambda(p-2)})dx \le C.$$

Setting now $\sigma = s^\kappa$, $\tau = t^\kappa$, and

$$\kappa = 1 + \frac{\lambda\gamma}{p - \lambda(p-2)},$$

we obtain

$$\int_{\widetilde{U}} \sigma^{\frac{2}{\kappa}-1}\tau^{\frac{2}{\kappa}-1}|\nabla_{(\sigma,\tau)}u^*|^{p-\lambda(p-2)}d\sigma d\tau \le C,$$

and taking $p = 3$, $\gamma = 3$ and $\lambda = 3/4$ (and thus $\kappa = 2$), we obtain

$$\int_{\widetilde{U}} |\nabla_{(\sigma,\tau)}u^*|^{9/4} d\sigma d\tau \le C.$$

Finally, applying Sobolev's inequality in the 2 dimensional plane (σ,τ), $u^* \in L^\infty(\Omega)$.

It remains to prove that $\|u^*\|_{L^\infty(\Omega_\delta)} \le C$ for some $\delta > 0$. Since $u^* \in W^{1,p}(\Omega)$ for every $p < 4$, we have

$$\int_{\Omega_\delta} st|\nabla u^*|^p dsdt \le C.$$

Since the domain is smooth, we must have $0 \notin \partial\Omega$ (otherwise the boundary would have an isolated point) and hence, there exist $r_0 > 0$ and $\delta > 0$ such that $\Omega_\delta \cap B_{r_0}(0) = \emptyset$. Thus, $s \ge r_0/\sqrt{2}$ in $\Omega_\delta \cap \{s > t\}$ and $t \ge r_0/\sqrt{2}$ in $\Omega_\delta \cap \{s < t\}$. It follows that

$$\int_{\Omega_\delta\cap\{s>t\}} t|\nabla u^*|^p dsdt \le C \qquad \text{and} \qquad \int_{\Omega_\delta\cap\{s<t\}} s|\nabla u^*|^p dsdt \le C.$$

Taking $p \in (3,4)$, we can apply Sobolev's inequality in dimension 3 (as explained in Remark 5.1.8), to obtain $u^* \in L^\infty(\Omega_\delta \cap \{s > t\})$ and $u^* \in L^\infty(\Omega_\delta \cap \{s < t\})$. Note

that u^* does not vanish through all $\partial(\Omega_\delta \cap \{s > t\})$ and $\partial(\Omega_\delta \cap \{s < t\})$, but it vanishes on their intersection with $\partial\Omega$ —a sufficiently large part of $\partial(\Omega_\delta \cap \{s > t\})$ and $\partial(\Omega_\delta \cap \{s < t\})$ to apply the Sobolev inequality. Therefore $u^* \in L^\infty(\Omega_\delta)$, as claimed.

To prove part c) in the non-convex case, let $n \leq 6$. By Proposition 5.1.6, it suffices to prove that $u^* \in H^1(\Omega_\delta)$ for some $\delta > 0$. Take r_0 and δ such that $\Omega_\delta \cap B_{r_0}(0) = \emptyset$, as in part a).

In [225] it is proved that $u^* \in W^{2,p}(\Omega)$ for $p < \frac{n}{n-2}$. Thus, by the previous lower bounds for s and t in $\{s > t\}$ and $\{s < t\}$ respectively,

$$\int_{\Omega_\delta \cap \{s>t\}} t^{k-1} |D^2 u^*|^p ds dt \leq C \qquad \text{and} \qquad \int_{\Omega_\delta \cap \{s<t\}} s^{m-1} |D^2 u^*|^p ds dt \leq C.$$

Since $n \leq 6$, $m \geq 2$, and $k \geq 2$, we have that $k \leq 4$ and $m \leq 4$. It follows that $\frac{2k+2}{k+3} < \frac{n}{n-2}$ and $\frac{2m+2}{m+3} < \frac{n}{n-2}$. Thus, we may take $p = \frac{2k+2}{k+3}$ and $p = \frac{2m+2}{m+3}$ respectively in the two previous estimates. Now applying Sobolev's inequality in dimension $k+1$ and $m+1$ respectively, we obtain $\nabla u^* \in L^2(\Omega_\delta \cap \{s > t\})$ and $\nabla u^* \in L^2(\Omega_\delta \cap \{s < t\})$. Therefore, $u^* \in H^1(\Omega_\delta)$. □

5.4 Weighted Sobolev inequality

It is well known that the classical Sobolev inequality can be deduced from the isoperimetric inequality. This is done by applying first the isoperimetric inequality to the level sets of the function and then using the coarea formula. In this way one deduces the Sobolev inequality with exponent 1 on the gradient. Then, by applying Hölder's inequality one deduces the general Sobolev inequality. Here, we will proceed in this way to prove the Sobolev inequality of Proposition 5.1.7.

Recall that we will apply this Sobolev inequality to the function u defined on the (σ, τ)-plane, where $\sigma = s^{2+\alpha}$ and $\tau = t^{2+\beta}$. Recall also that this application will be in convex domains, and thus u satisfies the hypothesis of Proposition 5.1.7, i.e., $u_\sigma \leq 0$ and $u_\tau \leq 0$, with strict inequality whenever $u > 0$. Hence, since the isoperimetric inequality will be applied to the level sets of u, it suffices to prove a weighted isoperimetric inequality for bounded domains $\widetilde{U} \subset (\mathbb{R}_+)^2 = (0, \infty)^2$ satisfying the following property:

(P) For all $(\sigma, \tau) \in \widetilde{U}$, $\widetilde{U}(\cdot, \tau) := \{\sigma' > 0 : (\sigma', \tau) \in \widetilde{U}\}$ and $\widetilde{U}(\sigma, \cdot) := \{\tau' > 0 : (\sigma, \tau') \in \widetilde{U}\}$ are intervals which are strictly decreasing in τ and σ, respectively.

We denote

$$m(\widetilde{U}) = \int_{\widetilde{U}} \sigma^a \tau^b d\sigma d\tau \qquad \text{and} \qquad m(\partial\widetilde{U} \cap (\mathbb{R}_+)^2) = \int_{\partial\widetilde{U} \cap (\mathbb{R}_+)^2} \sigma^a \tau^b d\sigma d\tau.$$

Note that in the weighted perimeter $m(\partial\widetilde{U} \cap (\mathbb{R}_+)^2)$ the part of $\partial\widetilde{U}$ on the σ and τ coordinate axes is not counted. The following isoperimetric inequality holds in domains satisfying property (P) above, under no further regularity assumption on them.

Proposition 5.4.1. *Let $\widetilde{U} \subset (\mathbb{R}_+)^2$ be a bounded domain satisfying (P) above, $a > -1$ and $b > -1$ be real numbers, being positive at least one of them, and*

$$D = a + b + 2.$$

Then, there exists a constant C depending only on a and b such that

$$m(\widetilde{U})^{\frac{D-1}{D}} \leq Cm(\partial\widetilde{U} \cap (\mathbb{R}_+)^2).$$

Proof. First, by symmetry we can suppose $a > 0$.

Property (P) ensures that there exists a unique well defined decreasing, bounded, and continuous function $\psi : (0, \overline{\sigma}) \to (0, \infty)$ for some $\overline{\sigma} > 0$ such that

$$\widetilde{U} = \{(\sigma, \tau) \in (\mathbb{R}_+)^2 : \tau < \psi(\sigma)\}. \tag{5.25}$$

In addition, extending ψ by zero in $[\overline{\sigma}, \infty)$, ψ is continuous and nonincreasing. Even that we could have $\psi' = -\infty$ at some points, $|\psi'| = -\psi'$ is integrable (since ψ is bounded) and thus $\psi \in W^{1,1}(\mathbb{R})$. We have that

$$m(\widetilde{U}) = \frac{1}{b+1}\int_0^{+\infty} \sigma^a \psi^{b+1} d\sigma \quad \text{and} \quad m(\partial\widetilde{U} \cap (\mathbb{R}_+)^2) = \int_0^{+\infty} \sigma^a \psi^b \sqrt{1+\psi'^2} d\sigma.$$

Let $\mu > 0$ be such that

$$m(\widetilde{U}) = \frac{\mu^D}{(a+1)(b+1)}. \tag{5.26}$$

We claim that

$$\psi(\sigma) < \mu \quad \text{for} \quad \sigma > \mu.$$

Assume that this is false. Then, we would have $\psi(\sigma') \geq \mu$ for some $\sigma' > \mu$, and hence

$$m(\widetilde{U}) \geq \frac{1}{b+1}\int_0^{\sigma'} \sigma^a \psi^{b+1} d\sigma > \frac{1}{b+1}\int_0^{\mu} \sigma^a \mu^{b+1} d\sigma = \frac{\mu^D}{(a+1)(b+1)},$$

a contradiction. On the other hand, since $a > 0$, $b + 1 > 0$, and $\psi' \leq 0$,

$$\begin{aligned} m(\partial\widetilde{U} \cap (\mathbb{R}_+)^2) &= \int_0^{+\infty} \sigma^a \psi^b \sqrt{1+\psi'^2} d\sigma \\ &\geq c\int_0^{+\infty} \sigma^a \psi^b \left\{1 - \frac{b+1}{a}\psi'\right\} d\sigma \\ &= c\int_0^{+\infty} \sigma^a \left\{\psi^b - \frac{d}{d\sigma}\left(\frac{\psi^{b+1}}{a}\right)\right\} d\sigma \\ &= c\int_0^{+\infty} \sigma^a \psi^{b+1}\left(\frac{1}{\psi} + \frac{1}{\sigma}\right) d\sigma, \end{aligned}$$

for some constant c depending only on a and b.

Finally, taking into account that $\psi(\sigma) < \mu$ for $\sigma > \mu$, we obtain that $\frac{1}{\psi} + \frac{1}{\sigma} \geq \frac{1}{\mu}$ for each $\sigma > 0$. Thus, recalling (5.26),

$$m(\partial\widetilde{U} \cap (\mathbb{R}_+)^2) \geq c\int_0^{+\infty} \sigma^a \psi^{b+1}\left(\frac{1}{\psi} + \frac{1}{\sigma}\right) d\sigma \geq \frac{c}{\mu} m(\widetilde{U}) = cm(\widetilde{U})^{\frac{D-1}{D}},$$

as claimed. □

Now we are able to prove our Sobolev inequality from the previous isoperimetric inequality. We follow the proof given in [120] for the classical unweighted case.

Proof of Proposition 5.1.7. We will prove first the case $q = 1$.

Letting χ_A denote the characteristic function of the set A, we have

$$u(\sigma,\tau) = \int_0^{+\infty} \chi_{[u(\sigma,\tau)>\lambda]} d\lambda.$$

Thus, by Minkowski's integral inequality

$$\begin{aligned}\left(\int_{(\mathbb{R}_+)^2} \sigma^a \tau^b |u|^{\frac{D}{D-1}} d\sigma d\tau\right)^{\frac{D-1}{D}} &\leq \int_0^{+\infty} \left(\int_{(\mathbb{R}_+)^2} \sigma^a \tau^b \chi_{[u(\sigma,\tau)>\lambda]} d\sigma d\tau\right)^{\frac{D-1}{D}} d\lambda \\ &= \int_0^{+\infty} m(\{u(\sigma,\tau) > \lambda\})^{\frac{D-1}{D}} d\lambda.\end{aligned}$$

Since $u_\sigma \leq 0$ and $u_\tau \leq 0$, with strict inequality when $u > 0$, the level sets $\{u(\sigma,\tau) > \lambda\}$ satisfy property (P) in the beginning of Section 4. In fact, since $u_\tau < 0$ at points where $u = \lambda > 0$, the implicit function theorem gives that the function ψ in (5.25) when $\widetilde{U} = \{u(\sigma,\tau) > \lambda\}$ is C^1 in $(0,\overline{\sigma})$. Thus, Proposition 5.4.1 leads to

$$\begin{aligned} m\left(\{u(\sigma,\tau) > \lambda\}\right)^{\frac{D-1}{D}} &\leq Cm\left(\partial\{u(\sigma,\tau) > \lambda\} \cap (\mathbb{R}_+)^2\right) \\ &= Cm\left(\{u(\sigma,\tau) = \lambda\} \cap (\mathbb{R}_+)^2\right), \end{aligned}$$

whence

$$\left(\int_{(\mathbb{R}_+)^2} \sigma^a \tau^b |u|^{\frac{D}{D-1}} d\sigma d\tau\right)^{\frac{D-1}{D}} \leq C \int_0^{+\infty} m\left(\{u(\sigma,\tau) = \lambda\} \cap (\mathbb{R}_+)^2\right) d\lambda.$$

Let u_{ev} be the even extension of u with respect to σ and τ in $\mathbb{R}^2$. Then,

$$\int_0^{+\infty} m\left(\{u(\sigma,\tau) = \lambda\} \cap (\mathbb{R}_+)^2\right) d\lambda = \frac{1}{4} \int_0^{+\infty} m\left(\{u_{ev}(\sigma,\tau) = \lambda\}\right) d\lambda,$$

and by the coarea formula

$$\int_0^{+\infty} m\left(\{u_{ev}(\sigma,\tau) = \lambda\}\right) d\lambda = \int_{\mathbb{R}^2} \sigma^a \tau^b |\nabla u_{ev}| d\sigma d\tau.$$

Thus, we obtain

$$\left(\int_{(\mathbb{R}_+)^2} \sigma^a \tau^b |u|^{\frac{D}{D-1}} d\sigma d\tau\right)^{\frac{D-1}{D}} \leq C \int_{(\mathbb{R}_+)^2} \sigma^a \tau^b |\nabla u| d\sigma d\tau,$$

and the proposition is proved for $q = 1$.

Finally, let us prove the case $1 < q < D$. Take u satisfying the hypotheses of Proposition 5.1.7, and define $v = u^\gamma$, where $\gamma = \frac{q^*}{1^*}$. Since $\gamma > 1$, we have that v also

satisfies the hypotheses of the proposition, and we can apply the weighted Sobolev inequality with $q = 1$ to get

$$\begin{aligned}\left(\int_{(\mathbb{R}_+)^2} \sigma^a \tau^b |u|^{q^*} d\sigma d\tau\right)^{1/1^*} &= \left(\int_{(\mathbb{R}_+)^2} \sigma^a \tau^b |v|^{\frac{D}{D-1}} d\sigma d\tau\right)^{\frac{D-1}{D}} \\ &\leq C \int_{(\mathbb{R}_+)^2} \sigma^a \tau^b |\nabla v| d\sigma d\tau.\end{aligned}$$

Now, $|\nabla v| = \gamma u^{\gamma-1} |\nabla u|$, and by Hölder's inequality it follows that

$$\int_{(\mathbb{R}_+)^2} \sigma^a \tau^b |\nabla v| d\sigma d\tau \leq C \left(\int_{(\mathbb{R}_+)^2} \sigma^a \tau^b |\nabla u|^q d\sigma d\tau\right)^{1/q} \left(\int_{(\mathbb{R}_+)^2} \sigma^a \tau^b |u|^{(\gamma-1)q'} d\sigma d\tau\right)^{1/q'}.$$

But from the definition of γ and q^* it follows that

$$\frac{\gamma - 1}{q^*} = \frac{1}{1^*} - \frac{1}{q^*} = \frac{1}{q'}, \qquad (\gamma - 1)q' = q^*,$$

and hence

$$\left(\int_{(\mathbb{R}_+)^2} \sigma^a \tau^b |u|^{q^*} d\sigma d\tau\right)^{1/q^*} \leq C \left(\int_{(\mathbb{R}_+)^2} \sigma^a \tau^b |\nabla u|^q d\sigma d\tau\right)^{1/q},$$

as desired. □

Chapter Six

THE EXTREMAL SOLUTION FOR THE FRACTIONAL LAPLACIAN

We study the extremal solution for the problem $(-\Delta)^s u = \lambda f(u)$ in Ω, $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, where $\lambda > 0$ is a parameter and $s \in (0,1)$. We extend some well known results for the extremal solution when the operator is the Laplacian to this nonlocal case. For general convex nonlinearities we prove that the extremal solution is bounded in dimensions $n < 4s$. We also show that, for exponential and power-like nonlinearities, the extremal solution is bounded whenever $n < 10s$. In the limit $s \uparrow 1$, $n < 10$ is optimal. In addition, we show that the extremal solution is $H^s(\mathbb{R}^n)$ in any dimension whenever the domain is convex.

To obtain some of these results we need L^q estimates for solutions to the linear Dirichlet problem for the fractional Laplacian with L^p data. We prove optimal L^q and C^β estimates, depending on the value of p. These estimates follow from classical embedding results for the Riesz potential in $\mathbb{R}^n$.

Finally, to prove the H^s regularity of the extremal solution we need an L^∞ estimate near the boundary of convex domains, which we obtain via the moving planes method. For it, we use a maximum principle in small domains for integro-differential operators with decreasing kernels.

6.1 Introduction and results

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and $s \in (0,1)$, and consider the problem

$$\begin{cases} (-\Delta)^s u &= \lambda f(u) & \text{in } \Omega \\ u &= 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{6.1}$$

where λ is a positive parameter and $f : [0,\infty) \longrightarrow \mathbb{R}$ satisfies

$$f \text{ is } C^1 \text{ and nondecreasing, } f(0) > 0, \text{ and } \lim_{t\to+\infty} \frac{f(t)}{t} = +\infty. \tag{6.2}$$

Here, $(-\Delta)^s$ is the fractional Laplacian, defined for $s \in (0,1)$ by

$$(-\Delta)^s u(x) = c_{n,s} PV \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy, \tag{6.3}$$

where $c_{n,s}$ is a constant.

It is well known —see [36] or the excellent monograph [120] and references therein— that in the classical case $s = 1$ there exists a finite extremal parameter λ^* such that if $0 < \lambda < \lambda^*$ then problem (6.1) admits a minimal classical solution u_λ, while for $\lambda > \lambda^*$ it has no solution, even in the weak sense. Moreover, the family of functions $\{u_\lambda : 0 < \lambda < \lambda^*\}$ is increasing in λ, and its pointwise limit $u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$ is a weak solution of problem (6.1) with $\lambda = \lambda^*$. It is called the extremal solution of (6.1).

When $f(u) = e^u$, we have that $u^* \in L^\infty(\Omega)$ if $n \leq 9$ [102], while $u^*(x) = \log \frac{1}{|x|^2}$ if $n \geq 10$ and $\Omega = B_1$ [177]. An analogous result holds for other nonlinearities such as powers $f(u) = (1+u)^p$ and also for functions f satisfying a limit condition at infinity; see [256]. In the nineties H. Brezis and J.L. Vázquez [36] raised the question of determining the regularity of u^*, depending on the dimension n, for general nonlinearities f satisfying (6.2). The first result in this direction was proved by G. Nedev [225], who obtained that the extremal solution is bounded in dimensions $n \leq 3$ whenever f is convex. Some years later, X. Cabré and A. Capella [43] studied the radial case. They showed that when $\Omega = B_1$ the extremal solution is bounded for all nonlinearities f whenever $n \leq 9$. For general nonlinearities, the best known result at the moment is due to X. Cabré [42], and states that in dimensions $n \leq 4$ then the extremal solution is bounded for any convex domain Ω. Recently, S. Villegas [295] have proved, using the results in [42], the boundedness of the extremal solution in dimension $n = 4$ for all domains, not necessarily convex. The problem is still open in dimensions $5 \leq n \leq 9$.

The aim of this paper is to study the extremal solution for the fractional Laplacian, that is, to study problem (6.1) for $s \in (0, 1)$.

The closest result to ours was obtained by Capella-Dávila-Dupaigne-Sire [80]. They studied the extremal solution in $\Omega = B_1$ for the spectral fractional Laplacian A^s. The operator A^s, defined via the Dirichlet eigenvalues of the Laplacian in Ω, is related to (but different from) the fractional Laplacian (6.3). We will state their result later on in this introduction.

Let us start defining weak solutions to problem (6.1).

Definition 6.1.1. *We say that $u \in L^1(\Omega)$ is a* weak solution *of* (6.1) *if*

$$f(u)\delta^s \in L^1(\Omega), \tag{6.4}$$

where $\delta(x) = dist(x, \partial\Omega)$, and

$$\int_\Omega u(-\Delta)^s \zeta dx = \int_\Omega \lambda f(u)\zeta dx \tag{6.5}$$

for all ζ such that ζ and $(-\Delta)^s\zeta$ are bounded in Ω and $\zeta \equiv 0$ on $\partial\Omega$.

Any bounded weak solution is a classical solution*, in the sense that it is regular in the interior of Ω, continuous up to the boundary, and* (6.1) *holds pointwise; see Remark 6.2.1.*

Note that for $s = 1$ the above notion of weak solution is exactly the one used in [35, 36].

In the classical case (that is, when $s = 1$), the analysis of singular extremal solutions involves an intermediate class of solutions, those belonging to $H^1(\Omega)$; see [36, 212].

These solutions are called [36] energy solutions. As proved by Nedev [226], when the domain Ω is convex the extremal solution belongs to $H^1(\Omega)$, and hence it is an energy solution; see [69] for the statement and proofs of the results in [226].

Similarly, here we say that a weak solution u is an *energy solution* of (6.1) when $u \in H^s(\mathbb{R}^n)$. This is equivalent to saying that u is a critical point of the energy functional

$$\mathcal{E}(u) = \frac{1}{2}\|u\|_{\dot{H}^s}^2 - \int_\Omega \lambda F(u)dx, \qquad F' = f, \tag{6.6}$$

where

$$\|u\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^n} \left|(-\Delta)^{s/2}u\right|^2 dx = \frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy = (u,u)_{\dot{H}^s} \tag{6.7}$$

and

$$(u,v)_{\dot{H}^s} = \int_{\mathbb{R}^n} (-\Delta)^{s/2}u(-\Delta)^{s/2}v\,dx = \frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}dxdy. \tag{6.8}$$

Our first result, stated next, concerns the existence of a minimal branch of solutions, $\{u_\lambda,\ 0<\lambda<\lambda^*\}$, with the same properties as in the case $s=1$. These solutions are proved to be positive, bounded, increasing in λ, and semistable. Recall that a weak solution u of (6.1) is said to be *semistable* if

$$\int_\Omega \lambda f'(u)\eta^2 dx \le \|\eta\|_{\dot{H}^s}^2 \tag{6.9}$$

for all $\eta \in H^s(\mathbb{R}^n)$ with $\eta \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. When u is an energy solution this is equivalent to saying that the second variation of energy $\mathcal{E}$ at u is nonnegative.

Proposition 6.1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $s \in (0,1)$, and f be a function satisfying* (6.2)*. Then, there exists a parameter $\lambda^* \in (0,\infty)$ such that:*

(i) If $0<\lambda<\lambda^*$, problem (6.1) *admits a minimal classical solution u_λ.*

(ii) The family of functions $\{u_\lambda : 0<\lambda<\lambda^*\}$ is increasing in λ, and its pointwise limit $u^* = \lim_{\lambda\uparrow\lambda^*} u_\lambda$ is a weak solution of (6.1) *with $\lambda=\lambda^*$.*

(iii) For $\lambda>\lambda^*$, problem (6.1) *admits no classical solution.*

(iv) These solutions u_λ, as well as u^*, are semistable.

The weak solution u^* is called the extremal solution of problem (6.1).

As explained above, the main question about the extremal solution u^* is to decide whether it is bounded or not. Once the extremal solution is bounded then it is a classical solution, in the sense that it satisfies equation (6.1) pointwise. For example, if $f \in C^\infty$ then u^* bounded yields $u^* \in C^\infty(\Omega) \cap C^s(\overline{\Omega})$.

Our main result, stated next, concerns the regularity of the extremal solution for problem (6.1). To our knowledge this is the first result concerning extremal solutions for (6.1). In particular, the following are new results even for the unit ball $\Omega = B_1$ and for the exponential nonlinearity $f(u) = e^u$.

Theorem 6.1.3. *Let Ω be a bounded smooth domain in $\mathbb{R}^n$, $s \in (0,1)$, f be a function satisfying* (6.2)*, and u^* be the extremal solution of* (6.1)*.*

(i) Assume that f is convex. Then, u^* is bounded whenever $n < 4s$.

(ii) Assume that f is C^2 and that the following limit exists:

$$\tau := \lim_{t \to +\infty} \frac{f(t) f''(t)}{f'(t)^2}. \tag{6.10}$$

Then, u^* is bounded whenever $n < 10s$.

(iii) Assume that Ω is convex. Then, u^* belongs to $H^s(\mathbb{R}^n)$ for all $n \geq 1$ and all $s \in (0,1)$.

Note that the exponential and power nonlinearities e^u and $(1+u)^p$, with $p > 1$, satisfy the hypothesis in part (ii) whenever $n < 10s$. In the limit $s \uparrow 1$, $n < 10$ is optimal, since the extremal solution may be singular for $s = 1$ and $n = 10$ (as explained before in this introduction).

Note that the results in parts (i) and (ii) of Theorem 6.1.3 do not provide any estimate when s is small (more precisely, when $s \leq 1/4$ and $s \leq 1/10$, respectively). The boundedness of the extremal solution for small s seems to require different methods from the ones that we present here. Our computations in Section 6.3 suggest that the extremal solution for the fractional Laplacian should be bounded in dimensions $n \leq 7$ for all $s \in (0,1)$, at least for the exponential nonlinearity $f(u) = e^u$. As commented above, Capella-Dávila-Dupaigne-Sire [80] studied the extremal solution for the *spectral* fractional Laplacian A^s in $\Omega = B_1$. They obtained an L^∞ bound for the extremal solution in a ball in dimensions $n < 2\left(2 + s + \sqrt{2s+2}\right)$, and hence they proved the boundedness of the extremal solution in dimensions $n \leq 6$ for all $s \in (0,1)$.

To prove part (i) of Theorem 6.1.3 we borrow the ideas of [225], where Nedev proved the boundedness of the extremal solution for $s = 1$ and $n \leq 3$. To prove part (ii) we follow the approach of M. Sanchón in [256]. When we try to repeat the same arguments for the fractional Laplacian, we find that some identities that in the case $s = 1$ come from local integration by parts are no longer available for $s < 1$. Instead, we succeed to replace them by appropriate inequalities. These inequalities are sharp as $s \uparrow 1$, but not for small s. Finally, part (iii) is proved by an argument of Nedev [226], which for $s < 1$ requires the Pohozaev identity for the fractional Laplacian, recently established by the authors in [254]. This argument requires also some boundary estimates, which we prove using the moving planes method; see Proposition 6.1.8 at the end of this introduction.

An important tool in the proofs of the results of Nedev [225] and Sanchón [256] is the classical L^p to $W^{2,p}$ estimate for the Laplace equation. Namely, if u is the solution of $-\Delta u = g$ in Ω, $u = 0$ in $\partial\Omega$, with $g \in L^p(\Omega)$, $1 < p < \infty$, then

$$\|u\|_{W^{2,p}(\Omega)} \leq C \|g\|_{L^p(\Omega)}.$$

This estimate and the Sobolev embeddings lead to $L^q(\Omega)$ or $C^\alpha(\overline{\Omega})$ estimates for the solution u, depending on whether $1 < p < \frac{n}{2}$ or $p > \frac{n}{2}$, respectively.

Here, to prove Theorem 6.1.3 we need similar estimates but for the fractional Laplacian, in the sense that from $(-\Delta)^s u \in L^p(\Omega)$ we want to deduce $u \in L^q(\Omega)$ or $u \in C^\alpha(\overline{\Omega})$. However, L^p to $W^{2s,p}$ estimates for the fractional Laplace equation, in which $-\Delta$ is replaced by the fractional Laplacian $(-\Delta)^s$, are not available for all p, even when $\Omega = \mathbb{R}^n$; see Remarks 6.7.1 and 6.7.2.

Although the L^p to $W^{2s,p}$ estimate does not hold for all p in this fractional framework, what will be indeed true is the following result. This is a crucial ingredient in the proof of Theorem 6.1.3.

Proposition 6.1.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain, $s \in (0,1)$, $n > 2s$, $g \in C(\overline{\Omega})$, and u be the solution of*

$$\begin{cases} (-\Delta)^s u &= g \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{6.11}$$

(i) For each $1 \le r < \frac{n}{n-2s}$ there exists a constant C, depending only on n, s, r, and $|\Omega|$, such that

$$\|u\|_{L^r(\Omega)} \le C \|g\|_{L^1(\Omega)}, \quad r < \frac{n}{n-2s}.$$

(ii) Let $1 < p < \frac{n}{2s}$. Then there exists a constant C, depending only on n, s, and p, such that

$$\|u\|_{L^q(\Omega)} \le C \|g\|_{L^p(\Omega)}, \quad \text{where} \quad q = \frac{np}{n-2ps}.$$

(iii) Let $\frac{n}{2s} < p < \infty$. Then, there exists a constant C, depending only on n, s, p, and Ω, such that

$$\|u\|_{C^\beta(\mathbb{R}^n)} \le C \|g\|_{L^p(\Omega)}, \quad \text{where} \quad \beta = \min\left\{ s, 2s - \frac{n}{p} \right\}.$$

We will use parts (i), (ii), and (iii) of Proposition 6.1.4 in the proof of Theorem 6.1.3. However, we will only use part (iii) to obtain an L^∞ estimate for u, we will not need the C^β bound. Still, for completeness we prove the C^β estimate, with the optimal exponent β (depending on p).

Remark 6.1.5. Proposition 6.1.4 does not provide any estimate for $n \le 2s$. Since $s \in (0,1)$, then $n \le 2s$ yields $n = 1$ and $s \ge 1/2$. In this case, any bounded domain is of the form $\Omega = (a,b)$, and the Green function $G(x,y)$ for problem (6.14) is explicit; see [24]. Then, by using this expression it is not difficult to show that $G(\cdot, y)$ is $L^\infty(\Omega)$ in case $s > 1/2$ and $L^p(\Omega)$ for all $p < \infty$ in case $s = 1/2$. Hence, in case $n < 2s$ it follows that $\|u\|_{L^\infty(\Omega)} \le C\|g\|_{L^1(\Omega)}$, while in case $n = 2s$ it follows that $\|u\|_{L^q(\Omega)} \le C\|g\|_{L^1(\Omega)}$ for all $q < \infty$ and $\|u\|_{L^\infty(\Omega)} \le C\|g\|_{L^p(\Omega)}$ for $p > 1$.

Proposition 6.1.4 follows from Theorem 6.1.6 and Proposition 6.1.7 below. The first one contains some classical results concerning embeddings for the Riesz potential, and reads as follows.

Theorem 6.1.6 (see [278])**.** *Let $s \in (0,1)$, $n > 2s$, and g and u be such that*

$$u = (-\Delta)^{-s} g \quad \text{in } \mathbb{R}^n, \tag{6.12}$$

in the sense that u is the Riesz potential of order $2s$ of g. Assume that u and g belong to $L^p(\mathbb{R}^n)$, with $1 \le p < \infty$.

(i) If $p = 1$, then there exists a constant C, depending only on n and s, such that

$$\|u\|_{L^q_{weak}(\mathbb{R}^n)} \leq C\|g\|_{L^1(\mathbb{R}^n)}, \quad \text{where} \quad q = \frac{n}{n-2s}.$$

(ii) If $1 < p < \frac{n}{2s}$, then there exists a constant C, depending only on n, s, and p, such that

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|g\|_{L^p(\mathbb{R}^n)}, \quad \text{where} \quad q = \frac{np}{n-2ps}.$$

(iii) If $\frac{n}{2s} < p < \infty$, then there exists a constant C, depending only on n, s, and p, such that

$$[u]_{C^\alpha(\mathbb{R}^n)} \leq C\|g\|_{L^p(\mathbb{R}^n)}, \quad \text{where} \quad \alpha = 2s - \frac{n}{p},$$

where $[\,\cdot\,]_{C^\alpha(\mathbb{R}^n)}$ denotes the C^α seminorm.

Parts (i) and (ii) of Theorem 6.1.6 are proved in the book of Stein [278, Chapter V]. Part (iii) is also a classical result, but it seems to be more difficult to find an exact reference for it. Although it is not explicitly stated in [278], it follows for example from the inclusions

$$I_{2s}(L^p) = I_{2s-n/p}(I_{n/p}(L^p)) \subset I_{2s-n/p}(\mathrm{BMO}) \subset C^{2s-\frac{n}{p}},$$

which are commented in [278, p.164]. In the more general framework of spaces with non-doubling n-dimensional measures, a short proof of this result can also be found in [147].

Having Theorem 6.1.6 available, to prove Proposition 6.1.4 we will argue as follows. Assume $1 < p < \frac{n}{2s}$ and consider the solution v of the problem

$$(-\Delta)^s v = |g| \ \ \text{in } \mathbb{R}^n,$$

where g is extended by zero outside Ω. On the one hand, the maximum principle yields $-v \leq u \leq v$ in $\mathbb{R}^n$, and by Theorem 6.1.6 we have that $v \in L^q(\mathbb{R}^n)$. From this, parts (i) and (ii) of the proposition follow. On the other hand, if $p > \frac{n}{2s}$ we write $u = \tilde{v} + w$, where $\tilde{v}$ solves $(-\Delta)^s \tilde{v} = g$ in $\mathbb{R}^n$ and w is the solution of

$$\begin{cases} (-\Delta)^s w &= 0 \quad \text{in } \Omega \\ w &= \tilde{v} \quad \text{in } \mathbb{R}^n \backslash \Omega. \end{cases}$$

As before, by Theorem 6.1.6 we will have that $\tilde{v} \in C^\alpha(\mathbb{R}^n)$, where $\alpha = 2s - \frac{n}{p}$. Then, the C^β regularity of u will follow from the following new result.

Proposition 6.1.7. *Let Ω be a bounded $C^{1,1}$ domain, $s \in (0,1)$, $h \in C^\alpha(\mathbb{R}^n \setminus \Omega)$ for some $\alpha > 0$, and u be the solution of*

$$\begin{cases} (-\Delta)^s u &= 0 \quad \text{in } \Omega \\ u &= h \quad \text{in } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{6.13}$$

Then, $u \in C^\beta(\mathbb{R}^n)$, with $\beta = \min\{s, \alpha\}$, and

$$\|u\|_{C^\beta(\mathbb{R}^n)} \leq C\|h\|_{C^\alpha(\mathbb{R}^n \setminus \Omega)},$$

where C is a constant depending only on Ω, α, and s.

To prove Proposition 6.1.7 we use similar ideas as in [249]. Namely, since u is harmonic then it is smooth inside Ω. Hence, we only have to prove C^β estimates near the boundary. To do it, we use an appropriate barrier to show that

$$|u(x) - u(x_0)| \leq C\|h\|_{C^\alpha}\delta(x)^\beta \quad in\ \Omega,$$

where x_0 is the nearest point to x on $\partial\Omega$, $\delta(x) = dist(x, \partial\Omega)$, and $\beta = \min\{s, \alpha\}$. Combining this with the interior estimates, we obtain C^β estimates up to the boundary of Ω.

Finally, as explained before, to show that when the domain is convex the extremal solution belongs to the energy class $H^s(\mathbb{R}^n)$ —which is part (iii) of Theorem 6.1.3— we need the following boundary estimates.

Proposition 6.1.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, $s \in (0,1)$, f be a locally Lipschitz function, and u be a bounded positive solution of*

$$\begin{cases} (-\Delta)^s u &= f(u) \quad in\ \Omega \\ u &= 0 \quad in\ \mathbb{R}^n \backslash \Omega. \end{cases} \tag{6.14}$$

Then, there exists constants $\delta > 0$ and C, depending only on Ω, such that

$$\|u\|_{L^\infty(\Omega_\delta)} \leq C\|u\|_{L^1(\Omega)},$$

where $\Omega_\delta = \{x \in \Omega : dist(x, \partial\Omega) < \delta\}$.

This estimate follows, as in the classical result of de Figueiredo-Lions-Nussbaum [110], from the moving planes method. There are different versions of the moving planes method for the fractional Laplacian (using the Caffarelli-Silvestre extension, the Riesz potential, the Hopf lemma, etc.). A particularly clean version uses the maximum principle in small domains for the fractional Laplacian, recently proved by Jarohs and Weth in [175]. Here, we follow their approach and we show that this maximum principle holds also for integro-differential operators with decreasing kernels.

The paper is organized as follows. In Section 6.2 we prove Proposition 6.1.2. In Section 6.3 we study the regularity of the extremal solution in the case $f(u) = e^u$. In Section 6.4 we prove Theorem 6.1.3 (i)-(ii). In Section 6.5 we show the maximum principle in small domains and use the moving planes method to establish Proposition 6.1.8. In Section 6.6 we prove Theorem 6.1.3 (iii). Finally, in Section 6.7 we prove Proposition 6.1.4.

6.2 Existence of the extremal solution

In this section we prove Proposition 6.1.2. For it, we follow the argument from Proposition 5.1 in [43]; see also [120].

Proof of Proposition 6.1.2. Step 1. We first prove that there is no weak solution for large λ.

Let $\lambda_1 > 0$ be the first eigenvalue of $(-\Delta)^s$ in Ω and $\varphi_1 > 0$ the corresponding eigenfunction, that is,

$$\begin{cases} (-\Delta)^s \varphi_1 &= \lambda_1 \varphi_1 & \text{in } \Omega \\ \varphi_1 &> 0 & \text{in } \Omega \\ \varphi_1 &= 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

The existence, simplicity, and boundedness of the first eigenfunction is proved in [265, Proposition 5] and [267, Proposition 4]. Assume that u is a weak solution of (6.1). Then, using φ_1 as a test function for problem (6.1) (see Definition 6.1.1), we obtain

$$\int_\Omega \lambda_1 u\, \varphi_1 dx = \int_\Omega u(-\Delta)^s \varphi_1 dx = \int_\Omega \lambda f(u)\varphi_1 dx. \tag{6.15}$$

But since f is superlinear at infinity and positive in $[0, \infty)$, it follows that $\lambda f(u) > \lambda_1 u$ if λ is large enough, a contradiction with (6.15).

Step 2. Next we prove the existence of a classical solution to (6.1) for small λ. Since $f(0) > 0$, $\underline{u} \equiv 0$ is a strict subsolution of (6.1) for every $\lambda > 0$. The solution $\overline{u}$ of

$$\begin{cases} (-\Delta)^s \overline{u} &= 1 & \text{in } \Omega \\ \overline{u} &= 0 & \text{on } \mathbb{R}^n \backslash \Omega \end{cases} \tag{6.16}$$

is a bounded supersolution of (6.1) for small λ, more precisely whenever $\lambda f(\max \overline{u}) < 1$. For such values of λ, a classical solution u_λ is obtained by monotone iteration starting from zero; see for example [120].

Step 3. We next prove that there exists a finite parameter λ^* such that for $\lambda < \lambda^*$ there is a classical solution while for $\lambda > \lambda^*$ there does not exist classical solution.

Define λ^* as the supremum of all $\lambda > 0$ for which (6.1) admits a classical solution. By Steps 1 and 2, it follows that $0 < \lambda^* < \infty$. Now, for each $\lambda < \lambda^*$ there exists $\mu \in (\lambda, \lambda^*)$ such that (6.1) admits a classical solution u_μ. Since $f > 0$, u_μ is a bounded supersolution of (6.1), and hence the monotone iteration procedure shows that (6.1) admits a classical solution u_λ with $u_\lambda \le u_\mu$. Note that the iteration procedure, and hence the solution that it produces, are independent of the supersolution u_μ. In addition, by the same reason u_λ is smaller than any bounded supersolution of (6.1). It follows that u_λ is minimal (i.e., the smallest solution) and that $u_\lambda < u_\mu$.

Step 4. We show now that these minimal solutions u_λ, $0 < \lambda < \lambda^*$, are semistable.

Note that the energy functional (6.6) for problem (6.1) in the set $\{u \in H^s(\mathbb{R}^n) : u \equiv 0 \text{ in } \mathbb{R}^n \setminus \Omega,\ 0 \le u \le u_\lambda\}$ admits an absolute minimizer $u_{\min}$. Then, using that u_λ is the minimal solution and that f is positive and increasing, it is not difficult to see that $u_{\min}$ must coincide with u_λ. Considering the second variation of energy (with respect to nonpositive perturbations) we see that $u_{\min}$ is a semistable solution of (6.1). But since $u_{\min}$ agrees with u_λ, then u_λ is semistable. Thus u_λ is semistable.

