Models for bacteriophage systems, Weak convergence of Gaussian processes and L^2 modulus of Brownian local time

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CERTIFICO que la present Memòria ha estat realitzada per en David Bascompte Viladrich, sota la direcció del Dr. Xavier Bardina Simorra.

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Introduction

In this dissertation three different problems are treated. In Chapter 1 we construct two families of processes that converge, in the sense of the finite dimensional distributions, towards two independent Gaussian processes. Chapter 2 is devoted to the study of a model of bacteriophage treatments for bacterial infections. Finally, in Chapter 3 we study some aspects of the L^2 modulus of continuity of Brownian local time.

The Wiener process, or standard Brownian motion, plays an important role both in pure and applied mathematics. In probability theory one of the most fundamental concept is the convergence in law, and a well known result is the following one, due to Stroock (see [48])

Theorem (Stroock). *Let $N = \{N_s, s \geq 0\}$ be a standard Poisson process. Then the family of processes*

$$\left\{ x_\varepsilon(t) = \frac{1}{\varepsilon} \int_0^t (-1)^{N_{\frac{s}{\varepsilon^2}}} \, ds, \quad t \in [0,T] \right\},$$

defined from the kernels $\theta_\varepsilon = \frac{1}{\varepsilon}(-1)^{N_{\frac{s}{\varepsilon^2}}}$, converges in law in $\mathcal{C}([0,T])$ to a standard Brownian motion.

Many generalizations of this result have been studied (see, e.g., [21] or [6]). The main result we introduce in Chapter 1 is another generalization, in which we consider two independent Gaussian processes that can be represented in terms of a stochastic integral of a deterministic kernel with respect to the Wiener process, i.e., we consider

$$Y^f = \left\{ \int_0^\infty f(t,s) \mathrm{d}W_s, t \in [0,T] \right\}$$

and

$$\tilde{Y}^g = \left\{ \int_0^\infty g(t,s) \mathrm{d}\tilde{W}_s, t \in [0,T], \right\}$$

where $W = \{W_s, s \geq 0\}$ and $\tilde{W} = \{\tilde{W}_s, s \geq 0\}$ are independent standard Brownian motions, and we construct, from a single Poisson process, two families of processes that converge, in the sense of the finite dimensional distributions, towards the processes Y^f and $\tilde{Y}^g$.

We will use this result to prove convergence in law results towards some other processes, like sub-fractional Brownian motion, which is a centered Gaussian process with covariance function

$$\mathrm{Cov}(S_t^H, S_s^H) = s^H + t^H - \frac{1}{2}\left[(s+t)^H + |s-t|^H\right]$$

where $H \in (0, 2)$.

Bacteria are prokaryotic organisms (i.e., organisms whose cells lack a membrane-bound nucleus), typically a few micrometres in length, and can take a wide range of shapes and inhabit in most habitats on the planet. Some bacteria can interact with animals. Some of them form a beneficial interaction (mutualism) and some others, called pathogens, form a parasitic association. They are the cause of human and animal death and diseases.

A virus is a small infectious agent that replicates only inside the living cells of other organisms. Those that use bacteria to replicate are called bacteriophages (or phages for short). With lytic bacteriophages bacterial cells are broken open (lysed) and destroyed after the replication of the virion. As such, they were found to be antibacterial agents and can be used as a tool to treat bacterial infections or to prevent them in food, animals or even humans.

In Chapter 2 we construct and study several models that pretend to study how will behave a treatment of bateriophages in some farm animals. This problem has been brought to our attention by the Molecular Biology Group of the Department of Genetics and Microbiology at the *Universitat Autònoma de Barcelona*.

Starting from a basic model, we will study several variations, first from a deterministic point of view, finding several results on equilibria and stability, and later in a noisy context, producing concentration type results.

Let $B = \{B_t, t \geq 0\}$ be a standard Brownian motion. The local time of a Brownian motion characterizes the amount of time a particle has spent at a given level and can be defined as follows

$$L_t^x := L^x(t) = \int_0^t \delta(x - B_s)\mathrm{d}s.$$

Then we may define the L^2 modulus of continuity of the Brownian local time as

$$G_t(h) = \int_{\mathbb{R}} (L_t^{x+h} - L_t^x)^2 \mathrm{d}x.$$

In [18] the authors prove the following Central Limit Theorem for the L^2 modulus of continuity of Brownian local time.

Theorem ([18], Theorem 1.1 or [26], Theorem 1)**.** *For each fixed $t > 0$,*

$$h^{-\frac{3}{2}}\left(\int_{\mathbb{R}}(L_t^{x+h} - L_t^x)^2 \mathrm{d}x - 4th\right) \xrightarrow{\mathcal{L}} 8\sqrt{\frac{\alpha_t}{3}}\eta,$$

as h tends to zero, where

$$\alpha_t = \int_{\mathbb{R}}(L_t^x)^2 \mathrm{d}x,$$

and η is a $N(0,1)$ random variable independent of B.

In this work we will prove the following result concerning the Wiener chaos decomposition of $G_t(h)$.

Theorem. *Let $G_t(h)$ be the random variable defined in* (3.1.1) *and denote the n-th Wiener chaos element of $G_t(h)$ by $\tilde{I}_n(G_t(h))$. Then, for $n = 2k$, $k \in \mathbb{N}^*$,*

$$\frac{1}{h^2\sqrt{\log(1/h)}}\,\tilde{I}_n(G_t(h)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_n^2)$$

as h tends to zero, where $\mathcal{N}(0,\sigma_n^2)$ is a centered Normal random variable with variance $\sigma_n^2 = \frac{2^6 t(2(k-1))!}{\pi 2^{2(k-1)}((k-1)!)^2}$. For $n = 2k-1$, $k \in \mathbb{N}^*$, the limit is zero.

This result is a joint work with Professor David Nualart, and in particular provides us with an example of a family of random variables that is convergent in law to a Normal distribution, but its chaos elements of even order do not converge.

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Chapter 1

Weak convergence of Gaussian processes

In this chapter, which is based on the paper [7], we consider two independent Gaussian processes that admit a representation in terms of a stochastic integral of a deterministic kernel with respect to a standard Wiener process. We construct two families of processes, from a unique Poisson process, the finite dimensional distributions of which converge in law towards the finite dimensional distributions of the two independent Gaussian processes. As an application of this result we obtain families of processes that converge in law towards fractional Brownian motion and sub-fractional Brownian motion.

We will also present a decomposition result of the sub-fractional Brownian motion, originally due to J. R. de Chávez and C. Tudor (see [43]), that we obtained independently and we will use it to obtain the convergence in law result towards sub-fractional Brownian motion mentioned before.

1.1 Introduction and preliminaries

Let $f(t,\cdot)$ and $g(t,\cdot)$ be functions of $\mathrm{L}^2(\mathbb{R}^+)$ for all $t \in [0,T]$, $T > 0$ and consider the processes given by

$$Y^f = \left\{ \int_0^\infty f(t,s)\mathrm{d}W_s, t \in [0,T] \right\} \tag{1.1.1}$$

and

$$\tilde{Y}^g = \left\{ \int_0^\infty g(t,s)\mathrm{d}\tilde{W}_s, t \in [0,T] \right\} \tag{1.1.2}$$

where $W = \{W_s, s \geq 0\}$ and $\tilde{W} = \{\tilde{W}_s, s \geq 0\}$ are independent standard Brownian motions.

The first result we introduce here is the construction of two families of processes, from a unique Poisson process, that converge, in the sense of the finite dimensional distributions, to the processes Y^f and $\tilde{Y}^g$. We will use this result later in order to prove weak convergence results towards different kinds of processes such as fractional Brownian motion and sub-fractional Brownian motion.

It is well known the result by Stroock (see [48]) where it is shown that the family of processes

$$\left\{x_\varepsilon(t) = \frac{1}{\varepsilon}\int_0^t (-1)^{N_{\frac{s}{\varepsilon^2}}}\,\mathrm{d}s, \quad t\in[0,T]\right\},$$

defined from the kernels $\theta_\varepsilon = \frac{1}{\varepsilon}(-1)^{N_{\frac{s}{\varepsilon^2}}}$, converges in law in $\mathcal{C}([0,T])$ to a standard Brownian motion, where $N=\{N_s, s\geq 0\}$ is a standard Poisson process. This kind of processes were introduced by Kac in [28] in order to write the solution of telegrapher's equation in terms of Poisson process.

On the other hand, Delgado and Jolis (see [21]) extend this result to processes represented by a stochastic integral, with respect to a standard Wiener process, of a deterministic kernel that satisfies some regularity conditions.

A generalization of Stroock's result can be found in [6], where it is proved that the family

$$\left\{x_\varepsilon^\theta(t) = \frac{2}{\varepsilon}\int_0^t e^{i\theta N_{\frac{2s}{\varepsilon^2}}}\,\mathrm{d}s, \quad t\in[0,T]\right\} \tag{1.1.3}$$

converges in law in $\mathcal{C}([0,T])$ to a complex Brownian motion, for $\theta\in(0,\pi)\cup(\pi,2\pi)$. Particularly, the real part and the imaginary part of (1.1.3) tend to independent standard Brownian motions.

Given $\{N_s, s\geq 0\}$ a standard Poisson process and $\theta\in(0,\pi)\cup(\pi,2\pi)$, we consider the following families of approximating processes

$$Y_\varepsilon^f = \left\{\frac{2}{\varepsilon}\int_0^\infty f(t,s)\cos\left(\theta N_{\frac{2s}{\varepsilon^2}}\right)\mathrm{d}s, \quad t\in[0,T]\right\} \tag{1.1.4}$$

and

$$\tilde{Y}_\varepsilon^g = \left\{\frac{2}{\varepsilon}\int_0^\infty g(t,s)\sin\left(\theta N_{\frac{2s}{\varepsilon^2}}\right)\mathrm{d}s, \quad t\in[0,T]\right\}. \tag{1.1.5}$$

The main result of [7] is the proof that the finite dimensional distributions of the processes Y_ε^f and $\tilde{Y}_\varepsilon^g$ converge in law to the finite dimensional distributions of the processes Y^f and $\tilde{Y}^g$ given by (1.1.1) and (1.1.2), respectively.

It is important to note that the processes Y_ε^f and $\tilde{Y}_\varepsilon^g$ are both functionally dependent. Nevertheless, integrating and taking limits, we obtain two independent processes.

As an application of this result it can be obtained approximations for different examples of centered Gaussian processes, among others, fractional Brownian motion and sub-fractional Brownian motion.

Recall that *fractional Brownian motion* (fBm for short) $B^H = \{B^H(t), t \geq 0\}$ is a centered Gaussian process with covariance function

$$\operatorname{Cov}(B_t^H, B_s^H) = \frac{1}{2}\left(s^H + t^H - |s-t|^H\right) \tag{1.1.6}$$

where $H \in (0,2)$. Usually fBm is defined with Hurst parameter belonging to the interval $(0,1)$ with the corresponding covariance, but in order to compare it with sub-fBm (as defined in[13]) we use the stated representation with $H \in (0,2)$.

On the other hand, *sub-fractional Brownian motion* (sub-fBm for brevity) $S^H = \{S^H(t), t \geq 0\}$ is a centered Gaussian process with covariance function

$$\operatorname{Cov}(S_t^H, S_s^H) = s^H + t^H - \frac{1}{2}\left[(s+t)^H + |s-t|^H\right] \tag{1.1.7}$$

where $H \in (0,2)$.

This process was introduced by Bojdecki *et al.* in 2004 (see [13]) as an intermediate process between standard Brownian motion and fractional Brownian motion. Note that both fBm and sub-fBm are standard Brownian motions for $H = 1$.

For $H \neq 1$, sub-fBm preserves some of the main properties of fBm, such as long-range dependence, but its increments are not stationary; they are more weakly correlated on non-overlapping intervals than fBm ones, and their covariance decays polynomially at a higher rate as the distance between the intervals tends to infinity. For a more detailed discussion of sub-fBm and its properties we refer the reader to [13]. Some properties of this process have also been studied in [50] and [49]. On the other hand there are some extensions of sub-fBm in [14] and [45].

In [43] (see Theorem 1.3.4 below) the authors obtain a decomposition of the sub-fBm in terms of fBm and another process with absolutely continuous trajectories, $X^H = \{X_t^H, t \geq 0\}$, which is defined by Lei and Nualart in [33] by

$$X_t^H = \int_0^\infty (1 - e^{-rt}) r^{-\frac{1+H}{2}} \, \mathrm{d}W_r \tag{1.1.8}$$

where W is a standard Brownian motion. Lei and Nualart introduce this process in order to obtain a decomposition of bifractional Brownian motion into the sum of a transformation of X_t^H and a fBm.

The decomposition of sub-fractional Brownian motion is different for $H \in (0,1)$ and $H \in (1,2)$. In the first case, sub-fBm is obtained as a sum of two independent processes, namely fBm and the process defined by (1.1.8), while for $H \in (1,2)$ is fBm that is decomposed into the sum of the process (1.1.8) and sub-fBm, these being independent.

In Section 2 we will prove the general result of weak convergence, in the sense of the finite dimensional distributions, towards integrals of functions of $L^2(\mathbb{R}^+)$ with respect to two independent standard Brownian motions. This theorem permits us to obtain, in Section 3, results of convergence in law, in the space $\mathcal{C}([0,T])$, towards fBm, the process defined in (1.1.8) and, finally, sub-fBm with parameter $H \in (0,1)$ using the decomposition of this process as a sum of two independent processes.

Positive constants, denoted by C, with possible subscripts indicating appropriate parameters, may vary from line to line.

1.2 General convergence result

Let $N = \{N_t, t \geq 0\}$ denote a standard Poisson process defined on a probability space $(\Omega, \mathcal{F}, P)$ and $W = \{W_t, t \geq 0\}$ a standard Brownian motion defined on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$. We will also denote the expectation of the respective probability spaces by $\mathbb{E}$ and $\hat{\mathbb{E}}$.

In this section we prove the main result of weak convergence in the sense of the finite dimensional distributions. We will use this result later in order to prove weak convergence results towards fractional Brownian motion and sub-fractional Brownian motion.

Theorem 1.2.1. *Let $f(t,\cdot)$ and $g(t,\cdot)$ be functions of $\mathrm{L}^2(\mathbb{R}^+)$ for all $t \in [0,T]$, $T > 0$, let $\{N_s, s \geq 0\}$ be a standard Poisson process and $\theta \in (0,\pi) \cup (\pi, 2\pi)$. Define the processes Y^f and $\tilde{Y}^g$, which are given by $Y^f = \{\int_0^\infty f(t,s) \mathrm{d}W_s, t \in [0,T]\}$ and $\tilde{Y}^g = \{\int_0^\infty g(t,s) \mathrm{d}\tilde{W}_s, t \in [0,T]\}$ and where $W = \{W_s, s \geq 0\}$ and $\tilde{W} = \{\tilde{W}_s, s \geq 0\}$ are independent standard Brownian motions. We also define the following processes*

$$Y_\varepsilon^f = \left\{ \frac{2}{\varepsilon} \int_0^\infty f(t,s) \cos\left(\theta N_{\frac{2s}{\varepsilon^2}}\right) ds, \quad t \in [0,T] \right\} \tag{1.2.1}$$

and

$$\tilde{Y}_\varepsilon^g = \left\{ \frac{2}{\varepsilon} \int_0^\infty g(t,s) \sin\left(\theta N_{\frac{2s}{\varepsilon^2}}\right) ds, \quad t \in [0,T] \right\}. \tag{1.2.2}$$

Then, the finite dimensional distributions of the family of processes $\{(Y_\varepsilon^f, \tilde{Y}_\varepsilon^g)\}_\varepsilon$ converge in law to the finite dimensional distributions of the process $(Y^f, \tilde{Y}^g)$ as ε goes to zero. In particular, $\{Y_\varepsilon^f\}$ and $\{\tilde{Y}_\varepsilon^g\}$ converge toward two independent Gaussian processes.

Taking into account that any statement and proof that follows in this section is valid for any fixed $t \in [0,T]$, by abuse of notation we will write $f(s)$ and $g(s)$ instead of $f(t,s)$ and $g(t,s)$, respectively.

The following Lemma gives a bound for the L^2 and L^4 norm of the approximating processes defined by (1.2.1) and (1.2.2). We will use the L^2 bound to prove Theorem 1.2.1 and the L^4 bound to prove weak convergence results in the following section.

Lemma 1.2.2. *Let Y_ε^f and $\tilde{Y}_\varepsilon^g$ be defined by (1.2.1) and (1.2.2) respectively, and assume we are under the conditions of Theorem 1.2.1. Then we have the following bounds for the L^2 norms of Y_ε^f and $\tilde{Y}_\varepsilon^g$*

$$\mathbb{E}\left[(Y_\varepsilon^f)^2\right] \leq C\left(\int_0^\infty f^2(s)\,ds\right), \quad \mathbb{E}\left[(\tilde{Y}_\varepsilon^g)^2\right] \leq C\left(\int_0^\infty g^2(s)\,ds\right). \tag{1.2.3}$$

We also have the following bounds for the L^4 norms

$$\mathbb{E}\left[(Y_\varepsilon^f)^4\right] \leq C\left(\int_0^\infty f^2(s)\,ds\right)^2, \quad \mathbb{E}\left[(\tilde{Y}_\varepsilon^g)^4\right] \leq C\left(\int_0^\infty g^2(s)\,ds\right)^2. \tag{1.2.4}$$

Proof of Lemma 1.2.2. We will proceed to prove the result only for Y_ε^f since the proof is exactly the same for $\tilde{Y}_\varepsilon^g$.

Observe that defining

$$Z_\varepsilon^f = Y_\varepsilon^f + i\tilde{Y}_\varepsilon^f = \frac{2}{\varepsilon}\int_0^\infty f(s)e^{i\theta N_{\frac{2s}{\varepsilon^2}}}\mathrm{d}s$$

we have $\mathbb{E}[Z_\varepsilon^f \bar{Z}_\varepsilon^f] = \mathbb{E}[(Y_\varepsilon^f)^2 + (\tilde{Y}_\varepsilon^f)^2]$, where $\bar{Z}$ denotes the complex conjugate of Z. Therefore if we prove $\mathbb{E}[Z_\varepsilon^f \bar{Z}_\varepsilon^f] \leq C\|f\|_2^2$, where $\|\cdot\|_2$ is the $L^2(\mathbb{R}^+)$ norm, the stated convergence follows.

$$\begin{aligned}
\mathbb{E}[Z_\varepsilon^f \bar{Z}_\varepsilon^f] &= \mathbb{E}\left[\frac{2}{\varepsilon}\int_0^\infty f(s)e^{i\theta N_{\frac{2s}{\varepsilon^2}}}\mathrm{d}s\frac{2}{\varepsilon}\int_0^\infty f(r)e^{-i\theta N_{\frac{2r}{\varepsilon^2}}}\mathrm{d}r\right] \\
&= \frac{4}{\varepsilon^2}\mathbb{E}\left[\int_0^\infty\int_0^\infty f(s)f(r)e^{i\theta\left(N_{\frac{2s}{\varepsilon^2}} - N_{\frac{2r}{\varepsilon^2}}\right)}\mathrm{d}s\,\mathrm{d}r\right] \\
&= \frac{4}{\varepsilon^2}\int_0^\infty\int_0^\infty \mathbb{1}_{\{r\leq s\}}f(s)f(r)\mathbb{E}\left[e^{i\theta\left(N_{\frac{2s}{\varepsilon^2}} - N_{\frac{2r}{\varepsilon^2}}\right)}\right]\mathrm{d}r\,\mathrm{d}s \\
&\quad + \frac{4}{\varepsilon^2}\int_0^\infty\int_0^\infty \mathbb{1}_{\{s\leq r\}}f(s)f(r)\mathbb{E}\left[e^{-i\theta\left(N_{\frac{2r}{\varepsilon^2}} - N_{\frac{2s}{\varepsilon^2}}\right)}\right]\mathrm{d}s\,\mathrm{d}r.
\end{aligned}$$

Since $\mathbb{E}[e^{i\theta X}] = e^{-\lambda(1-e^{i\theta})}$ and $\mathbb{E}[e^{-i\theta X}] = e^{-\lambda(1-e^{-i\theta})}$, being X a Poisson random variable of parameter λ, we obtain

$$
\begin{aligned}
\mathbb{E}[Z_\varepsilon^f \bar{Z}_\varepsilon^f] &= \frac{4}{\varepsilon^2}\int_0^\infty\int_0^\infty \mathbb{1}_{\{r\leq s\}} f(s)f(r)e^{-2\frac{s-r}{\varepsilon^2}(1-e^{i\theta})}\mathrm{d}r\,\mathrm{d}s \\
&\quad + \frac{4}{\varepsilon^2}\int_0^\infty\int_0^\infty \mathbb{1}_{\{s\leq r\}} f(s)f(r)e^{-2\frac{r-s}{\varepsilon^2}(1-e^{-i\theta})}\mathrm{d}r\,\mathrm{d}s \\
&\leq \frac{4}{\varepsilon^2}\int_0^\infty\int_0^\infty \mathbb{1}_{\{r\leq s\}} |f(s)f(r)|\, e^{-2\frac{s-r}{\varepsilon^2}(1-\cos\theta)}\mathrm{d}r\,\mathrm{d}s \\
&\quad + \frac{4}{\varepsilon^2}\int_0^\infty\int_0^\infty \mathbb{1}_{\{s\leq r\}} |f(s)f(r)|\, e^{-2\frac{r-s}{\varepsilon^2}(1-\cos\theta)}\mathrm{d}r\,\mathrm{d}s.
\end{aligned}
$$

Using the inequality $|f(s)f(r)| \leq \frac{1}{2}\left(f^2(s)+f^2(r)\right)$ and noting that, by means of a change of variables, the last two integrals are the same we have that

$$
\begin{aligned}
\mathbb{E}[Z_\varepsilon^f \bar{Z}_\varepsilon^f] &\leq \frac{4}{\varepsilon^2}\int_0^\infty\int_0^\infty \mathbb{1}_{\{s\leq r\}} \left(f^2(s)+f^2(r)\right) e^{-2\frac{r-s}{\varepsilon^2}(1-\cos\theta)}\mathrm{d}r\,\mathrm{d}s \\
&= \frac{4}{\varepsilon^2}\Big(\int_0^\infty f^2(s)\int_s^\infty e^{-2\frac{r-s}{\varepsilon^2}(1-\cos\theta)}\mathrm{d}r\,\mathrm{d}s + \int_0^\infty f^2(r)\int_0^r e^{-2\frac{r-s}{\varepsilon^2}(1-\cos\theta)}\mathrm{d}s\,\mathrm{d}r\Big) \\
&= 2\left(\int_0^\infty f^2(s)\left(\frac{1}{1-\cos\theta}\right)\mathrm{d}s + \int_0^\infty f^2(r)\left(\frac{1-e^{-2\frac{r}{\varepsilon^2}(1-\cos\theta)}}{1-\cos\theta}\right)\mathrm{d}r\right) \\
&\leq \frac{4}{1-\cos\theta}\int_0^\infty f^2(s)\,\mathrm{d}s,
\end{aligned}
$$

giving the desired result (1.2.3).

To find the bounds for the L^4 norm, being Z_ε^f as before, it is enough to prove that $\mathbb{E}[(Z_\varepsilon^f \bar{Z}_\varepsilon^f)^2] \leq C\|f\|_2^4$.

$$
\begin{aligned}
\mathbb{E}[(Z_\varepsilon^f \bar{Z}_\varepsilon^f)^2] &= \frac{16}{\varepsilon^4}\mathbb{E}\left[\int_{[0,\infty)^4} f(s_1)\cdots f(s_4)e^{i\theta\left(N_{\frac{2s_1}{\varepsilon^2}}+N_{\frac{2s_2}{\varepsilon^2}}-N_{\frac{2s_3}{\varepsilon^2}}-N_{\frac{2s_4}{\varepsilon^2}}\right)}\mathrm{d}s_1\cdots\mathrm{d}s_4\right] \\
&= \frac{64}{\varepsilon^4}\int_{[0,\infty)^4}\mathbb{1}_{\{s_1\leq\cdots\leq s_4\}} f(s_1)\cdots f(s_4)\mathbb{E}\left[E_1+\cdots+E_6\right]\mathrm{d}s_1\cdots\mathrm{d}s_4
\end{aligned}
$$

where

$$
E_1 = e^{i\theta\left(N_{\frac{2s_1}{\varepsilon^2}}+N_{\frac{2s_2}{\varepsilon^2}}-N_{\frac{2s_3}{\varepsilon^2}}-N_{\frac{2s_4}{\varepsilon^2}}\right)} = e^{-i\theta\left(N_{\frac{2s_4}{\varepsilon^2}}-N_{\frac{2s_3}{\varepsilon^2}}+2\left(N_{\frac{2s_3}{\varepsilon^2}}-N_{\frac{2s_2}{\varepsilon^2}}\right)+N_{\frac{2s_2}{\varepsilon^2}}-N_{\frac{2s_1}{\varepsilon^2}}\right)},
$$

$$
E_2 = e^{-i\theta\left(N_{\frac{2s_4}{\varepsilon^2}}-N_{\frac{2s_3}{\varepsilon^2}}+N_{\frac{2s_2}{\varepsilon^2}}-N_{\frac{2s_1}{\varepsilon^2}}\right)},\ E_3 = e^{i\theta\left(N_{\frac{2s_4}{\varepsilon^2}}-N_{\frac{2s_3}{\varepsilon^2}}-\left(N_{\frac{2s_2}{\varepsilon^2}}-N_{\frac{2s_1}{\varepsilon^2}}\right)\right)},
$$

$E_4 = \overline{E_3}$, $E_5 = \overline{E}_2$, $E_6 = \overline{E}_1$. To obtain the last expression note that we can arrange s_1, s_2, s_3, s_4 in 24 different ways and due to the symmetry between s_1 and

s_2 and between s_3 and s_4 we have 6 possible different situations, $E_1, \ldots, E_6$, each one repeated 4 times. By means of the properties of Poisson process we have

$$\|\mathbb{E}[E_1]\|, \|\mathbb{E}[E_2]\|, \|\mathbb{E}[E_3]\| \leq e^{-2\frac{s_4-s_3}{\varepsilon^2}(1-\cos\theta)} e^{-2\frac{s_2-s_1}{\varepsilon^2}(1-\cos\theta)}$$

and we can conclude

$$\begin{aligned}
\mathbb{E}[(Z_\varepsilon^f \bar{Z}_\varepsilon^f)^2] &\leq \frac{384}{\varepsilon^4} \int_{[0,\infty)^4} \mathbb{1}_{\{s_1 \leq \cdots \leq s_4\}} |f(s_1)\cdots f(s_4)| \\
&\qquad\qquad e^{-2\frac{s_4-s_3}{\varepsilon^2}(1-\cos\theta)} e^{-2\frac{s_2-s_1}{\varepsilon^2}(1-\cos\theta)} \mathrm{d}s_1 \cdots \mathrm{d}s_4 \\
&\leq \frac{384}{2\varepsilon^2} \left(\int_{[0,\infty)^2} \mathbb{1}_{\{s_1 \leq s_2\}} |f(s_1) f(s_2)| e^{-2\frac{s_2-s_1}{\varepsilon^2}(1-\cos\theta)} \mathrm{d}s_1 \mathrm{d}s_2 \right)^2 \\
&\leq 3 \left(\frac{4}{1-\cos\theta} \int_0^\infty f^2(s)\mathrm{d}s \right)^2.
\end{aligned}$$

□

Remark 1.2.3. On the previous lemma we proved, in particular, that the family $\{Y_\varepsilon^f \tilde{Y}_\varepsilon^g\}_{\varepsilon>0}$ is uniformly integrable.

Indeed, $\{Y_\varepsilon^f \tilde{Y}_\varepsilon^g\}_{\varepsilon>0}$ will be uniformly integrable if $\sup_{\varepsilon>0} \mathbb{E}\left[(Y_\varepsilon^f \tilde{Y}_\varepsilon^g)^2\right] < \infty$. Using Hölder's inequality we have

$$\sup_{\varepsilon>0} \mathbb{E}\left[(Y_\varepsilon^f \tilde{Y}_\varepsilon^g)^2\right] \leq \sup_{\varepsilon>0} \left(\mathbb{E}[(Y_\varepsilon^f)^4]\right)^{\frac{1}{2}} \left(\mathbb{E}[(\tilde{Y}_\varepsilon^g)^4]\right)^{\frac{1}{2}}.$$

Proof of Theorem 1.2.1. Now we will proceed to proof the Theorem. By definition, the family $\{(Y_\varepsilon^f, \tilde{Y}_\varepsilon^g)\}_\varepsilon$ converges in law, in the sense of finite dimensional distributions, to the process $(Y^f, \tilde{Y}^g)$ as ε goes to zero if and only if for every $k \in \mathbb{N}$ and every $t_1, \ldots, t_k \in [0, T]$

$$(Y_\varepsilon^f, \tilde{Y}_\varepsilon^g)(t_1, \ldots, t_k) \xrightarrow{\mathcal{L}} (Y^f, \tilde{Y}^g)(t_1, \ldots, t_k). \tag{1.2.5}$$

By the isometry of the spaces $\mathbb{R}^2 \times \mathbb{R}^k$ and $\mathbb{R}^{2k}$ and from Theorem 7.7 in [12, page 49] we deduce that our result will follow if and only if

$$S_\varepsilon := \sum_{i=1}^k a_i Y_\varepsilon^f(t_i) + \sum_{j=1}^k b_j \tilde{Y}_\varepsilon^g(t_j) \xrightarrow{\mathcal{L}} S := \sum_{i=1}^k a_i Y^f(t_i) + \sum_{j=1}^k b_j \tilde{Y}^g(t_j), \tag{1.2.6}$$

for any $k \in \mathbb{N}$, $a_j, b_j \in \mathbb{R}$, $1 \leq i, j \leq k$ and $t_1, \ldots, t_k \in [0, T]$. In order to prove (1.2.6) we will show that the respective characteristic functions converge, namely, for any $x \in \mathbb{R}$

$$\mathbb{E}[e^{ixS_\varepsilon}] \longrightarrow \hat{\mathbb{E}}[e^{ixS}] \quad \text{as} \quad \varepsilon \to 0. \tag{1.2.7}$$

We first observe that we can write

$$S_\varepsilon = \frac{2}{\varepsilon}\int_0^\infty \left(F(s)\cos\left(\theta N_{\frac{2s}{\varepsilon^2}}\right) + G(s)\sin\left(\theta N_{\frac{2s}{\varepsilon^2}}\right)\right)\mathrm{d}s$$

and

$$S = \int_0^\infty F(s)\mathrm{d}W_s + \int_0^\infty G(s)\mathrm{d}\tilde{W}_s,$$

where $F(s) := \sum_{i=1}^k a_i f(t_i, s)$ and $G(s) := \sum_{i=1}^k b_i g(t_i, s)$.

Since $F(s)$ and $G(s)$ belong to $L^2(\mathbb{R}^+)$ they can be approximated, respectively, by the following sequences of step functions

$$F^n(s) := \sum_{j=0}^{m_n-1} f_j^n \mathbb{1}_{(s_j^n, s_{j+1}^n]}(s) \qquad \text{and} \qquad G^n(s) := \sum_{j=0}^{m_n-1} g_j^n \mathbb{1}_{(s_j^n, s_{j+1}^n]}(s),$$

where $0 = s_0^n < s_1^n < \ldots < s_{m_n-1}^n < s_{m_n}^n$, f_j^n and g_j^n are chosen such that

$$\int_0^\infty \left((F(s) - F^n(s))^2 + (G(s) - G^n(s))^2\right)\mathrm{d}s \le \frac{1}{n} \tag{1.2.8}$$

for any $n \in \mathbb{N}$. Let us now define

$$S_\varepsilon^n := \frac{2}{\varepsilon}\int_0^\infty \left(F^n(s)\cos\left(\theta N_{\frac{2s}{\varepsilon^2}}\right) + G^n(s)\sin\left(\theta N_{\frac{2s}{\varepsilon^2}}\right)\right)\mathrm{d}s$$

and

$$S^n := \int_0^\infty F^n(s)\mathrm{d}W_s + \int_0^\infty G^n(s)\mathrm{d}\tilde{W}_s.$$

We have that for any $x \in \mathbb{R}$, $\varepsilon > 0$ and $n \in \mathbb{N}$,

$$|\mathbb{E}[e^{ixS_\varepsilon}] - \hat{\mathbb{E}}[e^{ixS}]| \le \alpha_\varepsilon^n + \beta_\varepsilon^n + \gamma^n,$$

where $\alpha_\varepsilon^n = |\mathbb{E}[e^{ixS_\varepsilon}] - \mathbb{E}[e^{ixS_\varepsilon^n}]|$, $\beta_\varepsilon^n = |\mathbb{E}[e^{ixS_\varepsilon^n}] - \hat{\mathbb{E}}[e^{ixS^n}]|$ and $\gamma^n = |\hat{\mathbb{E}}[e^{ixS^n}] - \hat{\mathbb{E}}[e^{ixS}]|$.

We observe that

$$\begin{aligned}\mathbb{E}[(S_\varepsilon - S_\varepsilon^n)^2] = \frac{4}{\varepsilon^2}\mathbb{E}\Bigg[\Bigg(\int_0^\infty &(F(s) - F^n(s))\cos\left(\theta N_{\frac{2s}{\varepsilon^2}}\right) \\ &+ (G(s) - G^n(s))\sin\left(\theta N_{\frac{2s}{\varepsilon^2}}\right)\mathrm{d}s\Bigg)^2\Bigg].\end{aligned}$$

Now, using the inequality $(a+b)^2 \le 2a^2 + 2b^2, a, b \in \mathbb{R}$, and by virtue of (1.2.3) and (1.2.8), we get that

$$\mathbb{E}[(S_\varepsilon - S_\varepsilon^n)^2] \le C\int_0^\infty (F(s) - F(s)^n)^2\mathrm{d}s + C\int_0^\infty (G(s) - G^n(s))^2\mathrm{d}s \le C\frac{1}{n}. \tag{1.2.9}$$

With this inequality and the mean value theorem we conclude that there exists $C > 0$ such that for any $\varepsilon > 0$ and any $n \in \mathbb{N}$,

$$\alpha_\varepsilon^n \leq x\mathbb{E}[|S_\varepsilon - S_\varepsilon^n|] \leq Cx\frac{1}{\sqrt{n}}.$$

On the other hand, for fixed $n \in \mathbb{N}$,

$$S_\varepsilon^n = \sum_{j=0}^{m_n-1} f_j^n \int_{s_j^n}^{s_{j+1}^n} \cos\left(\theta N_{\frac{2s}{\varepsilon^2}}\right) \mathrm{d}s + \sum_{j=0}^{m_n-1} g_j^n \int_{s_j^n}^{s_{j+1}^n} \sin\left(\theta N_{\frac{2s}{\varepsilon^2}}\right) \mathrm{d}s.$$

In [6] Bardina proved that, for $\theta \in (0,\pi) \cup (\pi, 2\pi)$,

$$\int_{s_j^n}^{s_{j+1}^n} \cos\left(\theta N_{\frac{2s}{\varepsilon^2}}\right) \mathrm{d}s \xrightarrow{\mathcal{L}} \int_{s_j^n}^{s_{j+1}^n} \mathrm{d}W_s$$

and

$$\int_{s_j^n}^{s_{j+1}^n} \sin\left(\theta N_{\frac{2s}{\varepsilon^2}}\right) \mathrm{d}s \xrightarrow{\mathcal{L}} \int_{s_j^n}^{s_{j+1}^n} \mathrm{d}\tilde{W}_s,$$

where W_s and $\tilde{W}_s$ are independent standard Brownian motions. Therefore we obtain that S_ε^n converges in law, when ε tends to zero, to

$$S^n = \sum_{j=0}^{m_n-1} f_j^n \int_{s_j^n}^{s_{j+1}^n} \mathrm{d}W_s + \sum_{j=0}^{m_n-1} g_j^n \int_{s_j^n}^{s_{j+1}^n} \mathrm{d}\tilde{W}_s.$$

Finally, by again applying the mean value theorem and computing the variance of the stochastic integral, we obtain that, for any $n \in \mathbb{N}$, γ^n can be bounded by

$$x\hat{\mathbb{E}}[|S - S^n|] \leq Cx\left(\int_0^\infty \left((F(s) - F^n(s))^2 + (G(s) - G^n(s))^2\right) \mathrm{d}s\right)^{\frac{1}{2}} \leq Cx\frac{1}{\sqrt{n}}.$$

Then, both α_ε^n and γ^n become arbitrarily small by taking $n \geq n_0$, for some $n_0 \in \mathbb{N}$. Finally, by fixing $n = n_0$, the term $\beta_\varepsilon^{n_0}$ converges to zero as ε goes to zero, since S_ε^n converges in law to S^n as ε goes to zero. This completes the proof of the convergence of the characteristic functions. □

Remark 1.2.4. We can use this result to approximate two independent processes of many kinds, such as processes with a Gousart kernel (see for instance [21]) or the Holmgren-Riemann-Liouville fractional integral ([21]). In the next section we will see convergence results towards some other processes.

1.3 Weak approximation of some fractional processes

In this section we apply Theorem 1.2.1 to prove weak convergence results towards fractional Brownian motion and sub-fBm. We will also reproduce a result due to Bardina and Es-Sebayi of convergence towards an extension of bifractional Brownian motion.