Step 5. We now prove that the pointwise limit $u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$ is a weak solution of (6.1) for $\lambda = \lambda^*$ and that this solution u^* is semistable.

As above, let $\lambda_1 > 0$ the first eigenvalue of $(-\Delta)^s$, and $\varphi_1 > 0$ be the corresponding eigenfunction. Since f is superlinear at infinity, there exists a constant $C > 0$ such that

$$\frac{2\lambda_1}{\lambda^*} t \le f(t) + C \quad \text{for all} \quad t \ge 0. \tag{6.17}$$

Using φ_1 as a test function in (6.5) for u_λ, we find

$$\int_\Omega \lambda f(u_\lambda)\varphi_1 dx = \int_\Omega \lambda_1 u_\lambda \varphi_1 dx \leq \frac{\lambda^*}{2}\int_\Omega \left(f(u_\lambda)+C\right)\varphi_1 dx.$$

In the last inequality we have used (6.17). Taking $\lambda \geq \frac{3}{4}\lambda^*$, we see that $f(u_\lambda)\varphi_1$ is uniformly bounded in $L^1(\Omega)$. In addition, it follows from the results in [249] that

$$c_1\delta^s \leq \varphi_1 \leq C_2\delta^s \quad in\ \Omega$$

for some positive constants c_1 and C_2, where $\delta(x) = dist(x, \partial\Omega)$. Hence, we have that

$$\lambda\int_\Omega f(u_\lambda)\delta^s dx \leq C$$

for some constant C that does not depend on λ. Use now $\overline{u}$, the solution of (6.16), as a test function. We obtain that

$$\int_\Omega u_\lambda dx = \lambda\int_\Omega f(u_\lambda)\overline{u}dx \leq C_3\lambda\int_\Omega f(u_\lambda)\delta^s dx \leq C$$

for some constant C depending only on f and Ω. Here we have used that $\overline{u} \leq C_3\delta^s$ in Ω for some constant $C_3 > 0$, which also follows from [249].

Thus, both sequences, u_λ and $\lambda f(u_\lambda)\delta^s$ are increasing in λ and uniformly bounded in $L^1(\Omega)$ for $\lambda < \lambda^*$. By monotone convergence, we conclude that $u^* \in L^1(\Omega)$ is a weak solution of (6.1) for $\lambda = \lambda^*$.

Finally, for $\lambda < \lambda^*$ we have $\int_\Omega \lambda f'(u_\lambda)|\eta|^2 dx \leq \|\eta\|^2_{\dot{H}^s}$, where $\|\eta\|^2_{\dot{H}^s}$ is defined by (6.7), for all $\eta \in H^s(\mathbb{R}^n)$ with $\eta \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. Since $f' \geq 0$, Fatou's lemma leads to

$$\int_\Omega \lambda^* f'(u^*)|\eta|^2 dx \leq \|\eta\|^2_{\dot{H}^s},$$

and hence u^* is semistable. □

Remark 6.2.1. As said in the introduction, the study of extremal solutions involves three classes of solutions: classical, energy, and weak solutions; see Definition 6.1.1. It follows from their definitions that any classical solution is an energy solution, and that any energy solution is a weak solution.

Moreover, any weak solution u which is bounded is a classical solution. This can be seen as follows. First, by considering $u*\eta^\epsilon$ and $f(u)*\eta^\epsilon$, where η^ϵ is a standard mollifier, it is not difficult to see that u is regular in the interior of Ω. Moreover, by scaling, we find that $|(-\Delta)^{s/2}u| \leq C\delta^{-s}$, where $\delta(x) = \text{dist}(x, \partial\Omega)$. Then, if $\zeta \in C_c^\infty(\Omega)$, we can integrate by parts in (6.5) to obtain

$$(u,\zeta)_{\dot{H}^s} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\big(u(x)-u(y)\big)\big(\zeta(x)-\zeta(y)\big)}{|x-y|^{n+2s}}dx\,dy = \int_\Omega \lambda f(u)\zeta dx \tag{6.18}$$

for all $\zeta \in C_c^\infty(\Omega)$. Hence, since $f(u) \in L^\infty$, by density (6.18) holds for all $\zeta \in H^s(\mathbb{R}^n)$ such that $\zeta \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, and therefore u is an energy solution. Finally, bounded energy solutions are classical solutions; see Remark 2.11 in [249] and [268].

6.3 An example case: the exponential nonlinearity

In this section we study the regularity of the extremal solution for the nonlinearity $f(u) = e^u$. Although the results of this section follow from Theorem 6.1.3 (ii), we exhibit this case separately because the proofs are much simpler. Furthermore, this exponential case has the advantage that we have an explicit unbounded solution to the equation in the whole $\mathbb{R}^n$, and we can compute the values of n and s for which this singular solution is semistable.

The main result of this section is the following.

Proposition 6.3.1. *Let Ω be a smooth and bounded domain in $\mathbb{R}^n$, and let u^* the extremal solution of (6.1). Assume that $f(u) = e^u$ and $n < 10s$. Then, u^* is bounded.*

Proof. Let α be a positive number to be chosen later. Setting $\eta = e^{\alpha u_\lambda} - 1$ in the stability condition (6.9) (note that $\eta \equiv 0$ in $\mathbb{R}^n \setminus \Omega$), we obtain that

$$\int_\Omega \lambda e^{u_\lambda}(e^{\alpha u_\lambda} - 1)^2 dx \le \|e^{\alpha u_\lambda} - 1\|^2_{\dot{H}^s}. \tag{6.19}$$

Next we use that

$$\left(e^b - e^a\right)^2 \le \frac{1}{2}\left(e^{2b} - e^{2a}\right)(b-a) \tag{6.20}$$

for all real numbers a and b. This inequality can be deduced easily from the Cauchy-Schwarz inequality, as follows

$$\left(e^b - e^a\right)^2 = \left(\int_a^b e^t dt\right)^2 \le (b-a)\int_a^b e^{2t} dt = \frac{1}{2}\left(e^{2b} - e^{2a}\right)(b-a).$$

Using (6.20), (6.8), and integrating by parts, we deduce

$$\begin{aligned}
\|e^{\alpha u_\lambda} - 1\|^2_{\dot{H}^s} &= \frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left(e^{\alpha u_\lambda(x)} - e^{\alpha u_\lambda(y)}\right)^2}{|x-y|^{n+2s}} dxdy \\
&\le \frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\frac{1}{2}\left(e^{2\alpha u_\lambda(x)} - e^{2\alpha u_\lambda(y)}\right)\left(\alpha u_\lambda(x) - \alpha u_\lambda(y)\right)}{|x-y|^{n+2s}} dxdy \\
&= \frac{\alpha}{2}\int_\Omega e^{2\alpha u_\lambda}(-\Delta)^s u_\lambda dx.
\end{aligned}$$

Thus, using that $(-\Delta)^s u_\lambda = \lambda e^{u_\lambda}$, we find

$$\|e^{\alpha u_\lambda} - 1\|^2_{\dot{H}^s} \le \frac{\alpha}{2}\int_\Omega e^{2\alpha u_\lambda}(-\Delta)^s u_\lambda dx = \frac{\alpha}{2}\int_\Omega \lambda e^{(2\alpha+1)u_\lambda} dx. \tag{6.21}$$

Therefore, combining (6.19) and (6.21), and rearranging terms, we get

$$\left(1 - \frac{\alpha}{2}\right)\int_\Omega e^{(2\alpha+1)u_\lambda} - 2\int_\Omega e^{(\alpha+1)u_\lambda} + \int_\Omega e^{\alpha u_\lambda} \le 0.$$

From this, it follows from Hölder's inequality that for each $\alpha < 2$

$$\|e^{u_\lambda}\|_{L^{2\alpha+1}} \le C \tag{6.22}$$

for some constant C which depends only on α and $|\Omega|$.

Finally, given $n < 10s$ we can choose $\alpha < 2$ such that $\frac{n}{2s} < 2\alpha + 1 < 5$. Then, taking $p = 2\alpha + 1$ in Proposition 6.1.4 (iii) (see also Remark 6.1.5) and using (6.22) we obtain

$$\|u_\lambda\|_{L^\infty(\Omega)} \leq C_1 \|(-\Delta)^s u_\lambda\|_{L^p(\Omega)} = C_1 \lambda \|e^{u_\lambda}\|_{L^p(\Omega)} \leq C$$

for some constant C that depends only on n, s, and Ω. Letting $\lambda \uparrow \lambda^*$ we find that the extremal solution u^* is bounded, as desired. □

The following result concerns the stability of the explicit singular solution $\log \frac{1}{|x|^{2s}}$ to equation $(-\Delta)^s u = \lambda e^u$ in the whole $\mathbb{R}^n$.

Proposition 6.3.2. *Let $s \in (0,1)$, and let*

$$u_0(x) = \log \frac{1}{|x|^{2s}}.$$

Then, u_0 is a solution of $(-\Delta)^s u = \lambda_0 e^u$ in all of $\mathbb{R}^n$ for some $\lambda_0 > 0$. Moreover, u_0 is semistable if and only if

$$\frac{\Gamma\left(\frac{n}{2}\right)\Gamma(1+s)}{\Gamma\left(\frac{n-2s}{2}\right)} \leq \frac{\Gamma^2\left(\frac{n+2s}{4}\right)}{\Gamma^2\left(\frac{n-2s}{4}\right)}. \tag{6.23}$$

As a consequence:

- *If $n \leq 7$, then u is unstable for all $s \in (0,1)$.*
- *If $n = 8$, then u is semistable if and only if $s \lesssim 0'28206....$*
- *If $n = 9$, then u is semistable if and only if $s \lesssim 0'63237....$*
- *If $n \geq 10$, then u is semistable for all $s \in (0,1)$.*

Proposition 6.3.2 suggests that the extremal solution for the fractional Laplacian should be bounded whenever

$$\frac{\Gamma\left(\frac{n}{2}\right)\Gamma(1+s)}{\Gamma\left(\frac{n-2s}{2}\right)} > \frac{\Gamma^2\left(\frac{n+2s}{4}\right)}{\Gamma^2\left(\frac{n-2s}{4}\right)}, \tag{6.24}$$

at least for the exponential nonlinearity $f(u) = e^u$. In particular, u^* should be bounded for all $s \in (0,1)$ whenever $n \leq 7$. This is an open problem.

Remark 6.3.3. When $s = 1$ and when $s = 2$, inequality (6.24) coincides with the expected optimal dimensions for which the extremal solution is bounded for the Laplacian Δ and for the bilaplacian Δ^2, respectively. In the unit ball $\Omega = B_1$, it is well known that the extremal solution for $s = 1$ is bounded whenever $n \leq 9$ and may be singular if $n \geq 10$ [43], while the extremal solution for $s = 2$ is bounded whenever $n \leq 12$ and may be singular if $n \geq 13$ [106]. Taking $s = 1$ and $s = 2$ in (6.24), one can see that the inequality is equivalent to $n < 10$ and $n \lesssim 12.5653...$, respectively.

We next give the

Proof of Proposition 6.3.2. First, using the Fourier transform, it is not difficult to compute

$$(-\Delta)^s u_0 = (-\Delta)^s \log \frac{1}{|x|^{2s}} = \frac{\lambda_0}{|x|^{2s}},$$

where

$$\lambda_0 = 2^{2s} \frac{\Gamma\left(\frac{n}{2}\right) \Gamma(1+s)}{\Gamma\left(\frac{n-2s}{2}\right)}.$$

Thus, u_0 is a solution of $(-\Delta)^s u_0 = \lambda_0 e^{u_0}$.

Now, since $f(u) = e^u$, by (6.9) we have that u_0 is semistable in $\Omega = \mathbb{R}^n$ if and only if

$$\lambda_0 \int_{\mathbb{R}^n} \frac{\eta^2}{|x|^{2s}} dx \leq \int_{\mathbb{R}^n} \left|(-\Delta)^{s/2} \eta\right|^2 dx$$

for all $\eta \in H^s(\mathbb{R}^n)$.

The inequality

$$\int_{\Omega} \frac{\eta^2}{|x|^{2s}} dx \leq H_{n,s}^{-1} \int_{\mathbb{R}^n} \left|(-\Delta)^{s/2} \eta\right|^2 dx$$

is known as the fractional Hardy inequality, and the best constant

$$H_{n,s} = 2^{2s} \frac{\Gamma^2\left(\frac{n+2s}{4}\right)}{\Gamma^2\left(\frac{n-2s}{4}\right)}$$

was obtained by Herbst [169] in 1977; see also [142]. Therefore, it follows that u_0 is semistable if and only if

$$\lambda_0 \leq H_{n,s},$$

which is the same as (6.23). □

6.4 Boundedness of the extremal solution in low dimensions

In this section we prove Theorem 6.1.3 (i)-(ii).

We start with a lemma, which is the generalization of inequality (6.20). It will be used in the proof of both parts (i) and (ii) of Theorem 6.1.3.

Lemma 6.4.1. *Let f be a $C^1([0,\infty))$ function, $\widetilde{f}(t) = f(t) - f(0)$, $\gamma > 0$, and*

$$g(t) = \int_0^t \widetilde{f}(s)^{2\gamma-2} f'(s)^2 ds. \tag{6.25}$$

Then,

$$\left(\widetilde{f}(a)^\gamma - \widetilde{f}(b)^\gamma\right)^2 \leq \gamma^2 \big(g(a) - g(b)\big)(a-b)$$

for all nonnegative numbers a and b.

Proof. We can assume $a \leq b$. Then, since $\frac{d}{dt}\left\{\widetilde{f}(t)^{\gamma}\right\} = \gamma \widetilde{f}(t)^{\gamma-1} f'(t)$, the inequality can be written as

$$\left(\int_a^b \gamma \widetilde{f}(t)^{\gamma-1} f'(t) dt\right)^2 \leq \gamma^2 (b-a) \int_a^b \widetilde{f}(t)^{2\gamma-2} f'(t)^2 dt,$$

which follows from the Cauchy-Schwarz inequality. □

The proof of part (ii) of Theorem 6.1.3 will be split in two cases. Namely, $\tau \geq 1$ and $\tau < 1$, where τ is given by (6.10). For the case $\tau \geq 1$, Lemma 6.4.2 below will be an important tool. Instead, for the case $\tau < 1$ we will use Lemma 6.4.3. Both lemmas are proved by Sanchón in [256], where the extremal solution for the p-Laplacian operator is studied.

Lemma 6.4.2 ([256])**.** *Let f be a function satisfying* (6.2)*, and assume that the limit in* (6.10) *exists. Assume in addition that*

$$\tau = \lim_{t\to\infty} \frac{f(t)f''(t)}{f'(t)^2} \geq 1.$$

Then, any $\gamma \in (1, 1+\sqrt{\tau})$ satisfies

$$\limsup_{t\to+\infty} \frac{\gamma^2 g(t)}{f(t)^{2\gamma-1} f'(t)} < 1, \tag{6.26}$$

where g is given by (6.25)*.*

Lemma 6.4.3 ([256])**.** *Let f be a function satisfying* (6.2)*, and assume that the limit in* (6.10) *exists. Assume in addition that*

$$\tau = \lim_{t\to\infty} \frac{f(t)f''(t)}{f'(t)^2} < 1.$$

Then, for every $\epsilon \in (0, 1-\tau)$ there exists a positive constant C such that

$$f(t) \leq C(1+t)^{\frac{1}{1-(\tau+\epsilon)}}, \qquad \text{for all } \ t > 0.$$

The constant C depends only on τ and ϵ.

The first step in the proof of Theorem 6.1.3 (ii) in case $\tau \geq 1$ is the following result.

Lemma 6.4.4. *Let f be a function satisfying* (6.2)*. Assume that $\gamma \geq 1$ satisfies* (6.26)*, where g is given by* (6.25)*. Let u_λ be the solution of* (6.1) *given by Proposition 6.1.2 (i), where $\lambda < \lambda^*$. Then,*

$$\|f(u_\lambda)^{2\gamma} f'(u_\lambda)\|_{L^1(\Omega)} \leq C$$

for some constant C which does not depend on λ.

Proof. Recall that the seminorm $\|\cdot\|_{\dot{H}^s}$ is defined by (6.7). Using Lemma 6.4.1, (6.8), and integrating by parts,

$$\begin{aligned}\left\|\widetilde{f}(u_\lambda)^\gamma\right\|_{\dot{H}^s}^2 &= \frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\left(\widetilde{f}(u_\lambda(x))^\gamma-\widetilde{f}(u_\lambda(y))^\gamma\right)^2}{|x-y|^{n+2s}}dxdy\\ &\le \gamma^2\frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\big(g(u_\lambda(x))-g(u_\lambda(y))\big)\,(u_\lambda(x)-u_\lambda(y))}{|x-y|^{n+2s}}dxdy\\ &= \gamma^2\int_{\mathbb{R}^n}(-\Delta)^{s/2}g(u_\lambda)(-\Delta)^{s/2}u_\lambda\,dx\\ &= \gamma^2\int_\Omega g(u_\lambda)(-\Delta)^s u_\lambda\,dx\\ &= \gamma^2\int_\Omega f(u_\lambda)g(u_\lambda)dx.\end{aligned}\tag{6.27}$$

Moreover, the stability condition (6.9) applied with $\eta=\widetilde{f}(u_\lambda)^\gamma$ yields

$$\int_\Omega f'(u_\lambda)\widetilde{f}(u_\lambda)^{2\gamma}\le\left\|\widetilde{f}(u_\lambda)^\gamma\right\|_{\dot{H}^s}^2.$$

This, combined with (6.27), gives

$$\int_\Omega f'(u_\lambda)\widetilde{f}(u_\lambda)^{2\gamma}\le\gamma^2\int_\Omega f(u_\lambda)g(u_\lambda).\tag{6.28}$$

Finally, by (6.26) and since $\widetilde{f}(t)/f(t)\to 1$ as $t\to+\infty$, it follows from (6.28) that

$$\int_\Omega f(u_\lambda)^{2\gamma}f'(u_\lambda)\le C\tag{6.29}$$

for some constant C that does not depend on λ, and thus the proposition is proved. □

We next give the proof of Theorem 6.1.3 (ii).

Proof of Theorem 6.1.3 (ii). Assume first that $\tau\ge 1$, where

$$\tau=\lim_{t\to\infty}\frac{f(t)f''(t)}{f'(t)^2}.$$

By Lemma 6.4.4 and Lemma 6.4.2, we have that

$$\int_\Omega f(u_\lambda)^{2\gamma}f'(u_\lambda)dx\le C\tag{6.30}$$

for each $\gamma\in(1,1+\sqrt{\tau})$.

Now, for any such γ, we have that $\widetilde{f}^{2\gamma}$ is increasing and convex (since $2\gamma\ge 1$), and thus

$$\widetilde{f}(a)^{2\gamma}-\widetilde{f}(b)^{2\gamma}\le 2\gamma f'(a)\widetilde{f}(a)^{2\gamma-1}(a-b).$$

Therefore, we have that

$$\begin{aligned}(-\Delta)^s \widetilde{f}(u_\lambda)^{2\gamma}(x) &= c_{n,s}\int_{\mathbb{R}^n} \frac{\widetilde{f}(u_\lambda(x))^{2\gamma} - \widetilde{f}(u_\lambda(y))^{2\gamma}}{|x-y|^{n+2s}}dy \\ &\leq 2\gamma f'(u_\lambda(x))\widetilde{f}(u_\lambda(x))^{2\gamma-1} c_{n,s}\int_{\mathbb{R}^n} \frac{u_\lambda(x)-u_\lambda(y)}{|x-y|^{n+2s}}dy \\ &= 2\gamma f'(u_\lambda(x))\widetilde{f}(u_\lambda(x))^{2\gamma-1}(-\Delta)^s u_\lambda(x) \\ &\leq 2\gamma\lambda f'(u_\lambda(x)) f(u_\lambda(x))^{2\gamma},\end{aligned}$$

and thus,

$$(-\Delta)^s \widetilde{f}(u_\lambda)^{2\gamma} \leq 2\gamma\lambda f'(u_\lambda) f(u_\lambda)^{2\gamma} := v(x). \tag{6.31}$$

Let now w be the solution of the problem

$$\begin{cases} (-\Delta)^s w = v & \text{in } \Omega \\ w = 0 & \text{in } \mathbb{R}^n\backslash\Omega, \end{cases} \tag{6.32}$$

where v is given by (6.31). Then, by (6.30) and Proposition 6.1.4 (i) (see also Remark 6.1.5),

$$\|w\|_{L^p(\Omega)} \leq \|v\|_{L^1(\Omega)} \leq C \quad \text{for each } p < \frac{n}{n-2s}.$$

Since $\widetilde{f}(u_\lambda)^{2\gamma}$ is a subsolution of (6.32) —by (6.31)—, it follows that

$$0 \leq \widetilde{f}(u_\lambda)^{2\gamma} \leq w.$$

Therefore, $\|f(u_\lambda)\|_{L^p} \leq C$ for all $p < 2\gamma\frac{n}{n-2s}$, where C is a constant that does not depend on λ. This can be done for any $\gamma \in (1, 1+\sqrt{\tau})$, and thus we find

$$\|f(u_\lambda)\|_{L^p} \leq C \quad \text{for each} \quad p < \frac{2n(1+\sqrt{\tau})}{n-2s}. \tag{6.33}$$

Hence, using Proposition 6.1.4 (iii) and letting $\lambda \uparrow \lambda^*$ it follows that

$$u^* \in L^\infty(\Omega) \quad \text{whenever} \quad n < 6s + 4s\sqrt{\tau}.$$

Hence, the extremal solution is bounded whenever $n < 10s$.

Assume now $\tau < 1$. In this case, Lemma 6.4.3 ensures that for each $\epsilon \in (0, 1-\tau)$ there exist a constant C such that

$$f(t) \leq C(1+t)^m, \qquad m = \frac{1}{1-(\tau+\epsilon)}. \tag{6.34}$$

Then, by (6.33) we have that $\|f(u_\lambda)\|_{L^p} \leq C$ for each $p < p_0 := \frac{2n(1+\sqrt{\tau})}{n-2s}$.

Next we show that if $n < 10s$ by a bootstrap argument we obtain $u^* \in L^\infty(\Omega)$. Indeed, by Proposition 6.1.4 (ii) and (6.34) we have

$$f(u^*) \in L^p \iff (-\Delta)^s u^* \in L^p \implies u^* \in L^q \implies f(u^*) \in L^{q/m},$$

where $q = \frac{np}{n-2sp}$. Now, we define recursively

$$p_{k+1} := \frac{np_k}{m(n-2sp_k)}, \qquad p_0 = \frac{2n(1+\sqrt{\tau})}{n-2s}.$$

Now, since

$$p_{k+1} - p_k = \frac{p_k}{n-2sp_k}\left(2sp_k - \frac{m-1}{m}n\right),$$

then the bootstrap argument yields $u^* \in L^\infty(\Omega)$ in a finite number of steps provided that $(m-1)n/m < 2sp_0$. This condition is equivalent to $n < 2s + 4s\frac{1+\sqrt{\tau}}{\tau+\epsilon}$, which is satisfied for ϵ small enough whenever $n \leq 10s$, since $\frac{1+\sqrt{\tau}}{\tau} > 2$ for $\tau < 1$. Thus, the result is proved. □

Before proving Theorem 6.1.3 (i), we need the following lemma, proved by Nedev in [225].

Lemma 6.4.5 ([225])**.** *Let f be a convex function satisfying* (6.2)*, and let*

$$g(t) = \int_0^t f'(\tau)^2 d\tau. \tag{6.35}$$

Then,

$$\lim_{t\to+\infty} \frac{f'(t)\widetilde{f}(t)^2 - \widetilde{f}(t)g(t)}{f(t)f'(t)} = +\infty,$$

where $\widetilde{f}(t) = f(t) - f(0)$.

As said above, this lemma is proved in [225]. More precisely, see equation (6) in the proof of Theorem 1 in [225] and recall that $\widetilde{f}/f \to 1$ at infinity.

We can now give the

Proof of Theorem 6.1.3 (i). Let g be given by (6.35). Using Lemma 6.4.1 with $\gamma = 1$ and integrating by parts, we find

$$\begin{aligned}
\|f(u_\lambda)\|_{\dot{H}^s}^2 &= \frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left(f(u_\lambda(x)) - f(u_\lambda(y))\right)^2}{|x-y|^{n+2s}}dxdy \\
&\leq \frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left(g(u_\lambda(x)) - g(u_\lambda(y))\right)\left(u_\lambda(x) - u_\lambda(y)\right)}{|x-y|^{n+2s}}dxdy \\
&= \int_{\mathbb{R}^n} (-\Delta)^{s/2} g(u_\lambda)(-\Delta)^{s/2} u_\lambda dx \\
&= \int_{\mathbb{R}^n} g(u_\lambda)(-\Delta)^s u_\lambda dx \\
&= \int_\Omega f(u_\lambda) g(u_\lambda).
\end{aligned} \tag{6.36}$$

The stability condition (6.9) applied with $\eta = \widetilde{f}(u_\lambda)$ yields

$$\int_\Omega f'(u_\lambda)\widetilde{f}(u_\lambda)^2 \leq \|\widetilde{f}(u_\lambda)\|_{\dot{H}^s}^2,$$

which combined with (6.36) gives

$$\int_\Omega f'(u_\lambda)\widetilde{f}(u_\lambda)^2 \leq \int_\Omega f(u_\lambda)g(u_\lambda). \tag{6.37}$$

This inequality can be written as

$$\int_\Omega \left\{ f'(u_\lambda)\widetilde{f}(u_\lambda)^2 - \widetilde{f}(u_\lambda)g(u_\lambda) \right\} \leq f(0)\int_\Omega g(u_\lambda).$$

In addition, since f is convex we have

$$g(t) = \int_0^t f'(s)^2 ds \leq f'(t)\int_0^t f'(s)ds \leq f'(t)f(t),$$

and thus,

$$\int_\Omega \left\{ f'(u_\lambda)\widetilde{f}(u_\lambda)^2 - \widetilde{f}(u_\lambda)g(u_\lambda) \right\} \leq f(0)\int_\Omega f'(u_\lambda)f(u_\lambda).$$

Hence, by Lemma 6.4.5 we obtain

$$\int_\Omega f(u_\lambda)f'(u_\lambda) \leq C. \tag{6.38}$$

Now, on the one hand we have that

$$f(a) - f(b) \leq f'(a)(a-b),$$

since f is increasing and convex. This yields, as in (6.31),

$$(-\Delta)^s \widetilde{f}(u_\lambda) \leq f'(u_\lambda)(-\Delta)^s u_\lambda = f'(u_\lambda)f(u_\lambda) := v(x).$$

On the other hand, let w the solution of the problem

$$\left\{ \begin{array}{rcll} (-\Delta)^s w & = & v & \text{in } \Omega \\ w & = & 0 & \text{on } \partial\Omega. \end{array} \right. \tag{6.39}$$

By (6.38) and Proposition 6.1.4 (i) (see also Remark 6.1.5),

$$\|w\|_{L^p(\Omega)} \leq \|v\|_{L^1(\Omega)} \leq C \text{ for each } p < \frac{n}{n-2s}.$$

Since $\widetilde{f}(u_\lambda)$ is a subsolution of (6.39), then $0 \leq \widetilde{f}(u_\lambda) \leq w$. Therefore,

$$\|f(u^*)\|_{L^p(\Omega)} \leq C \quad \text{for each} \quad p < \frac{n}{n-2s},$$

and using Proposition 6.1.4 (iii), we find

$$u^* \in L^\infty(\Omega) \quad \text{whenever} \quad n < 4s,$$

as desired. □

6.5 Boundary estimates: the moving planes method

In this section we prove Proposition 6.1.8. This will be done with the celebrated moving planes method [156], as in the classical boundary estimates for the Laplacian of de Figueiredo-Lions-Nussbaum [110].

The moving planes method has been applied to problems involving the fractional Laplacian by different authors; see for example [94, 21, 129]. However, some of these results use the specific properties of the fractional Laplacian —such as the extension problem of Caffarelli-Silvestre [68], or the Riesz potential expression for $(-\Delta)^{-s}$—, and it is not clear how to apply the method to more general integro-differential operators. Here, we follow a different approach that allows more general nonlocal operators.

The main tool in the proof is the following maximum principle in small domains.

Recently, Jarohs and Weth [175] obtained a parabolic version of the maximum principle in small domains for the fractional Laplacian; see Proposition 2.4 in [175]. The proof of their result is essentially the same that we present in this section. Still, we think that it may be of interest to write here the proof for integro-differential operators with decreasing kernels.

Lemma 6.5.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying $\Omega \subset \mathbb{R}^n_+ = \{x_1 > 0\}$. Let K be a nonnegative function in $\mathbb{R}^n$, radially symmetric and decreasing, and satisfying*

$$K(z) \geq c|z|^{-n-\nu} \quad \text{for all} \quad z \in B_1$$

for some positive constants c and ν, and let

$$L_K u(x) = \int_{\mathbb{R}^n} \big(u(y) - u(x)\big) K(x-y) dy.$$

Let $V \in L^\infty(\Omega)$ be any bounded function, and $w \in H^s(\mathbb{R}^n)$ be a bounded function satisfying

$$\begin{cases} L_K w &= V(x)w & \text{in } \Omega \\ w &\geq 0 & \text{in } \mathbb{R}^n_+ \setminus \Omega \\ w(x) &\geq -w(x^*) & \text{in } \mathbb{R}^n_+, \end{cases} \tag{6.40}$$

where x^* is the symmetric to x with respect to the hyperplane $\{x_1 = 0\}$. Then, there exists a positive constant C_0 such that if

$$\big(1 + \|V^-\|_{L^\infty(\Omega)}\big) |\Omega|^{\frac{\nu}{n}} \leq C_0, \tag{6.41}$$

then $w \geq 0$ in Ω.

Remark 6.5.2. When L_K is the fractional Laplacian $(-\Delta)^s$, then the condition (6.41) can be replaced by $\|V^-\|_{L^\infty} |\Omega|^{\frac{2s}{n}} \leq C_0$.

Proof of Lemma 6.5.1. The identity $L_K w = V(x)w$ in Ω written in weak form is

$$(\varphi, w)_K := \int\int_{\mathbb{R}^{2n} \setminus (\mathbb{R}^n \setminus \Omega)^2} (\varphi(x) - \varphi(y))(w(x) - w(y)) K(x-y) dx\, dy = \int_\Omega V w \varphi \tag{6.42}$$

for all φ such that $\varphi \equiv 0$ in $\mathbb{R}^n \setminus \Omega$ and $\int_{\mathbb{R}^n} \big(\varphi(x) - \varphi(y)\big)^2 K(x-y)dx\,dy < \infty$. Note that the left hand side of (6.42) can be written as

$$\begin{aligned}(\varphi, w)_K = &\int_\Omega \int_\Omega (\varphi(x) - \varphi(y))(w(x) - w(y))K(x-y)dx\,dy \\ &+ 2\int_\Omega \int_{\mathbb{R}^n_+ \setminus \Omega} \varphi(x)(w(x) - w(y))K(x-y)dx\,dy \\ &+ 2\int_\Omega \int_{\mathbb{R}^n_+} \varphi(x)(w(x) - w(y^*))K(x-y^*)dx\,dy,\end{aligned}$$

where y^* denotes the symmetric of y with respect to the hyperplane $\{x_1 = 0\}$.

Choose $\varphi = -w^-\chi_\Omega$, where w^- is the negative part of w, i.e., $w = w^+ - w^-$. Then, we claim that

$$\int\int_{\mathbb{R}^{2n}\setminus(\mathbb{R}^n\setminus\Omega)^2} (w^-(x)\chi_\Omega(x) - w^-(y)\chi_\Omega(y))^2 K(x-y)dx\,dy \le (-w^-\chi_\Omega, w)_K. \quad (6.43)$$

Indeed, first, we have

$$\begin{aligned}(-w^-\chi_\Omega, w)_K = &\int_\Omega \int_\Omega \{(w^-(x) - w^-(y))^2 + w^-(x)w^+(y) + w^+(x)w^-(y)\}K(x-y)dxdy + \\ &+ 2\int_\Omega \int_{\mathbb{R}^n_+ \setminus \Omega} \{w^-(x)(w^-(x) - w^-(y)) + w^-(x)w^+(y)\}K(x-y)dx\,dy \\ &+ 2\int_\Omega \int_{\mathbb{R}^n_+} \{w^-(x)(w^-(x) - w^-(y^*)) + w^-(x)w^+(y^*)\}K(x-y^*)dx\,dy,\end{aligned}$$

where we have used that $w^+(x)w^-(x) = 0$ for all $x \in \mathbb{R}^n$.

Thus, rearranging terms and using that $w^- \equiv 0$ in $\mathbb{R}^n_+ \setminus \Omega$,

$$\begin{aligned}(-w^-\chi_\Omega, w)_K = &\int\int_{\mathbb{R}^{2n}\setminus(\mathbb{R}^n\setminus\Omega)^2} (w^-(x)\chi_\Omega(x) - w^-(y)\chi_\Omega(y))^2 K(x-y)dx\,dy \\ &+ \int_\Omega \int_\Omega 2w^-(x)w^+(y)K(x-y)dx\,dy + \\ &+ 2\int_\Omega \int_{\mathbb{R}^n_+ \setminus \Omega} \{w^-(x)w^+(y) - w^-(x)w^-(y)\}K(x-y)dx\,dy \\ &+ 2\int_\Omega \int_{\mathbb{R}^n_+} \{w^-(x)w^+(y^*) - w^-(x)w^-(y^*)\}K(x-y^*)dx\,dy \\ \ge &\int\int_{\mathbb{R}^{2n}\setminus(\mathbb{R}^n\setminus\Omega)^2} (w^-(x)\chi_\Omega(x) - w^-(y)\chi_\Omega(y))^2 K(x-y)dx\,dy + \\ &+ 2\int_\Omega \int_{\mathbb{R}^n_+} w^-(x)w^+(y)K(x-y)dx\,dy + \\ &+ 2\int_\Omega \int_{\mathbb{R}^n_+} -w^-(x)w^-(y^*)K(x-y^*)dx\,dy.\end{aligned}$$

We next use that, since K is radially symmetric and decreasing, $K(x-y^*) \le K(x-y)$ for all x and y in $\mathbb{R}^n_+$. We deduce

$$(-w^-\chi_\Omega, w)_K \ge \int\int_{\mathbb{R}^{2n}\setminus(\mathbb{R}^n\setminus\Omega)^2} (w^-(x)\chi_\Omega(x) - w^-(y)\chi_\Omega(y))^2 K(x-y)dx\,dy + \\ + 2\int_\Omega\int_{\mathbb{R}^n_+} w^-(x)w^+(y) - w^-(x)w^-(y^*)K(x-y)dx\,dy,$$

and since $w^-(y^*) \le w^+(y)$ for all y in $\mathbb{R}^n_+$ by assumption, we obtain (6.43).

Now, on the one hand note that from (6.43) we find

$$\int_\Omega\int_\Omega (w^-(x) - w^-(y))^2 K(x-y)dx\,dy \le (-w^-\chi_\Omega, w)_K.$$

Moreover, since $K(z) \ge c|z|^{-n-\nu}\chi_{B_1}(z)$, then

$$\|w^-\|^2_{\dot H^{\nu/2}(\Omega)} := \frac{c_{n,s}}{2}\int_\Omega\int_\Omega \frac{(w^-(x)-w^-(y))^2}{|x-y|^{-n-\nu}}dx\,dy \\ \le C\|w^-\|_{L^2(\Omega)} + C\int_\Omega\int_\Omega \big(w^-(x)-w^-(y)\big)^2 K(x-y)dx\,dy,$$

and therefore

$$\|w^-\|^2_{\dot H^{\nu/2}(\Omega)} \le C_1\|w^-\|_{L^2(\Omega)} + C_1(-w^-\chi_\Omega, w)_K. \tag{6.44}$$

On the other hand, it is clear that

$$\int_\Omega V w w^- = \int_\Omega V(w^-)^2 \le \|V^-\|_{L^\infty(\Omega)}\|w^-\|_{L^2(\Omega)}. \tag{6.45}$$

Thus, it follows from (6.42), (6.44), and (6.45) that

$$\|w^-\|^2_{\dot H^{\nu/2}(\Omega)} \le C_1\left(1 + \|V^-\|_{L^\infty}\right)\|w^-\|_{L^2(\Omega)}.$$

Finally, by the Hölder and the fractional Sobolev inequalities, we have

$$\|w^-\|^2_{L^2(\Omega)} \le |\Omega|^{\frac{\nu}{n}}\|w^-\|^2_{L^q(\Omega)} \le C_2|\Omega|^{\frac{\nu}{n}}\|w^-\|^2_{\dot H^{\nu/2}(\Omega)},$$

where $q = \frac{2n}{n-\nu}$. Thus, taking C_0 such that $C_0 < (C_1C_2)^{-1}$ the lemma follows. □

Now, once we have the nonlocal version of the maximum principle in small domains, the moving planes method can be applied exactly as in the classical case.

Proof of Proposition 6.1.8. Replacing the classical maximum principle in small domains by Lemma 6.5.1, we can apply the moving planes method to deduce $\|u\|_{L^\infty(\Omega_\delta)} \le C\|u\|_{L^1(\Omega)}$ for some constants C and $\delta > 0$ that depend only on Ω, as in de Figueiredo-Lions-Nussbaum [110]; see also [34].

Let us recall this argument. Assume first that all curvatures of $\partial\Omega$ are positive. Let $\nu(y)$ be the unit outward normal to Ω at y. Then, there exist positive constants s_0 and α depending only on the convex domain Ω such that, for every $y \in \partial\Omega$ and every $e \in \mathbb{R}^n$ with $|e| = 1$ and $e\cdot\nu(y) \ge \alpha$, $u(y-se)$ is nondecreasing in $s \in [0, s_0]$. This fact

follows from the moving planes method applied to planes close to those tangent to Ω at $\partial\Omega$. By the convexity of Ω, the reflected caps will be contained in Ω. The previous monotonicity fact leads to the existence of a set I_x, for each $x \in \Omega_\delta$, and a constant $\gamma > 0$ that depend only on Ω, such that

$$|I_x| \geq \gamma, \qquad u(x) \leq u(y) \quad \text{for all} \quad y \in I_x.$$

The set I_x is a truncated open cone with vertex at x.

As mentioned in page 45 of de Figuereido-Lions-Nussbaum [110], the same can also be proved for general convex domains with a little more of care. □

Remark 6.5.3. When $\Omega = B_1$, Proposition 6.1.8 follows from the results in [21], where Birkner, López-Mimbela, and Wakolbinger used the moving planes method to show that any nonnegative bounded solution of

$$\begin{cases} (-\Delta)^s u &= f(u) & \text{in } B_1 \\ u &= 0 & \text{in } \mathbb{R}^n \setminus B_1 \end{cases} \tag{6.46}$$

is radially symmetric and decreasing.

When u is a bounded semistable solution of (6.46), there is an alternative way to show that u is radially symmetric. This alternative proof applies to all solutions (not necessarily positive), but does not give monotonicity. Indeed, one can easily show that, for any $i \neq j$, the function $w = x_i u_{x_j} - x_j u_{x_i}$ is a solution of the linearized problem

$$\begin{cases} (-\Delta)^s w &= f'(u)w & \text{in } B_1 \\ w &= 0 & \text{in } \mathbb{R}^n \setminus B_1. \end{cases} \tag{6.47}$$

Then, since $\lambda_1\left((-\Delta)^s - f'(u); B_1\right) \geq 0$ by assumption, it follows that either $w \equiv 0$ or $\lambda_1 = 0$ and w is a multiple of the first eigenfunction, which is positive —see the proof of Proposition 9 in [265, Appendix A]. But since w is a tangential derivative then it can not have constant sign along a circumference $\{|x| = r\}$, $r \in (0,1)$, and thus it has to be $w \equiv 0$. Therefore, all the tangential derivatives $\partial_t u = x_i u_{x_j} - x_j u_{x_i}$ equal zero, and thus u is radially symmetric.

6.6 H^s regularity of the extremal solution in convex domains

In this section we prove Theorem 6.1.3 (iii). A key tool in this proof is the Pohozaev identity for the fractional Laplacian, recently obtained by the authors in [254]. This identity allows us to compare the interior H^s norm of the extremal solution u^* with a boundary term involving u^*/δ^s, where δ is the distance to $\partial\Omega$. Then, this boundary term can be bounded by using the results of the previous section by the L^1 norm of u^*, which is finite.

We first prove the boundedness of u^*/δ^s near the boundary.

Lemma 6.6.1. *Let Ω be a convex domain, u be a bounded solution of* (6.14)*, and $\delta(x) = dist(x, \partial\Omega)$. Assume that*

$$\|u\|_{L^1(\Omega)} \leq c_1$$

for some $c_1 > 0$. Then, there exists constants $\delta > 0$, c_2, and C such that

$$\|u/\delta^s\|_{L^\infty(\Omega_\delta)} \le C\left(c_2 + \|f\|_{L^\infty([0,c_2])}\right),$$

where $\Omega_\delta = \{x \in \Omega : dist(x, \partial\Omega) < \delta\}$. Moreover, the constants δ, c_2, and C depend only on Ω and c_1.

Proof. The result can be deduced from the boundary regularity results in [249] and Proposition 6.1.8, as follows.

Let $\delta > 0$ be given by Proposition 6.1.8, and let η be a smooth cutoff function satisfying $\eta \equiv 0$ in $\Omega \setminus \Omega_{2\delta/3}$ and $\eta \equiv 1$ in $\Omega_{\delta/3}$. Then, $u\eta \in L^\infty(\Omega)$ and $u\eta \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. Moreover, we claim that

$$(-\Delta)^s(u\eta) = f(u)\chi_{\Omega_{\delta/4}} + g \qquad \text{in } \Omega \tag{6.48}$$

for some function $g \in L^\infty(\Omega)$, with the estimate

$$\|g\|_{L^\infty(\Omega)} \le C\left(\|u\|_{C^{1+s}(\Omega_{4\delta/5}\setminus\Omega_{\delta/5})} + \|u\|_{L^1(\Omega)}\right). \tag{6.49}$$

To prove that (6.48) holds pointwise we argue separately in $\Omega_{\delta/4}$, in $\Omega_{3\delta/4} \setminus \Omega_{\delta/4}$, and in $\Omega \setminus \Omega_{3\delta/4}$, as follows:

- In $\Omega_{\delta/4}$, $g = (-\Delta)^s(u\eta) - (-\Delta)^s u$. Since $u\eta - u$ vanishes in $\Omega_{\delta/3}$ and also outside Ω, g is bounded and satisfies (6.49).
- In $\Omega_{3\delta/4} \setminus \Omega_{\delta/4}$, $g = (-\Delta)^s(u\eta)$. Then, using

  $$\|(-\Delta)^s(u\eta)\|_{L^\infty(\Omega_{3\delta/4}\setminus\Omega_{\delta/4})} \le C\left(\|u\eta\|_{C^{1+s}(\Omega_{4\delta/5}\setminus\Omega_{\delta/5})} + \|u\eta\|_{L^1(\mathbb{R}^n)}\right)$$

  and that η is smooth, we find that g is bounded and satisfies (6.49).
- In $\Omega \setminus \Omega_{3\delta/4}$, $g = (-\Delta)^s(u\eta)$. Since $u\eta$ vanishes in $\Omega \setminus \Omega_{2\delta/3}$, g is bounded and satisfies (6.49).

Now, since u is a solution of (6.14), by classical interior estimates we have

$$\|u\|_{C^{1+s}(\Omega_{4\delta/5}\setminus\Omega_{\delta/5})} \le C\left(\|u\|_{L^\infty(\Omega_\delta)} + \|u\|_{L^1(\Omega)}\right); \tag{6.50}$$

see for instance [249]. Hence, by (6.48) and Theorem 1.2 in [249], $u\eta/\delta^s \in C^\alpha(\overline{\Omega})$ for some $\alpha > 0$ and

$$\|u\eta/\delta^s\|_{C^\alpha(\overline{\Omega})} \le C\|f(u)\chi_{\Omega_{\delta/4}} + g\|_{L^\infty(\Omega)}.$$

Thus,

$$\begin{aligned}\|u/\delta^s\|_{L^\infty(\Omega_{\delta/3})} &\le \|u\eta/\delta^s\|_{C^\alpha(\overline{\Omega})} \le C\left(\|g\|_{L^\infty(\Omega)} + \|f(u)\|_{L^\infty(\Omega_{\delta/4})}\right)\\ &\le C\left(\|u\|_{L^1(\Omega)} + \|u\|_{L^\infty(\Omega_\delta)} + \|f(u)\|_{L^\infty(\Omega_{\delta/4})}\right).\end{aligned}$$

In the last inequality we have used (6.49) and (6.50). Then, the result follows from Proposition 6.1.8. □

We can now give the

Proof of Theorem 6.1.3 (iii). Recall that u_λ minimizes the energy $\mathcal{E}$ in the set $\{u \in H^s(\mathbb{R}^n) : 0 \le u \le u_\lambda\}$ (see Step 4 in the proof of Proposition 6.1.2 in Section 6.2). Hence,

$$\|u_\lambda\|^2_{\dot{H}^s} - \int_\Omega \lambda F(u_\lambda) = \mathcal{E}(u_\lambda) \le \mathcal{E}(0) = 0. \tag{6.51}$$

Now, the Pohozaev identity for the fractional Laplacian can be written as

$$s\|u_\lambda\|^2_{\dot{H}^s} - n\mathcal{E}(u_\lambda) = \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u_\lambda}{\delta^s}\right)^2 (x \cdot \nu) d\sigma, \tag{6.52}$$

see [254, page 2]. Therefore, it follows from (6.51) and (6.52) that

$$\|u_\lambda\|^2_{\dot{H}^s} \le \frac{\Gamma(1+s)^2}{2s} \int_{\partial\Omega} \left(\frac{u_\lambda}{\delta^s}\right)^2 (x \cdot \nu) d\sigma.$$

Now, by Proposition 6.6.1, we have that

$$\int_{\partial\Omega} \left(\frac{u_\lambda}{\delta^s}\right)^2 (x \cdot \nu) d\sigma \le C$$

for some constant C that depends only on Ω and $\|u_\lambda\|_{L^1(\Omega)}$. Thus, $\|u_\lambda\|_{\dot{H}^s} \le C$, and since $u^* \in L^1(\Omega)$, letting $\lambda \uparrow \lambda^*$ we find

$$\|u^*\|_{\dot{H}^s} < \infty,$$

as desired. □

6.7 L^p and C^β estimates for the linear Dirichlet problem

The aim of this section is to prove Propositions 6.1.4 and 6.1.7. We prove first Proposition 6.1.4.

Proof of Proposition 6.1.4. (i) It is clear that we can assume $\|g\|_{L^1(\Omega)} = 1$.

Consider the solution v of

$$(-\Delta)^s v = |g| \quad in\ \mathbb{R}^n$$

given by the Riesz potential $v = (-\Delta)^{-s}|g|$. Here, g is extended by 0 outside Ω.

Since $v \ge 0$ in $\mathbb{R}^n \setminus \Omega$, by the maximum principle we have that $|u| \le v$ in Ω. Then, it follows from Theorem 6.1.6 that

$$\|u\|_{L^q_{\text{weak}}(\Omega)} \le C, \quad \text{where} \quad q = \frac{n}{n-2s},$$

and hence we find that

$$\|u\|_{L^r(\Omega)} \le C \quad \text{for all} \quad r < \frac{n}{n-2s}$$

for some constant that depends only on n, s, and $|\Omega|$.

(ii) The proof is analogous to the one of part (i). In this case, the constant does not depend on the domain Ω.

(iii) As before, we assume $\|g\|_{L^p(\Omega)} = 1$. Write $u = \tilde{v} + w$, where $\tilde{v}$ and w are given by

$$\tilde{v} = (-\Delta)^{-s} g \ \ in\ \mathbb{R}^n, \tag{6.53}$$

and

$$\begin{cases} (-\Delta)^s w &= 0 \quad \text{in } \Omega \\ w &= \tilde{v} \quad \text{in } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{6.54}$$

Then, from (6.53) and Theorem 6.1.6 we deduce that

$$[\tilde{v}]_{C^\alpha(\mathbb{R}^n)} \leq C, \quad \text{where} \quad \alpha = 2s - \frac{n}{p}. \tag{6.55}$$

Moreover, since the domain Ω is bounded, then g has compact support and hence $\tilde{v}$ decays at infinity. Thus, we find

$$\|\tilde{v}\|_{C^\alpha(\mathbb{R}^n)} \leq C \tag{6.56}$$

for some constant C that depends only on n, s, p, and Ω.

Now, we apply Proposition 6.1.7 to equation (6.54). We find

$$\|w\|_{C^\beta(\mathbb{R}^n)} \leq C \|\tilde{v}\|_{C^\alpha(\mathbb{R}^n)}, \tag{6.57}$$

where $\beta = \min\{\alpha, s\}$. Thus, combining (6.56), and (6.57) the result follows. □

Note that we have only used Proposition 6.1.7 to obtain the C^β estimate in part (iii). If one only needs an L^∞ estimate instead of the C^β one, Proposition 6.1.7 is not needed, since the L^∞ bound follows from the maximum principle.

As said in the introduction, the L^p to $W^{2s,p}$ estimates for the fractional Laplace equation, in which $-\Delta$ is replaced by the fractional Laplacian $(-\Delta)^s$, are not true for all p, even when $\Omega = \mathbb{R}^n$. This is illustrated in the following two remarks.

Recall the definition of the fractional Sobolev space $W^{\sigma,p}(\Omega)$ which, for $\sigma \in (0,1)$, consists of all functions $u \in L^p(\Omega)$ such that

$$\|u\|_{W^{\sigma,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+p\sigma}} \, dx \, dy \right)^{\frac{1}{p}}$$

is finite; see for example [115] for more information on these spaces.

Remark 6.7.1. Let $s \in (0,1)$. Assume that u and g belong to $L^p(\mathbb{R}^n)$, with $1 < p < \infty$, and that

$$(-\Delta)^s u = g \ \ \text{in } \mathbb{R}^n.$$

(i) If $p \geq 2$, then $u \in W^{2s,p}(\mathbb{R}^n)$.

(ii) If $p < 2$ and $2s \neq 1$ then u may not belong to $W^{2s,p}(\mathbb{R}^n)$. Instead, $u \in B^{2s}_{p,2}(\mathbb{R}^n)$, where $B^\sigma_{p,q}$ is the Besov space of order σ and parameters p and q.

For more details see the books of Stein [278] and Triebel [290].

By the preceding remark we see that the L^p to $W^{2s,p}$ estimate does not hold in $\mathbb{R}^n$ whenever $p < 2$ and $s \neq \frac{1}{2}$. The following remark shows that in bounded domains Ω this estimate do not hold even for $p \geq 2$.

Remark 6.7.2. Let us consider the solution of $(-\Delta)^s u = g$ in Ω, $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. When $\Omega = B_1$ and $g \equiv 1$, the solution to this problem is

$$u_0(x) = \left(1 - |x|^2\right)^s \chi_{B_1}(x);$$

see [154]. For p large enough one can see that u_0 does not belong to $W^{2s,p}(B_1)$, while $g \equiv 1$ belongs to $L^p(B_1)$ for all p. For example, when $s = \frac{1}{2}$ by computing $|\nabla u_0|$ we see that u_0 does not belong to $W^{1,p}(B_1)$ for $p \geq 2$.

We next prove Proposition 6.1.7. For it, we will proceed similarly to the C^s estimates obtained in [249, Section 2] for the Dirichlet problem for the fractional Laplacian with L^∞ data.

The first step is the following:

Lemma 6.7.3. *Let Ω be a bounded domain satisfying the exterior ball condition, $s \in (0,1)$, h be a $C^\alpha(\mathbb{R}^n \setminus \Omega)$ function for some $\alpha > 0$, and u be the solution of* (6.13). *Then*

$$|u(x) - u(x_0)| \leq C\|h\|_{C^\alpha(\mathbb{R}^n \setminus \Omega)} \delta(x)^\beta \quad \text{in } \Omega,$$

where x_0 is the nearest point to x on $\partial\Omega$, $\beta = \min\{s, \alpha\}$, and $\delta(x) = dist(x, \partial\Omega)$. The constant C depends only on n, s, and α.

Lemma 6.7.3 will be proved using the following supersolution. Next lemma (and its proof) is very similar to Lemma 2.6 in [249].

Lemma 6.7.4. *Let $s \in (0,1)$. Then, there exist constants ϵ, c_1, and C_2, and a continuous radial function φ satisfying*

$$\begin{cases} (-\Delta)^s \varphi \geq 0 & \text{in } B_2 \setminus B_1 \\ \varphi \equiv 0 & \text{in } B_1 \\ c_1(|x| - 1)^s \leq \varphi \leq C_2(|x| - 1)^s & \text{in } \mathbb{R}^n \setminus B_1. \end{cases} \tag{6.58}$$

The constants c_1 and C_2 depend only on n, s, and β.

Proof. We follow the proof of Lemma 2.6 in [249]. Consider the function

$$u_0(x) = (1 - |x|^2)_+^s.$$

It is a classical result (see [154]) that this function satisfies

$$(-\Delta)^s u_0 = \kappa_{n,s} \quad \text{in } B_1$$

for some positive constant $\kappa_{n,s}$.

Thus, the fractional Kelvin transform of u_0, that we denote by u_0^*, satisfies

$$(-\Delta)^s u_0^*(x) = |x|^{-2s-n}(-\Delta)^s u_0\left(\frac{x}{|x|^2}\right) \geq c_0 \quad \text{in } B_2 \setminus B_1.$$

Recall that the Kelvin transform u_0^* of u_0 is defined by

$$u_0^*(x) = |x|^{2s-n} u_0\left(\frac{x}{|x|^2}\right).$$

Then, it is clear that

$$a_1(|x|-1)^s \le u_0^*(x) \le A_2(|x|-1)^s \ \ in\ B_2 \setminus B_1,$$

while u_0^* is bounded at infinity.

Let us consider now a smooth function η satisfying $\eta \equiv 0$ in B_3 and

$$A_1(|x|-1)^s \le \eta \le A_2(|x|-1)^s \ \ in\ \mathbb{R}^n \setminus B_4.$$

Observe that $(-\Delta)^s \eta$ is bounded in B_2, since $\eta(x)(1+|x|)^{-n-2s} \in L^1$. Then, the function

$$\varphi = C u_0^* + \eta,$$

for some big constant $C > 0$, satisfies

$$\begin{cases} (-\Delta)^s \varphi \ge 1 & \text{in } B_2 \setminus B_1 \\ \varphi \equiv 0 & \text{in } B_1 \\ c_1(|x|-1)^s \le \varphi \le C_2(|x|-1)^s & \text{in } \mathbb{R}^n \setminus B_1. \end{cases}$$

Indeed, it is clear that $\varphi \equiv 0$ in B_1. Moreover, taking C big enough it is clear that we have that $(-\Delta)^s \varphi \ge 1$. In addition, the condition $c_1(|x|-1)^s \le \varphi \le C_2(|x|-1)^s$ is satisfied by construction. Thus, φ satisfies (6.59), and the proof is finished. □

Once we have constructed the supersolution, we can give the

Proof of Lemma 6.7.3. First, we can assume that $\|h\|_{C^\alpha(\mathbb{R}^n \setminus \Omega)} = 1$. Then, by the maximum principle we have that $\|u\|_{L^\infty(\mathbb{R}^n)} = \|h\|_{L^\infty(\mathbb{R}^n)} \le 1$. We can also assume that $\alpha \le s$, since

$$\|h\|_{C^s(\mathbb{R}^n)} \le C \|h\|_{C^\alpha(\mathbb{R}^n \setminus \Omega)} \quad \text{whenever } s < \alpha.$$

Let $x_0 \in \partial\Omega$ and $R > 0$ be small enough. Let B_R be a ball of radius R, exterior to Ω, and touching $\partial\Omega$ at x_0. Let us see that $|u(x) - u(x_0)|$ is bounded by CR^β in $\Omega \cap B_{2R}$.

By Lemma 6.7.4, we find that there exist constants c_1 and C_2, and a radial continuous function φ satisfying

$$\begin{cases} (-\Delta)^s \varphi \ge 0 & \text{in } B_2 \setminus B_1 \\ \varphi \equiv 0 & \text{in } B_1 \\ c_1(|x|-1)^s \le \varphi \le C_2(|x|-1)^s & \text{in } \mathbb{R}^n \setminus B_1. \end{cases} \tag{6.59}$$

Let x_1 be the center of the ball B_R. Since $\|h\|_{C^\alpha(\mathbb{R}^n \setminus \Omega)} = 1$, it is clear that the function

$$\varphi_R(x) = h(x_0) + 3R^\alpha + C_3 R^s \varphi\left(\frac{x - x_1}{R}\right),$$

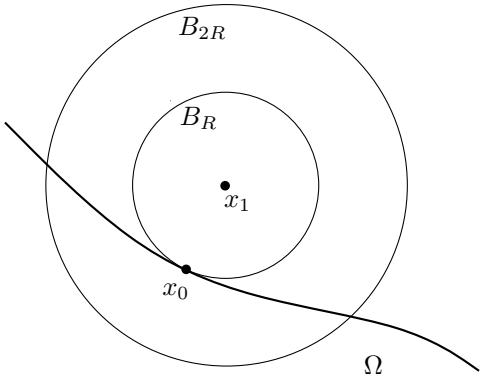

Figure 6.1:

with C_3 big enough, satisfies

$$\begin{cases} (-\Delta)^s \varphi_R \geq 0 & \text{in } B_{2R} \setminus B_R \\ \varphi_R \equiv h(x_0) + 3R^\alpha & \text{in } B_R \\ h(x_0) + |x - x_0|^\alpha \leq \varphi_R & \text{in } \mathbb{R}^n \setminus B_{2R} \\ \varphi_R \leq h(x_0) + C_0 R^\alpha & \text{in } B_{2R} \setminus B_R. \end{cases} \tag{6.60}$$

Here we have used that $\alpha \leq s$.

Then, since

$$(-\Delta)^s u \equiv 0 \leq (-\Delta)^s \varphi_R \quad \text{in } \Omega \cap B_{2R},$$
$$h \leq h(x_0) + 3R^\alpha \equiv \varphi_R \quad \text{in } B_{2R} \setminus \Omega,$$

and

$$h(x) \leq h(x_0) + |x - x_0|^\alpha \leq \varphi_R \quad \text{in } \mathbb{R}^n \setminus B_{2R},$$

it follows from the comparison principle that

$$u \leq \varphi_R \ \ in\ \Omega \cap B_{2R}.$$

Therefore, since $\varphi_R \leq h(x_0) + C_0 R^\alpha$ in $B_{2R} \setminus B_R$,

$$u(x) - h(x_0) \leq C_0 R^\alpha \ \ in\ \Omega \cap B_{2R}. \tag{6.61}$$

Moreover, since this can be done for each x_0 on $\partial\Omega$, $h(x_0) = u(x_0)$, and we have $\|u\|_{L^\infty(\Omega)} \leq 1$, we find that

$$u(x) - u(x_0) \leq C\delta^\beta \ \ in\ \Omega, \tag{6.62}$$

where x_0 is the projection on $\partial\Omega$ of x.

Repeating the same argument with u and h replaced by $-u$ and $-h$, we obtain the same bound for $h(x_0) - u(x)$, and thus the lemma follows. □

The following result will be used to obtain C^β estimates for u inside Ω. For a proof of this lemma see for example Corollary 2.4 in [249].

Lemma 6.7.5 ([249]). *Let $s \in (0,1)$, and let w be a solution of $(-\Delta)^s w = 0$ in B_2. Then, for every $\gamma \in (0, 2s)$*

$$\|w\|_{C^\gamma(\overline{B_{1/2}})} \le C\bigg(\|(1+|x|)^{-n-2s} w(x)\|_{L^1(\mathbb{R}^n)} + \|w\|_{L^\infty(B_2)} \bigg),$$

where the constant C depends only on n, s, and γ.

Now, we use Lemmas 6.7.3 and 6.7.5 to obtain interior C^β estimates for the solution of (6.13).

Lemma 6.7.6. *Let Ω be a bounded domain satisfying the exterior ball condition, $h \in C^\alpha(\mathbb{R}^n \setminus \Omega)$ for some $\alpha > 0$, and u be the solution of (6.13). Then, for all $x \in \Omega$ we have the following estimate in $B_R(x) = B_{\delta(x)/2}(x)$*

$$\|u\|_{C^\beta(\overline{B_R(x)})} \le C \|h\|_{C^\alpha(\mathbb{R}^n \setminus \Omega)}, \tag{6.63}$$

where $\beta = \min\{\alpha, s\}$ and C is a constant depending only on Ω, s, and α.

Proof. Note that $B_R(x) \subset B_{2R}(x) \subset \Omega$. Let $\tilde{u}(y) = u(x + Ry) - u(x)$. We have that

$$(-\Delta)^s \tilde{u}(y) = 0 \quad \text{in } B_1 \,. \tag{6.64}$$

Moreover, using Lemma 6.7.3 we obtain

$$\|\tilde{u}\|_{L^\infty(B_1)} \le C \|h\|_{C^\alpha(\mathbb{R}^n \setminus \Omega)} R^\beta. \tag{6.65}$$

Furthermore, observing that $|\tilde{u}(y)| \le C\|h\|_{C^\alpha(\mathbb{R}^n \setminus \Omega)} R^\beta (1 + |y|^\beta)$ in all of $\mathbb{R}^n$, we find

$$\|(1+|y|)^{-n-2s} \tilde{u}(y)\|_{L^1(\mathbb{R}^n)} \le C \|h\|_{C^\alpha(\mathbb{R}^n \setminus \Omega)} R^\beta, \tag{6.66}$$

with C depending only on Ω, s, and α.

Now, using Lemma 6.7.5 with $\gamma = \beta$, and taking into account (6.64), (6.65), and (6.66), we deduce

$$\|\tilde{u}\|_{C^\beta\left(\overline{B_{1/4}}\right)} \le C \|h\|_{C^\alpha(\mathbb{R}^n \setminus \Omega)} R^\beta,$$

where $C = C(\Omega, s, \beta)$.

Finally, we observe that

$$[u]_{C^\beta\left(\overline{B_{R/4}(x)}\right)} = R^{-\beta} [\tilde{u}]_{C^\beta\left(\overline{B_{1/4}}\right)}.$$

Hence, by an standard covering argument, we find the estimate (6.63) for the C^β norm of u in $\overline{B_R(x)}$. □

Now, Proposition 6.1.7 follows immediately from Lemma 6.7.6, as in Proposition 1.1 in [249].

Proof of Proposition 6.1.7. This proof is completely analogous to the proof of Proposition 1.1 in [249]. One only have to replace the s in that proof by β, and use the estimate from the present Lemma 6.7.6 instead of the one from [249, Lemma 2.9]. □

Chapter Seven

REGULARITY FOR THE FRACTIONAL GELFAND PROBLEM UP TO DIMENSION 7

We study the problem $(-\Delta)^s u = \lambda e^u$ in a bounded domain $\Omega \subset \mathbb{R}^n$, where λ is a positive parameter. More precisely, we study the regularity of the extremal solution to this problem.

Our main result yields the boundedness of the extremal solution in dimensions $n \leq 7$ for all $s \in (0,1)$ whenever Ω is, for every $i = 1, ..., n$, convex in the x_i-direction and symmetric with respect to $\{x_i = 0\}$. The same holds if $n = 8$ and $s \gtrsim 0'28206...$, or if $n = 9$ and $s \gtrsim 0'63237...$. These results are new even in the unit ball $\Omega = B_1$.

7.1 Introduction and results

Let $s \in (0,1)$ and Ω be a bounded smooth domain in $\mathbb{R}^n$, and consider the problem

$$\begin{cases} (-\Delta)^s u &= \lambda e^u & \text{in } \Omega \\ u &= 0 & \text{in } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{7.1}$$

Here, λ is a positive parameter and $(-\Delta)^s$ is the fractional Laplacian, defined by

$$(-\Delta)^s u(x) = c_{n,s} \mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy. \tag{7.2}$$

The aim of this paper is to study the regularity of the so-called extremal solution of the problem (7.1).

For the Laplacian $-\Delta$ (which corresponds to $s = 1$) this problem is frequently called the Gelfand problem [151], and the existence and regularity properties of its solutions are by now quite well understood [191, 177, 223, 212, 102]; see also [144, 234].

Indeed, when $s = 1$ one can show that there exists a finite extremal parameter λ^* such that if $0 < \lambda < \lambda^*$ then it admits a minimal classical solution u_λ, while for $\lambda > \lambda^*$ it has no weak solution. Moreover, the pointwise limit $u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$ is a weak solution of problem with $\lambda = \lambda^*$. It is called the extremal solution. All the solutions u_λ and u^* are stable solutions.

On the other hand, the existence of other solutions for $\lambda < \lambda^*$ is a more delicate question, which depends strongly on the regularity of the extremal solution u^*. More precisely, it depends on the boundedness of u^*.

It turns out that the extremal solution u^* is bounded in dimensions $n \leq 9$ for any domain Ω [212, 102], while $u^*(x) = \log \frac{1}{|x|^2}$ is the (singular) extremal solution in the unit ball when $n \geq 10$. This result strongly relies on the stability of u^*. In the case $\Omega = B_1$, the classification of all radial solutions to this problem was done in [201] for $n = 2$, and in [177, 223] for $n \geq 3$.

For more general nonlinearities $f(u)$ the regularity of extremal solutions is only well understood when $\Omega = B_1$. As in the exponential case, all extremal solutions are bounded in dimensions $n \leq 9$, and may be singular if $n \geq 10$ [43]. For general domains Ω the problem is still not completely understood, and the best result in that direction states that all extremal solutions are bounded in dimensions $n \leq 4$ [42, 295]. In domains of double revolution, all extremal solutions are bounded in dimensions $n \leq 7$ [49]. For more information on this problem, see [36] and the monograph [120].

For the fractional Laplacian, the problem was studied by J. Serra and the author [253] for general nonlinearities f. We showed that there exists a parameter λ^* such that for $0 < \lambda < \lambda^*$ there is a branch of minimal solutions u_λ, for $\lambda > \lambda^*$ there is no bounded solutions, and for $\lambda = \lambda^*$ one has the extremal solution u^*, which is a stable solution. Moreover, depending on the nonlinearity f and on n and s, we obtained L^∞ and H^s estimates for the extremal solution in general domains Ω. Note that, as in the case $s = 1$, once we know that u^* is bounded then it follows that it is a classical solution; see for example [249].

For the exponential nonlinearity $f(u) = e^u$, our results in [253] yield the boundedness of the extremal solution in dimensions $n < 10s$. Although this result is optimal as $s \to 1$, it is not optimal, however, for smaller values of $s \in (0, 1)$. More precisely, an argument in [253] suggested the possibility that the extremal solution u^* could be bounded in all dimensions $n \leq 7$ and for all $s \in (0, 1)$. However, our results in [253] did not give any L^∞ estimate uniform in s.

The aim of this paper is to obtain better L^∞ estimates for the fractional Gelfand problem (7.1) whenever Ω is even and convex with respect to each coordinate axis. Our main result, stated next, establishes the boundedness of the extremal solution u^* whenever (7.3) holds and, in particular, whenever $n \leq 7$ independently of $s \in (0, 1)$. As explained in Remark 7.2.2, we expect this result to be optimal.

Theorem 7.1.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^n$ which is, for every $i = 1, ..., n$, convex in the x_i-direction and symmetric with respect to $\{x_i = 0\}$. Let $s \in (0, 1)$, and let u^* be the extremal solution of problem (7.1). Assume that either $n \leq 2s$, or that $n > 2s$ and*

$$\frac{\Gamma\left(\frac{n}{2}\right)\Gamma(1+s)}{\Gamma\left(\frac{n-2s}{2}\right)} > \frac{\Gamma^2\left(\frac{n+2s}{4}\right)}{\Gamma^2\left(\frac{n-2s}{4}\right)}. \tag{7.3}$$

Then, u^* is bounded. In particular, the extremal solution u^* is bounded for all $s \in (0, 1)$ whenever $n \leq 7$. The same holds if $n = 8$ and $s \gtrsim 0'28206...$, or if $n = 9$ and $s \gtrsim 0'63237...$.

The result is new even in the unit ball $\Omega = B_1$.

We point out that, for $n = 10$ condition (7.3) is equivalent to $s > 1$.

Let us next comment on some works related to problem (7.1).

On the one hand, for the power nonlinearity $f(u) = (1 + u)^p$, $p > 1$, the problem has been recently studied by Dávila-Dupaigne-Wei [109]. Their powerful methods are

based on a monotonicity formula and a blow-up argument, using the ideas introduced in [108] to study the case of the bilaplacian, $s = 2$. For this case $s = 2$, extremal solutions with exponential nonlinearity have been also studied; see for example [106].

On the other hand, Capella-Dávila-Dupaigne-Sire [80] studied the extremal solution in the unit ball for general nonlinearities for a related operator but different than the fractional Laplacian (7.2). More precisely, they considered the *spectral* fractional Laplacian in B_1, i.e., the operator A^s defined via the Dirichlet eigenvalues of the Laplacian in B_1. They obtained an L^∞ bound for u^* in dimensions $n < 2\left(2 + s + \sqrt{2s+2}\right)$ and, in particular, their result yields the boundedness of the extremal solution in dimensions $n \leq 6$ for all $s \in (0, 1)$.

Another result in a similar direction is [107], where Dávila-Dupaigne-Montenegro studied the extremal solution of a boundary reaction problem. Recall that problems of the form (7.1) involving the fractional Laplacian can be seen as a local weighted problem in $\mathbb{R}^{n+1}_+$ by using the extension of Caffarelli-Silvestre. Similarly, the spectral fractional Laplacian A^s can be written in terms of an extension in $\Omega \times \mathbb{R}_+$. Thus, the boundary reaction problem studied in [107] is also related to a "fractional" problem on the boundary, in which $s = 1/2$. Although in this paper we never use the extension problem for the fractional Laplacian, we will use some ideas appearing in [107] to prove our results, as explained next.

Recall that the key property of the extremal solution u^* is that it is *stable* [120, 253], in the sense that

$$\int_\Omega \lambda e^{u^*} \eta^2 dx \leq \int_{\mathbb{R}^n} \left|(-\Delta)^{s/2}\eta\right|^2 dx$$

for all $\eta \in H^s(\mathbb{R}^n)$ satisfying $\eta \equiv 0$ in $\mathbb{R}^n \setminus \Omega$.

In the classical case $s = 1$, the main idea of the proof in [102] is to take $\eta = e^{\alpha u^*} - 1$ in the stability condition to obtain a $W^{2,p}$ bound for u^*. When $n < 10$, this $W^{2,p}$ estimate leads, by the Sobolev embeddings, to the boundedness of u^*. This is also the approach that we followed in [253] to obtain regularity in dimensions $n < 10s$.

Here, instead, we assume by contradiction that u^* is singular, and we prove a lower bound for u^* near its singular point. This is why we need to assume the domain Ω to be even and convex —in this case, the singular point is necessarily the origin. Then, in the stability condition we take an explicit function $\eta(x)$ with the same expected singular behavior as $e^{\alpha u^*(x)}$ (given by the previous lower bound). More precisely, we take as η a power function of the form $\eta(x) \sim |x|^{-\beta}$, with β chosen appropriately. This idea was already used in [107], where Dávila-Dupaigne-Montenegro studied the extremal solution for a boundary reaction problem.

The paper is organized as follows. First, in Section 7.2 we give some remarks and preliminary results that will be used in the proof of our main result. Then, in Section 7.3 we prove Theorem 7.1.1.

7.2 Some preliminaries and remarks

In this section we recall some facts that will be used in the proof of Theorem 7.1.1.

First, recall that a weak solution u of (7.1) is said to be stable when

$$\int_\Omega \lambda e^{u} \eta^2 dx \leq \int_{\mathbb{R}^n} \left|(-\Delta)^{s/2}\eta\right|^2 dx \tag{7.4}$$

for all $\eta \in H^s(\mathbb{R}^n)$ satisfying $\eta \equiv 0$ in $\mathbb{R}^n \setminus \Omega$; see [253] for more details. Note also that, integrating by parts on the right hand side, one can write (7.4) as

$$\int_\Omega \lambda e^u \eta^2 dx \leq \int_\Omega \eta (-\Delta)^s \eta \, dx. \tag{7.5}$$

We will use this form of the stability condition in the proof of Theorem 7.1.1.

Next we recall a computation done in [253] in which we can see that condition (7.3) arises naturally.

Proposition 7.2.1 ([253])**.** *Let $s \in (0,1)$, $n > 2s$, and $u_0(x) = \log \frac{1}{|x|^{2s}}$. Then, u_0 is a solution of*

$$(-\Delta)^s u_0 = \lambda_0 e^{u_0} \quad \text{in all of } \mathbb{R}^n,$$

with

$$\lambda_0 = 2^{2s} \frac{\Gamma\left(\frac{n}{2}\right) \Gamma(1+s)}{\Gamma\left(\frac{n-2s}{2}\right)}. \tag{7.6}$$

Moreover, setting

$$H_{n,s} = 2^{2s} \frac{\Gamma^2\left(\frac{n+2s}{4}\right)}{\Gamma^2\left(\frac{n-2s}{4}\right)}, \tag{7.7}$$

u_0 is stable if and only if $\lambda_0 \leq H_{n,s}$.

We point out that $H_{n,s}$ is the best constant in the fractional Hardy inequality, even though we will not use such inequality in this paper.

Remark 7.2.2. This proposition suggests that there could exist a stable singular solution to (7.1) in the unit ball whenever $\lambda_0 \leq H_{n,s}$. In fact, we may consider a larger family of problems than (7.1), by considering nonhomogeneous Dirichlet conditions of the form $u = g$ in $\mathbb{R}^n \setminus \Omega$. For all these problems, our result in Theorem 7.1.1 still remains true; see Remark 7.3.3. In the particular case $\Omega = B_1$ and $g(x) = \log |x|^{-2s}$ in $\mathbb{R}^n \setminus B_1$, the extremal solution to the new problem is exactly $u^*(x) = \log |x|^{-2s}$ in B_1 whenever $\lambda_0 \leq H_{n,s}$. Thus, when $\lambda_0 \leq H_{n,s}$ we have a singular extremal solution for some exterior condition g.

We expect the sufficient condition (7.3) of Theorem 7.1.1 to be optimal since it is equivalent to $\lambda_0 > H_{n,s}$.

The condition $\lambda_0 > H_{n,s}$, appeared and was discussed in Remark 3.3 in [253].

We next give a symmetry result, which is the analog of the classical result of Berestycki-Nirenberg [18]. It does not require any smoothness of Ω. From this result it will follow that, under the hypotheses of Theorem 7.1.1, the solutions $u_\lambda(x)$ attain its maxima at $x = 0$.

When $\Omega = B_R$, there are a number of papers proving the radial symmetry of solutions for nonlocal equations. However, we have not found any reference in which the following result is proved.

Lemma 7.2.3. *Let Ω be a bounded domain which is convex in the x_1-direction and symmetric with respect to $\{x_1 = 0\}$. Let f be a locally Lipschitz function, and u be a bounded positive solution of*

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Then, u is symmetric with respect to $\{x_1 = 0\}$, and it satisfies

$$\partial_{x_1} u < 0 \quad \text{in} \quad \Omega \cap \{x_1 > 0\}.$$

Proof. For the case of the Laplacian $-\Delta$, the result follows from the moving planes method and the maximum principle in small domains; see [18] and also, for example, [34, 41]. For the fractional Laplacian $(-\Delta)^s$ (or even for more general integro-differential operators), one can easily check that the same proof can be carried out by using the nonlocal maximum principle in small domains given by Lemma 5.1 in [253]. □

As said before, this lemma yields that solutions u_λ of (7.1) satisfy

$$\|u_\lambda\|_{L^\infty(\Omega)} = u_\lambda(0).$$

This allows us to locate the (possible) singularity of the extremal solution u^* at the origin, something that is essential in our proofs.

Finally, to end this section, we compute the fractional Laplacian on a power function, something needed in the proof of Theorem 7.1.1.

Proposition 7.2.4. *Let $(-\Delta)^s$ be the fractional Laplacian in $\mathbb{R}^n$, with $s > 0$ and $n > 2s$. Let $\alpha \in (0, n-2s)$. Then,*

$$(-\Delta)^s |x|^{-\alpha} = 2^{2s} \frac{\Gamma\left(\frac{\alpha+2s}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha-2s}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)} |x|^{-\alpha-2s},$$

where Γ is the Gamma function.

Proof. We use Fourier transform, defined by

$$\mathcal{F}[u](\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(x) e^{-i\xi\cdot x} dx.$$

Then, one has that

$$\mathcal{F}\big[(-\Delta)^s u\big](\xi) = |\xi|^{2s}\, \mathcal{F}[u](\xi). \tag{7.8}$$

On the other hand, the function $|x|^{-\alpha}$, with $0 < \alpha < n$, has Fourier transform

$$\kappa_\beta\, \mathcal{F}\big[|\cdot|^{-\beta}\big](\xi) = \kappa_{n-\beta} |\xi|^{\beta-n}, \qquad \kappa_\beta := 2^{\beta/2}\Gamma(\beta/2) \tag{7.9}$$

(see for example [197, Theorem 5.9], where another convention for the Fourier transform is used, however).

Hence, using (7.9) and (7.8), we find that

$$\begin{aligned}\mathcal{F}\big[(-\Delta)^s |\cdot|^{-\alpha}\big](\xi) &= |\xi|^{2s}\, \mathcal{F}\big[|\cdot|^{-\alpha}\big](\xi) \\ &= \frac{\kappa_{n-\alpha}}{\kappa_\alpha} |\xi|^{\alpha+2s-n} = \frac{\kappa_{n-\alpha}}{\kappa_\alpha} \frac{\kappa_{\alpha+2s}}{\kappa_{n-\alpha-2s}} \mathcal{F}\big[|\cdot|^{-\alpha-2s}\big](\xi).\end{aligned}$$

Thus, it follows that

$$(-\Delta)^s |x|^{-\alpha} = \frac{\kappa_{n-\alpha}}{\kappa_\alpha} \frac{\kappa_{\alpha+2s}}{\kappa_{n-\alpha-2s}} |x|^{-\alpha-2s} = 2^{2s} \frac{\Gamma\left(\frac{\alpha+2s}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha-2s}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)} |x|^{-\alpha-2s},$$

as claimed. □

7.3 Proof of the main result

The aim of this section is to prove Theorem 7.1.1. We start with two preliminary lemmas.

The first one gives a lower bound for the singularity of an unbounded extremal solution. As we will see, this is an essential ingredient in our proof of Theorem 7.1.1. A similar result was established in [107] in the case of the boundary reaction problem considered there.

Lemma 7.3.1. *Let n, s, and u^* as in Theorem 7.1.1, and assume that u^* is unbounded. Then, for each $\sigma \in (0,1)$ there exists $r(\sigma) > 0$ such that*

$$u^*(x) > (1-\sigma)\log\frac{1}{|x|^{2s}}$$

for all x satisfying $|x| < r(\sigma)$.