1.3.1 Weak approximation of fractional Brownian motion

We are going to prove a result of weak convergence in $\mathcal{C}([0,T])$ towards fBm, applying Theorem 1.2.1. In order to do so, we use the following representation of the fBm as the integral of a deterministic kernel with respect to standard Brownian motion (see for instance [20])

$$B_t^H = \int_0^t \tilde{K}^H(t,s)\,\mathrm{d}W_s, \tag{1.3.1}$$

where $H \in (0,2)$, $\tilde{K}^H(t,s)$ is defined on the set $\{0 < s < t\}$ and is given by

$$\tilde{K}^H(t,s) = d^H (t-s)^{\frac{H-1}{2}} + d^H \left(\frac{1-H}{2}\right) \int_s^t (u-s)^{\frac{H-3}{2}} \left(1 - \left(\frac{s}{u}\right)^{\frac{1-H}{2}}\right) \mathrm{d}u, \tag{1.3.2}$$

where the normalizing constant d^H is

$$d^H = \left(\frac{H\Gamma(\frac{3-H}{2})}{\Gamma(\frac{H+1}{2})\Gamma(2-H)}\right)^{\frac{1}{2}}.$$

Since in this section the domain of fBm is restricted to the interval $t \in [0,T]$, we can rewrite the integral representation as

$$B_t^H = \int_0^t \tilde{K}^H(t,s)\,\mathrm{d}W_s = \int_0^T K^H(t,s)\,\mathrm{d}W_s,$$

where $K^H(t,s) = \tilde{K}^H(t,s)\mathbb{1}_{[0,t]}(s)$.

Applying this representation, since $K^H(t,\cdot) \in \mathrm{L}^2(\mathbb{R}^+)$, the following result is a corollary of Theorem 1.2.1

Corollary 1.3.1. *Let* $K^H(t,s) = \tilde{K}^H(t,s)\mathbb{1}_{[0,t]}(s)$*, where* $\tilde{K}^H(t,s)$ *is defined by (1.3.2), let* $\{N_s, s \geq 0\}$ *be a standard Poisson process and let* $\theta \in (0,\pi) \cup (\pi, 2\pi)$*. Then the processes*

$$B_\varepsilon^H = \left\{ \frac{2}{\varepsilon} \int_0^T K^H(t,s) \cos\left(\theta N_{\frac{2s}{\varepsilon^2}}\right) ds, \quad t \in [0,T] \right\} \tag{1.3.3}$$

and

$$\tilde{B}_\varepsilon^H = \left\{ \frac{2}{\varepsilon} \int_0^T K^H(t,s) \sin\left(\theta N_{\frac{2s}{\varepsilon^2}}\right) ds, \quad t \in [0,T] \right\} \tag{1.3.4}$$

converge in law, in the sense of the finite dimensional distributions, towards two independent fractional Brownian motions.

We now proceed to prove the continuity and the tightness of the families of processes defined by (1.3.3) and (1.3.4), and consequently, proving the weak convergence in the space $\mathcal{C}([0,T])$.

The continuity can be deduced from Lemma 2.1 in [9] and the fact that B_ε^H and $\tilde{B}_\varepsilon^H$ equal zero if $t = 0$. We will reproduce a simplified version of this lemma here for the sake of conciseness.

Lemma 1.3.2 (Lemma 2.1, [9])**.** *For all* $s < t$*,*

$$|B_\varepsilon^H(t) - B_\varepsilon^H(s)| \leq C_\varepsilon^H (t-s)^{\left(\frac{H+1}{2}\right)\wedge 1}$$

and

$$|\tilde{B}_\varepsilon^H(t) - \tilde{B}_\varepsilon^H(s)| \leq C_\varepsilon^H (t-s)^{\left(\frac{H+1}{2}\right)\wedge 1},$$

where B_ε^H *and* $\tilde{B}_\varepsilon^H$ *are defined by (1.3.3) and (1.3.4) respectively.*

Proof. The proof is the same for both B_ε^H and $\tilde{B}_\varepsilon^H$, so we will only prove the result for B_ε^H.

From the definition of B_ε^H we have that

$$|B_\varepsilon^H(t) - B_\varepsilon^H(s)| \leq C_\varepsilon \int_0^T \left| K^H(t,u) - K^H(s,u) \right| \mathrm{d}u$$

and we can split this integral as follows

$$\begin{aligned}
&\int_0^T \left| K^H(t,u) - K^H(s,u) \right| \mathrm{d}u \\
= &\int_0^T \left| \tilde{K}^H(t,u) I_{[0,t)}(u) - \tilde{K}^H(s,u) I_{[0,s)}(u) \right| \mathrm{d}u \\
= &\int_s^t \left| \tilde{K}^H(t,u) \right| du + \int_0^s \left| \tilde{K}^H(t,u) - \tilde{K}^H(s,u) \right| \mathrm{d}u.
\end{aligned} \tag{1.3.5}$$

Let us begin with the first summand of the last expression. If $H < 1$ then we have that

$$\int_s^t \tilde{K}^H(t,u)^2 \mathrm{d}u = \int_0^{t-s} \tilde{K}^H(t, v+s)^2 \mathrm{d}v$$

where we set $v = u - s$. If we now consider the kernel $\tilde{K}^H(t, v+s)$ and using (1.3.2) we get that

$$\begin{aligned}
\tilde{K}^H(t, v+s) &= d^H (t-v-s)^{\frac{H-1}{2}} + \\
&\qquad d^H \left(\frac{1-H}{2}\right) \int_{v+s}^t (u-v-s)^{\frac{H-3}{2}} \left(1 - \left(\frac{v+s}{u}\right)^{\frac{1-H}{2}}\right) \mathrm{d}u \\
&= d^H((t-s)-v)^{\frac{H-1}{2}} + \\
&\qquad d^H \left(\frac{1-H}{2}\right) \int_v^{t-s} (y-v)^{\frac{H-3}{2}} \left(1 - \left(\frac{v+s}{y+s}\right)^{\frac{1-H}{2}}\right) \mathrm{d}y \\
&\leq \tilde{K}^H(t-s, v),
\end{aligned}$$

where in the last inequality we used that $(\frac{v+s}{y+s})^{\frac{1-H}{2}} \geq (\frac{v}{y})^{\frac{1-H}{2}}$ since $H < 1$ and $y \geq v$. With this, we obtain that

$$\int_s^t \tilde{K}^H(t,u)^2 \mathrm{d}u \leq \int_0^{t-s} \tilde{K}^H(t-s, v)^2 \mathrm{d}v = (t-s)^H$$

to finally conclude that

$$\int_s^t \left|\tilde{K}^H(t,u)\right| du \leq (t-s)^{\frac{1}{2}} \left(\int_s^t \tilde{K}^H(t,u)^2 \mathrm{d}u\right)^{\frac{1}{2}} \leq (t-s)^{\frac{1+H}{2}}$$

On the other hand, when $H \geq 1$,

$$\begin{aligned}
\tilde{K}^H(t,u) &= C^H u^{\frac{1-H}{2}} \int_u^t (x-u)^{\frac{H-3}{2}} x^{\frac{H-1}{2}} \mathrm{d}x \\
&\leq C^H u^{\frac{1-H}{2}} \int_u^t (x-u)^{\frac{H-3}{2}} \mathrm{d}x \\
&= C^H (t-u)^{\frac{H-1}{2}} u^{\frac{1-H}{2}}
\end{aligned}$$

and

$$\int_s^t |\tilde{K}^H(t,u)| \mathrm{d}u \leq C^H (t-s)^{\frac{H-1}{2}} \int_s^t u^{\frac{1-H}{2}} \mathrm{d}u \leq (t-s).$$

For the second summand of (1.3.5), notice that if $H = 1$, then

$$\int_0^s \left|\tilde{K}^H(t,u) - \tilde{K}^H(s,u)\right| \mathrm{d}u = 0.$$

When $H \neq 1$, we will use some bounds for the partial derivative of the kernel $\tilde{K}$. From (1.3.2) it is easy to check that the kernel $\tilde{K}^H(t,s)$ is differentiable with respect to the first variable in the set $\{0 < s < t\}$ and that

$$\frac{\partial}{\partial t}\tilde{K}^H(t,s) = d^H \left(\frac{H-1}{2}\right)\left(\frac{s}{t}\right)^{\frac{1-H}{2}} (t-s)^{\frac{H-3}{2}}.$$

Then, for $H > 1$ we have that

$$\left|\frac{\partial \tilde{K}^H}{\partial t}(t,s)\right| \leq C^H s^{\frac{1-H}{2}} (t-s)^{\frac{H-3}{2}}.$$

Therefore,

$$\begin{aligned}
\int_0^s \left|\tilde{K}^H(t,u) - \tilde{K}^H(s,u)\right| \mathrm{d}u &\leq \int_0^s \left(\int_s^t \left|\frac{\partial \tilde{K}^H}{\partial r}(r,u)\right| \mathrm{d}r\right) \mathrm{d}u \\
&\leq C^H \int_0^s u^{\frac{1-H}{2}} \left(\int_s^t (r-u)^{\frac{H-3}{2}} \mathrm{d}r\right) \mathrm{d}u \\
&\leq C^H (t-s) \int_0^s u^{\frac{1-H}{2}} (s-u)^{\frac{H-3}{2}} \mathrm{d}u \\
&= C^H B\left(\frac{3-H}{2}, \frac{H-1}{2}\right)(t-s),
\end{aligned}$$

where $B(x,y)$ is the Beta function defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t.$$

When $H < 1$ we have that

$$\left|\frac{\partial \tilde{K}^H}{\partial t}(t,s)\right| \leq C^H (t-s)^{\frac{H-3}{2}}.$$

In this situation, we obtain the following bound

$$\begin{aligned}
\int_0^s \left|\tilde{K}^H(t,u) - \tilde{K}^H(s,u)\right| \mathrm{d}u &\leq \int_0^s \left(\int_s^t \left|\frac{\partial \tilde{K}^H}{\partial r}(r,u)\right| \mathrm{d}r\right) \mathrm{d}u \\
&\leq C^H \int_0^s \left(\int_s^t (r-u)^{\frac{H-3}{2}} \mathrm{d}r\right) \mathrm{d}u \\
&= C^H \int_s^t \left((r-s)^{\frac{H-1}{2}} - r^{\frac{H-1}{2}}\right) \mathrm{d}r \\
&= C^H \left((t-s)^{\frac{H+1}{2}} - t^{\frac{H+1}{2}} + s^{\frac{H+1}{2}}\right) \\
&\leq C^H (t-s)^{\frac{H+1}{2}}.
\end{aligned}$$

The proof of the lemma is now complete. □

Theorem 1.3.3. *Under the hypothesis of Corollary 1.3.1, if moreover one of the following conditions is satisfied:*

1. $H \in (\frac{1}{2}, 2)$,
2. $H \in (0, \frac{1}{2}]$ *and* θ *satisfies* $\cos((2i+1)\theta) \neq 1$ *for all* $i \in \mathbb{N}$ *such that* $i \leq \frac{1}{2}\left[\frac{1}{H}\right]$,

then the processes B_ε^H *and* $\tilde{B}_\varepsilon^H$ *converge in law in* $\mathcal{C}([0,T])$ *towards two independent fractional Brownian motions.*

Proof. It only remains to prove the tightness of the families of processes defined by (1.3.3) and (1.3.4). Since $B_\varepsilon^H(0) = 0$, using Billingsley's criterion (see for instance [12]) it is enough to check that for some $m > 0$ and $\alpha > 1$

$$\mathbb{E}[|B_\varepsilon^H(t) - B_\varepsilon^H(s)|^m] \leq C(F(t) - F(s))^\alpha,$$

where F is a nondecreasing continuous function.

On the other hand, it is known that

$$\int_0^T \left(K^H(t,r) - K^H(s,r)\right)^2 \mathrm{d}r = \mathbb{E}\left[(B_t^H - B_s^H)^2\right] = (t-s)^H,$$

and then it is sufficient to show that

$$\mathbb{E}\left[(y_\varepsilon^f)^m\right] \leq C_m \left(\int_0^T f^2(r)\,\mathrm{d}r\right)^{\frac{m}{2}}, \quad \mathbb{E}\left[(\tilde{y}_\varepsilon^f)^m\right] \leq C_m \left(\int_0^T f^2(r)\,\mathrm{d}r\right)^{\frac{m}{2}} \tag{1.3.6}$$

holds for some m satisfying the condition $Hm/2 > 1$, where $f(r) := K^H(t,r) - K^H(s,r)$, $y_\varepsilon^f = \frac{2}{\varepsilon}\int_0^T f(r)\cos(\theta N_{\frac{2r}{\varepsilon^2}})\mathrm{d}r$ and $\tilde{y}_\varepsilon^f = \frac{2}{\varepsilon}\int_0^T f(r)\sin(\theta N_{\frac{2r}{\varepsilon^2}})\mathrm{d}r$.

Then, in the case (1), it is sufficient to prove (1.3.6) for $m = 4$, which can be seen proving that $\mathbb{E}[(z_\varepsilon^f \bar{z}_\varepsilon^f)^2] \leq C\|f\|_2^4$, where $\|\cdot\|_2$ is the $L^2[0,T]$ norm and $z_\varepsilon^f = y_\varepsilon^f + i\tilde{y}_\varepsilon^f$. If we extend f to $\mathbb{R}^+$ for zeros, i.e., if we consider $F(r) := f(r)\mathbb{1}_{[0,T]}(r)$, we have proved in Lemma 1.2.2 that

$$\mathbb{E}[(Z_\varepsilon^F \bar{Z}_\varepsilon^F)^2] \leq 3\left(\frac{4}{1-\cos\theta}\int_0^\infty F^2(s)\mathrm{d}s\right)^2.$$

Then,

$$\begin{aligned}\mathbb{E}[(z_\varepsilon^f \bar{z}_\varepsilon^f)^2] &= \mathbb{E}[(Z_\varepsilon^F \bar{Z}_\varepsilon^F)^2] \\ &\leq 3\left(\frac{4}{1-\cos\theta}\int_0^\infty F^2(s)\mathrm{d}s\right)^2 = 3\left(\frac{4}{1-\cos\theta}\int_0^T f^2(s)\mathrm{d}s\right)^2.\end{aligned}$$

To prove the result under the hypothesis (2) we can show that (1.3.6) is satisfied for some even m such that $\frac{Hm}{2} > 1$. If we proceed in the same way as in case (1) we obtain an expression that depends on $1-\cos((2i+1)\theta)$ for all $i = 0, 1, \ldots, \left[\frac{1}{2H}\right]$ and the constant C_m depends on $\max_{i=0,1,\ldots,\left[\frac{1}{2H}\right]} \frac{1}{1-\cos((2i+1)\theta)}$. □

1.3.2 Convergence towards sub-fractional Brownian motion

In order to obtain the convergence to sub-fractional Brownian motion, we will apply a decomposition result due to Ruiz de Chávez and Tudor in [43] that we obtained independently. To obtain this result we use a process X^H introduced by Lei and Nualart in [33] and defined in (1.1.8) by the equation

$$X_t^H = \int_0^\infty (1 - e^{-rt}) r^{-\frac{1+H}{2}} \, \mathrm{d}W_r,$$

where W is a standard Brownian motion. It can be proved (see [33] or [43]) that its covariance function is

$$\operatorname{Cov}(X_t^H, X_s^H) = \begin{cases} \frac{\Gamma(1-H)}{H} \left[t^H + s^H - (t+s)^H \right] & \text{if } H \in (0,1), \\ \frac{\Gamma(2-H)}{H(H-1)} \left[(t+s)^H - t^H - s^H \right] & \text{if } H \in (1,2), \end{cases} \tag{1.3.7}$$

and that X^H has a version with absolutely continuous trajectories on $[0, \infty)$.

The decomposition result can be stated and proved as follows:

Theorem 1.3.4 (Decomposition of sub-fBm)**.** *Let B^H be a fBm, S^H a sub-fBm and $W = \{W_t, t \geq 0\}$ a standard Brownian motion. Let X^H be the process given by (1.1.8). If for $H \in (0,1)$ we suppose that B^H and W are independents, then the processes $\{Y_t^H = C_1 X_t^H + B_t^H, t \geq 0\}$ and $\{S_t^H, t \geq 0\}$ have the same law, where $C_1 = \sqrt{\frac{H}{2\Gamma(1-H)}}$. If for $H \in (1,2)$ we suppose that S^H and W are independents, then the processes $\{Y_t^H = C_2 X_t^H + S_t^H, t \geq 0\}$ and $\{B_t^H, t \geq 0\}$ have the same law, where $C_2 = \sqrt{\frac{H(H-1)}{2\Gamma(2-H)}}$.*

Proof. It is clear that the process Y^H is centered and Gaussian in both cases. For $H \in (0,1)$, from (1.1.6), (1.3.7) and using the independence of X^H and B^H we have

$$\begin{aligned} \operatorname{Cov}(Y_t^H, Y_s^H) &= C_1^2 \operatorname{Cov}[X_t^H, X_s^H] + \operatorname{Cov}[B_t^H, B_s^H] \\ &= s^H + t^H - \frac{1}{2} \left[(s+t)^H + |s-t|^H \right], \end{aligned}$$

which completes the proof in this case, and for $H \in (1,2)$, from (1.1.7), (1.3.7) and using the independence of X^H and S^H we have

$$\begin{aligned} \operatorname{Cov}(Y_t^H, Y_s^H) &= C_2^2 \operatorname{Cov}[X_t^H, X_s^H] + \operatorname{Cov}[S_t^H, S_s^H] \\ &= \frac{1}{2} \left(s^H + t^H - |s-t|^H \right), \end{aligned}$$

which completes the proof. □

As an application of this result, we can study the space of integrable functions with respect to sub-fractional Brownian motion.

Let us consider $\mathcal{E}$ the set of simple functions on $[0,T]$. Generally, if $U := (U_t,\, t \in [0,T])$ is a continuous, centered Gaussian process, we denote by $\mathcal{H}_U$ the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathcal{H}} = \mathbb{E}\left(U_t U_s\right).$$

In the case of the standard Brownian motion W, the space $\mathcal{H}_W$ is $L^2([0,T])$. On the other hand, for the fractional Brownian motion B^H, the space $\mathcal{H}_{B^H}$ is the set of restrictions to the space of test functions $\mathcal{D}((0,T))$ of the distributions of $W^{\frac{1-H}{2},2}(\mathbb{R})$ with support contained in $[0,T]$ (see [27]). In the case $H \in (0,1)$ all the elements of the domain are functions, and the space $\mathcal{H}_{B^H}$ coincides with the fractional Sobolev space $I_{0^+}^{\frac{1-H}{2}}(L^2([0,T]))$ (see for instance [20]), but in the case $H \in (1,2)$ this space contains distributions which are not given by any function.

As a direct consequence of Theorem 1.3.4 we have the following relation between $\mathcal{H}_{B^H}$, $\mathcal{H}_{S^H}$ and $\mathcal{H}_{X^H}$, where S^H is the sub-fBm and X^H is the process introduced by Lei and Nualart in [33] and defined by (1.1.8).

Proposition 1.3.5. *For $H \in (0,1)$ the following equality*

$$\mathcal{H}_{X^H} \cap \mathcal{H}_{B^H} = \mathcal{H}_{S^H}$$

holds. On the other hand, for $H \in (1,2)$ we have that

$$\mathcal{H}_{X^H} \cap \mathcal{H}_{S^H} = \mathcal{H}_{B^H}.$$

Proof. This proposition is a direct consequence of the two decompositions into the sum of two independent processes proved in Theorem 1.3.4. □

In order to apply Theorem 1.2.1 to prove weak convergence to sub-fBm, we have to prove weak convergence to fBm and the process X^H introduced by Lei and Nualart. Then, applying the decomposition theorem and the independence of the limit laws, we can state the weak convergence to sub-fBm for $H \in (0,1)$.

So, it just remains to prove for the process X^H defined by (1.1.8) the same results we have obtained for fBm.

Corollary 1.3.6. *Let X^H be the process defined by (1.1.8), let $\{N_s, s \geq 0\}$ be a standard Poisson process and let $\theta \in (0,\pi) \cup (\pi, 2\pi)$. Then the processes*

$$X_\varepsilon^H = \left\{ \frac{2}{\varepsilon} \int_0^\infty (1 - e^{-st}) s^{-\frac{1+H}{2}} \cos\left(\theta N_{\frac{2s}{\varepsilon^2}}\right) ds, \quad t \in [0,T] \right\} \tag{1.3.8}$$

and

$$\tilde{X}^H_\varepsilon = \left\{ \frac{2}{\varepsilon} \int_0^\infty (1-e^{-st}) s^{-\frac{1+H}{2}} \sin\left(\theta N_{\frac{2s}{\varepsilon^2}}\right) ds, \quad t \in [0,T] \right\} \tag{1.3.9}$$

converge in law, in the sense of the finite dimensional distributions, towards two independent processes with the same law that X^H*.*

Theorem 1.3.7. *Under the hypothesis of Corollary 1.3.6 the processes defined by (1.3.8) and (1.3.9) converge in law, in* $\mathcal{C}([0,T])$*, towards two independent processes with the same law that the process defined by (1.1.8).*

Proof. We first need to show that the processes X^H_ε and $\tilde{X}^H_\varepsilon$ are continuous. In fact, they are absolutely continuous. Let us consider for all $r > 0$ the process

$$Y_r = \frac{2}{\varepsilon} \int_0^\infty s^{\frac{1-H}{2}} e^{-sr} \cos\left(\theta N_{\frac{2s}{\varepsilon^2}}\right) \mathrm{d}s.$$

This integral exists because, using inequality (1.2.3), we have

$$\mathbb{E}[Y_r^2] \le C \left(\int_0^\infty s^{1-H} e^{-2sr} \mathrm{d}s \right) = C r^{H-2} \Gamma(2-H).$$

On the other hand,

$$\mathbb{E}\left[\int_0^t |Y_r| \mathrm{d}r \right] \le \int_0^t (\mathbb{E}[Y_r^2])^{\frac{1}{2}} \mathrm{d}r \le C \int_0^t r^{\frac{H-2}{2}} \mathrm{d}r < \infty$$

since $H \in (0,2)$.

Let us now observe that $X^H_\varepsilon = \int_0^t Y_r \mathrm{d}r$. Indeed, applying Fubini's theorem,

$$\begin{aligned} \int_0^t Y_r \mathrm{d}r &= \frac{2}{\varepsilon} \int_0^\infty s^{\frac{1-H}{2}} \left(\int_0^t e^{-sr} \mathrm{d}r \right) \cos\left(\theta N_{\frac{2s}{\varepsilon^2}}\right) \mathrm{d}s \\ &= \frac{2}{\varepsilon} \int_0^\infty s^{-\frac{1+H}{2}} (1-e^{-st}) \cos\left(\theta N_{\frac{2s}{\varepsilon^2}}\right) \mathrm{d}s \\ &= X^H_\varepsilon. \end{aligned}$$

The same proof shows that the process $\tilde{X}^H_\varepsilon$ is continuous.

Next, we prove the convergence only for (1.3.8). For (1.3.9) the result is proved similarly.

It suffices to prove the tightness of the family $\{X^H_\varepsilon\}_\varepsilon$. Since $X^H_\varepsilon(0) = 0$, using Billingsley's criterion we only need to prove that

$$\mathbb{E}\left[|X^H_\varepsilon(t) - X^H_\varepsilon(s)|^4 \right] \le |F(t) - F(s)|^2$$

where F is a continuous, non-decreasing function. We observe that

$$\mathbb{E}\left[|X^H_\varepsilon(t) - X^H_\varepsilon(s)|^4 \right] = \mathbb{E}\left[\frac{2}{\varepsilon} \int_0^\infty \left(\Phi^H(t,r) - \Phi^H(s,r) \right) \cos(\theta N_{\frac{2r}{\varepsilon^2}}) \mathrm{d}r \right]^4$$

where $\Phi^H(t,r) = (1-e^{-rt})r^{-\frac{1+H}{2}} \in \mathrm{L}^2(\mathbb{R}^+)$.

Since $\Phi^H \in \mathrm{L}^2(\mathbb{R}^+)$, applying the bound (1.2.4), which is proved in Theorem 1.2.1, we obtain

$$\begin{aligned}\mathbb{E}\left[|X_\varepsilon^H(t) - X_\varepsilon^H(s)|^4\right] &\leq C\left(\int_0^\infty \left(\Phi^H(t,r) - \Phi^H(s,r)\right)^2 \mathrm{d}r\right)^2 \\ &= C\Bigg(\int_0^\infty \Big((1-e^{-rt})^2 r^{-(1+H)} + (1-e^{-rs})^2 r^{-(1+H)} \\ &\quad - 2(1-e^{-rt})(1-e^{-rs})r^{-(1+H)}\Big)\mathrm{d}r\Bigg)^2.\end{aligned}$$

Using (1.3.7) and assuming $s<t$ we obtain for $H \in (0,1)$

$$\begin{aligned}\mathbb{E}\left[|X_\varepsilon^H(t) - X_\varepsilon^H(s)|^4\right] &\leq C\left(2(t+s)^H - (2t)^H - (2s)^H\right)^2 \\ &\leq C\left((2t)^H - (2s)^H\right)^2,\end{aligned}$$

since $s+t<2t$. In the same way, if $H \in (1,2)$,

$$\begin{aligned}\mathbb{E}\left[|X_\varepsilon^H(t) - X_\varepsilon^H(s)|^4\right] &\leq C\left((2t)^H + (2s)^H - 2(t+s)^H\right)^2 \\ &\leq C\left((2t)^H - (2s)^H\right)^2,\end{aligned}$$

since $s+t>2s$. In both cases we have proved the result with $F(x) = (2x)^H$. □

Finally, we obtain the result of weak convergence to sub-fractional Brownian motion, as a direct conclusion of the previous results.

Theorem 1.3.8. *Let $H \in (0,1)$, let $\{X_\varepsilon^H(t), t \in [0,T]\}$ be the family of processes defined by (1.3.8), let $\{\tilde{B}_\varepsilon^H(t), t \in [0,T]\}$ be the family of processes defined by (1.3.4) and $C_1 = \sqrt{\frac{H}{2\Gamma(1-H)}}$. Let us assume $\theta \in (0,\pi)\cup(\pi,2\pi)$ and, for $H \in (0,\frac{1}{2}]$, that θ is such that $\cos((2i+1)\theta) \neq 1$ for all $i \in \mathbb{N}$ such that $i \leq \frac{1}{2}\left[\frac{1}{H}\right]$. Then, $\{Y_\varepsilon^H(t) = C_1 X_\varepsilon^H(t) + B_\varepsilon^H(t), t \in [0,T]\}$ weakly converges in $\mathcal{C}([0,T])$ to a sub-fractional Brownian motion.*

Proof. Applying Theorems **1.3.3** and **1.3.7** we know that, respectively, the processes $\tilde{B}_\varepsilon^H$ and X_ε^H converge in law in $\mathcal{C}([0,T])$ towards a fBm and the process defined by (1.1.8). Moreover, applying Theorem **1.2.1**, we know that the limit laws are independent. Hence, we are under the hypothesis of Theorem **1.3.4**, which proves the stated result. □

Remark 1.3.9. Obviously we can also obtain the same result using the families of processes defined by (1.3.9) and (1.3.3).

Remark 1.3.10. We may also notice that we cannot use our approximation result (Theorem 1.2.1) to construct a family of approximating processes towards sub-fBm when $H \in (1,2)$ since the decomposition result (Theorem 1.3.4) for sub-fBm when $H \in (1,2)$ only states that fBm can be decomposed as a sum of sub-fBm and another independent process and not otherwise. The same situation applies for the decomposition result of bifractional Brownian motion by Lei and Nualart in [33].

1.3.3 Convegence towards bifractional Brownian motion with parameter $K \in (1,2)$

In [8] X. Bardina and K. Es-Sebaiy proved that bifractional Brownian motion can be extended for $K \in (1,2)$ and they also proved a convergence in law result towards bifractional Brownian motion for $K \in (1,2)$ from a Poisson process.

The extension they provide is based on the decomposition of bifractional Brownian motion that Lei and Nualart introduce in [33] and can be stated as follows.

Theorem 1.3.11. *Assume $H \in (0,2)$ and $K \in (1,2)$ with $HK \in (0,2)$. Let B^{HK} be a fractional Brownian motion, and $W = \{W_t, t \geq 0\}$ a standard Brownian motion. Let $X^{K,H}$ the process defined in (1.1.8). If we suppose that B^{HK} and W are independent, then the process*

$$B_t^{H,K} = aB_t^{HK} + bX_t^{H,K}, \tag{1.3.10}$$

where $a = \sqrt{2^{1-K}}$ and $b = \sqrt{\frac{K(K-1)}{2^K \Gamma(2-K)}}$ is a centered Gaussian process with covariance function

$$E\left(B_t^{H,K} B_s^{H,K}\right) = \frac{1}{2^K}\left(\left(t^H + s^H\right)^K - |t-s|^{HK}\right); \quad s,\ t \geq 0.$$

With this extension, defined by equation (1.3.10), Bardina and Es-Sebaiy observed that the following convergence result can be directly concluded from Theorems 1.2.1, 1.3.3 and 1.3.7.

Theorem 1.3.12. *Let $H \in (0,2)$ and $K \in (1,2)$ with $HK \in (0,2)$. Consider $\theta \in (0,\pi) \cup (\pi, 2\pi)$ such that if $HK \in (0, \frac{1}{2}]$ then θ satisfies that* $\cos((2i+1)\theta) \neq 1$

for all $i \in \mathbb{N}$ *such that* $i \leq \frac{1}{2}\left[\frac{1}{H}\right]$. *Set* $a = \sqrt{2^{1-K}}$ *and* $b = \sqrt{\frac{K(K-1)}{2^K\Gamma(2-K)}}$. *Define the processes*

$$
\begin{aligned}
B_\epsilon^{HK} &= \left\{\frac{2}{\epsilon}\int_0^T K^{HK}(t,s)\sin\left(\theta N_{\frac{2s}{\epsilon^2}}\right)ds, \quad t \in [0,T]\right\}, \\
X_\epsilon^{H,K} &= \left\{\frac{2}{\epsilon}\int_0^\infty (1-e^{-st^H})s^{-\frac{1+K}{2}}\cos\left(\theta N_{\frac{2s}{\epsilon^2}}\right)ds, \quad t \in [0,T]\right\},
\end{aligned}
$$

where $K^{HK}(t,s)$ *is the kernel defined in (1.3.2). Then,* $\{Y_\epsilon^H(t) = aB_\epsilon^{HK}(t) + bX_\epsilon^{H,K}(t), t \in [0,T]\}$ *weakly converges in* $\mathcal{C}([0,T])$ *to a bifractional Brownian motion.*

Chapter 2

Models for bacteriophage systems

This chapter is devoted to the study of a model of bacteriophage treatments for bacterial infections. This problem has been brought to our attention by the Molecular Biology Group of the Department of Genetics and Microbiology at the *Universitat Autònoma de Barcelona*.

It was first studied from a deterministic point of view in a joint work with professor Àngel Calsina, the results of which form the research work [10], and later it was studied in a noisy context together with professors Xavier Bardina, Carles Rovira and Samy Tindel in a work that can be found in [5].

We will begin the chapter by introducing some general notions on the problem and a basic model. Then we will study the deterministic case, with n strains of bacteriophages, and finally we will proceed to study a stochastic model.

2.1 Introduction

Lately Bacteriophage therapies are attracting the attention of several scientific studies. They can be seen as a new and powerful tool to treat bacterial infections or to prevent them in food, animals or even humans. Generally speaking, they consist in inoculating a (benign) virus in order to kill the bacteria known to be responsible for a certain disease. This kind of treatment is known since the beginning of the 20^{th} century, but has been in disuse in the Western world, erased by antibiotic therapies. However, a small activity in this domain has survived in the USSR, and it is now re-emerging (at least at an experimental level). Among the reasons for this re-emergance we can find the progressive slowdown in antibiotic efficiency (antibiotic resistance). Reported recent experiments include animal diseases like hemorrhagic septicemia in cattle or atrophic rhinitis in swine, and a need for suitable mathematical models is now expressed by the community.

Let us be a little more specific about the (lytic) bacteriophage mechanism:

after attachment, the virus' genetic material penetrates into the bacteria and uses the host's replication mechanism to self-replicate. Once this is done, the bacteria is completely spoiled while new viruses are released, ready to attack other bacteria. It should be noticed at this point that among the advantages expected from the therapy is the fact that it focuses on one specific bacteria, while antibiotics also attack autochthonous microbiota. Generally speaking, it is also believed that viruses are likely to adapt themselves to mutations of their host bacteria.

At a mathematical level, whenever the mobility of the different biological actors is high enough, bacteriophage systems can be modeled by a kind of predator-prey equation. Namely, set $S(t)$ (resp. $Q(t)$) for the non-infected bacteria (resp. bacteriophages) concentration at time t. Then a model for the evolution of the couple (S, Q) is as follows

$$\begin{cases} \mathrm{d}S(t) = [\alpha - kQ(t)]\, S(t)\mathrm{d}t \\ \mathrm{d}Q(t) = \left[d - mQ(t) - kQ(t)S(t) + k\, b\, e^{-\mu\zeta} Q(t-\zeta)S(t-\zeta)\right] \mathrm{d}t, \end{cases} \tag{2.1.1}$$

where α is the reproducing rate of the bacteria and k is the adsorption rate. In equation (2.1.1), d also stands for the quantity of bacteriophages inoculated per unit of time, m is their death rate, we denote by b the number of bacteriophages which are released after replication within the bacteria cell, sometimes known as burst size, ζ is the delay necessary to the reproduction of bacteriophages (called latency time) and the coefficient $e^{-\mu\zeta}$ represents an attenuation in the release of bacteriophages (given by the expected number of bacteria cell's deaths during the latency time, where μ is the bacteria's death rate). A given initial condition $\{S_0(\tau), Q_0(\tau); -\zeta \leq \tau \leq 0\}$ is also specified.

Several models describing phages dynamics have already been considered in the literature (see, for instance [15, 34, 37, 41, 52]), many elaborated variants being introduced for instance in [42, 46]. To the best of our knowledge, none of the articles mentioned above contemplates the possibility of a continuous injection of phages into the system (represented by us by the constant d in (2.1.1)). This variant corresponds to the practical problem we are starting from, which has been brought to our attention by the Molecular Biology Group of the Department of Genetics and Microbiology at *Universitat Autònoma de Barcelona*. This situation corresponds to a treatment for cattle against Salmonella[1], for which phages are inoculated through food, with an approximate constant rate d.

[1]We refer to [16] for a preliminary study on this topic lead at *Universitat Autònoma de Barcelona*, and to the PhD theses [4] and [47] where the bacteriophages have been characterized.

2.2 Deterministic model. Equilibria and stability.

In this section we will proceed to study a generalization of the deterministic model (2.1.1) introduced before, where we will consider n different strains of phages, namely

$$\begin{cases} \dot{S}(t) = \Big(\alpha - \sum_{i=1}^{n} k_i Q_i(t)\Big) S(t) \\ \dot{Q}_i(t) = d_i - m_i Q_i(t) - k_i Q_i(t) S(t) + k_i\, b_i\, e^{-\mu\zeta_i} Q(t-\zeta_i) S(t-\zeta_i), \end{cases} \tag{2.2.1}$$

for $i = 1, \ldots, n$, where k_i, m_i, bi, ζ_i and $Q_i(t)$ are, respectively, the adsorption rate, mortality, burst size, latency time and concentration at time t of the i-th strain of bacteriophages. The constant d_i also stands for the quantity of bacteriophages inoculated per unit of time of the i-th strain of bacteriophages.

We shall start studying the positivity and existence of solutions, making use of the well-known theorems of existence for delay differential equations. Then we will study the equilibria system (2.2.1) may have depending on certain relation of the parameters. Once we know about the equilibria we will proceed to study their stability. To make this study simpler, we will first suppose that $\zeta_i = 0, i = 1, \ldots, n$, i.e., converting system (2.2.1) to an ordinary differential equation, and then we will use some bifurcation results to obtain some knowledge on the general case. Finally, we will proceed to a deeper study of some particular cases which are easier to treat.

2.2.1 Global existence and positivity of solutions

Before any study of the behavior of solutions of (2.2.1) one must ensure their existence and positivity for all positive time or, in other words, that the solutions will not lose their biological meaning. Let us first prove that the (local) solutions will remain positive on their maximal interval of existence. This fact will be used later to prove global existence.

Proposition 2.2.1 (Positivity)**.** *Let us assume that the initial condition satisfies*

$$\begin{cases} S(t) = S^0(t) \geq 0, \quad t \in [-\zeta, 0] \text{ with } \zeta = \max_{i=1\ldots n} \zeta_i, \\ Q_i(t) = Q_i^0(t) \geq 0, \quad t \in [-\zeta_i, 0]. \end{cases} \tag{2.2.2}$$

Then the solution is such that $S(t) \geq 0$, $Q_i(t) \geq 0$ for all t in the interval of existence and $i = 1, \ldots, n$.

Proof. The first equation in (2.2.1) implies the positivity of $S(t)$. To show the positivity of $Q_i(t)$, notice that $Q_i'(0)$ is positive when $Q_i(0) = 0$ (d_i is positive by hypothesis) so there exists a positive ε such that $Q_i(t) > 0$ for $t \in (0, \varepsilon)$ (this is clear if $Q_i(0) > 0$, by continuity). This fact implies the positivity of $Q_i(t)$ for all $t > 0$ (in the interval of existence). Indeed, if $Q_i(t)$ was not positive, then there would be some positive time such that Q_i vanishes. Let t_0 be the minimum of such times. By continuity, $Q_i(t_0) = 0$ and then $Q_i'(t_0)$ is positive because of the non-negativity of S and Q_i until time t_0 and hence of $L_i(t_0)$, leading to a contradiction since $Q_i(t)$ is positive for $t < t_0$. □

Next we will prove the existence and uniqueness of solution of the initial value problem for (2.2.1), using the standard results on local existence, uniqueness and continuation of solutions for delay differential equations (see [25] or [31] for instance), and Gronwall's Lemma.