Proof. We will argue by contradiction. Assume that there exist $\sigma \in (0,1)$ and a sequence $\{x_k\} \to 0$ for which

$$u^*(x_k) \le (1-\sigma)\log\frac{1}{|x_k|^{2s}}. \tag{7.10}$$

Recall that, by Lemma 7.2.3, we have $u_\lambda(0) = \|u_\lambda\|_{L^\infty}$. Thus, since u^* is unbounded by assumption, we have

$$\|u_\lambda\|_{L^\infty(\Omega)} = u_\lambda(0) \longrightarrow +\infty \quad \text{as} \quad \lambda \to \lambda^*.$$

In particular, there exists a sequence $\{\lambda_k\} \to \lambda^*$ such that

$$u_{\lambda_k}(0) = \log\frac{1}{|x_k|^{2s}}.$$

Define now the functions

$$v_k(x) = \frac{u_{\lambda_k}(|x_k|x)}{\|u_{\lambda_k}\|_{L^\infty}} = \frac{u_{\lambda_k}(|x_k|x)}{\log\frac{1}{|x_k|^{2s}}}, \qquad x \in \Omega_k = \frac{1}{|x_k|}\Omega.$$

These functions satisfy $0 \le v_k \le 1$, $v_k(0) = 1$, and

$$(-\Delta)^s v_k \longrightarrow 0 \quad \text{uniformly in } \Omega_k \text{ as } k \to \infty.$$

Indeed,

$$(-\Delta)^s v_k(x) = \frac{1}{\log\frac{1}{|x_k|^{2s}}}|x_k|^{2s}\lambda_k e^{u_{\lambda_k}(|x_k|x)} \le \frac{\lambda_k}{\log\frac{1}{|x_k|^{2s}}} \le \frac{\lambda^*}{\log\frac{1}{|x_k|^{2s}}} \longrightarrow 0.$$

Note also that the functions v_k are uniformly Hölder continuous in compact sets of $\mathbb{R}^n$, since $|(-\Delta)^s v_k|$ are uniformly bounded. Hence, it follows from the Arzelà-Ascoli

theorem that, up to a subsequence, v_k converges uniformly in compact sets of $\mathbb{R}^n$ to some function v satisfying

$$(-\Delta)^s v \equiv 0 \quad \text{in } \mathbb{R}^n, \quad 0 \le v \le 1, \quad v(0) = 1.$$

Thus, it follows from the strong maximum principle that $v \equiv 1$.

Therefore, we have that

$$v_k(x) \longrightarrow 1 \quad \text{uniformly in compact sets of } \mathbb{R}^n,$$

and in particular

$$\frac{u_{\lambda_k}(x_k)}{\log \frac{1}{|x_k|^{2s}}} = v_k\left(x_k/|x_k|\right) \longrightarrow 1.$$

This contradicts (7.10), and hence the lemma is proved. □

In the next lemma we compute the fractional Laplacian of some explicit functions in all of $\mathbb{R}^n$. The constants appearing in these computations are very important, since they are very related to the ones in (7.3).

Lemma 7.3.2. *Let $s \in (0,1)$, $n > 2s$, and $\varepsilon > 0$ be small enough. Then*

$$(-\Delta)^s |x|^{\frac{2s-n+\varepsilon}{2}} = \left(H_{n,s} + O(\varepsilon)\right) |x|^{\frac{-2s-n+\varepsilon}{2}}$$

and

$$(-\Delta)^s |x|^{2s-n+\varepsilon} = \left(\lambda_0 \frac{\varepsilon}{2s} + O(\varepsilon^2)\right) |x|^{-n+\varepsilon},$$

where $H_{n,s}$ and λ_0 are given by (7.7) and (7.6), respectively.

Proof. To prove the result we use Proposition 7.2.4 and the properties of the Γ function, as follows.

First, using Proposition 7.2.4 with $\alpha = \frac{1}{2}(n-2s-\varepsilon)$ and with $\alpha = n-2s-\varepsilon$, we find

$$(-\Delta)^s |x|^{\frac{2s-n+\varepsilon}{2}} = 2^{2s} \frac{\Gamma\left(\frac{n+2s-\varepsilon}{4}\right)\Gamma\left(\frac{n+2s+\varepsilon}{4}\right)}{\Gamma\left(\frac{n-2s+\varepsilon}{4}\right)\Gamma\left(\frac{n-2s-\varepsilon}{4}\right)} |x|^{\frac{-2s-n+\varepsilon}{2}}$$

and

$$(-\Delta)^s |x|^{2s-n+\varepsilon} = 2^{2s} \frac{\Gamma\left(\frac{n-\varepsilon}{2}\right)\Gamma\left(\frac{2s+\varepsilon}{2}\right)}{\Gamma\left(\frac{\varepsilon}{2}\right)\Gamma\left(\frac{n-2s-\varepsilon}{2}\right)} |x|^{-n+\varepsilon},$$

where Γ is the Gamma function.

Since $\Gamma(t)$ is smooth and positive for $t > 0$, then it is clear that

$$2^{2s} \frac{\Gamma\left(\frac{n+2s-\varepsilon}{4}\right)\Gamma\left(\frac{n+2s+\varepsilon}{4}\right)}{\Gamma\left(\frac{n-2s+\varepsilon}{4}\right)\Gamma\left(\frac{n-2s-\varepsilon}{4}\right)} = 2^{2s} \left(\frac{\Gamma\left(\frac{n+2s}{4}\right)}{\Gamma\left(\frac{n-2s}{4}\right)}\right)^2 + O(\varepsilon) = H_{n,s} + O(\varepsilon).$$

Thus, the first identity of the Lemma follows.

To prove the second identity, we use also that $\Gamma(1+t) = t\Gamma(t)$. We find,

$$\begin{aligned}
2^{2s}\frac{\Gamma\left(\frac{n-\varepsilon}{2}\right)\Gamma\left(\frac{2s+\varepsilon}{2}\right)}{\Gamma\left(\frac{\varepsilon}{2}\right)\Gamma\left(\frac{n-2s-\varepsilon}{2}\right)} &= 2^{2s}\frac{\Gamma\left(\frac{n-\varepsilon}{2}\right)\Gamma\left(\frac{2s+\varepsilon}{2}\right)}{\Gamma\left(1+\frac{\varepsilon}{2}\right)\Gamma\left(\frac{n-2s-\varepsilon}{2}\right)}\frac{\varepsilon}{2} \\
&= 2^{2s}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma(s)}{\Gamma(1)\Gamma\left(\frac{n-2s}{2}\right)}\left(1+O(\varepsilon)\right)\frac{\varepsilon}{2} \\
&= 2^{2s}\frac{\Gamma\left(\frac{n}{2}\right)s\Gamma(s)}{\Gamma\left(\frac{n-2s}{2}\right)}\left(\frac{\varepsilon}{2s}+O(\varepsilon^2)\right) \\
&= 2^{2s}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma(1+s)}{\Gamma\left(\frac{n-2s}{2}\right)}\left(\frac{\varepsilon}{2s}+O(\varepsilon^2)\right) \\
&= \frac{\lambda_0}{2s}\,\varepsilon + O(\varepsilon^2).
\end{aligned}$$

Thus, the lemma is proved. □

We can now give the proof of our main result.

Proof of Theorem 7.1.1. First, note that when $n \leq 2s$ the result follows from [253], since we proved there the result for $n < 10s$. Thus, from now on we assume $n > 2s$.

To prove the result for $n > 2s$ we argue by contradiction, that is, we assume that u^* is unbounded and we show that this yields $\lambda_0 \leq H_{n,s}$. As we will see, Lemma 7.3.1 plays a very important role in this proof.

Let u_λ, with $\lambda < \lambda^*$, be the minimal stable solution to (7.1). Using ψ in the stability condition (7.5), we obtain

$$\int_\Omega \lambda e^{u_\lambda}\psi^2 \leq \int_\Omega \psi(-\Delta)^s\psi.$$

Moreover, ψ^2 as a test function for the equation (7.1), we find

$$\int_\Omega u_\lambda(-\Delta)^s(\psi^2) = \int_\Omega \lambda e^{u_\lambda}\psi^2.$$

Thus, we have

$$\int_\Omega u_\lambda(-\Delta)^s(\psi^2) \leq \int_\Omega \psi(-\Delta)^s\psi \quad \text{for all} \quad \lambda < \lambda^*. \tag{7.11}$$

Next we choose ψ appropriately so that (7.11) combined with Lemma 7.3.1 yield a contradiction. This function ψ will be essentially a power function $|x|^{-\beta}$, as explained in the Introduction.

Indeed, let ρ_0 be small enough so that $B_{\rho_0}(0) \subset \Omega$. For each small $\varepsilon > 0$, let us consider a function ψ satisfying

1. $\psi(x) = |x|^{\frac{2s-n+\varepsilon}{2}}$ in $B_{\rho_0}(0) \subset \Omega$.
2. ψ has compact support in Ω.
3. ψ is smooth in $\mathbb{R}^n \setminus \{0\}$.

Now, since the differences $\psi(x) - |x|^{\frac{2s-n+\varepsilon}{2}}$ and $\psi^2(x) - |x|^{2s-n+\varepsilon}$ are smooth and bounded in all of $\mathbb{R}^n$ (by definition of ψ), then it follows from Lemma 7.3.2 that

$$(-\Delta)^s \psi(x) \leq (H_{n,s} + C\varepsilon)\, |x|^{\frac{-2s-n+\varepsilon}{2}} + C \tag{7.12}$$

and

$$(-\Delta)^s (\psi^2)(x) \geq \left(\lambda_0 \frac{\varepsilon}{2s} - C\varepsilon^2\right) |x|^{-n+\varepsilon} - C, \tag{7.13}$$

where C is a constant that depends on ρ_0 but not on ε.

In the rest of the proof, C will denote different constants, which may depend on ρ_0, n, s, Ω, and σ, but not on ε. Here, σ is any given number in $(0,1)$.

Hence, we deduce from (7.11)-(7.12)-(7.13), that

$$\left(\lambda_0 \frac{\varepsilon}{2s} - C\varepsilon^2\right) \int_\Omega u_\lambda |x|^{\varepsilon-n} dx \leq (H_{n,s} + C\varepsilon) \int_\Omega |x|^{\varepsilon-n} dx + C. \tag{7.14}$$

We have used that $\int_\Omega u_\lambda \leq C$ uniformly in λ. Since the right hand side does not depend on λ, we can let $\lambda \longrightarrow \lambda^*$ to find that (7.14) holds also for $\lambda = \lambda^*$.

Next, for the given $\sigma \in (0,1)$, we apply Lemma 7.3.1. Since u^* is unbounded by assumption, we deduce that there exists $r(\sigma) > 0$ such that

$$u^*(x) \geq (1-\sigma) \log \frac{1}{|x|^{2s}} \quad \text{in } B_{r(\sigma)}.$$

Thus, we find

$$(1-\sigma)\left(\lambda_0 \frac{\varepsilon}{2s} - C\varepsilon^2\right) \int_{B_{r(\sigma)}} |x|^{\varepsilon-n} \log \frac{1}{|x|^{2s}} dx \leq (H_{n,s} + C\varepsilon) \int_\Omega |x|^{\varepsilon-n} dx + C. \tag{7.15}$$

Now, we have

$$\begin{aligned}
\int_{B_{r(\sigma)}} |x|^{\varepsilon-n} \log \frac{1}{|x|^{2s}} dx &= 2s|S^{n-1}| \int_0^{r(\sigma)} r^{\varepsilon-1} \log \frac{1}{r}\, dr \\
&= 2s|S^{n-1}|\, (r(\sigma))^\varepsilon\, \frac{1 - \varepsilon \log \frac{1}{r(\sigma)}}{\varepsilon^2} \\
&\geq \left\{2s|S^{n-1}|\, (r(\sigma))^\varepsilon - C\varepsilon\right\} \frac{1}{\varepsilon^2}
\end{aligned}$$

and

$$\int_\Omega |x|^{\varepsilon-n} dx \leq |S^{n-1}| \int_0^1 r^{\varepsilon-1} dr + C = |S^{n-1}| \frac{1}{\varepsilon} + C.$$

Therefore, by (7.15),

$$(1-\sigma)\left(\lambda_0 \frac{\varepsilon}{2s} - C\varepsilon^2\right) \left\{2s|S^{n-1}|\, (r(\sigma))^\varepsilon - C\varepsilon\right\} \frac{1}{\varepsilon^2} \leq (H_{n,s} + C\varepsilon)\, |S^{n-1}| \frac{1}{\varepsilon} + C.$$

Hence, multiplying by ε and rearranging terms,

$$(1-\sigma)\lambda_0\, (r(\sigma))^\varepsilon \leq H_{n,s} + C\varepsilon.$$

Letting now $\varepsilon \to 0$ (recall that $\sigma \in (0,1)$ is an arbitrary given number), we find

$$(1-\sigma)\lambda_0 \leq H_{n,s}.$$

Finally, since this can be done for each $\sigma \in (0,1)$, we deduce that

$$\lambda_0 \leq H_{n,s},$$

a contradiction. □

Remark 7.3.3. Note that in our proof of Theorem 7.1.1 the exterior condition $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$ plays no role. Thus, the same result holds true for (7.1) with any other exterior condition $u = g$ in $\mathbb{R}^n \setminus \Omega$.

On the other hand, note that the nonlinearity $f(u) = e^u$ plays a very important role in our proof. Indeed, to establish (7.11) we have strongly used that $f'(u) = f(u)$, since we combined the stability condition (in which $f'(u)$ appears) with the equation (in which only $f(u)$ appears). It seems difficult to extend our proof to the case of more general nonlinearities. Even for the powers $f(u) = (1+u)^p$, it is not clear how to do it.

III

Part Three

ISOPERIMETRIC INEQUALITIES WITH DENSITIES

Introduction to Part III

In this last part of the thesis we study weighted Sobolev and isoperimetric inequalities.

Let us first recall what is an isoperimetric problem with a weight —also called density. Given a weight w (that is, a positive function w), one wants to characterize minimizers of the weighted perimeter $\int_{\partial E} w$ among those sets E having weighted volume $\int_E w$ equal to a given constant. A set solving the problem, if it exists, is called an isoperimetric set (or simply a minimizer). This question, and the associated isoperimetric inequalities with weights, have attracted much attention recently; see for example [222], [207], [98], [134], and the nice survey [218].

An important motivation for studying such isoperimetric inequalities with weights are their applications to Analysis and PDEs [162, 204, 49], in Geometry [91, 218, 221], and in Probability [194, 32].

As explained in Part II, while studying reaction-diffusion equations we were led to some Sobolev and isoperimetric inequalities with monomial weights. More precisely, by using the stability property of solutions u we obtained control on some integrals of the form

$$\int_{\Omega_2} \left(s^{-\alpha} u_s^2 + t^{-\beta} u_t^2\right) ds dt,$$

where $\Omega_2 \subset (\mathbb{R}_+)^2$ and $u \equiv 0$ on $\partial\Omega_2 \cap (\mathbb{R}_+)^2$ (note that u may not vanish on the axes of the s-t plane). From this, we wanted to deduce an L^p bound for u.

After the change of variables $\sigma = s^{2+\alpha}$, $\tau = t^{2+\beta}$, the problem transforms into the following: given nonnegative a and b, find the largest exponent $q > 2$ for which the weighted inequality

$$\left(\int_{\widetilde{\Omega}_2} \sigma^a \tau^b |u|^q d\sigma d\tau\right)^{1/q} \leq C \left(\int_{\widetilde{\Omega}_2} \sigma^a \tau^b |\nabla u|^2 d\sigma d\tau\right)^{1/2} \tag{7.16}$$

holds for all smooth functions u vanishing on $\partial\widetilde{\Omega}_2 \cap (\mathbb{R}_+)^2$. These weights are not in the Muckenhoupt class and the inequality (7.16) had not been proved in the literature.

In Chapter 5 of Part II (the work on domains of double revolution), we already established this embedding in $(\mathbb{R}_+)^2$ by proving first a weighted isoperimetric inequality. However, we did not find there its best constant, neither the extremal functions. In this Part III we accomplish this (also in the corresponding isoperimetric inequalities) not only in dimension 2 as above, but also in all dimensions $n \geq 1$, and also for all exponents p in the right hand side $|\nabla u|^p$. More precisely, in Chapter 8 we establish the following Sobolev inequality with monomial weights.

Theorem 6. *Let $n \geq 1$, and let us consider a monomial weight of the form $x^A = |x_1|^{A_1} \cdots |x_n|^{A_n}$ with every $A_i \geq 0$ a real number.*

Let $D = n + A_1 + \cdots + A_n$, and let also

$$\mathbb{R}^n_* = \{x \in \mathbb{R}^n \,:\, x_i > 0 \text{ for all } i \text{ such that } A_i > 0\}. \tag{7.17}$$

Then, for each $1 \le p < D$*, we have*

$$\left(\int_{\mathbb{R}^n_*} x^A |u|^{p_*} dx\right)^{1/p_*} \le C_p \left(\int_{\mathbb{R}^n_*} x^A |\nabla u|^p dx\right)^{1/p}, \tag{7.18}$$

where $p_* = \frac{pD}{D-p}$.

We also obtain an explicit expression for the best constant C_p in inequality (7.18), as well as extremal functions for which the best constant is attained.

For $p > D$ and $p = D$, we prove weighted versions of the classical Morrey and Trudinger inequalities, respectively.

The proof of inequality (7.18) is based on a new weighted isoperimetric inequality,

$$\left(\int_E x^A dx\right)^{\frac{D-1}{D}} \le C \int_{\partial E} x^A d\sigma(x) \qquad \text{for all } E \subset \mathbb{R}^n_*,$$

with the optimal constant C depending on $n, A_1, ..., A_n$. Note that the part $\partial E \cap \partial \mathbb{R}^n_*$ has zero weighted perimeter, since x^A vanishes on $\partial \mathbb{R}^n_*$. We establish it by adapting a proof of the classical Euclidean isoperimetric inequality due to X. Cabré. Our proof uses a linear Neumann problem for the operator $x^{-A}\mathrm{div}(x^A \nabla \cdot)$ combined with the Alexandroff contact set method (or ABP method). The best constant is attained by domains of the form $E = B_R(0) \cap \mathbb{R}^n_*$ —recall that $\mathbb{R}^n_*$ is defined by (7.17). In other words, this solves the isoperimetric problem in $\mathbb{R}^n$ for monomial weights $w(x) = |x_1|^{A_1} \cdots |x_n|^{A_n}$.

The solution to the isoperimetric problem in $\mathbb{R}^n$ with a weight w is known only for few weights, even in the case $n = 2$. For example, in $\mathbb{R}^n$ with the Gaussian weight $w(x) = e^{-|x|^2}$ all the minimizers are half-spaces [32, 96], and with $w(x) = e^{|x|^2}$ all the minimizers are balls centered at the origin [247]. For more general radial weights $w(|x|)$ in $\mathbb{R}^n$, the log-convex density conjecture states that balls about the origin are isoperimetric whenever $\log w(r)$ is convex. The conjecture is sustained by the fact that the convexity of $\log w(r)$ is equivalent to the stability of balls about the origin. The conjecture was formulated in 2006 [247], and remained open for some years —see [207, 134, 185] for some partial results on this problem. It has been recently solved by Chambers [84].

Other isoperimetric problems with radial weights $w(|x|)$ have also been solved. In the plane ($n = 2$) with the homogeneous weight $|x|^\alpha$, the minimizers depend on the values of α. On the one hand, Carroll-Jacob-Quinn-Walters [82] showed that when $\alpha < -2$ all minimizers are $\mathbb{R}^2 \setminus B_r(0)$, $r > 0$, and that when $-2 \le \alpha < 0$ minimizers do not exist. On the other hand, when $\alpha > 0$ Dahlberg-Dubbs-Newkirk-Tran [104] proved that all minimizers are circles passing through the origin (in particular, not centered at the origin).

Hence, radial homogeneous weights may lead to nonradial minimizers. Our isoperimetric inequality with monomial weights $w(x) = |x_1|^{A_1} \cdots |x_n|^{A_n}$ gives a nontrivial example in which the contrary happens: nonradial weights lead to radial minimizers.

In Chapter 9 we study more general isoperimetric problems with densities. We obtain a family of sharp isoperimetric inequalities with homogeneous weights in convex cones $\Sigma \subset \mathbb{R}^n$. We prove that Euclidean balls centered at the origin solve the

isoperimetric problem in any open convex cone Σ of $\mathbb{R}^n$ (with vertex at the origin) for a whole class of nonradial homogeneous weights. More precisely, our main result reads as follows.

Theorem 7. *Let $\Sigma \subset \mathbb{R}^n$ be any open convex cone. Let w be continuous, positively homogeneous of degree $\alpha \geq 0$, and such that $w^{1/\alpha}$ is concave in the cone Σ. Then,*

$$\frac{P_w(E;\Sigma)}{w(E\cap\Sigma)^{\frac{D-1}{D}}} \geq \frac{P_w(B_1;\Sigma)}{w(B_1\cap\Sigma)^{\frac{D-1}{D}}}$$

for all sets E with finite measure, where $D = n + \alpha$.

Here, $w(E\cap\Sigma)$ and $P_w(E;\Sigma)$ denote the weighted volume and weighted perimeter of the set E inside Σ, that is,

$$w(E\cap\Sigma) = \int_{\Omega\cap\Sigma} w(x)dx \qquad \text{and} \qquad \int_{\partial E\cap\Sigma} w(x)dS.$$

Note that the part of the boundary of E that lies on the boundary of the cone $\partial\Sigma$ is not counted.

When $w \equiv 1$, this inequality is known as the Lions-Pacella isoperimetric inequality in convex cones [200]. On the other hand, when w is a monomial weight and $\Sigma = \mathbb{R}^n_*$, we recover our isoperimetric inequality with monomial weights.

As before, the proof of this result consists of applying the ABP method to a linear Neumann problem involving now the operator $w^{-1}\mathrm{div}(w\nabla u)$, where w is the weight. More precisely, we essentially solve the following Neumann problem in $E \subset \Sigma$

$$\begin{cases} w^{-1}\mathrm{div}(w\nabla u) = b_E & \text{in } E \\ \dfrac{\partial u}{\partial \nu} = 1 & \text{on } \partial E\cap\Sigma \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial E\cap\partial\Sigma, \end{cases} \tag{7.19}$$

where the constant b_E is chosen, after integrating by parts, so that the problem admits a solution (b_E depends only on weighted perimeter and volume of E). If u is C^1 up to the boundary ∂E —which is not always the case, and this leads to technical difficulties—, then by touching the graph of u by below with planes (as in the ABP method) we find that $B_1\cap\Sigma \subset \nabla u(E)$. From this, using the area formula, an appropriate weighted geometric-arithmetic means inequality, and the concavity condition on the weight w, we deduce our weighted isoperimetric inequality. Since the solution of (7.19) is $u(x) = \frac{1}{2}|x|^2$ when $E = B_1\cap\Sigma$, in this radial case all the chain of inequalities in our proof become equalities, and this yields the sharpness of the constant in our inequality.

We also solve weighted *anisotropic* isoperimetric problems in cones for the same class of weights. In these anisotropic problems, the perimeter of a smooth domain E is given by

$$\int_{\partial E\cap\Sigma} H(\nu(x))w(x)dS,$$

where $\nu(x)$ is the unit outward normal to ∂E at x, and H is a nonnegative, positively homogeneous of degree one, and convex function. For these problems, we prove that the Wulff set

$$W = \left\{ x \in \mathbb{R}^n \,:\, x \cdot \nu < H(\nu) \ \text{ for all } \nu \in S^{n-1} \right\}$$

is the minimizer of the weighted anisotropic quotient. In particular, the solution of such weighted isoperimetric problems does not depend on the weight w. For the unweighted case $w \equiv 1$, this anisotropic isoperimetric problem is known as the Wulff inequality, and was established by Taylor [284, 285] in 1974.

It is worth saying that our proof of Theorem 7 follows a totally different approach from those of Lions-Pacella [200] and Taylor [284, 285]. Thus, as a particular case of our results of Chapter 9, we provide with new proofs of both the isoperimetric inequality in convex cones of Lions-Pacella and of the Wulff inequality.

Chapter Eight

SOBOLEV AND ISOPERIMETRIC INEQUALITIES WITH MONOMIAL WEIGHTS

We consider the monomial weight $|x_1|^{A_1} \cdots |x_n|^{A_n}$ in $\mathbb{R}^n$, where $A_i \geq 0$ is a real number for each $i = 1, ..., n$, and establish Sobolev, isoperimetric, Morrey, and Trudinger inequalities involving this weight. They are the analogue of the classical ones with the Lebesgue measure dx replaced by $|x_1|^{A_1} \cdots |x_n|^{A_n} dx$, and they contain the best or critical exponent (which depends on A_1, ..., A_n). More importantly, for the Sobolev and isoperimetric inequalities, we obtain the best constant and extremal functions.

When A_i are nonnegative *integers*, these inequalities are exactly the classical ones in the Euclidean space $\mathbb{R}^D$ (with no weight) when written for axially symmetric functions and domains in $\mathbb{R}^D = \mathbb{R}^{A_1+1} \times \cdots \times \mathbb{R}^{A_n+1}$.

8.1 Introduction and results

In this paper we establish Sobolev, Morrey, Trudinger, and isoperimetric inequalities in $\mathbb{R}^n$ with the weight x^A, where $A = (A_1, ..., A_n)$ and

$$x^A := |x_1|^{A_1} \cdots |x_n|^{A_n}, \qquad A_1 \geq 0, \ ..., \ A_n \geq 0. \tag{8.1}$$

They were announced in our previous article [49]. In fact, their interest and applications arose in [49], where we had $n = 2$ in (8.1). In that paper we studied the regularity of stable solutions to reaction-diffusion problems in bounded domains of double revolution in $\mathbb{R}^N$. That is, domains of $\mathbb{R}^N$ which are invariant under rotations of the first m variables and of the last $N - m$ variables, i.e.,

$$\overline{\Omega} = \{(x^1, x^2) \in \mathbb{R}^m \times \mathbb{R}^{N-m} : (s = |x^1|, t = |x^2|) \in \overline{\Omega}_2\},$$

where $\Omega_2 \subset (\mathbb{R}_+)^2$ is a bounded domain.

The first step towards the results in [49] consisted of obtaining bounds for some integrals of the form

$$\int_{\Omega_2} \left\{ s^{-\alpha} u_s^2 + t^{-\beta} u_t^2 \right\} ds\, dt,$$

where u is any stable solution and s and t are, as above, the two radial coordinates describing Ω. Then, from this bound we needed to deduce that $u \in L^q(\Omega)$, with q as

large as possible. After a change of variables of the form $s = \sigma^{\gamma_1}$, $t = \tau^{\gamma_2}$, what we needed to establish is the following Sobolev inequality. Given $a > -1$ and $b > -1$, find the greatest exponent q for which

$$\left(\int_{\widetilde{\Omega}_2} \sigma^a \tau^b |u|^q d\sigma d\tau\right)^{1/q} \leq C \left(\int_{\widetilde{\Omega}_2} \sigma^a \tau^b |\nabla u|^2 d\sigma d\tau\right)^{1/2}$$

holds for all smooth functions u vanishing on $\partial\widetilde{\Omega}_2 \cap (\mathbb{R}_+)^2$, where $\widetilde{\Omega}_2 = \{(\sigma,\tau) \in (\mathbb{R}_+)^2 : (s = \sigma^{\gamma_1}, t = \tau^{\gamma_2}) \in \Omega_2\}$ is an arbitrary bounded domain of $(\mathbb{R}_+)^2$.

On the one hand, we obtained that $u \in L^\infty(\widetilde{\Omega}_2)$ whenever the right hand side is finite for some a, b with $a+b<0$. On the other hand, in case $a+b>0$ we established the following.

Throughout the paper, $C_c^1(\mathbb{R}^n)$ denotes the space of C^1 functions with compact support in $\mathbb{R}^n$.

Proposition 8.1.1 ([49]). *Let a and b be real numbers such that*

$$a > -1, \quad b > -1, \quad \text{and} \quad a+b > 0.$$

Let u be a nonnegative $C_c^1(\mathbb{R}^2)$ function such that

$$u_\sigma \leq 0 \quad \text{and} \quad u_\tau \leq 0 \quad \text{in} \quad \{\sigma > 0, \tau > 0\}. \tag{8.2}$$

with strict inequalities in the set $\{u > 0\}$. Then, there exists a constant C, depending only on a and b, such that

$$\left(\int_{\{\sigma>0,\,\tau>0\}} \sigma^a \tau^b |u|^{2_*} d\sigma d\tau\right)^{1/2_*} \leq C \left(\int_{\{\sigma>0,\,\tau>0\}} \sigma^a \tau^b |\nabla u|^2 d\sigma d\tau\right)^{1/2}, \tag{8.3}$$

where $2_* = \frac{2D}{D-2}$ and $D = a+b+2$.

In [49] we also obtained Sobolev inequalities with other powers $|\nabla u|^p$, $1 \leq p < D$. By a standard scaling argument one sees that the exponent $2_* = \frac{2D}{D-2}$ in (8.3) is optimal, in the sense that (8.3) can not hold with any other exponent larger than this one. In addition, when $a < 0$ or $b < 0$ inequality (8.3) is not valid without assumption (8.2); see Remark 8.3.3 for more details.

Remark 8.1.2. When a and b are positive *integers*, inequality (8.3) is exactly the classical Sobolev inequality in $\mathbb{R}^D = \mathbb{R}^{a+1} \times \mathbb{R}^{b+1}$ for functions which are radially symmetric on the first $a+1$ variables and on the last $b+1$ variables.

Indeed, for each $z \in \mathbb{R}^D$ write $z = (z^1, z^2)$, with $z^1 \in \mathbb{R}^{a+1}$ and $z^2 \in \mathbb{R}^{b+1}$, and define $(\sigma,\tau) = (|z^1|, |z^2|) \in \{\sigma \geq 0, \tau \geq 0\}$. Now, for each function u in $(\mathbb{R}_+)^2$ we define $\tilde{u}(z) = u(|z_1|, |z_2|)$. We have that $|\nabla_z \tilde{u}| = |\nabla_{(\sigma,\tau)} u|$. Moreover, an integral over $\mathbb{R}^D$ of a function depending only on $|z^1|$ and $|z^2|$ can be written as an integral in $(\mathbb{R}_+)^2$ with $dz = c_{a,b}\sigma^a\tau^b d\sigma d\tau$ for some constant $c_{a,b}$. Therefore, writing in the coordinates (σ,τ) the classical Sobolev inequality in $\mathbb{R}^D$ for the function $\tilde{u}$, we obtain the validity of (8.3). Note that if $a > 0$ and $b = 0$ then we obtain the inequality in $\{\sigma > 0\}$ instead of $\{\sigma > 0, \tau > 0\}$, that is, $\left(\int_{\{\sigma>0\}} \sigma^a |u|^{2_*} d\sigma d\tau\right)^{1/2_*} \leq C \left(\int_{\{\sigma>0\}} \sigma^a |\nabla u|^2 d\sigma d\tau\right)^{1/2}$ — and this motivates definition (8.4) below in the case of a general monomial x^A.

The same argument as in the previous remark, but now with multiple axial symmetries, shows the following. When $A_1, ..., A_n$ are nonnegative *integers*, the Sobolev, isoperimetric, Morrey, and Trudinger inequalities with the monomial weight

$$x^A = |x_1|^{A_1} \cdots |x_n|^{A_n}$$

are exactly the classical ones in

$$\mathbb{R}^{A_1+1} \times \cdots \times \mathbb{R}^{A_n+1}$$

when written in radial coordinates for functions which are radially symmetric with respect to the first A_1+1 variables, also with respect to the next A_2+1 variables, and so on until radial symmetry with respect to the last A_n+1 variables.

The aim of this paper is to extend inequality (8.3) in $\mathbb{R}^2$ to the case of $\mathbb{R}^n$ with any weight of the form (8.1), i.e., of the form $x^A = |x_1|^{A_1} \cdots |x_n|^{A_n}$. When A_i are nonnegative *real* numbers, we prove that this weighted Sobolev inequality holds for any function $u \in C_c^1(\mathbb{R}^n)$ — and thus assumption (8.2) is not necessary. We obtain also Sobolev inequalities with $|\nabla u|^2$ replaced by other powers $|\nabla u|^p$. More importantly, we find the best constant and extremal functions in these inequalities. For this, a crucial ingredient is a new isoperimetric inequality involving the weight x^A and with best constant. This is Theorem 8.1.4 below, a main result of this paper. In addition, we prove Morrey and Trudinger type inequalities involving the monomial weight. All these results were announced in our previous paper [49].

The first result of the paper is the Sobolev inequality with a monomial weight, and reads as follows. Here, and in the rest of the paper, we denote

$$\mathbb{R}^n_* = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_i > 0 \text{ whenever } A_i > 0\} \tag{8.4}$$

and

$$B_r^* = B_r(0) \cap \mathbb{R}^n_*.$$

For each $1 \le p < \infty$, let $W_0^{1,p}(\mathbb{R}^n, x^A dx)$ be the closure of the space of $C_c^1(\mathbb{R}^n)$ under the norm $\left(\int_{\mathbb{R}^n} x^A(|u|^p + |\nabla u|^p)dx\right)^{1/p}$.

Theorem 8.1.3. *Let A be a nonnegative vector in $\mathbb{R}^n$, $D = A_1 + \cdots + A_n + n$, and $1 \le p < D$ be a real number. Then,*

(a) There exists a constant C_p such that for all $u \in C_c^1(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n_*} x^A |u|^{p_*} dx\right)^{\frac{1}{p_*}} \le C_p \left(\int_{\mathbb{R}^n_*} x^A |\nabla u|^p dx\right)^{\frac{1}{p}}, \tag{8.5}$$

where $p_* = \frac{pD}{D-p}$ and x^A is given by (8.1).

(b) The best constant C_p is given by the explicit expression (8.31)-(8.32). *When $p = 1$, this constant is not attained in $W_0^{1,1}(\mathbb{R}^n, x^A dx)$. Instead, when $1 < p < D$ it is attained in $W_0^{1,p}(\mathbb{R}^n, x^A dx)$ by the functions*

$$u_{a,b}(x) = \left(a + b|x|^{\frac{p}{p-1}}\right)^{1-\frac{D}{p}}, \tag{8.6}$$

where a and b are any positive constants.

Note that the exponent p_* is exactly the same as in the classical Sobolev inequality, but in this case the "dimension" is given by D instead of n. Note also that when $A_1 = ... = A_n = 0$ then $D = n$ and (8.5) is exactly the classical Sobolev inequality. As before, a scaling argument shows that the exponent p_* is optimal, in the sense that (8.5) can not hold with any other exponent.

Note that the integrals in (8.5) are computed over $\mathbb{R}^n_*$ but the functions u involved need not vanish on the coordinate hyperplanes on $\partial\mathbb{R}^n_*$. Let us mention that $u_{a,b}$ are extremal functions for inequality (8.5), but we do not know if these are all extremal functions for the inequality — except in the case when all A_i are integers.

The Sobolev inequalities in all of $\mathbb{R}^n$ follow easily (without the best constant) from the ones in $\mathbb{R}^n_*$ by applying them at most 2^n times (one for each hyperoctant of $\mathbb{R}^n$, that is, for each set $\{\epsilon_i x_i > 0,\ i = 1, ..., n\}$, where $\epsilon_i \in \{-1, 1\}$) and adding up the obtained inequalities. Consider now functions $u \in C^1_c(\mathbb{R}^n)$ that are even with respect to those variables x_i for which $A_i > 0$. They arise naturally in nonlinear problems in $\mathbb{R}^D$ whenever D is an integer (see [49]). Among these functions, the Sobolev inequality in all of $\mathbb{R}^n$ has also as extremals the functions $u_{a,b}$ in (8.6).

After a change of variables of the form $x_i = y_i^{\gamma_i}$, (8.5) yields new inequalities of the form

$$\|u\|_{L^{p_*}(\mathbb{R}^n_*)} \leq C \sum_{i=1}^{n} \|x_i^{\alpha_i} u_{x_i}\|_{L^p(\mathbb{R}^n_*)},$$

where α_i are arbitrary exponents in $[0, 1)$; see Corollary 8.3.5. In these inequalities, the exponent on the left hand side is given by $p_* = \frac{pD}{D-p}$, where $D = n + \frac{\alpha_1}{1-\alpha_1} + \cdots + \frac{\alpha_n}{1-\alpha_n}$.

When $p > 1$ and $A_i < p - 1$ for all $i = 1, ..., n$, the weight (8.1) belongs to the Muckenhoupt class A_p, and thus part (a) — without the best constant and for bounded domains — can be deduced from some classical results on weighted Sobolev inequalities. Indeed, it follows from a classical result of Fabes-Kenig-Serapioni [126] that for any bounded domain $\Omega \subset \mathbb{R}^n$ there exists $q > p$ for which $\|u\|_{L^q(\Omega, x^A dx)} \leq C\|u\|_{W^{1,p}_0(\Omega, x^A dx)}$ holds. Moreover, the optimal exponent $q = p_*$ can be found by using a result of Hajlasz [166, Theorem 6]. However, in general the monomial weight (8.1) does not satisfy the Muckenhoupt condition A_p and Theorem 8.1.3 cannot be deduced from these results on weighted Sobolev inequalities, even without the best constant in the inequality.

The main ingredient in the proof of Theorem 8.1.3 is a new weighted isoperimetric inequality with best constant, given by Theorem 8.1.4 below. Let us mention that if one is willing not to have the best constant in the Sobolev inequality, we give an alternative and more elementary proof of part (a) of Theorem 8.1.3 under some additional hypotheses. Namely, we assume $A_i > 0$ for all i and $u_{x_i} \leq 0$ in $\{x_i > 0,\ i = 1, ..., n\}$ — an assumption equivalent to (8.2) in Proposition 8.3 and which suffices for some applications to nonlinear problems.

The following is the new isoperimetric inequality with a monomial weight.

Theorem 8.1.4. *Let A be a nonnegative vector in $\mathbb{R}^n$, x^A given by* (8.1)*, and $D = A_1 + \cdots + A_n + n$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Denote*

$$m(\Omega) = \int_\Omega x^A dx \qquad \text{and} \qquad P(\Omega) = \int_{\partial\Omega} x^A d\sigma.$$

Then,

$$\frac{P(\Omega)}{m(\Omega)^{\frac{D-1}{D}}} \geq \frac{P(B_1^*)}{m(B_1^*)^{\frac{D-1}{D}}}\ , \tag{8.7}$$

where $B_1^* = B_1(0) \cap \mathbb{R}^n_*$ *is the unit ball intersected with* $\mathbb{R}^n_*$*, and* $\mathbb{R}^n_*$ *is given by* (8.4).

It is a surprising fact that the weight x^A is not radially symmetric but still Euclidean balls centered at the origin (intersected with $\mathbb{R}^n_*$) minimize this isoperimetric quotient.

Recently, these type of isoperimetric inequalities with weights (also called "with densities") have attracted much attention; see the nice survey of F. Morgan in the Notices of the AMS [218]. In a forthcoming paper [52] we will prove new weighted isoperimetric inequalities in convex cones of $\mathbb{R}^n$ that extend Theorem 8.1.4; some of them have been announced in [51].

Equality in (8.7) holds when $\Omega = B_r^* = B_r(0) \cap \mathbb{R}^n_*$, where r is any positive number. We expect these balls centered at the origin intersected with $\mathbb{R}^n_*$ to be the unique minimizers of the isoperimetric quotient. However, our proof involves the solution of an elliptic equation and due to an issue on its regularity we need to regularize slightly the domain Ω. This is why we can not obtain that B_r^* are the unique minimizers of (8.7). In a future paper [53] (still in progress) we will study the non uniformly elliptic operator (8.9) below and prove some regularity results in $\overline{\mathbb{R}^n_*}$ which may lead to the characterization of equality in the isoperimetric inequality (8.7).

Remark 8.1.5. Note that, when $A \neq 0$, the entire balls $B_r = B_r(0)$ are not minimizers of the isoperimetric quotient. This is because

$$\frac{P(B_1^*)}{m(B_1^*)^{\frac{D-1}{D}}} = 2^{-\frac{k}{D}} \frac{P(B_1)}{m(B_1)^{\frac{D-1}{D}}} < \frac{P(B_1)}{m(B_1)^{\frac{D-1}{D}}},$$

where k is the number of positive entries in the vector A. However, if we look for the minimizers of the isoperimetric quotient $P(\Omega)/m(\Omega)^{\frac{D-1}{D}}$ among all sets Ω which are symmetric with respect to each plane $\{x_i = 0\}$ with i such that $A_i > 0$, then the balls $B_r(0)$ solve this isoperimetric problem.

As explained below in Remark 8.2.2, the fact that $P(\Omega)/m(\Omega)^{\frac{D-1}{D}} \geq c$ for some constant $c > 0$ smaller than the one in (8.7) (and hence, nonoptimal) is an interesting consequence of the isoperimetric inequality in product manifolds of A. Grigor'yan [161].

As said before, our sharp isoperimetric inequality (8.7) is the crucial ingredient needed to prove Theorem 8.1.3 on the Sobolev inequality, especially part (b) on the best constant and on extremals. Indeed, we prove part (b) by applying our isoperimetric inequality with best constant together with two results of Talenti. The first one is a radial symmetrization result, which applies since our isoperimetric inequality (8.7) gives the best constant and the sets $B_r(0) \cap \mathbb{R}^n_*$ are extremal sets for any $r > 0$. The second one is a result in dimension 1, which characterizes the minimizers of the functional

$$J(u) = \frac{\left(\int_0^\infty r^{D-1}|u'|^p\right)^{1/p}}{\left(\int_0^\infty r^{D-1}|u|^{p_*}\right)^{1/p_*}},$$

where $p_* = \frac{pD}{D-p}$.

When $n = 2$ and $A_1 = 0$, our Sobolev and isoperimetric inequalities with best constant were already obtained by Maderna and Salsa [204] in 1981. Namely, they proved the sharp isoperimetric inequality in $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ with weight y^k, $k > 0$, and from it they deduced the Sobolev inequality with weight y^k. These inequalities arose in the study of an elliptic problem which involved the operator $y^{-k}\text{div}(y^k \nabla u)$ in $\{(x, y) \in \mathbb{R}^2 : y > 0\}$, where k is any positive number. Using symmetrization techniques and their weighted isoperimetric inequality, they obtained sharp estimates for the solution of the problem. To prove the isoperimetric inequality with weight y^k they first established the existence of a minimizer for the perimeter functional under constraint of fixed area, then computed the first variation of this functional, and finally solved the obtained ODE to deduce that minimizers must be half balls. Their result can be seen as a particular case of Theorem 8.1.4 by setting $n = 2$ and $A_1 = 0$. Our proof of the weighted isoperimetric inequality will be completely different from the one in [204], as explained next.

The proof of Theorem 8.1.4 follows the ideas introduced by the first author in a new proof of the classical isoperimetric inequality; see [40, 41] or the last edition of Chavel's book [91]. It is quite surprising (and fortunate) that this proof (which gives the best constant) can be adapted to the case of monomial weights.

The proof of the classical isoperimetric inequality from [40, 41] considers the linear problem

$$\begin{cases} \Delta u = c & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 1 & \text{on } \partial\Omega, \end{cases} \tag{8.8}$$

where c is the unique constant for which the problem has a solution. Then, one uses an argument similar to the Alexandroff-Bakelman-Pucci method (also called ABP method; see for example [157]) applied to this solution u. Using this argument and the classical inequality between the arithmetic mean (AM) and the geometric mean (GM), the isoperimetric inequality follows. When $\Omega = B_1$, the solution of (8.8) is $u(x) = |x|^2/2$ and all inequalities in the proof become equalities. Here we consider a similar problem to (8.8) but where the Laplacian is replaced by the operator

$$x^{-A} div(x^A \nabla u) = \Delta u + A_1 \frac{u_{x_1}}{x_1} + \cdots + A_n \frac{u_{x_n}}{x_n}. \tag{8.9}$$

Now, using the same ABP argument with this new problem and a weighted version of the AM-GM inequality, we obtain (8.7). An essential fact in our proof (and this is why $B_1(0) \cap \mathbb{R}^n_*$ is the minimizer) is that the function $u(x) = |x|^2/2$ also solves the equation $x^{-A} div(x^A \nabla u) = c$ for some constant $c > 0$. In addition, it has normal derivative $u_\nu = 1$ on ∂B_1, as in problem (8.8).

When $A_1, ..., A_n$ are nonnegative integers, the operator (8.9) is the Laplacian in the space $\mathbb{R}^D = \mathbb{R}^{A_1+1} \times \cdots \times \mathbb{R}^{A_n+1}$ written in radial coordinates. Thus, if instead A_i are not integers, (8.9) can be seen as some kind of Laplacian in a fractional dimension D. This class of operators was studied by A. Weinstein and others for $n = 2$, and the theory on these equations is called "Generalized Axially Symmetric Potential Theory"; see for example [298]. In case $A_1 = \cdots = A_{n-1} = 0$ and $A_n = a \in (-1, 1)$, the operator $x^{-A} div(x^A \nabla u)$ appears in the re-interpretation of the fractional Laplacian as a local problem in one higher dimension; see [69].

The paper [174] by Ivanov and Nazarov establishes some weighted Sobolev inequalities for $W^{1,p}$ functions with multiple radial symmetries — a space of functions denoted by $W^{1,p}_{sym}$. Their result is related to ours in the case in which all the exponents A_i are nonnegative *integers*. They prove that for functions with multiple radial symmetries in $\mathbb{R}^D$, the embedding $W^{1,p}_{sym}(B_1) \subset L^q(B_1; |x|^\alpha)$, with $p < D$ and $\alpha > 0$, holds for some exponents q depending on α that are greater than $p^* = pD/(p-D)$.

Some theorems of trace and interpolation type for functional spaces with weights of the form (8.1) were proved by A. Cavallucci [83] in 1969. Namely, he established some inequalities of the form

$$\|D^\lambda f\|_{L^p((\mathbb{R}_+)^m \times \{0\}, y^B dy)} \leq C \left(\|f\|_{L^p((\mathbb{R}_+)^n, x^A dx)} + \|D^l f\|_{L^p((\mathbb{R}_+)^n, x^A dx)} \right),$$

where $m \leq n$, $y^B = y_1^{B_1} \cdots y_m^{B_m}$ and $x^A = x_1^{A_1} \cdots x_n^{A_n}$ are two monomial weights, and λ and l are multiindices satisfying a certain condition involving A, B, m, n, and p. Note that in these inequalities the exponent p is the same in both sides, and thus they are not Sobolev-type inequalities. To obtain his results, the author used a representation of $D^\lambda f$ in terms of integral transforms of $D^l f$.

The third result of our paper is the weighted version of the Morrey inequality, which reads as follows.

Theorem 8.1.6. *Let A be a nonnegative vector in $\mathbb{R}^n$, $D = A_1 + \cdots + A_n + n$, and $p > D$ be a real number. Then, there exists a constant C, depending only on p and D, such that*

$$\sup_{x \neq y,\ x,y \in \mathbb{R}^n_*} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq C \left(\int_{\mathbb{R}^n_*} x^A |\nabla u|^p dx \right)^{1/p} \tag{8.10}$$

for all $u \in C^1_c(\mathbb{R}^n)$, where $\alpha = 1 - \frac{D}{p}$.

As a consequence, if $\Omega \subset \mathbb{R}^n$ is a bounded domain and $u \in C^1_c(\Omega)$ then

$$\sup_\Omega |u| \leq C\, diam(\Omega)^{1-\frac{D}{p}} \left(\int_\Omega x^A |\nabla u|^p dx \right)^{1/p}. \tag{8.11}$$

This weighted Morrey inequality will be deduced from the bound

$$|u(y) - u(0)| \leq C \int_{B^*_r} \frac{|\nabla u(x)|}{|x|^{D-1}}\, x^A dx, \tag{8.12}$$

which holds for each $y \in B^*_{r/2}$, Recall that we denote $B^*_r = B_r(0) \cap \mathbb{R}^n_*$. This bound is well known for $A = 0$ and $D = n$; see for example Lemma 7.16 in Gilbarg-Trudinger [157]. We prove (8.12) in two steps. First, we show that it suffices to prove it for integers A_i, $i = 1, ..., n$. Then, we deduce the integer case from the classical one $A = 0$ with an argument as in Remark 8.1.2.

The next result is the weighted version of the classical Trudinger inequality.

Theorem 8.1.7. *Let A be a nonnegative vector in $\mathbb{R}^n$, $D = A_1 + \cdots + A_n + n$, and $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then, for each $u \in C^1_c(\Omega)$,*

$$\int_\Omega \exp\left\{ \left(\frac{c_1 |u|}{\|\nabla u\|_{L^D(\Omega, x^A dx)}} \right)^{\frac{D}{D-1}} \right\} x^A dx \leq C_2 m(\Omega),$$

where $m(\Omega) = \int_\Omega x^A dx$, and c_1 and C_2 are constants depending only on D.

Our proof of this result is based on a bound for the best constant (8.32) in the weighted Sobolev inequality as p goes to D. Then, the Trudinger inequality will follow by expanding $\exp(\cdot)$ as a power series and applying the weighted Sobolev inequality to each term of the series. The obtained series is convergent thanks to the mentioned bound for the best constant (8.32).

Finally, adding up the results of Theorems 8.1.3, 8.1.6, and 8.1.7 we obtain the following continuous embeddings, which are weighted versions of the classical Sobolev embeddings.

Recall that the Orlicz space $L^{\varphi}(X, d\mu)$ is defined as the space of measurable functions $u : X \to \mathbb{R}$ such that

$$\|u\|_{L^{\varphi}(X,d\mu)} = \inf \left\{ K > 0 : \int_X \varphi\left(\frac{|u|}{K}\right) d\mu \leq 1 \right\}$$

is finite. Setting $\varphi(t) = t^p$ we recover the definition of the L^p spaces.

Corollary 8.1.8. *Let A be a nonnegative vector in $\mathbb{R}^n$, x^A be given by (8.1), and $D = A_1 + \cdots + A_n + n$. Let $k \geq 1$ be an integer and $p \geq 1$ be a real number. Then, for any bounded domain $\Omega \subset \mathbb{R}^n$ we have the following continuous embeddings:*

(i) If $kp < D$ then

$$W_0^{k,p}(\Omega, x^A dx) \subset L^q(\Omega, x^A dx),$$

where q is given by $\frac{1}{q} = \frac{1}{p} - \frac{k}{D}$.

(ii) If $kp = D$ then

$$W_0^{k,p}(\Omega, x^A dx) \subset L^{\varphi}(\Omega, x^A dx),$$

where

$$\varphi(t) = \exp\left(t^{\frac{D}{D-1}}\right) - 1.$$

(iii) If $kp > D$ then

$$W_0^{k,p}(\Omega, x^A dx) \subset C^{r,\alpha}(\overline{\Omega}),$$

where $r = k - [\frac{D}{p}] - 1$, and $\alpha = [\frac{D}{p}] + 1 - \frac{D}{p}$ whenever $\frac{D}{p}$ is not an integer, or α is any positive number smaller than 1 otherwise.

The paper is organized as follows. In section 2 we give the proof of the weighted isoperimetric inequality. Section 3 establishes the weighted Sobolev inequalities, while in section 4 we obtain their best constants and extremal functions. Section 5 deals with the weighted Morrey inequality. Finally, in section 6 we prove the weighted Trudinger inequality and Corollary 8.1.8.

8.2 Proof of the Isoperimetric inequality

In this section we prove the isoperimetric inequality with a monomial weight. Our proof extends the one of the classical isoperimetric inequality due to the first author [40, 41] (see also the last edition of [91]). In fact, setting $A = 0$ in the following proof we obtain exactly the original one. It is quite surprising (and fortunate) that this proof (which gives the best constant) can be adapted to the case of monomial weights. A

crucial fact in being able to obtain the sharp constant in the isoperimetric inequality is that

$$u(x) = |x|^2/2,$$

$x \in B_1 \cap \mathbb{R}^n_*$, is the solution of

$$\begin{cases} \div(x^A \nabla u) = b_\Omega x^A & \text{in } \Omega \\ x^A \dfrac{\partial u}{\partial \nu} = x^A & \text{on } \partial\Omega, \end{cases} \tag{8.13}$$

for some constant $b_\Omega > 0$ when $\Omega = B_1 \cap \mathbb{R}^n_*$.

In a forthcoming paper [52] we will use similar ideas to prove new sharp isoperimetric inequalities with homogeneous weights in open convex cones Σ of $\mathbb{R}^n$. We have already announced some of them in [51]. Note that monomial weights are homogeneous functions in the convex cone $\Sigma = \mathbb{R}^n_*$. In fact, the results in [52] extend the present isoperimetric inequality with a monomial weight.

Proof of Theorem 8.1.4. By symmetry, we can assume that $A = (A_1, ..., A_k, 0, ..., 0)$, with $A_i > 0$ for $i = 1, ..., k$, where $0 \le k \le n$.

Moreover, we can also suppose that Ω is contained in $\mathbb{R}^n_*$. Indeed, split the domain Ω in at most 2^k disjoint subdomains Ω_j, $j = 1, ..., J$, each one of them contained in the cone $\{\epsilon_i x_i > 0,\ i = 1, ..., k\}$ for different $\epsilon_i \in \{-1, 1\}$, and with $\overline{\Omega} = \overline{\Omega}_1 \cup \cdots \cup \overline{\Omega}_J$. Then, since the weight is zero on $\{x_i = 0\}$ for each $i = 1, ..., k$, we have that $P(\Omega) = \sum_{j=1}^J P(\Omega_j)$ and $m(\Omega) = \sum_{j=1}^J m(\Omega_j)$. Therefore

$$\frac{P(\Omega)}{m(\Omega)^{\frac{D-1}{D}}} \ge \min_{1 \le j \le J} \left\{ \frac{P(\Omega_j)}{m(\Omega_j)^{\frac{D-1}{D}}} \right\} =: \frac{P(\Omega_{j_0})}{m(\Omega_{j_0})^{\frac{D-1}{D}}},$$

with strict inequality unless $J = 1$. After some reflections, we may assume that $\Omega_{j_0} \subset \mathbb{R}^n_*$. Moreover, since Ω_{j_0} is the intersection of a Lipschitz domain of $\mathbb{R}^n$ with $\mathbb{R}^n_*$, Ω_{j_0} can be approximated in weighted area and perimeter by smooth domains Ω_ε with $\overline{\Omega}_\varepsilon \subset \Omega_{j_0} \subset \mathbb{R}^n_*$.

Therefore, from now on we assume:

$$\Omega \text{ is smooth and } \overline{\Omega} \subset \mathbb{R}^n_*.$$

In particular, $x^A \ge c$ in $\overline{\Omega}$ for some positive constant c.

Let u be a solution of the Neumann problem

$$\begin{cases} \div(x^A \nabla u) = b_\Omega x^A & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 1 & \text{on } \partial\Omega, \end{cases} \tag{8.14}$$

where the constant b_Ω is chosen so that the problem has a unique solution up to an additive constant, i.e.,

$$b_\Omega = \frac{P(\Omega)}{m(\Omega)}. \tag{8.15}$$

Since the equation in (8.14),

$$x^{-A} \div (x^A \nabla u) = \Delta u + \frac{A_1}{x_1} u_{x_1} + \cdots + \frac{A_n}{x_n} u_{x_n} = b_\Omega \tag{8.16}$$

is uniformly elliptic in Ω, u is smooth in $\overline{\Omega}$. The $C^{1,1}$ regularity of u up to $\overline{\Omega}$ will be crucial in the rest of the proof.

The following comment is not necessary to complete the proof, but it is useful to notice it here. Problem (8.14) is equivalent to (8.13) since $\partial\Omega \subset \mathbb{R}^n_*$. At the same time, when $\Omega = B_1^* = B_1 \cap \mathbb{R}^n_*$ the solution to (8.13) is given by $u(x) = |x|^2/2$, and we will have that all inequalities in the rest of the proof are equalities for $\Omega = B_1^*$ (see Remark 8.2.1 for more details).

Coming back to the solution u of (8.14), consider the lower contact set of u, defined by

$$\Gamma_u = \{x \in \Omega \ : \ u(y) \geq u(x) + \nabla u(x) \cdot (y - x) \ \text{ for all } y \in \overline{\Omega}\}.$$

It is the set of points where the tangent hyperplane to the graph of u lies below u in all $\overline{\Omega}$. Define also

$$\Gamma_u^* = \{x \in \Gamma_u \ : \ u_{x_1}(x) > 0, ..., u_{x_k}(x) > 0\} = \Gamma_u \cap (\nabla u)^{-1}(\mathbb{R}^n_*).$$

We claim that

$$B_1^* \subset \nabla u(\Gamma_u^*), \tag{8.17}$$

where $B_1^* = B_1(0) \cap \mathbb{R}^n_*$.

To show (8.17), take any $p \in \mathbb{R}^n$ satisfying $|p| < 1$. Let $x \in \overline{\Omega}$ be a point such that

$$\min_{y \in \overline{\Omega}} \{u(y) - p \cdot y\} = u(x) - p \cdot x$$

(this is, up to a sign, the Legendre transform of u). If $x \in \partial\Omega$ then the exterior normal derivative of $u(y) - p \cdot y$ at x would be nonpositive and hence $(\partial u/\partial \nu)(x) \leq p \cdot \nu \leq |p| < 1$, a contradiction with (8.14). It follows that $x \in \Omega$ and, therefore, that x is an interior minimum of the function $u(y) - p \cdot y$. In particular, $p = \nabla u(x)$ and $x \in \Gamma_u$. Thus $B_1 \subset \nabla u(\Gamma_u)$. Intersecting now both sides of this inclusion with $\mathbb{R}^n_*$, claim (8.17) follows. It is interesting to visualize geometrically the proof of the claim (8.17), by considering the graphs of the functions $p \cdot y + c$ for $c \in \mathbb{R}$. These are parallel hyperplanes which lie, for c close to $-\infty$, below the graph of u. We let c increase and consider the first c for which there is contact or "touching" at a point x. It is clear geometrically that $x \notin \partial\Omega$, since $|p| < 1$ and $\partial u/\partial \nu = 1$ on $\partial\Omega$.

Now, from (8.17) we deduce

$$\begin{aligned} m(B_1^*) &\leq \int_{\nabla u(\Gamma_u^*)} p^A dp \leq \int_{\Gamma_u^*} (\nabla u(x))^A \det D^2 u(x) dx \\ &= \int_{\Gamma_u^*} \frac{(\nabla u(x))^A}{x^A} \det D^2 u(x) x^A dx. \end{aligned} \tag{8.18}$$

We have applied the area formula to the smooth map $\nabla u : \Gamma_u^* \to \mathbb{R}^n$, and we have used that its Jacobian, $\det D^2 u$, is nonnegative in Γ_u by definition of this set.

We use now the weighted version of the arithmetic-geometric mean inequality,

$$w_1^{\lambda_1} \cdots w_m^{\lambda_m} \leq \left(\frac{\lambda_1 w_1 + \cdots + \lambda_m w_m}{\lambda_1 + \cdots + \lambda_m} \right)^{\lambda_1 + \cdots + \lambda_m}. \tag{8.19}$$

Here λ_i and w_i are arbitrary nonnegative numbers. To prove this inequality, take logarithms on both sides and use the concavity of the logarithm. We apply (8.19) to the numbers $w_i = u_{x_i}/x_i$ and $\lambda_i = A_i$ for $i = 1, ..., k$, and to the eigenvalues of $D^2u(x)$ and $\lambda_j = 1$ for $j = k+1, ..., k+n$. They are all nonnegative when $x \in \Gamma_u^*$. We obtain

$$\left(\frac{u_{x_1}}{x_1}\right)^{A_1} \cdots \left(\frac{u_{x_k}}{x_k}\right)^{A_k} \det D^2u \leq \left(\frac{A_1\frac{u_{x_1}}{x_1} + \cdots + A_k\frac{u_{x_k}}{x_k} + \Delta u}{A_1 + \cdots + A_k + n}\right)^{A_1+\cdots+A_k+n} \quad \text{in } \Gamma_u^*.$$

This, combined with (8.16)

$$A_1\frac{u_{x_1}}{x_1} + \cdots + A_k\frac{u_{x_k}}{x_k} + \Delta u = \frac{\div(x^A\nabla u)}{x^A} \equiv b_\Omega,$$

yields

$$\int_{\Gamma_u^*} \frac{(\nabla u(x))^A}{x^A} \det D^2u(x)x^A dx \leq \int_{\Gamma_u^*} \left(\frac{b_\Omega}{D}\right)^D x^A dx.$$

Therefore, by (8.18) and (8.15),

$$m(B_1^*) \leq \left(\frac{P(\Omega)}{Dm(\Omega)}\right)^D m(\Gamma_u^*) \leq \left(\frac{P(\Omega)}{Dm(\Omega)}\right)^D m(\Omega).$$

Thus, we conclude that

$$Dm(B_1^*)^{\frac{1}{D}} \leq \frac{P(\Omega)}{m(\Omega)^{\frac{D-1}{D}}}. \tag{8.20}$$

Finally, an easy computation — using that $|x|^2/2$ solves (8.13) with $b_\Omega = D$ in $\Omega = B_1^*$ — gives $P(B_1^*) = Dm(B_1^*)$. Thus,

$$Dm(B_1^*)^{\frac{1}{D}} = P(B_1^*)/m(B_1^*)^{\frac{D-1}{D}} \tag{8.21}$$

and the isoperimetric inequality (8.7) follows. □

Remark 8.2.1. An alternative (and more instructive) way to finish the proof goes as follows. When $\Omega = B_1^*$ we consider $u(x) = |x|^2/2$ and $\Gamma_u = B_1^*$. Now, $\partial u/\partial\nu = 1$ is only satisfied on $\mathbb{R}^n_* \cap \partial\Omega$ but, since $x^A \equiv 0$ on $\partial\mathbb{R}^n_* \cap \partial\Omega$, we have $b_{B_1^*} = P(B_1^*)/m(B_1^*)$ — as in (8.15). This is because $|x|^2/2$ solves problem $\div(x^A\nabla u) = b_\Omega x^A$ in Ω, $x^A u_\nu = x^A$ on $\partial\Omega$ for $\Omega = B_1^*$. For these concrete Ω and u one verifies that all inequalities in the proof are equalities, and therefore from (8.20) we deduce the isoperimetric inequality (8.7).

Remark 8.2.2. The fact that $P(\Omega)/m(\Omega)^{\frac{D-1}{D}} \geq c$ for some nonoptimal constant c is an interesting consequence of the following result of A. Grigor'yan [161] (see also [219]).

We say that a manifold M satisfies the m-isoperimetric inequality if there exists a positive constant c such that $\mu(\partial\Omega) \geq c\mu(\Omega)^{\frac{m-1}{m}}$ for each $\Omega \subset M$. In [161], he proved that if M_1 and M_2 are manifolds that satisfy the m_1-isoperimetric and m_2-isoperimetric inequalities, respectively, then the product manifold $M_1 \times M_2$ satisfies the $(m_1 + m_2)$-isoperimetric inequality. By applying this result to $M_i = (\mathbb{R}, x_i^{A_i}dx_i)$, this allows us to reduce the problem to $n = 1$, and in this case the isoperimetric inequality is easy to verify.

8.3 Weighted Sobolev inequality

The aim of this section is to prove the Sobolev inequality with a monomial weight, that is, part (a) of Theorem 8.1.3.

As in the classical inequality in $\mathbb{R}^n$, we can deduce any weighted Sobolev inequality from the isoperimetric inequality with the same weight via the coarea formula. Moreover, if the isoperimetric inequality has the sharp constant then this procedure gives the optimal constant for the Sobolev inequality when the exponent is $p = 1$ (see the following proof and also Remark 8.3.1). This classical argument is valid even on Riemannian manifolds; see for example [91]. We use it to prove part (a) of Theorem 8.1.3.

Proof of Theorem 8.1.3 (a). We prove first the case $p = 1$. By density arguments, we can assume $u \geq 0$ and also $u \in C_c^\infty(\mathbb{R}^n)$. Moreover, by approximation we can suppose $u \in C_c^\infty(\mathbb{R}^n_*)$. Indeed, consider $\tilde{u}_\varepsilon = u\eta_\varepsilon$, where $\eta_\varepsilon \in C_c^\infty(\mathbb{R}^n_*)$ is a function satisfying $\eta_\varepsilon \equiv 1$ in the set $\{x_i > \varepsilon \text{ whenever } A_i > 0\}$ and $|\nabla \eta_\varepsilon| \leq C/\varepsilon$. Then, it is clear that

$$\|u\eta_\varepsilon\|_{L^{\frac{D}{D-1}}(\mathbb{R}^n_*, x^A dx)} \longrightarrow \|u\|_{L^{\frac{D}{D-1}}(\mathbb{R}^n_*, x^A dx)}$$

as $\varepsilon \to 0$. Moreover,

$$\|\nabla \eta_\varepsilon\|_{L^1(\mathbb{R}^n_*, x^A dx)} \leq \sum_{A_i > 0} \int_{\{0 \leq x_i \leq \varepsilon\}} \frac{C}{\varepsilon} x^A dx \leq \sum_{A_i > 0} C\varepsilon^{A_i} \longrightarrow 0,$$

and thus

$$\|\nabla(u\eta_\varepsilon)\|_{L^1(\mathbb{R}^n_*, x^A dx)} \longrightarrow \|\nabla u\|_{L^1(\mathbb{R}^n_*, x^A dx)}.$$

Thus, we now have $u \in C_c^\infty(\mathbb{R}^n_*)$. For each $t \geq 0$, define

$$\{u > t\} := \{x \in \mathbb{R}^n_* \,:\, u(x) > t\} \quad \text{and} \quad \{u = t\} := \{x \in \mathbb{R}^n_* \,:\, u(x) = t\}.$$

By Theorem 8.1.4 and Sard's Theorem, we have

$$m(\{u > t\})^{\frac{D-1}{D}} \leq C_1 P(\{u > t\}) = C_1 \int_{\{u=t\}} x^A d\sigma \tag{8.22}$$

for almost all t (those t for which $\{u = t\}$ is smooth). Here, C_1 is the optimal constant in (8.7), i.e., recalling (8.21)

$$C_1 = \frac{P(B_1^*)}{m(B_1^*)^{\frac{D-1}{D}}} = D m(B_1^*)^{\frac{1}{D}}. \tag{8.23}$$

Letting χ_A be the characteristic function of the set A, we have

$$u(x) = \int_0^{+\infty} \chi_{\{u(x) > \tau\}} d\tau.$$

Thus, by Minkowski's integral inequality

$$\begin{aligned} \left(\int_{\mathbb{R}^n_*} x^A u^{\frac{D}{D-1}} dx \right)^{\frac{D-1}{D}} &\leq \int_0^{+\infty} \left(\int_{\mathbb{R}^n_*} \chi_{\{u(x) > \tau\}} x^A dx \right)^{\frac{D-1}{D}} d\tau \\ &= \int_0^{+\infty} m(\{u > \tau\})^{\frac{D-1}{D}} d\tau. \end{aligned}$$

Inequality (8.22), together with the coarea formula, yield

$$\left(\int_{\mathbb{R}^n_*} x^A u^{\frac{D}{D-1}} dx\right)^{\frac{D-1}{D}} \leq c_0 \int_0^{+\infty} \int_{\{u=t\}} x^A d\sigma \, d\tau = c_0 \int_{\mathbb{R}^n_*} x^A |\nabla u| dx,$$

and the theorem is proved for $p = 1$.

It remains to prove the case $1 < p < D$. Take $u \in C^1_c(\mathbb{R}^n)$, and define $v = |u|^\gamma$, where $\gamma = \frac{p_*}{1_*}$. Since, $\gamma > 1$, we have $v \in C^1_c(\mathbb{R}^n)$, and we can apply the weighted Sobolev inequality with exponent $p = 1$ (proved above) to get

$$\left(\int_{\mathbb{R}^n_*} x^A |u|^{p_*} dx\right)^{1/1_*} = \left(\int_{\mathbb{R}^n_*} x^A |v|^{\frac{D}{D-1}} dx\right)^{\frac{D-1}{D}} \leq c_0 \int_{\mathbb{R}^n_*} x^A |\nabla v| dx.$$

Now, $|\nabla v| = \gamma |u|^{\gamma-1} |\nabla u|$, and by Hölder's inequality we deduce

$$\int_{\mathbb{R}^n_*} x^A |\nabla v| dx \leq C \left(\int_{\mathbb{R}^n_*} x^A |\nabla u|^p dx\right)^{1/p} \left(\int_{\mathbb{R}^n_*} x^A |u|^{(\gamma-1)p'} dx\right)^{1/p'}.$$

Finally, from the definition of γ and p_* it follows that

$$\frac{1}{1_*} - \frac{1}{p'} = \frac{1}{p_*}, \qquad (\gamma - 1)p' = p_*,$$

and hence,

$$\left(\int_{\mathbb{R}^n_*} x^A |u|^{p_*} dx\right)^{1/p_*} \leq C \left(\int_{\mathbb{R}^n_*} x^A |\nabla u|^p dx\right)^{1/p}.$$

□

Remark 8.3.1. Since the constant appearing in (8.22) is optimal, this proof gives the optimal constant for the weighted Sobolev inequality for $p = 1$. This is because for each Lipschitz open set E there exists an increasing sequence of smooth functions $u_\varepsilon \to \chi_E$, such that $\|\nabla u_\varepsilon\|_{L^1(\mathbb{R}^n_*, x^A dx)} \to P(E)$.

Moreover, for $p = 1$ it follows from the previous proof (in fact from the use of Minkowski's inequality) that if equality is attained by a function u, then all the sets $\{u > t\}$ must coincide for $t \in (0, \max u)$. That is, the extremal function must be a characteristic function. This proves that the optimal constant is not attained by any $W^{1,1}_0(\mathbb{R}^n, x^A dx)$ function for $p = 1$.

We give now an alternative and short proof of part (a) of Theorem 8.1.3 — without best constant — under some additional assumptions. Indeed, under the hypotheses $A_i > 0$ for all i and $u_{x_i} \leq 0$ in $\{x_i > 0,\ i = 1, ..., n\}$, we establish the weighted Sobolev inequality (8.5) following the ideas used in [49] to prove the isoperimetric inequality in dimension $n = 2$ (without best constant) with the weight $\sigma^a \tau^b$. The following proof is much more elementary than the previous one, which used the weighted isoperimetric inequality. It does not use any elliptic problem nor the coarea formula, and it is also shorter. However, it does not give the best constant in the inequality, even for $p = 1$. The monotonicity hypotheses $u_{x_i} \leq 0$ in $\{x_i > 0,\ i = 1, ..., n\}$ are equivalent to (8.2) in Proposition 8.3. As said before, the weighted Sobolev inequality under these monotonicity assumptions suffices for some applications to nonlinear problems.

Proposition 8.3.2. *Let A be a positive vector in $\mathbb{R}^n$ and $1 \le p < D$ be a real number. Then, there exists a constant C such that for all $u \in C_c^1(\mathbb{R}^n)$ satisfying*

$$u_{x_i} \le 0 \quad in \quad (\mathbb{R}_+)^n \quad for \quad i = 1, ..., n, \tag{8.24}$$

we have

$$\left(\int_{(\mathbb{R}_+)^n} x^A |u|^{p_*} dx \right)^{1/p_*} \le C \left(\int_{(\mathbb{R}_+)^n} x^A |\nabla u|^p dx \right)^{\frac{1}{p}},$$

where $p_* = \frac{pD}{D-p}$ and $D = A_1 + \cdots + A_n + n$.

Proof. It suffices to prove the case $p = 1$, since the inequality for $1 < p < D$ follows from it by applying Hölder's inequality — see the previous proof of Theorem 8.1.3 (a).

From assumption (8.24), we deduce $u \ge 0$ in $(\mathbb{R}_+)^n$. Now, integrating by parts we have

$$\begin{aligned} \int_{(\mathbb{R}_+)^n} x^A (|u_{x_i}| + \cdots + |u_{x_n}|) dx &= - \int_{(\mathbb{R}_+)^n} x^A (u_{x_1} + \cdots + u_{x_n}) dx \\ &= \int_{(\mathbb{R}_+)^n} x^A u \left(\frac{A_1}{x_1} + \cdots + \frac{A_n}{x_n} \right) dx, \end{aligned}$$

and thus

$$\int_{(\mathbb{R}_+)^n} x^A u \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right) dx \le K \int_{(\mathbb{R}_+)^n} x^A |\nabla u| dx, \tag{8.25}$$

where $K = \sqrt{n} / \min_i A_i$.

Let now $\lambda > 0$ be such that

$$\int_{(\mathbb{R}_+)^n} x^A u^{\frac{D}{D-1}} dx = b\lambda^D,$$

where $b = \int_{\{0 \le x_i \le 1\}} x^A dx$. Here $\{0 \le x_i \le 1\} = \{x \in \mathbb{R}^n : 0 \le x_i \le 1 \text{ for } i = 1, ..., n\}$.

We claim that, for each $x \in (\mathbb{R}_+)^n$ we have $u(x)^{\frac{1}{D-1}} \le \frac{\lambda}{x_i}$ for some $i \in \{1, ..., n\}$. Indeed, otherwise there would exist $y \in (\mathbb{R}_+)^n$ such that $u(y)^{\frac{1}{D-1}} > \frac{\lambda}{y_i}$ for each i, and therefore

$$u(y)^{\frac{D}{D-1}} > \frac{\lambda^D}{y^{A+1}},$$

where $A + 1 = A + (1, ..., 1) = (A_1 + 1, ..., A_n + 1)$. But, by (8.24), $u(x) \ge u(y)$ if $0 \le x_i \le y_i$ for all $i = 1, ..., n$. We deduce

$$\int_{\{0 \le x_i \le y_i\}} x^A u(x)^{\frac{D}{D-1}} dx > \lambda^D \int_{\{0 \le x_i \le y_i\}} x^A y^{-A-1} dx = \lambda^D \int_{\{0 \le z_i \le 1\}} z^A dz = b\lambda^D,$$

a contradiction.

Hence,

$$u(x)^{\frac{1}{D-1}} \le \lambda \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right) \qquad \text{in } (\mathbb{R}_+)^n,$$

and therefore

$$\int_{(\mathbb{R}_+)^n} x^A u^{\frac{D}{D-1}} dx \leq \lambda \int_{(\mathbb{R}_+)^n} x^A u \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right) dx. \tag{8.26}$$

Finally, taking into account the value of λ

$$\lambda = b^{-\frac{1}{D}} \left(\int_{(\mathbb{R}_+)^n} x^A u^{\frac{D}{D-1}} dx \right)^{\frac{1}{D}},$$

we deduce from (8.26) and (8.25) that

$$\begin{aligned} \left(\int_{(\mathbb{R}_+)^n} x^A u^{\frac{D}{D-1}} dx \right)^{\frac{D-1}{D}} &\leq b^{-\frac{1}{D}} \int_{(\mathbb{R}_+)^n} x^A u \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right) dx \\ &\leq K b^{-\frac{1}{D}} \int_{(\mathbb{R}_+)^n} x^A |\nabla u| dx. \end{aligned}$$

This completes the proof and gives as constant $Kb^{-\frac{1}{D}}$, computed explicitly within the proof. □

This proof can not be used to establish the classical Sobolev inequality. Indeed, the constant on the right hand side blows up as $A_i \to 0$ for some i. It is surprising that the above proof of the Sobolev inequality with the monomial weight x^A, $A > 0$, seems more elementary than those of the classical Sobolev without weight.

The following remark justifies our assumption $A \geq 0$ in the weighted Sobolev inequality (8.5). It is related to the monotonicity assumption (8.2) in Proposition 8.3.

Remark 8.3.3. When $a < 0$ or $b < 0$ inequality (8.3) is not valid without the monotonicity assumption (8.2). To prove it, we only need to take functions u with support away from the origin, as follows. Assume that $a < 0$, $a + b > 0$ (and thus $b > 0$), and that (8.3) holds for functions u with support in the ball $B_1(x_0)$, with $x_0 = (2, 0)$. Then, since σ^a is bounded in this ball from above and below by positive constants, the same inequality holds — with a larger constant C — with the weight $\sigma^a \tau^b$ replaced by τ^b. But, since $a < 0$, we have $q' := \frac{2D'}{D'-2} < \frac{2D}{D-2}$, where $D' = b + 2$. This is a contradiction with the fact that the exponent q' is optimal for the weight τ^b (which can be seen by a scaling argument, i.e., considering the rescaled functions $u_\lambda(x) = u(x_0 + \lambda(x - x_0))$, with $\lambda \geq 1$). Of course, when a and b are both nonnegative this argument does not work.

Remark 8.3.4. One can think on adapting the classical proof of the Sobolev inequality by Gagliardo and Nirenberg (see for example [124]) to the case of monomial weights. As we show next, this leads to inequality

$$\left(\int_{\mathbb{R}^n} x^A |u|^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} x^{\frac{n-1}{n}A} |\nabla u| dx, \tag{8.27}$$

but not to our Sobolev inequality (8.5) with the same weight x^A in both integrals. The constant C (which does not appear) on the right hand side equals 1. To prove (8.27), one shows first that

$$|x_i|^{\frac{n-1}{n}A_i} |u(x)| \leq \int_{\mathbb{R}} |y_i|^{\frac{n-1}{n}A_i} |\nabla u(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)| dy_i. \tag{8.28}$$

This follows by integrating u_{y_i} on $(x_i, +\infty)$ if $x_i > 0$ and on $(-\infty, x_i)$ if $x_i < 0$, and using $|x_i| \leq |y_i|$ in these halflines. Then, (8.28) yields

$$|x_1|^{\frac{A_1}{n}} \cdots |x_n|^{\frac{A_n}{n}} |u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{+\infty} |\nabla u(x_1, ..., y_i, ..., x_n)| |y_i|^{\frac{n-1}{n} A_i} dy_i \right)^{\frac{1}{n-1}}.$$

Integrating both sides with respect to the measure $x^{\frac{n-1}{n}A} dx$ we deduce

$$\int_{\mathbb{R}^n} x^A |u(x)|^{\frac{n}{n-1}} dx \leq \int_{\mathbb{R}^n} \prod_{i=1}^{n} \left(\int_{-\infty}^{+\infty} |\nabla u(x_1, ..., y_i, ..., x_n)| |y_i|^{\frac{n-1}{n} A_i} dy_i \right)^{\frac{1}{n-1}} x^{\frac{n-1}{n}A} dx,$$

and the proof of (8.27) is completed in the same way as the classical one with the measures dx_i and dy_i replaced by $d\mu_i(x_i) = |x_i|^{\frac{n-1}{n} A_i} dx_i$ and $d\mu_i(y_i) = |y_i|^{\frac{n-1}{n} A_i} dy_i$.

Different from (8.5), inequality (8.27) is the Sobolev inequality for the Riemannian manifold conformal to $\mathbb{R}^n$ with conformal factor $g = x^A$. Indeed, the Riemannian gradient in $\mathbb{R}^n$ with this metric is given by $\nabla_R u = x^{-\frac{A}{n}} \nabla u$, and hence it holds

$$x^{\frac{n-1}{n}A} |\nabla u| = x^A |\nabla_R u|.$$

Moreover, from this Sobolev inequality one can deduce the following isoperimetric inequality (with nonoptimal constant) on this manifold

$$\left(\int_\Omega x^A dx \right)^{\frac{n-1}{n}} \leq \int_{\partial\Omega} x^{\frac{n-1}{n}A} d\sigma.$$

To end this section, we give an immediate consequence of Theorem 8.1.3. Recall that in [49] we wanted to prove inequality (8.29) for $n = 2$ and that, after a change of variables, we saw that it is equivalent to the Sobolev inequality (8.5) with a monomial weight.

Corollary 8.3.5. *Let $\alpha_1, ..., \alpha_n$ be real numbers such that $\alpha_i \in [0, 1)$. There exists a constant C such that for all $u \in C_c^1(\mathbb{R}^n)$,*

$$\left(\int_{\mathbb{R}^n_*} |u|^{p_*} dx \right)^{\frac{1}{p_*}} \leq C \left(\int_{\mathbb{R}^n_*} \left\{ |x_1|^{p\alpha_1} |u_{x_1}|^p + \cdots + |x_n|^{p\alpha_n} |u_{x_n}|^p \right\} dx \right)^{\frac{1}{p}}, \tag{8.29}$$

where $p_* = \frac{pD}{D-p}$ and $D = n + \frac{\alpha_1}{1-\alpha_1} + \cdots + \frac{\alpha_n}{1-\alpha_n}$.