Theorem 2.2.2 (Global existence of solutions)**.** *For all initial condition like in* (2.2.2) *there exists a unique solution of* (2.2.1)*, which is defined for all positive time.*

Proof. Since the right hand side of (2.2.1) is of the polynomial type we have local existence and uniqueness. Then, there only rests to prove the boundedness of the solution for all positive time in order to prove global existence.

Since $S'(t) \leq \alpha S(t)$ (here we use the positivity result) we clearly have $S(t) \leq S^0(0)e^{\alpha t}$, $t > 0$. In order to show the boundedness of $Q_i(t)$ for $i = 1, \ldots, n$ we use Proposition 2.2.1 to obtain

$$Q_i'(t) \leq d_i + k_i b_i e^{-\mu \zeta_i} S(t - \zeta_i) Q_i(t - \zeta_i).$$

Notice that when $t \in [0, \zeta_i]$, $S(t - \zeta_i) = S^0(t - \zeta_i) \leq S^0(t - \zeta_i)e^{\alpha t}$ and when $t > \zeta_i$, $S(t - \zeta_i) \leq S^0(0)e^{\alpha(t - \zeta_i)}$. Then, defining $\tilde{S} := \max\{S^0(0)e^{-\alpha \zeta_i}, \max_{t \in [-\zeta, 0]} S^0(t)\}$, we see that for all $t > 0$

$$Q_i'(t) \leq d_i + k_i b_i e^{-\mu \zeta_i} \tilde{S} e^{\alpha t} Q_i(t - T_i) =: d_i + C_i e^{\alpha t} Q_i(t - \zeta_i).$$

Integrating we have

$$Q_i(t) \leq Q_i^0(0) + d_i t + C_i \int_0^t e^{\alpha s} Q_i(s - \zeta_i)\, ds.$$

Now changing variables and using the positiveness of the integrand we obtain

$$Q_i(t) \leq Q_i^0(0) + d_i t + C_i \int_{-\zeta_i}^{t - \zeta_i} e^{\alpha s} Q_i(s)\, ds \leq Q_i^0(0) + d_i t + C_i \int_{-\zeta_i}^{t} e^{\alpha s} Q_i(s)\, ds.$$

The hypotheses of Gronwall's Lemma hold for all interval $[0, b]$, $b > 0$, with $\alpha(t) = Q_i^0(0) + d_i t$, $\beta(s) = C_i e^{\alpha s}$ and $u(t) = Q_i(t)$ (see [31]). Moreover, $\alpha(t)$ is non-decreasing, and then

$$Q_i(t) \leq (Q_i^0(0) + d_i t) \exp\left(\int_{-\zeta_i}^{t} C_i e^{\alpha s}\, ds \right),$$

proving the boundedness for all positive time and hence the global existence. □

2.2.2 Equilibria

Our first step in the study of the model (2.2.1) will consist in the search of equilibria, having in mind that for the sake of biological meaning we want steady states where the population densities of virus and bacteria are non-negative. For this, one must solve the system of equations

$$\begin{cases} 0 = \left(\alpha - \sum_{i=1}^{n} k_i Q_i\right) S \\ 0 = d_i - m_i Q_i - k_i Q_i S + k_i b_i e^{-\mu\zeta_i} Q_i S\,, \quad i = 1, \ldots, n. \end{cases} \tag{2.2.3}$$

On the one hand, for any value of the parameters, there is only one equilibrium without bacteria, say E_0 (if $S = 0$, then $Q_i = \frac{d_i}{m_i}$). On the other hand, if there is some coexistence equilibrium, say $E_c = (\hat{S}, \hat{Q}_1, \ldots, \hat{Q}_n)$ with $\hat{S} > 0$, it must satisfy the following equations,

$$\alpha = \sum_{i=1}^{n} k_i \hat{Q}_i \tag{2.2.4}$$

$$\hat{Q}_i = \frac{d_i}{m_i - k_i\left(b_i e^{-\mu\zeta_i} - 1\right)\hat{S}}\,, \quad i = 1, \ldots, n, \tag{2.2.5}$$

and, therefore, placing (2.2.5) in (2.2.4) one obtains the following condition for the existence of $\hat{S}$,

$$F(\hat{S}) := \sum_{i=1}^{n} k_i \hat{Q}_i = \sum_{i=1}^{n} \frac{k_i d_i}{m_i - k_i\left(b_i e^{-\mu\zeta_i} - 1\right)\hat{S}} = \alpha. \tag{2.2.6}$$

Now, we look for solutions of (2.2.6) such that $\hat{S}$, $\hat{Q}_i > 0$ and we obtain the following result.

Theorem 2.2.3 (Equilibria). *If $\sum_{i=1}^{n} \frac{k_i d_i}{m_i} \geq \alpha$ the system (2.2.1) has a unique steady state, $E_0 = (0, \frac{d_1}{m_1}, \ldots, \frac{d_n}{m_n})$. When $\sum_{i=1}^{n} \frac{k_i d_i}{m_i} < \alpha$ then model has two equilibria, E_0 and the coexistence equilibrium $E_c = (\hat{S}, \hat{Q}_1, \ldots, \hat{Q}_n)$, where $\hat{S}$ is the unique solution of*

$$F(S) = \sum_{i=1}^{n} \frac{k_i d_i}{m_i - k_i\left(b_i e^{-\mu\zeta_i} - 1\right) S} = \alpha, \quad S \in (0, S^*)\,,$$

with $S^* := \min_{i=1\ldots n} \frac{m_i}{k_i\left(b_i e^{-\mu\zeta_i} - 1\right)}$, and $\hat{Q}_i$, $i = 1, \ldots, n$, are given by

$$\hat{Q}_i = \frac{d_i}{m_i - k_i\left(b_i e^{-\mu\zeta_i} - 1\right)\hat{S}}, \quad i = 1, \ldots, n.$$

Proof. We have already shown the existence of E_0 for any value of the parameters. Furthermore, we know that there are coexistence equilibria if and only if there are solutions of (2.2.6) such that $\hat{S}$, $\hat{Q}_i > 0$.

From the condition of positivity of $\hat{Q}_i$ and (2.2.5) we have

$$\hat{S} < \frac{m_i}{k_i\left(b_i e^{-\mu\zeta_i} - 1\right)} \quad \text{for all } i = 1 \dots n$$

and therefore we must have $\hat{S} \in (0, S^*)$. Now, taking the derivative of F

$$F'(S) = \sum_{i=1}^{n} \frac{k_i^2 d_i \left(b_i e^{-\mu\zeta_i} - 1\right)}{\left[m_i - k_i(b_i e^{-\mu\zeta_i} - 1)S\right]^2} > 0$$

we observe that, for $S \in (0, S^*)$, F is a positive function, strictly increasing with infinite limit in S^*. Therefore, if $F(0) = \sum_{i=1}^{n} \frac{k_i d_i}{m_i} \geq \alpha$ there is no positive solution of $F(S) = \alpha$ and there is one and only one positive solution when $F(0) = \sum_{i=1}^{n} \frac{k_i d_i}{m_i} < \alpha$. This gives the claim.

□

2.2.3 Local study, $\zeta_i = 0$

In this section and the next one we will study the behavior of the solution near equilibria. First of all we will study the no delayed case ($\zeta_i = 0$) and making use of the results when $\zeta_i = 0$ we will proceed to study the more general (and difficult) case with delays in the following section.

So in this section we will let $\zeta_i = 0$ for $i = 1, \dots, n$ and we can write system (2.2.1) as follows

$$\begin{cases} S'(t) = \left(\alpha - \sum_{i=1}^{n} k_i Q_i(t)\right) S(t) \\ Q_i'(t) = d_i - m_i Q_i(t) - k_i Q_i(t) S(t) + k_i b_i Q_i(t) S(t), \quad i = 1, \dots, n. \end{cases} \tag{2.2.7}$$

As shown in Theorem 2.2.3, there exists a unique equilibrium E_0 when $\sum_{i=1}^{n} \frac{k_i d_i}{m_i} \geq \alpha$ and two equilibria E_0 and E_c when $\sum_{i=1}^{n} \frac{k_i d_i}{m_i} < \alpha$. The differential matrix of (2.2.7) is

$$A := \begin{pmatrix} \alpha - \sum_{i=1}^{n} k_i Q_i & -k_1 S & \dots & -k_n S \\ k_1 Q_1 (b_1 - 1) & -m_1 + (b_1 - 1) k_1 S & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ k_n Q_n (b_n - 1) & 0 & \dots & -m_n + (b_n - 1) k_n S \end{pmatrix}.$$

The next result states when the bacteria-free equilibrium E_0 is locally stable.

Proposition 2.2.4. *The bacteria-free equilibrium* $E_0 = (0, \frac{d_1}{m_1}, \ldots, \frac{d_n}{m_n})$ *of system* (2.2.7) *is asymptotically stable when* $\sum_{i=1}^n \frac{k_i d_i}{m_i} > \alpha$ *and unstable when* $\sum_{i=1}^n \frac{k_i d_i}{m_i} < \alpha$.

Proof. The differential matrix of system (2.2.7) in E_0 is

$$A_0 := \begin{pmatrix} \alpha - \sum_{i=1}^n k_i \dfrac{d_i}{m_i} & 0 & \ldots & 0 \\ k_1 \dfrac{d_1}{m_1}(b-1) & -m_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ k_n \dfrac{d_n}{m_n}(b-1) & 0 & \ldots & -m_n \end{pmatrix}.$$

Then, the eigenvalues are $\lambda_0 = \alpha - \sum_{i=1}^n k_i \frac{d_i}{m_i}$ and $\lambda_i = -m_i$ for $i = 1, \ldots, n$. Clearly λ_i is negative for $i = 1, \ldots, n$. Moreover λ_0 is positive for $\sum_{i=1}^n \frac{k_i d_i}{m_i} < \alpha$, giving the instability of E_0, and λ_0 is negative for $\sum_{i=1}^n \frac{k_i d_i}{m_i} < \alpha$, which in turn implies that E_0 is asymptotically stable for $\sum_{i=1}^n \frac{k_i d_i}{m_i} > \alpha$. $\square$

In particular, the previous result shows the instability of E_0 when the coexistence equilibrium exists. The following result states the stability of E_c whenever it exists, only for $n = 1$, 2 and 3, that is, considering no more than three different varieties of phages. Along the proof we will extensively use a well-known criterion due to Routh-Hurwitz which can be found, for example, in [23]. We reproduce it here for sake of clarity.

Theorem 2.2.5 (Routh-Hurwitz Criterion)**.** *Given an equation with real coefficients of the form* $p(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0$, $a > 0$, *consider the following* $n \times n$ *matrix*

$$H := \begin{pmatrix} a_1 & a_3 & a_5 & \ldots & 0 \\ a_0 & a_2 & a_4 & \ldots & 0 \\ 0 & a_1 & a_3 & \ldots & 0 \\ 0 & a_0 & a_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & a_n \end{pmatrix}$$

where $a_j = 0$ *for* $j > n$. *Then all the roots of* $p(\lambda)$ *have negative real part if and only if all the principal minors of* H *are positive, i.e.,*

$$\Delta_1 = a_1 > 0,\ \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} > 0, \dots, \Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & \dots & 0 \\ a_0 & a_2 & a_4 & \dots & 0 \\ 0 & a_1 & a_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & a_n \end{vmatrix} > 0.$$

Proposition 2.2.6. *Whenever the equilibrium E_c of system* (2.2.7) *exists (that is, when $\sum_{i=1}^n \frac{k_i d_i}{m_i} > \alpha$) it would be asymptotically stable, for $n = 1$, 2 and 3.*

Proof. From the equations that define E_c we get the equalities $\alpha - \sum_{i=1}^n k_i \hat{Q}_i = 0$ and $-m_i + k_i(b_i - 1)\hat{S} = -d_i/\hat{Q}_i$. Using these equalities we get that the differential matrix of system (2.2.7) in E_c is

$$A := \begin{pmatrix} 0 & -k_1\hat{S} & \dots & -k_n\hat{S} \\ k_1\hat{Q}_1(b_1 - 1) & -d_1/\hat{Q}_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ k_n\hat{Q}_n(b_n - 1) & 0 & \dots & -d_n/\hat{Q}_n \end{pmatrix},$$

and therefore the characteristic polynomial is

$$\begin{aligned} p(\lambda) &= (-1)^{n+1}\lambda \prod_{i=1}^n \left(\lambda + \frac{d_i}{\hat{Q}_i}\right) - \sum_{i=1}^n k_i^2(b_i - 1)\hat{Q}_i\hat{S} \prod_{j\neq i} \left(\lambda + \frac{d_j}{\hat{Q}_j}\right)(-1)^n \\ &= (-1)^{n+1}\left[\lambda \prod_{i=1}^n (\lambda + D_i) + \sum_{i=1}^n K_i \prod_{j\neq i}(\lambda + D_j)\right], \end{aligned}$$

where $K_i := k_i^2(b_i - 1)\hat{Q}_i\hat{S}$ and $D_i := d_i/\hat{Q}_i$. We note that K_i and D_i are positive.

We will use the Routh-Hurwitz Criterion to show that all the eigenvalues of the characteristic polynomial are negative and, therefore, E_c is asymptotically stable.

- Letting $\boldsymbol{n = 1}$ we have

$$p(\lambda) = \lambda(\lambda + D) + K = \lambda^2 + D\lambda + K,$$

and the associated Routh-Hurwitz matrix is

$$H = \begin{pmatrix} D & 0 \\ 1 & K \end{pmatrix}.$$

The conditions of the Routh-Hurwitz Criterion are

(i). $a_1 = D > 0$

(ii). $\det(H) = DK > 0$,

which are clearly fulfilled for any values of the parameters. Then, all the eigenvalues of the characteristic polynomial are negative.

- Letting $\boldsymbol{n = 2}$ we have

$$
\begin{aligned}
p(\lambda) &= -\Big[\lambda(\lambda + D_1)(\lambda + D_2) + K_1(\lambda + D_2) + K_2(\lambda + D_1)\Big] \\
&= -\lambda^3 - (D_1 + D_2)\lambda^2 - (D_1D_2 + K_1 + K_2)\lambda - (K_1D_2 + K_2D_1),
\end{aligned}
$$

and the associated Routh-Hurwitz matrix is

$$
H = \begin{pmatrix} D_1 + D_2 & K_1D_2 + K_2D_1 & 0 \\ 1 & D_1D_2 + K_1 + K_2 & 0 \\ 0 & D_1 + D_2 & K_1D_2 + K_2D_1 \end{pmatrix}.
$$

Letting $a_0 = 1$, $a_1 = D_1 + D_2$, $a_2 = D_1D_2 + K_1 + K_2$ and $a_3 = K_1D_2 + K_2D_1$ the conditions of the Routh-Hurwitz Criterion are

(i). $a_1 = D_1 + D_2 > 0$

(ii). $a_1a_2 - a_0a_3 = (D_1 + D_2)(D_1D_2 + K_1 + K_2) - (K_1D_2 + K_2D_1) > 0$

(iii). $\det(H) = (a_1a_2 - a_0a_3)a_3 > 0$.

Clearly $a_i > 0$ for $i = 0, \ldots, 3$. Then, condition (i) is satisfied, and condition (iii) would be satisfied if condition (ii) is checked, which in turn can be checked as follows

$$
\begin{aligned}
a_1a_2 - a_0a_3 &= (D_1 + D_2)(D_1D_2 + K_1 + K_2) - (K_1D_2 + K_2D_1) \\
&= D_1(D_1D_2 + K_1) + D_2(D_1D_2 + K_2) > 0.
\end{aligned}
$$

- Letting $\boldsymbol{n = 3}$ we have

$$
\begin{aligned}
p(\lambda) &= \lambda(\lambda + D_1)(\lambda + D_2)(\lambda + D_3) + K_1(\lambda + D_2)(\lambda + D_3) \\
&\qquad + K_2(\lambda + D_1)(\lambda + D_3) + K_3(\lambda + D_1)(\lambda + D_2) \\
&= \lambda^4 + (D_1 + D_2 + D_3)\lambda^3 - (D_1D_2 + D_1D_3 + D_2D_3 + K_1 + K_2 + K_3)\lambda^2 \\
&\qquad + (K_1D_2 + K_1D_3 + K_2D_1 + K_2D_3 + K_3D_1 + K_3D_2 + D_1D_2D_3)\lambda \\
&\qquad + (K_1D_2D_3 + K_2D_1D_3 + K_3D_1D_2).
\end{aligned}
$$

Letting $a_0 = 1$, $a_1 = D_1 + D_2 + D_3$, $a_2 = D_1D_2 + D_1D_3 + D_2D_3 + K_1 + K_2 + K_3$, $a_3 = K_1D_2 + K_1D_3 + K_2D_1 + K_2D_3 + K_3D_1 + K_3D_2 + D_1D_2D_3$, $a_4 = K_1D_2D_3 + K_2D_1D_3 + K_3D_1D_2$, $a_j = 0$, $j > 4$ the associated Routh-Hurwitz matrix is

$$H = \begin{pmatrix} a_1 & a_3 & 0 & 0 \\ a_0 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_0 & a_2 & a_4 \end{pmatrix}.$$

The conditions of the Routh-Hurwitz Criterion are

(i). $a_1 = D_1 + D_2 + D_3 > 0$

(ii). $a_1a_2 - a_0a_3 = a_1a_2 - a_3 > 0$

(iii). $\Delta_3 := a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 = (a_1a_2 - a_3)a_3 - a_1^2a_4 > 0$

(iv). $\det(H) = \Delta_3a_4 > 0$.

Again, we have $a_i > 0$ for $i = 0, \dots, 4$ and obviously condition (i) is satisfied. Condition (iv) would be satisfied if condition (iii) is checked. Condition (ii) can be checked as follows

$$\begin{aligned} a_1a_2 - a_3 &= (D_1 + D_2 + D_3)(D_1D_2 + D_1D_3 + D_2D_3 + K_1 + K_2 + K_3) \\ &\quad - (K_1D_2 + K_1D_3 + K_2D_1 + K_2D_3 + K_3D_1 + K_3D_2 + D_1D_2D_3) \\ &= D_1(D_1D_2 + D_1D_3) + (D_2 + D_3)(D_1D_2 + D_1D_3 + D_2D_3) \\ &\quad + K_1D_1 + K_2D_2 + K_3D_3 > 0, \end{aligned}$$

and now, after some tedious computations, condition (iii) is checked to be

$$\begin{aligned} &(a_1a_2 - a_3)a_3 - a_1^2a_4 = \\ &\quad = K_1(D_2 + D_3)\Big(D_1(D_2^2 + D_3^2 + D_1D_2) + K_1D_1 + K_2D_2 + K_3D_3\Big) + K_1D_1^2D_3^2 \\ &\qquad + K_2(D_1 + D_3)\Big(D_2(D_1^2 + D_3^2 + D_1D_2) + K_1D_1 + K_2D_2 + K_3D_3\Big) + K_2D_2^2D_3^2 \\ &\qquad + K_3(D_1 + D_2)\Big(D_3(D_1^2 + D_2^2 + D_1D_3) + K_1D_1 + K_2D_2 + K_3D_3\Big) + K_3D_2^2D_3^2 \\ &\qquad + D_1D_2D_3(a_1a_2 - a_3) > 0, \end{aligned}$$

giving the claim □

This result, and the computations done to demonstrate it, justifies the following conjecture.

Conjecture 2.2.1. Whenever the equilibrium E_c of system (2.2.7) exists (that is, when $\sum_{i=1}^{n} \frac{k_id_i}{m_i} > \alpha$) it would be asymptotically stable.

2.2.4 Local study, general case

Stability for the bacteria-free equilibrium

First of all we shall linearize system (2.2.1) near the equilibrium E_0. Assuming $S(t) = 0 + s(t)$ and $Q_i(t) = d_i/m_i + q_i(t)$ we obtain

$$\begin{aligned} s'(t) &= \Big(\alpha - \sum_{i=1}^{n} k_i\Big(\frac{d_i}{m_i} + q_i(t)\Big)\Big)s(t) \\ &\cong \Big(\alpha - \sum_{i=1}^{n} k_i\frac{d_i}{m_i}\Big)s(t) \end{aligned}$$

and

$$\begin{aligned} q_i'(t) &= d_i - m_i\Big(\frac{d_i}{m_i} + q_i(t)\Big) - k_i\Big(\frac{d_i}{m_i} + q_i(t)\Big)s(t) \\ &\quad + k_i b_i e^{-\mu\zeta_i}\Big(\frac{d_i}{m_i} + q_i(t-\zeta_i)\Big)s(t-\zeta_i) \\ &\cong -m_i q_i(t) - k_i\frac{d_i}{m_i}s(t) + k_i b_i e^{-\mu\zeta_i}\frac{d_i}{m_i}s(t-\zeta_i) \end{aligned}$$

for $i = 1, \dots, n$.

Supposing exponential solutions of the from $s(t) = e^{\lambda t}s$ and $q_i(t) = e^{\lambda t}q_i$ we get that

$$\lambda e^{\lambda t}s = \Big(\alpha - \sum_{i=1}^{n} k_i\frac{d_i}{m_i}\Big)e^{\lambda t}s \Longrightarrow \Big(\lambda - \Big(\alpha - \sum_{i=1}^{n} k_i\frac{d_i}{m_i}\Big)\Big)s = 0$$

$$\begin{aligned} \lambda e^{\lambda t}q_i &= -m_i e^{\lambda t}q_i + k_i\frac{d_i}{m_i}(b_i e^{-(\mu+\lambda)\zeta_i} - 1)e^{\lambda t}s \\ &\qquad\qquad \Longrightarrow (\lambda + m_i)q_i - k_i\frac{d_i}{m_i}(b_i e^{-(\mu+\lambda)\zeta_i} - 1)s = 0, \end{aligned}$$

so we can write the characteristic equation as follows

$$p(\lambda) = \begin{vmatrix} \lambda - \Big(\alpha - \sum_{i=1}^{n} k_i\frac{d_i}{m_i}\Big) & 0 & \cdots & 0 \\ -k_1\frac{d_1}{m_1}(b_1 e^{-(\mu+\lambda)\zeta_1} - 1) & \lambda + m_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -k_n\frac{d_n}{m_n}(b_n e^{-(\mu+\lambda)\zeta_n} - 1) & 0 & \cdots & \lambda + m_n \end{vmatrix} = 0.$$

Therefore the eigenvalues of the linearized system near E_0 are $\lambda_0 = \alpha - \sum_{i=1}^{n} k_i\frac{d_i}{m_i}$, $\lambda_1 = -m_1, \dots, \lambda_n = -m_n$ and we can state the same result we have obtained for the non-delayed case.

Proposition 2.2.7. *The bacteria-free equilibrium* $E_0 = (0, \frac{d_1}{m_1}, \ldots, \frac{d_n}{m_n})$ *of system* (2.2.7) *is asymptotically stable when* $\sum_{i=1}^n \frac{k_i d_i}{m_i} > \alpha$ *and unstable when* $\sum_{i=1}^n \frac{k_i d_i}{m_i} < \alpha$.

Theorem 2.2.8 (E_0 stability)**.** *Equilibrium* $E_0 = (0, \frac{d_1}{m_1}, \ldots, \frac{d_n}{m_n})$ *of system* (2.2.1) *is stable for* $\sum_{i=1}^n \frac{k_i d_i}{m_i} > \alpha$ *(moreover, it is an attractor node and therefore asymptotically stable) and it is unstable (a saddle point with an unstable manifold of dimension 1) for* $\sum_{i=1}^n \frac{k_i d_i}{m_i} < \alpha$. *In particular if there exists the coexistence equilibrium* E_c *then* E_0 *is unstable.*

Bifurcations and stability for the coexistence equilibrium

In this section we try to draw some conclusions about the stability of the coexistence equilibrium. Note that since we assume the existence of E_c then the equilibrium E_0 is unstable.

Let us proceed as in the previous section. First of all we linearize system (2.2.1) near the equilibrium $E_c = (\hat{S}, \hat{Q}_1, \ldots, \hat{Q}_n)$. We recall that E_c exists if and only if $\left(\alpha - \sum_{i=1}^n k_i \hat{Q}_i\right) \hat{S} = 0$ and $d_i - m_i \hat{Q}_i - k_i \hat{Q}_i \hat{S} + k_i b_i e^{-\mu T_i} \hat{Q}_i \hat{S} = 0$, so we will assume this two conditions hold along this section. Let $S(t) = \hat{S} + s(t)$ and $Q_i(t) = \hat{Q}_i + q_i(t)$ and we obtain

$$\begin{aligned} s'(t) &= \left(\alpha - \sum_{i=1}^n k_i \left(\hat{Q}_i + q_i(t)\right)\right)\left(\hat{S} + s(t)\right) \\ &\cong \alpha s(t) - \sum_{i=1}^n k_i \left(\hat{Q}_i s(t) + \hat{S} q_i(t)\right) \end{aligned}$$

and

$$\begin{aligned} q_i'(t) &= d_i - m_i \left(\hat{Q}_i + q_i(t)\right) - k_i \left(\hat{Q}_i + q_i(t)\right)\left(\hat{S} + s(t)\right) \\ &\quad + k_i b_i e^{-\mu \zeta_i} \left(\hat{Q}_i + q_i(t - \zeta_i)\right)\left(\hat{S} + s(t - \zeta_i)\right) \\ &\cong -m_i q_i(t) - k_i \left(\hat{Q}_i s(t) + \hat{S} q_i(t)\right) + k_i b_i e^{-\mu \zeta_i} \left(\hat{Q}_i s(t - \zeta_i) + \hat{S} q_i(t - \zeta_i)\right) \end{aligned}$$

for $i = 1, \ldots, n$.

Supposing exponential solutions of the from $s(t) = e^{\lambda t} s$ and $q_i(t) = e^{\lambda t} q_i$ we get that

$$\begin{aligned} \lambda e^{\lambda t} s &= \alpha e^{\lambda t} s - \sum_{i=1}^n k_i e^{\lambda t} (\hat{Q}_i s + \hat{S} q_i) \\ \Longrightarrow (\lambda - \alpha) s &= - \sum_{i=1}^n k_i (\hat{Q}_i s + \hat{S} q_i) \end{aligned} \tag{2.2.8}$$

and

$$\begin{aligned}\lambda e^{\lambda t} q_i &= -m_i e^{\lambda t} q_i + k_i(b_i e^{-(\mu+\lambda)\zeta_i} - 1)e^{\lambda t}(\hat{Q}_i s + \hat{S} q_i) \\ \implies (\lambda + m_i) q_i &= h_i(\lambda)(\hat{Q}_i s + \hat{S} q_i)\end{aligned} \tag{2.2.8i}$$

for $i = 1, \dots, n$, where we have introduced the notation $h_i(\lambda) := k_i(b_i e^{-(\mu+\lambda)\zeta_i} - 1)$.

If we sum equations (2.2.8i) multiplied by $\frac{k_i}{h_i(\lambda)}$, $i = 1, \dots, n$, we see that

$$\sum_{i=1}^{n} \frac{k_i(\lambda + m_i) q_i}{h_i(\lambda)} = \sum_{i=1}^{n} k_i(\hat{Q}_i s + \hat{S} q_i) \tag{2.2.9}$$

and we can replace (2.2.9) in (2.2.8) to obtain

$$(\lambda - \alpha) s = -\sum_{i=1}^{n} \frac{k_i(\lambda + m_i) q_i}{h_i(\lambda)}. \tag{2.2.10}$$

On the other hand, from (2.2.8i) we have

$$q_i = \frac{h_i(\lambda) \hat{Q}_i s}{(\lambda + m_i) - h_i(\lambda)\hat{S}}$$

and we get from (2.2.10) that

$$\lambda - \alpha = -\sum_{i=1}^{n} \frac{k_i(\lambda + m_i)\hat{Q}_i}{(\lambda + m_i) - h_i(\lambda)\hat{S}}.$$

Finally, using expression (2.2.5) for $\hat{Q}_i$ we have $\hat{Q}_i = \frac{d_i}{m_i - h_i(0)\hat{S}}$ which leads us to

$$\lambda - \alpha = -\sum_{i=1}^{n} \frac{k_i(\lambda + m_i) d_i}{((\lambda + m_i) - h_i(\lambda)\hat{S})(m_i - h_i(0)\hat{S})}, \tag{2.2.11}$$

and with the notation $H_i(\lambda, \hat{S}) := \lambda + m_i - h_i(\lambda)\hat{S}$ we obtain

$$\lambda - \alpha = -\sum_{i=1}^{n} \frac{k_i d_i H_i(\lambda, 0)}{H_i(\lambda, \hat{S}) H_i(0, \hat{S})}. \tag{2.2.11$'$}$$

We have obtained a form of the characteristic equation for system (2.2.1). We note that this is a transcendental equation (we recall that $h_i(\lambda)$ contains the term $e^{\lambda\zeta_i}$) which makes its study very difficult. Therefore we shall not look for the zeros of this equation, i.e., the eigenvalues of the linearized system. Instead, we shall try to use some previous results to extract some information on the stability (or instability) of E_c through bifurcation theory.

A bifurcation occurs when a change made to the parameter values (the bifurcation parameters) of a system causes a change in its behavior (like the stability of the equilibria). We have already seen a change in the stability of E_0: if we move the parameter α we can observe a change in the stability of E_0 $\Big($when $\alpha < F(0) = \sum_{i=1}^{n} k_i \frac{d_i}{m_i}$ E_0 is stable and when $\alpha > F(0)$ E_0 it is unstable$\Big)$. So we know there is a bifurcation for $\alpha = F(0) = \sum_{i=1}^{n} k_i \frac{d_i}{m_i}$.

This is called a transcritical bifurcation, which occurs when two different equilibrium curves intersect at the same point. In our case we have the equilibrium E_0, which is fixed with respect to α, and $E_c := E_c(\alpha) = (\hat{S}(\alpha), \hat{Q}_1(\alpha), \ldots, \hat{Q}_n(\alpha))$ that intersect when $\alpha = F(0)$. Indeed, recall that in Theorem 2.2.3 we used the function $F(S)$ defined by (2.2.6) and looked for solutions $\hat{S}$ of $F(S) = \alpha$. We proved that there exists a unique solution $\hat{S}$ such that $\hat{Q}_i > 0$, $i = 1, \ldots, n$, if and only if $\alpha > F(0)$. It is also easy to see that when $\alpha = F(0)$ we have $\hat{S} = 0$ and $E_c = E_0$. Moreover, when $\alpha < F(0)$ there exists a unique solution $\hat{S}$ of equation $F(S) = \alpha$ such that $\hat{Q}_i > 0$, $i = 1, \ldots, n$, and this solution $\hat{S}$ is negative. Therefore we have an equilibrium $E_c(\alpha)$ for all $\alpha > 0$, which coincides with E_0 when $\alpha = F(0) = \sum_{i=1}^{n} k_i \frac{d_i}{m_i}$.

Now that we have identified the bifurcation we can extract some conclusions. We know that the equilibrium E_0 changes its stability for $\alpha = F(0)$. We can use a well known result, that can be found, for example, in [30] pages 26–27, which tells us that, locally, the two equilibria involved in the transcritical bifurcation interchange their stability, leading us to the following result.

Proposition 2.2.9. *There exists an $\varepsilon > 0$ such that the coexistence equilibrium E_c of system* (2.2.1) *is stable for $\alpha \in (\sum_{i=1}^{n} k_i \frac{d_i}{m_i}, \sum_{i=1}^{n} k_i \frac{d_i}{m_i} + \varepsilon)$.*

Remark. We can also conclude that E_c is unstable when

$$\alpha \in (\sum_{i=1}^{n} k_i \frac{d_i}{m_i} - \varepsilon_1, \sum_{i=1}^{n} k_i \frac{d_i}{m_i})$$

for some $\varepsilon_1 > 0$, but we are not interested in this case because E_c has not biological meaning.

Another kind of bifurcation we can expect to find is the Hopf bifurcation, which happens when a fixed point loses its stability when a pair of complex conjugate eigenvalues of the characteristic equation cross the imaginary axis. It usually leads to the appearance of a small-amplitude limit cycle. This type of bifurcation is commonly found in delayed systems when increasing the delay time.

Taking $n = 2$, some numerical computations show that, for some suitable parameter values, the equilibrium E_c loses its stability when the value of the delays ζ_1, ζ_2 increase, as the pair of complex eigenvalues of the characteristic equation (2.2.11′) cross the imaginary axis.

These numerical results seem to indicate that we certainly have a Hopf bifurcation, but a more theoretical study should be conducted to have a certain proof.

2.2.5 Particular case: a single virus strain

In this section we will continue the study of the same model, but taking $n = 1$, i.e., a single strain of viruses. This case is easier to study because we can find the explicit form of the equilibrium of coexistence E_c. This allows us to get more detailed results than in the general case. We begin, however, by re-stating the results already obtained.

Model and equilibria

In this section we rewrite the system (2.2.1) for $n = 1$ and find the equilibria. Although we will not reproduce the results of sections 2.2.1 and 2.2.3 they are still valid in this case.

The system we will study is

$$\begin{cases} S'(t) = \Big(\alpha - kQ(t)\Big)S(t) \\ Q'(t) = d - mQ(t) - kQ(t)S(t) + kbe^{-\mu\zeta}Q(t-\zeta)S(t-\zeta) \end{cases} \tag{2.2.12}$$

where $S(t)$ is the density of healthy bacteria at time t and $Q(t)$ the density of bacteriophages at time t.

We recall that when studying equilibria of system (2.2.1) we considered equations (2.2.3). The equivalent of these equations for the case with a single virus strain is

$$\begin{cases} 0 = (\alpha - kQ)S \\ 0 = d - Q\Big(m - k(be^{-\mu\zeta} - 1)S\Big). \end{cases} \tag{2.2.13}$$

As in the general case, we see that the bacteria-free equilibrium $E_0 = (0, m/d)$ exists for all the parameter values, and the conditions of a possible coexistence equilibrium $E_c = (\hat{S}, \hat{Q})$ are

$$\begin{cases} 0 = (\alpha - k\hat{Q}) \\ 0 = d - \hat{Q}\Big(m - k(be^{-\mu\zeta} - 1)\hat{S}\Big), \end{cases}$$

which can be solved and we obtain $\hat{Q} = \alpha/k$ and $\hat{S} = \frac{m-kd/\alpha}{k(be^{-\mu\zeta}-1)}$. We observe that $\hat{Q} > 0$ and $\hat{S} > 0$ if and only if $\alpha > \frac{kd}{m}$ (which makes the numerator of $\hat{S}$ positive). Thus, as expected, we obtain the same result as in the general case (Theorem 2.2.3), but now we can explicitly write the equilibrium E_c.

Theorem 2.2.10 (Equilibria)**.** *For* $\frac{kd}{m} \geq \alpha$ *the model* (2.2.12) *has a unique equilibrium* $E_0 = (0, \frac{d}{m})$*. When* $\frac{kd}{m} < \alpha$ *there exists two equilibria,* E_0 *and the coexistence equilibrium* $E_c = (\hat{S}, \hat{Q}) = \Big(\frac{m-kd/\alpha}{k(be^{-\mu\zeta}-1)}, \frac{\alpha}{k}\Big)$*.*

Asymptotic behavior of the solutions for the case without delay

In what follows we show some results concerning the boundedness of the solutions of system (2.2.12) assuming latency time ζ negligible, i.e., we will consider the following system

$$\begin{cases} S'(t) = \Big(\alpha - kQ(t)\Big)S(t) \\ Q'(t) = d - mQ(t) + k(b-1)Q(t)S(t) \end{cases} \tag{2.2.14}$$

These results tell us that there are no solutions that 'escape' to infinity, or in biological terms that the administration of phages controls the population of bacteria (it does not allow their population to increase indefinitely as it would happen without phages). First of all we will study the case where there exists the coexistence equilibrium and finally we will obtain a result in the case where there is no coexistence equilibrium. This last result tells us that the bacteria-free equilibrium is a global attractor, i.e., the population of bacteria will eventually extinguish.

Remark. Equations (2.2.14) define a vector field on $\mathbb{R}^+ \times \mathbb{R}^+$. We will denote the vector associated to a point (S, Q) by (S', Q'), where $S' = (\alpha - kQ)S$ and $Q' = d - mQ + k(b-1)QS$

Theorem 2.2.11 (Boundedness)**.** *Let* $\zeta = 0$ *and suppose that* $\frac{kd}{m} < \alpha$*, i.e., there exists the coexistence equilibrium* E_c *for system* (2.2.14)*. Then, given any initial condition* (S_0, Q_0) *such that* $S_0 \geq 0$ *and* $Q_0 \geq 0$*, the solution of system* (2.2.14) *is bounded.*

Proof. First of all we recall that, under the previous conditions, there exist two equilibria, say the bacteria-free equilibrium $E_0 = (0, Q_*)$ with $Q_* = d/m$ and the coexistence equilibrium $E_c = (S_c, Q_c)$ where $S_c = \frac{m-kd/\alpha}{k(b-1)}$ and $Q_c = \alpha/k$. We can observe that $Q_c > Q_*$ due to the condition of existence of E_c.

Since $S' = 0$ over the axis $\{S = 0, Q > 0\}$ we know that it is invariant. Moreover, for an initial condition $(0, Q_0)$ with $Q_0 > d/m$ we have $Q' < 0$ and if $Q_0 < d/m$ then $Q' > 0$. Therefore every solution that starts on this axis will get closer to the bacteria-free equilibrium as we increase time, giving its boundedness.