Proof. It suffices to make the change of variables $y_i = x_i^{1-\alpha_i}$ in (8.29) and then apply Theorem 8.1.3 with $A_i = \frac{\alpha_i}{1-\alpha_i}$. □

The optimal exponent in (8.29) is $p_* = \frac{pD}{D-p}$, as in (8.5). However, in (8.29) the constant D has no clear interpretation in terms of any "dimension".

8.4 Best constant and extremal functions

In this section we obtain the best constant and the extremal functions in the weighted Sobolev inequality (8.5).

The first step is to compute the measure of the unit ball in $\mathbb{R}^n_*$ with the weight x^A. From this, we will obtain the optimal constant in the isoperimetric inequality and, therefore, the optimal constant in Sobolev inequality for $p = 1$ (see Remark 8.3.1).

Lemma 8.4.1. *Let A be a nonnegative vector in $\mathbb{R}^n$ and $B_1^* = B_1(0) \cap \mathbb{R}^n_*$. Then,*

$$\int_{B_1^*} x^A dx = \frac{\Gamma\left(\frac{A_1+1}{2}\right)\Gamma\left(\frac{A_2+1}{2}\right)\cdots\Gamma\left(\frac{A_n+1}{2}\right)}{2^k\Gamma\left(1+\frac{D}{2}\right)},$$

where $D = A_1 + \cdots + A_n + n$ and k is the number of strictly positive entries of A.

Proof. We will prove by induction on n that

$$\int_{B_1} x^A dx = \frac{\Gamma\left(\frac{A_1+1}{2}\right)\Gamma\left(\frac{A_2+1}{2}\right)\cdots\Gamma\left(\frac{A_n+1}{2}\right)}{\Gamma\left(1+\frac{D}{2}\right)},$$

where B_1 is the unit ball in $\mathbb{R}^n$. After this, the the lemma follows by taking into account that $m(B_1^*) = m(B_1)/2^k$.

For $n = 1$ it is immediate. Assume that this is true for $n-1$ and let us prove it for n. Let us denote $x = (x', x_n)$, $A = (A', A_n)$, with $x', A' \in \mathbb{R}^{n-1}$, and $D' = A_1 + \cdots + A_{n-1} + n - 1$. Then,

$$\begin{aligned}
\int_{B_1} x^A dx &= \int_{-1}^{1} |x_n|^{A_n} \left(\int_{|x'|\le\sqrt{1-x_n^2}} x'^{A'} dx' \right) dx_n \\
&= \int_{-1}^{1} |x_n|^{A_n} \left((1-x_n^2)^{\frac{D'}{2}} \int_{|y'|\le 1} y'^{A'} dy' \right) dx_n \\
&= \int_{|y'|\le 1} y'^{A'} dy' \int_{-1}^{1} |x_n|^{A_n} (1-x_n^2)^{\frac{D'}{2}} dx_n,
\end{aligned}$$

and hence it remains to compute $\int_{-1}^{1} |x_n|^{A_n}(1-x_n^2)^{\frac{D'}{2}} dx_n$.

Making the change of variables $x_n^2 = t$ one obtains

$$\begin{aligned}
\int_{-1}^{1} |x_n|^{A_n}(1-x_n^2)^{\frac{D'}{2}} dx_n &= 2\int_0^1 x_n^{A_n}(1-x_n^2)^{\frac{D'}{2}} dx_n \\
&= \int_0^1 t^{\frac{A_n-1}{2}}(1-t)^{\frac{D'}{2}} dt \\
&= B\left(\frac{A_n+1}{2}, 1+\frac{D'}{2}\right),
\end{aligned}$$

where B is the Beta function. Now, since

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \tag{8.30}$$

then

$$\begin{aligned}\int_{B_1} x^A dx &= \int_{|y'|\le 1} y'^{A'} dy' \int_{-1}^{1} x_n^{A_n} (1-x_n^2)^{\frac{D'}{2}} dx_n \\ &= \frac{\Gamma\left(\frac{A_1+1}{2}\right)\cdots\Gamma\left(\frac{A_{n-1}+1}{2}\right)}{\Gamma\left(1+\frac{D'}{2}\right)} \cdot \frac{\Gamma\left(\frac{A_n+1}{2}\right)\Gamma\left(1+\frac{D'}{2}\right)}{\Gamma\left(1+\frac{D}{2}\right)} \\ &= \frac{\Gamma\left(\frac{A_1+1}{2}\right)\Gamma\left(\frac{A_2+1}{2}\right)\cdots\Gamma\left(\frac{A_n+1}{2}\right)}{\Gamma\left(1+\frac{D}{2}\right)},\end{aligned}$$

and the lemma follows. □

Now, as in the classical Sobolev inequality, we find the extremal functions in our weighted Sobolev inequality by reducing it to the radial case. To do this, we use a weighted version of a rearrangement inequality due to Talenti [282]. His result states that, whenever balls minimize the isoperimetric quotient with a weight w, there exists a radial rearrangement (of u) which preserves $\int f(u)w\,dx$ and decreases $\int \Phi(|\nabla u|)w\,dx$ (under some conditions on Φ). When $w = x^A$, this is stated in the following.

Proposition 8.4.2. *Let u be a Lipschitz continuous function in $\mathbb{R}^n_*$ with compact support in $\overline{\mathbb{R}^n_*}$. Then, denoting $m(E) = \int_E x^A dx$, there exists a radial rearrangement u_* of u such that*

(i) $m(\{|u| > t\}) = m(\{u_* > t\})$ *for all* t,

(ii) u_* *is radially decreasing,*

(iii) for every Young function Φ *(i.e., convex and increasing function that vanishes at* 0*),*

$$\int_{\mathbb{R}^n_*} \Phi(|\nabla u_*|) x^A dx \le \int_{\mathbb{R}^n_*} \Phi(|\nabla u|) x^A dx.$$

Proof. It is a direct consequence of the main theorem in [282] and our isoperimetric inequality (8.7). □

We can now find the best constant in the weighted Sobolev inequality (8.5). The proof is based on Proposition 8.4.2, which allows us to reduce the problem to radial functions in $\mathbb{R}^n_*$. Then, the functional that we must minimize is exactly the same as in the classical Sobolev inequality but with a noninteger exponent D in the 1D weight, and the proof finishes by applying another result of Talenti in [283].

Proposition 8.4.3. *The best constant in the Sobolev inequality* (8.5) *is given by*

$$C_1 = D\left(\frac{\Gamma\left(\frac{A_1+1}{2}\right)\Gamma\left(\frac{A_2+1}{2}\right)\cdots\Gamma\left(\frac{A_n+1}{2}\right)}{2^k\Gamma\left(1+\frac{D}{2}\right)}\right)^{\frac{1}{D}} \qquad \text{for } p=1 \tag{8.31}$$

and by

$$C_p = C_1 D^{\frac{1}{D}-1-\frac{1}{p}} \left(\frac{p-1}{D-p}\right)^{\frac{1}{p'}} \left(\frac{p'\Gamma(D)}{\Gamma\left(\frac{D}{p}\right)\Gamma\left(\frac{D}{p'}\right)}\right)^{\frac{1}{D}} \qquad \text{for } 1<p<D. \tag{8.32}$$

Here, $p' = \frac{p}{p-1}$ and k is the number of positive entries in the vector A.

Moreover, this constant is not attained by any function in $W_0^{1,1}(\mathbb{R}^n, x^A dx)$ when $p = 1$. Instead, when $1 < p < D$ this constant is attained in $W_0^{1,p}(\mathbb{R}^n, x^A dx)$ by

$$u_{a,b}(x) = \left(a + b|x|^{\frac{p}{p-1}}\right)^{1-\frac{D}{p}},$$

where a and b are arbitrary positive constants.

Before giving the proof of Proposition 8.4.3, we recall Lemma 2 from [283], where the best constant for the classical Sobolev inequality is obtained.

Lemma 8.4.4 ([283])**.** *Let m, p, and q be real numbers such that*

$$1 < p < m \qquad \text{and} \qquad q = \frac{mp}{m-p}.$$

Let u be any real-valued function of a real variable r, which is Lipschitz continuous and such that

$$\int_0^{+\infty} r^{m-1}|u'(r)|^p dr < +\infty \qquad \text{and} \qquad u(r) \to 0 \quad \text{as} \quad r \to +\infty.$$

Then,

$$\frac{\left(\int_0^{+\infty} r^{m-1}|u(r)|^q dr\right)^{\frac{1}{q}}}{\left(\int_0^{+\infty} r^{m-1}|u'(r)|^p dr\right)^{\frac{1}{p}}} \leq \frac{\left(\int_0^{+\infty} r^{m-1}|\varphi(r)|^q dr\right)^{\frac{1}{q}}}{\left(\int_0^{+\infty} r^{m-1}|\varphi'(r)|^p dr\right)^{\frac{1}{p}}} =: J(\varphi),$$

where φ is any function of the form

$$\varphi(r) = (a + br^{p'})^{1-\frac{m}{p}}$$

with a and b positive constants. Here $p' = p/(p-1)$.

Moreover,

$$J(\varphi) = m^{-\frac{1}{p}} \left(\frac{p-1}{m-p}\right)^{\frac{1}{p'}} \left[\frac{1}{p'} B\left(\frac{m}{p}, \frac{m}{p'}\right)\right]^{-\frac{1}{m}},$$

where B is the Beta function.

We can now give the

Proof of Proposition 8.4.3. For $p = 1$, the best constant in Sobolev inequality is the same than in the isoperimetric inequality (see Remark 8.3.1). Recalling (8.23), it is given by $C_1 = Dm(B_1^*)^{1/D}$. Thus, the value of C_1 follows from Lemma 8.4.1. That C_1 is not attained by any $W_0^{1,1}(\mathbb{R}^n, x^A dx)$ function was explained in Remark 8.3.1.

Let now $1 < p < D$, u be a $C^1(\overline{\mathbb{R}^n_*})$ function with compact support in $\overline{\mathbb{R}^n_*}$, and u_* be its radial rearrangement given by Proposition 8.4.2. Then, by the proposition,

$$\frac{\|\nabla u_*\|_{L^p(\mathbb{R}^n_*, x^A dx)}}{\|u_*\|_{L^{p_*}(\mathbb{R}^n_*, x^A dx)}} \leq \frac{\|\nabla u\|_{L^p(\mathbb{R}^n_*, x^A dx)}}{\|u\|_{L^{p_*}(\mathbb{R}^n_*, x^A dx)}}.$$

Moreover,

$$\begin{aligned}\int_{\mathbb{R}^n_*} x^A|u_*|^{p_*}dx &= \int_0^\infty \left(\int_{\partial B_r^*} x^A|u_*|^{p_*}d\sigma\right)dr \\ &= \int_0^\infty r^{D-1}|u_*|^{p_*}\left(\int_{\partial B_1^*} x^A d\sigma\right)dr \\ &= P(B_1^*)\int_0^\infty r^{D-1}|u_*|^{p_*}dr\end{aligned}$$

and, analogously,

$$\int_{\mathbb{R}^n_*} x^A|\nabla u_*|^p dx = P(B_1^*)\int_0^\infty r^{D-1}|u_*'|^p dr.$$

Therefore, the best constant in the Sobolev inequality can be computed as

$$\inf_{u\in C_c^1(\mathbb{R}^n)}\frac{\|\nabla u\|_{L^p(\mathbb{R}^n_*,x^A dx)}}{\|u\|_{L^{p_*}(\mathbb{R}^n_*,x^A dx)}} = P(B_1^*)^{\frac{1}{D}}\inf_{u\in C_c^1(\mathbb{R})}\frac{\left(\int_0^\infty r^{D-1}|u'|^p dr\right)^{1/p}}{\left(\int_0^\infty r^{D-1}|u|^{p_*}dr\right)^{1/p_*}},$$

where we have used that $\frac{1}{p}-\frac{1}{p_*}=\frac{1}{D}$. Recalling (8.21) and (8.23), we have

$$P(B_1^*)^{\frac{1}{D}} = D^{\frac{1}{D}}m(B_1^*)^{\frac{1}{D}} = D^{\frac{1}{D}-1}C_1.$$

The value of C_p follows from Lemma 8.4.4, using (8.31) and (8.30). From Lemma 8.4.4 it also follows that the functions $u_{a,b}$ in (8.6) attain the best constant C_p. □

To end this section, we prove part (b) of Theorem 8.1.3.

Proof of Theorem 8.1.3 (b). For $p=1$ this was proved in Section 8.3; see Remark 8.3.1. For $p>1$ the result is proved in Proposition 8.4.3. □

8.5 Weighted Morrey inequality

In this section we prove Theorem 8.1.6. The main ingredient to establish the result is the following lemma.

Lemma 8.5.1. *Let A be a nonnegative vector in $\mathbb{R}^n$ and $D=A_1+\cdots+A_n+n$. Let $u\in C_c^1(\mathbb{R}^n)$ and $y\in\mathbb{R}^n_*$. Then,*

$$|u(y)-u(0)| \le C\int_{B^*_{2|y|}}\frac{|\nabla u(x)|}{|x|^{D-1}}x^A dx,$$

where $B^*_{2|y|}=B_{2|y|}(0)\cap\mathbb{R}^n_*$ and C is a constant depending only on D.

Proof. By symmetry, we can assume that $A=(A_1,...,A_k,0,...,0)$ with $A_i>0$ for all $i=1,...,k$.

Let us define $B_i=\lceil A_i\rceil$ (the smallest integer greater than or equal to A_i), $B=(B_1,...,B_n)$, and $N=B_1+\cdots+B_k+n$. For each Lipschitz function u in $\mathbb{R}^n$, define $\tilde{u}$ in $\mathbb{R}^N=\mathbb{R}^{B_1+1}\times\cdots\times\mathbb{R}^{B_k+1}\times\mathbb{R}^{n-k}$ as

$$\tilde{u}(X)=u(|X_1|,...,|X_k|,X_{k+1}),$$

with $X_i \in \mathbb{R}^{B_i+1}$ for $i = 1, ..., k$ and $X_{k+1} \in \mathbb{R}^{n-k}$. Notice that $\tilde{u} \in Lip(\mathbb{R}^N)$.

We use next the following classical inequality in $\mathbb{R}^N$ (see for example Lemma 7.16 in [157]). For all X and Y in $\mathbb{R}^N$,

$$|\tilde{u}(Y) - \tilde{u}(X)| \leq C \int_{B_{2R}(X)} \frac{|\nabla \tilde{u}(Z)|}{|Z|^{N-1}} dZ, \tag{8.33}$$

where $R = |X - Y|$. Setting $X = 0$ in (8.33) and writing the integral over $\mathbb{R}^N$ in radial coordinates — as in Remark 8.1.2 —, we deduce

$$|u(y) - u(0)| \leq C \int_{B_{2|y|}(0)} \frac{|\nabla \tilde{u}(Z)|}{|Z|^{N-1}} dZ = \tilde{C} \int_{B^*_{2|y|}} \frac{|\nabla u(z)|}{|z|^{N-1}} z^B dz. \tag{8.34}$$

It is important here to have $X = 0$, otherwise the inequality over $\mathbb{R}^N$ can not be written in radial coordinates as an integral over $\mathbb{R}^n$. In addition, we have used also that $R = 2|Y| = 2|y|$.

Now, clearly $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \leq |x|^{\alpha_1 + \cdots + \alpha_n}$ whenever α is a nonnegative vector in $\mathbb{R}^n$ and $x \in \mathbb{R}^n_*$. Thus, taking $\alpha = B - A$ we obtain

$$\frac{x^B}{|x|^{B_1 + \cdots + B_k}} \leq \frac{x^A}{|x|^{A_1 + \cdots + A_k}}. \tag{8.35}$$

Finally, from (8.34) and (8.35) we deduce

$$|u(y) - u(0)| \leq C \int_{B^*_{2|y|}} \frac{|\nabla u(x)|}{|x|^{N-1}} x^B dx \leq C \int_{B^*_{2|y|}} \frac{|\nabla u(x)|}{|x|^{D-1}} x^A dx,$$

as desired. Note that we can choose the constant C to depend only on D, since for each D there exist only a finite number of possible integer values for $n, B_1, ..., B_n$. □

As said before, our proof of Lemma 8.5.1 requires to take $X = 0$, since otherwise one can not write (8.33) in $\mathbb{R}^N$ as an inequality in $\mathbb{R}^n$.

We can now give the:

Proof of Theorem 8.1.6. Step 1. We first prove

$$\frac{|u(y) - u(z)|}{|y - z|^{1 - \frac{D}{p}}} \leq C \left(\int_{\mathbb{R}^n_*} x^A |\nabla u|^p dx \right)^{\frac{1}{p}} \tag{8.36}$$

for $z = 0$. Let $y \in \mathbb{R}^n_*$ and $r = 2|y|$. By Lemma 8.5.1 and by Hölder's inequality, we have that

$$\begin{aligned}
|u(y) - u(0)| &\leq C \int_{B^*_r} \frac{|\nabla u|}{|x|^{D-1}} x^A dx \\
&\leq C \left(\int_{B^*_r} x^A |\nabla u|^p dx \right)^{\frac{1}{p}} \left(\int_{B^*_r} \frac{x^A}{|x|^{p'(D-1)}} dx \right)^{\frac{1}{p'}} \\
&\leq C \left(\int_{\mathbb{R}^n_*} x^A |\nabla u|^p dx \right)^{\frac{1}{p}} \left(\int_0^r \rho^{D-1-p'(D-1)} d\rho \right)^{\frac{1}{p'}} \\
&= C \left(\int_{\mathbb{R}^n_*} x^A |\nabla u|^p dx \right)^{\frac{1}{p}} r^{1 - \frac{D}{p}},
\end{aligned}$$

where $p' = p/(p-1)$ and C denotes different constants depending only on p and D. Hence, (8.36) is proved for $z = 0$ and $y \in \mathbb{R}^n_*$.

Step 2. We now prove (8.36) for y and z in $\mathbb{R}^n_*$ such that $y - z \in \mathbb{R}^n_*$. Applying the inequality proved in Step 1 to the function $v(\tilde{y}) = u(\tilde{y} + z)$, $\tilde{y} \in \mathbb{R}^n$, at the point $\tilde{y} = y - z \in \mathbb{R}^n_*$, we deduce

$$|u(y) - u(z)| \leq C \left(\int_{z+\mathbb{R}^n_*} (x-z)^A |\nabla u(x)|^p dx \right)^{\frac{1}{p}} |y-z|^{1-\frac{D}{p}},$$

where $z + \mathbb{R}^n_* = \{x \in \mathbb{R}^n : x - z \in \mathbb{R}^n_*\}$. Therefore, since $(x-z)^A \leq x^A$ if x and $x - z$ belong to $\mathbb{R}^n_*$, this case of (8.36) follows.

Step 3. We finally prove (8.36) for all y and z in $\mathbb{R}^n_*$. Define $w \in \mathbb{R}^n_*$ as $w_i = \min\{y_i, z_i\}$ for all i. Then, it is clear that $y - w \in \mathbb{R}^n_*$ and $z - w \in \mathbb{R}^n_*$. Hence, we can apply the inequality proved in Step 2 to obtain

$$|u(y) - u(w)| \leq C \left(\int_{\mathbb{R}^n_*} x^A |\nabla u|^p dx \right)^{\frac{1}{p}} |y-w|^{1-\frac{D}{p}}$$

and

$$|u(z) - u(w)| \leq C \left(\int_{\mathbb{R}^n_*} x^A |\nabla u|^p dx \right)^{\frac{1}{p}} |z-w|^{1-\frac{D}{p}}.$$

Since $|y-w|^2 + |z-w|^2 = |y-z|^2$, from these two inequalities we deduce that

$$|u(y) - u(z)| \leq 2C \left(\int_{\mathbb{R}^n_*} x^A |\nabla u|^p dx \right)^{\frac{1}{p}} |y-z|^{1-\frac{D}{p}}$$

for all $y, z \in \mathbb{R}^n_*$. This finishes the proof of (8.36).

Let us prove now (8.11). Let $x_0 \in \Omega \subset \mathbb{R}^n$ be such that $\sup_\Omega |u| = |u(x_0)|$. After a finite number of reflections with respect to the coordinate hyperplanes, we may assume that $x_0 \in \overline{\mathbb{R}^n_*}$. Call $\tilde{u}$ the function u after doing such reflections, defined in the reflected domain $\widetilde{\Omega}$. Since $\tilde{u} \equiv 0$ on $\partial\widetilde{\Omega}$, we have

$$\sup_\Omega |u| \cdot diam(\Omega)^{-1+\frac{D}{p}} = |\tilde{u}(x_0)| \cdot diam(\widetilde{\Omega})^{-1+\frac{D}{p}} \leq \sup_{x,\, y \in \mathbb{R}^n_*} \frac{|\tilde{u}(x) - \tilde{u}(y)|}{|x-y|^{1-\frac{D}{p}}}.$$

The right hand side of this inequality is now bounded using (8.10). The proof is finished controlling the integral over $\mathbb{R}^n_*$ in (8.10) by an integral over $\Omega \subset \mathbb{R}^n$. This is needed because of the reflections done initially. □

8.6 Weighted Trudinger inequality and proof of Corollary 8.1.8

In this section we prove Theorem 8.1.7 and Corollary 8.1.8. The proof of the weighted Trudinger inequality is based on a bound for the best constant of the weighted Sobolev inequality as p goes to D. Then, the result follows by expanding $\exp(\cdot)$ as a power series and using the weighted Sobolev inequality in each term. To prove the convergence of this series we need the mentioned bound, which is stated in the following result.

Lemma 8.6.1. *Let A be a nonnegative vector in $\mathbb{R}^n$, $D = A_1 + \cdots + A_n + n$, and p be such that $1 < p < D$. Let C_p be the optimal constant of the Sobolev inequality* (8.5)*, given by* (8.31)-(8.32)*. Then,*

$$C_p \leq C_0 p_*^{1-\frac{1}{D}},$$

where $p_* = \frac{pD}{D-p}$ and C_0 is a constant which depends only on D.

Proof. The optimal constant is given by

$$C_p = C_1 D^{1-\frac{1}{D}-\frac{1}{p}} \left(\frac{p-1}{D-p}\right)^{\frac{1}{p'}} \left(\frac{p'\Gamma(D)}{\Gamma\left(\frac{D}{p}\right)\Gamma\left(\frac{D}{p'}\right)}\right)^{\frac{1}{D}},$$

where $p' = p/(p-1)$ and C_1 is a constant which only depends on A and n. It is easy to see that the constant C_p is bounded as $p \downarrow 1$. Thus, we only have to look at the limit $p \uparrow D$. It follows from the above expression that

$$C_p \leq C(D-p)^{-\frac{1}{p'}},$$

where C does not depend on p. Hence, taking into account that $\frac{1}{p'} = 1 - \frac{1}{D} - \frac{1}{p_*}$ and $D - p = pD/p_*$, we deduce

$$C_p \leq C_0 p_*^{1-\frac{1}{D}-\frac{1}{p_*}} \leq C_0 p_*^{1-\frac{1}{D}}.$$

Finally, it is easy to see that C_1 — which is given by (8.31) — can be bounded by a constant depending only on D, and therefore we can choose the constant C_0 to depend only on D. □

We can now give the:

Proof of Theorem 8.1.7. Let $u \in C_c^1(\Omega)$. From Theorem 8.1.3 and Lemma 8.6.1 we deduce that

$$\int_\Omega x^A |u|^q dx \leq C_0^q q^{q-\frac{q}{D}} \left(\int_\Omega x^A |\nabla u|^{\frac{qD}{q+D}} dx\right)^{\frac{q+D}{D}}$$

for each $q > 1$, where C_0 is a constant which depends only on D. Moreover, by Hölder's inequality,

$$\int_\Omega x^A |\nabla u|^{\frac{qD}{q+D}} dx \leq \left(\int_\Omega x^A dx\right)^{\frac{D}{q+D}} \left(\int_\Omega x^A |\nabla u|^D dx\right)^{\frac{q}{q+D}},$$

and thus

$$\int_\Omega x^A |u|^q dx \leq m(\Omega) C_0^q q^{q\frac{D-1}{D}} \|\nabla u\|^q_{L^D(\Omega, x^A dx)}. \tag{8.37}$$

Now, dividing the function u by some constant if necessary, we can assume

$$\|\nabla u\|_{L^D(\Omega, x^A dx)} = 1.$$

Let c_1 be a positive constant to be chosen later. Then, using (8.37) with $q = \frac{kD}{D-1}$, $k = 1, 2, 3, ...$, we obtain

$$\begin{aligned}\int_\Omega \exp\left\{(c_1|u|)^{\frac{D}{D-1}}\right\} x^A dx &= m(\Omega) + \sum_{k\geq 1} \frac{c_1^{\frac{kD}{D-1}}}{k!} \int_\Omega |u|^{\frac{kD}{D-1}} x^A dx \\ &\leq m(\Omega) + m(\Omega) \sum_{k\geq 1} \frac{c_1^{\frac{kD}{D-1}}}{k!} (C_0)^{\frac{kD}{D-1}} \left(\frac{kD}{D-1}\right)^k \\ &= m(\Omega) + m(\Omega) \sum_{k\geq 1} \frac{k^k}{k!} \left(\frac{D}{D-1}(c_1 C_0)^{\frac{D}{D-1}}\right)^k . \quad (8.38)\end{aligned}$$

Choose c_1 (depending only on D) satisfying $\frac{D}{D-1}(c_1 C_0)^{\frac{D}{D-1}} < \frac{1}{e}$. Then, by Stirling's formula

$$k! \sim \left(\frac{k}{e}\right)^k \sqrt{2\pi k},$$

we deduce that the series (8.38) is convergent, and thus

$$\int_\Omega \exp\left\{\left(\frac{c_1|u|}{\|\nabla u\|_{L^D(\Omega, x^A dx)}}\right)^{\frac{D}{D-1}}\right\} x^A dx \leq C_2 m(\Omega),$$

as claimed. Note that the constants c_1 and C_2 depend only on D. □

To end the paper, we give the

Proof of Corollary 8.1.8. It follows from Theorems 8.1.3, 8.1.6, and 8.1.7. For a domain $\Omega \subset \mathbb{R}^n$ that is not contained in $\mathbb{R}^n_*$, these results need to be applied to the intersections of Ω with each of the 2^k quadrants, where k is the number of positive entries of the vector A — see the proof of (8.11) in Theorem 8.1.6. □

9

Chapter Nine

SHARP ISOPERIMETRIC INEQUALITIES VIA THE ABP METHOD

We prove some old and new isoperimetric inequalities with the best constant using the ABP method applied to an appropriate linear Neumann problem. More precisely, we obtain a new family of sharp isoperimetric inequalities with weights (also called densities) in open convex cones of $\mathbb{R}^n$. Our result applies to all nonnegative homogeneous weights satisfying a concavity condition in the cone. Remarkably, Euclidean balls centered at the origin (intersected with the cone) minimize the weighted isoperimetric quotient, even if all our weights are nonradial —except for the constant ones.

We also study the anisotropic isoperimetric problem in convex cones for the same class of weights. We prove that the Wulff shape (intersected with the cone) minimizes the anisotropic weighted perimeter under the weighted volume constraint.

As a particular case of our results, we give new proofs of two classical results: the Wulff inequality and the isoperimetric inequality in convex cones of Lions and Pacella.

9.1 Introduction and results

In this paper we study isoperimetric problems with weights —also called densities. Given a weight w (that is, a positive function w), one wants to characterize minimizers of the weighted perimeter $\int_{\partial E} w$ among those sets E having weighted volume $\int_E w$ equal to a given constant. A set solving the problem, if it exists, is called an isoperimetric set or simply a minimizer. This question, and the associated isoperimetric inequalities with weights, have attracted much attention recently; see for example [222], [207], [98], [134], and [218].

The solution to the isoperimetric problem in $\mathbb{R}^n$ with a weight w is known only for very few weights, even in the case $n = 2$. For example, in $\mathbb{R}^n$ with the Gaussian weight $w(x) = e^{-|x|^2}$ all the minimizers are half-spaces [32, 96], and with $w(x) = e^{|x|^2}$ all the minimizers are balls centered at the origin [247]. Instead, mixed Euclidean-Gaussian densities lead to minimizers that have a more intricate structure of revolution [145]. The radial homogeneous weight $|x|^\alpha$ has been considered very recently. In the plane ($n = 2$), minimizers for this homogeneous weight depend on the values of α. On the one hand, Carroll-Jacob-Quinn-Walters [82] showed that when $\alpha < -2$ all minimizers are $\mathbb{R}^2 \setminus B_r(0)$, $r > 0$, and that when $-2 \leq \alpha < 0$ minimizers do not exist. On the other hand, when $\alpha > 0$ Dahlberg-Dubbs-Newkirk-Tran [104] proved that all minimizers are

circles passing through the origin (in particular, not centered at the origin). Note that this result shows that even radial homogeneous weights may lead to nonradial minimizers.

Weighted isoperimetric inequalities in cones have also been considered. In these results, the perimeter of E is taken relative to the cone, that is, not counting the part of ∂E that lies on the boundary of the cone. In [114] Díaz-Harman-Howe-Thompson consider again the radial homogeneous weight $w(x) = |x|^\alpha$, with $\alpha > 0$, but now in an open convex cone Σ of angle β in the plane $\mathbb{R}^2$. Among other things, they prove that there exists $\beta_0 \in (0, \pi)$ such that for $\beta < \beta_0$ all minimizers are $B_r(0) \cap \Sigma$, $r > 0$, while these circular sets about the origin are not minimizers for $\beta > \beta_0$.

Also related to the weighted isoperimetric problem in cones, the following is a recent result by Brock-Chiaccio-Mercaldo [37]. Assume that Σ is any cone in $\mathbb{R}^n$ with vertex at the origin, and consider the isoperimetric problem in Σ with any weight w. Then, for $B_R(0) \cap \Sigma$ to be an isoperimetric set for every $R > 0$ a necessary condition is that w admits the factorization

$$w(x) = A(r)B(\Theta), \tag{9.1}$$

where $r = |x|$ and $\Theta = x/r$. Our main result —Theorem 9.1.3 below— gives a sufficient condition on $B(\Theta)$ whenever Σ is convex and $A(r) = r^\alpha$, $\alpha \geq 0$, to guarantee that $B_R(0) \cap \Sigma$ are isoperimetric sets.

Our result states that Euclidean balls centered at the origin solve the isoperimetric problem in any open convex cone Σ of $\mathbb{R}^n$ (with vertex at the origin) for a certain class of nonradial weights. More precisely, our result applies to all nonnegative continuous weights w which are positively homogeneous of degree $\alpha \geq 0$ and such that $w^{1/\alpha}$ is concave in the cone Σ in case $\alpha > 0$. That is, using the previous notation, $w = r^\alpha B(\Theta)$ must be continuous in $\overline{\Sigma}$ and $rB^{1/\alpha}(\Theta)$ must be concave in Σ. We also solve weighted *anisotropic* isoperimetric problems in cones for the same class of weights. In these weighted anisotropic problems, the perimeter of a domain Ω is given by

$$\int_{\partial\Omega\cap\Sigma} H(\nu(x))w(x)dS,$$

where $\nu(x)$ is the unit outward normal to $\partial\Omega$ at x, and H is a positive, positively homogeneous of degree one, and convex function. Our results were announced in the recent note [51].

In the isotropic case, making the first variation of weighted perimeter (see [247]), one sees that the (generalized) mean curvature of $\partial\Omega$ with the density w is

$$H_w = H_{eucl} + \frac{1}{n}\frac{\partial_\nu w}{w}, \tag{9.2}$$

where ν is is the unit outward normal to $\partial\Omega$ and H_{eucl} is the Euclidean mean curvature of $\partial\Omega$. It follows that balls centered at the origin intersected with the cone have constant mean curvature whenever the weight is of the form (9.1). However, as we have seen in several examples presented above, it is far from being true that the solution of the isoperimetric problem for all the weights satisfying (9.1) are balls centered at the origin intersected with the cone. Our result provides a large class of nonradial

weights for which, remarkably, Euclidean balls centered at the origin (intersected with the cone) solve the isoperimetric problem.

In Section 9.2 we give a list of weights w for which our result applies. Some concrete examples are the following:

$$\operatorname{dist}(x, \partial\Sigma)^\alpha \quad \text{in } \Sigma \subset \mathbb{R}^n,$$

where Σ is any open convex cone and $\alpha \geq 0$ (see example (ii) in Section 9.2);

$$x^a y^b z^c, \ (ax^r + by^r + cz^r)^{\alpha/r}, \ \text{ or } \ \frac{xyz}{xy + yz + zx} \quad \text{in } \Sigma = (0,\infty)^3,$$

where a, b, c are nonnegative numbers, $r \in (0, 1]$ or $r < 0$, and $\alpha > 0$ (see examples (iv), (v), and (vii), respectively);

$$\frac{x - y}{\log x - \log y}, \ \frac{x^{a+1} y^{b+1}}{(x^p + y^p)^{1/p}}, \ \text{ or } \ x \log\left(\frac{y}{x}\right) \quad \text{in } \Sigma = (0,\infty)^2,$$

where $a \geq 0$, $b \geq 0$, and $p > -1$ (see examples (viii) and (ix));

$$\left(\frac{\sigma_l}{\sigma_k}\right)^{\frac{\alpha}{l-k}}, \quad 1 \leq k < l < n, \quad \text{in } \Sigma = \{\sigma_1 > 0, \ ..., \sigma_l > 0\},$$

where σ_k is the elementary symmetric function of order k and $\alpha > 0$ (see example (vii)).

Our isoperimetric inequality with an homogeneous weight w of degree α in a convex cone $\Sigma \subset \mathbb{R}^n$ yields as a consequence the following Sobolev inequality with best constant. If $D = n + \alpha$, $1 \leq p < D$, and $p_* = \frac{pD}{D-p}$, then

$$\left(\int_\Sigma |u|^{p_*} w(x) dx\right)^{1/p_*} \leq C_{w,p,n} \left(\int_\Sigma |\nabla u|^p w(x) dx\right)^{1/p} \tag{9.3}$$

for all smooth functions u with compact support in $\mathbb{R}^n$ —in particular, not necessarily vanishing on $\partial\Sigma$. This is a consequence of our isoperimetric inequality, Theorem 9.1.3, and a weighted radial rearrangement of Talenti [292], since these two results yield the radial symmetry of minimizers.

The proof of our main result follows the ideas introduced by the first author [40, 41] in a new proof of the classical isoperimetric inequality (the classical isoperimetric inequality corresponds to the weight $w \equiv 1$ and the cone $\Sigma = \mathbb{R}^n$). Our proof consists of applying the ABP method to an appropriate linear Neumann problem involving the operator

$$w^{-1} \operatorname{div}(w \nabla u) = \Delta u + \frac{\nabla w}{w} \cdot \nabla u,$$

where w is the weight.

9.1.1 The setting

The classical isoperimetric problem in convex cones was solved by P.-L. Lions and F. Pacella [200] in 1990. Their result states that among all sets E with fixed volume inside an open convex cone Σ, the balls centered at the vertex of the cone minimize the perimeter relative to the cone (recall that the part of the boundary of E that lies on the boundary of the cone is not counted).

Throughout the paper Σ is an open convex cone in $\mathbb{R}^n$. Recall that given a measurable set $E \subset \mathbb{R}^n$ the relative perimeter of E in Σ is defined by

$$P(E;\Sigma) := \sup\left\{ \int_E div\, \sigma \, dx \, : \, \sigma \in C_c^1(\Sigma, \mathbb{R}^n), \ |\sigma| \leq 1 \right\}.$$

When E is smooth enough,

$$P(E;\Sigma) = \int_{\partial E \cap \Sigma} dS.$$

The isoperimetric inequality in cones of Lions and Pacella reads as follows.

Theorem 9.1.1 ([200])**.** *Let Σ be an open convex cone in $\mathbb{R}^n$ with vertex at 0, and $B_1 := B_1(0)$. Then,*

$$\frac{P(E;\Sigma)}{|E \cap \Sigma|^{\frac{n-1}{n}}} \geq \frac{P(B_1;\Sigma)}{|B_1 \cap \Sigma|^{\frac{n-1}{n}}} \tag{9.4}$$

for every measurable set $E \subset \mathbb{R}^n$ with $|E \cap \Sigma| < \infty$.

The assumption of convexity of the cone can not be removed. In the same paper [200] the authors give simple examples of nonconvex cones for which inequality (9.4) does not hold. The idea is that for cones having two disconnected components, (9.4) is false since it pays less perimeter to concentrate all the volume in one of the two subcones. A connected (but nonconvex) counterexample is then obtained by joining the two components by a conic open thin set.

The proof of Theorem 9.1.1 given in [200] is based on the Brunn-Minkowski inequality

$$|A+B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}},$$

valid for all nonempty measurable sets A and B of $\mathbb{R}^n$ for which $A+B$ is also measurable; see [154] for more information on this inequality. As a particular case of our main result, in this paper we provide a totally different proof of Theorem 9.1.1. This new proof is based on the ABP method.

Theorem 9.1.1 can be deduced from a degenerate case of the classical Wulff inequality stated in Theorem 9.1.2 below. This is because the convex set $B_1 \cap \Sigma$ is the Wulff shape (9.6) associated to some appropriate anisotropic perimeter. As explained below in Section 9.3, this idea is crucial in the proof of our main result. This fact has also been used recently by Figalli and Indrei [133] to prove a quantitative isoperimetric inequality in convex cones. From it, one deduces that balls centered at the origin are the unique minimizers in (9.4) up to translations that leave invariant the cone (if they exist). This had been established in [200] in the particular case when $\partial\Sigma \setminus \{0\}$ is smooth (and later in [246], which also classified stable hypersurfaces in smooth cones).

The following is the notion of anisotropic perimeter. We say that a function H defined in $\mathbb{R}^n$ is a *gauge* when

$$H \text{ is nonnegative, positively homogeneous of degree one, and convex.} \tag{9.5}$$

Somewhere in the paper we may require a function to be homogeneous; by this we always mean positively homogeneous.

Any norm is a gauge, but a gauge may vanish on some unit vectors. We need to allow this case since it will occur in our new proof of Theorem 9.1.1 —which builds from the cone Σ a gauge that is not a norm.

The anisotropic perimeter associated to the gauge H is given by

$$P_H(E) := \sup\left\{\int_E div\,\sigma\,dx \,:\, \sigma \in C_c^1(\mathbb{R}^n,\mathbb{R}^n),\ \sup_{H(y)\le 1}(\sigma(x)\cdot y)\le 1 \text{ for } x\in\mathbb{R}^n\right\},$$

where $E \subset \mathbb{R}^n$ is any measurable set. When E is smooth enough one has

$$P_H(E) = \int_{\partial E} H\big(\nu(x)\big)dS,$$

where $\nu(x)$ is the unit outward normal at $x \in \partial E$.

The Wulff shape associated to H is defined as

$$W = \{x \in \mathbb{R}^n \,:\, x\cdot\nu < H(\nu) \ \text{ for all } \nu \in S^{n-1}\}. \tag{9.6}$$

We will always assume that $W \neq \varnothing$. Note that W is an open set with $0 \in \overline{W}$. To visualize W, it is useful to note that it is the intersection of the half-spaces $\{x\cdot\nu < H(\nu)\}$ among all $\nu \in S^{n-1}$. In particular, W is a convex set.

From the definition (9.6) of the Wulff shape it follows that, given an open convex set $W \subset \mathbb{R}^n$ with $0 \in \overline{W}$, there is a unique gauge H such that W is the Wulff shape associated to H. Indeed, it is uniquely defined by

$$H(x) = \inf\big\{t \in \mathbb{R} :\ W \subset \{z \in \mathbb{R}^n \,:\, z\cdot x < t\}\big\}. \tag{9.7}$$

Note that, for each direction $\nu \in S^{n-1}$, $\{x\cdot\nu = H(\nu)\}$ is a supporting hyperplane of W. Thus, for almost every point x on ∂W —those for which the outer normal $\nu(x)$ exists— it holds

$$x\cdot\nu(x) = H(\nu(x)) \quad \text{a.e. on } \partial W. \tag{9.8}$$

Note also that, since W is convex, it is a Lipschitz domain. Hence, we can use the divergence theorem to find the formula

$$P_H(W) = \int_{\partial W} H(\nu(x))dS = \int_{\partial W} x\cdot\nu(x)dS = \int_W div(x)dx = n|W|, \tag{9.9}$$

relating the volume and the anisotropic perimeter of W.

When H is positive on S^{n-1} then it is natural to introduce its dual gauge H°, as in [4]. It is defined by

$$H^\circ(z) = \sup_{H(y)\le 1} z\cdot y.$$

Then, the last condition on σ in the definition of $P_H(\cdot)$ is equivalent to $H^\circ(\sigma) \leq 1$ in $\mathbb{R}^n$, and the Wulff shape can be written as $W = \{H^\circ < 1\}$.

Some typical examples of gauges are

$$H(x) = \|x\|_p = \left(|x_1|^p + \cdots + |x_n|^p\right)^{1/p}, \qquad 1 \leq p \leq \infty.$$

Then, we have that $W = \{x \in \mathbb{R}^n : \ \|x\|_{p'} < 1\}$, where p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$.

The following is the celebrated Wulff inequality.

Theorem 9.1.2 ([299, 284, 285])**.** *Let H be a gauge in $\mathbb{R}^n$ which is positive on S^{n-1}, and let W be its associated Wulff shape. Then, for every measurable set $E \subset \mathbb{R}^n$ with $|E| < \infty$, we have*

$$\frac{P_H(E)}{|E|^{\frac{n-1}{n}}} \geq \frac{P_H(W)}{|W|^{\frac{n-1}{n}}}. \tag{9.10}$$

Moreover, equality holds if and only if $E = aW + b$ for some $a > 0$ and $b \in \mathbb{R}^n$ except for a set of measure zero.

This result was first stated without proof by Wulff [299] in 1901. His work was followed by Dinghas [117], who studied the problem within the class of convex polyhedra. He used the Brunn-Minkowski inequality. Some years later, Taylor [284, 285] finally proved Theorem 9.1.2 among sets of finite perimeter; see [286, 139, 209] for more information on this topic. As a particular case of our technique, in this paper we provide a new proof of Theorem 9.1.2. It is based on the ABP method applied to a linear Neumann problem. It was Robert McCann (in a personal communication around 2000) who pointed out that the first author's proof of the classical isoperimetric inequality also worked in the anisotropic case.

9.1.2 Results

The main result of the present paper, Theorem 9.1.3 below, is a weighted isoperimetric inequality which extends the two previous classical inequalities (Theorems 9.1.1 and 9.1.2). In particular, in Section 9.4 we will give a new proof of the classical Wulff theorem (for smooth domains) using the ABP method.

Before stating our main result, let us define the weighted anisotropic perimeter relative to an open cone Σ. The weights w that we consider will always be continuous functions in $\overline{\Sigma}$, positive and locally Lipschitz in Σ, and homogeneous of degree $\alpha \geq 0$. Given a gauge H in $\mathbb{R}^n$ and a weight w, we define (following [16]) the weighted anisotropic perimeter relative to the cone Σ by

$$P_{w,H}(E;\Sigma) := \sup\left\{ \int_{E\cap\Sigma} div(\sigma w)dx \ : \ \sigma \in X_{w,\Sigma}\,, \ \sup_{H(y)\leq 1} (\sigma(x)\cdot y) \leq 1 \text{ for } x \in \Sigma \right\},$$

where $E \subset \mathbb{R}^n$ is any measurable set with finite Lebesgue measure and

$$X_{w,\Sigma} := \left\{ \sigma \in \left(L^\infty(\Sigma)\right)^n \ : \ div(\sigma w) \in L^\infty\big(\Sigma\big) \ \ and \ \ \sigma w = 0 \ \ on \ \ \partial\Sigma \right\}.$$

It is not difficult to see that

$$P_{w,H}(E;\Sigma) = \int_{\partial E\cap\Sigma} H\big(\nu(x)\big)w(x)dS \tag{9.11}$$

whenever E is smooth enough.

The definition of $P_{w,H}$ is the same as the one given in [16]. In their notation, we are taking $d\mu = w\chi_\Sigma\, dx$ and $X_{w,\Sigma} = X_\mu$.

Moreover, when H is the Euclidean norm we denote

$$P_w(E;\Sigma) := P_{w,\|\cdot\|_2}(E;\Sigma).$$

When $w \equiv 1$ in Σ and H is the Euclidean norm we recover the definition of $P(E;\Sigma)$; see [16].

Given a measurable set $F \subset \Sigma$, we denote by $w(F)$ the weighted volume of F

$$w(F) := \int_F w\, dx.$$

We also denote

$$D = n + \alpha.$$

Note that the Wulff shape W is independent of the weight w. Next we use that if ν is the unit outward normal to $W \cap \Sigma$, then $x \cdot \nu(x) = H(\nu(x))$ a.e. on $\partial W \cap \Sigma$, $x \cdot \nu(x) = 0$ a.e. on $\overline{W} \cap \partial\Sigma$, and $x \cdot \nabla w(x) = \alpha w(x)$ in Σ. This last equality follows from the homogeneity of degree α of w. Then, with a similar argument as in (9.9), we find

$$\begin{aligned} P_{w,H}(W;\Sigma) &= \int_{\partial W\cap\Sigma} H(\nu(x))w(x)dS = \int_{\partial W\cap\Sigma} x\cdot\nu(x)\, w(x)dS \\ &= \int_{\partial(W\cap\Sigma)} x\cdot\nu(x)w(x)dS = \int_{W\cap\Sigma} div(xw(x))dx \\ &= \int_{W\cap\Sigma} \{nw(x) + x\cdot\nabla w(x)\}\, dx = D\, w(W\cap\Sigma). \end{aligned} \tag{9.12}$$

Here —and in our main result that follows— for all quantities to make sense we need to assume that $W \cap \Sigma \neq \varnothing$. Recall that both W and Σ are open convex sets but that $W \cap \Sigma = \varnothing$ could happen. This occurs for instance if $H|_{S^{n-1}\cap\Sigma} \equiv 0$. On the other hand, if $H > 0$ on all S^{n-1} then $W \cap \Sigma \neq \varnothing$.

The following is our main result.

Theorem 9.1.3. *Let H be a gauge in $\mathbb{R}^n$, i.e., a function satisfying* (9.5)*, and W its associated Wulff shape defined by* (9.6)*. Let Σ be an open convex cone in $\mathbb{R}^n$ with vertex at the origin, and such that $W \cap \Sigma \neq \varnothing$. Let w be a continuous function in $\overline{\Sigma}$, positive in Σ, and positively homogeneous of degree $\alpha \geq 0$. Assume in addition that $w^{1/\alpha}$ is concave in Σ in case $\alpha > 0$.*

Then, for each measurable set $E \subset \mathbb{R}^n$ with $w(E\cap\Sigma) < \infty$,

$$\frac{P_{w,H}(E;\Sigma)}{w(E\cap\Sigma)^{\frac{D-1}{D}}} \geq \frac{P_{w,H}(W;\Sigma)}{w(W\cap\Sigma)^{\frac{D-1}{D}}}, \tag{9.13}$$

where $D = n + \alpha$.

Remark 9.1.4. Our key hypothesis that $w^{1/\alpha}$ is a concave function is equivalent to a natural curvature-dimension bound (in fact, to the nonnegativeness of the Bakry-Émery Ricci tensor in dimension $D = n + \alpha$). This was suggested to us by Cédric Villani, and has also been noticed by Cañete and Rosales (see Lemma 3.9 in [79]). More precisely, we see the cone $\Sigma \subset \mathbb{R}^n$ as a Riemannian manifold of dimension n equipped with a reference measure $w(x)dx$. We are also given a "dimension" $D = n + \alpha$. Consider the Bakry-Émery Ricci tensor, defined by

$$\mathrm{Ric}_{D,w} = \mathrm{Ric} - \nabla^2 \log w - \frac{1}{D-n} \nabla \log w \otimes \nabla \log w.$$

Now, our assumption $w^{1/\alpha}$ being concave is equivalent to

$$\mathrm{Ric}_{D,w} \geq 0. \tag{9.14}$$

Indeed, since $\mathrm{Ric} \equiv 0$ and $D - n = \alpha$, (9.14) reads as

$$-\nabla^2 \log w^{1/\alpha} - \nabla \log w^{1/\alpha} \otimes \nabla \log w^{1/\alpha} \geq 0,$$

which is the same condition as $w^{1/\alpha}$ being concave. Condition (9.14) is called a curvature-dimension bound; in the terminology of [294] we say that $\mathrm{CD}(0, D)$ is satisfied by $\Sigma \subset \mathbb{R}^n$ with the reference measure $w(x)dx$.

In addition, C. Villani pointed out that optimal transport techniques could also lead to weighted isoperimetric inequalities in convex cones; see Section 9.1.3.

Note that the shape of the minimizer is $W \cap \Sigma$, and that W depends only on H and not on the weight w neither on the cone Σ. In particular, in the isotropic case $H = \|\cdot\|_2$ we find the following surprising fact. Even that the weights that we consider are not radial (unless $w \equiv 1$), still Euclidean balls centered at the origin (intersected with the cone) minimize this isoperimetric quotient. The only explanation that one has a priori for this fact is that Euclidean balls centered at 0 have constant generalized mean curvature when the weight is homogeneous, as pointed out in (9.2). Thus, they are candidates to minimize perimeter for a given volume.

Note also that we allow w to vanish somewhere (or everywhere) on $\partial\Sigma$.

Equality in (9.13) holds whenever $E \cap \Sigma = rW \cap \Sigma$, where r is any positive number. However, in this paper we do not prove that $W \cap \Sigma$ is the unique minimizer of (9.13). The reason is that our proof involves the solution of an elliptic equation and, due to an important issue on its regularity, we need to approximate the given set E by smooth sets. In a future work with E. Cinti and A. Pratelli we will refine the analysis in the proof of the present article and obtain a quantitative version of our isoperimetric inequality in cones. In particular, we will deduce uniqueness of minimizers (up to sets of measure zero). The quantitative version will be proved using the techniques of the present paper (the ABP method applied to a linear Neumann problem) together with the ideas of Figalli-Maggi-Pratelli [135].

In the isotropic case, a very recent result of Cañete and Rosales [79] deals with the same class of weights as ours. They allow not only positive homogeneities $\alpha > 0$, but also negative ones $\alpha \leq -(n-1)$. They prove that if a smooth, compact, and orientable hypersurfaces in a smooth convex cone is stable for weighted perimeter (under variations preserving weighted volume), then it must be a sphere centered at

the vertex of the cone. In [79] the stability of such spheres is proved for $\alpha \leq -(n-1)$, but not for $\alpha > 0$. However, as pointed out in [79], when $\alpha > 0$ their result used together with ours give that spheres centered at the vertex are the unique minimizers among smooth hypersurfaces.

Theorem 9.1.3 contains the classical isoperimetric inequality, its version for convex cones, and the classical Wulff inequality. Indeed, taking $w \equiv 1$, $\Sigma = \mathbb{R}^n$, and $H = \|\cdot\|_2$ we recover the classical isoperimetric inequality with optimal constant. Still taking $w \equiv 1$ and $H = \|\cdot\|_2$ but now letting Σ be any open convex cone of $\mathbb{R}^n$ we have the isoperimetric inequality in convex cones of Lions and Pacella (Theorem 9.1.1). Moreover, if we take $w \equiv 1$ and $\Sigma = \mathbb{R}^n$ but we let H be some other gauge we obtain the Wulff inequality (Theorem 9.1.2).

A criterion of concavity for homogeneous functions of degree 1 can be found for example in [217, Proposition 10.3], and reads as follows. A nonnegative, C^2, and homogeneous of degree 1 function Φ on $\mathbb{R}^n$ is concave if and only if the restrictions $\Phi(\theta)$ of Φ to one-dimensional circles about the origin satisfy

$$\Phi''(\theta) + \Phi(\theta) \leq 0.$$

Therefore, it follows that a nonnegative, C^2, and homogeneous weight of degree $\alpha > 0$ in the plane $\mathbb{R}^2$, $w(x) = r^\alpha B(\theta)$, satisfies that $w^{1/\alpha}$ is concave in Σ if and only if

$$(B^{1/\alpha})'' + B^{1/\alpha} \leq 0.$$

Remark 9.1.5. Let w be an homogeneous weight of degree α, and consider the isotropic isoperimetric problem in a cone $\Sigma \subset \mathbb{R}^n$. Then, by the proofs of Proposition 3.6 and Lemma 3.8 in [247] the set $B_1(0) \cap \Sigma$ is stable if and only if

$$\int_{S^{n-1}\cap\Sigma} |\nabla_{S^{n-1}} u|^2 w \, dS \geq (n-1+\alpha) \int_{S^{n-1}\cap\Sigma} |u|^2 w \, dS \tag{9.15}$$

for all functions $u \in C_c^\infty(S^{n-1} \cap \Sigma)$ satisfying

$$\int_{S^{n-1}\cap\Sigma} uw \, dS = 0. \tag{9.16}$$

Stability being a necessary condition for minimality, from Theorem (9.1.3) we deduce the following. If $\alpha > 0$, Σ is convex, and $w^{1/\alpha}$ is concave in Σ, then (9.15) holds.

For instance, in dimension $n = 2$, inequality (9.15) reads as

$$\int_0^\beta (u')^2 w \, d\theta \geq (1+\alpha) \int_0^\beta u^2 w \, d\theta \qquad \textit{whenever} \qquad \int_0^\beta uw \, d\theta = 0, \tag{9.17}$$

where $0 < \beta \leq \pi$ is the angle of the convex cone $\Sigma \subset \mathbb{R}^2$. This is ensured by our concavity condition on the weight w,

$$\left(w^{1/\alpha}\right)'' + w^{1/\alpha} \leq 0 \qquad \textit{in } (0, \beta). \tag{9.18}$$

Note that, even in this two-dimensional case, it is not obvious that this condition on w yields (9.15)-(9.16). The statement (9.17) is an extension of Wirtinger's inequality (which corresponds to the case $w \equiv 1$, $\alpha = 0$, $\beta = 2\pi$). It holds, for example, with $w = \sin^\alpha \theta$ on S^1 —since (9.18) is satisfied by this weight. Another extension of Wirtinger's inequality (coming from the density $w = r^\alpha$) is given in [104].

In Theorem 9.1.3 we assume that w is homogeneous of degree α. In our proof, this assumption is essential in order that the paraboloid in (9.26) solves the PDE in (9.24), as explained in Section 9.3. Due to the homogeneity of w, the exponent $D = n+\alpha$ can be found just by a scaling argument in our inequality (9.13). Note that this exponent D has a dimension flavor if one compares (9.13) with (9.4) or with (9.10). Also, it is the exponent for the volume growth, in the sense that $w(B_r(0) \cap \Sigma) = Cr^D$ for all $r > 0$. The interpretation of D as a dimension is more clear in the following example that motivated our work.

Remark 9.1.6. The monomial weights

$$w(x) = x_1^{A_1} \cdots x_n^{A_n} \qquad \text{in} \quad \Sigma = \{x \in \mathbb{R}^n \,:\, x_i > 0 \text{ whenever } A_i > 0\}, \tag{9.19}$$

where $A_i \geq 0$, $\alpha = A_1 + \cdots + A_n$, and $D = n + A_1 + \cdots + A_n$, are important examples for which (9.13) holds. The isoperimetric inequality —and the corresponding Sobolev inequality (9.3)— with the above monomial weights were studied by the first two authors in [49, 50]. These inequalities arose in [49] while studying reaction-diffusion problems with symmetry of double revolution. A function u has symmetry of double revolution when $u(x, y) = u(|x|, |y|)$, with $(x, y) \in \mathbb{R}^D = \mathbb{R}^{A_1+1} \times \mathbb{R}^{A_2+1}$ (here we assume A_i to be positive integers). In this way, $u = u(x_1, x_2) = u(|x|, |y|)$ can be seen as a function in $\mathbb{R}^2 = \mathbb{R}^n$, and it is here where the Jacobian for the Lebesgue measure in $\mathbb{R}^D = \mathbb{R}^{A_1+1} \times \mathbb{R}^{A_2+1}$, $x_1^{A_1} x_2^{A_2} = |x|^{A_1} |y|^{A_2}$, appears. A similar argument under multiple axial symmetries shows that, when w and Σ are given by (9.19) and all A_i are nonnegative integers, and H is the Euclidean norm, Theorem 9.1.3 follows from the classical isoperimetric inequality in $\mathbb{R}^D$; see [50] for more details.

In [49] we needed to show a Sobolev inequality of the type (9.3) in $\mathbb{R}^2$ with the weight and the cone given by (9.19). As explained above, when A_i are all nonnegative integers this Sobolev inequality follows from the classical one in dimension D. However, in our application the exponents A_i were not integers —see [49]—, and thus the Sobolev inequality was not known. We showed a nonoptimal version (without the best constant) of that Sobolev inequality in dimension $n = 2$ in [49], and later we proved in [50] the optimal one in all dimensions n, obtaining the best constant and extremal functions for the inequality. In both cases, the main tool to prove these Sobolev inequalities was an isoperimetric inequality with the same weight.

An immediate consequence of Theorem 9.1.3 is the following weighted isoperimetric inequality in $\mathbb{R}^n$ for symmetric sets and even weights. It follows from our main result taking $\Sigma = (0, +\infty)^n$.

Corollary 9.1.7. *Let w be a nonnegative continuous function in $\mathbb{R}^n$, even with respect to each variable, homogeneous of degree $\alpha > 0$, and such that $w^{1/\alpha}$ is concave in $(0, \infty)^n$. Let $E \subset \mathbb{R}^n$ be any measurable set, symmetric with respect to each coordinate hyperplane $\{x_i = 0\}$, and with $|E| < \infty$. Then,*

$$\frac{P_w(E; \mathbb{R}^n)}{|E|^{\frac{D-1}{D}}} \geq \frac{P_w(B_1; \mathbb{R}^n)}{|B_1|^{\frac{D-1}{D}}}, \tag{9.20}$$

where $D = n + \alpha$ and B_1 is the unit ball in $\mathbb{R}^n$.

The symmetry assumption on the sets that we consider in Corollary 9.1.7 is satisfied in some applications arising in nonlinear problems, such as the one in [49] explained in Remark 9.1.6. Without this symmetry assumption, isoperimetric sets in (9.20) may not be the balls. For example, for the monomial weight $w(x) = |x_1|^{A_1} \cdots |x_n|^{A_n}$ in $\mathbb{R}^n$, with all A_i positive, $B_1 \cap (0,\infty)^n$ is an isoperimetric set, while the whole ball B_r having the same weighted volume as $B_1 \cap (0,\infty)^n$ is not an isoperimetric set (since it has longer perimeter).

We know only of few results where nonradial weights lead to radial minimizers. The first one is the isoperimetric inequality by Maderna-Salsa [204] in the upper half plane $\mathbb{R}^2_+$ with the weight x_2^α, $\alpha > 0$. To establish their isoperimetric inequality, they first proved the existence of a minimizer for the perimeter functional under constraint of fixed area, then computed the first variation of this functional, and finally solved the obtained ODE to find all minimizers. The second result is due to Brock-Chiacchio-Mercaldo [37] and extends the one in [204] by including the weights $x_n^\alpha \exp(c|x|^2)$ in $\mathbb{R}^n_+$, with $\alpha \geq 0$ and $c \geq 0$. In both papers it is proved that half balls centered at the origin are the minimizers of the isoperimetric quotient with these weights. Another one, of course, is our isoperimetric inequality with monomial weights [50] explained above (see Remark 9.1.6). At the same time as us, and using totally different methods, Brock, Chiacchio, and Mercaldo [38] have proved an isoperimetric inequality in $\Sigma = \{x_1 > 0, ..., x_n > 0\}$ with the weight $x_1^{A_1} \cdots x_n^{A_n} \exp(c|x|^2)$, with $A_i \geq 0$ and $c \geq 0$.

In all these results, although the weight $x_1^{A_1} \cdots x_n^{A_n}$ is not radial, it has a very special structure. Indeed, when all $A_1, ..., A_n$ are nonnegative *integers* the isoperimetric problem with the weight $x_1^{A_1} \cdots x_n^{A_n}$ is equivalent to the isoperimetric problem in $\mathbb{R}^{n+A_1+\cdots+A_n}$ for sets that have symmetry of revolution with respect to the first A_1+1 variables, the next A_2+1 variables, ..., and so on until the last A_n+1 variables; see Remark 9.1.6. By this observation, the fact that half balls centered at the origin are the minimizers in $\mathbb{R}^n_+$ with the weight $x_1^{A_1} \cdots x_n^{A_n}$ or $x_1^{A_1} \cdots x_n^{A_n} \exp(c|x|^2)$, for $c \geq 0$ and A_i nonnegative integers, follows from the isoperimetric inequality in $\mathbb{R}^{n+A_1+\cdots+A_n}$ with the weight $\exp(c|x|^2)$, $c \geq 0$ (which is a radial weight). Thus, it was reasonable to expect that the same result for noninteger exponents $A_1, ..., A_n$ would also hold —as it does.

After announcing our result and proof in [51], Emanuel Milman showed us a nice geometric construction that yields the particular case when α is a nonnegative *integer* in our weighted inequality of Theorem 9.1.3. Using this construction, the weighted inequality in a convex cone is obtained as a limit case of the unweighted Lions-Pacella inequality in a narrow cone of $\mathbb{R}^{n+\alpha}$. We reproduce it in Remark 9.6.1 —see also the blog of Frank Morgan [220].

9.1.3 The proof. Related works

The proof of Theorem 9.1.3 consists of applying the ABP method to a linear Neumann problem involving the operator $w^{-1}\mathrm{div}(w\nabla u)$, where w is the weight. When $w \equiv 1$, the idea goes back to 2000 in the works [40, 41] of the first author, where the classical isoperimetric inequality in all of $\mathbb{R}^n$ (here $w \equiv 1$) was proved with a new

method. It consisted of solving the problem

$$\begin{cases} \Delta u = b_\Omega & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 1 & \text{on } \partial\Omega \end{cases}$$

for a certain constant b_Ω, to produce a bijective map with the gradient of u, $\nabla u : \Gamma_{u,1} \longrightarrow B_1$, which leads to the isoperimetric inequality. Here $\Gamma_{u,1} \subset \Gamma_u \subset \Omega$ and $\Gamma_{u,1}$ is a certain subset of the lower contact set Γ_u of u (see Section 9.3 for details). The use of the ABP method is crucial in the proof.

Previously, Trudinger [291] had given a proof of the classical isoperimetric inequality in 1994 using the theory of Monge-Ampère equations and the ABP estimate. His proof consists of applying the ABP estimate to the Monge-Ampère problem

$$\begin{cases} \det D^2 u = \chi_\Omega & \text{in } B_R \\ u = 0 & \text{on } \partial B_R, \end{cases}$$

where χ_Ω is the characteristic function of Ω and $B_R = B_R(0)$, and then letting $R \to \infty$.

Before these two works ([291] and [40]), there was already a proof of the isoperimetric inequality using a certain map (or coupling). This is Gromov's proof, which used the Knothe map; see [294].

After these three proofs, in 2004 Cordero-Erausquin, Nazaret, and Villani [101] used the Brenier map from optimal transportation to give a beautiful proof of the anisotropic isoperimetric inequality; see also [294]. More recently, Figalli-Maggi-Pratelli [135] established a sharp quantitative version of the anisotropic isoperimetric inequality, using also the Brenier map. In the case of the Lions-Pacella isoperimetric inequality, this has been done by Figalli-Indrei [133] very recently. As mentioned before, the proof in the present article is also suited for a quantitative version, as we will show in a future work with Cinti and Pratelli.

After announcing our result and proof in [51], we have been told that optimal transportation techniques à la [101] could also be used to prove weighted isoperimetric inequalities in certain cones. C. Villani pointed out that this is mentioned in the Bibliographical Notes to Chapter 21 of his book [294]. A. Figalli showed it to us with a computation when the cone is a halfspace $\{x_n > 0\}$ equipped with the weight x_n^α.

9.1.4 Applications

Now we turn to some applications of Theorems 9.1.3 and Corollary 9.1.7.

First, our result leads to weighted Sobolev inequalities with best constant in convex cones of $\mathbb{R}^n$. Indeed, given any smooth function u with compact support in $\mathbb{R}^n$ (we do not assume u to vanish on $\partial\Sigma$), one uses the coarea formula and Theorem 9.1.3 applied to each of the level sets of u. This establishes the Sobolev inequality (9.3) for $p = 1$. The constant $C_{w,1,n}$ obtained in this way is optimal, and coincides with the best constant in our isoperimetric inequality (9.20).

When $1 < p < D$, Theorem 9.1.3 also leads to the Sobolev inequality (9.3) with best constant. This is a consequence of our isoperimetric inequality and a weighted radial

rearrangement of Talenti [292], since these two results yield the radial symmetry of minimizers. See [50] for details in the case of monomial weights $w(x) = |x_1|^{A_1} \cdots |x_n|^{A_n}$.

If we use Corollary 9.1.7 instead of Theorem 9.1.3, with the same argument one finds the Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |u|^{p_*} w(x) dx \right)^{1/p_*} \leq C_{w,p,n} \left(\int_{\mathbb{R}^n} |\nabla u|^p w(x) dx \right)^{1/p}, \tag{9.21}$$

where $p_* = \frac{pD}{D-p}$, $D = n + \alpha$, and $1 \leq p < D$. Here, w is any weight satisfying the hypotheses of Corollary 9.1.7, and u is any smooth function with compact support in $\mathbb{R}^n$ which is symmetric with respect to each variable x_i, $i = 1, ..., n$.

We now turn to applications to the symmetry of solutions to nonlinear PDEs. It is well known that the classical isoperimetric inequality yields some radial symmetry results for semilinear or quasilinear elliptic equations. Indeed, using the Schwartz rearrangement that preserves $\int F(u)$ and decreases $\int \Phi(|\nabla u|)$, it is immediate to show that minimizers of some energy functionals (or quotients) involving these quantities are radially symmetric; see [238, 292]. Moreover, P.-L. Lions [191] showed that in dimension $n = 2$ the isoperimetric inequality yields also the radial symmetry of all positive solutions to the semilinear problem $-\Delta u = f(u)$ in B_1, $u = 0$ on ∂B_1, with $f \geq 0$ and f possibly discontinuous. This argument has been extended in three directions: for the p-Laplace operator, for cones of $\mathbb{R}^n$, and for Wulff shapes, as explained next.

On the one hand, the analogue of Lions radial symmetry result but in dimension $n \geq 3$ for the p-Laplace operator was proved with $p = n$ by Kesavan and Pacella in [183], and with $p \geq n$ by the third author in [260]. Moreover, in [183] it is also proved that positive solutions to the following semilinear equation with mixed boundary conditions

$$\begin{cases} -\Delta_p u = f(u) & \text{in } B_1 \cap \Sigma \\ u = 0 & \text{on } \partial B_1 \cap \Sigma \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } B_1 \cap \partial \Sigma \end{cases} \tag{9.22}$$

have radial symmetry whenever $p = n$. Here, B_1 is the unit ball and Σ any open convex cone. This was proved by using Theorem 9.1.1 and the argument of P.-L. Lions mentioned above.

On the other hand, Theorem 9.1.2 is used to construct a Wulff shaped rearrangement in [4]. This yields that minimizers to certain nonlinear variational equations that come from anisotropic gradient norms have Wulff shaped level sets. Moreover, the radial symmetry argument in [191] was extended to this anisotropic case in [17], yielding the same kind of result for positive solutions of nonlinear equations involving the operator $Lu = div\,(H(\nabla u)^{p-1} \nabla H(\nabla u))$ with $p = n$. In the same direction, in a future paper [254] we will use Theorem 9.1.3 to obtain Wulff shaped symmetry of critical functions of weighted anisotropic functionals such as

$$\int \big\{ H^p(\nabla u) - F(u) \big\} w(x)\, dx.$$

Here, w is an homogeneous weight satisfying the hypotheses of Theorem 9.1.3 and H is any norm in $\mathbb{R}^n$. As in [260], we will allow $p \neq n$ but with some conditions on F in case $p < n$.

Related to these results, when f is Lipschitz, Berestycki and Pacella [19] proved that any positive solution to problem (9.22) with $p = 2$ in a convex spherical sector Σ of $\mathbb{R}^n$ is radially symmetric. They used the moving planes method.

9.1.5 Plan of the paper

The rest of the article is organized as follows. In Section 9.2 we give examples of weights for which our result applies. In Section 9.3 we introduce the elements appearing in the proof of Theorem 9.1.3. To illustrate these ideas, in Section 9.4 we give the proof of the classical Wulff theorem via the ABP method. In Section 9.5 we prove Theorem 9.1.3 in the simpler case $w \equiv 0$ on $\partial\Sigma$ and $H = \|\cdot\|_2$. Finally, in Section 9.6 we present the whole proof of Theorem 9.1.3.

9.2 Examples of weights

When $w \equiv 1$ our main result yields the classical isoperimetric inequality, its version for convex cones, and also the Wulff theorem. On the other hand, given an open convex cone $\Sigma \subset \mathbb{R}^n$ (different than the whole space and a half-space) there is a large family of functions that are homogeneous of degree one and concave in Σ. Any positive power of one of these functions is an admissible weight for Theorem 9.1.3. Next we give some concrete examples of weights w for which our result applies. The key point is to check that the homogeneous function of degree one $w^{1/\alpha}$ is concave.

(i) Assume that w_1 and w_2 are concave homogeneous functions of degree one in an open convex cone Σ. Then, $w_1^a w_2^b$ with $a \geq 0$ and $b \geq 0$, $(w_1^r + w_2^r)^{\alpha/r}$ with $r \in (0, 1]$ or $r < 0$, and $\min\{w_1, w_2\}^\alpha$, satisfy the hypotheses of Theorem 9.1.3 (with $\alpha = a + b$ in the first case). More generally, if $F : [0, \infty)^2 \to \mathbb{R}_+$ is positive, concave, homogeneous of degree 1, and nondecreasing in each variable, then one can take $w = F(w_1, w_2)^\alpha$, with $\alpha > 0$.

(ii) The distance function to the boundary of any convex set is concave when defined in the convex set. On the other hand, the distance function to the boundary of any cone is homogeneous of degree 1. Thus, for any open convex cone Σ and any $\alpha \geq 0$,

$$w(x) = \operatorname{dist}(x, \partial\Sigma)^\alpha$$

is an admissible weight. When the cone is $\Sigma = \{x_i > 0,\ i = 1, ..., n\}$, this weight is exactly $\min\{x_1, ..., x_n\}^\alpha$.

(iii) If the concavity condition is satisfied by a weight w in a convex cone Σ' then it is also satisfied in any convex subcone $\Sigma \subset \Sigma'$. Note that this gives examples of weights w and cones Σ in which w is positive on $\partial\Sigma \setminus \{0\}$.

(iv) Let $\Sigma_1, ..., \Sigma_k$ be convex cones and $\Sigma = \Sigma_1 \cap \cdots \cap \Sigma_k$. Let

$$\delta_i(x) = dist(x, \partial\Sigma_i).$$

Then, the weight

$$w(x) = \delta_1^{A_1} \cdots \delta_k^{A_k}, \quad x \in \Sigma,$$

with $A_1 \geq 0, ..., A_k \geq 0$, satisfies the hypotheses of Theorem 9.1.3. This follows from (i), (ii), and (iii). Note that when $k = n$ and $\Sigma_i = \{x_i > 0\}$, $i = 1, ..., n$, then $\Sigma = \{x_1 > 0, ..., x_n > 0\}$ and we obtain the monomial weight

$$w(x) = x_1^{A_1} \cdots x_n^{A_n}.$$

(v) In the cone $\Sigma = (0, \infty)^n$, the weights

$$w(x) = \left(A_1 x_1^{1/p} + \cdots + A_n x_n^{1/p} \right)^{\alpha p},$$

for $p \geq 1$, $A_i \geq 0$, and $\alpha > 0$, satisfy the hypotheses of Theorem 9.1.3. Similarly, one may take the weights

$$w(x) = \left(\frac{A_1}{x_1^r} + \cdots + \frac{A_n}{x_n^r} \right)^{-\alpha/r},$$

with $r > 0$, or the limit case

$$w(x) = \min\{A_1 x_1, \cdots, A_n x_n\}^{\alpha}.$$

This can be showed using the Minkowski inequality. More precisely, the first one can be showed using the classical Minkowski inequality with exponent $p \geq 1$, while the second one using a reversed Minkowski inequality that holds for exponents $p = -r < 0$.

In these examples $\Sigma = (0, \infty)^n$ and therefore by Corollary 9.1.7 we find that among all sets $E \subset \mathbb{R}^n$ which are symmetric with respect to each coordinate hyperplane, Euclidean balls centered at the origin minimize the isoperimetric quotient with these weights.

(vi) Powers of *hyperbolic polynomials* also provide examples of weights. An homogeneous polynomial $P(x)$ of degree k defined in $\mathbb{R}^n$ is called hyperbolic with respect to $a \in \mathbb{R}^n$ provided $P(a) > 0$ and for every $\lambda \in \mathbb{R}^n$ the polynomial in t, $P(ta+\lambda)$, has exactly k real roots. Let Σ be the component in $\mathbb{R}^n$, containing a, of the set $\{P > 0\}$. Then, Σ is a convex cone and $P(x)^{1/k}$ is a concave function in Σ; see for example [149] or [65, Section 1]. Thus, for any hyperbolic polynomial P, the weight

$$w(x) = P(x)^{\alpha/k}$$

satisfies the hypotheses of Theorem 9.1.3. Typical examples of hyperbolic polynomials are

$$P(x) = x_1^2 - \lambda_2 x_2^2 - \cdots - \lambda_n x_n^2 \ \text{ in } \Sigma = \left\{ x_1 > \sqrt{\lambda_2 x_2^2 + \cdots + \lambda_n x_n^2} \right\},$$

with $\lambda_2 > 0$,...,$\lambda_n > 0$, or the elementary symmetric functions

$$\sigma_k(x) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k} \quad \text{in } \Sigma = \{\sigma_1 > 0, ..., \sigma_k > 0\}$$

(recall that Σ is defined above as a component of $\{P > 0\}$). Other examples are

$$P(x) = \prod_{1 \le i_1 < \cdots < i_r \le n} \sum_{j=1}^{r} x_{i_j} \quad \text{in } \Sigma = \{x_i > 0,\ i = 1, ..., n\},$$

which have degree $k = \binom{n}{r}$ (this follows by induction from the first statement in example (i); see also [15]), or the polynomial $\det(X)$ in the convex cone of symmetric positive definite matrices —which we consider in the space $\mathbb{R}^{n(n+1)/2}$.

The interest in hyperbolic polynomials was originally motivated by an important paper of Garding on linear hyperbolic PDEs [148], and it is known that they form a rich class; see for example [149], where the same author showed various ways of constructing new hyperbolic polynomials from old ones.

(vii) If σ_k and σ_l are the elementary symmetric functions of degree k and l, with $1 \le k < l \le n$, then $(\sigma_l/\sigma_k)^{\frac{1}{l-k}}$ is concave in the cone $\Sigma = \{\sigma_1 > 0, ..., \sigma_k > 0\}$; see [206]. Thus,

$$w(x) = \left(\frac{\sigma_l}{\sigma_k}\right)^{\frac{\alpha}{l-k}}$$

is an admissible weight. For example, setting $k = n$ and $l = 1$ we find that we can take

$$w(x) = \left(\frac{x_1 \cdots x_n}{x_1 + \cdots + x_n}\right)^{\frac{\alpha}{n-1}}$$

in Theorem 9.1.3 or in Corollary 9.1.7.

(viii) If $f : \mathbb{R} \to \mathbb{R}_+$ is any continuous function which is concave in (a, b), then

$$w(x) = x_1 f\left(\frac{x_2}{x_1}\right)$$

is an admissible weight in $\Sigma = \{x = (r, \theta) \,:\, \arctan a < \theta < \arctan b\}$.

(ix) In the cone $\Sigma = (0, \infty)^2 \subset \mathbb{R}^2$ one may take

$$w(x) = \left(\frac{x_1 - x_2}{\log x_1 - \log x_2}\right)^{\alpha}$$

for $\alpha > 0$. In addition, in the same cone one may also take

$$w(x) = \frac{1}{e}\left(x_1^{x_1} x_2^{-x_2}\right)^{\frac{\alpha}{x_1 - x_2}}.$$

This can be seen by using (viii) and computing f in each of the two cases. When $\alpha = 1$, these functions are called the *logarithmic mean* and the *identric mean* of the numbers x_1 and x_2, respectively.

Using also (viii) one can check that, in the cone $\Sigma = (0, \infty)^2$, the weight $w(x) = xy(x^p + y^p)^{-1/p}$ is admissible whenever $p > -1$. Then, using (i) it follows that

$$w(x) = \frac{x^{a+1} y^{b+1}}{(x^p + y^p)^{1/p}}$$

is an admissible weight whenever $a \ge 0$, $b \ge 0$, and $p > -1$.

9.3 Description of the proof

The proof of Theorem 9.1.3 follows the ideas introduced by the first author in a new proof of the classical isoperimetric inequality; see [40, 41] or the last edition of Chavel's book [91]. This proof uses the ABP method, as explained next.

The Alexandroff-Bakelman-Pucci (or ABP) estimate is an L^∞ bound for solutions of the Dirichlet problem associated to second order uniformly elliptic operators written in nondivergence form,

$$Lu = a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + c(x)u,$$

with bounded measurable coefficients in a domain Ω of $\mathbb{R}^n$. It asserts that if Ω is bounded and $c \leq 0$ in Ω then, for every function $u \in C^2(\Omega) \cap C(\overline{\Omega})$,

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + C\, diam(\Omega)\, \|Lu\|_{L^n(\Omega)},$$

where C is a constant depending only on the ellipticity constants of L and on the L^n-norm of the coefficients b_i. The estimate was proven by the previous authors in the sixties using a technique that is nowadays called "the ABP method". See [41] and references therein for more information on this estimate.

The proof of the classical isoperimetric inequality in [40, 41] consists of applying the ABP method to an appropriate Neumann problem for the Laplacian —instead of applying it to a Dirichlet problem as customary. Namely, to estimate from below $|\partial\Omega|/|\Omega|^{\frac{n-1}{n}}$ for a smooth domain Ω, one considers the problem

$$\begin{cases} \Delta u = b_\Omega & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 1 & \text{on } \partial\Omega. \end{cases} \tag{9.23}$$

The constant $b_\Omega = |\partial\Omega|/|\Omega|$ is chosen so that the problem has a solution. Next, one proves that $B_1 \subset \nabla u(\Gamma_u)$ with a contact argument (for a certain "contact" set $\Gamma_u \subset \Omega$), and then one estimates the measure of $\nabla u(\Gamma_u)$ by using the area formula and the inequality between the geometric and arithmetic means. Note that the solution of (9.23) is

$$u(x) = |x|^2/2 \qquad \text{when } \Omega = B_1,$$

and in this case one verifies that all the inequalities appearing in this ABP argument are equalities. After having proved the isoperimetric inequality for smooth domains, an standard approximation argument extends it to all sets of finite perimeter.

As pointed out by R. McCann, the same method also yields the Wulff theorem. For this, one replaces the Neumann data in (9.23) by $\partial u/\partial\nu = H(\nu)$ and uses the same argument explained above. This proof of the Wulff theorem is given in Section 9.4.