Let us now look for the vertical and horizontal isoclines.

$$0 = (\alpha - kQ)S \quad \Rightarrow \quad Q = \frac{\alpha}{k}\,, \tag{2.2.15}$$

$$0 = (d - Q(m - k(b-1)S)) \quad \Rightarrow \quad Q = \frac{d}{m - k(b-1)S}\,, \tag{2.2.16}$$

$$\Rightarrow \quad S = \frac{m - d/Q}{k(b-1)}\,. \tag{2.2.17}$$

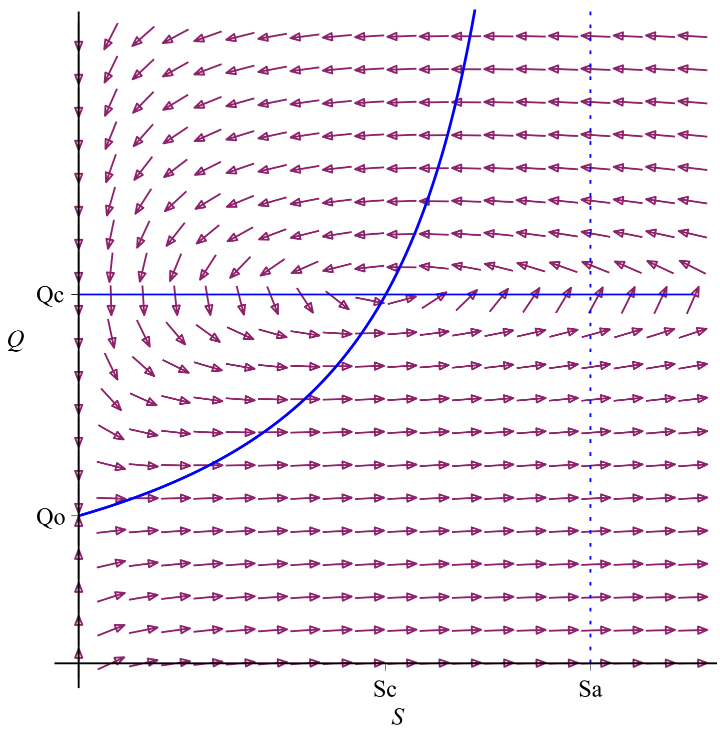

Figure 2.1: Field and isoclines

We observe that the coexistence equilibrium is the intersection of the two isoclines and that (2.2.16) have a vertical asymptote when $S_a = \frac{m}{k(b-1)}$ (which is greater than S_c). Moreover when $S > S_a$ we have $Q < 0$ and therefore we will only study equations (2.2.16) and (2.2.17) when $S < S_a$.

The isoclines divide the first quadrant into four sectors and we will label them as follows: A will be the region delimited by the two isoclines and the axis $S = 0$, with $Q > \alpha/k$; B will be the region that is also delimited by the two isoclines and the axis $S = 0$, but with $Q < \alpha/k$; C will be the region delimited by the two isoclines and both axes; and D will be the region that delimits only with the two isoclines.

Now we will observe the direction of the field. Since $S > 0$, the sign of S only depends on $\alpha - kQ$. Hence S' will be negative when $Q > \alpha/k$ and positive when $0 \leq Q < \alpha/k$. On the other hand, for $0 \leq S < S_a$ we have

$$Q > \frac{d}{m - k(b-1)S},$$

which gives $Q' < 0$ and when $S > S_a$ (and $Q > 0$) we have $Q' > 0$.

Therefore the solutions of our system cannot go to infinity on region A. Indeed, A is only unbounded from above and Q' is negative on A. B is bounded and the solutions must also remain bounded on this region.

Let us now focus on proving that solutions cannot go to infinity on region C. This region is only unbounded from the right, and we will prove that any solution in C will not 'escape' to the right.

Let $F(S, Q)$ be defined by $F(S, Q) = Q'/S'$, which describes the slope of the field at (S, Q). Note that $F(S, Q) > 0$ for every point in C. Moreover, along the axis $\{Q = 0, S > 0\}$ we have $Q' > 0$ and $F(S, Q) = \frac{d}{\alpha S}$.

We take a point $(S^*, Q^*) \in C$. Consider the point $(S^*, 0)$ and the straight line containing this point and slope $Q'/S' = \frac{d}{\alpha S^*}$, which comes defined by $r^* : Q = \frac{d}{\alpha S^*}(S - S^*)$. We will show that a solution containing the point (S^*, Q^*) cannot cross the straight line r^*, thus proving there is no solution that can escape to infinity on region C.

To this end it is enough to show that the vector field does not cross r^* 'to the right', which is equivalent to see that $F(S, Q) \geq \frac{d}{\alpha S^*}$ along r^*. Since $Q = \frac{d}{\alpha S^*}(S - S^*)$ over r^* we get

$$F(S) := F(S, Q)|_{Q \in r^*} = \frac{d - \Big(m - k(b-1)S\Big)\frac{d}{\alpha S^*}\Big(S - S^*\Big)}{\Big(\alpha - k\frac{d}{\alpha S^*}(S - S^*)\Big)S}. \tag{2.2.18}$$

So we want to prove that $F(S) \geq \frac{d}{\alpha S^*}$ for those S such that r^* is in the region C. Without loss of generality we can suppose S^* to be large enough, so we can take $S^* > S_c$ in order to ensure that r^* does not cross the isocline defined by (2.2.16). Therefore we will show that $F(S) \geq \frac{d}{\alpha S^*}$ for those values of S between S^* and the point where r^* intersects the isocline $Q = \alpha/k$, i.e., $S_i := S^* + \frac{\alpha^2 S^*}{kd}$.

From (2.2.18) we observe that $F(S) \geq \frac{d}{\alpha S^*}$ is equivalent to

$$\frac{d(S - S^*)\left(-\alpha - m + k(b-1)S + \frac{kdS}{\alpha S^*}\right)}{\alpha S^* S\left(\alpha - k\frac{d}{\alpha S^*}(S - S^*)\right)} \geq 0.$$

Since $S > S^* > 0$, the sign of this division only depends on the sign of the expressions $-\alpha - m + k(b-1)S + \frac{kdS}{\alpha S^*}$ and $\alpha - k\frac{d}{\alpha S^*}(S - S^*)$ for $S^* \leq S \leq S_i$.

The second expression vanishes for $S = S_i$ and it is decreasing on S, therefore it is positive for $S^* \leq S \leq S_i$. On the other hand, we notice that the first expression is increasing on S and it vanishes when $S = \tilde{S} := \frac{\alpha^2 S^* + m\alpha S^*}{k(b-1)\alpha S^* + dk}$. Taking $S^* \geq S_c + \frac{\alpha}{k(b-1)}$ we have $\tilde{S} \leq S^*$ and the positiveness of the first expression holds, proving the boundedness of solution on region C.

It only remains to prove the boundedness of the solution on region D. First of all notice that the vector field defined by (S', Q') crosses the vertical line $\{S = \hat{S}\}$ to the left for any $\hat{S} > S_a$, since $S' < 0$ and $Q' > 0$ on D. Hence there is no solution that 'escapes' to the right on region D.

To prove that the solutions are bounded from above on region D we take a point $(\hat{S}, \hat{Q})$ over the vertical line $S = \hat{S}$ such that $\hat{Q} > Q_c = \alpha/k$. We recall that $S' < 0$ and $Q' > 0$ on this point. Let us now consider the straight line $\hat{r}$ that contains this point and has slope $F(\hat{S}, \hat{Q})$.

We will proceed as before, proving that the vector field does not cross $\hat{r}$ upwards. In other words, we will see that $|F(S,Q)| \leq |F(\hat{S}, \hat{Q})|$ for any point (S, Q) in $\hat{r}$. To this end it is enough to see that $|F(S,Q)|$ is decreasing on Q and increasing on S or, since $|F(S,Q)| = -F(S,Q)$, that $\frac{\partial F}{\partial Q} > 0$ and $\frac{\partial F}{\partial S} < 0$ where

$$\begin{aligned}
\frac{\partial F}{\partial Q} &= \frac{-(m-k(b-1)S)(\alpha - kQ)S + kS(d - (m - k(b-1)S)Q)}{S^2(\alpha - kQ)^2} \\
&= \frac{kSd - (m - k(b-1)S)\alpha S}{S^2(\alpha - kQ)^2}, \\
\frac{\partial F}{\partial S} &= \frac{k(b-1)Q(\alpha - kQ)S - (\alpha - kQ)(d - (m - k(b-1)S)Q)}{S^2(\alpha - kQ)^2} \\
&= \frac{-(\alpha - kQ)(d - mQ)}{S^2(\alpha - kQ)^2}.
\end{aligned}$$

Since $Q > \alpha/k$ on D, and hence $Q > d/m$, we have that $\frac{\partial F}{\partial S} < 0$. It is also easy to see that $\frac{\partial F}{\partial Q} > 0$ for any $S > S_c$, condition which is fulfilled on D. And with this the prove is complete. □

Theorem 2.2.12. *Let the coexistence equilibrium E_c exist. Then there exists a bounded region which is positively invariant and global attractor.*

Proof. Let U be the unstable manifold of the bacteria free equilibrium E_0. With the notation used on the previous result, starting from E_0 we know that U will go through region C. Then it will cross the isocline $\{Q = \alpha/k\}$ and continue on region D until it crosses the other isocline.

Let $E^* = (S^*, Q^*)$ be this last point where U intersects the isocline defined by (2.2.16) and consider the closed region R, which comes delimited by the axis

$\{S = 0\}$, the straight line $\{Q = Q^*\}$ and the piece of U we just described, that goes from E_0 to E^*. It is clear that this region is positively invariant. Moreover, like we proved on the previous result, all the solutions will get in region A at some time, and once on this region they must get inside region R. □

Theorem 2.2.13 (Extinction). *Let $\frac{kd}{m} \geq \alpha$, i.e., there only exists the bacteria free equilibrium E_0 of system* (2.2.14). *Then, given any initial condition (S_0, Q_0) such that $S_0 \geq 0$ and $Q_0 \geq 0$, the solution of system* (2.2.14) *tends to E_0.*

Proof. First of all notice that, as in Theorem 2.2.11, we have two isoclines defined by (2.2.15) and (2.2.16). Though in this case, since $\alpha/k \leq d/m$, these two isoclines does not intersect in the first quadrant (otherwise the coexistence equilibrium would exist).

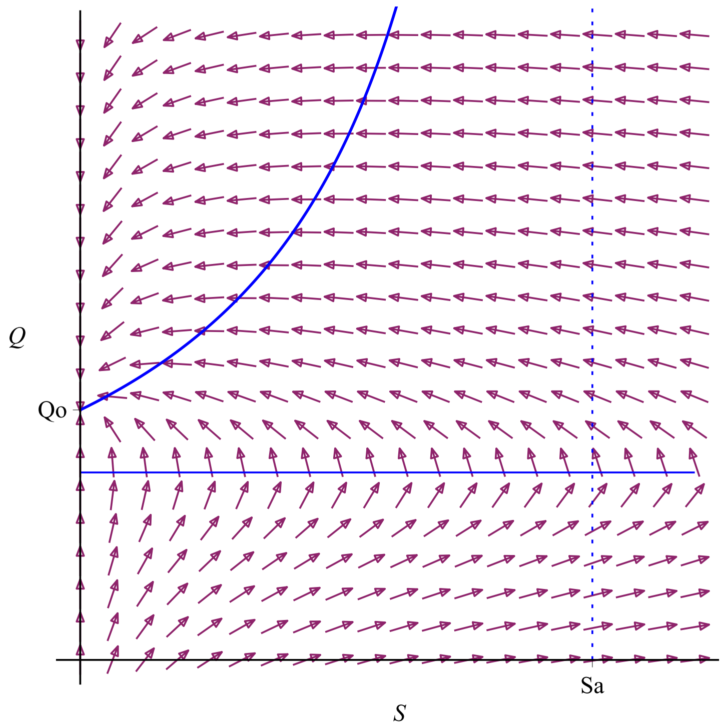

Figure 2.2: Field and isoclines

Therefore the situation on this case is slightly different since the first quadrant is divided into three regions only, namely: region A which is delimited by the isocline defined by (2.2.16) and the $\{S = 0\}$ axis; region B that limits with the isocline $\{Q = \alpha/k\}$ and both axes; and region C, delimited by the two isoclines and the $\{S = 0\}$ axis (when $\frac{kd}{m} = \alpha$ the two isoclines are enough to determine region C).

It is easy to see that the direction of the field on regions A, B and C is the same than the direction on regions A, C and D of theorem 2.2.11, respectively (see Figure 2.2.5).

Hence solution cannot escape to infinity on region A. Proceeding as we did for regions C and D in Theorem 2.2.11 it can be seen that solutions are also bounded on regions B and C respectively.

Moreover, we can observe that solutions will eventually get in region A. Indeed, solutions on region B, since they cannot escape to infinity nor cross the axes, and there is no fixed point on B, must then go to region C; similarly, solutions on C must go to region A since they cannot go to region B. Once they are on region A, solutions must remain on this region. Moreover, since there no interior equilibrium on A and $Q' < 0$, then all the solutions must tend to the E_0 equilibrium.

□

2.3 Stochastic model

2.3.1 Introduction

In this section, we will proceed to analyze a system modeling bacteriophage treatments for infections in a noisy context. In the small noise regime, we show that after a reasonable amount of time the system is close to a bacteria free equilibrium (which is a relevant biologic information) with high probability. Mathematically speaking, our study hinges on concentration techniques for delayed stochastic differential equations.

First of all, we consider a truncated identity function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$, such that $\sigma \in \mathcal{C}^\infty$, $\sigma(x) = x$ whenever $0 \le x \le M$ and $\sigma(x) = M+1$ for $x > M+1$, and we modify our basic model (2.1.1) as follows

$$\begin{cases} \mathrm{d}S(t) = \left[\alpha - k\sigma(Q(t))\right] S(t)\mathrm{d}t \\ \mathrm{d}Q(t) = \left[d - mQ(t) - k\sigma(Q(t))S(t) + k\, b\, e^{-\mu\zeta}\sigma(Q(t-\zeta))S(t-\zeta)\right] \mathrm{d}t. \end{cases} \tag{2.3.1}$$

Here all the parameters have the same meaning they had in (2.1.1), say, $S(t)$ (resp. $Q(t)$) denotes the non-infected bacteria (resp. bacteriophages) concentration at time t, α is the reproducing rate of the bacteria and k is the adsorption rate. Also, d stands for the quantity of bacteriophages inoculated per unit of time, m is their death rate, we denote by b the burst size, ζ the latency time and the coefficient $e^{-\mu\zeta}$ represents an attenuation in the release of bacteriophages (given by the expected number of bacteria cell's deaths during the latency time, where μ is the bacteria's death rate). A given initial condition $\{S_0(\tau), Q_0(\tau); -\zeta \le \tau \le 0\}$ is also specified.

We have considered here the truncation of the identity σ in order to manipulate bounded coefficients in our equations, but our parameter M can also be interpreted as a maximal phage attack rate. This feature is also present in [46], where the author argues that the saturation in the phage attack rate is due to multiple phage binding to a cell (the likelihood of this event being higher in case of high density of phages).

Let us point out that these changes with respect to the basic deterministic model (2.1.1) induce some additional mathematical difficulties, which are handled in Section 2.3.2. The results, though, are very similar to those of the previous section. Indeed, given a large enough M we shall see that when $kd/m > \alpha$ there exists a unique stable steady state $E_0 = (0, d/m)$ of system (2.3.1) (in particular bacteria have been eradicated), and when $kd/m < \alpha$ the point E_0 is still an equilibrium but it becomes unstable, while another coexistence equilibrium $E_c = \left(\frac{\frac{m\alpha}{k} - d}{\alpha(b-1)}, \frac{\alpha}{k}\right)$ emerges.

In Section 2.3.2 we will conduct a short study on the existence and stability of the equilibrium E_0 for any given $M > 0$, but we will not give any result on the other equilibrium E_c since we only study results concerning the bacteria-free equilibrium E_0 along this section. The case of a unique stable equilibrium E_0 makes the mathematical analysis easier and it corresponds to the main practical situation we have in mind, where high doses of phages are usually introduced in the cattle food. One should also mention a natural generalization of our problem, studied in the previous section for the model (2.1.1): Consider the action of several varieties of bacteriophages, which is an option widely considered among practitioners. We have restricted our analysis here to a simplified situation, and the case with n strains of bacteriophages is left for future works.

We are also interested in the exponential convergence of the solution of (2.3.1) towards its equilibrium E_0, which has to be worked out carefully.

This being recalled for the deterministic system, the main aim of this section is to deal with a noisy version of equation (2.3.1). This stochastic modeling can be justified by several effects:
(a) It is perfectly assumable that noise will appear when collecting data from laboratory tests.
(b) When one wishes to go from in vitro to in vivo modeling, it is commonly accepted that noisy versions of the differential systems at stake have to be considered. Indeed, random fluctuations in parameters like temperature or exposure to sun, rain and other environmental elements yield an important variability in the coefficients of our system. These fluctuations can be accurately summarized by a noisy random coefficient.
(c) Some quantities which were assumed to be deterministic in (2.3.1) are in fact random, such as the latency time ζ (see e.g. [11, 42] for contributions in this direction) and the number of phages b which are released from the lytic mechanism.

These random effects are present in other biological systems, and stochastic equations have been introduced for example in [19] for HIV dynamics and in [17] for bacteriophages in marine organisms. In these references it is always assumed that the noise enters in a bilinear way, which is quite natural in this situation and ensures positivity of the solution. We shall take up this strategy here, and consider system (2.3.1) with a small random perturbation of the form

$$\begin{cases} \mathrm{d}S^\varepsilon(t) = \left[\alpha - k\sigma(Q^\varepsilon(t))\right] S^\varepsilon(t)\mathrm{d}t + \varepsilon\sigma(S^\varepsilon(t)) \circ \mathrm{d}W^1(t) \\ \mathrm{d}Q^\varepsilon(t) = \Big[d - mQ^\varepsilon(t) - k\sigma(Q^\varepsilon(t))S^\varepsilon(t) + k\, b\, e^{-\mu\zeta}\sigma(Q^\varepsilon(t-\zeta))S^\varepsilon(t-\zeta)\Big]\, \mathrm{d}t \\ \qquad\qquad + \varepsilon\sigma(Q^\varepsilon(t)) \circ \mathrm{d}W^2(t), \end{cases} \tag{2.3.2}$$

where ε is a small positive coefficient and $W = (W^1, W^2)$ is a 2-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ equipped with the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ associated to the Wiener process W. Let us add the following remarks in order to further justify our model (2.3.2).

(i) Instead of giving a detailed model for all the random effects recalled above, we have decided to summarize them in a global stochastic term represented by the Wiener process W. This is obviously a first approximation, where one assumes that a sum of many small effects gives raise to a Gaussian random variable (as suggested by the central limit theorem). Let us mention however that more complex situations, were quantities like b are modeled e.g. by Ornstein-Uhlenbeck processes, might be the object of future extensions of the current work.

More specifically, let us examine the dynamics of S. According to the fact that this process can be expressed as an exponential, it is reasonable to think that its relative increments (namely $dS(t)/S(t)$) are governed by a trend $\alpha - k\sigma(Q(t))$ plus a small Gaussian perturbation $\varepsilon\, dW(t)$. We shall thus assume this additive noise perturbation for the relative increment $dS(t)/S(t)$, which yields the first equation in (2.3.2). The second relation of our system (2.3.2) can be obtained thanks to the same kind of hypothesis. Let us recall at this point that similar interpretations of random effects by an analysis of the relative increments are implicit in [17, 19]. Furthermore, it should also be mentioned that the curves $t \mapsto \log(S(t))$ and $t \mapsto \log(Q(t))$ based on real measurements are compatible with a stochastic model in which the noise enters in an additive way. We thus believe that our bilinear noisy model is a natural one, though the exploration of alternative stochastic modeling strategies as explained in [1, 2, 3] would obviously be extremely interesting. We defer these developments to a subsequent works.

(ii) We have chosen to work with Stratonovich differentials, denoted by $\circ\, \mathrm{d}W$, instead of Itô type differentials. This is harmless in terms of mathematical analysis and we believe this model to be physically accurate, in spite of the fact that it differs from the Itô type modeling of [17, 19]. Indeed, our starting point here is the macroscopic system of equations (2.3.1), in which the internal noise due to individual phage and bacteria fluctuations has already been averaged. Then all the randomness sources alluded to at points (a)-(b)-(c) above can be considered as external contributions. We refer to [51, Chapter 5] for a thorough justification of the fact that Stratonovich type noises are applicable in this kind of situation. Let us also stress the fact that Stratonovich equations can be seen as limits of smooth noisy equations, according to the celebrated Wong-Zakai theorem [53].

With these considerations in mind, the main aim of the current section can be summarized as follows: we wish to prove that for a time τ_0 within a reasonable range, the couple$Z^\varepsilon(\tau_0) := (S^\varepsilon(\tau_0), Q^\varepsilon(\tau_0))$ is not too far away from the stable equilibrium E_0 of equation (2.3.1). Note that *reasonable range* is meant here as a time which corresponds to the order of both the latency delay and the time when the immune system of the animal can cope with the remaining bacteria.

As we shall see in the sequel, the treatment of equation (2.3.2) involves the introduction of some rather technical assumptions on our coefficients. For sake of readability, we have thus decided to handle first the following system without delay:

$$\begin{cases} \mathrm{d}S^\varepsilon(t) = [\alpha - k\sigma(Q^\varepsilon(t))]\, S^\varepsilon(t)\mathrm{d}t + \varepsilon\sigma(S^\varepsilon(t)) \circ \mathrm{d}W^1(t) \\ \mathrm{d}Q^\varepsilon(t) = [d - mQ^\varepsilon(t) + k(b-1)\sigma(Q^\varepsilon(t))S^\varepsilon(t)]\, \mathrm{d}t + \varepsilon\sigma(Q^\varepsilon(t)) \circ \mathrm{d}W^2(t), \end{cases} \tag{2.3.3}$$

where we notice that the only difference between (2.3.2) and (2.3.3) is that we have set $\zeta = 0$ in the latter.

The main advantage of equation (2.3.3) lies in the fact that we are able to work under the following rather simple set of assumptions:

Hypothesis 2.3.1. *We will suppose that the coefficients of equation (2.3.3) satisfy:*
(i) *The initial condition (S_0, Q_0) of the system lies into the region*

$$R_0 := \left[0, \frac{mM - d}{k(b-1)M}\right] \times [d/m, M].$$

(ii) *The coefficient $\gamma = kd/m - \alpha$ is strictly positive and $M > d/m$.*

We shall also use extensively the following notation:

Notation 2.3.2. *The letters $c, c_1, c_2, \dots$ will stand for universal constants, whose exact value is irrelevant. For a continuous function f, we set*

$$\|f\|_{\infty,I} = \sup_{x\in I} |f(x)|.$$

Then the previous loose considerations about convergence to E_0 can be summarized in the following theorem, which is the main result of this section for our bacteriophage system without delay.

Theorem 2.3.3. *Given positive initial conditions, equation (2.3.3) admits a unique solution which is almost surely an element of $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+^2)$. Assume furthermore Hypothesis 2.3.1, set $\eta = m/2 \wedge \gamma$ and consider 3 constants $1 < \kappa_1 < \kappa_2 < \kappa_3$. Then there exists ρ_0 such that for any $\rho \le \rho_0$ and any interval of time of the form $I = [\kappa_1 \ln(c/\rho)/\eta, \kappa_2 \ln(c/\rho)/\eta]$, we have*

$$\mathbf{P}\left(\|Z^\varepsilon - E_0\|_{\infty,I} \ge 2\rho\right) \le \exp\left(-\frac{c_1\rho^{2+\lambda}}{\varepsilon^2}\right), \tag{2.3.4}$$

where λ is a constant satisfying $\lambda > \kappa_3/\eta$.

Remark 2.3.4. Relation (2.3.4) can be interpreted in the following manner: assume that we observe a noise with intensity ε. Then the kind of deviation we might expect from the noisy system (2.3.3) with respect to the equilibrium E_0 is of order $\varepsilon^{2\vartheta}$ with $\vartheta = 2\eta/\kappa_3$. This range of deviation happens at a time scale of order $\ln(\rho^{-1})/\eta$.

A second part of our analysis is then devoted to the more realistic delayed system (2.3.2), for which we have to impose some additional technical assumptions.

Hypothesis 2.3.5. *We will suppose that the coefficients of equation (2.3.3) satisfy the following conditions, valid for any $t \in [-\zeta, 0]$:*
(i) *The initial condition $(S_0(t), Q_0(t))$ of the system lies into the region*

$$R_0 := [0, M] \times \left[\frac{d}{m}, M\right].$$

(ii) *We have $b\, e^{-\mu\zeta} Q_0(t) S_0(t) > \frac{d}{m} S_0(0)$, and $b\, e^{-\mu\zeta} > 1$.*
(iii) *The condition $S_0(t) < \frac{mM-d}{kbe^{-\mu\zeta}M}$ is satisfied.*

With these hypotheses in hand, we obtain a result which is analogous to Theorem 2.3.3.

Theorem 2.3.6. *Equation* (2.3.4) *still holds for the delayed system* (2.3.2)*, under Hypothesis 2.3.5.*

Theorem 2.3.6 can be seen as the main result of the current section, and deserves some additional comments:
(1) We have produced a concentration type result instead of a large deviation principle for equation (2.3.3), because it seemed more adapted to our biological context. Indeed, in the current situation one wishes to know how far we might be from the desired equilibrium at a given fixed time, instead of producing asymptotic results as in the large deviation theory. At a technical level however, we rely on large deviation type tools, and in particular on an extensive use of exponential inequalities for martingales.
(2) Let us compare our result with [17, 19], which deal with closely related systems. The interesting article [17] is concerned with a predator-prey system similar to ours, but it assumes that a linearization procedure around equilibrium in the highly nonlinear situation (2.3.3) can be performed. The analysis relies then heavily on this unjustified step. As far as [19] is concerned, it roughly shows that if the noise intensity of the system is high enough, then HIV epidemics can be kept under control (in terms of exponential stability). This is valuable information, but far away from our point of view which assumes a low intensity for the noise. We should mention again the related thorough deterministic studies [42, 24, 32, 46], as well as the enlightening alternative stochastic modeling [1, 2, 3].
(3) Mathematically speaking, it would certainly be interesting to play with the rich picture produced by equation (2.3.1) and its perturbed version in terms of stable and unstable equilibria. We have not delved deeper into this direction because it did not seem directly relevant to the biological problem we are starting from.

This section is structured as follows: Section 2.3.2 is devoted to some preliminary considerations (convergence to equilibrium for the deterministic equations, and then existence and uniqueness results for our stochastic systems). Then we show our concentration results in Section 2.3.3. Finally, our theoretical results are illustrated by some numerical simulations presented in Section 2.3.4.

2.3.2 Preliminaries

In this section, we give some basic results concerning our competition system. This is done in increasing order for the complexity of the system under consideration.

1. Exponential convergence to equilibrium for the deterministic counterpart of the non delayed equation (2.3.3).

2. Same problem for the deterministic counterpart of the delayed equation (2.3.2).

3. Existence and uniqueness of the solution of the perturbed system (2.3.2), starting from the simpler system (2.3.3).

Before going on with our preliminary considerations, let us label the following set of hypothesis on our coefficient σ as well as the initial conditions.

Hypothesis 2.3.7. *The coefficients of our differential systems satisfy the following assumptions.*
(i) *The function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\sigma \in \mathcal{C}^\infty$, and satisfies $\sigma(x) = x$ for $0 \leq x \leq M$ and $\sigma(x) = M+1$ for $x > M+1$. We also assume that $0 \leq \sigma'(x) \leq C$ for all $x \in \mathbb{R}_+$, with a constant C such that $C > 1$.*
(ii) *As far as the initial condition is concerned, we assume that it is given as continuous positive functions $\{S_0(\tau), Q_0(\tau); -\zeta \leq \tau \leq 0\}$. In case of the non delayed systems, $\zeta = 0$, it is simply given by two positive constants (S_0, Q_0).*

Analysis of the deterministic non delayed system

This section is devoted to the analysis of the non perturbed system corresponding to (2.3.3). Namely, we shall consider the following dynamical system

$$\begin{cases} \mathrm{d}S(t) = [\alpha - k\sigma(Q(t))]\, S(t)\mathrm{d}t \\ \mathrm{d}Q(t) = [d - mQ(t) + k(b-1)\sigma(Q(t))S(t)]\, \mathrm{d}t. \end{cases} \tag{2.3.5}$$

We will give some sufficient conditions for the existence of a unique stable equilibrium E_0 and then show exponential convergence to this equilibrium.

Let us start with the basic results we shall need about equilibria of (2.3.5).

Theorem 2.3.8. *If either $M+1 < \frac{\alpha}{k}$ or $M > \frac{\alpha}{k}$ and $\frac{kd}{m} \geq \alpha$, system* (2.3.5) *has a unique (positive) steady state $E_0 = (0, \frac{d}{m})$. Moreover, the bacteria-free equilibrium E_0 is asymptotically stable for $\frac{kd}{m} > \alpha$ and $M > \frac{d}{m}$.*

Proof. To obtain the equilibria, we have to find the solution to the following equation

$$\begin{cases} 0 = (\alpha - k\sigma(\hat{Q}))\hat{S} \\ 0 = d - m\hat{Q} + k(b-1)\sigma(\hat{Q})\hat{S}, \end{cases} \tag{2.3.6}$$

where $\hat{S}$, $\hat{Q}$ are positive constants.

Owing to the first equation we have either $\hat{S} = 0$ or $\alpha - k\sigma(\hat{Q}) = 0$. $\hat{S} = 0$ and the second equation imply that $\hat{Q} = \frac{d}{m}$, and then we have that the bacteria-free equilibrium E_0 exists for any value of the parameters. In the case $M + 1 < \frac{\alpha}{k}$ one can observe that no other equilibrium exists (since $\alpha - k\sigma(\hat{Q}) > 0$ for any $\hat{Q}$).

Taking $M > \frac{\alpha}{k}$, $\alpha - k\sigma(\hat{Q}) = 0$ if and only if $\hat{Q} = \frac{\alpha}{k}$. Then, using the second equation in (2.3.6), we have

$$0 = d - m\frac{\alpha}{k} + (b-1)\alpha\hat{S} \quad \Longrightarrow \quad \hat{S} = \frac{m\alpha - kd}{k(b-1)\alpha},$$

which is positive only for $\alpha > \frac{kd}{m}$. Otherwise, this last equation gives us another equilibrium that we shall not consider along the stochastic case. So we have proved the first part of the result.

For the second part, the Jacobian matrix of system (2.3.5) at E_0 is

$$A_0 := \begin{pmatrix} \alpha - k\sigma(\frac{d}{m}) & 0 \\ k(b-1)\sigma(\frac{d}{m}) & -m \end{pmatrix}.$$

The eigenvalues of this matrix are easily shown to be $\lambda_0 = \alpha - k\sigma(\frac{d}{m})$ and $\lambda_1 = -m$, which are negative for $\frac{kd}{m} > \alpha$ and $M > \frac{d}{m}$.

□

Now we wish to study the rate of convergence towards the E_0 equilibrium in the stable case (i.e., when $kd/m > \alpha$ and $M > \frac{d}{m}$). The main result we obtain is the following one.

Theorem 2.3.9. *Under Hypothesis 2.3.1 and 2.3.7, the solution of system* (2.3.5) *with initial condition*

$$(S_0, Q_0) \in \left[0, \frac{mM - d}{k(b-1)M}\right] \times [d/m, M]$$

exponentially converges to the equilibrium E_0,

$$|(S(t), Q(t)) - E_0| \leq c\, e^{-\eta t}, \quad \text{with} \quad \eta = \gamma \wedge \frac{m}{2}, \tag{2.3.7}$$

where we recall that $\gamma = \frac{kd}{m} - \alpha > 0$.

Proof. In order to prove our claim, we first have to show that the region $R := [0, \frac{mM-d}{k(b-1)M}] \times [\frac{d}{m}, M] \subset [0, M]^2$ is left invariant by equation (2.3.5). Towards this aim, we can invoke the same method we will use in Proposition 2.3.10, and we let the reader check the details.

Now, since we have $Q(t) \leq M$ for all t, we can consider $\sigma(x) = x$ in equation (2.3.5). We will consider a version of this system centered at E_0 by means of the change of variables $\tilde{S} = S$, $\tilde{Q} = Q - d/m$. This leads to the system

$$\begin{cases} \tilde{S}'(t) = -\gamma\tilde{S}(t) - k\tilde{Q}(t)\tilde{S}(t) \\ \tilde{Q}'(t) = -m\tilde{Q}(t) + \dfrac{kd}{m}(b-1)\tilde{S}(t) + k(b-1)\tilde{Q}(t)\tilde{S}(t). \end{cases} \tag{2.3.8}$$

Notice that, according to our set of assumptions concerning the initial conditions, we have $\tilde{S}_0 \geq 0$ and $\tilde{Q}_0 \geq 0$. Thus the solution to (2.3.8) will remain positive for all $t > 0$ (it can be deduced from R being invariant, or can be proved just like in Proposition 2.3.14).

Now, from the first equation in (2.3.8), we have that $\tilde{S}'(t) \leq -\gamma\tilde{S}(t)$. This implies $\tilde{S}(t) \leq \tilde{S}_0 e^{-\gamma t}$, proving that $\tilde{S}(t)$ exponentially converges to zero.

Owing to the second equation in (2.3.8) and using positivity properties of the solution, we also get

$$\tilde{Q}'(t) \leq -m\tilde{Q}(t) + k(b-1)\tilde{S}_0 e^{-\gamma t}\left(\frac{d}{m} + \tilde{Q}(t)\right).$$

Finally, the variation of constants method will lead to the stated result, following the same steps we will detail later in the proof of Theorem 2.3.12. $\square$

Analysis of the deterministic delayed system

We now try to generalize the results of the previous section to our deterministic delayed system (2.3.1). To this aim, recall that we work under the additional assumptions 2.3.5.

A first step towards exponential stability is then the invariance of a certain region under our dynamical system.

Proposition 2.3.10. *Under Hypothesis 2.3.1, 2.3.7 and 2.3.5, the region*

$$R := \left[0, \frac{mM-d}{kbe^{-\mu\zeta}M}\right] \times \left[\frac{d}{m}, M\right] \subset [0, M]^2$$

is left invariant by equation (2.3.1).

Proof. We separate the analysis of S and Q in two steps.
Step 1: boundedness of S. Since S is obviously positive (along the same lines as for equation (2.3.11)) and owing to the fact that $S'(t) = (\alpha - k\sigma(Q(t)))\, S(t)$ we obtain that

$$S'(t) \leq 0 \text{ whenever } Q(t) > \frac{\alpha}{k}, \quad \text{and} \quad S'(t) \geq 0 \text{ whenever } Q(t) < \frac{\alpha}{k}.$$

Furthermore, our system starts from an initial condition $Q_0(0) \geq \frac{d}{m} > \frac{\alpha}{k}$. Thus S is non increasing as long as Q remains in the interval $[\frac{d}{m}, \infty)$.

Let us now observe what happens in the limiting case $Q_0(0) = \frac{d}{m}$. Recalling that our initial conditions are denoted by $S_0(t), Q_0(t)$ for $t \in [-\zeta, 0]$, we have

$$\begin{aligned} Q_0'(0) &= -k\frac{d}{m}S_0(0) + kbe^{-\mu\zeta}\sigma(Q_0(-\zeta))S_0(-\zeta) \\ &= k\left(be^{-\mu\zeta}Q_0(-\zeta)S_0(-\zeta) - \frac{d}{m}S_0(0)\right) > 0, \end{aligned}$$

where we have used the fact that $be^{-\mu\zeta}Q_0(-\zeta)S_0(-\zeta) > \frac{d}{m}S_0(0)$. According to this inequality, we obtain the existence of a strictly positive ε such that $Q(t) > \frac{d}{m}$ for all $t \in (0, \varepsilon)$. We thus introduce the quantity $t_0 = \inf\{t > 0 : \ Q(t) = \frac{d}{m}\}$, and notice that we have

$$Q'(t_0) = -k\frac{d}{m}S(t_0) + kbe^{-\mu\zeta}\sigma(Q(t_0 - \zeta))S(t_0 - \zeta).$$

Now we can distinguish two cases.

1. If $t_0 > \zeta$, since $S(t)$ is non-increasing in $[0, t_0]$, $S(t_0 - \zeta) \geq S(t_0)$ and hence

$$Q'(t_0) \geq kS(t_0)\left(be^{-\mu\zeta}\sigma(Q(t_0 - \zeta)) - \frac{d}{m}\right) > 0,$$

   due to the fact that $be^{-\mu\zeta} > 1$, $M > \frac{d}{m}$ and $Q(t_0 - \zeta) > \frac{d}{m}$.

2. If $t_0 \leq \zeta$, since $S(t_0) \leq S_0(0)$ we obtain

$$\begin{aligned} Q'(t_0) &\geq -k\frac{d}{m}S_0(0) + kbe^{-\mu\zeta}\sigma(Q_0(t_0 - \zeta))S_0(t_0 - \zeta) \\ &= k\left(be^{-\mu\zeta}Q_0(t_0 - \zeta)S_0(t_0 - \zeta) - \frac{d}{m}S_0(0)\right) > 0, \end{aligned}$$

   where we have used the fact that $be^{-\mu\zeta}Q_0(t)S_0(t) > \frac{d}{m}S_0(0)$ for all $t \in [-\zeta, 0]$.

This discussion allows to conclude that t_0 cannot be a finite time. Indeed, we should have $Q'(t_0) > 0$ and hence Q increasing in a neighborhood of t_0, while Q should be decreasing in a neighborhood of t_0 according to its very definition. We have thus reached the following partial conclusion,

$$Q(t) \geq \frac{d}{m}, \quad t \mapsto S(t) \text{ decreasing}, \quad S(t) \geq 0.$$

In particular, any interval of the form $[0, L]$ for $L \geq 0$ is left invariant by $t \mapsto S_t$.
Step 2: boundedness of Q. Our claim is now reduced to prove that for $(S_0(t), Q_0(t)) \in R$ we have $Q(t) \leq M$ for all $t \geq 0$.

To this aim notice that, whenever $Q_0(0) = M$ we have

$$\begin{aligned} Q'(0) &= d - mM - kMS_0(0) + kbe^{-\mu\zeta}\sigma(Q_0(-\zeta))S_0(-\zeta) \\ &\leq d - mM + kbe^{-\mu\zeta}MS_0(-\zeta) < 0, \end{aligned}$$

where we recall that $S_0(-\zeta) < \frac{mM-d}{kbe^{-\mu\zeta}M}$ according to Hypothesis 2.3.5. This yields the existence of $\varepsilon > 0$ such that $Q(t) < M$ for all $t \in (0, \varepsilon)$.

We now define $t_1 = \inf\{t > 0 : \ Q(t) = M\}$. It is readily checked that

$$\begin{aligned} Q'(t_1) &= d - mM - kMS(t_1) + kbe^{-\mu\zeta}\sigma(Q(t_1 - \zeta))S(t_1 - \zeta) \\ &= d - mM - kMS(t_1) + kbe^{-\mu\zeta}Q(t_1 - \zeta)S(t_1 - \zeta) \\ &\leq d - mM + kbe^{-\mu\zeta}MS(t_1 - \zeta), \end{aligned}$$

and we can distinguish again two cases.

1. If $t_1 > \zeta$, thanks to the fact that $t \mapsto S(t)$ is non-increasing on $[0, t_1]$, we have
$$Q'(t_1) \leq d - mM + kbe^{-\mu\zeta}MS_0(0) < 0,$$
since we have assumed that $S_0(0) < \frac{mM-d}{kbe^{-\mu\zeta}M}$.

2. If $t_1 \leq \zeta$ then
$$Q'(t_1) \leq d - mM + kbe^{-\mu\zeta}MS_0(t_1 - \zeta) < 0,$$
thanks to the fact that $S_0(t) < \frac{mM-d}{kbe^{-\mu\zeta}M}$ for all $t \in [-\zeta, 0]$.

As for the discussion of the previous step, this allows us to conclude that t_1 cannot be a finite time, due to the contradiction $Q'(t_1) < 0$ and $Q(t) < Q(t_1)$ for all $t \in (0, t_1)$. We have thus shown $Q(t) \leq M$ for all $t \geq 0$, which finishes the proof. □

Remark 2.3.11. Before stating the exponential convergence to the bacteria-free equilibrium result, let us observe that Theorem 2.3.8 still holds true for the delayed system (2.3.1). This can be easily checked, following the procedure as for the non-delayed system.

We are now ready to state our result on exponential convergence of the delayed system.

Theorem 2.3.12. *Assume Hypothesis 2.3.1, 2.3.7, and 2.3.5 are satisfied, and let R be the region defined at Proposition 2.3.10. Then the solution of system* (2.3.1) *with initial condition $(S_0, Q_0) \in R$ exponentially converges to the equilibrium E_0,*

$$|(S(t), Q(t)) - E_0| \leq c\, e^{-\eta t}, \quad \textit{with} \quad \eta = \gamma \wedge \frac{m}{2}, \tag{2.3.9}$$

where we recall that $\gamma = \frac{kd}{m} - \alpha > 0$.

Proof. According to Proposition 2.3.10, we have $Q(t) \leq M$ for all $-\zeta \leq t < \infty$ under our standing assumptions. Hence one can recast equation (2.3.1) as

$$\begin{cases} \mathrm{d}S(t) = (\alpha - kQ(t))\, S(t) dt \\ \mathrm{d}Q(t) = \Big(d - mQ(t) - kQ(t)S(t) + kbe^{-\mu\zeta} Q(t-\zeta)S(t-\zeta)\Big)\, dt \end{cases}$$

Let us perform now the change of variables $\tilde{Q} = Q - \frac{d}{m}$. This transforms the previous system into

$$\begin{cases} \mathrm{d}S(t) = \Big(\alpha - k(\tilde{Q}(t) + \frac{d}{m})\Big)\, S(t)\, \mathrm{d}t \\ d\tilde{Q}(t) = \Big(d - m(\tilde{Q}(t) + \frac{d}{m}) - k(\tilde{Q}(t) + \frac{d}{m})S(t) \\ \qquad\qquad + kbe^{-\mu\zeta}(\tilde{Q}(t-\zeta) + \frac{d}{m})S(t-\zeta)\Big)\, \mathrm{d}t. \end{cases}$$

Equivalently, our new system is

$$\begin{cases} \mathrm{d}S(t) = -\Big(\gamma S(t) + k\tilde{Q}(t)S(t)\Big)\, \mathrm{d}t \\ \mathrm{d}\tilde{Q}(t) = \Big(-m\tilde{Q}(t) - k\frac{d}{m}S(t) - k\tilde{Q}(t)S(t) + k\frac{d}{m}be^{-\mu\zeta}S(t-\zeta) \\ \qquad\qquad + kbe^{-\mu\zeta}\tilde{Q}(t-\zeta)S(t-\zeta)\Big)\, \mathrm{d}t. \end{cases}$$

Observe now that Proposition 2.3.10 asserts that $Q(t) \geq \frac{d}{m}$ for all $t \geq 0$, which means that $\tilde{Q}(t) \geq 0$. With our change of variables, we have also shifted our equilibrium to the point $(0, 0)$. We now wish to prove that $S(t)$ and $\tilde{Q}(t)$ exponentially converge to 0.

The bound on $S(t)$ is easily obtained, just note that

$$\mathrm{d}S(t) \leq -\gamma S(t)\, \mathrm{d}t,$$

which yields $S(t) \le S_0(0)\, e^{-\gamma t}$. As far as $\tilde{Q}(t)$ is concerned, one gets the bound

$$\begin{aligned}\frac{\mathrm{d}\tilde{Q}(t)}{\mathrm{d}t} &\le -m\tilde{Q}(t) + k\frac{d}{m}be^{-\mu\zeta}S_0(0)\, e^{-\gamma(t-\zeta)} + kbe^{-\mu\zeta}\tilde{Q}(t-\zeta)S_0(0)\, e^{-\gamma(t-\zeta)} \\ &\le -m\tilde{Q}(t) + kbe^{-\mu\zeta}S_0(0)\, e^{-\gamma(t-\zeta)}\left(\frac{d}{m} + M - \frac{d}{m}\right) \\ &= -m\tilde{Q}(t) + c\, e^{-\gamma t},\end{aligned}$$

with $c = kbMS_0(0)\, e^{(\gamma-\mu)\zeta}$, and where we have used the fact that $Q(t) \le M$ uniformly in t.

Invoking now the variation of constant method, it is readily checked that equation $\dot{x}(t) = -mx(t) + c\, e^{-\gamma t}$ with initial condition $x_0 = \tilde{Q}_0(0)$ can be explicitly solved as

$$\begin{aligned}x(t) &= e^{-mt}\left(\tilde{Q}_0(0) + \frac{c}{m-\gamma}\left(e^{(m-\gamma)t} - 1\right)\right) \\ &= \left(\tilde{Q}_0(0) - \frac{c}{m-\gamma}\right)e^{-mt} + \frac{c}{m-\gamma}e^{-\gamma t}.\end{aligned}$$

By comparison, this entails the inequality $\tilde{Q}(t) \le c_1\, e^{-\eta t}$, where $c_1 = \max(\tilde{Q}_0(0) - \frac{c}{m-\gamma}, \frac{c}{m-\gamma})$ and $\eta = m \wedge \gamma$. Our proof is now finished.

□

Properties of the stochastic system

Recall that we are considering the perturbed problem (2.3.2), with a coefficient σ and some initial conditions satisfying Hypothesis 2.3.7. In particular, due to the fact that we have assumed a bounded coefficient σ, the existence and uniqueness of the solution to our differential system is a matter of standard considerations.

Theorem 2.3.13 (Global existence of solution)**.** *For any positive initial condition there exists a unique solution of* (2.3.2)*, which is defined for all* $t \ge 0$.

Proof. It is readily checked that the coefficients of the equation are locally Lipschitz with linear growth. The existence and uniqueness of the solution is then a direct consequence of classical results (see e.g. [29, Section 5.2] for the non delayed system and [38] for the delayed one). □

Positivity of the solution is also an important feature, if we want the quantities $S(t), Q(t)$ to be biologically meaningful. Moreover, part of our analysis will rely on this property, that we label for further use.

Proposition 2.3.14 (Positivity). *If we take positive initial conditions $S_0(t) \geq 0$, $Q_0(t) \geq 0$ for all $t \in [-\zeta, 0]$ for the system* (2.3.2)*, then the solution fulfills $S^\varepsilon(t) \geq 0$, $Q^\varepsilon(t) \geq 0$ for all $t > 0$.*

Proof. Let us first consider the system with $\sigma(x) = x$ for all x, namely

$$\begin{cases} \mathrm{d}S^\varepsilon(t) = \left[\alpha - kQ^\varepsilon(t)\right] S^\varepsilon(t)\mathrm{d}t + \varepsilon S^\varepsilon(t) \circ \mathrm{d}W^1(t) \\ \mathrm{d}Q^\varepsilon(t) = \Big[d - mQ^\varepsilon(t) - kQ^\varepsilon(t)S^\varepsilon(t) + k\, b\, e^{-\mu\zeta} Q^\varepsilon(t-\zeta)S^\varepsilon(t-\zeta)\Big]\,\mathrm{d}t \\ \qquad\qquad + \varepsilon Q^\varepsilon(t) \circ \mathrm{d}W^2(t), \end{cases} \tag{2.3.10}$$

with initial condition $(S_0(t), Q_0(t))$. Assuming existence and uniqueness of the solution to (2.3.10), we shall prove that $S^\varepsilon(t), Q^\varepsilon(t) \geq 0$ for all $t \geq 0$ almost surely.

Indeed, after the change of variables $x(t) = e^{-\varepsilon W^1(t)} S^\varepsilon(t)$, $y(t) = e^{-\varepsilon W^2(t)} Q^\varepsilon(t)$, we can recast (2.3.10) into the following system of differential equations with random coefficients:

$$\begin{cases} x'(t) = \left(\alpha - k e^{\varepsilon W^2(t)} y(t)\right) x(t) \\ y'(t) = d e^{-\varepsilon W^2(t)} - m y(t) - k e^{\varepsilon W^1(t)} x(t) y(t) \\ \qquad\qquad + k\, b\, e^{-\mu\zeta - \varepsilon(W^2(t) - W^2(t-\zeta) - W^1(t-\zeta))} y(t-\zeta) x(t-\zeta), \end{cases} \tag{2.3.11}$$

with initial conditions $x^0(t) = S_0(t) \geq 0$, $y^0(t) = Q_0(t) \geq 0$ for all $t \in [-\zeta, 0]$. Then, the positivity of $x(t)$ is immediate from the representation

$$x(t) = x^0(0) \exp\left\{\int_0^t (\alpha - k e^{\varepsilon W^2(s)} y(s)) \mathrm{d}s\right\} \geq 0.$$

In order to see the positivity of $y(t)$ let us observe that for $y^0(0) = 0$ we have $y'(0) = d + k\, b\, e^{-\mu\zeta - \varepsilon(W^2(0) - W^2(-\zeta) - W^1(-\zeta))} y(-\zeta) x(-\zeta) > 0$. Therefore, for all initial condition $y(0) \geq 0$ there exists $\delta > 0$ such that $y(t) > 0$ for all $t \in (0, \delta)$. Let us suppose now that $y(t) < 0$ for some $t > 0$, and let $t_0 = \inf\{t > 0 \mid y(t) < 0\}$. Due to the continuity of the solution we have that $y(t_0) = 0$. Then

$$y'(t_0) = d e^{-\varepsilon W^2(t_0)} + k\, b\, e^{-\mu\zeta - \varepsilon(W^2(t_0) - W^2(t_0-\zeta) - W^1(t_0-\zeta))} y(t_0 - \zeta) x(t_0 - \zeta) > 0,$$

which is impossible since it would yield $y(t) > 0$ for $t \in (t_0, t_0 + \delta)$ for δ small enough. This contradiction means exactly that $y(t) \geq 0$ for all $t \geq 0$.

Now that we have the positivity for system (2.3.10), we can prove the positivity for (2.3.2) in the following way. Let us first handle the case of $S^\varepsilon(t)$, and assume that the initial condition is such that $S_0(0) \geq M$. Set then $\tau^0_{M,S} = \inf\{t \geq 0$ such that $S^\varepsilon(t) \leq M/2\}$, and observe that $\tau^0_{M,S}$ is a $\mathcal{F}_t$- stopping time (recall that $\mathcal{F}_t$ stands for the natural filtration of the Brownian motion W), such that S^ε

has remained positive until $\tau^0_{M,S}$. Furthermore, the strong Markov property for $(S^\varepsilon, Q^\varepsilon)$ entails that the process

$$\left\{\left(S^\varepsilon(\tau^0_{M,S}+t), Q^\varepsilon(\tau^0_{M,S}+t)\right);\, t \geq 0\right\}$$

also satisfies (2.3.2) on the set $\Omega_{M,S} = \{\omega \in \Omega;\, \tau^0_{M,S} < \infty\}$, with an initial condition $S_0(0) = M/2$. With these considerations in mind, we can assume that the initial condition of our differential system satisfies $S_0(0) < M$.

With such an initial condition we can conclude the positivity of $S^\varepsilon(t)$ until the stopping time $\hat{\tau}^0_{M,S} = \inf\{t \geq 0$ such that $S^\varepsilon(t) \geq M\}$ as we have done for the system (2.3.10), since up to time $\hat{\tau}^0_{M,S}$ we have $\sigma(S^\varepsilon(t)) = S^\varepsilon(t)$. Then, invoking again the strong Markov property, we can also guarantee positivity until time $\tau^1_{M,S} = \inf\{t \geq \hat{\tau}^0_{M,S}$ such that $S^\varepsilon(t) \leq M/2\}$ as above. We are now in a position to obtain the positivity of S^ε_t until time $\hat{\tau}^1_{M,S} = \inf\{t \geq \tau^1_{M,S}$ such that $S^\varepsilon(t) \geq M\}$, once again with the same reasoning than for the system (2.3.10). The global positivity of $S^\varepsilon(t)$ on any interval of the form $[\tau^k_{M,S}, \tau^{k+1}_{M,S}]$ for $k \geq 0$ now follows by iteration of this reasoning.

It remains to show that $\lim_{k\to\infty} \tau^k_{M,S} = \infty$. This is easily obtained by combining the following two ingredients.
(i) The increments $\{\tau^{k+1}_{M,S} - \tau^k_{M,S};\, k \geq 0\}$ form a i.i.d sequence by a simple application of the strong Markov property.
(ii) Owing to the specific coefficients we have for equation (2.3.2), it can be checked that for any $\eta_2 > 0$ one can find $\eta_1 > 0$ small enough such that $\mathbf{P}(\tau^1_{M,S} > \eta_1) \geq 1 - \eta_2$. Details of this assertion are omitted for sake of conciseness.

We let the reader check that the positivity of $Q^\varepsilon(t)$ can be obtained along the same lines, which ends the proof.

□

Remark 2.3.15. Using the a priori positivity properties stated above, we could have also obtained existence and uniqueness of the solution for system (2.3.10). We did not include these developments for sake of conciseness.

2.3.3 Fluctuations of the random system

Here again we shall proceed gradually, and work out the following cases.

1. Fluctuations for the non delayed system.

2. Extension to the delayed system.

Towards this aim, let us first summarize the information we have obtained up to now in the non delayed case. We are considering the system

$$\begin{cases} \mathrm{d}S^\varepsilon(t) = [\alpha - k\sigma(Q^\varepsilon(t))]\, S^\varepsilon(t)\mathrm{d}t + \varepsilon\sigma(S^\varepsilon(t)) \circ \mathrm{d}W^1(t) \\ \mathrm{d}Q^\varepsilon(t) = [d - mQ^\varepsilon(t) + k(b-1)\sigma(Q^\varepsilon(t))S^\varepsilon(t)]\, \mathrm{d}t + \varepsilon\sigma(Q^\varepsilon(t)) \circ \mathrm{d}W^2(t). \end{cases} \tag{2.3.12}$$

Under Hypothesis 2.3.1 and 2.3.7, we have shown the existence of a unique equilibrium $E_0 = (0, d/m)$ for the deterministic system (2.3.5), corresponding to (2.3.12) with $\varepsilon = 0$. Furthermore, we have constructed a region $R \in \mathbb{R}_+^2$ such that for any initial condition $(S_0, Q_0) \in R$, the solution converges exponentially to E_0, with a rate $\eta = \gamma \wedge \frac{m}{2}$. We now wish to obtain a concentration result for the perturbed system (2.3.12), that is give a proof of Theorem 2.3.3. To this aim, we shall divide our proof into several subsections.

Notation 2.3.16. *We will set $Z^\varepsilon(t)$ for the couple $(S^\varepsilon(t), Q^\varepsilon(t))$, and $Z^0(t)$ for the solution to the deterministic equation (2.3.5).*

Reduction of the problem

Recall that Theorem 2.3.3 states an exponential bound (valid for ρ small enough) of the form

$$\mathbf{P}\left(\|Z^\varepsilon - E_0\|_{\infty,I} \geq 2\rho\right) \leq \exp\left(-\frac{c_1\rho^{2+\lambda}}{\varepsilon^2}\right), \tag{2.3.13}$$

on any interval of the form $I = [\kappa_1 \ln(c/\rho)/\eta; \kappa_2 \ln(c/\rho)/\eta]$ and $1 < \kappa_1 < \kappa_2 < \kappa_3$ such that $\lambda > \kappa_3/\eta$.

A first step in this direction is to consider a generic interval of the form $\hat{I} = [a, b]$, and write

$$\begin{aligned} \mathbf{P}\left(\|Z^\varepsilon - E_0\|_{\infty,\hat{I}} \geq 2\rho\right) &= \mathbf{P}\left((\|Z^\varepsilon - E_0\|_{\infty,\hat{I}} \geq 2\rho) \cap (\|Z^0 - E_0\|_{\infty,\hat{I}} \geq \rho)\right) \\ &\quad + \mathbf{P}\left((\|Z^\varepsilon - E_0\|_{\infty,\hat{I}} \geq 2\rho) \cap (\|Z^0 - E_0\|_{\infty,\hat{I}} \leq \rho)\right), \end{aligned}$$

which yields

$$\mathbf{P}\left(\|Z^\varepsilon - E_0\|_{\infty,\hat{I}} \geq 2\rho\right) \leq A_1 + A_2,$$

with

$$A_1 = \mathbf{P}\left(\|Z^0 - E_0\|_{\infty,\hat{I}} \geq \rho\right), \quad \text{and} \quad A_2 = \mathbf{P}\left(\|Z^\varepsilon - Z^0\|_{\infty,\hat{I}} \geq \rho\right). \tag{2.3.14}$$

Moreover, the term A_1 is easily handled. Owing to (2.3.9), we have $A_1 = 0$ as soon as $a = \kappa_1 \ln(c/\rho)/\eta$ with $\kappa_1 > 1$. In order to prove (2.3.13), it is thus sufficient to check the following identity,

$$\mathbf{P}\left(\|Z^\varepsilon - Z^0\|_{\infty,I} \geq \rho\right) \leq \exp\left(-\frac{c_1\rho^{2+\lambda}}{\varepsilon^2}\right), \tag{2.3.15}$$

on any interval of the form $I = [\kappa_1 \ln(c/\rho)/\eta; \kappa_2 \ln(c/\rho)/\eta]$ and $1 < \kappa_1 < \kappa_2 < \kappa_3$. We shall focus on this inequality in the next subsection.

Exponential concentration of the stochastic equation

We will now give a general concentration result for $Z^\varepsilon - Z^0$ on suitable time scales as follows.

Proposition 2.3.17. *Let Z^ε be the solution to (2.3.12). Then there exists $\varepsilon_0 = \varepsilon_0(M, \tau)$ such that, for any $\rho \leq 1$ and $\varepsilon \leq \varepsilon_0$ we have*

$$\mathbf{P}\left(\|Z^\varepsilon - Z^0\|_{\infty,[0,\tau]} > \rho\right) \leq \exp\left(-\frac{c_2 \rho^2}{e^{\kappa_2 \tau} \varepsilon^2}\right), \tag{2.3.16}$$

where c_2, κ_2 are strictly positive constants which do not depend on ρ, ε, but both depend on our set of parameters $\alpha, k, \sigma, d, m, b, M$.

Proof. For notational sake, let us abbreviate $\|f\|_{\infty,[0,\tau]}$ into $\|f\|_\infty$ throughout the proof. In order to bound $Z^\varepsilon - Z^0$, we first seek a bound for $S^\varepsilon - S^0$. To this aim we notice that for the deterministic function S^0 and thanks to relation (2.3.9), one can find a constant $\kappa_1 = \kappa_1(\alpha, k, \sigma, d, m, b)$ such that $\|S^0\|_\infty \leq \kappa_1$. Set also $J^1(t) := \int_0^t \sigma(S^\varepsilon(s)) \circ \mathrm{d}W^1(s)$. Then

$$\begin{aligned}
|S^\varepsilon(t) - S^0(t)| &\leq \int_0^t \left|(\alpha - k\sigma(Q^\varepsilon(s)))\, S^\varepsilon(s) - \left(\alpha - k\sigma(Q^0(s))\right) S^0(s)\right| \mathrm{d}s + \varepsilon \left|J^1(t)\right| \\
&\leq \int_0^t \left|(\alpha - k\sigma(Q^\varepsilon(s)))\,(S^\varepsilon(s) - S^0(s))\right| \mathrm{d}s \qquad (2.3.17) \\
&\qquad + \int_0^t k \left|\sigma(Q^\varepsilon(s)) - \sigma(Q^0(s))\right| |S^0(s)| \mathrm{d}s + \varepsilon |J^1(t)| \\
&\leq \int_0^t (\alpha + kM) |S^\varepsilon(s) - S^0(s)| \mathrm{d}s \qquad (2.3.18) \\
&\qquad + \kappa_1 k \int_0^t |Q^\varepsilon(s) - Q^0(s)| \mathrm{d}s + \varepsilon |J^1(t)|.
\end{aligned}$$

Analogously, setting $J^2(t) := \int_0^t \sigma(Q^\varepsilon(s)) \circ \mathrm{d}W^2(s)$, we obtain

$$\begin{aligned}
|Q^\varepsilon(t) - Q^0(t)| &\leq \int_0^t (m + k(b-1)\kappa_1) |Q^\varepsilon(s) - Q^0(s)| \mathrm{d}s \\
&+ \int_0^t k(b-1) M |S^\varepsilon(s) - S^0(s)| \mathrm{d}s + \varepsilon |J^2(t)|. \qquad (2.3.19)
\end{aligned}$$

Hence, putting together (2.3.17) and (2.3.19), we get the existence of two positive constants κ_2, κ_3 such that

$$|Z^\varepsilon(t) - Z^0(t)|^2 \leq \kappa_2 \varepsilon^2 \left(|J^1(t)|^2 + |J^2(t)|^2\right) + \kappa_3 \int_0^t |Z^\varepsilon(s) - Z^0(s)|^2 \mathrm{d}s,$$

and by a standard application of Gronwall's lemma, we get for all $t \in [0,\tau]$:

$$
\begin{aligned}
|Z^\varepsilon(t) - Z^0(t)|^2 &\leq \kappa_2 \varepsilon^2 \left[|J^1(t)|^2 + |J^2(t)|^2 \right] \exp(\kappa_3 t) \\
&\leq \kappa_2 \varepsilon^2 \left[|J^1(t)|^2 + |J^2(t)|^2 \right] \exp(\kappa_3 \tau). \qquad (2.3.20)
\end{aligned}
$$

Let us now go back to our claim (2.3.16). Thanks to the inequality (2.3.20), we have

$$
\begin{aligned}
\mathbf{P}\left(\|Z^\varepsilon - Z^0\|_\infty > \rho \right) &= \mathbf{P}\left(\|Z^\varepsilon - Z^0\|_\infty^2 > \rho^2 \right) \\
&\leq \mathbf{P}\left(\|J^1\|_\infty^2 + \|J^2\|_\infty^2 > \frac{\rho^2}{\kappa_2 \varepsilon^2 \exp(\kappa_3 \tau)} \right) \leq T_1 + T_2,
\end{aligned}
$$

with

$$
T_1 = \mathbf{P}\left(\|J^1\|_\infty > \frac{\kappa_4 \rho}{\varepsilon \exp(\kappa_5 \tau)} \right), \quad \text{and} \quad T_2 = \mathbf{P}\left(\|J^2\|_\infty > \frac{\kappa_4 \rho}{\varepsilon \exp(\kappa_5 \tau)} \right).
$$

We now proceed to bound the quantity T_1. To this aim we first write $J^1(t)$ in terms of Itô's integrals. According to [29, Definition 3.13 p. 156],

$$
J^1(t) = \int_0^t \sigma(S^\varepsilon(s)) \mathrm{d}W^1(s) + \frac{1}{2} \left\langle \sigma(S^\varepsilon), W^1 \right\rangle_t ,
$$

where $\langle \cdot, \cdot \rangle$ stands for the bracket of two semi-martingales. Invoking equation (2.3.12) and ordinary rules of Stratonovich differential calculus, it is also readily checked that

$$
\sigma(S^\varepsilon(t)) = \sigma(S_0^\varepsilon) + \varepsilon \int_0^t \sigma\sigma'(S^\varepsilon(s)) \mathrm{d}W^1(s) + V(t),
$$

where V is a process with bounded variation. We thus end up with the expression $J^1(t) = \hat{M}^1(t) + V^1(t)$, where

$$
\hat{M}^1(t) = \int_0^t \sigma(S^\varepsilon(s)) \mathrm{d}W^1(s), \quad \text{and} \quad V^1(t) = \frac{\varepsilon}{2} \int_0^t \sigma\sigma'(S^\varepsilon(s)) \mathrm{d}s,
$$

and decompose T_1 accordingly into $T_1 \leq T_{1,1} + T_{1,2}$, with

$$
T_{1,1} = \mathbf{P}\left(\|\hat{M}^1\|_\infty > \frac{\kappa_4 \rho}{\varepsilon \exp(\kappa_3 \tau)} \right), \quad \text{and} \quad T_{1,2} = \mathbf{P}\left(\|V^1\|_\infty > \frac{\kappa_4 \rho}{\varepsilon \exp(\kappa_3 \tau)} \right).
$$

We will now bound the terms $T_{1,1}$ and $T_{1,2}$ separately.

The term $T_{1,2}$ is easily shown to be bounded thanks to some deterministic arguments. Indeed, since $\sigma\sigma'(x) \leq C(M+1)$ for any $x \in \mathbb{R}_+$, we have $\|V^1\|_\infty \leq$

$C(M+1)\varepsilon\tau$, so that for any $\rho \leq 1$ and $\varepsilon \leq \varepsilon_1 := (\kappa_4/(C(M+1)\tau\exp(\kappa_3\tau)))^{1/2}$, we have $T_{1,2} = 0$. As far as $T_{1,1}$ is concerned, one can apply the exponential martingale inequality (see, for instance, [22]) for stochastic integrals in order to get

$$T_{1,1} \leq \exp\left(-\frac{\kappa_4\rho^2}{M^2\exp(\kappa_3\tau)\varepsilon^2}\right).$$

Putting together the estimates for $T_{1,1}$ and $T_{1,2}$, we have thus obtained

$$T_1 \leq \exp\left(-\frac{\kappa_4\rho^2}{M^2\exp(\kappa_3\tau)\varepsilon^2}\right),$$

for any $\rho \leq 1$ and $\varepsilon \leq \varepsilon_1 := (\kappa_4/(C(M+1)\tau\exp(\kappa_3\tau)))^{1/2}$. We let the reader check that the term T_2 can be handled along the same lines, which finishes our proof.

□

Deviation from equilibrium

Let us now prove inequality (2.3.13). Recall that we have decomposed $\mathbf{P}(\|Z^\varepsilon - E_0\|_{\infty,I} \geq 2\rho)$ into $A_1 + A_2$ defined by (2.3.14). Furthermore, $A_1 = 0$ when $\hat{I}$ is of the form $[a,b]$ with $a = \kappa_1\ln(c/\rho)/\eta$.

In order to complete our result, let us analyze the term A_2 in the light of inequality (2.3.16). Indeed, in order to go from (2.3.16) to (2.3.15), it is sufficient to choose ρ, τ, λ such that

$$\rho^2\exp(-\kappa_2\tau) > \rho^{2+\lambda},$$

which is achieved for $\tau < b := \lambda\ln(1/\rho)/\kappa_2$. Hence our claim is satisfied on the interval $\hat{I} = [a,b]$. We now have to verify that this interval is nonempty, namely that $a < b$. This gives a linear equation in $\ln(1/\rho)$ of the form

$$\frac{\kappa_1}{\eta}\left[\ln(1/\rho) + \ln(c)\right] \leq \frac{\lambda}{\kappa_2}\ln(1/\rho),$$

and the reader might easily check that the following conditions are sufficient.
(i) The linear terms satisfy $\frac{\kappa_1}{\eta} < \frac{\lambda}{\kappa_2}$, that is $\lambda > \frac{\kappa_1\kappa_2}{\eta}$.
(ii) We take ρ small enough, namely $\rho \leq \rho_0$ in order to compensate the term $\ln(c)$.
The proof of (2.3.13) is now finished.

Extension to the delayed system

Let us deal now with the delayed case. As mentioned in the introduction, we consider the system

$$\begin{cases} \mathrm{d}S^\varepsilon(t) = [\alpha - k\sigma(Q^\varepsilon(t))]\, S^\varepsilon(t)\mathrm{d}t + \varepsilon\sigma(S^\varepsilon(t)) \circ \mathrm{d}W^1(t) \\ \mathrm{d}Q^\varepsilon(t) = \Big[d - mQ^\varepsilon(t) - k\sigma(Q^\varepsilon(t))S^\varepsilon(t) + k\, b\, e^{-\mu\zeta}\sigma(Q^\varepsilon(t-\zeta))S^\varepsilon(t-\zeta)\Big]\,\mathrm{d}t \\ \qquad\qquad\qquad + \varepsilon\sigma(Q^\varepsilon(t)) \circ \mathrm{d}W^2(t), \end{cases} \tag{2.3.21}$$

where for any $t \in [-\zeta, 0]$ and for any $\varepsilon > 0$, $(S^\varepsilon(t), Q^\varepsilon(t)) = (S^0(t), Q^0(t))$.

Under Hypothesis 2.3.1, 2.3.7 and 2.3.5 we have shown the existence of a unique equilibrium E_0 for the deterministic system (2.3.1), corresponding to (2.3.21) with $\varepsilon = 0$. Following the non-delayed case, we wish to obtain a concentration result for the perturbed system (2.3.21), as is given in Theorem 2.3.6.

The proof of this result can be carried out almost exactly as for Theorem 2.3.3. Let us point out that the main difference relies on how to get an equivalent of inequalities (2.3.17) and (2.3.19). To this aim, we set again $J^1(t) := \int_0^t \sigma(S^\varepsilon(s)) \circ \mathrm{d}W^1(s)$ and $J^2(t) := \int_0^t \varepsilon\sigma(Q^\varepsilon(s)) \circ \mathrm{d}W^2(s)$. Then in the delayed case, relations (2.3.17) and (2.3.19) become

$$|S^\varepsilon(t) - S^0(t)| \le \int_0^t (\alpha + kM)|S^\varepsilon(s) - S^0(s)|\mathrm{d}s + \kappa_1 k \int_0^t |Q^\varepsilon(s) - Q^0(s)|\mathrm{d}s + \varepsilon|J^1(t)|, \tag{2.3.22}$$

and

$$\begin{aligned} |Q^\varepsilon(t) - Q^0(t)| \;\le\; & \int_0^t (m + k\kappa_1)|Q^\varepsilon(s) - Q^0(s)|\mathrm{d}s + \int_0^t kM|S^\varepsilon(s) - S^0(s)|\mathrm{d}s \\ & + \varepsilon|J^2(t)| + \int_0^t kbMe^{-\mu\zeta}|S^\varepsilon(s-\zeta) - S^0(s-\zeta)|\mathrm{d}s \\ & + \int_0^t kbk_1e^{-\mu\zeta}|Q^\varepsilon(s-\zeta) - Q^0(s-\zeta)|\mathrm{d}s. \end{aligned} \tag{2.3.23}$$

Using that for any $t \in [-\zeta, 0]$ and for any $\varepsilon > 0$, $(S^\varepsilon(t), Q^\varepsilon(t)) = (S^0(t), Q^0(t))$ we can write the bounds

$$\begin{aligned} \int_0^t kbMe^{-\mu\zeta}|S^\varepsilon(s-\zeta) - S^0(s-\zeta)|\mathrm{d}s &= \int_0^{t-\zeta} kbMe^{-\mu\zeta}|S^\varepsilon(s) - S^0(s)|\mathrm{d}s \\ &\le \int_0^t kbMe^{-\mu\zeta}|S^\varepsilon(s) - S^0(s)|\mathrm{d}s, \\ \int_0^t kbk_1e^{-\mu\zeta}|Q^\varepsilon(s-\zeta) - Q^0(s-\zeta)|\mathrm{d}s &= \int_0^{t-\zeta} kbk_1e^{-\mu\zeta}|Q^\varepsilon(s) - Q^0(s)|\mathrm{d}s \\ &\le \int_0^t kbk_1e^{-\mu\zeta}|Q^\varepsilon(s) - Q^0(s)|\mathrm{d}s \end{aligned}$$

Then, putting these last bounds in (2.3.22) and (2.3.23) we get the existence of two positive constants κ_2, κ_3 such that

$$|Z^\varepsilon(t) - Z^0(t)|^2 \leq \kappa_2 \left(|J^1(t)|^2 + |J^2(t)|^2\right) + \kappa_3 \int_0^t |Z^\varepsilon(s) - Z^0(s)|^2 \mathrm{d}s.$$

Starting from this point, the proof follows exactly as for Theorem 2.3.3.

2.3.4 Numerical simulations

This final section is devoted to a presentation of some numerical simulations for the system described by equation (2.3.2). We have chosen the parameters $(\alpha, k, d, m, b, \zeta)$ according to some real data observed in vitro by the Molecular Biology Group of the Department of Genetics and Microbiology at *Universitat Autònoma de Barcelona*. We have also chosen to compare theoretical and noisy dynamics in order to see that the quantities S and Q are close to their equilibrium after a reasonable amount of time (in spite of randomness). We believe that this study is justified because the noise is expected to appear, either by the errors when collecting data, either by the appearance of several factors that may affect the behavior of the agents in vivo.

It is worth noticing at this point that the parameters we have chosen for our simulations do not meet the conditions stated at Hypothesis 2.3.5. Indeed, those conditions were imposed in order to obtain our theoretical large deviations type results with a reasonable amount of effort, but might be too restrictive to fit to real data experiments. Nevertheless, our simulations turn out to be satisfactory, since we observe that the solution $(S(t), Q(t))$ converges to E_0 for small values of ε in a reasonable amount of time, regardless of the violation of Hypothesis 2.3.5.

Specifically, we have simulated trajectories with parameters estimated on an experiment involving Salmonella ATCC14028 bacteria and UAB_Phi78 virus. From the experiments conducted by the mentioned group we have chosen the parameters as:

$$(\alpha, k, d, m, b, \zeta) = (12.1622, 27.36, 0.1, 0.1947, 61, 0.01875).$$

We have also put $M = 10$, $\mu = 0.5$, and we have taken the initial conditions $S_0(t) = 4.8e^{\alpha(t+\zeta)}$, $Q_0(t) = 0$ for $t \in [-\zeta, 0]$. The time is expressed in days and the amount of virus and bacteria are expressed in tens of millions of units.