We now sketch the proof of Theorem 9.1.3 in the isotropic case, that is, when $H = \|\cdot\|_2$. In this case, optimizers are Euclidean balls centered at the origin intersected with the cone. First, we assume that $E = \Omega$ is a bounded smooth domain. The key idea is to consider a similar problem to (9.23) but where the Laplacian is replaced by the operator

$$w^{-1} div(w\nabla u) = \Delta u + \frac{\nabla w}{w} \cdot \nabla u.$$

Essentially (but, as we will see, this is not exactly as we proceed —because of a regularity issue), we solve the following Neumann problem in $\Omega \subset \Sigma$:

$$\begin{cases} w^{-1} div\,(w\nabla u) = b_\Omega & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 1 & \text{on } \partial\Omega \cap \Sigma \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \cap \partial\Sigma, \end{cases} \tag{9.24}$$

where the constant b_Ω is again chosen depending on weighted perimeter and volume so that the problem admits a solution. Whenever u belongs to $C^1(\overline{\Omega})$ —which is not always the case, as discussed below in this section—, by touching the graph of u by below with planes (as in the proof of the classical isoperimetric inequality explained above) we find that

$$B_1 \cap \Sigma \subset \nabla u\big(\Omega\big). \tag{9.25}$$

Then, using the area formula, an appropriate weighted geometric-arithmetic means inequality, and the concavity condition on the weight w, we obtain our weighted isoperimetric inequality. Note that the solution of (9.24) is

$$u(x) = |x|^2/2 \qquad \text{when } \Omega = B_1 \cap \Sigma. \tag{9.26}$$

In this case, all the chain of inequalities in our proof become equalities, and this yields the sharpness of the result.

In the previous argument there is an important technical difficulty that comes from the possible lack of regularity up to the boundary of the solution to the weighted Neumann problem (9.24). For instance, if $\Omega \cap \Sigma$ is a smooth domain that has some part of its boundary lying on $\partial\Sigma$ —and hence $\partial\Omega$ meets tangentially $\partial\Sigma$—, then u can not be C^1 up to the boundary. This is because the Neumann condition itself is not continuous and hence $\partial_\nu u$ would jump from 1 to one 0 where $\partial\Omega$ meets $\partial\Sigma$.

The fact that u could not be C^1 up to the boundary prevents us from using our contact argument to prove (9.25). Nevertheless, the argument sketched above does work for smooth domains Ω well contained in Σ, that is, satisfying $\overline{\Omega} \subset \Sigma$. If, in addition, $w \equiv 0$ on $\partial\Sigma$ we can deduce the inequality for all measurable sets E by an approximation argument. Indeed, if $w \in C(\overline{\Omega})$ and $w \equiv 0$ on $\partial\Sigma$ then for any domain U with piecewise Lipschitz boundary one has

$$P_w(U;\Sigma) = \int_{\partial U \cap \Sigma} w\, dS = \int_{\partial U} w\, dS.$$

This fact allows us to approximate any set with finite measure $E \subset \Sigma$ by bounded smooth domains Ω_k satisfying $\overline{\Omega_k} \subset \Sigma$. Thus, the proof of Theorem 9.1.3 for weights w vanishing on $\partial\Sigma$ is simpler, and this is why we present it first, in Section 9.5.

Instead, if $w > 0$ at some part of (or everywhere on) $\partial\Sigma$ it is not always possible to find sequences of smooth sets with closure contained in the open cone and approximating in relative perimeter a given measurable set $E \subset \Sigma$. This is because the relative perimeter does not count the part of the boundary of E which lies on $\partial\Sigma$. To get around this difficulty (recall that we are describing the proof in the isotropic case, $H \equiv 1$) we need to consider an anisotropic problem in $\mathbb{R}^n$ for which approximation is

possible. Namely, we choose a gauge H_0 defined as the gauge associated to the convex set $B_1 \cap \Sigma$; see (9.7). Then we prove that $P_{w,H_0}(\,\cdot\,;\Sigma)$ is a calibration of the functional $P_w(\,\cdot\,;\Sigma)$, in the following sense. For all $E \subset \Sigma$ we will have

$$P_{w,H_0}(E;\Sigma) \leq P_w(E;\Sigma),$$

while for $E = B_1 \cap \Sigma$,

$$P_{w,H_0}(B_1;\Sigma) = P_w(B_1 \cap \Sigma;\Sigma).$$

As a consequence, the isoperimetric inequality with perimeter $P_{w,H_0}(\cdot;\Sigma)$ implies the one with the perimeter $P_w(\cdot;\Sigma)$. For $P_{w,H_0}(\cdot;\Sigma)$ approximation results are available and, as in the case of $w \equiv 0$ on $\partial\Sigma$, it is enough to consider smooth sets satisfying $\overline{\Omega} \subset \Sigma$ —for which there are no regularity problems with the solution of the elliptic problem.

To prove Theorem 9.1.3 for a general anisotropic perimeter $P_{w,H}(\cdot;\Sigma)$ we also consider a "calibrated" perimeter $P_{w,H_0}(\cdot;\Sigma)$, where H_0 is now the gauge associated to the convex set $W \cap \Sigma$. Note that, as explained above, even for the isotropic case $H = \|\cdot\|_2$ we have to consider an anisotropic perimeter (associated to $B_1 \cap \Sigma$) in order to prove Theorem 9.1.3.

9.4 Proof of the classical Wulff inequality

In this section we prove the classical Wulff theorem for smooth domains by using the ideas introduced by the first author in [40, 41]. When H is smooth on S^{n-1}, we show also that the Wulff shapes are the only smooth sets for which equality is attained.

Proof of Theorem 9.1.2. We prove the Wulff inequality only for smooth domains, that we denote by Ω instead of E. By approximation, if (9.10) holds for all smooth domains then it holds for all sets of finite perimeter.

By regularizing H on S^{n-1} and then extending it homogeneously, we can assume that H is smooth in $\mathbb{R}^n \setminus \{0\}$. For non-smooth H this approximation argument will yield inequality (9.10), but not the equality cases.

Let u be a solution of the Neumann problem

$$\begin{cases} \Delta u = \dfrac{P_H(\Omega)}{|\Omega|} & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = H(\nu) & \text{on } \partial\Omega, \end{cases} \tag{9.27}$$

where Δ denotes the Laplace operator and $\partial u/\partial\nu$ the exterior normal derivative of u on $\partial\Omega$. Recall that $P_H(\Omega) = \int_{\partial\Omega} H\big(\nu(x)\big)\, dS$. The constant $P_H(\Omega)/|\Omega|$ has been chosen so that the problem has a unique solution up to an additive constant. Since $H|_{S^{n-1}}$ and Ω are smooth, we have that u is smooth in $\overline{\Omega}$. See [224] for a recent exposition of these classical facts and for a new Schauder estimate for (9.27).

Consider the lower contact set of u, defined by

$$\Gamma_u = \big\{x \in \Omega \ :\ u(y) \geq u(x) + \nabla u(x)\cdot(y-x) \ \text{ for all } y \in \overline{\Omega}\big\}. \tag{9.28}$$

It is the set of points where the tangent hyperplane to the graph of u lies below u in all $\overline{\Omega}$. We claim that

$$W \subset \nabla u(\Gamma_u), \tag{9.29}$$

where W denotes the Wulff shape associated to H, given by (9.6).

To show (9.29), take any $p \in W$, i.e., any $p \in \mathbb{R}^n$ satisfying

$$p \cdot \nu < H(\nu) \ \text{ for all } \nu \in S^{n-1}.$$

Let $x \in \overline{\Omega}$ be a point such that

$$\min_{y \in \overline{\Omega}} \{u(y) - p \cdot y\} = u(x) - p \cdot x$$

(this is, up to a sign, the Legendre transform of u). If $x \in \partial\Omega$ then the exterior normal derivative of $u(y) - p \cdot y$ at x would be nonpositive and hence $(\partial u/\partial \nu)(x) \leq p \cdot \nu < H(\nu)$, a contradiction with the boundary condition of (9.27). It follows that $x \in \Omega$ and, therefore, that x is an interior minimum of the function $u(y) - p \cdot y$. In particular, $p = \nabla u(x)$ and $x \in \Gamma_u$. Claim (9.29) is now proved. It is interesting to visualize geometrically the proof of the claim, by considering the graphs of the functions $p \cdot y + c$ for $c \in \mathbb{R}$. These are parallel hyperplanes which lie, for c close to $-\infty$, below the graph of u. We let c increase and consider the first c for which there is contact or "touching" at a point x. It is clear geometrically that $x \notin \partial\Omega$, since $p \cdot \nu < H(\nu)$ for all $\nu \in S^{n-1}$ and $\partial u/\partial \nu = H(\nu)$ on $\partial\Omega$.

Now, from (9.29) we deduce

$$|W| \leq |\nabla u(\Gamma_u)| = \int_{\nabla u(\Gamma_u)} dp \leq \int_{\Gamma_u} \det D^2 u(x)\, dx. \tag{9.30}$$

We have applied the area formula to the smooth map $\nabla u : \Gamma_u \to \mathbb{R}^n$, and we have used that its Jacobian, $\det D^2 u$, is nonnegative in Γ_u by definition of this set.

Next, we use the classical inequality between the geometric and the arithmetic means applied to the eigenvalues of $D^2 u(x)$ (which are nonnegative numbers for $x \in \Gamma_u$). We obtain

$$\det D^2 u \leq \left(\frac{\Delta u}{n}\right)^n \quad \text{in } \Gamma_u. \tag{9.31}$$

This, combined with (9.30) and $\Delta u \equiv P_H(\Omega)/|\Omega|$, gives

$$|W| \leq \left(\frac{P_H(\Omega)}{n|\Omega|}\right)^n |\Gamma_u| \leq \left(\frac{P_H(\Omega)}{n|\Omega|}\right)^n |\Omega|. \tag{9.32}$$

Finally, using that $P_H(W) = n|W|$ —see (9.9)—, we conclude that

$$\frac{P_H(W)}{|W|^{\frac{n-1}{n}}} = n|W|^{\frac{1}{n}} \leq \frac{P_H(\Omega)}{|\Omega|^{\frac{n-1}{n}}}. \tag{9.33}$$

Note that when $\Omega = W$ then the solution of (9.27) is $u(x) = |x|^2/2$ since $\Delta u = n$ and $u_\nu(x) = x \cdot \nu(x) = H\big(\nu(x)\big)$ a.e. on ∂W —recall (9.8). In particular, $\nabla u = Id$ and all the eigenvalues of $D^2 u(x)$ are equal. Therefore, it is clear that all inequalities

(and inclusions) in (9.29)-(9.33) are equalities when $\Omega = W$. This explains why the proof gives the best constant in the inequality.

Let us see next that, when $H|_{S^{n-1}}$ is smooth, the Wulff shaped domains $\Omega = aW + b$ are the only smooth domains for which equality occurs in (9.10). Indeed, if (9.33) is an equality then all the inequalities in (9.30), (9.31), and (9.32) are also equalities. In particular, we have $|\Gamma_u| = |\Omega|$. Since $\Gamma_u \subset \Omega$, Ω is an open set, and Γ_u is closed relatively to Ω, we deduce that $\Gamma_u = \Omega$.

Recall that the geometric and arithmetic means of n nonnegative numbers are equal if and only if these n numbers are all equal. Hence, the equality in (9.31) and the fact that Δu is constant in Ω give that $D^2u = aId$ in all $\Gamma_u = \Omega$, where Id is the identity matrix and $a = P_H(\partial\Omega)/(n|\Omega|)$ is a positive constant. Let $x_0 \in \Omega$ be any given point. Integrating $D^2u = aId$ on segments from x_0, we deduce that

$$u(x) = u(x_0) + \nabla u(x_0) \cdot (x - x_0) + \frac{a}{2} |x - x_0|^2$$

for x in a neighborhood of x_0. In particular, $\nabla u(x) = \nabla u(x_0) + a(x - x_0)$ in such a neighborhood, and hence the map $\nabla u - aI$ is locally constant. Since Ω is connected, we deduce that the map $\nabla u - aI$ is indeed a constant, say $\nabla u - aI \equiv y_0$.

It follows that $\nabla u(\Gamma_u) = \nabla u(\Omega) = y_0 + a\Omega$. By (9.29) we know that $W \subset \nabla u(\Gamma_u) = y_0 + a\Omega$. In addition, these two sets have the same measure since equalities occur in (9.30). Thus, $y_0 + a\Omega$ is equal to W up to a set of measure zero. In fact, in the present situation, since W is convex and $y_0 + a\Omega$ is open, one easily proves that $W = y_0 + a\Omega$. Hence, Ω is of the form $\tilde{a}W + \tilde{b}$ for some $\tilde{a} > 0$ and $\tilde{b} \in \mathbb{R}^n$. □

9.5 Proof of Theorem 9.1.3: the case $w \equiv 0$ on $\partial\Sigma$ and $H = \|\cdot\|_2$

For the sake of clarity, we present in this section the proof of Theorem 9.1.3 under the assumptions $w \equiv 0$ on $\partial\Sigma$ and $H = \|\cdot\|_2$. The proof is simpler in this case. Within the proof we will use the following lemma.

Lemma 9.5.1. *Let w be a positive homogeneous function of degree $\alpha > 0$ in an open cone $\Sigma \subset \mathbb{R}^n$. Then, the following conditions are equivalent:*

- *For each $x, z \in \Sigma$, it holds the following inequality:*

$$\alpha \left(\frac{w(z)}{w(x)} \right)^{1/\alpha} \leq \frac{\nabla w(x) \cdot z}{w(x)}.$$

- *The function $w^{1/\alpha}$ is concave in Σ.*

Proof. Assume first $\alpha = 1$. A function w is concave in Σ if and only if for each $x, z \in \Sigma$ it holds

$$w(x) + \nabla w(x) \cdot (z - x) \geq w(z). \tag{9.34}$$

Now, since w is homogeneous of degree 1, we have

$$\nabla w(x) \cdot x = w(x). \tag{9.35}$$

This can be seen by differentiating the equality $w(tx) = tw(x)$ and evaluating at $t = 1$. Hence, from (9.34) and (9.35) we deduce that an homogeneous function w of degree 1 is concave if and only if

$$w(z) \leq \nabla w(x) \cdot z.$$

This proves the lemma for $\alpha = 1$.

Assume now $\alpha \neq 1$. Define $v = w^{1/\alpha}$, and apply the result proved above to the function v, which is homogeneous of degree 1. We obtain that v is concave if and only if

$$v(z) \leq \nabla v(x) \cdot z.$$

Therefore, since $\nabla v(x) = \alpha^{-1} w(x)^{\frac{1}{\alpha}-1} \nabla w(x)$, we deduce that $w^{1/\alpha}$ is concave if and only if

$$w(z)^{1/\alpha} \leq \frac{\nabla w(x) \cdot z}{\alpha w(x)^{1-\frac{1}{\alpha}}},$$

and the lemma follows. □

We give now the

Proof of Theorem 9.1.3 in the case $w \equiv 0$ on $\partial\Sigma$ and $H = \|\cdot\|_2$. For the sake of simplicity we assume here that $E = U \cap \Sigma$, where U is some bounded smooth domain in $\mathbb{R}^n$. The case of general sets will be treated in Section 9.6 when we prove Theorem 9.1.3 in its full generality.

Observe that since $E = U \cap \Sigma$ is piecewise Lipschitz, and $w \equiv 0$ on $\partial\Sigma$, it holds

$$P_w(E;\Sigma) = \int_{\partial U \cap \Sigma} w(x)dx = \int_{\partial E} w(x)dx. \tag{9.36}$$

Hence, using that $w \in C(\overline{\Sigma})$ and (9.36), it is immediate to prove that for any $y \in \Sigma$ we have

$$\lim_{\delta \downarrow 0} P_w(E + \delta y;\Sigma) = P_w(E;\Sigma) \quad \text{and} \quad \lim_{\delta \downarrow 0} w(E + \delta y) = w(E).$$

We have denoted $E + \delta y = \{x + \delta y\,,\ x \in E\}$. Note that $P_w(E + \delta y;\Sigma)$ could not converge to $P_w(E;\Sigma)$ as $\delta \downarrow 0$ if w did not vanish on the boundary of the cone Σ.

By this approximation property and a subsequent regularization of $E + \delta y$ (a detailed argument can be found in the proof of Theorem 9.1.3 in next section), we see that it suffices to prove (9.13) for smooth domains whose closure is contained in Σ. Thus, from now on in the proof we denote by Ω, instead of E, any smooth domain satisfying $\overline{\Omega} \subset \Sigma$. We next show (9.13) with E replaced by Ω.

At this stage, it is clear that by approximating $w|_{\overline{\Omega}}$ we can assume $w \in C^\infty(\overline{\Omega})$.

Let u be a solution of the linear Neumann problem

$$\begin{cases} w^{-1}\mathrm{div}(w\nabla u) = b_\Omega & \text{in } \Omega \quad (\text{with } \overline{\Omega} \subset \Sigma) \\ \dfrac{\partial u}{\partial \nu} = 1 & \text{on } \partial\Omega. \end{cases} \tag{9.37}$$

The Fredholm alternative ensures that there exists a solution of (9.37) (which is unique up to an additive constant) if and only if the constant b_Ω is given by

$$b_\Omega = \frac{P_w(\Omega;\Sigma)}{w(\Omega)}. \tag{9.38}$$

Note also that since w is positive and smooth in $\overline{\Omega}$, (9.37) is a uniformly elliptic problem with smooth coefficients. Thus, $u \in C^\infty(\overline{\Omega})$. For these classical facts, see Example 2 in Section 10.5 of [171], or the end of Section 6.7 of [157].

Consider now the lower contact set of u, Γ_u, defined by (9.28) as the set of points in Ω at which the tangent hyperplane to the graph of u lies below u in all $\overline{\Omega}$. Then, as in the proof of the Wulff theorem in Section 9.4, we touch by below the graph of u with hyperplanes of fixed slope $p \in B_1$, and using the boundary condition in (9.37) we deduce that $B_1 \subset \nabla u(\Gamma_u)$. From this, we obtain

$$B_1 \cap \Sigma \subset \nabla u(\Gamma_u) \cap \Sigma \tag{9.39}$$

and thus

$$\begin{aligned} w(B_1 \cap \Sigma) &\leq \int_{\nabla u(\Gamma_u)\cap\Sigma} w(p)dp \\ &\leq \int_{\Gamma_u \cap (\nabla u)^{-1}(\Sigma)} w(\nabla u(x)) \det D^2 u(x)\, dx \\ &\leq \int_{\Gamma_u \cap (\nabla u)^{-1}(\Sigma)} w(\nabla u) \left(\frac{\Delta u}{n}\right)^n dx. \end{aligned} \tag{9.40}$$

We have applied the area formula to the smooth map $\nabla u : \Gamma_u \to \mathbb{R}^n$ and also the classical arithmetic-geometric means inequality —all eigenvalues of D^2u are nonnegative in Γ_u by definition of this set.

Next we use that, when $\alpha > 0$,

$$s^\alpha t^n \leq \left(\frac{\alpha s + nt}{\alpha + n}\right)^{\alpha+n} \quad \text{for all} \quad s > 0 \text{ and } t > 0,$$

which follows from the concavity of the logarithm function. Using also Lemma 9.5.1, we find

$$\frac{w(\nabla u)}{w(x)} \left(\frac{\Delta u}{n}\right)^n \leq \left(\frac{\alpha \left(\frac{w(\nabla u)}{w(x)}\right)^{1/\alpha} + \Delta u}{\alpha + n}\right)^{\alpha+n} \leq \left(\frac{\frac{\nabla w(x)\cdot \nabla u}{w(x)} + \Delta u}{D}\right)^D.$$

Recall that $D = n + \alpha$. Thus, using the equation in (9.37), we obtain

$$\frac{w(\nabla u)}{w(x)} \left(\frac{\Delta u}{n}\right)^n \leq \left(\frac{b_\Omega}{D}\right)^D \quad in\ \Gamma_u \cap (\nabla u)^{-1}(\Sigma). \tag{9.41}$$

If $\alpha = 0$ then $w \equiv 1$, and (9.41) is trivial.

Therefore, since $\Gamma_u \subset \Omega$, combining (9.40) and (9.41) we obtain

$$\begin{aligned} w(B_1 \cap \Sigma) &\leq \int_{\Gamma_u \cap (\nabla u)^{-1}(\Sigma)} \left(\frac{b_\Omega}{D}\right)^D w(x)dx = \left(\frac{b_\Omega}{D}\right)^D w(\Gamma_u \cap (\nabla u)^{-1}(\Sigma)) \\ &\leq \left(\frac{b_\Omega}{D}\right)^D w(\Omega) = D^{-D} \frac{P_w(\Omega;\Sigma)^D}{w(\Omega)^{D-1}}. \end{aligned} \tag{9.42}$$

In the last equality we have used the value of the constant b_Ω, given by (9.38).

Finally, using that, by (9.12), we have $P_w(B_1;\Sigma) = D\, w(B_1 \cap \Sigma)$, we obtain the desired inequality (9.13).

An alternative way to see that (9.42) is equivalent to (9.13) is to analyze the previous argument when $\Omega = B_1 \cap \Sigma$. In this case $\overline{\Omega} \not\subseteq \Sigma$ and therefore, as explained in Section 9.3, we must solve problem (9.24) instead of problem (9.37). When $\Omega = B_1 \cap \Sigma$ the solution to problem (9.24) is $u(x) = |x|^2/2$. For this function u we have $\Gamma_u = B_1 \cap \Sigma$ and $b_{B_1 \cap \Sigma} = P_w(B_1;\Sigma)/w(B_1 \cap \Sigma)$ —as in (9.38). Hence, for these concrete Ω and u one verifies that all inclusions and inequalities in (9.39), (9.40), (9.41), (9.42) are equalities, and thus (9.13) follows. □

9.6 Proof of Theorem 9.1.3: the general case

In this section we prove Theorem 9.1.3 in its full generality. At the end of the section, we include the geometric argument of E. Milman that provides an alternative proof of Theorem 9.1.3 in the case that the exponent α is an integer.

Proof of Theorem 9.1.3. Let

$$W_0 := W \cap \Sigma,$$

an open convex set, and nonempty by assumption. Since $\lambda W_0 \subset W_0$ for all $\lambda \in (0,1)$, we deduce that $0 \in \overline{W}_0$. Therefore, as commented in subsection 9.1.1, there is a unique gauge H_0 such that its Wulff shape is W_0. In fact, H_0 is defined by expression (9.7) (with W and H replaced by W_0 and H_0).

Since $H_0 \leq H$ we have

$$P_{w,H_0}(E;\Sigma) \leq P_{w,H}(E;\Sigma) \quad \textit{for each measurable set } E,$$

while, using (9.11),

$$P_{w,H_0}(W_0;\Sigma) = P_{w,H}(W;\Sigma) \quad \text{and} \quad w(W_0) = w(W \cap \Sigma).$$

Thus, it suffices to prove that

$$\frac{P_{w,H_0}(E;\Sigma)}{w(E)^{\frac{D-1}{D}}} \geq \frac{P_{w,H_0}(W_0;\Sigma)}{w(W_0)^{\frac{D-1}{D}}} \tag{9.43}$$

for all measurable sets $E \subset \Sigma$ with $w(E) < \infty$.

The definition of H_0 is motivated by the following reason. Note that H_0 vanishes on the directions normal to the cone Σ. Thus, by considering H_0 instead of H, we will

be able (by an approximation argument) to assume that E is a smooth domain whose closure is contained in Σ. This approximation cannot be done when H does not vanish on the directions normal to the cone —since the relative perimeter does not count the part of the boundary lying on $\partial\Sigma$, while when $\overline{E} \subset \Sigma$ the whole perimeter is counted.

We split the proof of (9.43) in three cases.

Case 1. Assume that $E = \Omega$, where Ω is a smooth domain satisfying $\overline{\Omega} \subset \Sigma$.

At this stage, it is clear that by regularizing $w|_{\overline{\Omega}}$ and $H_0|_{S^{n-1}}$ we can assume $w \in C^\infty(\overline{\Omega})$ and $H_0 \in C^\infty(S^{n-1})$.

Let u be a solution to the Neumann problem

$$\begin{cases} w^{-1}\mathrm{div}(w\nabla u) = b_\Omega & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = H_0(\nu) & \text{on } \partial\Omega, \end{cases} \tag{9.44}$$

where $b_\Omega \in \mathbb{R}$ is chosen so that the problem has a unique solution up to an additive constant, that is,

$$b_\Omega = \frac{P_{w,H_0}(\Omega;\Sigma)}{w(\Omega)}. \tag{9.45}$$

Since w is positive and smooth in $\overline{\Omega}$, and H_0, ν, and Ω are smooth, we have that $u \in C^\infty(\overline{\Omega})$. See our comments following (9.37)-(9.38) for references of these classical facts.

Consider the lower contact set of u, defined by

$$\Gamma_u = \{x \in \Omega \ : \ u(y) \geq u(x) + \nabla u(x)\cdot(y-x) \ \text{ for all } y \in \overline{\Omega}\}.$$

We claim that

$$W_0 \subset \nabla u(\Gamma_u) \cap \Sigma. \tag{9.46}$$

To prove (9.46), we proceed as in the proof of Theorem 9.1.2 in Section 9.4. Take $p \in W_0$, that is, $p \in \mathbb{R}^n$ satisfying $p\cdot\nu < H_0(\nu)$ for each $\nu \in S^{n-1}$. Let $x \in \overline{\Omega}$ be a point such that

$$\min_{y\in\overline{\Omega}} \{u(y) - p\cdot y\} = u(x) - p\cdot x.$$

If $x \in \partial\Omega$ then the exterior normal derivative of $u(y) - p\cdot y$ at x would be nonpositive and, hence, $(\partial u/\partial\nu)(x) \leq p\cdot\nu < H_0(p)$, a contradiction with (9.44). Thus, $x \in \Omega$, $p = \nabla u(x)$, and $x \in \Gamma_u$ —see Section 9.4 for more details. Hence, $W_0 \subset \nabla u(\Gamma_u)$, and since $W_0 \subset \Sigma$, claim (9.46) follows.

Therefore,

$$w(W_0) \leq \int_{\nabla u(\Gamma_u)\cap\Sigma} w(p)dp \leq \int_{\Gamma_u \cap (\nabla u)^{-1}(\Sigma)} w(\nabla u)\det D^2u \, dx. \tag{9.47}$$

We have applied the area formula to the smooth map $\nabla u : \Gamma_u \to \mathbb{R}^n$, and we have used that its Jacobian, $\det D^2u$, is nonnegative in Γ_u by definition of this set.

We proceed now as in Section 9.5. Namely, we first use the following weighted version of the inequality between the arithmetic and the geometric means,

$$a_0^\alpha a_1 \cdots a_n \leq \left(\frac{\alpha a_0 + a_1 + \cdots + a_n}{\alpha + n}\right)^{\alpha+n},$$

applied to the numbers $a_0 = \left(\frac{w(\nabla u)}{w(x)}\right)^{1/\alpha}$ and $a_i = \lambda_i(x)$ for $i = 1, ..., n$, where $\lambda_1, ..., \lambda_n$ are the eigenvalues of D^2u. We obtain

$$\frac{w(\nabla u)}{w(x)} \det D^2 u \leq \left(\frac{\alpha \left(\frac{w(\nabla u)}{w(x)}\right)^{1/\alpha} + \Delta u}{\alpha + n} \right)^{\alpha+n} \leq \left(\frac{\frac{\nabla w(x) \cdot \nabla u}{w(x)} + \Delta u}{\alpha + n} \right)^{\alpha+n}. \tag{9.48}$$

In the last inequality we have used Lemma 9.5.1. Now, the equation in (9.44) gives

$$\frac{\nabla w(x) \cdot \nabla u}{w(x)} + \Delta u = \frac{div(w(x)\nabla u)}{w(x)} \equiv b_\Omega,$$

and thus using (9.45) we find

$$\begin{aligned} \int_{\Gamma_u \cap (\nabla u)^{-1}(\Sigma)} w(\nabla u) \det D^2 u \, dx &\leq \int_{\Gamma_u \cap (\nabla u)^{-1}(\Sigma)} w(x) \left(\frac{b_\Omega}{D}\right)^D dx \\ &\leq \int_{\Gamma_u} w(x) \left(\frac{b_\Omega}{D}\right)^D dx = \left(\frac{P_{w,H_0}(\Omega;\Sigma)}{D\, w(\Omega)}\right)^D w(\Gamma_u). \end{aligned} \tag{9.49}$$

Therefore, from (9.47) and (9.49) we deduce

$$w(W_0) \leq \left(\frac{P_{w,H_0}(\Omega;\Sigma)}{D\, w(\Omega)}\right)^D w(\Gamma_u) \leq \left(\frac{P_{w,H_0}(\Omega;\Sigma)}{D\, w(\Omega)}\right)^D w(\Omega). \tag{9.50}$$

Finally, using that, by (9.12), we have $P_{w,H_0}(W;\Sigma) = D\, w(W_0)$, we deduce (9.43).

An alternative way to see that (9.50) is equivalent to (9.43) is to analyze the previous argument when $\Omega = W_0 = W \cap \Sigma$. In this case $\overline{\Omega} \not\subseteq \Sigma$ and therefore, as explained in Section 9.3, we must solve problem

$$\begin{cases} w^{-1} div\,(w \nabla u) = b_\Omega & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = H_0(\nu) & \text{on } \partial\Omega \cap \Sigma \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \cap \partial\Sigma \end{cases} \tag{9.51}$$

instead of problem (9.44). When $\Omega = W_0$, the solution to problem (9.51) is

$$u(x) = |x|^2/2.$$

For this function u we have $\Gamma_u = W_0$ and $b_{W_0} = P_{w,H_0}(W_0;\Sigma)/w(W_0)$ —as in (9.45). Hence, for these concrete Ω and u one verifies that all inclusions and inequalities in (9.46), (9.47), (9.48), (9.49), and (9.50) are equalities, and thus (9.43) follows.

Case 2. Assume now that $E = U \cap \Sigma$, where U is a bounded smooth open set in $\mathbb{R}^n$. Even that both U and Σ are Lipschitz sets, their intersection might not be Lipschitz (for instance if ∂U and $\partial\Sigma$ meet tangentially at a point). As a consequence, approximating $U \cap \Sigma$ by smooth sets converging in perimeter is a more subtle issue.

However, we claim that there exists a sequence $\{\Omega_k\}_{k\geq 1}$ of smooth bounded domains satisfying

$$\overline{\Omega_k} \subset \Sigma \qquad \text{and} \qquad \lim_{k\to\infty} \frac{P_{w,H_0}(\Omega_k;\Sigma)}{w(\Omega_k)^{\frac{D-1}{D}}} \leq \frac{P_{w,H_0}(E;\Sigma)}{w(E)^{\frac{D-1}{D}}}. \tag{9.52}$$

Case 2 follows immediately using this claim and what we have proved in Case 1. We now proceed to prove the claim.

It is no restriction to assume that e_n, the n-th vector of the standard basis, belongs to the cone Σ. Then, $\partial\Sigma$ is a convex graph (and therefore, Lipschitz in every compact set) over the variables $x_1, \ldots, x_{n-1}$. That is,

$$\Sigma = \{x_n > g(x_1, \ldots, x_{n-1})\} \tag{9.53}$$

for some convex function $g : \mathbb{R}^{n-1} \to \mathbb{R}$.

First we construct a sequence

$$F_k = \{x_n > g_k(x_1, \ldots, x_{n-1})\}, \quad k \geq 1 \tag{9.54}$$

of convex smooth sets whose boundary is a graph $g_k : \mathbb{R}^{n-1} \to \mathbb{R}$ over the first $n-1$ variables and satisfying:

(i) $g_1 > g_2 > g_3 > \ldots$ in $\overline{B}$, where B is a large ball $B \subset \mathbb{R}^{n-1}$ containing the projection of $\overline{U}$.

(ii) $g_k \to g$ uniformly in $\overline{B}$.

(iii) $\nabla g_k \to \nabla g$ almost everywhere in $\overline{B}$ and $|\nabla g_k|$ is bounded independently of k.

(iv) The smooth manifolds $\partial F_k = \{x_n = g_k(x_1, \ldots, x_{n-1})\}$ and ∂U intersect transversally.

To construct the sequence g_k, we consider the convolution of g with a standard mollifier

$$\tilde{g}_k = g * k^{n-1}\eta(kx) + \frac{C}{k}$$

with C is a large constant (depending on $\|\nabla g\|_{L^\infty(\mathbb{R}^{n-1})}$) to guarantee $\tilde{g}_k > g$ in $\overline{B}$. It follows that a subsequence of $\tilde{g}_k$ will satisfy (i)-(iii). Next, by a version of Sard's Theorem [164, Section 2.3] almost every small translation of the smooth manifold $\{x_n = \tilde{g}_k(x_1, \ldots, x_{n-1})\}$ will intersect ∂U transversally. Thus, the sequence

$$g_k(x_1, \ldots, x_{n-1}) = \tilde{g}_k(x_1 - y_1^k, \ldots, x_{n-1} - y_{n-1}^k) + y_n^k$$

will satisfy (i)-(iv) if $y^k \in \mathbb{R}^n$ are chosen with $|y^k|$ sufficiently small depending on k —in particular to preserve (i).

Let us show now that $P_{w,H_0}(U \cap F_k;\Sigma)$ converges to $P_{w,H_0}(E;\Sigma)$ as $k \uparrow \infty$. Note that (i) yields $F_k \subset F_{k+1}$ for all $k \geq 1$. This monotonicity will be useful to prove the convergence of perimeters, that we do next.

Indeed, since we considered the gauge H_0 instead of H, we have the following property

$$P_{w,H_0}(E;\Sigma) = \int_{\partial U \cap \Sigma} H_0(\nu(x))w(x)dx = \int_{\partial E} H_0(\nu(x))w(x)dx. \tag{9.55}$$

This is because $\partial E = \partial(U \cap E) \subset (\partial U \cap \Sigma) \cup (\overline{U} \cap \partial\Sigma)$ and

$$H_0(\nu(x)) = 0 \quad \text{for almost all } \ x \in \partial\Sigma. \tag{9.56}$$

Now, since $\partial(U \cap F_k) \subset (\partial U \cap F_k) \cup (\overline{U} \cap \partial F_k)$ we have

$$0 \leq P_{w,H_0}(U \cap F_k; \Sigma) - \int_{\partial U \cap F_k} H_0(\nu(x))w(x)dx \leq \int_{\overline{U} \cap \partial F_k} H_0(\nu_{F_k}(x))w(x)dx.$$

On one hand, using dominated convergence, (9.53), (9.54), (ii)-(iii), and (9.56), we readily prove that

$$\int_{\overline{U} \cap \partial F_k} H_0(\nu_{F_k}(x))w(x)dx \to 0.$$

On the other hand, by (i) and (ii), $F_k \cap (B \times \mathbb{R})$ is an increasing sequence exhausting $\Sigma \cap (B \times \mathbb{R})$. Hence, by monotone convergence

$$\int_{\partial U \cap F_k} H_0(\nu(x))w(x)dx \to \int_{\partial U \cap \Sigma} H_0(\nu(x))w(x)dx = P_{w,H_0}(E; \Sigma).$$

Therefore, the sets $U \cap F_k$ approximate $U \cap \Sigma$ in L^1 and in the (w,H_0)-perimeter. Moreover, by (iv), $U \cap F_k$ are Lipschitz open sets.

Finally, to obtain the sequence of smooth domains Ω_k in (9.52), we use a partition of unity and local regularization of the Lipschitz sets $U \cap F_k$ to guarantee the convergence of the (w,H_0)-perimeters. In case that the regularized sets had more than one connected component, we may always choose the one having better isoperimetric quotient.

Case 3. Assume that E is any measurable set with $w(E) < \infty$ and $P_{w,H_0}(E; \Sigma) \leq P_{w,H}(E; \Sigma) < \infty$. As a consequence of Theorem 5.1 in [16], $C_c^\infty(\mathbb{R}^n)$ is dense in the space BV_{μ,H_0} of functions of bounded variation with respect to the measure $\mu = w\chi_\Sigma$ and the gauge H_0. Note that our definition of perimeter $P_{w,H_0}(E; \Sigma)$ coincides with the (μ, H_0)-total variation of the characteristic function χ_E, that is, $|D_\mu \chi_E|_{H_0}$ in notation of [16]. Hence, by the coarea formula in Theorem 4.1 in [16] and the argument in Section 6.1.3 in [208], we find that for each measurable set $E \subset \Sigma$ with finite measure there exists a sequence of bounded smooth sets $\{U_k\}$ satisfying

$$\lim_{k\to\infty} w(U_k \cap \Sigma) = w(E) \quad \textit{and} \quad \lim_{k\to\infty} P_{w,H_0}(U_k; \Sigma) = P_{w,H_0}(E; \Sigma).$$

Then we are back to Case 2 above, and hence the proof is finished. □

After the announcement of our result and proof in [51], Emanuel Milman showed us a nice geometric construction that yields the weighted inequality in Theorem 9.1.3 in the case that α is a nonnegative *integer*. We next sketch this construction.

Remark 9.6.1 (Emanuel Milman's construction). When α is a nonnegative integer the weighted isoperimetric inequality of Theorem 9.1.3 (when $H = \|\cdot\|_2$) can be proved as a limit case of the Lions-Pacella inequality in convex cones of $\mathbb{R}^{n+\alpha}$. Indeed, let $w^{1/\alpha} > 0$ be a concave function, homogeneous of degree 1, in an open convex cone $\Sigma \subset \mathbb{R}^n$. For each $\varepsilon > 0$, consider the cone

$$\mathcal{C}_\varepsilon = \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^\alpha \,:\, x \in \Sigma,\ |y| < \varepsilon w(x)^{1/\alpha}\right\}.$$

From the convexity of Σ and the concavity of $w^{1/\alpha}$ we have that $\mathcal{C}_\varepsilon$ is a convex cone. Hence, by Theorem 9.1.1 we have

$$\frac{P(\tilde{E};\mathcal{C}_\varepsilon)}{|\tilde{E}\cap\mathcal{C}_\varepsilon|^{\frac{n+\alpha-1}{n+\alpha}}} \geq \frac{P(\mathcal{B}_1;\mathcal{C}_\varepsilon)}{|\mathcal{B}_1\cap\mathcal{C}_\varepsilon|^{\frac{n+\alpha-1}{n+\alpha}}} \quad \text{for all } \tilde{E} \text{ with } |\tilde{E}\cap\mathcal{C}_\varepsilon|<\infty, \tag{9.57}$$

where $\mathcal{B}_1$ is the unit ball of $\mathbb{R}^{n+\alpha}$. Now, given a Lipschitz set $E\subset\mathbb{R}^n$, consider the cylinder $\tilde{E}=E\times\mathbb{R}^\alpha$ one finds

$$|\tilde{E}\cap\mathcal{C}_\varepsilon| = \int_{E\cap\Sigma} dx \int_{\{|y|<\varepsilon w(x)^{1/\alpha}\}} dy = \omega_\alpha\varepsilon^\alpha \int_{E\cap\Sigma} w(x)dx = \omega_\alpha\varepsilon^\alpha w(E\cap\Sigma)$$

and

$$P(\tilde{E};\mathcal{C}_\varepsilon) = \int_{\partial E\cap\Sigma} dS(x) \int_{\{|y|<\varepsilon w(x)^{1/\alpha}\}} dy = \omega_\alpha\varepsilon^\alpha \int_{\partial E\cap\Sigma} w(x)dS = \omega_\alpha\varepsilon^\alpha P_w(E;\Sigma).$$

On the other hand, one easily sees that, as $\varepsilon\downarrow 0$,

$$\frac{P(\mathcal{B}_1;\mathcal{C}_\varepsilon)}{|\mathcal{B}_1\cap\mathcal{C}_\varepsilon|^{\frac{n+\alpha-1}{n+\alpha}}} = (\omega_\alpha\varepsilon^\alpha)^{\frac{1}{n+\alpha}}\left(\frac{P_w(B_1;\Sigma)}{w(B_1\cap\Sigma)^{\frac{n+\alpha-1}{n+\alpha}}}+o(1)\right),$$

where B_1 is the unit ball of $\mathbb{R}^n$. Hence, letting $\varepsilon\downarrow 0$ in (9.57) one obtains

$$\frac{P_w(E;\Sigma)}{w(E\cap\Sigma)^{\frac{n+\alpha-1}{n+\alpha}}} \geq \frac{P_w(B_1;\Sigma)}{w(B_1\cap\Sigma)^{\frac{n+\alpha-1}{n+\alpha}}},$$

which is the inequality of Theorem 9.1.3 in the case that $H=\|\cdot\|_2$ and α is an integer.

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