Our simulations are summarized at Figure 2.3, in which different paths of the processes S and Q are computed. We have first expressed our Stratonovich type equation (2.3.2) into an Itô type equation plus corrections, and then used an Euler type discretization scheme for our equations implemented with the **R** software. We have then plotted the deterministic case ($\varepsilon = 0$) plus the curves corresponding to several values of ε (namely $\varepsilon = -3, 1$). As mentioned before, the fluctuations

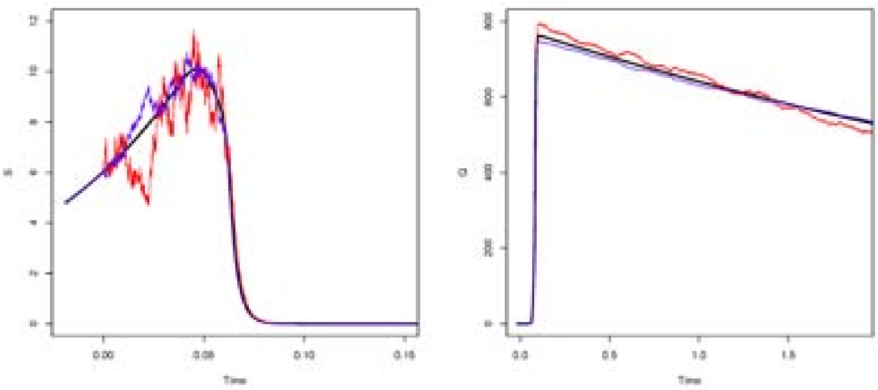

Figure 2.3: Simulation of the trajectories of S and Q with real parameters for the Salmonella ATCC14028 bacteria and UAB_Phi78 virus for the deterministic case ($\varepsilon = 0$), for $\varepsilon = -3$ (red curve) and $\varepsilon = 1$ (blue curve).

of S and Q (which are obviously due to the randomness we have introduced) do not prevent them to converge to equilibrium. Observe that there alternative ways to Euler discretizations in order to simulate Stratonovich type equations, such as the Runge-Kutta method introduced in [44]. Since our numerical context was not too demanding, we have chosen to resort to the Euler scheme based on Itô type equations for sake of simplicity.

Chapter 3

L^2 modulus of Brownian local time

3.1 Introduction

In [18] the authors prove the following Central Limit Theorem for the L^2 modulus of continuity of Brownian local time. Let $B = \{B_t, t \geq 0\}$ be a standard Brownian motion, denote by $\{L_t^x, t \geq 0, x \in \mathbb{R}\}$ its local time and consider

$$G_t(h) = \int_{\mathbb{R}} (L_t^{x+h} - L_t^x)^2 \mathrm{d}x. \tag{3.1.1}$$

Theorem 3.1.1 ([18], Theorem 1.1 or [26], Theorem 1)**.** *For each fixed $t > 0$,*

$$h^{-\frac{3}{2}} \left(\int_{\mathbb{R}} (L_t^{x+h} - L_t^x)^2 \mathrm{d}x - 4th \right) \xrightarrow{\mathcal{L}} 8\sqrt{\frac{\alpha_t}{3}}\eta,$$

as h tends to zero, where

$$\alpha_t = \int_{\mathbb{R}} (L_t^x)^2 \mathrm{d}x,$$

and η is a $N(0,1)$ random variable independent of B.

They were motivated to try to find this result by the interest in the expression

$$H_n = \sum_{i,j=1, i\neq j}^{n} \mathbb{1}_{\{S_i = S_j\}} - \frac{1}{2} \sum_{i,j=1, i\neq j}^{n} \mathbb{1}_{\{|S_i - S_j| = 1\}},$$

where S is a random walk on $\mathbb{Z}$. This expression appears as the Hamiltonian in a model for a polymer in a repulsive medium, and can be written as

$$H_n = \frac{1}{2} \sum_{x \in \mathbb{Z}} (l_n^x - l_n^{x+1})^2,$$

where $l_n^x = \sum_{i=1}^n \mathbb{1}_{\{S_i=x\}}$ is the local time for S.

There are various proofs of Theorem 3.1.1. While in [18] the authors use the method of moments, [26] provide a proof based on an asymptotic version of Knight's theorem combined with some other techniques of stochastic calculus and Malliavin calculus, like Clark-Ocone's formula.

Previous to this result, almost sure limits for the L^p moduli of continuity of local times of a very wide class of symmetric Lévy processes were obtained in [36]. This result uses, among others, Eisenbaum Isomorphism theorem (see [35]), and for the Browninan motian case, $p = 2$, can be written in the form

$$\lim_{h\downarrow 0} \int_{-\infty}^{\infty} \frac{(L_t^{x+h} - L_t^x)^2}{h} \mathrm{d}x = 4t \qquad \text{a.s.}$$

which is used in Theorem 3.1.1 to obtain the term $4th$.

In this chapter we shall study the decomposition on Wiener chaos of $G_t(h)$. More precisely, we shall find a CLT for each Wiener chaos element of $G_t(h)$ as states the following theorem.

Theorem 3.1.2. *Let $G_t(h)$ be the random variable defined in* (3.1.1) *and denote the n-th Wiener chaos element of $G_t(h)$ by $\tilde{I}_n(G_t(h))$. Then, for $n = 2k$, $k \in \mathbb{N}^*$,*

$$\frac{1}{h^2\sqrt{\log(1/h)}} \tilde{I}_n(G_t(h)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_n^2) \tag{3.1.2}$$

as h tends to zero, where $\mathcal{N}(0, \sigma_n^2)$ is a centered Normal random variable with variance $\sigma_n^2 = \frac{2^6 t(2(k-1))!}{\pi 2^{2(k-1)}((k-1)!)^2}$. For $n = 2k-1$, $k \in \mathbb{N}^*$, the limit is zero.

Remark 3.1.3. This result provides us with an example of a family of random variables that is convergent in law to a Normal distribution, but its chaos elements of even order do not converge. Moreover, we find a normalization where these chaos elements do converge in law to a Normal distribution. One can easily see that the sum of σ_n^2 diverges.

3.2 Proof of Theorem 3.1.2

The proof is based on a result due to Nualart and Pecatti [40], where the authors characterize the convergence in distribution to a normal law $\mathcal{N}(0, 1)$ for a sequence of random variables belonging to a fixed Wiener chaos. Let us now state their result here, for sake of readability.

Suppose that H is the Hilbert space $H = L^2(T, \mathcal{B}, \mu)$, where $(T, \mathcal{B})$ is a measurable space and μ is a σ-finite and non-atomic measure, and consider a Gaussian family of random variables $W = \{W(h), h \in H\}$ defined in a complete probability space $(\Omega, \mathcal{F}, P)$.

Let f be a symmetric element of $L^2(T^n)$. For any $p = 0, \dots, n$, we define the contraction of f of order p to be the element of $L^2(T^{2(n-p)})$ given by

$$f^{\otimes_p}(t_{p+1}, \dots, t_n, s_{p+1}, \dots, s_n) = \int_{T^p} f(t_1, \dots, t_p, t_{p+1}, \dots, t_n) \\ \times f(t_1, \dots, t_p, s_{p+1}, \dots, s_n) \mathrm{d}t_1 \dots \mathrm{d}t_p. \quad (3.2.1)$$

Theorem 3.2.1 ([40], Theorem 1)**.** *Fix $n \geq 2$. Consider a sequence $\{F_k = I_n(f_k), k \geq 1\}$ of square integrable random variables belonging to the nth Wiener chaos such that*

$$\mathbb{E}[F_k^2] = n! \|f_k\|^2_{H^{\otimes n}} \underset{k \to \infty}{\longrightarrow} \sigma^2. \quad (3.2.2)$$

The following statements are equivalent.

(1) As k goes to infinity, the sequence $\{F_k, k \geq 1\}$ converges in distribution to the normal law $N(0, \sigma^2)$.

(2) $\lim_{k \to \infty} \mathbb{E}[F_k^4] = 3\sigma^4$.

(3) For all $1 \leq p \leq n-1$, $\lim_{k \to \infty} f_k^{\otimes_p} = 0$, in $H^{\otimes 2(n-p)}$.

(4) $\|DF_k\|^2_H \to n\sigma^2$ in $L^2(\Omega)$.

In the following lemmas we shall prove that condition (3.2.2) and statement (3) hold true for $F_h = \frac{1}{h^2 \sqrt{\log(1/h)}} \tilde{I}_n(G_t(h))$ when h goes to zero, thus proving Theorem 3.1.2. First of all we derive the following representation of $\tilde{I}_n(G_t(h))$.

Lemma 3.2.2. *For $n = 2k$, $k \in \mathbb{N}^*$, the random variable $\tilde{I}_n(G_t(h))$ can be expressed as the sum of 4 terms, namely:*

$$\tilde{I}_n(G_t(h)) = \frac{16}{n!} \sum_{i=1}^{4} I_n \left(\Psi^i_{n,h} \left(\bigwedge_{j=1}^{n} t_j, \bigvee_{j=1}^{n} t_j \right) \right),$$

where I_n stands for the multiple stochastic integral and $\Psi^i_{n,h}$ will be defined along the proof (see (3.2.8)). For $n = 2k-1$, $k \in \mathbb{N}^*$, $\tilde{I}_n(G_t(h))$ equals zero.

Proof. Let us use the following expression of G (see [26])

$$G_t(h) = -2 \int_0^t \int_0^v (\delta(B_v - B_u + h) + \delta(B_v - B_u - h) - 2\delta(B_v - B_u)) \mathrm{d}u \mathrm{d}v, \quad (3.2.3)$$

where δ denotes the Dirac's delta function.

We know that $\delta(B_v - B_u + h) = \sum_{n=0}^{\infty} I_n(\varphi)$ where, for $u \le t_1, \dots, t_n \le v$,

$$\varphi(t_1, \dots, t_n) = \frac{1}{n!}\mathbb{E}(D^n_{t_1,\dots,t_n}(\delta(B_v - B_u + h))) \tag{3.2.4}$$

(see [39] for further details). Then we have

$$\varphi(t_1, \dots, t_n) = \frac{1}{n!}\prod_{j=1}^{n} \mathbb{1}_{[u,v]}(t_j)\mathbb{E}(\delta^{(n)}(B_v - B_u + h)) \tag{3.2.5}$$

where if we consider $p_t(x) = \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}$ the Gaussian distribution, we obtain

$$\begin{aligned}\mathbb{E}(\delta^{(n)}(B_v - B_u + h)) &= \int_{\mathbb{R}} \delta^{(n)}(x+h)p_{v-u}(x)\mathrm{d}x \\ &= (-1)^n \int_{\mathbb{R}} \delta(x+h)p^{(n)}_{v-u}(x)\mathrm{d}x = p^{(n)}_{v-u}(h). \end{aligned} \tag{3.2.6}$$

Therefore, we obtain the following expression

$$\begin{aligned} G_t(h) &= -2\sum_{n=0}^{\infty} I_n\left(\frac{1}{n!}\int_0^t\int_0^v \prod_{j=1}^{n}\mathbb{1}_{[u,v]}(t_j)(p^{(n)}_{v-u}(h) + p^{(n)}_{v-u}(-h) - 2p^{(n)}_{v-u}(0))\mathrm{d}u\mathrm{d}v\right) \\ &= \frac{-2}{n!}\sum_{n=0}^{\infty} I_n\left(\int_{\bigvee_{j=1}^n t_j}^t \int_0^{\bigwedge_{j=1}^n t_j} (p^{(n)}_{v-u}(h) + p^{(n)}_{v-u}(-h) - 2p^{(n)}_{v-u}(0))\mathrm{d}u\mathrm{d}v\right) \\ &= \frac{-2}{n!}\sum_{n=0}^{\infty} I_n\left(\Psi_{n,h}\left(\bigwedge_{j=1}^n t_j, \bigvee_{j=1}^n t_j\right)\right), \end{aligned} \tag{3.2.7}$$

defining $\Psi_{n,h}$ in the obvious way.

One can easily see that $(p^{(n)}_{v-u}(h) + p^{(n)}_{v-u}(-h) - 2p^{(n)}_{v-u}(0)) = 0$ when n is odd, proving the result for $n = 2k-1$, $k \in \mathbb{N}^*$. Then, from now on n will be a non-zero even number, i.e., $n = 2k$, $k \in \mathbb{N}^*$. Let us now rewrite $\Psi_{n,h}$ in the following way

$$\Psi_{n,h}(s_1, s_2) = 2\int_{s_2}^t \int_0^{s_1}\int_0^h\int_0^x p^{(n+2)}_{v-u}(y)\mathrm{d}y\mathrm{d}x\mathrm{d}u\mathrm{d}v,$$

and using the fact that $p^{(n+2)}_t(x) = 2\frac{\partial p^{(n)}_t}{\partial t}(x)$ we obtain

$$\begin{aligned}\Psi_{n,h}(s_1, s_2) &= 4\int_0^{s_1}\int_0^h\int_0^x (p^{(n)}_{t-u}(y) - p^{(n)}_{s_2-u}(y))\mathrm{d}y\mathrm{d}x\mathrm{d}u \\ &= -8\int_0^h\int_0^x (p^{(n-2)}_{t-s_1}(y) - p^{(n-2)}_t(y) - (p^{(n-2)}_{s_2-s_1}(y) - p^{(n-2)}_{s_2}(y)))\mathrm{d}y\mathrm{d}x \\ &= -8\int_0^h \Big(p^{(n-2)}_{t-s_1}(y) - p^{(n-2)}_t(y) - p^{(n-2)}_{s_2-s_1}(y) + p^{(n-2)}_{s_2}(y)\Big)(h-y)\mathrm{d}y \\ &:= -8\sum_{i=1}^{4}\Psi^i_{n,h}(s_1, s_2). \end{aligned} \tag{3.2.8}$$

□

Lemma 3.2.3. *For $n = 2k$, $k \in \mathbb{N}^*$,*

$$\frac{1}{h^4 \log(1/h)} \mathbb{E}\left(\tilde{I}_n(G_t(h))^2\right) \longrightarrow \sigma_n^2 := \frac{2^6 t(2(k-1))!}{\pi 2^{2(k-1)}((k-1)!)^2}$$

and for $n = 2k-1$, $k \in \mathbb{N}^*$, the limit is zero.

Proof. Let us fix an even $n \in \mathbb{N}^*$ and denote

$$I_i := \frac{16}{n! h^4 \log(1/h)} I_n\left(\Psi^i_{n,h}\left(\bigwedge_{j=1}^n t_j, \bigvee_{j=1}^n t_j\right)\right),$$

where $\Psi^i_{n,h}$, $i = 1 \dots 4$ is defined by (3.2.8). We will show the convergence of $\mathbb{E}(I_3^2)$ to σ_n^2 when h tends to zero. To complete the proof the reader can check that the limit of $\mathbb{E}(I_i^2)$, $i = 1, 2, 4$, is zero when h tends to zero, which can done similarly to the limit we will compute.

To start with, let us write $\mathbb{E}(I_3^2)$ in the following way

$$\begin{aligned}
\mathbb{E}(I_3^2) &= \frac{2^8}{h^2 \log(1/h)} \int_{\Delta_t^n} \left(\int_0^h p^{(n-2)}_{t_n - t_1}(y)(\frac{h-y}{h}) \mathrm{d}y\right)^2 \mathrm{d}t_1 \cdots \mathrm{d}t_n \\
&= \frac{2^8}{h^2 \log(1/h)} \int\int_{0 \le t_1 < t_n \le t} \frac{(t_n - t_1)^{n-2}}{(n-2)!} \\
&\quad \times \left(\int_0^h \frac{(n-2)!}{\sqrt{(t_n - t_1)^{n-1}}} H_{n-2}\left(\frac{y}{\sqrt{t_n - t_1}}\right) \frac{e^{\frac{-y^2}{2(t_n - t_1)}}}{\sqrt{2\pi}} \left(\frac{h-y}{h}\right) \mathrm{d}y\right)^2 \mathrm{d}t_1 \mathrm{d}t_n \qquad (3.2.9) \\
&= \frac{2^8 (n-2)!}{2\pi h^2 \log(1/h)} \int\int_{0 \le t_1 < t_n \le t} \frac{1}{t_n - t_1} \\
&\quad \times \left(\int_0^h H_{n-2}\left(\frac{y}{\sqrt{t_n - t_1}}\right) e^{\frac{-y^2}{2(t_n - t_1)}} \left(\frac{h-y}{h}\right) \mathrm{d}y\right)^2 \mathrm{d}t_1 \mathrm{d}t_n,
\end{aligned}$$

where $H_n(x)$ denote the n-th Hermite polynomial.

First letting $\tau = t_n - t_1$ and then letting $x = \frac{y}{\sqrt{\tau}}$ we get

$$\begin{aligned}
\mathbb{E}(I_3^2) &= \frac{2^8 (n-2)!}{2\pi h^2 \log(1/h)} \int_0^t \int_0^{t-\tau} \frac{1}{\tau} \left(\int_0^h H_{n-2}\left(\frac{y}{\sqrt{\tau}}\right) e^{\frac{-y^2}{2\tau}} \left(\frac{h-y}{h}\right) \mathrm{d}y\right)^2 \mathrm{d}t_1 \mathrm{d}\tau \\
&= \frac{2^8 (n-2)!}{2\pi h^2 \log(1/h)} \int_0^t \frac{1}{\tau} \left(\int_0^{\frac{h}{\sqrt{\tau}}} H_{n-2}(x) e^{-x^2/2} \left(1 - \frac{x\sqrt{\tau}}{h}\right) \sqrt{\tau} \mathrm{d}x\right)^2 (t-\tau) \mathrm{d}\tau,
\end{aligned}$$

and finally allowing $\frac{h}{\sqrt{\tau}} = u$ we obtain

$$\mathbb{E}(I_3^2) = \frac{2^8(n-2)!}{2\pi h^2 \log(1/h)} \int_{\frac{h}{\sqrt{t}}}^{\infty} \left(\int_0^u H_{n-2}(x) e^{-x^2/2} \left(1 - \frac{x}{u}\right) \mathrm{d}x \right)^2 \left(t - \frac{h^2}{u^2}\right) \frac{2h^2}{u^3} \mathrm{d}u.$$

Now we compute the limit when h tends to zero, using Hôpital's Theorem twice

$$\begin{aligned}\lim_{h\to 0} \mathbb{E}(I_3^2) &= \lim_{h\to 0} \frac{2^9(n-2)!}{2\pi} \frac{\int_{h/\sqrt{t}}^{\infty} \left(\int_0^u H_{n-2}(x) e^{-x^2/2} \left(1 - \frac{x}{u}\right) \mathrm{d}x \right)^2 \frac{(-2)h}{u^5} \mathrm{d}u}{-1/h} \\ &= \lim_{h\to 0} \frac{2^{10}(n-2)!}{2\pi} \frac{\frac{-1}{\sqrt{t}} \left(\int_0^{\frac{h}{\sqrt{t}}} H_{n-2}(x) e^{-x^2/2} \left(1 - \frac{x\sqrt{t}}{h}\right) \mathrm{d}x \right)^2 \frac{t^{5/2}}{h^5}}{-2/h^3},\end{aligned}$$

and to finish this part we consider

$$\frac{\int_0^{\frac{h}{\sqrt{t}}} H_{n-2}(x) e^{-x^2/2} \left(1 - \frac{x\sqrt{t}}{h}\right) \mathrm{d}x}{h}$$

and using again Hôpital's Theorem twice we compute its limit, when h tends to zero, which finally leads to the desired result

$$\lim_{h\to 0} \mathbb{E}(I_3^2) = \frac{2^6 t(n-2)!}{\pi} \left(\lim_{h\to 0} H_{n-2}\left(\tfrac{h}{\sqrt{t}}\right) \right)^2 = \frac{2^6 t(n-2)!}{\pi 2^{n-2} \left(\left(\frac{n-2}{2}\right)! \right)^2}.$$

$\square$

Lemma 3.2.4. *Let the above notation prevail. Then, for each $n \geq 0$,*

$$\left(\frac{16}{n! h^2 \sqrt{\log(1/h)}} \Psi_{n,h}^3 (t_1 \wedge \ldots \wedge t_n, t_1 \vee \ldots \vee t_n) \right)^{\otimes p} \xrightarrow{h\to 0} 0 \tag{3.2.10}$$

for $p = 1, \ldots, n-1$.

Proof. Since $\Psi_{n,h}^3$ is zero for any odd n, we will consider n to be a non-zero even number along the proof, i.e. $n = 2k, k \in \mathbb{N}^*$. Let us first recall that the contraction of order p of $\Psi_{n,h}^3$ is defined to be

$$\begin{aligned}\left(\Psi_{n,h}^3 \left(\bigwedge_{j=1}^n t_j, \bigvee_{j=1}^n t_j\right) \right)^{\otimes p} = &\int_{[0,t]^p} \Psi_{n,h}^3 \left(\bigwedge_{j=1}^p t_j \wedge \bigwedge_{j=p+1}^n t_j, \bigvee_{j=1}^p t_j \vee \bigvee_{j=p+1}^n t_j\right) \\ &\times \Psi_{n,h}^3 \left(\bigwedge_{j=1}^p t_j \wedge \bigwedge_{j=p+1}^n s_j, \bigvee_{j=1}^p t_j \vee \bigvee_{j=p+1}^n s_j\right) \mathrm{d}t_1 \ldots \mathrm{d}t_p.\end{aligned}$$

Let us now observe that

$$\Psi^3_{n,h}\Big(\bigwedge_{j=1}^{p} t_j \wedge \bigwedge_{j=p+1}^{n} t_j, \bigvee_{j=1}^{p} t_j \vee \bigvee_{j=p+1}^{n} t_j)\Big) = \\ p!(n-p)!\mathbb{1}_{\{t_1\leq\ldots\leq t_p\}}\mathbb{1}_{\{t_{p+1}\leq\ldots\leq t_n\}}\Psi^3_{n,h}(t_1\wedge t_{p+1}, t_p\vee t_n).$$

Taking into account the expression (3.2.9) and since $H_{n-2}(x)e^{-x^2}$ has a uniform bound C_n, it follows that

$$\begin{aligned}|\Psi^3_{n,h}(t_1\wedge t_{p+1}, t_p\vee t_n)| &\leq C_n(t_p\vee t_n - t_1\wedge t_{p+1})^{-\frac{n-1}{2}}\int_0^h (h-y)\mathrm{d}y \\ &= C_n\frac{h^2}{2}(t_p\vee t_n - t_1\wedge t_{p+1})^{-\frac{n-1}{2}}.\end{aligned}$$

Therefore we can bound the $L^2([0,t]^{2(n-p)})$ norm of (3.2.10) by

$$\int_{\Delta_t^{n-p}\times\Delta_t^{n-p}}\Bigg(\int_{\Delta_t^p}\frac{8^2C_n^2}{\binom{n}{p}^2\log 1/h}(t_p\vee t_n - t_1\wedge t_{p+1})^{-\frac{n-1}{2}} \\ \times (t_p\vee s_n - t_1\wedge s_{p+1})^{-\frac{n-1}{2}}\mathrm{d}t_1\cdots\mathrm{d}t_p\Bigg)^2\mathrm{d}t_{p+1}\cdots\mathrm{d}t_n\mathrm{d}s_{p+1}\cdots\mathrm{d}s_n,$$

and since $\log 1/h$ goes to zero as h tends to zero, we can prove the result by showing the boundedness of

$$T_n := \int_{\Delta_t^{n-p}\times\Delta_t^{n-p}}\Bigg(\int_{\Delta_t^p}(t_p\vee t_n - t_1\wedge t_{p+1})^{-\frac{n-1}{2}} \\ \times (t_p\vee s_n - t_1\wedge s_{p+1})^{-\frac{n-1}{2}}\mathrm{d}t_1\cdots\mathrm{d}t_p\Bigg)^2\mathrm{d}t_{p+1}\cdots\mathrm{d}t_n\mathrm{d}s_{p+1}\cdots\mathrm{d}s_n. \quad (3.2.11)$$

The study of the boundedness of T_n involves a lot of computations. The reader can find these computations in the following Appendix. $\square$

Appendix A

Further details on the proof of Lemma 3.2.4

In this appendix we shall provide more details on the proof of Lemma 3.2.4. Recall that we reduced our problem to bound the following expression

$$T_n := \int_{\Delta_t^{n-p}\times\Delta_t^{n-p}} \left(\int_{\Delta_t^p} (t_p \vee t_n - t_1 \wedge t_{p+1})^{-\frac{n-1}{2}} \right.$$
$$\left. \times (t_p \vee s_n - t_1 \wedge s_{p+1})^{-\frac{n-1}{2}} \mathrm{d}t_1 \cdots \mathrm{d}t_p \right)^2 \mathrm{d}t_{p+1} \cdots \mathrm{d}t_n \mathrm{d}s_{p+1} \cdots \mathrm{d}s_n, \quad \text{(A.0.1)}$$

where n is an even number and $p = 1, \dots, n-1$. First of all we will prove the boundedness of (A.0.1) for $n = 2$, $p = 1$ and $p = n-1$. Then we will provide the computations of the 'general case' $p = 2, \dots, n-2$, $n \neq 2$.

Along this appendix ε will be a number in the interval $(0, \frac{1}{2})$.

A.1 Case $n = 2$

In this case, $p = 1$ and from (A.0.1) we get that

$$\begin{aligned} T_2 &= \int_{[0,t]^2} \left(\int_0^t (t_1 \vee t_2 - t_1 \wedge t_2)^{-\frac{1}{2}} (t_1 \vee s_2 - t_1 \wedge s_2)^{-\frac{1}{2}} \mathrm{d}t_1 \right)^2 \mathrm{d}t_2 \mathrm{d}s_2 \\ &= 2 \int_0^t \int_{t_2}^t \left(\int_0^t (t_1 \vee t_2 - t_1 \wedge t_2)^{-\frac{1}{2}} (t_1 \vee s_2 - t_1 \wedge s_2)^{-\frac{1}{2}} \mathrm{d}t_1 \right)^2 \mathrm{d}s_2 \mathrm{d}t_2, \end{aligned}$$

where the last equality holds true thanks to the symmetry with respect to s_2 and t_2. Now this case is reduced to study the following 3 possibilities.

Subcase 1 $t_1 \leq t_2 \leq s_2$.

In this situation, we can easily see that

$$\begin{aligned}
T_2 &= 2\int_0^t \int_{t_2}^t \left(\int_0^{t_2} (t_2 - t_1)^{-\frac{1}{2}}(s_2 - t_1)^{-\frac{1}{2}} \mathrm{d}t_1\right)^2 \mathrm{d}s_2 \mathrm{d}t_2 \\
&\leq 2\int_0^t \int_{t_2}^t \left(\int_0^{t_2} (t_2 - t_1)^{-\frac{1}{2}}(s_2 - t_2)^{-\frac{1}{4}}(t_2 - t_1)^{-\frac{1}{4}} \mathrm{d}t_1\right)^2 \mathrm{d}s_2 \mathrm{d}t_2 < \infty.
\end{aligned}$$

Subcase **2** $t_2 \leq t_1 \leq s_2$.

We will start by noticing that

$$\begin{aligned}
T_2 &= 2\int_0^t \int_{t_2}^t \left(\int_{t_2}^{s_2} (t_1 - t_2)^{-\frac{1}{2}}(s_2 - t_1)^{-\frac{1}{2}} \mathrm{d}t_1\right)^2 \mathrm{d}s_2 \mathrm{d}t_2 \\
&= 2\int_0^t \int_{t_2}^t \int_{t_2}^{s_2} \int_{t_2}^{s_2} (t_1 - t_2)^{-\frac{1}{2}}(s_2 - t_1)^{-\frac{1}{2}}(t' - t_2)^{-\frac{1}{2}}(s_2 - t')^{-\frac{1}{2}} \mathrm{d}t_1 \mathrm{d}t' \mathrm{d}s_2 \mathrm{d}t_2 \\
&= 4\int_{\{t_2 \leq t_1 \leq t' \leq s_2\}} (t_1 - t_2)^{-\frac{1}{2}}(s_2 - t_1)^{-\frac{1}{2}}(t' - t_2)^{-\frac{1}{2}}(s_2 - t')^{-\frac{1}{2}} \mathrm{d}t_2 \mathrm{d}t_1 \mathrm{d}t' \mathrm{d}s_2,
\end{aligned}$$

where we used the symmetry with respect to t_1 and t' in the last step. We notice that $(t' - t_2)^{-\frac{1}{2}} \leq (t' - t_1)^{-\frac{1}{2}}$ and we obtain that

$$\begin{aligned}
T_2 &\leq 4\int_{\{t_2 \leq t_1 \leq t' \leq s_2\}} (t_1 - t_2)^{-\frac{1}{2}}(s_2 - t_1)^{-\frac{1}{2}}(t' - t_1)^{-\frac{1}{2}}(s_2 - t')^{-\frac{1}{2}} \mathrm{d}t_2 \mathrm{d}t_1 \mathrm{d}t' \mathrm{d}s_2 \\
&= 4\int_{\{t_1 \leq t' \leq s_2\}} \left[-2(t_1 - t_2)^{\frac{1}{2}}\right]_0^{t_1} (s_2 - t_1)^{-\frac{1}{2}}(t' - t_1)^{-\frac{1}{2}}(s_2 - t')^{-\frac{1}{2}} \mathrm{d}t_1 \mathrm{d}t' \mathrm{d}s_2.
\end{aligned}$$

Since $2\sqrt{t_1} \leq 2\sqrt{t}$ and $(s_2 - t_1)^{-\frac{1}{2}} \leq (s_2 - t')^{-\frac{1}{4}}(t' - t_1)^{-\frac{1}{4}}$ we finally obtain that

$$T_2 \leq 8\sqrt{t}\int_{\{t_1 \leq t' \leq s_2\}} (s_2 - t')^{-\frac{3}{4}}(t' - t_1)^{-\frac{3}{4}} \mathrm{d}t_1 \mathrm{d}t' \mathrm{d}s_2 < \infty$$

Subcase **3** $t_2 \leq s_2 \leq t_1$.

This situation can be dealt with an argument almost identical to the first situation.

A.2 Case $p = 1$ ($n \neq 2$)

In this case, since n is an even number greater or equal than four, if we first compute the integrals with respect to $t_3, \ldots, t_{n-1}, s_3, \ldots, s_{n-1}$ we obtain that

$$\begin{aligned}
T_n = &\int_{\Delta_t^{n-1} \times \Delta_t^{n-1}} \left(\int_0^t (t_1 \vee t_n - t_1 \wedge t_2)^{-\frac{n-1}{2}}\right. \\
&\qquad \left. \times (t_1 \vee s_n - t_1 \wedge s_2)^{-\frac{n-1}{2}} \mathrm{d}t_1\right)^2 \mathrm{d}t_2 \cdots \mathrm{d}t_n \mathrm{d}s_2 \cdots \mathrm{d}s_n \\
\leq &\, C_n \int_{\Delta_t^2 \times \Delta_t^2} \left(\int_0^t (t_1 \vee t_n - t_1 \wedge t_2)^{-1}(t_1 \vee s_n - t_1 \wedge s_2)^{-1} \mathrm{d}t_1\right)^2 \mathrm{d}t_2 \mathrm{d}t_n \mathrm{d}s_2 \mathrm{d}s_n,
\end{aligned}$$

where we used that $(t_n - t_2)^{n-3} \leq (t_1 \vee t_n - t_1 \wedge t_2)^{n-3}$ and $(s_n - s_2)^{n-3} \leq (t_1 \vee s_n - t_1 \wedge s_2)^{n-3}$. Thanks to the symmetry with respect to t_2, t_n and s_2, s_n we can reduce the problem to study the boundedness in the following 3 cases

Subcase 1 $t_2 \leq t_n \leq s_2 \leq s_n$,

Subcase 2 $t_2 \leq s_2 \leq t_n \leq s_n$,

Subcase 3 $t_2 \leq s_2 \leq s_n \leq t_n$

and for each one of these cases we have 5 possible positions for t_1. We shall proceed to show some of the computations involved in the study of these cases.

Along this section we will omit the constant C_n for the sake of readability.

1 A) $0 \leq t_1 \leq t_2 \leq t_n \leq s_2 \leq s_n \leq t$.

With this order we have

$$T_n \leq \int_{\Delta_t^4} \left(\int_0^{t_2} (t_n - t_1)^{-1} (s_n - t_1)^{-1} \mathrm{d}t_1 \right)^2 \mathrm{d}t_2 \mathrm{d}t_n \mathrm{d}s_2 \mathrm{d}s_n \tag{A.2.1}$$

and taking into account that $(s_n - t_1)^{-1} \leq (t_n - t_1)^{-\varepsilon} (s_n - t_n)^{-1+\varepsilon}$ we obtain that

$$\begin{aligned} T_n &\leq \int_{\Delta_t^4} \left(\int_0^{t_2} (t_n - t_1)^{-1-\varepsilon} \mathrm{d}t_1 \right)^2 (s_n - t_n)^{-2+2\varepsilon} \mathrm{d}t_2 \mathrm{d}t_n \mathrm{d}s_2 \mathrm{d}s_n \\ &\leq \int_{\Delta_t^3} \frac{(t_n - t_2)^{-2\varepsilon}}{\varepsilon^2} (s_n - t_n)^{-1+2\varepsilon} \mathrm{d}t_2 \mathrm{d}t_n \mathrm{d}s_n < \infty, \end{aligned}$$

where we used that $0 \leq (t_n - t_2)^{-\varepsilon} - t_n^{-\varepsilon} \leq (t_n - t_2)^{-\varepsilon}$.

The following situations

1 E) $0 \leq t_2 \leq t_n \leq s_2 \leq s_n \leq t_1 \leq t$,

2 A) $0 \leq t_1 \leq t_2 \leq s_2 \leq t_n \leq s_n \leq t$ and

2 E) $0 \leq t_2 \leq s_2 \leq t_n \leq s_n \leq t_1 \leq t$

can be studied with a reasoning similar to situation **1 A)**.

1 B) $0 \leq t_2 \leq t_1 \leq t_n \leq s_2 \leq s_n \leq t$.

Now we can write T_n as follows

$$T_n \leq \int_{\Delta_t^4} \left(\int_{t_2}^{t_n} (t_n - t_2)^{-1} (s_n - t_1)^{-1} \mathrm{d}t_1 \right)^2 \mathrm{d}t_2 \mathrm{d}t_n \mathrm{d}s_2 \mathrm{d}s_n.$$

After integrating with respect to s_2 we use that $(s_n - t_1)^{-1} \leq (t_n - t_1)^{-\varepsilon} (s_n - t_n)^{-1+\varepsilon}$ to get that

$$\begin{aligned} T_n &\leq \int_{\Delta_t^3} \left(\int_{t_2}^{t_n} (s_n - t_1)^{-1} \mathrm{d}t_1 \right)^2 (t_n - t_2)^{-2} (s_n - t_n) \mathrm{d}t_2 \mathrm{d}t_n \mathrm{d}s_n \\ &\leq \int_{\Delta_t^3} \left(\int_{t_2}^{t_n} (t_n - t_1)^{-\varepsilon} \mathrm{d}t_1 \right)^2 (t_n - t_2)^{-2} (s_n - t_n)^{-1+2\varepsilon} \mathrm{d}t_2 \mathrm{d}t_n \mathrm{d}s_n, \end{aligned}$$

and finally integrating with respect to t_1 we conclude that

$$T_n \leq \int_{\Delta_t^3} \frac{(t_n - t_2)^{2-2\varepsilon}}{(1-\varepsilon)^2} (t_n - t_2)^{-2} (s_n - t_n)^{-1+2\varepsilon} < \infty$$

In the situation

1 D) $0 \leq t_2 \leq t_n \leq s_2 \leq t_1 \leq s_n \leq t$

T_n can also be proved to be bounded in a similar way to **1 B)**.

1 C) $0 \leq t_2 \leq t_n \leq t_1 \leq s_2 \leq s_n \leq t.$

In this situation we have

$$\begin{aligned} T_n &\leq \int_{\Delta_t^4} \left(\int_{t_n}^{s_2} (t_1 - t_2)^{-1} (s_n - t_1)^{-1} \mathrm{d}t_1 \right)^2 \mathrm{d}t_2 \mathrm{d}t_n \mathrm{d}s_2 \mathrm{d}s_n \\ &= \int_{\Delta_t^4} \int_{t_n}^{s_2} \int_{t_n}^{s_2} (t_1 - t_2)^{-1} (s_n - t_1)^{-1} (t' - t_2)^{-1} (s_n - t')^{-1} \mathrm{d}t_1 \mathrm{d}t' \mathrm{d}t_2 \mathrm{d}t_n \mathrm{d}s_2 \mathrm{d}s_n. \end{aligned}$$

Using the symmetry with respect to t_1, t' and integrating with respect to t_n and s_2 we obtain that

$$\begin{aligned} T_n &\leq 2 \int_{\Delta_t^4} (t_1 - t_2)^{-1} (s_n - t_1)^{-1} (t' - t_2)^{-1} (s_n - t')^{-1} (t_1 - t_2)(s_n - t') \mathrm{d}t_2 \mathrm{d}t_1 \mathrm{d}t' \mathrm{d}s_n \\ &\leq 2 \int_{\Delta_t^4} (t_1 - t_2)^{\frac{-1}{2}} (s_n - t_1)^{\frac{-1}{2}} (t' - t_2)^{\frac{-1}{2}} (s_n - t')^{\frac{-1}{2}} \mathrm{d}t_2 \mathrm{d}t_1 \mathrm{d}t' \mathrm{d}s_n, \end{aligned}$$

where we used that $(t_1 - t_2) \leq (t_1 - t_2)^{\frac{1}{2}} (t' - t_2)^{\frac{1}{2}}$ and $(s_n - t') \leq (s_n - t_1)^{\frac{1}{2}} (s_n - t')^{\frac{1}{2}}$ to obtain the last inequality. It only remains to observe that we have already studied this integral in situation **2** of case $n = 2$.

(2) **B)** $0 \leq t_2 \leq t_1 \leq s_2 \leq t_n \leq s_n \leq t$

With this order T_n can be written as

$$T_n \leq \int_{\Delta_t^4} \left(\int_{t_2}^{s_2} (t_n - t_2)^{-1} (s_n - t_1)^{-1} \mathrm{d}t_1 \right)^2 \mathrm{d}t_2 \mathrm{d}s_2 \mathrm{d}t_n \mathrm{d}s_n.$$

Since in this case we cannot directly integrate with respect s_2 as in case (1) **B)** we shall start by using that $(s_n - t_1)^{-1} \leq (t_n - t_1)^{-\varepsilon}(s_n - s_2)^{-1+\varepsilon}$ to obtain that

$$T_n \leq \int_{\Delta_t^4} \left(\int_{t_2}^{s_2} (t_n - t_1)^{-\varepsilon} \mathrm{d}t_1 \right)^2 (t_n - t_2)^{-2} (s_n - s_2)^{-2+2\varepsilon} \mathrm{d}t_2 \mathrm{d}s_2 \mathrm{d}t_n \mathrm{d}s_n.$$

Now integrating with respect t_1 and s_n (we recall that $T_n \geq 0$) we finally obtain that

$$T_n \leq \int_{\Delta_t^3} \frac{(t_n - t_2)^{2-2\varepsilon}}{(1-\varepsilon)^2} (t_n - t_2)^{-2} (t_n - s_2)^{-1+2\varepsilon} \mathrm{d}t_2 \mathrm{d}s_2 \mathrm{d}t_n < \infty.$$

Again, situation

(2) **D)** $0 \leq t_2 \leq s_2 \leq t_n \leq t_1 \leq s_n \leq t$

can be dealt with a similar argument.

(2) **C)** $0 \leq t_2 \leq s_2 \leq t_1 \leq t_n \leq s_n \leq t$

We first observe that

$$\begin{aligned} T_n &\leq \int_{\Delta_t^4} \left(\int_{s_2}^{t_n} (t_n - t_2)^{-1} (s_n - s_2)^{-1} \mathrm{d}t_1 \right)^2 \mathrm{d}t_2 \mathrm{d}s_2 \mathrm{d}t_n \mathrm{d}s_n \\ &\leq \int_{\Delta_t^4} (t_n - s_2)^2 (t_n - t_2)^{-2} (s_n - s_2)^{-2} \mathrm{d}t_2 \mathrm{d}s_2 \mathrm{d}t_n \mathrm{d}s_n. \end{aligned}$$

Considering that $(t_n - s_2)^2 \leq (t_n - t_2)^{1+\varepsilon}(s_n - s_2)^{1-\varepsilon}$ we finally obtain that

$$\begin{aligned} T_n &\leq \int_{\Delta_t^4} (t_n - t_2)^{-1+\varepsilon} (s_n - s_2)^{-1-\varepsilon} \mathrm{d}t_2 \mathrm{d}s_2 \mathrm{d}t_n \mathrm{d}s_n \\ &\leq \int_{\Delta_t^3} (t_n - t_2)^{-1+\varepsilon} \frac{(t_n - s_2)^{-\varepsilon}}{\varepsilon} \mathrm{d}t_2 \mathrm{d}s_2 \mathrm{d}t_n < \infty. \end{aligned}$$

(3) **A)** $0 \leq t_1 \leq t_2 \leq s_2 \leq t_n \leq s_n \leq t.$

Now T_n becomes as follows

$$T_n \leq \int_{\Delta_t^4} \left(\int_0^{t_2} (t_n - t_1)^{-1} (s_n - t_1)^{-1} \mathrm{d}t_1 \right)^2 \mathrm{d}t_2 \mathrm{d}s_2 \mathrm{d}s_n \mathrm{d}t_n$$

and using that $(t_n - t_1)^{-1} \leq (s_n - t_1)^{-\varepsilon}(t_n - s_2)^{-1+\varepsilon}$ we obtain that

$$\begin{aligned} T_n &\leq \int_{\Delta_t^4} \left(\int_0^{t_2} (s_n - t_1)^{-1-\varepsilon} \mathrm{d}t_1 \right)^2 (t_n - s_2)^{-2+2\varepsilon} \mathrm{d}t_2 \mathrm{d}s_2 \mathrm{d}s_n \mathrm{d}t_n \\ &\leq \int_{\Delta_t^3} \frac{(s_n - t_2)^{-2\varepsilon}}{\varepsilon^2} \frac{(t_n - s_n)^{-1+2\varepsilon}}{1 - 2\varepsilon} \mathrm{d}t_2 \mathrm{d}s_n \mathrm{d}t_n < \infty. \end{aligned}$$

Situations

3 B) $0 \leq t_2 \leq t_1 \leq s_2 \leq s_n \leq t_n \leq t$,

3 D) $0 \leq t_2 \leq s_2 \leq s_n \leq t_1 \leq t_n \leq t$ and

3 E) $0 \leq t_2 \leq s_2 \leq s_n \leq t_n \leq t_1 \leq t$

can be studied in a similar way.

3 C) $0 \leq t_2 \leq s_2 \leq t_1 \leq s_n \leq t_n \leq t$

Observing that

$$\begin{aligned} T_n &\leq \int_{\Delta_t^4} \left(\int_{s_2}^{s_n} (t_n - t_2)^{-1} (s_n - s_2)^{-1} \mathrm{d}t_1 \right)^2 \mathrm{d}t_2 \mathrm{d}s_2 \mathrm{d}s_n \mathrm{d}t_n \\ &\leq \int_{\Delta_t^4} (s_n - s_2)^2 (t_n - t_2)^{-2} (s_n - s_2)^{-2} \mathrm{d}t_2 \mathrm{d}s_2 \mathrm{d}s_n \mathrm{d}t_n \end{aligned}$$

we can conclude that $T_n \leq \frac{t^2}{4} < \infty$.

A.3 Case $p = n - 1$ ($n \neq 2$)

In this case we can rewrite T_n as follows

$$\begin{aligned} T_n = C_n \int_{\Delta_t^2} \Bigg(\int_{\Delta_t^2} &(t_{n-1} \vee t_n - t_1 \wedge t_n)^{-\frac{n-1}{2}} \\ &\times (t_{n-1} \vee t_n - t_1 \wedge t_n)^{-\frac{n-1}{2}} (t_{n-1} - t_1)^{n-3} \mathrm{d}t_1 \mathrm{d}t_{n-1} \Bigg)^2 \mathrm{d}t_n \mathrm{d}s_n \\ \leq C_n \int_{\Delta_t^2} \Bigg(\int_{\Delta_t^2} &(t_{n-1} \vee t_n - t_1 \wedge t_n)^{-1} (t_{n-1} \vee s_n - t_1 \wedge s_n)^{-1} \mathrm{d}t_1 \mathrm{d}t_{n-1} \Bigg)^2 \mathrm{d}t_n \mathrm{d}s_n, \end{aligned}$$

where we used that $(t_{n-1}-t_1)^{n-3} \leq (t_{n-1} \vee t_n - t_1 \wedge t_n)^{\frac{n-3}{2}} (t_{n-1} \vee s_n - t_1 \wedge s_n)^{\frac{n-3}{2}}$.

We have 6 different ways of sorting t_1, t_{n-1} with respect to t_n, s_n. We shall now study each one of this orderings.

Along this section we will omit the constant C_n for the sake of readability.

Case 1 $0 \leq t_1 \leq t_{n-1} \leq t_n \leq s_n \leq t$

We first observe that

$$\begin{aligned} T_n &\leq \int_{\Delta_t^2} \left(\int_{\Delta_t^2} \mathbb{1}_{\{t_{n-1} \leq t_n\}} (t_n - t_1)^{-1} (s_n - t_1)^{-1} \mathrm{d}t_1 \mathrm{d}t_{n-1} \right)^2 \mathrm{d}t_n \mathrm{d}s_n \\ &\leq \int_{\Delta_t^2} \left(\int_0^{t_n} (t_n - t_1)^{-1} (s_n - t_1)^{-1} (t_n - t_1) \mathrm{d}t_1 \right)^2 \mathrm{d}t_n \mathrm{d}s_n. \end{aligned}$$

And now, by using that $(t_n - t_1) \leq (t_n - t_1)^{\varepsilon} (s_n - t_1)^{1-\varepsilon}$ and $(s_n - t_1)^{-\varepsilon} \leq (s_n - t_n)^{-\varepsilon}$ we conclude that

$$T_n \leq \int_{\Delta_t^2} \left(\int_0^{t_n} (t_n - t_1)^{-1+\varepsilon} \mathrm{d}t_1 \right)^2 (s_n - t_n)^{-2\varepsilon} \mathrm{d}t_n \mathrm{d}s_n < \infty$$

Also,

Case 6 $0 \leq t_n \leq s_n \leq t_1 \leq t_{n-1} \leq t$

can be studied in a similar fashion.

Case 2 $0 \leq t_1 \leq t_n \leq t_{n-1} \leq s_n \leq t$

Taking into account the order we have, we get that

$$T_n \leq \int_{\Delta_t^2} \left(\int_{\Delta_t^2} \mathbb{1}_{\{t_1 \leq t_n \leq t_{n-1}\}} (t_{n-1} - t_1)^{-1} (s_n - t_1)^{-1} \mathrm{d}t_1 \mathrm{d}t_{n-1} \right)^2 \mathrm{d}t_n \mathrm{d}s_n.$$

After using that $(s_n - t_1)^{-1} \leq (s_n - t_n)^{-\frac{1}{2}+\varepsilon} (t_{n-1} - t_1)^{-\frac{1}{2}-\varepsilon}$ we obtain that

$$\begin{aligned} T_n &\leq \int_{\Delta_t^2} \left(\int_{\Delta_t^2} \mathbb{1}_{\{t_1 \leq t_n \leq t_{n-1}\}} (t_{n-1} - t_1)^{-\frac{3}{2}-\varepsilon} \mathrm{d}t_1 \mathrm{d}t_{n-1} \right)^2 (s_n - t_n)^{-1+2\varepsilon} \mathrm{d}t_n \mathrm{d}s_n \\ &\leq \int_{\Delta_t^2} \left(\int_0^{t_n} \frac{(t_n - t_1)^{-\frac{1}{2}-\varepsilon}}{\frac{1}{2}+\varepsilon} \mathrm{d}t_1 \right)^2 (s_n - t_n)^{-1+2\varepsilon} \mathrm{d}t_n \mathrm{d}s_n < \infty \end{aligned}$$

Again we notice that

Case 5 $0 \leq t_n \leq t_1 \leq s_n \leq t_{n-1} \leq t$

can be dealt with a similar procedure.

Case **3** $0 \leq t_1 \leq t_n \leq s_n \leq t_{n-1} \leq t$

In this case we have that

$$T_n \leq \int_{\Delta_t^2} \left(\int_{\Delta_t^2} \mathbb{1}_{\{t_1 \leq t_n \leq s_n \leq t_{n-1}\}} (t_{n-1} - t_1)^{-1} (t_{n-1} - t_1)^{-1} \mathrm{d}t_1 \mathrm{d}t_{n-1} \right)^2 \mathrm{d}t_n \mathrm{d}s_n.$$

Now using that $(t_{n-1} - t_1)^{-2} \leq (t_{n-1} - t_1)^{-2+\varepsilon} (s_n - t_n)^{-\varepsilon}$ we get that

$$\begin{aligned} T_n &\leq \int_{\Delta_t^2} \left(\int_{\Delta_t^2} \mathbb{1}_{\{t_1 \leq t_n \leq s_n \leq t_{n-1}\}} (t_{n-1} - t_1)^{-2+\varepsilon} \mathrm{d}t_1 \mathrm{d}t_{n-1} \right)^2 (s_n - t_n)^{-2\varepsilon} \mathrm{d}t_n \mathrm{d}s_n \\ &\leq \int_{\Delta_t^2} \left(\int_0^{t_n} \frac{(s_n - t_1)^{-1+\varepsilon}}{1-\varepsilon} \mathrm{d}t_1 \right)^2 (s_n - t_n)^{-2\varepsilon} \mathrm{d}t_n \mathrm{d}s_n < \infty. \end{aligned}$$

Case **4** $0 \leq t_n \leq t_1 \leq t_{n-1} \leq s_n \leq t$

In this case we can bound T_n by

$$\int_{\Delta_t^2} \left(\int_{\Delta_t^2} \mathbb{1}_{\{t_n \leq t_1 \leq t_{n-1} \leq s_n\}} (t_{n-1} - t_n)^{-1} (s_n - t_1)^{-1} \mathrm{d}t_1 \mathrm{d}t_{n-1} \right)^2 \mathrm{d}t_n \mathrm{d}s_n$$

and using the inequality $(a+b)^2 \leq 2(a^2+b^2)$ we can split this expression as follows

$$\begin{aligned} &2 \int_{\Delta_t^2} \Big(\int_{\Delta_t^2} \mathbb{1}_{\{t_n \leq t_1 \leq t_{n-1} \leq s_n\}} \mathbb{1}_{\{t_{n-1} - t_n \leq s_n - t_1\}} (t_{n-1} - t_n)^{-1} (s_n - t_1)^{-1} \mathrm{d}t_1 \mathrm{d}t_{n-1} \Big)^2 + \\ &\quad + \Big(\int_{\Delta_t^2} \mathbb{1}_{\{t_n \leq t_1 \leq t_{n-1} \leq s_n\}} \mathbb{1}_{\{t_{n-1} - t_n \geq s_n - t_1\}} (t_{n-1} - t_n)^{-1} (s_n - t_1)^{-1} \mathrm{d}t_1 \mathrm{d}t_{n-1} \Big)^2 \mathrm{d}t_n \mathrm{d}s_n \\ &:= (I) + (II). \end{aligned}$$

We shall prove the boundedness of (I), and the reader can check that similar computations proof the boundedness of (II).

Since in (I) we have $t_{n-1} - t_n \leq s_n - t_1$ we can see that $(s_n - t_1)^{-1} \leq (s_n - t_1)^{-\frac{1}{2}} (t_{n-1} - t_n)^{-\frac{1}{2}}$ and we obtain that

$$\begin{aligned} (I) &\leq 2 \int_{\Delta_t^2} \left(\int_{\Delta_t^2} \mathbb{1}_{\{t_n \leq t_1 \leq t_{n-1} \leq s_n\}} (t_{n-1} - t_n)^{-\frac{3}{2}} (s_n - t_1)^{-\frac{1}{2}} \mathrm{d}t_1 \mathrm{d}t_{n-1} \right)^2 \mathrm{d}t_n \mathrm{d}s_n \\ &\leq 2 \int_{\Delta_t^2} \left(\int_{t_n}^{s_n} 2 (t_1 - t_n)^{-\frac{1}{2}} (s_n - t_1)^{-\frac{1}{2}} \mathrm{d}t_1 \right)^2 \mathrm{d}t_n \mathrm{d}s_n \\ &\leq 8 \int_{\Delta_t^4} (t_1 - t_n)^{-\frac{1}{2}} (s_n - t_1)^{-\frac{1}{2}} (t' - t_n)^{-\frac{1}{2}} (s_n - t')^{-\frac{1}{2}} \mathrm{d}t_n \mathrm{d}t_1 \mathrm{d}t' \mathrm{d}s_n, \end{aligned}$$

where we used the symmetry with respect t_1, t' in the last step. To finish, we notice that we have just found the same integral we studied in situation **2** of case $n = 2$.

A.4 Case $p = 2, \dots, n-2$ $(n \neq 2)$

In this general case we consider the parameters $p = 2, \dots, n-2$ and $n \neq 2$, i.e., n is an even number greater or equal than 4.

Integrating (A.0.1) with respect to $t_2, \dots, t_{p-1}, t_{p+2}, \dots, t_{n-1}, s_{p+2}, \dots, s_{n-1}$ and taking into account that

- $(t_p - t_1)^{p-2} \leq (t_p \vee t_n - t_1 \wedge t_{p+1})^{\frac{p-2}{2}} (t_p \vee s_n - t_1 \wedge s_{p+1})^{\frac{p-2}{2}}$
- $(t_n - t_{p+1})^{n-p-2} \leq (t_p \vee t_n - t_1 \wedge t_{p+1})^{n-p-2}$
- $(s_n - s_{p+1})^{n-p-2} \leq (t_p \vee s_n - t_1 \wedge s_{p+1})^{n-p-2}$

we can bound T_n (omitting constants depending on n, p) as follows

$$T_n \leq \int_{\Delta_t^2 \times \Delta_t^2} \Big(\int_{\Delta_t^2} (t_p \vee t_n - t_1 \wedge t_{p+1})^{-\frac{3}{2}} \times (t_p \vee s_n - t_1 \wedge s_{p+1})^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \Big)^2 \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_{p+1} \mathrm{d}s_n. \quad \text{(A.4.1)}$$

Let us now observe that we can sort $t_{p+1}, t_n, s_{p+1}, s_n$ in 6 different ways that, by symmetry, can be reduced to the following 3

(1) $t_{p+1} \leq t_n \leq s_{p+1} \leq s_n$

(2) $t_{p+1} \leq s_{p+1} \leq t_n \leq s_n$

(3) $t_{p+1} \leq s_{p+1} \leq s_n \leq t_n$.

Moreover, for each one of these three cases we can sort t_1, t_p in 15 different ways. Below we shall provide a sketch of the computations involving most of these 45 cases.

(1) **A)** $0 \leq t_1 \leq t_p \leq t_{p+1} \leq t_n \leq s_{p+1} \leq s_n \leq t$

Considering (A.4.1), taking into account the order we have in this case and then integrating with respect to t_p and s_{p+1} we obtain

$$\begin{aligned} T_n &\leq \int_{\Delta_t^4} \Big(\int_{\Delta_t^2} \mathbb{1}_{\{t_p \leq t_{p+1}\}} (t_n - t_1)^{-\frac{3}{2}} (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \Big)^2 \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_{p+1} \mathrm{d}s_n \\ &\leq \int_{\Delta_t^3} \Big(\int_0^{t_{p+1}} (t_n - t_1)^{-\frac{3}{2}} (s_n - t_1)^{-\frac{3}{2}} (t_{p+1} - t_1) \mathrm{d}t_1 \Big)^2 (s_n - t_n) \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_n. \end{aligned}$$

We now observe that $(t_{p+1} - t_1) \leq (t_n - t_1)^{\varepsilon}(s_n - t_1)^{1-\varepsilon}$ and $(s_n - t_1)^{-\frac{1}{2}-\varepsilon} \leq (s_n - t_n)^{-\frac{1}{2}-\varepsilon}$ to conclude that

$$\begin{aligned} T_n &\leq \int_{\Delta_t^3} \left(\int_0^{t_{p+1}} (t_n - t_1)^{-\frac{3}{2}+\varepsilon} \mathrm{d}t_1 \right)^2 (s_n - t_n)^{-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_n \\ &\leq \int_{\Delta_t^3} \frac{(t_n - t_{p+1})^{-1+2\varepsilon}}{(\frac{1}{2} - \varepsilon)^2} (s_n - t_n)^{-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_n < \infty. \end{aligned}$$

1 B) $0 \leq t_1 \leq t_{p+1} \leq t_p \leq t_n \leq s_{p+1} \leq s_n \leq t$

Considering the order in this case we have

$$\begin{aligned} T_n \leq \int_{\Delta_t^4} \Bigg(\int_{\Delta_t^2} \mathbb{1}_{\{t_1 \leq t_{p+1} \leq t_p \leq t_n\}} &(t_n - t_1)^{-\frac{3}{2}} \\ &\times (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \Bigg)^2 \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_{p+1} \mathrm{d}s_n, \end{aligned}$$

and integrating with respect to t_p and s_{p+1}

$$T_n \leq \int_{\Delta_t^3} \left(\int_0^{t_{p+1}} (t_n - t_1)^{-\frac{3}{2}} (s_n - t_1)^{-\frac{3}{2}} (t_n - t_{p+1}) \mathrm{d}t_1 \right)^2 (s_n - t_n) \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_n.$$

We now point out that $(t_n - t_{p+1}) \leq (t_n - t_1)^{\varepsilon}(s_n - t_1)^{1-\varepsilon}$ and $(s_n - t_1)^{-\frac{1}{2}-\varepsilon} \leq (s_n - t_n)^{-\frac{1}{2}-\varepsilon}$ to see that

$$T_n \leq \int_{\Delta_t^3} \left(\int_0^{t_{p+1}} (t_n - t_1)^{-\frac{3}{2}+\varepsilon} \mathrm{d}t_1 \right)^2 (s_n - t_n)^{-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_n,$$

which turns out to be the same integral we studied in case 1 **A)**.

1 C) $0 \leq t_1 \leq t_{p+1} \leq t_n \leq t_p \leq s_{p+1} \leq s_n \leq t$

In this case,

$$\begin{aligned} T_n \leq \int_{\Delta_t^4} \Bigg(\int_{\Delta_t^2} \mathbb{1}_{\{t_1 \leq t_{p+1} \leq t_n \leq t_p \leq s_{p+1}\}} &(t_p - t_1)^{-\frac{3}{2}} \\ &\times (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \Bigg)^2 \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_{p+1} \mathrm{d}s_n. \end{aligned}$$

We first notice that we can integrate with respect to t_p and s_{p+1}

$$T_n \leq \int_{\Delta_t^3} \left(\int_0^{t_{p+1}} 2(t_n - t_1)^{-\frac{1}{2}} (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \right)^2 (s_n - t_n) \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_n,$$

and now, using that $(s_n - t_1)^{-\frac{3}{2}} \leq (s_n - t_n)^{-1+\varepsilon}(t_n - t_1)^{-\frac{1}{2}-\varepsilon}$, we conclude that

$$\begin{aligned} T_n &\leq 4\int_{\Delta_t^3}\left(\int_0^{t_{p+1}}(t_n - t_1)^{-1-\varepsilon}\mathrm{d}t_1\right)^2(s_n - t_n)^{-1+2\varepsilon}\mathrm{d}t_{p+1}\mathrm{d}t_n\mathrm{d}s_n \\ &\leq 4\int_{\Delta_t^3}\frac{(t_n - t_{p+1})^{-2\varepsilon}}{\varepsilon^2}(s_n - t_n)^{-1+2\varepsilon}\mathrm{d}t_{p+1}\mathrm{d}t_n\mathrm{d}s_n < \infty. \end{aligned}$$

1 **D)** $0 \leq t_1 \leq t_{p+1} \leq t_n \leq s_{p+1} \leq t_p \leq s_n \leq t$

In this situation, (A.4.1) becomes

$$\begin{aligned} T_n \leq \int_{\Delta_t^4}\Bigg(\int_{\Delta_t^2}\mathbb{1}_{\{t_1\leq t_{p+1}\}}\mathbb{1}_{\{s_{p+1}\leq t_p\leq s_n\}}(t_p - t_1)^{-\frac{3}{2}} \\ \times (s_n - t_1)^{-\frac{3}{2}}\mathrm{d}t_1\mathrm{d}t_p\Bigg)^2\mathrm{d}t_{p+1}\mathrm{d}t_n\mathrm{d}s_{p+1}\mathrm{d}s_n. \end{aligned}$$

Again, integrating with respect to t_p and t_n we obtain

$$T_n \leq \int_{\Delta_t^3}\left(\int_0^{t_{p+1}}2(s_{p+1} - t_1)^{-\frac{1}{2}}(s_n - t_1)^{-\frac{3}{2}}\mathrm{d}t_1\right)^2(s_{p+1} - t_{p+1})\mathrm{d}t_{p+1}\mathrm{d}s_{p+1}\mathrm{d}s_n$$

and using that $(s_{p+1} - t_{p+1}) \leq (s_n - t_{p+1})$ and $(s_n - t_1)^{-\frac{3}{2}} \leq (s_{p+1} - t_1)^{-\frac{1}{2}-\varepsilon}(s_n - t_{p+1})^{-1+\varepsilon}$ we conclude that

$$\begin{aligned} T_n &\leq 4\int_{\Delta_t^3}\left(\int_0^{t_{p+1}}(s_{p+1} - t_1)^{-1-\varepsilon}\mathrm{d}t_1\right)^2(s_n - t_{p+1})^{-1+2\varepsilon}\mathrm{d}t_{p+1}\mathrm{d}s_{p+1}\mathrm{d}s_n \\ &\leq 4\int_{\Delta_t^3}\frac{(s_{p+1} - t_{p+1})^{-2\varepsilon}}{\varepsilon^2}(s_n - t_{p+1})^{-1+2\varepsilon}\mathrm{d}t_{p+1}\mathrm{d}s_{p+1}\mathrm{d}s_n < \infty. \end{aligned}$$

1 **E)** $0 \leq t_1 \leq t_{p+1} \leq t_n \leq s_{p+1} \leq s_n \leq t_p \leq t$

Taking into account this order, we have that

$$T_n \leq \int_{\Delta_t^4}\left(\int_{\Delta_t^2}\mathbb{1}_{\{t_1\leq t_{p+1}\}}\mathbb{1}_{\{s_n\leq t_p\}}(t_p - t_1)^{-3}\mathrm{d}t_1\mathrm{d}t_p\right)^2\mathrm{d}t_{p+1}\mathrm{d}t_n\mathrm{d}s_{p+1}\mathrm{d}s_n.$$

If we now use that $(t_p - t_1)^{-3} \leq (t_p - t_1)^{-3+\varepsilon}(s_n - s_{p+1})^{-\varepsilon}$ we get

$$\begin{aligned} T_n &\leq \int_{\Delta_t^4}\left(\int_0^{t_{p+1}}\frac{(s_n - t_1)^{-2+\varepsilon}}{2-\varepsilon}\mathrm{d}t_1\right)^2(s_n - s_{p+1})^{-2\varepsilon}\mathrm{d}t_{p+1}\mathrm{d}t_n\mathrm{d}s_{p+1}\mathrm{d}s_n \\ &\leq \int_{\Delta_t^4}\frac{(s_n - t_{p+1})^{-2+2\varepsilon}}{(2-\varepsilon)^2(1-\varepsilon)^2}(s_n - s_{p+1})^{-2\varepsilon}\mathrm{d}t_{p+1}\mathrm{d}t_n\mathrm{d}s_{p+1}\mathrm{d}s_n. \end{aligned}$$

To conclude we integrate with respect to t_{p+1} to obtain that

$$T_n \leq \int_{\Delta_t^3} \frac{(s_n - t_n)^{-1+2\varepsilon}}{(2-\varepsilon)^2(1-\varepsilon)^2(1-2\varepsilon)} (s_n - s_{p+1})^{-2\varepsilon} \mathrm{d}t_n \mathrm{d}s_{p+1} \mathrm{d}s_n < \infty.$$

1 F) $0 \leq t_{p+1} \leq t_1 \leq t_p \leq t_n \leq s_{p+1} \leq s_n \leq t$

In this situation we have

$$\begin{aligned} T_n \leq \int_{\Delta_t^4} \Bigg(\int_{\Delta_t^2} & \mathbb{1}_{\{t_{p+1} \leq t_1 \leq t_p \leq t_n\}} (t_n - t_{p+1})^{-\frac{3}{2}} \\ & \times (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \Bigg)^2 \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_{p+1} \mathrm{d}s_n \\ \leq \int_{\Delta_t^4} \Bigg(\int_{t_{p+1}}^{t_n} & (t_n - t_{p+1})^{-\frac{3}{2}} (s_n - t_1)^{-\frac{3}{2}} (t_n - t_1) \mathrm{d}t_1 \Bigg)^2 \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_{p+1} \mathrm{d}s_n. \end{aligned}$$

First using that $(t_n - t_1) \leq (t_n - t_{p+1})^{1-\varepsilon} (s_n - t_1)^{\varepsilon}$ and then writing the square of the integral as a double integral we obtain that

$$\begin{aligned} T_n &\leq \int_{\Delta_t^4} \left(\int_{t_{p+1}}^{t_n} (t_n - t_{p+1})^{-\frac{1}{2}-\varepsilon} (s_n - t_1)^{-\frac{3}{2}+\varepsilon} \mathrm{d}t_1 \right)^2 \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_{p+1} \mathrm{d}s_n \\ &\leq 2 \int_{\Delta_t^6} (t_n - t_{p+1})^{-1-2\varepsilon} (s_n - t_1)^{-\frac{3}{2}+\varepsilon} (s_n - t')^{-\frac{3}{2}+\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}t_1 \mathrm{d}t' \mathrm{d}t_n \mathrm{d}s_{p+1} \mathrm{d}s_n. \end{aligned}$$

To conclude, we integrate with respect to t_{p+1} and s_{p+1}

$$T_n \leq 2 \int_{\Delta_t^4} \frac{(t_n - t_1)^{-2\varepsilon}}{2\varepsilon} (s_n - t_1)^{-\frac{3}{2}+\varepsilon} (s_n - t')^{-\frac{3}{2}+\varepsilon} (s_n - t_n) \mathrm{d}t_1 \mathrm{d}t' \mathrm{d}t_n \mathrm{d}s_n$$

and we finally use that $(s_n - t_1)^{-\frac{3}{2}+\varepsilon} \leq (s_n - t_n)^{-\frac{3}{2}+\varepsilon}$ to obtain that

$$\begin{aligned} T_n &\leq 2 \int_{\Delta_t^4} \frac{(t_n - t_1)^{-2\varepsilon}}{2\varepsilon} \frac{(s_n - t_n)^{-\frac{1}{2}+\varepsilon}}{\frac{1}{2} - \varepsilon} (s_n - t_n)^{-\frac{1}{2}+\varepsilon} \mathrm{d}t_1 \mathrm{d}t_n \mathrm{d}s_n \\ &\leq 2 \int_{\Delta_t^4} \frac{(t_n - t_1)^{-2\varepsilon}}{2\varepsilon} \frac{(s_n - t_n)^{-1+2\varepsilon}}{\frac{1}{2} - \varepsilon} \mathrm{d}t_1 \mathrm{d}t_n \mathrm{d}s_n < \infty. \end{aligned}$$

1 G) $0 \leq t_{p+1} \leq t_1 \leq t_n \leq t_p \leq s_{p+1} \leq s_n \leq t$

We first observe that

$$\begin{aligned} T_n \leq \int_{\Delta_t^4} \Bigg(\int_{\Delta_t^2} & \mathbb{1}_{\{t_{p+1} \leq t_1 \leq t_n \leq t_p \leq s_{p+1}\}} (t_p - t_{p+1})^{-\frac{3}{2}} \\ & \times (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \Bigg)^2 \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_{p+1} \mathrm{d}s_n, \end{aligned}$$

and by means of inequality $(a+b)^2 \leq 2(a^2+b^2)$ we can split T_n as follows

$$
\begin{aligned}
&2\int_{\Delta_t^4}\left(\int_{\Delta_t^2} \mathbb{1}_{\{t_{p+1}\leq t_1\leq t_n\leq t_p\leq s_{p+1}\}}\mathbb{1}_{\{t_p-t_{p+1}\leq s_n-t_1\}}(t_p-t_{p+1})^{-\frac{3}{2}}(s_n-t_1)^{-\frac{3}{2}}\mathrm{d}t_1\mathrm{d}t_p\right)^2 \\
&\qquad+\left(\int_{\Delta_t^2} \mathbb{1}_{\{t_{p+1}\leq t_1\leq t_n\leq t_p\leq s_{p+1}\}}\mathbb{1}_{\{t_p-t_{p+1}\geq s_n-t_1\}}(t_p-t_{p+1})^{-\frac{3}{2}}\right. \\
&\qquad\qquad\qquad\left.\times (s_n-t_1)^{-\frac{3}{2}}\mathrm{d}t_1\mathrm{d}t_p\right)^2 \mathrm{d}t_{p+1}\mathrm{d}t_n\mathrm{d}s_{p+1}\mathrm{d}s_n \\
&:= (I)+(II).
\end{aligned}
$$

We shall prove the boundedness of (I), and the reader can check that similar computations prove the boundedness of (II). In case (I), we have $t_p - t_{p+1} \leq s_n - t_1$ and hence $(s_n-t_1)^{-\frac{3}{2}} \leq (t_n-t_{p+1})^{-\varepsilon}(s_n-t_1)^{-\frac{3}{2}+\varepsilon}$. Using this inequality and then integrating with respect to t_p we see that

$$
\begin{aligned}
(I) &\leq 2\int_{\Delta_t^4}\left(\int_{\Delta_t^2} \mathbb{1}_{\{t_{p+1}\leq t_1\leq t_n\leq t_p\leq s_{p+1}\}}(t_p-t_{p+1})^{-\frac{3}{2}-\varepsilon}\right. \\
&\qquad\qquad\left.\times (s_n-t_1)^{-\frac{3}{2}+\varepsilon}\mathrm{d}t_1\mathrm{d}t_p\right)^2 \mathrm{d}t_{p+1}\mathrm{d}t_n\mathrm{d}s_{p+1}\mathrm{d}s_n \\
&\leq C_\varepsilon \int_{\Delta_t^4}\left(\int_{t_{p+1}}^{t_n}(t_n-t_{p+1})^{-\frac{1}{2}-\varepsilon}(s_n-t_1)^{-\frac{3}{2}+\varepsilon}\mathrm{d}t_1\right)^2 \mathrm{d}t_{p+1}\mathrm{d}t_n\mathrm{d}s_{p+1}\mathrm{d}s_n,
\end{aligned}
$$

where C_ε is a constant that depends only on ε. We finally observe that we studied this last integral along the proof of case 1 **F)**.

1 **H)** $0 \leq t_{p+1} \leq t_1 \leq t_n \leq s_{p+1} \leq t_p \leq s_n \leq t$

In this case,

$$
\begin{aligned}
T_n &\leq \int_{\Delta_t^4}\left(\int_{\Delta_t^2} \mathbb{1}_{\{t_{p+1}\leq t_1\leq t_n\leq s_{p+1}\leq t_p\leq s_n\}}(t_p-t_{p+1})^{-\frac{3}{2}}\right. \\
&\qquad\qquad\left.\times (s_n-t_1)^{-\frac{3}{2}}\mathrm{d}t_1\mathrm{d}t_p\right)^2 \mathrm{d}t_{p+1}\mathrm{d}t_n\mathrm{d}s_{p+1}\mathrm{d}s_n.
\end{aligned}
$$

If we now integrate with respect to t_1 and t_p we obtain that

$$
\begin{aligned}
T_n &\leq 4\int_{\Delta_t^4}(s_{p+1}-t_{p+1})^{-1}(s_n-t_n)^{-1}\mathrm{d}t_{p+1}\mathrm{d}t_n\mathrm{d}s_{p+1}\mathrm{d}s_n \\
&\leq 4\int_{\Delta_t^4}(s_{p+1}-t_{p+1})^{-1}(s_n-t_n)^{-1}\left(\mathbb{1}_{\{s_{p+1}-t_{p+1}\leq s_n-t_n\}}\right. \\
&\qquad\qquad\left.+\mathbb{1}_{\{s_n-t_n\leq s_{p+1}-t_{p+1}\}}\right)\mathrm{d}t_{p+1}\mathrm{d}t_n\mathrm{d}s_{p+1}\mathrm{d}s_n,
\end{aligned}
$$

where we have split the integral into two parts. We will again study the first part, leaving the details of the second part to the reader. In the first part, since $s_{p+1} - t_{p+1} \leq s_n - t_n$ we have that $(s_n - t_n)^{-1} \leq (s_n - t_n)^{-1+\varepsilon}(s_{p+1} - t_{p+1})^{-\varepsilon}$ and we can conclude that it can be bounded by

$$
\begin{aligned}
&4 \int_{\Delta_t^4} (s_{p+1} - t_{p+1})^{-1-\varepsilon}(s_n - t_n)^{-1+\varepsilon} \mathrm{d}t_{p+1}\mathrm{d}t_n\mathrm{d}s_{p+1}\mathrm{d}s_n \\
&\leq 4 \int_{\Delta_t^3} \frac{(s_{p+1} - t_n)^{-\varepsilon}}{\varepsilon} (s_n - t_n)^{-1+\varepsilon} \mathrm{d}t_n\mathrm{d}s_{p+1}\mathrm{d}s_n < \infty.
\end{aligned}
$$

1 **I)** $0 \leq t_{p+1} \leq t_1 \leq t_n \leq s_{p+1} \leq s_n \leq t_p \leq t$

This situation can be studied in a similar way than situation 1 **D)**.

1 **J)** $0 \leq t_{p+1} \leq t_n \leq t_1 \leq t_p \leq s_{p+1} \leq s_n \leq t$

In this case,

$$
\begin{aligned}
T_n &\leq \int_{\Delta_t^4} \left(\int_{\Delta_t^2} \mathbb{1}_{\{t_n \leq t_1 \leq t_p \leq s_{p+1}\}} (t_p - t_{p+1})^{-\frac{3}{2}} (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \right)^2 \mathrm{d}t_{p+1}\mathrm{d}t_n\mathrm{d}s_{p+1}\mathrm{d}s_n \\
&\leq 2 \int_{\Delta_t^4} \left(\int_{\Delta_t^2} \mathbb{1}_{\{t_n \leq t_1 \leq t_p \leq s_{p+1}\}} \mathbb{1}_{\{t_p - t_{p+1} \leq s_n - t_1\}} (t_p - t_{p+1})^{-\frac{3}{2}} (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \right)^2 \\
&\quad + \left(\int_{\Delta_t^2} \mathbb{1}_{\{t_n \leq t_1 \leq t_p \leq s_{p+1}\}} \mathbb{1}_{\{t_p - t_{p+1} \geq s_n - t_1\}} \right. \\
&\qquad \left. \times (t_p - t_{p+1})^{-\frac{3}{2}} (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \right)^2 \mathrm{d}t_{p+1}\mathrm{d}t_n\mathrm{d}s_{p+1}\mathrm{d}s_n \\
&:= (I) + (II).
\end{aligned}
$$

We shall prove the boundedness of (I), and the reader can check that similar computations prove the boundedness of (II).

Since $(t_p - t_{p+1}) \leq (s_n - t_1)$, we have $(s_n - t_1)^{-\varepsilon} \leq (t_p - t_{p+1})^{-\varepsilon}$ and we can bound (I) by

$$
\begin{aligned}
&\int_{\Delta_t^4} \Big(\int_{\Delta_t^2} \mathbb{1}_{\{t_n \leq t_1 \leq t_p \leq s_{p+1}\}} (t_p - t_{p+1})^{\frac{-3}{2}-\varepsilon} (s_n - t_1)^{\frac{-3}{2}+\varepsilon} \mathrm{d}t_1 \mathrm{d}t_p \Big)^2 \mathrm{d}t_{p+1}\mathrm{d}t_n\mathrm{d}s_{p+1}\mathrm{d}s_n \\
&\leq C_\varepsilon \int_{\Delta_t^4} \Big(\int_{t_n}^{s_{p+1}} (t_1 - t_{p+1})^{\frac{-1}{2}-\varepsilon} (s_n - t_1)^{\frac{-3}{2}+\varepsilon} \mathrm{d}t_1 \Big)^2 \mathrm{d}t_{p+1}\mathrm{d}t_n\mathrm{d}s_{p+1}\mathrm{d}s_n.
\end{aligned}
$$

We now develop the square of the integral with respect to t_1 as a double integral with respect to t_1, t_1'. Moreover, by symmetry, we can assume $t_1 \leq t_1'$ and we obtain

$$(I) \leq C_\varepsilon \int_{\Delta_t^6} (t_1 - t_{p+1})^{\frac{-1}{2}-\varepsilon}(s_n - t_1)^{\frac{-3}{2}+\varepsilon}$$
$$\times (t_1' - t_{p+1})^{\frac{-1}{2}-\varepsilon}(s_n - t_1')^{\frac{-3}{2}+\varepsilon} \mathrm{d}t_{p+1}\mathrm{d}t_n\mathrm{d}t_1\mathrm{d}t_1'\mathrm{d}s_{p+1}\mathrm{d}s_n$$
$$= C_\varepsilon \int_{\Delta_t^4} (t_1 - t_{p+1})^{\frac{1}{2}-\varepsilon}(s_n - t_1)^{\frac{-3}{2}+\varepsilon}(t_1' - t_{p+1})^{\frac{-1}{2}-\varepsilon}(s_n - t_1')^{\frac{-1}{2}+\varepsilon} \mathrm{d}t_{p+1}\mathrm{d}t_1\mathrm{d}t_1'\mathrm{d}s_n.$$

Finally, using the fact that $(t_1 - t_{p+1})^{\frac{1}{2}-\varepsilon} \leq (t_1' - t_{p+1})^{\frac{1}{2}-\varepsilon}$, we can conclude that

$$(I) \leq C_\varepsilon \int_{\Delta_t^4} (s_n - t_1)^{\frac{-3}{2}+\varepsilon}(t_1' - t_{p+1})^{-2\varepsilon}(s_n - t_1')^{\frac{-1}{2}+\varepsilon} \mathrm{d}t_{p+1}\mathrm{d}t_1\mathrm{d}t_1'\mathrm{d}s_n$$
$$\leq C_\varepsilon \int_{\Delta_t^3} (s_n - t_1')^{\frac{-1}{2}+\varepsilon}(t_1' - t_{p+1})^{-2\varepsilon}(s_n - t_1')^{\frac{-1}{2}+\varepsilon} \mathrm{d}t_{p+1}\mathrm{d}t_1'\mathrm{d}s_n < \infty.$$

1 K) $0 \leq t_{p+1} \leq t_n \leq t_1 \leq s_{p+1} \leq t_p \leq s_n \leq t$

This situation can be studied in a similar way than situation **1 G)**.

1 L) $0 \leq t_{p+1} \leq t_n \leq t_1 \leq s_{p+1} \leq s_n \leq t_p \leq t$

This situation can be studied in a similar way than situation **1 C)**.

1 M) $0 \leq t_{p+1} \leq t_n \leq s_{p+1} \leq t_1 \leq t_p \leq s_n \leq t$

This situation can be studied in a similar way than situation **1 F)**.

1 N) $0 \leq t_{p+1} \leq t_n \leq s_{p+1} \leq t_1 \leq s_n \leq t_p \leq t$

This situation can be studied in a similar way than situation **1 B)**.

1 O) $0 \leq t_{p+1} \leq t_n \leq s_{p+1} \leq s_n \leq t_1 \leq t_p \leq t$

This situation can be studied in a similar way than situation **1 A)**.

2 A) $0 \leq t_1 \leq t_p \leq t_{p+1} \leq s_{p+1} \leq t_n \leq s_n \leq t$

Considering (A.4.1), taking into account the order we have in this case and then integrating with respect to t_p and s_{p+1} we obtain

$$\begin{aligned} T_n &\leq \int_{\Delta_t^4} \left(\int_{\Delta_t^2} \mathbb{1}_{\{t_p \leq t_{p+1}\}} (t_n - t_1)^{-\frac{3}{2}} (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \right)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n \\ &\leq \int_{\Delta_t^3} \left(\int_0^{t_{p+1}} (t_n - t_1)^{-\frac{3}{2}} (s_n - t_1)^{-\frac{3}{2}} (t_{p+1} - t_1) \mathrm{d}t_1 \right)^2 (t_n - t_{p+1}) \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_n. \end{aligned}$$

We now observe that $(t_{p+1} - t_1) \leq (s_n - t_1)^{\varepsilon} (t_n - t_1)^{1-\varepsilon}$ and $(t_n - t_1)^{-\frac{1}{2}-\varepsilon} \leq (t_n - t_{p+1})^{-\frac{1}{2}-\varepsilon}$ to conclude that

$$\begin{aligned} T_n &\leq \int_{\Delta_t^3} \left(\int_0^{t_{p+1}} (s_n - t_1)^{-\frac{3}{2}+\varepsilon} \mathrm{d}t_1 \right)^2 (t_n - t_{p+1})^{-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_n \\ &\leq \int_{\Delta_t^3} \frac{(s_n - t_{p+1})^{-1+2\varepsilon}}{(\frac{1}{2} - \varepsilon)^2} (t_n - t_{p+1})^{-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_n < \infty. \end{aligned}$$

2 B) $0 \leq t_1 \leq t_{p+1} \leq t_p \leq s_{p+1} \leq t_n \leq s_n \leq t$

In this situation,

$$\begin{aligned} T_n \leq \int_{\Delta_t^4} \Bigg(\int_{\Delta_t^2} & \mathbb{1}_{\{t_1 \leq t_{p+1} \leq t_p \leq s_{p+1}\}} (t_n - t_1)^{-\frac{3}{2}} \\ & \times (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \Bigg)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n. \end{aligned}$$

If we now integrate with respect to t_p

$$T_n \leq \int_{\Delta_t^4} \left(\int_0^{t_{p+1}} (t_n - t_1)^{-\frac{3}{2}} (s_n - t_1)^{-\frac{3}{2}} (s_{p+1} - t_{p+1}) \mathrm{d}t_1 \right)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n,$$

and we use that $(s_{p+1} - t_{p+1}) \leq (s_n - t_1)^{\varepsilon} (t_n - t_1)^{1-\varepsilon}$ to get that

$$\begin{aligned} T_n &\leq \int_{\Delta_t^4} \left(\int_0^{t_{p+1}} (t_n - t_1)^{-\frac{1}{2}-\varepsilon} (s_n - t_1)^{-\frac{3}{2}+\varepsilon} \mathrm{d}t_1 \right)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n \\ &\leq \int_{\Delta_t^3} \left(\int_0^{t_{p+1}} (t_n - t_1)^{-\frac{1}{2}-\varepsilon} (s_n - t_1)^{-\frac{3}{2}+\varepsilon} \mathrm{d}t_1 \right)^2 (t_n - t_{p+1}) \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_n. \end{aligned}$$

Finally we notice that $(t_n - t_1)^{-\frac{1}{2}-\varepsilon} \leq (t_n - t_{p+1})^{-\frac{1}{2}-\varepsilon}$ to conclude that

$$T_n \leq \int_{\Delta_t^3} \frac{(s_n - t_{p+1})^{-1+2\varepsilon}}{(\frac{1}{2} - \varepsilon)^2} (t_n - t_{p+1})^{-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_n < \infty.$$

2 C) $0 \le t_1 \le t_{p+1} \le s_{p+1} \le t_p \le t_n \le s_n \le t$

Considering the order in this case we have that

$$\begin{aligned} T_n &\le \int_{\Delta_t^4} \Bigg(\int_{\Delta_t^2} \mathbb{1}_{\{t_1 \le t_{p+1} \le s_{p+1} \le t_p \le t_n\}} (t_n - t_1)^{-\frac{3}{2}} \\ &\qquad\qquad\qquad \times (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \Bigg)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n \\ &\le \int_{\Delta_t^4} \left(\int_0^{t_{p+1}} (t_n - t_1)^{-\frac{3}{2}} (s_n - t_1)^{-\frac{3}{2}} (t_n - s_{p+1}) \mathrm{d}t_1 \right)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n. \end{aligned}$$

By means of the inequality $(t_n - s_{p+1}) \le (s_n - t_1)^{\varepsilon} (t_n - t_1)^{1-\varepsilon}$ we obtain that

$$T_n \le \int_{\Delta_t^3} \left(\int_0^{t_{p+1}} (t_n - t_1)^{-\frac{1}{2}-\varepsilon} (s_n - t_1)^{-\frac{3}{2}+\varepsilon} \mathrm{d}t_1 \right)^2 (t_n - t_{p+1}) \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_n,$$

which we have already studied in case **2 B)**.

2 D) $0 \le t_1 \le t_{p+1} \le s_{p+1} \le t_n \le t_p \le s_n \le t$

In this situation, (A.4.1) becomes

$$\begin{aligned} T_n &\le \int_{\Delta_t^4} \Bigg(\int_{\Delta_t^2} \mathbb{1}_{\{t_1 \le t_{p+1}\}} \mathbb{1}_{\{t_n \le t_p \le s_n\}} (t_p - t_1)^{-\frac{3}{2}} \\ &\qquad\qquad\qquad \times (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \Bigg)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n \\ &\le \int_{\Delta_t^3} \left(\int_0^{t_{p+1}} 2(t_n - t_1)^{-\frac{1}{2}} (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \right)^2 (t_n - t_{p+1}) \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_n. \end{aligned}$$

We now notice that $(t_n - t_{p+1}) \le (t_n - t_{p+1})^{1-2\varepsilon} (s_n - t_1)^{2\varepsilon}$ and $(t_n - t_1)^{-\frac{1}{2}} \le (t_n - t_{p+1})^{-\frac{1}{2}}$ to obtain that

$$\begin{aligned} T_n &\le 2 \int_{\Delta_t^3} \left(\int_0^{t_{p+1}} (s_n - t_1)^{-\frac{3}{2}+\varepsilon} \mathrm{d}t_1 \right)^2 (t_n - t_{p+1})^{-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_n \\ &\le 2 \int_{\Delta_t^3} \frac{(s_n - t_{p+1})^{-1+2\varepsilon}}{(\frac{1}{2} - \varepsilon)^2} (t_n - t_{p+1})^{-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_n < \infty. \end{aligned}$$

2 E) $0 \le t_1 \le t_{p+1} \le s_{p+1} \le t_n \le s_n \le t_p \le t$

Taking into account this order, we have that

$$T_n \leq \int_{\Delta_t^4} \left(\int_{\Delta_t^2} \mathbb{1}_{\{t_1 \leq t_{p+1}\}} \mathbb{1}_{\{s_n \leq t_p\}} (t_p - t_1)^{-3} \mathrm{d}t_1 \mathrm{d}t_p \right)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n.$$

If we now use that $(t_p - t_1)^{-3} \leq (t_p - t_1)^{-3+\varepsilon} (s_n - t_n)^{-\varepsilon}$ we get

$$\begin{aligned} T_n \leq & \int_{\Delta_t^4} \left(\int_{\Delta_t^2} \mathbb{1}_{\{t_1 \leq t_{p+1}\}} \mathbb{1}_{\{s_n \leq t_p\}} (t_p - t_1)^{-3+\varepsilon} \mathrm{d}t_1 \mathrm{d}t_p \right)^2 \\ & \qquad \times (s_n - t_n)^{-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n \\ \leq & \int_{\Delta_t^3} \left(\int_0^{t_{p+1}} \frac{(s_n - t_1)^{-2+\varepsilon}}{2-\varepsilon} \mathrm{d}t_1 \right)^2 (s_n - t_n)^{-2\varepsilon} (t_n - t_{p+1}) \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_n. \end{aligned}$$

To conclude we notice that $(t_n - t_{p+1}) \leq (s_n - t_{p+1})$ to obtain that

$$T_n \leq \int_{\Delta_t^3} \frac{(s_n - t_{p+1})^{-2+2\varepsilon}}{(2-\varepsilon)^2 (1-\varepsilon)^2} (s_n - t_n)^{-2\varepsilon} (s_n - t_{p+1}) \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_n < \infty.$$

2 F) $0 \leq t_{p+1} \leq t_1 \leq t_p \leq s_{p+1} \leq t_n \leq s_n \leq t$

In this situation we have

$$\begin{aligned} T_n \leq & \int_{\Delta_t^4} \Bigg(\int_{\Delta_t^2} \mathbb{1}_{\{t_{p+1} \leq t_1 \leq t_p \leq s_{p+1}\}} (t_n - t_{p+1})^{-\frac{3}{2}} \\ & \qquad \times (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \Bigg)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n \\ \leq & \int_{\Delta_t^4} \left(\int_{t_{p+1}}^{s_{p+1}} (t_n - t_{p+1})^{-\frac{3}{2}} (s_n - t_1)^{-\frac{3}{2}} (s_{p+1} - t_1) \mathrm{d}t_1 \right)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n. \end{aligned}$$

Using that $(s_{p+1} - t_1) \leq (t_n - t_{p+1})^{1-\varepsilon} (s_n - t_1)^{\varepsilon}$ we conclude that

$$\begin{aligned} T_n \leq & \int_{\Delta_t^4} \left(\int_{t_{p+1}}^{s_{p+1}} (s_n - t_1)^{-\frac{3}{2}+\varepsilon} \mathrm{d}t_1 \right)^2 (t_n - t_{p+1})^{-1-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n \\ \leq & \int_{\Delta_t^3} \frac{(s_n - s_{p+1})^{-1+2\varepsilon}}{(\frac{1}{2} - \varepsilon)^2} \frac{(s_{p+1} - t_{p+1})^{-2\varepsilon}}{2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n < \infty. \end{aligned}$$

2 G) $0 \leq t_{p+1} \leq t_1 \leq s_{p+1} \leq t_p \leq t_n \leq s_n \leq t$

We first observe that

$$\begin{aligned}T_n \leq & \int_{\Delta_t^4} \left(\int_{\Delta_t^2} \mathbb{1}_{\{t_{p+1} \leq t_1 \leq s_{p+1} \leq t_p \leq t_n\}} (t_n - t_{p+1})^{-\frac{3}{2}} \right. \\ & \qquad\qquad \left. \times (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \right)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n \\ \leq & \int_{\Delta_t^4} \left(\int_{t_{p+1}}^{s_{p+1}} (t_n - t_{p+1})^{-\frac{3}{2}} (s_n - t_1)^{-\frac{3}{2}} (t_n - s_{p+1}) \mathrm{d}t_1 \right)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n.\end{aligned}$$

By means of the inequality $(t_n - s_{p+1}) \leq (s_n - t_1)^{\varepsilon} (t_n - t_{p+1})^{1-\varepsilon}$ we obtain that

$$\begin{aligned}T_n &\leq \int_{\Delta_t^4} \left(\int_{t_{p+1}}^{s_{p+1}} (s_n - t_1)^{-\frac{3}{2}+\varepsilon} \mathrm{d}t_1 \right)^2 (t_n - t_{p+1})^{-1-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n \\ &\leq \int_{\Delta_t^3} \frac{(s_n - s_{p+1})^{-1+2\varepsilon}}{(\frac{1}{2} - \varepsilon)^2} \frac{(s_{p+1} - t_{p+1})^{-2\varepsilon}}{2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n < \infty.\end{aligned}$$

2 **H)** $0 \leq t_{p+1} \leq t_1 \leq s_{p+1} \leq t_n \leq t_p \leq s_n \leq t$

In this situation we have

$$\begin{aligned}T_n \leq & \int_{\Delta_t^4} \left(\int_{\Delta_t^2} \mathbb{1}_{\{t_{p+1} \leq t_1 \leq s_{p+1} \leq t_n \leq t_p \leq s_n\}} (t_p - t_{p+1})^{-\frac{3}{2}} \right. \\ & \qquad\qquad \left. \times (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \right)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n.\end{aligned}$$

After integrating with respect to t_1 and t_p we find that

$$T_n \leq \int_{\Delta_t^4} 16 (t_n - t_{p+1})^{-1} (s_n - s_{p+1})^{-1} \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n,$$

and now we use that $\mathbb{1}_{\{(t_n - t_{p+1}) \leq (s_n - s_{p+1})\}} + \mathbb{1}_{\{(s_n - s_{p+1}) < (t_n - t_{p+1})\}} = 1$ to split the integral into two parts.

In the first one, we have that $(s_n - s_{p+1})^{-1} \leq (s_n - s_{p+1})^{-1+\varepsilon} (t_n - t_{p+1})^{-\varepsilon}$ and we can see that

$$\begin{aligned}&16 \int_{\Delta_t^4} (t_n - t_{p+1})^{-1} (s_n - s_{p+1})^{-1} \mathbb{1}_{\{(t_n - t_{p+1}) \leq (s_n - s_{p+1})\}} \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n \\ &\leq 16 \int_{\Delta_t^4} (t_n - t_{p+1})^{-1-\varepsilon} (s_n - s_{p+1})^{-1+\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n \\ &\leq 16 \int_{\Delta_t^3} \frac{(t_n - s_{p+1})^{-\varepsilon}}{\varepsilon} (s_n - s_{p+1})^{-1+\varepsilon} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n < \infty.\end{aligned}$$

The other term can be proved to bounded using a similar reasoning.

2 I) $0 \leq t_{p+1} \leq t_1 \leq s_{p+1} \leq t_n \leq s_n \leq t_p \leq t$

This situation can be studied in a similar way than situation **2 D)**.

2 J) $0 \leq t_{p+1} \leq s_{p+1} \leq t_1 \leq t_p \leq t_n \leq s_n \leq t$

In this case,

$$\begin{aligned} T_n &\leq \int_{\Delta_t^4} \Bigg(\int_{\Delta_t^2} \mathbb{1}_{\{s_{p+1} \leq t_1 \leq t_p \leq t_n\}} (t_n - t_{p+1})^{-\frac{3}{2}} \\ &\qquad\qquad \times (s_n - s_{p+1})^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \Bigg)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n \\ &\leq \int_{\Delta_t^4} \frac{1}{4} (t_n - s_{p+1})^4 (t_n - t_{p+1})^{-3} (s_n - s_{p+1})^{-3} \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n. \end{aligned}$$

We observe that $(t_n - s_{p+1})^4 \leq (t_n - t_{p+1})^{2+\varepsilon} (s_n - s_{p+1})^{2-\varepsilon}$ and we conclude that

$$\begin{aligned} T_n &\leq \frac{1}{4} \int_{\Delta_t^4} (t_n - t_{p+1})^{-1+\varepsilon} (s_n - s_{p+1})^{-1-\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \mathrm{d}s_n \\ &\leq \frac{1}{4} \int_{\Delta_t^3} (t_n - t_{p+1})^{-1+\varepsilon} \frac{1}{\varepsilon} (s_n - t_n)^{-\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}t_n \mathrm{d}s_n < \infty. \end{aligned}$$

2 K) $0 \leq t_{p+1} \leq s_{p+1} \leq t_1 \leq t_n \leq t_p \leq s_n \leq t$

This situation can be studied in a similar way than situation **2 G)**.

2 L) $0 \leq t_{p+1} \leq s_{p+1} \leq t_1 \leq t_n \leq s_n \leq t_p \leq t$

This situation can be studied in a similar way than situation **2 C)**.

2 M) $0 \leq t_{p+1} \leq s_{p+1} \leq t_n \leq t_1 \leq t_p \leq s_n \leq t$

This situation can be studied in a similar way than situation **2 F)**.

2 N) $0 \leq t_{p+1} \leq s_{p+1} \leq t_n \leq t_1 \leq s_n \leq t_p \leq t$

This situation can be studied in a similar way than situation **2 B)**.

2 O) $0 \leq t_{p+1} \leq s_{p+1} \leq t_n \leq s_n \leq t_1 \leq t_p \leq t$

This situation can be studied in a similar way than situation **2 A)**.

3 A) $0 \le t_1 \le t_p \le t_{p+1} \le s_{p+1} \le s_n \le t_n \le t$

Considering (A.4.1), taking into account the order we have in this case and then integrating with respect to t_p and s_{p+1} we obtain

$$\begin{aligned} T_n &\le \int_{\Delta_t^4} \left(\int_{\Delta_t^2} \mathbb{1}_{\{t_p \le t_{p+1}\}} (t_n - t_1)^{-\frac{3}{2}} (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \right)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n \\ &\le \int_{\Delta_t^3} \left(\int_0^{t_{p+1}} (t_n - t_1)^{-\frac{3}{2}} (s_n - t_1)^{-\frac{3}{2}} (t_{p+1} - t_1) \mathrm{d}t_1 \right)^2 (s_n - t_{p+1}) \mathrm{d}t_{p+1} \mathrm{d}s_n \mathrm{d}t_n. \end{aligned}$$

We now observe that $(t_{p+1} - t_1) \le (s_n - t_1)^{\varepsilon} (t_n - t_1)^{1-\varepsilon}$ and $(t_n - t_1)^{-\frac{1}{2}-\varepsilon} \le (t_n - t_{p+1})^{-\frac{1}{2}-\varepsilon}$. Moreover, $(s_n - t_{p+1}) \le (t_n - t_{p+1})$. Taking all these inequalities into account, we can conclude that

$$\begin{aligned} T_n &\le \int_{\Delta_t^3} \left(\int_0^{t_{p+1}} (s_n - t_1)^{-\frac{3}{2}+\varepsilon} \mathrm{d}t_1 \right)^2 (t_n - t_{p+1})^{-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_n \mathrm{d}t_n \\ &\le \int_{\Delta_t^3} \frac{(s_n - t_{p+1})^{-1+2\varepsilon}}{(\frac{1}{2} - \varepsilon)^2} (t_n - t_{p+1})^{-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_n \mathrm{d}t_n < \infty. \end{aligned}$$

3 B) $0 \le t_1 \le t_{p+1} \le t_p \le s_{p+1} \le s_n \le t_n \le t$

In this situation,

$$\begin{aligned} T_n \le \int_{\Delta_t^4} \Bigg(\int_{\Delta_t^2} \mathbb{1}_{\{t_1 \le t_{p+1} \le t_p \le s_{p+1}\}} (t_n - t_1)^{-\frac{3}{2}} \\ \times (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \Bigg)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n. \end{aligned}$$

If we now integrate with respect to t_p

$$T_n \le \int_{\Delta_t^4} \left(\int_0^{t_{p+1}} (t_n - t_1)^{-\frac{3}{2}} (s_n - t_1)^{-\frac{3}{2}} (s_{p+1} - t_{p+1}) \mathrm{d}t_1 \right)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n,$$

and we use that $(s_{p+1} - t_{p+1}) \le (s_n - t_1)^{\varepsilon} (t_n - t_1)^{1-\varepsilon}$ to get that

$$\begin{aligned} T_n &\le \int_{\Delta_t^4} \left(\int_0^{t_{p+1}} (t_n - t_1)^{-\frac{1}{2}-\varepsilon} (s_n - t_1)^{-\frac{3}{2}+\varepsilon} \mathrm{d}t_1 \right)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n \\ &\le \int_{\Delta_t^3} \left(\int_0^{t_{p+1}} (t_n - t_1)^{-\frac{1}{2}-\varepsilon} (s_n - t_1)^{-\frac{3}{2}+\varepsilon} \mathrm{d}t_1 \right)^2 (s_n - t_{p+1}) \mathrm{d}t_{p+1} \mathrm{d}s_n \mathrm{d}t_n. \end{aligned}$$

Finally we notice that $(t_n - t_1)^{-\frac{1}{2}-\varepsilon} \le (t_n - t_{p+1})^{-\frac{1}{2}-\varepsilon}$ and $(s_n - t_{p+1}) \le (t_n - t_{p+1})$ to conclude that

$$T_n \le \int_{\Delta_t^3} \frac{(s_n - t_{p+1})^{-1+2\varepsilon}}{(\frac{1}{2} - \varepsilon)^2} (t_n - t_{p+1})^{-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_n \mathrm{d}t_n < \infty.$$

③ **C)** $0 \leq t_1 \leq t_{p+1} \leq s_{p+1} \leq t_p \leq s_n \leq t_n \leq t$

Considering the order in this case we have that

$$\begin{aligned} T_n &\leq \int_{\Delta_t^4} \Bigg(\int_{\Delta_t^2} \mathbb{1}_{\{t_1 \leq t_{p+1} \leq s_{p+1} \leq t_p \leq s_n\}} (t_n - t_1)^{-\frac{3}{2}} \\ &\qquad\qquad \times (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \Bigg)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n \\ &\leq \int_{\Delta_t^4} \left(\int_0^{t_{p+1}} (t_n - t_1)^{-\frac{3}{2}} (s_n - t_1)^{-\frac{3}{2}} (s_n - s_{p+1}) \mathrm{d}t_1 \right)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n. \end{aligned}$$

By means of the inequalities $(s_n - s_{p+1}) \leq (s_n - t_1)^{\varepsilon} (t_n - t_1)^{1-\varepsilon}$ and $(t_n - t_1)^{-\frac{1}{2}-\varepsilon} \leq (t_n - t_{p+1})^{-\frac{1}{2}-\varepsilon}$ we obtain that

$$T_n \leq \int_{\Delta_t^3} \left(\int_0^{t_{p+1}} (s_n - t_1)^{-\frac{3}{2}+\varepsilon} \mathrm{d}t_1 \right)^2 (t_n - t_{p+1})^{-1-2\varepsilon} (s_n - t_{p+1}) \mathrm{d}t_{p+1} \mathrm{d}s_n \mathrm{d}t_n,$$

and finally using that $(s_n - t_{p+1}) \leq (t_n - t_{p+1})$ we conclude that

$$T_n \leq \int_{\Delta_t^3} \frac{(s_n - t_{p+1})^{-1+2\varepsilon}}{(\frac{1}{2} - \varepsilon)^2} (t_n - t_{p+1})^{-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_n \mathrm{d}t_n < \infty.$$

③ **D)** $0 \leq t_1 \leq t_{p+1} \leq s_{p+1} \leq s_n \leq t_p \leq t_n \leq t$

In this situation, (A.4.1) becomes

$$\begin{aligned} T_n &\leq \int_{\Delta_t^4} \Bigg(\int_{\Delta_t^2} \mathbb{1}_{\{t_1 \leq t_{p+1}\}} \mathbb{1}_{\{s_n \leq t_p \leq t_n\}} (t_n - t_1)^{-\frac{3}{2}} \\ &\qquad\qquad \times (t_p - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \Bigg)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n \\ &\leq \int_{\Delta_t^3} \left(\int_0^{t_{p+1}} (t_n - t_1)^{-\frac{3}{2}} 2 (s_n - t_1)^{-\frac{1}{2}} \mathrm{d}t_1 \right)^2 (s_n - t_{p+1}) \mathrm{d}t_{p+1} \mathrm{d}s_n \mathrm{d}t_n. \end{aligned}$$

We now notice that $(s_n - t_{p+1}) \leq (s_n - t_{p+1})^{1-2\varepsilon} (t_n - t_1)^{2\varepsilon}$ and $(s_n - t_1)^{-\frac{1}{2}} \leq (s_n - t_{p+1})^{-\frac{1}{2}}$ to obtain that

$$\begin{aligned} T_n &\leq 4 \int_{\Delta_t^3} \left(\int_0^{t_{p+1}} (t_n - t_1)^{-\frac{3}{2}+\varepsilon} \mathrm{d}t_1 \right)^2 (s_n - t_{p+1})^{-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_n \mathrm{d}t_n \\ &\leq 4 \int_{\Delta_t^3} \frac{(t_n - t_{p+1})^{-1+2\varepsilon}}{(\frac{1}{2} - \varepsilon)^2} (s_n - t_{p+1})^{-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_n \mathrm{d}t_n < \infty. \end{aligned}$$

3 E) $0 \le t_1 \le t_{p+1} \le s_{p+1} \le s_n \le t_n \le t_p \le t$

Taking into account this order, we have that

$$T_n \le \int_{\Delta_t^4} \left(\int_{\Delta_t^2} \mathbb{1}_{\{t_1 \le t_{p+1}\}} \mathbb{1}_{\{t_n \le t_p\}} (t_p - t_1)^{-3} \mathrm{d}t_1 \mathrm{d}t_p \right)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n.$$

If we now use that $(t_p - t_1)^{-3} \le (t_p - t_1)^{-\frac{5}{2}+\varepsilon} (s_n - t_{p+1})^{-\frac{1}{2}-\varepsilon}$ we get

$$\begin{aligned} T_n &\le \int_{\Delta_t^4} \left(\int_{\Delta_t^2} \mathbb{1}_{\{t_1 \le t_{p+1}\}} \mathbb{1}_{\{t_n \le t_p\}} (t_p - t_1)^{-\frac{5}{2}+\varepsilon} \mathrm{d}t_1 \mathrm{d}t_p \right)^2 \\ &\qquad\qquad \times (s_n - t_{p+1})^{-1-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n \\ &\le \int_{\Delta_t^3} \left(\int_0^{t_{p+1}} \frac{(t_n - t_1)^{-\frac{3}{2}+\varepsilon}}{\frac{3}{2} - \varepsilon} \mathrm{d}t_1 \right)^2 \frac{(s_{p+1} - t_{p+1})^{-2\varepsilon}}{2\varepsilon} (t_n - t_{p+1}) \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n \\ &\le \frac{1}{2\varepsilon} \int_{\Delta_t^3} \frac{(t_n - t_{p+1})^{-1+2\varepsilon}}{(\frac{3}{2} - \varepsilon)^2 (\frac{1}{2} - \varepsilon)^2} (s_{p+1} - t_{p+1})^{-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}t_n < \infty. \end{aligned}$$

3 F) $0 \le t_{p+1} \le t_1 \le t_p \le s_{p+1} \le s_n \le t_n \le t$

In this situation we have

$$\begin{aligned} T_n &\le \int_{\Delta_t^4} \left(\int_{\Delta_t^2} \mathbb{1}_{\{t_{p+1} \le t_1 \le t_p \le s_{p+1}\}} (t_n - t_{p+1})^{-\frac{3}{2}} \right. \\ &\qquad\qquad \left. \times (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \right)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n \\ &\le \int_{\Delta_t^4} \left(\int_{t_{p+1}}^{s_{p+1}} (t_n - t_{p+1})^{-\frac{3}{2}} (s_n - t_1)^{-\frac{3}{2}} (s_{p+1} - t_1) \mathrm{d}t_1 \right)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n. \end{aligned}$$

Using that $(s_{p+1} - t_1) \le (t_n - t_{p+1})^{1-\varepsilon} (s_n - t_1)^{\varepsilon}$ we conclude that

$$\begin{aligned} T_n &\le \int_{\Delta_t^4} \left(\int_{t_{p+1}}^{s_{p+1}} (s_n - t_1)^{-\frac{3}{2}+\varepsilon} \mathrm{d}t_1 \right)^2 (t_n - t_{p+1})^{-1-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n \\ &\le \int_{\Delta_t^3} \frac{(s_n - s_{p+1})^{-1+2\varepsilon}}{(\frac{1}{2} - \varepsilon)^2} \frac{(s_n - t_{p+1})^{-2\varepsilon}}{2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n < \infty. \end{aligned}$$

3 G) $0 \le t_{p+1} \le t_1 \le s_{p+1} \le t_p \le s_n \le t_n \le t$

We first observe that

$$T_n \leq \int_{\Delta_t^4} \left(\int_{\Delta_t^2} \mathbb{1}_{\{t_{p+1} \leq t_1 \leq s_{p+1} \leq t_p \leq s_n\}} (t_n - t_{p+1})^{-\frac{3}{2}} \right.$$
$$\left. \times (s_n - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \right)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n$$
$$\leq \int_{\Delta_t^4} \left(\int_{t_{p+1}}^{s_{p+1}} (t_n - t_{p+1})^{-\frac{3}{2}} (s_n - t_1)^{-\frac{3}{2}} (s_n - s_{p+1}) \mathrm{d}t_1 \right)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n.$$

By means of the inequality $(s_n - s_{p+1}) \leq (s_n - t_1)^{\varepsilon} (t_n - t_{p+1})^{1-\varepsilon}$ we obtain that

$$T_n \leq \int_{\Delta_t^4} \left(\int_{t_{p+1}}^{s_{p+1}} (s_n - t_1)^{-\frac{3}{2}+\varepsilon} \mathrm{d}t_1 \right)^2 (t_n - t_{p+1})^{-1-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n$$
$$\leq \int_{\Delta_t^3} \frac{(s_n - s_{p+1})^{-1+2\varepsilon}}{(\frac{1}{2} - \varepsilon)^2} \frac{(s_n - t_{p+1})^{-2\varepsilon}}{2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n < \infty.$$

3 H) $0 \leq t_{p+1} \leq t_1 \leq s_{p+1} \leq s_n \leq t_p \leq t_n \leq t$

In this situation we have

$$T_n \leq \int_{\Delta_t^4} \left(\int_{\Delta_t^2} \mathbb{1}_{\{t_{p+1} \leq t_1 \leq s_{p+1} \leq s_n \leq t_p \leq t_n\}} (t_n - t_{p+1})^{-\frac{3}{2}} \right.$$
$$\left. \times (t_p - t_1)^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \right)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n.$$

After integrating with respect to t_p we find that

$$T_n \leq \int_{\Delta_t^4} \left(\int_{t_{p+1}}^{s_{p+1}} (t_n - t_{p+1})^{-\frac{3}{2}} 2 (s_n - t_1)^{-\frac{1}{2}} \mathrm{d}t_1 \right)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n,$$

and now using that $(t_n - t_{p+1})^{-\frac{3}{2}} \leq (t_n - t_{p+1})^{-\frac{1}{2}-\varepsilon} (s_n - t_1)^{-1+\varepsilon}$ we finally obtain that

$$T_n \leq 4 \int_{\Delta_t^4} \left(\int_{t_{p+1}}^{s_{p+1}} (s_n - t_1)^{-\frac{3}{2}+\varepsilon} \mathrm{d}t_1 \right)^2 (t_n - t_{p+1})^{-1-2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n$$
$$\leq 4 \int_{\Delta_t^3} \frac{(s_n - s_{p+1})^{-1+2\varepsilon}}{(\frac{1}{2} - \varepsilon)^2} \frac{(s_n - t_{p+1})^{-2\varepsilon}}{2\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n < \infty.$$

3 I) $0 \leq t_{p+1} \leq t_1 \leq s_{p+1} \leq s_n \leq t_n \leq t_p \leq t$

This situation can be studied in a similar way than situation **3 D)**.

3 J) $0 \le t_{p+1} \le s_{p+1} \le t_1 \le t_p \le s_n \le t_n \le t$

In this case,

$$\begin{aligned} T_n &\le \int_{\Delta_t^4} \bigg(\int_{\Delta_t^2} \mathbb{1}_{\{s_{p+1} \le t_1 \le t_p \le s_n\}} (t_n - t_{p+1})^{-\frac{3}{2}} \\ &\qquad\qquad \times (s_n - s_{p+1})^{-\frac{3}{2}} \mathrm{d}t_1 \mathrm{d}t_p \bigg)^2 \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n \\ &\le \int_{\Delta_t^4} \frac{1}{4} (s_n - s_{p+1})^4 (t_n - t_{p+1})^{-3} (s_n - s_{p+1})^{-3} \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n. \end{aligned}$$

We observe that $(s_n - s_{p+1})^4 \le (t_n - t_{p+1})^{1+\varepsilon} (s_n - s_{p+1})^{3-\varepsilon}$ and we conclude that

$$\begin{aligned} T_n &\le \frac{1}{4} \int_{\Delta_t^4} (t_n - t_{p+1})^{-2+\varepsilon} (s_n - s_{p+1})^{-\varepsilon} \mathrm{d}t_{p+1} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n \\ &\le \frac{1}{4} \int_{\Delta_t^3} \frac{(t_n - s_{p+1})^{-1+\varepsilon}}{1-\varepsilon} (s_n - s_{p+1})^{-\varepsilon} \mathrm{d}s_{p+1} \mathrm{d}s_n \mathrm{d}t_n < \infty. \end{aligned}$$

3 K) $0 \le t_{p+1} \le s_{p+1} \le t_1 \le s_n \le t_p \le t_n \le t$

This situation can be studied in a similar way than situation **3 G)**.

3 L) $0 \le t_{p+1} \le s_{p+1} \le t_1 \le s_n \le t_n \le t_p \le t$

This situation can be studied in a similar way than situation **3 C)**.

3 M) $0 \le t_{p+1} \le s_{p+1} \le s_n \le t_1 \le t_p \le t_n \le t$

This situation can be studied in a similar way than situation **3 F)**.

3 N) $0 \le t_{p+1} \le s_{p+1} \le s_n \le t_1 \le t_n \le t_p \le t$

This situation can be studied in a similar way than situation **3 B)**.

3 O) $0 \le t_{p+1} \le s_{p+1} \le s_n \le t_n \le t_1 \le t_p \le t$

This situation can be studied in a similar way than situation **3 A)**.

This concludes all the cases, finishing the proof of Lemma 3.2.4.

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